BUCKLING OF REINFORCED RINGS

by

ELCIO DE SA\' FREITAS

Guarda-Marinha, Escola Naval
(1956)

Submitted in partial fulfillment of the requirements for the degree of Naval Engineer and Master of Science in Civil Engineering at the Massachusetts Institute of Technology 1964

Signature redacted

Signature redacted

Signature redacted

Chairman, Departmental Committee on Graduate Students
RECORDING OF RINNORED RINGS

Thesis

W.A.

INCO HOPPER PLATING

Grade-Marketing, Ceramic Industry

Nov. 8
(1980)

Experimenter in Partial Fulfillment
of the Requirements for the Degree of

M.S. in Engineering, Civil Engineering

H. J. T. Institute of Technology

1980

Certificate of Acceptance of Graduate Committee

Department of Civil Engineering, May 28, 1980

Certificate of Acceptance of Graduate Committee

Department of Civil Engineering, May 28, 1980
ABSTRACT

BUCKLING OF REINFORCED RINGS

by

ELCIO DE SA FREITAS

Submitted on May 22, 1964 in partial fulfillment of the requirements for the Degrees of Naval Engineer and Master of Science in Civil Engineering

The buckling of circular rings under uniform external pressure has been studied by many authors (see Bibliography) and there is agreement on the value of the critical pressure. However, to the author's knowledge, systematic studies on the effect of reinforcements on that pressure have not been carried out.

It is the objective of this work to investigate the effect of reinforcements on the critical pressure of circular rings. Also it is our purpose to leave clear the extent to which are reasonable certain assumptions generally used for the buckling of rings. Among these assumptions, those involving the strain displacement relations and the so-called inextensional buckling are the most important. In order to leave clear the former, we include, as a chapter of this work, a study of planar strain displacement relations. This chapter was drawn essentially from class notes of course 1.561, at M.I.T. As to the inextensible buckling, it is discussed in Section III.1.

This work is limited to the cases where \( h/R \ll 1 \). The results for circular arches and circular rings are under the further restriction of \( \frac{po}{E} \ll \frac{h}{R} \). Elastic behavior is always assumed.
The fundamental results are:

--critical pressure for plain ring is a lower limit for critical pressures for strutted rings.
--buckling strength is not improved if struts are hinged to the ring and all parallel.
--strength may be improved by fitting non-parallel struts. Critical pressures for this case come out from stability conditions developed in this work.

June, 22, 1964


Title: Assistant Professor of Civil Engineering
ACKNOWLEDGMENT

The author is grateful to Professor Jerome Joseph Connon, Jr. for his valuable guidance, as thesis Supervisor. Also to Professor John Bernard Caldwell, of the Royal Navy College, who suggested this work, the author wants to express his thanks.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>TITLE PAGE</td>
<td>1</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>2</td>
</tr>
<tr>
<td>ACKNOWLEDGMENT</td>
<td>4</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>5</td>
</tr>
</tbody>
</table>

## CHAPTER

### I. THE PROBLEM: ITS IMPORTANCE, LIMITATIONS, METHOD AND RESULTS

- I.1 Background | 7
- I.2 Objective | 8
- I.3 Scope | 8
- I.4 Method | 9
- I.5 Results | 10

### II. STRAIN DISPLACEMENT AND FORCE DISPLACEMENT RELATIONS. EQUILIBRIUM AND STABILITY EQUATIONS

- II.1 Strain Displacement Relations | 11
  - II.1-1 Geometry of Undeformed Member | 11
  - II.1-2 Geometry of Deformed Member | 12
  - II.1-3 Analysis of Deformation of Reference Axis | 14
  - II.1-4 Analysis of Deformation-General | 19
  - II.1-5 Summary of Results for the Case of Small Strains and Small Rotations | 29
Table of Contents - Cont'd

Chapter | Page
------- | ----
II. | 30
II.2 Equilibrium Equations and Stability Equations | 30
II.2-1 Equilibrium Equations | 30
II.2-2 Force Displacement Relations | 35
II.2-3 Stability Equations | 37

III. | 39
BUCKLING OF RINGS UNDER UNIFORM EXTERNAL PRESSURE | 39
III.1 Buckling of a Thin Circular Ring Under Uniform External Pressure | 39
III.2 Buckling of Circular Arches | 46
III.2-1 Arches with Clamped Edges Under Uniform External Pressure | 46
III.2-2 Arches With Hinged Edges Under Uniform External Pressure | 53
III.3 Buckling of Circular Rings, Fitted With Struts | 56
III.3-1 Preliminary Considerations | 56
III.3-2 Ring Fitted with Struts Parallel To each Other | 58
III.3-3 Ring fitted with Several Struts At Arbitrary Positions | 61

IV. | 71
RECOMMENDATIONS FOR FUTURE WORK | 71

BIBLIOGRAPHY | 73

APPENDIX; DEFINITION OF SYMBOLS | 74
CHAPTER I

THE PROBLEM: ITS IMPORTANCE, LIMITATIONS, METHOD AND RESULTS

I.1 Background

Although rings are not common as structural members, they may appear as important elements of structural designs, particularly those designs where the expected load is fundamentally due to uniform pressure. That is the case of submarines, where rings are used as stiffeners for the cylindrical shell which constitutes the pressure hull.

As structural members under compressive forces, the buckling strength of rings is of importance. Many authors have investigated the critical pressure for the elastic buckling of thin rings, under uniform pressure (see bibliography) and they agree that it is equal to $\frac{3EI}{R^2}$. However, to our knowledge, there is no studies on how this value is changed by the presence of any reinforcement on the ring. Referring again to submarines, the decks could act as reinforcements to the ring stiffeners and cylindrical shell.
Therefore, an investigation on the action of such reinforcements is worthwhile. This was suggested to the author by Professor John Bernard Caldwell, of the Royal Navy College, when he stayed at the Institute as a visiting professor.

1.2 Objective

It is our objective to study the behavior of reinforced circular rings, under uniform external pressure. At the same time, we intend to establish a method which will leave clear the main features of the buckling of rings so that it can be applied to the analysis of other load conditions and geometries which we didn’t consider. We also intend to examine the buckling problem of circular arches, since it is very clearly within the approach we will use to investigate reinforced rings.

1.3 Scope

We will restrict ourselves to thin arches for which $\frac{h}{R} \ll 1$. We also assume that we are within the elastic range. Reinforcements are taken as struts, hinged to the ring. It will be seen, however, that this latter condition can be modified, if needed, with little change in the development of the solution.
1.4 Method

The buckling problem consists on the determination of the load for which the equilibrium configuration is not unique. This means that, at the buckling load, given one equilibrium configuration, it is possible to have an adjacent one, without any change of the load. The method adopted in this work aims to determine the critical load by establishing and solving a set of equations, the stability equations. Such equations will arise as a direct consequence of the above definition for the buckling problem.

The method consists on:

a - establishment of strain displacement and force displacement relations. Forces are defined as resultants of stresses over the cross section, at a reference point.

b - substitution of strain displacement relations in the work equation. The work equation is a necessary and sufficient condition for the structure being in equilibrium.

c - derivation of stability equations, from the work equation, involving forces and displacements.

d - variation of the equilibrium equations and force displacement relations. The variations of the terms in the equilibrium equations is a consequence of the variation of displacements.
when the structure passes to an adjacent equilibrium position. The resulting equations are the stability equations.

e - solution of the stability equations.

I.5 Results

The fundamental results we got, under the assumptions stated before, are:

1 - the buckling pressure $\frac{3EI}{R^3}$ is a lower limit for reinforced rings.

2 - this lower limit is reached when all the struts are parallel to a same direction, no matter how many struts there are.

3 - if the struts are not all parallel to a same direction, a stability condition can be derived, involving length, position and cross sectional area of the struts and the geometry of the ring. This condition is taken from equations (IV.90) and (IV.97).

4 - for circular arches, under uniform pressure, the buckling pressure is given by expressions (III.52) and (III.52a), for clamped edges, and by (III.56) for hinged edges. These expressions are the same Timoshenko obtained, using an entirely different approach (see bibliography).
CHAPTER II

STRAIN DISPLACEMENT AND FORCE DISPLACEMENT RELATIONS.
EQUILIBRIUM AND STABILITY RELATIONS

II.1 Strain Displacement Relations

II.1.1 Geometry of Undeformed Member

Consider Fig. II.1, where we represent an arbitrary plane member, supposed to have no inflection points. Let \( \mathbf{r} \) be the position vector to a point on the reference axis and \( \mathbf{r}_z \) be the position vector to a point having coordinates \( y_z \) and \( s \) the curvilinear coordinate for the reference axis and \( z \) the coordinate in the principal normal direction. We have

\[
\mathbf{r}_z = \mathbf{r} + z \mathbf{n} \tag{II.1}
\]

The unit tangent vector \( \mathbf{t} \) is given by

\[
\mathbf{t} = \frac{1}{\mathbf{r}} \frac{d\mathbf{r}}{dy} \tag{II.2}
\]

where

\[
\frac{d\mathbf{r}}{dy} \tag{II.3}
\]
Let the positive direction of arc length correspond to the direction of increasing $y$. Then:

\[ ds = \alpha dy \]  

II.4

The normal vector $\vec{n}$ is given by

\[ \vec{n} = \frac{l}{\frac{|dt}{dy}|} \frac{dt}{dy} \]  

II.5

We let

\[ \left| \frac{dt}{dy} \right| = \frac{\alpha}{R} = \left| \frac{d}{dy} \left( \frac{1}{\alpha} \frac{d\vec{r}}{dy} \right) \right| \]  

II.6

where $R$ is the radius of curvature of the plane curve defining the centroidal axis

\[ \frac{dt}{dy} = \frac{\alpha}{R} \vec{n} \]  

II.7

The expression for $\frac{dn}{dy}$ is

\[ \frac{dn}{dy} = -\frac{\alpha}{R} \vec{t} \]  

II.8

Equations (I.7) and (I.8) are the Frenet formulas, specialized for a plane curve.

Finally we let $ds_z$ be the differential arc length at the point $(y,z)$. We have (holding $z$ constant):

\[ ds_z = \alpha \left( 1 - \frac{z}{R} \right) \]  

II.9

and $\vec{t}_z = \vec{t}$  \quad $R_z = R - z$

II.1-2 Geometry of Deformed Member

We suppose the member is deformed in its plane. Let
us define:

\( \mathbf{F}_t \): position vector to a point \((y,0)\) on the deformed reference curve.

\( \mathbf{F}_z \): position vector to a point \((y,z)\) on the deformed state.

\( \mathbf{t}_t, \mathbf{n}_t \): unit vectors for the deformed reference curve.

\( \mathbf{d}_t, \mathbf{s}_t, \mathbf{R}_t \): scale factor, arc length and radius of curvature for the deformed reference curve.

In general, we will use a prime subscript for the quantities associated with the deformed curve. The undeformed and deformed curves are shown in Figure 1.2. Points \( P_t(y) \) and \( Z_t(y,z) \) correspond to \( P(y) \) and \( Q(y,z) \), in the undeformed state.

The various geometric quantities associated with the deformed reference curve are listed below:

\[
\begin{align*}
t_t &= \frac{1}{\alpha_t} \frac{d\mathbf{r}_t}{dy} \\
\alpha_t &= \frac{d\mathbf{r}_t}{dy} \\
ds_t &= \alpha_t dy \\
\frac{d\mathbf{t}_t}{dy} &= \frac{\alpha_t}{R_t} \mathbf{n}_t \\
\frac{d\mathbf{n}_t}{dy} &= -\frac{\alpha_t}{R_t} \mathbf{t}_t \\
\frac{\alpha_t}{R_t} &= \frac{d\mathbf{t}_t}{dy}
\end{align*}
\]

We have similar quantities for the parallel curve.
through point Q:

\[ t' = \frac{1}{\alpha} \frac{dr_z}{dy}, \quad \text{etc.} \]

II.1-3 Analysis of Deformation of the Reference Axis

Let \( \ddot{u}(y) \) be the displacement vector for the point \( P(y) \) on the reference axis. Then:

\[ \ddot{F}' = \ddot{F} + \ddot{u} \quad \text{(II.11)} \]

We will evaluate the geometric quantities associated with the reference axis:

\[ \frac{d\ddot{r}}{dy} = \frac{d\ddot{r}}{dy} + \frac{d\ddot{u}}{dy} = \alpha \ddot{t} + \frac{1}{\alpha} \frac{d\ddot{u}}{dy} = \alpha \left[ \ddot{t} + \frac{1}{\alpha} \frac{d\ddot{u}}{dy} \right] \]

We have

\[ \alpha' = \left| \ddot{t} + \frac{1}{\alpha} \frac{d\ddot{u}}{dy} \right| = \frac{|d\ddot{r}|}{dy} \quad \text{(II.12)} \]

and

\[ \ddot{t}' = \frac{\alpha}{\alpha'} \left( \ddot{t} + \frac{1}{\alpha} \frac{d\ddot{u}}{dy} \right) \quad \text{(II.13)} \]

Now we consider two adjacent points on the reference axis, \( P(y) \) and \( P_1(y+dy) \) (see Fig (II.3)).

\[ PP_1 = \alpha dy \ddot{t} = dr \quad \text{(II.14)} \]

\[ PP_1 = \alpha' \ddot{t}' dy = d\ddot{r}' \]

Let \( \xi \) be the extensional strain of the line element on the reference axis. Then:

\[ PP_1 \left| PP_1 \right| = (1 + \xi) \left| PP_1 \right| \]
Figure II.3

Figure II.4
Using (I.14) we have:

\[ \alpha' = \alpha (1 + \delta) \quad (II.15a) \]

This leads to:

\[ \delta = \frac{\alpha'}{\alpha} - 1 = \left| \frac{t + \frac{1}{R} \frac{du}{dy}}{\alpha} \right| - 1 \quad (II.15b) \]

Now, we must express \( \bar{u} \) in terms of its components.

Write:

\[ \bar{u} (y) = \bar{u}t + w \bar{n} \quad (II.16) \]

Therefore:

\[ \frac{d\bar{u}}{dy} = \frac{du}{dy} t + u \frac{\alpha}{R} \bar{n} + \frac{dw}{dy} \bar{n} - w \frac{\alpha}{R} \bar{t} \quad (II.17a) \]

\[ t + \frac{1}{\alpha} \frac{d\bar{u}}{dy} = (1 + \frac{1}{\alpha} \frac{du}{dy} - w) \bar{t} + \left( \frac{u}{R} + \frac{1}{\alpha} \frac{dw}{dy} \right) \bar{n} \quad (II.17b) \]

Let

\[ e_1 = \frac{1}{\alpha} \frac{du}{dy} - w = \frac{1}{\alpha} \bar{t} \cdot \frac{d\bar{u}}{dy} \quad (II.18a) \]

\[ e_2 = \frac{u}{R} + \frac{1}{\alpha} \frac{dw}{dy} = (1 + e_1) \bar{t} + e_2 \bar{n} \quad (II.18b) \]

This gives:

\[ t + \frac{1}{\alpha} \frac{d\bar{u}}{dy} = (1 + e_1) \bar{t} + e_2 \bar{n} \quad (II.19) \]

As a consequence:

\[ \frac{t'}{1+\delta} = \frac{1+e_1}{1+\delta} t + \frac{e_2}{1+\delta} \bar{n} \quad (II.20) \]

Let \( \phi \) be the angle that \( t' \) makes with \( \bar{t} \) (Fig. II.4).
We can write:

\[
\begin{align*}
\tilde{t}^\prime &= \cos \phi \tilde{t} + \sin \phi \tilde{n} \\
\tilde{n}^\prime &= -\sin \phi \tilde{t} + \cos \phi \tilde{n}
\end{align*}
\]

(II.19)

is the rotation of \( \tilde{t} \) in the \( \tilde{n} \) direction.

It follows that:

\[
\begin{align*}
\cos \phi &= \frac{1 + e_1}{1 + \epsilon} \\
\sin \phi &= \frac{e_2}{1 + \epsilon}
\end{align*}
\]

(II.20)

According to (I.15b) and (I.17):

\[
\epsilon = \left[ (1 + e_1)^2 + e_2^2 \right]^{1/2} - 1
\]

(II.21)

Now we will restrict ourselves to the case of small strains, that is, we assume \( \epsilon \ll 1 \) which it is reasonable for engineering materials. Expanding (I.21) we get:

\[
\epsilon = e_1 \left( 1 + \frac{1}{2} e_1 \right) + \frac{1}{2} (e_2)^2 + \text{higher order terms}
\]

(II.22)

Now consider equations (I.20). We have:

\[
\begin{align*}
e_2 &= (1 + \epsilon) \sin \phi \approx \sin \phi \\
e_1 &= (1 + \epsilon) \cos \phi - 1 = \cos \phi - 1 + \epsilon \cos \phi
\end{align*}
\]

(II.23)

This shows that the nonlinear terms in (II.22) are associated with the rotation of the tangent vector. We will consider, first, the case where \( \phi \approx 0 \), which is called the linear geometric case. The relations for this case are,
neglecting $\xi$ with respect to unit:

$$e_2 \approx \theta(0)$$

$$\xi \approx e_1$$

$$\bar{t}^* \approx \bar{t} + e_2 \bar{n}$$

$$\bar{n}^* = -e_2 \bar{t} + \bar{n}$$

Now consider the case where we have small rotations $\phi$, such that

$$\sin \phi \approx \phi$$

$$\cos \phi \approx 1 + 0(\phi^2)$$

$$e_2 \approx 0(\phi)$$

$$e_1 \approx \xi + 0(\phi^2)$$

The governing relations (neglecting and $e_2$ with respect to unit) are:

$$\xi \approx e_1 + \frac{1}{2} e_2^2$$

$$\bar{t}^* \approx \bar{t} + e_2 \bar{n}$$

$$\bar{n}^* = -\frac{e}{2} \bar{t} + \bar{n}$$

We will now determine an expression for $R^*$. Take equations (II.19) and differentiate $\xi^*$ with respect to $y$. 
We get:

\[
\frac{dt'}{dy} = \left( \frac{\alpha}{R} + \frac{d\phi}{dy} \right) n'
\] (II.27)

For the undeformed curve, we took \( \mathbf{n} \) pointing in the direction of \( \frac{d\mathbf{r}}{dy} \) and \( R \) always positive. For the deformed curve it is more convenient to fix the direction of \( \mathbf{n}' \). Therefore, there will be a sign associated with \( R' \). From (II.27) and (II.10d) have:

\[
\frac{\alpha}{R'} = \frac{\alpha}{R} + \frac{du}{dy}
\] (II.28)

\[
\frac{dt'}{dy} = \frac{\alpha}{R'} n'
\]

\( \mathbf{n}' \) is given by the second of equations (II.19).

We have:

\[
\alpha' = \alpha(1 + \delta)
\]

Therefore

\[
\frac{1}{R'} = \frac{1}{1+\delta} \left\{ \frac{1}{R} + \frac{1}{2} \frac{\alpha}{dy} \right\}
\] (II.29)

We still consider \( \phi \ll 1 \). From (II.20) it follows:

\[
\sin \phi \ll e_2
\]

Then:

\[
\frac{d\phi}{dy} \ll \frac{1}{\cos \phi} \frac{de_2}{dy}
\] (II.30)

For the case where \( \phi^2 \ll 1 \), we get:

\[
\frac{d\phi}{dy} \ll \frac{de_2}{dy}
\]
Let us summarize the results we got for the case where $\phi^2$ and $\dot{\phi}$ are neglected with respect to unit:

$$\dot{\phi} = e_1 + \frac{1}{2} e_2^2$$  \hspace{1cm} (a)

$$\frac{\alpha^i}{R^i} = \frac{\alpha}{R} + \frac{de_2}{dy}$$  \hspace{1cm} (b)

$$\ddot{t} = \dot{t} + e_2 \dot{n}$$  \hspace{1cm} (c)

$$\ddot{n} = -e_2 \dot{t} + \ddot{n}$$  \hspace{1cm} (d)

$$\phi = e_2$$  \hspace{1cm} (e)

$$\frac{d\ddot{t}^i}{dy} = \frac{\alpha^i}{R^i} \ddot{n}^i$$  \hspace{1cm} (f)  \hspace{1cm} (II.31)

$$\frac{d\ddot{n}^i}{dy} = -\frac{\alpha^i}{R^i} \dddot{t}^i$$  \hspace{1cm} (g)

$$\frac{1}{R^i} = \frac{1}{R} + \frac{1}{\alpha} \frac{de_2}{dy}$$  \hspace{1cm} (h)

$$e_1 = \frac{1}{\alpha} \frac{du}{dy} - \frac{w}{R}$$  \hspace{1cm} (i)

$$e_2 = \frac{u}{R} + \frac{1}{\alpha} \frac{dw}{dy}$$  \hspace{1cm} (j)

II.1-4 Analysis of Deformation - General

In the previous section we studied the deformations of the reference axis. Now we will study the deformation of elements off the reference axis. Let $\bar{u}_z$ be the displacement vector for the point $(y,z)$. The initial and final position
vectors are:

\[ \vec{r}_z = \vec{r} + z\vec{n} \quad (\text{II.32}) \]

\[ \vec{r}'_z = \vec{r} + z\vec{n} + \vec{u}_z \]

Express \( \vec{u}_z \) as:

\[ \vec{u}_z = \vec{u} + z\vec{B} + z^2 \vec{N} + \ldots \quad (\text{II.33}) \]

\( \vec{B}, \vec{N}, \ldots \) are vectors which are functions only of \( y \). Then:

\[ \vec{r}'_z = \vec{r}' + z(\vec{n} + \vec{B}) + z^2 \vec{N} + \ldots \quad (\text{II.34}) \]

Now consider the straight line originally normal to the reference axis. Take \( z \) as the curvilinear coordinate for this line. The relative position vector is \( \vec{z}\vec{n} \), for the undeformed state and \( z(\vec{n} + \vec{B}) + z^2 \vec{N} + \ldots \) for the deformed state. The contribution of \( \vec{u} \) gives only a rigid body translation. If we keep only linear terms in \( z \), the line is also straight in the deformed state. We have:

\[ \frac{\partial \vec{r}}{\partial z} = \vec{n} \]

\[ \frac{\partial \vec{r}'}{\partial z} = \vec{n} + \vec{B} + z \vec{N} + \ldots \]

The tangent vector \( n^* \), to the deformed line is:
If \( \bar{N} = 0 \), \( \bar{n}^* \) is independent of \( z \) and the line is straight. We can consider the linear expansion in \( z \) as a first approximation (see Figure II.5).

Consider two line elements at \( Q(y,z) \) (see Figure II.6):

\[
\overline{QQ_1} = \frac{\partial \bar{F}_z}{\partial y} \ dy = \alpha_z \ dy \ 
\]

\[
\overline{QQ_2} = \frac{\partial r_z}{\partial z} \ dz = \ dz \bar{n} \]  

\[
\alpha_z = \alpha(1 - \frac{z}{R}) \]

Consider first the parallel curve. We have:

\[
\bar{t}_z = \bar{t} \quad \alpha_z = \alpha(1 - \frac{z}{R}) 
\]

\[
\alpha_z^i = \left| \frac{\partial \bar{F}_z}{\partial y} \right| = \left| \alpha_z \bar{t} + \frac{\partial \bar{u}_z}{\partial y} \right| \]  

\[
\bar{t}_z^i = \frac{1}{\alpha_z^i} \frac{\partial \bar{F}_z}{\partial y} = \frac{1}{\alpha_z^i} \left( \alpha_z \bar{t} + \frac{\partial \bar{u}_z}{\partial y} \right) \]

Define \( \bar{\epsilon}_z \) as the extensional strain of the parallel line element:

\[
\bar{\epsilon}_z = \left| \frac{\overline{QQ_2}}{\overline{QQ_1}} \right| - 1 \]  

(II.38)
Then
\[ \alpha^*_z = (1 + \xi_z) \quad (II.39) \]

From (II.39) and the second of equations (II.37):
\[ \xi_z = \left| \bar{t} + \frac{1}{\alpha_z} \frac{\partial \bar{u}_z}{\partial y} \right| - 1 \quad (II.40) \]

Substitution of (II.40) in the last of equations (II.37) yields:
\[ \bar{t}_z^* = \frac{1}{1 + \xi_z} \left( \bar{t} + \frac{1}{\alpha_z} \frac{\partial \bar{u}_z}{\partial y} \right) \quad (II.41) \]

Now, take a linear expansion for \( \bar{u}_z \):
\[ \bar{u}_z = \bar{u} + z \bar{B} \quad (II.42) \]

This gives:
\[ \frac{\partial \bar{u}_z}{\partial y} = \frac{d\bar{u}}{dy} + z \frac{d\bar{B}}{dy} \quad (II.43) \]

Now, let
\[ \bar{u} = u \bar{t} + w \bar{n} \]
\[ \bar{B} = B \bar{t} + w \bar{\lambda} \bar{n} \quad (II.44) \]

Then:
\[ \frac{d\bar{B}}{dy} = \left( \frac{d\bar{B}}{dy} - \frac{w_1 \alpha}{R} \right) \bar{t} + \left( \frac{dw_1}{dy} + \frac{\alpha \bar{B}}{R} \right) \bar{n} \]

Collecting terms we get
\[
\frac{\partial \tilde{u}_z}{\partial y} = \tilde{t} \left( \frac{du}{dy} - \frac{w \alpha}{R} \right) + z \left( \frac{dB}{dy} - \frac{w_1 \alpha}{R} \right) + \\
+ n \left[ \left( \frac{u \alpha}{R} + \frac{dw}{dy} \right) + z \left( \frac{B \alpha}{R} + \frac{dw_1}{dy} \right) \right]
\] (II.45)

Then
\[
\frac{1}{\alpha z} \frac{\partial \tilde{u}_z}{\partial y} = \frac{1}{1 - \frac{z}{R}} \left[ (e_1 + ze_3) \tilde{t} + (e_2 + ze_4) n \right]
\] (II.46)

where \(e_1\) and \(e_2\) are defined by equations (II.16), and:

\[
e_3 = \frac{1}{\alpha} \frac{dB}{dy} - \frac{w_1}{R}
\] (II.47)

\[
e_4 = \frac{B}{R} + \frac{1}{\alpha} \frac{dw_1}{dy}
\]

Now, we follow the same procedure as for the line elements on the reference axis. Let \(\phi_z\) be the rotation of the initial tangent, \(\tilde{t}_z = \tilde{t}\) (Figure II.7).

Then
\[
\tilde{t}_z = \cos \phi_z \tilde{t} + \sin \phi_z \tilde{n}
\] (II.48)

Then by comparing with the last of equations (II.37):

\[
(1 + \xi_z) \cos \phi_z = 1 + e_1 + ze_3 \cdot \frac{1}{1 - \frac{z}{R}}
\] (II.49)

\[
(1 + \xi_z) \sin \phi_z = e_2 + ze_4 \cdot \frac{1}{1 - \frac{z}{R}}
\]
Figure II.7

Figure II.8
We restrict the discussion to the case where $\phi_z^2 \ll 1$, for which we have:

$$\frac{e_1 + z e_3}{1 - \frac{z}{R}} \approx z + 0 \left( \phi_z^2 \right)$$

(II.50)

$$\frac{e_2 + z e_4}{1 - \frac{z}{R}} \approx w_z$$

The corresponding consistent expressions for $\xi_z$ and $\xi_z^i$ are (neglecting $\phi_z^2$ and $\xi_z$ with respect to unit):

$$\xi_z = \frac{e_1 + z e_3}{1 - \frac{z}{R}} + 1 \left( \frac{e_2 + z e_4}{1 - \frac{z}{R}} \right)^2$$

(II.51)

$$\xi_z^i = \xi + e_2 + z e_4 \frac{\bar{n}}{1 - \frac{z}{R}}$$

We must determine the geometrical significance of $e_3$ and $e_4$. To do this, consider the line element $\overline{QQ_2}$. Let $\xi_n$ be the extensional strain in the normal direction.

$$\xi_n = \frac{\overline{Q'Q_2}'}{\overline{QQ_2}} - 1 = \left| \begin{array}{c} \bar{n} + \frac{\partial \bar{u}}{\partial z} \\ -1 \\
\end{array} \right|$$

(II.52)

$$\bar{n}^* = \frac{1}{1 + \xi_n} \left( \bar{n} + \frac{\partial \bar{u}}{\partial z} \right)$$

Taking a linear expansion for $\bar{u}_z$, we have:

$$\frac{\partial \bar{u}_z}{\partial z} = \bar{B} = B \bar{t} + w_1 \bar{n}$$
Then:

$$\varepsilon_n = \left| B \hat{t} + (1 + w_1) \bar{n} \right| - 1 \quad (II.54)$$

$$n* = \frac{1}{1+\varepsilon_n} \left[ B \hat{t} + (1 + w_1) \bar{n} \right]$$

Now, consider the normal line element at the reference axis. Let $\Omega$ be the rotation of $\bar{n}$ in the $\hat{t}$ direction. The deformed normal vector is:

$$\bar{n}* = \sin \Omega \hat{t} + \cos \Omega \bar{n} \quad (II.55)$$

From (II.54) it follows that:

$$B = (1 + \varepsilon_n) \sin \Omega \quad (II.56)$$

$$1 + w_1 = (1 + \varepsilon_n) \cos \Omega$$

For the case where $\Omega^2 \ll 1$, and negligible extensional strain, we have:

$$\varepsilon_n \approx \frac{w_1}{(\varepsilon_n)} \approx 0 \quad (II.57)$$

$$\varepsilon_n \approx w_1 + \frac{1}{2} B^2$$

$$\bar{n}* = B \hat{t} + \bar{n}$$

It is of interest, at this point, to consider the case of negligible shear deformation. Let $\gamma$ be the transverse shear strain at the reference axis.

$$\gamma = \phi + \Omega \quad (II.58)$$

where $\sin \phi = \frac{e}{1+\varepsilon}$
For \( \gamma = 0 \), \( \phi = -\Lambda \) and \( \sin \phi = -\sin \Lambda \).

Then,

\[
\frac{B}{1+\varepsilon_n} = -e_2
\]

and, for negligible extensional strain with respect to unit:

\[
B \approx -e_2 \quad \text{(II.59)}
\]

Note that, as we are using \( G = -6+\varepsilon \), \( \mathbf{n}^* \) coincides with \( \mathbf{n}^f \).

In general, when squares of rotations are neglected with respect to unit, we have:

\[
\phi \approx e_2
\]

\[
\Lambda \approx B \quad \text{(II.60)}
\]

\[
\gamma \approx e_2 + B
\]

It follows that

\[
e_2 = \gamma - B \quad \text{(II.61)}
\]

\[
B = \gamma - e_2
\]

Now, consider the terms \( e_3 \) and \( e_4 \), which are given by equations \((II.47)\). For a member element, the transverse normal stress is negligible with respect to the tangential normal stress. Then:
Actually \( \varepsilon_n \sim 0(\varepsilon) \) \hspace{1cm} \text{(II.62)}

\( \varepsilon_n \sim -\nu \varepsilon \)

\( \nu \) is the Poisson's ratio.

Using (II.58) we can eliminate \( w_1 \) by substitution in the third of equations (II.57):

\[
\begin{align*}
w_1 + \frac{1}{2} B^2 \sim & -\nu e_1 - \frac{\nu}{2} e_2^2
\end{align*}
\]

The transverse shear strain \( \gamma \) is usually of order of magnitude lower than the extensional strain. In simplifying the expressions for the extensional strain and the deformed vectors, we can neglect the effect of shear strain, that is, we take \( e_2 \sim -B \). Then:

\[
\begin{align*}
w_1 \sim & -\frac{1}{2} B^2 (1 + \nu) - \nu e_1 \sim -\frac{1}{2} e_2^2 (1 + \nu) - \nu e_1
\end{align*}
\]

Now, consider the first of equations (II.47). We have:

\[
\frac{1}{\alpha} \frac{dB}{dy} \sim -\frac{1}{\alpha} \frac{de_2}{dy} = \frac{1}{R} - \frac{1}{R^\nu}
\]

\hspace{1cm} \text{(II.63)}

Then:

\[
\frac{e_3}{\alpha} = \frac{1}{\alpha} \frac{dB}{dy} - \frac{w_1}{R} \sim \frac{1}{R} \left\{ \frac{1}{R} - \frac{w_1}{R} \right\} - \frac{1}{R^\nu}
\]

\hspace{1cm} \text{(II.64)}

Neglecting strains and squares of rotations with respect to unit, we obtain:

\[
\frac{e_3}{\alpha} = \frac{1}{\alpha} \frac{dB}{dy} \sim \frac{1}{R} - \frac{1}{R^\nu}
\]

\hspace{1cm} \text{(II.65)}
Now, consider the second of equations (II.47).

We have:

\[
\frac{dw_1}{dy} \sim -B (1 + \gamma) \frac{dB}{dy} - \gamma \frac{de_1}{dy}
\]

Assuming \(\delta \approx 0\) and \(\delta_n \approx 0\) last equation becomes:

\[
\frac{dw_1}{dy} \sim -B \frac{dB}{dy}
\]

Then

\[
e_4 \sim \frac{B}{R} - B \left[ \frac{1}{\infty} \frac{dB}{dy} \right] \sim \frac{B}{R^f}
\]

and

\[
e_2 + ze_4 \sim -B \left( 1 - \frac{z_f}{R_f} \right) \quad (II.66)
\]

It remains to evaluate the expressions for \(\delta_z\) and \(\tilde{t}'_z\). We have:

\[
\delta_z = \frac{1}{1 - \frac{z}{R}} \left\{ e_1 + \frac{ze_3 + \frac{1}{2} B^2 \left[ \frac{1 - \frac{2}{R} + \left( \frac{z_f}{R_f} \right)^2}{1 - \frac{z}{R}} \right]} \right\}
\]

If we neglect \(\left( \frac{z}{R} \right)^2\) and \(\left( \frac{z_f}{R_f} \right)^2\) with respect to unit, we get:

\[
\delta_z = \frac{1}{1 - \frac{z}{R}} \left\{ \delta + z \left[ e_3 + \frac{1}{2} B^2 \left( \frac{1}{R} - \frac{2}{R_f} \right) \right] \right\}
\]

If we go further and neglect \(B^2\) with respect to unit,
We get:

\[ \varepsilon_z = \frac{1}{1 - \frac{z}{R}} \left[ \varepsilon + z\varepsilon_3 \right] \]  

(II.67)

where

\[ \varepsilon = e_1 + \frac{1}{2} B^2 \]

\[ \varepsilon_3 = \frac{1}{\alpha} \frac{dB}{dy} \]

Also

\[ \varepsilon_z' \approx \varepsilon - Bn \]  

(II.68)

II.1-5 Summary of Results for the Case of Small Strains and Small Rotations.

Listed below are the results obtained in the previous sections, considering the strains and squares of rotations as negligible, with respect to unit.

--- strain displacement relations:

\[ \varepsilon_z = \frac{1}{1 - \frac{z}{R}} \varepsilon + z\varepsilon_3 \]

\[ \gamma_z = e_2 + B \]

\[ \varepsilon = e_1 + \frac{1}{2} B^2 \]

\[ e_1 = \frac{1}{\alpha} \frac{du}{dy} - \frac{w}{R} \]  

(II.69)

\[ e_2 = \frac{u}{R} + \frac{1}{\alpha} \frac{dw}{dy} \]

\[ e_3 = \frac{1}{\alpha} \frac{dB}{dy} \]

\[ \ddot{u}_z = (u + zB)\ddot{e} + (w + zw_1)\ddot{n} \]
---unit vectors

We neglect the effect of shear deformation on the deformed unit vectors. Then, taking \( e_z \sim -B \), we have:

\[
\vec{e}_z \sim \vec{e} \sim \vec{e} - B\vec{n}
\]

\[
\vec{n}^* \sim \vec{n} \sim B\vec{e} + \vec{n}
\]

The differentiation formulas for the unit vectors take the form:

\[
\frac{d\vec{e}^*}{dy} = \frac{\alpha^*}{R^*} \vec{n}^*
\]

\[
\frac{d\vec{n}^*}{dy} = -\frac{\alpha^*}{R^*} \vec{e}^*
\]

\[
\frac{\alpha^*}{R^*} = \frac{\alpha}{R} - \frac{dB}{dy}
\]

\[
\frac{1}{R^*} \sim \frac{1}{R} - \frac{1}{\alpha} \frac{dB}{dy}
\]

II. 2 Equilibrium Equations and Stability Equations

II.2-1 Equilibrium Equations

We will now proceed to the development of the equilibrium equations for a plane member, subjected to a load in its plane. We will restrict ourselves to thin members, that is, we will consider \( \frac{z}{R} \) and \( \frac{z^*}{R^*} \) much smaller than unit.

This will lead us to certain simplifications in the geometrical
Consider an element of member in its deformed configuration (Fig. II.7). The normal stresses in the transverse direction will be neglected before the normal stresses in the transverse direction.

Now, let us use the work equation. It states that, for an equilibrium configuration, the work of deformation done by the stresses, during an arbitrary compatible virtual deformation, is equal to the work done by the external forces. We will look first for an expression for the work of deformation. Let \( \delta \varepsilon_z \) and \( \delta \gamma_z \) be variations in extensional strains due to an arbitrary compatible virtual deformation. Let us call \( d(dW_D) \) the work of deformation, for an element of ring bounded by faces \( y, y + dy, z \) and \( z + dz \). We have:

\[
d (dW_D) = \xi z \delta \varepsilon_z \alpha^z \text{dy} \text{dz} + \xi z \delta \gamma_z \alpha^z \text{dy} \text{dz} \tag{II.70}
\]

Neglecting \( \frac{z}{R} \) and with respect to unit we have:

\[
\alpha'_z = \alpha_z \left(1 + \varepsilon_z \right) \approx \alpha_z
\]

\[
\alpha_z = \alpha \left(1 - \frac{z}{R} \right) \approx \alpha
\]

\[
\delta \varepsilon_z = \frac{1}{1 - \frac{z}{R}} \left[ \delta \varepsilon + \frac{z}{\alpha} \frac{d\varepsilon}{dy} \right] \approx \delta \varepsilon + \frac{z}{\alpha} \delta B^2
\]

\[
\delta \gamma_z = \delta \gamma = \frac{\delta u}{R} + \frac{1}{\alpha} \frac{d(\delta w)}{dy} + \delta B
\]
Substituting (II.71) in (II.70) we get:
\[ d(dW_D) = \alpha \frac{d(dy)}{dy} \left\{ C_z \left[ \frac{\delta \epsilon}{\alpha} + \int \frac{\delta B}{\alpha} \left( \frac{\delta B}{dy} \right) \right] + C_z \left[ \frac{\delta u}{R} + \frac{1}{\alpha} \frac{\delta (dw)}{dy} + \frac{\delta B}{\alpha} \right] \right\} dz \]  

(II.72)

Integrating along \( z \) we get:

\[ dW_D = \alpha \frac{d(dy)}{dy} \left[ N \delta \epsilon + \frac{M}{\alpha} \left( \frac{\delta B}{dy} \right) + Q \left[ \frac{\delta u}{R} + \frac{1}{\alpha} \frac{\delta (dw)}{dy} + \frac{\delta B}{\alpha} \right] \right] \]  

(II.73)

\( dW \) is the work of deformation for a length \( ds \) of the ring, measured along the reference axis.

In (II.73)

\[ N = \int_{-h/2}^{h/2} C_z dz \]  

(II.74)

\[ Q = \int_{-h/2}^{h/2} \frac{C_z}{\alpha} dz \]

and \( M = \int_{-h/2}^{h/2} \frac{C_z}{\alpha} z dz \),

are the force resultants and couple resultant, at the reference axis.

The quantities which now appear are referred to the undeformed reference axis. From now on, we will use a prime to indicate differentiation with respect to \( y \). Rewriting (II.73) we get:

\[ dW_D = \alpha \frac{d(dy)}{dy} \left[ N \delta \epsilon + \frac{M}{\alpha} \left( \frac{\delta B}{dy} \right) + Q \left( \frac{\delta u}{R} + \frac{1}{\alpha} \frac{\delta (dw)}{dy} + \frac{\delta B}{\alpha} \right) \right] \]  

(II.75)

In the problems which we will discuss, the undeformed
member is a circular ring. The coordinate \( y \) is an angle, measured from a reference radius, clockwise (see Fig. II.8). Then: \( \alpha = R - R_0 \), where \( R_0 \) is the radius of the centroidal axis of the undeformed ring. We will take this axis as the reference axis.

Therefore, we can write, noting that:

\[
\delta E = \frac{u^2}{\alpha} - \frac{w}{R} + \frac{1}{2} \left( \frac{u}{R} + \frac{w^2}{\alpha} \right)^2
\]

\[
dW_D = dy \left[ N (\delta u^2 - \delta w) + N \left( u + w^2 \right) \left( \delta u + \delta w^2 \right) + M \delta B^2 + Q (\delta u + \delta w^2 + R \delta B) \right]
\]

Integrating along \( y \) we get:

\[
W_D = \int_{y_1}^{y_2} \left[ N (\delta u^2 - \delta w) + N \left( u + w^2 \right) \left( \delta u + \delta w^2 \right) + M \delta B^2 + Q (\delta u + \delta w^2 + R \delta B) \right] dy \quad \text{(II.76)}
\]

using integration by parts we get:

\[
\int_{y_1}^{y_2} \frac{N}{R} \left( u + w^2 \right) \delta w^2 dy = \left. \frac{N}{R} \left( u + w^2 \right) \delta w \right|_{y_1}^{y_2} - \int_{y_1}^{y_2} \frac{N(u+w^2)^2}{R} w dy
\]

\[
\int_{y_1}^{y_2} M \delta B^2 \delta u dy = \left. M \delta B \delta u \right|_{y_1}^{y_2} - \int_{y_1}^{y_2} M \delta B \delta B dy
\]

\[
\int_{y_1}^{y_2} Q \delta w^2 dy = \left. Q \delta w \right|_{y_1}^{y_2} - \int_{y_1}^{y_2} Q \delta w \delta w dy
\]

\[
\int_{y_1}^{y_2} N \delta u^2 dy = \left. N \delta u \right|_{y_1}^{y_2} - \int_{y_1}^{y_2} N \delta u \delta u dy
\]

Substitution in (II.76) gives:
Let us find the expression for the work done by the external forces. Let:

\( \bar{p} = \text{external load, per unit length of arc.} \)

In vector form:

\[ \bar{p} = q \, \bar{t} + p \, \bar{n} \quad \text{(II.78)} \]

Therefore, the external load is referred to the axis \( \bar{t} \) and \( \bar{n} \) correspondent to the undeformed position.

Calling \( N_1, N_2, Q_1, Q_2, M_1 \) and \( M_2 \) as end forces, we have:

\[ W_e = \text{work of external forces: } \int R \left( p \, \delta w + q \, \delta u \right) \, dy + \]

+ work done by end forces. \quad \text{(II.79)}

As mentioned before, if the deformed configuration is an equilibrium one, we have:

\[ W_D = W_e \]

It follows that we can equate (II.79) and (II.77). Doing that and noting that \( \delta u, \delta w \) and \( \delta B \) are linearly independent, we get:
There are the equilibrium equations, when we consider non zero shear strains.

Now we will specialize these equations for the case of zero shear strain. Then, we have, according to (II.69):

\[ e_2 = \frac{u + w'}{R} = -B \]  

The equilibrium equations take the form:

\[ N - \left[ \frac{N}{B} \right] u + Q' + Rp = 0 \]  

(II.81)

\[ N' + Nb = Q + Rq = 0 \]  

(II.82)

\[ M' = QR \]  

(II.83)

### II.2-2 Force Displacement Relations

In the previous section we defined the force and couple resultants, at the centroidal axis, by equations (II.74). Now we will derive the relations between these resultants and the displacements.

Take

\[ N = \int_{-h/2}^{h/2} \sigma_z dz = \int_{-h/2}^{h/2} E \varepsilon_z dz = \int_{-h/2}^{h/2} E \left( \varepsilon + \frac{z}{R} B' \right) dz \]
Noting that we choose the centroidal axis as reference axis, we can conclude:

\[ N = E A \hat{\gamma} \tag{II.84} \]

where \( E \) is the Young's modulus and \( A \) is the cross sectional area, per unit length perpendicular to the plane of the ring.

Now consider:

\[ Q = \int_{-h/2}^{h/2} z \, dz = \int_{-h/2}^{h/2} G \hat{\gamma} \, dz, \text{ where } G = \frac{E}{2(1+\nu)}. \]

At this instant we are assuming that the transverse shear strain is not zero. According to (II.69) we have:

\[ \hat{\gamma}_z = \gamma \]

Then:

\[ Q = G A \gamma \tag{II.85} \]

When we consider zero transverse shear strain, the force \( Q \) is any longer given by (II.85) and must be determined from the equilibrium equations.

Finally, let us consider:

\[ M = \int_{-h/2}^{h/2} z \, dz = \int_{-h/2}^{h/2} E \left( \frac{G + z \, B'}{R} \right) \, dz \]

As the reference axis is a centroidal axis:

\[ M = E I B' \tag{II.86} \]

where \( I \) is the moment of inertia of the cross section about the centroidal axis.
II. 2-3 The Stability Equations

The equilibrium equations and the force displacement relations involve six unknowns and six equations. Therefore they suffice to determine the equilibrium configuration. In the case of zero shear strain we lose one force displacement relation but we got one more relation from the condition of zero shear (equation II.80). Now, we want to solve the problem of stability. This means that we want to know when it will be possible to have adjacent equilibrium configurations for one same value of the load. Starting from one equilibrium configuration, if we pass to an adjacent one, the force and couple resultants will present variations, as a consequence of variations in displacements. However, as equilibrium still exists, these new forces and couple satisfy the equilibrium equations. As a result, the force and couple variations satisfy the equilibrium equations. Therefore, this is the condition for buckling and the equations by which it is expressed are the stability equations. From now on we will consider only the case of zero shear strain. Therefore, take variations of equations (II.83):

\[
\delta N - [N \delta B + B \delta N] + \delta Q^t + R \delta q = 0 \quad (II.87)
\]
\[
N^t + N \delta B + B \delta N - \delta Q + R \delta q = 0 \quad (II.88)
\]
\[
\frac{\delta M^t}{R} = \delta Q \quad (II.89)
\]

Take variations of equations (II.84) and (II.86):

\[
\delta N = E A \delta \xi \quad (II.90)
\]
\[
\delta M = E I \frac{\delta B^t}{R} \quad (II.91)
\]
Take variations of (II.80):

\[ \delta B = - \frac{1}{R} (\delta u + \delta w^2) \]  

(II.92)

Take variations of the third of equations (II.69):

\[ \delta \delta \xi = \frac{1}{R} (\delta u^2 - \delta w) + B \delta B \]  

(II.93)

Equations (II.87) to (II.93) are the stability equations. We see that, to solve them, we must first determine N and B, by means of the equilibrium equations.
III.1 Buckling of a Thin Circular Ring Under Uniform External Pressure

Let us illustrate the method by solving the problem of a thin circular ring, under uniform external pressure. The solution is known to be:

\[ p_{ocr} = \frac{3EI}{R^3} \]

where \( p_{ocr} \) is the buckling pressure and the remaining symbols are as already defined.

Let us use the stability equations. For the present case we know that:

\[ N = -p_0 R \]

\[ B = 0 \] \hspace{1cm} (III.1)

\[ N' = 0 \]

\[ B' = 0 \]
Equations (II.87) and (II.88) become:

\[
\delta N - N \delta B + \frac{\delta M_{tt}}{R} + R \delta p = 0 \\
N' + N \delta B - \frac{\delta M_{tt}}{R} + R \delta q = 0 ,
\]

after substituting (II.89).

Taking into account (II.91), we write:

\[
\delta N - N \delta B + \frac{EI \delta B_{tt}}{R^2} + R \delta p = 0 \tag{III.2}
\]

\[
N' + N \delta B - \frac{EI \delta B_{tt}}{R^2} + R \delta q = 0 \tag{III.3}
\]

We can easily determine the variations \( \delta p \) and \( \delta q \) by looking at Figure III.1. It shows the rotation of the normal when the member deforms. As the external pressure force is always normal to the surface, it changes its direction in the same way as the normal. Noting that \( p \) and \( q \) were defined as the components along \( \mathbf{t} \) and \( \mathbf{n} \), we have:

\[
\delta q = p_o \delta B \tag{III.4}
\]

\[
\delta p = 0
\]

Substituting in (III.2) and (III.3) and noting that \( N = -p_o R \), we get:

\[
N + p_o R \delta B + \frac{EI}{R^2} \delta B_{tt} = 0 \tag{III.5}
\]
\[ \frac{\delta N'}{EI} - \frac{ET}{R^2} \delta B^{**} = 0 \]  

(III.6)

Differentiating the first equation, we have:

\[ \frac{R^2}{EI} \delta N' + \frac{p_o R^3}{EI} \delta B^{**} + \delta B^{IV} = 0 \]  

(III.7)

Substituting (III.6) into (III.7) we have:

\[ \delta B^{IV} + (K^2 + 1) \delta B^{**} = 0 \]

where \( K^2 = \frac{p_o R^3}{EI} \)  

(III.9)

The solution for (III.8) is:

\[ \delta B = C_1 + C_2 y + C_3 \sin ky + C_4 \cos ky \]  

(III.10)

where \( k = K^2 + 1 \)

(III.11)

We will neglect \( C_1 \), since it is the component of a rigid body motion. Then:

\[ B = C_2 y + C_3 \sin ky + C_4 \cos ky \]  

(III.12)

We must impose the boundary condition:

\[ B_y = B_{y+2\pi} \]  

(III.13)

Therefore:

\[ C_2 = 0 \]

\[ R = \pm 1, \pm 2, ... \]

This leads to:

\[ B = C_3 \sin ky + C_4 \cos ky \]  

(III.14)
Also, according to (III.9):

\[ p_0 = \frac{k^2 EI}{R^3} \]

From (III.11):

\[ K^2 = k^2 - 1 \]

Therefore:

\[ P_{ocr} = \frac{3EI}{R^3} \quad \text{(III.15)} \]

Now we will proceed to determine the displacements \( \delta u \) and \( \delta w \). Substitute (II.93) into (II.90) noting that \( B = 0 \):

\[ \delta N = \frac{EA}{R} \left( \delta u' - \delta w \right) \quad \text{(III.15a)} \]

From (II.92):

\[ \delta B = -\frac{1}{R} \left( \delta u + \delta w' \right) \quad \text{(III.15b)} \]

Then:

\[ R \frac{\delta N'}{EA} = \delta u'' - \delta w' \quad \text{or} \quad \delta w' = -\frac{R}{EA} \delta N' + \delta u'' \]

Substitution into the expression for \( \delta B \) leads to:

\[ \delta B = -\frac{1}{R} \left( \delta u - \frac{R}{EA} \delta N' + \delta u'' \right) \quad \text{(III.16)} \]

Substitute (III.6) into (III.15):

\[ \delta B = -\frac{1}{R} \left( \delta u - \frac{1}{AR} \delta B'' + \delta u'' \right) \quad \text{(III.17)} \]

Therefore:

\[ \delta u'' + \delta u + R \delta B - \frac{1}{AR} \delta B'' = 0 \quad \text{(III.18)} \]
Substitution of (III.14) into (III.18) gives:
\[ \delta u'' + \delta u = \left( -R \frac{Ik^2}{AR} \right) C_3 \sin ky + \left( -R \frac{Ik^2}{AR} \right) C_4 \cos ky \]

(III.19)

It is of interest to look at the order of magnitude of \( \frac{Ik^2}{AR} \). We have:

\[ \frac{I}{AR} = \frac{h^2}{12R} \]

\[ \frac{I}{AR} \frac{R}{R} = \frac{h^2}{12R^2} \]

In section (II.2-1) we have considered \( \frac{z}{R} \ll 1 \), or \( \frac{h}{R} \ll 1 \).

Then \( \frac{h^2}{R^2} \ll 1 \) which leads to

\[ \frac{I}{AR} \ll 1 \text{ or } \frac{Ik^2}{AR} \ll R, \text{ up to values of } k^2 \text{ equal to } R \].

Therefore, in general, we can determine \( \delta u \) using the equation:

\[ \delta u'' + \delta u = -R \left( C_3 \sin ky + C_4 \cos ky \right) \]

or

\[ \delta u'' + \delta u = -R \delta B \]

(III.20)
It must be noted that neglecting $\frac{Ik^2}{AR}$ before $R$ is to neglect the effect of extension of the centroidal axis. In other words, equation (III.20) is readily obtained by letting $\delta u = 0$ in equation (II.93), which leads to $\delta u = \delta w$. Therefore, the buckling which occurs for a thin ring is extensional, except possibly for very high modes.

The solution for (III.20) is:

$$\delta u = C_1 \sin y + C_2 \cos y - \frac{RC_3 \sin ky}{1-k^2} - \frac{RC_4 \cos ky}{1-k^2} \quad (III.21)$$

Let us determine $\delta w$. From (III.15b):

$$\delta w^t = - R \delta B - \delta u \quad (III.22)$$

But $- R \delta B = \delta u'' + \delta u$

Then:

$$\delta w^t = \delta u''$$

$$\delta w = \delta u^t + C_5 \quad (III.23)$$

$$w = C_1 \cos y - C_2 \sin y - \frac{RC_3 k \cos ky + RC_4 k \sin ky + C_5}{1 - k^2} \quad (III.24)$$

The boundary conditions for $\delta u$ and $\delta w$ are:

$$\delta u_y = \delta u_{y+2\pi}$$

$$\delta w_y = \delta w_{y+2\pi}$$

The constant $C_5$ represents the contribution to $u$ due to the extension of centroidal axis; as we are neglecting
this extension we set \( C_5 = 0 \); also, noting that the terms in \( \sin y \) and \( \cos y \), in (III.21) and (III.24) represent rigid body motion, we must drop them out, when considering deformations.

Now we summarize the results for the buckling of a thin circular ring:

\[
P_{ocr} = \frac{(k^2-1)EI}{R^3}
\]

\[
\delta_B = C_3 \sin ky + C_4 \cos ky
\]

\[
\delta_u = \frac{RC_3}{k^2-1} \sin ky + \frac{RC_4}{k^2-1} \cos ky
\]

\[
\delta_w = \frac{RC_3}{k^2-1} \cos ky - \frac{RC_4}{k^2-1} \sin ky
\]

For the smallest buckling pressure, \( k = 2 \), so that:

\[
\delta_B = C_3 \sin 2y + C_4 \cos 2y
\]

\[
\delta_u = \frac{3}{3} RC_3 \sin 2y + \frac{3}{3} RC_4 \sin 2y
\]

\[
\delta_w = \frac{2}{3} RC_3 \cos 2y - \frac{2}{3} RC_4 \sin 2y
\]

\[
P_{ocr} = \frac{2EI}{R^3}
\]

It is interesting to note that the buckled shape has two orthogonal axis of symmetry and one axis of anti-symmetry at \( 45^\circ \) from those axis. As a matter of fact, it
can be shown that \( w \) may be expressed as:

\[
\delta w = (C \cos 2\phi - D \sin 2\phi) \cos 2\psi
\]  

(III.26)

where

\[
C = \frac{2}{3} RC_3
\]

\[
D = -\frac{2}{3} RC_4
\]  

(III.27)

\[
\tan 2\phi = -\frac{D}{C}
\]

\[
y^* = y + \phi
\]

The existence of two orthogonal symmetry axis can be seen from (III.26), as well as that of the anti-symmetry axis.

### III.2 Buckling of Circular Arches

#### III.2-1 Arches with Clamped Edges Under Uniform External Pressure

Consider the arch shown in Figure III.2. The geometrical quantities defined for circular ring, in the previous sections, remain the same for the arch.

We will follow essentially the same method adopted to find the buckling pressure for the complete circular ring. In this case however, and in general, we don't know \( N, B, N' \) and \( B' \), necessary for the solution of the stability equations, as pointed out at the end of section II.2-3.

In order to determine \( N, B, N' \) and \( B' \) we use the
Figure III.2

Figure III.3
equilibrium equations, which maybe, for this purpose, linearized. Equations (II.81) to (II.83) become:

\[ N + Q + R_p = 0 \]
\[ N' - Q + R_q = 0 \]  \hspace{1cm} (III.28)
\[ M' = QR \]

The linearized strain displacement relations are:

\[ \xi = \frac{1}{R} (u' - w) \]  \hspace{1cm} (III.29)
\[ \gamma = e_2 + B = \frac{u' + w'}{R} + B = 0 \]  \hspace{1cm} (III.30)

The correspondent force displacement relations are, from (II.84) and (II.86):

\[ N = \frac{EA}{R} (u' - w) \]  \hspace{1cm} (III.31)
\[ M = \frac{EI}{R} B' \]  \hspace{1cm} (III.32)

Also:

\[ p = p_0 \]
\[ q = p_0 B \]

We will examine the possibility of no extension of the centroidal axis, that is, we will look for the existence of a solution for the case where \( u' - w = 0 \). In this case \( N \) can't be obtained from (III.31).

Differentiate the first of equations (III.28). We
get:
\[ N' + Q'' = 0 \]  \hspace{1cm} (III.33)

From the second of equations (III.28):
\[ N' = Q - Rq = Q - RpoB \]

Then:
\[ Q - RpoB = Q'' = 0 \]
But \( M' = QR \)

Then:
\[ \frac{M'}{R} - RpoB + \frac{M''}{R} = 0 \]  \hspace{1cm} (III.34)

Take into account (III.32):
\[ \frac{EI}{R^2} B'' - RpoB + \frac{EI}{R} B^{IV} = 0 \]  \hspace{1cm} (III.35)
\[ B^{IV} + B'' - K^2 B = 0 \]

where
\[ K^2 = \frac{poR^2}{EI} \]

It is seen that \( B = 0 \) is a solution for (III.35).

Besides, it satisfies the boundary conditions, which are:

a) \( B = 0 \) for \( y = \alpha \)
b) \( B(y) = -B(\alpha) \) \hspace{1cm} (symmetry about \( y = 0 \)).

So far, we haven't used the assumption \( u^t - w = 0 \). We will use it now.

\[ u^t - w = 0 \]  \hspace{1cm} (III.36)
\[ u + w^t = 0 \hspace{1cm} \text{, from (III.30), with } B = 0 \]
Then: \[ u'' + u = 0 \]  
\[ u = C \sin y + D \cos y \]  
(III.37)

Boundary conditions are:

a) \( u(y) = -u(-y) \) (symmetry about \( y = 0 \))

b) \( u(\alpha) = 0 \)

From (III.36): \( w = C \cos y - D \sin y \)  
(III.38)

Boundary conditions are:

a) \( w(y) = w(-y) \) (symmetry about \( y = 0 \))

b) \( w(\alpha) = 0 \)

This leads to: \( C = 0 \), \( D = 0 \)  
(III.39)

We have shown that the assumption of no extension of centroidal axis yields a solution which satisfies all the boundary conditions. Now we must investigate the conditions for which this assumption is adequate. In order to do this, substitute \( N = -pR \) in (III.31) and get:

\[ -pR = \frac{1}{EA} (u'' - w) = 0 \]  
(III.40)

Therefore, we must have:

\[ \frac{-pR}{EA} \lessgtr 0 \]  
or  
\[ \frac{-p}{R} \frac{R}{E} \lessgtr 0 \]  

or

\[ \frac{p}{E} \ll \frac{h}{R} \]  
(III.41)

It may seem, at a first glance, that the above condition is incompatible with the other we have imposed earlier, namely \( h \ll l \). However, considering that \( E \) is usually a high value, compared with common pressures, we
see that we have a wide range of values for \( p_0 \) which satisfies the two apparently incompatible conditions.

Now we can proceed and work with the stability equations. Note that the values of \( N, N', B \) and \( B' \) we just got are exactly those given by equations (III.1) for the circular ring. As a result, we can follow the same path we followed in section (III.1) and find:

\[
\delta B = C_1 + C_2y + C_3 \sin ky + C_4 \cos ky \quad (III.42)
\]

\[
k = k^2 + 1
\]

\[
K^2 = \frac{p_0 R^3}{EI}
\]

It has been observed that the lowest buckling mode is anti-symmetric. Therefore, we will assume anti-symmetry for the buckled shape. The following boundary conditions must be imposed on \( B \):

a) \( \delta B(y) = \delta B(-y) \)

b) \( \delta B(\alpha) = 0 \)

It follows that:

\[
C_3 = 0 \quad (III.43)
\]

\[
C_2 = 0 \quad (III.44)
\]

\[
C_1 = -C_4 \cos k \quad (III.44a)
\]

Let us determine \( \delta u \). In the same way as we did in section (III.1) we can get equation (III.18):

\[
\delta u'' + \frac{1}{AR} \delta B'' = 0
\]
Also, in the same way as in section (III.1),
we can conclude that, for \( \frac{k^2}{R^2} \ll 1: \)

\[
\delta u'' + \delta u = -R \delta B \quad \text{(equation (III.20)).}
\]

Also:

\[
\delta u = \delta w \quad \text{(III.45)}
\]

We must remark that the relations (II.20) and (II.45) can always be used, for circular arches and rings,
when \( \frac{k}{R} \ll 1, \ B = B^* = N^* = 0, \) and \( N = -poR. \)

In our present case we have:

\[
\delta u'' + \delta u = -R \delta B
\]

\[
\delta B = C_4 \cos ky + C_1 \quad \text{(III.46)}
\]

Then

\[
\delta u'' + \delta u = -RC_4 \cos ky - RC_1 \quad \text{(III.47)}
\]

The solution for this equation is:

\[
\delta u = a_1 \sin y + a_2 \cos y - \frac{RC_4 \cos ky - RC_1}{1-R^2}
\]

Boundary conditions are:

a) \( \delta u(y) = \delta u(-y) \)

b) \( \delta u(\alpha) = 0 \)

It follows that:

\[
a_1 = 0
\]

\[
a_2 = \frac{RC_4 \cos k\alpha + RC_1}{1-k^2 \cos \alpha} = \frac{RC_4 \cos k - RC_4 \cos k\alpha}{1-k^2 \cos \alpha} = \frac{RC_4 \cos k\alpha(1-1)}{1-k^2 \cos \alpha}
\]

\[
a_2 = \frac{RC_4 \cos k\alpha(1-1)}{\cos \alpha} = \frac{RC_4 \cos k\alpha(k^2)}{1-k^2 \cos \alpha} \quad \text{(III.48)}
\]
Then:
\[ \delta u = a_2 \cos y - \frac{RC_4 \cos ky - RC_1}{1-k^2} \]  

(III.49)

From (III.45):
\[ \delta w = \delta u^t = -a_2 \sin y + \frac{RC_4 k \sin ky}{1-k^2} \]

Boundary conditions are:

a) \( \delta w(y) = -\delta w(-y) \)
b) \( \delta w(\alpha) = 0 \)

It follows that:
\[ a_2 = + \frac{RC_4 k \sin k\alpha}{1-k^2} \frac{\cos \alpha}{\sin \alpha} \]  

(III.50)

From (III.48) and (III.50):
\[ C_4 \left( \frac{k^2}{1-k^2} \frac{\cos k\alpha}{\cos \alpha} - \frac{k}{1-k^2} \frac{\sin k\alpha}{\sin \alpha} \right) = 0 \]

For buckling, we must have:
\[ C_4 \neq 0 \]

Then:
\[ k \frac{\cos k\alpha}{\cos \alpha} - \frac{\sin k\alpha}{\sin \alpha} = 0 \]  

(III.51)

Multiplying through by \( \frac{\sin \alpha}{\sin k\alpha} \) we get:
\[ k \cot k\alpha \tan \alpha = +1 \]  

(III.52)
The lower value $k$ which satisfied equation (III.52) yields the buckling pressure:

$$P_{cr0} = \frac{(k^2-1) \frac{E}{R}}{R^3}$$  \hspace{1cm} (III.52a)

### III.2-2 Arches with Hinged Edges Under Uniform External Pressure

In dealing with this problem we can follow the same path taken in the previous section.

When determining the coefficients $N$, $B$, $N'$ and $B'$, for the stability equations, we note that the solution of the equilibrium equations for the clamped edges case still satisfies the present boundary conditions. Therefore, we still have:

- $N = -p_o R$
- $N' = 0$
- $B = 0$
- $B' = 0$

As a consequence, we get:

$$\delta B = C_1 + C_2 y + C_3 \sin k y + C_4 \cos k y \quad (III.53)$$

In the same way as in the previous sections, we look for anti-symmetric buckled shapes. The boundary conditions are, therefore:

- a) $\delta B(y) = \delta B(-y)$
- b) $\delta B' (\alpha) = 0$
Condition \( \frac{1}{r} \) imposes zero moments at the hinges.

It follows that:

\[
C_2 = C_3 = 0 \quad \text{(from the first boundary condition)}.
\]

\[-C_4 \sin k \alpha = 0 \quad \text{(from the second boundary condition).} \quad (III.54)\]

Buckling occurs when \( C_4 \neq 0 \). Therefore:

\[
\sin k \alpha = 0 \quad \text{or} \quad k = \pi, 2\pi, 3\pi, \ldots \quad (III.55)
\]

Buckling pressure is:

\[
P_{ocr} = \frac{k^2EI}{R^3} = (k^2 - 1) \frac{EI}{R^3}
\]

The lowest possible value for \( k^2 \) is:

\[
k^2 = \pi^2
\]

Therefore:

\[
P_{ocr} = (\pi^2 - 1) \frac{EI}{R^3} \quad (III.56)
\]

\[
\delta B = C_1 + C_4 \cos ky \quad (III.57)
\]

It remains to show that the above expression for \( \delta B \), when \( k \) is given by (III.55) yields displacements \( \delta u \) and \( \delta w \) which satisfies the boundary conditions.

We have:

\[
\delta u'' + \delta u = -R \delta B = -RC_1 - RC_4 \cos ky
\]

\[
\delta j = a_1 \sin y + a_2 \cos y - \frac{RC_4 \cos ky - RC_1}{1 - k^2}
\]

This is the same expression we got for the clamped edge case,
and the boundary conditions to be imposed are the same, which leads to:

\[ a_2 = \frac{RC_4}{1-k^2} \cos k\alpha + \frac{RC_1}{\cos \alpha} \]

\[ a_1 = 0 \]

Then:

\[ \delta u = a_2 \cos y - \frac{RC_4}{1-k^2} \cos ky - \frac{RC_1}{\cos \alpha} \]

For \( \delta w \) we have:

\[ \delta w = \delta u^* = -a_2 \sin y + \frac{RC_4}{1-k^2} k \sin ky \]

The boundary conditions are:

a) \( \delta w(y) = -\delta w(-y) \)

b) \( \delta w(\alpha) = 0 \)

Then

\[ a_2 = \frac{RC_4}{1-k^2} \frac{k \sin k\alpha}{\sin \alpha} \]

(III.58) and (III.59) give:

\[ \frac{RC_4}{1-k^2} \cos k\alpha + \frac{RC_1}{\cos \alpha} = \frac{RC_4}{1-k^2} \frac{k \sin k\alpha}{\sin \alpha} \]
In this case $k$ is already fixed by (III.55) but we still have $C_1$ available to satisfy (III.61). Therefore, every condition is satisfied and the buckling pressure is given by (III.56).

III.3 Buckling of Circular Rings Fitted with Struts

III.3-1 Preliminary Considerations

We consider struts hinged to the circular ring. As we already pointed out in the previous sections, in order to solve the stability equations we must first determine $N$, $N'$, $B$ and $B'$, using the equilibrium equations. We have done this for circular arches, assuming inextensibility of the centroidal axis. This approximation was shown to be reasonable when condition (III.41) is satisfied, together with the condition $\frac{h}{R} \ll 1$. When dealing with strutted rings we also will use this approximation which leads again to zero values of $N'$, $B$ and $B'$ and gives $-poR$ as the value for the force resultant $N$. It follows, as we have remarked in section III.2-1, that the relations (III.45) and (III.20) can be used, when $\frac{h}{R} \ll 1$.

Now let us see what conclusions, about the buckling of strutted rings, we can advance. We will use the studies we have done so far. Assume an arbitrarily strutted ring buckles. The struts may or may not be deformed.
If there is deformation of any strut, it will act with a concentrated force on the ring, causing a discontinuity in the force resultants, at the point where it meets the ring. This point we will call joint, from now on. The equations we derived don't allow for such concentrated forces so that, to handle the problem, we have to integrate the stability equations for each section between two joints. Also we must impose, as boundary conditions, the continuity of the displacements at the joints and the joint equilibrium. The joint equilibrium relates the force in the strut to the force resultants at each side of the joint. Therefore, as a result of this integration for each section, we will have different coefficients. These coefficients must be referred to the correspondent section. A second subscript will be used for this purpose so that, in the expression for $\delta B_i$, for example, we will have:

$$\delta B_i = C_{1i} + C_{2i} y + C_{3i} \sin k y + C_{4i} \cos k y,$$

the subscript $i$ being used to refer the quantities to the section $i$.

We will now show that there is a lower limit for the critical pressure of a strutted ring. This limit is equal to the buckling pressure of the same ring without struts (plain ring).

The existence of the lower limit can easily be drawn
by examining the problem under an energy point of view. In the buckling process there must be conservation of energy. Considering the process as adiabatic, the changes in energy come from the work of the loads and the work of deformation. For no change in energy, these two must balance each other. Now, consider any possible buckling of a strutted ring. It follows that the work done by the pressure forces, in the process, was equal to the work of deformation of the ring plus the struts. Therefore, if there were no struts or no deformation of the struts, a smaller pressure could bring the ring to the same configuration. This shows that the buckling pressure for the plane ring is always smaller than that for the strutted ring, being equal when struts are not deformed. We will see that this is the case when the struts are all parallel to each other.

III.3-2 Ring fitted with Several Struts, Parallel to Each Other

Consider a ring fitted with several struts parallel to each other. In Figure III.4 we represented just three struts but, from the discussion that will follow, it will be seen that the number of struts is arbitrary.

According to what we said in the previous section, the lowest limit for the buckling pressure of a strutted
Figure III. 4

Figure III. 5
ring is the buckling pressure for the same ring without any deformation of the struts, that is

\[
P_{ocr} > \frac{3EI}{R^3} \quad \text{(III.62)}
\]

Now we will show that, when all struts are parallel to each other, the lower limit is reached.

We have shown, in section III.1 that the buckled shape for a plain circular ring has one axis of anti-symmetry. Then, assume that our ring buckles at the lower limit given by equation (III.62) and that the anti-symmetry axis coincides with the reference axis. The assumption will be correct if no deformation of struts occur.

Take, for example, strut number 2, in Figure III.4. Supposing that the reference axis is an axis of anti-symmetry, we have:

\[
\begin{align*}
  u (\alpha_1) &= u_1 = u (\alpha_2) = u_2 = u \quad \text{(III.63)} \\
  w (\alpha_1) &= w_1 = -w (\alpha_2) = -w_2 = -w \quad \text{(III.64)}
\end{align*}
\]

Figure (III.5) shows the displacements for strut number 2, with a reference frame XY which has its center at the left end of the strut. Let \( l \) be the length after buckling. To these lengths we associate vectors \( \vec{l} \) and \( \vec{l}' \). The change in length then will be:

\[
\Delta l = |\vec{l}'| - |\vec{l}| \quad \text{(III.65)}
\]
Let
\[ \bar{l} = i \bar{x} \]
and
\[ \bar{l}' = i \bar{x}' + j \gamma' \] (III.66)

We see that
\[ X' = X \]
\[ Y' = 2 \delta u \sin \phi_1 + 2 \delta w \cos \phi_1 \] (III.67)

Therefore
\[ |\bar{F}'| = \sqrt{X^2 + (\sin^2 \phi_1 \delta u^2 + 2 \sin \phi_1 \cos \phi_1 \delta u \delta w + \delta w^2 \cos^2 \phi_1)} \]

Expanding:
\[ |\bar{F}'| = X_0 + \frac{1}{2X_0} \left( 4 \sin^2 \phi_1 \delta u^2 + 2 \sin \phi_1 \cos \phi_1 \delta u \delta w + \delta w^2 \cos^2 \phi_1 \right) \]
\[ \Delta l = \frac{1}{2X_0} \left( 4 \sin^2 \phi_1 \delta u^2 + 2 \sin \phi_1 \cos \phi_1 \delta u \delta w + \delta w^2 \cos^2 \phi_1 \right) \] (III.68)

We see that the change in length is of the second order in the displacements. As a consequence the forces with which the strut reacts to buckling are of second order in the displacements. The work of deformation of the strut will be of third order in the displacements and won't influence the buckling of the ring, since only first and second order terms in the displacements play role in the buckling process. Then, the ring buckles as if it had no struts (plain ring). Note that strut number 2 was taken arbitrarily and the same
reasoning applies to all struts, provided they are all parallel to each other.

III.3-3 Ring Fitted with Several Struts at Arbitrary Positions

We consider two cases:

a) buckling without deformation of the struts.

b) buckling with deformation of the struts.

a) buckling without deformation of the struts

In the previous section, we saw that, if all struts are parallel to each other the work of deformation of the struts is of third order in the displacements which makes it irrelevant to the buckling problem. The reciprocal is not taken as a truth, that is, the parallelism of the struts must not be taken as a necessary condition for negligible deformation of the struts. Therefore, even if the struts are not parallel, deformation of the struts may be negligible.

Now, assume we have a buckling mode for which the deformation of the struts is negligible. The ring buckled shape may be such that the work of deformation is very high. Then the lowest mode may be one where relevant deformation of the struts exists. Therefore, it must be kept in mind that the buckling pressure we are going to obtain is the minimum one for this kind of buckling. The more general case will be treated in the next section.
Consider the ring in Figure III.6 and take, for instance, strut B. We use a set of reference axis XY, with origin at one end of the strut, we are studying.

Let \( l \) be the initial length and \( l' \) the final length of the strut, and let us associate the vectors \( \vec{l} \) and \( \vec{l}' \) to the lengths \( l \) and \( l' \):

\[
\vec{l} = \vec{i} x \quad \text{(III.69)}
\]

\[
\vec{l}' = \vec{i} x' + j \gamma' \quad \text{(III.70)}
\]

It follows that:

\[
X' = X + \delta w_1 \sin\left(\frac{\alpha_1 - \alpha_2}{2}\right) + \delta w_2 \sin\left(\frac{\alpha_1 - \alpha_2}{2}\right) - \delta u_1 \cos\left(\frac{\alpha_1 - \alpha_2}{2}\right) + \delta u_2 \cos\left(\frac{\alpha_1 - \alpha_2}{2}\right), \quad \text{or}
\]

\[
X' = X + (\delta w_2 - \delta w_1) \sin\left(\frac{\alpha_1 - \alpha_2}{2}\right) + (\delta u_2 - \delta u_1) \cos\left(\frac{\alpha_1 - \alpha_2}{2}\right)
\]

\[
\text{(III.71)}
\]

We let:

\[
X' = X + m
\]

\[
\text{(III.72)}
\]

where \( m = (\delta w_2 - \delta w_1) \sin\left(\frac{\alpha_1 - \alpha_2}{2}\right) + (\delta u_2 - \delta u_1) \cos\left(\frac{\alpha_1 - \alpha_2}{2}\right)
\]

\[
\text{(III.73)}
\]

We also have:

\[
Y' = (\delta w_2 - \delta w_1) \cos\left(\frac{\alpha_1 - \alpha_2}{2}\right) + (\delta u_2 + \delta u_1) \sin\left(\frac{\alpha_2 - \alpha_1}{2}\right) = n
\]

\[
\text{III.74)}
\]
Figure III.6
Then:

$$|\tilde{t}| = \sqrt{x^2 + 2xm + m^2 + n^2}$$  \hspace{1cm} (III.75)

Developing in power series about \(n = 0, \ m = 0\), we get:

$$|\tilde{t}| = x + m + \text{higher order terms}$$  \hspace{1cm} (III.76)

The change in length is:

$$|\tilde{t}^1| - |\tilde{t}| = x + m - x = +m$$  \hspace{1cm} (III.77)

The condition of no strain in the struts may be expressed as:

$$m = 0$$  \hspace{1cm} (III.78)

Let

$$\frac{\alpha_1 - \alpha_2}{2} = \phi_1$$  \hspace{1cm} (III.79)

Therefore we must impose:

$$(\delta w_2 + \delta w_1) \sin \phi_1 + (\delta u_2 - \delta u_1) \cos \phi_1 = 0$$  \hspace{1cm} (III.79a)

For strut A an identical condition can be imposed:

$$(\delta w_3 + \delta w_4) \sin \phi_2 + (\delta u_3 - \delta u_4) \cos \phi_2 = 0$$  \hspace{1cm} (III.80)

where $$\phi_4 = \frac{\alpha_3 - \alpha_4}{2}$$  \hspace{1cm} (III.81)

The \(\delta w\) and \(\delta u\) are given by expressions (III.25).

Substitution in (II.79) and (II.80) gives:
The stability condition is that $C_3$ and $C_4$ are not all zero. Therefore, it can be written as:

$$
egin{vmatrix}
-k(\cos k\alpha_4 + \cos k\alpha_3) \sin \phi_2 + k(\sin k\alpha_4 + \sin k\alpha_3) \sin \phi_2^+ \\
-(\sin k\alpha_4 - \sin k\alpha_3) \cos \phi_2 - (\cos k\alpha_4 - \cos k\alpha_3) \cos \phi_2^+ \\
-k(\cos k\alpha_2 + \cos k\alpha_1) \sin \phi_1^+ + k(\sin k\alpha_2 + \sin k\alpha_1) \sin \phi_1^+
\end{vmatrix} = 0
$$

(III.82)
The buckling pressure is:

\[ p_{cr} = \frac{K^2EI}{R^3} \]

where \( K^2 = k^2 - 1 \) is the smallest value yielded by the equation above.

In the case of a number \( n \) of struts we would get a determinant corresponding to \( \text{III.82} \) with dimensions \( n \) by \( n \).

b) buckling with deformation of the struts

As we have already seen in section III.3-1, the deformation for this case is not denoted by the same expression along the whole ring, but rather by a different expression for each buckled ring piece, between two consecutive struts. One such piece will be referred, from now on, simply as a piece. Since the stability equations are the same, the expressions for the displacements are, for a piece \( i \) of ring:

\[ \delta B_i = C_{1,i} + C_{2,i} y + C_{3,i} \sin ky + C_{4,i} \cos ky \]

(III.83)

\[ \delta u_i = C_{1,i} \sin y + C_{2,i} \cos y \frac{\gamma - RC_{2,i}}{1-k^2} y - \frac{RC_{3,i}}{1-k^2} \sin ky \frac{\gamma - RC_{2,i}}{1-k^2} \]

(III.84)

\[ \delta w_i = C_{1,i} \cos y - C_{2,i} \sin y \frac{-RC_{2,i}}{1-k^2} - \frac{RC_{3,i}}{1-k^2} k \cos ky \]

(III.85)

\[ + RC_{4,i} \frac{k \sin ky}{1-k^2} \]
Now we must look at the boundary conditions of the problem. As noted in section (III.3-1) we must require the continuity of displacements at the joints and the joint equilibrium.

Note that this procedure takes into account the cases of relevant deformations of all struts, some of the struts, or none of the struts. Then the buckling pressure it will yield will be the minimum one.

In the previous section we got the stability equations for no deformation of struts (Equations III.80a, III.81a). We will see that, for the same number of struts, the number of stability equations is doubled, so that equations (III.80a) and (III.81a) are more convenient when we are just looking for that kind of buckling.

We will look first at the joint equilibrium. Let us now establish subscript conventions necessary to the distinction of the several parameters which will appear. Consider Figure III.7. Choose a direction (clockwise, or anti-clockwise) and give numbers to the joints and the pieces, so that a piece i is between joints i-1 and i. Designate each strut by $S_{ij}$, where i and j are the joints at its extremities. $F_{ij}$ is the force from strut $S_{ij}$ acting on the ring (positive where strut is compressed).

Now, consider the equilibrium of a joint element Figure III.8. Let us designate any force resultant by
two subscripts, the first referring to the joint and the second to the piece of ring.

Equilibrium of the joint in the radical and tangential directions yields respectively:

\[
F_{ij} \sin \left( \frac{\alpha_j - \alpha_i}{2} \right) - \delta Q_{i,i+1} + \delta Q_{i,i} = 0 \quad (III.86)
\]

\[
F_{ij} \cos \left( \frac{\alpha_j - \alpha_i}{2} \right) + \delta N_{i,i} - \delta N_{i,i+1} = 0 \quad (III.87)
\]

\[
\delta M_{i,i+1} = \delta M_{i,i} \quad (III.88)
\]

Note that these equilibrium equations neglect the rotation \( \delta B \) of the section, since this would just add second order terms in the displacements.

Equations III.86-88 will give us three boundary conditions in the displacements, for each strut, after substitution of forces in terms of displacements. If we have \( n \) struts, we will get \( 3n \) such conditions.

Now, let us impose the continuity of displacements. We must have:

\[
\delta B_{i,i} = \delta B_{i,i+1} \quad (III.89a)
\]

\[
\delta u_{i,i} = \delta u_{i,i+1} \quad (III.89b)
\]

\[
\delta w_{i,i} = \delta w_{i,i+1} \quad (III.89c)
\]
Now look at equations (III.83) to (II.85) and note that the constant $C_{11}$ is the contribution of a rigid body rotation; then we will drop it when considering deformation. We will also do the same thing with the coefficients $C_{11}$ and $C_{21}$ which represent rigid body translation.

Substitution of the displacements by its expressions, in equations (III.89) leads to:

\[
C_{2,i} \alpha_i + C_{3,i} \sin k\alpha_i + C_{4,i} \cos k\alpha_i = C_{2,i+1} \alpha_i
\]

\[
- C_{3,i+1} \sin k\alpha_i - C_{4,i+1} \cos k\alpha_i = 0
\]

(III.90a)

\[
\frac{C_{3,i}}{1-k^2} \alpha_i + \frac{C_{3,i}}{1-k^2} \sin k\alpha_i + \frac{C_{4,i}}{1-k^2} \cos k\alpha_i = C_{2,i+1} \alpha_i
\]

\[
- \frac{C_{3,i+1}}{1-k^2} \sin k\alpha_i - \frac{C_{4,i+1}}{1-k^2} \cos k\alpha_i = 0
\]

(III.90b)

\[
-\frac{C_{2,i}}{1-k^2} \kappa \cos k\alpha_i + \frac{C_{3,i}}{1-k^2} \kappa \sin k\alpha_i + \frac{C_{2,i+1} + C_{3,i+1}}{1-k^2} \kappa \cos k\alpha_i = 0
\]

(III.90c)

Now we must express equations (III.86) to (III.88) in terms of displacements.
From (III.89) and (III.91) we have:

$$\delta Q_{i,i} = \frac{EI}{R^2} \delta B^y_{i,i}$$  \hspace{2cm} \text{(III.91)}$$

$$\delta M_{i,i} = \frac{EI}{R} \delta B^o_{i,i}$$  \hspace{2cm} \text{(III.92)}$$

From (III.5):

$$\delta N_{i,i} = p_0 R \delta B^o_{i,i} - \frac{EI}{R^2} \delta B^y_{i,i}$$  \hspace{2cm} \text{(III.93)}$$

On the other hand:

$$F_{ij} = E \frac{\Delta l_{ij}}{l_{ij}} A_{ij}$$  \hspace{2cm} \text{(III.94)}$$

$A_{ij} =$ cross-sectional area of strut $S_{ij}$.  

$l_{ij} =$ length of strut $S_{ij}$.  

$\Delta l_{ij} =$ change in length

Let us define:

$$\varphi_{ij} = \frac{\alpha_i - \alpha_j}{2}$$  \hspace{2cm} \text{(III.95)}$$

Using equations (III.95), (III.77) and (III.73), equations (III.94) becomes, using our convention:

$$F_{ij} = \frac{EA_{ij}}{l_{ij}} \left[ (\delta w_{j,j} + \delta w_{i,i}) \sin \varphi_{ij} + (\delta u_{j,j} - \delta u_{i,i}) \cos \varphi_{ij} \right]$$  \hspace{2cm} \text{(III.96)}$$
Substitution in equations (III.86) will give us:

\[
\begin{align*}
\frac{EA_{ij}}{l_{ij}} \left( \delta w_{ij,j} + \delta w_{i+1,j} \right) \sin \varphi_{ij} + \left( \delta u_{ij,j} - \delta u_{i,j} \right) \cos \varphi_{ij} \\
-\sin \varphi_{ij} - \frac{EI}{R^2} \left[ \frac{\delta B_{ij,i+1}}{l_{ij}} - \delta B_{ij,i} \right] = 0
\end{align*}
\]

(III.97a)

\[
\begin{align*}
\frac{EA_{ij}}{l_{ij}} \left( \delta w_{ij,j} + \delta w_{i+1,j} \right) \sin \varphi_{ij} + \left( \delta u_{ij,j} - \delta u_{i,j} \right) \cos \varphi_{ij} \\
\cos \varphi_{ij} - P_0 R \left[ \delta B_{ij,i} - \delta B_{ij,i+1} \right] - \frac{EI}{R^2} \left[ \frac{\delta B_{ij,i}}{l_{ij}} - \delta B_{ij,i+1} \right] = 0
\end{align*}
\]

(III.97b)

\[
\delta B_{ij,i+1} - \delta B_{ij,i} = 0
\]

(III.97c)

Equations (III.90 and (III.97) constitute a set of 6 homogeneous linear equations. If we will have \( n \) struts, we will write such equations for \( 2n \) joints. In this way, we will have \( 12xn \) homogeneous linear equations. At the same time we will have \( 2n \) pieces, each piece involving six coefficients of displacements. Therefore we end up with \( 12xn \) linear homogeneous equations and twelve unknowns. The stability condition is that the determinant of this system be zero. The lowest value of \( k \) which satisfies this condition gives the critical pressure:

\[
P_{ocr} = (k^2 - 1) \frac{EI}{R^3}
\]
CHAPTER IV

RECOMMENDATIONS FOR FUTURE WORK

In section 1.2 we stated that one of our objectives was to develop a method which could be employed for the solution of other buckling problems which we didn't consider here. These cases are, primarily, those where the geometry is not that of a circumference, and the load is not a uniform pressure.

The first step to broaden the scope of the method we used, is to develop the equilibrium equations for any planar member, instead of doing it for a circular member. This can be done following the same path we took in this work. The main difference will be that the scale factors will have to be kept general, throughout the development, which fact won't permit certain simplifications we had.

After having written equilibrium equations for a general plane member, stability equations will follow, in the same way we had them.

If we allow for any load, even if we have a circular
ring, the expressions for \( N, N', B \) and \( B' \) will be, in general, functions of the curvilinear coordinate \( y \). They can be yielded by the linearized equilibrium equations, as in this work. However, the dependence of \( N, N', B \) and \( B' \) or \( y \), will make the coefficients of the stability equations functions of \( y \). Therefore, the stability equations won't present any longer constant coefficients. As a result, the solution of the buckling problem will call for numerical methods of integration and possibly the use of computers.

Experimental work is also needed, as a verification of the results obtained in section III.3. When performing experiments, the imperfections in circularity should be reduced to a possible minimum, since we didn't allow for them, in this work. A theoretical study of such imperfections on the results we got in section III.3 is also recommended.

2. Buckling Strength of Metal Structures, by Friederich Bleich and Cdr. Leyle B. Ramsey (USN).


DEFINITION OF SYMBOLS

R  radius of curvature of reference axis of undeformed member. Radius of centroidal axis of undeformed circular ring.

R' radius of curvature of deformed reference axis of a member.

y  curvilinear coordinate along reference axis.

z  coordinate along the principal normal to the reference axis.

u  displacement, along y coordinate, for a point on the reference axis.

w  displacement, along z coordinate, for a point on the reference axis.

B  rotation angle of the normal to the reference axis, in the tangential direction.

\[ |K| = \frac{1}{R} \frac{\partial B}{\partial y} \]

\( \varepsilon \)  extensional strain for an element in the y direction, on the reference axis.

\( \gamma \)  transverse shear strain, at the reference axis.

\( \varepsilon_z \)  extensional strain for an element parallel to the reference axis at a distance z from the axis, along the z coordinate.
transverse shear strain, at a point of coordinates $y$ and $z$.

$N$ force resultant of extensional stresses, over the cross section, in the positive direction of the $y$ coordinate line.

$Q$ force resultant of shear stresses over the cross section in the positive direction of the $z$ coordinate line.

$M$ couple resultant of extensional stresses, over the cross section, computed at the reference axis, in the positive direction of the vector $\mathbf{\tau} \times \mathbf{n}$.

$\mathbf{\tau}$ unit vector in the positive direction of the undeformed $y$ coordinate line.

$\mathbf{\tau}'$ unit vector in the positive direction of the deformed $y$ coordinate line.

$\mathbf{n}$ unit vector in the positive direction of the undeformed $z$ coordinate line.

$\mathbf{n}'$ unit vector in the positive direction of the deformed $z$ coordinate line.

$p_0$ uniform external pressure.

$p$ component of external force along $\mathbf{\tau}$.

$q$ component of external pressure along $\mathbf{n}$.

$I$ moment of inertia of unit width cross section about centroidal axis.

$A$ cross sectional area of ring.
$A_{ij}$ cross sectional area of a strut between joints $i$ and $j$.

$E$ Young modulus.

$G$ transverse shear strain modulus $= \frac{E}{2(1+\nu)}$

$\nu$ Poisson's ratio.

$K = \sqrt{\frac{p_0 R^3}{EI}}$

$k^2 = K^2 + 1$

$\alpha$ scale factor along curvilinear coordinate $y$, for a point on the reference axis; also a constant angle.

$\phi$ rotation angle, in the direction of the principal normal, of an element on the reference axis.

$\phi_z$ rotation angle, in the direction of the principal normal, of an element parallel to the reference axis, at a distance from this axis.

$h$ distance between inner and outer circumferences of the ring.