A Feasibility Analysis of Single-Sensor Active Noise Cancellation in the Interior of an Automobile

by

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B.E., Vanderbilt University (1992)

Submitted to the
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Abstract

The problem of single-sensor active noise cancellation (ANC) within the interior of an automobile is explored, using spectral analysis of experimental data and computer simulations based upon Linear Quadratic Gaussian (LQG) optimal control. Simulations are constrained to work within the existing acoustic environment of an automobile; one of the car's stereo speakers, for example, is designated the cancelling loudspeaker. After incorporating stochastic models for the unwanted road noise inside an automobile, and for the speaker-to-microphone transfer function, the design procedure formulates an infinite-horizon optimal LQG regulator, which generates a cancelling signal based upon a two-step process: Bayes' Linear Least Squares (BLSE) state estimation, followed by a linear control law.

A number of experiments are conducted to assess the predictability of road noise. Next, the effectiveness of the LQG formulation for ANC within an automobile is analyzed by determining the maximum attenuation in overall noise power for a given speaker-microphone pair, and by spectral analysis of the original road noise and the residual error signal. Extensions of the basic algorithm are proposed to address the issues of large plant delays characteristic within an automobile, as well as the perceived effectiveness of the algorithm to a passenger seated near the error sensor.

Thesis Supervisor: Alan V. Oppenheim
Title: Distinguished Professor of Electrical Engineering
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Chapter 1

Introduction

Acoustic noise, which is widely recognized as a source of environmental pollution, is an unwanted byproduct of numerous industrial and transportation processes and systems. Researchers and engineers have devoted considerable effort to designing both passive and, more recently, active systems for noise attenuation. Active noise cancellation (ANC) is an approach to noise reduction in which an acoustic field is generated to interfere destructively with the unwanted noise field measured at one or more pressure sensors.

Passive silencers, in the form of low-pass acoustical filters, are primarily limited in their effectiveness to high frequency noise attenuation, since the wavelengths of low-frequency acoustic disturbances—typically on the order of several meters—are much larger than the thickness of filter materials. To achieve low-frequency noise reduction, researchers have recently devoted considerable attention to active noise cancellation strategies.

Almost all existing active noise cancellation systems are “pointwise” in nature; the ANC controller is designed to minimize some performance index involving the acoustic noise pressure measured at a finite number of spatial locations [1]. As we discuss in Section 1.1, pointwise active noise cancellation systems may generally be classified as either feedforward systems, in which there is no feedback from the “plant” output to the inputs of the controller, or feedback systems, in which all controller inputs contain a part of the plant output. Section 1.1 and Chapter 2 elaborate on the notion of “plant” as it applies to the interior of an automobile.

The goal of this thesis is to assess the feasibility of performing pointwise active noise
cancellation within the interior of an automobile, without modifying the automobile's existing acoustic environment. We will focus upon a particular feedback ANC controller derived from linear quadratic optimal control, the infinite-horizon optimal Linear Quadratic Gaussian (LQG) regulator. In our feedback configuration, a single noise pressure sensor is mounted within the automobile, near the likely location of a passenger's ears, to estimate both the original road noise and the cancelled noise field, and one of the stereo speakers in the automobile is used to broadcast the cancelling signal generated by the controller. Our experiments will consist of computer simulations, in which we use recordings of road noise within an automobile, as well as models based upon our measurements of the acoustic system from the input of the cancelling speaker to the input of the error microphone inside the automobile.

The LQG formulation which we have chosen for our ANC simulations has several significant advantages over previous feedback and feedforward ANC schemes. Unlike earlier feedback ANC formulations, the regulator design fully exploits the statistical characteristics of the noise [1]. In addition, the formulation will work, albeit with inherent performance limitations, even when the transfer function has non-minimum phase zeros; in contrast, the system outlined in [2] assumes the speaker-to-microphone transfer function to be minimum-phase. The feedback formulation also sidesteps two problems inherent in feedforward ANC systems, which require at least one input microphone to measure the original noise field in addition to the error microphone measuring the residual signal: destabilizing effects of feedback from the cancelling speaker to the input microphone; and difficulties in acquiring an appropriate reference signal, a particularly vexing problem when the acoustic noise, such as that within an automobile, is omnidirectional [3].

1.1 Previous Work in Active Noise Cancellation

1.1.1 Feedforward ANC

A representative pointwise feedforward ANC controller is depicted in Figure 1-1, in which \( m(t) \) is the input to the controller, \( r(t) \) is the cancelling signal generated by the controller, and \( n(t) \) is the unwanted acoustic noise. The "plant" in this design encompasses the entire
acoustic system which alters a signal as it passes from the output of the controller to the spatial location at which noise attenuation is desired, including the transfer function of the cancelling loudspeaker as well as the geometry of the acoustic enclosure containing the loudspeaker and error microphone. The goal of the controller is to generate \( r(t) \) so as to minimize some performance index involving \( e(t) = n(t) + c(t) \), the residual error signal measured by the error microphone.

![Diagram of Feedforward ANC](image)

Figure 1-1: A generic feedforward active noise cancellation system.

Much of the early work in active noise cancellation focused on unidirectional duct noise, for which the feedforward implementation is particularly well suited [4, 5]. These designs typically rely upon several sensors: at least one input microphone to measure \( m(t) \), the acoustic pressure upstream of the location in the duct where attenuation is desired; as well as an error microphone to measure the residual error signal \( e(t) \) at the cancellation location.

Feedforward designs may also be employed when the acoustic noise is not unidirectional, so long as the reference signal \( m(t) \) is correlated with the unwanted noise, \( n(t) \). The adaptive, discrete-time finite impulse response (FIR) controller proposed by Burgess, for example, does not require unidirectional noise [5]. Nevertheless, feedforward ANC systems must address two potential limitations in practical applications: obtaining an appropriate reference signal, particularly when the acoustic noise is omnidirectional; and minimizing feedback from the cancelling speaker to the input microphones. To address the second problem, which may lead to instability in the feedforward ANC system, researchers often utilize a configuration of cancelling speakers which minimizes feedback, or an adaptive filter which accounts for feedback in the production of the cancelling signal [5, 6].
1.1.2 Feedback ANC

A generic feedback controller for performing pointwise ANC is depicted in Figure 1-2, in which the controller input, \( e(t) \), is the sum of the plant output \( c(t) \) and the unwanted acoustic disturbance \( n(t) \). In contrast to feedforward designs, feedback controllers are characterized by inputs which contain a part due to the plant output, \( c(t) \), and by error microphones which are used to estimate both the original acoustic noise and the residual error signal. The goal is to generate the cancelling signal \( r(t) \) based upon past and present observations of \( e(t) \), so as to minimize some performance criterion based upon the residual signal.

![Feedback ANC Controller diagram](image)

Figure 1-2: A feedback controller for pointwise active noise cancellation.

A significant advantage of feedback ANC controllers, particularly for omnidirectional noise sources, is that they sidestep the problems of obtaining an appropriate reference signal and minimizing feedback from the cancelling speaker to the input microphones. Moreover, feedback controllers often incorporate the statistics of the unwanted acoustic noise in order to enhance attenuation. Graupe, for example, proposed a discrete-time Box-Jenkins controller for which he claimed 54 dB of compressor noise cancellation in computer simulations [2, 1]. The simulations were unrealistic, however, in that they assumed minimum phase plant dynamics as well as zero propagation delay between the loudspeaker and the error microphone.

More recently, feedback ANC designs have drawn heavily upon results from modern control literature. Zangi proposed an optimal controller derived from linear quadratic optimal
control theory which achieved considerable success in noise cancelling headphones designed to attenuate aircraft cabin noise [1]. In this thesis we apply the formulation, which incorporates an infinite-horizon optimal LQG regulator, to attenuate road noise within the interior of an automobile in computer simulations.

In subsequent chapters we show that the plant dynamics play a crucial role in determining the degree of noise attenuation achievable by a feedback ANC controller. In contrast to the acoustic environment within a headphone, the typical speaker-to-microphone transfer function within an automobile is characterized by large propagation delays, small magnitudes over some of the frequencies in which much of the road noise power is concentrated, and non-minimum phase zeros. All of these factors, which tend to inhibit the controller's ability to attenuate noise, make active noise cancellation within an automobile a difficult problem open to future research.

1.2 Thesis Organization and Development

In Chapter 2, we review the development and application of the infinite-horizon optimal LQG regulator to the problem of pointwise active noise cancellation within an automobile. After introducing a stochastic model for the observed road noise and a nominal pole-zero model for the plant dynamics between the cancelling speaker and the error microphone, for which uncertainty is modelled as additive Gaussian noise at the system states, we reformulate the linear system model into an LQG regulator and review why the optimal feedback controller exhibits a separation structure characterized by state estimation followed by a deterministic control law. Through a frequency domain analysis, we gain insight into factors which affect the performance of the feedback controller, such as the predictability of road noise and the presence of non-minimum phase zeros in the speaker-to-microphone transfer function. Additionally, we offer extensions of the algorithm to deal with the large plant delays characteristic within an automobile as well as the perceived loudness of sound as a function of frequency.

Chapter 3 focuses upon the process of obtaining accurate models for the road noise and the speaker-to-microphone transfer functions within an automobile. We begin by de-
scribing the physical experiments in which we measured road noise inside an automobile while driving, followed by each of the speaker-to-microphone transfer functions while the vehicle was stationary. Next, we obtain autoregressive (AR) stochastic models to describe the road noise over limited time intervals, and employ autoregressive moving average with exogenous input (ARMAX) modelling to obtain nominal models for the transfer functions. Using spectral properties of the data and our models, we show that road noise within an automobile is indeed highly predictable, but that the typical transfer function inside an automobile tends to inhibit active noise cancellation.

In Chapter 4, we analyze the results of our computer simulations, which were designed to determine the feasibility of single-sensor active noise cancellation within the interior of an automobile, without modifying the existing acoustic environment. Our first simulation focuses upon the predictability of road noise within an automobile using a non-adaptive feedback controller. Further experiments, which incorporate our models for the speaker-to-microphone transfer functions, simulate ANC within an automobile using the infinite-horizon optimal LQG regulator and one of the automobile's stereo speakers to broadcast the cancelling signal. The simulations show that although road noise is highly predictable, particularly for prediction times under 2.1 ms, the success of a feedback controller in performing active noise cancellation depends heavily upon the speaker-to-microphone transfer function, not the speaker-to-microphone separation distance. Furthermore, although the LQG controller is able to achieve attenuation in the overall noise road noise power, particularly at the low frequencies where much of the power of road noise is concentrated, high frequency noise, which may be perceived as more annoying to a listener, is generally amplified. Our last set of simulations employ the frequency-weighted LQG formulation to address the perceived effectiveness of the algorithm to a passenger seated within the automobile.

Finally, Chapter 5 briefly summarizes the work presented in this thesis, advances conclusions regarding the most significant experimental results, and gives directions for future research in the area of active noise cancellation in the interior of an automobile.
Chapter 2

Applying LQG Optimal Control to Pointwise Active Noise Cancellation

To address our inquiries regarding the feasibility of active noise cancellation within the interior of an automobile, we adopted the discrete-time, pointwise optimal control formulation originally proposed by Zangi and implemented in active noise cancelling headphones to attenuate aircraft cabin noise [1]. Using nonrestrictive modelling assumptions, this procedure creates a multiple-input single-output (MISO) feedback ANC controller to attenuate the acoustic pressure at a finite number of error microphones using a single cancelling speaker.

In this chapter we review the single-input single-output (SISO) controller formulation. The error microphone is mounted where noise attenuation is desired, near the probable location of a passenger's head, and one of the automobile's speakers is designated the cancelling speaker. Given linear stochastic models for the observed road noise, the transfer function measured from the input of the car speaker to the input of the error microphone, and the error microphone itself, the procedure formulates an optimal Linear Quadratic Gaussian Regulator. This controller minimizes a performance criterion involving the mean-square sum of the residual signal at the error microphone and the mean-square sum of the control signal.
The following sections outline two fundamental stages in designing the feedback ANC controller: linear stochastic modelling of the acoustic environment, followed by reformulation of the closed-loop system model into an optimal LQG regulator for ANC. Specifically, Section 2.1 models the residual signal at the error microphone as the sum of the outputs of two linear stochastic systems. An autoregressive moving-average (ARMA) process driven by white Gaussian noise models the observed road noise; and a nominal pole-zero system function, with uncertainty characterized by additive white Gaussian noise at the system states, models the transfer function from the input of the cancelling speaker to the input of the error microphone. Section 2.2 reformulates our system model into a linear quadratic regulator, and reviews why the optimal feedback controller exhibits a separation structure characterized by state estimation followed by a deterministic control law. A Kalman filter estimates the system state, and the control gain is computed by solving an algebraic Riccati equation.

Additional insight into the performance of our feedback controller, including its ability to exploit statistical properties of the noise and performance limitations caused by the speaker-to-microphone transfer function, is given in the frequency domain analysis of Section 2.3. Section 2.4 develops extensions to the basic algorithm to address issues of large transfer function delays and perceived loudness of sound as a function of frequency.

2.1 Modelling the Single-Sensor ANC Problem

The physical system we must model inside the automobile is the single-microphone, single-speaker active noise cancellation system depicted in Figure 2-1. The microphone is placed at the location where noise attenuation is desired, typically near the probable location of a passenger’s head, and the cancellation speaker is simply one of the automobile’s stereo speakers. The acoustic objective of the system is to minimize the sound pressure at the error microphone by broadcasting a secondary sound field from the car speaker which interferes destructively with the primary noise field measured at the microphone. Through the remainder of the chapter we will designate the unwanted ambient noise at the error microphone \( n(t) \), the control signal generated by the ANC algorithm \( r(t) \), the cancelling
signal immediately outside the error microphone $c(t)$, the total acoustic pressure measured by the microphone $e(t) = c(t) + n(t)$, and the microphone output $m(t)$.

![Diagram of ANC system](image)

Figure 2-1: Single microphone feedback ANC system within an automobile.

Three of the physical systems depicted in Figure 2-1 require modelling before we may present an ANC algorithm: the microphone transfer function; the ambient noise $n(t)$; and the transfer function from the input of the automobile speaker to the input of the error microphone, which governs the input/output relationship between $r(t)$ and $c(t)$. In each case, our approximation of the physical system will be a nominal linear system having uncertainty arising from independent, identically distributed (iid.) Gaussian random variables.

We assume that the output of the error microphone is a direct feedthrough of the input, plus additive zero-mean white Gaussian measurement noise, $v(t)$, having variance $\sigma_v^2$. This approximation is reasonable for most practical pressure microphones at frequencies below 1 kHz [7]; furthermore, we demonstrate in Chapter 3 that nearly all of the power of road noise is concentrated in frequency bands below 400 Hz.

Our model for the unwanted acoustic noise within the interior of the automobile, $n(t)$, is an ARMA stochastic process. At time "$t$," therefore, we assume that the road noise is the output of a stable linear filter driven by a zero-mean white Gaussian process $w(t)$ having covariance $\sigma_w^2 \delta(t - \tau)$:

$$n(t) = -\sum_{k=1}^{q} \alpha_k n(t - k) + \sum_{j=1}^{p} \beta_k w(t - j) \quad p \leq q, \quad (2.1)$$
\[ E\{w(t)w(\tau)\} = \sigma_w^2 \delta(t - \tau). \]

Equivalently, the state space representation for the disturbance \( n(t) \), in observability canonical form, is:

\[
x_d(t + 1) = \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & 0 & \cdots & 0 \\ -\alpha_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{q-1} & 0 & 0 & 0 & \cdots & 1 \\ -\alpha_q & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} x_d(t) + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} w(t) \quad (2.2)
\]

\[
n(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x_d(t), \quad (2.3)
\]

where \( x_d(t) \) is the \( q \times 1 \) state vector and \( \Phi_d \) is the \( q \times q \) state transition matrix. The dimensions of the matrices \( G_d \) and \( H_d \) are \( q \times 1 \) and \( 1 \times q \), respectively. In subsequent figures, we denote the noise system function:

\[
\Gamma(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\sum_{j=1}^{p} \beta_j}{1 + \sum_{k=1}^{q} \alpha_k},
\]

(2.4)

Since the road noise and the microphone are independent physical systems, we will assume that \( w(t) \) is uncorrelated with \( v(t) \), the microphone measurement noise. Furthermore, in Chapter 3 we demonstrate that a 5th order autoregressive (AR) process in fact accurately describes \( n(t) \).

Next, we designate the overall transfer function from the input of the cancelling loudspeaker to the input of the error microphone as \( G(z) \), and refer to \( G(z) \) as the “plant,” following standard terminology from control literature [8, 9, 10]. Our model for the plant is the nominal pole-zero system function \( G_0(z) \),

\[
G_0(z) = \frac{B_0(z)}{A_0(z)} = \frac{\sum_{j=1}^{m} b_k z^{-j}}{1 + \sum_{k=1}^{n} a_k z^{-k}} \quad m \leq n, \quad (2.5)
\]

22
plus uncertainty arising from an additive white Gaussian random vector at the system states [1, 11]. The corresponding observability canonical state space representation for \( G(z) \) is therefore:

\[
x_p(t + 1) = \begin{bmatrix}
-a_1 & 1 & 0 & 0 & \cdots & 0 \\
-a_2 & 0 & 1 & 0 & \ddots & 0 \\
-a_3 & 0 & 0 & 1 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
-a_{n-1} & 0 & 0 & 0 & \ddots & 1 \\
-a_n & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_p(t) \\
\end{bmatrix}
\begin{bmatrix}
\sigma_\eta & 0 \\
\sigma_\eta & \ddots \\
0 & \ddots & \sigma_\eta \\
\end{bmatrix}
\begin{bmatrix}
\eta(t) \\
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_m \\
0 \\
0 \\
\end{bmatrix} r(t)
\]

(2.6)

\[
c(t) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_p(t) \\
\end{bmatrix}.
\]

(2.7)

where \( x_p(t) \) is the \( n \times 1 \) state vector for the plant, \( \Phi_p \) is the \( n \times n \) state transition matrix, and \( \eta(t) \) is an \( n \times 1 \) zero-mean Gaussian random process having covariance function

\[
E \{ \eta(t) \eta^T(\tau) \} = \delta(t - \tau) I_{n \times n}.
\]

The dimensions of the matrices \( G_p, B_p, \) and \( H_p \) are \( q \times q, q \times 1, \) and \( 1 \times q, \) respectively.

Again, since the plant is independent of both the error microphone and the processes generating the unwanted road noise, we assume that \( \eta(t) \) is uncorrelated with \( u(t) \) and \( w(t) \).

After replacing the physical structures of Figure 2-1, which are in general nonlinear, with our linear stochastic models, we may analyze the single-microphone feedback ANC system depicted in Figure 2-2. The feedback controller must solve a standard disturbance rejection problem: attenuating the power of \( e(t) \), the residual error signal outside the microphone.
2.2 Incorporating LQG Optimal Control

The controller which we chose to assess the feasibility of active noise cancellation within an automobile is an LQG controller, capable of driving the output of the plant \( G(z) \) along the desired trajectory \( E \{-n(t) \mid t = 0, 1, \ldots, N, \ldots\} \) in some optimal sense. Specifically, the controller must solve an optimal stochastic servo problem by minimizing the general quadratic performance index:

\[
J_{LQT}(N) = E \left\{ \sum_{t=0}^{N-1} \left\{ r^T(t) R(t) r(t) + e^T(t) Q(t) e(t) \right\} + e^T(N) R(N) e(N) \right\} \tag{2.8}
\]

where \( R(t) \) is positive definite and \( Q(t) \) is positive semidefinite.

The standard technique for obtaining an explicit solution to an optimal tracking problem, such as the stochastic servo configuration depicted in Figure 2-2, is to reformulate the problem into an optimal regulator by augmenting the plant [8, 9, 10]. In the deterministic case, the qualitative goal of an optimal regulator is to drive the plant from an unknown initial state to the zero state while balancing appropriate penalties for control cost and state excitation costs. For a stochastic plant, as depicted in the feedback configuration of Figure

---

Figure 2-2: Block diagram of the single-microphone feedback ANC system within an automobile.
2-3, the optimal LQ regulator minimizes the quadratic performance criterion:

\[
J_{LQR}(N) = E \left\{ \sum_{t=0}^{N-1} \left[ r^T(t) R(t) r(t) + \hat{x}^T(t) Q(t) \hat{x}(t) \right] + \hat{x}^T(N) R(N) \hat{x}(N) \right\}, \tag{2.9}
\]

where \( R(t) \) is positive definite, \( Q(t) \) is positive semidefinite, and the \( \hat{x}(t) \) refers to the estimated plant state at time \( t \).

![Figure 2-3: The standard regulator arrangement.](image)

### 2.2.1 Recasting the ANC Problem Through Plant Augmentation

To design an LQG regulator for the feedback ANC system within an automobile, we augment the original plant, \( G(z) \), by including the ARMA noise process \( n(t) \) in parallel, followed by the cascade addition of the microphone transfer function. The augmented plant, \( \tilde{G}(z) \), which is depicted in Figure 2-4, models the entire acoustic system from the output of the regulator, \( r(t) \), to the output of the error microphone, \( m(t) \).

The state vector for the augmented plant, which has dimension \( (q + n) \times 1 \), consists of the states from both the disturbance model \( \Gamma(z) \) and the original plant model \( G(z) \):

\[
x = \begin{bmatrix} x_d \\ x_p \end{bmatrix} . \tag{2.10}
\]

Furthermore, the observability canonical state-space representation for \( \tilde{G}(z) \) reflects its
Figure 2-4: Reformulation of the feedback ANC system to the standard LQR configuration.

inherent parallel structure:

\[
\begin{align*}
x(t + 1) &= \begin{bmatrix} \Phi_d & 0 \\ 0 & \Phi_p \end{bmatrix} x(t) + \begin{bmatrix} G_d & 0 \\ 0 & G_p \end{bmatrix} \zeta(t) + \begin{bmatrix} 0 \\ B_p \end{bmatrix} r(t) \\
m(t) &= \begin{bmatrix} H_d & H_p \end{bmatrix} x(t) + v(t)
\end{align*}
\]  

(2.11)  
(2.12)

The dimensions of the state-space matrices for the augmented plant, encompassing all of our linear stochastic models outside of the ANC regulator, are: $\Phi$, $(q + n) \times (q + n)$; $G$, $(q + n) \times (q + n)$; $B$, $(q + n) \times 1$; and $H$, $1 \times (q + n)$.

Finally, we note that the zero-mean white Gaussian vector process producing uncertainty
in the augmented plant states,

\[
\zeta(t) = \begin{bmatrix} w(t) \\ \eta(t) \end{bmatrix}_{(n+1) \times 1},
\]

has covariance function

\[
E \left\{ \zeta(t) \zeta^T(\tau) \right\} = \begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_\eta I_{n \times n} \end{bmatrix} \delta(t - \tau);
\]

furthermore, \( \zeta(t) \) is uncorrelated with the microphone measurement noise, \( v(t) \).

Through the remainder of this chapter, we will use the term "plant" to denote to our augmented plant.

### 2.2.2 The Optimal Linear Quadratic Gaussian Regulator

The optimal strategy for solving the stochastic regulator problem is to divide the problem into two well-documented, independent steps [8]:

1. Compute the causal Bayes' Least Squares Estimate (BLSE) of the augmented plant state \( x(t) \) at time "\( t \)" using observations of the microphone output \( m(t) \) up until time "\( t \)" and knowledge of the control \( r(t) \) up to time "\( t - 1 \)." We denote this optimal state estimate:

\[
\hat{x}(t|t) = E \left\{ x(t) \mid m(1), m(2), \ldots, m(t), r(1), r(2), \ldots, r(t - 1) \right\},
\]

(2.13)

and its associated \((q + n) \times (q + n)\) error covariance matrix:

\[
P(t|t) = E \left\{ [\hat{x}(t|t) - x(t)] [\hat{x}(t|t) - x(t)]^T \mid m(1), \ldots, m(t), r(1), \ldots, r(t - 1) \right\}.
\]

(2.14)

Since all random variables are assumed Gaussian, we may in fact utilize a Kalman filter to compute the causal Linear Least Squares Estimate (LLSE).
2. Calculate the optimal control law \( r(t) = K_C(t) \hat{x}(t) \), for the corresponding deterministic LQ problem \((\zeta(t) = 0, \nu(t) = 0)\) with weighting matrices \( R(t) \) and \( Q(t) \).

Next, we cascade the two steps by replacing \( x(t) \) with \( \hat{x}(t|t) \) in the optimal control law:

\[
    r(t) = K_C(t) \hat{x}(t|t). \tag{2.15}
\]

Note that the optimal control \( r(t) \) is simply a linear feedback of the optimal state estimates \( \hat{x}(t|t) \); Figure 2-5 depicts the resulting separation structure exhibited by our ANC regulator.

![Diagram](image)

**Figure 2-5:** Separation structure exhibited by the optimal LQG regulator.

This separation structure, which is formally termed the *Separation Theorem* or *Certainty Equivalence Principle*, is a consequence of the orthogonality of the estimation errors \( x(t) - \hat{x}(t|t) \) and \( \nu(t) = m(t) - H\hat{x}(t|t) \) with the measurements \( m(t) \), the optimal state estimates \( \hat{x}(t|t) \), and all linear functions of \( \hat{x}(t|t) \), including our optimal feedback control law.

Zangi developed both finite- and infinite-horizon LQG regulators for ANC [1]. However, practical LQG controller implementations generally incorporate the time-invariant, infinite-horizon formulation; consequently, we shall review only the infinite-horizon LQG regulator for ANC.
Infinite-Horizon Formulation

A computationally efficient implementation of the LQG regulator exists for the special case of time-invariant system matrices characterized by the stabilizability of \([\Phi, B]\) and \([\Phi, G]\) and the detectability of \([\Phi, H]\), and constant weighting factor \(\rho\).

To minimize the acoustic pressure outside the error microphone,

\[
e(t) = n(t) + c(t) = Hx(t),
\]

we substitute \(R(t) = \rho\) and \(Q(t) = H^T H\) into the performance measure of Equation (2.9), and allow \(N \to \infty\). The resulting infinite-horizon optimal LQG regulator minimizes

\[
J_{\infty} = \lim_{N \to \infty} \frac{1}{N} E \left\{ \sum_{t=0}^{N-1} \left\{ x^T(t)(H^T H)x(t) + r^T(t)\rho r(t) \right\} + x^T(N)(H^T H)x(N) \right\}
= \lim_{N \to \infty} \frac{1}{N} E \left\{ \sum_{t=0}^{N-1} \left\{ e^2(t) + \rho r^2(t) \right\} + e^2(N) \right\},
\]

which is a weighted sum of the power of the residual noise \(e(t)\) and the control \(r(t)\), over all time. By increasing \(\rho\), we penalize control effort at the expense of noise attenuation.

To implement the corresponding optimal control law of Equation (2.15), one may use a steady-state Kalman filter to compute \(\hat{x}(t|t)\), and solve for the unique positive definite solution of an algebraic Riccati equation to find the optimal steady-state control gain \(K_c(\infty)\).

Specifically, if \([\Phi, G]\) is stabilizable and \([\Phi, H]\) is detectable, then the Kalman filter error covariance \(P(t|t-1)\) converges to the constant matrix \(P(\infty)\), which is the unique positive definite solution of the following algebraic Riccati equation

\[
P(\infty) = \Phi P(\infty) \Phi^T - \Phi P(\infty) H^T \left[ HP(\infty) H^T + \sigma_v^2 \right]^{-1} HP(\infty) \Phi^T + GG^T.
\]

Using the discrete-time steady state Kalman filter, one computes \(\hat{x}(t|t)\) sequentially in time:

**Measurement Update Step:**

\[
\hat{x}(t|t) = \hat{x}(t|t-1) + K_c(\infty) [m(t) - H\hat{x}(t|t-1)]
\]
**Prediction Step:**

\[
\hat{x}(t+1|t) = \Phi \hat{x}(t|t) + B r(t),
\]

where the Kalman gain \(K_f(\infty)\) is the constant matrix:

\[
K_f(\infty) = \left[ H P(\infty) H^T + \sigma_v^2 \right]^{-1} P(\infty) H^T.
\]  

Similarly, if \([\Phi, B]\) is stabilizable and \([\Phi, H]\) is detectable, then the optimal control gain \(K_c(\infty)\) converges to the constant matrix

\[
K_c(\infty) = - \left( B^T P_c(\infty) B + \rho \right)^{-1} B^T P_c(\infty) \Phi,
\]

where \(P_c(\infty)\) is the unique positive definite solution of the algebraic Riccati equation:

\[
P_c(\infty) = \Phi^T P_c(\infty) \Phi - \Phi^T P_c(\infty) B \left( B^T P_c(\infty) B + \rho \right)^{-1} B^T P_c(\infty) \Phi + H^T H.
\]

The optimal feedback control sequence \(\{r(t + \tau) \mid \tau = 0, 1, 2, \ldots\}\) is then [1, 9]:

\[
r(t) = K_c(\infty) \hat{x}(t|t).
\]

Since the steady-state Kalman filter is both linear and time-invariant, and our linear optimal control law utilizes a time-invariant control gain, then the infinite-horizon optimal LQG regulator is in fact a linear time-invariant (LTI) system. In Chapter 4, we present results of simulating an infinite-horizon LQG regulator for single-sensor active noise cancellation in the interior of an automobile.

### 2.3 Frequency Domain Analysis of the LQG Regulator

Additional insight into the feedback ANC system depicted in Figure 2-2 can be obtained by analyzing the infinite-horizon optimal LQG regulator, whose system function we denote \(K(z)\), in the frequency domain. Since all systems are LTI and \(\eta(t), w(t),\) and \(v(t)\) are assumed wide-sense stationary (WSS) Gaussian processes, then all physical signals are jointly WSS.
Analysis of Figure 2-2 reveals that the method by which the ANC regulator alters $P_{ee}(e^{j\omega})$, the power spectrum of the acoustic pressure at the error microphone, is by appropriately modifying the sensitivity transfer function

$$S(z) = \frac{1}{1 - G(z)K(z)},$$  \hspace{1cm} (2.25)

which defines the closed-loop input/output relationship between $n(t)$ and $e(t)$, and consequently the relationship between their respective power spectra $P_{nn}(e^{j\omega})$ and $P_{ee}(e^{j\omega})$:

$$P_{ee}(e^{j\omega}) = \left| \frac{1}{1 - G(e^{j\omega})K(e^{j\omega})} \right|^2 P_{nn}(e^{j\omega}) = |S(e^{j\omega})|^2 P_{nn}(e^{j\omega}).$$  \hspace{1cm} (2.26)

To minimize $E\{e^2(t)\}$, or equivalently to minimize the performance measure $\lim_{\rho \to 0} J_\alpha$, the feedback controller must attenuate $|S(e^{j\omega})|$ over the frequencies for which the unwanted noise $n(t)$ has significant power.

Equation (2.26) underscores the importance of the noise model, the spectral characteristics of the noise, and the transfer function $G(z)$ in determining the achievable attenuation of the unwanted noise $n(t)$.

Since the controller relies upon our noise model to determine frequencies over which $n(t)$ has significant power, and therefore frequencies over which $|S(e^{j\omega})|$ should be minimized, one must accurately measure and model the true noise spectrum $P_{nn}(e^{j\omega})$. Without a good model, we must assume that $P_{nn}(e^{j\omega})$ is flat over a wide range of frequencies; as a result, the controller can at best achieve low attenuation over the frequency band [1].

A closely related issue is how “peaky,” or narrowly concentrated, is the actual noise spectrum. If $P_{nn}(e^{j\omega})$ is highly localized, then the autocorrelation function

$$R_{nn}(\tau) = E\{n(t)n(t - \tau)\}$$  \hspace{1cm} (2.27)

will exhibit significant correlation over long time intervals; consequently, the noise will be highly predictable. In fact, referring back to Figure 1-2, we observe that if the plant $P(z)$ is invertible except for delays, then the essential function of the feedback controller is simply prediction of the unwanted acoustic noise $n(t)$. 

31
The transfer function from the input of the cancelling speaker to the input of the error microphone, \( G(z) \), also plays a key role in determining the achievable attenuation. If \( |G(e^{j\omega})| \) is small over frequencies for which \( n(t) \) exhibits significant power, then our regulator must apply very large gains \( |K(e^{j\omega})| \) to drive \( |S(e^{j\omega})| \leq 1 \). Yet the weighting factor \( \rho \) in the performance criterion \( J_\infty \) may preclude making \( |K(e^{j\omega})| \) large over these frequencies.

In addition, nonminimum phase zeros in the plant \( G(z) \) place inherent performance limitations on the optimal feedback ANC controller by forcing \( |S(e^{j\omega})| > 1 \), or in other words forcing noise amplification, over some frequencies. This performance barrier arises from the fact that \( K(z) \) must be a stabilizing controller, precluded from attempting an unstable pole/zero cancellation to make \( |S(z)| \neq 1 \) at the nonminimum phase zero, and from an algebraic constraint known as the maximum modulus principle, which states that a stable function that is analytic outside the unit circle achieves its maximum value over this region when evaluated on the unit circle itself:

\[
|S(z)| \leq \sup_{\omega} |S(e^{j\omega})| \leq \|S\|_\infty \quad \forall \ z \text{ s.t. } |z| > 1. \tag{2.28}
\]

In Chapter 3, we show that our plant models for an automobile are nonminimum phase, thereby limiting the noise attenuation achieved by our ANC simulations in Chapter 4.

### 2.4 Practical Issues and Extensions of the Basic Algorithm

After applying the LQG regulator for ANC to the acoustic environment within an automobile, we immediately observe that the basic formulation fails to address two key issues: numerical problems arising from the large delays inherent in the speaker-to-microphone transfer functions in a car, and perceived loudness of noise as a function of frequency.

Section 2.4.1 presents alternatives for dealing with the first problem, while Section 2.4.2 addresses the second issue by reviewing the frequency-weighted optimal controller proposed by Zangi [1].
2.4.1 Working with Large Plant Delays

If the number of delays in the nominal plant model $G_0(z)$ forces the degree of the numerator to be greater than that of the denominator, then numerical problems in traditional Hamiltonian methods for computing $P(\infty)$ and $P_c(\infty)$ often preclude one from implementing the infinite-horizon optimal LQG regulator described in Section 2.2.2. Recall that Hamiltonian eigenvalue methods require the augmented plant transition matrix $\Phi$ to be nonsingular [12, 13], which is not possible if plant delays force $m > n$, and that generalized eigenvalue techniques, which do not demand the state transition matrix to be invertible [12], will nevertheless suffer from numerical instability if delays make $m \gg n$.

We chose two methods to bypass numerical difficulties: modification of the Kalman filter in the LQG regulator to an $M$-step predictor, where $M$ is an appropriate subset of the total number of delays in $G_0(z)$, or in other words a number less than or equal to the total number of sampling periods required for sound to travel from the stereo speaker to the error microphone; and increasing the number of poles in our model $G_0(z)$ so that $m \leq n$. Not surprisingly, the second method outperformed the first, albeit with increased computational complexity, in our ANC simulations; a Kalman filter designed for filtering will always perform at least as well as one designed for prediction.

Consider a nominal speaker-to-microphone transfer function $G_0(z)$ of the form

$$G_0(z) = z^{-M} \tilde{G}_0(z) = z^{-M} \left[ \frac{\tilde{B}_0(z)}{\tilde{A}_0(z)} \right] = \frac{\sum_{j=0}^{n} b_j z^{-j}}{1 + \sum_{k=1}^{n} a_k z^{-k}} \quad m \leq n,$$

(2.29)

which corresponds to the cascade structure depicted in Figure 2-6. When $m + M > n$, we suffer from the aforementioned numerical problems in computing $P(\infty)$ and $P_c(\infty)$.

One alternative, depicted in Figure 2-7, is to simply account for the delays, $z^{-M}$, in the first stage of the regulator by modifying the Kalman filter into an $M$-step predictor, and utilizing $\tilde{G}_0(z)$, not $G_0(z)$, to compute the canonical state space representation for the plant, $\{ \Phi_p, G_p, B_p, H_p, 0 \}$.

The basic Kalman filter in Equations (2.19)-(2.21) remains unchanged; however, we add one additional step to compute $\tilde{\mathcal{R}} (t + M | t)$. Using the augmented plant dynamics from Equation (2.11), coupled with our knowledge that the noise vector $\zeta(t)$ is zero mean and
that the optimal control is given by the linear Equa- tion (2.15), we arrive at $\hat{x}(t + M|t)$ by
induction:

$$
\begin{align*}
\hat{x}(t + 1|t) &= \Phi \hat{x}(t|t) + B \hat{r}(t|t) = \Phi \hat{x}(t|t) + B K_c(\infty) \hat{x}(t|t) \\
&= [\Phi + B K_c(\infty)] \hat{x}(t|t) \\
\hat{x}(t + 2|t) &= \Phi \hat{x}(t + 1|t) + B \hat{r}(t + 1|t) = [\Phi + B K_c(\infty)] \hat{x}(t + 1|t) \\
&= [\Phi + B K_c(\infty)]^2 \hat{x}(t|t) \\
&\vdots \\
\hat{x}(t + M|t) &= [\Phi + B K_c(\infty)]^M \hat{x}(t|t), 
\end{align*}
$$

(2.30)

where the notation $\hat{r}(s|\tau), s > \tau,$ designates the predicted optimal control for time “$s$” using observations of the microphone output $m(t)$ up until time “$\tau$” and knowledge of the control $r(t)$ up to time “$\tau - 1$.” At time “$t$,” therefore,

$$
r(t) = \hat{r}(t + M|t) = K_c(\infty) \hat{x}(t + M|t) = K_c(\infty) [\Phi + B K_c(\infty)]^M \hat{x}(t|t). \quad (2.31)
$$

Modifying the Kalman filter bypassed the numerical instabilities for large $M$ inherent in the partial generalized eigenanalysis [12]:

$$
\begin{bmatrix}
I & \Psi_1 \\
0 & -A^T
\end{bmatrix}
\begin{bmatrix}
T_{11} \\
T_{21}
\end{bmatrix}
L =
\begin{bmatrix}
A & 0 \\
\Psi_2 & -I
\end{bmatrix}
\begin{bmatrix}
T_{11} \\
T_{21}
\end{bmatrix},
$$

(2.32)

where $L$ is in Jordan form and includes all generalized eigenvalues outside the unit circle,
and $P = T_1T_2^{-1}$ is the positive definite solution of the Algebraic Riccati Equation (2.18) or (2.23). Note that to calculate $P(\infty)$ one chooses $A = \Phi$, $\Psi_1 = -GG^T$, and $\Psi_2 = -\frac{1}{\sigma^2}HTH$, whereas to find $P_c(\infty)$ we substitute $A = \Phi^T$, $\Psi_1 = -HT^TH$, and $\Psi_2 = -\frac{1}{\rho}BB^T$.

The more effective solution, however, is to simply increase the number of poles in $G_0(z)$ to ensure $m + M \leq n$. Although this may result in as many as twenty extra poles in our nominal transfer function, as well as the associated modelling difficulties and increased numerical complexity, we show in Chapter 4 that this procedure actually outperforms the first. The inclusion of the additional plant states, as well as the elimination of the $M$-step predictor in the first stage of the regulator, likely caused the performance improvement. In fact, the almost linear phase plots of the speaker-to-microphone transfer functions in Chapter 3 suggest that the actual continuous-time transfer functions contain a delay term.
\( e^{-T_d s} \), which requires an infinite number of states to model:

\[
e^{-T_d s} = \frac{1}{1 - T_d s + \frac{T_d^2}{2} s^2 - \frac{T_d^2}{3!} s^3 + \ldots}.
\] (2.33)

### 2.4.2 Frequency-Weighting the LQG Regulator

Another issue our basic ANC formulation does not address is perceived loudness of sound; two frequencies having the same sound pressure level (SPL) may not seem equally annoying to a listener. Sound meters, for example, generally weight the measured SPL with a filter that matches the subjective annoyance for numerous industrial and transportation noises [14], such as the A-filter plotted in Figure 2-8.

![A-Filter Magnitude Response](image)

Figure 2-8: An A-Filter weights sound pressure level to address perceived loudness of sound.

By incorporating a frequency-weighting filter, whose transfer function we denote \( F(z) \), into the augmented plant as shown in Figure 2-9, and appropriately modifying our state-space formulation, our simulations in Chapter 4 can provide insight into the perceived effectiveness of the ANC algorithm within an automobile. Since \( F(z) \) is a mathematical, not a physical, construct, we choose to simplify our overall state-space representation and the corresponding Kalman filter by conceptualizing the frequency-weighting filter prior to the microphone in our augmented plant.

If we assume that the frequency-weighting filter has the state-space representation
Figure 2.9: Frequency-weighted optimal LQG regulator for ANC.

\( \{\Phi_w, B_w, H_w, 0\} \), then the input/output relationship between \( e(t) \) and \( e_w(t) \) is defined by

\[
\begin{align*}
x_w(t + 1) &= \Phi_w x_w(t) + B_w e(t) \\
e_w(t) &= H_w x_w(t),
\end{align*}
\]  

(2.34) (2.35)

and the state-space representation for the frequency-weighted augmented plant becomes:

\[
\begin{align*}
x_f(t + 1) &= \begin{bmatrix}
\Phi & 0 \\
B_wH & \Phi_w
\end{bmatrix} x_f(t) + \begin{bmatrix}
G \\
0
\end{bmatrix} \zeta(t) + \begin{bmatrix}
B \\
0
\end{bmatrix} r(t) \\
m(t) &= \begin{bmatrix}
0 & H_w
\end{bmatrix} x_f(t) + v(t),
\end{align*}
\]  

(2.36) (2.37)

where the state vector consists of states from our original augmented plant, defined by
Equations (2.10)-(2.12), plus states from the frequency weighting filter \( F(z) \)

\[
\mathbf{x}_f(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_w(t) \end{bmatrix}.
\]  

(2.38)

To derive the optimal finite- and infinite-horizon LQG regulators for frequency-weighted ANC, we follow the same procedure outlined in Section 2.2.2, substituting \( \Phi_f, G_f, B_f, \) and \( H_f \) for \( \Phi, G, B, \) and \( H \), respectively.

Therefore, to minimize the frequency-weighted acoustic pressure outside the error microphone,

\[
e_w(t) = H_w \mathbf{x}_w(t) = H_f \mathbf{x}_f(t),
\]  

(2.39)

we substitute \( R(t) = \rho \) and \( Q(t) = H_f^T H_f \) into the quadratic performance measure of Equations (2.9). If \([\Phi_f, B_f]\) and \([\Phi_f, G_f]\) are stabilizable and \([\Phi_f, H_f]\) is detectable, then we may define

\[
J_{\infty,f} = \lim_{N \to \infty} \frac{1}{N} E \left\{ \sum_{t=0}^{N-1} \left\{ \mathbf{x}_f^T(t) (H_f^T H_f) \mathbf{x}_f(t) + r^T(t) \rho r(t) \right\} + \mathbf{x}_f^T(N) (H_f^T H_f) \mathbf{x}_f(N) \right\}
\]

\[
= \lim_{N \to \infty} \frac{1}{N} E \left\{ \sum_{t=0}^{N-1} \left\{ e_w^2(t) + \rho r^2(t) \right\} + e_w^2(N) \right\},
\]  

(2.40)

to obtain a frequency-weighted, infinite-horizon optimal LQG regulator for ANC. One simply substitutes the new state-space description for the augmented plant, \( \{\Phi_f, G_f, B_f, H_f, 0\}\), into Equations (2.18)-(2.23).
Chapter 3

Experimental Issues: Obtaining a Model for the ANC Environment Inside an Automobile

Accurately modelling the acoustic environment within the interior of an automobile is crucial to designing meaningful simulations which seek to determine the feasibility of active noise cancellation. Since we have chosen the infinite-horizon LQG formulation to explore ANC within a car, we must first adopt a stochastic model for the observed road noise as well as a nominal LTI transfer function modelling speaker-to-microphone acoustics.

Using data collected from experiments performed at the Bose Corporation in Framingham, Massachusetts, we argue in this chapter that a 5th-order AR process adequately describes the observed road noise, \( n(t) \), in the interior of an automobile over limited time intervals, and that autoregressive moving-average with exogenous input (ARMAX) modelling is appropriate for describing \( G(z) \), the transfer function from the input of the automobile stereo speaker to the input of the error microphone. Recall from Equations (2.5) and (2.6) in Chapter 2, though, that we will simplify plant uncertainty in our ANC simulations. Instead of the moving average of Gaussian noise in the ARMAX model, which necessitates a Kalman filter designed for correlated process and measurement noise, we will model uncertainty as additive white Gaussian noise at the plant states.
Two key points will be clear by the end of the chapter. First, road noise is highly predictable. Most of the power of \( n(t) \) is restricted to narrow frequency bands under 400 Hz; as a result, road noise correlation times, typically on the order of \( 1/10 \) s, are long enough to allow prediction in accord with the time required for sound to travel from an automobile speaker to an error microphone.

Secondly, the typical speaker-to-microphone transfer function \( G(z) \) within an automobile exhibits characteristics which make active noise cancellation difficult. Not only do the transfer functions require high-order pole-zero models to adequately describe, but they also suffer from several ANC-inhibiting characteristics, including small magnitudes at some of the frequencies in which much of the power of \( n(t) \) is concentrated and nonminimum phase zeros.

### 3.1 Measuring and Analyzing the Acoustic Environment

Our test vehicle for measuring the acoustic environment within an automobile was a 1990 Chevrolet Lumina Euro. As depicted in Figure 3-1, we focused upon two error microphone locations, chosen for their proximity to the likely ear locations of a passenger sitting in the front or rear passenger seats. Using box-like "dummies" to securely mount the microphones and a portable, two-channel 48-kHz digital audio tape (DAT) machine to record data, we simultaneously sampled acoustic noise at microphones "1" and "2" while driving the Lumina, and measured the transfer functions from the input of each automobile speaker to the output of each error microphone while the vehicle was stationary. The second set of measurements, in which we directly connected the speaker to a pink noise generator producing 20-kHz bandlimited colored noise and simultaneously recorded the speaker input and microphone output, were designed to facilitate correlation estimates of each speaker-to-microphone transfer function. Since we assume nominal microphone transfer functions of unity, these estimates form each \( G_0(z) \), the nominal system function from speaker input to microphone input.

Our notation for referring to the speakers and microphones is based upon their positions within the automobile. For the cancelling speakers depicted in Figure 3-1, therefore, "RP"
designates the rear passenger-side speaker, "RD" the rear driver-side speaker, "FP" the front passenger-side speaker, and "FD" the front driver-side speaker. The microphones are simply labelled "1" and "2," based upon distance from the front of the car.

We have circled speaker "RP" and microphone "1," which are separated by about 1.27 m, because the closed-loop system comprising these devices resulted in the highest noise attenuation for any speaker-microphone pair. The corresponding plant and road noise are denoted $G_{RP\to1}(z)$ and $n_1(t)$, respectively.

In Section 3.1.1, we show that nearly all of the power of road noise is localized to frequencies below 400 Hz; consequently, we decimated all data designated for modelling and simulations to a sampling rate of 4.8 kHz.

### Road Noise Spectral Characteristics

Figure 3-2 depicts a 45-second interval of road noise scaled to variance 1, which was recorded at microphone "1" while the Lumina slowed to a stop and then accelerated to approximately 30 miles per hour (m.p.h.), as well as the associated narrowband spectrogram computed using a Hanning window of duration 0.375 s. To study the spectral characteristics of the road
noise, we have highlighted three segments of $n_1(t)$ exhibiting nearly stationary behavior, according to the spectrogram: time segment “A,” during which the Lumina is moving at 15 m.p.h.; interval “B,” when the engine is idling; and “C,” a period when the Lumina is travelling about 30 m.p.h.

**Acoustic Pressure Recorded at Microphone “1”, and its Narrowband Spectrogram**

![Spectrogram Image]

Figure 3-2: Acoustic pressure recorded at microphone “1,” and the associated narrowband spectrogram calculated using a Hamming window of duration $3/8$ s.

Both the spectrogram and the estimated power spectra of $n_1(t)$ during intervals “A,” “B,” and “C,” which we have plotted in Figure 3-3, show that nearly all of the power of unwanted road noise is localized to frequency bands under 400 Hz, and that much of the power is in fact restricted to frequencies below 50 Hz. To estimate the power spectra, we utilized a modified Cooley-Tukey approach by averaging overlapped, smoothed periodograms and dividing by the window energy:

$$
\hat{P}_{n_1n_1}(e^{i\omega}) = \frac{1}{M} \frac{1}{\sum_{\tau=0}^{N-1} w(\tau)} \sum_{k=0}^{M-1} \frac{1}{N} \left| S_{n_1}^{(k)}(e^{i\omega}) \right|^2,
$$

(3.1)
Figure 3-3: Estimated power spectra, $\hat{\Phi}_{n_1}$, and autocorrelation functions, $\hat{R}_{n_1}(e^{j\omega})$, during selected time intervals of $n_1(t)$, the acoustic pressure recorded at microphone “1.” Top to bottom: interval “A,” 0-5.5 s; interval “B,” 7.5-11 s; and interval “C,” 27-40.5 s.
where

\[ S_{n_1}^{(k)}(e^{j\omega}) = \sum_{\tau=0}^{N-1} w(\tau) n_1(t_i+\tau+\lambda N k) \ e^{j\omega \tau}, \]  

(3.2)

We chose a 4800-point Hanning window for \( w(t) \), and the overlap factor \( \lambda = 1/2 \). Our choice for \( N \) sought to balance the need to make \( w(t) \) long, to ensure that the window mainlobe width was narrow enough to prevent frequency smearing near harmonics, and the conflicting requirement of making \( N \) short to allow the periodogram averaging in Equation (3.1) designed to reduce the variance in the spectral estimates.

From the localized, "peaky" power spectra during each of the three time intervals, we should expect that road noise is highly predictable. Our simulations, in fact, verify this; in Section 3.2.1 we plot normalized prediction error versus prediction time using an adaptive algorithm for various AR model orders, and in Chapter 4 we repeat the experiment with similar success using the non-adaptive LQG formulation as an \((M-1)\)-step predictor (the "plant" for this experiment is simply the delay \( z^{-M} \)). Of course, the autocorrelation functions depicted in Figure 3-3, which exhibit correlation times longer than 60 ms, also suggest the feasibility of experiments designed to perform prediction of road noise. For example, predicting \( n_1(t) \) 4.2 ms into the future, as required for the 1.27-meter separation between speaker "RP" and microphone "1," is significantly less difficult if \( n_1(t) \) remains correlated for at least 1/20 s. Intervals dominated by harmonics are even more predictable; the 29-Hz and 32-Hz tones produced by the idling engine during interval "B" generate the beating, sinusoidal autocorrelation function in Figure 3-3 which exhibits significant correlation for nearly 2/10 s.

### 3.1.2 Estimation of the Speaker-to-Microphone Transfer Functions

To measure each of the speaker-to-microphone transfer functions within the Lumina, we directly applied bandlimited colored noise to the speaker and simultaneously sampled the resulting signals measured at the speaker input, \( u(t) \), and the error microphone output, \( y(t) \). Our correlation estimate for the transfer function is then the ratio of two power spectral estimates:

\[ \hat{G}(e^{j\omega}) = \frac{\hat{P}_{uu}(e^{j\omega})}{\hat{P}_{uu}(e^{j\omega})}, \]  

(3.3)
where $\hat{P}_{nn}^N(e^{j\omega})$ is the power spectral estimate of $u(t)$, and $\hat{P}_{yy}^N(e^{j\omega})$ is the cross-power spectral estimate for $y(t)$ and $u(t)$. Both estimates were obtained using the scaled Cooley-Tukey approach outlined in Equations (3.1) and (3.2); for example, to compute $\hat{P}_{yy}^N(e^{j\omega})$ we replaced

$$\left|S_{n1}^{(k)}(e^{j\omega})\right|^2 \text{ with } S_{y}^{(k)}(e^{j\omega}) S_{y}^{(k)*}(e^{j\omega})$$

in Equation (3.2).

For the remainder of this thesis we will focus upon two of the transfer functions, $G_{RP\rightarrow 1}(e^{j\omega})$ and $G_{RP\rightarrow 2}(e^{j\omega})$. Not only are these system functions representative of the other speaker-to-microphone transfer functions within the Lumina, but they also illustrate that the transfer function plays a more important role than the speaker-to-microphone separation distance in determining the achievable noise attenuation using the infinite-horizon optimal LQG regulator. Although microphone "1" was nearly three times farther from speaker "RP" than microphone "2," the maximum noise attenuation we achieved at the first microphone was 8.58 dB, compared to only 1.77 dB for the closer sensor.

![Graph](image)

**Figure 3-4:** The estimated transfer function from the input of speaker "RP" to the input of microphone "1."

Figures 3-4 and 3-5 depict $\hat{G}_{RP\rightarrow 1}(e^{j\omega})$ and $\hat{G}_{RP\rightarrow 2}(e^{j\omega})$, our estimates for the system.
functions from the input of speaker "RP" to the input of microphone "1" and to the input of microphone "2," respectively. We immediately observe that the transfer functions share several common features, including highly irregular shapes, which necessitate high-order models, and unwrapped phases which are highly linear over large frequency bands, suggesting that multipath effects are minimal for the propagation of sound within an automobile from a cancelling speaker to an error sensor. In fact, fitting $\hat{G}_{RP\rightarrow 1}(e^{j\omega})$ to a line suggests that $G_{RP\rightarrow 1}(e^{j\omega})$ contains the delay $e^{-22}$, which corresponds to a 1.38 m separation between speaker "RP" and microphone "1," compared to the actual 1.27 m distance.

![Estimated Transfer Function, $\hat{G}_{RP\rightarrow 2}(e^{j\omega})$, from Speaker "RP" to Microphone "2"](image)

Figure 3-5: The estimated transfer function from the input of speaker "RP" to the input of microphone "2."

Another important characteristic is the behavior of the transfer function magnitudes over low frequencies, where much of the power of road noise is concentrated. At least part of the explanation for the great disparity in noise attenuation we achieved using models of $\hat{G}_{RP\rightarrow 1}(e^{j\omega})$ and $\hat{G}_{RP\rightarrow 2}(e^{j\omega})$ in our feedback controller is likely due to behavior of the magnitudes under 100 Hz. Whereas $|\hat{G}_{RP\rightarrow 1}(e^{j\omega})|$ is greater than 0 dB and relatively flat over the frequency range 20-100 Hz, $|\hat{G}_{RP\rightarrow 2}(e^{j\omega})|$ is not only strictly less than 0 dB below 40 Hz, but also varies sharply between -10 dB and 17 dB over the frequency range 20-100 Hz.
3.2 Modelling the Acoustic Environment within an Automobile

3.2.1 Modelling Road Noise as an AR Process

We concluded from the narrowband spectrogram in Figure 3-2 and the power spectra depicted in Figure 3-3 that over limited time intervals, the observed road noise within an automobile may be assumed stationary and modelled as an AR process:

\[ n(t) = -\sum_{k=1}^{p} \alpha_k n(t - k) + Gw(t), \quad (3.4) \]

where \( p \) is the order of the all-pole linear filter, and \( w(t) \) is a zero-mean, unit variance white Gaussian process.

To determine an appropriate model order \( p \), we tested the single-microphone recursive/adaptive identification algorithm described in [1, 15], using various AR filter orders in the context of the ideal feedback active noise cancellation system depicted in Figure 3-6. Since the speaker-to-microphone transfer function in this experiment is simply the ideal delay \( z^{-M} \), \( M \geq 1 \), we were also testing our earlier hypothesis regarding the predictability of road noise within an automobile.

![Diagram](image)

Figure 3-6: Closed-loop system with ideal delays used to determine an appropriate AR model order for the observed road noise within the Lumina.
Figure 3-7 plots attenuation in the acoustic pressure at the error microphone as a function of prediction time when we adaptively model \( n(t) \) as a 5\(^{th} \) and a 10\(^{th} \)-order AR process using the aforementioned algorithm. Observing that 5 system states capture nearly all of the information obtained using 10 states for prediction times as high as 4.17 ms, we modelled the road noise at microphones "1" and "2" during each of intervals "A," "B," and "C" as fixed, independent 5\(^{th} \) order AR processes for our ANC simulations in Chapter 4. Note that our definition for attenuation is:

\[
\text{Attenuation (dB)} = -10 \log_{10} \left( \frac{E\{e^2(t)\}}{E\{n^2(t)\}} \right) = - \text{Normalized Prediction Error (dB)},
\]

where \( E\{n^2(t)\} \) is the average power of the original road noise at the error microphone, and \( E\{e^2(t)\} \) is the average power of the residual error signal.

Figure 3-8 depicts the squared magnitude of the AR filter we used to model \( n_1(t) \) during interval "A." As expected, the all-pole filter, defined by the parameters \( G = 0.0143 \) and

\[
\alpha = \begin{bmatrix} -2.275 & 1.997 & -1.291 & 0.751 & -0.181 \end{bmatrix}^T,
\]

captures the coarse characteristics of \( \hat{P}_{n_1}(e^{j\omega}) \).
Figure 3-8: Our 5\textsuperscript{th}-order AR model for $n_1(t)$ during interval "A" captures the coarse behavior in the power spectrum.

3.2.2 Modelling the Speaker-to-Sensor Transfer Functions

To obtain nominal ARMA models for each of the speaker-to-microphone transfer functions within the Lumina, we experimented with various ARMAX model structures, in which plant uncertainty, or in other words error in the difference equation relating system output to exogenous input, is described by a moving average of white noise:

$$c(t) = -\sum_{i=1}^{n_a} a_i c(t - i) + \sum_{j=1}^{n_b} b_j r(t - j - nk) + \left\{ e(t) + \sum_{k=1}^{n_c} c_k e(t - k) \right\}$$

$$n_k \geq 0; \quad n_b, n_c \leq n_a,$$  \quad (3.7)

where $n_k$ is the number of delays in the nominal plant. As we mentioned at the beginning of the chapter, however, we retained only the nominal pole-zero transfer functions in our ANC simulations, so that we could use a Kalman filter designed for uncorrelated process and measurement noise. Plant uncertainty was simply modelled as additive white Gaussian noise at the plant states, as in Equation (2.6); these mutually independent random processes were characterized by mean zero and variance $\sigma^2$.

The lower half of Figure 3-9 depicts $G_{0,RP \rightarrow 1}^{(LO)}(e^{j\omega})$, the nominal ARMA system function having the fewest poles which we found to give a reasonable description of the transfer function from the input of speaker "RP" to the input of microphone "1." Observe that
even with 12 zeros and 16 poles, the model has difficulty rendering $G_{RP \rightarrow 1}(e^{j\omega})$ over low frequencies, where much of the power of road noise is localized. Using the low-order model in our ANC simulations, therefore, could lead to inaccurate results. Furthermore, from our discussion in Section 2.4.1 we know that since $n_s + n_k = 31 > n_a = 16$, then the Kalman filter in the LQG regulator must predict the augmented plant state, $x(t)$, 15 steps into the future, rather than estimate the state at the current time $t$. As a result, we expect that the feedback ANC system using $G_{0,RP \rightarrow 1}^{(LO)}(e^{j\omega})$ will achieve significantly lower noise attenuation than one employing an ARMA model with 31 poles.

Figure 3-9: High- and low-order nominal ARMA models for the estimated transfer function from speaker “RP” to microphone “1.”

Figure 3-9 also depicts our high-order nominal plant model, $G_{0,RP \rightarrow 1}^{(HO)}(e^{j\omega})$, which is comprised of 12 zeros, 31 poles, and 19 delays. Not only does this model more accurately describe the low-frequency behavior of $G_{RP \rightarrow 1}(e^{j\omega})$, but it also permits the LQG regulator to perform filtering rather than prediction and thereby achieve significantly higher noise attenuation at microphone “1,” as we show in Chapter 4. In fact, we found that to achieve
the highest noise attenuation for a given cancelling speaker and error microphone using the LQG regulator, the nominal plant model must be characterized by \( n_b + n_k \leq n_a \).

An important ANC-inhibiting characteristic in our nominal ARMAX models is the existence of nonminimum phase zeros, which ensures that a feedback ANC controller will amplify acoustic noise at some frequencies, as a result of the maximum modulus principle discussed in Section 2.3. In fact, we show in Chapter 4 that the road noise is amplified at almost all frequencies above 50 Hz, a potentially troublesome result since humans generally find high frequency disturbances much more annoying than low frequency noise. Figure 3-10 depicts the complex conjugate, nonminimum phase zeros \( 1.72 e^{\pm j1.54} \) in our high-order model \( G_{0,RP \rightarrow 1}^{(HO)}(e^{j\omega}) \).

![Pole-Zero Plot of High-Order Nominal Plant](image)

Figure 3-10: The pole-zero plot of our high-order model \( G_{0,RP \rightarrow 1}^{(HO)}(e^{j\omega}) \) depicts nonminimum phase zeros at \( 1.72 e^{\pm j1.54} \).

Lastly, for some speaker-microphone pairs within the Lumina, nominal transfer function models characterized by \( n_b + n_k \geq n_a \) were actually necessary to achieve noise attenuation. One example involved speaker "RP" and microphone "2," for which we achieved the lowest noise attenuation in any of our simulations. Figure 3-11 depicts our nominal model \( G_{0,RP \rightarrow 2}(e^{j\omega}) \), consisting of 9 zeros, 16 poles, and 7 delays. Of course, the necessity of using high-order models in the LQG regulator increases the computational cost of the
Figure 3-11: Our nominal plant model $G_{0,RP→2}(e^{jω})$ for the estimated transfer function from speaker "RP" to microphone "2."
Chapter 4

Analysis of ANC Simulations

The underlying goal of the experiments we present in this chapter is to assess the feasibility of performing active noise cancellation within an automobile using the infinite-horizon optimal LQG regulator. All simulations of ANC are constrained to use the existing acoustic environment within the automobile; we may not modify the speaker locations nor the non-minimum phase speaker-to-microphone transfer functions discussed in Chapter 3.

Our simulations will address three topics: the predictability of road noise within an automobile, the ability of the feedback controller to attenuate the road noise outside an error microphone once we introduce models of the speaker-to-microphone acoustics in our ANC simulations, and the perceived effectiveness of the algorithm to a passenger seated near the error sensor. In all simulations, we characterize plant uncertainty by $\sigma_n = 0.001$.

By the end of the chapter it will be clear that although road noise is highly predictable, the transfer function from the cancelling speaker to the error microphone within the automobile plays as important a role in determining the achievable noise attenuation as the speaker-to-microphone separation distance. Moreover, the LQG controller tends to focus exclusively on low frequencies, where most of the power of the road noise is concentrated, while *amplifying* acoustic power above 100 Hz. Although the overall noise power is in fact reduced, a passenger may nevertheless find the error residual *more annoying* than the original road noise, since humans find high frequency disturbances more annoying than low frequency noise.
4.1 Prediction of Road Noise Within an Automobile

We already know from the experiments presented in Chapter 3 that road noise within an automobile is highly predictable. Figure 3-3, for example, displayed the "peaky" power spectra of the road noise recorded at microphone "1" during intervals "A," "B," and "C," as well as the corresponding autocorrelation functions, characterized by correlation times significantly longer than the average propagation time of sound from a cancelling speaker to an error sensor. We also plotted attenuation as a function of prediction time in Figure 3-7, for a closed-loop feedback ANC system using the recursive/adaptive identification algorithm described in [1, 15].

In this section we also argue that road noise may be successfully predicted over limited time intervals by using the non-adaptive infinite-horizon LQG regulator as an \((M-1)\)-step predictor. Figure 4-1 describes the experimental design, in which the plant \(G(z)\) is simply the ideal delay \(z^{-M} = z^{-(M-1)} \cdot z^{-1}, M \geq 1.\) As we described in Section 2.4.1, the controller uses \(z^{-1}\) in the augmented plant to formulate the steady-state Kalman gain \(K_f(\infty)\) and control gain \(K_c(\infty),\) and then predicts the augmented plant state \((M-1)\) steps into the future.

![Diagram of LQG regulator design](image)

Figure 4-1: LQG regulator design which allows one to assess the predictability of road noise within an automobile, by incorporating an ideal \(M\)-delay plant.
Figure 4-2 shows that the non-adaptive, infinite-horizon LQG controller performs quite comparably to the adaptive algorithm of Section 3.2.1. The plot of attenuation, defined previously by Equation (3.5), versus prediction time of the unwanted road noise recorded at microphone "1," during time interval "A," shows that we achieve over 15 dB of noise attenuation for delays under 2.1 ms, and nearly 11.3 dB of attenuation for a 3.3 ms delay, which corresponds to approximately a 1-m separation distance between the cancelling speaker and the error microphone. For this experiment, we characterized plant uncertainty by $\sigma_n = 0.001$ and chose the control weighting $\rho = 0.0001$.

![Noise Attenuation Achieved for an Ideal M-Delay Plant, $x^M$, using the Infinite-Horizon Optimal LQG Regulator ($\eta(t)$, interval "A")](image.png)

Figure 4-2: Noise attenuation as a function of prediction time achieved for an ideal $M$-delay plant using the infinite-horizon optimal LQG controller.

Since speaker "RP" and microphone "2" are only 0.43 m apart—the smallest separation distance, and therefore the shortest prediction time, for any speaker-microphone pair in our test vehicle—we would expect to achieve the highest noise cancellation, perhaps as high as 19.5 dB, using $G_{RP\rightarrow 2}(e^{j\omega})$ in our ANC simulation. Section 4.2, however, quickly dispels this notion; the ANC-inhibiting characteristics in the transfer function from speaker "RP" to microphone "2" produced the lowest noise attenuation in any of our ANC simulations.
4.2 Pointwise Active Noise Cancellation Within an Automobile

Our simulations of single-sensor active noise cancellation within an automobile were designed to elicit a number of important facts regarding the effectiveness of an LQG optimal controller constrained to use the existing acoustic environment of an automobile. All of our simulations were conducted during the aforementioned time intervals “A,” “B,” and “C,” when road noise may be assumed stationary.

We begin by confirming that high-order transfer function models, for which the number of finite poles is at least as large as the number of delays plus finite zeros, are necessary to achieve the highest noise attenuation, since they allow the Kalman filter in the LQG regulator to perform filtering rather than prediction. More importantly, though, we will show that the behavior of the transfer function from the cancelling speaker to the error microphone, not the separation distance, is what ultimately determines the effectiveness of a feedback ANC algorithm. Thirdly, the control weighting \( \rho \) plays an important role in the success of the LQG regulator; we found that \( \rho \approx 0.0001 \) was a good choice for transfer functions conducive to ANC, whereas \( \rho \approx 0.1 \) was a reasonable value for transfer functions which tended to inhibited the controller's ability to perform noise attenuation. Recall from Equation (2.17) that increasing \( \rho \) penalizes large control efforts. Lastly, nearly all noise attenuation at the error microphone using the LQG formulation is restricted to low frequencies, at the expense of high-frequency noise amplification; since the human ear is more sensitive to high frequencies, the error residual may in fact be more annoying than the original road noise.

Figure 4-3 depicts the magnitude of sensitivity transfer function \( S(\tau) \), which describes the input/output relationship between the unwanted road noise \( n(t) \) and the error residual \( e(t) \), when we use a high- and a low-order transfer function model for \( G_{RP-1}(e^j\omega) \) in the feedback ANC controller. Note that for these experiments, the rear passenger-side speaker is the cancelling speaker and microphone “1” is the error sensor; the AR model for \( n_1(t) \) is identical for the two simulations. We observe that when the Kalman filter in the LQG regulator is allowed to perform filtering, as is the case for the high-order model
Figure 4-3: Sensitivity transfer function magnitudes for a single-sensor, feedback ANC system which: a) incorporates a high-order nominal model for the transfer function from speaker "RP" to microphone "1," and b) uses a low-order model to describe $G_{RP\rightarrow 1}(e^{j\omega})$.

$G_{0,RP\rightarrow 1}^{(HO)}(e^{j\omega})$, we achieve attenuation of 8.58 dB over interval “A,” whereas forcing the regulator to perform prediction of the augmented plant state by using $G_{0,RP\rightarrow 1}^{(LO)}(e^{j\omega})$ causes the attenuation to drop to 3.21 dB. The large delays characterizing speaker-to-microphone transfer functions within an automobile, therefore, force one to choose complex plant models in which the number of finite poles is at least equal to the number of delays plus finite zeros, in order to maximize noise attenuation. A real-time implementation of the LQG controller must confront this increased numerical complexity.

A more important point highlighted by these closed-loop transfer functions is that attenuation in the unwanted road noise is almost wholly restricted to frequencies under 100 Hz; higher frequencies are generally amplified in power. This result is not wholly unexpected, given our discussion in Chapter 3 regarding the non-minimum phase transfer function from speaker "RP" to microphone "1" as well as the low-frequency characteristics of the observed road noise. Figure 4-4 depicts the power spectra of the original noise $n_1(t)$ and the error.
residual $e_1(t)$, when we ran the algorithm using the recorded road noise at microphone "1" during interval "A," as well as the high-order transfer function model $G_{0,RP \rightarrow 1}(e^{j\omega})$. Although low frequencies are attenuated by as much as 15 dB, all frequencies above 70 Hz are amplified by as much as 19 dB.

Figure 4-4: Estimated power spectra of the original road noise and the residual error signal, for the ANC simulation designed to attenuate the noise at microphone "1" using speaker "RP" to broadcast the cancelling signal.

Since the human ear is less sensitive to low frequency disturbances, a performance criterion which seeks to minimize the average power of a signal, such as that defined by Equation (2.17), is probably the wrong criterion for the low-frequency road noise and the non-minimum phase transfer functions within an automobile. In Section 4.3, we attempt to address these concerns by using the frequency-weighted LQG formulation.

The importance of the speaker-to-sensor transfer function in determining the achievable noise cancellation is further highlighted by comparing Figures 4-4 and 4-5, which depict the maximum noise attenuation achieved at microphones "1" and "2," respectively, by using the rear passenger-side speaker to broadcast the cancelling signal $r(t)$. Although microphone "1" is nearly three times farther from speaker "RP" than microphone "2," we were nev-
Figure 4-5: Results of the ANC simulation designed to attenuate the noise at microphone "2," using speaker "RP" to broadcast the cancelling signal: a) sensitivity transfer function magnitude, and b) power spectra of the original road noise, $n_2(t)$, and the residual error signal, $e_2(t)$.

Nevertheless, able to achieve nearly five times more attenuation at the first microphone—8.58 dB during interval "A"—than at the second: 1.77 dB during interval "C." Studying Figure 4-5, we see that the undesirable low frequency characteristics of $G_{RP \rightarrow 2}(e^{j\omega})$, previously discussed in Section 3.1.2, allow only 5 dB of attenuation at frequencies below 25 Hz; the non-minimum phase zeros result in as much as 12 dB of noise amplification at frequencies outside the bands 0-25 Hz and 340-460 Hz. Table 4.1 lists the maximum noise attenuation we achieved at microphones "1" and "2" during all three time intervals, using cancelling speaker "RP."

These ANC results highlight one more interesting observation regarding the application of an LQG regulator to active noise control. We found that to maximize noise attenuation, choosing the control weighting $\rho \approx 0.0001$ was reasonable for a speaker-to-microphone
<table>
<thead>
<tr>
<th>MICROPHONE</th>
<th>ATTENUATION</th>
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<tbody>
<tr>
<td></td>
<td>Interval &quot;A&quot;</td>
</tr>
<tr>
<td>&quot;1&quot;</td>
<td>8.58 dB</td>
</tr>
<tr>
<td></td>
<td>($\rho = 0.0004$)</td>
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<tr>
<td>&quot;2&quot;</td>
<td>0.729 dB</td>
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<tr>
<td></td>
<td>($\rho = 0.075$)</td>
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Table 4.1: Maximum attenuation in noise power achieved by the feedback ANC controller at microphones "1" and "2" using cancelling speaker "RP."

transfer function which was conductive to ANC, such as $G_{RP\rightarrow 1}(e^{j\omega})$, whereas fixing $\rho \approx 0.1$ was appropriate for a transfer function which tended to inhibit ANC, such as $G_{RP\rightarrow 2}(e^{j\omega})$. Although additional study is required to determine what makes a "good" transfer function for ANC within an automobile, we may nevertheless conclude that when the plant makes generating an appropriate cancelling signal $r(t)$ difficult for the controller, then we should penalize large control efforts.

4.3 Perceived Effectiveness of the ANC Algorithm

A very real concern regarding the ANC simulations we presented in Section 4.2 is the high-frequency noise amplification, illustrated in Figures 4-4 and 4-5, which is caused by the LQG regulator. Since most people find high-frequency noise more annoying than low frequencies, it is likely that a passenger seated near the error microphone would find the residual signal more irritating than the original road noise, despite attenuation of as much as 8.5 dB in the overall power.

To address these concerns, we experimented with the frequency-weighted, infinite-horizon optimal LQG formulation discussed in Section 2.4.2. Our frequency-weighting filter was an A-filter, whose magnitude response, depicted in Figure 2-8, matches the frequency-dependent, subjective annoyance of humans for a wide range of industrial and transportation noises [14].

Figure 4-6 depicts the sensitivity transfer function magnitude from $n_1(t)$, the road noise recorded at microphone "1" during interval "A," to $e_1(t)$, as well as the estimated power
spectra of the two signals, \( \hat{P}_{n_1}(e^{j\omega}) \) and \( \hat{P}_{e_1}(e^{j\omega}) \). Although the frequency-weighted formulation has eliminated noise amplification above 250 Hz, it has also introduced a dominant tone at 200 Hz and significantly reduced the attenuation in overall noise power, from 8.58 dB to 0.7867 dB. In fact, comparison of the two power spectra reveals that a passenger seated near the error microphone would almost certainly find the residual signal more annoying than the original road noise.

![Sensitivity Transfer-Function from Speaker "RP" to Microphone "1", using the Frequency-Weighted LQG Formulation, Interval "A"](image1)

![Power Spectra of the Road Noise and the Residual Error Signal at Microphone "1", using Cancelling Speaker "RP" and the Frequency-Weighted LQG Regulator, Interval "A"](image2)

Figure 4-6: Results of the frequency-weighted ANC simulation designed to attenuate the road noise at microphone "1," using speaker "RP" to broadcast the cancelling signal: a) sensitivity transfer function magnitude, and b) estimated power spectra of the original noise, \( n_1(t) \), and the residual error signal, \( e_1(t) \).

The frequency-weighted LQG formulation produced similarly poor results for the other speaker-to-microphone transfer functions within the Lumina, often producing overall noise power amplification rather than attenuation. To determine whether the formulation was well-posed for the large plant delays characteristic within an automobile, we repeated the prediction experiment of Section 4.1, after incorporating an A-filter into the feedback controller. Figure 4-7 shows that the attenuation achieved by the frequency-weighted
Figure 4-7: Noise attenuation as a function of prediction time for an ideal $M$-delay plant using the frequency-weighted, infinite-horizon optimal LQG regulator for ANC. Note the overall amplification in noise power for prediction times larger than 3.7 ms.

controller was generally 10-18 dB less than that achieved using the basic LQG formulation. Although it is still possible to achieve significant attenuation for small plant delays, the frequency-weighted controller performed poorly for the large prediction times characteristic within an automobile, producing amplification in overall road noise power for speaker-to-microphone separation distances larger than 1.1 m.
Chapter 5

Conclusions and Future Directions

The research presented in this thesis addressed the fundamental issue of whether single-sensor active noise cancellation in the interior of an automobile is feasible, given the constraints of the existing acoustic system. Using spectral analysis of experimental data, as well as computer simulations in which we designed infinite-horizon optimal LQG regulators for ANC by incorporating models of the road noise and the speaker-to-microphone transfer functions within an automobile, we demonstrated that road noise is indeed highly predictable, and that it possible to achieve attenuation in the overall noise power using the LQG formulation. Our simulations also uncovered some serious hurdles for future research, however, in the form of transfer function characteristics which tend to inhibit ANC, such as large plant delays and nonminimum phase zeros, as well as power spectral modification of the noise field outside the error sensor which may in fact make the residual signal more irritating than the original road noise.

Based upon our experimental results, we would argue that effective, single-sensor ANC within an automobile, could be technically feasible, even under the existing acoustic constraints of an automobile cabin, but that it is likely not feasible using the LQG formulation described in this thesis. In fact, given the frequency dependence of listener’s subjective annoyance to noise described in Section 2.4.2, any controller which is based upon minimization
of a time-domain performance criterion, such as a weighted sum of the power of the error residual and the control effort, is probably the wrong formulation for ANC inside an automobile. Moreover, a feedback controller should be adaptive, as well as robust to uncertainty in the plant and the disturbance. The LQG formulation, by contrast, is only optimal under the admittedly unrealistic assumption that we have completely and accurately modelled the real-world road noise and speaker-to-microphone transfer function, as well as the set of all uncertainties governing these systems.

Future inquiries into the feasibility of pointwise ANC within an automobile should proceed along a number of fronts. Issues we did not address in this thesis include: assessing the effectiveness of modern control techniques based upon frequency-domain performance objectives, such as $H_\infty$ design methods; determining the effect of the cancelling signal upon the noise field at other spatial locations inside the automobile; and exploring what exactly constitutes a "desirable" speaker-to-microphone transfer function for active noise cancellation within an automobile. In their studies of ANC of harmonics within rectangular enclosures, Nelson et al. demonstrated that the production of a node in a noise field is often accompanied by the unintentional production of antinodes at other spatial locations, unless considerable forethought is given to the placement of cancelling speakers [16, 17, 18]. Furthermore, it is quite likely that significant single-sensor noise reduction inside an automobile will not be attained until researchers modify the existing speaker-to-microphone transfer functions, or even alter the speaker locations to be more conducive to ANC at a number of preselected spatial locations inside the automobile. Noise cancelling headphones, for example, require considerable design effort for the transfer function inside each earpiece from the cancelling speaker to the error sensor.

The problem of multi-sensor ANC within in automobile remains an open research question. If future efforts on single-sensor ANC prove fruitful, one could explore multiple-input multiple-output (MIMO) modern control methods; to model the noise field at the multiple error sensors, researchers could employ the stochastic realization techniques developed by Aoki [13]. Alternatively, researchers could extend the work in distributed active noise cancellation recently developed by Zangi [1].
References


