GUIDANCE PARAMETERS AND CONSTRAINTS FOR
CONTROLLED ATMOSPHERIC ENTRY

by

Robert C. Duncan, Lieutenant Commander, U.S. Navy
B.S. United States Naval Academy, 1945
B.S. United States Naval Postgraduate School, 1953
S.M. Massachusetts Institute of Technology, 1954

Volume I of II
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Signature of Author

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GUIDANCE PARAMETERS AND CONSTRAINTS FOR CONTROLLED ATMOSPHERIC ENTRY

by

Robert C. Duncan

Submitted to the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, on January 11, 1960, in partial fulfillment of the requirements for the degree of Doctor of Science.

ABSTRACT

Entry of astronautical vehicles into planetary atmospheres is examined in this thesis with respect to interactions of the guidance function, vehicle performance, trajectory prediction, and mission objectives.

All entry missions which originate from planetary reconnaissance orbits are classified into two broad categories: the direct entry profile and the degenerate orbital profile. These classifications are distinguished by the fact that some form of thrust generating mechanism is required to effect controlled entry in the direct entry profile, while engines are not required for controlled entry in the degenerate orbital profile.

Approximate analytical solutions of guidance parameters and constraints are derived for both classes of profiles. As used in this thesis, guidance parameters include:

1. Predicted values of distance flown, range, range-to-go, altitude, velocity, time of flight, and specific force level;
2. Sensitivity of the foregoing quantities to errors and uncertainties in the specification of aerodynamic characteristics of the vehicle, density characteristics of the atmosphere, magnitude and direction of the engine thrust vector.

Constraints are defined as trajectory limitations resulting from:

1. Permissible specific force levels of the vehicle and its human occupants;
2. Heat flow rates to the skin of the vehicle, and stagnation point temperatures;
3. Radiation hazards.
A major shortcoming of numerical studies performed in the conceptual and early design phases of astronautical entry systems is that an infinite number of possible trajectories and guidance schemes must be explored. In order that the engineer understand the effect of changing various design parameters and in order to compare different guidance schemes, simple analytical results, even though only approximate, can be far more informative than a long series of machine computations. The philosophy advanced in this thesis, therefore, emphasizes the use of dynamical approximations, with specified limitations, to derive simple analytical solutions of important guidance quantities.

It is shown in this thesis that the trajectories of both the direct and degenerate orbital profiles possess three distinct operational regimes, defined as the Keplerian, Intermediate, and Gas-Dynamic Phases. A parameter, called the Conservation Parameter, is defined to specify precisely the boundary conditions between these operational phases. The Conservation Parameter is an index of the influence of the operating environment on vehicular motion and may be used as a switching function for the guidance computer and as an external aid for adaptive control of the vehicle.

Three-dimensional dynamical equations of motion are developed for entry of lifting or non-lifting vehicles in banking or wings-level flight with variable thrust capabilities into the atmosphere of oblate, rotating planets with or without atmospheric winds. The figure of the planet, the gravitational model, and the atmospheric model for important bodies of the solar system are summarized. Special problems associated with first-time entry into the atmospheres of strange planets are discussed.

Thesis Supervisors:  Dr. Walter Wrigley
Professor of Instrumentation and Astronautics

Paul E. Sandorff
Associate Professor of Aeronautics and Astronautics

Dr. Winston R. Markey
Assistant Professor of Aeronautics and Astronautics
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The publication of this report does not constitute approval by the Navy, the Air Force, or the Instrumentation Laboratory of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.
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Chapter 1

INTRODUCTION

1.1 Object

The object of this thesis is to formulate analytical techniques for specifying important parameters and constraints necessary in the preliminary development of guidance concepts for a wide class of entry missions into planetary atmospheres.

To be of maximum efficacy, analytical techniques and solutions formulated must be of sufficient generality as to apply to many vehicles and missions, of sufficient utility to enable a clear understanding of the relation between trajectory and performance, of sufficient simplicity to enable rapid determination of numerical answers from a diversity of initial conditions, and of sufficient accuracy when compared to more exact machine computations that engineering decisions can be deduced with confidence.

1.2 Guidance Parameters and Constraints

As used in this thesis, guidance parameters include:

(1) Predicted values of distance flown, range, range-to-go, altitude, velocity, time of flight, and specific force level.

(2) Sensitivity of the foregoing quantities to errors and uncertainties in the specification of aerodynamic characteristics
of the vehicle, density characteristics of the atmosphere, magnitude and direction of the engine thrust vector.

Guidance constraints are defined as trajectory limitations imposed by:

(1) Permissible specific force levels of the vehicle and its human occupants.
(2) Heat flow rates to the skin of the vehicle, and stagnation point temperatures.
(3) Radiation hazards.

1.3 The Family of Astronautical Vehicles

Astronautical vehicles existing today may be classified into various types according to the specific function for which each is designed. Examples of these classes are sounding rockets, ballistic missiles, Earth satellite research vehicles to accumulate data concerning the Earth and its environment, meteorological satellites, manned research vehicles (such as the X-15), lunar and solar probes.

Vehicles under development for use in the immediate future extend the above list to manned orbital vehicles (such as Projects Mercury and Dyna-Soar), unmanned planetary probes, navigation satellites, reconnaissance satellites, communication-relay satellites, and special purpose satellites to map more extensively the recently discovered radiation patterns in space, to verify Einstein's relativity theories, etc. A host of other astronautical vehicles have been suggested, most of which depend on successful completion of projects already in progress. Among these are international inspection satellites, manned orbital space stations, manned lunar and planetary reconnaissance vehicles, hypersonic
gliders for transportation and other logistic functions, astronautical vehicles designed to perform specific military functions such as to destroy missiles and satellites, to perform military reconnaissance functions, to jam communication facilities, to serve as bombing platforms, to exercise supply and logistic functions.

The above enumeration of astronautical vehicles frequently referenced in the unclassified literature is by no means a complete tabulation; yet, it is a dramatic demonstration of the rapid growth of the space family in the few years since the birth of the V-2 rocket, which must surely be classified as the father of this modern family. The spectacular growth of this family suggests a truly fertile expansion in the future.

This thesis is devoted to an investigation of a special portion of the trajectory of some of these vehicles, namely that portion in or near the atmosphere of a planet or a natural planetary satellite. The investigation is sufficiently general as to apply to manned and unmanned vehicles, to a diversity of mission concepts, and to a wide class of vehicles both lifting and non-lifting. The study is oriented toward determination of optimal guidance requirements for directing the vehicle to a particular landing site selected prior to entry. Emphasis is placed on the manned vehicle and the guidance constraints resulting from its human occupants. General constraints imposed by the operational environment are considered, with special emphasis on atmospheric constraints.

It is intended in this introductory chapter to show how the work of this thesis fits into the framework of the entry mission concept and into the large body of previous work recorded in the literature.
Specific conclusions and recommendations for further study are advanced in Chapter 2. Where appropriate in relating the work of this thesis to the broad problem of guidance and control systems design, general conclusions are proffered in this introductory chapter.

1.4 Manned Planetary Missions

Taylor and Blockley (1) have stated eloquently man's motivation for space exploration:

"The feat of putting man into orbit and the final triumph of interplanetary and even interstellar exploration are supreme human goals, transcending the purely pragmatic question of which performs more efficiently, man or machine."

A competent team of trained and motivated men cannot be surpassed by machine to perform exploratory functions, as contrasted to reasonably well-established routine functions (such as data collection and computation). The machine can do only those functions for which it is designed; it cannot seek new functions beyond the scope of its sensory transducers. The design of the machine may be elaborate in order to encompass limited decision-making capabilities, the ability to adapt its performance to a changing environment, the faculty for rapid determination, computation, cataloging, and storing of data. A machine or group of machines that can perform an almost unlimited diversity of tasks such as the following, however, will probably never be built in a package weighing less than 200 pounds:

(1) Correlate many non-related observations.

* Superscript numerals refer to similarly numbered references in the Bibliography.
(2) Readily perform multifarious physical tasks, in any order, such as moving from place to place, replacing or repairing a faulty engine and electronic component, controlling a flying vehicle, collecting minerals and plant life, comparing observations with data previously accumulated, etc.

(3) Make rapid decisions based on seemingly independent events.

(4) Has intuition*.

Man has a unique facility for exercising judgment; he can reason inductively and has the ability to draw inferences from isolated elements of one situation and apply them to another. The trained human being has the capacity to analyze problems never before encountered and to make decisions on the basis of general rather than specific experience. He is a valuable technical trouble shooter and can better assure reliable operation of all equipment on the astronautical vehicle by continuous checking, repairing, and replacing of faulty equipment. It has been demonstrated over a half-century of flight that he is a versatile flight controller; there is much expert opinion to substantiate the belief that he can exercise this function to a useful extent even at escape velocities.

The planning of an astronautical mission and the design of the vehicle and its guidance and control system is a problem of optimum system design to effect a successful mission with maximum flexibility.

* "Immediate apprehension or cognition; the power of knowing or the knowledge obtained without recourse to inference or reasoning; insight; familiarity, a quick or ready apprehension." (Webster's Collegiate Dictionary).
and reliability, minimum initial weight, and the most efficient power utilization. A realistic approach to analyzing the mission problem is to assign to humans those functions performed best by them and to relegate to machines the functions that they can perform more efficiently. Taylor and Blockley state man's role:

"Man will perform primarily as a strategist, as a correlator of data at many probability levels, a maker of insightful interpretations from these data, and a formulator of effective plans of action. Although it is possible to conceive of a machine that could perform some or all of these functions, it would be very large, complicated, and expensive.

Man is needed as a scientific observer of space phenomena. He can act to control the instrumentation toward the most significant and reliable readings, to make the unexpected observation, and to gain basic insights into the pattern of results. Although machines could conceivably do this job, man will want to do the job personally, whether required or not."

Before any attempt is made to explore the surface of a strange planet, sufficient data regarding the planet must be accumulated in advance to demonstrate that the potential benefits of such an exploration are great enough to justify not only the tremendous cost but also the attendant risk to human life. Preceding the manned mission will be a certain number of unmanned exploratory probes in order to collect data on radiation hazards, gravity, magnetic field, atmospheric properties such as temperature, density and wind characteristics, terrain mapping, etc. Manned exploration of the vicinity of the planet, which may or may not culminate in immediate atmospheric penetration to a landing, should follow relatively few unmanned probes, in the opinion of the author.

It is interesting to compare some of the debits and credits which enter the engineering ledger as a result of designing a mission system
to include the human crew. The man excels over the machine in:

1. Detecting minimal changes in visual or auditory stimuli;
2. Perceiving, in the presence of noise, meaningful patterns and information;
3. Choosing new courses of action with great flexibility and adaptability when circumstances change unexpectedly;
4. Storing tremendous quantities of data for long periods of time and recalling the required relevant information rapidly.*

Engineering liabilities incurred by incorporating man in the system include the following:

1. Much engineering effort is required to make the astronautical vehicle habitable and safe due to the inherent vulnerability of man to the space environment.
2. Engineering provision must be made for living, rest, and recreation that go beyond a utilitarian minimum because more is required of the man than mere survival. He must perform at top efficiency, a requirement closely coupled with morale and physiological considerations.
3. Equipment required** to make man effective in his duties are costly in weight, size, complexity, and dollars.

The net cost of each unmanned planetary probe, in terms of man

* It was estimated in reference (1) that the brain's storage capacity is of the order of 2 to 100 times \(10^6\) bits. This compares to storage of the order of \(10^5\) bits in a typical modern high-speed computer.

** Such as human-operator stations, data-processing and storage facilities, sensory displays, etc.
hours and natural resources expended versus data collected, is sufficiently great that emphasis should be placed on the manned mission as early as possible. The potential wealth of knowledge that may be obtained versus expenditure of resources and money is greater, by orders of magnitude, in manned exploration as contrasted to unmanned probes, even though each manned vehicle must itself be more elaborate and costly. Therefore, there is strong justification for manned exploration to follow relatively few unmanned probes; the latter should be undertaken in such numbers only to establish that the risk factor* to man is within acceptable bounds.

In the investigation described subsequently in this thesis, a specific guidance system is not considered in detail, hence the advantages and disadvantages of using man in contrast to machine to perform specific guidance functions are not debated. Major emphasis is placed, however, on an important consideration in systems design of the manned entry vehicle; viz., constraints on the trajectory resulting from allowable human acceleration loads.

1.5 Dynamics of Entry

The analysis of the flight of the entry vehicle is a particular application of the general theory of the dynamics of rigid bodies in three dimensions. It is usual in such theory to separate the motion of the center of mass from the motion of the body around the center of mass. The theory of flight performance embodies the former, the theory of stability and control the latter.

* Such as from radiation and meteor hazards, launch and recovery failure, operational reliability, etc.
The analytical work of this thesis is concerned specifically with performance analysis and the relation of guidance to performance. Stability and control functions are recognized where it is considered appropriate to emphasize their coupling with performance*. In this thesis, therefore, the entry vehicle appears as a point particle of varying mass subject to various forces. Research was commenced with investigation of the three-dimensional equations of motion of a variable mass particle in the vicinity of an oblate rotating planet possessing an atmosphere with winds. The particle was generalized to one capable of generating lift, drag, and thrust forces and capable of generating side forces by banking in order to induce trajectory curvature in the horizontal plane.** As a practical matter, the trajectory was intentionally confined to a fixed plane in much of the analytical work of the thesis. Assumptions required and limitations on the resulting solutions as a consequence of using the theory of planar motion are discussed in detail.

One of the difficulties encountered in any practical study of the dynamics of entry results from the uncertain knowledge of many of the important parameters which influence the trajectory. The lift and drag coefficients of the vehicle at high velocities may differ considerably from expected values since wind tunnel data and other experimental data are fragmentary for velocities in excess of Mach 10 to 15. Atmospheric properties, particularly the densities existing at higher altitudes,

* See, for example, Appendix F.

** In order to simplify the difficult task of the reader in keeping track of the many coordinate frames, symbols, and constants used in this thesis, the author has defined and tabulated all such quantities in Appendix A.
are subject to much uncertainty.

The density of the Earth's atmosphere decreases with altitude until it merges with the very thin solar atmosphere. Chapman\(^{(2)}\) advances the theory that the sun possesses a tenuous atmosphere which extends through interplanetary space beyond the distance of the Earth's orbit. He suggests that this solar atmosphere consists mainly of ionized hydrogen, protons, and electrons, with a density of about \(10^3\) particles per cubic centimeter in the vicinity of the Earth. The solar atmosphere has little effect on the entry mission as conceived in this thesis since the planetary atmosphere is of considerably greater significance at the altitudes considered herein. The solar atmosphere, however, may have a significant effect on interplanetary trajectories. All studies of interplanetary trajectories performed to date have neglected any atmosphere in space; the minute drag terms may be important when integrated over many months.

Lift and drag are the primary gas-dynamic forces influencing the trajectory of the entry vehicle. It is usual to express these in terms of a coefficient based on the frontal area \(S\) of the vehicle. For example:

\[
\text{Drag} = \frac{1}{2} \rho C_D S V^2
\]

\[
\text{Lift} = \frac{1}{2} \rho C_L S V^2
\]

where \(\rho\) is the local free-stream atmospheric density and \(V\) is the magnitude of the vehicle's velocity vector with respect to the atmosphere. Lift and drag coefficients depend on the configuration of the vehicle, on the angle of attack and angle of yaw, and on the flight Mach number, \(M_a\).

Even if \(C_L\) and \(C_D\) were independent of velocity, the insertion of
the lift and drag terms in the equations of motion make it impossible to integrate the dynamical equations of motion except under certain specific restrictions. A number of methods for circumventing this difficulty are discussed in this thesis.

Drag perturbations on Vanguard I and Sputnik III which coincided* with solar flares have led to the suggestion that electrical drag of satellites may be important\(^{(3)}\). Electrical drag may be produced by the motion of a charged satellite through clouds of charged particles. The vehicle can pick up a negative charge, for example, by colliding with high energy electrons in a radiation band. The charge on the vehicle repels other electrons, creating a shell of net positive charge around the vehicle. The area of this shell is the effective drag area of the vehicle. The increased drag area leads to increased atmospheric resistance to motion of the vehicle due to collision with other particles. Electrical drag may be important in perturbing the motion of long lifetime satellites and interplanetary vehicles; it is a higher order effect on the entry mission and is not considered in this analysis.

It is shown in this thesis that the altitude at which the planetary atmosphere is important in vehicle guidance depends strongly on flight path angle, lift, drag, and density characteristics of the entry vehicle. The heating and deceleration loads which accompany high velocity entry into the atmosphere are strong functions of initial penetration angle and vehicle design characteristics.

* The drag increased about a day or so after the intense solar activity was observed. This lends credence to the theory that particles emitted from the sun, and not radiation, caused the drag perturbations.
1.6 The Entry Mission

The entry mission, as defined in this thesis, is that portion of an astronautical vehicle's consignment which starts from a reconnaissance orbit in the vicinity of a planet, asteroid, or natural satellite of a planet, and ends with a landing on the surface of the body.

The Reconnaissance Orbit

The reconnaissance orbit may fall into one of two main categories, called in this thesis the stable orbit and the degenerate orbit. These two major classes of reconnaissance orbits are shown in Fig. 1.1. The significant difference between the two types of reconnaissance orbits is that an engine or some other form of external thrust generating mechanism is required in order to initiate controlled entry from the stable orbit, while entry may be effected without an external thrust generating system in the degenerate orbit.

It is difficult to draw a clear line of demarkation between the two classes of reconnaissance orbits, except in terms of total satellite lifetimes. A vehicle in a stable orbit experiences little or no atmospheric perturbations, hence it has a long lifetime (months or years). A vehicle in a degenerate orbit encounters significant perturbations in the vicinity of perigee*; therefore, its apogee altitude decays rapidly until the orbit is near circular. This phase of orbital flight is frequently called the "circularization" process; i.e., the ellipse decays to a circle. The lifetime of the vehicle in a degenerate orbit is short (hours or days). Gravitational perturbations, which lead to

---

* Definition of terms frequently used to describe elliptical orbital motion is given in Appendix A.
Fig. 1.1: Two Major Classes of Reconnaissance Orbits.

(a) A stable reconnaissance orbit.

(b) A degenerate reconnaissance orbit.
phenomena such as regression of the line of nodes and movement of the line of apsides (advance of perigee), are considered briefly in this thesis. The major perturbation of interest in this work, however, is due to the planetary atmosphere; all other perturbations are generally higher order effects.

Methods by which the vehicle establishes the reconnaissance orbit are beyond the scope of this thesis. It is noted that both classes of reconnaissance orbits are planned for manned Earth orbital missions. Eggers\(^4\) suggests using the degenerate reconnaissance orbit as a transition phase between the transfer ellipse and the entry trajectory in interplanetary missions. The interplanetary transfer ellipse terminates in a series of braking passes through the atmosphere of the planet; each pass reduces the total energy of the vehicle by transfer of energy to the planetary atmosphere. In this way, the need for using engines to reduce total vehicle energy is avoided; higher payloads are possible because very little vehicle mass must be invested in propellent fuel.

There are serious disadvantages to Egger's mission concept:

1. Extreme guidance accuracy is required in order to graze the planetary atmosphere at just the right altitude for proper energy transfer and, concurrently, not incur excessive deceleration and heating loads. The perigee altitude of the first pass, the total energy transferred, vehicle heating rates, and deceleration loads are extremely sensitive to small aiming errors in the interplanetary transfer ellipse. Trajectory corrections of this transfer ellipse near the destination planet are almost mandatory.
The apogee of the first few orbital ellipses may be sufficiently high above the planet to encounter severe radiation hazards from the Van Allen belts.* Though Van Allen belts around planets other than Earth have not been detected, it is reasonable to assume that they may exist around any planet possessing a magnetic field. Drake(5) has advanced the theory that the radio waves from Jupiter originate from the Jovian version of the Van Allen belts. He estimates radiation intensities to be 100 times as strong in the Jovian belts as in the Earth belts.

Hazards from planetary radiation belts conceivably could make multiple atmospheric braking passes an unpleasant experience. It may be more profitable to accept the weight penalty of rocket braking in the interest of radiation safety.

The stable reconnaissance orbit can be established either by using rocket braking directly, or by suitable application of rocket thrust after energy is reduced in one braking pass. The second method has the advantage of reducing the total propellant mass required, the disadvantage of

* The radiation belts named after Dr. James A. Van Allen, who first disclosed their discovery, were detected by the Army's Pioneer III lunar probe launched in December 1958. There are two main belts containing energetic particles trapped in the Earth's magnetic field. Pioneer III showed the inner belt to be centered approximately 2400 miles above the Earth and the outer belt centered at 10,000 miles. Maximum radiation rate was estimated to be 5-10 Roentgens per hour. A comparatively free area exists between the two belts; for example, the measured radiation rate at 6000 miles was 0.3 Roentgens per hour. Pioneer IV (March 1959) reported particle densities in the outer belt to be many times greater than those measured by Pioneer III. It is believed that the particles in these belts are of solar origin.
requiring extreme guidance accuracy in setting up the braking pass.

A distinct advantage of establishing a stable orbit is that the time and location for initiating final entry by retro-thrust may be freely chosen. Extensive reconnaissance of the surface of the planet is possible and a wise selection made of a site for eventual landing. If a landing appears unwise for any reason, the stable reconnaissance orbit is a relatively easy staging point for starting the return mission. An attractive feature in establishing a stable reconnaissance orbit, particularly for first time entry into strange planets, is that a navigational satellite can be left in this orbit as a source of external tracking information in the absence of ground based tracking stations.*

A single-pass landing has also been suggested for planetary entry(6). The reconnaissance trajectory is omitted in this concept of the interplanetary mission. The vehicle is braked by the atmosphere to a landing in its initial encounter with the atmosphere. The vehicle must be guided down a narrow corridor if gas-dynamic drag is to reduce the velocity sufficiently for landing without producing accelerations beyond human tolerances. The Earth's landing corridor for a non-lifting vehicle is less than seven miles wide, for a lifting vehicle the corridor may be broadened to 60 miles. The corridor for Venus is about like that of the Earth, for Mars the corridor is broader, and for Jupiter the corridor is much more narrow.

The guidance problem of traversing such a corridor during the first pass from an interplanetary journey is obviously severe and the ability to land at a predetermined location is subject to many

* See Appendix D.
uncertainties. The single-pass entry mission is a special case of the direct entry profile.* It is special in the sense that:

(1) The initial velocity for the entry mission is greater than orbital velocity;

(2) It is the only method for generating the direct entry profile without using engines.

If a reconnaissance orbit is part of the planetary mission, it is considered herein as prescribing the initial conditions of the entry phase of the mission. Analytical techniques are developed for approximate determination of guidance parameters and constraints for entry from both elliptical and circular, stable and degenerate, reconnaissance orbits.

1.7 Entry Profiles

In general, entry systems may be classified into two broad types: lifting and non-lifting. Precision landing point control is obtainable with either system, providing the non-lifting vehicle has variable drag capabilities. No comparison is made in this thesis with respect to the relative merits of one system versus the other; an analytical framework is presented which will enable analysis of either type.

Two dissimilar entry profiles are analyzed in this thesis:

(1) The direct entry profile is generated in either of the following two ways:

(a) Large scale perturbations of the stable reconnaissance orbit by means of retro-rocket thrust forces;

* See section 1.7 for definitions of the classes of entry profiles.
(b) First pass termination of an interplanetary transfer ellipse.

(2) The degenerate orbital profile consists of a series of braking passes through the outer reaches of the atmosphere in order to reduce the energy level preparatory to final direct entry. The degenerate orbital profile is the consequence of entering from (a) the degenerate reconnaissance orbit, or (b) minor retro-rocket perturbations of the stable reconnaissance orbit.

The entry trajectory may be viewed as made up of a number of separate functional phases. Fig. 1.2 shows the functional phases for both classes of entry profiles.

The functional phases of the direct entry profile are:

(1) **Orbital phase:** The vehicle is in a stable reconnaissance orbit; this is at all times beyond altitudes where the planetary atmosphere has a significant effect on the trajectory. This phase is omitted if entry is made during the first atmospheric perturbation of an interplanetary transfer ellipse; viz., the "first pass" entry mission concept.

(2) **Departure Phase:** Entry is initiated at the trajectory modification point by application of retro-rocket thrust forces. The departure phase terminates when engines have flamed out. This phase may be omitted if entry is made during the first pass of an interplanetary mission.

(3) **Free-fall phase:** Consists of that portion of the trajectory from the point where departure thrust is "turned off" to the point where significant gas-dynamic forces (e.g., lift, drag) are generated.
Fig. 1.2: Functional Phases of Entry

(a) The Direct Entry Profile
Note: Only two perturbation passes are shown for convenience; perigee separation is grossly exaggerated for pictorial clarity.

(b) The Degenerate Orbital Entry Profile

Fig. 1.2 (continued) Functional Phases of Entry
(4) **Approach phase:** The atmospheric portion of the trajectory traversed from the point where gas-dynamic forces are measurable to the point where landing is initiated.

(5) **Landing phase:** That portion of the trajectory, near the intended destination and characterized by low altitudes and velocities, which terminates in either impact with the surface or a glide landing to rest.

The functional phases of the degenerate orbital profile are shown in Fig. 1.2 (b). These are:

1. **Orbital phase:** The vehicle is in a degenerate reconnaissance orbit consisting of two sub-phases:
   - **Free-fall phase:** Consists of that portion of the elliptical orbit beyond the sensible atmosphere.
   - **Perturbation phase:** Consists of that portion of the elliptical orbit near perigee where significant energy is transferred from the vehicle to the atmosphere.

2. **Approach phase:** Same as for the direct entry profile.

3. **Landing phase:** Same as for the direct entry profile.

In this thesis, it is shown that a more convenient division of the trajectory for guidance analysis is in terms of the operational regimes or operational phases. The three operational phases were defined as follows:

1. **Keplerian Phase:** That segment of the trajectory, at high altitudes, where gas-dynamic forces are insignificant. ($\mathcal{O} \approx 0$)

2. **Intermediate Phase:** That segment of the trajectory where accelerations due to gas-dynamic forces are of comparable
magnitude with other terms in the dynamical equations of motion.

(3) Gas-Dynamic Phase: That segment of the trajectory where gas-dynamic accelerations are the predominant terms affecting the shape of the trajectory.

In order to specify precisely the boundary conditions between phases, an auxiliary quantity was defined. It was shown in Chapters 8 and 9 that this auxiliary quantity, called the Conservation Parameter*, has a sharp behavior at certain critical points in the entry mission and is a valuable index of the influence of the atmosphere on the vehicle and its trajectory. The Conservation Parameter was defined as follows:

\[
\text{Conservation Parameter} = \left( \frac{1}{\xi} - \frac{\ddot{R}}{R} \right) \frac{V_{I\phi}}{V_{I\phi}}
\]

\[
\text{where } \xi = \frac{\ddot{R}}{R} \frac{V_{I\phi}}{V_{I\phi}}
\]

This is equivalent to the following definition:

\[
\text{Conservation Parameter} = \left| 1 - \frac{\ddot{R}}{R} \right|
\]

\[
\text{where } \xi = \frac{\ddot{R}}{R} \frac{V_{I\phi}}{V_{I\phi}}
\]

R is the distance of the entry vehicle from the planet center and \( V_{I\phi} \)

* The name Conservation Parameter was selected because of its close relation to conservation of energy and angular momentum:

Time rate of transfer of angular momentum: \( p' = -(\text{Cons. Parameter})h'v_{I\phi} \)

Time rate of transfer of energy: \( \frac{E'(\text{kin})}{E'(\text{pot})} \approx -[\text{Cons. Parameter}]v_{I\phi}^2 r + 1 \)
is the horizontal component of velocity with respect to inertial coordinates.

It is shown in this thesis that the Conservation Parameter may be computed from navigational data. It has potential application as an environmental index for adaptive control systems and as a switching function for the guidance computer.

In terms of the Conservation Parameter, the operational phases of the entry trajectory were defined as follows:

(1) **Keplerian Phase:**
A true Keplerian trajectory exists only when the Conservation Parameter is zero. The trajectory is near-Keplerian when the magnitude of the Conservation Parameter is less than 0.1.

(2) **Intermediate Phase:**
The magnitude of the Conservation Parameter is greater than 0.1 and less than 10.

(3) **Gas-Dynamic Phase:**
The magnitude of the Conservation Parameter is greater than 10.

The phase boundary between the Keplerian and Intermediate phases is at the point where the magnitude of the Conservation Parameter is 0.1. The phase boundary between the Intermediate and Gas-Dynamic Phases is at the point where the magnitude of the Conservation Parameter is 10. Fig. 1.3 shows the operational phases for the two classes of entry profiles.

1.8 Guidance

The guidance process consists of measurements of vehicle position
Boundary Conditions between Keplerian and Intermediate Phases:
Conservation Parameter = 0.1

Boundary Conditions between Intermediate and Gas-Dynamic Phases:
Conservation Parameter = 10

(a) The Direct Entry Profile

Fig. 1.3 Operational Phases of Entry
Boundary Condition between Keplerian and Intermediate Phases:
Conservation Parameter $= 0.1$

Boundary Condition between Intermediate and Gas-Dynamic Phases:
Conservation Parameter $= 10$

Note: Only two perturbation passes are shown for convenience; perigeeal separation is greatly exaggerated for pictorial clarity.

(b) The Degenerate Orbital Entry Profile

Fig. 1.3 (continued) Operational Phases of Entry
and velocity, computation of control actions necessary to properly adjust position and velocity, and delivery of suitable adjustment commands to the vehicle’s control system. The guidance system, of which an human pilot may or may not be an integral part, operates as a force vector control system; i.e., the system must control the direction and magnitude of the force vector* in such a way that the vehicle eventually arrives safely at the desired destination.

Any method for guiding the entry vehicle to a selected geographic point on the surface of the planet must necessarily involve a perturbation of its original orbit. Because of the tremendous energy possessed by the vehicle in orbit, it is difficult to perturb the orbit greatly without investing a large portion of the total mass of the vehicle in rocket fuel.

The trajectory that follows the initiation of entry from a satellite orbit beyond the sensible atmosphere is a part of the general problem of transfer between orbits that has received considerable attention in the literature. The special nature of the entry problem, however, requires solutions that are not available in general treatises on the subject of transfers between orbits.

Various methods are available for perturbing the satellite orbit to bring about controlled entry:

1. Impulsive application of forces at the trajectory modification point by means of chemical rockets or other high thrust propulsive systems.

* The controllable force vector is the vector sum of all external forces (lift, drag, thrust) that are capable of being adjusted by the guidance system.
(2) Continuous application of low thrust (e.g., ion rockets);
(3) Multiple impulses at intermediate* thrust levels;
(4) Drag modulation, if portions of the orbit pass through the planetary atmosphere.

Fig. 1.4(a) shows a generalized functional diagram of the guidance and control system for the entry mission. It is to be emphasized that this is a functional diagram and not an operational component diagram; for example, the digital computer, a key component of the system, performs computations for all four functional elements shown in Fig. 1.4(a).

The generalized entry trajectory is shown in Fig. 1.4(b). This trajectory may be controlled by use of continuous, piecewise continuous, or impulsive thrust forces as commanded, and, in the atmospheric portion of flight, by suitable adjustment of lift and drag forces. Attitude may be controlled by gas-dynamic moments and/or reaction control**. In the Keplerian phase of the mission, thrust is the only means by which substantial trajectory alterations can be made. If the trajectory modification point shown on Fig. 1.4(b) is at sufficiently high altitudes for lift and drag forces to be negligible, then thrust is the only means by which entry can be initiated.

A logical step in the direction of simplifying the generalized guidance and control system of Fig. 1.4 is to assume that high thrust impulses are utilized early in the entry mission to initiate and adjust

---

* Thrust levels between those obtainable with chemical rockets and with continuous ion propulsion.

** Reaction control, in the form of expulsion of gases from strategically located nozzles, is a useful method of generating forces to supplement weak aerodynamic forces in the maintenance of vehicular attitude. It is necessary to use reaction control at high altitudes, or in the maintaining of certain extremes in attitude.
Interferences (Atmospheric Disturbances)

(1) Stored Data
(2) Guidance Computer
(3) Vehicle, Engines Control System
(4) Navigation and Tracking System

Flight Path

Computed or Measured Position

Command Input

(a) Guidance System.

Fig. 1.4: Functional Diagram of Generalized Guidance and Control System for the Entry Mission
Trajectory Modification Point

Initial Reconnaissance Orbit

Entry Trajectory
(Trajectory controlled by use of continuous, piecewise continuous, or impulsive thrust forces and by suitable adjustment of lift and drag forces.)

(b) Generalized Entry Trajectory.

Fig 1.4 (cont.): Functional Diagram of Generalized Guidance and Control System for the Entry Mission.
the recovery trajectory, after which the engines are cut off. Gas-
dynamic forces are utilized for all subsequent trajectory modifications. 
This scheme permits separation of the computation functions for engine 
control and the computation functions for control of the vehicle's 
aerodynamic surfaces.

Fig. 1.5 shows a function diagram of a guidance and control system 
for which engine commands and control of the aerodynamic surfaces are 
uncoupled. Guidance and control systems of this type are generally 
adopted for almost all classes of astronautical missions. Ballistic 
missiles, for example, omit the guidance computer, box (2), except for 
attitude stability and control; the ballistic missile's trajectory is 
not adjusted after engine cut-off. The basic differences in guidance 
system design among the various classes of ballistic missiles are in the 
stored data, the computations performed to determine engine command 
inputs, and the navigation or tracking scheme employed. Some systems 
rely on precomputation of many coefficients which do not vary radically 
with time and storing this information, thus reducing significantly the 
computational load required of the computer. The navigation and tracking 
system generally falls into one of three major categories; inertial, 
radio, or a combination of the two.

Satellite launching systems also utilize the guidance system of 
Fig. 1.5. For trajectories which require extreme precision, such as 
lunar probes, some of the computations are performed on the ground and 
command signals are transmitted by radio to the vehicle for mid-course 
vernier corrections.

The entry mission differs from the examples cited previously in 
one major respect; i.e., the most important period of the flight is in
Fig 1.5: Functional Diagram of Guidance and Control System in Which Engine Commands and Control of Aerodynamic Surfaces are Uncoupled.
the atmosphere of the planet where gas-dynamic forces are extremely influential. The atmosphere, like fire and water, can be of invaluable aid if properly harnessed and can lead to disaster if permitted to reign unchecked; witness the parachute on one hand versus the flaming meteor on the other. The relative importance of boxes (2) and (5) in Fig. 1.5 is reversed in the case of the entry mission as compared to launching a satellite or ballistic missile.

Most papers on entry vehicle guidance are devoted to an investigation of the aerodynamic loop in Fig. 1.5, neglecting entirely the engine loop. One of the purposes of this thesis was to consider the entire system as a unity. Major consideration is given to optimal determination of inputs to box (3). For example, Chapter 9 presents graphical and analytical methods for determining optimum engine gimbal angle* for entry by impulsive thrust application at the trajectory modification point from both circular and elliptical reconnaissance orbits.

1.9 Some Suggested Guidance Systems

Various guidance schemes have been proposed for entry missions. Among the more interesting ones are:

(1) Eggleston and Cheatham(8) suggest entry in a variable-lift vehicle flying at near 90° angle of attack in order that large drag forces may be experienced, thus ensuring rapid entry. By use of small amounts of lift, the decelerations may be maintained at moderate levels**

* Optimum engine gimbal angle is defined as that angle for which errors in gimbal orientation result in minimum range error.

** See Chapter 9 of this thesis for a quantitative discussion of the effect of lift in reducing accelerations.
and, at the same time, navigational errors may be corrected. By keeping the total drag high, prolonged periods of time and heating in the atmosphere are avoided.

Two trajectory control schemes were proposed by these authors, one involving acceleration control and the other range control. Figs. 1.6 and 1.7 show functionally these two concepts. Analytical solutions derived in this thesis are readily adapted for preliminary investigation of either of these guidance schemes. Range expressions as a function of time derived in Chapter 9, for example, provide a simple method for determining range-to-go.

(2) Detra, Riddle, and Rose(9) envision a guidance scheme for the non-lifting vehicle which is identical in functional form to that shown in Fig. 1.7. The computation requirements are considerably different, however, since range control is effected through varying the drag parameter \( \frac{M}{C_D S} \). Changes of drag parameter by 20 to 1 are possible with a variable high-drag umbrella. The input \( \Delta \alpha \) to the control system in Fig. 1.7 is replaced by \( \Delta(\frac{M}{C_D S}) \) in this particular system.

(3) Webber(10) suggests the use of ballistic perturbation theory to simplify the complexity of the guidance computer during the atmospheric gliding portion of the entry trajectory. The object of the perturbation theory is to calculate the behavior of the entry vehicle near a nominal trajectory.* If the actual flight conditions differ from those for the nominal trajectory, the trajectory of the vehicle will not lie along the nominal path. If the disturbing influences are small,

* The nominal trajectory is a certain trajectory with specified initial conditions, atmospheric conditions, and programmed lift and drag.
\( \Delta \alpha \) = change in angle of attack from trim condition.

\( F_z \) = normal accelerations, external specific force measured along z-axis.

\( \dot{F}_z \) = computed rate of change of normal accelerations, computed from measured values of \( F_z, V, \dot{V}, H \), and stored data on atmospheric properties as a function of altitude.

\( K_1, K_2 \) = constants

\( V, \dot{V} \) = velocity and time rate of change of velocity

\( H \) = time rate of change of altitude.

Fig. 1.6 Guidance System which Commands Angle of Attack from Normal Specific Force Measurements
Guidance Computer

$\Delta \alpha = k_1 (\Delta X_{TG}) + k_2 (\Delta \dot{X}_{TG})$

Fig. 1.7: Guidance System which Commands Angle of Attack from Error in Range-to-go from Reference Trajectory
the perturbed trajectory will be in the vicinity of the nominal trajectory. The nearness to a calculated, known trajectory is the basis of linearization of the equations of motion for the perturbed trajectory; the perturbed system is then represented as a linear system with time-varying coefficients.

Fig. 1.8 is a functional representation of Webber's guidance scheme. For convenience, the stored data and the guidance computer boxes are shown in two parts. Dashed lines represent information flow early in the flight; solid lines represent information flow during the major portion of the mission. This guidance system is conceived as one for which the nominal trajectory is specified in terms of acceleration components in the lift and drag directions. Because inaccuracies in specification of vehicle aerodynamic characteristics and in specification of atmospheric properties are anticipated, an exact nominal trajectory is not prescribed in advance of the mission. Instead, it is initially stored as a mathematical model which requires determination of eight parameters before the nominal trajectory is specified.

Early in-flight measurements of vehicle temperatures and accelerations are used to compute the parameters needed to specify the nominal trajectory. This concept is attractive for entry into atmospheres in which the density characteristics are subject to considerable uncertainties. In essence, the nominal trajectory is determined in the early phase of the mission where the ability to alter the trajectory is limited due to small gas-dynamic forces, but where forces and temperatures are of sufficient magnitudes to be measurable. These measurements are used to determine atmospheric properties actually being encountered.

Once the nominal trajectory is computed, this information is used
Fig. 1.8: Guidance System Which Utilizes Ballistic Perturbation Theory
to determine the time-varying coefficients of the perturbation equations. The guidance problem cannot be solved, however, without additional information. By using adjoint functions, a guidance equation is derived which is amenable to solution. A fundamental end condition of the guidance equation is that range error vanish at the vehicle's destination.

The nominal trajectory, the perturbation equation coefficients, and the adjoint functions are computed and stored early in the mission. These quantities, together with measurements of vehicle position, velocity, and accelerations from the navigation and tracking system enable solution of the guidance equation.

A particular guidance system is not investigated or proposed in this thesis. Approximate analytical techniques are developed which may be applied to any proposed system, particularly early in the design stages where an understanding of the physical relationships are required and where approximate answers (with a known degree of accuracy) are needed.

Design of instrumentation of any physical problem is composed of three fundamental phases:

(1) **Statement of the problem** - investigation and analysis of the problem to determine the physical quantities and relationships involved.

(2) **Determination of instrumentation requirements** - translation and interpretation of physical relations into requirements for design of free and flexible components. These requirements must be determined subject to constraints imposed by the physical problem and by any fixed components in the system.

(3) **Design of instrumentation** - design and testing of specific
components and equipment to realize the previously determined requirements.

In this thesis, items 1 and 2 above are considered for the general entry mission into the atmosphere of a planet.

1.10 Control System

Detailed examination of the control system and vehicle, lumped together in box (3) of Figs. 1.4 and 1.5, is not a subject of this thesis. The design of these elements is a strong function of the interaction of their stability and control characteristics. It should be noted, however, that guidance and control system design cannot proceed efficiently without unified consideration of all elements of the system because of the complex inter-relationships existing among them.

The control system and vehicle present a severe design problem because of the operational extremes experienced during the course of a typical entry mission. Prior to atmospheric penetration, vehicle skin temperatures are low, gas-dynamic forces and moments are negligible, and the vehicle is in a free-fall condition (zero accelerations). Reaction jets, or their equivalent, offer an effective means for attitude stability and control in this, the Keplerian portion of the trajectory. In the Intermediate Phase of the trajectory, vehicle accelerations due to gas-dynamic forces are large enough to be detectable, yet small enough that only minor trajectory alterations are possible without thrust augmentation. A combination of reaction and gas-dynamic moments for orientation control is feasible in this operational regime. In the Gas-Dynamic Phase, vehicle temperatures rise sharply to a maximum value as the vehicle velocity decelerates to
about 0.8 of circular satellite velocity. Gas-dynamic accelerations increase to maximum values at velocities substantially lower* than that for maximum temperatures and heat flow rates.

The operational extremes experienced during entry suggest a control system that must adapt itself to the changing environment. The Conservation Parameter, defined in section 1.7, is a measurable function that has sharp behavior near the operational phase boundaries. The Conservation Parameter may be useful either as a switching function or a variable sensitivity factor to improve control system operation.

Fig. 1.9 is a partial functional diagram of one channel of a control system which uses the Conservation Parameter as an input to a logic system. The environmental logic system selects the most efficient method to implement the command input; i.e., it selects either aerodynamic controls, reaction controls, or a combination of both, and adjusts loop gains for near-optimum performance. The operation of this particular system is explained in Appendix F.

The system of Fig. 1.9 is not a "self-adaptive" control system; i.e., it does not monitor its own performance and automatically adjust itself for optimum performance in a changing environment. The system is "adaptive", however, in the sense that it uses environmental data from the guidance computer to adjust its performance characteristics. Augmenting the adaptive features of a system like that of Fig. 1.9 with

* For example, it is shown in Chapter 9 that a vehicle with lift-drag ratio of 1.0 experiences a maximum of 0.9 g's at about 0.2 circular satellite velocity for gliding entry into the Earth's atmosphere. A non-lifting vehicle, on the other hand, experiences more than 8 g's at about 0.5 circular satellite velocity (zero initial entry angle). If the nonlifting vehicle's entry angle is increased to 6°, maximum accelerations rise to more than 18 g's.
Fig. 1.9: Partial Functional Diagram of Normal Acceleration Control System which Adapts to the Operating Environment
self-adaptive inner loops to optimize control system operation may be worthy of further study.

1.11 Navigation and Tracking System

The guidance of vehicles involves the functions of indication and control (11). The indication aspect of guidance is navigation. The correct course of action required at any given time is a function of the instantaneous position and velocity vectors; determination of these vectors is the task of the navigation and tracking system. The control aspect of guidance requires that a comparison be made of navigational data with programmed (stored) data, selection of the most appropriate action (decision-making), and executing the action prescribed.

There are three fundamental techniques commonly used for navigating astronomical vehicles:

1. Inertial Systems;
2. Systems relying entirely on external radiation data; viz., optical, radio, radar, and infra-red systems;
3. Externally aided inertial systems (combination of (1) and (2)).

Methods (1) and (2) are pure systems, method (3) results in a hybrid system. The advantages of using any one of these systems in preference to any other is not debated in this thesis. It is well to consider, however, some of the characteristics of these systems.

An inertial navigation system consists basically of three specific force measuring subsystems (accelerometers) mounted on a gyro-stabilized platform and some form of computer. The stabilized platform may be either a physical platform or a mathematical (phantom) platform. One
of the difficulties encountered in using an inertial system in the free-fall phase of flight is that in the "weightless" environment of free space, the accelerometers indicate nothing, regardless of the trajectory. Disturbances due to imperfections of the gyros and accelerometers ("drift") are primary sources of error in inertial systems.

The major difference between the characteristics of an inertial system used in a low-velocity vehicle (aircraft, ships, submarines) and one used in a vehicle which operates at orbital or near-orbital velocities is that the familiar instability in the vertical (altitude) channel of low-velocity systems disappears in orbital vehicles and is replaced by instability in the horizontal (range) channel. Initial condition errors such as in altitude or velocity, concomitant with error in orbital energy, induce diverging errors in range. Constant accelerometer errors produce unbounded range errors. Young (12) shows that the order of range divergence is less for an inertially oriented accelerometer than for a range-oriented accelerometer.

Major advantages of using an inertial system are:

(1) Navigational information is based entirely on measurements made from within the vehicle; no contact with the outside world is required. Communication difficulties resulting from ionized atmospheric layers, weather, etc., have no deleterious influence on the operation of the pure inertial system.

(2) Streamlined guidance and control system design is feasible since certain measurements required in the navigation function may be used for other guidance and control functions.
The inertial navigation system may be the most desirable of the three types of navigation systems for the direct entry profile since flight times are generally short (15 min. - 2 hr.) and the major portion of the trajectory is in the Gas-Dynamic operational regime where external specific forces are measurable.

External radiation systems for navigating or tracking the vehicle use electromagnetic energy as the information medium. These methods suffer from line-of-sight limitations and propagation disturbances (such as ionized layers in the atmosphere and weather). Radio-radar systems are susceptible to man-made jamming.

In a radio-radar tracking system, the target must radiate an appropriate signal either by carrying a radio transmitter or, in the case of a radar system, by reradiating a part of the incident energy as a reflected signal. The basic measurements that can be made with a single radio or radar guidance station are range, radial velocity, and direction of the target relative to the receiving station. The entry vehicle may be either the target or the control point, depending on whether tracking is performed from within the vehicle or at some remote station.

Radio navigational systems may be of two types:

1. **Passive**: A coded pulse emanating from the transmitter in the control station is received at the target, which in turn generates a signal (after a known time delay) that is returned to the control station. The target's radio beacon, in this case, is called a transponder; i.e., it transmits a signal in response to an input signal of the proper type. The time delay between the
transmission and reception at the control point serves as a measure of the target's distance from the control point. The direction of the target can be deduced from the incoming signal or by optical means. Observation of the Doppler effect on the target's signal gives the relative velocity of the target with respect to the control point along the line of sight.

(2) **Active:** The target generates a radio signal either continuously or intermittently. The direction to the target can be determined from this signal; the relative velocity of the target with respect to the control point along the line of sight can be determined from the Doppler shift if the transmitting frequency is known precisely. The distance to the target cannot be determined from a single control point.

Radar, too, may be of two major types: pulse radar for position determination and continuous-wave radar (Doppler) for velocity determination.

The antenna system measures the direction of arrival of the signal. Antennas can be divided into two classes, those using a single mirror to focus the energy into a small region and those using spaced antennas for interferometer measurements. The long distances traveled by the signals makes it necessary for the energy-collecting area of the antenna to be as large as possible. In general, the beam-width decreases and angular accuracy improves as the antenna area is increased.

First-time entry into the atmospheres of strange planets presents special problems in specifying position when compared to navigating
over a well-mapped planet such as the Earth. The choice of a suitable landing site must necessarily be based on reconnaissance of the planetary surface while in orbit around the planet. The orbital altitude for the reconnaissance phase must be high enough such that a prolonged orbit may persist, yet low enough that fairly accurate mapping of the terrain is feasible.

Ground-based tracking stations are of great utility for navigating vehicles over Earth. The crew of the *Explorer* vehicle may find it greatly to their advantage prior to initiating entry to launch a network of radio and/or radar beacon missiles suitably dispersed over the planetary surface. Much valuable data on the atmospheric density characteristics and winds can be accumulated by tracking the missiles during their descent. The observed location on the planetary surface where these missiles ultimately land will be valuable landmarks for entry navigation. Another valuable aid to entry navigation will be from a navigation satellite left in the reconnaissance orbit prior to initiating entry. This navigation satellite may be simply a radio beacon, transponder, radar beacon, or balloon; or it may be an elaborate tracking and control center such as an orbital space platform which is the parent or carrier of the entry vehicle. The concept of a navigational satellite for external tracking data is discussed further in Appendix D of this thesis.

* The term "Explorer" vehicle is used in this thesis to denote a vehicle executing first-time entry into strange planets. "Strange planets" signifies natural bodies of the solar system (planets, asteroids, moons) on which a vehicle with human occupants has not previously landed. Therefore, ground-based tracking information is not available during Explorer missions.
Radio and radar tracking systems have the advantage of not being penalized by long periods of operation. This feature suggests the externally-aided inertial system as a hybrid technique for selecting the best features of each of the pure navigating and tracking systems.

1.12 System Optimization

The determination of guidance requirements for the entry mission involves a rather complicated analysis of trajectories, range sensitivity to changes in controllable parameters, uncertainties in specification of parameters such as atmospheric properties and vehicle's aerodynamic characteristics, guidance system capabilities, propulsion system, payload requirements, and system cost. It is of paramount importance in every consideration of the guidance problem that constraints, primarily those due to gas-dynamic heating and deceleration, be recognized in order not to exceed the operational limitations of the vehicle or to endanger its occupants.

One of the most important questions that must be answered is simply: "What trajectory should the entry vehicle follow; i.e., how should it be steered?" A careful formulation of this leads to problems of optimization. What choice of trajectory will maximize or minimize some criterion of performance, where this criterion might be range, payload, flight time, landing point accuracy, cost, etc.? The selection of feasible systems, optimized with respect to guidance accuracy, propulsion requirements, payload, and mission cost requires a comprehensive analysis of the interaction of these parameters. These relationships might be presented as trade-off curves, showing some advantage to be gained in one parameter by varying another.
It is of fundamental importance to recognize that there is no single optimum entry trajectory that holds for all entry missions. The optimum trajectory is a strong function of the purpose of the mission and the initial conditions of the problem. If maximum payload weight is of over-riding importance, then one family of trajectories is superior to all others. If landing point accuracy is of fundamental importance, then a vastly different trajectory is dictated. If both factors are of equal importance, certain compromises are obviously required. In particular entry missions the selection of feasible systems, optimized with respect to guidance accuracy, payload capabilities, and mission cost, requires a detailed analysis of the interaction of these parameters. In this thesis the generalized guidance problem alone is considered; other mission factors are recognized when they have an important modifying effect on the requirements of the guidance system.

In many engineering problems optimization procedures are of small importance, the performance not being particularly sensitive to the choice of operating parameters. This is not the case for the entry mission because system performance is extremely sensitive to small changes in design, the system is complex with strong interactions among the different subsystems, and the number of variables influencing any single quantity is generally large enough to make intuitive decisions difficult.

Many optimization problems involve straight-forward application of

* For example, Chapter 9 demonstrates that large reductions in deceleration loads encountered during entry result from employing small amounts of lift.
the calculus of variations. In this thesis, one-sided constraint**
expressions for stagnation point temperature and vehicle accelerations
are developed for application of the techniques of the calculus of
variations to the determination of extremals of arbitrary guidance
functions. It should be noted, however, that many important results
can be deduced without invoking the full power of the calculus of
variations.

The prediction of the entry trajectory is difficult at best due to
the inherent non-linearities in the statement of the problem. Analytical
techniques are developed in this thesis for predicting parameters that
are important in selecting the optimum trajectory and for preliminary
specification of the guidance system, parameters such as the predicted
time history of distance flown, range-to-go, time of flight, altitude,
velocity, specific force level, atmospheric density variation, heating
rates, and stagnation point temperatures under a diversity of initial
conditions. Results are presented in non-dimensional form for general-
ized application. Accuracies and limitations of the solutions derived
were determined by comparing them to more exact numerical solutions.

One of the primary purposes of this study was to provide analytical
solutions of known accuracy to the engineer who is faced with the
difficult task of designing an entry system. It is possible, of course,
for the designer to write down complete equations of motion for compli-
cated vehicle models, including gas-dynamic forces and moments, rigid
body rotations, feedback loops in the control system, stochastic

** Minimum allowable acceleration loads and temperature levels are not
constrained; maximum levels alone are constrained.
interferences, etc.; and digital computers such as the IBM 704 can provide solutions in a short time. A straight numerical approach, however, has many shortcomings in the conceptual and early design phases where an infinite number of possible trajectories and guidance schemes must be explored. One of the major difficulties is that a numerical "run" is required for each set of initial conditions. Machine calculations not only provide more detail than is generally required at this stage, but also fail to indicate straightforwardly the effect of changing various design parameters. In order to compare different guidance concepts and in order to determine the effect of various parameters on system performance, a numerical approach is time consuming, expensive, and unwieldy.

In order that the engineer understand the effect of changing various design parameters and in order to compare different guidance schemes, simple analytical results can be far more informative than a long series of machine computations. The philosophy advanced in this thesis, therefore, emphasizes the use of dynamical approximations, with specified limitations, to derive simple analytical solutions of important guidance quantities. In this way, the requirement for numerical studies may be delayed until later stages in the design process of many entry systems.

1.13: Guidance Trajectories

Related to the concept of guidance is the requirement for specifying a standard, reference, or nominal trajectory. This trajectory is the path described by a standard or nominal vehicle under standard atmospheric conditions. If the atmospheric model of the planet is
accurate, a particular entry vehicle should trace out a spatial path which is close to the nominal trajectory. Inaccurate specification of the atmospheric model and uncertain knowledge of atmospheric winds, particularly in the case of entry into the atmosphere of strange planets, are likely to be the most significant factors leading to departures from the nominal trajectory during a particular entry mission. The nominal trajectory should be specified as one for which errors and uncertainties in the atmospheric model contribute to minimum total range error. This trajectory must be chosen to optimize other requirements besides guidance accuracy, however; among these additional requirements are:

1. Payload weight;
2. Computational complexity required of the computer, and its effect on weight, power consumption, and reliability;
3. Vehicle range;
4. Flexibility in changing trajectories;
5. Gas-dynamic heating and specific force levels encountered.

These are often conflicting requirements; hence compromises in guidance system design are generally necessary. It was emphasized in Section 1.8 that guidance is a closed loop process which includes the vehicle and its control system as part of the loop. This loop must have high enough gains such that dynamic lags are not excessive, and must have acceptable stability characteristics. The design of the guidance and control system for atmospheric entry is an excellent example of a systems problem for which the specific mission, physical constraints, available components, and detailed characteristics of the vehicle must be considered.
A number of interesting papers have been published concerning a suitable choice of entry trajectory consistent with reducing the severe heating and deceleration loads encountered in the atmosphere. Eggers\(^{(4)}\), Gazley\(^{(14)}\), and Chapman\(^{(15)}\) have suggested entry by means of grazing* trajectories with vehicles of moderate lift-drag ratios. It is shown in this thesis that such trajectories may result in large errors in the location of landing for either small errors in the flight path angle or for perturbations in atmospheric density due to incomplete knowledge of the properties of the upper atmosphere.

Lees\(^{(16)}\) suggests improvement of guidance accuracy by penetrating the atmosphere at flight path angles greater than 4°. The vehicle is then permitted to partially skip until a design maximum g-load is incurred, at which time lift is either "cut-off" or programmed such that maximum allowable acceleration loads are subsequently not exceeded. Lees' trajectory is attractive for interplanetary missions because it greatly broadens the allowable corridor for the "first-pass" entry mission**. For direct entry from stable reconnaissance orbits, Lees' trajectory has certain guidance advantages: it shifts the location of the Intermediate Phase to lower altitudes and the steep penetration angle reduces considerably the range errors which may accumulate. It is unattractive for direct entry from reconnaissance orbits because a

* Grazing trajectories are these trajectories for which the flight path angle, \(\gamma\), is approximately zero. In the atmospheric portions of grazing trajectories, it is assumed in most analytical studies that the drag force predominates over the component of gravity forces in the flight direction.

** That is, final direct entry is achieved during the first atmospheric perturbation of the interplanetary transfer ellipse.
large amount of the initial mass of the vehicle must be invested in propellant fuel in order to achieve a substantial flight path angle in the region of initial atmospheric penetration; i.e., a large scale perturbation of the reconnaissance trajectory is required. For both first-pass entry and entry from a stable reconnaissance orbit, Lees' trajectory has the disadvantage that the lift program required in order to obtain accurate landing point control may be difficult to achieve in practice.

Kepler suggests an entry trajectory which embodies many of the advantages of Lees' trajectory, plus some additional guidance advantages in the case of direct entry from the stable reconnaissance orbit. This trajectory, shown in Fig. 1.10, is similar to Lee's trajectory through Phase 2 (except lift is cut-off in horizontal flight in the Kepler trajectory rather than at a certain g-level as suggested by Lees). Kepler's phase 3 has guidance advantages over the trajectory envisaged by Lees since a lift program to hold constant altitude flight is achieved more easily than a lift program to ensure peak accelerations remain below a certain value.*

Sufficient range control may be achieved in the Kepler trajectory with vehicles of moderate lift by adjusting the phase boundary between phase 3 and 4 to correct for range errors which accumulate over the first three phases, even if atmospheric density differs substantially from predicted values. For example, Kepler shows that in entering the Earth's atmosphere from a 200 nautical mile circular reconnaissance

* Prediction of expected future g-levels is required of the guidance system in this instance since the acceleration loads encountered in the future are a function of present trajectory adjustments.
Fig. 1.10: Kepler's Model Trajectory for Lifting Entry.
orbit, a 2% velocity impulse error, a 5 degree error in alignment of the engine gimbals*, and a 10% error in Phase 2 lift-drag ratio results in a total RMS error of 153 nautical miles at the boundary between phases 2 and 3. He shows that these accumulated errors are easy to erase over phases 3 and 4 without incurring excessive accelerations and heating loads.

1.14 Synopsis

Chapter 1 discusses some general concepts of guidance and control for various types of entry missions and correlates the work of this thesis with the entry mission and previous investigations in this area.

Chapter 2 advances certain specific conclusions and recommendations for further study.

Chapter 3 describes the three-dimensional kinematics of entry in two separate sets of position reference frames, the "latitude-longitude" triad, and the "instantaneous great circle" triad.

Chapter 4 discusses the three-dimensional kinematics of entry in elliptical parameters and the possible advantages of performing in-flight position and velocity computations for Explorer vehicles in these parameters.

Chapter 5 formulates a mathematical description of all forces acting on the entry vehicle which affect the three-dimensional characteristics of its trajectory. These forces are described in component form suitable for specifying the dynamical equations of motion in either of the reference frames of Chapter 3 or the elliptical parameters.

* Kepler uses a 600 ft/sec velocity impulse directed anti-parallel to the velocity vector in orbit. From Fig. 9.5 of this thesis, it is seen that rotating the engines a few degrees toward the orbital center would be more appropriate.
of Chapter 4. It should be noted that the three-dimensional dynamical equations of motion of Chapters 3 through 5 are sufficiently general to permit computer analysis of the trajectory of a lifting or non-lifting vehicle in banking or wings-level flight with variable thrust capabilities entering the atmosphere of an oblate, rotating planet with or without atmospheric winds. The figure of the planet, the gravitational model*, and the atmospheric model for important bodies of the solar system are discussed in Appendix B through E.

Solutions in closed form of the three-dimensional equations of motion for atmospheric entry cannot be obtained by any analytical techniques known at this time. As a practical matter, the trajectory was intentionally confined to a fixed plane in Chapters 6 through 10. The procedure of this thesis departs from that generally followed in dynamical studies; the usual practice is to apply the theory of planar motion before considering the additional complexity of the third dimension. The reverse procedure, as carried out here, has the advantage of clearly delineating those conditions under which planar motion is an accurate approximation.

Derivation of the two-dimensional equations of motion in a number of useful forms, including both differential and integral forms with energy and angular momentum as dependent variables, is described in Chapter 6.

Chapter 7 discusses trajectory constraints imposed by vehicle

* Quantitative description of gravitational harmonics of the pear-shaped model of the Earth are contained in reference (18).
heating rates, stagnation point temperatures, and human acceleration tolerances. These one-sided constraints are represented in both analytical form and graphical form (in the velocity-altitude plane) for entry into atmospheres of terrestrial planets. Altitudes at which specific force level exceeds minimum detectable levels are also discussed in Chapter 7 for various vehicle, accelerometer, and planet combinations.

The generalized entry trajectory is examined in Chapter 8 from a standpoint not presented previously in dynamical studies of atmospheric entry. A "Conservation Parameter"* was selected on the basis of its sharp and predictable behavior as an index of the influence of the atmosphere on the trajectory. This parameter is used to separate the trajectory into three separate operational regimes - the Keplerian, Intermediate, and Gas-Dynamic phases - and to define boundary conditions between phases. All known analytical studies conducted to date omit the Intermediate Phase.** Methods for computing the Conservation Parameter are suggested, and uses for this parameter as a switching function for guidance computations and as an environmental index for adaptive control systems are advanced.

Chapter 9 is devoted to a comprehensive investigation of each of the three separate phases of flight. Closed form analytical solutions of guidance parameters and constraints are obtained for the Keplerian phase.**

* This parameter is defined in section 1.7 and is used frequently in Chapter 8 and subsequent chapters.

** It is shown in Chapter 8 that the Intermediate Phase spans an altitude band approximately 20 miles wide (for Earth) at an initial altitude which is a sensitive function of flight path angle. Specific force levels encountered in traversing the Intermediate Phase increase approximately 100 times from an initial value of $10^{-4}$ to $10^{-2}$ g's (depending on flight path angle).
Phase (for vehicles launched from both circular and elliptical reconnaissance orbits) and for the Gas-Dynamic Phase under initial conditions which induce skipping, gliding, and ballistic flight. Closed form solutions cannot generally be derived in the Intermediate Phase; special solutions in this phase for particular flight programs are discussed.

Chapter 10 describes a method original with this thesis for obtaining a continuous solution of guidance parameters for lifting and non-lifting vehicles initially in circular and degenerate elliptical reconnaissance orbits. Some features of entry trajectories not generally recognized are disclosed in this analysis, which leads to solutions in terms of modified Bessel functions of the first kind.
Chapter 2

Conclusions and Recommendations for Further Study

2.1 Conclusions

The investigation described in subsequent chapters of this thesis analyzes atmospheric entry from both a general and specific point of view. The investigation is general in the sense that a particular entry vehicle, guidance system, or entry trajectory is not examined or proposed. General analytical techniques are developed which may be useful in preliminary analysis of guidance requirements for any particular system. The investigation is specific in the sense that elements which are common to all atmospheric entry systems are examined in detail. All entry missions are characterized, for example, by the fact that the vehicle traverses the three separate operational regimes defined in Section 1.7, the Keplerian, Intermediate and Gas-Dynamic Phases. The time spent in each of these flight regimes is very much a function of the initial conditions of the problem, the trajectory that is to be flown, and the characteristics of both the vehicle and the planetary atmosphere.

The Intermediate Phase of flight has not previously been examined in the literature. This flight regime is defined in this thesis and its importance for a general understanding of the dynamics of entry is
discussed in considerable detail. It is shown that, from the guidance standpoint, the Intermediate Phase cannot be ignored in the direct entry profile and is the most important single phase in the degenerate orbital profile.

All entry missions may be viewed as falling into one of two general categories:

1) **The direct entry profile:**
   
   This atmospheric entry trajectory is characterized by sequential transition from Keplerian flight through the Intermediate Phase into Gas-Dynamic flight.

2) **The degenerate orbital profile:**
   
   This atmospheric entry trajectory consists of a series of braking passes through the outer reaches of the atmosphere in order to reduce the energy level preparatory to final direct entry.

The direct entry profile is the result of either:

1) Large scale perturbations of a stable reconnaissance orbit by means of retro-rocket thrust.

2) Controlled first-pass entry at the termination of the interplanetary transfer ellipse.

The degenerate orbital profile results if the vehicle at perigee possesses an energy excess over the energy level corresponding to circular orbital flight. This situation may occur under a variety of mission concepts:

1) If the vehicle is launched from the surface of the planet for the purpose of making a number of orbits around the planet and re-entering, the degenerate
orbital profile results when the energy at engine cut-off in the launch phase exceeds circular orbital energy.

(2) In the case of interplanetary operations in which the high energy level of the vehicle in the interplanetary transfer ellipse is to be reduced by atmospheric braking, the degenerate orbital profile follows first perigeeal passage through the planetary atmosphere if the energy transfer is sufficient during this passage for the planet to capture the vehicle.

Flight times in the direct entry profile are short, generally ranging from 15 minutes to two hours, while in the degenerate orbital profile the time of flight may run from two hours to a number of days.

Guidance parameters and constraints for the direct entry profile were developed in Chapter 9 of this thesis by examining separately each of the three operational phases. The Conservation Parameter* was used for purposes of matching boundary conditions between phases.

It was shown in Chapter 9 that the Intermediate Phases is the most difficult phase from the guidance standpoint because none of the simplifications in the equations of motion permitted in the Keplerian and Gas-Dynamic Phases of the trajectory are warranted.

Chapter 9 shows that the trajectory in the Gas-Dynamic Phase is a strong function of the initial flight conditions at the onset of this phase and of the lift-drag characteristics of the vehicle.

Three basic trajectory patterns are possible:

* The Conservation Parameter was defined in Section 1.7 and is discussed in detail in Chapter 8.
(1) **Ballistic trajectory**: Non-lifting vehicle with large initial flight path angle.

(2) **Glide trajectory**: Lifting vehicle with zero initial flight path angle.

(3) **Skip trajectory**: Lifting vehicle with finite initial flight path angle.

Approximate analytical solutions were developed in this thesis for each of these classes of trajectories. Fig. 9.7 shows the conditions under which these approximate analytical solutions are an accurate representation of the trajectory when compared to numerical solutions.

It is interesting to note that the range, stagnation temperatures, and vehicle decelerations are independent of the exponential decay parameter, $K$, of the planetary atmosphere in the special case of the gliding trajectory.

Chapter 10 examines the trajectory of a vehicle in the degenerate orbital profile. Results were presented in dimensionless form suitable for analysis of entry into the atmosphere of any planet. Two separate trajectory phases were examined:

(1) The circularization phase for the degenerate elliptical orbit.

(2) The entry phase which follows the circularization process.

In the special case where the vehicle is initially at circular orbital velocity, the first of these two phases obviously does not exist.

The solution for altitude during the circularization process was derived as equation (10-79). This solution was written in terms of a number of constants which depend on the initial conditions of the problem and on the characteristics of the vehicle. Some of the
important constants in this equation were written in terms of Bessel functions. Since these functions are tabulated, numerical answers to the resulting equations are easily obtained. The time variation of perigeal and apogeal altitudes were given in equations (10-81) and (10-83) and the rate of decay for most practical problems is described by equations (10-84) and (10-85). Analytical representation of other quantities which are important in the conceptual and preliminary design stages of guidance systems are summarized in equations (10-86) through (10-94).

It was shown in the analysis of the circularization phase that the drag characteristics of the vehicle are important in specifying the resulting trajectory and that the lift characteristics of the vehicle are relatively unimportant. It was shown that, within the approximations made in the analysis, lift does not enter in the specification of the decay rates of apogee and perigee. Perigal altitude, to a first order, remains essentially constant. The major difference between the decay rate of apogee and perigee is that the perigal decay rate is proportional to the difference of two near-equal large numbers* while apogeal decay rate is proportional to the sum of these same numbers.

The solution for altitude in circular orbital entry was given by equation (10-41). This solution was written in simplified form suited to almost all practical problems as equation (10-47) and (10-48). It was shown in this thesis that a true circular orbit or a linear decaying

* Perigal decay is proportional to \((I_0 - I_1)\), where \(I_0\) and \(I_1\) are the first two modified Bessel functions of the first kind with an argument that is generally greater than 10.
circular orbit cannot exist even under idealized conditions of injecting a vehicle exactly at circular orbital velocity above a spherical planet. The influence of the atmosphere on the dynamics of energy transfer result in undamped oscillatory motion in altitude and flight path angle.

The analytical solutions derived in Chapter 10 were compared to machine-computed numerical solutions under a variety of initial conditions. It was shown that the analytical solutions give an accurate history of the resulting motion except after relatively long periods of time. The analytical solutions depart from more accurate computer solutions only after there is a significant increase in atmospheric density from the assumed initial value as a result of altitude loss during entry. This limitation on the solution may be largely overcome by periodically starting the problem over again under a new set of initial conditions in accordance with the procedure outlined in Chapter 10.

One of the troublesome factors arising in the design of guidance systems for missions of the degenerate orbital type is the undamped oscillations in altitude and flight path angle due to the dynamics of energy transfer between vehicle and atmosphere. Quantities such as total energy, orbital eccentricity, and angular momentum, which have monotonic behavior (generally as a series of steps in the vicinity of perigee), are more suited for minimal guidance system design than are quantities which characteristically behave in an oscillatory manner. Management of the energy history, for example, is feasible by command changes in the aerodynamic configuration of the vehicle; i.e., the lift and drag characteristics are varied by changing the vehicle geometry or angle of attack.
Chapters 4 and 5 examine the three-dimensional dynamics of entry in terms of parameters of a time-varying ellipse which instantaneously matches the position and velocity vectors of the vehicle. This technique is not new\(^{(20)(21)}\). The particular elliptical parameters chosen in this thesis, however, are original and are shown to have the distinct advantage of considerable simplification in the resulting equations of motions and at the same time are free of singularities in the special case of circular orbits (eccentricity equal to zero). The parameters selected were as follows:

(a) **Orientation of the instantaneous ellipse:**
   1. Inclination of the orbital plane with respect to the equatorial plane.
   2. Longitude of the ascending line of nodes.

(b) **Specification of the instantaneous ellipse:**
   3. Angular momentum.
   4. \( \xi_1 = \xi \cos \theta \)
   5. \( \xi_2 = \xi \sin \theta \)

   where \( \xi \) is eccentricity of the instantaneous ellipse and \( \theta \) is true anomaly.

(c) **Position of vehicle in the instantaneous ellipse:**
   6. Angle in the plane of the ellipse from the line of nodes to vehicular position.

   It is shown in Chapter 6 that the entry trajectory is confined to a single plane in space only if lift, drag, and thrust are programmed such as to maintain \( \dot{\psi} \) and \( \dot{\lambda}_{IT} \) equal to zero. In the special case of
an equatorial trajectory \((\psi = 0)\), planar motion follows if:

1. Components of atmospheric winds normal to the trajectory plane are zero.
2. The angle of bank of the vehicle is zero.
3. There are no components of thrust normal to the trajectory plane.

It was concluded in this thesis that the optimum engine thrust program for entry is a strong function of mission objectives. There is no single optimum thrust program that is applicable to all entry missions.

If the vehicle is in a stable reconnaissance orbit, and if maximum payload weight is an overriding requirement for the mission, then the minimum energy trajectory is desired; i.e., it is desired that entry be effected with minimum expenditure of propellant mass. In the stable reconnaissance orbit, minimum propellant mass is expended by generating retro-thrust tangentially at apogee. As a result of this reduction in velocity at apogee, the perigee next following will be at a lower altitude. By generating enough thrust for the perigee to drop within the sensible atmosphere, the degenerate reconnaissance orbit is established. Controlled entry is possible from the degenerate reconnaissance orbit by varying the drag parameter.

If payload requirements are not as critical as the requirement for a relatively short time of flight, or if the vehicle has only modest capabilities for changing its aerodynamic characteristics, then the minimum energy profile is not necessarily the most efficient method for satisfying mission objectives. Under these conditions, a second retro-thrust application at perigee, in addition to the initial thrust
perturbation at apogee, is a more suitable mission concept than the minimum energy profile. This scheme for the entry mission includes the direct entry profile.

If payload capabilities of the entry vehicle are secondary to stringent requirements on landing point accuracy, the initial retro-thrust application at apogee in the reconnaissance orbit may be increased a sufficient amount to cause the following perigee to fall beneath the surface of the planet. For example, if the propellant mass expended at apogee is approximately twice that required to cause the perigee to fall within the atmosphere, then the entry transfer ellipse will intersect the atmosphere after the vehicle travels approximately one-fourth of the distance around the planet rather than one-half. The flight path angle at atmospheric penetration is considerably greater under these conditions than when lesser amounts of fuel are expended, hence variations in atmospheric density from standard will result in much lower range errors. This mission concept has advantages from the guidance standpoint, but these advantages are paid for in terms of payload capabilities.

Chapter 9 discusses analytical and graphical methods for determination of range sensitivity to errors in magnitude and direction of the thrust vector generated by the retro-rocket system for entry from both circular and elliptical reconnaissance orbits. Figs. 9.5 and 9.6 show the vacuum ground range for the particular case of entry from a circular reconnaissance orbit from a dimensionless altitude of approximately 0.0758 (corresponds to 300 mi. altitude in the case of entry to the surface of Earth). These figures show that a given range may be obtained with a particular velocity impulse at two distinct engine
deflection angles. Minimum errors in range accrue, however, if the engine is aligned in such a direction as to obtain minimum range; engine deflection angle is unique and single-valued for minimum flight range and is a two-valued function for all ranges in excess of this minimum. These figures also show that the capability to reduce total range is less effective as the impulsive velocity level is increased. At high impulsive velocity levels, however, range is less sensitive to errors in engine alignment.

It is shown in this thesis that, from the guidance standpoint, the altitude of the sensible planetary atmosphere depends strongly on flight path angle, lift, drag, and density characteristics of the vehicle. The heating and deceleration loads which accompany high velocity entry into the atmosphere are strong functions of the initial penetration angle and the design characteristics of the vehicle.

The build-up of appreciable heating and deceleration loads are two of the most important effects encountered during entry of a vehicle into the planetary atmosphere. These manifestations of energy transfer from the vehicle to the planetary atmosphere are examined in this thesis from the standpoint of the restrictions they impose on the allowable guidance trajectories. Both the deceleration loads and the heating rates are most severe when there is a combination of high atmospheric density and high vehicular velocity. It is therefore necessary that the guidance system operate in such a way as to prohibit high velocities from persisting down to low altitudes. This undesirable condition most likely will occur if the flight path angle is large and the approach velocity is very great.

The velocity of the vehicle at atmospheric penetration depends on
the type of entry mission. If atmospheric penetration follows interplanetary transfer, then the velocity at penetration is of the order of escape velocity. If entry is initiated from a planetary reconnaissance orbit, then the velocity at atmospheric penetration is of the order of orbital velocity. The velocity at initial atmospheric penetration can be changed a significant amount only by large scale thrust perturbations; this leads to severe penalties in terms of payload capabilities.

The flight path angle, on the other hand, may generally be selected at a smaller weight penalty, provided thrust perturbations are generated at the proper time and in the proper direction. For example, a small thrust perturbation during the interplanetary transfer ellipse (when the vehicle is far from its destination planet) can change the flight path angle at initial atmospheric penetration a very large amount. It must be emphasized, however, that small errors in thrust perturbations at this point can lead to gross errors in penetration angle; guidance accuracy requirements are more stringent when the system is highly sensitive to small control actions.

Selection of the best entry profile for a given mission requires analysis of the coupling of payload requirements, aerodynamic characteristics, range sensitivity to control actions, vehicular heating rates, deceleration loads, and effects of errors and uncertainties in the knowledge of the physical characteristics of the planet and the vehicle.

A shallow flight path angle (tangential approach) tends to limit the region of high velocities to high altitudes. Consequently, choosing a shallow trajectory tends to minimize heating rates and deceleration loads encountered. Inaccurate knowledge of the density characteristics
of the planetary atmosphere, however, may lead to considerably greater range errors for entry at shallow flight path angles than for steep flight path angles. It is shown in this thesis that the flight path angle at atmospheric penetration must be selected as a compromise between two conflicting mission requirements:

1. Heating rates and deceleration loads are appreciably reduced at shallow flight path angles.
2. Range accuracy is generally greater for steep flight path angles.

It is shown in this thesis that both heating and acceleration constraints must be examined in arriving at a suitable entry profile. Specific force level generally is more restrictive on the allowable operating region at lower altitudes while heating considerations are more restrictive at higher altitudes. The lower the maximum allowable heating rates, skin temperatures, and g-tolerances, the more severe are the limitations of these constraints on permissible guidance trajectories.

Measured temperatures at the surface of the vehicle may be useful as inputs to the guidance system. The temperature history for a particular entry profile may be predicted. Guidance information, particularly with respect to atmospheric properties, may be obtained by comparing measured and predicted values. Temperature readings may be more useful than accelerometer outputs in early determination of atmospheric conditions actually being encountered in flight. It is shown in this thesis that temperatures generally attain maximum values much earlier in the mission than the maximum specific force level. In most situations, the temperature level is a maximum before the specific force level exceeds a small fraction of one g.
Among the special problems associated with first time entry into the atmospheres of strange planets are requirements for:

1. Terrain mapping and selection of a suitable landing site.
2. Accumulation of data on the atmosphere of the planet, wind and climatic characteristics, the figure of the planet, and its gravitational properties.
3. Provision for augmentation of guidance measurements made from within the vehicle with data derived via external radiations.

It was concluded in this thesis that the above requirements suggest an entry system that embodies the following features:

1. Setting the entry vehicle up in a stable reconnaissance orbit in order to:
   a. Map the surface of the planet as thoroughly as required prior to initiating entry.
   b. Provide considerable freedom in the selection of the time and location of the point for initiating controlled entry.

2. Launching a navigation satellite from the entry vehicle while it is still in the reconnaissance orbit; this satellite may be tracked during the course of entry in order to provide position data on the entry craft.

3. Launching radar and/or radio beacon missiles from the reconnaissance orbit to the planetary surface before initiating entry in order to:
   a. Provide data on atmospheric density and wind characteristics by tracking the missiles as they
descend to the surface of the planet.

(b) Establish a network of navigational landmarks on the planetary surface.

It was concluded in this thesis that there is no single optimum entry profile which applies to all entry missions. The optimum trajectory is a strong function of the purpose of the mission and the initial conditions of the problem.

2.2 Recommendations for Further Work

The investigations of this thesis provide an analytical explanation for many of the important consequences of entry of an astronautical vehicle into a planetary atmosphere under a variety of initial conditions. Emphasis was placed on features which are common to all entry missions. Many additional problems associated with atmospheric entry are worthy of extensive study. Some of these problem areas are suggested below.

(1) Constraints on Entry Imposed by Van Allen Radiation Belts

The discovery in December 1958 of belts of energetic particles trapped in the Earth's magnetic field has suggested re-evaluation of certain entry mission concepts. The radiation belts may be viewed in the context of this thesis as restricting the initial conditions of the entry mission. The direction of approach to the planet prescribes the initial orientation of the trajectory plane. If a reconnaissance orbit is part of the mission concept, the radiation belts restrict the initial values of parameters of the reconnaissance ellipse to certain maximums.

Quantitative constraints on the entry trajectory resulting from
radiation belts cannot be accurately determined until more data on these belts is published. The presence of planetary radiation belts may require that severe restrictions be established on allowable eccentricities of the first few braking passes. Because of Van Allen radiation hazards, the direct first-pass entry profile is worthy of serious consideration, even though it presents a more serious guidance problem with respect to heating and acceleration problems.

(2) Gaseous Radiation Contribution to Total Vehicular Thermal Input

In the analysis performed in this thesis, thermal inputs to the surface of the vehicle as a result of gaseous radiation were ignored in the specification of heating constraints on allowable guidance trajectories (Chapter 7); convective heating alone was considered in this thesis. In the case of small vehicles, gaseous radiation contributes very little to total thermal input (37). For large vehicles, particularly winged vehicles at angles of attack near 90°, gaseous radiation may contribute as much as 10% of the total thermal input. Gaseous radiation heating rate was represented as equation (7-2); solution of this equation requires knowledge of the stagnation temperature of the gas and the radiation emissivity of the gas per unit path length. Simplified methods for computing thermal contributions due to gaseous radiation are not available at present. Development of analytical methods for incorporating gaseous radiation in the specification of thermal constraints on entry guidance is recommended for further study.

(3) Effects of Solar Atmosphere on Interplanetary Trajectories

Chapman (2) advances the theory that the Sun possesses a tenuous
atmosphere which extends through interplanetary space beyond the distance of the Earth's orbit. This atmosphere is believed to consist mainly of ionized hydrogen, protons, and electrons with a density of $10^3$ particles per cubic centimeter in the vicinity of the Earth, considerably greater density in the vicinity of Venus, and much less density near Mars. All known studies of interplanetary trajectories performed to date have neglected any atmosphere in space; the minute drag terms may be important, however, when integrated over flight times of many months.

(4) Externally Aided Adaptive Control

The operational extremes experienced by a vehicle during entry into a planetary atmosphere suggest a control system that must adapt itself to the changing environment. Self-adaptive control of the entry vehicle was not investigated in this thesis; it was suggested, however, that data which is measured or computed by the guidance system may conceivably be utilized by the control system to augment self-adaptive features of this system. The Conservation Parameter, defined in Section 1.7, was shown in Chapter 8 to be a measurable function which has sharp and predictable behavior during the course of entry. It was suggested that this parameter may be useful as a switching function and as a variable sensitivity factor to improve control system operation by augmenting self-adaptive features of the system. A logical extension of the present work is that of examining motions about the center of mass in order to specify stability and control requirements during the course of atmospheric entry.
Chapter 3

THREE-DIMENSIONAL KINEMATICS OF ENTRY

3.1 Statement of the Entry Problem

The general problem of guidance of astronautical vehicles entering planetary atmospheres can be stated as follows:

How can a vehicle be guided from an initial state (position, velocity) in or near a planetary atmosphere to a final state on or near the surface of the planet without compromising the structural integrity of the vehicle or endangering its human occupants (if any)?

This statement at once implies that the initial and final conditions of the problem are fundamental elements of the problem statement. Physical constraints are also fundamental to the statement of the problem and must be considered on all solutions to the problem.

3.2 A General Theory of Controlled Atmospheric Entry

The basic problem of entry appears in various forms in the literature: it may involve lifting or non-lifting vehicles, manned or unmanned missions, entry from stable or degenerate reconnaissance orbits, direct entry from an interplanetary transfer ellipse*; it may

* Called "first-pass entry" in Section 1.6.
involves various trajectory schemes such as glide decay or a trajectory separated into various phases each with a specific guidance program**. The diversity of approaches taken in studying the entry problem has resulted in the growth of philosophies with many elements in common and other elements at odds. In general, the written matter describing a general or specific technique has dealt with one or a few entry mission concepts. It is sometimes very difficult to compare these approaches on any common ground because of the differing approaches taken toward solving the same basic problem.

All of these concepts of the entry mission are simply variations of the same single basic situation: the guiding of a manned or unmanned vehicle from some initial point at which the decision is made to initiate controlled entry into the planetary atmosphere to the point at which its dynamical state is suitable for either impact or landing on the planetary surface. Underlying this situation is a certain element of prediction; control actions instituted at one time have a distinct influence on the dynamical state of the vehicle at some later time.

It was shown in Chapter 1 that all entry profiles may be considered as falling into one of two major classes:

1. The direct entry profile;
2. The degenerate orbital profile.

This thesis advances a generalized treatment of controlled entry into planetary atmospheres in the following manner:

1. The entry guidance problem is stated. This problem is the same regardless of the mission concept or the instrumentation

** See Fig. 1.10, for example.
system chosen for the purpose of solving the problem.

(2) The fundamental physical parameters that are available to solve the problem are discussed. The many different mission concepts and instrumentations systems are distinguished one from another by the:

(a) Initial conditions chosen in setting up the problem;
(b) Methods by which these fundamental physical parameters are utilized in arriving at a solution.

(3) Physical constraints on the problem of atmospheric entry are discussed. These are truly part of the statement of the problem since they must be recognized at all times in its solution; discussion of physical constraints are enough of a subject in themselves, however, to warrant a separate chapter in this thesis.

(4) Dynamical elements which are common to all entry missions are examined. The investigations performed in this thesis led to the separation of all trajectories into three operational regimes and to the definition of an environmental index, the Conservation Parameter, to define precisely the boundary conditions between phases.

(5) Approximate methods for predicting quantities which are important in the conceptual and early design phases of guidance systems are discussed for both of the major classes of entry profiles.

Item (1) above is discussed in chapters 3 and 4, item (2) in chapter 5, item (3) in chapter 7, item (4) in chapter 8, and item (5) in chapters 9 and 10.
3.3 Kinematics of Entry

The atmospheric entry problem is expressed in terms of its dynamics (forces) and kinematics (velocities). The dynamics of various mission concepts are different; this subject is considered in subsequent chapters of this thesis. The forces may be measured and controlled in many different ways. The major portion of the current unclassified literature which is concerned with the problem of atmospheric entry is devoted to advancing schemes for controlling these forces. A particular lift or drag program, for example, may induce a trajectory which has some desirable qualities under a certain set of initial conditions. Instrumentation methods for

1. Measuring these forces,
2. Controlling the forces,
3. Establishing stabilized reference frames,
4. Selecting the most convenient reference frame for computation purposes

are dealt with much less frequently in the unclassified literature. Comprehensive systems studies are generally concerned with a specific mission concept and/or vehicle; as a result, military security classifications are usually assigned to such studies.

Kinematics is the study of motion of particles and rigid bodies without consideration of the forces required to produce these motions. The kinematical statement of the entry problem is common to all mission concepts; it is basically a description of the geometry of the problem. The three-dimensional kinematics of the entry vehicle, considered as a point mass, is described in two separate guidance grids in this chapter.
3.4 Coordinate Systems

Inherent with any investigation of vehicle guidance is the requirement that a suitable set of coordinate frames* and trajectory parameters be chosen to describe the motion.

Fig. 3.1 shows three separate planet centered coordinate systems:

(1) **Planet Centered Inertial Frame** (subscript I):

Newton's laws are valid in an inertial frame. The use of a planet centered inertial frame is justified because gravitation and acceleration effects are indistinguishable.

(2) **Planet Reference Frame** (subscript 0):

Establishes the orientation of the planet with respect to the inertial frame.

(3) **Instantaneous Trajectory Plane Frame** (subscript T):

Establishes the orientation of the instantaneous plane of the trajectory with respect to the inertial frame. The orientation of the trajectory plane changes with time; the plane is normal to the vector cross product of the position and velocity vectors of the vehicle (\(\mathbf{R}\) and \(\mathbf{V}_I\), respectively**).

In order to analyze the trajectory of the entry vehicle, it is convenient to describe the motion in spherical polar coordinates. Reference frames for this purpose are usually selected with one axis along the radius vector from the center of the planet to the point at

---

* In this thesis, "frame", "reference space", "frame of reference", etc., are used synonymously.

** \(\mathbf{V}_I\) denotes velocity measured with respect to inertially fixed coordinates.
Planet Centered Inertial Frame: Centered at the center of the planet and sidereally nonrotating relative to the "fixed stars". $X_I$ and $Y_I$ are in the equatorial plane and $Z_I$ is directed along the polar axis (North).

Planet Reference Frame: Centered at the center of the planet and nonrotating with respect to the planet. This frame rotates about the planet's polar axis relative to the inertial frame at the planet's daily sidereal rate. $X_0$ and $Y_0$ are in the equatorial plane and fixed with respect to the planet. $Z_0$ is directed along the polar axis.

Instantaneous Trajectory Plane Frame: Centered at the center of the planet. $X_T$ is the line of intersection of equatorial plane with the instantaneous plane of the trajectory; $Y_T$ is in equatorial plane perpendicular to $X_T$; $Z_T$ is directed along the polar axis.

Fig. 3.1: Definition of the Inertial, Planet, and Instantaneous Trajectory Plane Coordinate Systems.
which the guidance is taking place, $\mathbf{r}_\text{r}$. This radius vector, for the planet, Earth, is the geocentric radius. In this thesis, the term "geocentric" is used in a more general sense (i.e., "planetocentric") to identify quantities associated with the spherical polar coordinates of any planet.

Two separate sets of guidance grids are shown in Fig. 3.2, the Geocentric Latitude-Longitude Triad ($\mathbf{r}_\text{r}$, $\mathbf{l}_\lambda$, $\mathbf{l}_\Delta$) and the Instantaneous Great-Circle Triad ($\mathbf{r}_\text{r}$, $\mathbf{l}_\phi$, $\mathbf{l}_\psi$). The following angles are defined in Fig. 3.2:

- $\psi$: inclination of the instantaneous trajectory plane with the equatorial plane (measured from equator to trajectory plane).
- $\lambda_{IT}$: "inertial" geocentric longitude of the ascending line of nodes; i.e., angle measured in equatorial plane from $X_I$ to $X_T$.
- $\lambda_{IP}$: "inertial" geocentric longitude of the vehicle; i.e., angle measured in equatorial plane from $X_I$ to entry vehicle at $P$.
- $\Lambda$: geocentric latitude of the vehicle.
- $\phi$: angle measured in the plane of the trajectory (in the direction of vehicle motion) from the ascending line of nodes to the vehicle.
- $\beta$: bearing angle of entry vehicle as seen by an inertially fixed observer in the vehicle; angle between North ($\mathbf{l}_\Delta$) and the horizontal component of velocity (directed along $\mathbf{l}_\phi$).

The geocentric latitude of the vehicle, $\lambda$, is not shown in Fig. 3.2. This is defined as the angle measured in the equatorial plane from $X_0$ to the vehicle.

Angular relations that are useful for converting quantities from one guidance grid to the other are summarized in Table 3.1:

---

* The interpretation of "geocentric" in this thesis may be considered to be a shortening of "geometric-centric" rather than the more precise "geoid-centric".
Geocentric Latitude-Longitude Triad: Orthogonal set of unit vectors centered at the entry vehicle center of mass and directed in radial, longitude, and latitude directions respectively.

Instantaneous Great Circle Triad: Orthogonal set of unit vectors centered at the vehicle center of mass and directed in radial, along-track, and across-track directions respectively.

Fig. 3.2: Definition of Latitude-Longitude and Great Circle Guidance Grids.
Table 3.1: Angular relations among guidance grids shown in Fig. 3.2.

\[
\sin \Lambda = \sin \phi \sin \psi \quad (3.1.1)
\]
\[
\sin (\lambda_{IP} - \lambda_{IT}) = \frac{\sin \phi \cos \psi}{\sin \Lambda} \quad (3.1.2)
\]
\[
\cos (\lambda_{IP} - \lambda_{IT}) = \frac{\cos \phi}{\cos \Lambda} \quad (3.1.3)
\]
\[
\sin \psi \cos \phi = \cos \Lambda \cos \beta \quad (3.1.4)
\]
\[
\cos \psi = \cos \Lambda \sin \beta \quad (3.1.5)
\]
\[
\tan \psi = \frac{\tan \Lambda}{\sin (\lambda_{IP} - \lambda_{IT})} \quad (3.1.6)
\]

Direction cosines between \( \overline{1}_r, \overline{1}_\phi, \overline{1}_\psi \) system and \( \overline{1}_r, \overline{1}_\Lambda, \overline{1}_\Phi \) system:

<table>
<thead>
<tr>
<th></th>
<th>( \overline{1}_r )</th>
<th>( \overline{1}_\phi )</th>
<th>( \overline{1}_\psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{1}_r )</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \overline{1}_\Lambda )</td>
<td>0</td>
<td>\sin \beta</td>
<td>- \cos \beta</td>
</tr>
<tr>
<td>( \overline{1}_\Phi )</td>
<td>0</td>
<td>\cos \beta</td>
<td>\sin \beta</td>
</tr>
</tbody>
</table>

(3.1.7)

Direction cosines between \( \overline{1}_{XT}, \overline{1}_{YT}, \overline{1}_{ZT} \) system and \( \overline{1}_r, \overline{1}_\phi, \overline{1}_\psi \) system:

<table>
<thead>
<tr>
<th></th>
<th>( \overline{1}_r )</th>
<th>( \overline{1}_\phi )</th>
<th>( \overline{1}_\psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{1}_{XT} )</td>
<td>\cos \phi</td>
<td>- \sin \phi</td>
<td>0</td>
</tr>
<tr>
<td>( \overline{1}_{YT} )</td>
<td>\sin \phi \cos \psi</td>
<td>\cos \phi \cos \psi</td>
<td>- \sin \psi</td>
</tr>
<tr>
<td>( \overline{1}_{ZT} )</td>
<td>\sin \phi \sin \psi</td>
<td>\cos \phi \sin \psi</td>
<td>\cos \psi</td>
</tr>
</tbody>
</table>

(3.1.8)
Navigational parameters are often measured in coordinates that are not spherical polar coordinates; examples are latitude and longitude measured with respect to astronomic and geographic* reference frames. Appendix D discusses these and other navigational reference frames, which are related to the figure of the planet. Mathematical methods for relating geographic and geocentric angles for the planets of the solar system are also derived in Appendix D. It will be noted that a numerical value of the eccentricity of the planet is required in order to convert angles from geocentric to geographic angles and vice-versa. The eccentricity of the planet is readily determined from the physical data of major bodies of the solar system summarized in Appendix B.

3.5 Kinematics of Entry in Geocentric Latitude-Longitude Coordinates

It is usual in the theory of the dynamics of rigid bodies to separate the motion of the center of mass from the motion about the center of mass. The former constitutes the study of flight performance while the latter embodies the theory of stability and control. The analytical work of this thesis is concerned specifically with performance analysis, hence motion about the center of mass is not considered.

Geocentric latitude longitude coordinates \((R, \lambda_{IP}, \Lambda)\) are a special set of spherical polar coordinates. Position of the vehicle is as follows:

\[
\bar{R} = (R \cos \Lambda \cos \lambda_{IP}) \bar{X} + (R \cos \Lambda \sin \lambda_{IP}) \bar{Y} + (R \sin \Lambda) \bar{Z} \tag{3-1}
\]

The vector angular velocity of the center of mass of the vehicle (with

* "Geographic" is used in this thesis as a shortening of "geometric-graphic" rather than the more precise "geoid-graphic."
The linear velocity (with respect to inertial space) expressed in latitude-longitude coordinates is:

$$V_I = R \dot{\lambda} + (R \dot{\lambda} \cos \Lambda) \cos \alpha + (R \ddot{\lambda}) \sin \alpha$$  \hspace{1cm} (3-3)

The Law of Coriolis is used to derive an expression for the acceleration of the vehicle:

$$\vec{a} = \left[ \frac{dV_I}{dt} \right] = \left[ \frac{dV_I}{dt} \right] + \vec{w}_{IP} \times \vec{V}_{IP}$$  \hspace{1cm} (3-4)

Performing the operations indicated in Eq. (3-4) gives the kinematic equation of the entry vehicle in the latitude-longitude triad:

$$\ddot{\lambda} = \left[ \frac{R}{R^2 - R \dot{\lambda}^2 - R \dot{\lambda}^2 \cos^2 \Lambda} \right] \dot{\lambda} + \left[ \frac{1}{R \cos \Lambda} \frac{d}{dt} (R^2 \dot{\lambda} \cos^2 \Lambda) \right] \cos \alpha + \left( R \ddot{\lambda} + 2R \dot{\lambda} \right) + R \dot{\lambda}^2 \cos \Lambda \sin \alpha \right) \sin \alpha$$  \hspace{1cm} (3-5)

The inertial geocentric longitude (\( \lambda_{IP} \)) differs from geocentric longitude (\( \lambda \)) by an angle equal to the time integral of the planet's angular velocity about its polar axis, \( \vec{w}_{IO} \). Eq. (3-5) may be written in terms of \( \lambda \) by making the following substitution:

$$\lambda_{IP} = \lambda + \vec{w}_{IO}$$  \hspace{1cm} (3-6)

3.6 Kinematics of Entry in Instantaneous Great-Circle Coordinates:

Most of the investigations described subsequently in this thesis
are based on equations of motion written in terms of components in the \( \mathbf{I}_r, \mathbf{I}_\phi, \mathbf{I}_\psi \) triad. In these coordinates, four quantities must be specified in order to position the vehicle: \( R, \phi, \psi \) and \( \lambda_{IT} \).

The angular velocity of the vehicle is:

\[
\dot{\mathbf{W}}_{IP} = \dot{\phi} \mathbf{I}_\psi + \dot{\psi} \mathbf{I}_X + \dot{\lambda}_{IT} \mathbf{I}_Z
\]  

(3-7)

Using the angular relations given in Table 3-1, this is written:

\[
\dot{\mathbf{W}}_{IP} = \left[ \dot{\lambda}_{IT} \sin \phi \sin \psi + \dot{\psi} \cos \phi \right] \mathbf{I}_r + \left[ \dot{\lambda}_{IT} \cos \phi \sin \psi - \dot{\psi} \sin \phi \right] \mathbf{I}_\phi \\
+ \left[ \dot{\phi} + \dot{\lambda}_{IT} \cos \psi \right] \mathbf{I}_\psi
\]  

(3-8)

The velocity of the vehicle is determined by applying the law of Coriolis to the radius vector:

\[
\mathbf{R} = \mathbf{R} \mathbf{I}_r
\]  

(3-9)

Thus:

\[
\mathbf{V}_I = \left( \frac{d\mathbf{R}}{dt} \right)_I = \left( \frac{d\mathbf{R}}{dt} \right)_{\text{moving}} + \mathbf{W}_{IP} \times \mathbf{R}
\]  

(3-10)

Carrying out the operations indicated by equation (3-10) gives:

\[
\mathbf{V}_I = \dot{\mathbf{R}} \mathbf{I}_r + \left[ \mathbf{R} \dot{\phi} + \mathbf{R} \dot{\lambda}_{IT} \cos \psi \right] \mathbf{I}_\phi - \left[ \mathbf{R} \dot{\lambda}_{IT} \cos \phi \sin \psi - \mathbf{R} \dot{\psi} \sin \phi \right] \mathbf{I}_\psi
\]  

(3-11)

The instantaneous plane of the trajectory is defined as a plane normal to \( \mathbf{R} \times \mathbf{V}_I \); therefore, there is no velocity component normal to this plane. From Fig. 3.2:

\[
\mathbf{V}_I = \dot{\mathbf{R}} \mathbf{I}_r + \mathbf{V}_{I\phi} \mathbf{I}_\phi
\]  

(3-12)

Comparing Equation (3-11) and (3-12) gives the following kinematic
relations:

\[ \frac{V_{I\phi}}{R} = \dot{\phi} + \lambda_{IT}\cos \psi \]  
(3-13)

\[ \lambda_{IT}\cos \phi \sin \psi = \dot{\psi} \sin \phi \]  
(3-14)

Differentiating equation (3-12) in the manner prescribed by

Eq. (3-4) gives:

\[ \ddot{A} = \left[ \ddot{R} - \frac{V_{I\phi}^2}{R} \right] \bar{I}_r + \left[ \ddot{V}_{I\phi} + \frac{R}{R} \ddot{V}_{I\phi} \right] \bar{I}_\phi + \left[ V_{I\phi} \left( \lambda_{IT}\sin \phi \sin \psi + \dot{\psi} \cos \phi \right) \right] \bar{I}_\psi \]  
(3-15)

Equations (3-13) through (3-15) are a set of five differential equations which, under arbitrary initial conditions, describe the time history of vehicular motion.*

* Noting that the \( \bar{I}_\psi \) component of \( \ddot{A} \) in Eq. (3-15) is \( V_{I\phi} W_{IP} \), where \( W_{IP} \) is the \( I_r \) component of \( \ddot{W}_{IP} \), these five equations may be described more simply as follows:

(a) Eq. (3-14) is replaced by two equations:

\[ W_{(IP)\_r} = \lambda_{IT} \sin \frac{\psi}{\cos \phi} \]  
(3-16)

(b) Eq. (3-13) is unchanged.

(c) Eq. (3-15) becomes:

\[ \ddot{A} = \left[ \ddot{R} - \frac{V_{I\phi}^2}{R} \right] \bar{I}_r + \left[ \ddot{V}_{I\phi} + \frac{R}{R} \ddot{V}_{I\phi} \right] \bar{I}_\phi + V_{I\phi} W_{(IP)\_r} \bar{I}_\psi \]  
(3-17)
4.1 The Instantaneous Ellipse

It is common in celestial mechanics to describe the motion of heavenly bodies in terms of parameters of conic sections. There are many definitions of conics; the following definitions are convenient:

1. **Ellipse**: The locus of points the sum of whose distances from two fixed points (foci) is constant.

2. **Circle**: The locus of points equidistant from a single point (special case of the ellipse).

3. **Hyperbola**: The locus of points the difference of whose distances from two fixed points (foci) is constant.

4. **Parabola**: The locus of points equally distant from a fixed point (focus) with a fixed straight line (directrix).

When a body is in motion under the action of an attractive central force that varies as the inverse square of the distance, and if no external forces act on the body, the path described will be a conic whose focus is at the center of attraction. A particle moving according to such a force obeys Kepler's laws. Kepler's three laws of planetary motion, published about 1610, were the result of his pioneering analysis of planetary observations, and laid the groundwork for many of the
important contributions of Sir Isaac Newton. Kepler's laws may be summarized as follows:

A particle moving under the action of an attractive central force that varies as the inverse square of the distance

(1) travels in an ellipse or hyperbola (or their special cases, a circle or parabola) with the attracting center at one of the foci;

(2) the radius vector from the center to the particle sweeps out equal areas in equal time*;

(3) for the elliptic orbit, which results in periodic motions, the squares of the periods of rotation are proportional to the cubes of the major axis of the orbits.

The computational complexity involved in solving any physical problem depends to a large extent on the coordinates chosen for describing the problem. It is clear, for example, that a rectilinear problem may be solved more easily in rectilinear coordinates than in spherical polar coordinates or oblate spheroidal coordinates. Much of the trajectory of a vehicle entering the atmosphere of a planet is elliptical in nature, hence the motion is most easily described in terms of elliptical parameters.

Elliptical parameters are useful for analyzing the motion of the entry vehicle during the orbital phases and in the early phases of entry where gas-dynamic forces are negligible in comparison to other terms in

* Kepler's second law, the conservation of areal velocity, is a general theorem for central force motion(19) since angular momentum is always conserved. The first and third laws are restricted specifically to the inverse square law of force.
the dynamical equations of motion*. Particular portions of the flight where it is appropriate to describe the trajectory in elliptical parameters are:

1. In the reconnaissance orbit (whether stable or degenerate);
2. In the Keplerian portion of the direct entry profile;
3. During circularization of the degenerate orbital profile.

It may be desirable to perform computations in terms of elliptical parameters during these phases of the trajectory, then to switch to a more conventional computational scheme for the atmospheric portion of flight. The Conservation Parameter discussed in Chapter 8 may be used as a switching function to convert from one guidance mode to another.

Elliptical parameters may be used throughout the trajectory by the "instantaneous ellipse" technique. At each instant an ellipse is constructed which matches the dynamical state of the vehicle; that is, the ellipse is oriented to pass through the present position of the vehicle and the ellipse has the angular momentum, energy, and velocity characteristics of the vehicle. The trajectory is described, therefore, as a time-varying ellipse; both the orientation and shape of the ellipse change with time. Fig. 4.1 shows the instantaneous ellipse and defines certain important parameters used frequently herein.

It should be noted that even during the stable reconnaissance orbit the trajectory is not planar, nor is the orientation of the ellipse fixed in space. By definition, the atmosphere does not perturb the motion in this case. The non-spherical component of gravitation does perturb the motion, however, such that even the stable orbit is

* This segment of the trajectory is called the Keplerian phase in this thesis.
Line of apsides: The major axis of the ellipse, connecting the apsides (apocenter and pericenter).

- Major axis = 2a
- Minor axis = 2b
- Latus rectum = 2l
- $l = a(1 - \varepsilon^2)$
- Eccentricity = $\varepsilon$ (Always less than 1.0 for ellipse)
- $\varepsilon = \sqrt{1 - (b/a)^2}$
- True anomaly = $\theta$
- Area of ellipse = $\pi ab$
- $n = \theta - \theta$
- Equation of ellipse: $R = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \theta}$
- $T = \text{period of orbit} = \frac{2 \pi a^{3/2}}{\sqrt{\gamma g Mo}}$

Fig. 4.1: The Instantaneous Ellipse.
slowly moving with respect to inertial coordinates. These perturbations result in secular effects* (regression of the line of nodes, $\dot{\lambda}_\text{IT} \neq 0$; and movement of the line of apsides (advance of perigee), $\dot{\eta} \neq 0$) and in periodic effects (oscillation of the angle of inclination $\psi$).

Atmospheric perturbations of the degenerate orbital profile are generally much more significant than gravitational perturbations. Because of atmospheric and gravitational perturbations, the Keplerian Phase may best be described as a slowly changing instantaneous ellipse and the Intermediate and Gas-Dynamic Phases as a rapidly changing instantaneous ellipse. It is the slowly changing characteristics of the instantaneous ellipse which leads to the conclusion that performing computations in elliptical parameters is the most desirable method in the Keplerian Phase.

4.2 Entry Kinematics in Elliptical Parameters

There are many possible combinations of elliptical parameters which may be used to describe the kinematics of the entry trajectory. The kinematical equations are readily written as six first order equations in terms of elliptical parameters instead of three second order equations as described in Chapter 3. Taratynova(20) selected the latus rectum and eccentricity as two of the primary dependent variables and described the kinematics with either time or $\theta$ as independent variables. His solution contains singularities in the special case of circular orbits; no discussion is given with respect to how singularities were handled in obtaining his results. Nielsen, Goodwin, and

---

* Effects which accumulate with time; i.e., non-periodic components.
Mersman(21) defined two special parameters, p and q, to obtain kinematical equations which were found to be convenient for numerical studies of the trajectories of non-lifting satellites. Singularities were removed by separating q into two additional parameters, v and w.

Many combinations of elliptical parameters were examined during the course of this investigation to determine a particular set which would lead to the most simple kinematical description and still retain much of the geometry of the problem. Of the sets examined, the following was found most convenient:

(1) **Independent variable**: time, t.

(2) **Dependent variables**:

(a) **Parameters of the instantaneous ellipse**:

(1) \( P = \) angular momentum = \( |\mathbf{R} \times \mathbf{V}_I| = RV_I \phi \) (4-1)

(2) \( \dot{\epsilon}_1 = \dot{\epsilon} \cos \theta \) (4-2)

(3) \( \dot{\epsilon}_2 = \dot{\epsilon} \sin \theta \) (4-3)

In equations (4-2) and (4-3), \( \dot{\epsilon} \) is eccentricity of the instantaneous ellipse and \( \theta \) is the angle from perigee to vehicular position (see Fig. 4.1).

(b) **Orientation of the instantaneous ellipse**:

(4) \( \psi \) : inclination of orbital plane

(5) \( \lambda_{IT} \) : inertial longitude of ascending line of nodes

(c) **Position of vehicle in instantaneous ellipse**:

(6) \( \phi \) : angle measured in elliptical plane from line of nodes to vehicle.

The foregoing set does not suffer from singularities in the case of
circular orbits, as will be shown subsequently in this chapter. The six parameters listed are sufficient to describe the state of the entry vehicle and its matching ellipse.

In obtaining the kinematical equations of the above set of quantities, an additional set was derived which may be used for elliptical motion ($\mathcal{E} \neq 0$):

1. Independent variable: $t$
2. Dependent variables:
   a. Parameters of instantaneous ellipse:
      1. $P$
      2. $\mathcal{E}$
      3. $\eta$ or $\theta$
   b. Orientation of instantaneous ellipse:
      4. $\psi$
      5. $\lambda_{IT}$
   c. Position of vehicle:
      6. $\phi$

In performing machine-aided numerical studies of entry trajectories involving a number of orbits, it is often desirable to examine the motion from the standpoint of the history of various quantities per orbit instead of with time. This is particularly true for deducing periodic and secular trends (21). If it is desired to examine trends from perigee to perigee, $\theta$ is a convenient choice for the independent variable. Similarly, if it is desired to examine trends per orbit, $\phi$ is a convenient choice. The equations of the foregoing sets may be easily converted to these or any other particular independent variable.
Nielsen(21) shows that the independent variable for computer studies must be chosen with care. In the terminal phase of a non-lifting satellite's trajectory or throughout a ballistic trajectory, for example, the vehicle is traveling in a near-vertical path. Many of the quantities change rapidly for very small changes in $\theta$. Time is a much more suitable choice than $\theta$ as an independent variable in studies of steep trajectories.

The areal velocity of the instantaneous ellipse is constant (by Kepler's second law). The areal velocity is one half the angular momentum. The areal velocity of the ellipse is(22):

$$\frac{P}{2} = \frac{\pi ab}{T} \quad (4-4)$$

where $T$ is the period of the orbit and is equal to(22):

$$T = \frac{3}{2} \frac{2\pi a}{\sqrt{\gamma g M_0}} \quad (4-5)$$

$\gamma_g$ is the gravitational constant and $M_0$ is the mass of the planet.

From equations (4-4) and (4-5):

$$\frac{P^2}{\gamma g M_0} = \frac{b^2}{a} \quad (4-6)$$

The definition of eccentricity as a function of $a$ and $b$ is given in Fig. 4.1:

$$1 - \epsilon^2 = \left(\frac{b}{a}\right)^2 \quad (4-7)$$

Substituting this into the equation of the ellipse:

$$R = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos \theta} \quad (4-8)$$
and eliminating \( a \) and \( b \) with equation (4-6) gives:

\[
R = \frac{P^2}{\gamma G M o (1 + \epsilon \cos \theta)} \quad (4-9)
\]

The instantaneous ellipse represents a constant energy trajectory. It is desirable to derive certain energy relations which are useful in obtaining the kinematical description of the ellipse. The total energy per unit mass is:

\[
\frac{\mathcal{E}_{tot}}{M} = \frac{\mathcal{E}_{pot}}{M} + \frac{\mathcal{E}_{kin}}{M} = - \frac{\gamma M o}{R} + \frac{\mathbf{v}_I \cdot \mathbf{v}_I}{2} \quad (4-10)
\]

Since the total energy of the instantaneous ellipse is constant, apogeeal or perigeal values may be used for \( \mathbf{v}_I \) and \( R \) in equation (4-10). Value of a quantity at perigee is denoted in this thesis by the subscript \( \eta \) and at apogee by the subscript \( \alpha \). The following equations may be written:

\[
V_{I\eta} = \sqrt{\frac{R_\eta}{R_\gamma}} \sqrt{\frac{2 \gamma M o}{R_\xi + R_\eta}} \quad (4-11)
\]

\[
V_{I\alpha} = \sqrt{\frac{R_\gamma}{R_\xi}} \sqrt{\frac{2 \gamma M o}{R_\xi + R_\eta}} \quad (4-12)
\]

\[
R_\xi = a (1 + \epsilon) \quad (4-13)
\]

\[
R_\eta = a (1 - \epsilon) \quad (4-14)
\]

Substituting equations (4-14) and (4-11) into (4-10) gives:

\[
\frac{\mathcal{E}_{tot}}{M} = - \frac{\gamma M o}{2a} \quad (4-16)
\]

Solving equation (4-10) for \( v_I^2 \), with equation (4-16) for \( \frac{\mathcal{E}_{tot}}{M} \), equation (4-8) for \( a \), and equation (4-9) for \( R \) gives:
\[ v_{1}^{2} = \frac{\gamma_{M}^{2}}{R (1 + \varepsilon \cos \theta)} (1 + 2 \varepsilon \cos \theta + \varepsilon^{2}) = \left( \frac{\gamma_{M}^{2}}{P} \right) (1 + 2 \varepsilon \cos \theta + \varepsilon^{2}) \]  

(4-17)

Equation (3-12) showed that:

\[ v_{1}^{2} = R^{2} + v_{I\phi}^{2} \]  

(4-18)

Using equation (4-17) and (4-1) in (4-18) gives:

\[ R = \frac{\gamma_{M}^{2}}{P} \varepsilon \sin \theta \]  

(4-19)

The foregoing equations provide enough information to determine first order kinematic equations for all of the elliptical parameters. From equation (4-1):

\[ p = R V_{I\phi} + R V_{I\phi} \]  

(4-20)

Comparing this to equation (3-15) shows:

\[ p = RA_{\phi} \]  

(4-21)

From equations (3-14) and (3-15):

\[ \psi = \frac{\cos \phi}{V_{I\phi}} A_{\psi} = \frac{R}{P} \cos \phi A_{\psi} \]  

(4-22)

\[ \lambda_{IT} = \frac{\sin \phi}{V_{I\phi} \sin \psi} A_{\psi} = \frac{R}{P \sin \psi} A_{\psi} \]  

(4-23)

From equations (3-13) and (4-23):

\[ \dot{\phi} = \frac{V_{I\phi}}{R} - \frac{\cot \psi \sin \phi}{V_{I\phi}} A_{\psi} = \frac{P}{R^{2}} - \frac{R \cot \psi \sin \phi}{P} A_{\psi} \]  

(4-24)

The determination of \( \dot{\xi} \) and either \( \dot{\theta} \) or \( \dot{\eta} \) to complete the set of
six first order equations is considerably more involved than that required to derive the previous four equations. From equation (4-9):

\[ \mathcal{E} = \frac{1}{\cos \theta} \left[ \frac{P^2}{g_M R} - 1 \right] \]  

(4-25)

Differentiating this, with equation (4-21) for \( \dot{P} \) and (4-19) for \( \dot{R} \), gives:

\[ \dot{\mathcal{E}} = \mathcal{E} \tan \theta \dot{\theta} + \frac{2PA_P}{\gamma g_M \cos \theta} - \frac{PC \sin \theta}{R^2 \cos \theta} \]  

(4-26)

This equation contains both \( \dot{\mathcal{E}} \) and \( \dot{\theta} \). It is necessary to determine another independent equation containing \( \dot{\mathcal{E}} \) and \( \dot{\theta} \), then solve the equations simultaneously. From equation (3-15):

\[ \ddot{R} = A_P + \frac{V \Theta}{R} \]  

(4-27)

Differentiating equation (4-19), setting it equal to equation (4-27), and solving explicitly for \( \dot{\theta} \) gives:

\[ \dot{\theta} = \frac{P}{\mathcal{E} \gamma g_M \cos \theta} (A_P + \frac{P^2}{R^2}) - \tan \theta \frac{\mathcal{E}}{\mathcal{E}} + \frac{R \tan \theta}{P} A_P \]  

(4-28)

Substituting equation (4-28) into (4-26) gives:

\[ \dot{\mathcal{E}} = \left[ \frac{R(1 + \mathcal{E} \cos \theta) \sin \theta}{P} \right] A_P + \frac{R \sin \theta}{F} \left[ \mathcal{E} + 2 \cos \theta + \mathcal{E} \cos^2 \theta \right] A_P + \frac{R \sin \theta}{P^2} \]  

(4-29)

Using equation (4-29) in (4-28) gives:

\[ \dot{\theta} = \left[ \frac{R \cos \theta (1 + \mathcal{E} \cos \theta)}{\mathcal{E} P} \right] A_P - \left[ \frac{R \sin \theta (2 + \mathcal{E} \cos \theta)}{\mathcal{E} P} \right] A_P + \frac{P}{\mathcal{E} R^2} (\mathcal{E} + \cos \theta) \]  

(4-30)
From Fig. 4.1:

\[ \eta = \phi - \theta \]  \hspace{1cm} (4-31)

Therefore, \( \eta \) can be written from equations (4-24) and (4-30) as:

\[ \dot{\eta} = - \left[ \frac{R \cos \theta}{\epsilon P} (1 + \epsilon \cos \theta) \right] A_r + \left[ \frac{R \sin \theta}{\epsilon P} (2 + \epsilon \cos \theta) \right] A_\phi 
- \left[ \frac{R \cot \psi \sin \phi}{P} \right] A_\psi - \frac{P \cos \theta}{\epsilon R^2} \]  \hspace{1cm} (4-32)

The spherical component of gravitational mass attraction may be written:

\[ G_{sp} = \frac{\gamma \frac{M}{R^2}}{R^2} = \frac{P^2}{R^3 (1 + \epsilon \cos \theta)} \]  \hspace{1cm} (4-33)

Thus equation (4-32) becomes:

\[ \dot{\eta} = - \left[ \frac{R \cos \theta}{\epsilon P} (1 + \epsilon \cos \theta) \right] [A_r + G_{sp}] + \left[ \frac{R \sin \theta}{\epsilon P} (2 + \epsilon \cos \theta) \right] A_\phi 
- \left[ \frac{R \cot \psi \sin \phi}{P} \right] A_\psi \]  \hspace{1cm} (4-34)

Equation (4-30) becomes:

\[ \dot{\theta} = \left[ \frac{R \cos \theta}{\epsilon P} (1 + \epsilon \cos \theta) \right] [A_r + G_{sp}] - \left[ \frac{R \sin \theta}{\epsilon P} (2 + \epsilon \cos \theta) \right] A_\phi + \frac{P}{R^2} \]  \hspace{1cm} (4-35)

Equation (4-29) is written:

\[ \dot{\epsilon} = \left[ \frac{R \sin \theta}{P} (1 + \epsilon \cos \theta) \right] [A_r + G_{sp}] + \frac{R}{P} \left[ \epsilon + 2 \cos \theta + \epsilon \cos^2 \theta \right] A_\phi \]  \hspace{1cm} (4-36)
The six kinematical equations are therefore:

(1) \( \dot{P} \) \hspace{1cm} \text{Equation (4-21)}

(2) \( \dot{\epsilon} \) \hspace{1cm} \text{Equation (4-36)}

(3) \( \eta \) or \( \dot{\theta} \) \hspace{1cm} \text{Equations (4-34) or (4-35)}

(4) \( \dot{\psi} \) \hspace{1cm} \text{Equation (4-22)}

(5) \( \dot{N}_{IT} \) \hspace{1cm} \text{Equation (4-23)}

(6) \( \dot{\phi} \) \hspace{1cm} \text{Equation (4-24)}

Equation (4-31) is used to relate \( \eta \) and \( \phi \) in these equations. Equation (4-9) gives an expression for \( R \) and equation (4-33) an expression for \( G_{sp} \). It will be shown later that \( G_{sp} \) in equations (4-34), (4-35) and (4-36) cancels one of the terms in \( \Lambda_r \).

If numerical studies are performed in which it is desired to use \( \phi \) or \( \theta \) as the independent variable instead of time, the above equations are easily converted as follows:

(1) \( \frac{dP}{d\phi} = \frac{dP}{dt} \frac{dt}{d\phi} = \frac{\dot{P}}{\dot{\phi}} \) \hspace{1cm} (4-37)

(2) \( \frac{d\epsilon}{d\phi} = \frac{\dot{\epsilon}}{\dot{\phi}} \) \hspace{1cm} (4-38)

(3) \( \frac{d\eta}{d\phi} = \frac{\dot{\eta}}{\dot{\phi}} ; \frac{d\theta}{d\phi} = \frac{\dot{\theta}}{\dot{\phi}} \) \hspace{1cm} (4-39)

(4) \( \frac{d\psi}{d\phi} = \frac{\dot{\psi}}{\dot{\phi}} \) \hspace{1cm} (4-40)

(5) \( \frac{dN_{IT}}{d\phi} = \frac{\dot{N}_{IT}}{\dot{\phi}} \) \hspace{1cm} (4-41)
A similar procedure may be followed for $\theta$.

Examination of the six kinematic relations shows that singularities exist if:

1. $P = 0$ (V_I$\phi = 0$)
2. $\Psi = 0$
3. $R = 0$
4. $\xi = 0$

The angular momentum can be zero only in vertical flight, a condition that will generally not exist during the entry vehicle's trajectory.

$\Psi = 0$ causes the $\sin \Psi$ term in the denominator of equation (4-23) to vanish; this is not troublesome because, as will be seen in Chapter 5, $A\Psi$ in the numerator also has a $\sin \Psi$ term which cancels the sine term in the denominator. $R$ does not go to zero in a realizable trajectory. Therefore, the only troublesome singularity in the above list is that existing in eccentricity (for the special case of a circular orbital trajectory).

Eccentricity is in the denominator in the expressions for $\eta$ and $\dot{\theta}$ only; one or the other of these equations is required to complete the set of kinematic equations. This singularity can be removed by replacing $\theta$ and $\xi$ by two new quantities:

\[
\xi_1 = \xi \cos \theta
\]

\[
\xi_2 = \xi \sin \theta
\]
Effectively, eccentricity is replaced by its components along the major and minor axis respectively. Differentiating:

\[ \dot{E}_1 = \dot{\epsilon} \cos \theta - \dot{\epsilon} \dot{\theta} \sin \theta \]  
\[ (4-43) \]

\[ \dot{E}_2 = \dot{\epsilon} \sin \theta + \dot{\epsilon} \dot{\theta} \cos \theta \]  
\[ (4-44) \]

Substituting \( \dot{\theta} \) and \( \dot{\epsilon} \) from equations (4-35) and (4-36) into these two equations gives:

\[ \dot{E}_1 = \left[ \frac{P}{R^2} \right] E_2 + \frac{2R}{P} (1 + E_1) \lambda_\phi \]  
\[ (4-45) \]

\[ \dot{E}_2 = \frac{R}{P} (1 + E_1) \left[ a_r + G_{sp} \right] + \frac{R}{P} E_2 \lambda_\phi + \frac{P \epsilon_1}{R^2} \]  
\[ (4-46) \]

It is clear that the troublesome singularity arising in the special case of a circular orbit \( (\epsilon = 0) \) has been removed. Considerable simplification in the resulting equations has been effected in the process.

The six kinematical equations become:

1. \( \dot{P} \) Equation (4-21)
2. \( \dot{E}_1 \) Equation (4-45)
3. \( \dot{E}_2 \) Equation (4-46)
4. \( \dot{\gamma} \) Equation (4-22)
5. \( \dot{\lambda}_{IT} \) Equation (4-23)
6. \( \dot{\phi} \) Equation (4-24)

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This set of equations, original with this thesis, is considerably more simple than other sets cited in the literature or examined during the course of this investigation.

Energy as a function of angular momentum and eccentricity is determined by substituting Equation (4-9) and (4-17) into equation (4-10):

\[
\frac{\mathcal{E}_{\text{tot}}}{M} = \frac{1}{2} \left( \frac{\gamma g_0 M}{p} \right)^2 \left( \mathcal{E}_1^2 + \mathcal{E}_2^2 - 1 \right) = \frac{p^2}{2R^2} \left( \frac{\mathcal{E}_2^2 - 1}{1 + \mathcal{E}_1^2} \right)^2
\]

(4-47)

The rate of change of energy transfer is:

\[
\frac{d}{dt} \left( \frac{\mathcal{E}_{\text{tot}}}{M} \right) = \frac{p \mathcal{E}_2}{R (1 + \mathcal{E}_1^2)} \left[ A_r + G_{sp} \right] + \left( \frac{p}{R} \right) A_\phi
\]

(4-48)

4.3 Computation of Navigation Parameters

The six first order kinematic equations in terms of parameters of the instantaneous ellipse may be easily transformed into latitude-longitude coordinates, or any other desired guidance grid. Fig. 4.2 is a functional diagram of an open loop system which computes altitude, latitude, longitude, velocity, altitude rate, total energy, and time rate of transfer of total energy to the planetary atmosphere. The latter two quantities may be used for predicting total range or lifetime remaining.

It may be seen that the complexity of this three-dimensional system is not great due to the simplicity of the computations for \( p \), \( \mathcal{E}_1 \), \( \mathcal{E}_2 \), \( \Psi \), \( \lambda_{IT} \), and \( \phi \). Stored data includes the gravitational model of the planet; the gravitational model is discussed in Appendix C.
Fig 4.2 Open Loop Navigation System

Note:
Number in each box refers to equation which defines computational function of that box.
of this thesis. The input to the system is the external specific force vector, which is the vector sum of lift, drag, and thrust forces. These are discussed in detail in Chapter 5.

The external specific force vector is measured by three orthogonal accelerometers which must be stabilized in some coordinate frame. Outputs of the specific force measuring system are components of external specific force in the $\bar{I}_r$, $\bar{I}_\phi$, and $\bar{I}_\psi$ directions. For numerical trajectory studies, the external specific force may be computed from altitude, velocity, and stored data on the atmospheric model and vehicle characteristics. Appendix E discusses various atmospheric models for the planets.

The acceleration vector $\bar{A}$ represented as outputs of the adder in Fig. 4.2 is the vector sum of gravitational specific force and external specific force. Chapter 5 discusses this further.

4.4 Gravitational and Atmospheric Perturbations

It is possible to deduce much information regarding the general characteristics of the vehicle's motion from the kinematic equations of section 4.2.

For a reconnaissance orbit that is at all times beyond the sensible atmosphere, the $A_r$, $A_\phi$, $A_\psi$ terms in the kinematic equations contain gravitational terms only. It may be seen from these equations that $A_r$ always exists in combination with $G_{sp}$; the sum of these two quantities, in this case, is the radial component of non-spherical gravitational effects.

It is shown in Chapter 5 that for an oblate spheroidal planet:

$$G_r \approx G_{sp} - \frac{\beta G_{sp} R^2 (eq)}{R^2} (1 - 3 \sin^2 \phi \sin^2 \psi) \quad (4.49)$$
\[ G_\phi \approx -\frac{12 \nu G_{sp} R^2(\text{eq})_0}{R^2} \sin \phi \cos \phi \sin^2 \psi \]  
(4-50)

\[ G_\psi \approx -\frac{12 \nu G_{sp} R^2(\text{eq})_0}{R^2} \sin \phi \cos \theta \sin \psi \]  
(4-51)

In these equations:

(1) \( \nu \) is a constant representing the quadrupole strength of the planet's gravitational field.

(2) \[ G_{sp} = \frac{\gamma_{m} M_0}{R^2} \]

(3) \( R(\text{eq})_0 \) is the planet's equatorial radius.

For the case of the vacuum trajectory, lift and drag forces are zero. If no thrust is generated by the vehicle's propulsive system:

\[ A_r = G_r \]

\[ A_\phi = G_\phi \]

\[ A_\psi \neq G_\psi \]  
(4-52)

Inclination of orbital plane for the vacuum trajectory:

Combining equations (4-22) and (4-51) gives:

\[ \dot{\psi} = -\frac{\sqrt{3}}{2} G_{sp} \frac{R^2(\text{eq})_0}{PR} \sin 2\phi \cos \psi \]  
(4-53)

The inclination of the orbital plane has a very small periodic oscillation due to the non-spherical component of gravitation.
Regression of the line of nodes for the vacuum trajectory:

Combining equations (4-23) and (4-51) gives:

\[ \dot{\lambda}_{IT} = -\frac{12\nu G_{sp} R^2_{(eq)o}}{pr} \sin^2\phi \cos\psi \]  
(4-54)

Roberson\(^{23}\) shows that the regression of the line of nodes per nodal period\(^*\) can be expressed as:

\[ \left[ \Delta \lambda_{IT} \right]_{\text{per nodal period}} = -12\nu \left( \frac{\gamma_{M_o}}{P^2} \right)^2 R^2_{(eq)o} \cos\psi \]  
(4-55)

The line of nodes therefore rotates slowly in the equatorial plane opposite in direction (regression) to the projection of the vehicle's motion on the equatorial plane.

Movement of line of apsides (advance of perigee) for the vacuum trajectory:

Combining equation (4-34) with equations (4-49) through (4-51) gives:

\[ \dot{\eta} = \frac{6\nu R^2_{(eq)o}}{R^3} \left\{ \frac{P\cos\theta}{\epsilon R} \left( 1 - 3 \sin^2\phi \sin^2\psi \right) \right. \]
\[ \left. - \frac{2}{\epsilon} \frac{\gamma_{M_o}}{P} \left[ \left( \frac{2 + \epsilon_4}{2\epsilon} \right) \sin\theta \sin^2\psi \sin^2\phi - \sin^2\phi \cos^2\psi \right] \right\} \]  
(4-56)

Roberson\(^{23}\) shows that the change of \( \eta \) per nodal period is:

\[ \left[ \Delta \eta \right]_{\text{per nodal period}} = 6\nu R^2_{(eq)o} \left[ \frac{\gamma_{M_o}}{P^2} \right]^2 (4 - 5 \sin^2\psi) \]  
(4-57)

Comparing (4-51) with (4-55) gives some interesting results. For near

\* The nodal period, sometimes called the synodic period, is the time required to go from one ascending node to the next.
equatorial orbits ($\psi$ small), $\eta$ advances twice as rapidly as $\lambda_T$ regresses. Thus, the accumulated advance of perigee on the equatorial plane with respect to inertial coordinates is equal to (4-55), or one-half of equation (4-57).*

For a vehicle at an altitude of 300 nautical miles in a circular orbit around the Earth at an orbital inclination of $23^\circ 27'$ (ecliptic plane), the regression of the line of nodes is less than 9.5 milliradians per nodal period (less than 32.5 minutes of arc). The rate for Moon is about $1.5^\circ$ per nodal period. The advance of perigee is about 65 minutes of arc per nodal period (approximately 12.5 milliradians per hour).

**Atmospheric perturbations on angle of inclination**

The primary quantity causing the angle of inclination to change is $A\psi$. Since drag is in the plane of the trajectory, it has little effect on the angle of inclination. The most significant effect in causing changes in $\psi$ is side forces generated by a lifting vehicle in banking flight.

**Atmospheric perturbations on line of nodes:**

Nielsen$^{(21)}$ shows that drag forces, being in the plane of the trajectory, have very small influence on the regression of the line of nodes. He shows that equation (4-55) generally applies to zero-lift vehicles to three significant figures. The movement of the line of nodes is a sensitive function of $A\psi$, which is significant only for

* The line of nodes cannot be visualized for a true equatorial orbit, hence it is desirable to envision a very small angle of inclination.
lifting vehicles in banking flight.

Atmospheric perturbations on line of apsides

Equation (4-34) shows that $\eta$ is sensitive to $A_r$, $A\phi$, and $A\gamma$. The small motion of the line of apsides due to the non-spherical component of gravity is completely overshadowed when drag forces are significant. Nielsen shows that the introduction of drag terms in the motion of a non-lifting vehicle (initially in a circular, equatorial orbit) causes the line of apsides to move around the orbit with a variable lag at the average speed of the satellite. Lift also influences the motion of the line of apsides, but not as markedly as drag.

4.5 A Navigational Satellite for Position Reference During First-time Entry into the Atmospheres of Strange Planets

First-time entry into the atmospheres of strange planets presents special problems in specifying position when compared to navigating over a well-mapped planet such as Earth. The choice of a suitable landing site must necessarily be based on reconnaissance of the planetary surface while in orbit around the planet. The orbital altitude for the reconnaissance phase must be high enough such that a prolonged orbit may persist, yet low enough that fairly accurate mapping of the terrain is feasible.

The navigation of a vehicle flying from one point to another point on the surface of a planet is generally based on navigational parameters measured with respect to the planet, i.e., latitude, longitude, and altitude. It is less common to use parameters identified with a particular mission, such as angular displacements measured with respect to
the great circle course along-track and across-track.

The entry mission to the surface of a strange planet, on the other hand, does not originate from a point on the planet's surface. The entry mission generally originates from a reconnaissance orbit which, if the perigeeal altitude is sufficiently great, is very slowly changing with time. The non-spherical component of the planet's gravitational field causes the line of nodes and the line of apsides to rotate slowly with respect to an inertial framework. Drag forces result in energy transfer from the vehicle to the planetary atmosphere, but for sufficiently high orbits, this transfer causes negligible change in the satellite orbit over periods of time comparable to that required for entry once retro-rocket thrust is generated.

A basic position reference available during the course of entry is the original reconnaissance orbit. If the entry vehicle is launched from a mother satellite, then the mother satellite, which remains in the reconnaissance orbit, may track both the entry vehicle and the pre-selected landing site and transmit this tracking information to the guidance computer of the entry vehicle. In this way, the parent satellite replaces ground tracking stations which are used as an external source of tracking information for Earth satellites and entry vehicles.

In the event that no parent satellite exists, then a navigational satellite to serve the same purpose may be deposited in the reconnaissance orbit by the entry vehicle prior to initiating the entry phase.

Since a navigational scheme such as outlined briefly above uses the reconnaissance orbit as the basic reference from which to measure positions, it may be found convenient to express position and to carry out the guidance computations in terms of elliptical parameters. The
landing site may be considered to be a target moving in threedimensional space with respect to the near-stable reconnaissance trajectory represented by the mother or navigational satellite. The entry vehicle is also moving with respect to the initial reconnaissance trajectory. The entry problem is therefore similar to the fire control problem, with the entry vehicle (projectile) fired from the parent satellite (gun) to hit the moving landing site (target). The problem is much more severe than the conventional fire control problem, however, because the projectile must be constrained to paths for which it will not burn up or encounter accelerations beyond tolerable levels.

The nominal or programmed path of the entry vehicle may be computed in advance as one which the vehicle would fly under standard atmospheric conditions starting from the particular initial point and ending at the landing site selected in advance. This trajectory must be consistent with tolerable accelerations and heating rates.

It was shown previously in this chapter that there are six elliptical elements required to specify the position and path of the vehicle. One set of six such elements are:

1. $\psi, \Lambda_{TT}$, (to specify the instantaneous orientation of the plane of the trajectory);
2. $P, \varepsilon_1, \varepsilon_2$ (to specify the ellipse which matches the instantaneous radius and velocity vectors of the vehicle);
3. $\phi$ (to specify the position of the vehicle in this ellipse).

$P, \psi$, and $\Lambda_{TT}$ are constant or very slowly varying with time in the reconnaissance orbit. $\varepsilon_1$ and $\varepsilon_2$ are sinusoidal with a very slowly changing magnitude of oscillation. $\phi$ increases monotonically.
The instantaneous dynamical state of the entry vehicle, with respect to the navigational satellite in the reconnaissance orbit, may conveniently be specified in terms of the six elliptical quantities. The orbit of the navigation satellite should be predictable to a fairly high degree of accuracy. The predicted values of six elliptical elements for this orbit are part of the data stored for use during the entry mission.
5.1 Introduction

The forces acting on the entry vehicle as it moves along its trajectory are of two major types: field forces and non-field forces.

(1) Field forces

The most important field forces which act on the entry vehicle are those arising from the planet's gravitational field and magnetic field. The weight of the vehicle is a field force. If the vehicle should acquire a static charge, a force will act on it as it passes through the magnetic field surrounding the planet; this, too, is a field force.

The planet's gravitational field is the predominant field force acting on the entry vehicle and is the only one considered in this thesis. Gravitational effects of any natural satellites of the planet, the Sun, and other planets in the solar system are considered herein to be negligibly small in the vicinity of the planet in comparison to the gravitational field of the planet itself.

(2) Non-field forces

Non-field forces acting on the entry vehicle are of two
major types: kinetic reaction forces and external forces.

(a) Kinetic reaction forces are apparent forces resulting from accelerations of the reference space which is used in specifying the problem with respect to inertial space (space in which Newton's laws are valid). If the reference space is either fixed or moving at a constant linear velocity relative to a Newtonian frame, then the reference space itself is Newtonian. If the reference space has a translation with constant acceleration \( \mathbf{a} \) relative to a Newtonian frame, then the reference space may be treated as Newtonian if a fictitious force, \(-\mathbf{Ma}\), is applied to each particle. If the reference space is rotating with constant angular velocity relative to a Newtonian frame, then the reference frame may be treated as Newtonian, if, to each particle, two kinetic reaction forces are applied: Coriolis force and centrifugal force.

(b) External forces are generated primarily as follows:

(1) Gas-dynamic forces: lift and drag forces which are produced as the vehicle flies through the planetary atmosphere. The magnitude of these forces depends on the characteristics of the atmosphere, the velocity of the vehicle with respect to the atmosphere, and the gas-dynamic characteristics of the vehicle.

(2) Thrust forces: forces produced by the propulsive elements of the vehicle.

Lift and drag forces exist only when the vehicle is flying through
an atmosphere while thrust forces may be generated in a vacuum. In this thesis, lower order external forces are neglected. These may include electric drag, solar radiation pressure, etc.

Application of Newton's laws of motion to the vehicle gives:

$$\ddot{\mathbf{A}} = \ddot{\mathbf{F}}_{\text{li}} + \ddot{\mathbf{F}}_{\text{dr}} + \ddot{\mathbf{F}}_{\text{th}} + \ldots + \ddot{\mathbf{G}} + \ldots$$  (5-1)

where $\ddot{\mathbf{A}}$ is the acceleration of the vehicle in the Newtonian frame.

This term includes all kinetic reaction forces when the reference space is not Newtonian.

$\ddot{\mathbf{F}}_{\text{li}}$, $\ddot{\mathbf{F}}_{\text{dr}}$, $\ddot{\mathbf{F}}_{\text{th}}$ are external specific forces (force per unit mass) resulting from lift, drag, and thrust, respectively.

Other external specific force terms are considered to be higher order effects and are neglected in this thesis.

$\ddot{\mathbf{G}}$ is the field force due to the planet's gravitational field.

Other field forces are considered to be higher order effects and are neglected in this thesis.

The derivation of the acceleration vector $\ddot{\mathbf{A}}$ was discussed in Chapter 3 in two separate guidance grids, the latitude-longitude triad and the instantaneous great-circle triad. It is to be noted that both of these reference frames are rotating with respect to inertial space, hence Coriolis and centrifugal force terms are implicit in $\ddot{\mathbf{A}}$. Derivation of field and external forces in component form are discussed in subsequent sections of the present chapter to complete the dynamical statement of the entry problem.

It should be emphasized that the right-hand side of equation (5-1) has two distinct roles. In the instrumentation of the guidance system for an entry mission, the vector sum of the external specific force terms
is measured by the specific force measuring subsystem (three orthogonal accelerometer units). In trajectory studies leading to the establishment of a reference or nominal trajectory for the mission, the external specific forces are mathematically derived from the gas-dynamic model of the vehicle, from the atmospheric model of the planet, and from a mathematical model of the engines. Errors in these models when compared to conditions actually encountered during a particular mission lead to departures of the vehicle from its planned path in space and time. It is clear that accurate information is needed on the physical characteristics of both the vehicle and the planet.

The reference trajectory should be selected as one for which errors in the vehicle model and the atmospheric model will not jeopardize the successful completion of the mission; that is, the reference trajectory should be the mean path in a corridor established by upper and lower estimates of errors in these models.* The reference trajectory should also be selected such that errors in the models result in minimum range errors, if landing point accuracy is important to the mission. It is not necessary that the guidance system always return the vehicle to a predetermined reference trajectory when position errors accumulate; a more efficient concept is that of computing during the course of the mission, either continuously or intermittently, new reference trajectories. These trajectories should be based on measured atmospheric data and observed winds and must be at all times consistent with

* The corridor must also be consistent with heating, acceleration, and radiation constraints on the trajectory, discussed in Chapter 7.
mission objectives and physical constraints on the trajectory. For this guidance concept, which is particularly desirable in the case of Explorer missions, the force terms on the right-hand side of equation (5-1) are measured and computed; i.e., the reference trajectory is revised periodically by computations based on measurements accumulated. Other independent measurements, such as vehicle skin temperature, may be used to augment specific force measurements.

5.2 The Gravitational Model of the Oblate Planet

A discussion of gravitational mass attraction and the acceleration of gravity is contained in Appendix C. The gravitational potential of the planet, including the quadrupole contribution of the non-spherical component, is as follows*:

\[
\phi = - \frac{GM}{R} \left[ 1 - \nu \frac{R_{eq}^2}{R^2} (1 - 3 \cos^2 \Delta) \right]
\]

(C-7)

The gravitational field vector was defined as:

\[
\vec{G} = - \nabla \phi
\]

(C-8)

The following equation defines the gradient in spherical polar coordinates\(^{(24)}\):

\[
\nabla \phi = \frac{\partial \phi}{\partial R} \hat{r} + \frac{1}{R \sin (90 - \Delta)} \frac{\partial \phi}{\partial \lambda} \hat{\lambda} + \frac{1}{R} \frac{\partial \phi}{\partial \Delta} \hat{\Delta}
\]

(5-2)

Carrying out the indicated operations on equation (C-7) gives:

* A quantitative description of the gravitational model for the pear-shaped Earth is contained in reference (18).
\[ G_r = -\frac{\partial \bar{\phi}}{\partial r} = -G_{sp} + 3\nu G_{sp} \left(\frac{R_{eq}}{R}\right)^2 (1 - 3 \cos 2\lambda) \quad (5-3) \]

\[ G_\lambda = -\frac{1}{R \sin (90 - \lambda)} \frac{\partial \bar{\phi}}{\partial \lambda_{IP}} = 0 \quad (5-4) \]

\[ G_\phi = -\frac{1}{R} \frac{\partial \bar{\phi}}{\partial \phi} = -6\nu G_{sp} \left(\frac{R_{eq}}{R}\right)^2 \sin 2\lambda \quad (5-5) \]

In these equations, \( G_{sp} \) is the radial component of the gravitational mass attraction for a spherical planet:

\[ G_{sp} = \frac{\gamma M}{R^2} \quad (5-6) \]

The gravitational components in the latitude-longitude triad are readily converted to the great-circle triad by using the angular relations given in Table 3.1 together with the following trigonometric identities:

\[ \cos 2\lambda = 1 - 2 \sin^2 \lambda \quad (5-7) \]

\[ \sin 2\lambda = 2 \sin \lambda \cos \lambda \]

The results of this transformation are as follows:

\[ G_r = -G_{sp} - 6\nu G_{sp} \left(\frac{R_{eq}}{R}\right)^2 (1 - 3 \sin^2 \phi \sin^2 \psi) \quad (5-8) \]

\[ G_\phi = -6\nu G_{sp} \left(\frac{R_{eq}}{R}\right)^2 \sin^2 \psi \sin 2\phi \quad (5-9) \]

\[ G_\psi = -6\nu G_{sp} \left(\frac{R_{eq}}{R}\right)^2 \sin \phi \sin 2\psi \quad (5-10) \]
5.3 Gas-Dynamic Forces

The interaction of the vehicle and the atmosphere may be both electrical and mechanical in nature. Electrical effects, which may increase the drag over gas-dynamic values, are sensitive to the charge accumulated on the vehicle. Electrical drag has greatest relative influence on the vehicle at high altitudes where gas-dynamic forces are small. The general concensus is that electrical drag contributes less than 10\% of the total drag of long-lifetime satellites presently in orbit around Earth\(^{(25)}\); in the atmospheric portion of the entry trajectory, electrical drag should contribute even less of the total drag. Electrical contributions to vehicular drag are therefore not included in the force terms of this thesis.

The entry vehicle passes through a number of flight regimes. Mechanical interaction of the vehicle and atmosphere is a function of the mean free path of the molecules of the planetary atmosphere. Three types of flow regimes are generally traversed by the entry vehicle:

1. **Free molecular flow regime**: the mean free path of atoms and molecules which rebound from the entry vehicle is great when compared to vehicle dimensions. In this regime, the atmospheric particles are unaffected by the moving vehicle until they strike its surface.

2. **Slip-flow regime**: the mean free path of the atmospheric particles rebounding from the vehicle is not large compared to the dimensions of the vehicle. Outgoing atoms may collide with incoming atoms, thus preventing them from striking the vehicle.
(3) **Continuum flow regime:** the region where the atmospheric particles are affected before striking the vehicle. In this regime, the mean free path of atmospheric particles is very much less than the dimensions of the vehicle.

For Earth, the mean free path at 300,000 feet is approximately 0.1 ft.; at 500,000 feet, the mean free path is approximately 100 feet.

Di Taranto (26) suggests these altitudes as approximate boundaries between the above flow regimes.

The drag coefficient of the entry vehicle decreases as it passes from the free-molecular flow regime to the continuum flow regime.*

Fig. 5.1 shows the drag force as a vector having a direction opposite that of the velocity vector of the vehicle with respect to the atmosphere, \( \overrightarrow{V(AM)} \). The lift force vector acts in a direction normal to \( \overrightarrow{V(AM)} \) in the plane of symmetry of the vehicle. In this analysis, it is assumed that the atmosphere surrounding the planet rotates with the

---

* Henry (27) estimates the average molecular mass of particles at high altitudes above the Earth to be of the order of 14.5 on the atomic scale, since the air is assumed largely monatomic; he estimates the velocities of the particles to be of the order of \( 1.5 \times 10^5 \) cm/sec. This velocity is about 0.2 the velocity of a circular orbital satellite. Henry estimates the temperature of a satellite at high altitudes to be approximately 280-300° Kelvin when exposed to the Sun and about 50° cooler when in the Earth's shadow. Assuming the particles bouncing off the surface of the satellite have a temperature roughly the same as the vehicle, reflected particles should therefore have velocities in the neighborhood of \( 10^5 \) cm/sec. Since speeds of atmospheric particles both before and after collision with the satellite are much less than the satellite's velocity, it is reasonable to assume that, in a coordinate frame in which the satellite is at rest, the molecules approach with a relative velocity equal to the satellite's velocity and rebound with zero relative velocity. The change in momentum of all of the free molecules striking the satellite in unit time is \( PS \overrightarrow{V(AM)}^2 \), which is equal to the drag of the vehicle. Hence the drag coefficient in the free molecular flow regime is \( CD = 2 \). Spheres in hypersonic (attached shock) continuum flow, on the other hand, have drag coefficients of the order of 0.7.
Fig. 5.1: Lift and Drag Specific Force Vectors.
planet (when wind* is zero). The analysis is generalized to include the effects of wind. The vectors $\overrightarrow{F_{(dr)}}$, $\overrightarrow{F_{(li)}}$, and $\overrightarrow{V_{(AM)}}$ are always coplanar, the vector $\overrightarrow{T_r}$ does not, in general, lie in this plane.

The magnitude of the lift and drag specific forces are:

$$|\overrightarrow{F_{(dr)}}| = \frac{D}{M} = \frac{1}{2} \rho \overrightarrow{V_{(AM)}}^2 \frac{C_D S}{M}$$  \hspace{1cm} (5-11)

$$|\overrightarrow{F_{(li)}}| = \frac{L}{M} = \frac{1}{2} \rho \overrightarrow{V_{(AM)}}^2 \frac{C_L S}{M}$$  \hspace{1cm} (5-12)

where:

- $D$ is the drag force
- $L$ is the lift force
- $\rho$ is the free stream atmospheric density
- $\overrightarrow{V_{(AM)}}$ is the vehicular velocity relative to the atmosphere
- $S$ is a reference area for the vehicle
- $C_L$ is lift coefficient
- $C_D$ is drag coefficient

The lift and drag coefficients vary with the angle of attack, Mach number, and configuration of the vehicle. Atmospheric density, viscous, elastic, and turbulence properties are significant in determination of lift and drag coefficients.

Analytical determination of lift and drag coefficients for other than the most simple shapes is difficult. Wind tunnel tests and other experimental methods for measuring lift and drag cannot be performed at this time for Mach numbers in excess of 10 to 15. Thus the specification of $C_L$ and $C_D$ for a particular entry vehicle may be subject to errors.

---

* Wind is defined as relative movement of the atmosphere with respect to coordinates fixed in the planet, $\overrightarrow{V_{O(AM)}}$.
In most of the analytical work of this thesis, \((C_D S)\) is assumed constant irrespective of the flow regimes traversed by the vehicle. In some missions, transition between flow regimes may occur in the Intermediate Phase. Should this situation occur, the altitude span of the Intermediate Phase may be slightly greater than that given in Chapter 8*.

The lift and drag characteristics of the entry vehicle depend strongly on its configuration. Many vehicle designs are under consideration for entry missions. Among these are\(^{(29)}\):

1. **High performance gliders**
   Vehicles with high lift-to-drag ratios that are similar in appearance to current high performance delta wing fighter aircraft.

2. **Giders with wings having rounded leading edges**
   These are similar to (1) except sharp structural edges are avoided to prevent localized heating during entry.

3. **Lifting and non-lifting capsules**

4. **Variable geometry vehicles with triangular planforms**
   Horizontal control surfaces are folded into the back of the vehicle until velocities are reduced below those values corresponding to maximum heating. The

* For example, if the drag coefficient and flight path angles are constant, the Intermediate Phase spans about 20 miles of altitude for entry into Earth's atmosphere. If the drag coefficient is reduced to one-half its free-molecular flow value during the time that the vehicle crosses the Intermediate Phase, the width of this phase increases to about 24 miles.
vehicles enter the atmosphere at near 90° angle of attack (high drag configuration); the nose is lowered to conventional angles of attack in low speed flight.

(5) **Inflatable vehicles with large wing areas**

These vehicles have exceptionally low wing loadings.

(6) **Collapsible Wings**

The wings for these vehicles resemble kites that are extended to increase lift capabilities.

The lift and drag coefficients are coupled in a different manner for different vehicle configurations. The general lift-drag polar may be approximated by (30):

\[
C_D = C_{D0} + K_{ind} C_L^2
\]

(5-13)

- **$C_{D0}$** is zero-lift drag coefficient
- **$K_{ind}$** is induced drag factor

$C_{D0}$ and $K_{ind}$ are generally functions of Mach number.

Although this thesis was not concerned with a particular vehicle, it was necessary in many instances to select certain quantitative values of lift and drag coefficients either to illustrate the solutions derived or to compare the results with more exact numerical solutions. The following simplifying assumptions were made throughout this thesis:

(1) **Lift and drag coefficients are independent of Mach number.**

This is a reasonable assumption only for high Mach numbers *.

---

* That is, at Mach numbers greater than approximately five.
Lift and drag coefficients are independent of altitude.

As pointed out previously, the drag coefficient in the free molecular flow regime may be two or more times as great as that in the continuum flow regime. Since the Gas-Dynamic Phase is generally characterized by continuum flow throughout, and since lift and drag forces are of greatest significance in this particular phase, this assumption should not introduce limitations on the solutions. The altitude span of the Intermediate Phase discussed in Chapter 8 may be slightly conservative as a result of this assumption.

The lift and drag coefficients for bodies at small angles of attack in hypersonic flight may be expressed as:

\[ C_L = 2\alpha \]  \hspace{1cm} (5-14)

\[ C_D = C_{D_0} + \alpha C_L = C_{D_0} + C_L^2/2 \]  \hspace{1cm} (5-15)

where \( \alpha \) is angle of attack.

The zero-lift drag coefficient is approximately constant for a particular vehicle. From equations (5-15), (5-11) and (5-12):

\[ \frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + C_L^2/2} \]  \hspace{1cm} (5-16)

This equation has a maximum value when:

\[ C_{D_0} = C_L^2/2 \]  \hspace{1cm} (5-17)

Thus:

\[ (L/D)_{\text{max}} = \frac{1}{2\alpha_{\text{max}}} = \frac{1}{C_{L_{\text{max}}}} = \frac{1}{\sqrt{2C_{D_0}}} = \frac{1}{\sqrt{C_{D_{\text{max}}}}} \]  \hspace{1cm} (5-18)
Table 5.1 summarizes lift and drag coefficients for three representative vehicle classes computed from Eq. (5-18).

<table>
<thead>
<tr>
<th>(L/D)$_{\text{max}}$</th>
<th>C$_{D0}$</th>
<th>C$_{D\text{max}}$</th>
<th>C$_{L\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.0</td>
<td>4.0</td>
<td>2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.125</td>
<td>0.250</td>
<td>0.5</td>
</tr>
<tr>
<td>4.0</td>
<td>0.03125</td>
<td>0.0625</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The vehicle classes summarized in Table 5.1, which have the lift-drag polar of equation (5-15), are used in the following sections of this thesis:

1. **Chapter 7: trajectory constraints imposed by human acceleration tolerances**
   
   Constraint bands are shown for each of the three vehicle classes in the velocity-altitude plane. The upper line of each band corresponds to flight at C$_{L\text{max}}$, the lower line corresponds to flight at C$_{L} = 0$.

2. **Chapter 8: phase boundaries between the Keplerian, Intermediate, and Gas-Dynamic Phases**

   It should be noted that for flight path angles less than about 10°, which includes most conceivable entry trajectories, the location of these boundaries is a strong function of the instantaneous drag characteristics of the vehicle and is essentially independent of
lift. The results presented graphically in Chapter 8, therefore, may be applied to a broad class of vehicles by entering the curves with the appropriate drag coefficient.

A particular specification of lift and drag coefficients were not required in the analytical solutions of Chapters 9 and 10. In order to determine quantitatively the accuracy of closed form solutions determined, however, comparison is frequently made to numerical solutions for vehicles with particular aerodynamic characteristics. The aerodynamic qualities of the vehicles chosen is clearly indicated in each case.

5.4 Vehicle Coordinate Triad and Engine Gimbal Triad:

The drag vector is directed anti-parallel to the velocity vector of the vehicle with respect to the atmosphere. The lift vector is perpendicular to this velocity vector and acts in the plane of symmetry of the vehicle. In the analysis performed herein, it is assumed that the entry vehicle has the capability of rotating the lift vector (i.e., banking) in order to produce curvature of the trajectory in a controlled manner to enable the vehicle to reach landing sites at some distance from the "no-bank" trajectory. For military vehicles, such as a reconnaissance platform, the banking capability may permit adequate military coverage of an entire hostile nation.

Two vehicle-centered coordinate triads are defined in Fig. 5.2; the Wings Level triad $\vec{l}_{xWL}, \vec{l}_{yWL}, \vec{l}_{zWL}$ and the Vehicle triad $\vec{l}_x, \vec{l}_y, \vec{l}_z$. It is noted that both the Wings Level and the Vehicle triad are oriented along the velocity vector of the vehicle with respect
Fig. 5.2: "Wings Level" and "Vehicle" Coordinate Triads.
to the atmosphere. The left wing unit vector in the zero bank condition
\( \overline{v}_{WL} \) always lies in the \( \overline{I} - \overline{I}_\Lambda \) plane\(^*\). Rotation of the vehicle
about \( \overline{V}(AM) \) (coincident with \( \overline{I}_{xWL} \), \( \overline{I}_x \)) causes the lift vector to
move out of the vertical plane, hence causes a side force which tends
to curve the trajectory. The amount of rotation of the vehicle from the
wings level condition is defined as the bank angle, \( B \).

Following is a brief description of the coordinate systems shown
in Fig. 5.2:

\( \overline{I}_{xWL}, \overline{I}_{yWL}, \overline{I}_{zWL} \) **Wings Level Triad:** Orthogonal set of unit vectors
centered at vehicle center of gravity. \( \overline{I}_{xWL} \) is along the
velocity vector of the vehicle with respect to the atmos-
phere. \( \overline{I}_{yWL} \) is along the left wing of the vehicle in the
zero bank condition. \( \overline{I}_{zWL} \) is along the lift vector in the
zero bank condition.

\( \overline{I}_x, \overline{I}_y, \overline{I}_z \) **Vehicle Coordinate Frame:** Orthogonal set of unit vectors
centered at vehicle center of gravity. \( \overline{I}_x \) is along the
velocity vector of the vehicle with respect to the atmos-
phere. \( \overline{I}_y \) is along the left wing of the vehicle; \( \overline{I}_z \) is in the
plane of symmetry of the vehicle and is a unit vector in the
lift direction.

It is noted that \( \overline{I}_x \) and \( \overline{I}_{xWL} \) are always coincident. \( \overline{I}_y \) differs
from \( \overline{I}_{yWL} \) and \( \overline{I}_z \) differs from \( \overline{I}_{zWL} \) by the angle of bank \( B \). In the
zero bank condition, both triads are coincident.

* \( \overline{I}_\Lambda \) is positive in the geocentric East direction. \( \overline{I}_\Lambda \) is positive
in the North direction. The \( \overline{I}_\Lambda - \overline{I}_\Lambda \) plane is the geocentric horizontal
plane, or "level plane" - hence the term "Wings Level" coordinate system.
The following angles are defined in Fig. 5.2:

\[ \alpha_H = \alpha \left( \mathbf{i}_{\lambda} - \mathbf{i}_{\lambda_1} \right) \]  
angle between East (\( \mathbf{i}_{\lambda} \)) and horizontal component of vehicular velocity with respect to the atmosphere.

\[ \gamma = \alpha \left( \mathbf{i}_{\lambda_1} - \mathbf{i}_x \right) \]  
angle between the horizontal component of vehicular velocity with respect to the atmosphere (i.e., the component in the \( \mathbf{i}_{\lambda} - \mathbf{i}_{\lambda_1} \) plane) and the total velocity vector of the vehicle with respect to the atmosphere.

\[ \beta = \alpha \left( \mathbf{i}_{y_{WL}} - \mathbf{i}_y \right) = \alpha \left( \mathbf{i}_{z_{WL}} - \mathbf{i}_z \right) \]  
bank angle of vehicle. Corresponds to rotations about the \( l_x \) axis from the wings level condition.

Following are definitions of \( \alpha_H \) and \( \gamma \) in terms of velocity components:

\[ \alpha_H = \arctan \left( \frac{V(AM)_x}{V(AM)} \right) = \arccos \left( \frac{V(AM)}{\sqrt{V^2(AM)_x + V^2(AM)}} \right) \]  
(5-19)

\[ \gamma = \arctan \left( \frac{V(AM)_y}{\sqrt{V^2(AM)_x + V^2(AM)}} \right) = \arccos \left( \frac{\sqrt{V^2(AM)_x + V^2(AM)}}{V(AM)} \right) \]  
(5-20)

Rotations from the \( \mathbf{i}_r, \mathbf{i}_{\lambda}, \mathbf{i}_{\lambda_1} \) triad to the \( \mathbf{i}_z, \mathbf{i}_x, \mathbf{i}_y \) triad are performed in the following order:

(1) Rotate \( \left( \mathbf{i}_r, \mathbf{i}_{\lambda}, \mathbf{i}_{\lambda_1} \right) \) to \( \left( \mathbf{i}_r, \mathbf{i}_{\lambda_1}, \mathbf{i}_{y_{WL}} \right) \) by angle \( \alpha_H \) about \( \mathbf{i}_r \) axis.
(2) Rotate \((\mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_y)\) to \((\mathbf{I}_{zWL}, \mathbf{I}_{xWL}, \mathbf{I}_{yWL})\) by angle \(\gamma\) about \(\mathbf{I}_{yWL}\) axis.

(3) Rotate \((\mathbf{I}_{zWL}, \mathbf{I}_{xWL}, \mathbf{I}_{yWL})\) to \((\mathbf{I}_z, \mathbf{I}_x, \mathbf{I}_y)\) by angle \(B\) about \(\mathbf{I}_{xWL}\) axis.

Direction cosines between \(\mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_\Lambda\) triad and the \(\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z\) triad are given in Table 5.2.

<table>
<thead>
<tr>
<th></th>
<th>(\mathbf{I}_r)</th>
<th>(\mathbf{I}_\lambda)</th>
<th>(\mathbf{I}_\Lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbf{I}_x)</td>
<td>(\sin \gamma)</td>
<td>(\cos \gamma \cos A_H)</td>
<td>(\cos \gamma \sin A_H)</td>
</tr>
<tr>
<td>(\mathbf{I}_y)</td>
<td>(\cos \gamma \sin B)</td>
<td>(- \sin A_H \cos B)</td>
<td>(\cos A_H \cos B)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(- \sin \gamma \cos A_H \sin B)</td>
<td>(- \sin \gamma \sin A_H \sin B)</td>
</tr>
<tr>
<td>(\mathbf{I}_z)</td>
<td>(\cos \gamma \cos B)</td>
<td>(\sin A_H \sin B)</td>
<td>(- \cos A_H \sin B)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(- \sin \gamma \cos A_H \cos B)</td>
<td>(- \sin \gamma \sin A_H \cos B)</td>
</tr>
</tbody>
</table>

The coordinate systems defined above are not those used in conventional aircraft analysis where it is common to define a set of axes fixed to the airframe. It is necessary to define a set of vehicle-fixed axes if the stability and control characteristics are to be studied - a subject beyond the scope of this thesis.
Thrust forces generated by the propulsion system of the entry vehicle will, in general, be applied in such a direction as to slow the vehicle down* (retro-thrust). It is assumed that the thrust vector passes at all times through the vehicle center of gravity in order to avoid excessive moments. The engine gimbal angles shown in Fig. 5.3 were selected to describe the thrust direction.

Rotation from the $\mathbf{I}_x$, $\mathbf{I}_y$, $\mathbf{I}_z$ to the thrust triad $(\mathbf{I}_{x2}, \mathbf{I}_{y2}, \mathbf{I}_{z2})$ is performed in the following order:

1. Rotate $(\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z)$ to $(\mathbf{I}_{x1}, \mathbf{I}_{y1}, \mathbf{I}_{z1})$ by angle $A_e$ about $-\mathbf{I}_y$ axis.

2. Rotate $(\mathbf{I}_{x1}, \mathbf{I}_{y1}, \mathbf{I}_{z1})$ to $(\mathbf{I}_{x2}, \mathbf{I}_{y2}, \mathbf{I}_{z2})$ by angle $A_d$ about $\mathbf{I}_{z1}$ axis.

Direction cosines between $\mathbf{I}_{x2}', \mathbf{I}_{y2}', \mathbf{I}_{z2}$ triad and $\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z$ triad are given in Table 5.3.

<table>
<thead>
<tr>
<th>$\mathbf{I}_{x2}$</th>
<th>$\mathbf{I}_{y2}$</th>
<th>$\mathbf{I}_{z2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{I}_x$</td>
<td>$\mathbf{I}_y$</td>
<td>$\mathbf{I}_z$</td>
</tr>
<tr>
<td>$\cos A_e \cos A_d$</td>
<td>$\sin A_d$</td>
<td>$\sin A_e \cos A_d$</td>
</tr>
<tr>
<td>$\cos A_e \sin A_d$</td>
<td>$\cos A_d$</td>
<td>$- \sin A_e \sin A_d$</td>
</tr>
<tr>
<td>$-\sin A_e$</td>
<td>0</td>
<td>$\cos A_e$</td>
</tr>
</tbody>
</table>

* Except possibly in the landing phase.
Fig. 5.3: Engine Gimbal Angles
The following angles are shown in Fig. 5.3:

$A_e$: Engine gimbal angle generated by rotations about the negative $I_y$ axis.

$A_d$: Engine gimbal angle generated by rotations about the displaced $I_a$ axis (after rotating through angle $A_e$).

5.5 Components of Drag.

The velocity vector of the vehicle with respect to the atmosphere, $\vec{V}_{(AM)}$, is:

$$\vec{V}_{(AM)} = \vec{V}_I - \vec{V}_{I(AM)}$$ (5-21)

where: $\vec{V}_I$ is the velocity vector of the vehicle with respect to fixed inertial coordinates.

$\vec{V}_{I(AM)}$ is the velocity vector of the atmosphere with respect to inertial coordinates.

Under no-wind conditions, the air mass rotates with respect to inertial space with the same angular velocity as the planet. Winds are the result of relative movement of the atmospheric mass with respect to coordinates rotating with the planet. In terms of geographic coordinates, wind may be considered to have a vertical component, a north-south component, and an east-west component. Winds encountered are generally not constant, particularly for a vehicle whose geographic position may be changing rapidly. Updrafts and downdrafts (vertical winds) may, to a first order, be considered as essentially cancelling each other over the long ranges traversed by the entry vehicle; they are generally localized effects. Neglecting vertical components of atmospheric winds, the wind vector is:
\[
\bar{V}_o(AM) = V_o(AM)_\lambda \bar{1}_\lambda + V_o(AM)_\Lambda \bar{1}_\Lambda
\]  
(5-22)

where:

\(\bar{V}_o(AM)\) is the velocity vector of the moving atmosphere with respect to coordinates fixed in the planet (wind).

\(V_o(AM)_\lambda\) is the east component of wind (may be positive or negative).

\(V_o(AM)_\Lambda\) is the north component of wind (may be positive or negative).

The velocity vector of the atmosphere with respect to inertially fixed coordinates is:

\[
\bar{V}_I(AM) = \bar{V}_o(AM) + W_{10} R \cos \Lambda \bar{1}_\lambda
\]  
(5-23)

where \(W_{10}\) is the angular velocity of the planet with respect to inertial coordinates.

Substituting equations (5-22) and (5-23) into equation (5-21), and using \(\bar{V}_I\) (in latitude-longitude coordinates) derived in Chapter 3 gives the following equation for the velocity of the vehicle with respect to the moving atmosphere:

\[
\bar{V}(AM) = \dot{R}\bar{1}_r + \left[R(\dot{\lambda}_P - W_{10}) \cos \Lambda - V_o(AM)_\lambda\right] \bar{1}_\lambda + \left[R\dot{\Lambda} - V_o(AM)_\Lambda\right] \bar{1}_\Lambda
\]  
(5-24)

Equation (5-24) is transformed to the instantaneous great circle triad with the aid of Table 3.1:

\[
\bar{V}(AM) = \dot{R}\bar{1}_r + \left[V_I\dot{\phi} - (W_{10} R \cos \psi + V_o(AM) \phi)\right] \bar{1}_\phi + \left[W_{10} R \sin \psi \cos \phi - V_o(AM) \psi\right] \bar{1}_\psi
\]  
(5-25)
The drag specific force vector is:

$$ F_{(dr)} = - \frac{\rho C_D S}{2M} V^2_{(AM)} \overline{r} $$

(5-26)

By definition:

$$ \overline{r} = \frac{\overline{V}_{(AM)}}{V_{(AM)}} $$

(5-27)

Therefore:

$$ F_{(dr)} = - \frac{\rho C_D S}{2M} V_{(AM)} \overline{V}_{(AM)} $$

(5-28)

The drag components are readily written from equations (5-24), (5-25), and (5-28):

I. Geocentric Latitude-Longitude Coordinates:

A. No wind conditions ($V_{0(AM)} = 0$):

$$ F_{(dr)}_r = - \frac{\rho C_D S}{2M} V_{(AM)} \dot{R} $$

(5-29)

$$ F_{(dr)}_\lambda = - \frac{\rho C_D S}{2M} V_{(AM)} R \cos \Lambda (\dot{\tau}_{IP-WIO}) $$

(5-30)

$$ F_{(dr)}_\phi = - \frac{\rho C_D S}{2M} V_{(AM)} R \dot{\Lambda} $$

(5-31)

where:

$$ V^2_{(AM)} = R^2 + [R \cos \Lambda (\dot{\tau}_{IP-WIO})]^2 + (R \dot{\Lambda})^2 $$

(5-32)

B. With atmospheric winds ($V_{0(AM)} \neq 0$):

$$ F_{(dr)}_r = - \frac{\rho C_D S}{2M} V_{(AM)} \dot{R} $$

(5-33)
\[
F_{(dr)\lambda} = -\rho \frac{C_D S}{2M} V_{(AM)} \left[ R \cos \Lambda (\dot{\lambda}_{IP} - W_{IO}) - V_o(AM) \lambda \right] \quad (5-34)
\]

\[
F_{(dr)\Lambda} = -\rho \frac{C_D S}{2M} V_{(AM)} \left[ R \dot{\Lambda} - V_o(AM) \Lambda \right] \quad (5-35)
\]

where:

\[
V^2_{(AM)} = \dot{r}^2 + \left( R \cos \Lambda (\dot{\lambda}_{IP} - W_{IO}) - V_o(AM) \lambda \right)^2 \quad (5-36)
\]

\[
+ \left( R \dot{\Lambda} - V_o(AM) \Lambda \right)^2
\]

II. **Instantaneous Great-Circle Coordinates:**

A. **No wind conditions** ($V_o(AM) = 0$):

\[
F_{(dr)r} = -\rho \frac{C_D S}{2M} V_{(AM)} \dot{r} \quad (5-37)
\]

\[
F_{(dr)\phi} = -\rho \frac{C_D S}{2M} V_{(AM)} \left[ V_{i\phi} - W_{IO} \cos \psi \right] \quad (5-38)
\]

\[
F_{(dr)\psi} = -\rho \frac{C_D S}{2M} V_{(AM)} \left[ W_{IO} R \sin \psi \cos \phi \right] \quad (5-39)
\]

where:

\[
V^2_{(AM)} = \dot{r}^2 + (V_{i\phi} - W_{IO} \cos \psi)^2 + (W_{IO} R \sin \psi \cos \phi)^2 \quad (5-40)
\]

B. **With atmospheric winds** ($V_o(AM) \neq 0$):

\[
F_{(dr)r} = -\rho \frac{C_D S}{2M} V_{(AM)} \dot{r} \quad (5-41)
\]

\[
F_{(dr)\phi} = -\rho \frac{C_D S}{2M} V_{(AM)} \left[ V_{i\phi} - W_{IO} \cos \psi - V_o(AM) \phi \right] \quad (5-42)
\]

\[
F_{(dr)\psi} = -\rho \frac{C_D S}{2M} V_{(AM)} \left[ W_{IO} R \sin \psi \cos \phi - V_o(AM) \psi \right] \quad (5-43)
\]
where:

\[
V^2_{(AM)} = R^2 + \left[ V_{I\phi} - W_{IO} R \cos \psi - V_o(AM) \phi \right]^2 + \left[ W_{IO} R \sin \psi \cos \phi - V_o(AM) \psi \right]^2
\]

(5-44)

5.6 Components of Lift

The lift force is directed along the vehicle's \( \mathbf{I}_z \) axis shown in Fig. 5.2:

\[
\mathbf{F}_{(li)} = \frac{C_{LS}}{2M} v^2(AM) \mathbf{I}_z
\]

(5-45)

By using the angular relations of Tables 3.1 and 5.2, the components of lift may be written as follows:

I. Latitude-Longitude Coordinates

\[
F_{(li)r} = \frac{C_{LS}}{2M} \left[ v^2(AM) + v^2(AM) \right]^{1/2} v_{(AM)} \cos B
\]

(5-46)

\[
F_{(li)} = \frac{C_{LS}}{2M} v_{(AM)} \frac{V_{(AM)} V_{(AM)} \sin B - \hat{R} V_{(AM)} \cos B}{\sqrt{v^2(AM) + v^2(AM)}^{1/2}}
\]

(5-47)

\[
F_{(li)} = -\frac{C_{LS}}{2M} v_{(AM)} \frac{V_{(AM)} V_{(AM)} \sin B + \hat{R} V_{(AM)} \cos B}{\sqrt{v^2(AM) + v^2(AM)}^{1/2}}
\]

(5-48)

In these equations

\[
\bar{V}_{(AM)} = \hat{R} \mathbf{I}_r + \left[ R ( \lambda_{IP} - W_{IO} ) \cos \Lambda - V_o(AM) \lambda \right] \mathbf{I}_\lambda + \left[ R \hat{\lambda} - V_o(AM) \lambda \right] \mathbf{I}_\lambda
\]

(5-49)
II. Great-Circle Coordinates:

\[
F_{(1)i}^r = \frac{C_L S}{2M} \left( \frac{V_{(AM)}^2 \phi + V_{(AM)}^\psi}{V_{(AM)} \cos B} \right)^{1/2}
\]

\[
F_{(1)i}^\phi = \frac{C_L S}{2M} \frac{V_{(AM)}}{V_{(AM)}} \left( \frac{V_{(AM)} \sin B - \dot{R} V_{(AM)} \phi \cos B}{V_{(AM)}^2 \phi + V_{(AM)}^2 \psi} \right)^{1/2}
\]

\[
F_{(1)i}^\psi = \frac{C_L S}{2M} \frac{V_{(AM)}}{V_{(AM)}} \left( \frac{V_{(AM)} \sin B + \dot{R} V_{(AM)} \psi \cos B}{V_{(AM)}^2 \phi + V_{(AM)}^2 \psi} \right)^{1/2}
\]

In these equations:

\[
\vec{V}_{(AM)} = \dot{R} \frac{\vec{I}}{r} + \left[ \vec{v}_{I\phi} - \vec{w}_{IO} R \cos \psi - \vec{V}_{o(AM)} \phi \right] \frac{\vec{I}}{\dot{\phi}}
\]

\[
+ \left[ \vec{w}_{IO} R \sin \psi \cos \phi - \vec{V}_{o(AM)} \psi \right] \frac{\vec{I}}{\dot{\psi}}
\]

In the special case that atmospheric winds are zero, \( \vec{V}_{o(AM)} = 0 \) in the foregoing equations.

5.7 Thrust Forces

Thrust or its equivalent must be generated in many phases of astronomical missions in order to perturb the trajectory of the vehicle. For example, in the mid-course phase of an interplanetary ellipse, engines or solar pressure techniques must be used if adjustments of the path of the vehicle are required. In the particular case of the entry mission, thrust forces must be generated for substantial perturbation of the stable reconnaissance orbit since atmospheric perturbations of
this orbit are negligible.

Optimum use of thrust depends very much on the requirements established for the mission. If the vehicle is in a stable reconnaissance orbit initially, and if maximum payload weight is an over-riding requirement for the mission, then the minimum energy trajectory is desired; i.e., it is desired that entry be effected with minimum expenditure of propellant mass. Minimum propellant mass is expended by generating retro-thrust tangentially at apogee in the stable reconnaissance orbit. As a result of this reduction in velocity at apogee, the perigee next following will be at a lower altitude. By generating just enough thrust for the perigee to drop within the sensible atmosphere, the degenerate reconnaissance orbit is established. Controlled entry is possible from the degenerate reconnaissance orbit, particularly if the vehicle has variable drag capabilities(9).

If payload requirements are not as critical as the requirement for a relatively short time of flight, or if the vehicle has only modest capabilities for changing its aerodynamic characteristics due to structural design limitations, then the minimum energy profile as mentioned above is not necessarily the most efficient method for satisfying mission objectives. Under these conditions, a second retro-thrust application at perigee is a more suitable mission concept than the minimum energy profile of the previous paragraph(33). This scheme for the entry mission induces the direct entry profile. Considerable landing point control is available if the vehicle is capable of generating lift.

If payload capabilities of the entry vehicle are secondary to stringent requirements on landing point accuracy, the initial retro-
thrust application at apogee in the reconnaissance orbit may be increased a sufficient amount to cause the following perigee to drop beneath the surface of the planet. If the propellant mass expended at apogee in the stable reconnaissance orbit is approximately twice that required for the minimum energy profile, then the transfer ellipse will intersect the atmosphere after the vehicle travels roughly one-fourth of the distance around the planet rather than one-half. The flight path angle at atmospheric penetration is considerably greater under these conditions than when lesser amounts of fuel are expended, hence variations in atmospheric density from standard will result in much lower range errors. This mission concept has advantages from the guidance standpoint, but these advantages are paid for in terms of payload capabilities.

The foregoing qualitative discussion is presented for the purpose of emphasizing that the optimum use of the vehicle's engines is a strong function of mission objectives. There is no single optimum engine program applicable to all entry missions.

The type or configuration of the propulsion system is not considered in detail in this thesis. Among propulsion systems suitable for astronautical vehicles are:

(1) Chemical propellants:
   (a) Liquid propellants
      1. Bi-propellants
      2. Mono-propellants
   (b) Solid propellants
(2) Nuclear propulsion systems:
   (a) Fission
(b) Fusion
(c) Radioactive decay isotopes

(3) Electrical propulsion systems:
(a) Arc heating
(b) Ion propulsion
(c) Magneto plasma systems

(4) Solar heating rocket.

(5) Light wave propulsion:
(a) Solar sail
(b) Photon rocket

It is generally planned at this time to use chemical propellants for entry missions. In the atmospheric portions of the flight, air breathing engines may be used in addition to those listed above.

The mass of chemical propellants required to perturb the trajectory a given amount depends on the specific impulse of the propellant. The higher the specific impulse, the better the propellant*. Specific impulse is determined experimentally by measuring the thrust generated and dividing it by the fuel flow rate:

\[ I_{sp} = \frac{\text{Thrust}}{G} = \frac{V_e}{G} \text{ seconds} \]  

(5-54)

where \( I_{sp} \) is specific impulse,
\( G \) is mass flow rate of fuel
\( V_e \) is the effective exhaust velocity of the gases.

* This statement must be qualified because other characteristics must be considered besides the specific impulse. For example, if the density of the propellant is low, then a larger volume of space may be required to achieve a given total impulse than is required by a more dense competitive fuel of lower specific impulse.
The following relation must be satisfied between vehicle mass, time, and propellant mass flow:

\[ \dot{M} + \int (t) = 0 \]  \hspace{1cm} (5-55)

Exhaust velocity is usually computed from measurements of \( I_{sp} \), not vice-versa. Specific impulse may be predicted fairly accurately by applying the laws of thermodynamics to the chemicals involved.

It is shown in reference (34) that values of specific impulse presently attainable or expected in the foreseeable future for chemical rockets range from 200 to 420 seconds.

The total impulse, \( I_t \), is the integral of the thrust over the burning time of the engine:

\[ I_t = \int_0^{t_b} (\text{Thrust}) \, dt = \int_0^{t_b} T \, V_e \, dt \]  \hspace{1cm} (5-56)

where \( t_b \) is propellant burning time. For constant exhaust velocity:

\[ I_t = V_e (M_i - M_f) \]  \hspace{1cm} (5-57)

where \( M_i \) and \( M_f \) denote initial and final mass of the vehicle.

Requirements for astronautical missions are often stated in terms of the change of velocity required:

\[ \delta V = V_e \ln \frac{M_i}{M_f} \]  \hspace{1cm} (5-58)

where \( \delta V \) is the magnitude of the velocity change of the vehicle resulting from the thrust impulse. The velocity impulse must be added vectorially to the vehicular velocity prior to "turning on" thrust in order to determine final velocity of the craft after thrust perturbation.
In this thesis the power plant is assumed to be an ideal rocket engine with a constant equivalent exit velocity. It is assumed that both the magnitude and direction of the applied thrust may be controlled; that is, the engine may be rotated by means of a gimbal system and the engine is capable of being throttled. The magnitude of the applied thrust is \( V_\text{e} \Gamma(t) \), where \( \Gamma(t) \) is a controllable propellant mass flow rate. \( \Gamma(t) \) may vary between zero (coasting flight) and \( \Gamma_{\text{max}} \) (maximum engine output). Miele\(^{(30)}\) represents the propellant mass flow rate in parametric form as shown in Fig. 5.4.

The following three operation regimes are possible:

(1) \( -\infty \leq u \leq u_1 ; \quad \Gamma = 0 ; \quad \frac{d\Gamma}{dt} = 0 ; \) coasting flight.

(2) \( u_2 \leq u \leq +\infty ; \quad \Gamma = \Gamma_{\text{max}} ; \quad \frac{d\Gamma}{dt} = 0 ; \) maximum engine output.

(3) \( u_1 \leq u \leq u_2 ; \quad \frac{d\Gamma}{du} \neq 0 ; \) variable thrust

Referring to the engine gimbal triad in Fig. 5.3, the following may be written for the thrust vector:

\[
\overline{F}_{(\text{th})} = - \frac{V_\text{e}}{M} \overline{\Gamma}_x \overline{\Gamma}_y
\]

(5-59)

Using the angular relations of Table 5.3, this may be written:

\[
\overline{F}_{(\text{th})} = - \frac{V_\text{e}}{M} \left[ \cos A_e \cos A_d \overline{I}_x + \sin A_d \overline{I}_y + \sin A_e \cos A_d \overline{I}_z \right]
\]

(5-60)

Thrust components may be written in component form in the \( \overline{I}_x \), \( \overline{I}_\lambda \), \( \overline{I}_\Lambda \) triad by using the direction cosines given in Table 5.2.
Propellant Mass Flow Rate

\[ \Gamma \]

Parameter \( u \)

\[ u \quad : \text{A parameter having no physical meaning: } -\infty < u < +\infty \]

\[ \Gamma(u) \quad : \text{Variable propellant mass flow rate. A controllable quantity that is a function of } u. \]

Fig. 5.4: Parametric Representation of Propellant Mass Flow. (30)
These components may, in turn, be transformed to the $\mathbf{I}_r, \mathbf{I}_\phi, \mathbf{I}_\psi$ triad by the table of cosines and angular conversions of Table 3.1.

Once the velocity requirements of the entry vehicle are established, the propulsion system must be designed in such a way that thrust may be applied in the proper direction with the proper magnitude and for the proper length of time; i.e., thrust vector control. Thrust vector control is also important for generation of orientation torques for attitude stability; problems connected with thrust vector control for attitude orientation and rotational moment stability are not considered in this thesis.
Chapter 6

THE THEORY OF PLANAR MOTION

A two-dimensional model of the trajectory is generally assumed in most dynamical studies of astronautical missions before the additional complexity of the third dimension is introduced into the problem. In this way, the problem statement is made successively more complete and complex.

An investigation of the three-dimensional characteristics of the trajectory was undertaken first in this thesis, however, before simplifying the problem statement to two dimensions in order to emphasize the limitations on assuming planar motion. In some situations, the trajectory of the vehicle may remain in a plane essentially fixed with respect to inertial coordinates. In other situations, the orientation of the plane of the trajectory is rapidly changing. Section 6.1 discusses conditions under which the trajectory may be accurately described by a two-dimensional model.

6.1 Limitations of the Theory of Planar Motion

Determination of three-dimensional dynamical equations of motion of the entry vehicle in two separate guidance grids was discussed in chapters 3 through 5. These equations were generalized to describe entry of lifting or non-lifting vehicles in banking or wings-level
flight with variable thrust capabilities into the atmosphere of oblate, rotating planets with or without atmospheric winds. The resulting vector equation was:

$$\bar{F}(li) + \bar{F}(dr) + \bar{F}(th) = \bar{A} - \bar{G} \quad (6-1)$$

Even if the left-hand side of equation (6-1) vanishes, which corresponds to powerless flight in a vacuum, it was shown in Chapter 4 that the resulting motion of the vehicle is not confined to a single plane in space because of non-spherical components of gravitational mass attraction. Gravitational perturbations lead to periodic oscillation of the angle of inclination and to secular effects such as regression of the line of nodes and advance of perigee. During the powered portions of the trajectory ($\bar{F}(th) \neq 0$) and during the atmospheric phase of flight ($\bar{F}(li)$ and $\bar{F}(dr) \neq 0$), the left-hand side of equation (6-1) is of great importance in specifying the ultimate path of the vehicle.

Variables which define the orientation of the orbital plane are $\psi$, the inclination of the trajectory plane with respect to the equatorial plane, and $\lambda_{IT}$, the longitude of the line of nodes measured as an angle with respect to fixed inertial coordinates. The plane of the entry vehicle's trajectory remains fixed in inertial space under the following conditions:

1. In the special case of an equatorial trajectory ($\psi = 0$), the requirement for planar motion is simply:

$$\dot{\psi} = 0 \quad (6-2)$$

2. If the trajectory has a finite inclination to the equatorial plane ($\psi \neq 0$), both the longitude of the
line of nodes, \( \lambda_{IT} \), and the inclination of the orbital plane, \( \psi \), must remain constant in order that the trajectory be confined to a single plane in space. Therefore, if \( \psi \neq 0 \), then

\[
\dot{\lambda}_{IT} = 0
\]

(6-3)

\[
\dot{\psi} = 0
\]

must simultaneously be satisfied in order to apply the theory of planar motion.

The line of nodes cannot be visualized in the special case of an equatorial trajectory since it is defined as the line of intersection of the equatorial plane with the trajectory plane. Movement of any line in the equatorial plane does not contribute to a change in orientation of the trajectory plane with respect to inertial coordinates as long as the trajectory plane is coincident with the equatorial plane; therefore, there is no restriction on \( \dot{\lambda}_{IT} \) in case (1) above.

Equation (3-14) showed that:

\[
\dot{\lambda}_{IT} \cos \phi \sin \psi = \dot{\psi} \sin \phi
\]

(3-14)

From equations (4-23) and (6-1), the time rate of change of longitude of the line of nodes is:

\[
\dot{\lambda}_{IT} = \frac{\sin \phi}{V \sin \psi} (F_\psi + G_\psi)
\]

(6-4)

Using equations (3-14) and (6-4), the time rate of change of the inclination of the trajectory plane is as follows:
\[ \dot{\psi} = \frac{\cos \phi}{V_{\phi}} (F_\psi + G_\psi) \] (6-5)

In powerless flight, thrust terms in \( F_\psi \) are equal to zero. Substituting lift*, drag*, and gravitational terms as written in Chapter 5 into equations (6-4) and (6-5) gives, under zero bank conditions (i.e., \( B = 0 \)):

\[ \lambda_{\text{IT}} = -\frac{\sin \phi}{V_{\phi}} \left\{ \frac{12}{7} \frac{G_{\text{ap}}}{R_{\text{reg}}} \sin \phi \cos \psi \right. \]

\[ + \frac{\rho G S}{2M} VR W_{\text{i0}} \cos \phi \left\{ 1 - \frac{L}{D} \left[ \frac{R}{(V_{\phi} - R W_{\text{i0}} \cos \psi)^2} + \frac{R}{(R W_{\text{i0}} \sin \psi \cos \phi)^2} \right] \right\} \] (6-6)

\[ \dot{\psi} = -\frac{\cos \phi \sin \psi}{V_{\phi}} \left\{ \text{Same as curly bracket term in Eq. (6-6)} \right\} \] (6-7)

The theory of planar motion therefore requires the right-hand side of both equations (6-6) and (6-7) to be equal to zero in the case of a finite angle of inclination (\( \psi \neq 0 \)). In the special case of an equatorial trajectory (\( \psi = 0 \)), it is sufficient that the right-hand side of equation (6-7) alone vanish. Because of the \( \sin \psi \) term in the numerator of equation (6-7), it is seen that planar motion results for powerless (no thrust), wings-level flight in the equatorial plane in the special case that wind components normal to the trajectory plane do not exist.

The theory of planar motion is therefore valid under the following

* Atmospheric winds were assumed zero in this substitution, i.e., \( V_{\text{o(AM)}} = 0 \). When wind is zero, the velocity vector of the vehicle with respect to the atmosphere \( V_{\text{(AM)}} \) is the same as the velocity vector of the vehicle with respect to coordinates rotating with the planet, \( \bar{V} \). Thus, under no-wind conditions, \( V = V_{\text{(AM)}} \).
limiting circumstances:

(1) **Equatorial trajectory** (Ψ = 0)
   
   (a) Components of atmospheric winds normal to the trajectory plane are zero (V_{o(AM)} = 0).
   
   (b) The angle of bank of the vehicle is zero (B = 0).
   
   (c) There are no components of thrust generated normal to the trajectory plane (A_d = 0).
   
   (d) Gravity anomalies do not exist.

(2) **Trajectory that is not in the equatorial plane** (Ψ ≠ 0)

   Lift, drag, and thrust must be programmed such that
   
   (a) \( \dot{\psi} = 0 \)
   
   (b) \( \dot{\lambda}_{IT} = 0 \)

In much of the analytical work described in subsequent chapters of this thesis, planar motion is assumed. For purposes of generality, the angle of inclination \( \psi \) is retained in these equations and is assumed constant. The following factors contribute to changing the orientation of the trajectory plane if \( \psi \neq 0 \):

(1) Non-spherical components of the planet's gravitational field.

(2) Gravity anomalies.

(3) Lift and drag components normal to the \( \mathbf{I}_r - \mathbf{I}_\phi \) plane resulting from:

   (a) The rotating atmosphere
   
   (b) Atmospheric winds
   
   (c) Banking the vehicle.

(4) Thrust components normal to the \( \mathbf{I}_r - \mathbf{I}_\phi \) plane.
6.2 The Two-Dimensional Equations of Motion

Fig. 6.1 shows the forces acting at the center of mass of the entry vehicle and defines the positive direction of the various angles, velocities and forces. From the kinematic equations of Chapter 3:

\[ A_r = \ddot{R} - \frac{V_{I\phi}^2}{R} \]  \hspace{1cm} (6-8)

\[ A_\phi = \dot{V}_{I\phi} + \left( \frac{R}{R} \right) \frac{V}{V_{I\phi}} \]  \hspace{1cm} (6-9)

The acceleration components are written in terms of components in the tangential \((I_x)\) direction and the normal \((I_z)\) direction by noting that:

\[ A_x = A_r \sin\gamma + A_\phi \cos\gamma \]  \hspace{1cm} (6-10)

\[ A_z = A_r \cos\gamma - A_\phi \sin\gamma \]  \hspace{1cm} (6-11)

The horizontal component of the velocity of the vehicle with respect to the rotating atmosphere, \(V_\phi\), is related to the horizontal component of velocity with respect to fixed Newtonian coordinates, \(V_{I\phi}\), by:

\[ V_{I\phi} = V_\phi + \dot{W}_{I0} R \cos\psi = V \cos\gamma + \dot{W}_{I0} R \cos\psi \]  \hspace{1cm} (6-12)

The rate of change of radius is related to velocity and flight path angle by:

\[ \dot{R} = V \sin\gamma \]  \hspace{1cm} (6-13)

Equation (6-12) is used to eliminate \(V_{I\phi}\) and equation (6-13) is used to eliminate \(\dot{R}\) and \(\ddot{R}\) from equations (6-8) and (6-9). The resulting acceleration components are written in the tangential and normal directions by using equations (6-10) and (6-11):

\[ A_x = \dot{V} - \dot{W}_{I0}^2 R \cos^2\psi \sin\gamma \]  \hspace{1cm} (6-14)
Definitions:

$X$ = distance flown

$\mathbf{V}$ = velocity of vehicle with respect to coordinates rotating with the planet (same as velocity of vehicle with respect to the planetary atmosphere, $\mathbf{V}_{(AM)}$, for no-wind condition)

$H$ = altitude of vehicle above planet

$\gamma$ = flight path angle with respect to local geocentric horizon - positive as shown.

$\Gamma$ = mass flow rate of propellant

$V_e$ = equivalent exit velocity of rocket propellant (assumed constant)

$M$ = instantaneous mass of vehicle

$D$ = drag

$L$ = lift

$A_e$ = engine gimbal angle of retro-rocket; angle the thrust vector makes with negative $\mathbf{I}_x$ axis.

Fig. 6.1: The Two-Dimensional Trajectory
\[ A_z = V \dot{Y} - \frac{V^2 \cos \gamma}{R} - W_{10}^2 R \cos^2 \psi \cos \gamma - 2 W_{10} V \cos \psi \] \quad (6-15)

The following terms in these kinematic equations are the consequence of stating the problem in a reference space which is rotating with respect to inertial space:

1. \( 2 W_{10} V \cos \psi \) is Coriolis acceleration due to coupling of vehicular velocity with angular velocity of the planet.
2. The \( W_{10}^2 R \cos^2 \psi \) terms in both equations are centrifugal components associated with the rotation of the planet with respect to inertial coordinates.

The quantity \( \frac{V^2 \cos \gamma}{R} \) is centrifugal force resulting from motion of the vehicle with respect to the planet.

Assuming a spherical planet, the gravitational specific force vector reduces to:

\[ \bar{G} = - G_{sp} \bar{I}_r = - G_{sp} \left( \sin \gamma \bar{I}_x + \cos \gamma \bar{I}_z \right) \] \quad (6-16)

The vector sum of external specific forces are readily written from Fig. 6.1:

\[ \bar{F} = - \left( \frac{D}{M} + \frac{V_e \cos A_e}{M} \right) \bar{I}_x + \left( \frac{L}{M} - \frac{V_e \sin A_e}{M} \right) \bar{I}_z \] \quad (6-17)

Substituting equations (6-14) through (6-17) into equation (6-1) and rearranging gives:

\[ \dot{V} + (G_{sp} - W_{10}^2 R \cos^2 \psi) \sin \gamma + \frac{D}{M} + \frac{V_e \cos A_e}{M} = 0 \] \quad (6-18)

\[ V \dot{\gamma} - \frac{V^2 \cos \gamma}{R} + (G_{sp} - W_{10}^2 R \cos^2 \psi) \cos \gamma - \frac{L}{M} \]
\[ + \frac{V_e}{M} \sin A_e - 2 W_{10} V \cos \psi = 0 \] \quad (6-19)
The following kinematic equations are required in addition to the two dynamical equations in order to complete the theory of planar motion.

From equation (5-55):
\[ \dot{M} + \Gamma = 0 \] (6-20)

From equation (6-13):
\[ R - V \sin \gamma = 0 \] (6-21)

In establishing guidance requirements for the entry system, the range capability under given flight conditions is a fundamental quantity that must be predicted. Defining \( X \) on Fig. 6.1 as distance flown, range rate may be readily written as follows:
\[ \dot{X} = \frac{R(m) \dot{\theta}}{R} V \cos \gamma = 0 \] (6-22)

6.3 Dimensionless Planar Equations of Motion

Numerical solutions to guidance parameters and constraints may be applied to a broad class of entry missions if the solutions are presented in dimensionless form. Subsequent chapters of this thesis are devoted exclusively to examination of the entry problem in dimensionless form.

The convention generally followed in this thesis is to represent dimensional quantities by capital letters and the same quantity in dimensionless form by lower case letters. It will be noted that there are certain exceptions to this rule because, at times, conflicts in notation arise; however, in most cases this convention is obeyed.

The following parameters were chosen for purposes of converting the various dimensional quantities into non-dimensional form:
(1) The reference length chosen was the mean radius of the planet, \( R_{(m)0} \).

(2) The reference velocity chosen was the satellite velocity at the surface of the planet:
\[
V_s = \sqrt{G_{(m)0} \ R_{(m)0}}
\]  
(6-23)

(3) The reference acceleration chosen was the mean gravitational acceleration at the surface of the planet, \( G_{(m)0} \):
\[
G_{(m)0} = \frac{\gamma \ M_{(m)0}}{R_{(m)0}^2}
\]  
(6-24)

(4) The reference mass chosen was the initial mass of the entry vehicle prior to the expenditure of rocket fuel during the particular segment of the trajectory being examined.

(5) Reference time was chosen from items (1) and (2) above as follows:
\[
\tau = \text{dimensionless time}
\]
\[
\tau = \frac{V_s}{R_{(m)0} \ t} = \sqrt{\frac{G_{(m)0}}{R_{(m)0}^2} \ t}
\]  
(6-25)

With these reference quantities, the following dimensionless terms were defined:

(1) \( v = \text{dimensionless velocity of the vehicle with respect to coordinates rotating with the planet.} \)
\[
v = \frac{V}{V_s}
\]  
(6-26)

(2) \( v_I = \text{dimensionless velocity of the vehicle with respect to planet centered inertial coordinates.} \)
\[ \frac{d(\_)}{d\tau} \equiv \text{dimensionless time rate of change of the quantity in parenthesis.} \]

\[ \frac{d(\_)}{d\tau} = \frac{R(m)_0}{\sqrt{G(m)_0}} \frac{d(\_)}{dt} = \frac{R(m)_0}{G(m)_0} (\ast) \] (6-28)

(4) \( m \equiv \text{dimensionless mass of the vehicle.} \)

\[ m = \frac{M}{M_i} \] (6-29)

(5) \( r \equiv \text{dimensionless radius from the center of the planet to the vehicle.} \)

\[ r = \frac{R}{R(m)_0} \] (6-30)

(6) \( h \equiv \text{dimensionless altitude} \)

\[ h = \frac{H}{R(m)_0} \] (6-31)

(7) \( X_N \equiv \text{dimensionless distance flown.} \)

\[ X_N = \frac{X}{R(m)_0} \] (6-32)

(8) \( \Gamma_N \equiv \text{dimensionless propellant mass flow rate.} \)

\[ \Gamma_N = \frac{\Gamma}{M_i} \frac{R(m)_0}{\sqrt{G(m)_0}} \] (6-33)

(9) \( v_e \equiv \text{dimensionless equivalent exit velocity of exhaust gases.} \)

\[ v_e = \frac{V_e}{V_S} \] (6-34)

(10) \( \Omega \equiv \text{dimensionless angular velocity of the planet about} \)

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its polar axis.

\[ \alpha = w_0 \frac{\sqrt{R(m) / G(m) \omega}}{\sqrt{G(m) \omega}} \]  \hspace{1cm} (6-35)

(11) \( n_D \) = drag load factor

\[ n_D = \frac{F(dr)}{G(m) \omega} = \frac{D}{M G(m) \omega} \]  \hspace{1cm} (6-36)

(12) \( n_L \) = lift load factor.

\[ n_L = \frac{F(l)}{G(m) \omega} = \frac{L}{M G(m) \omega} \]  \hspace{1cm} (6-37)

Gravitational mass attraction appears in the following form in the dimensionless equations of motion.

\[ \text{Dimensionless} \ G_{sp} = \frac{G_{sp}}{G(m) \omega} = \left( \frac{R(m) \omega}{R} \right)^2 = \frac{1}{r^2} \]  \hspace{1cm} (6-38)

Dimensionless radius and altitude are related as follows:

\[ r = 1 + h \]  \hspace{1cm} (6-39)

With the foregoing dimensionless quantities, the equations of motion given in section 6.2 are written as follows:

\[ \nu' + \left( \frac{1}{r^2} - r \Omega^2 \cos^2 \psi \right) \sin \gamma + n_D + \frac{N}{m} v_e \cos \alpha_e = 0 \]  \hspace{1cm} (6-40)

\[ \nu \gamma' - \frac{v^2}{r} \cos \gamma + \left( \frac{1}{r^2} - r \Omega^2 \cos^2 \psi \right) \nu \cos \gamma - n_L \]  
\[ + \frac{N}{m} v_e \sin \alpha_e - 2 \nu \Omega \cos \psi = 0 \]  \hspace{1cm} (6-41)

\[ m' + \Gamma_N = 0 \]  \hspace{1cm} (6-42)
\[ r' - v \sin \gamma = h' - v \sin \hat{\gamma} = 0 \]  
\[ X'_{N} - (v/r) \cos \gamma = 0 \]

### 6.4 An Alternate Set of Planar Equations:

Equations (6-40) and (6-41) describe the dynamics of entry in the \( \overline{I}_x \) and \( \overline{I}_z \) directions. In some of the investigations described in subsequent chapters of this thesis, it was found more convenient to work with the tangential and radial components of velocity instead of velocity and flight path angle as primary dependent variables. For this statement of the entry problem the following quantities were defined:

\[ v_r = v \sin \gamma \]  
\[ v_{\phi} = v \cos \gamma \]

Differentiating these equations and substituting for \( v' \) from equation (6-40) and \( \gamma' \) from equation (6-41) gives:

\[ v_{r}' - \frac{\nu_{\phi}^2}{r} + \eta \sin \gamma \left( 1 - \frac{L}{D} \cot \gamma \right) + \left( \frac{1}{r^2} - r \frac{\nu_{r}^2}{2} \cos^2 \Psi \right) \]
\[ + \frac{\nu_{e}}{m} \int_{N} \sin(\gamma + A_{e}) - 2 \nu_{\phi} \Omega \cos \Psi = 0 \]  
\[ v_{\phi}' + \frac{v_{r} v_{\phi}}{r} + \eta \cos \gamma \left( 1 + \frac{L}{D} \tan \gamma \right) + \frac{\nu_{e}}{m} \int_{N} \cos(\gamma + A_{e}) \]
\[ + 2 \nu_{r} \Omega \cos \Psi = 0 \]

Equations (6-47) and (6-48) are the equations of motion in the \( \overline{I}_r \) and \( \overline{I}_{\phi} \) directions respectively. The following additional equations complete this planar set of equations:

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Dimensionless total energy per unit mass is defined as follows:

\[ E_{(\text{tot})} = \frac{C_{(\text{tot})}^2}{M V_s^2} \]  

(6-53)

Equation (6-52) therefore reduces to:

\[ E_{(\text{tot})} = \frac{v_I^2}{2} - \frac{1}{r} \]  

(6-54)

The time rate of change of total energy is the dot product of the non-conservative force vector and the velocity vector of the vehicle. In dimensionless form, this is equivalent to:

\[ E'_{(\text{tot})} = \overline{F} \cdot \overline{v_I} \]  

(6-55)

where \( \overline{F} \) is the vector sum of all external specific forces in mean surface g's of the planet.

\[ f = \overline{F} / G(m)_0 = \frac{1}{G(m)_0} \left( \overline{F}(dr) + \overline{F}(ll) + \overline{F}(th) \right) \]  

(6-56)

The dot product of force and velocity in equation (6-55) is the rate of working of the force, or the power. The rate at which the vehicle is transferring energy, given by equation (6-55), may be measured by the instrumentation system as follows:

\[ E'_{(\text{tot})} = v_{IR} \text{(Accelerometer output in } \overline{I}_R \text{ direction)} \]

\[ + v_{I\phi} \text{(Accelerometer output in } l_\phi \text{ direction)} \]  

(6-57)

or

\[ E'_{(\text{tot})} = v_{IX} \text{(Accelerometer output in } \overline{I}_X \text{ direction)} \]
Equation (6-55) may be written as follows:

\[
\frac{d}{d\gamma} \left( \frac{v_I^2}{2} - \frac{1}{r} \right) = -\left[ (n_D + \frac{v_e}{m} \sum A_e) (v + r \cos \gamma \sum \cos \psi) + (n_L - \frac{v_e}{m} \sum \sin A_e) (r \sin \gamma \sum \cos \psi) \right]
\]

Equation (6-55) can be written in integral form as follows:

\[
E(tot) - E(tot)_1 = \int_0^{\gamma} ( \vec{f} \cdot \vec{v}_I ) \, d\gamma
\]

Equation (6-60) is equivalent to:

\[
\left( \frac{v_I^2}{2} - \frac{1}{r} \right)_{\gamma=0} = \int_0^{\gamma} d\gamma \left[ (n_D + \frac{v_e}{m} \sum A_e) (v + r \cos \gamma \sum \cos \psi) + (n_L - \frac{v_e}{m} \sum \sin A_e) (r \sin \gamma \sum \cos \psi) \right]
\]

Angular momentum was defined in equation (4-1) as:

\[
P = R \, v_I \phi
\]

The dimensionless angular momentum is defined as follows:

\[
P = \frac{P}{R(m)_0 V_S}
\]

Therefore, dimensionless angular momentum may be written:

\[
p = r \left( v_0 + r \Omega \cos \psi \right)
\]
The time rate of change of angular momentum is equal to the external torque applied, therefore:

\[
\vec{p}' = \vec{r} \times \vec{f}
\]  
(6-64)

The rate at which the vehicle is transferring angular momentum because of applied external forces may be measured by the instrumentation system as follows:

\[
p' = r \text{(Accelerometer output in } \vec{1}_\phi \text{ direction)}
\]  
(6-65)

Equation (6-65) is equivalent to:

\[
(r \vec{v}_o)' = -r \left[ n_c \cos \gamma \left( 1 + \frac{L}{D} \tan \gamma \right) + \frac{I_n}{m} \nu_c \cos \left( \gamma + A_c \right) \right]
\]  
(6-66)

Equation (6-65) can be written in integral form as:

\[
p - p_i = \int_0^\gamma r f \phi \, d\gamma
\]  
(6-67)

Equation (6-67) is equivalent to:

\[
r(n_0 + \nu_c \cos \Psi) \bigg|_{\gamma=0}^{\gamma} = -\int_0^\gamma \, d\gamma \left\{ r \left[ n_c \cos \gamma \left( 1 + \frac{L}{D} \tan \gamma \right) + \frac{I_n}{m} \nu_c \cos \left( \gamma + A_c \right) \right] \right\}
\]  
(6-68)

The dynamical equations of motion were written as dimensionless equations in the \( \vec{I}_x \) and \( \vec{I}_z \) directions as equations (6-40) and (6-41) and in the \( \vec{I}_r \) and \( \vec{I}_\phi \) directions as equations (6-47) and (6-48). These sets are equivalent to the energy and angular momentum equations derived in differential form as equations (6-59) and (6-66) and in integral form as equations (6-61) and (6-68).
6.6 Instrumentation of Planar Entry

Section 6.5 showed that the energy and angular momentum levels of the entry vehicle can be determined from velocity, altitude, and specific force measurements as follows:

\[ E_{\text{tot}} = E_{\text{tot}}^{\text{(i)}} + \int_0^r d\gamma (f_r v_{Ir} + f_\phi v_{I\phi}) \]  

\[ p = p_1 + \int_0^r d\gamma (r f_\phi) \]  

where: \( r = 1 + h \): radius to the vehicle, non-dimensionalized with respect to mean radius of the planet.

\( f_r, f_\phi \) are specific force measurements in the geocentric vertical and horizontal directions respectively (non-dimensionalized with respect to mean surface g's of the planet).

\( v_{Ir}, v_{I\phi} \) are vertical and horizontal components of velocity with respect to planet-centered inertial coordinates (non-dimensionalized with respect to circular satellite velocity at the surface of the planet).

If \( r \) is measured by some external means, such as with radar, and if initial values of energy and angular momentum are known, then it is possible to design a system to compute position and velocity based on equations (6-69) and (6-70) and the definitions of energy and angular momentum:

\[ p = r v_{I\phi} \]  

\[ E_{\text{tot}} = \frac{v_I^2}{2} - \frac{1}{r} \]  

The operation of this system depends, of course, on the ability...
of the system to measure specific force components in the vertical and horizontal directions; this is equivalent to saying that some means must be available for indicating the vertical. Angular momentum may be continuously computed in accordance with equation (6-70) from externally derived altitude data and from measurements of the horizontal component of specific force. With the computed value of angular momentum and the observed altitude, the horizontal component of the velocity is determined directly by equation (6-71). Equations (6-69) and (6-72) may be written:

\[ \frac{P^2}{2r^2} + \frac{V_{lr}^2}{2} - \frac{1}{r} = E_{(tot)} + \int_0^r \left( f_r V_{lr} + f_\phi \frac{P}{r} \right) \]  

(6-73)

Assuming that the system is capable of measuring \( f_r \) and \( f_\phi \), and with the computed value of \( \rho \) and the measured value of \( r \), the radial component of velocity is determined from (6-73). The system has therefore determined all quantities important for navigating the vehicle: \( V_{lr} \), \( V_{l\phi} \), and \( r \). From these quantities the flight path angle \( \gamma \), distance flown \( X \), position, etc., may be computed. The instantaneous energy level is useful in predicting the number of orbits (range) capable of being traveled by the vehicle. Range prediction from energy levels is discussed further in Chapter 10.
7.1 Energy Transfer During Entry

A vehicle approaching a planetary atmosphere possesses a large amount of energy; for example, the kinetic energy of a vehicle in a circular orbit around the Earth is of the order of 13,500 BTU per pound. All of the mechanical energy possessed by the vehicle need not be converted to heat within the body itself during its passage through the atmosphere; there is sufficient energy to vaporize the entire vehicle unless a large percentage of this energy is transferred to the atmosphere. The energy must be transferred in a manner that will not prove disastrous either to the vehicle or its human occupants. The original energy of the vehicle is transformed, through the mechanism of gas-dynamic drag, into thermal energy of the gas surrounding the craft; the fraction of the original energy that ultimately appears as thermal energy of the vehicle itself depends on the characteristics of the gaseous flow.

The build-up of appreciable heating and deceleration loads are two of the most important effects encountered during entry of a vehicle into a planetary atmosphere. Both the deceleration loads and the heating rates are most severe when there is a combination of high atmospheric density and high vehicular velocity; it is therefore
necessary that the guidance system operate in such a way as to prohibit high velocities from persisting down to low altitudes. This undesirable condition most likely will occur if the flight path angle $\gamma$ is large and if the approach velocity is very great.

The velocity of the vehicle at atmospheric penetration depends on the mission concept. If atmospheric penetration is permitted at the termination of an interplanetary transfer ellipse, then the velocity at penetration is of the order of escape velocity. If entry is initiated from a reconnaissance orbit around the planet, then the velocity at atmospheric penetration is near circular orbital velocity. The velocity at initial atmospheric penetration can be changed a significant amount only by large scale thrust perturbations; this leads to severe penalties in terms of payload capabilities. The flight path angle, on the other hand, may generally be selected at a lesser cost in terms of weight, provided thrust perturbations are applied at the proper time and in the proper direction. For example, a small thrust perturbation during the interplanetary transfer ellipse (when the vehicle is far from its destination planet) can change the flight path angle at initial atmospheric penetration a very large amount. It must be emphasized that a small error in thrust perturbation at this point, however, can lead to gross errors in penetration angle; thus guidance accuracy requirements go up with an increase in the sensitivity of the system to control actions.

A shallow flight path angle (tangential approach) tends to limit the region of high velocities to high altitudes. Consequently, choosing a shallow trajectory tends to minimize heating rates and deceleration loads encountered. Inaccurate knowledge of the density characteristics
of the planetary atmosphere, however, may lead to considerably greater range errors for entry at shallow flight path angles than for steep flight path angles. Therefore, it may be seen that the flight path angle at atmospheric penetration must be selected as a compromise between two conflicting mission requirements:

1. Heating rates and deceleration loads are appreciably reduced at shallow flight path angles.
2. Range accuracy is generally greater for steep flight path angles.

The coupling of payload requirements, sensitivity to control actions, vehicular heating rates, acceleration loads, and guidance accuracy in determination of the best entry profile are not the only conflicting considerations that must be resolved by the design engineer of the entry mission system. The mechanism of energy transfer to the planetary atmosphere is also a function of the configuration of the vehicle. High drag (blunt) vehicles cause deceleration to take place at higher altitudes than is the case for low drag shapes. Lifting vehicles allow more gradual descents; hence the high velocity portion of the flight can be restricted to higher altitudes with lift in order to reduce deceleration loads and heating rates. Table 7.1, taken from reference (7), shows the wide range of deceleration loads which may be encountered during entry. The deceleration loads are a sensitive function of the initial conditions of the problem and of the gas-dynamic characteristics of the vehicle.

In subsequent sections of this chapter, the manifestation of energy transfer from the vehicle to the planetary atmosphere in the form of heating and deceleration are discussed. The general approach taken in
Table 7.1: Maximum Decelerations Encountered During Atmospheric Entry (Values given in Earth g's)

<table>
<thead>
<tr>
<th>Direct Entry at Escape Velocity (Zero-Lift Vehicle)</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1 = -5^\circ$</td>
<td>28.6</td>
<td>28.3</td>
<td>1.6</td>
</tr>
<tr>
<td>$\gamma_1 = -20^\circ$</td>
<td>112</td>
<td>111</td>
<td>6.3</td>
</tr>
<tr>
<td>$\gamma_1 = -90^\circ$</td>
<td>326</td>
<td>324</td>
<td>18.3</td>
</tr>
</tbody>
</table>

Direct Entry at Orbital Velocity:

A. Zero-Lift Vehicle

<table>
<thead>
<tr>
<th>$\gamma_1$</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.9</td>
<td>9.5</td>
<td>1.4</td>
</tr>
<tr>
<td>-5$^\circ$</td>
<td>14.3</td>
<td>14.2</td>
<td>0.8</td>
</tr>
<tr>
<td>-20$^\circ$</td>
<td>56</td>
<td>55.5</td>
<td>3.2</td>
</tr>
<tr>
<td>-90$^\circ$</td>
<td>163</td>
<td>162</td>
<td>9.2</td>
</tr>
</tbody>
</table>

B. Lifting Vehicles at $\gamma_1 = 0^\circ$

<table>
<thead>
<tr>
<th>L/D</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.88</td>
<td>1.0</td>
<td>0.38</td>
</tr>
<tr>
<td>2</td>
<td>0.44</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>0.18</td>
<td>0.2</td>
<td>0.07</td>
</tr>
</tbody>
</table>
this thesis is that the choice of a reference or nominal trajectory should not necessarily be that trajectory for which heating and decelerations are minimized. Instead, these materializations of energy transfer between vehicle and atmosphere should be viewed as establishing constraints on allowable profiles for any given mission. Trade-offs may be made among other mission factors in arriving at an optimum entry profile for any particular entry system as long as these constraints are not violated. These are one-sided constraints on the entry guidance problem since maximum allowable values alone are of concern to the designer. Maximum allowable accelerations for the human crew, of course, are a function of the direction in which they are applied.

In this chapter, heating and acceleration constraints are presented in analytical and in graphical form as a series of velocity-altitude plots for entry into the atmosphere of Venus, Earth, and Mars. In order to determine quantitative values for graphical representation of these one-sided constraints, it is necessary that the density characteristics of the atmosphere and the lift and drag characteristics of the vehicle be specified. The exponential model of the planetary atmosphere* and the lift-drag characteristics of the three classes of vehicles summarized in Table 5.1 were chosen as sufficiently general to be representative of many possible entry situations.

In addition to trajectory constraints resulting from energy transfer considerations, viz. heating and acceleration loads encountered during the atmospheric phase of flight, radiation bands surrounding the planet may offer enough of a hazard to manned entry to influence the selection

* Atmospheric models of the various planets are discussed in Appendix E.
of a suitable mission profile. Possible constraints on the entry trajectory as a result of radiation belts around the planet are discussed briefly in section 7.2.

7.2 Radiation Constraints on Entry

The discovery in December, 1958, of belts of energetic particles trapped in the Earth's magnetic field has suggested re-evaluation of certain entry mission concepts. These radiation belts, named after Dr. James A. Van Allen who first disclosed their discovery, were detected by the Army's Pioneer III lunar probe. Pioneer III showed two distinct belts; the inner belt is centered approximately 2400 miles and the outer belt centered at approximately 10,000 miles above the surface of the Earth. Maximum radiation rate was estimated to be of the order of 10 Roentgens per hour. A comparatively free area exists between the two belts; for example, the measured radiation rate at 6000 miles was approximately 0.3 Roentgens per hour. Pioneer IV (March 1959) reported particle densities in the outer belt to be many times greater than those measured by Pioneer III. It is believed that the particles in these belts are of solar origin.

It is reasonable to assume that Van Allen radiation belts may exist around any planet possessing a magnetic field. Drake (15) suggests that radio waves from Jupiter originate from the Jovian version of the Van Allen belts. He estimates radiation intensities to be 100 times as strong in the Jovian belts as in the Earth belts. Experimental verification of Venusian, Martian, and other planetary belts should be one of the important functions of unmanned planetary probes, the first of which has not yet been launched.
The effects of radiation on human beings depend not only on the total amount absorbed, but also on the area of the body exposed and on the rate of absorption. The biological effect of a given dose of radiation decreases as the rate of exposure decreases; for example, 600 Roentgens would almost certainly be fatal if absorbed by the whole body in one day, but would probably have no noticeable consequences if spread over 20 years. If the radiation rate is small, the damaged tissues have a chance to recover.

Because large doses have been accepted by human beings only as a result of accidents of one kind or another, it is not possible to state definitely that a particular amount of radiation will have certain consequences. The following table\(^{35}\) gives an approximate indication of the early effects on humans of occasional large doses of radiation absorbed in a short period of time over the whole body. Somewhat larger doses may be accepted if the exposure is protracted over several days or weeks.

<table>
<thead>
<tr>
<th>Dose</th>
<th>Probable Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-25 Roentgens</td>
<td>No obvious injury</td>
</tr>
<tr>
<td>25-50 R.</td>
<td>Possible blood changes. No severe injury.</td>
</tr>
<tr>
<td>50-100 R.</td>
<td>Blood-cell changes, some injury, no disability.</td>
</tr>
<tr>
<td>100-200 R.</td>
<td>Injury, possible disability</td>
</tr>
<tr>
<td>200-400 R.</td>
<td>Injury and disability certain, death possible.</td>
</tr>
<tr>
<td>400 R.</td>
<td>Fatal to 50%.</td>
</tr>
<tr>
<td>600 R. or more</td>
<td>Fatal.</td>
</tr>
</tbody>
</table>
Quantitative constraints on the entry trajectory resulting from planetary radiation belts cannot be accurately determined until more data on these belts is accumulated. The most severe consequence of these belts should be restrictions on the allowable altitudes of the planetary reconnaissance orbits. The path of the reconnaissance orbit must be such that the crew does not accumulate total radiation dosages beyond prescribed human radiation maximums. The allowable maximum, of course, must be well below lethal levels; the successful completion of the mission is jeopardized if the pilot or crew incur mild radiation sickness to the extent of experiencing feelings of discomfort or uneasiness (malaise), depression, or bodily fatigue.

For reconnaissance orbits with perigees below the inner radiation belt, the eccentricity of the orbit must be low enough that radiations incurred by penetration of this belt in the vicinity of apogee are mild. Since entry is effected in the degenerate orbital profile by atmospheric perturbations in the vicinity of perigee (which is well below the inner radiation belt), the energy possessed by the vehicle in excess of the energy level corresponding to circular orbital energy at perigee altitude must be constrained. Chapter 10 discusses methods for predicting apogee altitudes from prescribed perigee flight conditions for the degenerate orbit.

The major advantage of the degenerate orbital profile as a concept for the entry phase of interplanetary probes is the high payload weights possible (since a thrust generating mechanism is not required). The presence of planetary radiation belts may make it mandatory, however, that severe restrictions be established on allowable eccentricities of the first few braking passes. It is for this reason that the direct
first-pass entry profile is worthy of serious consideration, even though it presents a more severe guidance problem in combating heating and acceleration hazards.

Data accumulated on the radiation belts around Earth show the belts to resemble distorted donuts with comparatively free areas along the polar axis. Trajectories from outer space which are oriented in polar directions will encounter considerably less radiation than trajectories oriented in equatorial directions. Radiation considerations may be important in prescribing the most favorable direction for approaching the planet.

The radiation belts may be viewed in the context of this thesis as presenting constraints on the initial conditions of the entry problem. The direction of approach to the planet prescribes the initial orientation of the trajectory plane. If a reconnaissance orbit is part of the mission concept, the radiation belts constrain the initial values of the parameters of the reconnaissance ellipse to certain maximums. Once the initial conditions of the entry phase of the mission are prescribed, the primary trajectory constraints which must be considered are heating and acceleration loads resulting from the passage of the vehicle through the planetary atmosphere. These are considered in subsequent sections of this chapter.

7.3 Heating of the Entry Vehicle

The reduction of the total energy of a vehicle during its passage through a planetary atmosphere is accompanied by an increase in thermal energy of the surrounding gas. Most of the critical problems associated with this heat transfer occur in the continuum flow regime. Some
of this thermal energy is transferred to the surface of the vehicle; the fraction of thermal energy which reaches the vehicle's surface is of primary concern to the designer. This fraction depends on the mechanism of heat transfer between the hot gas and the vehicle surface and on vehicle shape, velocity, and altitude. At extremely high altitudes, the heat energy is developed directly at the vehicle's surface. At lower altitudes, thermal energy appears in the gas between the shock wave and the vehicle. Heat is transferred from this hot gas to the body by conduction and convection through the viscous boundary layer. Radiation from the hot gas may also contribute appreciably to the surface heating.

The net rate of heat input to an element of the surface of the vehicle is:

1. the rate of heat entering by convection ($Q_c$)
2. plus the rate of heat entering by gaseous radiation ($Q_r$)
3. minus the rate of heat re-radiated from the surface ($Q_{rad}$)
4. minus the rate of heat conducted to adjacent elements of the surface
5. minus the rate of heat conducted to the interior of the vehicle
6. minus the rate of heat radiated to the interior of the vehicle.

The net rate of heat input to this surface element causes its temperature to change. The temperature change may be sufficient for the material either to melt or to sublimate. The heating of the vehicle determines the type of surface protection required. Methods for protecting a payload from high external heating include:

1. Thickening of skin: to absorb heat in a greater mass of material (in the case of transient heating) and to
compensate for decreased material strength at elevated temperatures (in the case of equilibrium conditions).

(2) **Insulation of the outer surface**: to reduce transmission of external heat into the interior of the vehicle.

(3) **Cooling of the inner skin surface**: absorbing transmitted heat by heating a coolant fluid.

(4) **Transpiration cooling**: pumping of gas or vapor through a porous skin to carry heat away from and to insulate the vehicle.

(5) **Ablation cooling**: carrying away heat and insulating by vaporizing the surface material of the vehicle.

(6) **Combination of methods (1) through (5).**

The basic mechanism of gas-dynamic heating is influenced by the geometry of the vehicle. Fig. 7.1 shows the flow pattern around a blunt vehicle and around a slender vehicle.

(1) **Blunt vehicle**: The shock wave is detached and ahead of the vehicle is nearly normal to the velocity vector. The gas between the body and the shock wave is hot and moves slowly with respect to the vehicle. The velocities of the gas in the boundary layer are small enough that little viscous dissipation occurs. The temperature in the boundary layer rises smoothly from a relatively low value at the surface to the high value characteristic of the gas between the vehicle and shock wave. Heat is convected from this gas to the surface of the vehicle.

(2) **Slender vehicle**: The shock wave is oblique to the stream,
Fig. 7.1: Effect of Body Shape on the Mechanism of Gas-dynamic Heating\(^{(37)}\).
hence the gas between the shock wave and vehicle is moving fast with respect to the vehicle and is relatively cool. Since the boundary layer has a steep velocity shear, the viscosity of the gas causes mechanical energy to be transformed to thermal energy within this layer. The maximum temperature occurs within the boundary layer and is convected both toward and away from the vehicle's surface.

It is apparent from these examples that the rate of heat transfer by convection is different for vehicles of different configurations.

The convective heating rate at a stagnation point \( (\dot{Q}_c)_s \), with radius of curvature \( R \), in hypersonic flow can be expressed as:

\[
(\dot{Q}_c)_s = \frac{C_{(atm)}}{\sqrt{R}} \left( \frac{\rho}{\rho_{(5L)}} \right)^n \left( \frac{V_{(AM)}}{G R} \right)^m Y \frac{BTU}{ft^2 \cdot sec}
\]

The dimensionless constants \( m \) and \( n \) depend on the type of boundary layer flow and the dimensional constant \( C_{(atm)} \) (dimensions: \( \text{BTU-ft}^{-3/2} \text{ sec}^{-1} \)) is a function of the planetary atmosphere. \( Y \) is unity for a sphere and is a function of angle of attack, angle of sweep, and Mach number if the vehicle has lifting surfaces.

For laminar flow*: \( n = \frac{3}{2} \)

If the viscosity of the gas is proportional to the square root of the temperature: \( m = 3 \). Other values of \( m \) determined theoretically and experimentally are: 3.1 (ref. 38); 3.15 (ref. 39); 3.22 (references 40 and 41).

* During entry, the Reynolds numbers in the regions important for heat transfer are almost always low enough to insure laminar flow.
C(atm) for the Earth's atmosphere is variously given as follows:

16,800 BTU/ft.\(^{3/2}\) - sec. (ref. 38)
17,000 (ref. 15)
17,600 (refs. 37 and 39)
19,800 (refs. 40 and 41).

Radiation heating rates from the hot gas to the surface of the vehicle may be expressed as follows:

\[ \dot{Q}_r = \left(\frac{K_g}{l_g}\right) K_{bolz} T_t l_g \frac{BTU}{ft^2 \text{sec.}} \]  

(7-2)

where \( \frac{K_g}{l_g} \) = radiation emissivity of the gas per unit path length
\( K_{bolz} \) = Stefan-Boltzman constant \((4.81 \times 10^{-13} \frac{\text{BTU}}{\text{sec. ft.}^2 \text{°R}^4})\)
\( T_t \) = stagnation temperature of the gas (°R).
\( l_g \) = path length of gas.

Blunt bodies experience more heating by gaseous radiation than do slender bodies. Rubensin\(^{(37)}\) shows that the magnitudes of \( K_g \) and \( T_t \) in equation (7-2) must be determined for the real gas rather than using ideal gas relations*.

The total heat input to a surface element of the vehicle is the sum of equations (7-1) and (7-2). In the analysis performed in this thesis, thermal contributions from gaseous radiation to the surface of the vehicle are ignored. Gaseous radiation contributes a small percentage of the total heat for small vehicles, hence the neglect of this

* Ideal gas stagnation temperatures for entry into the Earth's atmosphere at satellite speeds are approximately four times as great as real gas temperatures. Emissivity, \( K_g \), results directly from real gas effects.\(^{(37)}\)
quantity is not serious in this case. For example, Rubensin shows that gaseous radiation contributes less than 1% of the total heat input for the following specific vehicles:

1. Non-lifting hemisphere with diameter of one foot, weight 82 lbs., and $W/C_pS = 26$ lbs./ft.$^2$

2. Non-lifting hemisphere with diameter of one foot, weight 45.5 lbs., and $W/C_pS = 14.5$ lbs./ft.$^2$

3. Lifting vehicle with triangular wings of blunt leading edges and thickness equal to 10% of total length. Other characteristics of vehicle:
   (a) Angle of sweep = $60^\circ$
   (b) $C_L = 0.59$
   (c) $L/D = 0.7$
   (d) $\alpha = 42^\circ$
   (e) $W = 57$ lbs.
   (f) Length of wing: 3.6 ft.
   (g) Area of wing: 7.4 ft.$^2$

Vehicles five times as large as the foregoing, with the same M/S, are more representative of small manned entry vehicles. Gaseous radiation for these larger vehicles contributes approximately 10% of the total heat input$^{(37)}$. For winged vehicles at very large angles of attack (near $90^\circ$), gaseous radiation should contribute more than the percentage estimated above. Neglect of thermal inputs due to gaseous radiation should lead to inaccuracies of less than 20% in the computations described subsequently in this thesis. Simplified methods for computing this radiation are not available at present. Determination of accurate analytical methods for including gaseous radiation warrants further study.
Four aspects of gas-dynamic heating of the entry vehicle may be important to the design engineer:

1. **The total heat input:**
   - Total coolant weight* required depends on the total heat input.

2. **The maximum time rate of average heat input per unit area:**
   - Peak average rates of fluid coolants and the required structural strength (where thermal stresses predominate) depend on the maximum time rate of average heat input per unit area.

3. **The maximum time rate of local heat input per unit area.**

4. **The maximum local skin temperature.**
   
   Local heating is a serious problem with hypervelocity vehicles. "Hot spots" occur at the stagnation region of the nose and leading edges of planar surfaces used for developing lift and obtaining stable and controlled flight.

For entry missions of short duration, such as non-lifting entry at a steep flight path angle, the maximum heating rates are high but the total heat input is low. Re-radiation from the surface of the vehicle plays a minor role in cooling the craft. Vehicles designed for this type of entry generally must rely on a structure that:

1. Absorbs heat by raising the temperature of a mass of material**;

---

* The coolant may be solid (e.g., the structure), liquid, gas, or combination thereof.

** Beryllium is a favorable structural material in this case.
Absorbs heat with a heat shield that decomposes and changes phase in a self-controlled process (ablation)*.

For entry missions of long duration, the total heat input is large but the heating rates are generally small enough that re-radiation may be adequate for cooling the vehicle. Internal cooling systems may be desired for special areas of large vehicles. Strong structural materials can generally withstand the maximum temperatures encountered; weak refractory materials may be used at leading edges and tips of planar surfaces where hot spots develop and where the capability of withstanding heavy structural loads is not required.

Re-radiation from the surface of the vehicle may be written as follows:

\[ Q_{\text{rad}} = K_{\text{rad}} K_{\text{boltz}} T_{\text{surf}}^4 \frac{\text{BTU}}{\text{ft}^2 \text{sec}} \]  (7-3)

where

- \( Q_{\text{rad}} \) = rate of heat re-radiated from the surface.
- \( K_{\text{rad}} \) = surface radiation emissivity.
- \( K_{\text{boltz}} \) = Stefan-Boltzman constant.
- \( T_{\text{surf}} \) = temperature of the surface in degrees Rankine.

\( K_{\text{rad}} \) is of the order of 0.8 at high surface temperatures for representative metals of vehicles with thin shell structure.

For missions of short duration, which generally involve heat-sink type vehicles, the total heat input must not exceed design maximums. The total heat absorbed during entry is the sum of convective heating

* Teflon (polytetrafluoroethylene) is commonly referenced in the unclassified literature as a suitable material for this purpose.
and gaseous radiation thermal inputs:

\[ Q_{\text{tot}} = \int_0^{t_f} Q \, dt \, dS_w \] \hspace{1cm} (7-4)

where: \( S_w \) is the wetted area

\[ Q = Q_c + Q_r \]

and \( t_f \) is total time of flight.

The heating rate at any point on the entry vehicle is a certain fraction, \( k_1 \), of the stagnation point heating rate. The magnitude of \( k_1 \) varies with the location of the unit area and the geometry of the vehicle.

\[ k_1 = \frac{Q}{Q_s} \] \hspace{1cm} (7-5)

\( Q_s \) is heating rate per unit area at the stagnation point and is independent of the wetted area.

Thus equation (7-4) may be written with the aid of equation (7-5):

\[ Q_{\text{tot}} = \left[ \frac{1}{S_w} \int k_1 \, dS_w \right] S_w \int_0^{t_f} Q_s \, dt \] \hspace{1cm} (7-6)

The quantity in square brackets is a function of the geometry of the vehicle. For hemispheres, it is approximately equal to \( 0.5(15)(37) \).

For missions of long duration, which generally involve vehicles which are cooled by re-radiation, the heating rates must not exceed design maximums. In this case, the mission is usually conceived as one for which the vehicle operates at radiation equilibrium temperatures throughout the trajectory. The stagnation surface temperature experienced during entry of a vehicle designed to operate at radiation
equilibrium temperatures is calculated by equating the input heating rate to the re-radiation heating rate:

\[ \dot{Q}_c + \dot{Q}_r = \dot{Q}_{\text{rad}} \tag{7-7} \]

Neglecting the input heating due to gaseous radiation since it generally contributes a small percentage of the total heat, Eq. (7-7) is written as follows from equations (7-1) and (7-3):

\[ \frac{C_{\text{atm}}}{\sqrt{R}} \left( \frac{\rho}{\rho_{\text{SL}}} \right)^{\frac{1}{2}} \left( \frac{V_{\text{AM}}}{\sqrt{GR}} \right)^3 = K_{\text{rad}} K_{\text{bolt}} \frac{T_s^4}{\lambda} \tag{7-8} \]

In equation (7-8), \( T_s \) is stagnation point temperature; laminar flow is assumed \((n = \frac{1}{2})\); \( m \) is assumed equal to 3 from the ideal gas relation; and \( Y \) is assumed equal to 1.0.

7.4 Analytical Representation of Trajectory Constraints Resulting from Convective Heating of Vehicles Designed to Operate at Radiation Equilibrium Temperatures

Equation (7-8) is the fundamental equation used in this thesis for expressing constraints on the trajectory resulting from gas-dynamic heating. Temperatures in the vicinity of the stagnation point or stagnation line may be useful from the guidance standpoint because:

(1) Temperatures conceivably can be measured either directly or indirectly in flight.

(2) The temperature history for a given nominal profile of a particular vehicle entering the atmosphere of a particular planet may be predicted as shown in subsequent chapters of this thesis. Guidance information can be deduced by comparing measured and predicted values.
(3) Temperatures change rapidly early in the mission while changes in specific force level are more abrupt later in the mission.* Temperature readings may be useful in early determination of atmospheric conditions actually being encountered in flight.

In order to write an analytical expression for stagnation point temperature from equation (7-8) for entry into the atmosphere of an arbitrary planet, it is necessary to investigate the atmospheric model of the planet. Lees [40] shows that:

\[
C_{(atm)} = (\text{Constant}) \left( \frac{\rho_{(sl)} \mu_{(sl)}}{(GR)^2} \right)^{\frac{1}{2}} Pr^{\frac{3}{2}} \left( \frac{\gamma - 1}{\gamma_0} \right)^{\frac{1}{2}} \quad (7-9)
\]

The following symbols are used in equation (7-9):

- \( Pr \) = Prandtl number
- \( \gamma_0 \) = \( C_p / C_v \) = ratio of atmospheric heat capacity
- \( C_p \) = heat capacity at constant pressure
- \( C_v \) = heat capacity at constant volume
- \( \mu_{(sl)} \) = sea level coefficient of viscosity of the planetary atmosphere (slug/ft.-sec.).

Assuming no atmospheric winds \( (V_0(AM) = 0) \), then:

\[
V(AM) = V \quad (7-10)
\]

Non-dimensionalizing velocity in accordance with the procedure of Chapter 6:

\[
v = \frac{V}{V_s} = \frac{V}{\sqrt{G(m) \circ R(m) \circ}} \quad (7-11)
\]

* Maximum temperature levels are encountered when the vehicle velocity has slowed to about 0.8 circular satellite velocity, while maximum accelerations occur at much slower velocities (see Chapt. 9).
Substituting equations (7-9) through (7-11) into equation (7-8) gives:

\[ K_{rocl} K_{bol} T_s^4 = (\text{Constant}) \left( \frac{\rho M(\text{SL})}{m} \right)^{\frac{2}{3}} V_s^2 \rho^3 \frac{1}{r^2} \left( \frac{y_o - 1}{y_o} \right)^{\frac{3}{2}} \]

(7-12)

Various models of planetary atmospheres are discussed in appendix E. Assuming the exponential model:

\[ \rho = \rho(\text{SL}) e^{-KH} \]

(7-13)

where \( \rho(\text{SL}) \) is the "assumed" mean sea level atmospheric density. This is not necessarily equal to the true sea level density \( \rho(\text{SL}) \). The "assumed" quantity is the intercept of the straight line which best fits the curve \( \log \rho \) vs. altitude.

\( K \) is the atmospheric density decay parameter, ft.\(^{-1}\).

\( H \) is the altitude of the vehicle above the surface of the planet, ft.

The following constants were assumed for the planetary radius and for the parameters of the exponential atmospheric model (see Appendix E):

<table>
<thead>
<tr>
<th>Planet</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(\text{m}) ), ft.</td>
<td>20.27 ( \times ) 10^6</td>
<td>20.89 ( \times ) 10^6</td>
<td>11.08 ( \times ) 10^6</td>
</tr>
<tr>
<td>( \rho(\text{SL}) ), slugs/ft.(^3)</td>
<td>0.0326</td>
<td>0.0027</td>
<td>0.000193</td>
</tr>
<tr>
<td>( K ), ft.(^{-1})</td>
<td>4.89 ( \times ) 10(^{-5})</td>
<td>4.25 ( \times ) 10(^{-5})</td>
<td>1.67 ( \times ) 10(^{-5})</td>
</tr>
</tbody>
</table>

Assuming for the Earth's atmosphere the following values:

\[ C(\text{atm}) = 17,000 \text{ BTU ft.}^{-3/2} \text{ sec}^{-1} \]

\[ P_r = 1.0^* \]

\[ \gamma_0 = 1.4 \]

* The assumption that \( P_r = 1 \) for hypersonic flow in the Earth's atmosphere is generally made(31).
then the (Constant) in equation (7-12) is written:

\[
\text{(Constant)} = \frac{590}{\sqrt{2}} \mu_{(5L)}^2 \gamma_{(m)}^2 R_{(m)}^2 K_{E}^2 \gamma_{E}^2 \left( \frac{\gamma_{E} - 1}{\gamma_{E}} \right)^{-\frac{1}{2}}
\]

\[(7-14)\]

The subscript \( E \) in equation (7-14) denotes values of the various quantities for the Earth and its atmosphere.

Substituting Eq. (7-13) and (7-14) into equation (7-12) gives:

\[
K_{\text{rad}} K_{\text{bol}} T_s^4 = \frac{590}{\sqrt{2}} \left( \frac{R_{(5L)}}{R} \right)^2 \left( \frac{R_{(m)}}{R_{(m)}} \right)^\frac{1}{4} \left( \frac{K_{E}}{K_{E}} \right)^\frac{1}{4} \left( \frac{P_{(m)}}{P_{(m)}} \right)^\frac{1}{2} \left( \frac{\gamma_{E} - 1}{\gamma_{E}} \right)^{\frac{1}{4}} e^{-\frac{KH}{2}}
\]

\[(7-15)\]

where \((HR)_{EO}\) represents the "heating ratio" of the atmosphere of planet 0 with respect to the Earth:

\[
(HR)_{EO} = \left( \frac{\mu_{(5L)}}{\mu_{(5L)}} \right)^\frac{1}{2} \left( \frac{G_{(m)}}{G_{(m)}} \right)^\frac{1}{2} \left( \frac{R_{(m)}}{R_{(m)}} \right)^\frac{5}{4} \left( \frac{K_{E}}{K_{E}} \right)^\frac{1}{4} \left( \frac{P_{(m)}}{P_{(m)}} \right)^\frac{1}{2} \left( \frac{\gamma_{E} - 1}{\gamma_{E}} \right)^{\frac{1}{4}}
\]

\[(7-16)\]

Equation (7-15) may be further simplified to:

\[
K_{\text{rad}} K_{\text{bol}} T_s^4 = \frac{417}{\sqrt{R}} (HF)_0 T_s^3 e^{-\frac{KH}{2}}
\]

\[(7-17)\]

In equation (7-17):

- \( k = KR_{(m)}^2 \): dimensionless atmospheric density decay parameter
- \( h = H/R_{(m)} \): dimensionless altitude
- \((HF)_0\) is the "heating function" of the planetary atmosphere.

\[
(HF)_0 = (HR)_{EO} \left( \frac{R_{(m)}}{K} \right)^\frac{1}{4}
\]

Equation (7-17) is written in the following form:

201
\[ K_{\text{rad}} K_{\text{bol}} T_0^4 = 18,000 \frac{(HF)_{EO}}{\sqrt{R}} \nu^3 e^{-\frac{kh}{2}} \]  

\[ \text{BTU} \text{ ft.}^2\text{-sec.} \]  

(7-19)

where \((HF)_{EO}\) is the ratio of the heating function of the planetary atmosphere with respect to the heating function of Earth's atmosphere:

\[
(HF)_{EO} = \frac{(HF)_0}{(HF)_E}
\]

(7-20)

Table 7.3 summarizes approximate values of the heating function for the terrestrial planets.

<table>
<thead>
<tr>
<th>Planet</th>
<th>((HF)_0)</th>
<th>((HF)_{EO})</th>
<th>((HF)_{EO}^{1/4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>105</td>
<td>2.41</td>
<td>1.25</td>
</tr>
<tr>
<td>Earth</td>
<td>43.5</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Mars</td>
<td>1.1</td>
<td>0.0253</td>
<td>0.4</td>
</tr>
</tbody>
</table>

It should be noted that the right-hand side of equation (7-19) expresses the convective heat flow rate as a function of vehicular velocity, radius of curvature at the stagnation point, and flight altitude for entry into an arbitrary planetary atmosphere.

Equation (7-19) is solved for stagnation point temperature as follows:

\[
T_s = 1.392 \times 10^{4} (HF)_{EO}^{1/4} (VF)^{1/4} \nu^{3/2} e^{-kh/8}
\]

(7-21)

where \((VF)\) represents a "vehicle design function":

\[
(VF) = \frac{1}{K_{\text{rad}} R^{5/2}}
\]

(7-22)
$K_{rad}$ is of the order of 0.8 at high surface temperatures for representative thin shell structures.

Within the approximations made in arriving at equation (7-21), the following stagnation temperatures are derived for a vehicle with $(VF) = 1.0$ flying at circular orbital velocity at the surface of the terrestrial planets:

$$(T_s)_{Earth} = 14,000^\circ \text{ Rankine}$$

$$(T_s)_{Venus} = 17,300^\circ \text{ Rankine}$$

$$(T_s)_{Mars} = 5,600^\circ \text{ Rankine}$$

The same vehicle traveling at 0.8 circular orbital velocity at 50 nautical miles altitude above Earth has a stagnation temperature of 2310° R. and at 40 statute miles a temperature of 3830° R. Rubensin(37) computed the following approximate maximum temperatures for hemispheres entering the Earth's atmosphere: For circular orbital decay, $(T_s)_{max}$ is equal to 3700° R; for entry with initial flight path angle of minus 4°, $(T_s)_{max} = 4850^\circ \text{ R}$.

Stagnation temperature is a useful guidance parameter for vehicles designed to operate at radiation equilibrium temperatures. From the guidance standpoint, the trajectory must be constrained such that maximum design heat flow rates are not exceeded.

$$0 < \dot{Q}_c < (\dot{Q}_c)_{max}$$ (7-23)

where

$$\dot{Q}_c = 18,000 \left( \frac{(H+\nu)e}{\sqrt{\nu}} \right) \frac{\dot{H}}{2} \frac{\text{BTU}}{\text{ft}^2\text{sec}}$$ (7-24)

If input convective heat flow rate and re-radiation heat flow rate are
in equilibrium, equation (7-23) may be written in terms of stagnation point temperature, which is a measurable manifestation of the heat transfer.

$$0 < T_s < (T_s)_{\text{max}}$$  \hspace{1cm} (7-25)

$T_s$ is given by equation (7-21) and $(T_s)_{\text{max}}$ is the maximum allowable stagnation temperature. The inequality (7-25) may be written as an equality which must be satisfied by all acceptable* trajectories. Since stagnation temperatures must not exceed a certain design maximum while no restriction is placed on minimum temperatures, this temperature limitation on trajectories is a "one-sided" constraint. One-sided constraints may be handled analytically as follows:

1. Introducing two auxiliary real variables $\xi_1$ and $\beta_1$: \hspace{1cm} (7-26)

$$T_s - (T_s)_{\text{max}} + \xi_1^2 = 0$$

$$T_s - \beta_1^2 = 0$$

2. Introducing one auxiliary variable $\eta_1$: \hspace{1cm} (7-27)

$$(T_{s\text{max}} - T_s)T_s - \eta_1^2 = 0$$

3. Representing stagnation temperatures in parametric form such as given previously in Chapter 5 for propellant mass flow rate. In the first method of representing the temperature constraint, $\xi_1$ is real for all negative values of $T_s$ and for all positive values with a magnitude less than $(T_s)_{\text{max}}$. The expression involving $\beta_1$ would be

* Acceptable solutions in this case are guidance trajectories which do not compromise the structural integrity of the vehicle as a result of atmospheric heating.
required if minimum values of \( T_s \) were also to be constrained (such as to prevent negative temperatures). Since negative temperatures cannot enter if positive roots of equation (7-21) are the only solutions admitted, the first of equation (7-26) is the only one required.

Representing this in dimensionless form by dividing through by \((T_s)_{\text{max}}\) and substituting for \( T_s \) from equation (7-21) gives:

\[
(TC) \left( \frac{3}{5} e^{-\frac{4h}{5}} - 1 + \xi_1(\tau) \right) = 0
\]

(7-28)

where \((TC)\) represents a "temperature constraint constant", a constant for a particular vehicle entering the atmosphere of a particular planet.

\[
(TC) = \frac{1.392 \times 10^4 (HF)_{E0} (VF)_{(T_s)}^{\frac{1}{4}}}{(T_s)_{\text{max}}}
\]

(7-29)

For example, if \((VF) = 1\) and if \((T_s)_{\text{max}} = 3000^\circ \text{R}\), then \((TC)\) equals 5.8 for Venus, 4.64 for Earth, and 1.86 for Mars. \( \xi_1(\tau) \) in equation (7-28) is some real function of dimensionless time which must satisfy this equation.

Equation (7-28) may be used as an analytic function for specifying temperature constraints in the application of the calculus of variations to determine extremals of guidance functions. Graphical representation of temperature constraints is described in Section 7.7.

### 7.5 Human Acceleration Tolerances

During the entry mission as well as during maneuvers in the free-fall phase of flight, the vehicle will encounter rolling, pitching, yawing and accelerations in the longitudinal and lateral directions. The effect of forces on the human occupants of the craft as a result of
these motions will depend on the position, posture, and adaptability of
the individual crew members.

Human tolerances to specific forces encountered during the mission
depend on the following factors:

(1) Absolute specific force level (accelerations)
(2) Duration of exposure
(3) Time rate of change of specific forces*.

During the manned entry mission, certain guidance and control functions
may be required of the pilot. His performance generally deteriorates
under high g-loads; even though these acceleration levels may be well
below critical levels for survival or permanent physical damage. If the
control functions are of the "off-on" type, the tolerable g-levels are
greater than if the control functions require continuous precision
corrections such as matching pointers. Auditory stimuli may be used to
a limited extent to augment visual operations; however, auditory
hallucinations are frequent at high acceleration levels.

The acceleration levels encountered by the entry vehicle must be
constrained to values below that for which there is deterioration of the
human control function. If the guidance system does not require any
human actions, then the tolerable levels are limited only to the extent
required to prevent physical damage to the human occupants. If the
functions of the pilot involve movements of hands and arms and require
that he make tactical decisions and operate complex equipment, then the
accelerations that may be tolerated are small.

Studies have shown that humans can withstand considerably greater

* That is, the rise time to peak g-loads.
g-forces in the transverse direction (chest-to-back) than in the longitudinal direction (head-to-toe). The direction of g-forces are generally defined as follows:

1. Transverse Prone: back to chest
2. Transverse Supine: chest to back
3. Longitudinal positive: head to toe
4. Longitudinal negative: toe to head.

Table 7.4 summarizes the gross effects of acceleration forces on the human being:

<table>
<thead>
<tr>
<th>Effects:</th>
<th>g's</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weightlessness</td>
<td>0</td>
</tr>
<tr>
<td>Earth normal</td>
<td>1</td>
</tr>
<tr>
<td>Hands and feet heavy; walking and climbing difficult</td>
<td>2</td>
</tr>
<tr>
<td>Walking and climbing impossible; crawling difficult</td>
<td>3</td>
</tr>
<tr>
<td>Movement only with great effort; crawling almost impossible</td>
<td>4</td>
</tr>
<tr>
<td>Only slight movements of arms and head possible</td>
<td>5</td>
</tr>
</tbody>
</table>

Positive longitudinal g's, short duration. (blood forced from head toward feet)

<table>
<thead>
<tr>
<th>Effects:</th>
<th>g's</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visual symptoms appear</td>
<td>2.5 - 7.0</td>
</tr>
<tr>
<td>Blackout</td>
<td>3.5 - 8.0</td>
</tr>
<tr>
<td>Confusion, loss of consciousness</td>
<td>4.0 - 8.5</td>
</tr>
<tr>
<td>Structural damage, especially to spine</td>
<td>(greater than 18-23)</td>
</tr>
</tbody>
</table>

Transverse g's, short duration. (Head and heart at same hydrostatic level)

<table>
<thead>
<tr>
<th>Effects:</th>
<th>g's</th>
</tr>
</thead>
<tbody>
<tr>
<td>No visual symptoms or loss of consciousness</td>
<td>0 - 17</td>
</tr>
<tr>
<td>Tolerated</td>
<td>28 - 30</td>
</tr>
<tr>
<td>Structural damage may occur</td>
<td>(greater than 30-45)</td>
</tr>
</tbody>
</table>

It is seen from Table 7.4 that considerably greater forces may be tolerated in the transverse direction than in the longitudinal direction.
Fig. 7.2 shows human tolerances to steady-state positive longitudinal accelerations as a function of time-duration of exposure. Data for this graph was taken from references (7) and (44). From this figure it may be observed that 4 g's may be tolerated for only 10 seconds, 3 g's may be tolerated for about 2 minutes, and 2 g's may be tolerated indefinitely.

The reaction of man to transverse acceleration loads is summarized in Figs. 7.3 and 7.4. The first of these figures covers the time interval up to 9 seconds of exposure, the second covers exposure times from 9 to about 4000 seconds. Data for Fig. 7.3 was taken from reference (44) and data for Fig. 7.4 was taken primarily from reference (7). These figures show that man can safely withstand 10 g's transverse acceleration for 2 minutes and 5 g's for about 25-30 minutes.

A study of Figs. 7.2, 7.3, and 7.4 shows clearly that the tolerance of the crew of the entry vehicle to accelerations is a strong function of the direction of the applied forces and the time interval over which the force level is maintained. An important factor not shown in these figures is the decrement of the human operator in carrying out control functions; this factor is strongly dependent on the complexity of the control functions required of the man. As mentioned earlier in this section, precision operations involving movements of the hands and arms restrict tolerable g-levels to much lower levels than indicated by Figs. 7.2 through 7.4.

Woodling and Clark (44) describe recent studies with the centrifuge which were performed in order to determine pilot physiological tolerances to accelerations. One phase of the study involved programmed accelerations which were unaffected by pilot control actions, the second phase involved a series of runs in which the pilot actually controlled...
Fig. 7.2: Human Tolerances to Steady State Positive Longitudinal Accelerations.
Fig. 7.3: Human Tolerances to Steady State Transverse Prone Accelerations Over Short Time Intervals. Equivalent Force Applied From Back to Chest.
Fig. 7.4: Human Tolerances to Steady State Transverse Accelerations. Duration of Exposure = 7 to 4,000 seconds.
the motion of the centrifuge and the resulting accelerations to which he
was subjected. In this study, pilots wore standard Navy Z-2-G-suits and
were seated in contoured couches in order to minimize body pressure
points and jostling effects due to oscillatory accelerations. Direction
of g-forces were primarily from chest to back.

In the first phase of this study, pilots were assigned an arbitrary
visual tracking task. Among the interesting results of this study were
the following:

An important factor in increasing the tolerance level
during the runs was the technique with which the subject
strained, as if he were trying to support someone standing
on his chest. No chest pain was reported as long as proper
straining was maintained, although breathing became difficult
around the 12-g level. Blurring of vision was reported by
some subjects at levels of g near 16. Subjects were able to
operate a small, right-hand control stick under accelerations
as high as 25-g. A thumb-operated stick was also used
effectively. Although conclusive objective data was not
obtained, performance of the tracking task showed deteriora-
tion with increasing accelerations but improved with experi-
ence. It was felt that the contoured type couch offered
satisfactory means of support for entry accelerations as
high as 25-g for a trained subject.

In the second phase of the studies in which the pilot controlled
the centrifuge, a computer provided aerodynamic simulation of the
vehicle under consideration. Woodling and Clark\(^{(44)}\) reported:

One extreme re-entry condition involved normal accel-
erations of 7 g's and longitudinal deceleration of 4 g's
which lasted as long as 25 seconds, during which time the
pilot was able to maintain adequate control. The blood
pressure in the limbs increased and petechia (small skin
hemorrhages) were noted in the forearm and ankles. Tingling
and subsequent numbness of limbs were noted and in a few
cases definite pain was reported.

In addition to the absolute specific force level and the duration
of exposure, the rise time to peak levels is significant. Col. John
Stapp, Air Force Missile Development Center, has investigated loadings
up to 45 g's sustained over a period of about 1/3 second. He concluded that for brief high g-loadings, the rate of change of g is a more important consideration than the peak magnitude. Studies of the ability of a human to withstand a rapid rate of change of loading indicates the rate of onset of g-loadings should not exceed 200-250 g's per second.

During the Keplerian Phase of the entry mission, the crew will experience zero g's. There is limited information to date about the effects of weightlessness on the human being. Man has been exposed to weightlessness for periods of the order of only seconds or minutes to date; some individuals find it unpleasant while others seem to enjoy it. It is believed that some individuals will be able to adapt to a weightless condition for long periods of time.

7.6 Analytical Representation of Trajectory Constraints Imposed by Human Acceleration Tolerances

The vector sum of external specific forces measured by a set of orthogonal accelerometers in the vehicle is equivalent to the total accelerations experienced by the human crew. If it is required that the total specific force level be restricted to values below a certain maximum number of Earth g's, then:

\[ (\text{Magnitude of total external specific force vector}) \leq c G(m)E \quad (7-30) \]

where \( G(m)E \) = mean surface gravitational acceleration of Earth

\[ c = \text{maximum allowable specific force level (number of } G(m)E) \].

Since it is likely that significant retro-thrust will be generated in bursts of short duration, the contribution to total specific force
due to thrust is not considered in this analysis.* Constraints in this thesis are imposed only on force levels resulting from lift and drag accelerations. Thus, equation (7-30) is written in dimensionless form as:

\[
\left( n_D^2 + n_L^2 \right)^{\frac{1}{2}} \leq c \frac{G(m) \xi}{G(m) \theta} 
\]  

(7-31)

In accordance with the procedure previously discussed for temperature constraints in section 7.4, inequality (7-31) may be written as an equation which must be satisfied throughout the entry trajectory by introducing a new real variable \( \xi_2 \). Assuming the exponential model of the planetary atmosphere and no atmospheric winds, the following equation may be written:

\[
(AC) \frac{\rho^2 S}{c M} e^{-\frac{h}{h}} (c_D^2 + c_L^2)^{\frac{1}{2}} - 1 + \xi_2^2(\tau) = 0 
\]  

(7-32)

where \( \xi_2(\tau) \) is some real function of dimensionless time which satisfies this equation.

(AC) represents an "acceleration constraint constant", a constant for entry into the atmosphere of a particular planet.

\[
(AC) = \frac{\rho(\Sigma) G(m) \theta R(m) \theta}{2 G(m) \xi} 
\]  

(7-33)

The acceleration constraint constant is equal to the satellite dynamic pressure at the surface of the planet divided by the mean surface

* Figs. 7.2 and 7.3 show that high specific force levels may be tolerated for short periods of time provided the rate of change of specific force does not exceed 200-250 g/sec. Since high thrust levels will be generated over short periods of time only, the additional force level incurred during periods of engine operation can generally be tolerated.
gravitational force of the Earth. Table 7.5 gives approximate values of the acceleration constant for the terrestrial planets.

<table>
<thead>
<tr>
<th>Table 7.5: Values of Acceleration Constraint Constant for Terrestrial Planets</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AC)</td>
</tr>
<tr>
<td>Venus</td>
</tr>
<tr>
<td>Earth</td>
</tr>
<tr>
<td>Mars</td>
</tr>
</tbody>
</table>

Equation (7-32) is used as an analytic function for specifying trajectory constraints resulting from human acceleration tolerances. This equation may be used in the application of the calculus of variations to determining extremals of arbitrary guidance functions. Graphical representation of both heating and deceleration constraints are described in Section 7.7.

7.7 Graphical Representation of Heating and Acceleration Constraints

A velocity-altitude graph of the design maximum vehicular temperature and maximum permissible specific force level is instructive as an aid to visualizing trajectory restraints imposed by these manifestations of energy transfer from the vehicle to the planetary atmosphere. The guidance system must operate in such a way that the resulting trajectory does not violate these constraints; an "allowable operating region" is thereby established. Any guidance trajectory conceived for the mission can be plotted in the velocity-altitude plane and is an
acceptable trajectory with respect to mission constraints as long as
the path remains entirely within the allowable operating region. The
optimum guidance trajectory may be selected from among all permissible
trajectories based on evaluation of the other mission factors which are
important in system design — factors such as payload requirements, range
accuracy, flight time, system reliability, computational complexity, etc.

Either maximum allowable heat flow rates or stagnation point tem-
peratures are readily represented in the velocity-altitude plane. To
illustrate this method of representing constraints, stagnation point
temperature was selected here; this presupposes a vehicle which is
designed to operate at radiation equilibrium temperatures.

Assuming a certain maximum stagnation point temperature is pre-
scribed, equation (7-21) is written:

$$h = \frac{18.44}{k} \left\{ \frac{3}{4} \log \nu - \log \left[ \frac{(T_s)_{\text{max}}}{1.392 \times 10^7 (HF)^{\frac{1}{4}} (VF)^{\frac{1}{4}}} \right]\right\}$$ (7-34)

where $\log (\ )$ represents the logarithm to the base 10.

Using Equation (7-29), equation (7-34) is written:

$$h = \frac{18.44}{k} \left[ \frac{3}{4} \log \nu + \log(TC) \right]$$ (7-35)

Values of $k$ are approximately as follows for the atmospheres of the
terrestrial planets:

<table>
<thead>
<tr>
<th>Planet</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>1013</td>
</tr>
<tr>
<td>Earth</td>
<td>889</td>
</tr>
<tr>
<td>Mars</td>
<td>184.6</td>
</tr>
</tbody>
</table>

Equation (7-35) is plotted on Figs. 7.5, 7.6, and 7.7 for Earth, Venus,
and Mars, respectively. The following combinations of $\frac{(T_s)_{\text{max}}}{(VF)^{1/4}}$ are represented in these figures:

$\frac{(T_s)_{\text{max}}}{(VF)^{1/4}} = 1000, 2000, 2500, \text{ and } 3000 \, \circ\text{R} \times \text{ft}.$

The similarity between entry into Earth and Venus from the atmospheric heating standpoint is clearly evident by comparing Figs. 7.6 and 7.5. Temperature constraints for entry into the atmosphere of Mars, on the other hand, are nearly constant velocity lines.

Representation of acceleration constraints require that the lift, drag, and mass characteristics of the vehicle be specified. For graphical representation in this thesis, the three families of vehicles described in Table 5.1 were selected. Maximum tolerable specific force level was chosen in this thesis to be at the level of six Earth g's. Judicious choice of the maximum specific force level is strongly dependent on the decrement of the human operator in performing guidance and control functions; choosing the 6-g level assumes that his control actions require little movement of hand and arms and are restricted primarily to "off-on" type operations.

Equation (7-31) is written:

$$h = \frac{1}{k} \log e \left[ 2 \log v + \frac{1}{2} \log (C_L^2 + C_D^2) + \log \left[ \frac{(AC)}{c M/\text{s}} \right] \right]$$

Equation (7-36) is plotted in Figs. 7.5, 7.6, and 7.7. for the terrestrial planets with $c = 6$ Earth g's. The acceleration constraints are represented as bands in these figures rather than lines. The bands result because:

(I) It was assumed that the lift-coefficient may vary between
Fig. 7.5: Operating Regions Permitted by Constraints on Stagnation Temperatures and Human Accelerations for Entry into Earth's Atmosphere.
Fig. 7.6: Operating Regions Permitted by Constraints on Stagnation Temperature and Human Accelerations for Entry into Venusian Atmosphere.
Fig. 7.7: Operating Regions Permitted by Constraints on Stagnation Temperature and Human Accelerations for Entry into Martian Atmosphere.
zero and $C_{L_{\text{max}}}$

(2) A variable mass vehicle was assumed.

Mass variation was taken to be $M_{\text{initial}}$ to $0.7 M_{\text{initial}}$ in Figures 7.5 through 7.7. This corresponds to an expenditure of propellant mass equivalent to seven separate 500 ft./sec. velocity impulses* if the specific impulse of the propellant is approximately 300 seconds. If the vehicle mass, lift, and drag coefficients were constant during entry, the acceleration bands of these figures would reduce to lines.

Acceleration constraints place a more severe restriction on allowable velocity-altitude loci than shown in Figs. 7.5 through 7.7 if the magnitude of $c$ is reduced. If the vehicle flies in a high drag condition, such as near 90° angle of attack, the drag accelerations are more severe than indicated by these figures. It should be emphasized that the lift-drag polar of the vehicles plotted in Figs. 7.5 through 7.7 are valid only in the small angle of attack regions; that is, $\alpha$ may vary from zero to a value corresponding to $(L/D)_{\text{max}}$. The allowable operating region is reduced if the vehicle is operated on the other side of the lift curve due to the higher drag characteristics of this configuration.

For most vehicle-planet combinations, both heating and acceleration constraints must be examined in arriving at a suitable entry profile. Specific force levels are generally more restrictive on the mission at lower altitudes while heating considerations are predominant at high altitudes. The lower the maximum g-tolerance, the more severe are the effects of acceleration constraints on restricting the family of allowable

* See equation (5-58).
guidance trajectories. The higher the drag characteristics of the vehicle, the higher the altitude at which a given g-level is observed. If M/S is increased while drag coefficient is fixed, a particular specific force level is encountered at lower altitudes.

7.8 Locus, in Velocity-Altitude Plane, of Points where External Specific Forces Equal or Exceed Minimum Detectable Levels of Specific Force Measuring Subsystem.

Lift and drag forces may be utilized in a controlled manner to modify the trajectory of the entry vehicle when these forces are of such magnitudes as to be significant. These external forces become significant, from the guidance standpoint, when they are of such magnitudes as to be detectable by accelerometers carried in the vehicle. Until the vehicle has penetrated a sufficient distance into the planetary atmosphere for accelerations to exceed the threshold value detectable by the specific force measuring subsystem, the entry trajectory is basically that of the Keplerian transfer ellipse.

Defining:

\[ C_{th} = \text{threshold value of accelerations detectable by specific force measuring subsystem in Earth g's.} \]

Equation (7-36) can be written:

\[ K_{E0} h + C_{th} = \frac{1}{k_E} \log e \left\{ 2 \log \nu + \frac{1}{2} \log (C_L^2 + C_D^2) + \log \left[ \frac{(AC)_{E}}{C_{th} M/S} \right] \right\} \] (7-37)

where \( k_E \) = dimensionless atmospheric exponential decay parameter for Earth \( \approx 889 \).

\( K_{E0} \) = ratio of dimensionless planetary atmospheric exponential decay parameter with respect to the decay parameter of the
Earth; i.e.,

\[ K_{EO} = k/k_E \]  

(7-38)

The magnitude of \( K_{EO} \) for Venus is 1.14, for Earth 1.0, and for Mars 0.207.

\( C_a \) is a constant for each planet and is defined as follows:

\[ C_a = - \frac{\log (AC)_{EO}}{k_E \log e} \]  

(7-39)

\((AC)_{EO}\) was tabulated in Table 7.5.

The left side of equation (7-37) is dimensionless altitude when entry is made into the Earth's atmosphere. It is readily converted to altitude for entry into the atmospheres of other planets by using the conversion quantities listed in Table 7.6.

<table>
<thead>
<tr>
<th>Table 7.6: Conversion Quantities for Use in Equation (7-37)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>( K_{EO} )</td>
</tr>
<tr>
<td>( C_a )</td>
</tr>
</tbody>
</table>

Equation (7-37) is plotted on Figures 7.8 through 7.10. For these graphs, the three classes of vehicles summarized in Table 5.1 were used and the mass characteristics of the vehicles were taken to be \( M_i/S \) equal to 1.0 slug/ft.\(^2\). Values of threshold accelerations measurable by the specific force measuring subsystem were arbitrarily assumed as follows:
Fig. 7.8: \( C_{th} = 10^{-4} \) Earth g's

Fig. 7.9: \( C_{th} = 10^{-5} \) Earth g's

Fig. 7.10: \( C_{th} = 10^{-6} \) Earth g's

For convenience, the information plotted in Figs. 7.5 through 7.7 was also plotted in the same manner in Figs. 7.8 through 7.10. The acceleration constraint bands are easily plotted by replacing \( C_{th} \) with the value of 6.0 previously chosen as representative for missions during which the human operator plays a relatively minor role in controlling the vehicle.

The stagnation temperature lines are represented by writing equation (7-35) as:

\[
K_{E0} n + C_t = \frac{18.44}{K_E} \left[ \frac{3}{4} \log v + \log \left( \frac{1.392 \times 10^{14} (VF)^{4/4}}{(V_f)_{max}^{1392}} \right) \right] \quad (7-40)
\]

where

\[
C_t = -\frac{18.44}{4K_E} \log (HF)_{E0} \quad (7-41)
\]

Equation (7-40) is plotted on Figs. 7.8 through 7.10 for

\[
\frac{(T_s)_{max}}{(VF)^{1/4}} = 2000, 2500, \text{ and } 3000 \quad \text{°R ft.}^{1/8}
\]

The left side of equation (7-40) is dimensionless altitude for entry into the atmosphere of the planet Earth. Dimensionless altitudes for entry into the atmosphere of other terrestrial planets is readily determined with the aid of conversion quantities listed in Table 7.7:
Fig. 7.8: Threshold of Specific Force Measuring Subsystem in Altitude-Velocity Plane. Specific Force Measuring Subsystem Threshold = $10^{-4}$ Earth g's.
Fig. 7.9: Threshold of Specific Force Measuring Subsystem in Altitude-Velocity Plane. Specific Force Measuring Subsystem Threshold = $10^{-5}$ Earth g's.
Fig. 7.10: Threshold of Specific Force Measuring Subsystem in Altitude-Velocity Plane.
Specific Force Measuring Subsystem Threshold = 10^6 Earth g's.

<table>
<thead>
<tr>
<th></th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{e0}$</td>
<td>1.14</td>
<td>1.0</td>
<td>0.207</td>
</tr>
<tr>
<td>$C_a$</td>
<td>-2.75X10^-3</td>
<td>0</td>
<td>+4.62X10^-3</td>
</tr>
<tr>
<td>$C_t$</td>
<td>-1.98X10^-3</td>
<td>0</td>
<td>+8.28X10^-3</td>
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Plots exemplified by Figs. 7.8 through 7.10 are particularly useful as a background graph for comparing various guidance trajectories which are selected on the basis of optimizing other mission factors such as payload capabilities, range accuracy, flight time, etc. If a series of trajectory plots are made on Fig. 7.8, for example*, each having particular advantages for the mission under study, a straightforward comparison of the trajectories is readily made with respect to the altitude and velocity points at which the threshold value of the specific force measuring system is exceeded. The temperature and acceleration constraint loci on the same plot indicate the favorable or unfavorable character of each of the trajectories from the standpoint of energy transfer limitations.

---

* This plot would be used if the temperature and constraint lines are appropriate for the particular vehicle system under consideration and if the specific force measuring subsystem had a design threshold of $10^{-4}$ g's.

---

| Table 7.7: Conversion Quantities for Use in Equation (7-40) |
|-----------------|-----------------|-----------------|
|                | Venus           | Earth           | Mars            |
| $K_{EO}$       | 1.14            | 1.0             | 0.207           |
| $C_t$          | $-1.981 \times 10^{-3}$ | 0              | $8.28 \times 10^{-3}$ |
| $(HF)_{EO}$    | 2.41            | 1.0             | 0.0253          |
GUIDANCE PARAMETERS AND CONSTRAINTS FOR
CONTROLLED ATMOSPHERIC ENTRY

by

Robert C. Duncan, Lieutenant Commander, U.S. Navy
B.S. United States Naval Academy, 1945
B.S. United States Naval Postgraduate School, 1953
S.M. Massachusetts Institute of Technology, 1954

Volume II of II
Chapters 8 through 10
Appendix A through G

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Signature redacted

Dept. of Aeronautics and Astronautics, January 11, 1960

Certified by

Signature redacted

Thesis Supervisor

Signature redacted

Thesis Supervisor

Signature redacted

Thesis Supervisor

Accepted by

Signature redacted

Chairman, Departmental Committee on Graduate Students
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8.1 Definition of Trajectory Phases

The function of a guidance system in a vehicle entering the atmosphere of a planet is complicated by the radically different operational environment it encounters during the course of entry. For example, the density of the atmosphere changes from an insignificantly small quantity at orbital altitudes to the dominating factor affecting the shape of the trajectory at low altitudes.

It is convenient to consider the entry trajectory as made up of three separate phases:

1. **Keplerian Phase** (Free-Fall Phase):
   That segment of the trajectory, at high altitudes, where gas-dynamic forces are insignificantly small ($\rho \approx 0$).

2. **Intermediate Phase**:
   That segment of the trajectory where accelerations due to gas-dynamic terms are of comparable magnitude with other terms in the dynamical equations of motion.

3. **Gas-Dynamic Phase**:
   That segment of the trajectory where gas-dynamic accelerations
are the predominant terms affecting the shape of the trajectory.

Figure 8.1 shows the various phases of the entry trajectory and gives definitions of boundaries between phases.

In the Intermediate Phase, either a pure Keplerian description of the trajectory or the gas-dynamic simplifications* in the equations of motion are not warranted. Chapman(15) considers the entry trajectory as made up of two phases only** which are fitted together by matching boundary conditions. He defines the onset of the Gas-Dynamic Phase as that point at which drag has reduced the vehicle velocity by about 0.01 of the initial velocity. It is shown in this chapter that the Intermediate Phase may span an operating band of sufficient width that three, rather than two, separate segments of the trajectory should be considered in a piecewise analysis.

The investigation described in this chapter is unique in that three separate operating regions are suggested for guidance analysis; the boundaries between these operational phases are defined on the basis of the rate at which angular momentum and energy are being transferred from the vehicle to the planetary atmosphere. For this purpose, a Conservation Parameter is defined which is related directly to the rate of transfer of energy and angular momentum. The Conservation Parameter may be determined in flight from measurements of specific force levels and flight path angle. Quantitative estimates are given for the specific force level and altitude at the phase boundaries between the three operating regimes for vehicles with various lift and

---

* See Chapter 9

** Keplerian and Gas-Dynamic
Gas-dynamic forces exceed threshold value detectable by Specific Force Measuring Subsystem

Magnitude of the percentage change in distance of the entry vehicle from planet center is very much smaller (0.1 times) the percentage change in horizontal component of vehicle velocity with respect to inertial space.

Magnitude of the percentage change in distance of the vehicle from the planet center is approximately equal to the percentage change in horizontal component of vehicle velocity with respect to inertial space.

Key
- K: Keplerian Phase
- I: Intermediate Phase
- G: Gas-Dynamic Phase

Fig. 8.1: Phases of the Entry Trajectory and Definition of Boundary Conditions
drag characteristics.

The Keplerian Phase and the Gas-Dynamic Phase have been examined extensively in the current literature. The Intermediate Phase, on the other hand, has been largely ignored in trajectory studies. From the guidance standpoint, this is a very important phase, particularly in the degenerate orbital entry profile where perturbations in the Intermediate Phase in the vicinity of perigee are the mechanism by which eventual entry is induced. The Intermediate Phase is also important in the direct entry profile; this transition from Keplerian motion to Gas-Dynamic flight spans an altitude band of the order of 20 miles in the case of entry into the Earth's atmosphere*. The distance flown while in the Intermediate Phase may be a significant portion of the total vehicular range.

8.2 Conservation of Energy and Angular Momentum

Angular momentum was defined in equation (4-1) as:

\[ P = RV_I\phi \]  

(4-1)

The time rate of change of angular momentum is therefore:

\[ \dot{P} = \dot{RV}_I\phi + \dot{R}V_I\phi \]  

(8-1)

Defining the following auxiliary parameter:

\[ \xi = -\frac{\dot{R}}{V_I\phi} \]  

(8-2)

* Assuming constant flight path angle and constant drag coefficient. If the drag coefficient is reduced by a factor of \( \frac{1}{2} \) during the course of this phase, such as might occur if the flow regime changes from free molecular to continuum, the altitude band of this phase increases to approximately 24 miles.
With equation (8-2), equation (8-1) is written:

\[ \dot{P} = \frac{(1-\xi)}{\xi} \cdot RV_I \phi \]  

(8-3)

Using the non-dimensionalizing procedure described in Chapter 6, equation (8-3) is written in dimensionless form as follows:

\[ P' = \frac{(1-\xi)}{\xi} r' v_I \phi \]  

(8-4)

The Conservation Parameter was defined as follows:

\[ (\text{Conservation Parameter}) \equiv \left| \frac{1-\xi}{\xi} \right| \]  

(8-5)

The name Conservation Parameter was chosen because of the close relation of this quantity to conservation of energy and angular momentum.

The magnitude of the rate of transfer of angular momentum from the vehicle to the atmosphere is the product of the Conservation Parameter, the rate of change of altitude, and the horizontal component of velocity with respect to inertially fixed coordinates. During the Intermediate Phase, the change of velocity is small, hence the \( v_I \phi \) term in equation (8-4) is nearly constant in this phase.

Total energy per unit mass was defined in equation (4-10) as:

\[ \frac{E_{\text{tot}}}{M} = -\frac{\gamma g M_o}{R} + \frac{\overline{V}_I^2}{2} \]  

(4-10)

In dimensionless form, this was written in equation (6-54) as:

\[ E_{\text{(tot)}} = \frac{v_I^2}{2} - \frac{1}{r} \]  

(6-54)

Equation (6-55) showed that the rate of change of total energy may be expressed as:
\[ E'(\text{kin}) + E'(\text{pot}) = \vec{f} \cdot \vec{v} = r f_r + v I \phi \] (8-6)

where \( f_r \) and \( f_\phi \) are the radial and horizontal components of external specific forces in mean surface g's of the planet.

Differentiating equation (6-54), substituting into equation (8-6), and rearranging gives:

\[ 1 + \frac{E'(\text{kin})}{E'(\text{pot})} = r^2 \left[ f_r + \frac{v_I \phi}{r} f_\phi \right] \] (8-7)

It was shown in Chapter 6 that for a spherical planet:

\[ f_\phi = \frac{\dot{v}_I \phi}{\dot{R}} + \frac{R}{R} v_I \phi \] (8-8)

Equation (8-8) is written in dimensionless form as:

\[ f_\phi = (1 - \xi) \frac{v'_I \phi}{\xi} \] (8-9)

Therefore, equation (8-7) may be written:

\[ 1 + \frac{E'(\text{kin})}{E'(\text{pot})} = r^2 \left[ f_r - \left( \frac{1 - \xi}{\xi} \right) \frac{v_I \phi}{r} \right] \] (8-10)

Equation (8-10) shows that in a Keplerian trajectory (\( f_r = 0 \), \( \frac{\phi}{r} = +1.0 \)):

\[ E'(\text{kin}) = - E'(\text{pot}) \] (8-11)

This corresponds to a continuous trade-off of kinetic and potential energy throughout the trajectory; total energy remains constant. For small flight path angles, the radial component of external specific force is approximately equal to \( n_L \) and the horizontal component is
approximately equal to \(-n_D\). The predominant term in the bracket on the right-hand-side of equation (8-7) is \(\frac{v_I \phi}{r'}\) because \(r'\) is generally very much smaller than \(v_I \phi\) until near the terminal phase of the mission. Therefore, in the vicinity of atmospheric perturbations where velocities are generally high and flight path angles are relatively small, equation (8-10) reduces to the following approximate form:

\[
1 + \frac{E'_\text{(kin)}}{E'_\text{(pot)}} = -\left(1 - \frac{E}{E}\right) \frac{v^2_I \phi}{r} \tag{8-12}
\]

Equation (8-12) is plotted in Fig. 8.2. The Keplerian Phase of the trajectory corresponds to \(\xi\) in the vicinity of +1 while the Gas-Dynamic Phase corresponds to \(\xi\) in the vicinity of zero. All other values of \(\xi\) correspond to flight in the Intermediate Phase. In the Keplerian Phase, the rate of change of kinetic energy is equal to the negative of the rate of change of potential energy. In the Gas-Dynamic Phase, the magnitude of kinetic energy transfer rate is very much greater than the magnitude of the potential energy transfer rate.

### 8.3 Relating the Flight Trajectory and the Conservation Parameter

Gas-Dynamic forces become significant, from the guidance standpoint, when they are of sufficient magnitude to be detectable by accelerometers carried within the vehicle. The trajectory is described by a pure Keplerian transfer ellipse as long as the angular momentum of the entry trajectory is conserved. Under these conditions, \(F_\phi = 0\) in equation (8-8); therefore:

\[
\frac{\dot{v}_I \phi}{v_I \phi} = -\frac{R}{R} \tag{8-13}
\]

or
Fig. 8.2: Kinetic-Potential Energy Transfer Rate as Function of Ratio of Percentage Change in Radial Distance of the Entry Vehicle from the Planet Center to the Percentage Change in Horizontal Velocity.
Equation (8-14) shows that in order for angular momentum of the entry trajectory to be conserved, the percentage change in distance of the entry vehicle from the center of the planet is equal to the negative of the percentage change in horizontal component of vehicle velocity with respect to inertial space.

Angular momentum is not conserved when external forces (e.g., lift, drag, and thrust forces) exist in the $\vec{r}$ direction. The auxiliary parameter $\xi$ was defined in equation (8-2). The magnitude of $\xi$ is an expression of the ratio of the percentage change in radial distance to the percentage change in horizontal component of vehicle velocity with respect to fixed inertial coordinates. $\xi$ must be equal to unity for conservation of angular momentum.

During the course of entry, the magnitude of the ratio of percentage change in radial distance to the percentage change in horizontal velocity will decrease toward zero as gas-dynamic forces become predominant factors in specifying the ultimate path of the vehicle. Hence, the magnitude of $\xi$ is an indication of the degree to which angular momentum of the trajectory is conserved; as $\xi$ becomes very much smaller than 1.0, angular momentum is transferred rapidly to the gaseous envelope of the atmosphere through the mechanism of lift and drag.

From equation (8-9), the horizontal component of external specific force is:

$$f_\phi = -\left(1 - \frac{\xi}{\xi}\right) \frac{r'}{r} v_I\phi$$

(8-15)

In powerless flight, $f_\phi$ results entirely from lift and drag gas-dynamic
forces:

\[ f_\phi = - (n_D \cos \gamma + n_L \sin \gamma) = - n_D \cos \gamma (1 + (L/D) \tan \gamma) \quad (8-16) \]

For small flight path angles, \( f_\phi \) is always negative*. With the left side of equation (8-15) negative, the following conditions are required of the right hand side:

- If \( V_{f\phi} < 0 \): \( \xi \) must be less than +1.0 or must be negative.
- If \( V_{f\phi} > 0 \): \( \xi \) must be greater than +1.0.
- If \( R < 0 \) (i.e., descending flight): \( \xi \) must be greater than +1.0 or it must be negative.
- If \( R > 0 \) (i.e., ascending flight): \( \xi \) must lie between 0 and +1.0.

Fig. 8.3 shows the resulting flight regimes for a vehicle entering the planetary atmosphere. Boundaries separating the various phases of the trajectory are shown on Fig. 8.3 in terms of the auxiliary parameter \( \xi \). It is noted that a true Keplerian description of the trajectory applies only for \( \xi = 1.0 \); it was assumed in this analysis that the Keplerian description of the trajectory is adequate for small departures of \( \xi \) from 1.0 (i.e., 0.9 < \( \xi \) < 1.1).

Fig. 8.3 can best be visualized as a cylindrical graph; i.e., the points at \( \xi = \pm \infty \) should be connected together. The Keplerian

* For descending flight (i.e., \( \gamma \) negative), it is seen from equation (8-16) that \( f_\phi \) is positive only when \(|(L/D) \tan \gamma| > 1.0\), a condition which could arise only for relatively large negative flight path angles with a vehicle having a relatively high lift-drag ratio.
Fig. 8.3: Flight Regimes of Entry Trajectory

Key:

K: Keplerian Phase
I: Intermediate Phase
G: Gas-Dynamic Phase
Phase is a small region in the neighborhood of $\xi = +1.0$. The region within $\pm 0.1$ of $\xi = 1.0$ is the region in which the gas-dynamic terms in the equations of motion are negligibly small in comparison to the other terms; i.e., the terms involving $n_L$ and $n_D$ in equations (6-47) and (6-48) are zero for $\xi = 1.0$ and are negligible for $0.9 < \xi < 1.1$. The Gas-Dynamic Phase, on the other hand, is a small region in the vicinity of $\xi = 0$. In the band $-0.1 < \xi < +0.1$, the $\frac{v_r v_\phi}{r}$ term in equation (6-48) is less than 0.1 of the lift and drag terms and can reasonably be ignored. The Intermediate Phase spans all other values of $\xi$ not included in the Gas-Dynamic Phase and the Keplerian Phase. In the Intermediate Phase, all terms in the equations of motion must be retained for a reasonably accurate guidance analysis of entry.

The fact that the parameter $\xi$ has a sharp behavior during the course of an entry mission can be inferred from Fig. 8.3. For this purpose, the $R$ and $v_{I\phi}$ bands shown in this figure are instructive. $\xi$ lies between zero and 1.0 for climbing flight; i.e., $\gamma$ is positive in this region. For $\xi$ negative and for $\xi > +1.0$, the vehicle is in descending flight ($\gamma =$ negative). The vehicle is always slowing down ($v_{I\phi} < 0$) in ascending flight; if it is slowing down at the same time it is descending, then $\xi$ must be negative. The speed of the vehicle increases only in descending flight and only when $\xi > 1.0$. At these points where $v_{I\phi} = 0$, $\xi$ is $\pm \infty$. The behavior is sharp because $\xi$ moves from infinity to zero during the time interval between the instant that $v_{I\phi} = 0$ and $\dot{R} = 0$. In elliptical flight with small drag forces, for example, these two points occur very close to each other in the vicinity of perigee and apogee. They would occur simultaneously in
pure vacuum flight. In the vicinity of perigee when small drag forces exist, \( \dot{V}_{\text{I} \phi} \) goes to zero shortly before \( \dot{R} \) goes to zero. In the vicinity of apogee if some drag exists, \( \dot{V}_{\text{I} \phi} \) goes to zero a very short time after \( \dot{R} = 0 \).

To illustrate the behavior of \( \xi \) as a consequence of the foregoing conditions, it is instructive to consider a couple of simple examples. Fig. 8.4 shows a typical braking pass in the planetary atmosphere such as occurs in the vicinity of perigee during the unstable orbital profile. When the vehicle is approaching the planet at a very high altitude, gas-dynamic forces are negligible and \( \xi \) is near 1.0. In this, the Keplerian region, the vehicle is descending \( (\dot{R} < 0) \) and velocity is increasing \( (\dot{V}_{\text{I} \phi} > 0) \) due to kinetic-potential energy trade-off. As drag forces increase, \( \xi \) increases to 1.1 when the Keplerian-Intermediate Phase boundary is crossed; velocity is still increasing due to energy trade-off, but the rate of increase decreases as drag forces become more pronounced. Ultimately in the Intermediate Phase, the effects of drag and the effects of energy trade-off cancel; at this point \( \dot{V}_{\text{I} \phi} = 0 \) and \( \xi \) passes from \(+\infty\) to \(-\infty\). At some later time perigee is crossed \( (\dot{R} = 0) \); \( \xi \) has increased rapidly from \(-\infty\) to its value of zero at perigeeal passage. As the vehicle now ascends through the outer reaches of the atmosphere, \( \dot{R} \) is positive and \( \dot{V}_{\text{I} \phi} \) is negative; here \( \xi \) continues to increase as atmospheric drag forces become less significant until at about \( \xi = 0.9 \), the vehicle is again effectively in Keplerian flight. During this braking pass, a pulse of energy and a pulse of angular momentum are transferred to the planetary atmosphere.

Fig. 8.5 shows a gliding direct entry profile. Unless skipping
Fig. 8.4: Behavior of $\xi$ Parameter During a Braking Pass

Key
K: Keplerian Phase
I: Intermediate Phase
G: Gas-Dynamic Phase

Planetary Atmosphere
Perigee ($R=0$)

$V_{I\phi} > 0$
$R < 0$

$V_{I\phi} < 0$
$R > 0$
Fig. 8.5: Behavior of $\xi$-Parameter During the Direct Entry Profile

Key:
- K: Keplerian Phase
- I: Intermediate Phase
- G: Gas-Dynamic Phase
motion is induced by the planetary atmosphere*, the radius is always decreasing; i.e., \( \dot{R} < 0 \). During the Keplerian portion of flight and during part of the Intermediate Phase, velocity is increasing due to kinetic-potential energy trade-off. \( \dot{\xi} \) increases from its characteristic value of 1.0 in Keplerian flight to \(+\infty\) as drag forces increase. When drag forces reduce \( \dot{V}_I \) to zero, \( \dot{\xi} \) moves from \(+\) to \(-\) at infinity. \( \dot{\xi} \) increases from \(-\infty\) toward zero as gas-dynamic drag builds up with altitude loss; when \( \dot{\xi} \approx -0.1 \), the vehicle enters the Gas-Dynamic phase and remains in this phase of flight thereafter.

Equations (8-4), (8-12) and (8-15) expressed rate of change of angular momentum, energy, and horizontal force level as functions of \( \dot{\xi} \). In all cases, \( \dot{\xi} \) appears in the combination \( \left( \frac{1-\xi}{\xi} \right) \). The magnitude of this combination was defined in this thesis as the Conservation Parameter; this is a convenient quantity for defining the boundary conditions between phases. At the Keplerian-Intermediate Phase boundary, the Conservation Parameter is approximately equal to 0.1; and at the Intermediate-Gas-Dynamic phase boundary, the Conservation Parameter is approximately equal to 10. The information of Fig. 8.3 is shown on Fig. 8.6 as a function of \( \left( \frac{1-\xi}{\xi} \right) \). From this figure it is seen that the vehicle is in ascending flight when \( \left( \frac{1-\xi}{\xi} \right) \) is positive and in descending flight when this quantity is negative. \( \dot{V}_I \) is negative in all regimes except \(-1.0 < \left( \frac{1-\xi}{\xi} \right) < 0\); i.e., the horizontal velocity of the vehicle can increase only in the Keplerian Phase and during a portion of the Intermediate Phase.

* See Chapter 9 for a discussion of skipping flight.
Fig. 8.6: Flight Regimes of Entry Trajectory as a Function of the Conservation Parameter
8.4 Altitude Description of Boundaries Between Operational Phases of the Entry Trajectory

The horizontal component of specific force was written as a function of the Conservation Parameter in equation (8-15) and as a function of lift and drag forces in equation (8-16). Eliminating $f_\phi$ from these two equations gives:

$$n_D \cos \gamma + n_L \sin \gamma = \left(\frac{1-\frac{\xi}{\xi}}{\xi}\right) \frac{r}{r} \nu \psi$$

(8-17)

The following relations were given in Chapter 6 for the dimensionless quantities required in Equation (8-17) (assuming the exponential atmospheric model and no atmospheric winds):

$$n_D = \left(\frac{C_D}{M}\right) \frac{\rho_{(5L)}}{2} R(m)_0^2 \nu e^{-kh}$$

(8-18)

$$n_L = n_D (L/D)$$

(8-19)

$$r' = \nu \sin \gamma$$

(8-20)

$$\nu \psi = \nu \cos \gamma + r \nu \cos \psi$$

(8-21)

Substituting equations (8-18) through (8-21) into equation (8-17) gives:

$$\frac{R(m)_0 \rho_{(5L)}}{2} \frac{S_e}{M} e^{-kh} \left[\frac{C_D}{\sin \gamma} + \frac{C_L}{\cos \gamma}\right] = \left(\frac{1-\frac{\xi}{\xi}}{\xi}\right) \frac{1}{r} + \frac{\nu \cos \psi}{\nu \cos \gamma}$$

(8-22)

This is written in logarithmic form as follows:

$$K_{eo} h + C_e + 2.59 \times 10^{-3} \log \left[\frac{1}{r} + \frac{\nu \cos \psi}{\nu \cos \gamma}\right]$$

$$= 2.59 \times 10^{-3} \left\{ \log \left(5.64 \times 10^4 \frac{M}{S}\right) + \log \left[\frac{C_D}{\sin \gamma} + \frac{C_L}{\cos \gamma}\left(\frac{\xi}{\xi}\right)\right] \right\}$$

(8-23)
where

\[ C_e = - \frac{1}{k_E} \log_e \left( \frac{R(m)\omega}{R(m)E} \right) \]  

(8-24)

The following table gives values of quantities on the left side of equation (8-23):

<table>
<thead>
<tr>
<th></th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{EO} )</td>
<td>1.14</td>
<td>1.0</td>
<td>0.207</td>
</tr>
<tr>
<td>( C_e )</td>
<td>(-2.77 \times 10^{-3})</td>
<td>0</td>
<td>(3.87 \times 10^{-3})</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>(2.69 \times 10^{-4})</td>
<td>(5.87 \times 10^{-2})</td>
<td>(6.75 \times 10^{-2})</td>
</tr>
</tbody>
</table>

The left side of equation (8-23) expresses dimensionless altitude directly for the Earth (for \(v > 0.1\)). It is plotted as "altitude function" on subsequent graphs. At very low velocities, the term on the left side of this equation involving planetary rotation becomes significant. For example:

If: \(v \cos \gamma = 0.01\)

\[ \psi = 0^\circ \text{ (equatorial trajectory)} \]

\[ \frac{1}{r} \approx 1.0 \]

\[ 2.59 \times 10^{-3} \log \left( \frac{1}{r} + \frac{\Omega \cos \psi}{v \cos \gamma} \right) \approx 2.16 \times 10^{-3} \]

\[ \approx 3.7 \text{ miles (Earth)} \]

At velocities actually existing in the altitude range for which equation
(8-23) is useful (i.e., in specifying boundaries of the Intermediate Phase), the third term on the left side is negligible.

In order to illustrate the behavior of equation (8-23), the vehicle with \((L/D)_{\text{max}} = 2.0\) listed in Table 5.1 was plotted on Fig. 8.7. Mass characteristics chosen for this graph was \(M/S = 1.0\) slugs/ft\(^2\). The effect on the location of the resulting curves for other \(M/S\) is indicated in the figure. Four separate flight path angles were selected; these ranged in value from \(\gamma = \pm 0.01^\circ\) to \(\pm 10^\circ\). The effects of varying the angle of attack from zero to the value corresponding to \((L/D)_{\text{max}}\) is shown for each flight path angle.

The curves of Fig. 8.7 were plotted for constant flight path angles. In general, the trajectory of the lifting entry vehicle will be characterized by a changing flight path angle as it passes through the Intermediate Phase, hence the true altitude band in this phase may be greater or less than that indicated by a constant \(\gamma\) curve. For example, if the vehicle has a flight path angle of \(-10^\circ\) at an altitude of \(15 \times 10^{-3}\) on Fig. 8.7, it corresponds to a near Keplerian trajectory (abscissa near 1.0). If the magnitude of the flight path angle rapidly decreases to \(-0.01^\circ\), as in the early phase of a skip, the trajectory (at the same altitude) may be described as in the Gas-Dynamic Phase (abscissa less than 0.1). These figures show that for a given flight path angle, angular momentum is essentially conserved down to an altitude that is strongly dependent on the characteristics of the vehicle and on the flight path angle. Once the vehicle has descended to this altitude, angular momentum rapidly is transferred to the atmosphere. The altitude range over which the abscissa of Fig. 8.7 moves from approximately 1.0 to approximately 0.1 is called herein the
Fig. 8.7: Behavior of $\zeta$ -Parameter as a Function of Altitude at Various Flight Path Angles for $(L/D)_{\text{max}} = 2.0$

Note: Angular Momentum is Conserved at abscissa = 1.0
Intermediate Phase. Below this altitude, the trajectory is described as in the Gas-Dynamic Phase.

At flight path angles with magnitudes less than 10°, the upper and lower altitude bounds of the Intermediate Phase is a very weak function of lift coefficient. This weak dependence arises from the fact that (at $\gamma$ small) the horizontal component of external specific force is made up almost entirely of gas-dynamic drag. This weak dependence on lift was verified by plotting curves similar to Eq. 8.7 for many classes of vehicles with and without the lift term included. The lift term generally caused such slight displacement of the phase boundaries as to be virtually unperceptible. Lift terms become significant only with large lift-drag ratios at flight path angles greater than five to ten degrees.

With the weak dependence of the locus of the Intermediate Phase boundaries on lift having been established, Fig. 8.8 was plotted to show the phase boundaries for various drag parameters.

The curves of Fig. 8.8 show the Keplerian-Intermediate phase boundary and the Intermediate-Gas-dynamic phase boundary for four separate flight path angles, $\gamma = \pm 0.01^\circ, 0.1^\circ, 1^\circ,$ and $10^\circ$. The altitude band spanned by the Intermediate Phase at a given drag parameter and a given flight path angle is approximately 25 miles for Venus, 20 miles for Earth, and 10 miles for Mars.

8.5 Description of Boundaries Between Operational Phases of the Entry Trajectory in Terms of the Horizontal ($\hat{1}_\phi$) Component of Specific Force

The horizontal component of the specific forces measured by an accelerometer oriented in the $\hat{1}_\phi$ direction was given in equation (8-16).
Fig. 8.8: Phase Boundaries of Intermediate Phase for Various Drag Parameters
Using equations (8-18) and (8-19) this is written:

\[
f_\phi = -\left(\frac{c_D s}{M}\right)\frac{\theta(S)}{t^2} R(m)_o \quad v^2 e^{-kh} (1 + \frac{L}{D \tan \gamma}) \cos \gamma \quad (8-25)
\]

In order to express \( f_\phi \) in Earth g's, the following is used:

\[
f_{E\phi} = \frac{G(m)_o}{G(m)_E} f_\phi \quad \text{Earth g's} \quad (8-26)
\]

Using equation (8-26), equation (8-25) is written in logarithmic form as follows:

\[
K_{E0} h + C_e + C_g = 2.59 \times 10^{-3} \left[ \log (1 + \frac{L}{D \tan \gamma}) + \log \left(\frac{5.64 \times 10^4}{M/S}\right) + \log C_D \cos \gamma - \log \left(\frac{2 f_{E\phi}}{v^2}\right) \right] 
\]

(8-27)

where \( C_e \) is given by equation (8-24) and:

\[
C_g = \frac{1}{k_E} \log \left( \frac{G(m)_o}{G(m)_E} \right) 
\]

(8-28)

The following table lists values of quantities given on the left side of equation (8-27).

| Table 8.2: Conversion Quantities for use in Equation (8-27) |
|-------------------------------|------------------|------------------|
| \( K_{E0} \)                | \( \text{Venus} \) | \( \text{Earth} \) | \( \text{Mars} \) |
| \( C_e \)                   | \(-2.77 \times 10^{-3}\) | 0                | \(3.87 \times 10^{-3}\) |
| \( C_g \)                   | \(0.0342 \times 10^{-3}\) | 0                | \(0.713 \times 10^{-3}\) |

It is noted that the left side of equation (8-27) expresses dimensionless altitude directly, in the case of the planet Earth.
Equation (8-27) is plotted on Fig. 8.9 for the classes of vehicles listed in Table 5.1 with \( M/S = 1.0 \) slug/\( ft^2 \) and for a zero-lift vehicle with \( C_D S/M = 20 \). For small flight path angles and moderate lift-drag ratios, the curves are essentially independent of \( \gamma \). The curves are drawn for values of \( C_D \) and \( C_L \) at \( (L/D)_{\text{max}} \) for the lifting vehicles.

Equation (8-15) may be written as follows:

\[
I_\phi = - (\frac{1 - \xi}{\xi}) v^2 \sin \gamma \cos \gamma \left[ \frac{1}{r} + \frac{\Omega \cos \psi}{v \cos \gamma} \right] \tag{8-29}
\]

At the altitudes and velocities which are important at the boundaries of the Intermediate Phase, the quantity in square brackets in equation (8-29) is approximately equal to 1.0. Therefore:

\[
I_\phi \cong - (\frac{1 - \xi}{\xi}) v^2 \sin \gamma \cos \gamma \cong (\frac{1 - \xi}{\xi}) v^2 \gamma \tag{8-30}
\]

Equation (8-29), and its simplified version (8-30), show the interesting result that transition from Keplerian flight through the Intermediate Phase to Gas-Dynamic flight is independent of the surface loading and lift-drag characteristics of the vehicle. The magnitude of the horizontal component of specific force in transition depends only on velocity and flight path angle. Hence, if flight path angle, velocity, and horizontal specific force are measured or computed by the guidance system, then the magnitude of the Conservation Parameter can be specified; this is an index of the degree to which angular momentum is being conserved and is indicative of whether the vehicle is in the Keplerian, Intermediate, or Gas-Dynamic Phases of the trajectory.

Equation (8-30) is plotted in Fig. 8.10 to show the boundaries between the various operational regimes as a function of the horizontal specific force measured by the accelerometers of the vehicle. This
Fig. 8.9: Horizontal Specific Force vs Altitude
figure shows that the force level changes by two orders of magnitude in traversing the Intermediate Phase and that the specific force level at transition between phases is a sensitive function of flight path angle.

The horizontal specific force level cannot give accurate information on the relative rate of transfer of angular momentum from the vehicle to the planetary atmosphere without accurate specification of flight path angle, which is tantamount to requiring precision measurements of vertical velocity (altitude rate).
Fig. 8.10: Level of Horizontal Specific Force at Boundaries Between Operational Phases as a Function of Flight Path Angle.
Chapter 9

APPROXIMATE ANALYTICAL SOLUTION OF GUIDANCE PARAMETERS AND CONSTRAINTS FOR THE DIRECT ENTRY PROFILE; RANGE SENSITIVITY TO ERRORS IN CONTROL SYSTEM OPERATION

9.1 Introduction

The investigation described in Chapter 8 demonstrated that the trajectory of a vehicle entering a planetary atmosphere from an initial point beyond the sensible atmosphere traverses three distinct operational regimes; these regimes are called in this thesis, for convenience in identification, the Keplerian, Intermediate, and Gas-Dynamic Phases.

Studies of entry vehicle dynamics in the past have largely ignored the existence of the Intermediate Phase as a separate entity. In some trajectories, however, the Intermediate Phase may be the most important single phase in specifying the ultimate destination of the vehicle. As an example, consider the degenerate orbit of Fig. 9.1. The trajectory of Fig. 9.1 consists of a series of "braking passes" through the outer reaches of the atmosphere; energy is transferred to the planetary atmosphere during each perigeeal passage. During most of each orbit, the vehicle obeys Kepler's laws of planetary motion to a high degree of accuracy. However, in the vicinity of perigee, the vehicle enters the Intermediate Phase of operation momentarily and returns again to the
Fig. 9.1: The Degenerate Orbit of a Vehicle About a Planet Which Possesses an Atmosphere
Keplerian Phase. As a result of the impulses in the Intermediate Phase each successive apogee occurs at a lower altitude and eventual entry is assured. If the pilot desires to control the number of orbits remaining, he may change the drag characteristics of the vehicle during these impulses in the Intermediate Phase to bring about increased or decreased total energy transfer per orbit, and therefore shorten or lengthen, as desired, the flight time remaining. The fact that the vehicle enters the Intermediate Phase during trajectories such as sketched in Fig. 9.1 makes eventual entry possible without the use of external thrust; otherwise, the flight would continue indefinitely.

The Conservation Parameter $\left| \frac{1 - \epsilon}{\sqrt{\epsilon}} \right|$ was selected as convenient for defining boundaries between the Keplerian, Intermediate, and Gas-Dynamic Phases of the trajectory. $\bar{\epsilon}$ was defined as the negative of the ratio of percentage change in radial distance of the entry vehicle from the planet center to the percentage change in horizontal component of vehicle velocity with respect to inertial coordinates.

The phases of the entry trajectory were defined in Chapter 8 in terms of the Conservation Parameter as follows:

1. **Keplerian Phase:**
   A true Keplerian trajectory exists when $\bar{\epsilon} = 1.0$. The trajectory is near-Keplerian when
   \[
   \left| \frac{1 - \epsilon}{\sqrt{\epsilon}} \right| < 0.1
   \]

2. **Intermediate Phase:**
   \[
   10 > \left| \frac{1 - \epsilon}{\sqrt{\epsilon}} \right| > 0.1
   \]

3. **Gas-Dynamic Phase:**
   \[
   \left| \frac{1 - \epsilon}{\sqrt{\epsilon}} \right| > 10
   \]
The phase boundary between the Keplerian and Intermediate Phases was defined as \( \left| \frac{1-e_{\text{Kepler}}}{e_{\text{Kepler}}} \right| = 0.1 \); and between the Intermediate and Gas-Dynamic Phases as \( \left| \frac{1-e_{\text{Gas-Dyn}}}{} \right| = 10.0 \). It is noted that the Conservation Parameter changes by two orders of magnitude in spanning the Intermediate Phase.

Methods of altering the trajectory of the entry vehicle in accordance with guidance commands may be vastly different in each operational regime. In a pure Keplerian orbit, perturbations may be introduced by the use of rocket or ion thrust or by means of external pressure techniques, such as through streaming a solar radiation sail. Thrust forces may be used equally well in the Intermediate and Gas-Dynamic Phases.

Varying the lift and drag coefficients*, however, has no effect on the Keplerian trajectory but may have a striking influence on the trajectory in the Intermediate and Gas-Dynamic operational phases.

The investigation described in subsequent sections of this chapter considers each phase of the entry trajectory separately. Solutions obtained are continuous or piecewise-continuous within each phase; the Conservation Parameter serves as a unifying quantity for matching solutions at boundaries and for relating the solution of one phase to the solutions of other phases. The accuracy of approximations made in each

* Lift and drag coefficients may be controlled by changing the angle of attack or by introducing auxiliary high-lift or high-drag devices. Since lift and drag coefficients are coupled through angle of attack, auxiliary devices must be used if independent lift and/or drag control is necessary. It is important to note that the coupling of lift and drag with angle of attack is different depending on which side of the lift curve the vehicle is operating. For example, the vehicle may be flown in the "high drag configuration"; i.e., angle of attack near 90°. Decreasing the angle of attack reduces drag and increases lift in this configuration. In the "low drag configuration" (small angles of attack), however, reduction in angle of attack causes both lift and drag to decrease.
phase are discussed, and the accuracy of the solutions derived are compared to numerical solutions obtained from the non-linear equations of motion. Limitations of the approximate solutions were established as a result of this comparison.

The feasibility of using the Conservation Parameter as a switching function for the guidance system and as a prediction function for range control in degenerate orbits is discussed. In-flight computation of the Conservation Parameter from navigational data is shown to be possible.

The solutions of this chapter are oriented toward the determination of guidance quantities. It may be noted that guidance considerations of this chapter are relatively unsophisticated, however, consisting primarily of determination of range capabilities and sensitivity in range to control system errors. The hazards of exceeding tolerable acceleration and temperature levels are considered.

9.2 The Conservation Parameter as a Switching Function:

Considerable simplification is possible in the design of the guidance system if:

1. Computations are based on simplified* equations of motion.
2. An adequate means is available to shift from one simplified guidance mode to another as phase boundaries are crossed.

The switching parameter should be based on in-flight measurements and/or computations.

At any given time as the entry vehicle passes through the planetary atmosphere, the dominant terms in the equations of motion depend upon

---

* Provided reasonable accuracy is not sacrificed in the process of simplifying the equations.
the phase of the trajectory in which the vehicle is operating. The parameter \( \frac{1 - \frac{v}{v_c}}{r} \) may be a convenient switching function to shift the guidance computer from one set of simplified equations (Keplerian Phase), through the region in which few simplifications in the dynamical equations of motion are permitted (Intermediate Phase), into the region where another set of simplified equations provide an adequate description of the trajectory (Gas-Dynamic Phase).

It was shown in Chapter 8 that:

\[
  f_E \phi = - \frac{G(m)_o}{G(m)_E} \left(1 - \frac{G(m)_o}{r}\right) \frac{v^2}{r} \sin \gamma \cos \phi \tag{9-1}
\]

where

- \( f_E \phi \) = tangential component of specific force (in Earth G's) measured by accelerometers carried by the vehicle.
- \( v \) = velocity of the vehicle with respect to coordinates rotating with the planet (non-dimensionalized with respect to circular satellite velocity at the surface of the planet).
- \( \gamma \) = flight path angle; angle between the local horizon and \( v \) -- positive for climbing flight.
- \( r \) = distance from center of planet to vehicle, non-dimensionalized with respect to mean planetary radius.
- \( \frac{G(m)_o}{G(m)_E} \) = ratio of mean gravitational acceleration at the surface of the planet 0 to that at the surface of the Earth.

Equation (9-1) shows that if flight path angle, velocity, altitude and tangential specific force are measured or computed by the guidance system, it is an easy matter for the system to determine \( \frac{1 - \frac{v}{v_c}}{r} \), which is a measure of the degree to which energy and angular momentum are being conserved and is indicative of whether the vehicle is in the Keplerian, Intermediate, or Gas-Dynamic Phases of the trajectory. Equation (9-1) is significant because it demonstrates that the specific force level at
transition between phases is independent of the lift-drag characteristics of the vehicle, the frontal loading of the vehicle (M/S), and the atmospheric density variation with altitude.

The variation of atmospheric density with altitude is not known or measurable in advance to a high degree of accuracy, particularly for the first-time entry into the atmosphere of a strange planet. Atmospheric density may vary more than two to one between day and night, may vary significantly with latitude, and may change considerably with the seasons of the year. The specification of the aerodynamic properties of the vehicle may also be subject to error. Determination of aerodynamic characteristics at near-orbital velocities is presently not experimentally feasible; these properties are generally specified on a theoretical rather than an experimental basis.

Equation (9-1) has serious disadvantages as a suitable means for computing \( \frac{1 - \frac{6}{\alpha}}{3} \). The direction of the horizon must be known in order to measure the horizontal component of specific force. An extremely troublesome factor is \( \sin \gamma \), which is small for small flight path angles. If \( \gamma \) is measured as 0.1°, but is actually 1°, then the computed value of \( \frac{1 - \frac{6}{\alpha}}{3} \) is off by a factor of ten. Strong dependence on flight path angle is the most serious handicap on the practical use of equation (9-1).

A method for computing the Conservation Parameter that is independent of the sine of flight path angle was derived in Chapter 8 by relating it to Energy transfer rates:

\[
\frac{E'_{\text{kin}}}{E'_{\text{pot}}} = \frac{1 - \frac{6}{\alpha}}{3} \frac{V^2}{I_0} \rho_r - 1 \quad (9-2)
\]

In equation (9-2):

\[
E_{\text{kin}} = \text{dimensionless kinetic energy per unit mass} = \frac{V_x^2}{2}
\]
\[ v_1 = \text{dimensionless velocity of the vehicle with respect to inertial coordinates.} \]

\[ E_{(pot)} = \text{dimensionless potential energy per unit mass } = \frac{GR}{(Gm_o Rm_o)} \]

(zero level at infinity).

\[ R = \text{radius from center of planet to vehicle} \]

\[ G = \text{local gravitational acceleration at vehicle.} \]

The prime denotes differentiation with respect to dimensionless time \( \tau \).

Equation (9-2) has all the advantages of equation (9-1), independence of atmospheric density and vehicle aerodynamic characteristics, plus the additional strong advantage of being essentially independent of flight path angle. The energy quantities in equation (9-2) depend on velocity, altitude, and gravitation; these quantities may be measured or computed by the navigation system from data obtained either within the vehicle or from external tracking stations.*

9.3 The Conservation Parameter as a Prediction Function

One of the problems encountered in the guidance of vehicles undergoing braking passes through the atmosphere is control of the point at which final entry is to be initiated. A vehicle entering the atmosphere of a strange planet from an interplanetary transfer ellipse may make many braking passes through the atmosphere prior to final entry. A

* For first-time entry into a planetary atmosphere, a parent satellite from which the entry vehicle is launched appears to be the best external source of navigational information in the absence of ground-based tracking stations. If there is no parent satellite, then a navigational satellite may be left in the original orbit prior to initiating entry.
similar trajectory may result if the vehicle is launched from the surface of a planet for the purpose of orbiting a few turns and re-entering. The latter trajectory pattern is planned for the Mercury "man in space" program scheduled for operational firings in the early 1960's.

The most significant characteristic of the degenerate elliptical orbit is that the perigeal altitude remains essentially constant* while the apogeal altitude drops after each pass through the atmosphere. The vehicle transfers energy to the planetary atmosphere in the vicinity of perigee during each pass until the total energy level is reduced to a point where further orbits may not persist.

General characteristics of the time-varying nature of altitude, total energy, and the Conservation Parameter are shown in Fig. 9.2. This figure does not represent quantitatively any particular trajectory, but it does contain all the features of a typical trajectory. During most of each orbit, the vehicle obeys Kepler's laws of planetary motion to a high degree of accuracy. A near pulse of energy is transferred to the surrounding atmosphere as perigee is approached. Here, the vehicle enters the Intermediate Phase momentarily, then returns to the Keplerian Phase.

After each perigeal passage, the total energy level remains essentially constant for most of the following orbit at a lower level than that of the preceding orbit. The elliptical orbit consequently undergoes a "circularization" process. The apogee of the elliptical orbit is slowly reduced by drag, primarily exerted near perigee, eventually to become a circle. Thereafter, the circle decays spirally as the vehicle

* See Chapter 10 for a discussion of perigeal altitude decay rates.
Fig. 9.2: Altitude, Total Energy, and Conservation Parameter vs Time for a Typical Degenerate Orbit.
glides through the Gas-Dynamic Phase of flight to a landing. When the total energy level is reduced below that required for a circular orbit at perigeal altitude, final entry is inevitable.

It is interesting to examine the nature of the energy transfer. Since each succeeding perigee occurs at approximately the same altitude, the potential energy is nearly constant. Therefore, the energy transferred at perigee is basically kinetic energy. At the following apogee, however, kinetic energy is greater than it was at the preceding apogee since the velocity of the vehicle at apogee increases as the apogeal altitude is reduced. Therefore, the energy transferred during the perigean passage results in an increase in kinetic energy and a decrease in potential energy at the following apogee when compared to the preceding apogee. During the circularization process, the changes in levels of energy when viewed at perigee and apogee may be summarized as follows:

1. **At perigee**: Potential energy level remains constant; kinetic energy level decreases with each impulsive energy transfer.
2. **At apogee**: Potential energy level drops as altitude of apogee decreases; kinetic energy level rises.

The length of time that the vehicle is in the Intermediate Phase increases in successive perigean passes; this is due to the fact that:

1. The velocity of the vehicle is slower near perigee during each succeeding pass (less kinetic energy).
2. The trajectory continually becomes more shallow.

The energy transferred per orbit becomes greater as the orbit becomes more circular. The width* of the Intermediate Phase pulses on the graph

* More precisely, the area enclosed by the pulses. Practically all of the energy is transferred in the vicinity of perigee.
of Conservation Parameter vs. Time is an indication of the total energy transferred per orbit.

If the total energy level at the first perigee passage is computed, the amount of energy in excess of circular orbital energy (corresponding to the perigee altitude) is easily determined. The magnitude of the impulse of energy transferred during the next orbit (i.e., during next perigee passage) shows the rate of energy decay. Energy decays as a series of successively larger steps (see Fig. 9.2-b). For given vehicle drag characteristics, and for a particular initial energy level, the measurement of the magnitude of one energy step uniquely specifies all future steps (for constant altitude at perigee and constant drag characteristics). Therefore, the number of orbits remaining until the energy level has decayed to the circular orbital level may be predicted from measurements and computations during the first complete orbit.

Energy decay per orbital period may serve as a useful method for predicting how many orbits remain before final entry; i.e., the number of orbits until the onset of the Gas-Dynamic Phase of flight. With the relation between energy transfer rates and the Conservation Parameter of equation (9-2), the time-varying characteristics of the parameter \( \left( \frac{1 - \frac{L}{c}}{c} \right) \) may serve equally well as a prediction function for determining the number of orbits remaining.

Control of the number of orbits remaining may be implemented either by the use of thrust or by changing the drag characteristics of the vehicle. If no control action is taken by the pilot, the width of the Intermediate Phase pulses observed during successive orbits is an indication of how long the flight will last. The flight may be extended or shortened only through suitable control action within the limits of
thrust or drag modulation available.

A framework therefore has been established for the utilization of the Conservation Parameter for prediction and control of degenerate orbits based on its relation to energy transfer rates (Eq. 9-2) and predicated on the fact that knowledge of the energy decay per period and the initial energy level uniquely specifies the number of orbits remaining. Control of the number of orbits remaining is vested either in the engines of the vehicle or in the pilot's facility to vary the magnitude of the energy transferred during each braking pass by means of drag adjustments.

The range capability of the vehicle in the Gas-Dynamic Phase is small compared to the total range of the orbital phase. If the pilot changes the number of orbits remaining by one, through suitable control action, he has changed the total range of the vehicle more than the distance normally traversed throughout the entire Gas-Dynamic Phase. Range corrections in the Gas-Dynamic Phase may be adjudged vernier corrections when compared to range corrections in terms of changing the number of orbits.

The function of the guidance system in a vehicle which is undergoing the circularization process such as sketched in Fig. 9.1 may be summarized as follows:

1. To determine in advance a suitable geographic point for entry; i.e., the point for initiating the Gas-Dynamic Phase of flight.

2. To adjust the orbital characteristics of the vehicle in order to hit this entry point.

3. To make range corrections in the Gas-Dynamic Phase to correct for position errors in the initial entry point and to correct for range errors which may arise as a
result of perturbations from the nominal trajectory in the Gas-Dynamic Phase.

The geographic location of the entry point is determined by solving the trajectory backwards from the geographic location of the predetermined landing site. The maximum and minimum range capabilities of the vehicle in the Gas-Dynamic Phase may be determined from the approximate range expressions derived in section 9.6. The entry point is selected as a mean position between the limits imposed by backing off from the landing point the maximum and minimum ranges obtainable. If the actual entry point lies within the corridor thus prescribed, adequate control is available to the pilot to prevent overshooting or undershooting his destination.

9.4 The Keplerian Phase

Any method for guiding the entry vehicle to a selected geographical point on the surface of the planet must necessarily involve a perturbation of its original orbit. Because of the tremendous energy possessed by the vehicle in orbit, it is difficult to perturb the orbit greatly without investing a large portion of the total mass of the vehicle in rocket fuel.

The vacuum phase of the trajectory that follows the initiation of entry from a satellite orbit beyond the sensible atmosphere is a part of the general problem of transfer between orbits that has received considerable attention in the literature. The special nature of the entry problem, however, requires solutions that are not available in general treatises on the subject of transfers between orbits.

Excellent examples of published papers concerning investigations of transfer maneuvers between orbits are the works of Hohmann(48), Lawden(49),
and Battin\(^{(50)}\). Hohmann and Lawden have shown that the optimum transfer maneuver between elliptical orbits involves the application of impulsive forces at departure from one orbit and at arrival in the other orbit. Battin has shown that several important features of the transfer problem are basically three-dimensional in nature and that two-dimensional models are inadequate. Very little has been published concerning guidance requirements for the special problem of transfer from a satellite orbit to a particular landing point on the surface of the planet.

The development carried out herein is unique in that the transfer ellipse is solved in terms of geometric quantities at the trajectory modification point\(^*\). These quantities are:

1. Inertial flight path angle (see Figs. 9.3 and 9.4)
2. Altitude
3. Velocity

The solution is valid for entry from either circular or elliptical orbits.

In addition to the general solutions described above, special solutions were obtained for entry from circular orbits in terms of:

1. Velocity Impulse
2. Engine Gimbal Angle
3. Circular Orbital Velocity.

It should be noted that the relation between \{inertial flight path angle, altitude, and velocity\} at the trajectory modification point and \{velocity impulse, engine gimbal angle, and orbital velocity\} are uniquely defined only if the entry vehicle is launched from a

\* The trajectory modification point is that point in the original orbit at which entry is initiated through generation of thrust forces, see Fig. 9.3.
circular orbit. For transfer from an elliptical orbit, the relation between these two sets of quantities depends on the particular point in the original elliptical orbit where perturbations are introduced. The first set of quantities are more useful for generalized study of transfer from elliptical orbits, the second set for specialized study of transfer from circular orbits.

Results are presented in dimensionless form for general application to any planet. A generalized range expression is developed in terms of the foregoing quantities, and range sensitivity to errors in these quantities is discussed.

Various methods are available for perturbing the satellite orbit to bring about controlled entry:

1. Impulsive application of forces at departure point by means of chemical rockets or other high thrust propulsive systems.
2. Continuous application of low thrust (e.g., ion rockets).
3. Multiple impulses at intermediate* thrust levels.
4. Drag modulation, if portions of the orbit pass through the planetary atmosphere.

The analysis of this section is restricted to the first method. The analysis is simplified to the two-dimensional trajectory near a spherical planet.

The Keplerian Phase of the trajectory was defined previously as that portion of the trajectory where \( \left| \frac{1 - \frac{\mu}{r}}{\mu} \right| < 0.1 \). The equations of motion developed in Section 6.4 may be written for the Keplerian Phase as:

\[
X' = \frac{V}{\mu} \frac{\phi}{r} \quad \text{(9-3)}
\]

* Thrust levels between those obtainable with chemical rockets and with continuous ion propulsion.
\[ h' = r' = v_r = v_{Ir} \quad (9-4) \]

\[ v_r' = \frac{v_{Ir}}{r} - \frac{1}{r^2} \quad (9-5) \]

\[ v_{\phi}' = \frac{v_r}{r} (v_{\phi} - 2 v_{Ir}) \quad (9-6) \]

Fig. 9.3 describes the geometry of the entry trajectory. Quantities at the trajectory modification point are denoted by the subscript "m". The transfer ellipse is solved in Derivation Summary (9-1). A brief summary of the solutions obtained are tabulated in sections 9.4.1, 9.4.2, 9.4.3, and 9.4.4 following.

9.4.1 Description of the Entry Transfer Ellipse in Terms of Geometric Quantities at the Trajectory Modification Point (Applicable to Entry from Elliptical and Circular Orbits).

a) Eccentricity:

\[ \epsilon = \left( 1 - v_{Im}^2 r_m \cos^2 \gamma_{Im} (2 - v_{Im}^2 r_m) \right)^{\frac{1}{2}} \quad (9-7) \]

b) Dimensionless semi-major axis:

\[ a_N = \frac{r_m}{\left( 2 - v_{Im}^2 r_m \right)} \quad (9-8) \]

c) Dimensionless semi-minor axis:

\[ b_N = a_N \sqrt{1 - \epsilon^2} \quad (9-9) \]

d) Dimensionless latus rectum:

\[ l_N = a_N (1 - \epsilon^2) \quad (9-10) \]

e) Dimensionless area of ellipse:

\[ \text{Area} = \pi a_N b_N \quad (9-11) \]
\( v_c \) = dimensionless circular orbital velocity prior to modification
\( \delta \) = dimensionless velocity impulse imparted
\( \Theta_m \) = true anomaly of trajectory modification point
\( \nu_{I_m} \) = dimensionless velocity of vehicle with respect to inertial coordinates immediately after modification
\( \gamma_{I_m} \) = inertial flight path angle immediately after modification
\( A_c \) = engine deflection angle
\( \chi_{RN} \) = dimensionless distance traversed from modification point.

**ANGULAR RELATIONS**

\[
(9.3-1) \quad \sin(-\gamma_{I_m}) = (\delta v)\sin A_e / \nu_{I_m}
\]
\[
(9.3-2) \quad \cos(-\gamma_{I_m}) = \left[ v_c - (\delta v)\cos A_e \right] / \nu_{I_m}
\]
\[
(9.3-3) \quad \tan(-\gamma_{I_m}) = (\delta v)\sin A_e / \left[ v_c - (\delta v)\cos A_e \right]
\]

**VELOCITY RELATIONS**

\[
(9.3-4) \quad \nu_{I_m} = \left[ (\delta v)^2 + v_c^2 - 2(\delta v)v_c \cos A_e \right]^{1/2}
\]
\[
(9.3-5) \quad v_c = \sqrt{\frac{G R}{G(m)oR(m)o}} = \sqrt{\frac{1}{R_m}}
\]

Fig. 9.3: Geometry of the Transfer Ellipse; Definitions of Quantities and Geometric Identities.
f) Dimensionless period of orbit:

\[ T_N = 2\pi a_N^{3/2} \]  
\[ (9-12) \]

g) Dimensionless total energy:

\[ E_{(tot)} = -1/2a_N \]  
\[ (9-13) \]

h) Dimensionless angular momentum:

\[ p = r_m v_{Im} \cos \varphi_{Im} \]  
\[ (9-14) \]

i) Dimensionless radius at perigee:

\[ r_{\eta} = a_N(1-\xi) \]  
\[ (9-15) \]

j) Dimensionless velocity at perigee:

\[ v_{I\eta} = \sqrt{1+\xi} \]  
\[ \frac{r_{\eta}}{r_{\theta}} \]  
\[ (9-16) \]

k) Dimensionless radius at apogee:

\[ r_\alpha = a_N(1+\xi) \]  
\[ (9-17) \]

l) Dimensionless velocity at apogee:

\[ v_{I\alpha} = \sqrt{1-\xi} \]  
\[ \frac{r_\alpha}{r_{\theta}} \]  
\[ (9-18) \]

\[ 9.4.2 \text{ Guidance Quantities in Terms of Geometric Quantities at the Trajectory Modification Point (Applicable to Entry from Elliptical and Circular Orbits)} \]

\[ a) \text{ Time of flight } (T_m = 0) \]

\[ \tau = \frac{p^3}{(1-\xi^2)} \left[ \frac{-\xi \sin \varphi}{1+\xi \cos \varphi} + 2 \frac{\tan^{-1} \left( (1-\xi) \tan \varphi / 2 \right)}{\sqrt{1-\xi^2}} \right] Q_m \]  
\[ (9-19) \]
b) Dimensionless velocity:

\[ \nu_I = \left[ \frac{2}{r} - \left( \frac{2 - \frac{v_I}{v_m}}{r_m} \right)^{\frac{1}{2}} \right] \quad (9-20) \]

c) Inertial flight path angle:

\[ \gamma_I = \arccos \left( \frac{r_m v_I}{r v} \cos \gamma \right) \quad (9-21) \]

Using equation (9-20), this may be written:

\[ \gamma_I = \arccos \left( \frac{r_m v_I}{r v} \cos \gamma \right) \left( \frac{2/r - (2 - \frac{v_I}{v_m})}{r_m} \right)^{\frac{1}{2}} \quad (9-22) \]

d) Range:

\[ X_N = \theta - \theta_m \quad (9-23) \]

where

\[ \cos \theta = \left( \frac{p^2}{r^2} - 1 \right) \quad (9-24) \]

\[ \cos \theta_m = \left( \frac{p^2}{r_m^2} - 1 \right) \quad (9-25) \]

Range may be expressed in closed form as:

\[ \cos X_N = \frac{\{ \}}{\xi^2 r r_m} \quad (9-26) \]

where:

\[ \{ \} = \left( p^2 - r \right) \left( p^2 - r_m \right) + \left[ \xi^2 r^2 - (p^2 - r)^2 \right] \left( \frac{r_m}{r} \right)^2 \left( \xi^2 r_m^2 - (p^2 - r_m)^2 \right)^{\frac{1}{2}} \]

(9-27)
9.4.3: Description of Transfer Ellipse in Terms of Velocity Impulse and Engine Gimbal Angle. Applicable to Entry from Circular Orbits Only.

The variables used in the following equations are defined in Fig. 9.3. Equations (9.3-1) through (9.3-5) given in Fig. 9.3 give various relations for vehicle velocity and inertial flight path angle at the trajectory modification point in terms of velocity impulse and engine gimbal angle.

The following quantity is defined:

\[ \delta v_c = \frac{\delta V}{V_c} = \text{velocity impulse non-dimensionalized with respect to circular orbital velocity at the trajectory modification point.} \] (9-28)

a) Eccentricity:

\[ \epsilon = \left[ 1 - (1 - \delta v_c \cos A_e)^2 \right] \left( 1 - \delta v_c (\delta v_c - 2 \cos A_e) \right) \] (9-29)

b) Dimensionless semi-major axis:

\[ a_N = \frac{v_c^2}{v_c^2 \left( 1 - \delta v_c (\delta v_c - 2 \cos A_e) \right)} \] (9-30)

All other quantities listed in Section 9.4.1 are the same except the following:

h) Dimensionless angular momentum:

\[ p = \frac{(1 - \delta v_c \cos A_e)}{v_c} \] (9-31)

9.4.4: Guidance Quantities in Terms of Velocity Impulse and Engine Gimbal Angle. Applicable to Entry from Circular Orbits Only.

a) Time of Flight:

Same as Equation (9-19)
b) Dimensionless Velocity:

\[ v_I = \left\{ \frac{2}{r} - v_c^2 \left[ 1 - \delta v_c \left( \delta v_c - 2 \cos A_e \right) \right] \right\}^{1/2} \]  \hspace{1cm} (9-32)

c) Flight Path Angle:

\[ \gamma_I = \arccos \left( \frac{1 - \delta v_c \cos A_e}{v_c r} \right) \]  \hspace{1cm} (9-33)

or

\[ \gamma_I = \arccos \left\{ \frac{1 - \delta v_c \cos A_e}{2v_c^2 r - v_c^4 \left( 1 - \delta v_c \left[ \delta v_c - 2 \cos A_e \right] \right)^{1/2}} \right\} \]  \hspace{1cm} (9-34)

d) Range:

Equations (9-23) through (9-25) apply. An alternate expression for range is:

\[ \cos X_n = \frac{v_c^2}{\xi^2 r} \{ \} \]  \hspace{1cm} (9-35)

where:

\[ \{ \} = \left\{ \left( p^2 - r \right) \left( p^2 - \frac{1}{v_c^2} \right) + \left[ \xi^2 r^2 - \left( p^2 - r \right)^2 \right]^{1/2} \left[ \frac{\xi^2 r^2}{v_c^2} - \left( p^2 - \frac{1}{v_c^2} \right)^2 \right]^{1/2} \right\} \]  

Fig. 9.4 shows the geometric relation between flight path angle, \( \gamma \), and inertial flight path angle, \( \gamma_I \). Equations (9.4-1) through (9.4-8) given as part of this figure are useful for relating velocities and angles. The solutions for the Intermediate and Gas Dynamic Phases given in later sections of this chapter are generally written in terms of \( \gamma \) because this angle is more convenient when drag and lift terms are important (since these quantities involve velocities with respect to the atmosphere of the planet). Fig. 9.4 and the equations listed therein enable conversion of angles at phase boundaries such that the most
\[ \nu_1 \phi = \nu_1 \cos \gamma = \nu \cos \gamma + \nu \Omega \cos \psi \]  \hspace{1cm} (9.4-1)

\[ r' = h' = \nu_1 \sin \gamma = \nu \sin \gamma \]  \hspace{1cm} (9.4-2)

\[ p = r \nu_1 \cos \gamma \]  \hspace{1cm} (9.4-3)

\[ \sin \gamma = \frac{\nu_1}{\nu} \sin \gamma \]  \hspace{1cm} (9.4-4)

\[ \nu^2 = \nu_1^2 - 2 \nu_1 \cos \gamma \nu \Omega \cos \psi + \nu^2 \Omega^2 \cos^2 \psi \]  \hspace{1cm} (9.4-5)

\[ \nu_1^2 = \nu^2 + 2 \nu \cos \gamma \nu \Omega \cos \psi + \nu^2 \Omega^2 \cos^2 \psi \]  \hspace{1cm} (9.4-6)

\[ \cos \gamma = \frac{\nu_1 \cos \gamma - \nu \Omega \cos \psi}{\nu} \]  \hspace{1cm} (9.4-7)

\[ \cos \gamma = \frac{\nu_1 \cos \gamma + \nu \Omega \cos \psi}{\nu_1} \]  \hspace{1cm} (9.4-8)

Fig. 9.4: Geometric Relations Between Flight Path Angle and Inertial Flight Path Angle.
9.4.5: Optimum Engine Gimbal Angle:

A thorough graphical representation of equations (9-7) through (9-35) was not undertaken in this thesis. An example of a solution to the circular orbital equations (given in subsections 9.4.3 and 9.4.4) in order to determine optimum engine gimbal angle is presented graphically in Figures 9.5 and 9.6. The circular orbital altitude was chosen as \( h_m = 0.07575 \), corresponding to an altitude of 300 miles above the Earth. The final altitude was selected to be \( h_f = 0 \), corresponding to impact with the planet. No atmospheric effects were considered.

The following velocity impulses were chosen:

\[
\delta v = 0.0385; \ 0.0772; \ 0.1156; \text{ and } 0.1542
\]

These correspond to velocity impulses of 1000, 2000, 3000, and 4000 ft/sec, respectively, for the planet Earth.

Figs. 9.5 and 9.6 show that there is one value of engine deflection angle for which range is minimized with a given magnitude of velocity impulse.* At this engine deflection angle, any errors in alignment of the engine gimbals result in minimum range error. For example, if \( \delta v = 0.0385 \), the optimum engine angle is \( \theta_e = +34^\circ \); if \( \delta v = 0.1542 \), optimum engine angle is \( \theta_e = +60^\circ \).

It is interesting to note in Fig. 9.5 that at velocity impulses of large magnitudes, range is relatively insensitive to errors in engine gimbal angle from the optimum angle. The range sensitivity to errors in

* Labeled "Optimum Engine Deflection Angle" on the graphs.
Fig. 9.5: Vacuum Ground Range vs Engine Deflection Angle.
Optimum Engine Deflection Angle

$x_N = 0.4$
$x_N = 0.6$
$x_N = 0.8$
$x_N = 1.0$
$x_N = 1.5$
$x_N = 2.0$

$\delta v$: Dimensionless Velocity Impulse

$\delta v$ for Earth in Ft/sec

1000 2000 3000 4000

0 .0386 .0772 .1156 .1542

Fig. 9.6: Velocity Impulse vs Engine Deflection Angle for Vacuum Trajectories of Fixed Ground Range
engine gimbal angle increases as the magnitude of the velocity impulse is reduced.* It might be concluded that large thrust capability for extended periods (i.e., big engines with large quantities of fuel) is the cure-all for improving impact accuracy of the entry vehicle. A compromise is generally dictated in fuel and engine weight allowances, however, by payload requirements.

It may be observed that with a given velocity impulse and range, the engine deflection angle is a two-valued function (except at \( A_{\text{e, optimum}} \)). A given range may be obtained with either of two values of engine deflection angle.

It is significant to note in Fig. 9.5 that the range interval between curves decreases as \( \delta_v \) is increased.** Therefore, the magnitude of reduction in range decreases for a given step increase in \( \delta_v \) at high impulsive velocity levels.

Graphical determination of range sensitivity to errors in either engine deflection angle or velocity impulse exemplified by Figs. 9.5 and 9.6 is cumbersome and inefficient. A particular set of initial conditions must be assumed; then the trajectory must be solved repeatedly using systematically chosen values of engine gimbal angle and velocity impulse. A more effective analytical method of obtaining the same information was derived in Derivation Summary 9.1. The results of this derivation are discussed briefly in the next two sections.

---

* That is, the curves are more pointed in the vicinity of \( A_{\text{e, optimum}} \) for small \( \delta_v \).

** On fig. 9.5, the curves are more closely spaced at high values of \( \delta_v \).

A single equation relating altitude \( h \), altitude at which entry is initiated \( h_m \), velocity and inertial flight path angle at the trajectory modification point, and range was derived in Derivation Summary 9.1 as follows (note: \( r = 1 + h \)):

\[
\frac{r_m^2}{r} \nu_{I_m}^2 \cos^2 \gamma_{I_m} - 1 \nu_m \nu_{I_m}^2 \cos^2 \gamma_{I_m} \cos (X_N + \gamma_{I_m}) - \cos X_N
\]

This equation was written in other forms in equations (28) and (29) of Derivation Summary 9.1. Implicit partial differentiation of equation (9-36) yields equations which may be used to evaluate range sensitivity to errors in either altitude, velocity, or flight path angle.

Equation (9-37) gives range sensitivity to variations in:

\[
\frac{\partial X_N}{\partial \nu_{I_m}} = \frac{2 \nu_m \nu_{I_m}^2 \cos^2 \gamma_{I_m} \left( \frac{r_m}{r} - \cos (X_N + \gamma_{I_m}) \right)}{\sin X_N - \nu_m \nu_{I_m}^2 \cos^2 \gamma_{I_m} \sin (X_N + \gamma_{I_m})}
\]

Equation (9-38) gives range sensitivity to variations in \( \gamma_{I_m} \):

\[
\frac{\partial X_N}{\partial \gamma_{I_m}} = \frac{\nu_{I_m}^2 \nu_m \cos \gamma_{I_m} \left\{ 2 \sin \gamma_{I_m} \left[ \cos (X_N + \gamma_{I_m}) - \frac{r_m}{r} \right] + \cos \gamma_{I_m} \sin (X_N + \gamma_{I_m}) \right\}}{\sin X_N - \nu_m \nu_{I_m}^2 \cos^2 \gamma_{I_m} \sin (X_N + \gamma_{I_m})}
\]


A single equation relating altitude, altitude of the circular orbit, velocity impulse, engine gimbal angle, and range was derived in the
following forms by the methods of Derivation Summary 9.1.

\[
\left( S_{v_c} \right)^2 \cos A_e \left( \frac{\Gamma_m}{r} - \cos X_N + \sin X_N \tan A_e \right) - \delta v_c \cos A_e \left( 2 \left( \frac{\Gamma_m}{r} - \cos X_N \right) + \sin X_N \tan A_e \right) + \frac{\Gamma_m}{r} - 1 = 0
\]

(9-39)

or

\[
\delta v_c = \left\{ \left[ 2 \left( \frac{\Gamma_m}{r} - \cos X_N \right) + \sin X_N \tan A_e \right] - \left[ \sin^2 X_N \tan^2 A_e + 4 \left( 1 - \cos X_N \right) \left( \frac{\Gamma_m}{r} - \cos X_N + \sin X_N \tan A_e \right) \right] ^{1/2} \right\} / 2 \cos A_e \left( \frac{\Gamma_m}{r} - \cos X_N + \sin X_N \tan A_e \right)
\]

(9-40)

Range sensitivity to retro-rocket system operation was determined by implicity differentiation of equations (9-39) or (9-40):

\[
\frac{\delta X_N}{\delta A_c} = \frac{\left[ \sin A_c \left( \cos X_N - \frac{\Gamma_m}{r} - \sin X_N \tan A_c \right) + \sin X_N \tan A_c \right] \frac{1}{\cos^2 A_c} \left( \sin X_N - \frac{\sin A_e}{\delta v_c} \left( \frac{\Gamma_m}{r} - 1 \right) \right)}{- \delta v_c \sin (X_N + A_c) + 2 \sin X_N + \cos X_N \tan A_c}
\]

(9-41)

\[
\frac{\delta X_N}{\delta \delta v_c} = \frac{2 \cos A_c \left( \frac{\Gamma_m}{r} - \cos X_N + \sin X_N \tan A_c \right) - \frac{1}{\delta v_c} \left[ 2 \left( \frac{\Gamma_m}{r} - \cos X_N \right) + \sin X_N \tan A_c \right]}{(2 \sin X_N + \cos X_N \tan A_c) - \delta v_c \sin (X_N + A_c)}
\]

(9-42)

With the aid of equations (9-36) through (9-42), a description of range sensitivity to errors in retro-rocket system operation is available for analysis of entry into any planetary atmosphere from elliptical or circular reconnaissance orbits.

9.5: The Intermediate Phase

The Intermediate Phase presents the most complex computing phase for the guidance system. In this phase accelerations due to gas-dynamic
terms are of comparable magnitude with other terms in the dynamical equations of motion. Some of the simplifying assumptions made by most authors* analyzing entry trajectories are not permissible in the Intermediate Phase. A partial list of typical assumptions and their accuracy with respect to the Intermediate Phase are summarized below:

1. **Lift and Drag coefficients are independent of Mach Number and Reynolds Number.** This assumption is generally accurate at the high Mach numbers of the Intermediate Phase.

2. **Gravitational acceleration is constant.** This is clearly a poor assumption in the Keplerian Phase, and a questionable assumption in the Intermediate Phase. For example, the gravitational acceleration of the Earth decreases about 1% for each 20 miles increase in altitude. It was shown in Chapter 8 that, for constant $\gamma$, the Intermediate Phase spans an altitude band of 20 miles for the Earth's atmosphere. If a value of gravitational acceleration is used that corresponds to the average altitude of the Intermediate Phase, then variations of gravitational acceleration of less than 1% from this local average value may be expected in the Intermediate Phase.

3. **An isothermal atmosphere is assumed;** therefore, atmospheric density decays exponentially with altitude. This assumption is reasonably accurate for Earth below altitudes of 80 miles (see Fig. E.2); the accuracy of this assumption for other planets cannot be predicted at this time. It was shown in

* See, for example, Chapman(15), Eggers, Allen and Neice(31), and Allen and Eggers(51).
Chapter 8 that the onset of the Intermediate Phase may be higher than 80 miles. In general, the altitude of the Keplerian-Intermediate Phase boundary increases as:

1. Drag coefficient is increased;
2. Surface loading (M/S) is decreased;
3. The magnitude of flight path angle is decreased.

Since the altitude of the Intermediate Phase may be above the region where the exponential approximation of the atmosphere is accurate, either a power approximation or a variable decay parameter $k$ may be more accurate representation for analytical studies of this Phase.

4. **Planetary rotation is neglected**; therefore, Coriolis forces are not included in the equations of motion (even though these equations are written in a coordinate system that rotates with respect to inertial space). This assumption is reasonably accurate in the Intermediate and Gas-Dynamic Phases for terrestrial planets.

5. $(1 + L/D \tan \gamma) \approx 1.0$. This assumption is accurate in all phases for moderate lift-drag ratios and normal flight path angles (except in steep terminal phases near the end of the Gas-Dynamic Phase).

6. Quantities in the equations of motion involving $v_r v_\phi$ are neglected. This is equivalent to assuming $|1 - \frac{v_{\phi}}{v_r}| > 10$ (i.e., $|\frac{v_{\phi}}{v_r}| << 1.0$). It was shown in Chapter 8 that this assumption is accurate only in the Gas-Dynamic Phase. This

* Or terms equivalent to this quantity.
assumption cannot be made in the Intermediate and Keplerian Phases. Solutions to the simplified equations of motion in the Keplerian and Gas-Dynamic Phases are discussed in other sections of this chapter. Few simplifying approximations are warranted in the Intermediate Phase, however, and closed form solutions were not obtained for this phase.

In powerless flight*, the two-dimensional equations of motion may be written in terms of velocity, altitude, and the Conservation Parameter as follows:

\[ X' \frac{\dot{\phi}}{N} = \frac{V_\phi}{r} \]
\[ h' = r' = v_r \]
\[ v_r' = v_r \frac{r^2}{r} - v_r \frac{\phi}{r} \frac{1 - \frac{1}{\xi_0}}{\xi_0} \]
\[ v_\phi' = v_\phi \left[ v_\phi - (1 + \frac{1}{\xi_0}) \frac{r}{v_r} \right] \]

where

\[ v_r = v_r + r \Omega \cos \phi \]

Equation (9-45) may be simplified slightly by assumption (5) discussed previously:

\[ \left( 1 + \frac{L}{D} \frac{v_r}{v_\phi} \right) = \left( 1 + \frac{L}{D} \tan \gamma \right) \approx 1.0 \]

The Intermediate Phase may be traversed impulsively by using certain

* Thrust terms are zero in powerless flight, i.e., \[ v_N = 0. \]
trajectory control techniques at the onset of this phase; i.e., at the point where \( \left| \frac{1 - \frac{\rho}{\rho_0}}{\rho} \right| = 0.1 \). Examples of such techniques are:

1. **Application of thrust:**

   When the phase boundary between the Keplerian and Intermediate Phases is approached, the Conservation Parameter rises from near zero to 0.1. At the point where this parameter reaches 0.1, impulsive retro-thrust may be applied in such a manner as to cause \( \left| \frac{1 - \frac{\rho}{\rho_0}}{\rho} \right| \) to step to 10 or greater, corresponding to flight in the Gas-Dynamic Regime.

2. **Use of Auxiliary High-Drag Device**

   Fig. 8.8 showed upper and lower phase boundaries of the Intermediate Phase as a function of drag coefficient. It may be observed from this figure that a step change in drag coefficient of approximately two orders of magnitude is required to move the operating phase of the vehicle from the Keplerian-Intermediate Phase boundary to the Intermediate-Gas-Dynamic Phase boundary (with constant altitude). This step drag increase may be obtained at the onset of the Intermediate Phase, for example, by streaming a drogue chute or umbrella. If the horizontal component of external specific force steps two orders of magnitude as the drag device is extended, then the Intermediate Phase is traversed during the step drag increase.

   Certain lift-drag programs** may be performed in the Intermediate

* Such as a drogue chute or umbrella.

** Programmed lift and drag, such as discussed in this section, are not easily realized in practice.
Phase to induce a particular trajectory pattern. One example of such a
lift-drag program is that of holding constant flight path angle through-
out the Intermediate Phase.

The required lift program to hold constant flight path angle, $\gamma$, may be determined by setting $\gamma' = 0$ in equation (6-41). The drag
program is determined by integrating $\frac{dv}{dh}$ from equations (6-40) and (6-43) with $\gamma$ constant. It may be seen that the flight path angle remains
constant if lift and drag are programmed as follows:

$$L/D = \frac{\cot \gamma (1 - v^2) (1 - e^{-k\dot{h}})}{kv^2 \ln \left(\frac{v}{v_1}\right)}$$

(9-48)

$$C_L = \frac{\cos \gamma (1 - v^2)}{(C(SL)R(m)\rho)\sqrt{2M/S}v^2 e^{-k\dot{h}}}$$

(9-49)

$$C_D = \frac{k \sin \gamma \ln \left(\frac{v}{v_1}\right)}{(C(SL)R(m)\rho)\frac{2M/S}{(e^{-k\dot{h}} e^{-k\dot{hi}})}}$$

(9-50)

It is noted that the exponential atmospheric model was assumed and that
terms involving planetary rotation were dropped because they are higher
order effects in comparison to the quantities retained.

By switching to the above lift-drag program as $|\frac{1 - \frac{\dot{h}}{\dot{h}}}{|}$ rises to 0.1
and maintaining it until $|\frac{1 - \frac{\dot{h}}{\dot{h}}}{|} = 10$, the Intermediate Phase is traversed
at constant flight path angle. Closed form expressions for range and
other quantities developed in section 9.6 for the "Ballistic Trajectory"
apply in the Intermediate Phase if this lift-drag program is maintained.

9.6 The Gas-Dynamic Phase:

The final portion of any entry mission to the surface of a planet
through its atmosphere necessarily involves flight in the Gas-Dynamic
Phase. The analysis of motion of vehicles in this operational regime has been a substantial percentage of man's total scientific effort during the past half-century and resulted in the evolution of an entirely new branch of science and engineering, viz. "aeronautics". With each advance in propulsive system design, the increased velocity and altitude capabilities of the vehicle has brought on problems of increasing complexity.

The German V-2 program was the first substantial step up the ladder toward flight beyond the sensible atmosphere at hypervelocities. This beginning has been greatly extended in the IREM and ICBM programs in the United States and the Soviet Union.

The character of the atmospheric phase of flight at near orbital velocities is, in many ways, far removed from that of flight at sonic or subsonic velocities. The vehicle possesses tremendous energy and angular momentum at near-orbital velocities. Most of this energy must be transferred to the atmospheric envelope surrounding the planet prior to initiating a safe landing. Two of the primary dangers encountered during the process of transferring this energy are manifested as heating of the vehicle and deceleration of the vehicle and its occupants. It was not until the feasibility of space activities was clearly demonstrated in the post World War II period that analytical investigation of hypersonic flight was carried out on a scale more extensive than the occasional publication of a paper touching the subject*. Since 1955, the interest in this problem has grown rapidly, as evidenced by the steadily increasing frequency of published work. Among the papers

* One of the earliest papers of significance is that of Sanger(52), published in 1933.
published in recent years, some of the best are the works of Sanger and Bredt (53) (1944); Eggers, Allen, and Neice (31) (1957); Allen and Eggers (51) (1957); Allen (54) (1957); and Chapman (15) (1958). Chapman considers atmospheric entry into not only Earth but also other planets of the solar system; the other authors cited above restrict their analysis to the Earth system. Much of the results presented in the following pages are based on conclusions of the foregoing papers. The results that follow, however, go beyond the scope of the above treatises to the derivation of guidance quantities in terms of certain independent variables (such as time and atmospheric density ratio) not previously considered.

Three basic trajectory patterns are possible in the Gas-Dynamic Phase. The trajectory profile which is traced out on entry depends strongly on the initial flight conditions and on the lift-drag characteristics of the vehicle. These trajectories are classified as follows:

1. **Ballistic trajectory**: Non-lifting vehicle with large initial flight path angle.
2. **Glide trajectory**: Lifting vehicle with zero initial flight path angle.
3. **Skip trajectory**: Lifting vehicle with finite initial flight path angle.

Approximate analytical solutions were derived in this thesis for each of these classes of trajectories. Fig. 9.7 shows the regions in which these approximate analytical solutions are an accurate representation of the trajectory when compared to numerical solutions obtained from the non-linear dynamical equations of motion.

Chapman (15) discussed trajectories in the Gas-Dynamic Phase by transforming the equations of motion to a single, ordinary, non-linear differential equation of second order. A transformation similar to
Initial Conditions: Zero lift vehicle with steep flight path; or programmed lift-drag to maintain constant flight path angle.

Limitations of Approximate Solution:

Approximate solution is accurate

(1) Throughout the velocity spectrum for $\frac{k^{1/2}V}{c} \geq 2.5$ (this corresponds to entry at $|\gamma| \geq 5^\circ$ for Earth).

(2) For flight path angles less than listed in (1) above, approximate solution is accurate down to $v$ equal to about 0.8.

(a) BALLISTIC TRAJECTORY

Fig. 9.7: Regions Where Approximate Solutions of This Thesis are Accurate.
Initial Conditions: Lifting vehicle with zero initial flight path angle.

Limitations of Approximate Solution:

Approximate solution is accurate:
(1) For \( v > 0.2 \) if vehicle characteristics are:
\[
\frac{1}{2} \sqrt{k} \frac{L}{D} \geq 30 \quad (\text{i.e.}, \ (L/D)_{Earth} \geq 1.0)
\]
(2) For \( v > 0.5 \) if vehicle characteristics are:
\[
30 > \frac{1}{2} \sqrt{k} \frac{L}{D} \geq 15 \quad (\text{i.e.}, \ 1.0 > (L/D)_{Earth} \geq 0.5)
\]
(3) For \( v > 0.7 \) if vehicle characteristics are:
\[
15 > \frac{1}{2} \sqrt{k} \frac{L}{D} \geq 7.5 \quad (\text{i.e.}, \ 0.5 > (L/D)_{Earth} \geq 0.25)
\]

(b) GLIDE TRAJECTORY

Fig. 9.7: Regions where approximate solutions of this thesis are accurate.
Initial Conditions: Lifting vehicle with finite initial flight path angle.

Limitations of approximate solution:

Approximate solution is accurate if $2 k \left| \gamma_1 \frac{L}{D} \right| \gg 1.0$

(i.e., $\left| \gamma_1 \frac{L}{D} \right| \text{ Earth} \gg 5.6 \times 10^{-4}$)

Approximate solution can be applied only to a single skip. Subsequent skips may be analyzed by considering each skip individually with a new set of initial conditions.

(c) SKIP TRAJECTORY

Fig. 9.7: Regions where approximate solutions of this thesis are accurate.
Chapman's Z-transformation may be applied to equations (9-45) and (9-46) with the following assumptions:

\[ 1 + (L/D) \tan \gamma \approx 1.0 \]

\[ v_{IF} \approx v_{\phi} \]

\[ \xi \ll 1.0 \]

\[ r \approx 1.0 \]

The equivalent Z-transformation is:

\[ Z = \frac{v_{r} v_{\phi}}{\xi k_{v}^{\frac{3}{2}}} \approx \frac{v \cos^{2} \gamma \sin \gamma}{k_{v}^{\frac{3}{2}} \xi} \]  \hspace{1cm} (9-51)

The single equation derived by performing this transformation is:

\[ n \phi \frac{d^{2} Z}{d n^{2}} - \left( \frac{d Z}{d n_{\phi}} - \frac{Z}{n_{\phi}} \right) = (1 - n_{\phi}^{2}) \frac{\cos^{2} \gamma}{Z n_{\phi}} - k_{v}^{\frac{3}{2}} \frac{L}{D} \cos^{3} \gamma \]  \hspace{1cm} (9-52)

\begin{align*}
\text{Vertical Acceleration} & \quad \text{Vertical Component of drag} & \quad \text{Gravity minus Centrifugal Force} & \quad \text{Lift Force} \\
\end{align*}

Alternate forms of expressing the Z-function are as follows:

\[ Z = n_{D} \cos \gamma = - \frac{f_{\theta}}{k_{v}^{\frac{3}{2}}} \]  \hspace{1cm} (9-53)

Chapman's numerical solutions are compared to the approximate solutions of the dynamical equations of motion derived in this thesis in Figs. 9.8, 9.9, and 9.10 for the ballistic, glide, and skip trajectories respectively. The abscissa on each of these graphs is the horizontal component of dimensionless velocity with respect to the atmosphere; The ordinate is the horizontal component of specific force.
Fig. 9.8: Comparison of Approximate "Ballistic" Solution to Machine Computed Numerical Solutions.
Fig. 9.9: Comparison of Approximate "Glide" Solution to Machine Computed Numerical Solutions.
Fig. 9.10: Comparison of Approximate "Skip" Solution to Machine Computed Numerical Solutions.
in surface g's of the planet concerned.*

9.6.1 Solution of the Ballistic Trajectory

The ballistic trajectory is characterized geometrically by a constant flight path angle. Prerequisites for this trajectory may be summarized as follows:

(1) Zero-Lift Vehicles:

If entry is made along such a steep path that the sum of vertical components of lift, gravity, and centrifugal forces are small in comparison to the vertical component of drag force.

(2) Programmed Lift and Drag:

If lift and drag are programmed in such a manner as to maintain constant flight path angle. (See equations (9-48) through (9-50).

The limitations of the approximate solution developed in Derivation Summary (9-2) when compared to more exact computer solutions of the non-linear system of equations are briefly:

(1) The solution derived here is an excellent approximation for $|\sqrt{k}Y| \geq 2.5$ throughout the velocity spectrum. This corresponds to entry into the Earth's atmosphere at angles equal to or greater than 5°.

(2) The approximate solution is accurate for flight path angles less than $|\sqrt{k}Y| = 2.5$ over the limited velocity spectrum $\eta > 0.8$. Therefore, the solution is accurate for all flight path angles down to velocities corresponding to maximum

* Ordinate is (L/D) times horizontal specific force in Fig. 9.9 for glide vehicle.
Comparison of the approximate solution derived herein to numerical solutions computed by Chapman (15) is presented graphically in Fig. 9.8. The derivation carried out in Derivation Summary (9-2) and summarized below is based on the same approximations made by Chapman in deriving his $Z_I$ solution and by Eggers, Allen, and Neice (31) for their "Ballistic Trajectory". The solutions of this thesis, however, extend the work of the cited papers to include guidance quantities expressed with either density ratio, range, velocity, or time as the independent variable.

(1) **Density ratio $\sigma$ as independent variable**: \( \sigma \equiv \frac{\rho}{\rho_{(SL)}} \approx e^{-kh} \)

(a) **Altitude**: \( h(\sigma) = -(1/k)\ln \sigma \) 

(b) **Velocity**: \( v(\sigma) = \frac{U C_p}{V_c k \sin \gamma_b} \left( \frac{(\sigma - \sigma_1)}{(\sigma - \sigma_1)} \right) \) 

(c) **Range**: \( X_N(\sigma) = X_{Ni} + \cot \gamma_b \ln \left( \frac{(1-(1/k)\ln \sigma)}{(1-(1/k)\ln \sigma_1)} \right) \approx X_{Ni} - \cot \gamma_b \ln \frac{\sigma_1}{\sigma_1} \)

(2) **Altitude as independent variable**:

(a) **Density ratio**: \( \sigma(h) = e^{-kh} \)

* Maximum stagnation point temperature occurs when \( v = 0.846 \, v_1 \) (See Derivation Summary 9-2).

** For convenience in writing this and all subsequent equations, the following quantity is defined:

\[ U = \frac{C(\bar{S}L)R(m)Q}{2M} \]
(b) **Velocity:**
\[ v(h) = v_1 e^{k \sin \gamma_b} (e^{-kh} - e^{-kh_i}) \]  
\[ \text{Velocity as independent variable:} \]
\[ \sigma(\nu) = \frac{k \sin \gamma_b}{U C_D} \frac{1}{\nu} \sin \kappa \ln \left( \frac{1+h}{1+h_i} \right) + \sigma_i \]  
\[ \text{Density ratio:} \]
\[ h(\nu) = -\frac{1}{k} \ln \left( \frac{k \sin \gamma_b}{U C_D} \frac{1}{\nu} \ln \left( \frac{\kappa}{\nu_i} + e^{-kh_i} \right) \right) \]  
\[ \text{Altitude:} \]
\[ X_N(\nu) = X_{N_i} + \cot \gamma_b \ln \left( \frac{1}{k} \ln \left( \frac{k \sin \gamma_b}{U C_D} \frac{1}{\nu} \ln \left( \frac{\kappa}{\nu_i} + \sigma_i \right) \right) \right) \]  
\[ \text{Range:} \]
\[ \text{Dimensionless time } \tau \text{ as independent variable:} \]
\[ -k \nu_i \sin \gamma_b \left( e^{k \sin \gamma_b} \right) (\tau - \tau_i) = \ln \left| \frac{\sigma}{\sigma_i} \right| + \sum_{n=1}^{\infty} \left( \frac{-U C_D}{k \sin \gamma_b} \right)^n \frac{(\sigma^n - \sigma_i^n)}{n \cdot n!} \]  
\[ \text{Note: Solution is valid if } \left| \frac{U C_D}{k \sin \gamma_b} \right| < 1.0 \]
(b) **Altitude:**

\[
-(\tau - \tau_i) k \sigma_i \sin \beta_b \ e^{k \sin \beta_b} e^{-kh_i} = k(h_i - h) + \sum_{n=1}^{\infty} \left( \frac{-UC_D}{k \sin \beta_b} \right)^n \left( e^{nkh} - e^{-nkh_i} \right) \frac{n!}{n} \tag{9-65}
\]

(c) **Velocity:**

\[
-(\tau - \tau_i) k \sigma_i \sin \beta_b \ e^{k \sin \beta_b} = \ln \left( \frac{k \sin \beta_b}{UC_D \sigma_i} \right) \left( \frac{\ln \frac{\eta}{n+1}}{n!} \right) + \sum_{n=1}^{\infty} \left( \frac{-UC_D \sigma_i}{k \sin \beta_b} \right)^n \left( -\ln \frac{\eta}{n+1} \right) \frac{n!}{n} \tag{9-66}
\]

(d) **Range:**

\[
-(\tau - \tau_i) k \sigma_i \sin \beta_b \ e^{k \sin \beta_b} = \ln \left( \frac{k \sin \beta_b}{UC_D \sigma_i} \right) \left( \frac{\ln \frac{\eta}{n+1}}{n!} \right) + \sum_{n=1}^{\infty} \left( \frac{-UC_D \sigma_i}{k \sin \beta_b} \right)^n \left( -\ln \frac{\eta}{n+1} \right) \frac{n!}{n} \tag{9-67}
\]

(5) **Range** \( X_N \) as independent variable:

(a) **Density ratio:**

\[
\sigma(X_N) = \sigma_i e^{-k(X_N - X_i) \tan \beta_b} \tag{9-68}
\]

(b) **Altitude:**

\[
h(X_N) = (1 + h_i) e^{(X_N - X_{N1}) \tan \beta_b} - 1 \tag{9-69}
\]

(c) **Velocity:**

\[
\eta'(X_N) = \eta_i e^{\frac{UC_D}{k \sin \beta_b}} \left\{ -\sigma_i e^{k \tan \beta_b (X_N - X_d)} \right\} \tag{9-70}
\]

\[
\eta^-(X_N) = \eta_i e^{\frac{UC_D}{k \sin \beta_b}} \left\{ -k \tan \beta_b (X_N - X_{N1}) - h_i \right\} \tag{9-70a}
\]

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As a typical example of a ballistic trajectory solution, consider a vehicle entering the Earth's atmosphere with:

\[ h_1 = 80 \text{ miles} \]
\[ \gamma_b = -6^\circ \]
\[ X_{N1} = 0 \]

The solution for total range from Equation (9-59) is:

\[ X_N = 0.192 \approx 760 \text{ statute miles} \]

From Fig. 9.6, it may be seen that the vehicle would experience about 18 g's maximum specific force at \( v \approx 0.6 \). The velocity spectrum and time of flight depend on the drag and density characteristics of the vehicle.

9.6.2 Solution of The Glide Trajectory

The glide trajectory is characterized geometrically by a monotonically increasing negative flight path angle. The flight path angle initially starts at zero and remains small throughout most of the trajectory. The flight path angle increases to appreciable magnitudes in the terminal phases of the trajectory after velocity has decayed to subsonic values. The gliding trajectory occurs when the flight path angle is shallow enough and velocity is small enough that vertical accelerations and the vertical component of drag force are negligible in comparison to other terms in the dynamical equations of motion. In equilibrium gliding flight, there is a balance of lift, gravity, and centrifugal forces.
The limitations of the approximate solution presented here when compared to more exact computer solutions of the complete system of equations are summarized below:

(1) The approximate solution derived here is accurate for
\[ \sqrt{k \frac{L}{D}} \geq 30 \] for dimensionless velocities greater than 0.2.
For the planet Earth, this corresponds to \((L/D)\) equal to or greater than 1.0 and velocities greater than 5200 ft/sec. (Mach \( \geq 4.5\)). The assumption that lift and drag coefficients are independent of Mach number is generally accurate over this velocity spectrum only*.

(2) The approximate solution is accurate over the velocity spectrum \( v > 0.5 \) for \( \sqrt{k \frac{L}{D}} = 15 \). This corresponds to \((L/D)_{\text{Earth}} = 0.5\).

(3) The approximate solution is accurate for \( v > 0.7 \) for \( \sqrt{k \frac{L}{D}} = 7.5 \). This corresponds to \((L/D)_{\text{Earth}} = 0.25\).

Comparison of the approximate solution derived herein to numerical solutions computed by Chapman is presented graphically in Fig. 9.9.

The derivation carried out in Derivation Summary 9.3 and summarized below is based on approximations originally made by Sanger(52). These approximations were also used by Eggers, Allen, and Neice(31) in discussing the "glide trajectory", and by Chapman(15) in deriving his

* The worth of generalized numerical solutions of entry trajectories for velocities less than Mach 5.0 may be questioned since Mach Number variations of lift and drag coefficients become significant at these lower velocities. The variation of \( C_L \) and \( C_D \) with Mach Number for Mach less than 5.0 is difficult to generalize because of the strong dependence on design characteristics of the vehicle. Most authors assume lift and drag coefficients are independent of Mach Number effects throughout the entire velocity spectrum. Exceptions, of course, are studies of a particular vehicle configuration using \( C_L \) and \( C_D \) data from experimental tests.
solution. The derivation summarized here, however, extends the solutions of the above authors to expressing guidance quantities in terms of independent variables such as dimensionless time, density ratio, etc.

(1) Density ratio $\sigma$ as independent variable:

(a) Altitude:

Equation (9-54)

(b) Velocity:

\[
v_{\phi}(\sigma) = \frac{1}{(1 + UC_D \frac{L}{D} \sigma)^{\frac{1}{2}}}
\]  

(9-71)

(c) Flight path angle:

\[
\tan \gamma = \frac{-2}{k} \frac{(1 + UC_D \frac{L}{D} \sigma)}{k \frac{L}{D}}
\]  

(9-72)

(d) Range:

\[
X_N(\sigma) = X_{N_i} + \frac{L}{2D} \left( \ln \frac{\sigma}{\sigma_i} - \ln \left| \frac{1 + UC_D \frac{L}{D} \sigma}{1 + UC_D \frac{L}{D} \sigma_i} \right| \right)
\]  

(9-73)

(2) Altitude $h$ as independent variable:

(a) Density ratio:

Equation (9-57)

(b) Velocity:

\[
v_{\phi}(h) = \frac{1}{(1 + UC_{DD} e^{-kh})^{\frac{1}{2}}}
\]  

(9-74)

(c) Flight Path Angle:

\[
\tan \gamma = \frac{-2 (1 + UC_{DD} \frac{L}{D} e^{-kh})}{k \frac{L}{D}}
\]  

(9-75)
(d) Range:

\[ X_N(h) = X_i + \frac{k}{2} \frac{L}{D} \left( (h_i - h) - \frac{1}{k} \ln \left| \frac{1 + U C_D \frac{L}{D} e^{-kh}}{1 + U C_D \frac{L}{D} e^{-kh_i}} \right| \right) \]  \hspace{1cm} (9-76)

(3) Flight Path Angle as independent variable:

(a) Density ratio:

\[ \sigma(\gamma) = -\frac{k \gamma}{2 U C_D} - \frac{1}{U C_D \frac{L}{D}} \]  \hspace{1cm} (9-77)

(b) Altitude:

\[ h(\gamma) = -\frac{1}{k} \ln \left( \frac{-(k \frac{L}{D} + 2)}{2 U C_D \frac{L}{D}} \right) \]  \hspace{1cm} (9-78)

(c) Velocity:

\[ v^2(\gamma) = \left( \frac{-2}{k (\frac{L}{D}) \tan \gamma} \right) \]  \hspace{1cm} (9-79)

(d) Range:

\[ X_N(\gamma) = X_{N_i} + \frac{1}{2} \frac{L}{D} \ln \left( 1 + \frac{2}{k (\frac{L}{D}) \tan \gamma} \right) \]  \hspace{1cm} (9-80)

(4) Velocity \((v_\phi)\) as independent variable:

(a) Density ratio:

\[ \sigma(v_\phi) = \frac{(1 - v_\phi^2)}{U C_D \frac{L}{D} v_\phi^2} \]  \hspace{1cm} (9-81)

(b) Altitude:

\[ h(v_\phi) = -\frac{1}{k} \ln \left( \frac{(1 - v_\phi^2)}{U C_D \frac{L}{D} v_\phi^2} \right) \]  \hspace{1cm} (9-82)
(c) Flight Path Angle:
\[ \gamma(\nu_0) = \tan^{-1} \left( \frac{-2}{k \left( \frac{L}{D} \right) \nu_0^2} \right) \]  
\[ (9-83) \]

(d) Range:
\[ X_N(\nu_0) = X_{N_i} + \frac{1}{2} \frac{L}{D} \ln \left( \frac{(1-\nu_0^2)}{(1-\nu_{0_i}^2)} \right) \]  
\[ (9-84) \]

(5) Dimensionless time as independent variable: \( \tau = 0 \)

(a) Density ratio:
\[ \sigma(\tau) = \frac{1}{\frac{U_C D}{L}} \left\{ \frac{1 - \left( \frac{1+\nu_{0_i}}{(1+\nu_{0_i})} e^{\frac{2 \tau}{L/D}} \right)^2}{\left( \frac{1+\nu_{0_i}}{(1+\nu_{0_i})} e^{\frac{2 \tau}{L/D}} \right)^2} \right\} \]  
\[ (9-85) \]

(b) Altitude:
\[ h(\tau) = -\frac{1}{k} \ln \left[ \sigma(\tau) \right] \]  
\[ (9-86) \]

(c) Velocity:
\[ \nu_{\phi}(\tau) = \frac{(1-\nu_{0_i}) e^{\frac{2 \tau}{L/D}}}{1 - \frac{(1+\nu_{0_i})}{(1+\nu_{0_i})} e^{\frac{2 \tau}{L/D}}} \]  
\[ (9-87) \]

(d) Flight Path Angle:
\[ \tan \gamma = \frac{-2}{k \left( \frac{L}{D} \right) \nu_0^2} \left[ \left( \frac{1+\nu_{0_i}}{(1+\nu_{0_i})} e^{\frac{2 \tau}{L/D}} \right)^2 \right] \]  
\[ (9-88) \]
(e) Range:

\[
X_N(\tau) = X_{N_i} + \frac{1}{2} \frac{L}{D} \ln \left\{ \left( \frac{\left(1 + \nu_{0_i} \right) - \left(1 - \nu_{0_i} \right) e^{\frac{2\tau}{L_D}} \right)^2}{\left(1 - \nu_{0_i}^2 \right)} \right\}
\]  
(9-89)

(6) Range as independent variable:

(a) Density ratio:

\[
\sigma(X_N) = \left\{ \frac{(1 - \nu_{0_i}^2) e^{\frac{2}{L_D}} (X_N - X_{N_i})}{UC_D \frac{L}{D} \left[ 1 - (1 - \nu_{0_i}^2) e^{\frac{2}{L_D}} (X_N - X_{N_i}) \right]} \right\}
\]  
(9-90)

(b) Altitude:

\[
h(X_N) = - \frac{1}{k} \ln \left( \sigma \left( X_{N_i} \right) \right)
\]  
(9-91)

(c) Velocity:

\[
\nu_{0_i} (X_N) = \sqrt{1 - (1 - \nu_{0_i}^2) e^{\frac{2}{L_D}} (X_N - X_{N_i})}
\]  
(9-92)

(d) Flight path angle:

\[
\tan \gamma = - \frac{2}{k \frac{L}{D} \left[ 1 - (1 - \nu_{0_i}^2) e^{\frac{2}{L_D}} (X_N - X_{N_i}) \right]}
\]  
(9-93)

9.6.3 Solution of The Skipping Trajectory

The glide solution discussed in section 9.6.2 is valid only when a lifting vehicle enters the atmosphere under very special initial conditions, viz. zero initial flight path angle at sub-orbital velocities. The glide solution does not apply to trajectories with non-zero initial
flight path angles because a finite initial vertical velocity induces a trajectory in which the vertical acceleration term in the equations of motion is not small compared to lift force.

A skipping trajectory is the consequence of a lifting vehicle penetrating the atmosphere at high velocity with a finite initial flight path angle. The magnitude of the skip in terms of the spectrum of altitude, velocity, and specific force spanned during the skip is primarily a function of the initial conditions. A number of skips in the Gas-Dynamic Phase may be anticipated; the number of skips and the severity of the skips depend strongly on the initial conditions and the lift-drag characteristics of the vehicle.

The derivation of an approximate analytical solution of the Skip Trajectory is carried out in Derivation Summary (9.4). This derivation is based on the following assumptions:

1. The difference between the components of gravitation and centrifugal force in the lift direction is negligible when compared to lift force.
2. Coreolis force is negligible when compared to lift force.
3. The difference between the components of gravitation and centrifugal forces in the drag direction is negligible when compared to drag force.
4. \( \sin \gamma \approx \gamma \); \( \cos \gamma \approx 1 - \frac{\gamma^2}{2} \)

The approximate solution of this thesis is applicable only for flight in the Gas-Dynamic Phase and may be applied only to one skip with a given set of initial conditions. Subsequent skips may be analyzed by considering each skip individually with a new set of initial conditions.
Fig. 9.7 shows the regions in which the approximate skip solution is applicable. The approximate solution is accurate for lift-drag ratios and initial flight path angles that satisfy the following:

$$2k\left|\gamma\frac{L}{D}\right| \gg 1.0$$

A comparison of the accuracy of the approximate solution of this thesis to a numerical solution for the first skip of a vehicle entering the Earth's atmosphere is presented graphically in Fig. 9.10. Initial conditions for this trajectory were:

$$\frac{L}{D} = 1.0$$

Planet: Earth

$$\gamma_i = -2^\circ$$

$$v\phi_i = 1.0$$

Note: $$2k\left|\gamma\frac{L}{D}\right| \approx 62$$

The fact that analytical approximate solutions to the skipping trajectory must be performed in a series of individual piece-wise continuous solutions with proper matching of end conditions for each skip may appear to relegate such a solution to the "academically interesting but impractical" category. This is not entirely true. In general, a skipping trajectory for the manned entry vehicle is undesirable because of a number of aerodynamic and thermodynamic reasons.*

* One of the disadvantages of the skip vehicle is the relatively high lateral loads that the vehicle would be required to withstand during a skip. These loads, coupled with high thermal stresses due to high convective heating rates, would require the vehicle structure to be stronger and heavier than that of a comparable glide vehicle.

Eggers(31) concludes that skip vehicles with $$\frac{L}{D} = 4.0$$ and 6.0 cannot radiate heat at rates comparable to the maximum convective
There may be guidance advantages, however, to inducing a partial skip early in the Gas-Dynamic Phase in order to reduce a large flight path angle** to zero, or near zero, then "turn off" lift to maintain either a constant altitude or slowly descending flight path in order that an appreciable percentage of the total energy be transferred at high altitudes where heating and deceleration loads are less severe. The skipping solution derived in Derivation Summary (9.4) and summarized below may be used to analyze this partial skip with a reasonably high degree of accuracy.

(1) Density ratio $\sigma$ as independent variable

(a) Altitude:
Equation (9-54)

(b) Velocity

$$V(\sigma) = V_0 \left\{ \frac{f}{V_0} - \left[ \left( \frac{v}{V_0} \right)^2 - \frac{2UC_D}{k(\%)} (\sigma - \sigma_0) \right]^{1/2} \right\}$$

(9-94)

heating rate. The skip vehicle with lift-drag ratios in the neighborhood of 2.0 absorbs less heat than vehicles developing higher lift-drag ratios; the former vehicle, however, still absorbs more heat than a glide vehicle or a comparable high-pressure-drag ballistic vehicle.

** In order to control accurately the geographic position at which the vehicle enters the Intermediate Phase, it is desired that the Keplerian transfer ellipse intersect the atmosphere at as large an angle as possible. This may be visualized on the physical grounds that if the flight path is nearly horizontal, the slightest error in altitude or a perturbation in the density of the upper atmosphere may cause a large position error at the penetration point.

It was shown in Chapter 8 that the effective altitude of the sensible atmosphere decreases for steep flight paths; therefore, a bonus advantage of entering with a large flight path angle is that the exponential atmospheric approximation is a more accurate representation of actual conditions.
(c) Flight Path Angle:

\[ \gamma(\sigma) = \pm \left( \sigma_i - \sigma \right) \frac{2UC_D L_D}{k} + \frac{\sigma_i^2}{k} \]  
(9-95)

(d) Range:

\[ X_N(\sigma) = X_{N_i} \pm \frac{1}{k\Theta} \left\{ \ln \left( \frac{(\gamma-\Theta)(\gamma+\Theta)}{(\gamma+\Theta)(\gamma-\Theta)} \right) \right\} \]  
(9-96)

where \( \gamma(\sigma) \) is given by equation (9-95) and the following quantity is defined:

\[ \Theta = \left( \sigma_i^2 + \frac{2UC_D L_D}{k} \sigma_i \right)^{\frac{1}{2}} \]  
(9-97)

(2) Altitude \( h \) as independent variable

(a) Density ratio:

Equation (9-57)

(b) Velocity:

\[ \nu_r(h) = \nu_i e^{\frac{h}{L_D} - \left( \frac{\nu_i}{L_D} - \frac{2UC_D}{kL_D} \left( e^{-kh} - e^{-kh_i} \right) \right)^{\frac{1}{2}}} \]  
(9-98)

(c) Flight Path Angle:

\[ \gamma(h) = \pm \left( \nu_i^2 - (e^{-kh} - e^{-kh_i}) \frac{2UC_D L_D}{k} \right)^{\frac{1}{2}} \]  
(9-99)

(d) Range:

Same as equation (9-96) with equation (9-99) for \( \gamma \)
and equation (9-97) for $\Theta$.

(3) **Flight Path Angle as independent variable**

(a) Density ratio:

$$\sigma(\gamma) = \sigma_i + \frac{k}{2UC_D} \left( \gamma_i^2 - \gamma^2 \right)$$

(9-100)

(b) Altitude:

$$h(\gamma) = -\frac{1}{k} \ln \left[ \sigma(\gamma) \right]$$

(9-101)

(c) Velocity:

$$v(\gamma) = \frac{\gamma_i - \gamma}{L/D}$$

(9-102)

(d) Range:

Same as equation (9-96).

(4) **Velocity $v$ as independent variable**

(a) Density ratio:

$$\sigma(v) = \sigma_i + \frac{k \ln \left( \frac{C_v}{C_{vi}} \right)}{2UC_D} \left( 2C_v - \frac{L}{D} \ln \frac{C_v}{C_{vi}} \right)$$

(9-103)

(b) Altitude:

$$h(v) = -\frac{1}{k} \ln \left[ \sigma(v) \right]$$

(9-104)

(c) Flight Path Angle:

$$\gamma(v) = C_v - \frac{L}{D} \ln \frac{C_v}{C_{vi}}$$

(9-105)
(d) Range

Same as equation (9-96) with equation (9-105) for $\tau$.

(5) Dimensionless time $\tau$ as independent variable ($\tau_i = 0$)

The basic equation is:

$$k \nu_i \tau \approx \frac{1}{\Theta} \ln \left( \frac{(\gamma - \Theta)(\gamma_1 + \Theta)}{(\gamma + \Theta)(\gamma_1 - \Theta)} \right)$$  \hspace{1cm} (9-106)

where $\Theta$ is given by equation (9-97).

From equation (9-106):

$$\gamma_i(\tau) = \Theta \left[ 1 + \left( \frac{\gamma_i - \Theta}{\gamma_i + \Theta} \right) e^{k \nu_i \Theta \tau} \right] \left[ 1 - \left( \frac{\gamma_i - \Theta}{\gamma_i + \Theta} \right) e^{k \nu_i \Theta \tau} \right]^{-1}$$  \hspace{1cm} (9-107)

With equation (9-107), the following explicit expressions were determined:

(a) Density ratio:

$$\sigma(\tau) \approx \sigma_i + \frac{k \nu_i}{2 U C_D D} \left\{ \Theta^2 \left[ 1 + \left( \frac{\gamma_i - \Theta}{\gamma_i + \Theta} \right) e^{k \nu_i \Theta \tau} \right]^2 \right\} \left[ 1 - \left( \frac{\gamma_i - \Theta}{\gamma_i + \Theta} \right) e^{k \nu_i \Theta \tau} \right]^2$$  \hspace{1cm} (9-108)

(b) Altitude:

$$h(\tau) = -\frac{1}{k} \ln \left[ \sigma(\tau) \right]$$  \hspace{1cm} (9-109)

(c) Velocity:

$$nr(\tau) \approx \nu_i + \exp \left\{ \frac{D}{L} \left[ \Theta \left( \frac{1 + \left( \frac{\gamma_i - \Theta}{\gamma_i + \Theta} \right) e^{k \nu_i \Theta \tau}}{1 - \left( \frac{\gamma_i - \Theta}{\gamma_i + \Theta} \right) e^{k \nu_i \Theta \tau}} \right) \right] \right\}$$  \hspace{1cm} (9-110)
(d) Flight Path Angle:

Equation (9-107)

(e) Range:

\[ X_N(t) \approx X_{N_1} + v_1 t \]  \hspace{1cm} (9-111)

(6) Range \( X_{N_1} \) as independent variable:

(a) Density ratio:

\[ \sigma(X_N) = \sigma_i + \frac{k}{2U_D L_B} \left\{ \Theta^2 - \left\{ \frac{1 + \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}}{1 - \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}} \right\}^2 \right\} \]  \hspace{1cm} (9-112)

(b) Altitude:

\[ h(X_N) = -\frac{1}{k} \ln \left[ \sigma(X_N) \right] \]  \hspace{1cm} (9-113)

(c) Velocity:

\[ v(X_N) = v_i \left\{ \frac{D}{t} - \left\{ \Theta \left\{ \frac{1 + \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}}{1 - \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}} \right\} \right\} \]  \hspace{1cm} (9-114)

(d) Flight Path Angle:

\[ \gamma(X_N) = \frac{\Theta \left\{ \frac{1 + \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}}{1 - \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}} \right\}}{\left\{ \frac{1 + \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}}{1 - \frac{(y_i - \Theta)}{(y_i + \Theta)} e^{k\Theta(X_N - X_{N_i})}} \right\}} \]  \hspace{1cm} (9-115)
9.6.4 Specific Forces Acting on the Entry Vehicle:

The specific force in Earth g's measured by accelerometers carried by the entry vehicle is:

\[ f_E = \frac{G(m)_0}{G(m)_E} U v^2 C_D \sigma \left[ 1 + (L/D)^2 \right]^{\frac{1}{2}} \]  

(9-116)

Specific force may be written as a function of any one of the independent variables given in equations (9-54) through (9-115) by substituting the appropriate expressions for v and \( \sigma \) as functions of the single independent variable selected. As an example, the specific force may be written as a function of velocity by substituting \( \sigma(v) \) as follows:

1. **Ballistic Trajectory** (\( L = 0; \gamma = \text{constant} = \gamma_b \)):

\[ f(v) = \frac{G(m)_0}{G(m)_E} U n^2 C_D \left[ \sigma_i + \frac{k \sin \gamma_b}{U C_D} \ln \frac{v}{v_i} \right] \]  

(9-117)

2. **Glide Trajectory**:

\[ f(v) = \frac{G(m)_0}{G(m)_E} \frac{(1-v^2)}{(L/D)^2} \left[ 1 + (L/D)^2 \right]^{\frac{1}{2}} \]  

(9-118)

3. **Skip Trajectory**:

\[ f(v) = \frac{G(m)_0}{G(m)_E} n^2 \left[ 1 + (L/D)^2 \right]^{\frac{1}{2}} \left[ \sigma_i U C_D + \frac{k}{2} \ln \frac{n}{\sigma_0} \left( 2 T_i - \frac{L}{D} \ln \frac{v}{v_i} \right) \right] \]  

(9-119)

Since \( \sigma_i \ll \sigma \) during most of the **ballistic trajectory**, Equation (9-117) shows that the total specific force is essentially independent of vehicle drag characteristics, vehicle mass to area ratio, and planetary surface satellite dynamic pressure. It is a function primarily of the exponential decay characteristics of the planetary atmosphere, velocity, and flight path angle.
It is interesting to note that the total specific force for the glide trajectory, on the other hand, is independent of the decay characteristics of the planetary atmosphere. Specific force decreases as (L/D) is increased for the glide vehicle.

The approximation δ₁ << δ cannot generally be made in the case of the skipping vehicle since altitude may not change by orders of magnitude during any one skip. At the bottom of the skip, however, this approximation is reasonably accurate; hence, the specific force experienced at the bottom of the skip is independent of C_D, vehicle frontal loading, and surface atmospheric density. It is a strong function of the planetary atmospheric decay characteristics, velocity, and flight path angle at the beginning of the skip.

9.6.5 Maximum Specific Force

It may be seen from equation (9-116) that the specific force is a maximum when:

$$\frac{d(v^2 \sigma)}{dT} = 0$$

(9-120)

Therefore, specific force is a maximum when:

$$v' = \frac{kv^2}{2} \sin \gamma$$

(9-121)

Using equation (3) of Derivation Summary 9.2, it is seen that maximum specific force of the ballistic vehicle occurs when:

$$v \approx 0.607 v_i$$

(9-122)

The magnitude of the maximum specific force for the 760 mile ballistic vehicle discussed in section 9.6.1 is readily determined by substituting equation (9-122) into equation (9-117) (assuming v_i = 1.0). A maximum
force of 17.3 g's at \( v = 0.607 \) was computed in this manner; this result compares well with a value of 18.3 g's at \( v = 0.59 \) determined from numerical solution of the non-linear equations of motion.

Solving equation (9-121) for the glide vehicle results in maximum specific force at \( v = 0 \). This result could have been predicted by noting on Fig. 9.9 that the approximate solution of this thesis has its maximum ordinate value when the abscissa equals zero. Numerical calculations, on the other hand, show that:

1. For \( \sqrt{k} \, L/D = 30 \), \( v_\phi \) at \( f_{\text{max}} \approx 0.23 \) (Corresponds to \( (L/D)_{\text{Earth}} \approx 1.0 \))
2. For \( \sqrt{k} \, L/D = 15 \), \( v_\phi \) at \( f_{\text{max}} \approx 0.29 \) (Corresponds to \( (L/D)_{\text{Earth}} \approx 0.5 \))
3. For \( \sqrt{k} \, L/D = 7.5 \), \( v_\phi \) at \( f_{\text{max}} \approx 0.35 \) (Corresponds to \( (L/D)_{\text{Earth}} \approx 0.25 \))
4. For \( \sqrt{k} \, L/D = 3.0 \), \( v_\phi \) at \( f_{\text{max}} \approx 0.45 \) (Corresponds to \( (L/D)_{\text{Earth}} \approx 0.1 \))

In general, the velocity at maximum specific force for glide vehicles in excess of \( (L/D)_{\text{Earth}} = 1.0 \) occurs at velocities less than 0.2.

It was shown in Derivation Summary (9.4) that flight path angle is a convenient parameter for determining the point of maximum specific force for the skip vehicle.

\[
(\gamma)_{f_{\text{max}}} = \frac{L}{2D} \left[ 1 - \sqrt{1 + \left( \frac{2D}{L} \right)^2 \left( \gamma^2 + 2UCD \frac{L}{D} \sigma_1 \right) } \right] \approx \frac{1}{2D} \left[ 1 - \sqrt{1 + \left( \frac{\gamma}{L/2D} \right)^2 } \right] \quad (9-123)
\]

For small initial flight path angles and large lift-drag ratios, the maximum specific force occurs when \( \gamma \approx 0 \). The velocity at maximum
specific force for the skip vehicle is:

\[(n)_r = \frac{n_i}{\gamma_{1D}} - \frac{1}{2}\left[1 - \sqrt{1 + \left(\frac{n_i}{\gamma_{2D}}\right)^2}\right]\]  

(9-124)

9.6.6 Stagnation Point Temperature

The stagnation point temperature was written in equation (7-21) as follows:

\[T_s = 1.392 \times 10^4 \left(\frac{HF}{EO}\right)^{\frac{1}{4}} \left(\frac{VF}{\sigma}\right)^{\frac{1}{4}} \frac{v_0}{v} \text{°Rankine} \]  

(9-125)

where \((HF)_{EO}\) is a "heating function ratio" of the atmosphere of planet 0 with respect to that of the Earth, and \((VF)\) is a vehicle function depending on the emissivity of the skin structure and on the radius of curvature in the vicinity of the stagnation point.

Stagnation point temperature may be written as a function of any one of the independent variables given in equations (9-54) through (9-115) by substituting the appropriate expressions for \(v\) and \(\sigma\) as functions of the single independent variable selected. As an example, the stagnation point temperature may be written as a function of velocity by substituting \(\sigma (v)\) as follows:

(1) Ballistic Trajectory \((L = 0; \gamma = \text{constant} = \gamma_b)\):

\[T_s (\nu_r) = 1.392 \times 10^4 \left(\frac{HF}{EO}\right)^{\frac{1}{4}} \left(\frac{VF}{\sigma}\right)^{\frac{1}{4}} \left(\frac{\sin \gamma_b \nu_{C_D}^{\frac{1}{2}} \ln \frac{\nu_{C_D}}{\nu_i}}{k}\right)^{\frac{1}{4}} \text{°R} \]  

(9-126)

(2) Glide Trajectory:

\[T_s (\nu_\phi) = 1.392 \times 10^4 \left(\frac{HF}{EO}\right)^{\frac{1}{4}} \left(\frac{VF}{\sigma}\right)^{\frac{1}{4}} \left(\frac{\nu_{C_D}^{\frac{1}{2}} (1 - \nu_\phi^2)^{\frac{1}{8}}}{(U_C_L)^{\frac{1}{8}}}\right) \text{°R} \]  

(9-127)
(3) Skip Trajectory:

\[ T_S = 1.392 \times 10^4 \left( \frac{H_F}{E_0} \right)^{1/4} \left( \frac{V_F}{V} \right)^{1/4} \rho^{3/4} \left( \sigma_i + \frac{k \ln \frac{\sigma}{\sigma_i}}{2UC_D} \left( 2 \gamma - \frac{L}{D} \ln \frac{\sigma}{\sigma_i} \right) \right)^{1/6} \]  

\(^{\circ}R \quad (9-128)\)

The stagnation point temperature for all three trajectory profiles is a function of \( C_L \) and/or \( C_D \), vehicle mass to area characteristics, and surface satellite dynamic pressure. It is interesting to note, however, that the stagnation point temperature of the glide vehicle is independent of the exponential decay characteristics of the planetary atmosphere.

9.6.7 Maximum Stagnation Point Temperature

From equation (9-125), stagnation point temperature is a maximum when:

\[ \frac{d}{dt}(v^6 \sigma) = 0 \]  

\((9-129)\)

This corresponds to:

\[ v' = \frac{kv^2 \sin \theta}{6} \]  

\((9-130)\)

For the ballistic vehicle, stagnation point temperature is a maximum when

\[ v = 0.846 \, v_1 \]  

\((9-131)\)

This result may be used in equation (9-126) to give the maximum temperature that may be expected during the ballistic trajectory. For example:

(1) \( v_1 = 1.0 \)

\( C_D = 1.0 \)

\( \gamma = -6^\circ \)
(VF) = 1.0

M/S = 1.0

Planet Earth

\( T_{s_{\text{max}}} \) for this situation is 4850° Rankine.

(2) If all conditions are identical as in (1) above except \( C_D = 10 \), then \( T_{s_{\text{max}}} = 3650° \) Rankine.

By differentiating equation (9-127) with respect to \( \tau \) and setting the result equal to zero, the velocity at maximum stagnation point temperature of the glide vehicle is determined. This occurs when:

\[ \nu_{\max} \phi = 0.815 \]  
(9-132)

The maximum temperature for the skip vehicle is incurred at the following flight path angle:

\[ \gamma_{\max} = \frac{L}{\nu_{\max}} \left[ 1 - \sqrt{1 + \left( \frac{6D}{L} \gamma \right)^2} \right] \]  
(9-133)

\[ \gamma_{\max} = \frac{L}{\nu_{\max}} \left[ 1 - \sqrt{1 + \left( \frac{6D}{L} \gamma \right)^2} \right] \]  
(9-133a)

This corresponds to a velocity of:

\[ \nu_{\max} = \nu_{\max} \left\{ \frac{1}{\sqrt{c_1} - \frac{L}{6D} \left[ 1 - \sqrt{1 + \left( \frac{6D}{L} \gamma \right)^2} \right] \} \right. \]  
(9-134)

Comparing equations (9-133) and (9-123) it is seen that for the skip vehicle:

\[ \gamma_{\max} = \gamma_{\max} \]  
(9-135)

Equation (9-135) shows that in a typical skip trajectory, the maximum temperature level is incurred before the flight path angle reduces to the
value corresponding to maximum specific force level. Comparison of

equations (9-124) and (9-134) shows that:

\[ \frac{v'(T_{s_{max}})}{v'(f_{max})} = e \left( \frac{-2 r_{(f_{max})}}{(L/D)} \right) = e \left( \frac{-\frac{23}{3} r_{(T_{s_{max}})}}{(L/D)} \right) \]  

Equation (9-136) demonstrates that the velocity at maximum temperature
is greater than the velocity at maximum specific force level during the
skip.

9.7 Summary

Atmospheric entry trajectories were examined in Chapter 8 from a
unique and instructive vantage point, namely by studying the behavior
of the Conservation Parameter \(|\frac{1-\xi}{\xi}|\). Examination of the conduct of
this parameter with respect to altitude, specific force level, energy,
and angular momentum led to the conclusion that the entry trajectory
could profitably be analyzed in three distinct operational regimes;
these regimes were named the Keplerian, Intermediate, and Gas-Dynamic
Phases.

The investigation discussed in Chapter 9 was oriented toward
utilizing the Conservation Parameter for determination of guidance
quantities. Two dissimilar entry profiles were considered:

1. The direct entry profile results from perturbation of a stable satellite
orbit by means of retro-rocket thrust. In this profile, the vehicle generally traverses in sequence the
Keplerian, Intermediate, and Gas-Dynamic Phases to a landing.

* Or relatively stable.
(2) The degenerate orbital profile consists of a series of braking passes through the outer reaches of the atmosphere in order to reduce the energy level preparatory to final direct entry. In this profile, the vehicle enters the Intermediate Phase in a series of near impulses until the energy level is reduced sufficiently for final transition to the Gas-Dynamic Phase.

The feasibility of using the Conservation Parameter as a switching function for profile (1) and as a prediction function for profile (2) was discussed.

Range capabilities and range accuracy in the Keplerian Phase were discussed in terms of geometric quantities at the trajectory modification point for entry from elliptical satellite orbits, and in terms of velocity impulse and engine gimbal angle for entry from circular satellite orbits. Examples of range sensitivity to errors in operation of the retro-rocket system were computed and presented in both graphical and analytical form.

Approximate solutions of various guidance quantities were derived for ballistic, glide, and skip profiles in the Gas-Dynamic Phase. Limitations on the approximate solutions were established after comparing them to numerical solutions obtained from the non-linear equations of motion. Analytical determination of instantaneous and maximum deceleration loads and temperature levels was discussed.
9.8 Derivation Summaries

All Derivation Summaries referred to in the previous sections of this chapter are appended.

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Solution of Keplerian Phase of Entry Trajectory in Terms of Geometric Quantities at the Trajectory Modification Point.

Angular momentum is conserved in a pure Keplerian trajectory; \( p \) is a convenient constant of integration to use in solving the equations of motion for the Keplerian Phase of the trajectory.

\[
p = rv_\phi
\]

Assuming planar motion*, the horizontal component of velocity with respect to inertial coordinates is:

\[
v_\phi = r\phi'
\]

Therefore, angular momentum is written:

\[
p = r^2\phi'
\]

Using equations (3) and (9-4), equation (9-5) is written:

\[
\frac{2}{r^3} + \frac{1}{r^2} = 0
\]

This equation may be written in a form suitable for integrating by making the following transformation:

\[
u = \frac{1}{r}
\]

Therefore:

\[
du = -dr/r^2
\]

\[
r' = -r^2u' = -r^2\phi' = -\frac{du}{d\phi}
\]

* See Section 6.1 for a quantitative discussion of the effects of non-spherical gravitational components in limiting the assumption of planar motion.
Derivation Summary 9.1 (cont.)

\[ r'' = -p^2 u^2 \frac{d^2 u}{d\phi^2} \]  

(8)

Equation (4) is written with the aid of equation (8) as:

\[ \frac{d^2 u}{d\phi^2} + \frac{1}{p^2} u = \text{constant} \]  

(9)

The solution of equation (9) is as follows:

\[ u(\phi) = C_1 \cos(\phi + C_2) + \frac{1}{p^2} \]  

(10)

It is convenient to express \( u \) in terms of the true anomaly:

\[ \phi = \gamma + \theta \]  

(11)

\( \gamma \) is the angle measured from the ascending line of nodes to the line of apsides; \( \theta \) is true anomaly. If the entry transfer ellipse is fixed in inertial space, then \( \gamma \) is constant. Therefore:

\[ u(\theta) = C_1 \cos(\theta + C_3) + \frac{1}{p^2} \equiv \frac{1}{r} \]  

(12)

Constant \( C_3 \) is determined by noting that the time rate of change of radius (altitude) is zero at perigee and apogee. From equation (7):

\[ r' = -r^2 u' = r^2 C_1 \sin(\theta + C_3) \theta' \]  

(13)

Therefore:

\[ \theta + C_3 = 0 + \eta \pi (\eta = 0,1,2...) \]  

(14)

From Fig. 4.1, it may be seen that \( \theta = \pi \) at apogee and \( \theta = 0 \)
at perigee. The equation of the ellipse is given in Fig. 4.1 in terms of $R$, $a$, $\theta$, and $\varepsilon$. Writing this in dimensionless form:

$$\frac{1}{r} = \frac{1 + \varepsilon \cos \theta}{a_N(1 - \varepsilon^2)} \tag{15}$$

In equation (15), $a_N$ is the semi-major axis non-dimensionalized with respect to $R(m_0)$. Equation (15) may be written:

$$\frac{1}{r} = \frac{a_N}{b_N^2} (1 + \varepsilon \cos \theta) \tag{16}$$

where $b_N$ is dimensionless semi-minor axis of the ellipse. With $C_3 = 0$, equation (12) and equation (16) are used to eliminate $r$:

$$p^2 = \frac{b_N^2}{a_N} \tag{17}$$

$$C_1 = \frac{a_N}{b_N^2} \varepsilon \tag{18}$$

The constant $C_1$ can be determined by differentiating (12):

$$C_1 \sin \theta = \frac{r'}{r^2 \theta'} = \frac{r'}{p} \tag{19}$$

Using equations (9.4-2) and (9.4-3) given in Fig. 9.4, this may be written:

$$C_1 \sin \theta = \frac{\tan \gamma_I}{r} \tag{20}$$

From equation (12):

$$C_1 \cos \theta = \frac{1}{r} - \frac{1}{r^2 v_I^2 \cos^2 \gamma_I} \tag{21}$$
Squaring and adding equations (20) and (21) makes it possible to eliminate \( \theta \). Using the result in equations (17) and (18) gives:

\[
\varepsilon^2 = 1 - v_r^2 r \cos^2 \gamma(2 - v_r^2 r) \tag{22}
\]

Equation (22) expresses the eccentricity of the entry transfer ellipse in terms of velocity, radius, and flight path angle. The magnitude of these quantities at the trajectory modification point is denoted by the subscript \( m \).

The semi-major axis is determined from equations (17) and (22) and the following relation:

\[
\varepsilon^2 = 1 - \frac{b_N^2}{a_N^2} \tag{23}
\]

The resulting equation in terms of quantities at the trajectory modification point is given as equation (9-8). Equations (9-9) through (9-18) are readily determined from the foregoing results and from the equations listed in Fig. 4.1.

Time of flight may be derived in terms of \( \theta \) by noting from equation (3) and (11), with \( \eta \) = constant:

\[
\theta' = \frac{p}{r^2} \tag{24}
\]

Using equations (16) and (17), this is written:

\[
\theta' = \frac{(1 + \varepsilon \cos \theta)^2}{p^3} \tag{25}
\]
Derivation Summary 9.1 (cont.)

Integrating equation (25) gives the time of flight listed as equation (9-19).

The velocity at any instant in terms of quantities at the trajectory modification point is written from equation (9-8) with \( a_N = \text{constant} \). Flight path angle may be written in terms of quantities at the trajectory modification point from equation (9.4-3) with angular momentum equal to a constant.

Range is defined by equation (9-23). With the aid of equations (17) and (18), equations (9-24) and (9-25) are written. The true anomaly may be eliminated from the range expression by taking the cosine of equation (9-23) and substituting equations (9-24) and (9-25) into the result. The range expression so derived is given by equations (9-26) and (9-27).

With equations (9.3-1) through (9.3-5) given in Fig. 9.3, the foregoing results are written in terms of velocity impulse \( \delta v \), and engine gimbal angle \( A_e \) for the special case of a circular reconnaissance orbit. The results are summarized as equations (9-29) through (9-35).

Equations (9-36) through (9-42) are derived by writing for equation (9-3):

\[
\cos \theta = \cos X_N \cos \theta_m - \sin X_N \sin \theta_m
\] (26)

\( \theta \) and \( \theta_m \) are eliminated from equation (26) by using equations (9-24) and (9-25):
\[
\frac{r_m}{r} \left[ p^2 - r \right] = \cos X_N (p^2 - r_m) - \sin X_N \left[ \varepsilon^2 r_m^2 - (p^2 - r_m)^2 \right]^{\frac{1}{2}}
\]

(27)

Substituting for eccentricity from equation (9-7) and for angular momentum from equations (9.4-3) gives equation (9-36). This may be written in the following alternate forms:

\[
v_{\text{Im}}^2 = \frac{r (1 - \cos X_N)}{\cos \gamma_\text{I} \left[ r_m^2 \cos \gamma_\text{Im} - r r_m \cos \left( X_N + \gamma_\text{Im} \right) \right]}
\]

(28)

\[
\frac{r_m}{r} \cos (2 \gamma_\text{Im}) - \cos (2 \gamma_\text{Im} + X_N) = \frac{2(1 - \cos X_N)}{v_{\text{Im}}^2 r_m} - \left[ \frac{r_m}{r} - \cos X_N \right]
\]

(29)

Substituting equations (9-29), (9-31) and (9.3-5) into equation (26) gives equation (9-39) or its alternate form (9-40). Equations (9-41) and (9-42) were derived by implicit differentiation of equation (9-39).
Solution of the Ballistic Trajectory

The most convenient set of planar equations for solving the ballistic trajectory are equations (6-40) through (6-44); these were written in terms of components in the $\vec{i}_x$ and $\vec{i}_z$ directions. Assuming powerless flight ($\Gamma_N = 0$), it may be seen from equation (6-41) that if the centrifugal force and Coriolis force terms in the $\vec{i}_z$ direction are balanced by gravitational force, then:

$$\gamma' = 0$$ (1)

Therefore:

$$\gamma = \text{constant} = \gamma_b \text{ ("ballistic" flight path angle)}$$ (2)

If the gravity force in the $\vec{i}_x$ direction is small in comparison to drag force, equation (6-40) gives:

$$v' = -n_D = -UC_Dv^2\delta$$ (3)

where:

$$U = \frac{\rho(SL)}{2} \frac{R(m)\sigma}{M}$$ (4)

The time rate of change of density ratio is:

$$\sigma' = -kv\sigma\sin\gamma$$ (5)

Therefore, from equations (3) and (5):

$$\frac{dv}{d\sigma} = \frac{UC_D}{k\sin\gamma}v$$ (6)
Integrating equation (6) gives:

\[
\ln \frac{v}{v_i} = \frac{U_C D}{k \sin \gamma_b} (\sigma - \sigma_i) \tag{7}
\]

The solution for range is readily determined from equations (6-43) and (6-44) with \( \gamma = \gamma_b \):

\[
X_N - X_{Ni} = \cot \gamma_b \ln \frac{r}{r_1} = \cot \gamma_b (h-h_i) \tag{8}
\]

Atmospheric density ratio may be solved as a function of time by using equation (5) and (7):

\[
\frac{\sigma' - U_C D \sigma_i}{k \sin \gamma_b} = -k \nu_i \sin \gamma_b e \tag{9}
\]

Integrating equation (9) gives:

\[
-(\tau - \tau_i) k \nu_i \sin \gamma_b e = \ln \left| \frac{\sigma}{\sigma_i} \right| - \frac{U_C D}{k \sin \gamma_b} (6 - \sigma_i)
\]

\[
+ \left( \frac{U_C D}{k \sin \gamma_b} \right)^2 \frac{(\sigma^2 - \sigma_i^2)}{4} + \cdots \left( \frac{U_C D}{k \sin \gamma_b} \right)^n \frac{(\sigma^n - \sigma_i^n)}{n \cdot n!}
\]  \tag{10}

Equations (7), (8), and (10) are three fundamental equations from which all of the equations (9-54) through (9-70) are derived.
Solution of the Glide Trajectory

If the flight path angle, $\gamma$, at the onset of the Gas-Dynamic Phase is zero, then the vertical velocity at this point is zero. Basic assumptions required for equilibrium gliding flight to exist throughout the trajectory are:

(a) Vertical accelerations are small, i.e., $v_r' \rightarrow 0$.

(b) Vertical component of drag force is small; i.e. $UC_D v_r \sigma \rightarrow 0$

With the above assumptions and the following approximations:

$$|\frac{\Delta \phi}{\phi}| \ll 1.0$$

(1)

$$(1 + (L/D)\frac{v_r}{v_\phi}) \cong 1.0$$

(2)

$$v_{\phi} \cong v_\phi$$

(3)

$$r = (1 + h) \cong 1.0$$

(4)

equation (9-45) is written as follows:

$$\frac{(1 - \frac{\phi^2}{v_\phi})}{v_\phi} = \frac{L}{D} C_D v_\phi U$$

(5)

Substituting equation (5) into equation (9-46) gives:

$$v_\phi' = \frac{\frac{v_\phi^2}{L/D} - 1}{L/D}$$

(6)
For L/D equal to a constant, equation (6) may be integrated to give:

\[
\frac{(v-1)(v_{\phi}+1)}{(v+1)(v_{\phi}-1)} = e^{\frac{2(\tau_{-1})}{L/D}}
\]

Equation (7) holds only when \( v_{\phi} \leq 1.0 \). Solving this explicitly for \( v_{\phi} \) gives equation (9-87).

Flight path angle may be computed by differentiating equation (5) with respect to \( v_{\phi} \):

\[
\frac{1}{D} \frac{d\sigma}{d\nu} = k(\frac{\nu}{\nu_{\phi}}) \nu_{\phi} \frac{1}{1-\nu_{\phi}^2} \tan \gamma
\]

Using equation (6) above and equation (5) of Derivation Summary 9.2:

\[
\frac{1}{d\nu} \frac{d\gamma}{d\nu_{\phi}} = \frac{k(\frac{\nu}{\nu_{\phi}}) \nu_{\phi} \frac{1}{1-\nu_{\phi}^2} \tan \gamma}{(1-\nu_{\phi}^2)}
\]

Also, the following may be written:

\[
\frac{1}{\nu} \frac{d\nu}{d\nu_{\phi}} = \frac{1}{\nu_{\phi}} \left(1 + \nu_{\phi} \frac{d\gamma}{d\nu_{\phi}}\right)
\]

The second term in the parenthesis of equation (10) is the product of two small quantities and can be neglected in comparison to 1.

Using equation (9) and (10) in equation (8) gives:

\[
\tan \gamma = \gamma = \frac{-2}{k(L/D)\nu_{\phi}^2}
\]

Range is determined as a function of altitude from equations (9-43) and (9-44):
Using equation (11) for $\tan \gamma$ and equation (5) for $v_\phi (h)$ gives:

$$\frac{dX_N}{dh} = \frac{1}{r \tan \gamma} \frac{1}{\tan \gamma}$$  \hspace{1cm} (12)

Integrating equation (13) gives equation (9-76).

Range as a function of velocity is determined from equation (9-43) and (6):

$$\frac{dX_N}{dv_\phi} = \frac{v_\phi}{r} \frac{L/D}{(v_\phi^2 - 1)} \approx \frac{(L/D)v_\phi}{(v_\phi^2 - 1)}$$  \hspace{1cm} (14)

Integrating equation (14) gives equation (9-84). Velocity as a function of range may be written by solving equation (9-84) explicitly for $v_\phi$.

Equations (5), (7), (11), (9-76), and (9-84) are fundamental equations from which equations (9-71) through (9-93) were derived by algebraic operations.
Solution of the Skip Trajectory

The glide solution discussed in Derivation Summary 9.3 is valid only when a lifting vehicle enters the planetary atmosphere under very special initial conditions, viz. zero initial flight path angle at velocities equal to or less than orbital velocity. The glide solution does not apply to trajectories with non-zero flight path angles because a finite initial vertical velocity induces a trajectory in which the vertical acceleration term in the equation of motion is not small compared to lift force.

The following basic assumptions are made in deriving the solutions for the skipping trajectory; i.e., the trajectory of a lifting vehicle entering the planetary atmosphere with a finite initial flight path angle:

1. The difference between the components of gravity and centrifugal force in the lift direction is negligible when compared to lift force.
2. Coriolis force is negligible when compared to lift force.
3. The difference between the components of gravity and centrifugal forces in the drag direction is negligible when compared to drag force.

Using assumptions (1) and (2) in equation (6-41), and assuming powerless flight ($\Gamma_N = 0$), gives:

$$v \gamma' = nL \quad (1)$$
Using assumption (3) in equation (6-40) gives:

\[ v' = -n_D \]  \hspace{1cm} (2)

Dividing equation (2) by equation (1) gives:

\[ \frac{1}{v} \frac{dv}{d\gamma} = -\frac{D}{L} \]  \hspace{1cm} (3)

Assuming \((L/D) = \text{constant}\) and integrating gives:

\[ v = v_1 e^{-\frac{(\gamma - \gamma_1)}{L/D}} \]  \hspace{1cm} (4)

Equation (4) gives velocity as a function of flight path angle.

Dividing equation (1) by \(\gamma'\) given by equation (5) of Derivation Summary 9.2 gives:

\[ \frac{d\gamma}{d\sigma} = -\frac{UC_D(L/D)}{k \sin \gamma} \]  \hspace{1cm} (5)

Integrating equation (5) for a constant lift-drag ratio gives:

\[ \cos \gamma - \cos \gamma_1 = \frac{UC_D}{k} (L/D)(\sigma - \sigma_1) \]  \hspace{1cm} (6)

For small flight path angles:

\[ \cos \gamma \approx 1 - \frac{\gamma^2}{2} \]  \hspace{1cm} (7)

With equation (7), equation (6) is written:

\[ \sigma = \sigma_1 + \frac{k}{2UC_D L/D} (\gamma_1^2 - \gamma^2) \]  \hspace{1cm} (8)
Derivation Summary 9.4 (cont.)

Equation (8) relates density ratio (altitude) and flight path angle. The rate of change of range with respect to density ratio is determined by dividing equation (6-44) by density ratio given by equation (5) of Derivation Summary 9.2:

$$\frac{dX_N}{d\sigma} = -\cot \gamma \frac{\sigma}{k r}$$  \hspace{1cm} (9)

At altitudes where skipping may occur, the distance from the center of the planet to the vehicle differs little from the mean planetary radius, hence \( r \approx 1.0 \). With this assumption and the small angle assumption on \( \cot \gamma \), equation (9) is written:

$$\frac{dX_N}{d\sigma} \approx -\frac{1}{\sigma k \gamma}$$  \hspace{1cm} (10)

Equation (8) is used to eliminate \( \gamma \) from equation (10). Integrating the resulting equation gives range as a function of density ratio. This integrated expression is listed as equations (9-96) and (9-97).

Flight path angle as a function of time can be determined by writing equation (1) as:

$$\gamma' = \nu v_c L \sigma$$  \hspace{1cm} (11)

Assuming that the velocity change during a single skip is small and substituting \( \sigma (\gamma') \) from equation (8) into equation (11) gives an expression for \( \gamma' \) which may be integrated. The resulting solution
for \( \gamma(t) \) is listed as equation (9-107).

Equations (4), (8), (9-96), and (9-107) are fundamental equations from which equations (9-94) through (9-115) were determined.

The flight path angle at maximum specific force level is determined by substituting equation (2) into equation (9-121):

\[
2\sigma U C_D = - k \sin \gamma \quad \text{at } f_{\max}
\]

Thus, at maximum specific force level:

\[
\gamma \approx \frac{-2 U C_D}{k} \delta
\]

Substituting equation (8) for \( \delta(\gamma) \) into equation (13) and solving explicitly for \( \gamma \) gives equation (9-123). The velocity at maximum specific force level is determined by substituting equation (9-123) into equation (4).

The flight path angle at maximum stagnation temperature level is determined by substituting equation (2) into equation (9-130). The velocity at maximum stagnation temperature level is determined by substituting the resulting solution for \( \gamma \) into equation (4).
Chapter 10

APPROXIMATE ANALYTICAL SOLUTION OF GUIDANCE PARAMETERS AND CONSTRAINTS FOR THE DEGENERATE ORBITAL ENTRY PROFILE

10.1 Introduction

The investigation described in Chapter 9 was devoted largely to derivation of guidance parameters and constraints for the direct entry profile. This atmospheric entry trajectory is characterized by sequential transition from Keplerian flight through the Intermediate Phase into the Gas-Dynamic flight. The Conservation Parameter \( \frac{1-\frac{\mu}{\ell}}{\mu} \) was convenient for defining the boundaries existing at transition points between the three separate operational regimes encountered during the course of entry.

The degenerate orbital profile is examined in Chapter 10 from an entirely different standpoint. This entry profile has a time history far removed from that of the direct entry profile. A series of braking passes through the outer reaches of the atmosphere are employed in this concept of entry for purposes of reducing the energy level of the vehicle until final direct entry is assured. The Intermediate Phase is entered in a series of near impulses in the vicinity of perigee; during each of these pulses, energy is transferred to the planetary atmosphere until ultimately the energy level is low enough that final transition to flight in the Gas-Dynamic Phase evolves.
The characteristics of powerless flight in the outer reaches of the atmosphere are examined in Chapter 10 for vehicles initially in circular and elliptical reconnaissance orbits. For this analysis, the entry mission was assumed to start at the first perigee. Analytical techniques are presented for predicting the range and time of flight of the craft, the rate of circularization of elliptical orbits, the rate of decay of circular orbits, the number of orbits remaining under specified initial conditions, perigeeal and apogeeal decay rates, and the effect of the vehicle's aerodynamic and mass characteristics on these quantities.

It is shown in this chapter that a true circular orbit or a linear decaying circular orbit cannot exist, even under idealized conditions of injecting a vehicle exactly at circular orbital velocity in the vicinity of a spherical planet. Because of the influence of the planetary atmosphere on the dynamics of energy transfer, the altitude and flight path angle in the high altitude circular orbit oscillate at orbital frequency with such small damping as to be essentially undamped.

The analysis of the circularization phase of elliptical reconnaissance orbits described in this chapter shows that the drag characteristics of the vehicle are important in specifying the resulting trajectory and that the lift characteristics of the vehicle are relatively unimportant. The altitude at apogee decays in proportion to the sum of two large constants which are almost equal in magnitude while the altitude at perigee decays in proportion to the difference of these two

* The constants are described by modified Bessel Functions of the first kind; i.e., Bessel Functions of the first kind with pure imaginary arguments.
near-equal constants. To a first order, therefore, perigee altitude remains constant during the circularization process.

Guidance parameters important in the conceptual phases of entry systems are derived in closed form in this chapter. The solutions obtained are compared with more exact machine-computed numerical solutions. As a result of this comparison, accuracy and limitations of the analytical solutions of this thesis are determined.

10.2 The Altitude Differential Equation

The planar equations of motion in the $\overline{I}_r$ and $\overline{I}_\phi$ directions were written as equations (6-47) and (6-48) in Chapter 6 of this thesis. In powerless flight ($\Gamma_N = 0$) and assuming $1 + L/D \tan \gamma = 1.0$, these equations reduce to:

\begin{align*}
  r'' &= \frac{v^2}{r} - \frac{1}{r^2} - n_D \sin \gamma + n_L \cos \gamma \quad (10-1) \\
  v_{I\phi}' &= -\frac{v_{I\phi}}{r} r' - n_D \cos \gamma \quad (10-2) \\
  X_N' &= \frac{v_{I\phi}}{r} \quad (10-3)
\end{align*}

Equation (10-1) shows that the acceleration in the $\overline{I}_r$ direction is equal to the centrifugal specific force minus gravitational acceleration plus lift and drag specific force components in this direction. For small flight path angles, the drag specific force in the radial direction may be neglected and $\cos \gamma \approx 1.0$. Substituting for $n_D$ and $n_L$ from equation (6-36) and (6-37) and assuming $v \approx v_{I\phi}$, these equations are written:

\begin{equation}
  r'' = \frac{v^2}{r} - \frac{1}{r^2} + \frac{(L/D)C}{2} \frac{v^2}{\rho_{I1}} \quad (10-4)
\end{equation}
In equations (10-4) and (10-5), \( C \) is a dimensionless constant for a particular vehicle at a given initial altitude above a particular planet:

\[
C = R_0 \left( \frac{C D}{M} \right) \rho_1
\]  

(10-7)

\( \rho_1 \) is the atmospheric density in slugs per cubic foot corresponding to the initial flight altitude.

It is convenient to write equations (10-4) and (10-5) in terms of \( \Delta r \) and \( \Delta v \), which are defined as changes in radius and velocity from given initial values \( r_i \) and \( v_i \):

\[
\Delta r = r - r_i \quad (10-8)
\]

\[
\Delta v = v - v_i \quad (10-9)
\]

Since \( r_i \) and \( v_i \) are constants in equations (10-8) and (10-9), the rate of change of \( \Delta r \) is equal to the rate of change of \( r \) and the rate of change of \( \Delta v \) is equal to the rate of change of \( v \), i.e.:

\[
\Delta r' = r'
\]

(10-10)

\[
\Delta v' = v'
\]

(10-11)

With equations (10-8) through (10-11), equation (10-4) is written:

\[
\frac{\Delta r''}{r_i} = \frac{\alpha_i^2 \left( \frac{2 \Delta v'}{\alpha_i r_i} + \left( \frac{\Delta r'}{\alpha_i r_i} \right)^2 \right)}{\left( 1 + \frac{\Delta r'}{r_i} \right)} - \frac{1}{r_i^2 \left[ 1 + \frac{2 \Delta v'}{r_i} + \left( \frac{\Delta r'}{r_i} \right)^2 \right]} + \frac{L}{D} C \frac{\alpha_i^2}{2} \frac{\rho}{\rho_i} \left[ 1 + \frac{2 \Delta v'}{\alpha_i r_i} + \left( \frac{\Delta r'}{r_i} \right)^2 \right]
\]

(10-12)
In the Keplerian and Intermediate Phases of the trajectory and through much of the Gas-Dynamic Phase of flight, \((\Delta v)^2\) and \((\Delta r)^2\) and all higher order terms can be neglected; therefore equation (10-12) is written:

\[
\Delta r'' = \frac{v_i^2}{r_i} \left[1 - \left(\frac{v_i}{v_i^*}\right)^2\right] - \Delta r\left(\frac{v_i}{r_i}\right)^2 \left[1 - 2\left(\frac{v_i}{v_i^*}\right)^2\right] + \Delta v \left[2\frac{v_i}{r_i} + \frac{l}{D} C_{D} \frac{\rho}{\rho_i}\right]
\]

\[+ \frac{l}{D} C_{r} \frac{v_i^2}{2} \frac{\rho}{\rho_i}\]  

(10-13)

\(v_{ci}\) in equation (10-13) represents the dimensionless circular orbital velocity at the initial altitude:

\[v_{ci}^2 = \frac{1}{r_i}\]  

(10-14)

There are three time varying quantities in equation (10-13): \(\Delta v\), \(\Delta r\), and \(\rho\). Equation (10-5) may be used to give an expression for \(\Delta v\):

\[
\frac{(rv)^1}{(rv)^2} = - \frac{C}{2} \frac{\rho}{\rho_i}
\]

(10-15)

Integrating equation (10-15) with \(\tau = 0\) gives:

\[
\Delta v = - \Delta r \frac{v_i}{r_i} = C \frac{v_i^2}{2} \int_{0}^{\tau} \left(1 - \frac{4r}{r_i^2} \frac{\rho}{\rho_i}\right) d\tau
\]

(10-16)

The first term on the right hand side of equation (10-16) represents the change in velocity resulting from kinetic-potential energy trade-off; the second term represents velocity loss due to drag.
Substituting equation (10-16) into equation (10-13) gives:

\[
\frac{d^2 \Delta r}{dr^2} = \frac{2}{r_i^2} \left[ 1 - \left( \frac{v_r}{v_i} \right)^2 \right] - \Delta r \left\{ \frac{v_r^2}{r_i^2} \left[ 3 - 2 \left( \frac{v_r}{v_i} \right)^2 \right] + \frac{1}{D} C_i \frac{\rho_i}{\rho_i} \right\} + \frac{1}{D} C \frac{v_r^2 \rho_i}{\rho_i}
\]

\[-C v_i^2 \left( \frac{1}{r_i} + \frac{C}{2} \frac{\rho_i}{\rho_i} \right) \int_0^\infty \left( 1 - \frac{\Delta r}{r_i} \right) \frac{\rho}{\rho_i} dr\]

(10-17)

Equation (10-17) is the basic equation which must be solved in order to determine the behavior of \( \Delta r \) with time. It is non-linear because of the dependence of \( \rho \) on \( r \). If the exponential model of the planetary atmosphere is assumed, then:

\[
\frac{\rho}{\rho_i} = e^{-k \Delta r}
\]

(10-18)

Solutions to the non-linear equation (10-17) with the atmospheric model (10-18) are difficult to obtain by conventional techniques. It is shown subsequently in this chapter that a surprisingly accurate analytical solution may be obtained which is valid over reasonably large time intervals by a sequence of approximations. This method may be described as follows:

1. Assume a reasonable approximation for \( \Delta r(T) \) based on the initial conditions established for the trajectory.
2. With this approximation of \( \Delta r(T) \) the atmospheric density behavior is specified as a function of time for the particular atmospheric model chosen.
3. Substitute \( \rho(T) \) from step (2) into equation (10-17) and solve for \( \Delta r(T) \). This gives a refined approximation for \( \Delta r(T) \) which should be more accurate than the assumed solution of step (1).
(4) Repeat steps (2) and (3) as necessary. Provided the solution converges, the degree of accuracy at any stage may be estimated by comparing successive approximations.

This method for obtaining analytical solutions is carried out in subsequent sections of this chapter for both circular and elliptical reconnaissance orbits. Accuracy of the analytical solutions obtained by this procedure is confirmed by comparing them with machine-computed numerical solutions for various vehicles and initial conditions.

10.3 Circular Orbital Entry

If the vehicle is in a circular reconnaissance orbit at zero time, then certain simplifications may be made in equation (10-17).

Specifically:

\[ v_i = v_{ci} \]  \hspace{1cm} (10-19)

Equation (10-17) therefore reduces to:

\[ \Delta r'' = -\Delta t \left( \frac{v_i^2}{\rho_i} \left[ 1 + \frac{L}{D} \frac{C_r}{\rho_i} \right] \right) + \frac{L}{D} \frac{v_i^2}{2} \frac{C}{\rho_i} \]

\[ -C \rho_i \left( \frac{1}{\rho_i} + \frac{C}{2} \frac{L}{D} \frac{\rho}{\rho_i} \right) \left( 1 - \frac{\Delta r}{\rho_i} \right) \frac{\rho}{\rho_i} \, d\tau \]  \hspace{1cm} (10-20)

Since the vehicle is initially in a circular orbit, a reasonable approximation to \( \Delta r(\tau) \) required in the first step of the method of solution outlined in section 10.2 is:

\[ \Delta r = 0 \]  \hspace{1cm} (10-21)

Equation (10-21) is the equation of a circle. If \( \Delta r \) does not change,
then regardless of the atmospheric model*:

$$\rho / \rho_1 = 1$$  \hspace{1cm} (10-22)

Substituting equation (10-22) into equation (10-20) gives:

$$\Delta r'' + \Delta \left[ (\frac{v_1}{r_1})^2 \left(1 + \frac{1}{D}C_1 r_1\right) + C_1^2 \left( \frac{1}{r_1} + \frac{C}{2} \frac{1}{D} \right)^2 \right] \left( \frac{\Delta r}{r_1} \right) = \frac{1}{D} \frac{C}{2} \omega_c^2$$

(10-23)

The following constants are defined:

$$\omega_c^{-2} = \left( \frac{v_1}{r_1} \right)^2 \left( 1 + \frac{1}{D}C_1 r_1 \right)$$  \hspace{1cm} (10-24)

$$A_c = (L/D) \frac{Cv_1^2}{2}$$  \hspace{1cm} (10-25)

$$B_c = Cv_1^3 \left[ \frac{1/r_1 + (L/D)C}{2} \right]$$  \hspace{1cm} (10-26)

$$C_c = \frac{B_c}{r_1}$$  \hspace{1cm} (10-27)

The subscript "c" in definitions (10-24) through (10-27) is used for purposes of identifying the constants as applying to the circular orbital case only. In terms of these constants, equation (10-23) is written:

$$\Delta r'' + \omega_c^{-2} \Delta r - C_c \int_0^\tau \Delta r'd\tau' = A_c - B_c\tau$$

(10-28)

It is noted that at the initial point ($\tau = 0$):  

* Equation (10-22) assumes a homogeneous atmosphere around the planet; i.e., the density at a specific altitude is constant. This, of course, neglects diurnal and latitude density variations.
\[ \Delta r''(0) = A_c \]

\[ \Delta r'(0) = 0 \]  \hspace{1cm} (10-29)

\[ \Delta r(0) = 0 \]

Equation (10-28) is solved by means of the Laplace transform:

\[ \mathcal{L}\{\Delta r(t)\} = R(s) = \int_{0}^{\infty} e^{-st} \Delta r(t) \, dt \]  \hspace{1cm} (10-30)

The transform of the first and second derivatives is given by:

\[ \mathcal{L}\{\Delta r'(t)\} = sR(s) - \Delta r(0) \]  \hspace{1cm} (10-31)

\[ \mathcal{L}\{\Delta r''(t)\} = s^2 R(s) - s \Delta r(0) - \Delta r(0) \]  \hspace{1cm} (10-32)

Applying the transformation to equation (10-28) gives:

\[ R(s) = \frac{A_c - B_c s}{s^3 + \omega_c^2 s - C_c} \]  \hspace{1cm} (10-33)

Equation (10-33) may be solved by setting:

\[ s^3 + \omega_c^2 s - C_c = (s + b_1)(s + b_2)(s + b_3) \]  \hspace{1cm} (10-34)

where:

\[ b_1 = -2x_1 \]  \hspace{1cm} (10-35)

\[ b_2 = x_1 - i y_1 \]  \hspace{1cm} (10-36)

\[ b_3 = x_1 + i y_1 \]  \hspace{1cm} (10-37)
The following constants are defined:

\[
x_1 = \frac{\omega_c}{\sqrt{3}} \sinh \left( \frac{1}{3} \sinh^{-1} \left( \frac{C_c \frac{3 \sqrt{3}}{2 \omega_c^3}}{10^{-38}} \right) \right)
\]

(10-38)

\[
y_1 = \omega_c \cosh \left( \frac{1}{3} \sinh^{-1} \left( \frac{C_c \frac{3 \sqrt{3}}{2 \omega_c^3}}{10^{-39}} \right) \right)
\]

(10-39)

\[
i^2 = -1
\]

(10-40)

Substituting equations (10-34) through (10-40) into equation (10-33), expanding by partial fractions, and taking the inverse transform of the result gives:

\[
\Delta r(\tau) = \frac{A_c}{(g_{x_1^2} + y_1^2)} \left[ e^{2x_1 \tau} - e^{-x_1 \tau} \left( \cos y_1 \tau + \frac{3x_1}{y_1} \sin y_1 \tau \right) \right]
\]

\[
+ \frac{B_c}{2x_1(x_1^2 + y_1^2)(g_{x_1^2} + y_1^2)} \left[ \left( g_{x_1^2} + y_1^2 \right) - e^{2x_1 \tau} (x_1^2 + y_1^2) \right]
\]

\[
- 2 \sinh^{-1} \left[ 4x_1^2 \cos y_1 \tau + \frac{x_1}{y_1} (3x_1^2 - y_1^2) \sin y_1 \tau \right] \right]
\]

(10-41)

Equation (10-41) is the solution of equation (10-17) for circular orbital motion. Considerable simplifications may be made in this equation by examining the magnitudes of the quantities involved for reasonable initial reconnaissance orbital conditions:

\[
x_1 \approx \frac{3}{2} \frac{B_c}{r_1 \omega_c^2} \ll 1.0
\]

(10-42)
\[ y_1 \approx \omega_c \] (10-43)

\[ x_1 \ll y_1 \] (10-44)

\[ e^{2x_1 \tau} \approx 1 + 2x_1 \tau \approx 1 + \frac{3}{r_1} \frac{B_c \tau}{\omega_c^2} \] (10-45)

\[ e^{-x_1 \tau} \approx 1 - \frac{3}{2} \frac{B_c \tau}{r_1 \omega_c^2} \] (10-46)

Substituting approximations (10-42) through (10-46) into equation (10-41), and noting that 3x_1 \ll \omega_c and 4x_1 \ll \omega_c gives:

\[ \Delta r(\tau) = \frac{A_c}{\omega_c^2} \left[ (1 + 2x_1 \tau) - (1 - x_1 \tau) \cos \omega_c \tau \right] - \frac{B_c}{\omega_c^2} \left[ \tau - (1 - x_1 \tau) \frac{\sin \omega_c \tau}{\omega_c} \right] \] (10-47)

Equation (10-41) and its simplified version, equation (10-47), correspond to oscillatory motion of \( \Delta r \); the frequency of the oscillations is equal to orbital frequency. These oscillations are damped at such a small rate* that, from the engineering standpoint in guidance system design, the damping terms can be ignored. Therefore, equation (10-47) reduces to:

\[ \Delta r(\tau) = \frac{A_c}{\omega_c^2} \left( 1 - \cos \omega_c \tau \right) - \frac{B_c}{\omega_c^2} \left( \tau - \frac{\sin \omega_c \tau}{\omega_c} \right) \] (10-48)

Equation (10-48) is the fundamental solution for the circular orbital trajectory. It may be observed that equation (10-48) is obtained directly by assuming for the integrand of equation (10-20) that:

\[ x_1 \] is generally an extremely small number.

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* \( x_1 \) is generally an extremely small number.
Therefore, equation (10-48) is an accurate solution to equation (10-23) for changes in altitude which are small in comparison to the initial distance of the vehicle from the center of the planet. This does not necessarily mean, of course, that it is an accurate solution of the true trajectory because of the approximations made in arriving at equation (23). Proof of the accuracy and limitations of equation (10-48) is carried out in Figs. 10.1 through 10.7. Before discussing these results, it is desirable to examine equation (10-48) in greater detail.

It is noted that both $A_c$ and $B_c$ in equation (10-48) contain lift terms. For the zero-lift vehicle, $A_c = 0$ and $B_c = \frac{C_v^3}{r_1}$. The altitude of the zero lift vehicle is characterized by a linear decay on which is superimposed a sinusoidal oscillation at orbital frequency. In the case of a lifting vehicle, $A_c > 0$, the trajectory also has a cosine term. Lift, from equation (10-48), has no effect on $\Delta r$ at the termination of each complete orbit (i.e., at $\omega_c T = n2\pi$). It may be seen from equation (10-47), however, that if $2x_1^{\gamma'}$ becomes significant in comparison to 1.0, lift causes successive perigees to be higher than in the zero-lift case.

Equation (10-48) is plotted on Fig. 10.1 for one complete orbit of a zero lift vehicle with $C_{DS/M} = 1$ ft. $^2$-slug$^{-1}$. Superimposed on this plot is Nielsen's numerical solution for the same vehicle. It may be observed that the solution of this thesis and Nielsen's numerical solution are practically identical throughout the orbit. The total
Fig. 10.1: Comparison of Analytical Solution to Machine Computed Numerical Solution for Zero-Lift Vehicle Initially in Circular Orbit at 120 Miles Altitude Above Earth.

\[ \frac{C_{DS}}{M} = \frac{1 \text{ ft}^2}{\text{slug}} \]

\[ H_i = 633,600 = 120 \text{ mi} \]

\[ V_i = \text{Circular Orbital} \]

Fraction of Orbit \( \frac{\omega c \tau}{2\pi} \)

Key:
- Analytical Solution of this Thesis
- Numerical Solution for Zero-Lift Vehicle (See Fig. 3(a) of Ref. 21)

\( \frac{L}{D} = 2.0 \)

\( \frac{L}{D} = 1.0 \)

\( \frac{L}{D} = 0 \)
altitude loss during the orbit is approximately 760 ft. Also shown in Fig. 10.1 are solutions to equation (10-48) for lifting vehicles with L/D = 1.0 and L/D = 2.0 having the same drag parameter \( \frac{C_p S}{M} = 1.0 \text{ ft.}^2 / \text{slug} \).

It is seen that the L/D = 1.0 vehicle climbs about 50 ft. before descending while the L/D = 2.0 climbs about 200 ft. prior to descending. In any case, Fig. 10.1 and equation (10-48) show that a true circular orbit or a linear decaying circular orbit cannot exist because of the dynamics of energy transfer even if the problem starts under precise circular orbital conditions.

Fig. 10.2 compares the solution given by equation (10-48) with the machine-computed numerical solution for the first ten orbits under the identical initial conditions used for Fig. 10.1. It should be noted that the altitude at the completion of each orbit alone is plotted in Fig. 10.2; the sinusoidal oscillations during each orbit have been omitted. It may be observed that the analytical solution and the numerical solution are remarkably close; at the completion of 10 orbits the analytical solution shows an altitude loss of 7,600 ft., while the numerical solution shows about 7,800 ft. altitude loss.

Fig. 10.3 compares the analytical solution of this thesis with numerical solutions for one orbit of a zero-lift vehicle with

\[
\frac{C_D S}{M} = 10 \text{ ft.}^2 / \text{slug}
\]

launched in a circular orbit at 120 miles above the Earth. The trajectory has the same general shape as that of Fig. 10.1 except the total altitude loss is considerably greater here because of the ten to one drag increase of the vehicle. The analytic solution derived a total altitude loss during the first orbit of approximately 7,600 ft. while the numerical solution computed the altitude loss to be
\[ \frac{C_D}{M} = 1.0 \text{ ft}^2/\text{slug} \]

\[ H_i = 633,600 \text{ ft} = 120 \text{ mi} \]

\[ V_i = \text{Circular Orbital} \]

Fig. 10.2: Comparison of Analytical Solution to Machine Computed Numerical Solution for 10 Orbits of Low-Drag Zero-Lift Vehicle Initially in Circular Orbit at 120 Miles Altitude above Earth.
**Fig. 10.3:** Comparison of Analytical Solution to Machine Computed Numerical Solution for High-Drag Zero-Lift Vehicle Initially in Circular Orbit at 120 Miles Altitude above the Earth.
about 7,750 ft.

The limitations of the analytical solution in the high drag case is demonstrated by Fig. 10.4. This figure represents the same information as Fig. 10.3 except the number of orbits has been extended from one to eight. The altitude at the completion of each orbit is plotted in Fig. 10.4; the sinusoidal oscillation which is seen during each orbit has not been represented in this graph. The analytical solution was carried through the first four orbits, at which time the atmospheric density was revised to conform with the altitude computed to exist at the completion of the fourth orbit. At the completion of two more orbits, the atmospheric density was again revised to agree with the altitude level derived at the completion of orbit number six. It may be seen that the analytical solution and numerical solution are essentially identical after eight complete orbits. Maximum difference between the two curves during the first eight orbits is less than two miles while the total altitude lost at the completion of eight orbits is approximately 27 miles.

Equation (10-48) was derived with the assumption that an exponential model atmosphere exists. The only density value required in obtaining quantitative answers from this equation, however, is the initial atmospheric density $\rho_1$. Since the machine-computed numerical data for Figs. 10.1 through 10.7 assumed the ARDC model atmosphere, the values chosen for $\rho_1$ for use in the analytical solutions plotted in these figures was also taken from the ARDC model.

Fig. 10.5 shows a much more severe test of the accuracy of equation (10-48). The initial altitude of the vehicle is 80 miles above the Earth and drag parameter is $\frac{C_D S}{M} = 1.0 \text{ ft.}^2 \text{ slug}^{-1}$. Fig. 10.5 shows that
Analytical Solution of this Thesis

Numerical Solution (Fig. 4(b) of Ref 27)

\[
\frac{C_{DS}}{M} = 10 \text{ ft}^2/\text{slug}
\]

\[
H_i = 633,600 \text{ ft} = 120 \text{ mi}
\]

\[
V_i = \text{Circular Orbital}
\]

Note:
Analytical solution assumed initial atmospheric density held for first 4 orbits. Atmospheric density was revised each two orbits thereafter.

Fig. 10.4: Comparison of Analytical Solution to Machine Computed Numerical Solutions for 8 Orbits of High-Drag Zero-Lift Vehicle Initially in Circular Orbit at 120 Miles Altitude above Earth.
the analytical solution and the numerical solution are essentially identical for the first 2200 seconds of the trajectory. The total time of flight computed numerically is in the vicinity of 6000 seconds; hence the analytic and numerical solutions agree for more than one-third of the total flight time. During this time, the total loss in altitude is approximately 30,000 ft. Also plotted in Fig. 10.5 are two curves representing Nielsen's (21) approximate solution of the trajectory. His approximate method involves solution of the differential equations of motions over suitably selected time intervals based on predictions of mean atmospheric density during this time interval. His approximate method is represented in Fig. 10.5 in two separate curves, one for a long time interval and the other for a short time interval between piecewise continuous trajectory segments. The solution of this thesis shown in Fig. 10.5 follows the numerical trajectory solution more closely than Nielsen's approximate solution over the time interval plotted. A distinct advantage of the solution advanced in this thesis by equation (10-48) is the piecewise continuous solutions are not required for reasonable accuracy except after much larger time intervals than required by the approximate methods of reference (21).

Fig. 10.6 shows the accuracy of equation (10-48) when the atmospheric density ratio $\bar{\rho}_1$ is revised after 2000 seconds for the entry trajectory previously considered in Fig. 10.4. The one-step solution of Fig. 10.5 is shown in this figure for comparison. The curve of Fig. 10.6 was plotted for the first 2000 seconds by using equation (10-48) with $\bar{\rho}_1$ corresponding to 80 miles altitude. This solution was extended to 3000 seconds in order to predict the mean altitude during the interval $2000 \text{ sec.} \leq t \leq 3000 \text{ sec.}$. With the mean altitude
For Nielsen's Solution, see Fig. 11 of Ref. 21

Fig. 10.5: Comparison of Analytical Solution to Machine Computed Numerical Solution for Zero-Lift Vehicle Initially in Circular Orbit at 80 Miles Altitude above Earth.
Fig. 10.6: Comparison of One-Step and Two-Step Analytical Solutions to Machine Computed Numerical Solution for Zero-Lift Vehicle Initially in Circular Orbit at 80 Miles Altitude above Earth.
thus predicted from the initial solution, the second step was plotted over the second time interval. The simple analytical solution and the machine-computed numerical solution are clearly very close for the first 3200 seconds of the trajectory. During this time the vehicle has completed more than half of its total flight time and lost approximately one-fourth of its total initial altitude.

The method of determining values of $\Delta r$ during the time interval greater than $t = 2000$ seconds in Fig. 10.6 requires some explanation. After carrying forth the original solution to 3000 sec., the average altitude during the interval 2000-3000 sec. was derived. $\bar{C}_1$ was determined for this altitude from the ARDC model atmosphere and the constant $B_c$ in equation (10-48) was adjusted accordingly*. It should be noted that in the derivation of equation (10-48), the initial altitude rate $\Delta r'$ was assumed zero at zero time. At time 2000 sec., which is zero time for the second step, $\Delta r'$ is not zero; on differentiating equation (10-48), it is seen that $\Delta r'$ is in the neighborhood of -30 ft./sec. at time 2000 seconds. This value of $\Delta r'$ must be carried through the second step of the solution. For example:

(1) At $t = 2500$ sec., equation (10-48) is used to determine $\Delta r$ by substituting $\tau$ equivalent to 500 seconds into this equation (i.e., $t - t_{12} = t_2 = 2500 - 2000 = 500$ sec., where $t_{12}$ is initial time of the second segment and $t_2$ is time measured in the second segment of the solution.) This gives the value of $\Delta r$ at 2500 seconds if and only if $\Delta r'$ were zero at 2000 seconds.

* $A_c = 0$ in equation (10-48) in this case since lift is zero.
(2) Since $\Delta r'(0) \approx -30 \text{ ft./sec.}$ for the second phase of the solution, approximately $30 \times 500 = 15,000 \text{ ft.}$ must be added in the negative direction to the $\Delta r$ computed in (1) above in order to determine the total $\Delta r$ at $t = 2500$ sec. This may be summarized as:

$$\Delta r(t) = \Delta r(t_2) + \Delta r'(t_{i2}) \times t_2 \quad (10-50)$$

Eq. (10-48) with $\tau$ corresponding to $t_2 = t - t_{i2}$

The time variation of $\Delta r'$ is plotted in Fig. 10.7. This is easily determined by differentiating equation (10-48):

$$\Delta r'(\tau) = \frac{AC}{\omega_C} \sin\omega_0 \tau - \frac{BC}{\omega_C} (1 - \cos\omega_0 \tau) \quad (10-51)$$

It should be noted that the value of $\Delta r'$ at 2000 seconds was carried through as an initial value for the second step in the two-step solution. It may be seen from this figure that the difference between the computer solution and the analytical two-step solution of this thesis differ by less than 5 ft./sec. over the first 3000 seconds of flight. During the first 1500 seconds of the trajectory, the numerical and analytical solutions agree exactly.

Velocity may be determined from equations (10-16), (10-48), and (10-49) as follows:

$$\Delta v = -\frac{n_i}{n_i} A \left(1 - \cos \omega_0 \tau \right) + \frac{n_i}{n_i} B \left(1 - \sin \omega_0 \tau \right) \frac{C n_i^2}{2} \int_0^{\tau} \phi \, d\tau \quad (10-52)$$

Using equation (10-18) and equation (10-48) without the oscillatory terms, the last quantity in the above equation may be integrated. The
Fig. 10.7: Comparison of Analytical Radial Velocity Solution to Machine Computed Numerical Solution for Zero-Lift Vehicle Initially in Circular Orbit at 80 miles Altitude above Earth.

Key:
- Two-step Analytical Solution
- One-step Analytical Solution
- Numerical Solution (See Fig. 11 (b) of Ref. 21)

Note: Second step of Analytical Solution begins at $t=2000$ sec.

- $C_D S = 1 \frac{ft^2}{slug}$
- $H_i = 80$ mi
- $V_i = \text{Circular Orbital}$
- $\rho_i = 2.96 \times 10^{-11} \text{slug/ft}^3$
resulting solution for $\Delta v$ is:

$$\Delta v = \frac{\rho r}{\rho_c} r_c \left[ B_c \left( \gamma - \sin \omega_c r \right) - A_c (1 - \cos \omega_c r) \right] - \frac{C_v^2 \omega_c^2}{2 \frac{k}{B} C_c} \left[ e \frac{B_c \rho}{B_c} - 1 \right]$$  \hspace{1cm} (10-53)

Kinetic-potential energy trade-off. Velocity loss due to atmospheric drag.

Range rate may be determined from equations (10-6), (10-8) and (10-9):

$$X_N' = \frac{v}{r} \approx \frac{1}{r_1} \left( 1 + \frac{\Delta v}{v_1} - \frac{\Delta r}{r_1} \right)$$  \hspace{1cm} (10-54)

In equation (10-54), $\Delta r(\tau)$ is given by equation (10-48) and $\Delta v(\tau)$ by equation (10-53):

$$X_N' = \frac{\rho r}{\rho_c} \frac{2v}{r_c} \left[ B_c \left( \gamma - \sin \omega_c r \right) - A_c (1 - \cos \omega_c r) \right] - \frac{C_v^2 \omega_c^2}{2 \frac{k}{B} C_c} \left[ e \frac{B_c \rho}{B_c} - 1 \right]$$  \hspace{1cm} (10-55)

This is integrated to give range as a function of time:

$$X_N = \frac{2v_c}{r_c^2} \omega_c \left[ B_c (\cos \omega_c r - 1) + A_c \sin \omega_c r \right] - \frac{C_v^2 \omega_c^4}{2 \frac{k}{B} C_c} \left( e \frac{B_c \rho}{B_c} - 1 \right)$$

$$+ \frac{\rho_c}{r_1} \frac{2v_c A_c}{r_c^2 \omega_c^2} + \frac{C_v^2 \omega_c^2}{2 \frac{k}{B} C_c} \frac{A_c}{r_c^2 \omega_c^2} \gamma^2$$  \hspace{1cm} (10-56)

The total range to impact determined by numerical methods (21) for a zero-lift vehicle with $\frac{\sigma v S}{M} = 1.0 \text{ ft}^2 \text{-slug}^{-1}$ initially at 80 miles in a circular orbit above the Earth is approximately 26,000 miles. Time of flight was determined to be 5991 seconds. The total range of the same vehicle computed from equation (56) in a one-step solution of the trajectory is approximately 28,000 miles.

Equation (7-19) showed that the convective heating rate at the stagnation point is:

$$\dot{Q}_S = 18,000 \left( \frac{HF}{\rho c} \right) \frac{v^3}{\sqrt{R}} e^{-\frac{kh}{v^2}} \text{ BTU ft.}^{-2} \text{-sec.}$$  \hspace{1cm} (10-57)
where \((HF)_{EO}\) is the heating function ratio tabulated in Table 7.3 for the terrestrial planets and \(R\) is the radius of curvature at the stagnation point.

Equation (10-57) may be written as follows:

\[
(Q_c)_S \approx \frac{18,000 (HF)_{EO}}{\sqrt{R}} v_1^3 \left( 1 + \frac{3 \Delta v}{v_1} \right) e^{-\frac{k h_i}{2}} e^{-\frac{k \Delta r}{2}}
\]  
\(10-58\)

Substituting for \(\Delta v(T)\) from equation (10-53) and \(\Delta r(V)\) from equation (10-48) gives analytical expression for convective heating rate as a function of time.

Stagnation point temperature in degrees Rankine as a function of time is represented analytically in similar fashion from equation (7-21):

\[
T_s \approx 1.392 \times 10^4 (HF)_{EO} (VF) v_1^3 \left( 1 + \frac{3 \Delta v}{v_1} \right) \left( 1 + \frac{3 \Delta r}{v_1} \right) e^{-\frac{k h_i}{8}} e^{-\frac{k \Delta r}{8}}
\]  
\(10-59\)

An approximate time history of vehicle accelerations is given by differentiating equation (10-53).

10.4 The Degenerate Elliptical Entry Trajectory

The initial point for the analysis of the elliptical trajectory is at the first perigee point. An elliptical trajectory results if the velocity of the vehicle at its first perigee passage is greater than that required to generate a circular orbit; i.e., the first perigee is characterized by an energy excess over circular orbital energy. In the specific case that a vehicle is launched from the planet for the purpose of orbiting the planet a certain desired number of orbits and re-entering, then an elliptical orbit results if the velocity at engine cut-off in the launch phase exceeds circular orbital velocity. In the
case of interplanetary operations in which the high energy level of the
vehicle in the interplanetary transfer ellipse is to be reduced by
atmospheric braking, the degenerate orbital profile follows the first
perigeal passage if the energy transfer is sufficient during this
passage for the planet to capture the vehicle.

The basic equation that must be solved in order to determine the
time behavior of $\Delta r$ was given as equation (10-17). It was shown in
section 10.3 that the approximation given by equation (10-49) may be
made in this equation. Therefore equation (10-17) is written for the
exponential atmospheric model as:

$$\Delta r'' + \omega^2 \Delta r = \frac{\alpha r^2}{r_i^3} \left[ 1 - \left( \frac{v_i}{v_r} \right)^2 \right] + \frac{L}{D} \frac{C}{Z} e^{-kr}$$

$$- \frac{C v_i^3}{r_i} \left( 1 + \frac{L}{D} \frac{C}{Z} e^{-kr} \right) \int_0^r e^{-kr} \, dr$$

(10-59)

The angular velocity term, $\omega$, in equation (10-59) represents a
slowly varying quantity during the circularization phase of the
degenerate orbital profile. For the purpose of this analysis, $\omega$
is assumed constant; however, by writing it in terms of vehicular velocity
and the distance of the vehicle from the planet center, then $\omega$
may be changed at each separate initial point chosen for piecewise continuous
solution of the trajectory. $\omega$ is defined as follows:

$$\omega = \frac{2\pi}{T_N}$$

(10-60)

where $T_N$ is the dimensionless period of the orbit. Using equations
(9-8) and (9-12), the average angular velocity $\omega$ is written in terms
of velocity and radius at the initial point as follows:

$$\omega^2 = \left( \frac{2}{r_1} - v_1^2 \right)^3$$  \hfill (10-61)

It is possible to make considerable simplification in equation (10-59) if:

$$\left[ 1 + \frac{L}{D}(C/2) r_i e \right]^{-k \Delta r} \leq 1.0$$  \hfill (10-62)

Equation (10-62) is automatically satisfied for zero-lift vehicles. This approximation is accurate if:

$$N \ll 1.0$$

where

$$N = \frac{L}{D}(C/2) r_i e$$  \hfill (10-63)

The quantity $L \frac{C_D S}{2D M}$ is plotted as an approximation parameter in Fig. 10.8 versus the minimum altitude for which the elliptical solution developed herein is accurate in the case of lifting vehicles entering the Earth's atmosphere.

As an example of the use of Fig. 10.8, consider a vehicle with the following characteristics:

$$L/D = 1.5$$

$$\frac{C_D S}{M} = 0.4625 \text{ ft.}^2/\text{slug}$$

$N = 0.1$ for this vehicle at an altitude of 54.5 miles above the surface of the Earth. By the time the vehicle has descended to this altitude, orbital flight conditions no longer exist, consequently the limitations on the solution as a result of assuming approximation (10-62) are

* This particular vehicle is used later in this section to compare the analytical solution derived herein to machine-computed numerical solutions.
Fig. 10.8: Minimum Earth Altitudes for Which Elliptical Solution is Accurate in the Case of Lifting Vehicles
generally not severe.

With approximation \( (10-62) \), equation \( (10-59) \) reduces to:

\[
\Delta r'' + \omega^2 \Delta r = A_1 e^{-k \Delta r} + A_2 - A_3 \int_0^\gamma e^{-k \Delta r} d\gamma
\]  
\[\text{(10-64)}\]

where the constants are defined as follows:

\[
A_1 = (L/D) \frac{Cv_1^2}{2} \]  
\[\text{(10-65)}\]

\[
A_2 = \frac{v_i^2}{r_1} \left[ 1 - \left( \frac{v_i}{v_1} \right)^2 \right] \]  
\[\text{(10-66)}\]

\[
A_3 = \frac{Cv_1^3}{r_1^3} \]  
\[\text{(10-67)}\]

In accordance with the method of solution outlined in section 10.2, a first estimate is made for \( r(\tau) \) in equation \( (10-64) \) by assuming a vacuum trajectory. With \( \rho_1 = 0 \), then \( C = 0 \) and equation \( (10-64) \) reduces to:

\[
\Delta r'' + \omega^2 \Delta r = A_2
\]  
\[\text{(10-68)}\]

The solution to equation \( (10-68) \), with \( \Delta r(0) = 0 \) and \( \Delta r'(0) = 0 \) is:

\[
\Delta r = \frac{A_2}{\omega^2} (1 - \cos \omega \tau)
\]  
\[\text{(10-69)}\]

At apogee of the vacuum trajectory:

\[
\Delta r_a = 2 (a_N - r_1)
\]  
\[\text{(10-70)}\]

where \( a_N \) is dimensionless semi-major axis of the ellipse. Comparing equations \( (10-70) \) and \( (10-69) \) at apogee \( (\omega \tau = \pi) \) shows that:

\[
A_2 = \omega^2 (a_N - r_1)
\]  
\[\text{(10-71)}\]
Therefore, by substituting for \( a_N \) from equation (9-8) \( A_2 \) may be written in terms of initial velocity, radius, and angular velocity as follows:

\[
A_2 = \omega^2 r_1 \frac{V_i^2 r_i - 1}{2 - V_i^2 r_i}
\]

(10-72)

With \( A_2 \) defined by equation (10-72) and the first estimate of \( \Delta r(T) \) given by equation (10-69), equation (10-64) is written:

\[
\Delta r'' + \omega^2 \Delta r = A_2 + A_1 e^{-A_4} + A_4 \cos \omega \tau - A_3 e^{-A_4} \int_0^\tau (e^{A_4 \cos \omega \tau}) d\tau
\]

(10-73)

where \( A_4 \) is defined as follows:

\[
A_4 = \frac{kA_2}{\omega^2} = k r_i \left( \frac{V_i^2 r_i - 1}{2 - V_i^2 r_i} \right)
\]

(10-74)

Equation (10-73) is the fundamental differential equation of the degenerate elliptical entry trajectory.

Taking the Laplace transform of equation (10-73) gives:

\[
R(s) = \frac{1}{s(s^2 + \omega^2)} \left[ A_2 + e^{-A_4}(sA_1 - A_3) F(s) \right]
\]

(10-75)

where \( R(s) \) is given by equation (10-30) and \( F(s) \) is the Laplace transform of \( e^{A_4 \cos \omega \tau} \). Before the inverse transform of this equation can be taken, \( F(s) \) must be determined. A search of the mathematical literature failed to disclose a representation of \( e^{A_4 \cos \omega \tau} \) suitable for this purpose. Derivation Summary 10.1 analyzes this function in terms of modified Bessel Functions of the first kind. This investigation led to the following representation:

* Note that \( \Delta r(0) = 0 \) and \( \Delta r'(0) = 0 \).
e^{A_4 \cos \omega \tau} = I_0(A_4) + 2 \sum_{n=1}^{\infty} I_n(A_4) \cos n \omega \tau \quad (10-76)

where \( I_n \) is the \( n \)th Bessel Function of the first kind with pure imaginary argument.

Fig. 10.9 is a graph of \( I_n \) for small values of \( A_4 \) taken from reference (80). Engineering analysis of most entry trajectories leads to fairly large values of \( A_4 \); for example, \( A_4 \) for the vehicle plotted in Fig. 10.10 is in the neighborhood of \( 58 \) for the first segment and \( 19 \) for the second segment. The following asymptotic series (79) is convenient for large values of \( A_4 \):

\[
I_n(A_4) \approx \frac{e^{A_4}}{\sqrt{2\pi A_4}} \left[ 1 - \frac{(4n^2 - 1)}{118A_4} + \frac{(4n^2 - 1)(4n^2 - 3^2)}{2! (8A_4)^2} - \cdots \right] \quad (10-77)
\]

The magnitude of \( I_n \) for small \( n \) is generally extremely large (e.g., of the order of \( 10^{15} \) or greater) due to the \( e^{+A_4} \) term in equation (10-77).

The Laplace transform of equation (10-76) is:

\[
\mathcal{F}(s) = \frac{I_0}{s} + 2s \sum_{n=1}^{\infty} \frac{I_n}{s^2 + (n\omega)^2} \quad (10-78)
\]

Substituting equation (10-77) into equation (10-75), expanding by partial fractions, and taking the inverse transform of the result gives:

\[
\Delta r(\tau) = \left( \frac{A_2}{\omega^2} + \frac{A_1 I_0 e^{-A_4}}{\omega^2} \right) \left( 1 - \cos \omega \tau \right) + \frac{e^{-A_4}}{\omega^2} \left[ -A_3 I_0 \left( \tau - \frac{\sin \omega \tau}{\omega} \right) + A_1 I_1 \omega \tau \sin \omega \tau - A_3 I_4 \left( \frac{\sin \omega \tau}{\omega} - \tau \cos \omega \tau \right) \right] + \sum_{n=2}^{\infty} \frac{2 I_n}{(n^2-1)} \left( A_4 \left[ \cos \omega \tau - \cos n \omega \tau \right] - \frac{A_3}{\omega} \left[ \sin \omega \tau - \frac{\sin n \omega \tau}{n} \right] \right) \]

\[
\quad (10-79)
\]
Fig. 10.9: Values of Coefficients in Elliptical Trajectory Solution for Small $A_4$
Equation (10-79) is the solution for altitude as a function of time for the degenerate orbital profile. Following is a summary of constants in this equation:

\[ I_n : \text{Fig. 10.9 for small } A_4; \text{equation } (10-77) \text{ for large } A_4. \]
\[ \omega : \text{Equation } (10-61) \]
\[ A_1 : \text{Equation } (10-65) \]
\[ A_2 : \text{Equation } (10-72) \]
\[ A_3 : \text{Equation } (10-67) \]
\[ A_4 : \text{Equation } (10-74) \]

The term in curly brackets appears at first glance to be formidable for engineering computations. Fig. 10.9 shows, however, that this solution converges rapidly due to the fact that terms involving \( \frac{2I_n}{n^2 - 1} \) become relatively unimportant for \( n \) greater than 2 or 3. For all reasonable initial problem conditions the asymptotic equation \( (10-77) \) is an accurate and easy method for computing \( I_n \). It should be noted that the exponential term in equation \( (10-77) \) cancels the exponential term in equation \( (10-79) \).

At perigee:
\[
\sin n\omega \gamma = 0 \\
\cos n\omega \gamma = 1 
\] (10-80)

Equation (10-79) reduces to the following for the time history of perigee altitude:
\[
\Delta r_p(\gamma) = -\frac{A_3}{\omega^2} e^{-\frac{A_2}{2}(I_0 - I_4)^2} \gamma 
\] (10-81)

At apogee:
\[
\sin n\omega \gamma = 0 \\
\cos n\omega \gamma = -1 
\] (10-82)
Equation (10-79) reduces to the following for apogee altitude as a function of time:

$$\Delta r_a(t) = \frac{2}{\omega^2} \left( A_2 + A_1 I_0 e^{-A_4 t} \right) - \frac{A_2}{\omega^2} \gamma (I_0 + I_1) e^{-A_4 t}$$

Equations (10-81) and (10-83) were plotted in Fig. 10.10 for a lifting vehicle launched in an elliptical orbit with initial velocity 3% greater than circular orbital velocity. Initial altitude was 350,000 feet above the Earth and the vehicle parameters were arbitrarily selected as follows:

$$L/D = 1.5$$

$$\frac{C_D S}{M} = 0.4625 \text{ ft.}^2/\text{slug}.$$

The analytical solution is represented in two segments with a revised atmospheric density \( \rho_1 \) chosen for the second segment; the first segment covered 10\( \frac{1}{2} \) orbits and the second segment used the atmospheric density corresponding to the altitude computed for the 11th perigee passage. Total flight time during circularization was computed to be 20.3 hours. The perigee decay rate was computed from equation (10-81) to be approximately 0.14 ft./sec. during the first segment and 0.61 ft./sec. in the second segment. Perigee decayed 8030 ft. over the first 960 minutes and an additional 8780 ft. between 960 and 1200 minutes. Total loss in perigee altitude during the 20 hour flight was about 3 miles. Apogee decay rate averaged 31 to 32 ft./sec. during the first segment and increased to 46 ft./sec. during the second segment. Apogee decayed more than 360 miles during the first segment and 170 miles during the second segment.
Fig. 10.10: Comparison of Perigeeal and Apogeeal Altitudes of Analytical Solution with Those Determined Numerically with IBM-704. Lifting Vehicle Launched with Initial Velocity 3% Greater than Circular Orbital Velocity
In order to estimate the accuracy of the analytical solution discussed in the preceding paragraph, Fig. 10.10 also shows decay of perigee and apogee computed numerically with the IBM-704. The numerical runs represented a trajectory initially inclined 30° to the equatorial plane about the oblate, pear-shaped Earth with an exponential atmospheric model. Atmospheric density was assumed zero above 600,000 ft. in the computer run. Total flight time in the circularization phase was computed to be about 19.9 hours.

Some rather interesting facts may be deduced from equations (10-81) and (10-83). First, it may be seen that lift has little effect on perigeal and apogeeal altitudes during the circularization phase. Equation (10-81) shows perigee decay rate to be independent of lift. The only term involving lift in equation (10-83) is the term containing \( A_1 \). For the particular example plotted in Fig. 10.10, the \( A_1 \) term was orders of magnitude smaller than the \( A_2 \) term. The apogeal decay rate given by the second term in equation (10-83) is independent of lift.

The major difference between the decay rate at apogee and that at perigee is the fact that perigee decay rate is proportional to the difference of two near-equal large numbers \( (I_0 - I_1) \), while apogeal decay rate is proportional to the sum of these two large numbers. It may be seen from Fig. 10.9 that the difference between these two numbers is essentially constant for \( A_4 > 5 \); the sum is very sensitive to the magnitude of \( A_4 \).

For most degenerate orbital trajectories, \( A_4 \) is large enough that the first two terms in equation (10-77) are sufficient for accurate estimation of perigeeal and apogeal decay rates. Differentiating equations (10-81) and (10-83), and substituting the first two terms
from equation (10-77) for $I_0$ and $I_1$ gives:

\[
\Delta r'_{\tau} = - \frac{A_3}{4\sqrt{2\pi} \omega^2 A_4^{3/2}}
\]  \hspace{1cm} (10-84)

\[
\Delta r'_\alpha = - \frac{2A_3}{\omega^2 \sqrt{2\pi} A_4}
\]  \hspace{1cm} (10-85)

Other guidance parameters may be computed from the foregoing results. Some of the quantities of interest are summarized below:

1. **Dimensionless semi-major axis $a_N$:**

\[
a_N = \frac{r_{\tau} + r_\alpha}{2} = r_i + \frac{\Delta r_{\tau} + \Delta r_\alpha}{2}
\]  \hspace{1cm} (10-86)

Using equation (10-81) and (10-83), equation (10-86) becomes:

\[
a_N(\tau) = r_i + \left( \frac{A_2 + A_1 I_0 e^{-A_4}}{\omega^2} \right) - \frac{A_3}{\omega^2} e^{-A_4 I_0 \tau}
\]  \hspace{1cm} (10-87)

The semi-major axis changes almost as a step function in the vicinity of perigee; hence equation (10-87) should not be viewed as a continuous time solution. Recomputation of $a_N$ should be performed at each perigeal passage with a near constant value during the following orbit.

2. **Period of orbit:**

\[
T_N = 2\pi a_N^{3/2}
\]  \hspace{1cm} (10-88)

where $a_N(\tau)$ is given by equation (10-87).

3. **Average angular velocity in orbit:**

\[
\omega(\tau) = a_N^{-3/2}
\]  \hspace{1cm} (10-89)

where $a_N(\tau)$ is given by equation (10-87).
(4) **Instantaneous velocity:**

In a Keplerian elliptical orbit, velocity is given by:

\[
\mathbf{v}^2 = \frac{2}{r} - \frac{1}{a_N} \quad (10-90)
\]

Velocity may be approximated during the circularization phase from equation (10-90) as follows:

\[
\mathbf{v}^2 = \frac{2}{r_1}(1 - \frac{\Delta r}{r_1}) - \frac{1}{a_N} \quad (10-91)
\]

\(\Delta r(\tau)\) is given by equation (10-79) and \(a_N(\tau)\) by equation (10-87).

Velocity at perigee is derived from equation (10-91) as follows:

\[
\nu_{\gamma}^2 = \frac{1}{r_1} \left[ 1 + \frac{1}{2r_1} \left( \Delta r - 3\Delta r_\gamma \right) \right] \quad (10-92)
\]

where \(\Delta r_\gamma(\tau)\) and \(\Delta r_\alpha(\tau)\) are given by equations (10-81) and (10-83).

Velocity at apogee is as follows:

\[
\nu_{\alpha}^2 = \frac{1}{r_\alpha} \left[ 1 + \frac{1}{2r_\alpha} \left( \Delta r_\gamma - 3\Delta r_\alpha \right) \right] \quad (10-93)
\]

(5) **Total energy:**

Total energy changes approximately as a step function in the vicinity of each perigeeal passage. The time behavior of energy level is given by:

\[
E(tot) = -\frac{1}{2a_N} \quad (10-94)
\]

where \(a_N(\tau)\) is given by equation (10-87).

(6) **Eccentricity**

Eccentricity, like the semi-major axis and energy level,
decreases in a series of near steps in the vicinity of perigee. The value between perigeeal steps is reasonably constant. With this step behavior recognized, the time behavior of eccentricity is given by equation (10-94) (which was derived from equations (9-16), (9-18), (10-92) and (10-93)):

\[ e(\tau) = \frac{1}{2r_e} \left( \Delta r_\alpha - \Delta r_\gamma \right) \]  

(10-95)

(7) Heating Rates, Stagnation Temperatures, and Accelerations:

During the circularization phase of the degenerate orbital profile, heating and accelerations are not of major concern in guidance analysis since maximum levels are generally encountered after the trajectory has degenerated to a circle. The methods of Chapter 9 and Section 10.3 may be used to predict these quantities during the Gas-Dynamic Phase of flight.

10.5 Summary

Chapter 10 examines the trajectory of a vehicle in the degenerate orbital profile. Initial conditions assumed in this analysis were that a lifting or non-lifting vehicle was injected in horizontal flight at some initial altitude above an arbitrary planet with \( \Delta r(0) \) and \( \Delta r'(0) \) equal to zero. The results were described in dimensionless form suitable for analysis of entry into the atmosphere of any planet. The exponential atmospheric density model was assumed.

Two separate trajectory phases were examined:

(1) The circularization phase for the degenerate orbit;

(2) The entry phase which follows the circularization process.

In the special case where the vehicle is initially at circular orbital
velocity, the first of these two phases obviously does not exist.

The solution for altitude during the circularization process was derived as equation (10-79). This solution was written in terms of a number of constants which depend on the initial conditions of the problem and on the characteristics of the vehicle. Some of the important constants in this equation were written in terms of Bessel Functions. Since these functions are tabulated, numerical answers to the resulting equations are easily obtained. The time variation of perigeeal and apogeeal altitudes was given in equations (10-81) and (10-83) and the rate of decay for most practical problems is described by equations (10-84) and (10-85). Analytical representation of other quantities which are important in the conceptual and preliminary design stages of guidance systems are summarized in equations (10-86) through (10-94).

It was shown in the analysis of the circularization phase that the drag characteristics of the vehicle are important in specifying the resulting trajectory and that the lift characteristics of the vehicle are relatively unimportant. It was shown that, within the approximations made in this analysis, lift does not enter in the specification of the decay rates of apogee and perigee. Perigeeal altitude, to a first order, remains essentially constant. The major difference between the decay rates of apogee and perigee is that perigeeal decay rate is proportional to the difference of two near-equal large numbers* while apogeeal decay rate is proportional to the sum of these same numbers.

The solution for altitude in circular orbital entry was given by

\[ \text{Perigeeal decay is proportional to } (I_0 - I_1), \text{ where } I_0 \text{ and } I_1 \text{ are the first two Bessel Functions with an argument that is generally greater than 10.} \]
equation (10-41). This solution was written in simplified forms suited to almost all practical problems as equations (10-47) and (10-48). It was shown in this chapter that a true circular orbit or a linear decaying circular orbit cannot exist even under idealized conditions of injecting a vehicle exactly at circular orbital velocity near a spherical planet. The influence of the atmosphere on the dynamics of energy transfer result in undamped oscillatory motion in altitude and flight path angle.

The analytical solutions derived in this chapter were compared to machine-computed numerical solutions under a variety of initial conditions. It was shown that the analytical solutions presented an accurate picture of the resulting motion except after long periods of time. The analytical solutions depart from more accurate computer solutions only after there is a significant increase in atmospheric density from the assumed initial value as a result of altitude loss during entry. This limitation on the solutions may be greatly alleviated by starting the solution over again under a new set of initial conditions. Examples of this method of solution were given.
Derivation Summary 10.1

Expansion of $e^a \cos x$ in Terms of Modified Bessel Functions of the First Kind.

In order to solve equation (10-75), the Laplace transform of $e^a \cos x$ is required. An expansion of this function in terms of Bessel Functions of the first kind with pure imaginary arguments was carried out in this thesis. This particular expansion is not given in the mathematical literature that the author has searched.

The following is written:

$$ e^{a \cos x} = 1 + a \cos x + \frac{a^2 \cos^2 x}{2!} + \frac{a^3 \cos^3 x}{3!} + \ldots \quad (1) $$

The following equation defines the $n^{th}$ power of $\cos x$:

$$ \cos^n x = \frac{1}{2^{n-1}} \left\{ \cos nx + n \cos((n-2)x) + \frac{n(n-1)}{2!} \cos((n-4)x) + \frac{n(n-1)(n-2)}{3!} \cos((n-6)x) + \ldots \right\} \quad (2) $$

Substituting equation (2) into equation (1) gives:

$$ e^{a \cos x} = 1 + a \cos x + \frac{a^2}{2!} \frac{1}{2} (\cos 2x + 1) + \frac{a^3}{3!} \frac{1}{4} (\cos 3x + 3 \cos x) $$

$$ + \frac{a^4}{4!} \frac{1}{8} (\cos 4x + 4 \cos 2x + \frac{6}{2}) + \frac{a^5}{5!} \frac{1}{16} (\cos 5x + 5 \cos 3x + 10 \cos x) $$

$$ + \frac{a^6}{6!} \frac{1}{32} (\cos 6x + 6 \cos 4x + 15 \cos 2x + \frac{20}{2}) $$

$$ + \frac{a^7}{7!} \frac{1}{64} (\cos 7x + 7 \cos 5x + 21 \cos 3x + 35 \cos x) + \ldots \quad (3) $$
Equation (3) is written:

\[ e^{\alpha \cos x} = I_0 + 2 I_1 \cos x + 2 I_2 \cos 2x + 2 \sum_{n=3}^{\infty} I_n \cos nx \]  

(4)

After a rather difficult process of mathematical bookkeeping is performed on equation (3), the coefficients in equation (4) are derived as follows:

\[ I_n(a) = \sum_{j=0}^{\infty} \frac{\alpha^{2j+n}}{2^{(2j+n)} j! (j+n)!} \]  

(5)

Equation (5) is the series representation of Bessel Functions of the first kind with pure imaginary argument:

\[ I_n(a) = i^{-n} J_n(ia) \]  

(6)

where \( J_n \) are Bessel Functions of the first kind.

\( I_n(a) \) are plotted in Fig. 10.9 for small values of \( a \). For large \( a \), the asymptotic series given as equation (10-77) is adequate.
Robert Clifton Duncan was born in Jonesville, Virginia, on November 21, 1923. He was educated in the public schools in the state of Ohio and was graduated from the Beavercreek High School, Alpha, Ohio, in 1941.

In the fall of 1941 he entered Ohio State University as an undergraduate student in chemical engineering. In July, 1942, he entered the U.S. Naval Academy and graduated with the degree of Bachelor of Science in June, 1945.

After serving for two years as Combat Information Officer in the heavy cruiser USS BREMERTON (CA-130) engaged in occupational duty in Japan and as flagship of the North China Patrol, he was assigned to Naval flight training. Upon completing flight training in 1948, specializing in fighter type aircraft, he was assigned to duty as a pilot in Fighting Squadron 41 (VF-41), Composite Squadron 5 (VC-5), and Composite Squadron 6 (VC-6). He was a member of the Navy's first carrier based squadrons equipped for delivery of atomic weapons.

He entered the U.S. Naval Postgraduate School in July, 1951, and graduated in June, 1953, with the degree Bachelor of Science in Aeronautical Engineering. In September, 1953, he entered M.I.T. and graduated in June, 1954, with the degree Master of Science in Aeronautical Engineering.

He was assigned in June, 1954, to the Heavy Attack Training Unit as an instructor in atomic weapons delivery for pilots of fighters, light attack, and heavy attack aircraft and for bombardier-navigators of heavy attack aircraft.

In September, 1957, he entered M.I.T. as a doctoral candidate in the Department of Aeronautics and Astronautics. He was elected to Sigma Gamma Tau and is a member of the American Ordnance Association, the U.S. Naval Institute, and the Institute of the Aeronautical Sciences.
Appendix A

COORDINATE FRAMES USED IN ENTRY TRAJECTORY ANALYSIS;
GLOSSARY OF SYMBOLS, CONSTANTS, AND DEFINITIONS

A.1 Coordinate Frames

The guidance process consists of measurements of vehicle position and velocity, computation of control actions necessary to properly adjust position and velocity, and delivery of suitable adjustment commands to the vehicle's control system. Guidance of vehicles through or above the atmosphere of a planet involves the twofold functions of indication and control. Navigation is the indication aspect of guidance.

Inherent with an investigation of guidance requirements of vehicles entering planetary atmospheres is the selection of suitable coordinate frames of reference* and trajectory parameters. Many sets of coordinate frames are required to completely specify the three dimensional nature of the entry trajectory under the following general conditions:

(1) The figure of the planet is non-spherical; i.e., it may be an oblate spheroid with local surface irregularities and large scale harmonics leading to a figure resembling that of a pear.

* In this thesis, "frame of reference", "frame", "reference space", etc., are used synonymously.
(2) The planet is rotating about an axis through its center of gravity.

(3) The gravitational mass attraction of the planet is not spherically symmetric.

(4) The atmosphere surrounding the planet is rotating with the planet. Relative motion between the atmosphere and planet may exist; i.e., winds may be present.

(5) The entry vehicle may have the capability of generating variable lift, drag, and thrust forces in order to control the shape of its trajectory.

(6) The vehicle may have the ability to bank; i.e., rotate the direction of the lift vector out of the plane of the trajectory.

(7) The propulsive system may be mounted on gimbals in order to vary the thrust direction as desired.

Seven sets of coordinate systems were chosen in order to analyze the three dimensional dynamical equations important in the study of guidance. One of these coordinate systems is the guidance grid, for which two alternatives are presented herein. The coordinate systems selected were as follows:

(1) **Planet centered Inertial Frame**: Newton's laws are valid in an inertial frame. The use of a planet centered inertial frame is justified because gravitation and acceleration effects are indistinguishable.

(2) **Planet Reference Frame**: Establishes the orientation of the planet with respect to the inertial frame.

(3) **Instantaneous Trajectory Plane Frame**: Establishes the
position and orientation of the instantaneous plane of the trajectory with respect to the inertial frame.

(4) Position Reference Frame (or Guidance Grid): Two alternative methods discussed in subsequent paragraphs of Appendix A are:

(a) The Latitude-Longitude Triad
(b) The Great Circle Triad.

(5) The Wings-Level Triad: This frame is basically orientated by the velocity vector of the vehicle with respect to the atmosphere when the lift vector is in the instantaneous plane of the trajectory.

(6) The Vehicle Coordinate Triad: Required for a vehicle that has the ability to "bank"; i.e., to rotate the lift vector out of the plane of the trajectory in order to generate trajectory curvature as desired.

(7) The Engine Gimbal Triad: Required for a vehicle that has an engine capable of being rotated with respect to the vehicle in order to control the direction in which thrust forces are generated.

Relations among the angles involved are presented for ready reference in converting quantities specified in one frame into components in any of the other frames.

Fig. A.1 shows a vehicle located at the point P entering the atmosphere of a planet. The destination of the vehicle is some landing site on the surface of the planet that was selected prior to initiation of entry. The instantaneous plane of the trajectory is not fixed with respect to an inertial reference. Following is a brief definition of
Planet Centered Inertial Frame: Centered at the center of the planet and sidereally nonrotating relative to the "fixed stars". \( X_I \) and \( Y_I \) are in the equatorial plane and \( Z_I \) is directed along the polar axis (North).

Planet Reference Frame: Centered at the center of the planet and nonrotating with respect to the planet. This frame rotates about the planet's polar axis relative to the inertial frame at the planet's daily sidereal rate. \( X_O \) and \( Y_O \) are in the equatorial plane and fixed with respect to the planet. \( Z_O \) is directed along the polar axis.

Instantaneous Trajectory Plane Frame: Centered at the center of the planet. \( X_T \) is the line of intersection of equatorial plane with the instantaneous plane of the trajectory; \( Y_T \) is in equatorial plane perpendicular to \( X_T \); \( Z_T \) is directed along the polar axis.

Fig. A.1: Definition of the Inertial, Planet, and Instantaneous Trajectory Plane Coordinate Systems.
the coordinate frames shown in Fig. A.1.

**$X_I, Y_I, Z_I$**

**Planet Centered Inertial Frame:** Centered at the center of the planet and sidereally nonrotating relative to the "fixed stars". $X_I$ and $Y_I$ are oriented in the equatorial plane and sidereally nonrotating; $Z_I$ is directed along the polar axis (North).

**$X_0, Y_0, Z_0$**

**Planet Reference Frame:** Centered at the center of the planet and nonrotating with respect to the planet. This frame rotates about the planet's polar axis relative to the inertial frame at the planet's daily sidereal rate. $X_0$ and $Y_0$ are oriented in the equatorial plane and fixed in the planet; $Z_0$ is directed along the planet's polar axis (North).

**$X_T, Y_T, Z_T$**

**Instantaneous Trajectory Plane Frame:** Centered at the center of the planet. $X_T$ is the line of intersection of the planet's equatorial plane with the plane of the trajectory (ascending line of nodes); $Y_T$ axis is in the planet's equatorial plane and is perpendicular to the ascending line of nodes. $Z_T$ axis coincides with the planet's polar axis (North).

In an analysis of the trajectory of entry vehicles, it is convenient to study the motion of the vehicle in spherical polar coordinates. The reference frame is usually selected with one axis along the radius vector from the center of the planet to the point at which the guidance is taking place. This radius vector, for the planet Earth, is the **geocentric** radius. In this thesis, the term "geocentric" is used in
a more general sense to identify quantities associated with the spherical polar coordinates of any planet*.

The following coordinate axes systems are shown in Fig. A.2:

\( \vec{r}, \vec{\lambda}, \vec{\lambda} \) Geocentric Latitude-Longitude Triad: Orthogonal set of unit vectors centered at the entry vehicle center of gravity and directed in radial, longitude, and latitude directions respectively.

\( \vec{r}, \vec{\phi}, \vec{\psi} \) Instantaneous Great Circle Triad: Orthogonal set of unit vectors centered at the vehicle center of gravity and directed in radial, along-track, and cross-track directions respectively.

The following quantities are defined in Fig. A.2:

Line of Nodes: The line of intersection of the plane of the instantaneous trajectory with the equatorial plane.

Ascending Node: The point of intersection of the line of nodes and the northerly-directed segment of the instantaneous ellipse describing the trajectory.

Descending Node: The point of intersection of the line of nodes and the southerly-directed segment of the instantaneous ellipse describing the trajectory.

\( \vec{R} \) : The position vector directed from the center of the planet to the instantaneous position of the vehicle.

* The interpretation of "geocentric" in this thesis may be considered to be a shortening of "geometric-centric" rather than the more precise "geoid-centric".
**Geocentric Latitude-Longitude Triad:** Orthogonal set of unit vectors centered at the entry vehicle center of mass and directed in radial, longitude, and latitude directions respectively.

**Instantaneous Great Circle Triad:** Orthogonal set of unit vectors centered at the vehicle center of mass and directed in radial, along-track, and across-track directions respectively.

*Fig. A.2: Definition of Latitude-Longitude and Great Circle Guidance Grids.*
\( \vec{v}_I \) : The instantaneous velocity vector of the vehicle with respect to inertial coordinates.

\( \psi \) : Inclination of the instantaneous trajectory plane with the equatorial plane (measured from equator to trajectory plane).

\( \lambda_{IT} \) : "Inertial" geocentric longitude of ascending line of nodes; i.e., angle measured in the equatorial plane from \( X_I \) to \( X_T \).

\( \lambda_{IP} \) : "Inertial" geocentric longitude of the vehicle; i.e., angle measured in equatorial plane from \( X_I \) to entry vehicle \( P \).

\( \lambda \) : (not shown in Fig. A.2) geocentric longitude; i.e., angle measured in equatorial plane from \( X_O \) to the vehicle.

\( \Lambda \) : geocentric latitude of the vehicle.

\( \Lambda_g \) : (not shown in Fig. A.2) geographic latitude of entry vehicle.

\( \phi \) : angle measured in the plane of trajectory (in direction of vehicle motion) from ascending line of nodes to the vehicle.

\( \beta \) : bearing angle of entry vehicle as seen by an observer directly under the vehicle on non-rotating planet; angle between North (\( \vec{I}_\Lambda \)) and the horizontal component of velocity (directed along \( \vec{I}_\phi \)).

The following angular relations are useful in conversion among the sets of angles defined in Fig. A.2.
Angular relations among coordinate systems given in Fig. A.2.

\[
\begin{align*}
\sin \Lambda &= \sin \phi \sin \psi \\
\sin(\lambda_{IP} - \lambda_{IT}) &= \frac{\sin \phi \cos \psi}{\sin \Lambda} \\
\cos(\lambda_{IP} - \lambda_{IT}) &= \frac{\cos \phi}{\cos \Lambda} \\
\sin \psi \cos \phi &= \cos \Lambda \cos \beta \\
\cos \psi &= \cos \Lambda \sin \phi \\
\tan \psi &= \frac{\tan \Lambda}{\sin(\lambda_{IP} - \lambda_{IT})}
\end{align*}
\]

Direction cosines between \( \bar{I}_r, \bar{I}_\phi, \bar{I}_\psi \) system and \( \bar{I}_r, \bar{I}_\lambda, \bar{I}_\Lambda \) system:

<table>
<thead>
<tr>
<th>( \bar{I}_r )</th>
<th>( \bar{I}_\phi )</th>
<th>( \bar{I}_\psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{I}_r )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{I}_\lambda )</td>
<td>0</td>
<td>sin ( \beta )</td>
</tr>
<tr>
<td>( \bar{I}_\Lambda )</td>
<td>0</td>
<td>cos ( \beta )</td>
</tr>
</tbody>
</table>

Direction cosines between \( x_T, y_T, z_T \) system and \( \bar{I}_r, \bar{I}_\phi, \bar{I}_\psi \) system:

<table>
<thead>
<tr>
<th>( x_T )</th>
<th>( \bar{I}_r )</th>
<th>( \bar{I}_\phi )</th>
<th>( \bar{I}_\psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_T )</td>
<td>cos ( \phi )</td>
<td>- sin ( \phi )</td>
<td>0</td>
</tr>
<tr>
<td>( z_T )</td>
<td>sin ( \phi ) cos ( \psi )</td>
<td>cos ( \phi ) cos ( \psi )</td>
<td>- sin ( \psi )</td>
</tr>
</tbody>
</table>

\( \bar{I}_r, \bar{I}_\phi, \bar{I}_\psi \) system:
The drag vector is directed anti-parallel to the velocity vector of the vehicle with respect to the atmosphere. The lift vector is perpendicular to this velocity vector and acts in the plane of symmetry of the vehicle. In the analysis performed herein, it is assumed that the entry vehicle has the capability of rotating the lift vector (i.e., banking) in order to produce curvature of the trajectory in a controlled manner to enable the vehicle to reach landing sites at some distance from the "no-bank" trajectory. For military vehicles, such as a reconnaissance platform, the banking capability may permit adequate military coverage of an entire hostile nation from a single parent satellite.

Two additional vehicle-centered coordinate triads are defined in Fig. A.3; the "Wings Level" triad $\overline{I}_x^{WL}, \overline{I}_y^{WL}, \overline{I}_z^{WL}$ and the "Vehicle" triad $\overline{I}_x, \overline{I}_y, \overline{I}_z$. It is noted that both the "Wings Level" triad and the "Vehicle" triad are basically oriented by the velocity vector of the vehicle with respect to the atmosphere. The left wing unit vector in the zero bank condition (iv) always lies in the $\overline{I}_x^{WL}$ plane.* Rotation of the vehicle about $\overline{V}_{(AM)}$ (coincident with $\overline{I}_x^{WL}, \overline{I}_x$) causes the lift vector to move out of the vertical plane, hence causes a side force which tends to curve the trajectory. The amount of rotation of the vehicle from the wings level condition is defined as the bank angle, $B$.

Following is a brief description of the coordinate systems shown in Fig. A.3:

- **Wings Level Triad**: Orthogonal set of unit vectors centered at entry vehicle center of gravity. $\overline{I}_x^{WL}$

* $\overline{I}_x$ is positive in the geocentric East direction. $\overline{I}_y$ is positive in the geocentric North direction. The $\overline{I}_x - \overline{I}_z$ plane is the geocentric horizontal plane, or "level plane" -- hence the term "Wings Level" coordinate system.
A.3: "Wings Level" and "Vehicle" Coordinate Triads.
is along the velocity vector of the vehicle with respect to the atmosphere. $\mathbf{i}_{WL}^x$ is along the left wing of the entry vehicle in the zero bank condition. $\mathbf{i}_{WL}^z$ is along the lift vector in the zero bank condition.

Vehicle Coordinate Frame: Orthogonal set of unit vectors centered at entry vehicle center of gravity. $\mathbf{i}_x$ is along the velocity vector of the vehicle with respect to the atmosphere. $\mathbf{i}_y$ is along the left wing of the vehicle; $\mathbf{i}_z$ is in the plane of symmetry of the vehicle and is a unit vector in the lift direction.

It is noted that $\mathbf{i}_x$ and $\mathbf{i}_{WL}^x$ are always coincident. $\mathbf{i}_y$ differs from $\mathbf{i}_{WL}^y$ and $\mathbf{i}_z$ differs from $\mathbf{i}_{WL}^z$ by the angle of bank, $B$. In zero bank condition, both triads are coincident.

The following angles are defined in Fig. A.3:

$$A_H = A (\mathbf{i}_\lambda - \mathbf{i}_x) : \text{angle between "East" (}\mathbf{i}_\lambda\text{) and horizontal component of vehicle velocity with respect to the atmosphere.}$$

$$\gamma = A (\mathbf{i}_\lambda - \mathbf{i}_x) : \text{angle between the horizontal component of vehicle velocity with respect to the atmosphere (i.e., the component in the } \mathbf{i}_\lambda - \mathbf{i}_x \text{ plane) and the total velocity vector of the vehicle with respect to the atmosphere.}$$

$$B = A (\mathbf{i}_{WL}^y - \mathbf{i}_y) = A (\mathbf{i}_{WL}^z - \mathbf{i}_z) : \text{bank angle of vehicle. Corresponds to rotations about the } \mathbf{i}_x \text{ axis from the wings level condition.}$$

Following are definitions of $A_H$ and $\gamma$ in terms of velocity components:
Rotations from the \( \mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_\Lambda \) triad to the \( \mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z \) triad are performed in the following order:

1. Rotate \( (\mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_\Lambda) \) to \( (\mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_y) \) by angle \( A_H \) about the \( \mathbf{I}_r \) axis.

2. Rotate \( (\mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_y) \) to \( (\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_y) \) by angle \( \gamma \) about the \( \mathbf{I}_y \) axis.

3. Rotate \( (\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_y) \) to \( (\mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z) \) by angle \( B \) about the \( \mathbf{I}_x \) axis.

Direction cosines between \( \mathbf{I}_r, \mathbf{I}_\lambda, \mathbf{I}_\Lambda \) triad and \( \mathbf{I}_x, \mathbf{I}_y, \mathbf{I}_z \) triad are as follows:

<table>
<thead>
<tr>
<th>( \mathbf{I}_r )</th>
<th>( \mathbf{I}_\lambda )</th>
<th>( \mathbf{I}_\Lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{I}_x )</td>
<td>( \sin \gamma )</td>
<td>( \cos \gamma \cos A_H )</td>
</tr>
<tr>
<td>( \mathbf{I}_y )</td>
<td>( \cos \gamma \sin B )</td>
<td>( -\sin A_H \cos B )</td>
</tr>
<tr>
<td>( \mathbf{I}_z )</td>
<td>( \cos \gamma \cos B )</td>
<td>( \sin A_H \sin B )</td>
</tr>
</tbody>
</table>

\[ (A-3) \]
The coordinate systems defined above are not those used in conventional aircraft analysis where it is common to define a set of axes fixed to the airframe. It is necessary to define a set of vehicle-fixed axes if the stability and control characteristics are to be studied—a subject beyond the scope of this thesis. Lift and drag forces are the primary gas-dynamic forces affecting the shape of the trajectory in the atmospheric phases of flight; secondary forces and moments are of lower order.

Thrust forces generated by the entry vehicle propulsion system will, in general, be applied in such a direction as to slow the vehicle down* (retor-thrust). It was assumed that the thrust vector passes at all times through the vehicle center of gravity in order to avoid excessive moments. The engine gimbal angles shown in Fig. A.4 were selected to describe the thrust direction.

Rotation from the $\begin{pmatrix} I_x \\ I_y \\ I_z \end{pmatrix}$ triad to the thrust triad $\begin{pmatrix} I_{x2} \\ I_{y2} \\ I_{z2} \end{pmatrix}$ is performed in the following order:

1. Rotate $\begin{pmatrix} I_x \\ I_y \\ I_z \end{pmatrix}$ to $\begin{pmatrix} I_{x1} \\ I_{y1} \\ I_{z1} \end{pmatrix}$ by angle $\alpha_e$ about $-I_y$ axis.

2. Rotate $\begin{pmatrix} I_{x1} \\ I_{y1} \\ I_{z1} \end{pmatrix}$ to $\begin{pmatrix} I_{x2} \\ I_{y2} \\ I_{z2} \end{pmatrix}$ by angle $\alpha_d$ about $I_{z1}$ axis.

Direction cosines between $\begin{pmatrix} I_{x2} \\ I_{y2} \\ I_{z2} \end{pmatrix}$ triad and $\begin{pmatrix} I_x \\ I_y \\ I_z \end{pmatrix}$ triad are as follows:

* Except possibly in the landing phase.
Fig. A.4: Engine Gimbal Angles
<table>
<thead>
<tr>
<th></th>
<th>$\bar{I}_x$</th>
<th>$\bar{I}_y$</th>
<th>$\bar{I}_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{I}_{x2}$</td>
<td>$\cos A_e \cos A_d$</td>
<td>$\sin A_d$</td>
<td>$\sin A_e \cos A_d$</td>
</tr>
<tr>
<td>$\bar{I}_{y2}$</td>
<td>$- \cos A_e \sin A_d$</td>
<td>$\cos A_d$</td>
<td>$- \sin A_e \sin A_d$</td>
</tr>
<tr>
<td>$\bar{I}_{z2}$</td>
<td>$- \sin A_e$</td>
<td>$0$</td>
<td>$\cos A_e$</td>
</tr>
</tbody>
</table>

The following angles are shown in Fig. A-4:

$A_e$ : Engine gimbal angle generated by rotations about the negative $\bar{I}_y$ axis.

$A_d$ : Engine gimbal angle generated by rotations about the displaced $\bar{I}_z$ axis (after rotating through angle $A_e$).

Thrust components may be written in component form in the $\bar{I}_r$, $\bar{I}_\lambda$, $\bar{I}_\phi$ triad by using the direction cosines given in (A-3). These components may, in turn, be transformed to the $\bar{I}_r$, $\bar{I}_\psi$, $\bar{I}_\psi$ triad by the table of cosines and angular conversion relations given in Table A-1.

Once the velocity requirements of the entry vehicle are established, the propulsion system must be designed in such a way that thrust may be applied in the proper direction with the proper magnitude and for the proper length of time; i.e., thrust vector control. Thrust vector control is also important for generation of orientation torques for attitude stability; problems connected with thrust vector control for attitude orientation and rotational moment stability are not considered in this thesis.
A.2: The Two-Dimensional Trajectory

Most of the analytical study described in this thesis was concerned with a two-dimensional approximation of the entry trajectory. Following is a brief summary of the principal assumptions made in postulating a two-dimensional trajectory:

1. The gravitational field of the planet is spherically symmetric and is free of anomalies.
2. The atmosphere rotates with the planet, and no wind exists. The velocity component of the rotating atmosphere in the $\bar{W}$ direction is negligible.
3. The vehicle flies in the zero bank condition with $A_d = 0$.

Figure A.5 defines the important forces and geometric quantities of the two-dimensional trajectory.

A.3: The Instantaneous Ellipse

It is common in celestial mechanics to describe the motion of heavenly bodies in terms of parameters of conic sections. There are many definitions of conics; the following definitions are convenient:

1. **Ellipse**: The locus of points the sum of whose distances from two fixed points (foci) is constant.
2. **Circle**: The locus of points equidistant from a single point (special case of the ellipse).
3. **Hyperbola**: The locus of points the difference of whose distances from two fixed points (foci) is constant.
4. **Parabola**: The locus of points equally distant from a fixed point (focus) with a fixed straight line (directrix).
Definitions:

- $X$ = distance flown
- $V$ = velocity of vehicle with respect to coordinates rotating with the planet (same as velocity of vehicle with respect to the planetary atmosphere, $V(AM)$, for no-wind condition)
- $H$ = altitude of vehicle above planet
- $\gamma$ = flight path angle with respect to local geocentric horizon - positive as shown.
- $\Gamma$ = mass flow rate of propellant
- $V_e$ = equivalent exit velocity of rocket propellant (assumed constant)
- $M$ = instantaneous mass of vehicle
- $D$ = drag
- $L$ = lift
- $A_e$ = engine gimbal angle of retro-rocket; angle the thrust vector makes with negative $I_x$ axis.

Fig. A.5: The Two-Dimensional Trajectory
When a body is in motion under the action of an attractive central force that varies as the inverse square of the distance, and if no external forces act on the body, the path described will be a conic whose focus is at the center of attraction. A particle moving according to such a force obeys Kepler's laws. Kepler's three laws of planetary motion, published around 1610, were the result of his pioneering analysis of planetary observations, and laid the ground work for many of the important contributions of Sir Isaac Newton. Kepler's laws may be summarized as follows:

A particle moving under the action of an attractive central force that varies as the inverse square of the distance

1. travels in an ellipse or a hyperbola (or their special cases, a circle or parabola) with the attracting center at one of the foci;
2. the radius vector from the center to the particle sweeps out equal areas in equal time*;
3. for the elliptic orbit, which results in periodic motions, the squares of the periods of rotation are proportional to the cubes of the major axis of the orbits.

Elliptical parameters are useful for analyzing the motion of the entry vehicle during the orbital phase and in the early phases of entry where gas-dynamic forces are negligible**. The use of elliptical parameters may be extended to the study of the entire trajectory by using the "instantaneous ellipse" technique; i.e., by describing the trajectory as a time-varying ellipse which matches the position and velocity vectors of the vehicle at each instant.

* Kepler's second law, the conservation of areal velocity, is a general theorem for central force motion(19) since angular momentum is always conserved. The first and third laws are restricted specifically to the inverse square law of force.

** This segment of the trajectory is called the "Keplerian Phase" in this thesis.
Frequent reference is made in this thesis to elliptical parameters, particularly in the study of retro-rocket system requirements described in Chapter 9. A description of the parameters of the ellipse are given in Fig. A.6.
Line of apsides: The major axis of the ellipse, connecting the apsides (apocenter and pericenter).

- Major axis \( = 2a \)
- Minor axis \( = 2b \)
- Latus rectum \( = 2l \)
- Latus rectum \( = 2l = a(1 - e^2) \)
- Eccentricity \( e \) (Always less than 1.0 for ellipse)
  \[ e = \sqrt{1 - (b/a)^2} \]
- True anomaly \( \theta \)
- Area of ellipse \( \approx \pi ab \)
  \[ \eta = \phi - \theta \]
- Equation of ellipse:
  \[ R = \frac{a(1 - e^2)}{1 + e \cos \theta} \]
- Period of orbit:
  \[ T = \frac{2 \pi a^{3/2}}{\sqrt{\frac{G}{2M_0}}} \]

Fig. A.6: The Instantaneous Ellipse.
A.4 Glossary of Symbols

A fundamental difficulty in the presentation of any mathematical description of physical phenomena is that of choosing a complete and workable system of symbols and notation. Following is a summary of symbols and definitions used in this thesis.

A.4.1 Mathematical Symbols

(\vec{\ )} : a line above a quantity designates a vector quantity; e.g., \vec{R}

(\dot{ )} : a dot above a quantity represents total time derivative \frac{d( )}{dt}

(\dot{\ )} : denotes total derivative with respect to dimensionless time \gamma, i.e., \frac{d( )}{d\gamma} Note: \gamma = \sqrt{\frac{G(m)\Omega}{R(m)\Omega}} t

\nabla(\ ) : del or gradient of the quantity in parenthesis.

(\nabla \times \ ) : vector cross product of quantities in parenthesis.

(\nabla \cdot \ ) : dot or scalar product of vector quantities in parenthesis.

The magnitude of a vector quantity is represented by the letter designation of the quantity with the vector symbol removed. Thus, the magnitude of \vec{R} is R.

\frac{\partial( )}{\partial x} : partial derivative of quantity in parenthesis with respect to x.

\delta( ) : variation of the quantity in parenthesis.

\Delta( ) : incremental change of the quantity in the parenthesis from its initial value; e.g., \Delta r = r - r_i

\sum_{n=1}^{\infty} ( )_n : summation of the quantity in parenthesis.

\log ( ) = \log_{10} ( )

\ln ( ) = \log_e ( )
A.4.2 Subscripts

\( r, \lambda, \Lambda \): Components along the \( \mathbf{i}_r, \mathbf{i}_\lambda, \mathbf{i}_\Lambda \) axes.

\( r, \phi, \psi \): Components along the \( \mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_\psi \) axes.

\( i \): Denotes initial value of the quantity.

\( f \): Denotes final value of the quantity.

\( 0 \): The subscript 0 refers to the generalized planet 0. When specialized to the planet Earth, "0" is replaced by "E".

\( m \): Denotes quantities at the "trajectory modification point".

The trajectory modification point is that point in the original satellite orbit at which entry is initiated through generation of thrust forces. (See Fig. 9.3)

\( I \): Measured with respect to inertial coordinates. For example, \( \vec{V}_I \) is the velocity of the vehicle with respect to inertial space, while \( \vec{V} \) denotes velocity of the vehicle with respect to coordinates rotating with the planet.

\( \alpha \): Value of the quantity at apogee; e.g., \( R_\alpha \) is the radius at apogee of an elliptical trajectory.

\( \gamma \): Value of the quantity at perigee; e.g., \( R_\gamma \) is the radius at perigee of an elliptical orbit.

A.4.3 Symbols

The author has endeavored to be consistent in the use of notation and symbols in this thesis. For example, the symbol \( \vec{V}_I \) represents the same quantity regardless of where it may appear in the text.

As is well known, certain particular symbols are used within each scientific field to represent quantities frequently encountered. Examples
of this are $\varrho$ for atmospheric density and $C_D$ for drag coefficient of aerodynamics, $Q$ for "true anomaly" of celestial mechanics, "p" or "h" for angular momentum in mechanics or physics, etc. Where possible, such conventions are followed in this thesis. Unfortunately in an analysis that spans many fields, one finds a conflict between the common symbols of one branch of science or engineering with those of another field. In such instances, the author had no alternative except to invent a symbol which he considered convenient: the choice may often be unsuited to the taste of the reader who is conversant with the more conventional representation.

A.4.3-1 English Letters

$\ddot{A}$ : acceleration vector of entry vehicle, ft./sec.$^2$

$A_e$ : Engine gimbal angle in $\overline{I}_x - \overline{I}_z$ plane (Figs. A.4 and A.5)

$(A_e)_{opt}$ : Optimum engine gimbal angle (i.e., that angle for which range errors are minimized with respect to errors in operation of retro-rocket system).

$A_d$ : Engine gimbal angle; consists of rotations out of the $\overline{I}_x - \overline{I}_z$ plane (Fig. A-4).

$A_H$ : Heading angle. Angle between "East" ($\overline{I}_\phi$) and horizontal component of vehicle velocity with respect to the atmosphere (Fig. A.3).

$A_c$ : Constant defined in circular trajectory study of section 10.3 as follows:

$$A_c = \left(\frac{I}{D}\right)C \frac{v_1^2}{2}$$

$A_1$ : Constant defined for degenerate elliptical orbital profile
in section 10.4 as follows:

\[ A_1 = \left( \frac{L^2}{D^2} \right) C \frac{V_1^2}{2} = A_c \]

\[ A_2 = \frac{\omega^2 \rho}{\rho_c} \left( \frac{V_1^2}{2} - \frac{1}{2} \right) \]

\[ A_3 = \frac{C V_{\infty}^3}{\rho_c} \]

\[ A_4 = \frac{\rho A_2}{\omega^2} \]

\( a \) : One-half the major axis of the elliptical trajectory

\( a_N \) : Dimensionless semi-major axis of the ellipse

\[ a_N = \frac{a}{R(m)0} \]

\( B \) : Angle of bank of the entry vehicle; angular rotation about the velocity vector of the vehicle with respect to the atmosphere (Fig. A.3)

\( B_c \) : Constant defined in section 10.3 as follows:

\[ B_c = C V_{\infty}^3 \left( \frac{1}{R_c} + \frac{L}{D} \frac{C}{2} \right) \]

\( b \) : One-half the minor axis of the ellipse.

\( b_N \) : Dimensionless semi-minor axis of the ellipse

\[ b_N = b/R(m)0 \]
b_1 : Constant defined in section 10.3 as follows:

$$b_1 = -2x_1$$

b_2 : Constant defined in section 10.3 as follows:

$$b_2 = x_1 - iy_1$$

b_3 : Constant defined in section 10.3 as follows:

$$b_3 = x_1 + iy_1$$

C : Trajectory constant. Defined as follows:

$$C = R^{(m)}_0 \rho_0 \frac{C_D S}{M}$$

C_c : Constant defined in section 10.3 as follows:

$$C_c = \frac{B_i}{r_i}$$

C_{(atm)} : Dimensional constant for the planetary atmosphere (BTU ft.-3/2 sec.-1)

C_D : Drag coefficient

C_{D_0} : Zero-lift drag coefficient.

C_L : Lift coefficient.

C_a : An acceleration constant defined as follows:

$$C_a = -\frac{\log (AC)_{ro}}{k_f \log e}$$

C_e : Auxiliary quantity defined as follows in Eq. (8-24)

$$C_e = -\frac{1}{k_f \log e} \log \left( \frac{\rho_{(SL)}}{\rho_{(SL)E}} \frac{R^{(m)}_0}{R^{(m)}_E} \right)$$
\( C_g \) : Auxiliary quantity defined as follows in Eq. (8-28):
\[
C_g = \frac{1}{k_E} \log \left( \frac{G(m)_{\text{E}}}{G(m)_{\text{P}}} \right)
\]

\( C_p \) : Heat capacity at constant pressure.

\( C_t \) : A temperature constant defined as follows:
\[
C_t = -\frac{18.44}{4k_E} \log \left( \frac{H}{F} \right)_{\text{EO}}
\]

\( C_v \) : Heat capacity at constant volume.

\( c \) : Maximum allowable specific force level in Earth g's

\( C_{th} \) : Threshold value of accelerations detectable by specific force measuring subsystem in Earth g's.

\( D \) : Drag force (lb.)

\( \mathcal{C}^{(\text{kin})} \): Kinetic Energy (ft.-lb.)
\[
\mathcal{E}^{(\text{kin})} = \frac{M}{2} \left( \overline{V}_1 \cdot \overline{V}_1 \right)
\]

\( \mathcal{C}^{(\text{pot})} \): Potential Energy (ft.-lb.)
\[
\mathcal{E}^{(\text{pot})} = -\frac{\gamma M_0 M}{R}
\]

\( \mathcal{C}^{(\text{tot})} \): Total Energy (ft.-lb.)
\[
\mathcal{E}^{(\text{tot})} = \mathcal{E}^{(\text{kin})} + \mathcal{E}^{(\text{pot})}
\]

\( E^{(\text{kin})} \) : Dimensionless kinetic energy per unit mass.
\[
E^{(\text{kin})} = \frac{\mathcal{C}^{(\text{kin})}}{MV_s^2}
\]

\( E^{(\text{pot})} \) : Dimensionless potential energy per unit mass.
\[
E^{(\text{pot})} = \frac{\mathcal{C}^{(\text{pot})}}{MV_s^2}
\]

\( E^{(\text{tot})} \) : Dimensionless total energy per unit mass:
\[
E^{(\text{tot})} = \frac{\mathcal{C}^{(\text{tot})}}{MV_s^2}
\]
\( \mathbf{F} \) : Specific force; i.e., force per unit mass. \( \text{ft./sec.}^2 \)

\( \mathbf{F}(\text{dr}) \) : Drag specific force vector. \( \text{ft./sec.}^2 \)

\( \mathbf{F}(\text{li}) \) : Lift specific force vector. \( \text{ft./sec.}^2 \)

\( \mathbf{F}(\text{th}) \) : Thrust specific force vector. \( \text{ft./sec.}^2 \)

\( \mathbf{f} \) : Dimensionless specific force (in surface g's of the planet).

\[ \mathbf{f} = \frac{\mathbf{\overline{F}}}{G(m)E} \]

\( \mathbf{f}_E \) : Dimensionless specific force in surface Earth g's.

\[ \mathbf{f}_E = \frac{\mathbf{\overline{F}}}{G(m)E} \]

\( \overline{G} \) : Gravitational field intensity. \( \text{(ft./sec.}^2) \)

\( G(m)O \) : Mean value of gravitational field intensity at surface of planet \( O \). \( \text{(ft./sec.}^2) \)

\( G(m)E \) : Mean value of gravitational field intensity at surface of Earth. \( G(m)E = 32.22848 \text{ ft./sec.}^2 \)

\( G_{sp} \) : Spherical component of gravitational mass attraction.

\[ G_{sp} = \frac{G_0 M_0}{R^2} \]

\( \overline{g} \) : Planet's gravity field. \( \text{(ft./sec.}^2) \)

\[ \overline{g} = \overline{G} - \overline{\mathbf{W}}_{10} \times (\overline{\mathbf{W}}_{10} \times \mathbf{R}) \]

\( g(m)E \) : Mean value of gravity field at surface of Earth.

\[ g(m)E = 32.17405 \text{ ft./sec.}^2 \]

\( H \) : Altitude of vehicle above planetary surface. \( \text{(ft.)} \)
h : Dimensionless altitude of vehicle above planetary surface:
\[ h = \frac{H}{R(m)} \]

\( I_n(x) \) : \( n \)th modified Bessel function of the first kind with argument \( x \).

\( \text{I}_{sp} \) : Specific impulse of the rocket propellant (sec.)

\( I_t \) : Total impulse of rocket thrust (lb.-sec.)

\( i \) : Symbol used in complex numbers; defined as follows:
\[ i^2 = -1 \]

\( K \) : Exponential decay parameter of planetary atmosphere (ft.\(^{-1}\))
\[ \rho = \rho_{(5L)} e^{-KH} \]

\( K_{\text{boltz}} \) : Stefan-Boltzman constant = \( 4.81 \times 10^{-13} \) BTU ft.\(^{-2}\) sec.\(^{-1}\) oR\(^{-4}\)
where oR represents degrees Rankine.

\( K_{rad} \) : Vehicle surface radiation emissivity.

\( K_g \) : Radiation emissivity of atmospheric gas.

\( k \) : Dimensionless decay parameter of planetary atmosphere
\[ k = KR(m) \]
\[ \rho = \rho_{(5L)} e^{-kH} \]

\( k_1 \) : Ratio of rate of total heat input at any point on the surface of the entry vehicle with respect to the rate of total heat input at the stagnation point.
\[ k_1 = \frac{Q}{\dot{Q}_s} \]

\( L \) : Lift force (lb.)
\[ l : \text{One-half the latus rectum of the ellipse: } l = a(1 - \epsilon^2) \]

\[ l_N : \text{Dimensionless semi-latus rectum of the ellipse.} \]

\[ l_N = \frac{1}{R(m)} \]

\[ l_e : \text{Unit path length of atmospheric gas.} \]

\[ \mathcal{F}\{\gamma\} : \text{Laplace transform of } f(\tau) \]

\[ M_a : \text{Mach number.} \]

\[ M : \text{Instantaneous mass of vehicle (slugs)} \]

\[ m : \text{Dimensionless instantaneous mass of vehicle: } m = \frac{M}{M_{\text{initial}}} \]

\[ M_0 : \text{Mass of planet.} \]

\[ M_E : \text{Mass of Earth.} \]

\[ N : \text{Constant defined as follows in section 10.4:} \]

\[ N = \frac{L C}{D^2} r_i e^{-k\Delta r} \]

This number must be less than about 0.3 for degenerate orbital solution of 10.4 to be accurate.

\[ n_L : \text{Dimensionless lift load factor: } n_L = \frac{L}{M_0(M)} \]

\[ n_D : \text{Dimensionless drag load factor: } n_D = \frac{D}{M_0(M)} \]

\[ P : \text{Angular momentum per unit mass of the vehicle; twice the areal velocity of the instantaneous ellipse.} \]

\[ P = RV_\phi = \frac{\gamma_{ab}}{T} \text{ ft.}^2/\text{sec.} \]

\[ p : \text{Dimensionless angular momentum.} \]

\[ p = \frac{P}{R(m)0^{3/2} G(m)0^{-1/2}} \]
Pr : Prandtl number.

Q : Total heat input per unit surface area of entry vehicle (BTU/ft.²)

\[ Q = Q_c + Q_r \]

Q_c : Total convective heat absorbed per unit area (BTU/ft.²)

Q_r : Total heat entering unit area by gaseous radiation (BTU/ft.²)

Q_rad : Total heat reradiated from unit surface area of vehicle (BTU/ft.²).

CR : Radius of curvature of vehicle in vicinity of stagnation point (ft.)

R_h : Universal Gas Constant = 8.31 X 10⁷ ergs/°Kelvin (i.e., work per degree Absolute temperature).

R : Radius vector from center of planet to vehicle.

r : Dimensionless radius vector from center of planet to vehicle.

\[ \bar{r} = \frac{\bar{R}}{R(m)0} \]

\Delta r : Incremental change in dimensionless radius from initial value.

\[ \Delta r = r - r_i \]

Re : Reynolds number.

\[ Re = \frac{\rho V \times \text{length}}{\mu} \]

R(eq)0 : Equatorial radius of planet 0.

R(m)0 : Mean radius of planet 0.

R(p)0 : Polar radius of planet 0.

S : Reference area of entry vehicle used in drag and lift computations (ft.²)
$S_w$ : Wetted area of entry vehicle (ft.$^2$)

$s$ : Complex variable used in the Laplace transform.

$T$ : Period of elliptical orbit:

$$T = \frac{2\pi a^3}{\sqrt{\mu M_0}}$$

$T_N$ : Dimensionless period of elliptical orbit:

$$T_N = \frac{T V_s}{R(m)0}$$

$t$ : time (sec.)

$t_b$ : Burning time of rocket (sec.)

$T_s$ : Stagnation point temperature (degrees Rankine)

$(T_s)_{\text{max}}$ : Maximum stagnation point temperature.

$T_{(m)0}$ : Mean temperature of planet's atmosphere (degrees Kelvin)

$T_t$ : Stagnation temperature of atmospheric gas (degrees Rankine)

$U$ : Auxiliary quantity defined as follows:

$$U = \frac{P(S)}{R(m)0} \frac{S}{M}$$

$u$ : Independent parameter having no physical meaning introduced to describe propellant mass flow.

$\vec{V}$ : Velocity vector of vehicle with respect to coordinates rotating with the planet.

$v$ : Dimensionless velocity with respect to coordinates rotating with the planet.

$$v = \frac{V}{V_s}$$

$\vec{v}_I$ : Velocity vector of entry vehicle with respect to inertial coordinates.
$v_I$: Dimensionless velocity of vehicle with respect to inertial coordinates. 

$$v_I = \frac{v_I}{V_S}$$

$\Delta v$: Incremental change in dimensionless velocity from initial value. 

$$\Delta v = v - v_I$$

$$\Delta v_I = v_I - v_{I_1}$$

$\overline{V}_{(AM)}$: Velocity vector of vehicle with respect to the planetary atmosphere (Note: $\overline{V} = \overline{V}_{(AM)}$ for no-wind condition).

$\overline{V}_{I(AM)}$: Velocity vector of planetary atmosphere with respect to inertial coordinates.

$\overline{V}_{O(AM)}$: Velocity vector of atmosphere with respect to coordinates rotating with the planet (i.e., wind).

$V_e$: Equivalent exit velocity of rocket propellant (assumed constant)

$$v_e = \frac{V_e}{V_S}$$

$V_C$: Circular orbital velocity: 

$$V_C = (GR)^{\frac{1}{2}}$$

$v_c$: Dimensionless circular orbital velocity: 

$$v_c = \frac{V_C}{V_S}$$

$V_S$: Circular orbital velocity at surface of planet:

$$V_S = (\frac{G(m)O}{R(m)O})^{\frac{1}{2}}$$

$\delta V$: Velocity impulse imparted at trajectory modification point by retrorocket system.
\( \delta v \) : Dimensionless velocity impulse imparted at trajectory modification point by retrorocket system:

\[
\delta v = \frac{\delta V}{V_S}
\]

\( \delta v_C \) : Dimensionless velocity impulse imparted at trajectory modification point by retrorocket system; non-dimensionalized with respect to circular satellite velocity at the trajectory modification point:

\[
\delta v_C = \frac{\delta V}{V_C}
\]

\( \bar{\omega}_{10} \) : Daily sidereal angular velocity of the planet; a vector quantity directed along the polar axis (North) of the planet.

\( \bar{\omega}_{1p} \) : Vector angular velocity of the center of gravity of the entry vehicle with respect to inertial coordinates.

\( X \) : Distance flown - measured at surface of planet.

\( X_N \) : Dimensionless distance flown:

\[
X_N = \frac{X}{R_{(m)0}}
\]

\( x_1 \) : Constant defined in section 10.3 as follows:

\[
x_1 = \frac{\omega_k \sinh \left[ \frac{1}{3} \sinh^{-1} \left( \frac{C_c \sqrt{3}}{2 \omega_k \sqrt{3}} \right) \right]}{\sqrt{3}}
\]

\( Y \) : A function of angle of attack, angle of sweep, and Mach number in convective heat transfer equation (7-1).

\( y_1 \) : A constant defined in section 10.3 as follows:

\[
y_1 = \frac{\omega_k \cosh \left[ \frac{1}{3} \sinh^{-1} \left( \frac{C_c \sqrt{3}}{2 \omega_k \sqrt{3}} \right) \right]}{\sqrt{3}}
\]

\( Z \) : Chapman's transformation variable. Equivalent to:

\[
Z = \frac{v \cos \gamma \sin \gamma}{\frac{\delta}{\sqrt{k}}}
\]
A.4.3-2 Multiple Letter Symbols

\( (AC) \): Dimensional acceleration constant for entry into the planetary atmosphere:

\[
(AC) = \frac{\rho_{(SL)} G(m) o R(m) o}{2 G(m) E} \text{ slug/s}^2
\]

\( (AC)_{EO} \): Ratio of acceleration constant of the planet to that of the Earth:

\[
(AC)_{EO} = \frac{(AC)}{(AC)_{E}}
\]

\( (HR)_{EO} \): "Heating Ratio" of the atmosphere of planet with respect to Earth:

\[
(HR)_{EO} = \left( \frac{\mu_{(SL)}}{\mu_{(SL)E}} \right)^{\frac{1}{2}} \left( \frac{G(m) o}{G(m) E} \right)^{\frac{5}{2}} \left( \frac{P(r) o}{P(r) E} \right)^{\frac{5}{4}} \left( \frac{K_{m}}{K_{E}} \right)^{\frac{1}{2}} \left( \frac{P_{r}}{P_{r} E} \right)^{\frac{2}{3}} \left( \frac{T_{0} - 1}{T_{0}} \right)^{\frac{1}{4}} \left( \frac{T_{0}}{T_{0} - 1} \right)^{\frac{1}{4}}
\]

\( (HF)_0 \): "Heating Function" of the planetary atmosphere.

\[
(HF)_0 = \frac{1}{2} \rho_{(SL)} \left( \frac{R(m) o}{K} \right)^{\frac{1}{4}} (HR)_{EO} \left( \text{slug/s}^2 \right)^{\frac{1}{2}}
\]

\( (HF)_{EO} \): Ratio of the heating function of the planetary atmosphere to the heating function of the Earth's atmosphere.

\[
(HF)_{EO} = \frac{(HF)_0}{(HF)_{E}}
\]

\( (TC) \): "Temperature Constraint Constant" -- a constant for a particular vehicle entering the atmosphere of a particular planet:

\[
(TC) = \frac{1.392 \times 10^4}{(T_s)_{max}^{\frac{1}{4}}} (HF)_{EO}^{\frac{4}{4}} (VF)^{\frac{1}{4}}
\]
(VF) : Vehicle function.

\[(VF) = \frac{1}{k_{\text{rad}} R^2}\]

Values of the above quantities which are constant for the planet are listed in Table A-2 for the terrestrial planets.

| Table A-2: Approximate values of Heating and Acceleration Constants for the Terrestrial Planets |
|--------------------------------------------------|----------------------------------|-----------------|
|                     |     |                         |                 |
|                     | Venus | Earth                  | Mars            |
| (AC)                |       |                         |                 |
| 2.86 x 10^5          | 2.49 x 10^4        | 3.44 x 10^2         |
| (AC)_{EO}            | 11.5  | 1.0                     | 0.0138          |
| (HR)_{EO}            | 0.7   | 1.0                     | 0.09            |
| (HF)_{O}             | 104.8 | 43.5                    | 1.1             |
| (HF)_{EO}            | 2.41  | 1.0                     | 0.0253          |
| (HF)_{EO}^{1/4}      | 1.25  | 1.0                     | 0.4             |

A.4.3-3: Greek Symbols

\(\alpha\) : angle of attack

\(\beta\) : Bearing angle of entry vehicle as seen by an inertial
observer in the vehicle; angle between North \((l_A)\) and
tangential velocity vector of vehicle (directed along \(l_\phi\)).

\(\beta_1\) : An auxiliary real variable (Eq. 7-26).

\(\gamma, \gamma\) : Flight path angle in inertial framework and planetary
axes, respectively.

\(\gamma_0\) : Ratio of heat capacity of the atmosphere of planet 0.

\[\gamma_0 = \frac{c_p}{c_v}\]

\(\gamma_E\) : Ratio of heat capacity of the Earth's atmosphere.

\(\gamma_g\) : Universal Gravitational Constant

\[\gamma_g = 6.658 \times 10^{-11} \frac{m^3}{kg \cdot sec^2}\]

\(\Gamma\) : Propellant mass flow rate (slugs per sec.)

\(\Gamma_N\) : Dimensionless propellant mass flow rate:

\[\Gamma_N = \frac{\Gamma}{M_t} \sqrt{\frac{2(m)}{G(m)G_0}}\]

\(\delta(\ )\) : Variation of the quantity in parenthesis.

\(\delta(\ )\) : Incremental change of quantity in parenthesis from initial
value; e.g., \(\Delta r = r - r_1\)

\(\varepsilon\) : Eccentricity of ellipse

\[\varepsilon = \sqrt{1 - (b/a)^2}\]

\(\varepsilon_1\) : Component of eccentricity along major axis:

\[\varepsilon_1 = \varepsilon \cos \theta\]

\(\varepsilon_2\) : Component of eccentricity along minor axis:

\[\varepsilon_2 = \varepsilon \sin \theta\]

\(\varepsilon_E\) : Eccentricity of the Earth:

\[\varepsilon_E = \sqrt{0.0067226700}\]

\(\xi_1\) : An auxiliary real variable (Eq. (7-26))

\(\xi_2\) : An auxiliary real variable (Eq. (7-32))
\( \eta \): angle measured in the plane of an elliptical trajectory from the ascending line of nodes to the line of apsides.

\( \eta_1 \): An auxiliary variable (Eq. (7-27))

\( \Theta \): True anomaly; angle measured from line of apsides in the ellipse to the position of the vehicle.

\( \Theta \): Angle defined as follows in equation (9-97):

\[
\Theta = \left( \chi_i^2 + \frac{2 U C_0 (\frac{1}{r}) \sigma_i}{k} \right)^{\frac{1}{2}}
\]

\( \lambda \): Geocentric longitude of the vehicle; i.e., equatorial angle measured from planet's \( \chi_0 \) axis to the vehicle.

\( \lambda_{ip} \): Inertial geocentric longitude of the vehicle; i.e., equatorial angle measured from planet's \( \chi_{\perp} \) axis to the vehicle.

\( \lambda_{in} \): Inertial geocentric longitude of ascending line of nodes; i.e., angle measured in equatorial plane between \( \chi_{\perp} \) and \( \chi_{\parallel} \).

\( \lambda_1 \ldots \lambda_n \): Lagrange Multipliers

\( \Lambda \): Geocentric latitude; angle measured between the radius vector from the planet center to the vehicle and the projection of this vector on the planet's equatorial plane.

\( \Lambda_g \): Geographic latitude; angle measured between the normal to the reference ellipsoid of the planet and the projection of this normal on the equatorial plane.

\( \mu \): Coefficient of viscosity of the planet's atmosphere (slug/ft\(_2\)sec)

\( \mu_E \): Coefficient of viscosity of the Earth's atmosphere.

\( \nu \): Dimensionless constant describing non-spherical component of planet's gravitational potential.

\( \xi \): The negative of the ratio of percentage change in radial distance of the entry vehicle from the planet center to the percentage change in horizontal component of the vehicle's velocity with respect to inertial coordinates.

\[
\xi = - \frac{R/R}{V_p/V_{Ip}}
\]

\( \xi \) is used in the definition of the "Conservation Parameter". Conservation Parameter = \( \left| 1 - \frac{\xi}{\xi} \right| \)
\[ \tau : \pi (3.1416) \]
\[ \rho : \text{Free stream atmospheric density (slug/ft.}^3) \]
\[ \rho_{(SL)} : \text{Sea Level or surface value of atmospheric density (slug/ft.}^3) \]
\[ \rho_{(SL)} : \text{The intercept of the straight line which best fits a curve of log } \rho \text{ vs altitude (not the same as true sea level density).} \]
\[ \sigma : \text{Atmospheric density ratio: } \sigma = \frac{\rho}{\rho_{(SL)}} \approx e^{-kh} \]
\[ \sum_{n=1}^{\infty} ( \theta_n ) = \text{Summation of quantities in parenthesis} \]
\[ \tau : \text{Dimensionless time } \tau = \frac{V_S}{R(m)_0}t = \sqrt{\frac{\Omega(m)_0}{R(m)_0}}t \]
\[ \phi : \text{angle measured in plane of trajectory in direction of motion from line of nodes to the vehicle.} \]
\[ \Phi : \text{Gravitational potential (potential energy per unit mass). In this thesis, the gravitational potential of the Earth is assumed to be:} \]
\[ \Phi = -\frac{Y_g M_e}{R} \left[1 - \frac{\nu R^2 E e}{R^2(1 - 3\cos 2\Lambda)}\right] \]
\[ \psi : \text{Angle of inclination of the trajectory; angle measured from equatorial plane to the trajectory plane (Fig. A-2).} \]
\[ \Theta : \text{A constant for the skipping trajectory solution:} \]
\[ \Theta = (\gamma^2 + \frac{2UCI(C_1)}{R})^2 \]
\[ \Omega : \text{Dimensionless angular velocity of the planet about its polar axis:} \]
\[ \Omega = \omega_0 \left(\frac{R(m)_0}{G(m)_0}\right) \]
\[ \omega : \text{Average angular velocity in the degenerate elliptical trajectory:} \]
\[ \omega^2 = \left(\frac{2}{r_i^2} - u_i^2\right)^3 \]
\[ \omega_c : \text{Angular velocity in the decaying circular orbital trajectory:} \]
\[ \omega_c^2 = \left(\frac{\nu}{r_i}^2\right)^2(1 + C \nu \frac{L}{D}) \]
A.5 Summary of Physical Constants of the Planets and Atmospheres

Table A.3 summarizes values of various physical characteristics of the planets and their atmospheres used in the numerical calculations in this thesis. New observations and calculations revise existing data frequently; hence complete agreement is seldom found between any two tables of this type. Much of the physical data concerning many of the planets is fragmentary; large scale revisions may be expected when additional data is obtained from artificial satellites and planetary probes.

### Table A.3: Summary of Planetary Atmospheric and Physical Data

(a) Data on the planet Earth:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{(eq)}^E )</td>
<td>6,378,388 ± 18 meters = 3963.3386 statute miles Ref. (55)</td>
</tr>
<tr>
<td>( R_{(p)}^E )</td>
<td>6,356,911.946 meters = 3949.9941 statute miles Ref. (55)</td>
</tr>
<tr>
<td>( R_{(m)}^E )</td>
<td>20,888,104 ft. Ref. (55)</td>
</tr>
<tr>
<td>Ellipticity</td>
<td>( \frac{R_{(eq)}^E - R_{(p)}^E}{R_{(eq)}^E} = \frac{1}{297} = .0033670034 ) Ref. (55)</td>
</tr>
<tr>
<td>( \epsilon^2_E )</td>
<td>0.0067226700 Ref. (55)</td>
</tr>
<tr>
<td>( W_{IE} )</td>
<td>7.2921159 \times 10^{-5} \text{ rad/sec.} Ref. (55)</td>
</tr>
<tr>
<td>( \gamma g M_E )</td>
<td>( (1 \pm 8 \times 10^{-5})(3.986329 \times 10^{20} \text{ cm}^3/\text{sec.}^2) ) Ref. (56)</td>
</tr>
<tr>
<td>( G(m)E )</td>
<td>32.22848 ft./sec.(^2)</td>
</tr>
<tr>
<td>( \epsilon(m)E )</td>
<td>32.17405 ft./sec.(^2)</td>
</tr>
<tr>
<td>( 6\nu )</td>
<td>1.638 \times 10^{-3} Ref. (56)</td>
</tr>
</tbody>
</table>
(b) Data on other planets

<table>
<thead>
<tr>
<th></th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
<th>Jupiter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(m)O$</td>
<td>20.27x10^6 ft.</td>
<td>20.89x10^6 ft.</td>
<td>11.08x10^6 ft.</td>
<td>230x10^6 ft.</td>
</tr>
<tr>
<td>$R(m)O/R(m)E$</td>
<td>0.97</td>
<td>1.0</td>
<td>0.5313</td>
<td>11.0</td>
</tr>
<tr>
<td>$G(m)O/G(m)E$</td>
<td>0.91</td>
<td>1.0</td>
<td>0.38</td>
<td>2.64</td>
</tr>
<tr>
<td>Atmospheric Gases</td>
<td>$CO_2, N_2$</td>
<td>$N_2, O_2$</td>
<td>$N_2, CO_2$</td>
<td></td>
</tr>
<tr>
<td>$\mu/O/\mu_E$</td>
<td>0.8</td>
<td>1.0</td>
<td>1.0</td>
<td>0.5</td>
</tr>
<tr>
<td>Mean molecular wgt. (gm/mole)</td>
<td>40</td>
<td>29</td>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>$T(m)O$ (°K)</td>
<td>270</td>
<td>240</td>
<td>220</td>
<td>170</td>
</tr>
<tr>
<td>l/K ft.</td>
<td>2.0x10^4</td>
<td>2.35x10^4</td>
<td>6x10^4</td>
<td>6x10^4</td>
</tr>
<tr>
<td>k/k_E</td>
<td>1.14</td>
<td>1.0</td>
<td>0.207</td>
<td>4.32</td>
</tr>
<tr>
<td>k</td>
<td>1013</td>
<td>889</td>
<td>184.6</td>
<td>3840</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>2.69x10^-4</td>
<td>.0587</td>
<td>.0675</td>
<td></td>
</tr>
<tr>
<td>$\rho_{(SL)}$</td>
<td>.0326</td>
<td>.002378</td>
<td>.000193</td>
<td></td>
</tr>
<tr>
<td>$\rho_{(SI)}$</td>
<td>.0027</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_{(SL)}/\rho_{(SL)E}$</td>
<td>12.1</td>
<td>0.88</td>
<td>.0604</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX B

PHYSICAL CHARACTERISTICS OF MAJOR BODIES OF THE SOLAR SYSTEM

The solar system is the collection of bodies, of which the Earth is one, that are distinguished from all other bodies in the universe by having the Sun as the center of motion. The bodies of this system may be classified under the following headings:

(1) **Sun**

(2) **Planets**

   (a) The four inner planets — Mercury, Venus, Earth, and Mars — are relatively small, dense bodies and are known as the "terrestrial" planets.

   (b) The next four in distance from the Sun — Jupiter, Saturn, Uranus, and Neptune — are often called the "major" or "giant" planets. These are relatively large bodies composed principally of gases with solid ice and rock cores at unknown depths below the visible upper surfaces of their atmospheres.

   (c) Pluto, recently discovered (1930), has relatively unknown physical characteristics.

(3) **Natural Satellites of the Planets**

Thirty-one moons have been discovered to date (1959). The system formed by a planet and its satellites is distinguished by the name of the planet, such as the Martian system, the Jovian system, the Saturnian system.

(4) **Asteroids**

Some 2000 minor planets have been discovered; others are being discovered each year. The largest, Ceres, has a diameter of 480 miles. Most of the rest are less than 50 miles wide. The
majority move between the orbits of Mars and Jupiter. Most families of asteroids seem strongly influenced by Jupiter, and move in orbits determined by the gravitational attraction of Jupiter and Sun.

(5) Comets

About 5-10 new comets are discovered yearly. Comets are very loose collections of orbital material that sweep into the inner regions of the solar system from space far beyond the orbit of Pluto. Some return periodically; some never do. The body of the comet probably consists of rarified gases and dust; the heads are thought to be frozen gases or "ices".

(6) Meteors

Meteors exist in untold millions. Meteor fragments that reach the ground are known as meteorites. The average meteor weighs 0.0005 ounce(57) and over 100 million strike the Earth's atmosphere daily. F.C. Leonard (1952) (58) estimated the amount of meteoric material falling upon the earth, including the ashes of burned out meteors and unburned micrometeorites, to be 5000 metric tons daily. This estimate was much larger than former ones, but is confirmed by the work of Hans Pettersson and Henri Rotschi in Sweden. A more recent estimate made by Harvard Observatory favors a value of 2,000 tons per day(59). Soviet scientists announced in August 1958 that their satellite data indicated an estimate of 8 to 10 x 10^5 tons per day(60).

(7) Micrometeorites and Dust

The smallest dust particles (micrometeorites) are concentrated primarily either in the ecliptic plane* or along the orbits of comets(61). Evidence that cosmic dust is concentrated in the plane of the ecliptic consists of observations of a faint tapered band of light centered along the ecliptic, called the zodiacal light.

A brief summary of physical data on principal bodies of the solar system is given in Table B-1. New discoveries together with more complete observations and calculations revise existing data frequently.

Table B-2 summarizes physical and astronomical data collected concerning the principal natural satellites of planets in the Solar System.

* Plane of the Earth's orbit about the Sun.
Table B.1: Summary of Physical Data on Principal Bodies of Solar System

Note: Superscript numerals refer to notes at end of table.

<table>
<thead>
<tr>
<th>Body</th>
<th>Mass $M_o$ (Earth = 1.0)</th>
<th>Volume (Earth = 1.0)</th>
<th>Density (Earth = 1.0)</th>
<th>Density (Water = 1.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>332,500</td>
<td>1,306,000</td>
<td>0.255</td>
<td>1.41</td>
</tr>
<tr>
<td>Mercury</td>
<td>0.0543</td>
<td>0.06</td>
<td>0.76</td>
<td>4.0</td>
</tr>
<tr>
<td>Venus</td>
<td>0.0136</td>
<td>0.92</td>
<td>0.89</td>
<td>4.9</td>
</tr>
<tr>
<td>Earth</td>
<td>0.1069 - 0.1076</td>
<td>1.0</td>
<td>1.00</td>
<td>5.52</td>
</tr>
<tr>
<td>Mars</td>
<td>318.35</td>
<td>1.318</td>
<td>0.241</td>
<td>1.32</td>
</tr>
<tr>
<td>Jupiter</td>
<td>93.3</td>
<td>376</td>
<td>0.13</td>
<td>0.72</td>
</tr>
<tr>
<td>Saturn</td>
<td>14.54</td>
<td>64</td>
<td>0.23</td>
<td>1.25</td>
</tr>
<tr>
<td>Uranus</td>
<td>17.2</td>
<td>60</td>
<td>0.29</td>
<td>1.6</td>
</tr>
<tr>
<td>Neptune</td>
<td>0.8</td>
<td>0.07</td>
<td>&gt; 0.9</td>
<td>-</td>
</tr>
<tr>
<td>Pluto</td>
<td>0.0123</td>
<td>0.02</td>
<td>0.607</td>
<td>-</td>
</tr>
</tbody>
</table>

Equatorial Gravitational Acceleration $g_{eq}$ (ft/sec^2)

<table>
<thead>
<tr>
<th>Body</th>
<th>$g_{eq}$ (ft/sec^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>4.679 x 10^21</td>
</tr>
<tr>
<td>Mercury</td>
<td>7.645 x 10^14</td>
</tr>
<tr>
<td>Venus</td>
<td>1.145 x 10^16</td>
</tr>
<tr>
<td>Earth</td>
<td>1.406 x 10^16</td>
</tr>
<tr>
<td>Mars</td>
<td>1.515 x 10^15</td>
</tr>
<tr>
<td>Jupiter</td>
<td>4.467 x 10^15</td>
</tr>
<tr>
<td>Saturn</td>
<td>1.338 x 10^18</td>
</tr>
<tr>
<td>Uranus</td>
<td>2.046 x 10^17</td>
</tr>
<tr>
<td>Neptune</td>
<td>2.423 x 10^17</td>
</tr>
<tr>
<td>Pluto</td>
<td>1.123 x 10^16</td>
</tr>
<tr>
<td>Moon</td>
<td>1.73 x 10^14</td>
</tr>
</tbody>
</table>

Circular Satellite Velocity at Surface $V_s = \sqrt{GM_o/R_{eq}}$ ft/sec

<table>
<thead>
<tr>
<th>Body</th>
<th>$V_s$ (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>1.43 x 10^6</td>
</tr>
<tr>
<td>Mercury</td>
<td>9.655</td>
</tr>
<tr>
<td>Venus</td>
<td>23.730</td>
</tr>
<tr>
<td>Earth</td>
<td>25.950</td>
</tr>
<tr>
<td>Mars</td>
<td>11.810</td>
</tr>
<tr>
<td>Jupiter</td>
<td>139.600</td>
</tr>
<tr>
<td>Saturn</td>
<td>84.170</td>
</tr>
<tr>
<td>Uranus</td>
<td>51.420</td>
</tr>
<tr>
<td>Neptune</td>
<td>54.350</td>
</tr>
<tr>
<td>Pluto</td>
<td>15.650</td>
</tr>
<tr>
<td>Moon</td>
<td>5.500</td>
</tr>
</tbody>
</table>

Surface Escape Velocity $V_e$ (ft/sec) and (km/sec)

<table>
<thead>
<tr>
<th>Body</th>
<th>$V_e$ (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>13.600</td>
</tr>
<tr>
<td>Mercury</td>
<td>33.600</td>
</tr>
<tr>
<td>Venus</td>
<td>36.700</td>
</tr>
<tr>
<td>Earth</td>
<td>16.700</td>
</tr>
<tr>
<td>Mars</td>
<td>197.000</td>
</tr>
<tr>
<td>Jupiter</td>
<td>36.700</td>
</tr>
<tr>
<td>Saturn</td>
<td>22.4</td>
</tr>
<tr>
<td>Uranus</td>
<td>25.5</td>
</tr>
<tr>
<td>Neptune</td>
<td>3 ?</td>
</tr>
<tr>
<td>Pluto</td>
<td>53.000</td>
</tr>
</tbody>
</table>

Minimum launch Velocity to reach Earth (ft/sec)

<table>
<thead>
<tr>
<th>Body</th>
<th>$V_{min}$ (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>44,000</td>
</tr>
<tr>
<td>Mercury</td>
<td>38,000</td>
</tr>
<tr>
<td>Venus</td>
<td>38,000</td>
</tr>
<tr>
<td>Earth</td>
<td>46,000</td>
</tr>
<tr>
<td>Mars</td>
<td>49,000</td>
</tr>
<tr>
<td>Jupiter</td>
<td>51,000</td>
</tr>
<tr>
<td>Saturn</td>
<td>52,000</td>
</tr>
<tr>
<td>Uranus</td>
<td>53,000</td>
</tr>
<tr>
<td>Neptune</td>
<td>53,000</td>
</tr>
<tr>
<td>Pluto</td>
<td>53,000</td>
</tr>
<tr>
<td>Moon</td>
<td>53,000</td>
</tr>
</tbody>
</table>

Transit Time

<table>
<thead>
<tr>
<th>Body</th>
<th>Transit Time</th>
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</thead>
<tbody>
<tr>
<td>Sun</td>
<td>110 days</td>
</tr>
<tr>
<td>Mercury</td>
<td>150 d</td>
</tr>
<tr>
<td>Venus</td>
<td>260 d</td>
</tr>
<tr>
<td>Earth</td>
<td>1.7 years</td>
</tr>
<tr>
<td>Mars</td>
<td>6 years</td>
</tr>
<tr>
<td>Jupiter</td>
<td>16 years</td>
</tr>
<tr>
<td>Saturn</td>
<td>31 years</td>
</tr>
<tr>
<td>Uranus</td>
<td>46 years</td>
</tr>
</tbody>
</table>

* 5.966 ± .001 x 10^27 gms.
<table>
<thead>
<tr>
<th></th>
<th>Mean Distance from Sun</th>
<th>Orbital Distance</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Astronomical Units</td>
<td>Astronomical Millions of miles</td>
<td>Perihelion (x 10^9 ft.)</td>
<td>Aphelion (x 10^9 ft.)</td>
</tr>
<tr>
<td>Sun</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mercury</td>
<td>0.387</td>
<td>36</td>
<td>151</td>
<td>229</td>
</tr>
<tr>
<td>Venus</td>
<td>0.723</td>
<td>67</td>
<td>353</td>
<td>357</td>
</tr>
<tr>
<td>Earth</td>
<td>1.00</td>
<td>92.9</td>
<td>483</td>
<td>499</td>
</tr>
<tr>
<td>Mars</td>
<td>1.524</td>
<td>142</td>
<td>679</td>
<td>817</td>
</tr>
<tr>
<td>Jupiter</td>
<td>5.203</td>
<td>483</td>
<td>2430</td>
<td>2680</td>
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<tr>
<td>Saturn</td>
<td>9.539</td>
<td>886</td>
<td>4520</td>
<td>4940</td>
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<tr>
<td>Uranus</td>
<td>19.19</td>
<td>1,782</td>
<td>8960</td>
<td>9860</td>
</tr>
<tr>
<td>Neptune</td>
<td>30.07</td>
<td>2,793</td>
<td>14610</td>
<td>14870</td>
</tr>
<tr>
<td>Pluto</td>
<td>39.46</td>
<td>3,670</td>
<td>14510</td>
<td>24100</td>
</tr>
<tr>
<td>Moon</td>
<td>1.00</td>
<td>92.9</td>
<td>1.170</td>
<td>1.334</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orbital Speed (ft./sec.)</td>
<td>Perihelion</td>
<td>Aphelion</td>
<td>Eccentricity of Orbit</td>
<td>Orbital Period</td>
</tr>
<tr>
<td>Sun</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mercury</td>
<td>176,000</td>
<td>143,000</td>
<td>.206</td>
<td>88 days</td>
</tr>
<tr>
<td>Venus</td>
<td>115,000</td>
<td>114,500</td>
<td>.007</td>
<td>224 d 17 hrs.</td>
</tr>
<tr>
<td>Earth</td>
<td>98,500</td>
<td>96,800</td>
<td>.017</td>
<td>365 d 6 hr. 9 min.</td>
</tr>
<tr>
<td>Mars</td>
<td>83,000</td>
<td>75,700</td>
<td>.093</td>
<td>687 days</td>
</tr>
<tr>
<td>Jupiter</td>
<td>43,900</td>
<td>41,800</td>
<td>.048</td>
<td>4326 days</td>
</tr>
<tr>
<td>Saturn</td>
<td>32,200</td>
<td>30,800</td>
<td>.056</td>
<td>29 years 167 days</td>
</tr>
<tr>
<td>Uranus</td>
<td>22,800</td>
<td>21,800</td>
<td>.0086</td>
<td>84.01 yrs.</td>
</tr>
<tr>
<td>Neptune</td>
<td>17,900</td>
<td>17,700</td>
<td>.055</td>
<td>248 yrs.</td>
</tr>
<tr>
<td>Pluto</td>
<td>3,480</td>
<td>3,260</td>
<td></td>
<td>27 d 7 h 43 m</td>
</tr>
<tr>
<td>Moon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orbital Inclination to Ecliptic Plane</td>
<td>Length of Day</td>
<td>Inclination of Equator</td>
<td>Number of Moons</td>
<td></td>
</tr>
<tr>
<td>Sun</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mercury</td>
<td>7° 1'</td>
<td>24.65d (5)</td>
<td>7° 10'</td>
<td>0</td>
</tr>
<tr>
<td>Venus</td>
<td>3° 24'</td>
<td>88.0d (6)</td>
<td>?</td>
<td>0</td>
</tr>
<tr>
<td>Earth</td>
<td>0° 0'</td>
<td>224.7d</td>
<td>24 h</td>
<td>1</td>
</tr>
<tr>
<td>Mars</td>
<td>1° 51'</td>
<td>23° 27'</td>
<td>25° 10'</td>
<td>1(?)</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1° 18'</td>
<td>24.6h</td>
<td>3° 7'</td>
<td>12(8)</td>
</tr>
<tr>
<td>Saturn</td>
<td>2° 29'</td>
<td>9h 50m</td>
<td>26° 45'</td>
<td>9(9)</td>
</tr>
<tr>
<td>Uranus</td>
<td>0° 46'</td>
<td>10h</td>
<td>98°</td>
<td>5(10)</td>
</tr>
<tr>
<td>Neptune</td>
<td>1° 45'</td>
<td>15.7h</td>
<td>29°</td>
<td>2(11)</td>
</tr>
<tr>
<td>Pluto</td>
<td>17° 9'</td>
<td>(?)</td>
<td>?</td>
<td>0 (?,?)</td>
</tr>
<tr>
<td>Moon</td>
<td>18° 19' to 28° 35' from Earth's equator</td>
<td>27°</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Distance from Earth (millions of miles)</td>
<td>Intensity of Sunlight at mean distance Earth = 1.0</td>
<td>Total Radiation Flux</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------------------------------</td>
<td>-----------------------------------------------</td>
<td>---------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Greatest</td>
<td>Least</td>
<td>watts/ft.²</td>
<td>kw/meter²</td>
<td></td>
</tr>
<tr>
<td>Sun</td>
<td>94.6</td>
<td>91.5</td>
<td>6.7</td>
<td>835</td>
</tr>
<tr>
<td>Mercury</td>
<td>138</td>
<td>48.(12)</td>
<td>1.9</td>
<td>238.6</td>
</tr>
<tr>
<td>Venus</td>
<td>162</td>
<td>26</td>
<td>1.0</td>
<td>125</td>
</tr>
<tr>
<td>Earth</td>
<td>-</td>
<td>-</td>
<td>0.43</td>
<td>54.1</td>
</tr>
<tr>
<td>Mars</td>
<td>234</td>
<td>35</td>
<td>0.37</td>
<td>4.62</td>
</tr>
<tr>
<td>Jupiter</td>
<td>602</td>
<td>367</td>
<td>0.11</td>
<td>1.37</td>
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Table B-1 (cont.)

Superscript numerical notes of Table B-1.

(1) These values are based on 1950 observations of Kuiper. This diameter of Neptune is 3,300 miles less than usually given. The diameter and volume of Pluto by these observations is considerably less than earlier values.

(2) Has no solid surface.

(3) Location of solid surface (below thousands of miles of dense atmospheric gases covering these planets) is not known; hence gravitational figures at surface are not reliable for the four giant planets.

(4) The "astronomical unit", a fundamental unit of astronomy, is the mean distance of the Sun from the Earth. "Solar parallax" is defined as an arc whose tangent is the radius of the Earth divided by the mean distance of the Earth from the Sun.

For over 2000 years, increasingly accurate determinations of the solar parallax have been made. From about 1900-1940, astronomers were confident that the value of solar parallax was 8.80", with the error in its determination not greater than 0.01". The asteroid Eros, roughly a cylindrical body 5 miles in diameter and 20 miles long, provided a means for recent readjustment in the value of solar parallax. The nearest approach of Eros to the Earth is approximately 10 million miles; its last opposition (1930) was about 16 million miles. Approximately 2500 photographs of Eros, taken in 1930-31, were analyzed in great detail by Sir Harold Spencer Jones over a 10-year period; his determination of solar parallax gave the unexpectedly low value of 8.790" with an estimated accuracy of 1 part in 10,000. Considering the scope of Jones' work, his figure hardly seemed open to question.

Recently, however, a new determination of solar parallax, along with other fundamental data, was published by Eugene Rabe. He made a full rediscussion of all the observations of Eros from 1926 to 1945 and secured a solar parallax value of 8.79835" ± 0.00039", which is much closer to the old value of 8.80" than that of Jones. As a byproduct, the ratio of the masses of the Earth to Moon is 81.375 and of Sun to Earth plus Moon is 328,452. He also secured improved values for the masses of Mercury, Venus, and Mars. There are indeed several very small asteroids which approach the Earth closer than does Eros, and hence might be expected to give better results. Thus far, however, they have proved unavailable because of their extreme faintness and the uncertainty of their orbits.

(5) Period of rotation on axis.
Table B-1 (cont.)

(6) Mercury's rotation period equals its period of revolution about the sun; hence one side always faces the sun. Temperatures here are as high as 750° F. above zero; on the other side, perhaps hundreds of degrees below zero.

(7) Mars' moons, Deimos and Phobos, are about 10 miles in diameter. The inner one (Phobos) revolves so fast that it rises and sets three times in a Martian day.

(8) Four of the moons have diameters of 2,300 to 3,200 miles and revolve about Jupiter in 2-17 days. The other 8 moons are less than 100 miles in diameter. One, very close to Jupiter, revolves at over 1,000 miles per minute. The 12th moon was discovered by S.B. Nicholson of Mt. Wilson and Palomar Observatories in 1951. It is only 14 miles in circumference, 2 x 10^6 miles from the planet, and circles Jupiter in 700 days retrograde motion (east-to-west). Four of Jupiter's moons are characterized by retrograde motion.

(9) One of Saturn's moons is larger than the Earth's moon and is believed to have an atmosphere. All nine moons are outside the rings characteristic of this planet.

(10) The fifth satellite of Uranus, Miranda, was discovered in February 1948 by Kuiper.

(11) The second satellite of Neptune (Nereid) was discovered on 1 May 1949 by Kuiper. It has a period of 359 days and ranges from 830,000 to 6,100,000 miles with major orbit axis of 0.074 a.u., eccentricity of 0.76 (greater than for any other satellite), and inclination of 5° to ecliptic plane. The first satellite of Neptune is called Triton.

(12) The closest approach of any planet to Earth.
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<tr>
<th>Body Name</th>
<th>Parent Planet</th>
<th>Distance from Planet (KM)</th>
<th>Period of Revolution</th>
<th>Orbital Eccentricity</th>
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<td>364,400</td>
<td>27° 7' 43.2&quot;</td>
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<td>Phobos</td>
<td>Mars</td>
<td>9.35 x 10^3</td>
<td>1° 6' 17.9&quot;</td>
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<tr>
<td>Deimos</td>
<td>Mars</td>
<td>2.35 x 10^4</td>
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<td>0.003</td>
</tr>
<tr>
<td>J I</td>
<td>Jupiter</td>
<td>4.22 x 10^5</td>
<td>1° 18' 27.6&quot;</td>
<td>0.0</td>
</tr>
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<td>J II</td>
<td>&quot;</td>
<td>6.714 x 10^5</td>
<td>3° 13' 13.7&quot;</td>
<td>0.0003</td>
</tr>
<tr>
<td>J III</td>
<td>&quot;</td>
<td>1.071 x 10^6</td>
<td>7° 3' 42.6&quot;</td>
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</tr>
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<td>J IV</td>
<td>&quot;</td>
<td>1.884 x 10^6</td>
<td>16° 16' 32.2&quot;</td>
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<tr>
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* Satellites indicated by asterisk are characterized by retrograde motion. Motion of Miranda and Nereid is not known.

** Most of this data was taken from reference (62).
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</table>

*** Mass of Earth = 5.966 ± .001 x 10^{27} gms = 4.09 x 10^{23} slugs
APPENDIX C

GRavitational MASS AtTRACTION AND THE

ACCELerATION OF GRAVITY

C.1: Gravitation

According to Newton's law of gravitation, every particle in the universe attracts every other particle with a force varying directly as the product of the two masses and inversely as the square of the distance between them.

\[ \text{Force} = \frac{GM M'}{R^2} \]  

\( \gamma \) is a universal constant whose numerical value is approximately:

\[ \gamma = 6.658 \times 10^{-11} \text{ m}^3 \text{ KG}^{-1} \text{ sec}^2 \]

A single fixed mass point of mass \( M \) will exert a force on any other particle according to equation (C-1). All space is affected by the mass \( M \); a field of force is set up around the mass. The same general statement is true for any system of bodies; at each point in space a force vector can be drawn which is the vector sum of the forces acting on the mass \( M \) by the system of bodies \( M_1, M_2, \ldots M_n \). The tangents to the force vectors at each point in space give the direction of the force exerted on the mass at every point. The curves map out the field.
and are called lines of force.

For the case of a single particle, the lines of force consist of straight lines radiating in all directions from the particle as a center. The lines of force give information as to the direction but not as to the magnitude of the gravitational forces. The latter is specified by introducing the intensity of the force field: the force per unit mass exerted on a particle at any given point. The intensity of the gravitational field is denoted by \( G \); for a single particle of mass \( M \):

\[
G = \frac{\text{Force}}{M} = \frac{\mathbf{F}}{M} = -\frac{GM}{R^2}
\]

\( G \) is a vector whose magnitude is given by (C-3) and whose direction is inward along the line connecting the masses. For a system of particles the intensity of the field is calculated by vector addition, the resultant intensity being the vector sum of the intensities due to the individual particles.

Gravitational forces are conservative\(^{(19)}\), and it is normal to call the fields conservative\(*\). It is convenient to introduce the gravitational potential \( \Phi \), defined as the potential energy per unit mass. For the case of the field of a single particle, the gravitational potential is

\[
\Phi = -\frac{GM}{R^2}
\]

\( \Phi \) and the gravitational intensity vector \( \mathbf{G} \) is the negative gradient of

\* A conservative field is defined as one in which the total work done is zero in moving a particle around any closed contour in the field. In moving a particle between any two points in a conservative field, the work done is independent of the path chosen between the points.
the potential $\Phi$.

$$\overline{G} = -\nabla\Phi$$  \hspace{1cm} (C-5)

The gravitational field intensity $\overline{G}$ at a point outside a spherical planet is determined by considering the planet to be composed of thin spherical shells, each of constant density. Each spherical shell acts as if its mass were concentrated at the center. Therefore, at any point outside the planet, the gravitational field intensity vector is directed toward the center of the planet with a magnitude directly proportional to the product of the gravitational constant times the mass of the planet and inversely proportional to the square of the distance from the center of the planet to the point.

$$\overline{G} = -\frac{\gamma g M_0}{R^2} \frac{R}{R}$$  \hspace{1cm} (C-6)

where $M_0$ is total mass of the planet. This result is identical with that for a single particle given in equation (C-3).

C.2 Gravitational Field of Oblate Spheroidal Planets

If the planet were a nonrotating fluid mass, it would form a sphere under the influence of its own gravitational field. Because of rotation about an axis through the planet, the associated centrifugal force field causes the planet to bulge at the equator. In the case of Mercury, the axial rate of rotation is extremely small*, hence the planet must be very nearly a perfect sphere (except for surface anomalies --

* Mercury's day and year are both equal to approximately 88 Earth days.
mountains and valleys). Little is known about the rotational rate of Venus since the surface is not visible through the opaque atmosphere. The best guess is that Venus, too, has equal days and years (224.7 Earth days). Consequently, Venus should be almost a perfect sphere (except for mountains, etc.)

Earth, Mars, and the giant planets have relatively high rotational velocities*. Mars rotates at almost Earth's rate, hence should have nearly the same equatorial bulge as Earth. Jupiter, Saturn, and Uranus have days that range from 9.8 to 10.7 hours, and Neptune's day is slightly under 16 hours. Consequently, the equatorial bulge of these giants is much more pronounced than Earth -- a fact verified by astronomic photographs.

Fig. C.1: Gravitational Field of Oblate Spheroidal Planet

* See Table B.1.
Due to the interaction of gravitational mass attraction and the centrifugal force due to axial rotation, the actual shape of rotating planets is nearly ellipsoidal, or oblate spheroidal. Because of the non-spherical mass distribution of the planet, its gravitational field is not spherically symmetric. A first order approximation of the non-spherical component of gravitational field intensity may be made by considering it as a quadrupole moment. Fig. C.1 shows two sources and two sinks of equal strength located on the polar axis of the planet near its center to represent the non-spherical mass component. The two sources are located at the center and the two sinks are symmetrically spaced on the polar axis above and below the center. The positions of the sinks approach the center, while the product of strength and spacing remains constant. The quadrupole thus is similar to a pair of dipoles that have mirror symmetry. The gravitational potential including this oblateness quadrupole may be considered to be (63):

\[
\bar{\Phi} = -\frac{\gamma_0 M_0}{R} \left[ 1 - \frac{\nu R_{\text{eq}}^2}{R^2} (1 - 3 \cos^2 2\Lambda) \right]
\]  

(C-7)

\[
\ddot{R} = -\nabla \bar{\Phi}
\]

(C-8)

In equation (C-7), \(\nu\) is a dimensionless constant for the planet, \(\Lambda\) is geocentric latitude, and \(R_{\text{eq}}\) is equatorial radius of the planet. For the planet Earth, reference (63) lists the value of \(\nu\) as follows:

\[
6 \nu_E = (1.637 \pm .004) \times 10^{-3}
\]

(C-9)
Ref. (56) gives a value of $v'$ for the international ellipsoid\* of Earth as follows:

$$6 \quad v'_E = 1.638 \times 10^{-3} \quad \text{(C-10)}$$

and the value of $\gamma_g M_e$ for Earth as follows:

$$\gamma_g M_e = (1 \pm 8 \times 10^{-5}) \left(3.986329 \times 10^{20} \text{ cm.}^3/\text{sec.}^2\right)$$

$$= 0.14077500 \times 10^{17} \text{ ft.}^3/\text{sec.}^2 \quad \text{(C-11)}$$

C.3 Gravity

The surfaces for which $\bar{g}$ is constant are not oblate spheroids, nor does one surface coincide with the assumed geometric figure of the planet. The surface $\bar{g}$ equal to a constant are everywhere normal to the gravitational field intensity vector of the planet. The vector sum of the planet's gravitational specific force\** and centrifugal force.

---

\* The Hayford Spheroid of 1909 is the international reference ellipsoid of the Earth. At the Internationale Geodetic and Geophysical Union of 1924, the dimensions of the Earth summarized in Table A.3 were adopted (see Table 716, page 570 of ref. 55).

\** Gravitational specific force $\bar{g}$ is identical with gravitational field intensity. Centrifugal specific force, a reaction force, is equal in magnitude but opposite in direction to the associated centripetal acceleration. Specific force is expressed in force-per-mass dimensions.
per unit mass is defined as the planet's *gravity field intensity.*

$$\bar{g} = - \nabla \Phi - \bar{W}_{I0} \times (\bar{W}_{I0} \times \bar{R})$$ (C-12)

where

- $\Phi$ is given by equation (C-7)
- $\bar{R}$ is the vector from the center of the planet to a point on or external to the surface of the planet
- $\bar{W}_{I0}$ is the angular velocity of the planet about its polar axis.

The tangential component of the first term in Equation (C-12) prevents the oceans of the Earth from flowing toward the equator. The direction of gravity is normal to the surface of the planet; if gravity anomalies are neglected, this direction is unique at each point on the surface. The time variation of the direction of gravity on the Earth is less than 0.05 microradians and is mainly due to tidal effects. Fig. C.2 shows the relation of Earth's gravitational specific force and gravity specific force. The angles are exaggerated in this figure for clarity.

* The specific force of gravity (the force of gravity acting on a unit mass) is identical with the gravity field intensity.
Note:
Angles are exaggerated for clarity in presentation. Numerical values are approximate values for Earth.

Fig. C.2: Relation of Gravitation and Gravity.

The angular velocity of the Earth about its polar axis is 

\[ W_{10} = 7.2921159 \times 10^{-5} \text{ rad./sec.} \]  

(C-13)

The international standard gravitational conversion factor is the apparent acceleration of gravity at latitude 45° 32' 40":

\[ g_{(m)E} = 32.17405 \text{ ft./sec}^2 \]  

(C-14)

Since

\[ g_{(m)E} = G_{(m)E} - W_{10}^2 R_{(m)E} \cos \Lambda \]  

(C-15)

the following is determined for \( G_{(m)E} \):

\[ G_{(m)E} = 32.17405 + 0.05443 = 32.22848 \text{ ft./sec}^2 \]  

(C-16)
Appendix D

FIGURE OF THE PLANET AND DEFINITION OF NAVIGATIONAL PARAMETERS

D.1 Figure of the Planet

The figure that would be taken by a fluid body with the mass distribution and axial rotation of the planet is defined as the figure of the planet. The figure of the planet is an equipotential surface of the planet's gravity field.

In connection with the actual configuration of the Earth, study of the motion of Vanguard I has shown the Earth to be slightly pear-shaped, with the more narrow half above the equator and the sag below. There is an approximate 50 ft. rise in mean sea level at the South pole\(^{(65)}\).

This deformation is interpreted to mean that the crystal layers of the Earth have more strength and less elasticity than previously assumed.

The figure of other planets in the solar system is not nearly as well known as that of the Earth. Unmanned planetary probes, which are planned for launch in the 1960's should provide the first quantitative information obtained from data sources near the planet to augment astronomical observations and calculations accumulated over many years. The oblateness of the planets is generally estimated from astronomical photographs and from calculations based on observed rotational rates. In general, the terrestrial planets are believed to be more spherical.
than the giant planets because of the higher rotational rates and smaller densities of the latter.

The figure of the planet may be approximated by a reference ellipsoid, an analytical figure closely approximating the planet. The Clark Spheroid (1866) is the reference ellipsoid of the Earth used for North American triangulations. The Hayford Spheroid (1909) is the international reference ellipsoid of the Earth adopted by the Internationale Geodetic and Geophysical Union of 1924. This is called the geoid, a "square shouldered" ellipsoid with a slightly undulated surface. The surface of the geoid is represented by mean sea level. Variations in the elevation of the geoid relative to the closest reference ellipsoid are approximately one percent of the topographic variations in elevation.

D.2 Navigation Parameters

A manned vehicle entering the atmosphere of a planet must have some means of identifying its present position and the location of its destination. Three coordinates are required in order to uniquely specify position in three-dimensional space. Of the infinite number of possible sets of coordinates that may be used to identify position near a planet, three are most common: altitude, latitude, and longitude.

Three navigational reference frames are discussed in this section and a comparison made between navigational parameters expressed in each of these frames:

(1) Astronomic Frame

Recognizes the actual planet configuration. The coordinate frame is centered at the guidance point and has one axis along the true vertical (gravity vector).
The other two axes depend on the guidance grid desired (e.g., astronomic latitude and longitude).

(2) Geographic Frame

Recognizes the ellipticity of the planet. The coordinate frame is centered at the guidance point and has one axis along the geographic vertical (normal to the surface of the reference ellipsoid). The other two axes depend on the choice of guidance grid (e.g., geographic latitude and longitude).

(3) Geocentric Frame

Utilizes spherical polar coordinates. The coordinate frame is centered at the guidance point and has one axis along the geocentric radius. The other two axes depend on the choice of guidance grid (e.g., $\mathbf{I}_r$, $\mathbf{I}_\lambda$, $\mathbf{I}_\phi$ triad or $\mathbf{I}_r$, $\mathbf{I}_\phi$, $\mathbf{I}_\psi$ triad discussed in Appendix A).

It was pointed out in Appendix C that, neglecting anomalies, the direction of the gravity vector $\mathbf{g}$ is unique at each point on the surface of the planet. The direction of the gravity vector is defined as the true vertical. The uniqueness of the true vertical is the basis for astronomical position. Two angles are sufficient to identify the position of any point on the surface of the planet:

(1) Astronomical latitude:

The complement of the angle between a line parallel to the polar axis of the planet and the local gravity vector.

(2) Astronomical longitude:

The angle about the polar axis of the planet between an arbitrarily chosen reference vertical and the local vertical. Due to the arbitrary choice of reference vertical, longitude is not inertially unique, although change of longitude is.

The geographic vertical* is defined as the normal to the surface.

* In the strictest sense, the term "geographic" refers to the Earth (geoid) only. In this thesis, "geographic" is used in a more general sense to refer to quantities associated with the reference ellipsoid of any planet.
of the reference ellipsoid. Because the figure of the planet does not have a smooth surface, the true vertical is not, in general, parallel to the normal to the reference ellipsoid at the same position. The angle measured from the normal of the reference ellipsoid to the true vertical is called the deflection of the vertical, or station error. For the Earth, station error is generally less than 0.3 milliradians; over continental land masses, it rarely exceeds 0.1 milliradians. (11)

Identification of position on the surface of the reference ellipsoid is accomplished with the following two angles:

(1) **Geographic Latitude** ($\Lambda_g$):

Angle between the normal to the reference ellipsoid and its projection on the equatorial plane.

(2) **Geographic Longitude**:

Angle between the projection of the normal to the reference ellipsoid on the equatorial plane of the planet and an arbitrarily chosen reference meridian.

The geocentric vertical is defined as a unit vector directed from the point at which the guidance is taking place toward the center of the planet. The angle measured from the geocentric vertical to the normal of the reference ellipsoid is called the deviation of the normal. The maximum deviation of the normal for Earth is approximately 11.5 minutes of arc.

Identification of position by means of spherical polar coordinates may be accomplished through specifying radius from planet center (or altitude above the surface of the planet) and by two independent angles.

One set of independent angles that may be used to specify position are great circle parameters, i.e., angular position along and across the great circle track. Even though the term great circle has no meaning
on any surface other than that of a sphere, it is a useful concept in navigation over non-spherical bodies. An astronomic great circle course is defined by the property that the normals to the planet's surface along such a course are parallel to a single, fixed plane. A geographic great circle course is defined as a course for which the normals to the reference ellipsoid of the planet are parallel to a single, fixed plane. The astronomic and geographic great circle courses are not geodesics, therefore a point on the planet's surface following such a course would experience a geodesic acceleration.*

Another set of spherical angles used to specify position are:

(1) Geocentric Latitude (A):
Angle measured between the geocentric radius and its projection on the equatorial plane of the planet.

(2) Geocentric Longitude (\lambda):
Angle in the equatorial plane of the planet between the projection of the geocentric radius on the equatorial plane and the longitudinal reference meridian.

When the equatorial plane is taken as the true equatorial plane, geocentric and geographic longitudes are identical and astronomic longitude differs by a component of station error. A comparison of geocentric and geographic latitude is made in the next section.

* A horizontal component of acceleration arising from the curvature of the course in the horizontal plane.
D.3 Comparison of Geocentric Latitude and Geographic Latitude

Most of the analysis of this thesis is based on geocentric angles as prime dependent variables in the statement of the guidance problem of vehicles entering planetary atmospheres. Since position on maps and charts is conventionally specified in terms of geographic latitude, it is instructive to compare geocentric and geographic latitudes.

\[ A = \text{Geocentric Latitude} \]
\[ A_g = \text{Geographic Latitude} \]

\[ R_{(c)0}, R_{(p)0} = \text{Equatorial and Polar radii, respectively.} \]

\[ \tan (90° + A_g - \Lambda) = R \frac{d\Lambda}{dR} \quad (D.1) \]

Fig. D.1: The Reference Ellipsoid
Therefore: \[ \tan (\Lambda_0 - \Lambda) = - \frac{1}{R} \frac{dR}{d\Lambda} \quad (D.2) \]

The equation of the ellipse in polar coordinates is:

\[
\left( \frac{R_x}{R(\text{eq})} \right)^2 + \left( \frac{R_z}{R(\text{eq})} \right)^2 = 1
\]

\[ (D.3) \]

where:

\[ R_x = R \cos \Lambda \]
\[ R_z = R \sin \Lambda \]

Therefore:

\[
R^2 \left[ \left( \frac{\cos \Lambda}{R(\text{eq})} \right)^2 + \left( \frac{\sin \Lambda}{R(\text{eq})} \right)^2 \right] = 1
\]

\[ (D.4) \]

Solving for R:

\[
R = \frac{R(\text{eq})}{\sqrt{\cos^2 \Lambda + \left( \frac{R(\text{eq})}{R(p)} \right)^2 \sin^2 \Lambda}}
\]

\[ (D.5) \]

The eccentricity of the planet, \( \varepsilon \), is defined* by (see Fig. A.6):

\[
\varepsilon^2 = 1 - \left( \frac{R(p)}{R(\text{eq})} \right)^2
\]

Equation D.5 thus becomes:

\[
R = \frac{R(\text{eq})}{\sqrt{1 + \frac{\varepsilon^2 \sin^2 \Lambda}{1 - \varepsilon^2}}}
\]

\[ (D.6) \]

* The ellipticity of the Hayford ellipsoid is given as:

\[
\text{Ellipticity} = \frac{R(\text{eq}) - R(p)}{R(\text{eq})} = 1/297
\]

Thus:

\[
\varepsilon_E = \sqrt{593} \times \text{Ellipticity}
\]

\[
\varepsilon_E^2 = 0.0067226700
\]
Using Eq. (D.6) to eliminate $R$ from Eq. (D.2) gives:

$$\tan(\Lambda_g - \Lambda) = \frac{\varepsilon^2 \sin \Lambda \cos \Lambda}{(1-\varepsilon^2)(1 + \frac{\varepsilon^2 \sin^2 \Lambda}{1-\varepsilon^2})} = \frac{\varepsilon^2 \sin \Lambda \cos \Lambda}{(1-\varepsilon^2 \cos^2 \Lambda)} \quad (D.7)$$

Equation (D.7) may be written:

$$\tan(\Lambda_g - \Lambda) = \varepsilon^2 \sin \Lambda \cos \Lambda + \frac{\varepsilon^4 \sin \Lambda \cos^3 \Lambda}{1-\varepsilon^2 \cos^2 \Lambda} \quad \{\text{All Planets}\} \quad (D.8)$$

This expression is an exact* relation between geocentric latitude $\Lambda$ and geographic latitude $\Lambda_g$. Examination of this equation shows that $(\Lambda_g - \Lambda) < 12$ minutes of arc over the Earth's surface. It is probably less than 12' for Mercury, Venus, and Mars; but $(\Lambda_g - \Lambda)$ is probably somewhat greater than 12' for the giant planets (high rotational speeds -- small density). Using the small angle approximation of $(\Lambda_g - \Lambda)$ for terrestrial planets give:

$$\Lambda_g \approx \Lambda + \varepsilon^2 \sin \Lambda \cos \Lambda \quad \{\begin{array}{l}
\text{Mercury} \\
\text{Venus} \\
\text{Earth} \\
\text{Mars}
\end{array}\} \quad \text{only} \quad (D.9)$$

D.4 Radius of Spheroidal Planets as Function of Latitude

Equation (D.6) gives an expression for the radius vector to a point on the surface of the ellipsoidal planet in terms of geocentric latitude, eccentricity, and equatorial radius. This may be expanded by the binomial theorem to give:

$$R_{\text{surface}} = R_{\text{equ}} \left[1 - \frac{1}{2} \left(\frac{\varepsilon \sin \Lambda}{1-\varepsilon^2}\right)^2 + \frac{3}{8} \left(\frac{\varepsilon \sin \Lambda}{1-\varepsilon^2}\right)^4 - \cdots \right] \quad \{\text{All Planets}\} \quad (D.10)$$

* "Exact" to the extent that the reference ellipsoid approximates the figure of the planet.
For the terrestrial planets where $R(\text{p})_\text{O}$ and $R(\text{eq})_\text{O}$ do not differ greatly, Equation (D.10) may be reduced to a first order in $\epsilon^2$:

$$R_{\text{surface}} \approx R(\text{eq})_\text{O} \left(1 - \frac{\epsilon^2}{2} \sin^2 \Lambda \right)$$

\begin{align*}
\text{Mercury} & \quad \text{Venus} \\
\text{Earth} & \quad \text{only} \\
\text{Mars} &
\end{align*}

(D.11)

D.5 Position Reference for First-Time Entry into the Atmospheres of Strange Planets

First-time entry into the atmospheres of strange planets presents special problems in specifying position when compared to navigation over a well-mapped planet such as the Earth. The choice of a suitable landing site must necessarily be based on reconnaissance of the planetary surface while in orbit around the planet. The orbital altitude for the reconnaissance phase must be high enough such that a prolonged orbit may persist, yet low enough that fairly accurate mapping of the terrain is feasible.

The navigation of a vehicle flying from one point to another point on the surface of a planet is generally based on navigational parameters measured with respect to the planet, i.e., latitude, longitude, and altitude. It is less common to use parameters identified with the particular mission, such as angular displacements measured with respect to the great circle course along-track and across-track.

The entry mission to the surface of a strange planet, on the other hand, does not originate from a point on the planet's surface. The entry mission originates from the reconnaissance orbit, which, if the perigee altitude is sufficiently great, is very slowly changing with time. The

* "Strange planets", as used in this thesis, signifies any planet on which a vehicle with human occupants has not landed.
non-spherical component of the planet's gravitational field causes the line of nodes to rotate slowly with respect to the inertial framework. Drag forces result in energy transfer from the vehicle to the planetary atmosphere, but for sufficiently high orbits, this transfer causes negligible change in the satellite orbit over periods of time comparable to that required for entry once retro-rocket thrust is generated.

A basic position reference available during the course of entry is the original reconnaissance orbit. If the entry vehicle is launched from a mother satellite, then the mother satellite, which remains in the reconnaissance orbit, may track both the entry vehicle and the pre-selected landing site and transmit this tracking information to the navigational computer of the entry vehicle. In this way, the parent satellite replaces ground tracking stations which are used as an external source of tracking information for Earth satellites and entry vehicles.

In the event that no parent satellite exists, then a navigational satellite to serve the same purpose may be deposited in the reconnaissance orbit by the entry vehicle prior to initiating the entry phase.

Since a navigational scheme such as outlined briefly above uses the reconnaissance orbit as the basic reference from which to measure positions, it may be found convenient to express position and to carry out the guidance computations in terms of elliptical parameters such as those presented in Chapter 4. The landing site may be considered to be a target moving in three dimensional space with respect to the near-stable reconnaissance trajectory represented by the mother or navigational satellite. The entry vehicle is also moving with respect to the reconnaissance trajectory. The entry problem is therefore similar to the Fire Control problem with the entry vehicle (projectile) fired from the
parent satellite (gun) to hit the moving landing site (target). The problem is much more severe than the conventional fire control problem, however, because the projectile must be constrained to paths for which it will not burn up or encounter accelerations beyond tolerable levels.

The nominal or programmed path of the entry vehicle may be computed in advance as one which the vehicle would fly under standard atmospheric conditions starting from the particular initial point and ending at the landing site selected in advance. This trajectory must be consistent with tolerable accelerations and heating rates (Chapter 7).

It was shown in Chapter 4 that there are six elliptical elements required to specify the position and path of the vehicle. One set of six such elements are:

1. $\psi$, $\lambda_{IT}$, (to specify the instantaneous orientation of the plane of the trajectory).
2. $P$, $\xi_1$, $\xi_2$ (to specify the ellipse which matches instantaneously the dynamical state of the vehicle)
3. $\phi$ (to specify the position of the vehicle in this ellipse).

$P$, $\psi$, and $\lambda_{IT}$ are constant or very slowly varying with time in the reconnaissance orbit. $\xi_1$ and $\xi_2$ are sinusoidal with a very slowly changing magnitude of oscillation. $\phi$ increases monotonically. The instantaneous state of the entry vehicle, with respect to the navigational satellite in the reconnaissance orbit, may conveniently be specified in terms of the six elliptical quantities. The orbit of the navigation satellite should be predictable to a fairly high degree of accuracy. The predicted values of six elliptical elements for this orbit are part of the data stored for use during the entry mission.
APPENDIX E

THE ATMOSPHERE OF THE PLANETS AND THEIR NATURAL SATELLITES

E.1 Composition of Atmosphere of the Planets and Their Natural Satellites

The presence and stability of planetary atmospheres can be predicted from the kinetic theory of gases\(^{(67)}\). For a stable atmosphere to exist, the root-mean-square molecular velocity of the atmosphere should be less than 20% of escape velocity. The RMS molecular velocity is a function of temperature (which depends on distance from the Sun), atmospheric composition, planetary rotational rates, etc. The relative likelihood of major bodies of the solar system possessing an atmosphere is given in the following list\(^{(67)}\). The higher a planet or moon is on the list, the higher is the probability that there exists an atmosphere around it.

(1) Jupiter*
(2) Saturn*
(3) Neptune*
(4) Uranus*
(5) Earth*
(6) Venus*
(7) Pluto
(8) Triton
(9) Mars*
(10) Titan*
(11) Jovian III
(12) Jovian I

* Proof of the existence of an atmosphere by spectroscopic or other means has been established for all bodies marked with an asterisk(*). Evidence for all other bodies is inconclusive.
A brief description of the atmosphere and climate of each of the planets and their natural satellites, adapted primarily from references (7) and (58), is summarized below:

(1) Mercury

Mercury is a small rocky sphere, about half again as large as Moon, that always has the same side exposed to the Sun. Maximum surface temperature on the "hot" side is estimated at 650-750° F. while that on the cold side approaches -400 to -415° F. Mercury is not known to have an atmosphere of any significance, nor would a permanent gaseous envelope be expected to occur under the condition existing on the planet. Its rocky surface is probably somewhat similar to that of Moon.

(2) Venus

A dense, dusty, turbulent atmosphere, containing much carbon dioxide and some nitrogen (but negligible free oxygen and water) conceals the planet's surface. The atmosphere of Venus contains white particles in suspension and is opaque to light of all wave lengths. On the basis of all available evidence, it may be presumed that the surface of Venus is probably hot, dry, dusty, windy, and dark beneath a continuous dust storm; that the atmospheric pressure is probably several times the normal barometric pressure at the surface of the Earth; and that carbon dioxide is probably the major atmospheric gas with nitrogen and argon also present as minor constituents (68).

(3) Mars

Many questions about surface conditions on Mars are still unanswered. It has an appreciable atmosphere and its surface markings exhibit seasonal changes in coloration. Its white polar caps* are apparently thin layers of frost of the order of inches or fractions of inches thick. The atmosphere is composed primarily of nitrogen, carbon dioxide, traces of water vapor, and is believed to be almost totally absent of free oxygen (67).

* The polar caps of Mars appear during its winter and disappear in summer.
Topographically, its surface is flat, with no abrupt changes in elevation and no prominent mountains. The climate is imagined to be like that which would be encountered on the Earth at 11 miles altitude. Noon summer temperatures in the tropics may reach 80° - 90° F, while predawn temperatures may drop to -100° F.

There is evidence that some indigenous life forms may exist on Mars. The seasonal color changes, from green in summer to brown in autumn, suggest vegetation. Recent spectroscopic studies give evidence of organic molecules being responsible for the dark areas. The objections raised concerning differences between the color and infra-red reflectivities of terrestrial organic matter and those of the dark areas on Mars have been explained by Tikhov. He has shown that arctic plants differ in infrared reflection from temperate and tropical plants, and an extrapolation to Martian conditions leads to the conclusion that the dark areas are vegetable life. Human life could not survive without vast environmental modifications, but a self-sustaining local animal colony is possible.

(4) The Giant Planets

The four giant planets are massive bodies of low density and large diameter. They all rotate rapidly, hence have considerable "flattening" at the poles ("bulges" at the equator). It is shown in Table B-1 that their densities vary from about 0.72 to 1.6 times the density of water. On the basis of this and spectral data, they are considered to be composed of a dense rocky core surrounded by a thick shell of ice and covered by thousands of miles of compressed hydrogen and helium.* Methane and ammonia are known to be present as minor constituents of the atmosphere, but no water has been observed.

Jupiter and Saturn show light and dark belts in the atmosphere parallel to the equator. These banded clouds are slowly changing in Jupiter; they are not as clear nor are they changing as rapidly on Saturn. Jupiter's great red spot, 20,000 miles long, seems more permanent than its cloud belts, but it is believed to be fading. Bright spots occasionally appear in the cloud banks of Saturn. Saturn's rings are its most prominent physical feature. They are probably composed of millions of tiny solid particles. They may be material which never formed into a satellite, or fragments of a close satellite torn asunder by the tidal pull of Saturn, or ice particles.

* A "rock-in-a-snowball" structure -- reference (?).
(5) Pluto

Little is known about Pluto beyond the physical characteristics listed in Table B-1 and the fact that it is extremely cold.

(6) Moon

The Moon has no appreciable atmosphere, and its surface is probably dry, dust-covered rock that is not homogeneous either in chemical composition or topography. The origin of the Moon's craters is still a matter of debate. Lunar mountains are higher than Earth mountains, presumably because of the absence of weathering effects.

(7) Planetary Satellites

A number of the satellites of Jupiter, Saturn, and Neptune are larger than the Earth's Moon; some may be as large as Mercury. At least one of Saturn's moons, Titan, is believed to have an atmosphere. Although reliable physical data on the satellites are lacking, it is possible that some of them may be more hospitable than their parent planets.

There are two principal experimental methods for obtaining data concerning the composition of the planetary atmospheres:

(1) Optical observations of the surface and atmosphere.

(2) Spectroscopic measurements of radiation emitting from the surface.

Both methods suffer severely because of interference with the Earth's atmosphere.

In addition to the above experimental methods, other constituents of the atmospheres of the planets can be inferred by correlating estimates of the composition of the atmosphere at the time of formation of the planet with present measurements of physical characteristics of the planet and its atmosphere. Table E.1 gives estimates of composition of the atmospheres of the planets and their natural satellites from observations and calculations recorded in reference (67). If the amount of the constituent is based on spectroscopic measurements, the amount is
recorded in Column 4 as the height in centimeters of an equivalent column of the gas at 0° C. and Earth standard sea level pressure (760 mm Hg). If the amount of the constituent is based on calculations and estimates, it is recorded in Column 3 as a percent by weight of the total planetary atmosphere.

Table E.1: Composition of the Atmospheres of the Planets and Their Major Natural Satellites.

Notes: (1) If the amount of the constituent is based on spectroscopic measurements, it is recorded in Column 4 as the height in centimeters of an equivalent column of the gas at 0° C. and at a pressure equal to a 760 mm column of Hg.

(2) If the amount of the constituent is based on estimates and calculations, it is recorded in Column 3 as a percent by weight of the total planetary atmosphere.

<table>
<thead>
<tr>
<th>Column 1 Body</th>
<th>Column 2 Gas</th>
<th>Column 3 Amount (%)</th>
<th>Column 4 Amount (see Note (1) above) cm</th>
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<tbody>
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<td>Mercury</td>
<td>(^{40}\text{A})</td>
<td>trace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Kr</td>
<td>trace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\text{Xe})</td>
<td>trace</td>
<td></td>
</tr>
<tr>
<td>Venus</td>
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<td></td>
<td>(\text{N}_2)</td>
<td>8-10%</td>
<td></td>
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<td></td>
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<td>trace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\text{CO})</td>
<td>trace</td>
<td></td>
</tr>
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<td></td>
<td>(\text{N}_2\text{O})</td>
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<td></td>
<td>(\text{O}_3)</td>
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<td>0.3</td>
</tr>
</tbody>
</table>
Table E.1 (Cont.): Composition of the Atmosphere of the Planets and Their Major Natural Satellites.

<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
<th>Column 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moon</td>
<td>SO₂</td>
<td></td>
<td>&lt;0.0003</td>
</tr>
<tr>
<td></td>
<td>O₃</td>
<td></td>
<td>&lt;0.005</td>
</tr>
<tr>
<td>Mars</td>
<td>N₂</td>
<td>97-98%</td>
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<tr>
<td></td>
<td>CO₂</td>
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</tr>
<tr>
<td></td>
<td>H₂O</td>
<td>trace</td>
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</tr>
<tr>
<td></td>
<td>O₂</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td></td>
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<td></td>
<td>O₃</td>
<td></td>
<td>&lt;0.05</td>
</tr>
<tr>
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<td>N₂O</td>
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<td>&lt;200</td>
</tr>
<tr>
<td></td>
<td>CH₄</td>
<td></td>
<td>&lt;10</td>
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<td>C₂H₆</td>
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<td>NH₃</td>
<td></td>
<td>&lt;2</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>1.2%</td>
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</tr>
<tr>
<td>Jupiter</td>
<td>He</td>
<td>62.8%</td>
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</tr>
<tr>
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<td>H₂</td>
<td>23.2%</td>
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<td>CH₄</td>
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<td>NH₃</td>
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<td>700</td>
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<td>Ne</td>
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<tr>
<td></td>
<td>H₂O</td>
<td>trace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>trace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SiH₄</td>
<td>trace</td>
<td></td>
</tr>
<tr>
<td>JI-JIV</td>
<td>CH₄</td>
<td></td>
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<td>NH₃</td>
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<td>&lt;40</td>
</tr>
<tr>
<td>Saturn</td>
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<td>&lt;0.01</td>
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<td>Titan</td>
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<td>NH₃</td>
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<td>&lt;300</td>
</tr>
<tr>
<td></td>
<td>A</td>
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</table>

* The total amount of atmosphere for Titan is estimated to be 60,000 cm.
Table E.1 (Cont.): Composition of the Atmosphere of the Planets and Their Major Natural Satellites.

<table>
<thead>
<tr>
<th>Column 1</th>
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<th>Column 3</th>
<th>Column 4</th>
</tr>
</thead>
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<td>58-75%</td>
<td>220,000</td>
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<tr>
<td></td>
<td>H₂</td>
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<tr>
<td></td>
<td>CH₄</td>
<td>2-21%</td>
<td>&lt; 0.01</td>
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<tr>
<td></td>
<td>O₃</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SO₂</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Neptune</td>
<td>He</td>
<td>58-75%</td>
<td>370,000</td>
</tr>
<tr>
<td></td>
<td>H₂</td>
<td>20-22%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CH₄</td>
<td>2-21%</td>
<td></td>
</tr>
</tbody>
</table>

E.2 Models of the Atmosphere

In much of the work of this thesis, the following two assumptions are made:

1. The planet and its atmosphere are spherically symmetric.
2. The atmospheric density varies exponentially with altitude.

The first assumption is reasonable for the terrestrial planets* because of their slow rotational speeds. The giant planets have high rotational speeds, hence this assumption is not nearly so valid for these planets.

The second assumption is based on the simple kinetic theory of an isothermal gas in a uniform gravitational field. This theory yields the well-known exponential approximation for the atmosphere:

\[ \rho = \rho_0 e^{-KH} \]  \hspace{1cm} (E-1)

* Mercury has no appreciable atmosphere.
where:

\( \rho = \text{free stream atmospheric density (slug/ft.}^3) \)

\( \rho_{(SL)} = \text{"assumed" mean sea level atmospheric density*} \)

\( K = \text{atmospheric density decay parameter, ft.}^{-1} \)

\( H = \text{altitude of vehicle above surface of planet (ft.)} \)

\( K \) is equal to the mean molecular weight of the planet's atmosphere times the local acceleration due to gravity divided by the product of mean atmospheric temperature and the universal gas constant:

\[ K = \frac{g \text{ (Mean molecular weight of atmospheric gases)}}{R_h T_{(m)o}} \tag{E-2} \]

where:

\( R_h = \text{universal gas constant} = 8.31 \times 10^7 \text{ ergs/}^0\text{Kelvin} \)

\( g = \text{acceleration due to gravity} \)

\( T_{(m)o} = \text{the mean temperature of the planetary atmosphere (}^0\text{K)} \)

Two additional models of the planetary atmosphere are sometimes used:

(1) An empirical matching of measured atmospheric data:

\[ \frac{\rho}{\rho_{(SL)}} = \sigma \overset{\approx}{=} (H)^n \tag{E-3} \]

This model is useful for the Earth's atmosphere only, since measurements of the density variation with altitude for other planetary atmospheres have not been obtained to date. The model represented by Eq. (E-3), with \( n \approx -5 \) to \(-8\), generally matches more closely at high altitudes the data accumulated from Vanguard, Sputnik, and Explorer.

* This is not the true sea level atmospheric density \( \rho_{(SL)} \). The "assumed" quantity \( \rho_{(SL)} \) is the intercept of the straight line which best fits the curve of \( \log \rho \) vs altitude.
Earth satellites than does the exponential atmosphere with $K$ constant. Since Eq. (E-3) is based on an empirical curve fit and not on any known physical law, the value of $n$ for other planets cannot be predicted at this time.

(2) The variation of density with altitude for an atmosphere in which temperature decreases linearly with altitude is:

$$\frac{\rho}{\rho_{(SL)}} = \sigma = \left(\frac{\text{Local temp.}}{\text{Surface temp.}}\right)^{\frac{1}{\gamma_0 - 1}} \left[\frac{(\text{Surface temp.}) - (\text{Lapse Rate})H}{\text{Surface Temperature}}\right]^{\frac{1}{\gamma_0 - 1}}$$

(E-4)

where:

$\gamma_0$ = ratio of specific heat $\frac{C_p}{C_v}$

If it is assumed that $\gamma_0$ and gravity are constant, the adiabatic lapse rate* is given by:

$$(\text{Adiabatic Lapse Rate}) = \frac{(\text{Mean Molecular Weight of atm. gases}) \frac{9}{2} \gamma_0 (\gamma_0 - 1)}{R_h \gamma_0}$$

(E-5)

E.3 The Venusian Model Atmosphere

It is generally assumed that the atmosphere of Venus originated from volcanic gases. On Venus, water was converted to molecular oxygen, atomic oxygen, ozone, and hydrogen via photodissociation at high altitudes. This action was speeded up because of the nearness of Venus to the Sun. The ozone layer on Venus was probably formed at greater altitudes than on Earth. The oxygen was probably removed from the atmosphere by chemical combination with surface material. This chemical process

* Lapse rate is the rate at which temperature decreases with increasing altitude.
should react about 30 times as fast on Venus as it does on Earth because of the higher surface temperatures on Venus.

Loss of water via photodissociation to oxygen probably stopped the chemical reaction of carbon dioxide with surface material. Once this situation prevailed, it may be expected that carbon dioxide started to appear in the atmosphere of Venus in ever-increasing quantities.

Venus should have about the same amount of nitrogen as Earth, and much greater quantities of carbon dioxide in the atmosphere. The much larger quantities of carbon dioxide are expected in the atmosphere of Venus when compared to Earth because of the high quantities of carbon dioxide combined with surface material on Earth.*

Because of the above reasoning, Dole(68) proposed an atmospheric model containing 90% carbon dioxide and 10% nitrogen. The values of the constants in Equation (E-1) for Venus proposed by Dole are:

\[ \rho_{(51)} = 0.0326 \text{ slug/ft.}^3 \]
\[ K = 4.88 \times 10^{-5} \text{ ft.}^{-1} \]

Romer(72), using Equation (E-4), determined \( \rho_{(51)} = 0.0348 \text{ slug/ft.}^3 \). He used an atmospheric model of 10% nitrogen and 90% carbon dioxide with a mean molecular weight of 42.4 and a specific heat ratio of \( \gamma_0 = 1.31 \) at 400 °K. The lapse rate was computed to be 10.45 °K/KM. Kuiper(8) suggests the lapse rate is 13° K/KM and \( \gamma_0 = 1.4 \), while de Vancouleurs(62) determines a lapse rate of 10.4 °K/KM.

Values given in (E-6) are used in this thesis for Venus.

* Carbon dioxide was able to combine with surface material on Earth because of the presence of water.
E.4 Martian Model Atmosphere

Evidence for the existence of a Martian atmosphere is conclusive. Clouds are frequently observed. Stars passing behind the disc of Mars are seen longer and appear earlier than possible without atmospheric refraction*. A twilight zone on Mars caused by atmospheric refraction has been observed. Mars appears to be larger when photographed by reflected ultraviolet light than by reflected infrared light. (The atmosphere of Mars is included in ultraviolet photographs). A lower limit of 60 miles can be placed on the altitude of the Martian atmosphere from these photographs. This is probably the altitude to which small ice crystals rise, thus creating a "blue haze". The blue haze often clears when Mars passes close to the Sun. On these occasions, damage is seen to occur to the primitive vegetation which is believed to exist on Mars. When present, the blue haze protects the vegetation from destructive ultraviolet radiation originating from the Sun.

Although nitrogen is probably the main constituent of the Martian atmosphere, it has not yet been detected. This is due to the fact that absorption bands for molecular nitrogen are in the low ultraviolet region. These lines are effectively obscured from observation on Earth by ozone in the upper atmosphere of Earth. Atomic nitrogen in the Martian atmosphere could theoretically be detected through the atmosphere.

* It is interesting to note that the only time in recorded history that occultation of a first magnitude star (Regulus) by Venus occurred in mid-July, 1959. The total time that Regulus was obscured was 11 minutes 4.8 seconds. This event was observed from a Spanish observatory by a team led by Dr. Allen Hynek from the Smithsonian Astrophysical Observatory of Cambridge, Mass. Revised information on the Venusian atmosphere (its density variation with altitude, temperature, chemical composition, etc.) based on the results of this observation has not been published at this time.
of the Earth. However, there is a larger quantity of atomic nitrogen in the atmosphere of the Earth than in the atmosphere of Mars. Mars probably had less molecular nitrogen to begin with than Earth, and definitely has less solar energy to produce atomic nitrogen from molecular nitrogen. These two factors have combined to provide a weak spectrum against a background that even sensitive Doppler techniques have proven futile for the detection of atomic nitrogen.

It is generally thought that the atmosphere of Mars, like that of Venus and Earth, originated from volcanic gases. These gases consisted of water vapor, carbon dioxide, nitrogen, and other minor constituents in that order. Nitrogen was probably the only major constituent of volcanic gas which was not readily removed from the atmosphere of Mars by chemical reaction with rock formations or by escape since nitrogen is inert and highly resistant to photodissociation.

Water was probably removed first from the Martian atmosphere. Atomic oxygen, ozone, molecular oxygen, and hydrogen are some of the gases produced as a result of photodissociation of water. Most of the hydrogen probably escaped from the Martian atmosphere. Since ozone would be produced at low altitudes on Mars, it could combine rapidly with surface material. Water probably remained close to the surface making the rapid combination of carbon dioxide with surface material possible. With the loss of oxygen via ozone formation and chemical reaction, there would result a shift in the chemical equilibrium to more ozone; this gas, in turn, could continue to combine with surface material until most of the water disappeared.

After the disappearance of most of the water from the Martian atmosphere, carbon dioxide could no longer combine with surface material
at an appreciable rate. Consequently, it may be reasoned that the atmosphere of Mars has more carbon dioxide than the atmosphere of Earth. This has been verified by spectroscopic measurements (see Table E.1). In addition, water and oxygen are known to be absent from the Martian atmosphere in quantities which are significant in gas-dynamic problems (less than a few hundredths of a per cent.) The reasoning outlined in the foregoing paragraphs may account qualitatively for the composition listed in Table E.1.

The Gazley(14) model for the atmosphere of Mars assumes that it consists of 95% nitrogen and 5% carbon dioxide. This model ignores minor constituents, especially argon, and is subject to a large error in the estimation of carbon dioxide content. Nevertheless, it is doubtful that a more accurate model would allow a significant improvement in the accuracy of gas-dynamic calculations.

The average surface temperature of Mars was estimated by Gazley from the Stefan-Boltzman relationship to be minus 40° F.* He made the following assumptions in determining the average surface temperature:

1. The ratio of the heat flux reaching Mars and Earth is determined by their respective solar constants (3/7 for Mars and 1.0 for Earth).
2. The emissivities are the same for both planets.
3. The temperature of the surroundings to which the planet radiates is sufficiently low to be neglected.

It is noted that a value of -40° F. was listed by Wanders(74) as an

* Kuiper(67) and de Vaucouleurs(73) use a surface temperature of 273° K. (0° C.) The maximum diurnal temperature change is about 50° C. This is also approximately the maximum seasonal average temperature variation (occurring in summer and fall).
average experimental value.

The ratio of the absolute mean-sea-level temperature of Earth to the absolute ground-level temperature of Mars is 1.23. The temperature lapse rate of the Martian atmosphere is lower than for the Earth*; therefore, this temperature ratio decreases with increasing altitude.

Gazley\(^{(14)}\) uses a surface density value of \(6.2 \times 10^{-3}\) lbs./ft.\(^3\) for the Martian atmosphere which is in close agreement with the value of \(1.94 \times 10^{-4}\) slugs/ft.\(^3\) given by Hess\(^{(75)}\). Gazley uses a value for K (see equations E-1 and E-2) of \(1.15 \times 10^{-5}\) ft.\(^{-1}\). Chapman\(^{(15)}\) uses a value for K of \(1.665 \times 10^{-5}\) ft.\(^{-1}\). In the numerical calculations of this thesis, the following are used for the values of the constants in Eq. (E-1) for the Martian atmosphere:

\[
\rho(\text{sl}) = 1.93 \times 10^{-4}\ \text{slugs/ft.}^3 \\
K = 1.665 \times 10^{-5}\ \text{ft.}^{-1}
\]  

(E-7)

The maximum error in decelerations computed in this thesis should be no greater than about 20% for Mars, and the maximum error in heating rates and stagnation point temperatures should not exceed 10%.

E.5 Model Atmosphere for the Giant Planets and Titan

Information on the density characteristics of the atmospheres of Jupiter, Saturn, Titan, Uranus, and Neptune is fragmentary at best. Because of the cloud layer which obscures the surface of the giant planets, extrapolation of density characteristics to the surface is not

* The lapse rate for Earth at sea level is 9.8 °K/KM. The average measured lapse rate at all altitudes is approximately 5 °K/KM, which indicates that the atmosphere is not completely in adiabatic equilibrium. Most authorities\(^{(67)}\)\(^{(73)}\) have computed a lapse rate of 3.7 - 3.9 °K/KM for Mars.
Table E.2 summarizes briefly Kuiper's model atmospheres for the giant planets and Titan.

<table>
<thead>
<tr>
<th>Table E.2: Model Atmospheres for Giant Planets and Titan</th>
</tr>
</thead>
</table>

1. **Jupiter**
   - Height of cloud tops from main cloud deck: 20-30 KM.
   - Exponential decay parameter: $K = 1.65 \times 10^{-5}$ ft.$^{-1}$.
   - Pressure at main cloud deck: 6-9 atmospheres.
   - Temperature at top of atmosphere: 80-90°K.

   **Atmospheric Model I:**
   - $\gamma_0 = 1.46$
   - Mean molecular weight 2.47
   - Lapse rate 2.64 °K/KM

   **Atmospheric Model II:**
   - $\gamma_0 = 1.56$
   - Mean molecular weight 3.26
   - Lapse rate 3.96 °K/KM

   Temperature at top of clouds:
   - Model I: 165° K
   - Model II: 170° K

   Temperature at main cloud deck:
   - Model I: 215-245° K
   - Model II: 250-290° K

2. **Saturn**
   - Height of cloud tops from main cloud deck: 20-30 KM
   - Temperature at top of atmosphere: 60-70° K.

   **Atmospheric Model I:**
   - $\gamma_0 = 1.46$
   - Mean molecular weight 2.47
   - Lapse rate 1.17 °K/KM

   **Atmospheric Model II:**
   - $\gamma_0 = 1.56$
   - Mean molecular weight 3.26
   - Lapse rate 1.75 °K/KM

   Temperature at top of clouds: 155° K.

   Temperature at main cloud deck:
   - Model I: 175-190° K.
   - Model II: 190-210° K.
Table E.2: (Cont.) Model Atmospheres for Giant Planets and Titan.

3. **Uranus and Neptune:**
   
   Mean molecular weight: 3.55.
   
   Mean temperature of atmosphere: 78 °K.
   
   Pressure at main cloud deck: 9 atmospheres
   
   Exponential decay parameter: \( K = 1.87 \times 10^{-5} \text{ ft.}^{-1} \)

4. **Titan**

   \( \gamma_0 = 1.5 \)

   Mean molecular weight: 20
   
   Mean temperature of atmosphere: 70-83 °K.
   
   Lapse rate 1.4 °K/KM
   
   Pressure at surface: \( 8.9 \times 10^{-4} \) atmospheres
   
   \( \rho_{(5L)} = 0.006 \text{ slugs/ft.}^3 \)

**E.6 Model Atmosphere for Earth**

The 1956 ARDC model atmosphere is generally accepted as a reasonably good representation of average atmospheric characteristics of the Earth below 100 miles. This profile was constructed with only one point above 100 miles. Data obtained from various sounding rockets and Earth satellites during 1957-1959, however, suggests that substantial revision of this model of the atmosphere is needed. Revised data obtained from satellites indicates that density of the ARDC model may be in error by an order of magnitude at low altitudes, and that densities at altitudes in excess of 100 miles are much higher than predicted.
by the ARDC model. Furthermore, satellite data shows that the density at high altitudes is approximately twice as high in summer as in winter, and twice as high in the daytime as at night. Therefore, any model of the atmosphere is at best, a crude representation of conditions that may actually be encountered by a vehicle entering the atmosphere.

Fig. E.1 compares atmospheric density at high altitudes predicted by the ARDC model atmosphere with actual measurements made by recent satellites and sounding rockets. A mean curve is shown on this figure which more closely matches measured densities at high altitudes than does the ARDC model; this mean curve matches the ARDC model at altitudes below 90 KM, and does not depart much from the ARDC model below 120 KM.

The exponential approximation to the Earth's atmosphere was used in most of the numerical computations of this thesis. Values used in Equation (E-1) for Earth are:

\[ \rho_{(SL)} = 0.0027 \text{ slug/ft.}^3 \]

\[ K = \frac{1}{23,500} \text{ ft.}^{-1} \]  

(E-8)

The measured sea level atmospheric density of the Earth is $2.38 \times 10^{-3}$ slug/ft.$^3$.

The exponential approximation for Earth using the values given in Eq. (E-8) are compared in Fig. E.2 with the 1956 ARDC Model atmosphere. It is clear that a single value of $K$ is a reasonable approximation below 400,000 ft. (approximately 80 miles)*. Numerical comparison of

* It is shown in ref. (15) that peak decelerations and maximum aerodynamic heating of an entry vehicle occurs well below this altitude. The region of most important heating and deceleration for a given vehicle occurs over a strip of altitude approximately 70,000 ft. thick across with the density changes by a factor of 20. Once the
Fig. E.1: Comparison of Recent Measurements of Atmospheric Density with ARDC Model Atmosphere.
Fig. E.2: Comparison of Exponential Atmosphere with ARDC Model of Earth Atmosphere (1956)
the exponential model and the ARDC model is given in Table E.3 to four or five significant figures.

Most of the derivations and numerical calculations of this thesis use dimensionless parameters in order that the results may be applied to any planet. Mean planetary radius is used as the reference length for non-dimensionalizing quantities having the dimensions of length or distance. Thus:

\[ \sigma = \frac{\rho}{\rho(3L)} = e^{-k h} \]  \hspace{1cm} (E-9)

where:

- \( k \) = dimensionless atmospheric decay parameter = \( KR(m)_0 \)  \hspace{1cm} (E-10)

- \( h \) = dimensionless altitude = \( H/R(m)_0 \)  \hspace{1cm} (E-11)

- \( \sigma \) = atmospheric density ratio  \hspace{1cm} (E-12)

Fig. E.3 shows a plot of the dimensionless exponential decay parameter \( \sqrt{k} \) taken from ref. (15). \( k \) was determined for the ARDC model atmosphere at each altitude point by determining the mean slope of a plot of \( \log \rho \) vs altitude for the 70,000 ft. strip of altitude immediately above the particular altitude in question. It is seen from Fig. E.3 that below 400,000 ft. the variation in \( \sqrt{k} \) is no more than about 10\% from a mean value of 30. These variations are due primarily to temperature changes with altitude (a constant exponential decay rate assumes an isothermal atmosphere). Since temperature changes as much as 15\% with season and latitude, than \( \sqrt{k} \) (which is pro-

altitude of this critical strip is determined for a given vehicle entering a given atmosphere, the exponential decay parameter \( K \) can be adjusted to correspond more nearly to that of the critical strip of altitude.
Fig. E.3: Dimensionless Exponential Parameter $k$ for ARDC Model of Earth Atmosphere.
portional to $T_{(m)0}^{-\frac{1}{2}}$, see equation (E-2)) may fluctuate by 7% due to seasonal changes and latitude changes alone.

For calculations of this thesis where the Earth is considered, $k = 889$ is used (corresponding to a mean atmospheric temperature of $240^\circ K$ ($432^\circ R$)) together with $\rho_{(SL)} = 0.0027$ slug/ft.$^3$.

E.7 Comparison of the Exponential Model Atmospheres of Venus, Earth, and Mars

Fig. E.4 shows the isothermal atmospheric models used in this thesis for the terrestrial planets. It is apparent that the atmospheres of Venus and Earth are distributed in a similar way. Even with large errors in the assumed atmospheric composition of Venus, gas-dynamic heating and deceleration loads would, to a first approximation, be the same as they are on the Earth. A parachute on Venus, however, should be about three times as effective as on Earth.

Maximum deceleration loads will be severe on Earth and Venus for some trajectories as pointed out in this thesis. These problems may be solved by resorting to lifting vehicles, by entering at lower entry angles (requires precision guidance) or coming in at slower velocities (requires large amounts of propellants).

The atmosphere of Mars is characterized by a gradual decrease in density with increasing altitude and low surface density as compared to that of the Earth. The low ground-level density of this atmosphere will reduce the effectiveness of a parachute and will require the use of a high drag (low density) entry vehicle. The gradual change in atmospheric density of the Martian atmosphere will reduce the severity of peak gas-dynamic accelerations and heating loads. For identical size, weight, shape, and velocity, a body entering the Martian atmosphere would be
Fig. E4: Isothermal Atmospheric Models for Terrestrial Planets
subjected to about 25% of the peak deceleration load and about 80% of the peak heating load that it would encounter during entry into the Earth's atmosphere.

Table E.3: Comparison of ARDC Model Atmosphere (1956) with Exponential Approximation

\[ \rho = \rho_{(5L)} e^{-KH} \]

\[ \rho_{(5L)} = 2.7 \times 10^{-3} \quad (1/K = 2.35 \times 10^4) \]

<table>
<thead>
<tr>
<th>H (feet)</th>
<th>ARDC (slug/ft.³)</th>
<th>( \rho = \rho_{(5L)} e^{-KH} ) (slug/ft.³)</th>
</tr>
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<td>2.3769 \times 10^{-3}</td>
<td>2.7 \times 10^{-3}</td>
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<td>10^4</td>
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</tr>
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EXTERNALLY-AIDED SELF ADAPTIVE CONTROL OF THE ENTRY VEHICLE

The operational extremes experienced by a vehicle during entry into a planetary atmosphere suggest a control system that adapts itself to the changing environment. The Conservation Parameter, defined in section 1.7, is a measurable function that has sharp behavior at the boundaries between operational phases; behavior of this function is discussed in some detail in Chapters 8 and 9. The conservation parameter may be useful as a switching function and as a variable sensitivity factor to improve control system operation.

The generalized guidance and control system represented functionally in Fig. 1.4 is a three-dimensional system. The command signal to the control system from the guidance computer may be in the form of acceleration components along three axes fixed to the entry vehicle. Fig. F.1 suggests a relatively simple functional instrumentation of the control system for one of these axes, the normal axis, which utilizes the conservation parameter for adapting to the operational environment. Similar adaptive techniques may be applied to other control channels.

The function of the acceleration control system of Fig. F.1 is to produce normal accelerations of the vehicle in response to command inputs, while minimizing changes in flight direction that result from
Fig. F.1: Functional Diagram of Normal Acceleration Flight Control System which Adapts to the Operating Environment
gust disturbances. The operation of this system is not unlike that used to control guided missiles and high performance aircraft except for the unique feature of incorporating the conservation parameter as an input. The environmental logic system compensates for environmental influences on the effectiveness of the elevators in controlling the vehicle.

The normal specific force measuring system in the feedback loop detects the sum of all specific force components in the normal direction. With measurements of bank and pitch angles and altitude, it is possible to introduce compensations for gravity. Such compensation is not shown on Fig. F.1.

After compensating for gravity, the measured normal acceleration is compared to the input command voltage. When a difference exists, a correction is introduced. The correction signal goes to the environmental logic system and the integrating gyro unit. The integrating gyro unit produces a signal containing two terms, one proportional to the integral of the correction voltage and one proportional to the integral of the pitch rate. In order to produce a large elevator angle for given pitch deviation such as may be required in the subsonic phase of flight near the termination of the entry trajectory, it is necessary to provide a signal to the elevator servo proportional to pitch rate\(^{(78)}\).

The environmental logic system takes the time derivative of the signal from the integrating gyro unit and combines it with the correction voltage. The output voltage of the environmental logic system is sent either to the reaction controls, to the elevator servo, or to both, depending on the magnitude of the conservation parameter and its time behavior. The conservation parameter, which comes from the guidance computer, indicates the phase of flight (see Chapters 8 and 9) and
reflects the degree of effectiveness that can be expected from the elevators due to flight conditions.

The system of Fig. F.1 is not a self-adaptive control system, i.e., it does not monitor its own performance and automatically adjust itself for optimum performance in a changing environment. The system is "adaptive", however, in the sense that it uses environmental data from the guidance computer to adjust its performance characteristics. Augmenting the adaptive features of a system similar to Fig. F.1 with self-adaptive inner loops to optimize control system operation is suggested for future investigations.
BIBLIOGRAPHY


(2) S. Chapman; "The Earth and the Sun's Atmosphere"; *Scientific American* (Oct. 1959)

(3) R. Jastrow; "Artificial Satellites and the Earth's Atmosphere" *Scientific American*, Aug. 1959


(6) Aviation Week, June 22, 1959.


(10) A.L. Webber; "Investigation of the Instrumentation of an Atmospheric Re-Entry"; Massachusetts Institute of Technology Instrumentation Laboratory Report T-218 (Sept. 1959)


(12) L.R. Young; "Inertial Navigation in an Orbital Vehicle", Massachusetts Institute of Technology Instrumentation Laboratory Report T-219 (Sept. 1959)


(14) C. Gazley, Jr.; "Deceleration and Heating of a Body Entering a Planetary Atmosphere from Space"; Rand Report P-955 (2-18-57)

(16) L. Lees, F.W. Hartwig, C.B. Cohen; "The Use of Aerodynamic Lift During Entry into the Earth's Atmosphere". ARS Paper 785-59 (April 1959)

(17) D.I. Kepler; "Concepts Influencing the Selection of a Configuration for Atmospheric Re-entry", ARS Paper 786-59 (April 1959)

(18) T. Alexander, Jr.; "Gravitational Potential and the Gravity Vector"; Report M1137, Massachusetts Institute of Technology Instrumentation Laboratory (April 1959)

(19) H. Goldstein; Classical Mechanics, Addison-Wesley Press, Inc. Cambridge 42, Mass (1951)


(24) F.B. Hildebrand; Advanced Calculus for Engineers; Prentice-Hall, Inc. N.Y. (1949)


(43) G. Leitmann: "On a Class of Variational Problems in Rocket Flight" Lockheed Missiles Systems Division Report No. LSMD-5067 (Sept. 1958)


(48) W. Hohmann: "Die Erreichbarkeit der Himmelskorper"; R. Oldenberg Munich (1925)


(52) E. Sanger: "Raketen-Flugtechnik"; R. Oldenbourge (Berlin) 1933


(58) The Encyclopedia Americana; The Americana Corporation, N.Y.


Bulletin 78: "Physics of the Earth-II, The Figure of the Earth". National Research Council; Washington, D.C. (1931)

"Astronautics" magazine; vol. 4, No. 3; March 1959; page 8.


G.P. Kuiper; The Atmospheres of the Earth and Planets; Univ. of Chicago Press (1952)


G.A. Tikhov; "Is Life Possible on Other Planets?"; Journal of the British Astronomical Association; Vol. 65, No. 3, pp. 193 (April 1955)

H.C. Urey; The Planets, Their Origin and Development, Yale Univ. Press, New Haven, Conn. (1952)


G. de Vancouleurs: Physics of the Planet Mars; Faber and Faber, London (1954)


(76) N.P. Carleton; "Bibliography on Upper Atmosphere; Summary of Current Knowledge of Basic Properties of the Upper Atmosphere"; AVCO Research Note 91, Sept. 1958.


(80) E. Jahnke and F. Emde: Funktionentafeln Mit Formeln und Kurven Dover Publications, N.Y. (1945)