COMPUTER AIDED MACRO-MODELING OF THE ELECTROMECHANICS OF THE TILTING ELASTICALLY SUPPORTED PLATE

by

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Abstract

Simulation of the non-linear dynamics of electrostatically actuated deformable
MEMS structures involves the computationally expensive task of self-consistently
determining the distribution of charges and the structure deformation that its
forces cause. However, there is a class of deformable MEMS structures for which
computation time can be reduced significantly by approximating the structure as
a network of macro-models, which supply the potential and kinetic energies of
the system as functions of a reduced set of degrees of freedom. This paper reports
techniques for taking a macro-modeled electromechanical system from concept to
dynamics simulation. These techniques are applied to the case of an
electromechanical circular plate suspended by idealized springs above an infinite
ground plane. First, software is developed that uses pre-existing tools available in
the MIT MEMCAD package in order to numerically extract capacitance and
stored potential energy as a function of the position and orientation of the plate.
Second, an algebraic model of capacitance is constructed for the suspended
circular plate using an approximate theoretical model, and a non-linear curve
fitting algorithm is used to fit a parameterized version of the algebraic model to
the capacitance data. Finally, a suite of tools is developed that use the
parameterized model in order to simulate and characterize the non-linear
dynamics of the suspended circular plate. The dynamics of the macro-modeled
suspended plate, as an illustration of the results of this approach, is simulated
under a variety of applied voltage waveforms.

Thesis Supervisor: Stephen D. Senturia
Title: Barton L. Weller Professor of Electrical Engineering
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Technology benefits from computer aided design (CAD); however, the appropriate CAD tools are not always available or well developed. This is the case for the field of microelectromechanical systems (MEMS) [1-4]. MEMS structures are machined on semiconductor wafers using existing VLSI technologies. The types of structures can range from gears and motors to deformable thin membranes. The dynamics of electrostatically actuated MEMS structures involves the tight coupling of electrostatic and mechanical forces. In the quasi-static limit, the distribution of charges and the effect of its forces upon the deformation of the structure must be determined self-consistently. In general, this solution only addresses stable states, where potential energy is minimized and kinetic energy is neglected. To extend to the non-linear dynamics case, kinetic energy must be included. Unfortunately, however, the numerical simulation of the general distributed non-linear dynamical system is computationally difficult.

In some cases, it may be practical to model a system as a network of lumped-element macro-models, each having only a small number of degrees of freedom. Each macro-model should be an analytical function that exhibits the
correct dependence upon structural dimensions and material properties while still agreeing with full three-dimensional physical simulation. The macro-model paradigm has three levels [5]. At the lowest level, there is the full three-dimensional physical simulation. In the approach taken here, the capacitance and mechanical stored energy of the system are simulated as functions of a restricted set of displacement and orientation coordinates. The resulting numerical data must then be synthesized into an algebraic form. Therefore, the next level is the development of a functional form for the system. The approach used here is to start with a simplified analytical expression derived from a physical model and enhance the functional form with additional parameters that permit a fit to the three-dimensional simulation. The highest level is full dynamical analysis of the system, which involves the construction of the dynamical equations of motion using the analytical representations of the capacitance and stored energy, and the direct integration of the non-linear equations of motion in order to simulate the dynamics of the system.

The goal of this research is to apply this macro-model paradigm to a particular case, and thus glean insight into the issues that concern the macro-modeling of MEMS structures. This research applies the paradigm to the case of a tilting, circular, capacitive plate suspended by compliant beams that are modeled as linear springs above an infinite ground plane, as depicted in Figure 1. Each beam is treated as having two modes of energy storage: bending and torsion. A voltage is applied between the plate and the ground plane and causes an electrostatic attractive force that draws the plate toward the ground plane. The plate is assumed to be rigid, and therefore the deformation of the system at any given time can be described by three parameters. The first is the distance from the ground plane to the center of the bottom face of the plate; this is referred to as the gap $g$. The remaining two parameters describe the tilt of the plate relative to an $x$-$y$ axis projected up to the plane $z = g$; these are referred to as the tilt angles $\phi_x$ and $\phi_y$. For any combination of $\phi_x$ and $\phi_y$ an axis can be found that remains in the plane of the untilted plate. Thus, the tilting can be characterized alternately by a principle tilt angle $\phi_p$ and a principle axis angle $\phi_l$. All of these parameters are
depicted in Figure 2. The state of the system at any given time can be expressed by six parameters: $g$, $\phi_x$, $\phi_y$ and their time derivatives. Throughout this report, the term "state" is used to refer to only the position parameters $g$, $\phi_x$, and $\phi_y$ in order to illustrate the dependence of the energy domains upon position, even though there are a total of six state variables, the positive coordinates and their time derivatives, as explained below.

In the suspended circular plate system, there are five energy storage
elements, four beams and one capacitor. These elements contribute to the number of state variables; however, there are two constraints to the system that restrict the independence of the storage elements. The first is the rigidness of the suspended plate. The energy stored in each of the beams is a function of the four coordinates of attachment of the beams with the plate. Because the plate is rigid, one of these coordinates can be determined by the other three. Thus, there are only three independent energy storage elements among the four beams. The second constraint is the fact that the capacitor is attached to a voltage source. The electrostatic energy in the capacitor is determined by the charge on the plate, but the attached voltage source fixes that charge. This eliminates the independence of the capacitor storage element. Thus, the remaining independent energy storage elements are the three beams, each of which contributes one state variable for position and one for an associated inertial term. This yields six state variables for the suspended circular plate system.

This paper presents the techniques for carrying the concept of a system through the various levels of macro-modeling using the suspended circular plate as an example. In Chapter 2, an efficient process is described that is designed to perform full three-dimensional simulation of the capacitance of the structure over a range of deformations in $g$ and $\phi_p$. In Chapter 3, an algebraic model is constructed that represents the capacitance of the suspended circular plate as a function of $g$ and $\phi_p$. In Chapter 4, a simulator is implemented that uses this algebraic model to predict the dynamics of the suspended circular plate.
In order to macro-model the electrostatic nature of an electromechanical structure, a numerical model must be constructed that represents the electrostatic energy of the system as a function of the deformational state of the structure. For a linear capacitor connected to a voltage source $V$, the electrostatic energy is equivalent to $\frac{1}{2}Cv^2$, where the capacitance $C$ is the only state dependent parameter in the function. Thus, the stored electrostatic energy can be completely described by modeling the capacitance as a function of state.

In order to model the capacitance of a structure, the capacitance is determined by three-dimensional simulation over a range of possible states. Then, a numerical model can be constructed to agree with this data. The process of extracting the capacitance of a structure at a single state can take an hour or more. This time comes not only from the actual three-dimensional simulation but also from the preparation of a model of the structure for simulation and the post-processing of the simulation output. It is therefore necessary to devise an efficient technique that will repeat the single state capacitance extraction process over multiple states.
This chapter presents the design of a multiple state capacitance extraction process using existing tools in the MIT MEMCAD package [6]. First, the single state capacitance extraction process is described. Then, three successive designs of a multiple state capacitance extraction process are presented. Finally, the last of these is used to perform multiple state capacitance extraction for the suspended circular plate structure.

2.1 Single State Capacitance Extraction

There are three tools from the MIT MEMCAD package that are used in the single state capacitance extraction process. The first is I-DEAS, a commercial CAD package, capable of constructing three-dimensional solid models and meshed models (FEMs) from these solid models [7]. The second is FastCap, a three-dimensional capacitance extraction program that, given a finite element model with a surface mesh of a set of conductors, determines the capacitance matrix for the conductors [8]. The third is MEMBase, a suite of tools and a library of C++ functions that enables the in-memory manipulation of meshed structures [9].

The process of single state capacitance extraction is outlined in Figure 3. First, a solid model of the structure is constructed in I-DEAS. This model is then surface meshed, creating a meshed model of the structure, which will be referred to here as a finite element model (FEM). This FEM is written to disk, and I-DEAS may be exited. I-DEAS can write an FEM according to several file formats. However, it is incapable of writing in a format that FastCap accepts. MEMBase includes an executable that can translate an I-DEAS Universal (UNV) FEM file format to the Patran Neutral File (PNF) FEM file format, which is accepted by FastCap. This executable is used to translate the FEM, and then the new FEM is passed to FastCap.

FastCap returns a capacitance matrix that describes charges in terms of applied voltages measured relative to a ground at infinity. This capacitance matrix must be post-processed, because the electrostatic forces on the structure are derived from the capacitance with voltages measured between conductors,
subject to the constraint of zero charge at infinity. Jansen and Lang presented a
discussion of this issue and developed a program, called PostCap [10]. This
program performs this post-processing and returns the desired capacitance
matrix. Thus, PostCap is used to obtain the desired capacitance information.

2.2 Multiple State Capacitance Extraction

Three designs of a multiple state capacitance extraction processes are
presented: a manually performed process, an automated version of the manual
process, and an optimized automated process. The optimized automated process
is used later for the multiple state capacitance extraction of the suspended circular
plate.

2.2.1 Process A - Manual

The first multiple state capacitance extraction process is outlined in Figure
4. First, a solid model of the structure is constructed in an arbitrary state $x$. This is followed by repeating the rest of the capacitance extraction process for the desired set of states. In each pass through the loop, the solid model is deformed to a new state $x$, the FEM is generated, and the rest of the single state capacitance extraction process is completed, yielding the capacitance at $x$.

Each step in this process must be executed manually. One advantage of this is that the mesh can be refined for each state of the structure. Overall, however, the manual process is unnecessarily slow and time consuming, taking weeks to extract the capacitances at tens of state points. Therefore, it is imperative that the bulk of this process be automated. Another disadvantage of this technique is that if there are $n$ states for which capacitance is to be extracted, then nearly all of the steps must be repeated $n$ times. In order to optimize this process, not only must it be automated, but the amount of $O(n)$ work, i.e. the number of steps that are repeated for each extraction state, must be reduced. This can accomplished by introducing the dependence upon state at a later point in the process.
2.2.2 Process B - AutoGen and AZ

The next design is an automated version of the previous process. Three programs were written for this purpose. The first is an I-DEAS macro, called AutoGen, that for each desired state in state space deforms the given structure to that state, meshes the deformed structure, and saves the FEM to disk. The second is a UNIX shell script, called AZ, that takes the generated FEMs and passes them through the rest of the capacitance extraction process. However, because PostCap requires user input, it cannot be incorporated directly into an automated process. In order to enable automation, a clone of PostCap was written, called ReCap. ReCap is identical to PostCap in function except that it can be controlled entirely by command line options. Furthermore, it can be used as a filter to the FastCap output. Effectively, FastCap and ReCap can be used in the same command, extracting the desired capacitance in one step. The code for AutoGen, AZ, and ReCap is presented in Appendix A.

Figure 5 depicts the path of this process. In spite of the improvement over
the manual method, there are still several disadvantages to this technique. First, because of the inflexibility of the I-DEAS macro language, AutoGen must be written specifically for each class of structure; e.g. the version of AutoGen developed for this research is written specifically for the suspended circular plate model. Next, there is no communication between AutoGen and AZ; e.g. it is not possible to dynamically choose the state points for capacitance extraction based upon the previous results. Finally, although the process has been automated, the amount of $O(n)$ work has not been reduced. While the dependence upon human interaction is eliminated, the process still takes several days of computation time to generate the FEMs and then several more days to extract the corresponding capacitances. In order to optimize this automated process, the amount of $O(n)$ work is reduced by removing the dependence upon I-DEAS for model manipulation, as explained below.

2.2.3 Process C - Churn

In the third and final design of a multiple state capacitance extraction process, the dependence upon I-DEAS as a model manipulator is eliminated by transferring the task to a new MEMBase application called Churn. The code for Churn is given in Appendix A. Churn is designed to accept any number of I-DEAS Universal files of named finite element models, assemble them into one internal FEM, and deform the internal model repeatedly, writing each deformation to a Patran Neutral File and passing it through FastCap and ReCap.

The process is depicted in Figure 6. By delaying the dependence upon state, the amount of $O(n)$ work is significantly reduced. Only one model needs to be generated for each part of the structure. Churn translates the FEMs only once as the files are read. Because all structure deformation is done in memory, the models never have to be re-read. Although this version of Churn is written with the assumption that a circular plate and a ground plane base will be submitted, and although the only structure deformations that it performs are the translation and rotation of the plane, MEMBase has the capacity to apply complex
deformations to meshes. Thus it will be possible to construct an advanced version of Churn that will allow multiple state capacitance extraction for arbitrary structures with arbitrary state complexity.

2.3 Suspended Circular Plate Multiple State Capacitance Extraction

In order to develop the macro-model for the electrostatic component of the suspended circular plate structure, Churn is used to perform the multiple state capacitance extraction. In this section, the state space sample points, the plate and ground plane FEM meshes, and the extracted capacitance information are presented.

2.3.1 State Space Sample Points

The model of the suspended circular plate system assumes that the plate is constrained to translation in gap and rotation (tilt). Although the plate is allowed
two degrees of freedom in tilt, the capacitance sees them as being degenerate. This effect is due to the circular symmetry of the plate, which makes the capacitance independent of $\phi_l$ and dependent only upon $\phi_p$. Thus, the extraction spans over two state parameters: the gap $g$ and the principle tilt $\phi_p$. The points in state space at which capacitance is to be extracted are shown in Figure 7. The plate cannot tilt beyond the point of contact with the ground plane, restricting the states by $R \sin \phi_p < g$. Because capacitance changes more rapidly as gap decreases, the density of state sample points is increased as gap is decreased in order to insure an accurate representation of the capacitance.

2.3.2 Mesh Construction

For the case of the suspended circular plate, two meshes are created. The first is the circular plate shown in Figure 8. This surface mesh contains 2756 triangular elements whose side lengths range from 2.5\(\mu m\) to 5\(\mu m\). The plate has dimensions of a radius of 50\(\mu m\) and a thickness of 5\(\mu m\). For the second mesh, it is
impossible to construct an infinite ground plane, so a finite ground plane is constructed with dimensions of 200x200µm. The mesh of this structure contains 4225 (65x65) square shell elements with side lengths of about 3µm as shown in Figure 9.

When constructing finite element models for FastCap, it is important to refine the mesh sufficiently to obtain an accurate representation of the capacitance, yet it must be coarse enough to minimize computation time. The
refinements of the plate and ground plane meshes are chosen by searching for the
coearest combination of refinements such that a slight increase in either
refinement does not change the extracted capacitance by more than 1%. An
example of an assembled plate and ground plane FEM created by Churn is shown
in Figure 10.

2.3.3 Extracted Capacitance Information

Figure 11 presents a sample of the capacitance extracted through Churn.
This multiple state capacitance extraction for over 500 data points took one and a
half days. In the following chapter, this data is compared to an approximate
algebraic model and used to fit a parameterized form of that model, thereby
creating a sufficiently accurate mathematical model of the capacitance of the
suspended circular plate as a function of state.

Figure 10: Assembled Plate and Ground Plane FEM
Figure 11: Suspended Circular Plate Extracted Capacitance
Figure 11 (cont): Suspended Circular Plate Extracted Capacitance
CHAPTER 3

Numerical Model

In order to simulate the dynamics of a macro-modeled system efficiently, each component of the system must be described by a quickly calculable function. These functions should be based upon physics and be modified to correspond with experiment or simulation. In this chapter, a numerical model of the capacitance of the suspended circular plate is constructed. First, an algebraic model is derived based upon an approximation of tilted plate capacitance. Then, this model is parameterized and fit to the data obtained in the multiple state capacitance extraction process performed in the previous chapter.

3.1 Algebraic Model

Two approximations are presented for calculating the capacitance of a flat tilted structure suspended above a ground plane; these are the parallel plate and the wedge approximations. These approximations are then shown to be related by a factor dependent only upon tilt. Finally, the wedge approximation is applied to the suspended circular plate to produce an algebraic model of its capacitance.
3.1.1 Parallel Plate Approximation

The parallel plate approximation is applied to an arbitrarily shaped flat conductor suspended above a ground plane. The x-z cross-section of this such system is shown in Figure 12. The coordinate system is chosen so that the plate is tilted about the y axis and the ground plane lies in the x-y plane. In this approximation, every differential element \( \Delta x \) of the plate is treated as a parallel plate capacitor, because all of the electric fields are perpendicular to the ground plane.

The capacitance is obtained by calculating the total stored energy in the electric fields. The energy is given by:

\[
U_E = \frac{1}{2} \varepsilon_0 \iint E^2 dv
\]

where \( E \) is the electric field strength. Assuming that the length of the plate in the y direction is given by \( L_y(x) \), where \( L_y(x) \) goes to zero at the limits \( x_1 \) and \( x_2 \), the integral becomes:

\[
U_E = \frac{1}{2} \varepsilon_0 \int_{x_1}^{x_2} \int_{0}^{x \tan \varphi_p} \int_{0}^{L_y(x)} E^2 dy dz dx
\]

In the parallel plate capacitor approximation, the electric field is constant between the plates, and its magnitude is given by:

\[
E = \frac{V}{d}
\]

where \( d \) is the distance between the plates. In this approximation, \( d = x \tan \varphi_p \). It
follows that

\[ U_E = \frac{1}{2} \varepsilon_0 \int_{x_1}^{x_2} \int_0^{x \tan \varphi_p} L_x(x) \left( \frac{V}{x \tan \varphi_p} \right)^2 dy dz dx \]  

(4)

Evaluating the two innermost integrals yields

\[ U_E = \frac{1}{2} V^2 \left[ \frac{\varepsilon_0}{\tan \varphi_p} \int_{x_1}^{x_2} \frac{L_x(x)}{x} dx \right] \]  

(5)

In this expression, the term in square brackets represents total capacitance between the plate and ground plane in the parallel plate approximation limit:

\[ C_{pp} = \frac{\varepsilon_0}{\tan \varphi_p} \int_{x_1}^{x_2} \frac{L_x(x)}{x} dx \]  

(6)

3.1.2 Wedge Approximation

The wedge approximation is also applied to an arbitrarily shaped flat conductor suspended above a ground plane. The wedge approximation is that all electric fields are perpendicular to the plate and curve downward to meet the ground plane perpendicularly. The x-z cross-section is shown in Figure 13. Again, the coordinate system is chosen such that the plate is tilted about the y axis and the ground plane lies in the x-y plane.

The capacitance is obtained by calculating the total stored energy in the electric fields. Assuming that the length of the plate in the y direction is given by
\( L_r(r) \), where \( L_r(r) \) goes to zero at the limits \( r_1 \) and \( r_2 \), the integral becomes

\[
U_E = \frac{1}{2} \varepsilon_0 \int_{r_1}^{r_2} \int_0^{\varphi_p} \int_0^{L_r(r)} \int_0^{E^2 r dy} \, dr \, d\theta
\]  
(7)

For the wedge approximation capacitor, the electric field is uniform along the path of the field lines. The distance that the field line travels is \( d = r \varphi_p \). Thus, the integral becomes:

\[
U_E = \frac{1}{2} \varepsilon_0 \int_{r_1}^{r_2} \int_0^{\varphi_p} \int_0^{L_r(r)} \left( \frac{V}{r \varphi_p} \right)^2 r dy d\theta dr
\]

(8)

Evaluating the two innermost integrals yields:

\[
U_E = \frac{1}{2} V^2 \left[ \frac{\varepsilon_0 \int_{r_1}^{r_2} L_r(r)}{\varphi_p} \right] \int_{r_1}^{r_2} \frac{r dy d\theta dr}{r}
\]

(9)

where the term in square brackets represents total capacitance between the plate and ground plane in the wedge approximation limit:

\[
C_w = \frac{\varepsilon_0 \int_{r_1}^{r_2} L_r(r)}{\varphi_p} \int_{r_1}^{r_2} \frac{r dy d\theta dr}{r}
\]

(10)

3.1.3 Relationship Between Parallel Plate and Wedge Approximations

The capacitance of a flat plate suspended above a ground plane in the parallel plate approximation has been shown to be

\[
C_{pp} = \frac{\varepsilon_0 \int_{x_1}^{x_2} L_x(x)}{\tan \varphi_p x_i} \, dx
\]

(11)

Similarly, the capacitance of such plate in the wedge approximation can be expressed as

\[
C_w = \frac{\varepsilon_0 \int_{r_1}^{r_2} L_r(r)}{\varphi_p} \int_{r_1}^{r_2} \frac{r dy d\theta dr}{r}
\]

(12)

These two approximations differ only by a constant factor. The relationship

\[
C_{pp} = \frac{\tan \varphi_p x_i}{x} \cdot C_w
\]
between $x$ and $r$ is given by

$$x = r \cos \varphi_p$$  \hspace{1cm} (13)$$

Thus, the differential elements are related by

$$dx = dr \cos \varphi_p$$  \hspace{1cm} (14)$$

By definition,

$$L_x(x) = L_r(r)$$  \hspace{1cm} (15)$$

and their limits of integration correspond:

$$x_1 = r_1 \cos \varphi_p \text{ and } x_2 = r_2 \cos \varphi_p$$  \hspace{1cm} (16)$$

Therefore,

$$\int_{x_1}^{x_2} \frac{L_x(x)}{x} \, dx = \int_{r_1}^{r_2} \frac{L_r(r)}{r} \, dr$$  \hspace{1cm} (17)$$

Thus, the parallel plate and wedge approximations for the capacitance of a tilted flat plate suspended above a ground plane are related by

$$(\tan \varphi_p) \, C_{pp} = (\varphi_p) \, C_w$$  \hspace{1cm} (18)$$

In the examples considered here, $\varphi_p$ never exceeds 0.1 radians, so these models differ by less than 0.4%.

### 3.1.4 Suspended Circular Plate Algebraic Model

The wedge approximation is chosen to determine the algebraic model of the capacitance of the suspended circular plate. The plate is oriented according to the wedge approximation as depicted in Figure 14. $L_r(r)$ is given by

$$L_r(r) = 2 \sqrt{R^2 - (r - r_0)^2} \text{ for } r_0 - R < r < r_0 + R$$  \hspace{1cm} (19)$$

and is zero elsewhere. Thus, the capacitance is expressed by

$$C_w = \frac{\varepsilon_0 r_0 + R}{\varphi_p r_0 - R} \int \frac{2 \sqrt{R^2 - (r - r_0)^2}}{r} \, dr$$  \hspace{1cm} (20)$$
The change of variables of \( r \to r + r_0 \) yields

\[
C_w = \frac{\varepsilon_0 R}{\phi_p R} \int_{r_0}^{r} \frac{2 \sqrt{R^2 - r^2}}{r + r_0} \, dr
\]

(21)

Another change of variables of \( r \to Rr \) yields

\[
C_w = 2 \frac{\varepsilon_0 R^2}{\phi_p R} \int_{1}^{-1} \frac{\sqrt{1 - r^2}}{r + (r_0/R)} \, dr
\]

(22)

According to Gradshteyn and Ryzhik [11]

\[
\int_{-1}^{1} \frac{\sqrt{1 - x^2}}{x \pm p} \, dx = \pm \pi \left[ p - \sqrt{p^2 - 1} \right] \text{ for } p \geq 1
\]

(23)

Using this identity, the capacitance becomes

\[
C_w = 2 \frac{\varepsilon_0 R^2}{\phi_p} \pi \left[ \left( \frac{r_0}{R} \right) - \sqrt{\left( \frac{r_0}{R} \right)^2 - 1} \right]
\]

(24)

Substituting \( r_0 = g / (\sin \phi_p) \) into this equation yields the final expression for capacitance:

\[
C = 2 \pi \frac{\varepsilon_0 R}{\phi_p \sin \phi_p} \left[ \left( \frac{g}{R} \right) - \sqrt{\left( \frac{g}{R} \right)^2 - \sin^2 \phi_p} \right]
\]

(25)

This is used as the algebraic model of the capacitance of the suspended circular plate, which is the starting point for the macro-model.

It should be noted that this capacitance function cannot be evaluated numerically at \( \phi_p = 0 \) because of the division by zero. Although this function has a limit as \( \phi_p \to 0 \), the \( \phi_p \) terms in the denominator cause extreme numerical error for small values of \( \phi_p \). In order to reduce this error, the first few terms of a Taylor
series expansion of the capacitance about $\varphi_p = 0$ are used to represent the capacitance for small values of $\varphi_p$. In this limit, the capacitance becomes

$$
C_{\varphi_p \to 0} = \frac{\varepsilon_0 \pi R^2}{g} - \frac{1}{12} \frac{\varepsilon_0 \pi R^2 (2g^2 - 3R^2)}{g^3} \varphi_p^2 + O(\varphi_p^4)
$$

(26)

The threshold at which the small $\varphi_p$ limit for capacitance is applied is determined by trial and error to be $\varphi_p = 10^{-4}$ radians.

### 3.2 Fitting an Algebraic Model to Numerical Data

An algebraic model based upon approximations does not necessarily match the data that would be obtained by experiment or three-dimensional simulation. Figure 15 depicts the algebraic model of the suspended circular plate capacitance alongside the multiple state capacitance extraction data obtained from the three-dimensional simulation of the plate by FastCap. It can be seen that although the algebraic model of Equation 25 has the same form as the data, the model has inaccuracies.

In order to improve the accuracy of Equation 25, fitting parameters are incorporated into the algebraic model. These parameters are chosen to have physical meaning. For the suspended circular plate model, three parameters are defined. The first two parameters are used to define an effective gap $\tilde{g} = \alpha_1 g$ and an effective plate radius $\tilde{R} = \alpha_2 R$. The third parameter is used to account for the fringing of electric fields. A fringing term based upon approximations of the capacitance per unit length of a parallel plate capacitor, $\left(1 + \alpha_3 \left(\frac{\tilde{g}}{R}\right)\right)$, is multiplied to the algebraic model [12,13]. The parameterized model becomes

$$
C = 2\pi \frac{\varepsilon_0 \tilde{R}}{\varphi_p \sin \varphi_p} \left[ \left( \frac{\tilde{g}}{R} \right) - \sqrt{\left( \frac{\tilde{g}}{R} \right)^2 - \sin^2 \varphi_p} \right] \left(1 + \alpha_3 \left(\frac{\tilde{g}}{R}\right)\right)
$$

(27)

and in the small $\varphi_p$ limit:

$$
C_{\varphi_p \to 0} = \frac{\varepsilon_0 \pi \tilde{R}^2}{\tilde{g}} \left(1 + \alpha_3 \left(\frac{\tilde{g}}{R}\right)\right) - \frac{1}{12} \frac{\varepsilon_0 \pi \tilde{R}^2 (2\tilde{g}^2 - 3\tilde{R}^2)}{\tilde{g}^3} \left(1 + \alpha_3 \left(\frac{\tilde{g}}{R}\right)\right) \varphi_p^2 + O(\varphi_p^4)
$$

(28)

The Levenberg-Marquardt non-linear fitting algorithm is used to find the
Figure 15: Comparison of Theoretically Based Algebraic Model with Capacitance Data Extracted by FastCap
Figure 15 (cont): Comparison of Theoretically Based Algebraic Model with Capacitance Data Extracted by FastCap
optimum values for these alpha parameters. A MATLAB script of this algorithm is written by porting the code presented by Press, Flannery, Teukolsky, and Vetterling [14]. The MATLAB code for this non-linear fitting algorithm is presented in Appendix A. This program requires that derivatives of the function to be fit are taken with respect to the fitting parameters. Two variables are defined to simplify the expressions for these derivatives:

\[ r \equiv \frac{\tilde{g}}{\tilde{R}} \]  

(29)

and

\[ s = \sqrt{r^2 - \sin^2 \phi} \]  

(30)

Differentiating the capacitance with respect to the fitting parameters yields

\[
\begin{bmatrix}
\frac{\partial}{\partial \alpha_1} \\
\frac{\partial}{\partial \alpha_2} \\
\frac{\partial}{\partial \alpha_3}
\end{bmatrix} C = 2\pi \varepsilon_0 \tilde{R} \left[ \begin{array}{c}
\frac{1}{\alpha_1} \left( r \left( \frac{1 - r}{s} \right) \left( 1 + \alpha_3 r \right) + \alpha_3 r \left( r - s \right) \right) \\
-\frac{1}{\alpha_2} \left( r \left( \frac{1 - r}{s} \right) \left( 1 + \alpha_3 r \right) + \alpha_3 r \left( r - s \right) \right) \\
r \left( r - s \right)
\end{array} \right]
\]  

(31)

and in the small \( \phi \) limit:

\[
\begin{bmatrix}
\frac{\partial}{\partial \alpha_1} \\
\frac{\partial}{\partial \alpha_2} \\
\frac{\partial}{\partial \alpha_3}
\end{bmatrix}_{\phi \to 0} C = \varepsilon_0 \pi \tilde{R} \left[ \begin{array}{c}
\frac{1}{\alpha_1} \left( -\frac{2r^2 - 6\alpha_3 r - 9\phi^2}{r} \right) \\
\frac{2 + \alpha_3 r}{r} - \frac{2\alpha_3 r + 4r^2 - 9\alpha_3 r - 12}{12r^3} \phi^2 \\
1 - \frac{2r^2 - 3}{12r^2} \phi^2
\end{array} \right]
\]  

(32)

This non-linear fitting algorithm also requires that each data point be assigned a standard deviation that represents its significance in the fitting. The multiple state capacitance extraction data for the suspended circular plate are assigned uniform standard deviations.
The fitting parameters are found to be:

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>1.097</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_2$</td>
<td>1.056</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>3.847</td>
</tr>
</tbody>
</table>

Table 1: Non-Linear Fit Results

Figure 16 shows the parameterized model using these results plotted alongside the capacitance data extracted by FastCap and the unparameterized model. This fit corresponds sufficiently for the purposes of this demonstration, and thus it is used as the numerical model of the suspended circular plate capacitance.

In the next chapter, it will be shown that the derivatives of the capacitance with respect to $g$ and $\varphi_p$ are also needed to determine the electrostatic forces. These derivatives are

$$
\left[ \begin{array}{c} \frac{\partial}{\partial g} \\ \frac{\partial}{\partial \varphi_p} \end{array} \right] C = 2\pi \frac{\varepsilon_0}{\varphi_p \sin \varphi_p} \left[ \alpha_1 \left( \left( 1 - \frac{r}{s} \right)(1 + \alpha_3 r) + \alpha_3 (r - s) \right) \right] \left( R (1 + \alpha_3 r) \left( \frac{\sin \varphi_p \cos \varphi_p}{s} - (r - s) \left( \frac{1}{\varphi_p} + \frac{1}{\tan \varphi_p} \right) \right) \right]$$

(33)

and in the small $\varphi_p$ limit:

$$
\left[ \begin{array}{c} \frac{\partial}{\partial g} \\ \frac{\partial}{\partial \varphi_p} \end{array} \right] C_{\varphi_p \to 0} = \varepsilon_0 \pi \left[ \begin{array}{c} \alpha_1 \left( - \frac{1}{r^2} + \frac{2r^2 - 6\alpha_3 r - 9}{12r^4} \varphi_p^2 \right) \\ R \left( 1 + \alpha_3 r \right) \left( \frac{2r^2 - 3}{6r^3} \varphi_p \right) \end{array} \right]
$$

(34)
Figure 16: Comparison of Unfitted and Fitted Algebraic Models with Capacitance Data Extracted by FastCap
Figure 16 (cont): Comparison of Unfitted and Fitted Algebraic Models with Capacitance Data Extracted by FastCap
In the preceding chapters, a numerical model of the capacitance of the suspended circular plate was developed. In this chapter, a suite of tools is developed that use this model to simulate and analyze the dynamics of suspended circular plate structures under arbitrary driving voltages. First, the theory of a modular dynamics simulator is presented. This theory is then applied to the suspended circular plate by the implementation of a simulator that depends upon modules that describe the system: the algebraic model of capacitance, a function of voltage with respect to time, and the dimensions and material parameters of the suspended plate and its beam supports. Next, a technique for determining the equilibrium state and normal modes about equilibrium of the system is presented. Finally, the dynamics simulator is used to simulate the complex dynamics of the suspended circular plate.
4.1 Modularizing the Equations of Motion

The dynamical equations of motion can be expressed as

\[ \ddot{x} = f(x, \dot{x}, t) \] (35)

where \( x \) is a vector that describes the displacement of the system as a function of time, and \( f \) is a vector function that returns the acceleration of the displacement as a function of position, velocity, and time. It is a misnomer to describe these as vectors, because the term implies that a common unit system exists among their components. For example, a macro-modeled system could have displacement coordinates in terms of position, angle, or bending, each of which would be defined in terms of a different dimensional unit. In actuality, these are arrays or multivalued variables, but for simplicity they are here referred to as vectors.

Solving the equations of motion on a computer is done by numerical integration. First, the second order differential equations of motion are reduced to standard state-equation form, where now both position \( x \) and velocity \( \dot{x} \) appear as true state variables:

\[ \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ f(x, \dot{x}, t) \end{bmatrix} \] (36)

If the acceleration function \( f(x, \dot{x}, t) \) can be implemented on a computer, this equation can be integrated numerically to solve for the motion of the system.

Here, an acceleration function that contains only stored energy terms is discussed; however, the functionality for including friction and driving forces exists. The force applied by stored energy is determined by linearizing the potential energy about the state of the system. Thus, the acceleration of each state component is derived from the derivative of the potential energy function with respect to that state component. The acceleration of the state can be written as

\[ \ddot{x} = \begin{bmatrix} \ddots \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = -M^{-1} \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} U(x, t) \] (37)
where $M$ is a matrix containing the inertial terms for the respective displacement coordinates, and $U(x,t)$ is the stored energy.

The above formulation implies that state acceleration is proportional to force by some inertial constant. This is true for the case of a force on a particle

$$F = \frac{dp}{dt} = m\frac{dv}{dt} = m\frac{d^2 x}{dt^2}$$

(38)

The force and state acceleration are proportional by the mass. However, torque and angular acceleration are not separated easily. For the case of a torque on an object with moment of inertia tensor $\{I\}$,

$$N = \frac{dL}{dt} = \frac{d}{dt} \{I\} \dot{\theta}$$

(39)

The torque $N$ and angular velocity $\dot{\theta}$ carry associated coordinate systems that must be transformed to the coordinate system of the object. By including matrices $\lambda_{ij}$ that transform the coordinate system from $j$ to $i$, the torque becomes

$$N = \frac{d}{dt} \left( \lambda_{N0} I_{00} \lambda_{00} \frac{d\theta}{dt} \right) = \frac{d\lambda_{N0}}{dt} I_{00} \lambda_{00} \frac{d\theta}{dt} + \lambda_{N0} I_{00} \frac{d\lambda_{00}}{dt} \frac{d\theta}{dt} + \lambda_{N0} I_{00} \lambda_{00} \frac{d^2 \theta}{dt^2}$$

(40)

The angular acceleration $\frac{d^2 \theta}{dt^2}$ is not proportional to torque, because the transforms are functions of state. For this dynamics simulator implementation it is assumed that the tilt angles are in the small deflection limit, and torque is approximated as

$$N = I_{00} \frac{d^2 \theta}{dt^2}$$

(41)

In the determination of forces by the differentiation of potential energy, it is not necessarily convenient to differentiate $U$ with respect to the state variables. The chain rule is applied to alleviate this problem by separating the differentiation vector into the product of a derivative matrix and a new
differentiation vector taken with respect to a preferred coordinate basis $\tilde{x}$:

$$
\frac{\partial}{\partial x_i} = \left[ \begin{array}{c} \frac{\partial \tilde{x}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{x}_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}_1}{\partial x_n} & \cdots & \frac{\partial \tilde{x}_n}{\partial x_n} \end{array} \right] \frac{\partial}{\partial \tilde{x}}
$$

(42)

This is expressed more simply by introducing a shorthand notation:

$$
\frac{\partial}{\partial x} = \frac{\partial \tilde{x}}{\partial x} \cdot \frac{\partial}{\partial \tilde{x}}
$$

(43)

Thus, the acceleration function for a system of potential energy functions are expressed by

$$
\ddot{x} = -M^{-1} \sum_i \frac{\partial \tilde{x}_i}{\partial x} \cdot \frac{\partial U_i}{\partial \tilde{x}_i}
$$

(44)

The preceding formulation describes a process for constructing a modular dynamics simulator. First, a set of parameters is chosen that fully describes the state of the system, $x$. Second, the components of the potential energy, $U_i$, are defined. Third, a set of preferred coordinates, $\tilde{x}_i$, is chosen for each potential energy function. Fourth, each potential energy function is differentiated with respect to its preferred coordinates, $\frac{\partial U_i}{\partial \tilde{x}_i}$. Fifth, a derivative matrix of the preferred coordinates with respect to the state coordinates is implemented, $\frac{\partial \tilde{x}_i}{\partial x}$. Finally, the inertia matrix $M$ is constructed and inverted. These modules combine to form a numerical model of the equations of motion, which can then be numerically integrated to yield the simulated dynamics.

### 4.2 Modular Equations of Motion for the Suspended Circular Plate

A dynamics simulator that solves for the motion of the macro-modeled suspended circular plate is implemented by following the process outlined above. The code developed for this simulator is presented in Appendix A.
First, the positional state of the system is defined in terms of three parameters:

\[
\mathbf{x} = \begin{bmatrix} g \\ \phi_x \\ \phi_y \end{bmatrix}
\]  \hspace{1cm} (45)

where \( g \), \( \phi_x \), and \( \phi_y \) are the gap and tilt angles of the suspended circular plate. Next, the potential energy of the plate is classified into two modules: an electrostatic and a mechanical energy component. These modules, in combination with the inverse of the inertia matrix, comprise the simulator.

4.2.1 Electrostatic Contribution

The potential energy stored in a linear capacitor attached to a voltage source is given by

\[
U_E = \frac{1}{2} CV^2
\]  \hspace{1cm} (46)

where \( C \) is the capacitance and \( V \) is the voltage across the capacitor. The only state dependent term in the energy equation is the capacitance; the voltage is purely time dependent. Thus, the voltage can be seen as a time dependent scaling factor to the energy, which is characterized by \( C \).

Two preferred coordinates are inherent within the macro-model of the suspended circular plate capacitance: the gap \( g \) and the primary tilt angle \( \phi_p \). The preferred coordinates thus become

\[
\tilde{\mathbf{x}}(\mathbf{x}) = \begin{bmatrix} g \\ \phi_p \end{bmatrix} = \begin{bmatrix} g \\ \frac{g}{\tan^{-1}\left(\frac{\tan \phi_x}{\sqrt{\tan^2 \phi_x^2 + \tan^2 \phi_y}}\right)} \end{bmatrix}
\]  \hspace{1cm} (47)
and the derivative matrix becomes

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}} = \begin{bmatrix} 1 & 0 \\ \tan \varphi_x \sec^2 \varphi_x & 0 \\ 0 & \tan \varphi_p \sec^2 \varphi_p \\ \tan \varphi_y \sec^2 \varphi_y & 0 \\ 0 & \tan \varphi_p \sec^2 \varphi_p \end{bmatrix}$$

(48)

The electrostatic energy must be declared carefully before it is differentiated. Although $\frac{1}{2}Cv^2$ is equivalent to the energy stored in the electric fields, it actually represents the co-energy [15]. The energy is written in terms of the charges on the conductors $Q$:

$$U_E = \frac{1}{2} \frac{Q^2}{C}$$

(49)

Because the electrostatic forces originate from charges and not voltage, the charge dependent formula is used for differentiation. Differentiating this with respect to the preferred coordinates yields

$$\frac{\partial U_E}{\partial \mathbf{x}} = -\frac{1}{2} \frac{Q^2 \partial C}{C^2 \partial \mathbf{x}} = -\frac{1}{2} V^2 \frac{\partial C}{\partial \mathbf{x}}$$

(50)

This derivative is exactly the negative of that which would be calculated if Equation 46 were interpreted as the energy instead of the co-energy. Thus, attention to the energy versus co-energy is important, even for linear capacitors.

The numerical model developed in the previous chapters is used to supply the derivative of the capacitance. Thus, the electrostatic contribution to the force on the plate is specified completely.

### 4.2.2 Mechanical Contribution

Each support beam is assumed to have two modes of energy storage, bending and torsion. The spring constants for each deformation mode are determined by applying beam theory in the linear regime to the dimensions and material parameters of the beams. Beam theory is a well documented field,
having linearized the restoring forces for many beam designs under various loads. The translational spring is treated as a linearized beam in pure bending that is fixed at one end but is free and guided with a concentrated load at the other, as depicted in Figure 17. The decision to have the support beam guided by

![Figure 17: Translational Beam Deflection](image)

the plate comes from the assumption that the plate is rigid, which implies that the plate will clamp the support beam. The rotational spring is treated as a linearized beam fixed at one end with concentrated torque at the other, as depicted in Figure 18. It is important to note that the rotational model does not take into account the

![Figure 18: Rotational Beam Deformation](image)

simultaneous deflection of the beam, and neither does the translational model take into account the twisting of the beam. Furthermore, neither model accounts for the minute stretching and shearing that occurs. To simplify this implementation, however, these issues are ignored.

The potential energy stored in the linear springs is given by

\[ U = \frac{1}{2} k (\Delta z)^2 + \frac{1}{2} \kappa (\Delta \theta)^2 \]  

(51)

where \( \Delta z \) is the distance by which the tip of the beam is deflected, and \( \Delta \theta \) is the angle by which the tip is rotated. Clearly, \( \Delta z \) and \( \Delta \theta \) are the preferred coordinates
of this energy function. The spring constants $k$ and $\kappa$ are defined from pre-existing models determined by beam theory [16]. For a beam of length $L$, width $w$, thickness $t$, Young's modulus $E$, and Poisson ratio $v$, the spring constant $k$ for a beam that is fixed at one end and free and guided at the other, with a concentrated load at the guided end, is given by

$$k = \frac{Ewt^3}{L^3}$$  \hspace{1cm} (52)

The spring constant $\kappa$ for a beam fixed at one end with concentrated torque at the other is given by

$$\kappa = \frac{Ewt^3}{2(1+v)L} \left( \frac{1}{3} \cdot \frac{3.36}{16} \left( \frac{t}{w} \right) \left( 1 - \frac{1}{12} \left( \frac{t}{w} \right)^4 \right) \right)$$  \hspace{1cm} (53)

The energy is differentiated with respect to the preferred coordinates to yield

$$\frac{\partial U}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial U}{\partial \Delta z} \\ \frac{\partial U}{\partial \Delta \theta} \end{bmatrix} = \begin{bmatrix} k\Delta z \\ \kappa \Delta \theta \end{bmatrix}$$  \hspace{1cm} (54)

Defining $\Delta z$ and $\Delta \theta$ in terms of the state coordinates can yield complicated formulas, in particular, $\Delta \theta$ for a beam whose axis is not parallel or perpendicular to the direction of principle tilt. For simplicity, only beams placed along the $x$ or $y$ axes are considered here. For the case of the $\pm x$ axis,

$$\vec{x}_x(x) = \begin{bmatrix} \Delta z \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} (g - g_0) \mp R \tan \varphi_y \\ \varphi_x \end{bmatrix}$$  \hspace{1cm} (55)

and for the case of the $\pm y$ axis,

$$\vec{x}_y(x) = \begin{bmatrix} \Delta z \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} (g - g_0) \pm R \tan \varphi_x \\ \varphi_y \end{bmatrix}$$  \hspace{1cm} (56)

where $g_0$ is the initial gap between the plate and the base, and $R$ is the radius of
the plate. The derivative matrices are respectively

\[
\frac{\partial \vec{x}}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \pm R (\sec \varphi_y)^2 & 0 \end{bmatrix}
\]

(57)

and

\[
\frac{\partial \vec{x}}{\partial \vec{x}} = \begin{bmatrix} 1 & 0 \\ \pm R (\sec \varphi_x)^2 & 0 \\ 0 & 1 \end{bmatrix}
\]

(58)

Thus, the spring contribution to the force on the plate is specified completely.

4.2.3 Inertia Matrix

In order to model the inertia of the suspended circular plate system, only the mass of the rigid circular plate is considered; the mass of the beams is neglected. The circular plate is treated here as a uniform cylinder with radius \( R \) and thickness \( t \). The mass of the plate is

\[
m = \rho \pi R^2 t
\]

(59)

where \( \rho \) is the mass density of the plate. The moment of inertia about an axis along the face of the plate through the center of the face is given by

\[
I = m \left( \frac{R^2}{4} + \frac{t^2}{3} \right)
\]

(60)

Therefore, the inertia matrix is given by

\[
M = \begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\]

(61)

This matrix is inverted easily and construction of the dynamics simulator is concluded.
4.3 Equilibrium and Normal Modes

It is desirable to be able to determine the equilibrium state and normal modes about equilibrium in order to characterize the behavior of macro-modeled systems. Numerical techniques are developed to determine the equilibrium and normal modes. The code that implements these techniques can be found in Appendix A.

4.3.1 Equilibrium

By definition, equilibrium occurs in a system when the accelerations for all of the state variables go to zero simultaneously. Unless the acceleration functions are readily available in an easily solvable form, finding equilibrium requires a relatively sophisticated zero searching algorithm. Of the many such algorithms available, the damped matrix-free Newton method is chosen.

Newton techniques use an initial guess and an estimated slope of the function in order to extrapolate to more exact guesses repeatedly. It follows that the initial guess passed to the Newton method determines which zero will be found. There are both stable and unstable equilibrium points in this class of electromechanical systems. The type of point the Newton method discovers depends upon the initial guess. The rest state, i.e. the undeflected state at zero voltage, is a good initial guess, because this state is closer to any stable equilibrium state than to any unstable equilibrium state. This is so because the unstable equilibrium states divide stable dynamics and pull-in. Thus, the rest state is used as the initial guess for the Newton method.

The Newton method may have difficulty converging, because the state coordinates and their corresponding accelerations have varied units and magnitudes. Newton methods applied to a function are most effective when the function output has the same order of magnitude as its input and when the individual coordinates share a common magnitude. The gap is recorded on the computer in units of meters, yielding magnitudes on the order of $10^{-6}$. However, the tilt angles, recorded in radians, have magnitudes on the order of $10^{-3}$. 
Furthermore, the accelerations have significantly greater magnitudes, with gap acceleration on the order of $10^3$ and tilt accelerations on the order of $10^8$. The solution to this problematic magnitude variation is to precondition the state and acceleration function. Two diagonal matrices, $P$ and $Q$, are defined to normalize the magnitudes of the state and acceleration respectively to be of order 1:

$$\dot{x} = Px$$  \hspace{1cm} (62) \\
$$\dot{f}(t, x) = Qf(t, x)$$  \hspace{1cm} (63)

A new acceleration function is defined whose inputs and outputs are of order 1:

$$g(t, \dot{x}) = Qf(t, P^{-1}\dot{x})$$  \hspace{1cm} (64)

This normalization enables the Newton solver to converge for $g(t, \dot{x})$ more easily. The initial guess passed to the Newton solver on $g(t, \dot{x})$ must be preconditioned by:

$$\dot{x}_0 = Px_0$$  \hspace{1cm} (65)

and the equilibrium state $\dot{x}_e$, that the algorithm returns must be reconditioned to its original units:

$$x_e = P^{-1}\dot{x}_e$$  \hspace{1cm} (66)

From this theory, a preconditioned acceleration module can be written around the original acceleration function, and an equilibrium determination module can be written to use the preconditioned acceleration module with a damped Jacobian-free Newton solver to find the equilibrium state [17].

4.3.2 Normal Modes

Normal modes are calculated by first expressing the acceleration function, $\ddot{x}$, as a Taylor series expansion about some $x_0$:

$$\ddot{x} = f(x) = f(x_0) + \left(\frac{d}{dx}f(x)|_{x_0}\right)(x - x_0) + \ldots$$  \hspace{1cm} (67)

At equilibrium, $\ddot{x} = 0$. Thus, by expanding about equilibrium and by neglecting
higher order terms, it is found that

\[
\tilde{x} = \left( \frac{d}{dx} f(x) \bigg|_{x_\epsilon} \right) (x - x_\epsilon)
\]  \hspace{1cm} (68)

This equation is modified to match the equation for a simple harmonic oscillator:

\[
\frac{d^2}{dt^2} (x - x_\epsilon) = \left( \frac{d}{dx} f(x) \bigg|_{x_\epsilon} \right) (x - x_\epsilon)
\]  \hspace{1cm} (69)

The eigenvectors \( v_i \) of the matrix \( \frac{d}{dx} f(x) \bigg|_{x_\epsilon} \) determine the normal modes of the dynamical system about equilibrium, and the frequencies of those modes can be determined from the eigenvalues \( \zeta_i \) by

\[
\zeta_i = -\omega_i^2
\]  \hspace{1cm} (70)

The \( \frac{d}{dx} f(x) \bigg|_{x_\epsilon} \) matrix is not Hermitian because, as previously stated, the state vector components do not share a common unit system. A metric would have to be constructed and applied to the state coordinate system before it would become Hermitian. Thus, the eigenvectors are not orthogonal by dot product. In order to extract the contributions of each mode to the total state, a contravariant eigenvector basis \( v^i \) to the covariant basis \( v_i \) must be generated. By definition,

\[
v^i \cdot v_j = \delta^i_j
\]  \hspace{1cm} (71)

where \( \delta^i_j \) is the Kronecker delta. Thus, by placing the contravariant vectors in the rows of a matrix \( V^{\text{row}} \) and the covariant vectors in the columns of another matrix \( V_{\text{col}} \), it follows that

\[
V^{\text{row}} V_{\text{col}} = I
\]  \hspace{1cm} (72)

and thus the contravariant vectors are determined by

\[
V^{\text{row}} = V^{-1}_{\text{col}}
\]  \hspace{1cm} (73)

Therefore, the projections of the state onto each mode are determined by the dot product of that state with the corresponding contravariant vector of the mode.

4.4 Suspended Circular Plate Simulation

In Appendix B, the developed simulator is used to explore the dynamics of
the suspended circular plate in detail; a sample of the results is presented here. The simulated structure is depicted in Figure 19. A rigid circular plate with radius 50µm and thickness 5µm is suspended 5µm above a ground plane by four thin beams. Each beam has length 100µm and width 10µm. Two beams have thickness 2µm, one has thickness 1.85µm and the last has thickness 1.9µm. The plate starts at rest at a gap of 5µm with voltage 0. Voltage is then linearly increased to 150 volts over 10^{-4} seconds, after which voltage is held constant at 150 volts.

Figure 20 shows the motion of the plate, and Figure 21 shows this motion projected onto the calculated normal modes about equilibrium at 150 volts. The dots on each plot represent the integration steps taken, and the horizontal lines represent the calculated equilibrium value at 150 volts. The non-sinusoidal behavior in modes 2 and 3 demonstrates the energy transfer between normal modes due to large displacements about equilibrium. This is an example of the complex non-linearity of electromechanical motion.

For an extended set of dynamical simulations, see Appendix B.
Figure 20: Sample Dynamics
Figure 20 (cont): Sample Dynamics
Figure 21: Sample Dynamics - Mode Projection
Figure 21 (cont): Sample Dynamics - Mode Projection
In this research, the ability to carry a structure concept through the macro-modeling process has been demonstrated. The suspended circular plate has undergone full three-dimensional simulation via the multiple state capacitance extraction process. A physically based algebraic model has been fit to the simulation data. Finally, this model has been used to predict and characterize the dynamics of a suspended circular plate system.

Several issues regarding this macro-modeling process remain untouched. One is that the multiple state capacitance extraction process can be further optimized by incorporating MEMBase into FastCap, thereby eliminating the time taken to write out FEM files for FastCap to read. Another is that developing an approximate theoretical model of a physical system is not always possible. It might be beneficial to derive some curve fitting scheme that will construct a functional form for the system using only the obtained data without any approximate function to base it on. It might be further possible to combine the capacitance extraction tools with the numerical modeling tools, thereby permitting a numerical modeling routine to dynamically control what data is
obtained via capacitance extraction. A final issue is that the inertia of the system relative to the state coordinates is treated as being constant, although it can depend upon state. Some thought must go into deriving a general way for extracting the state dependence from the inertia of the system.

The tools developed in this paper are far from being able to construct macro-models of a given structure automatically. However, they do provide the functionality to simulate the dynamics of a suspended plate system with arbitrary plate shape. These tools also provide a framework within which specialized tools for particular structures can be constructed. In particular, the various modules which automate highly repetitive tasks reduce the total time required to construct an accurate macro-model by several orders of magnitude. Thus, a promising start has been made in the practical linking of three-dimensional numerical simulation of meshed structures to corresponding lumped-element dynamical macro-models.
A.1 Capacitance Extraction

A.1.1 AutoGen

This code is written in the I-DEAS program file language.

```plaintext
K : #input "Enter Gap start: " GapStart
K : #input "Enter Gap step: " GapStep
K : #input "Enter Gap stop: " GapStop
K : #input "Enter Thick start: " ThickStart
K : #input "Enter Thick step: " ThickStep
K : #input "Enter Thick stop: " ThickStop
K : #input "Enter Phi start: " PhiStart
K : #input "Enter Phi step: " PhiStep
K : #input "Enter Phi stop: " PhiStop
C : =---------------------------------------------------------------
C : Turning off forms...
K : /o p p 2
K :
K : fd of
K : OKAY
K : OKAY
C : =---------------------------------------------------------------
C : Leave part parameters on after update...
K : /up uo ap no OKAY
C : =---------------------------------------------------------------
```
C : Loop 1 (Gap)
C : Loop 1 Init
K : #Gap = GapStart
C : -----------------------------------------------
C : Loop 1 Test
K : #L1TEST:
K : #if (NOT (Gap LE GapStop)) then goto L1FAIL
C : -----------------------------------------------
C : Loop 1 Block
C : Edit parameters...
K : /mo e
K : lab
K : Til
K : pt
K : di
K : v
C : Here is the gap (1phi, 2thick, 5gap)
K : 5
K :
K : EQ Gap
K : OKAY
K :
C : -----------------------------------------------
C : -----------------------------------------------
C : Loop 2 (Thick)
C : Loop 2 Init
K : #Thick = ThickStart
C : -----------------------------------------------
C : Loop 2 Test
K : #L2TEST:
K : #if (NOT (Thick LE ThickStop)) then goto L2FAIL
C : -----------------------------------------------
C : Loop 2 Block
C : Edit parameters...
K : /mo e
K : lab
K : Til
K : pt
K : di
K : v
C : Here is the thickness (1phi, 2thick, 5gap)
K : 2
K :
K : EQ Thick
K : OKAY
K :
C : -----------------------------------------------
C : -----------------------------------------------
C : Loop 3 (Phi)
C : Loop 3 Init
K : #Phi = PhiStart
C : -----------------------------------------------
C : Loop 3 Test
K : #L3TEST:
K : #if (NOT ((Phi LE PhiStop) AND (50*SIN(Phi) LT Gap))) then goto L3FAIL
C : -------------------------------
C : Loop 3 Block
C : Edit parameters...
K : /mo e
K : lab
K : Til
K : pt
K : di
K : v
C : Here is the phi (1phi, 2thick, 5gap)
K : 1
K :
K : EQ Phi
K : OKAY
K :
C : -------------------------------
K : #Filename = "g"+Gap++"t"+Thick++"p"+Phi++".unv"
C : Here is where you update
K : /up p
C : Here is where you save the file
K : /f exp s 1
K :
K : OKAY
K : EU 3
K : FN Filename
K : Y
K : OKAY
C : -------------------------------
C : Loop 3 Incr
K : #Phi = Phi+PhiStep
K : #goto L3TEST
K : #L3FAIL:
C : ----------------------------------------
C : Loop 2 Incr
K : #Thick = Thick+ThickStep
K : #goto L2TEST
K : #L2FAIL:
C : ----------------------------------------
C : Loop 1 Incr
K : #Gap = Gap+GapStep
K : #goto L1TEST
K : #L1FAIL:
C : ----------------------------------------
C : Turning forms back on...
K : /o p p 2
K :
K : fd on
K : OKAY
K : OKAY
E : **** END OF SESSION ****
A.1.2 AZ

This code is written in the UNIX tcsh shell script language.

```
#!/usr/local/bin/tcsh
unalias rm cp mv

set TARGETDIR = ~/tilted_plate
source ~/bin/HP_rsh
foreach i ( $* )
    echo "Analyzing $i"
    set SOURCEDIR = $i:h
    if( "$SOURCEDIR" == "$i" ) then
        set SOURCEDIR = .
    endif
    set ROOT = $i:t:r
    if( !( -r $TARGETDIR/archive/$ROOT.tgz ) ) then
        rm -f $TARGETDIR/$ROOT.unv
        rm -f $TARGETDIR/$ROOT.pat
        rm -f $TARGETDIR/$ROOT.fc
        rm -f $TARGETDIR/$ROOT.cap
        cp -f $SOURCEDIR/$ROOT.unv $TARGETDIR
        echo "Unv2Pnf"
        RSH gatekeeper "unalias cp rm mv; cd /usr/local/scratch/
        ldgababay; \n        cp $TARGETDIR/$ROOT.unv .; \n        $TARGETDIR/unv2pnf.sun $ROOT.unv $ROOT.pat; \n        rm $ROOT.unv; \n        mv -f $ROOT.pat $TARGETDIR" >& /dev/null
        echo "FastCap"
        RSH memcad3 "cd $TARGETDIR; fastcap.alpha $ROOT.pat > $ROOT.fc" >& /dev/null
    echo "Recap"
    RSH blofeld "cd $TARGETDIR; (recap.hp -e -15 -r 1 -F 1 0 -i $ROOT.fc -o -) | tee $ROOT.cap cap/$ROOT.cap"
    echo "Spawning Archiver..."
    RSH madonna "cd $TARGETDIR; tar cvf - $ROOT.unv $ROOT.pat $ROOT.fc $ROOT.cap | gzip -9 > archive/$ROOT.tgz; rm -f $ROOT.*" >& /dev/null &
    endif
    echo "Done! ($i)"
end
```

A.1.3 ReCap

This code is written in C++.

```
recap.C

#include <stdio.h>
#include <string.h>
```
#include <fstream.h>

#include "Matrix.H"
#include "FCinfo.H"

FCinfo fci;
Matrix<Double> C(0);
Matrix<int> I_floating(0), I_fixed(0);
int N_conductors, N_floating, N_fixed, ref, preferredExponent;

int isExponent=0;
int isInputSet=0;
int isOutputSet=0;
int isReferenceSet=0;
int isFloatSet=0;
int isFixedSet=0;

char infile[35];
char outfile[35];
istream *is;
ostream *os;

void processArgs(int argc, char *argv[]) {
    int current_arg = 1;

    while(current_arg<argc) {
        if(argv[current_arg][0] == '-')
            switch(argv[current_arg++][1]) {
            case 'e':
                isExponent=1;
                preferredExponent = atoi(argv[current_arg]);
                current_arg++;
                break;
            case 'r':
                isReferenceSet=1;
                ref = atoi(argv[current_arg]);
                current_arg++;
                break;
            case 'f': { // fixed
                isFixedSet=1;
                isFloatSet=0;
                I_floating.zero();
                N_fixed = atoi(argv[current_arg++]);
                int i;
                I_fixed.resize(1,N_fixed);
                for(i=0;i<N_fixed;i++)
                    I_fixed(0,i) = atoi(argv[current_arg++]);
                break;
            }
            case 'F': { // float
                isFloatSet=1;
                isFixedSet=0;
                I_fixed.zero();
            }
        }
    }
}
N_floating = atoi(argv[current_arg++]);
int i;
I_floating.resize(1,N_floating);
for(i=0;i<N_floating;i++)
  I_floating(0,i) = atoi(argv[current_arg++]);
break;
}
case 'i':
  isInputSet=1;
  strncpy(infile,argv[current_arg]);
  current_arg++;
  break;
case 'o':
  isOutputSet=1;
  strncpy(outfile,argv[current_arg]);
  current_arg++;
  break;
}
}
}

inline int isPipe(char *str)
{
  return ((str[0]=='-')&&(str[1]=='\0'));
}

void secureInput()
{
  if(!isInputSet) {
    printf("Enter fastcap output filename: ");
    gets(infile);
  }
  if(isPipe(infile))
    is = new ifstream(cin);
  else
    is = new ifstream(infile);
}

void secureFCinfo()
{
  (*is) >> fci;
  N_conductors = fci.getNumberOfConductors();
}

void securePreferredExponent()
{
  if(!isExponent)
    preferredExponent = fci.getUnitExponent();
}

void secureReference()
{
  if(!isReferenceSet) {

char refs[2];
printf("Reference conductor? ");
gets(refs);
ref = atoi(refs);
}

void secureFloatFixed()
{
#define DEBUG

cerr << "[ secureFloatFixed()]" << endl;
#endif

int i, j, k;
if(isFixedSet) {
    N_floating = N_conductors - N_fixed;
    I_floating.resize(1, N_floating);

    for(i=0, k=0; (i<N_conductors) && (k<N_floating); i++) {
        for(j=0; j<N_fixed; j++)
            if(I_fixed(0, j)==i)
                break;
        if(j==N_fixed)
            I_floating(0, k++) = i;
    }
} else if(isFloatSet) {
    N_fixed = N_conductors - N_floating;
    I_fixed.resize(1, N_fixed);

    for(i=0, k=0; (i<N_conductors) && (k<N_fixed); i++) {
        for(j=0; j<N_floating; j++)
            if(I_floating(0, j)==i)
                break;
        if(j==N_floating)
            I_fixed(0, k++) = i;
    }
} else {
    N_floating = 1;
    I_floating.resize(1, N_conductors);
    I_floating(0, 0) = 0;

    N_fixed = 1;
    I_fixed.resize(1, N_conductors);
    I_fixed(0, 0) = ref;

    char fofixs[1];
    for (i=1; (i==ref??i;i)<N_conductors;i++) {
        do {
            printf("Is conductor %d's potential floating or controlled (f,c)? ",i);
            gets(fofixs);
        } while((fofixs[0]!="c")||(fofixs[1]!="f"));
        if (fofixs[0] == 'f')
            I_floating(0, N_floating++) = i;
        /* "controlled" == "fixed" */
if (flofixs[0] == 'c')
    I_fixed(0,N_fixed++) = i;
}
}
#endif DEBUG
    cerr << " ] secureFloatFixed()" << endl;
#endif

} 

void secureOutput()
{
    if(!isOutputSet) {
        printf("Enter output filename: ");
        gets(outfile);
    }
    if(isPipe(outfile))
        os = new ofstream(cout);
    else
        os = new ofstream(outfile);
}

void floatConductor(int f, Matrix<
double>& C)
{
    #ifdef DEBUG
        cerr << " [ floatConductor()" << endl;
    #endif
    int i, j;

    for(i=0;(i==f?++i:i)<C.getM();i++)
        for(j=0;(j==f?++j:j)<C.getN();j++)
            C(i,j) -= C(i,f)*C(f,j)/C(f,f);
    for(i=0;i<C.getM();i++)
        C(i,f) = C(f,i) = 0;
    #ifdef DEBUG
        cerr << C << " ] floatConductor()" << endl;
    #endif
}

void processData()
{
    #ifdef DEBUG
        cerr << " [ processData()" << endl;
    #endif
    int i, j, k;

    C = fcl.getCapacitanceMatrix();
    for(i=0;i<N_floating;i++)
        floatConductor(I_floating(0,i),C);

    int N_out;
    N_out = N_fixed;
    for(i=0;i<N_fixed;i++)
        if(I_fixed(0,i)==ref)
            N_out--;
Matrix<double> S(N_conductors,N_out);
for(i=0,j=0;(i<N_fixed)&&(j<N_out);i++)
    if(I_fixed(0,i)!=ref)
        S(I_fixed(0,i),j++) = 1;

C = S.transpose()*C*S;

#ifdef DEBUG
    cerr << C << " ] processData()" << endl;
#endif

double pow10(int i)
{
    if(i) {
        if(i<0)
            return 1/pow10(-i);
        else
            return 10.0*pow10(i-1);
    } else {
        return 1;
    }
}

void exportData()
{
    int i = fci.getUnitExponent() - preferredExponent;
    double a = pow10(i);
    (*os) << a*C << " 10^" << preferredExponent << " farads" <<
    endl;
}

void initialize(int argc, char *argv[])
{
    processArgs(argc, argv);
    secureInput();
    secureFCinfo();
    securePreferredExponent();
    secureReference();
    secureFloatFixed();
    secureOutput();
}

int main(int argc, char *argv[])
{
    initialize(argc, argv);
    processData();
    exportData();
    delete is;
    delete os;
    return 0;
}
#include <iostream.h>
#include <math.h>

class FCinfo
{
    int number_of_conductors;
    int unit_exponent;
    Matrix<double> capacitance_matrix;
public:
    FCinfo() : number_of_conductors(0), unit_exponent(0),
            capacitance_matrix(1) {}  
    void setNumberOfConductors(int nc)
    {
        number_of_conductors = nc+1;
        capacitance_matrix.resize(number_of_conductors);
    }
    int getNumberOfConductors()
    {
        return number_of_conductors;
    }

    void setUnitExponent(int ue)
    {
        unit_exponent = ue;
    }
    int getUnitExponent()
    {
        return unit_exponent;
    }

    Matrix<double>& getCapacitanceMatrix() { return capacitance_matrix; }

    friend istream& operator>>(istream&, FCinfo&);
    friend ostream& operator<<(ostream&, FCinfo&);
};

int getUnits(const char *unitString)
{
    switch(unitString[0]) {
    case 'a': // atto
        return -18;
    case 'f': // farad, femto
        switch(unitString[1]) {
        case 'a': // farad
            return 0;
        case 'e': // femto
            return -15;
        }
    case 'm': // micro, milli
        switch(unitString[2]) {
    
}
case 'c': // micro
    return -6;
    case 'l': // milli
    return -3;
    }
    case 'n': // nano
    return -9;
    case 'p': // pico
    return -12;
    default: // i don't know
    return 0;
    }
}

#define FCLINE (30)
#define FCEOL (is.get(buf,FCLINE, '\n'); is.get(buf[0]));
#define FCSKIP (is.get(buf,FCLINE, ' ');

istream& operator>>(istream& is, FCom& fci)
{
    char buf[FCLINE];
    int foo;

    FCSKIP;
    is >> foo;
    fci.setNumberOfConductors(foo);
    FCEOL;

    fci.getCapacitanceMatrix().zero();
    int i, j;
    double maxfoo = 0.0;
    for (i=1;i<fci.getNumberOfConductors();i++) {
        FCSKIP;
        for (j=1;j<fci.getNumberOfConductors();j++) {
            is >> (fci.getCapacitanceMatrix()(i,j));
            if(fabs(fci.getCapacitanceMatrix()(i,j))>maxfoo)
                maxfoo = fabs(fci.getCapacitanceMatrix()(i,j));
            (fci.getCapacitanceMatrix())(0,j) -= (fci.getCapacitanceMatrix())(i,j);
            (fci.getCapacitanceMatrix())(i,0) -= (fci.getCapacitanceMatrix())(i,j);
        }
    (fci.getCapacitanceMatrix())(0,0)
       -= (fci.getCapacitanceMatrix())(i,0);
    FCEOL;
}

    int newfoo = (int) floor(log10(maxfoo));
    fci.setUnitExponent(-6+newfoo);
    fci.getCapacitanceMatrix() = fci.getCapacitanceMatrix()*pow(10,-newfoo);

    #ifdef DEBUG
    cerr << fci;
#endif
        return is;
    }
#undef FCLINE
#undef FCEOL

ostream& operator<< (ostream& os, FCIInfo& fci)
{
    os << "Conductors: " << fci.number_of_conductors << endl
    << "UnitExponent: " << fci.unit_exponent << endl
    << "Capacitance Matrix: " << endl
    << fci.capacitance_matrix;
    return os;
}

Matrix.H

#include <iostream.h>
#include <stdlib.h>

template <class T>
class Matrix
{
    int m, n;
    T** matrix;
    T dummy_element;
    int isExist() const { return (m&&n); }

    void initialize(int, int);
    void copy(const Matrix<T>&);
    void destroy();
    void swap(int, int);
public:
    Matrix(int _m, int _n) { initialize(_m,_n); }
    Matrix(int _m) { initialize(_m,_m); }
    Matrix(Matrix& A) : m(0), n(0) { copy(A); }
    ~Matrix() { destroy(); }
    T& operator[](int, int) const;
    void zero();
    void resize(int, int);
    void resize(int _m) { resize(_m,_m); }
    friend ostream& operator<<(ostream&, const Matrix<T>&);
    Matrix<T>& operator=(const Matrix<T>& x) { copy(x); return *this; }
    friend Matrix<T> operator+(const Matrix<T>&, const Matrix<T>&);
    friend Matrix<T> operator-(const Matrix<T>&, const Matrix<T>&);
    friend Matrix<T> operator*(const Matrix<T>&, const Matrix<T>&);
    friend Matrix<T> operator*(T, const Matrix<T>&);
    friend Matrix<T> operator*(const Matrix<T>& A, T c) { return c*A; }
    Matrix<T> inverse() const;
    Matrix<T> transpose() const;
    int getM() const { return m; }
int getN() const { return n; }
);

template <class T>
inline T& Matrix<T>::operator[](int i, int j)
{
  if((i<m)&&(j<n))
    return matrix[i][j];
  else
    return dummy_element;
}

template <class T>
inline void Matrix<T>::initialize(int _m, int _n)
{
  #ifdef DEBUG
  cerr << "[ Matrix::initialize(" << _m << "," << _n << ")" << endl;
  #endif
  m = _m;
  n = _n;
  if(isExist()) {
    int i;
    matrix = new (T*)[m];
    for(i=0;i<m;i++)
      matrix[i] = new T[n];
    zero();
  }
  #ifdef DEBUG
  cerr << "] Matrix::initialize(" << m << "," << n << ")" << endl;
  #endif
}

template <class T>
inline void Matrix<T>::copy(const Matrix<T>& x)
{
  #ifdef DEBUG
  cerr << "[ Matrix::copy" << endl;
  #endif
  resize(x.m,x.n);
  if(isExist()) {
    int i,j;
    for(i=0;i<m;i++)
      for(j=0;j<n;j++)
        (*this)(i,j) = x(i,j);
  }
  #ifdef DEBUG
  cerr << "] Matrix::copy" << endl;
  #endif
}
inline void Matrix<T>::destroy()
{
    #ifdef DEBUG
    cerr << "[ Matrix::destroy(" << m << "," << n << ")" << endl;
    #endif
    if(isExist()) {
        int i;
        for(i=0;i<m;i++)
            delete[] matrix[i];
        delete[] matrix;
    }
    m = n = 0;
    #ifdef DEBUG
    cerr << "] Matrix::destroy(" << m << "," << n << ")" << endl;
    #endif
}

template <class T>
inline void Matrix<T>::resize(int _m, int _n)
{
    if((_m!=_m)||(_n!=_n)) {
        destroy();
        initialize(_m,_n);
    }
    zero();
}

template <class T>
inline void Matrix<T>::swap(int i, int j)
{
    if(isExist()) {
        T *temp = matrix[i];
        matrix[i] = matrix[j];
        matrix[j] = temp;
    }
}

template <class T>
void Matrix<T>::zero()
{
    if(isExist()) {
        int i, j;
        for(i=0;i<m;i++)
            for(j=0;j<n;j++)
                (*this)(i,j) = 0;
    }
}

template <class T>
ostream& operator<<(ostream& os, const Matrix<T>& x)
{
    #ifdef DEBUG
    cerr << "[ operator<<(ostream&,Matrix<T>&)" << endl;
    #endif

if(x.isExist()) {
    int i, j;
    os.setf(ios::fixed, ios::floatfield);
    os.precision(10);
    for(i=0; i<x.m; i++) {
        for(j=0; j<x.n; j++)
            os << x(i, j) << " ";
        os << endl;
    }
}
#endif DEBUG
    cerr << "] operator<<\n\n\n\nMatrix<T> operator+(const Matrix<T>& a, const Matrix<T>& b) {
    if((a.m!=b.m) || (a.n!=b.n)) {
        cerr << "Matrices cannot be added\n";
        exit(1);
    }

    Matrix<T> new_matrix(a.m,a.n);

    if(new_matrix.isExist()) {
        int i, j;
        for(i=0; i<a.m; i++)
            for(j=0; j<a.n; j++)
                new_matrix(i,j) = a(i,j) + b(i,j);
    }

    return Matrix<T>(new_matrix);
}

template <class T>
Matrix<T> operator-(const Matrix<T>& a, const Matrix<T>& b) {
    if((a.m!=b.m) || (a.n!=b.n)) {
        cerr << "Matrices cannot be subtracted\n";
        exit(1);
    }

    Matrix<T> new_matrix(a.m,a.n);

    if(new_matrix.isExist()) {
        int i, j;
        for(i=0; i<a.m; i++)
            for(j=0; j<a.n; j++)
                new_matrix(i,j) = a(i,j) - b(i,j);
    return Matrix<T>(new_matrix);
}

template <class T>
Matrix<T> operator*(const Matrix<T>& a, const Matrix<T>& b)
{
    #ifdef DEBUG
    cerr << "[ Matrix::operator*(M&,M&)" << endl << a << b;
    #endif
    if(a.n!=b.m) {
        cerr << "Matrices cannot be multiplied\n";
        exit(1);
    }

    Matrix<T> new_matrix(a.m,b.n);
    if(new_matrix.isExist()) {
        int i, j, k;
        for(i=0;i<a.m;i++)
            for(j=0;j<b.n;j++)
                for(k=0;k<a.n;k++)
                    new_matrix(i,j) += a(i,k)*b(k,j);
    }
    #ifdef DEBUG
    cerr << new_matrix << "] Matrix::operator*(M&,M&)" << endl;
    #endif
    return Matrix<T>(new_matrix);
}

template <class T>
Matrix<T> operator*(T a, const Matrix<T>& b)
{
    Matrix<T> new_matrix(b.m,b.n);

    if(new_matrix.isExist()) {
        int i, j;
        for(i=0;i<b.m;i++)
            for(j=0;j<b.n;j++)
                new_matrix(i,j) = a*b(i,j);
    }
    return Matrix<T>(new_matrix);
}

template <class T>
Matrix<T> Matrix<T>::inverse()
{
    if(m!=n) {
        cerr << "Matrix cannot be inverted\n";
        exit(1);
    }
    if(!isExist()) {
        #ifdef DEBUG
        cerr << "Matrix 0x0 faking inversion\n";
        #endif
        return Matrix<T>();
    }

    Matrix<T> temp(b.m,b.n);
    if(temp.isExist()) {
        int i, j;
        for(i=0;i<b.m;i++)
            for(j=0;j<b.n;j++)
                temp(i,j) = b(i,j);

        Matrix<T> new_matrix(a.m,b.n);
        if(new_matrix.isExist()) {
            int i, j, k;
            for(i=0;i<a.m;i++)
                for(j=0;j<a.n;j++)
                    for(k=0;k<b.n;k++)
                        new_matrix(i,j) += a(i,k)*temp(k,j);
        }
        return Matrix<T>(new_matrix);
    }
    return Matrix<T>();
}
# endif
    
    return Matrix<T>({0});
}

int i, j, k;

Matrix<T> inv(m);
for(i=0;i<n;i++) {
    for(j=0;j<n;j++)
        inv(i,j) = 0;
    inv(i,i) = 1;
}

Matrix<T> lu(m);
lu = *this;

for(i=0;i<n-1;i++) {
    for(j=i+1;j<n;j++)
        if(lu(j,i)>1e-15)
            break;
    if(j>i) {
        lu.swap(i,j);
        inv.swap(i,j);
    }

    for(j=i+1;j<n;j++) {
        lu(j,i) /= lu(i,i);
        for(k=i+1;k<n;k++)
            lu(j,k) -= lu(j,i)*lu(i,k);
    }
}

for(k=0;k<n;k++) {
    for(i=0;i<n;i++)
        for(j=0;j<i;j++)
            inv(i,k) -= lu(i,j)*inv(j,k);
    for(i=n-1;i>=0;i--) {
        for(j=i+1;j<n;j++)
            inv(i,k) -= lu(i,j)*inv(j,k);
        inv(i,k) /= lu(i,i);
    }
}

return Matrix<T>(inv);
}

template <class T>
Matrix<T> Matrix<T>::transpose()
{
    Matrix<T> new_matrix(n,m);

    if(isExist()) {
        int i, j;
        for(i=0;i<m;i++)
            for(j=0;j<n;j++)
for (j = 0; j < n; j++)
    new_matrix(j,i) = (*this)(i,j);
}

return Matrix<T>(new_matrix);
}

A.14 Churn

This code is written in C++.

churn.c

#include <stdio.h>
#include <unistd.h>
#include <stdlib.h>
#include <sys/time.h>
#include "global_defs.h"
#include "PNF.h"
#include "UpdateNodes.h"
#include "transform.h"
#include "transforms.h"
#include "PNF_node_list.h"
#include "elementTempList.h"
#include "C_funics.h"
#include "churn_fns.h"

struct tms timestuff;
long timecount = 0;
#define SUMTMS(t) ((long)(t.tms_utime+t.tms_stime+t.tms_cutime+t.tms_cstime))
#define CLK_TCK (sysconf(_SC_CLK_TCK))
#define STARTTIME (times(&timestuff); timecount -= SUMTMS(timestuff);
#define STOPTIME (times(&timestuff); timecount += SUMTMS(timestuff);
#define TOTALTIME (((double)timecount)/CLK_TCK)
#define CURRENTTIME
    (times(&timestuff),((double)(timecount+SUMTMS(timestuff)))/
    CLK_TCK)
#define PRINTTIME(t) {printf("\t\t\tttime: %8.2f
",(t));

int main(int argc, char *argv[])
#define OUTPUT_FN "output.pat"
{
    STARTTIME;
    PRINTTIME(CURRENTTIME);

    PATTRAN_neutral_file *structure;
if(!(structure = assembleNewPNF(argv-1,argv+1))) {
    showUsage(argv[0]);
    exit(1);
}

STOP TIME;
Name *plateName = chooseName(structure);
START TIME;

PRINTTIME(CURRENTTIME);
printf("copying old nodes starting\n");
PNF_Node_List *oldNodes = structure->getListOfNodes()->copy();
printf("copying old nodes done\n");
PRINTTIME(CURRENTTIME);

structure->setFileName("output.pat");

FILE *outfile = fopen("testdata","a");
{
    int i;
    fprintf(outfile,"#files: ");
    for(i=1;i<argc;i++)
        fprintf(outfile," %s",argv[i]);
    fprintf(outfile,"\n");
    fflush(outfile);
}

double gap;
int gCount;
double phi;
int phiCount;
double phir;
double cap;
char capstring[80];

FILE *capfile;
PRINTTIME(CURRENTTIME);
printf("beginning to churn\n");
simulateExtractFaces1(structure);
phi=0;
for(gap=0.5;gap<=5;gap/=25*gap,gap+=0.33/sqrt(1+cap*cap))
    for(phiCount=0,php=0;phiCount<10;php=atan(gap/50)*((++phiCount)/10))
    {
        double innerlooptime = CURRENTTIME;

        structure->getListOfNodes()->updateCoordsFrom(oldNodes);
        plateTransform(plateName,gap,phi,phir);
        simulateExtractFaces2(structure);
        structure->writeSurfaces();

        unlink("fastcap.cap");
system("memcap output.pat");
        unlink("output.pat.qs");
unlink("output.cap");
system("rememcap.hp -e -15 -r 1 -F 1 0 -i fastcap.cap -o -
| tee output.cap");
capfile = fopen("output.cap","r");
scanf(capfile, "%s", capstring);
fclose(capfile);
if (php==0)
  cap = atof(capstring);

fprintf(outfile, "%13.10f\t%13.10f\t%13.10f\t%13.10f\t%4s\n", gap,php,phi
r, capstring);
fflush(outfile);

PRINTTIME(CURRENTTIME-innerlooptime);
PRINTTIME(CURRENTTIME);

unlink("assembled.pat");
fclose(outfile);
return 0;
}

churn_fns.h

PATRAN_neutral_file *assembleNewPNF(int numFileNames, char
 *FileNames[]);

void showUsage(char *executable);

Name *chooseName(PATRAN_neutral_file *structure);

void setM_plate(double mat[4][4], double gap, double php, double
 phi);

void plateTransform(Name *nameToTransform, double gap, double
 phi, double phi);

void simulateExtractFaces1(PATRAN_neutral_file *structure);

void simulateExtractFaces2(PATRAN_neutral_file *structure);

churn_fns.c

#include <stdio.h>
#include <unistd.h>
#include <stdlib.h>
#include <sys/times.h>
#include "global_def.h"
#include "PNF.h"
#include "UpdateNodes.h"
#include "transform.h"
#include "transforms.h"
#include "PNF_node_list.h"
#include "elementTempList.h"
#include "C_funcs.h"

PATRAN_neutral_file *assembleNewPNF(int numFileNames, char
    *FileNames[])
{
    if (!numFileNames)
        return NULL;

    PATRAN_neutral_file *mainPNF;

    mainPNF = new PATRAN_neutral_file;
    mainPNF->setFileName(FileNames[0]);
    mainPNF->readFile();

    if (numFileNames > 1) {
        PATRAN_neutral_file **PNFs;
        PNFs = (PATRAN_neutral_file **) malloc((numFileNames-1)*sizeof(PATRAN_neutral_file *));

        int count;
        for (count = 0; count < numFileNames-1; count++) {
            PNFs[count] = new PATRAN_neutral_file;
            PNFs[count]->setFileName(FileNames[count+1]);
            PNFs[count]->readFile();
        }

        mainPNF->addFiles(PNFs, numFileNames-1, RENUMBER);

        for (count = 0; count < numFileNames-1; count++)
            delete PNFs[count];
        free(PNFs);
    }

    mainPNF->setFileName("assembled.pat");
    mainPNF->writePNF(0);
    delete mainPNF;
    mainPNF = new PATRAN_neutral_file;
    mainPNF->setFileName("assembled.pat");
    mainPNF->readFile();

    return mainPNF;
}

void showUsage(char *executable)
{
    printf("syntax:\t%s <fem-file> [<fem-file>]\n", executable);
}

Name *chooseName(PATRAN_neutral_file *structure)
{
    Name *aName;

    int nameCount;
for(aName = (Name *) structure->getListOfNames() -
   >setCurrentToBeginning(), nameCount=1;
   aName;
   aName = (Name *) structure->getListOfNames()->Next(),
   nameCount++)
   printf("\t%d. \"%s\"\n",nameCount,aName->getNameString());

int whichCount;
whichCount=0;
while((whichCount<1)||(whichCount>=nameCount)) {
   printf("\nSelect plate name: ");
   scanf("%d",&whichCount);
}

for(aName = (Name *) structure->getListOfNames() -
   >setCurrentToBeginning(), nameCount=1;
   nameCount<whichCount;
   aName = (Name *) structure->getListOfNames()->Next(),
   nameCount++);

printf("%s chosen\n",aName->getNameString());
return aName;
}

void setM_plate(double mat[4][4], double gap, double phi, double phir)
{
   double rotr[4][4];
   double rotp[4][4];
   double tran[4][4];
   double temp[4][4];
   double A[3];

   phi *= 180/3.1415926;
   phir *= 180/3.1415926;

   id_mat(tran);
   makeTranslate(tran,A);

   id_mat(rotr);
   makeRotate(rotr,phir,A);

   id_mat(rotp);
   makeRotate(rotp,phi,A);

   multM(temp,rotp,rotr);
   multM(mat,tran,temp);
}

void plateTransform(Name *nameToTransform, double gap, double phi,
   double phir, double phir)
{  
double mat[4][4];
  Transformation *trans = new Transformation;

  setM_plate(mat,gap,phi,pphi);
  trans->setmatrix(mat);
  nameToTransform->applyNodeTransformToName(trans);
}

void simulateExtractFaces1(PATRAN_neutral_file *structure)  
{  
  structure->getFaces();
  structure->removeInteriorFaces();
  structure->organizeFaces();
}

void simulateExtractFaces2(PATRAN_neutral_file *structure)  
{  
  structure->checkFacePlanarity(1.0e-3);
  structure->tagSubPatchesbyNormal();
}

A.2 Numerical Model

This code is written in the MATLAB script language.

A.2.1 Non-Linear Fitting

nlfit.m

function []=nlfit

data = readCap('testdata.circ.55');
N = size(data,1);

x = data(:,1:2)*diag([1e-6 1]);
y = data(:,4)*1e-15;

sig = 0.01*ones(N,1);
a0 = [1;1;0];

a = mrqmin(x,a0,y,sig,'capfn');
y0 = capfn(x,a0);
yfit = capfn(x,a);

fid = fopen('output','w');
for i=1:length(a),
  fprintf(fid,'a(%d) = %15.13e
',i,a(i));
end
fprintf(fid,'gap\tphi_p\tfastcap\tideal\tfit\n');
for i=1:N,
    fprintf(fid,'%12.8e\t%12.8e\t%12.8e\t%12.8e\t%12.8e\n',x(i,1),
           x(i,2),y(i),y0(i),yfit(i));
end
fclose(fid);

mrqmin.m

function [a] = mrqmin(x,a0,y,sig,funcs);

a = a0;
[alpha,beta,minchisq] = mrqcoef(x,a,y,sig,funcs);
lambda = 0.001;

fprintf(1,'initial Chi^2 = %12.8f\n',minchisq);
for i=1:length(a),
    fprintf('a%d = %15.13e\n',i,a(i));
end

da = 1;
while norm(da)>1e-8,
    alphap = alpha + diag(lambda*diag(alpha));
    da = prec(alphap,beta);
    [alpha,beta,chisq] = mrqcoef(x,a + da,y,sig,funcs);
    if (chisq < minchisq),
        a = a + da;
        minchisq = chisq;
        lambda = lambda * 0.1;
    else
        lambda = lambda * 10.0;
    end
    fprintf(1,'Chi^2 = %12.8e Min(Chi^2) = 
            %12.8e\n',chisq,minchisq);
    for i=1:length(a),
        fprintf('a%d = %15.13e\n',i,a(i));
    end;
end

mrqcoef.m

function [alpha, beta, chisq] = mrqcoef(x,a,y,sig,func);

%alpha: #param x #param
%beta: #param x 1
%chisq: 1 x 1

[ymod,dyda] = feval(func,x,a);
dy = y - ymod;
sig2i = 1.0./((sig.^2));
wdyda = diag(sig2i)*dyda;

alpha = wdyda'*dyda;
beta = wdyda'*dy;
chisq = sum(dy.^2.*sig2i);

**prec.m**

function [x] = prec(A,b);

P = diag(1./sqrt(diag(A)));
x = P*((P*A*P)\(P*b));

**capfn.m**

function [C,dCda] = capfn(q,a);
%capfn:
% assumes q = [gap(m) phi_p(rad)]
% returns [C,dCda]

N = size(q,1);
C = zeros(N,1);
dCda = zeros(N,3);

eps0 = 8.8541878162e-12;
R = a(2)*50e-6;

for i=1:N,
g = a(1)*q(i,1);
phi = q(i,2);
r = g/R;

if abs(phi)<=1e-4,
    C(i) = eps0*pi*R * ( (1+a(3)*r)/(r - (2*r^2-3)*(1+a(3)*r)*phi^2/(12*r^3) ));
dCda(i,1) = eps0*pi*R * ( -1/r + (2*r^2-6*a(3)*r-9)*phi^2/(12*r^3) ) / a(1);
dCda(i,2) = eps0*pi*R * ( (2+a(3)*r)/(r - (4*r^2+2*a(3)*r^3-9*a(3)*r-12)*phi^2/(12*r^3) ) / a(2);
dCda(i,3) = eps0*pi*R * ( 1 - (2*r^2-3)*phi^2/(12*r^2) ));
elseif
    s = sqrt(r^2-sin(phi)^2);
    C(i) = 2*pi*eps0*R/(phi*sin(phi)) * (r-s)*(1+a(3)*r);
    dCda(i,1) = 2*pi*eps0*R/(phi*sin(phi)) * (r*(1-r/s)*(1+a(3)*r) + a(3)*r*(r-s))/a(1);
    dCda(i,2) = 2*pi*eps0*R/(phi*sin(phi)) * ((r^2/s-s)*(1+a(3)*r) - a(3)*r*(r-s))/a(2);
    dCda(i,3) = 2*pi*eps0*R/(phi*sin(phi)) * r*(r-s);
end
end
readCap.m

function [result] = readCap(file)

N = 0;
fid = fopen(file, 'r');
fgetl(fid);
while 1
    line = fgetl(fid);
    if ~isstr(line), break, end
    if length(line) == 0,
        N = N + 1;
        stuff = sscanf(line, '%lf %lf %lf %lf');
        result(N, :) = stuff;
    end
end

A.3 Dynamics Simulation

This code is written in the MATLAB script language.

A.3.1 Initialization

globals.m

global PLATE_INV_M PLATE_R PLATE_GAP;
global SPRING_CONST SPRING_AXIS SPRING_NUM;
global DRIVE_MAX_V DRIVE_MAX_T;

load_all.m

function [] = load_all

globals;
load_plate;
load_springs;
DRIVE_MAX_V = 150;
DRIVE_MAX_T = 1e-4;

load_plate.m

function [] = load_plate

globals;

plate_density = 2331;  % kg/m^3
plate_radius = 50e-6;  % m
plate_thickness = 5e-6;  % m
plate_mass = plate_density*pi*plate_radius^2*plate_thickness;
plate_i = plate_mass*(plate_radius^2/4+plate_thickness^2/3);

PLATE_INV_M = inv([ plate_mass 0 0
                  0 plate_i 0
                  0 0 plate_i]);

PLATE_R = plate_radius;
PLATE_GAP = 5e-6;

load_springs.m

function [] = load_springs

globals;
if (size(PLATE_R)==0)
   load_plate;
end

% springs = [ L w t axis(x=1,y=2)?
 100e-6 10e-6 1.8e-6 1
 100e-6 10e-6 2e-6  1
 100e-6 10e-6 2e-6  2
 100e-6 10e-6 1.9e-6 -2];
SPRING_NUM = size(springs,1);

L = springs(:,1);
w = springs(:,2);
t = springs(:,3);

E = 155e9; % Young’s modulus
nu = 0.3; % Poisson ratio

I = w.*t.^3/12;
k = 12*E*I./L.^3; % z axis translation spring constant

G = E/(2*(1+nu));
K = w.*t.^3.*(1/3-(3.36/16)*(t./w).^2)*(1-((t./w).^4)/12));
kappa = G*K./L; % beam axis rotational spring constant

SPRING_CONST(1,:) = k’;
SPRING_CONST(2,:) = kappa’;
SPRING_AXIS(:) = springs(:,4);

A.3.2 Acceleration Function
d2xdt2.m

function [d2xdt2] = d2xdt2(t,x);
globals;

d2xdt2 = -PLATE_INV_M * (capacitor_contrib(t,x) +
spring_contribution\(t, x\));

**capacitor_contrib.m**

```matlab
function [cc] = capacitor_contrib\(t, x\);

globals;

[q,dqdx] = q_cap\(x\);
[C,dCdq] = capacitance\(q\);
V = voltage\(t\);
dUdq = -(1/2)*dCdq*V^2;
cc = dqdx*dUdq;
```

**q_cap.m**

```matlab
function [q,dqdx] = q_cap\(x\);

% x = [ g phi_x phi_y ]
% q = [ g phi_p ]

q = zeros(2,1);
dqdx = zeros(3,2);

q(1) = x(1);
q(2) = atan(sqrt(tan(x(2))^2+tan(x(3))^2));

if (tan(q(2)) <= eps)
dq2dx2 = 1;
dq2dx3 = 1;
else
dq2dx2 = tan(x(2))*sec(x(2))^2/(tan(q(2))*sec(q(2))^2);
dq2dx3 = tan(x(3))*sec(x(3))^2/(tan(q(2))*sec(q(2))^2);
end

dqdx = [ 1 0
0 dq2dx2
0 dq2dx3 ];
```

**capacitance.m**

```matlab
function [C,dCdq] = capacitance\(q\)
%capfn:
% assumes q = [gap(m) phi_p(rad)]
% returns [C,dCda]

globals;

N = size(q,2);
C = zeros(1,N);
dCdq = zeros(2,N);

a = [1.0974241404426e+00 1.0558692307271e+00 3.8474147762451e+00];
```
eps0 = 8.8541878162e-12;
R = a(2)*PLATE_R;

for i=1:N,
    g = a(1)*q(1,i);
    phi = q(2,i);
    r = g/R;

if abs(phi)<=1e-4
    C(i) = eps0*pi*R * ( (1+a(3)*r)/r - (2*r^2-3)*(1+a(3)*r)*phi^2/(12*r^4) );
    dCdq(1,i) = eps0*pi*a(1) * ( -1/r^2 + (2*r^2-6*a(3)*r-9)*phi^2/(12*r^4) );
    dCdq(2,i) = - eps0*pi*R * (2*r^2-3)*(1+a(3)*r)*phi/(6*r^3);
else
    s = sqrt(r^2-sin(phi)^2);
    C(i) = 2*pi*eps0*R/(phi*sin(phi)) * (r-s)*(1+a(3)*r);
    dCdq(1,i) = 2*pi*eps0/(phi*sin(phi)) * a(1) * ((1-r/s)*(1+a(3)*r)+a(3)*(r-s));
    dCdq(2,i) = 2*pi*eps0*R/(phi*sin(phi)) * (1+a(3)*r)*(sin(phi)*cos(phi)/s-(r-s)*(1/phi+cot(phi)))
end
end

voltage.m

function [V] = voltage(t);

globals;

V = zeros(size(t));
for i=1:length(V),
    if t(i)<=0,
        V(i) = 0;
    elseif t(i)<=DRIVE_MAX_T,
        V(i) = DRIVE_MAX_V*t(i)/DRIVE_MAX_T;
    else
        V(i) = DRIVE_MAX_V;
    end
end

spring_contrib.m

function [sc] = spring_contrib(t,x);

globals;

sc = zeros(3,1);
for i=1:SPRING_NUM,
    [q,dqdx] = q_spr(x,SPRING_AXIS(i));
    dUdq = SPRING_CONST(:,i).* q;
    sc = sc + dqdx * dUdq;
end;
q_spr.m

function [q,dqdx] = q_spr(x,axis);
% x = [ g  phi_x  phi_y ]
% q = [ dz  dth ]

globals;

q = zeros(2,1);
dqdx = zeros(3,2);

if (abs(axis)==1),
r = sign(axis) * PLATE_R;
q(1) = (x(1)-PLATE_GAP)-r*tan(x(3));
q(2) = x(2);
dqdx = [ 1 0
0 1
-r*sec(x(3))\^2 0 ];
else
r = sign(axis) * PLATE_R;
q(1) = (x(1)-PLATE_GAP)+r*tan(x(2));
q(2) = x(3);
dqdx = [ 1 0
0 1
r*sec(x(2))\^2 0 ];
end

A.3.3 Equilibrium Determination

equilibrium.m

function [x] = equilibrium(x0);

x = djfnewton(x0-=0,'F',x0).*x0;

F.m

function [a] = F(x,x0);

globals;

if any(x0-=0),
a0 = d2xdt2(DRIVE_MAX_T,x0);
end

for i=1:length(x0),
   if x0(i)-=0,
      x(i) = x(i).*x0(i);
   end
end
a = d2xdt2(DRIVE_MAX_T,x);

for i=1:length(a0),
    if x0(i)==0,
        a(i) = a(i)./a0(i);
    end
end

djfnewton.m

function [x] = djfnewton(x0,F_fn,fixed);

tol_abs = 1e-6;
damp_trac = 0.5;
jf_alpha=0.001;
jf_tol_rel=0.001;

N = length(x0);
x = 0;
x = x0;
dx = zeros(N,1);
p = zeros(N,N);
Ap = zeros(N,N);

fprintf('start djfnewton...\n');
while 1,
    f = feval(F_fn,x,fixed);
    norm_f = norm(f);
    fprintf('|F| = %16.12f\n',norm_f);
    if(norm_f < tol_abs),
        break;
    end;

    r = -f;
    dx = 0*dx;

    for i = 1:N,
        p(:,i) = r;
        Ar = (feval(F_fn,x+jf_alpha*r,fixed)-f)/jf_alpha;
        Ap(:,i) = Ar;
        for j=1:i-1,
            beta = Ap(:,i)' * Ap(:,j);
            p(:,i) = p(:,i) - beta * p(:,j);
            Ap(:,i) = Ap(:,i) - beta * Ap(:,j);
        end;
    end;

    norm_ap = norm(Ap(:,i),2);
    p(:,i) = p(:,i)/norm_ap;
    Ap(:,i) = Ap(:,i)/norm_ap;

    alpha = r' * Ap(:,i);
    dx = dx + alpha * p(:,i);
    r = r - alpha * Ap(:,i);
if(norm(r) < jf_tol_rel*norm_f),
    break;
end;
end;
if(norm(r) >= jf_tol_rel*norm_f),
    fprintf('\nJF Newton GCR didn’t converge!\n');
end;

damp_alpha = 1;
damp_norm = norm(feval(F_fn,x+dx,\fixed));
fprintf('|damp_F| = %16.12f\n',damp_norm);
while damp_norm >= norm_f,
    damp_norm = norm(feval(F_fn,x+damp_alpha*dx,\fixed));
    fprintf('|damp_F| = %16.12f\n',damp_norm);
    damp_alpha = damp_frac*damp_alpha;
end;

x = x + damp_alpha*dx;
k = k + 1;
end;
fprintf('...end dfjnewton\n');

A.3.4 Normal Mode Determination

\texttt{modes.m}

\begin{verbatim}
function [w,v,k] = modes(x);

globals;

J = zeros(3,3);
for i=1:3,
    h=zeros(3,1);
    h(i)=x(i)*0.001+1e-8;
    old_dx = (d2xdx2(1,x+h)-d2xdx2(1,x-h))/(2*h(i));
    h=h/2;
    dx = (d2xdx2(1,x+h)-d2xdx2(1,x-h))/(2*h(i));
    while norm(old_dx-dx)>1e-3*norm(dx),
        old_dx = dx;
        h=h/2;
        dx = (d2xdx2(1,x+h)-d2xdx2(1,x-h))/(2*h(i));
    end;
    J(:,i) = dx;
end

[v,d]=eig(J);
w = sqrt(-diag(d));
k = inv(v)';
\end{verbatim}
In order to demonstrate the functionality of the dynamics simulator developed in this report, the simulator was used to determine the motion of the suspended circular plate system with a variety of support beam thicknesses and voltage inputs. The results of those simulations are presented in this appendix.

B.1 Structure Specification

The structure simulated is shown in Figure 22. The structure is a rigid circular plate suspended 5µm above a ground plane by four thin beams placed along the x and y axes. The plate and beams are composed of polysilicon, which is assigned a density of 2330kg/m³, a Young's modulus of 155GPa, and a Poisson ratio of 0.3. The circular plate has a radius of 50µm and a thickness of 5µm. Each beam has a length of 100µm, a width of 10µm, and a thickness that varies between experiments but is usually 2µm.

Depending upon how the beam thicknesses are chosen, motion can occur with different numbers of degrees of freedom and within any of the three
simulated parameters $g$, $\varphi_x$, and $\varphi_y$. Names are given to four structures with specially chosen beam thicknesses, each structure permitting more complex motion than the previous. For the simplest structure, henceforth called the 1D structure, all of the beam thicknesses are equal to $2\mu m$. The symmetry of this structure prevents an applied voltage from causing any tilting motion. Thus, all motion is exhibited within $g$. For the second structure, henceforth called the 2D structure, the $-y$ axis beam is weakened by reducing its thickness to $1.8\mu m$. Symmetry prevents tilting about the $y$ axis, but allows it about the $x$ axis. Thus, all motion occurs within two degrees of freedom exhibited within $g$ and $\varphi_x$. The third structure, henceforth known as the 2D+ structure, is that for which the $-y$ and $+x$ axis beam thicknesses are reduced equally to $1.9\mu m$. The structure is symmetric about the $y=-x$ axis; therefore an applied voltage will induce identical tilting about the $x$ and $y$ axes. Thus, two degrees of freedom are exhibited within all three simulated parameters, $g$, $\varphi_x$, and $\varphi_y$. The fourth structure, henceforth known as the 3D structure, is that for which the $-y$ and $+x$ axis beam thicknesses are reduced unequally to $1.9\mu m$ and $1.85\mu m$ respectively. This structure permits three degrees of freedom of motion exhibited within $g$, $\varphi_x$, and $\varphi_y$. 

Figure 22: Simulated Suspended Circular Plate Structure Diagram
B.2 Experiments and Results

Three experiments are presented. The first experiment is the simulation of dynamics about equilibrium at constant voltage. The second experiment is the simulation of dynamics induced by a linear voltage ramp. The final experiment is the simulation of dynamics induced by linear voltage ramps for varying ramp rates.

B.2.1 Near Equilibrium Dynamics

In this first experiment, the 1D, 2D, 2D+, and 3D structures are held at a constant voltage of 150 volts. Each structure is displaced initially from its calculated equilibrium by 1%. The results of these simulations are given in Figures 23 through 38. The straight line across some of the plots indicates the equilibrium value as estimated by the implemented Newton method, and the dots depict the integration steps of the simulation. Table 2 presents the equilibrium states estimated for the 1D, 2D, 2D+, and 3D structures. Using this estimation of equilibrium, the normal modes, frequencies, and periods of small oscillation about equilibrium are calculated. The results are presented in Tables 3 through 6.

<table>
<thead>
<tr>
<th>Equilibrium at 150v</th>
<th>1D</th>
<th>2D</th>
<th>2D+</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g ) component (( \mu m ))</td>
<td>3.997</td>
<td>3.787</td>
<td>3.793</td>
<td>3.708</td>
</tr>
<tr>
<td>( \varphi_x ) component (rad)</td>
<td>0.000</td>
<td>3.433 ( 10^{-3} )</td>
<td>1.733 ( 10^{-3} )</td>
<td>1.927 ( 10^{-3} )</td>
</tr>
<tr>
<td>( \varphi_y ) component (rad)</td>
<td>0.000</td>
<td>0.000</td>
<td>1.733 ( 10^{-3} )</td>
<td>2.874 ( 10^{-3} )</td>
</tr>
</tbody>
</table>

Table 2: Equilibrium State for 1D, 2D, 2D+, and 3D Structures at 150 Volts

<table>
<thead>
<tr>
<th>1D Modes at 150 Volts</th>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g ) component (m)</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>( \varphi_x ) component (rad)</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 3: Mode Information for 1D Structure at 150 Volts
Table 3: Mode Information for 1D Structure at 150 Volts

<table>
<thead>
<tr>
<th>1D Modes at 150 Volts</th>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ_y component (rad)</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>angular frequency (rad/sec)</td>
<td>5.195 $10^5$</td>
<td>1.106 $10^6$</td>
<td>1.106 $10^6$</td>
</tr>
<tr>
<td>period (sec)</td>
<td>1.209 $10^{-5}$</td>
<td>5.683 $10^{-6}$</td>
<td>5.683 $10^{-6}$</td>
</tr>
</tbody>
</table>

Table 4: Mode Information for 2D Structure at 150 Volts

<table>
<thead>
<tr>
<th>2D Modes at 150 Volts</th>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>g component (m)</td>
<td>2.204 $10^{-4}$</td>
<td>2.874 $10^{-6}$</td>
<td>0.000</td>
</tr>
<tr>
<td>φ_x component (rad)</td>
<td>-1.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>φ_y component (rad)</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>angular frequency (rad/sec)</td>
<td>4.162 $10^6$</td>
<td>1.017 $10^6$</td>
<td>1.052 $10^6$</td>
</tr>
<tr>
<td>period (sec)</td>
<td>1.509 $10^{-5}$</td>
<td>6.180 $10^{-6}$</td>
<td>5.973 $10^{-6}$</td>
</tr>
</tbody>
</table>

Table 5: Mode Information for 2D+ Structure at 150 Volts

<table>
<thead>
<tr>
<th>2D+ Modes at 150 Volts</th>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>g component (m)</td>
<td>-3.114 $10^{-4}$</td>
<td>2.034 $10^{-6}$</td>
<td>9.816 $10^{-13}$</td>
</tr>
<tr>
<td>φ_x component (rad)</td>
<td>7.071 $10^{-1}$</td>
<td>7.071 $10^{-1}$</td>
<td>-7.071 $10^{-1}$</td>
</tr>
<tr>
<td>φ_y component (rad)</td>
<td>7.071 $10^{-1}$</td>
<td>7.071 $10^{-1}$</td>
<td>7.071 $10^{-1}$</td>
</tr>
<tr>
<td>angular frequency (rad/sec)</td>
<td>4.227 $10^5$</td>
<td>1.033 $10^6$</td>
<td>1.031 $10^6$</td>
</tr>
<tr>
<td>period (sec)</td>
<td>1.486 $10^{-5}$</td>
<td>6.082 $10^{-6}$</td>
<td>6.097 $10^{-6}$</td>
</tr>
</tbody>
</table>

Table 6: Mode Information for 3D Structure at 150 Volts

<table>
<thead>
<tr>
<th>3D Modes at 150 Volts</th>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>g component (m)</td>
<td>-2.306 $10^{-4}$</td>
<td>1.887 $10^{-6}$</td>
<td>-1.984 $10^{-6}$</td>
</tr>
<tr>
<td>φ_x component (rad)</td>
<td>5.535 $10^{-1}$</td>
<td>-2.248 $10^{-1}$</td>
<td>-9.757 $10^{-1}$</td>
</tr>
<tr>
<td>φ_y component (rad)</td>
<td>8.328 $10^{-1}$</td>
<td>9.744 $10^{-1}$</td>
<td>-2.191 $10^{-1}$</td>
</tr>
<tr>
<td>angular frequency (rad/sec)</td>
<td>3.773 $10^5$</td>
<td>1.004 $10^6$</td>
<td>1.013 $10^6$</td>
</tr>
<tr>
<td>period (sec)</td>
<td>1.665 $10^{-5}$</td>
<td>6.261 $10^{-6}$</td>
<td>6.204 $10^{-6}$</td>
</tr>
</tbody>
</table>
Figure 23: Near Equilibrium Dynamics - 1D Structure - g
Figure 24: Near Equilibrium Dynamics - 1D Structure - Normal Modes
Figure 25: Near Equilibrium Dynamics - 2D Structure - $g$
Figure 26: Near Equilibrium Dynamics - 2D Structure - $\phi_x$
Figure 27: Near Equilibrium Dynamics - 2D Structure - Principle Tilt
Figure 28: Near Equilibrium Dynamics - 2D Structure - Normal Modes
Figure 28 (cont): Near Equilibrium Dynamics - 2D Structure - Normal Modes
Figure 29: Near Equilibrium Dynamics - 2D+ Structure - g
Figure 30: Near Equilibrium Dynamics - 2D+ Structure - $\phi_x$
Figure 31: Near Equilibrium Dynamics - 2D+ Structure - $\phi_y$
Figure 32: Near Equilibrium Dynamics - 2D+ Structure - Principle Tilt
Figure 33: Near Equilibrium Dynamics - 2D+ Structure - Normal Modes
Figure 33 (cont): Near Equilibrium Dynamics - 2D+ Structure - Normal Modes
Figure 34: Near Equilibrium Dynamics - 3D Structure - $g$
Figure 35: Near Equilibrium Dynamics - 3D Structure - $\phi_x$
Figure 36: Near Equilibrium Dynamics - 3D Structure - $\phi_y$
Figure 37: Near Equilibrium Dynamics - 3D Structure - Principle Tilt
Figure 38: Near Equilibrium Dynamics - 3D Structure - Normal Modes
Figure 38 (cont): Near Equilibrium Dynamics - 3D Structure - Normal Modes
B.2.2 Voltage Ramp

In the second experiment, a time dependance upon voltage is introduced. At time $t=0$, the plate is at rest and no voltage is applied. After time $t=0$, the voltage is linearly increased over $10^{-4}$ seconds to 150 volts. The voltage is then held at 150 volts in order to observe the oscillation of the state around the new equilibrium of the system. The results of these simulations are given in Figures 39 through 54. The equilibrium and mode information given in the previous section holds for this experiment, because the final voltage level is the same.
Figure 39: Voltage Ramp Dynamics - 1D Structure - $g$
Figure 40: Voltage Ramp Dynamics - 1D Structure - Normal Modes
Figure 40 (cont): Voltage Ramp Dynamics - 1D Structure - Normal Modes
Figure 41: Voltage Ramp Dynamics - 2D Structure - g
Figure 42: Voltage Ramp Dynamics - 2D Structure - $\varphi_x$
Figure 43: Voltage Ramp Dynamics - 2D Structure - Principle Tilt
Figure 44: Voltage Ramp Dynamics - 2D Structure - Normal Modes
Figure 44 (cont): Voltage Ramp Dynamics - 2D Structure - Normal Modes
Figure 45: Voltage Ramp Dynamics - 2D+ Structure - $g$
Figure 46: Voltage Ramp Dynamics - 2D+ Structure - $\phi_x$
Figure 47: Voltage Ramp Dynamics - 2D+ Structure - $\phi_y$
Figure 48: Voltage Ramp Dynamics - 2D+ Structure - Principle Tilt
Figure 49: Voltage Ramp Dynamics - 2D+ Structure - Normal Modes
Figure 49 (cont): Voltage Ramp Dynamics - 2D+ Structure - Normal Modes
Figure 50: Voltage Ramp Dynamics - 3D Structure - $g$
Figure 51: Voltage Ramp Dynamics - 3D Structure - ϕ_x
Figure 52: Voltage Ramp Dynamics - 3D Structure - $\phi_y$
Figure 53: Voltage Ramp Dynamics - 3D Structure - Principle Tilt
Figure 54: Voltage Ramp Dynamics - 3D Structure - Normal Modes
Figure 54 (cont): Voltage Ramp Dynamics - 3D Structure - Normal Modes
B.2.3 Varied Voltage Ramp Rate

The preceding voltage ramp situation demonstrates that momentum of the plate build as voltage is ramped, causing the plate to overshoot equilibrium and conceivably past pull-in. It follows that the rate at which voltage is ramped affects this overshoot. The third experiment repeats the 3D structure voltage ramp experiment, using ramp time values of $10^{-3}$ and $10^{-5}$ seconds. The results of these simulations are given in Figures 55 through 64. Notice that the $10^{-5}$ second ramp results in pull-in, yet the $10^{-3}$ second ramp brings the plate very close to equilibrium and only small oscillations occur. This implies that momentum has a significant effect upon the success of moving an electrostatically actuated suspended plate from one equilibrium position to another. The rate of voltage ramping must be chosen so as not to pull the plate past pull-in.
Figure 55: Ramp Time $10^{-3} \cdot g$
Figure 56: Ramp Time $10^{-3}$ - $\phi_x$
Figure 57: Ramp Time $10^{-3} - \varphi_y$
Figure 58: Ramp Time $10^{-3}$ - Principle Tilt
Figure 59: Ramp Time $10^{-3}$ - Normal Modes
Figure 59 (cont): Ramp Time $10^{-3}$ - Normal Modes
Figure 60: Ramp Time $10^{-5} - g$
Figure 61: Ramp Time $10^{-5} - \phi_x$
Figure 62: Ramp Time $10^{-5}$ - $\phi_y$
Figure 63: Ramp Time $10^{-5}$ - Principle Tilt
Figure 64: Ramp Time $10^{-5}$ - Normal Modes
Figure 64 (cont): Ramp Time $10^{-5}$ - Normal Modes
References


[7] I-DEAS Master’s Series Version 1.3c software, SDRC, Milford, Ohio.


