RANDOM THEORY OF TURBULENCE

by

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ABSTRACT

There has long been a need for a rational method of dealing with turbulent flow problems. Random Theory, especially the Calculus of Random Functionals, seems to provide a solid foundation for a rational theory; and this thesis attempts to both develop and apply such a theory. The work seems to divide naturally into two parts: First, a development of the Calculus of Random Functionals in sufficient detail that it may be applied to turbulence problems; and second, application of the Calculus to several problems in turbulent flow.

The principle results are the analyses contained in Chapters IV through VII. In these chapters, a number of methods of analysis are developed and applied to several turbulent flow problems. The correlation with experiment seems satisfactory, and methods of improvement are presented.

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Title: Associate Professor of Electrical Engineering
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CHAPTER I

Introduction

1. Historical Background

Many ordinary flow situations seem to contain velocities which are not fixed by the macroscopic surroundings. Because the macroscopic surroundings do not fully determine the velocities, the velocities are considered to have random components, and the existence of random components in a flow establishes a type of flow known as turbulence.

It was Osborne Reynolds* who, in 1883, made the first systematic investigation of turbulence. Experimenting with different fluids, pipes and flow rates, Reynolds found that one non-dimensional parameter was sufficient to define the flow characteristics. This number, known as the Reynolds number in his honor is defined as:

$$Re = \frac{V_o L_o}{\eta}$$

where $V_o$ is some characteristic velocity (say the flow rate velocity), $L_o$ is some characteristic length (say the pipe diameter), $\rho$ is the mass density, and $\eta$ is the dynamic viscosity. Reynolds found that, for pipe flow, there is a critical Reynolds number (in his case $Re = 2000$). If $Re$ were above this value, the flow was turbulent; but if $Re$ were below this value, the flow was laminar.

Since Reynolds' important contribution, much effort has been put into analyzing turbulent flows. The equations of motion, based on a continuum model, ** could be averaged, but in so doing

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* Superscript numerals refer to references listed in bibliography.

** The equations, originally done by Navier and later modified by Stokes are discussed in Chapter II.
the investigators were left with more unknowns than equations. For some particular flow situations, semi-empirical or phenomenological theories* have been developed which give a surprisingly accurate description of the mean velocity and friction law. However, these theories rely heavily on the numerical results of experiment and suffer from a number of physical defects (e.g. the velocities do not generally satisfy the boundary conditions), hence they are unsatisfactory as a rational theory of turbulence.

Statistical methods have also been employed with some success in the study of turbulent flows. Starting with several new notions introduced by Taylor, Batchelor et al. have developed the theory until today, some aspects of simple turbulent flows are well understood. The theory is not complete, however, and no technically interesting flow has been effectively treated by statistical methods.

2. Object and Scope of This Investigation

The title of this thesis, "Random Theory of Turbulence," is meant to convey the notion that the methods employed here come from Random Theory as compared with Statistical Theory. For this distinction, Statistical Theory is taken to be that sort of analysis which characterizes the random processes by various correlation functions and other averaged quantities, whereas Random Theory is taken to be that sort of analysis which characterizes the random quantities themselves in terms of some other random quantity taken as fundamental. Both descriptions are complete in the sense that a large class of random quantities (including all those appearing in turbulent flows) may be characterized. There is a large difference in methodology between the two methods, 

*The semi-empirical theories are developed by Harris and Pai.
however, since Statistical Theory deals with integral or global properties of the random quantities, whereas the Random Theory deals with differential or local properties of the random quantities. It is felt that the local nature of the Random Theory methods is more appropriate for the solution of differential equations, and the success of the method in describing turbulent Plane Poiseuille Flow seems to bear this out.

Turbulent flow situations seem to divide naturally into two classes*—decaying turbulent flows, and turbulent shear flows. Decaying turbulent flows are characterized by the fact that the turbulence is essentially injected into the flow, either by initial conditions or boundary conditions. Turbulent shear flows are characterized by the fact that the turbulence is essentially generated volumewise by the mean shear stresses in the flow. The Calculus of Random Functionals seems also to divide into two more-or-less distinct methods; Homogeneous Polynomial Functionals which are applicable to decaying turbulent flows, and Orthogonal Polynomial Functionals which are applicable to turbulent shear flows.

The fundamental objective of this thesis is thus to apply Random Theory to some well known turbulent flow situations and thereby develop a rational analysis. Thus, physical problems are treated by recently developed methods of analysis to provide a rational mathematical treatment of phenomena whose nature is otherwise understood.

There is some discussion, but little emphasis, on the magneto-fluid dynamic case of the problems considered in this work. The main reason for this lack of emphasis is as follows: turbulence may drastically alter the characteristics of a magneto-fluid

* Some flows, such as wakes and jets do not really fit either class.
dynamic device, but the addition of electromagnetic fields adds nothing essentially new to the problem of turbulence. Furthermore, there has been very little work done, either experimentally or theoretically, on such flows, the notable exception being the work of Harris. 8

3. Notation and Conventions

Many of the equations in this work are written in a sort of hybrid Cartesian tensor index notation. The index notation is applied to vector and tensor quantities in space. The correspondence between indexed variables and ordinary vector variables is illustrated by a typical vector:

\[ a^1 = a_x \]
\[ a^2 = a_y \]
\[ a^3 = a_z \]

Since the coordinate system is always Cartesian and transformation of coordinates is never performed, all tensor quantities are written in contravariant (superscripted indices) form and no distinction is made between covariant and contravariant forms. The summation convention is used, i.e. \( a^k b^k \) is the "dot" product of vectors \( a^i \) and \( b^j \). The Kronecker delta \( \delta_{ij} \) or \( \delta_i^j \) as well as the alternator \( e^{ijk} \) are used extensively and are defined by:

\[ \delta_{ij} = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \]

\[ e^{ijk} = -e^{jki} = -e^{kji} = -e^{jik} \]

and

\[ e^{123} = 1 \]

Since no coordinate reflections are involved and the metric is 1, \( e^{ijk} \) is a proper tensor.
Fourier transforms are occasionally employed and the particular form chosen is the pair:

\[ F(x) = \int_{-\infty}^{\infty} f(x)e^{-j \pi x} dx \]

and

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{j \pi x} dx \]

The symbols used in this work are, with a few exceptions, assigned no a priori meaning. Some of the exceptions are: \( u^i \) is always a velocity vector, \( R_e \) is always the Reynolds number, etc. For the most part, a particular symbol is defined and used in a (hopefully) consistent fashion--until redefined. Thus, such a parameter as \( \omega \) is used at times as the sampling parameter in probability space, and at times as the argument of a function, e.g., \( \cos (\omega(1-y)) \). The reason for this apparent disregard for uniform definition of symbols is simple: when three well established fields of endeavor--fluid dynamics, electromagnetics, and probability theory--none of which is noted for uniformity of symbol definition, are melded together, there is bound to be difficulty; e.g. \( p \) may be pressure, probability, or polarization. Generally, the symbols used here will individually conform to usage in at least one of the three fields.

One additional difficulty with symbols arises when both index summation, and \( \Sigma \) summation are employed. Thus, care must be exercised in distinguishing between the \( v \) and the \( i \) in such an expression as:

\[ p^i = \sum_{v} p^i_v \]

In addition, symbols indexed differently are distinct (though related), e.g. \( u^1 \neq u^2 \), and \( b^1 \neq b^1j \).
CHAPTER II

The Equations of Motion

1. The Dynamic Equations

The equations to be used in this investigation to study the problem of turbulence in fluids are derived from a continuum model. These equations will comprise a system of non-linear, partial differential equations whose solutions are field quantities (i.e. quantities which describe the properties of the matter at some position in space at a given instant of time).

Ordinarily, there would be no question about the use of such a system of equations, but from time to time the objection is raised that the velocities, etc. in a turbulent medium must be much too chaotic (too "turbulent"!) to be described by a continuum model and partial differential equations. The argument is not without merit and deserves further attention.

The continuum hypothesis is based on the assumption that there exist increments of time and distance which are small compared to the smallest fluctuation of the media, yet large compared to the molecular scale. If such increments exist, then the continuum model cannot be seriously questioned.

As an example, * consider the flow of air down a pipe of 0.01 meter diameter at a velocity of $10^2$ meters/sec. This corresponds to a Reynolds number of $Re = 6.6 \cdot 10^4$ at standard temperature and pressure. Under these conditions, the size of the smallest eddies is $5 \cdot 10^{-5}$ meters. However, the average distance between nearest

---

*This example is quoted from Townsend, page 24.
neighboring molecules under these conditions is $3 \cdot 10^{-9}$ meter, and the mean free path is only $2 \cdot 10^{-8}$ meter. Thus, for this case the continuum model is certainly valid. Surely then, for dense liquid such as water, mercury, or NaK, the continuum hypothesis holds at all but relativistic velocities (although there may be some question about the stress law).

Thus, the equations of motion used here to describe the fluid properties are those derived from the continuum model:

1.) Continuity of Mass (conservation of mass)

$$\frac{\partial}{\partial t} \rho u^i + \frac{\partial}{\partial x^j} \rho = 0$$  

and

2.) Navier-Stokes equation (momentum balance*)

$$\rho \left( \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = \frac{\partial \sigma^{ij}}{\partial x^j} + F^i$$  

where

$\rho$ - mass density

$u^i$ - velocity vector

$\sigma^{ij}$ - stress tensor

$F^i$ - external body forces

2. The Stress Tensor

The form of the stress tensor used in this investigation is:

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*The Navier-Stokes equations are not truly force balance equations since energy dissipating forces may exist even when the right side of 2.1.2 is zero term by term. Instead, the right hand side of 2.1.2 is the body force equivalent of that portion of all forces which go into changing the momentum per unit volume.
\[ \sigma^{ij} = \left( -p - \frac{2}{3} \eta \frac{\partial u^k}{\partial x^k} \right) \delta^{ij} + \eta \left( \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) \]

where \( p \) is by definition the arithmetic mean of the normal surface stress and \( \eta \) is the fluid viscosity (independent of \( x^1 \) and \( t \)). Although this stress law may be questioned for some fluids at extreme pressures and temperatures (or for any plastic or elastic substance), Reynolds \(^{24}\) pointed out that for ordinary fluids (water, air, mercury, etc.) this stress law has been put to a severe test by the experiments of Poiseuille \(^{22}\) and others. Furthermore, more recent experiments have provided a direct verification of the stress law. In the case of gases, the kinetic theory of Maxwell \(^{13}\) predicts this law. Considering the evidence, the stress law seems well established for ordinary fluids under the assumption of a continuum.

Thus, equations 2.1.1 and 2.1.2 become:

\[ \rho \left[ \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right] - \eta \nu^2 u^i - \frac{1}{3} \eta \frac{\partial}{\partial x^i} \frac{\partial u^j}{\partial x^j} + \frac{\partial p}{\partial x^i} = F^i \]

and:

\[ \frac{\partial}{\partial x^i} \left( \rho u^i \right) + \frac{\partial p}{\partial t} = 0 \]

These form a system of four nonlinear, partial-differential equations in four unknowns \((u^i, p)\) which can be solved (in principle) given \( F^i \) and suitable boundary conditions.

3. Compressibility and Homogeneity

In addition to the above assumptions, the fluid will be considered to be incompressible and homogeneous. Ordinary liquids such as water, etc. are incompressible to a very good approximation, and even gases behave as incompressible fluid at low Mach number.*

*See Shapiro \(^{26}\) for a discussion of compressible phenomena.
The additional assumption of homogeneity has the main effect of excluding turbulent mixing problems from the consideration of this investigation.

If the fluid is incompressible and homogeneous, then $\rho$ may not be a function of position or time. Hence, 2.2.3 becomes simply:

$$\frac{\partial u^1}{\partial x^1} = 0$$  \hspace{1cm} 2.3.1

and from this 2.2.2 becomes:

$$\rho [\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^1}{\partial x^1}] - \eta \frac{\partial^2 u^1}{\partial x^1} + \frac{\partial p}{\partial x^1} = F^1$$  \hspace{1cm} 2.3.2

4. The Kinematic Equations

The body forces represented by $F^1$ in 2.1.2 or 2.2.2 may arise from a number of sources. However, for the greater part of this investigation these forces will be taken as zero. In the one case where body forces are considered, $F^1$ will be attributed to electromagnetic forces, in which case:

$$F^1 = \rho_e E^1 + \varepsilon ^{ijk} J^j B^k$$  \hspace{1cm} 2.4.1

where:

- $\rho_e$ = electric charge density
- $E^1$ = electric intensity vector
- $J^1$ = current density vector
- $B^1$ = magnetic induction vector

Inherent in this equation is the assumption that polarization effects (both electric and magnetic) are negligible. Furthermore, for most physically interesting situations, the term $\rho_e E^1$ is much
smaller than \( e^{ijx} J^j B^k \) because of very large electrical conductivity. Considering only these cases*, the force law becomes simply:

\[
F^i = e^{ijx} J^j B^k
\]

Furthermore, the equations used to relate the electromagnetic quantities will be taken as Maxwell's equations, neglecting displacement currents (again because of high conductivity). **

These are:

\[
e^{ijx} \frac{\partial}{\partial x^j} B^k = \mu_0 J^i
\]

\[
e^{ijx} \frac{\partial}{\partial x^j} E^k = -\frac{\partial}{\partial t} B^i
\]

and

\[
\frac{\partial B^k}{\partial x^k} = 0
\]

(The last of the four Maxwell equations:

\[
\frac{\partial E^k}{\partial x^k} = \frac{\rho_e}{\varepsilon_0}
\]

will not be considered since it serves only to define the charge density once the \( E \) field is known.) Inherent in the above equations is the assumption that the permittivity (and permeability) of the media is that of free space.

---

*An important exception to these assumptions arises if the conductivity is identically zero. In this event the effects of permittivity and permeability are all important. See Melcher.**

**For a discussion of these equations as well as the effects of polarizable materials, see Fano, Chu and Adler, and Harris.
The electrical conductivity will be taken to be scalar and independent of position and time. Then, the constitutive relationship between $J^i$, and $E^i$ and $B^i$ is:

$$ J^i = \sigma (E^i + e^{ijk} B^j \times B^k) $$  \hspace{1cm} 2.4.6

i.e. "Ohm's Law".

5. Normalization of the Equations

It is often convenient to normalize the equations relating to turbulent flow problems by replacing the physical equations by equations in dimensionless form. Such a normalization replaces the physical parameters (such as $\rho$, $\sigma$, $\eta$, etc.) by dimensionless groups, thus making the setting of boundary conditions as well as the interpretation of results much simpler.

The physical equations of motion may be written:

$$ \rho \left( \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \right) - \eta \frac{\partial^2 v^i}{\partial x^k \partial x^k} + \frac{\partial p}{\partial x^i} = F^i $$ \hspace{1cm} 2.5.1

$$ \frac{\partial v^i}{\partial x^i} = 0 $$ \hspace{1cm} 2.5.2

$$ e^{ijk} \frac{\partial}{\partial x^j} B^k = - \frac{\partial}{\partial t} B^i $$ \hspace{1cm} 2.5.3

$$ e^{ijk} \frac{\partial}{\partial x^j} B^k = \mu J^i $$ \hspace{1cm} 2.5.4

$$ J^i = \sigma (E^i + e^{ijk} v^j B^k) $$ \hspace{1cm} 2.5.5

$$ F^i = e^{ijk} J^j B^k $$ \hspace{1cm} 2.5.6

and

$$ \frac{\partial B^k}{\partial x^k} = 0 $$ \hspace{1cm} 2.5.7

Then, the process of normalization consists of: 1. choosing a characteristic length, time, velocity, magnetic field, electric
field, current density, force density, and pressure for a given problem; 2. writing each of these variables in terms of a normalized variable; and, 3. determining the equations satisfied by the normalized variables. Of course, for some groups of problems, such characteristic quantities may not exist for the general problem (e.g. a length scale for isotropic turbulence). Nevertheless, for any particular physical problem, such scales will always exist. Thus the physical variable $X^i$ is written in terms of the normalized variable $x^i$ as:

$$X^i = L_0 x^i$$  \hspace{1cm} 2.5.8

where $L_0$ is a characteristic length (for channel flow $L_0 = \text{channel half width}$). Similarly:

$$V^i = V_0 u^i$$  \hspace{1cm} 2.5.9

$$P = \rho V_0^2$$  \hspace{1cm} 2.5.10

$$T = \frac{L_0}{V_0} t$$  \hspace{1cm} 2.5.11

$$B^i = B_0 b^i$$  \hspace{1cm} 2.5.12

$$J^i = \sigma V_0 B_0 j^i$$  \hspace{1cm} 2.5.13

and

$$E^i = V_0 B_0 e^i$$  \hspace{1cm} 2.5.14

The equations relating the normalized variables are then:

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} - \frac{1}{Re} \frac{\partial^2}{\partial x^k \partial x^k} u^i + \frac{\partial P}{\partial x^i} = R \frac{\partial}{\partial x^i} e^{il} m^j e^{jm} b^m$$  \hspace{1cm} 2.5.15

$$\frac{\partial u^i}{\partial x^i} = 0$$  \hspace{1cm} 2.5.16

$$\epsilon^{ijk} \frac{\partial}{\partial x^j} e^k = - \frac{\partial}{\partial t} b^i$$  \hspace{1cm} 2.5.17
\[ e^{ijx} \frac{\partial}{\partial x^j} b^x = R_m \mathbf{j}^i \]
\[ j^i = e^i + e^{ijx} u^j b^x \]
\[ \frac{\partial b^x}{\partial x^x} = 0 \]

where 2.5.1 and 2.5.6 have been incorporated in 2.5.15. The dimensionless groups are:
\[ R_e = \rho \frac{v_o L_o}{\eta} \]
\[ R_m = \mu_o \sigma L_o v_o \]
\[ R_h = \frac{B_o^2}{\mu_o \rho v_o^2} \]

These three dimensionless groups are known respectively as the Reynolds Number, the Magnetic Reynolds Number, and the Energy Ratio. The last of these is often conveniently replaced by the Hartman Number:
\[ M = [R_e R_m R_h]^{1/2} = L_o B_o (\sigma/\eta)^{1/2} \]

These dimensionless groups are often given physical meaning by dimensionless, or "order or magnitude" reasoning. Thus the Reynolds number \( R_e \) is interpreted as:

\[ \text{(fluid kinetic energy)/(viscous loss in unit time)} \]

or:
\[ \text{(inertia force)/(viscous force)} \]

or for a gas:
\[ \text{(characteristic length)/(mean free path)} \]

Of course, the last interpretation really has no place in a continuum theory, but it is a direct result of kinetic theory. The puzzling thing about the Reynolds number is that it is so
large ($R_e > 10^4$ is common for most channel situations). This does not mean that the viscous term $\frac{1}{R_e} \frac{\partial^2 u}{\partial x \partial x}$ in 2.5.15 is negligible--on the contrary, viscosity is responsible both for the generation and dissipation of turbulent fluctuations.

The magnetic Reynolds number $R_m$ is interpreted as:

- (magnetic field diffusion time)/(fluid transit time)

or:

- (ohmic loss in unit time)/(magnetic energy)

or:

- (magnetic energy)/(ohmic loss in unit time).

The last two interpretations are reciprocal and clearly demonstrate that the interpretation given to a dimensionless group depends critically on the physical situation.

The energy density $R_h$ is interpreted simply as:

- (magnetic energy)/(kinetic energy)

and the square of the Hartmann number $M^2$ is interpreted as:

- (ohmic loss)/(viscous loss).

The above interpretations are far from universal and in fact are often in considerable error. The interpretation valid in a given physical situation may often show trends, but little confidence should be placed in such results unless some definite numerical calculations can be used to verify them.*

6. Forms of the Equations

The above equations may be written in a number of forms by combining two or more of 2.5.15 - 24. For example, 2.5.15, 2.5.18, and 2.5.19 may be combined to give:

*See, for example, the discussion in Penhune Chapter 4.
\[
\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} - \frac{1}{R_e} \frac{\partial^2 u^i}{\partial x^k \partial x^k} + \frac{\partial p}{\partial x^i} = \frac{M^2}{R_e} e^{ij} b^k j b^k
\]

\[
= R_n [b^k \frac{\partial}{\partial x^k} b^i - \frac{1}{2} \frac{\partial}{\partial x^i} b^k b^k]
\]

\[
= \frac{M^2}{R_e} [e^{ij} b^k j b^k - (b^k b^k) u^i + (u^k b^k) b^i]
\]

Equations 2.5.17, 2.5.18, and 2.5.19 may be combined to form the dimensionless Bullard's equation:

\[
\frac{\partial b^i}{\partial t} + u^j \frac{\partial b^i}{\partial x^j} - \frac{1}{R_m} \frac{\partial^2 b^i}{\partial x^k \partial x^k} = b^j \frac{\partial u^i}{\partial x^j}
\]

This equation involves only \( b^i \) and \( u^i \) and (together with the forms of 2.6.1 which involve only \( b^i \) and \( u^i \)) forms a convenient starting point for calculations.

7. Boundary Conditions

The boundary conditions on a viscous fluid at a fixed impervious barrier are simply:

\[
u^i = 0 \quad \text{at the barrier.}
\]

Since the fluid is incompressible:

\[
\frac{\partial u^i}{\partial x^i} = 0
\]

and therefore:

\[
\frac{n^i}{n^j} n^j u^j = 0 \quad \text{at the barrier}
\]

where \( n^i \) is the unit surface normal vector.

The boundary conditions on the magnetic field are more difficult to establish. Generally, in turbulence problems there is a certain
applied field and an induced field which has both a mean and random part. The boundary conditions on the mean induced field will depend on the physical situation (especially the way in which current carrying leads are returned) and can generally be determined without difficulty. The boundary conditions on the random part of the induced magnetic field are more difficult to establish, primarily because these conditions depend greatly on the external environment (especially the way in which energy may be transported away from the fluid). For this work, however, the problem never really arises so, there is no difficulty (see Chapter V, Section 5).
CHAPTER III

Calculus of Random Functionals

The Calculus of Random Functionals is a mathematical tool by which the random solutions of partial differential equations may be expressed in a very convenient manner.

A method commonly used to obtain solutions to differential equations is to expand the solution in terms of some sequence of known functions. This technique is useful because a complicated function whose nature is not known may be expressed in terms of a sequence of functions whose nature is known. Familiar examples are the various Fourier transforms and numerous sets of orthogonal polynomials.

Similarly, the calculus of random functionals is useful because it makes possible the expansion of a complicated random functional whose nature is not known in terms of random functionals whose nature is known.

These random functionals form a sequence of integral functional operators dependent on some random process (say \( r(s) \)) which is taken as fundamental ("fundamental" in the sense that \( e^{j\omega} \) is "fundamental" to Fourier transforms). This sequence of functional operators is, in a sense, complete and, subject to certain restrictions, differentiable. Furthermore, the functionals may be made orthogonal in a certain sense, and when this is done, an inverse may be explicitly defined.

Such a representation of random functions seems well suited to the description of the velocities, etc. existing in turbulent flow situations, and succeeding chapters will exploit this possibility.
The purpose of this chapter is to present the development of the calculus of random functionals in sufficient detail that it may be applied to turbulent flow situations without difficulty. The treatment given here is similar to Weiner's\textsuperscript{32} (Weiner seems to have done most of the fundamental work in this area) and is an extension of a treatment given in another work.\textsuperscript{21}

The notation and language used to describe the probabilistic concepts in this work will adhere closely to those of ordinary probability theory. Thus, the terms \textit{expectation, probability, variance}, etc., will be used frequently, and references to measure theoretic concepts will be minimized.*

1. \textit{Fundamentals of the Brownian Process}

The most convenient choice of the fundamental random process on which the random functionals are to depend is the Brownian Motion Process (really the Einstein Process). One physical realization of this process (in fact, the one which motivated English botanist, Robert Brown) is the motion of a small particle immersed in a frictionless fluid medium and subject to very frequent collisions. The theory is very well developed, and only the

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*Crudely, the mathematical setting for this chapter is as follows: The concept of a random function of space (or time) carries with it the notion of a set of all possible realizations of the random function. Each possible realization of such a function is designated by including in the arguments of the function a parameter \(\omega\) (\(\omega\) will be consistently used as the sampling parameter). Thus, if \(f(t)\) is the symbol assigned to some random process on \(t\), then for each permissible \(\omega\), \(f(t,\omega)\) will represent a possible realization of the function. The total collection of all \(\omega\) will be named \(\Omega\), and "probability" is then interpreted as a certain measure of sets of \(\omega\) contained in \(\Omega\). For further details, see Doob\textsuperscript{5} and Wiener.\textsuperscript{32}
results* are presented here. With suitable normalization the distribution function of the displacement \( r \) at time \( t \) from an initial position \( r_0 \) at time \( t_0 \) is given by:

\[
W_2(r,t|r_0,t_0) = \left| \frac{1}{2\pi(t-t_0)} \right|^{\frac{1}{2}} \exp \left[ -\frac{(r-r_0)^2}{2|t-t_0|} \right]
\]

3.1.1

It is easy to show that this function composes in the proper fashion. For example,

\[
W_2(r,t|r_0,t_0) = \int_{-\infty}^{\infty} W_2(r,t|r_1,t_1)W_2(r_1,t_1|r_0,t_0)\,dr_1
\]

3.1.2

whenever \( t_0 < t_1 < t \).

It is convenient to choose the axes so that at time \( t = 0 \), \( r = 0 \). In this case the distribution function for the displacement \( r \) at time \( t = 0 \) is simply:

\[
W_1(r,t) = \frac{1}{(2\pi|t|)^{\frac{1}{2}}} \exp\left( - \frac{r^2}{2|t|} \right)
\]

3.1.3

Equations 3.1.1 and 3.1.3 completely describe the Brownian process and from them the set of random functions

\( r(t,\omega) \)

is easily defined.**

With probability 1, \( r(t,\omega) \) is continuous*** for all \( t \). Furthermore, the expectation values of various powers of \( r(t,\omega) \) are given by

---

*See, for example, Uhlenbeck and Ornstein**

**See Weiner, He takes \( \alpha \) as the real line \( \alpha \) from 0 to 1 and the probability measure as Lebesgue measure on the line.

***See Weiner and Doob.
\[ E\{[r(t, \omega)]^n\} = \int_{-\infty}^{\infty} r^n W_1(r, t) dr \]
\[ = \begin{cases} 
0 & \text{n odd} \\
\frac{n!}{(n/2)!} \left(\frac{t}{2}\right)^n & \text{n even}
\end{cases} \]

This \( r(t, \omega) \) will be used as the basis of the functional operators to be developed, and further details will be presented as needed.

2. **Stochastic Integrals**

The development of the calculus of random functionals is dependent to a large extent on the notion of stochastic integrals.

Consider for example the integral:

\[ F(\omega) = \int_{a}^{b} f(t) dr(t, \omega) \]

This integral is to be interpreted in the ordinary Reimann-Stieltjes sense. That is: let \( \{x_i\} \) be any finite set of \( n+1 \) points, such that:

\[ a = x_0 < x_1 < x_2 \ldots < x_n = b \]

and let \( \{\varepsilon_i\} \) be a set of \( n \) points such that:

\[ x_{i-1} \leq \varepsilon_i \leq x_i \]

and let:

\[ \delta_n = \max(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}) \]

then if:

\[ F(\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\varepsilon_i) [r(x_i, \omega) - r(x_{i-1}, \omega)] \]

exists for all possible sequences of \( \{x_i\} \) and \( \{\varepsilon_i\} \), then \( F(\omega) \) is the Reimann-Stieltjes integral of \( f(t) \) with respect to \( r(t, \omega) \).
over \([a,b]\). It can be shown \(^*\) that if, over \([a,b]\) one of \((f,r)\) is continuous and the other is of bounded variation, then \(F(\omega)\) exists. Thus, in all the work which follows the kernel \((f(t))\) will be considered to be of bounded variation. Then since \(r(t,\omega)\) is continuous with probability 1, \(F(\omega)\) exists with probability 1. \(^*\) Furthermore, the integral will exist if the limits are extended to infinity provided that the "tails" of \(f(t)\) behave properly. In this case the infinite integral is defined in the usual limiting sense.

These integrals have several very interesting properties. For example, the expectation value of \(F(\omega)\) is: \(^***\)

\[
E \left\{ F(\omega) \right\} = E \left\{ \lim \Sigma_i f(\epsilon_i) (r(x_i,\omega) - r(x_{i-1},\omega)) \right\} \\
= \lim \Sigma_i f(\epsilon_i) E \left\{ r(x_i,\omega) - r(x_{i-1},\omega) \right\} \\
= \lim \Sigma_i f(\epsilon_i) \cdot 0 = 0 \\
\]

Furthermore:

\[
E \left\{ (F(\omega))^2 \right\} = E \left\{ \left[ \lim \Sigma_i f(\epsilon_i) (r(x_i,\omega) - r(x_{i-1},\omega)) \right] \cdot \left[ \lim \Sigma_j f(\epsilon_j) (r(x_j,\omega) - r(x_{j-1},\omega)) \right] \right\} \\
= \lim \Sigma_i, j f(\epsilon_i) f(\epsilon_j) E \left\{ (r(x_j,\omega) - r(x_{j-1},\omega)) \cdot (r(x_i,\omega) - r(x_{i-1},\omega)) \right\} \\
= \lim \Sigma_i, j f(\epsilon_i) f(\epsilon_j) \cdot \delta^i_j \cdot (x_i - x_{i-1}) \\
= \lim \Sigma_i f(\epsilon_i) f(\epsilon_i) (x_i - x_{i-1}) \\
= \int_a^b f^2(t) dt \\
\]

\(^*\) See Notes for Course 18.21, "Analysis," MIT, (unpublished) or any text on advanced calculus.

\(^*\) It is not sufficient to take \(f(t)\) as continuous, because \(r(t,\omega)\) is not of bounded variation. See Doob\(^4\).

\(^***\) For details of these theorems, see Doob\(^5\).
Thus the transformation from \( f(t) \) to \( F(\omega) \) is, in a sense, **isometric** in that:

\[
E \left\{ [F(\omega)]^2 \right\} = \int_{\Omega} [F(\omega)]^2 \, dP(\omega) = \int_{a}^{b} f^2(t) \, dt
\]

The independent variable \( t \) may be reintroduced as follows:

\[
F(t, \omega) = \int_{a}^{b} f(\tau + t) \, d\tau(\tau, \omega)
\]

The most important situation occurs when the limits extend to infinity so that:

\[
F(t, \omega) = \int_{-\infty}^{\infty} f(\tau + t) \, d\tau(\tau, \omega)
\]

or alternately:

\[
F(t, \omega) = \int_{-\infty}^{\infty} f(\tau) \, d\tau(\tau - t, \omega)
\]

Now since the distribution of \([\tau(\tau + \Delta \tau - t, t, \omega) - \tau(\tau-t, \omega)]\) is independent of \( t \), it follows that the distribution of \( F(t, \omega) \) is independent of \( t \), i.e. \( F(t, \omega) \) is a stationary function of time.

And, as above:

\[
E \left\{ F(t, \omega) \right\} = 0
\]

\[
E \left\{ [F(t, \omega)]^2 \right\} = \int_{-\infty}^{\infty} f^2(\tau) \, d\tau
\]

The property of an integral such as 3.2.7 which makes them useful in the solution of differential equations is this: if \( \frac{df}{dt} \) exists for every \( t \) and \( \int_{-\infty}^{\infty} \frac{df}{dt} \bigg|_{t = \tau + \tau} \, d\tau(\tau, \omega) \) exists, then:

\[
\frac{d}{dt} F(t, \omega) = \int_{-\infty}^{\infty} \frac{df(t + \tau)}{dt} \, d\tau(\tau, \omega)
\]

This is the fundamental property of the random functional representation which distinguishes it from statistical methods, because in statistical methods knowledge of the distribution function does not imply knowledge of the derivative, whereas 3.2.7 implies all the
knowledge to be had concerning $F(t, \omega)$.

3. **Homogeneous Polynomial Functionals**

The notion of stochastic integrals can be extended to a whole class of functional operators of the form:

$$F_n(t, \omega) = \int_{-\infty}^{\infty} f_n(t + \tau_1, t + \tau_2, \ldots, t + \tau_n) \prod_{j=1}^{n} \mathrm{d}r_j(\tau_j, \omega)$$  \hspace{1cm} 3.3.1

for $n \geq 1$

and: $F_0(\omega) = \text{constant}$  \hspace{1cm} 3.3.2

The function $f_n$ may be taken as symmetric in its $n$ arguments without loss in generality, because if $f_n$ is not symmetric, only the symmetrical part of $f_n$ will contribute to the value of the integral.

The quantity $E\{F_n(\omega)\}$ is of considerable importance and its evaluation will be needed frequently in later sections of this chapter. The general rule is as follows:

$$E\{F_v(\omega)\} = \begin{cases} 0 & \text{if } v \text{ odd} \\ \Sigma L_v \int_{-\infty}^{\infty} f(t_{k_1}, t_{k_2}, \ldots, t_{k_v}) \prod_{j=1}^{v/2} \mathrm{d}t_j & \text{if } v \text{ even} \end{cases}$$ \hspace{1cm} 3.3.3

Where the sum extends over all ways of dividing the $t_{k_j}$'s into pairs. The integrations are then performed by associating both of the $t_{k_j}$'s of each pair with one $dt_j$ so that no two pairs are associated with the same $dt_j$. Expressed somewhat more concisely, the expectation of

$$F_v(\omega) = \int_{-\infty}^{\infty} f_v(t_1, t_2, \ldots, t_v) \prod_{j=1}^{v} \mathrm{d}r_j(\tau_j, \omega)$$ \hspace{1cm} 3.3.4

is obtained (for $v$ even) by pairing the $t_k$'s in all possible ways, integrating, and adding.
Thus for example, for $v = 4$, there are three ways of dividing four variables into pairs. Hence,

$$
E \left\{ F_4(\omega) \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ f_4(t_1, t_1, t_2, t_2) + f_4(t_1, t_2, t_1, t_2) + f_4(t_1, t_2, t_2, t_1) \right] \, dt_1 \, dt_2
$$

3.3.5

If the integrand is symmetric in its arguments then all integrals of the sum 3.3.3 are equal. There are $\frac{v!}{2^{v/2}(v/2)!}$ such integrals and therefore:

$$
E \left\{ F_v(\omega) \right\} = \begin{cases} 
0 & \text{for } v \text{ odd} \\
\frac{v!}{2^{v/2}(v/2)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_v(t_1, t_1, \ldots, t_{v/2}, t_{v/2}) \, dt_1 \cdots dt_{v/2} \right\}^{v/2} 
\end{cases}_j 3.3.6
$$

The proof of 3.3.3 or 3.3.6 is rather complicated and will not be given here. The essence of the proof, however, lies in the fact that:

$$
E \left\{ \sum_{j=1}^{2n} dr(t_j, \omega) \right\} = 0
$$

3.3.7

is zero unless the $t_j$ are at least paired. If the pairs are distinct so that no two pairs are the same, then the expectation is the product of the expectation of each pair, and:

$$
E \left\{ (r(t + dt, \omega) - r(t, \omega))^2 \right\} = dt
$$

3.3.8

or more cruelly:

$$
E \left\{ (dr(t_1, \omega))(dr(t_j, \omega)) \right\} = \delta(t_1 - t_j) dt_1 dt_j
$$

3.3.9

where $\delta(t_1 - t_j)$ is the Dirac delta function. The occurrence of two (or more) equal pairs contributes negligibly to the integral. The theorem follows from these considerations.

In terms of the above homogeneous polynomial functionals a very general random function may be represented as:
\[ A(t, \omega) = \sum_{v} \int_{-\infty}^{\infty} \int_{0}^{v} f_v(t + \tau_1, \ldots, t + \tau_v) \prod_{j=1}^{v} d\tau_j(\omega) \]  

3.3.10

Not all random functions may be represented in this way, however, because there are some difficulties with the convergence and summability of the series--much like the problem encountered in a Taylor series expansion of a function of a real variable. However, when such problems are absent, it is true that for a given \( A(t, \omega) \) the \( f_v \) in the expansion above are unique and:

\[ \sum_{v} \int_{0}^{v} \int_{-\infty}^{\infty} p_v(t+\tau) \prod_{j=1}^{v} d\tau_j = 0 \]  

3.3.11

implies that: \( p_v = 0 \), for all \( v \).  

3.3.12

For many problems, however, the above expansion will not work. In such instances recourse must be made to a more sophisticated expansion in the form of Orthogonal Polynomial Functionals developed below.

4. Orthogonal Polynomial Functionals

When the homogeneous polynomial expansion fails, the terms may often be regrouped in such a fashion as to insure convergence and summability. The same situation exists where the Taylor series representation of a function fails to exist. In this latter case, the powers of \( x \) may be regrouped into certain sets or orthogonal polynomials (e.g. Legendre polynomials over the range \([-1, 1]\)) in such a way that convergence and summability are assured.

A typical homogeneous polynomial functional of degree \( n \) is:

\[ F_n = \sum_{n} \int_{-\infty}^{\infty} f_n(t) \prod_{j=1}^{n} d\tau_j(\omega) \]  

3.4.1

A typical inhomogeneous polynomial functional of degree \( n \) is a sum of terms like the \( F_v \) for \( v \leq n \) such that \( F_n \) is not zero.
For example:

\[ I_{v_n} = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f_v^j \int_{-\infty}^{\infty} \frac{v}{\tau_j,\omega} \, dr(\tau_j,\omega) \]  

3.4.2

An Orthogonal Polynomial Functional is an inhomogeneous polynomial functional which is orthogonal to all functionals of lower order, i.e. if \( I_n \) is an orthogonal functional then:

\[ E \left\{ I_n \cdot F_v \right\} = 0 \quad \text{for every } v < n \]

The orthogonal polynomial functional of degree \( n \) will be represented operationally as:

\[ G_v[K_v,\omega] = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} K_v^n \int_{-\infty}^{\infty} \frac{v}{\tau_j,\omega} \, dr(\tau_j,\omega) \]

where:\n
\[ K_v^0 = K_v \]

and (evidently) the \( K_v^n \) for \( n < v \) are completely specified by \( K_v \).

The zeroth degree orthogonal functional is thus a constant and is represented by:

\[ G_0[K_0,\omega] = K_0 = \text{constant} \]

3.4.4

The first degree functional is of the form:

\[ G_1[K_1,\omega] = \int_{-\infty}^{\infty} K_1(t_1)dr(t_1,\omega) + K_1^0 \]

3.4.5

This functional is to be orthogonal to \( G_0 \), hence:

\[ E \left\{ K_0 \int_{-\infty}^{\infty} K_1(t_1)dr(t_1,\omega) + K_0 K_1^0 \right\} = 0 \]

3.4.6

The first term is an odd degree homogeneous polynomial and reference to 3.3.3 shows that its expectation is zero. Thus:

\[ K_0 \cdot K_1^0 = 0 \quad \text{for all } K_0. \]

3.4.7

hence, \( K_1^0 = 0 \) and:

\[ G_1[K_1,\omega] = \int_{-\infty}^{\infty} K_1(t_1)dr(t_1,\omega) \]

3.4.8
The second degree functional is of the form:

\[ G_2[K_2, \omega] = \int_{-\infty}^{\infty} \mathcal{K}_2(t_1, t_2) \, dt_1 \, dt_2 + \int_{-\infty}^{\infty} \mathcal{K}_2^1(t_1) \, dt_1 + \int_{-\infty}^{\infty} \mathcal{K}_2^2(t_1, t_2) \, dt_2 + \int_{-\infty}^{\infty} \mathcal{K}_2^0 \]

This functional must be orthogonal to both \( G_0 \) and \( G_1 \), hence:

\[ E \left\{ \mathcal{K}_0 \int_{-\infty}^{\infty} \mathcal{K}_2(t_1, t_2) \, dt_1 \, dt_2 + \mathcal{K}_0^1 \mathcal{K}_1^1(t_1) \, dt_1 + \mathcal{K}_0^2 \mathcal{K}_1^2(t_1, t_2) \right\} = 0 \]

and:

\[ E \left\{ \int_{-\infty}^{\infty} \mathcal{K}_2(t_1, t_2) \mathcal{K}_1(t_3) \, dt_1 \, dt_2 + \mathcal{K}_2^1 \mathcal{K}_1^1(t_1) \, dt_1 + \mathcal{K}_2^0 \mathcal{K}_1^0(t_1) \, dt_1 \right\} = 0 \]

Reference to 3.3.3 shows that 3.4.10 and 3.4.11 imply (respectively):

\[ \mathcal{K}_0 \int_{-\infty}^{\infty} \mathcal{K}_2(t_1, t_1) \, dt_1 + \mathcal{K}_0^1 \mathcal{K}_2^1 = 0 \]

and:

\[ \int_{-\infty}^{\infty} \mathcal{K}_2^1(t_1) \mathcal{K}_1^1(t_1) \, dt_1 = 0 \]

for all \( \mathcal{K}_0 \) and \( \mathcal{K}_1 \). Thus:

\[ \mathcal{K}_2^0 = -\int_{-\infty}^{\infty} \mathcal{K}_2(t_1, t_1) \, dt_1 \]

and:

\[ \mathcal{K}_2^1 = 0 \]

and therefore:

\[ G_2[K_2, \omega] = \int_{-\infty}^{\infty} \mathcal{K}_2(t_1, t_2) \, dt_1 \, dt_2 + \int_{-\infty}^{\infty} \mathcal{K}_2^0 \mathcal{K}_1^0(t_1, t_1) \, dt_1 \]

The higher order functionals may be developed in an analogous manner, and perhaps the general formula could be derived inductively, but it is more instructive to derive a generating function for the various coefficients, etc.

In the first place, it follows inductively from the first few functionals developed that the general \( G_v[K_v, \omega] \) must be of the form:
\[
G_v[K_v, \omega] = \sum_n a_n^v \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_j \int_{-\infty}^{v-2n} d\nu \int_{1=1}^{v-2n} dr(t_{1,\omega})
\]

\[K_v(t_1, \ldots, t_{v-2n}; \lambda_1, \ldots, \lambda_n, \lambda_{v})\]

where the \(K_v\) is assumed symmetrical in its arguments and:

\[
[v]_2 = \begin{cases} 
\frac{v}{2} & \text{v even} \\
\frac{v-1}{2} & \text{v odd}
\end{cases}
\]

and:

\[a_v^v \Delta 1\]

The problem then boils down to determining the \(a_n^v\). To compute the \(a_n^v\), however, it is necessary to understand the relationships among the quantities \(F_n, f_n,\) and \(G_v[H_v^n, \omega]\) in the expressions:

\[F_n(\omega) = \int_{-\infty}^{\omega} f_n(t_1, t_2, \ldots, t_n) \prod_{j=1}^{n} dr(t_j, \omega)\]

and:

\[F_n(\omega) = \sum_{n=0}^{n} G_v[H_v^n, \omega]\]

Consider the quantity:

\[I_n^\mu = E \left\{ G_{\mu}[Q_{\mu, \omega}] \cdot F_n(\omega) \right\}\]

Because of the presumed orthogonality of the \(G\)'s, this quantity is zero unless \(\mu \leq n\); in fact it is zero unless \(\mu = n-2\lambda\), where \(\lambda = 0, 1, \ldots, \left[\frac{n}{2}\right]\). Suppose that \(\mu\) is one of the values for which \(I_n^\mu\) is non-zero; then:

\[I_n^\mu = E \left\{ G_{\mu}[Q_{\mu, \omega}] \cdot G_{\mu}[H_{\mu}^n, \omega] \right\}\]

\[= E \left\{ [\int_{-\infty}^{\mu} \int_{-\infty}^{\mu} J_{\mu} \prod_{j=1}^{n} dr(t_j, \omega) + \text{lower order term}] \times [\int_{-\infty}^{\mu} \int_{-\infty}^{\mu} J_{\mu} \prod_{j=1}^{n} dr(t_j, \omega) + \text{lower order term}] \right\}\]

The lower order terms in the expansion of \(G_{\mu}[Q_{\mu, \omega}]\) will be orthogonal.
to \( G_{\mu} [H_{\mu}^n, \omega] \) and therefore 3.4.23 becomes:

\[
I_{\mu}^n = E \left\{ \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_{\mu}(t_1, t_2, \ldots, t_\mu) \frac{\mu}{j=1} \prod_{j=1}^{\infty} dt_j, \omega \right] \right. \\
\times \left. \left[ \int_{-\infty}^{\infty} H_{\mu}^n(\tau_1, \tau_2, \ldots, \tau_\mu) \prod_{j=1}^{\infty} d\tau_j, \omega \right] + \text{lower order terms} \right\} \tag{3.4.24}
\]

The rule for evaluating such an expression is to take all the \( t \) and \( \tau \) variables, associate by pairs in all possible ways, integrate and add up the results. However, if two \( t \)'s are associated together, the result would be equivalent to finding:

\[
E \left\{ \left[ \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} Q_{\mu}(\lambda, \lambda, t_1, t_2, \ldots, t_{\mu-2}) \prod_{j=1}^{\mu-2} dt_j, \omega \right] \right. \\
\times \left. G_{\mu} [H_{\mu}^n, \omega] \right\} \tag{3.4.25}
\]

The result is necessarily zero because \( G_{\mu} \) is orthogonal to lower order terms. The only thing left is to associate each \( t \) with a \( \tau \) in 3.4.24. This can only be done with the first term in the expansion for \( G_{\mu} [H_{\mu}^n, \omega] \). The result is therefore:

\[
J_{\mu}^{2n} = (\mu!) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_{\mu}(t_1, t_2, \ldots, t_\mu) H_{\mu}^n(t_1, t_2, \ldots, t_\mu) \prod_{j=1}^{\mu} dt_j \tag{3.4.26}
\]

because there are \( \mu! \) ways of associating each of the \( t \)'s with a \( \tau \), and by the symmetry of \( Q_{\mu} \) and \( H_{\mu}^n \), all integrals are the same.

Two important results follow from 3.4.26. First, if

\[ Q_{\mu} = H_{\mu}^n, \text{ it follows that:} \]

\[ E \left\{ F^2(\omega) \right\} = \sum_{v} v! \int_{-\infty}^{\infty} [K_{v}(t_1, t_2, \ldots, t_v)]^2 \prod_{v}^2 dt_j \tag{3.4.27} \]

Furthermore, if \( Q_{\mu} \) is the symmetrical part of:
\[ F(\omega) = \sum_{\nu} \mathbb{E}_{\nu}[K_{\nu},\omega] \]  

Then:

\[ K_{\nu}(t_1, \ldots, t_{\nu}) = \lim_{h \to 0} \frac{1}{\nu!} \lim_{h \to 0} \mathbb{E} \left\{ F(\omega) \cdot G_{\nu}[Q_{\nu},\omega] \right\} \]

These important relations define the mean square and the inverse of a function such as 3.4.30. It is interesting to note that both the mean square and the inverse of 3.4.30 have been defined without knowledge of the exact structure of the \( G_{\nu} \) operator.

Let \( Q_{\mu} \) be defined as in 3.4.29. Then, from 3.4.20 and 3.4.21 it follows that:

\[ \mu! H_{\mu}^{n}(t_1, \ldots, t_{\mu}) = \mathbb{E} \left\{ G_{\mu}[Q_{\mu},\omega] \cdot F_{n}(\omega) \right\} \]

\[ = \mathbb{E} \left\{ G_{\mu}[Q_{\mu},\omega] \cdot \int_{-\infty}^{\infty} \int \sum_{j=1}^{\nu} dr(\tau_j, \omega) \right\} \]

To evaluate this expression, first note that the value is zero unless \( \mu \) and \( n \) are both either even or odd, for otherwise the value would be the expectation of a sum of odd polynomials all of whose expectation are zero. Furthermore, if \( n < \mu \) the value is zero, because \( G_{\mu} \) is orthogonal to all lower order functionals. Thus for a given \( n \) the only values of \( \mu \) for which the expectation is non-zero are of the form \( n-2\lambda \) where \( \lambda \geq 0 \) and \( n-2\lambda \geq 0 \). Thus:

\[ (n-2\lambda)! H_{n-2\lambda}^{n} = \lim_{h \to 0} \mathbb{E} \left\{ \int_{-\infty}^{\infty} \sum_{j=1}^{n} \int f_{n}(\sigma_1, \sigma_2, \ldots, \sigma_n) \prod_{j=1}^{n} \right\} dr(\sigma_j, \omega) \]

\[ = \int_{-\infty}^{\infty} \sum_{j=1}^{n-2\lambda} \prod_{j=1}^{n-2\lambda} dr(\tau_j, \omega) + \text{lower order terms} \}

The rule for evaluating such an expression is as follows: associate the \( \sigma \)'s and \( \tau \)'s in pairs in all possible ways, integrate, and add.
Since there are $2\lambda$ more $\sigma$'s than $\tau$'s, at least $2\lambda$ of the $\sigma$'s must be associated with each other. If more than $2\lambda$ of the $\sigma$'s are associated, however, the result is equivalent to finding the expectation of $G_{\mu}^{n-2\lambda}$ with some lower order functional. The value of this is zero by the presumed orthogonality of the $G_{\mu}$'s. By these considerations, 3.4.33 is reduced to:

$$H_{n-2\lambda}^{\mu}(t_1, \ldots, t_{n-2\lambda}) = \lim_{h \to 0} \sum_{j=1}^{\lambda} \int_{-\infty}^{\infty} \prod_{k=1}^{n-2\lambda} d\tau_k (t_1, \ldots, t_{n-2\lambda})$$

$$f_n(\sigma_1, \sigma_1, \ldots, \sigma_{\lambda}, \sigma_{\lambda}; t_1, \ldots, t_{n-2\lambda}) \cdot Q_{n-2\lambda}(t_1, \ldots, t_{n-2\lambda})$$

where the sum extends over all ways of selecting $\lambda$ pair of $\sigma$'s from $f_n$, and then identifying the remaining $\sigma$'s of $f_n$ with the $\tau$'s of $Q_{n-2\lambda}$. There are $\frac{n!}{2^{\lambda}\lambda!}$ ways of doing this: $\binom{n}{2\lambda}$ ways of selecting the $2\lambda$ variables from $f_n$, $\frac{(2 \cdot \lambda)!}{2^{\lambda}\lambda!}$ ways of forming pairs from these variables, and $(n-2\lambda)!$ ways of associating the remaining variables of $f_n$ with those of the $Q_{n-2\lambda}$. Thus, since $f_n$ and $Q_{n-2\lambda}$ are symmetric, 3.4.34 becomes simply:

$$H_{n-2\lambda}^{\mu}(t_1, \ldots, t_{n-2\lambda}) = \frac{n!}{2^{\lambda}\lambda!(n-2\lambda)!} \lim_{h \to 0} \int_{-\infty}^{\infty} \prod_{k=1}^{\lambda} d\sigma_k$$

$$f_n(\tau_1, \ldots, \tau_{n-2\lambda}; \sigma_1, \sigma_1, \ldots, \sigma_{\lambda}, \sigma_{\lambda}) \cdot Q_{n-2\lambda}(\tau_1, \ldots, \tau_{n-2\lambda})$$

Thus if $f_n$ is continuous so that the limit behaves properly, 3.4.35 becomes:

$$H_{n-2\lambda}^{\mu}(t_1, \ldots, t_{n-2\lambda}) = \frac{n!}{2^{\lambda}\lambda!(n-2\lambda)!} \int_{-\infty}^{\infty} f_n(t_1, \ldots, t_{n-2\lambda}; \sigma_1, \sigma_1, \ldots, \sigma_{\lambda}, \sigma_{\lambda})$$

$$\cdot \prod_{k=1}^{\lambda} d\sigma_k$$

This important equation defines the conversion from orthogonal functionals to homogeneous functionals. Again it is interesting to note that this can be done without knowledge of the exact structure of the $G_{\mu}$ functional operator. Equations 3.4.20 and 3.4.21 may be written:
\[ \mathcal{P}_n(\omega) = \int_{-\infty}^{\infty} \int f_n(t_1, \ldots, t_n) \prod_{j=1}^{n} \mathrm{dr}(t_j, \omega) \]

\[ \left[ \begin{array}{c} 3.4.37 \\ \end{array} \right] 

\[ \mathcal{G}_v = \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{n!}{2^v v! (n-2v)!} \mathcal{G}_{n-2v} \left[ \prod_{j=1}^{n} \sigma_j \right] = \int_{-\infty}^{\infty} \int f_n(t_1, \ldots, t_{n-2v}; \sigma_1, \sigma_2, \ldots, \sigma_v, \sigma_v) \prod_{j=1}^{\nu} d\sigma_j ; \omega \]

The development of the \( \mathcal{G}_v \) may now be completed by considering the function:

\[ \exp[\int_{-\infty}^{\infty} \phi(t) \mathrm{dr}(t, \omega)] = \sum_{n=0}^{\infty} \frac{b^n}{n!} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \phi(t_j) \mathrm{dr}(t_j, \omega) \]

\[ \left[ \begin{array}{c} 3.4.38 \\ \end{array} \right] 

where \( \phi(t) \) is so chosen that:

\[ \int_{-\infty}^{\infty} [\phi(t)]^2 \mathrm{dt} = 1 \]

By 3.4.37, 3.4.38 may be written:

\[ \exp[\int_{-\infty}^{\infty} \phi(t) \mathrm{dr}(t, \omega)] = \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{n!}{2^v v! (n-2v)!} \mathcal{G}_{n-2v} \left[ \prod_{j=1}^{n} \phi(t_j) ; \omega \right] \]

\[ \left[ \begin{array}{c} 3.4.40 \\ \end{array} \right] 

\[ = \sum_{n=0}^{\infty} \frac{b^{n+2v}}{n!} \frac{v!}{2^v v! n!} \mathcal{G}_n \left[ \prod_{j=1}^{n} \phi(t_j) ; \omega \right] \]

\[ = \sum_{n=0}^{\infty} \frac{b^n}{n!} \exp\left(\frac{b^2}{2}\right) \mathcal{G}_n \left[ \prod_{j=1}^{n} \phi(t_j) ; \omega \right] \]

Now let:

\[ z = \int_{-\infty}^{\infty} \phi(t) \mathrm{dr}(t, \omega) \]

Then \( \mathcal{G}_v \left[ \prod_{j=1}^{n} \phi(t_j) ; \omega \right] \) is simply a polynomial of degree \( v \) in \( z \), and 3.4.40 becomes:

\[ \exp\left[\frac{b^2}{2} + bz\right] = \sum_{n=0}^{\infty} \frac{b^n}{n!} \mathcal{G}_v \left[ \prod_{j=1}^{v} \phi(t_j) ; \omega \right] \]

\[ \left[ \begin{array}{c} 3.4.42 \\ \end{array} \right] 

and by 3.4.17 this becomes:
\[ \exp\left[ -\frac{b^2}{2} + bz \right] = \sum_{o}^{\infty} b^o_v \sum_{n}^{\infty} a^n_v z^{v-2n} \]

However, the Hermite polynomials defined by:

\[ H_{\mu}(x) = e^{\frac{x^2}{2}} \left[ -\frac{d}{dx} \right]^{\mu} e^{-\frac{x^2}{2}} = \sum_{o}^{\infty} \left(\frac{x}{\mu} \right) \frac{(-1)^{\lambda}(\mu)}{\lambda!(\mu-2\lambda)!2^{\lambda}} x^{\mu-2\lambda} \]

have a generating function which satisfies the identity:

\[ \exp\left[ -\frac{b^2}{2} + bz \right] = \sum_{o}^{\infty} b^o_v \sum_{n}^{\infty} a^n_v H_v(z) \]

Comparing 3.4.45 with 3.4.42 and 3.4.43 it is clear that:

\[ G_v \left[ \prod_{j=1}^{v} \phi(t_j); \omega \right] \]

\[ = \sum_{o}^{\infty} b^o_v \sum_{n}^{\infty} a^n_v z^{v-2n} \]

\[ = \sum_{o}^{\infty} \left(\frac{x}{\mu} \right) \frac{(-1)^{\lambda}(\mu)}{\lambda!(\mu-2\lambda)!2^{\lambda}} x^{\mu-2\lambda} \]

and therefore:

\[ a^n_v = \frac{(-1)^{n}(v)!}{(v-2n)!2^{n}!} \]

From this determination of the \( a^n_v \) the \( G_v \) are completely specified by:

\[ G_n \left[ k_n, \omega \right] = \sum_{o}^{\infty} \left(\frac{x}{\mu} \right) \frac{(-1)^{\lambda}(\mu)!}{\lambda!(\mu-2\lambda)!2^{\lambda}} \int_{-\infty}^{+\infty} \prod_{j=1}^{v} d\sigma_j \int_{-\infty}^{+\infty} \prod_{j=1}^{n-2v} dr(t_j, \omega) \]

5. Completeness of the Orthogonal Polynomial Functionals

The preceding section of this chapter developed the expansion of a random function in terms of orthogonal polynomial functionals.
to the extent that it may be stated that:

\[ F(t, \omega) = \sum_{v=0}^{\infty} G_v \left[ K_v(t + t_1, \ldots, t + t_v); \omega \right] \]  
\[ \text{if:} \]

\[ E \left\{ F^2 \right\} = \sum_{v=0}^{\infty} v! \int_{-\infty}^{\infty} |K_v(t_1, t_2, \ldots, t_v)|^2 \prod_{j=1}^{v} dt_j \]  
\[ \text{then:} \]

\[ K_v(t + t_1, \ldots, t + t_v) = \frac{1}{v!} E \left\{ F(t, \omega) G_v(Q_v, \omega) \right\} \]

where \( Q_v \) is as defined in 3.4.29. Thus, any stationary random function of \( t \) which may be defined in terms of the \( G_v \)'s as in 3.5.1 has a second moment as defined in 3.5.2, and the sum may be inverted as in 3.5.3.

The question of completeness is thus: What class of stationary random functions of \( t \) may be expressed as a sum of \( G_v \)'s as in 3.5.1? The answer is that any strictly stationary random function of \( t \), i.e. any stationary function \( f(t) \) for which \( E \left\{ f^2 \right\} \) is bounded, may be so represented.

The purpose of this section is not to give a formal proof of this statement, but instead to give a construction which gives plausibility to the completeness statement except for some annoying details concerning convergence, etc. This path is chosen because it not only indicates completeness, but also it indicates how variables on which the function is stationary may be included, and furthermore indicates how the analysis may be extended to several dimensions.*

Consider a strictly stationary random function of \( t \) say \( f(t) \). Suppose that \( f(t) \) is smooth enough that it is accurately described

---

*The following development is extracted mainly from Chapter 12 and 13 of Weiner. Reference should also be made to Doob especially Chapters 9 and 12.
by its values at each of the points \( t_n \) of the sequence \( \{t_n\} \) where the sequence of \( t_n \) is defined by \( t_n = nh \) for some small \( h \) (later \( h \rightarrow 0 \)) and for all integer \( n \).

Now let \( h_n \) be the probability function for \( f_n \) given \( f_{n+1}, f_{n+2}, \ldots \), i.e.

\[
h_n(\lambda) = P \left\{ f_n(\omega) \leq \lambda \mid f_{n+1}(\omega) = \mu_1, f_{n-1}(\omega) = \mu_{-1}, \ldots \right\}
\]

Since \( f_n \) is strictly stationary, \( h_n \) is independent of \( n \), and it will be assumed that \( h_n \) is continuous and monotone strictly increasing. Thus \( h_n(\lambda) \) takes on all values from 0 to 1 and might look as shown in Fig. 3.1.

![Figure 3.1: The Conditional Distribution of \( f_n(\lambda) \).](image)

Now denote the set of all \( \omega \), such that the conditional part of 3.5.4 is met, as \( \{S\} \), i.e.:

\[
\{S\} = \left\{ \omega \mid f_{n+1}(\omega) = \mu_1, f_{n-1}(\omega) = \mu_{-1}, \ldots \right\}
\]

Then \( h_n(\lambda) \) is simply the conditional probability measure of all \( \omega \) contained in \( \{S\} \) such that \( f_n(\omega) \leq \lambda \).

Now pick an \( \omega \) from \( S \), say \( \omega_1 \); then \( f_n(\omega_1) \) will have a definite numerical value, and it is legitimate to ask for the value of \( h_n(\lambda) \). In fact, the function \( g_n(\omega) = h_n(f(\omega)) \) is a legitimate function of \( \omega \), since the sets \( \{S\} \) formed for various values of the \( \mu_k \) are non-intersecting. In fact the random function so formed is very interesting—suppose, for example, that:
\( f_n(\omega) \) may be defined in terms of the \( g_n(\omega), \ldots, g_{n+m}(\omega) \) for \( m \to \omega \), i.e., \( f_n(\omega) \) is a function of the \( g_n(\omega) \). Thus:

\[ f_n(\omega) = \text{func}(g_n(\omega), g_{n+1}(\omega), g_{n-1}(\omega), \ldots) \]  

and by the way in which the \( f_n(\omega) \) were constructed, it follows that:

\[ f_{n+k}(\omega) = \text{func}(g_{n+k}(\omega), g_{n+k+1}(\omega), g_{n+k-1}(\omega), \ldots) \]  

for any interger \( k \). But, the \( g_n(\omega) \) are independent, similarly distributed random functions; hence, by a simple mapping of the Gaussian law into the uniform law, the \( g_n(\omega) \) may be expressed in terms of increments of the Brownian motion process: for example, \( g_n(\omega) \) may be replaced by:

\[ \Delta r_n(\omega) = \Delta r(t_n, \omega) = r(t_n, \omega) - r(t_{n-h}, \omega) \]

so that \( f_{n+k}(\omega) \) may be expressed as:

\[ f_{n+k}(\omega) = \text{func}(\Delta r_{n+k}(\omega), \Delta r_{n+k+1}(\omega), \Delta r_{n+k-1}(\omega), \ldots) \]

Thus as the separation between the \( t_n \) tends to zero the representation of \( f_n(\omega) \) as a function of an infinite number of \( g_n(\omega) \) or \( \Delta r_n(\omega) \) will go into a representation of \( f(t, \omega) \) as a functional of \( dr(t + \tau, \omega) \) on the parameter \( \tau \), i.e.

\[ f(t, \omega) = \text{functional}[dr(t + \tau, \omega); \tau] \]

From this, the expansion in terms of orthogonal polynomial functionals follows as one possible expansion of 3.5.13 provided \( E \{[f(t, \omega)]^2\} \) is bounded so that the sum converges. The terms of the orthogonal polynomial expansion may be regrouped into homogeneous polynomial functionals, provided the necessary sums are summable and convergent.**

---

*For a discussion of the expansion of functionals, see George and Volterra.**

**If these conditions are met the functional is termed analytic, see Volterra.**
The essence of the completeness theorem is then first to show that a functional relationship such as 3.5.13 exists for a given random function. After the functional relationship is established, standard expansion techniques will provide a representation of the functional dependence.

6. Extension to Several Dimensions

In all the preceding discussion, the random functions described have been functions of one independent variable on which the function was stationary, and although this variable has been labeled $t$ there is no fundamental reason to associate it with time. If other parameters are involved in the definition of the random function, they may be introduced simply into the kernels of the functional, i.e. if $f(t,y,\omega)$ is a stationary random function of $t$ and depends on $y$ parametrically, then $f$ may be represented as:

$$f(t,y,\omega) = \sum_{v} G_v(t + t_1, \ldots t + t_v; y; \omega)$$  

If, however, the random function is stationary on more than one variable, then all of these variables should be included in the definition of the functional operator.

Suppose, for example, that $f$ is a random function of $x$ and $z$, and that it is stationary on both variables. This function would then be represented as $f(x,z,\omega)$.

The trick to defining the $G_v$ for such a function lies in mapping the total $x,z$ plane onto the entire line $s$ (from $-\infty$ to $\infty$) in such a way that the area is preserved, i.e. the points contained in the interval $[s,s+ds]$ go into points in the $x,z$ plane which occupy an area of $ds$. With such a mapping, a stochastic integral may be defined as:

$$F(\omega) = \int_{A} f(x,z) \, dr(s,\omega)$$
where the variables $x,z$ have been mapped into $s$ and $A$ is the set of $s$ over which the integration is to be performed. The integral of 3.6.2 cannot be treated as an ordinary Stieltjes integral, because $f$, treated as a function of $s$, is not generally of bounded variation, although it may well be continuous. The integral can be defined however in an l.i.m. sense, which is essentially a Lebesque-Stieltjes form.\footnote{The set of functions $f$ for which 3.6.2 is then defined will be those for which:

$$\int_A f^2 ds = \int_A \int f^2(x,z) dx dz < \infty$$

and with this definition it will be true that if:

$$F(x,z,\omega) = \int_{-\infty}^{\infty} f(x + x_1, z + z_1) dr(s_1, \omega)$$

(where the infinite limits imply the whole $x,z$ plane)

then:

$$E \left \{ F(x,z,\omega) \right \} = 0$$

and:

$$E \left \{ F^2(x,z,\omega) \right \} = \int_{-\infty}^{\infty} \int f^2(x,z) dx dz$$

and:

$$\frac{\partial F}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \bigg|_{x=x+x_1, z+z_1} dr(s, \omega)$$

and $F$ is a strictly stationary function of $x$ and $z$.}

From this definition of a multidimensional random functional, homogeneous polynomial functionals and orthogonal random functionals may be developed in precisely the same manner as was done for the one-dimensional case. Furthermore, the completeness theorem will follow the same lines indicated for the one-dimensional case.

\footnote{Doob considers this problem in detail, see Chapter 9. Also Wernikoff defines integration in a similar fashion.}
The three-dimensional case follows analogously from what has been said above and for the rest of this work, random functions which are stationary on three variables (either 3 space, or 2 space and 1 time) will be employed.

7. Summary

In the preceding sections of this chapter, the Calculus of Random Functionals has been developed in a manner, not at all rigorous, but hopefully, suited to conveying the ideas pertinent to applying this calculus to turbulence problems.

There are many extensions of what has been presented here, for example, Doob\(^5\) gives a generalization of stochastic integrals and Weiner\(^32\) discusses such subjects as F.M. spectra and stastical mechanics. The reader interested in more rigor and/or extensions of this work should consult the above mentioned works.
CHAPTER IV

Decaying Turbulent Flows

There are a number of decaying turbulent flow situations which involve such a high degree of symmetry that ordinary statistical methods can be applied with some success. These flows are generally easier to analyze and investigate experimentally than other flows of more technical interest and thus, lend themselves more readily to an explanation of the processes involved in turbulence.

Perhaps the simplest example of such a flow is decaying isotropic turbulence. This flow is characterized by complete spherical symmetry and positional independence of all statistical properties. Decaying isotropic turbulence will be analyzed in section two of this chapter.

A simple extension of the idea of isotropic turbulence is the decay of an initially isotropic turbulent velocity field in the presence of a uniform d.c. magnetic field. The magnetic field will of course tend to make the field non-isotropic, but the form of the energy decay law is not exponential in time as might have been suspected. The decay of an initially isotropic velocity field in the presence of a uniform d.c. magnetic field will be analyzed in section three of this chapter.

The method used to analyze the above two examples of decaying turbulent flows is an expansion of the velocities and pressure gradients in terms of Homogeneous Polynomial Functionals. A description of this method is contained in section one of this chapter.
1. Application of Homogeneous Polynomial Functionals

As presented in Chapter I (q.v.), the normalized equations of motion for the turbulent flow of an incompressible, viscous, Newtonian fluid with a high, scalar electrical conductivity in the presence of electro-magnetic fields are:

\[
\frac{\partial \mathbf{u}^i}{\partial t} + \frac{\partial}{\partial x^j} \mathbf{u}^i \mathbf{u}^j - \frac{1}{\text{Re}} \nabla^2 \mathbf{u}^i + \frac{\partial \mathbf{p}}{\partial x^i} = \frac{M^2}{\text{Re}} \epsilon_{ikl} \mathbf{b}^k \mathbf{b}^l
\]

4.1.1

\[\mathbf{j}^i = \mathbf{e}^i + \epsilon_{ijk} \mathbf{u}^j \mathbf{b}^k\]

4.1.2

\[\epsilon_{ijk} \frac{\partial \mathbf{e}^k}{\partial x^j} = - \frac{\partial \mathbf{b}^i}{\partial t}\]

4.1.3

\[\epsilon_{ijk} \frac{\partial \mathbf{b}^k}{\partial x^j} = \mathbf{R}^i\]

4.1.4

and \(\frac{\partial \mathbf{u}^k}{\partial x^k} = 0\), \(\frac{\partial \mathbf{b}^k}{\partial x^k} = 0\)

4.1.5

4.1.6

where: \(\mathbf{u}^i\), \(\mathbf{j}^i\), \(\mathbf{e}^i\), and \(\mathbf{b}^i\) are the normalized velocity, current density, electric field, and magnetic induction vectors, respectively.

For the moment, consider the ordinary hydrodynamic case, i.e. all electrical effects are zero (the analysis will shortly be extended to the magneto-fluid dynamic case). Then, the normalized equations of motion are simply:

\[\frac{\partial \mathbf{u}^i}{\partial t} + \frac{\partial}{\partial x^j} \mathbf{u}^i \mathbf{u}^j - \frac{1}{\text{Re}} \nabla^2 \mathbf{u}^i + \frac{\partial \mathbf{p}}{\partial x^i} = 0\]

4.1.7

and:

\[\frac{\partial \mathbf{u}^i}{\partial x^i} = 0\]

4.1.8
Now, suppose the flow situation is such that an expansion of the velocities and pressure gradients in terms of Homogeneous Polynomial Functionals is convergent and summable. Then the velocities and pressure may be written:

$$u^i(x^k,t) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} K_n^i(x^k,t;s_1,s_2,\ldots,s_n) \left| \right| \frac{dr(s_j,\omega)}{j=1}$$

and:

$$p(x^k,t) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} W_n(x^k,t;s_1,s_2,\ldots,s_n) \left| \right| \frac{dr(s_j,\omega)}{j=1}$$

where the $K_n^i$ and $W_n$ are symmetrical functions of the $s_j$ arguments.

The form of the $K_n^i$ and $W_n$ of equations 4.1.9 and 4.1.10 may be simplified in a manner depending on the flow situation. If, for example, the velocities and pressure are stationary random functions of all three space variables, but a decaying function of time, then the appropriate velocity expansion is:

$$u^i(x^k,t) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} K_n^i(x^k + x_1^k,\ldots,x_n^k + x_n^k,t) \left| \right| \frac{dr(s_j,\omega)}{j=1}$$

(and similarly for the pressure), where each $s_j$ is the measure ("volume") preserving mapping of the three space variables, $x_1^j$, $x_2^j$, and $x_3^j$ onto the line $s_j$ (see Chapter III, Section 6).

If on the other hand, the velocities and pressure are stationary random functions of two space variables (say $x_1^3$ and $x_2^3$) and time, but a decaying function of the third space variable ($x_3^2$), then the appropriate velocity expansion is:

$$u^i(x^k,t) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} K_n^i(x_1^3 + x_2^3 + x_3^3,t + t_1;\ldots; \left| \right| \frac{dr(s_j,\omega)}{j=1}$$

$x_1^1 + x_2^1 + x_3^1, x_1^3 + x_2^3 + x_3^3$ (and similarly for the pressure) where in this case, each $s_j$ is the measure preserving mapping of the variables $x_1^j, x_3^j$, and $t_j$ onto the line $s_j$. 
Now, if the first few derivatives of the expansion in equations 4.1.9 and 4.1.10 are assumed convergent and summable (but no assumption is made concerning the form of the $K_n^i$ and $W_n$ such as lead to equations 4.1.11 or 4.1.12), then the expansion may be inserted directly into the equations of motion to obtain:

$$
\sum_{n}^{\infty} \int_{-\infty}^{\infty} \int_{n}^{\infty} \int_{j=1}^{n-1} \frac{\partial K_n^i}{\partial t} + \frac{\partial}{\partial x^j} \left( \sum_{m}^{\infty} K_m^i K_{n-m}^j - \frac{1}{Re} \nu^2 K_n^i + \frac{\partial W_n}{\partial x^i} \right) = 0
$$

4.1.13

and:

$$
\sum_{n}^{\infty} \int_{-\infty}^{\infty} \int_{n}^{\infty} \int_{j=1}^{n-1} \frac{\partial K_n^k}{\partial x^j} = 0
$$

4.1.14

Thus, (referring to the results of Chapter III, Section 3):

$$
\frac{\partial K_n^i}{\partial t} + \sum_{m}^{n} \frac{\partial}{\partial x^j} \langle K_m^i K_n^{j} \rangle - \frac{1}{Re} \nu^2 K_n^i + \frac{\partial W_n}{\partial x^i} = 0
$$

4.1.15

and:

$$
\frac{\partial K_n^i}{\partial x^j} = 0; \text{for } n = 0, 1, 2, \ldots
$$

4.1.16

where the symbolism $\langle \ldots \rangle$ means "the symmetric part on the $s_j$ variables".

For $n = 0$, equations 4.1.15 and 4.1.16 yield:

$$
\frac{\partial K_0^i}{\partial t} + \frac{\partial}{\partial x^j} K_0^i K_0^j - \frac{1}{Re} \nu^2 K_0^i + \frac{\partial W_0}{\partial x^i} = 0
$$

4.1.17

and:

$$
\frac{\partial K_0^k}{\partial x^j} = 0
$$

4.1.18

These equations are simply the original equations of motion, and they describe the laminar motion associated with the particular flow geometry. It is important to note that equations 4.1.17 and
4.1.18 are completely independent of the higher order $K^1_n$ and therefore $K^1_0$ may be determined independently of $K^1_n$ for $n > 0$.

For $n = 1$, equations 4.1.15 and 4.1.16 yield:

$$\frac{\partial K^1_1}{\partial t} + \frac{\partial}{\partial x^1} \left[ K^1_0 K^1_1 + K^1_1 K^1_1 \right] - \frac{1}{R_e} v^2 K^1_1 + \frac{\partial W^1_1}{\partial x^1} = 0$$

4.1.19

and:

$$\frac{\partial K^k_1}{\partial x^k} = 0$$

4.1.20

These equations completely specify the $K^1_1$ and $W^1_1$ (subject to boundary and/or initial conditions) once the $K^1_0$ have been determined. It is interesting to note that equations 4.1.19 and 4.1.20 are precisely the stability equations studied by Lin et al. Consequently, for those flows which may (for certain values of $R_e$) have two (or more) stable states (e.g. one laminar and one turbulent as in Plane Poiseuille Flow) there will no doubt be problems with convergence and summability of the series in equations 4.1.9 and 4.1.10. However, for decaying turbulent flows, the turbulent state gradually and smoothly connects with the laminar flow as $t \to \infty$ (or as some coordinate (say $x^2$) $\to \infty$) and there are either boundary or initial conditions which specify the turbulent state. Thus, it is appropriate to seek solutions to decaying turbulent flow problems in terms of Homogeneous Polynomial Functionals.

The remaining sets of equations for $n > 1$ each consist of four simultaneous, linear, partial differential equations dependent only on solutions of equations of lower order. In principle then, these sets of equations generated by application of Homogeneous Polynomial Functionals are coupled only from below and may be solved in sequence. However, the equations become
progressively more difficult to solve, and if the first few terms are not sufficient, then some approximate technique will be needed.

The extension of the above equations to the magneto-fluid dynamic case is not difficult. The electric field, magnetic fields, and current density are expanded in terms of Homogeneous Polynomial Functionals as follows:

\[
\begin{aligned}
\left\{ e^i_n \right\} &= \frac{1}{\Sigma_0} \int_{-\infty}^{\infty} \int_{n}^{\infty} \left\{ E^i_n \right\} \frac{n}{W} \text{d}r(s_j, \omega) \\
\left\{ b^i_n \right\} &= \frac{1}{\Sigma_0} \int_{-\infty}^{\infty} \int_{n}^{\infty} \left\{ B^i_n \right\} j = 1 \\
\left\{ j^i_n \right\} &= \frac{1}{\Sigma_0} \int_{-\infty}^{\infty} \int_{n}^{\infty} \left\{ J^i_n \right\}
\end{aligned}
\]

where the arguments of the \( E^i_n \), \( B^i_n \), and \( J^i_n \) are constructed in the manner of equation 4.1.9. These expansions may be inserted into the equations of motion to yield:

\[
\frac{\partial e^i_n}{\partial t} + \frac{\partial}{\partial x^j} \frac{n}{\Sigma_0} <K^i_m K^j_{n-m}> - \frac{1}{Re} v^2 K^i_n + \frac{\partial W_n}{\partial x^i} = \frac{M^2}{Re} e^{ijk} \frac{n}{\Sigma_0} <J^j_m B^k_{n-m}>
\]

\[
J^i_n = e^i_n + e^{ijk} \frac{n}{\Sigma_0} <K^j_m B^k_{n-m}>
\]

\[
e^{ijk} \frac{\partial B^k_n}{\partial x^j} = \frac{1}{\Sigma_0} \frac{\partial e^i_n}{\partial t}
\]

\[
e^{ijk} \frac{\partial B^k_n}{\partial x^j} = \frac{1}{\Sigma_0} \frac{\partial e^i_n}{\partial t}
\]

and:

\[
\frac{\partial K^k_n}{\partial x^k} = 0 ; \quad \frac{\partial B^k_n}{\partial x^k} = 0
\]

where \( n \) takes on all positive integral values.
2. The Decay of Isotropic Turbulence

Consider a fluid of infinite extent which is viscous, continuous, homogeneous, and incompressible, and in which, at time \( t = 0 \), there exists an isotropic distribution of decaying turbulence.

For such a flow, all correlations and averages are independent of the space coordinates \( x^k \). Thus, according to the results of Section 1 of this chapter, it is appropriate to expand the velocities and pressure in terms of Homogeneous Polynomial Functionals as:

\[
\begin{align*}
\mathbf{u}^i(x^k, t) & = \sum_{n=0}^{\infty} \int \int_{-\infty}^{\infty} K_n^i(x^k + x^k_1, \ldots, x^k + x^k_n; t) \prod_{j=1}^{n} ds_j(d) \\
\mathbf{p}(x^k, t) & = \sum_{n=0}^{\infty} \int \int_{-\infty}^{\infty} W_n(x^k + x^k_1, \ldots, x^k + x^k_n; t) \prod_{j=1}^{n} ds_j(d)
\end{align*}
\]

where each \( s_j \) is the measure ("volume") preserving mapping of the three space variables \( x^1_j, x^2_j, \text{ and } x^3_j \) onto the line \( s_j \).

If all electrical effects are presumed zero, the normalized equations of motion become:

\[
\frac{\partial K_n^i}{\partial t} - \frac{1}{Re} \nabla^2 K_n^i + \frac{\partial W_n^i}{\partial x^i} = - \frac{\partial}{\partial x^j} \sum_{m}^{n} \langle K_{n-m}^i K_m^j \rangle
\]

and:

\[
\frac{\partial K_o^i}{\partial x^i} = 0
\]

For \( n = 0 \), the equations are:

\[
\frac{\partial K_o^i}{\partial t} - \frac{1}{Re} \nabla^2 K_o^i + \frac{\partial W_o^i}{\partial x^i} = - \frac{\partial}{\partial x^j} K_o^i K_o^j
\]
but the expansion form assumed allows $K_0^i$ and $\frac{\partial W_0}{\partial x^i}$ to be, at most, functions of time. Thus, the equations become:

\[
\frac{\partial K_0^i}{\partial t} + \frac{\partial W_0}{\partial x^i} = 0
\]

but, of course, the only isotropic vector function of time is $K_0^i = 0$. Thus the equations for $n > 0$ become:

\[
\frac{\partial K_1^i}{\partial t} - \frac{1}{R_e} \nabla^2 K_1^i + \frac{\partial W_1}{\partial x^i} = 0
\]

\[
\frac{\partial K_n^i}{\partial t} - \frac{1}{R_e} \nabla^2 K_n^i + \frac{\partial W_n}{\partial x^i} = \frac{n-1}{1m} \sum_{m} <K_{n-m}^i K_m^i> \; ; \; n > 1
\]

\[
\frac{\partial K_n^i}{\partial x^i} = 0
\]

The divergence of the $n = 1$ equation yields:

\[
\nabla^2 W_1 = 0
\]

However, the pressure gradients must be $L^2$ integrable -- a property that no solution of the Laplacian can have in the infinite domain -- thus $W_1 = 0$, and the pertinent $n = 1$ equations become:

\[
\frac{\partial K_1^i}{\partial t} - \frac{1}{R_e} \nabla^2 K_1^i = 0
\]

and:

\[
\frac{\partial K_1^i}{\partial x^i} = 0
\]

The equations for $n > 1$ are unchanged, i.e.:
\[
\frac{\partial K^i_n}{\partial t} - \frac{1}{Re} \nabla^2 K^i_n + \frac{\partial W^i_n}{\partial x^i} = - \frac{\partial}{\partial x^j} \sum_{l=1}^{n-1} \sum_{m=1}^{m} \langle K^i_{n-l} K^j_m \rangle
\]

and:
\[
\frac{\partial K^i_n}{\partial x^i} = 0
\]
and the derived pressure equation is:
\[
\nabla^2 W_n = - \frac{\partial^2}{\partial x^i \partial x^j} \sum_{l=1}^{n-1} \sum_{m=1}^{m} \langle K^i_{n-l} K^j_m \rangle
\]

At this point, it is worthwhile to collect all the results of this section and examine the possibilities of solution.

The appropriate expansion of the velocities and pressure gradients existing in isotropic turbulence is made in terms of homogeneous polynomial functionals in four dimensions \(x^1, x^2, x^3, t\). The expansion is made to be stationary on all three space variables and time is introduced parametrically. The zeroth order velocity and both the zeroth and first order pressure are known to be zero. Thus, the velocities and pressure gradients are expanded as:

\[
u^1(x^k, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K^i_n(x_1^k, x_2^k, x_3^k, \ldots, x_n^k, x^k; t) \frac{n}{j=1} \int dr(s_j; \omega)
\]

\[
p(x^k, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_n(x_1^k, x_2^k, x_3^k, \ldots, x_n^k, x^k; t) \frac{n}{j=1} \int dr(s_j; \omega)
\]

The equations of motion relating the \(K^i_n\), and \(W_n\) are derived in a straightforward manner and the result is:
\[
\frac{\partial K^i_n}{\partial t} - \frac{1}{Re} \nabla^2 K^i_n = 0
\]
\[
\frac{\partial K_i^n}{\partial t} - \frac{1}{Re} \nabla^2 K_i^n + \frac{\partial W_i^n}{\partial x^1} = - \frac{3}{\partial x^1} \sum_{m=1}^{n-1} \langle K_i^{n-m} K_i^m \rangle ; \quad n > 1
\] 4.2.20

\[
\frac{\partial K_i^n}{\partial x^1} = 0, \text{ for all } n.
\] 4.2.21

From the homogeneity of the above equations it is easily observed that: if equation 4.2.17 and equation 4.2.18 form a solution to 4.2.19 - 4.2.21, then so do:

\[
u^n(x,t) = \sum_{j=1}^{\infty} \lambda^n \int_n^\infty \int_{-\infty}^{\infty} K_i^n(j) \| dr(s_j,\omega)
\] 4.2.22

\[
p^n(x,t) = \sum_{j=1}^{\infty} \lambda^n \int_n^\infty \int_{-\infty}^{\infty} W_i^n(j) \| dr(s_j,\omega)
\] 4.2.23

where \(\lambda\) is some measure of the strength of the turbulent field. Moreover, for a study of the final decay of isotropic turbulence, \(\lambda\) should be taken quite small, and initial conditions should be placed on the leading term of the expansion. Thus the leading term will always dominate provided higher order terms decay at least as fast as the leading term.

The leading term in the velocity expansion is \(K_i^1\), the homogeneous solution of:

\[
\frac{\partial K_i^1}{\partial t} - \frac{1}{Re} \nabla^2 K_i^1 = 0
\] 4.2.24

The higher order terms; i.e. \(K_i^n, n > 1\); consist of two parts: the homogeneous solution satisfying:

\[
\frac{\partial K_i^n}{\partial t} - \frac{1}{Re} \nabla^2 K_i^n = 0
\] 4.2.25

and a particular solution satisfying:
\[
\frac{\partial K_n^i}{\partial t} - \frac{1}{Re} \nabla^2 K_n^i = -\frac{\partial W_n}{\partial x^i} - \sum_{m}^{n-1} \frac{\partial}{\partial x^j} \langle K_{n-m}^i K_m^j \rangle
\]

where the \( W_n \) is completely specified by \( \frac{\partial}{\partial x^j} \sum_{m}^{n-1} \langle K_{n-m}^i K_m^j \rangle \). Thus the homogeneous part of \( K_n^i \), \( n > 1 \), has the same decay law as does \( K_1^i \) and the particular part has the same decay law as its drive, viz. the product of the decay law of two lower order functions. Thus, the decay of the \( K_n^i \) (\( n > 1 \)) is at least as fast as the \( K_1^i \), and therefore, for the study of the decay of isotropic turbulence, it is sufficient to take:

\[
u^i(x, t) = \int_{-\infty}^{\infty} K_{1}^i(x^k, t+\tau) d\tau \]

where the \( K_1^i \) are solutions of:

\[
\frac{\partial K_1^i}{\partial t} - \frac{1}{Re} \nabla^2 K_1^i = 0
\]

These equations are most easily solved in terms of a three-dimensional Fourier Transform as follows:

\[
K_1^i(x^k, t) = (\frac{1}{2\pi})^3 \int_{-\infty}^{\infty} \int \Lambda_1^i(\lambda^k, t) e^{i\lambda^k x^k} d\lambda_k
\]

\[d\tau_\lambda = (d\lambda^1)(d\lambda^2)(d\lambda^3)\]

Then in terms of the \( A_1^i \) the equations of motion become:

\[
\frac{\partial A_1^i}{\partial t} + \frac{\mu}{Re} A_1^i = 0
\]

and:

\[
\lambda^j A_1^j = 0
\]
\[ \mu^2 = \lambda k \lambda \]

The general solution to these equations is:

\[ A^i_1(\lambda^k, t) = C^i_1(\lambda^k) \exp(-\mu^2 t/R_o) \]

where:

\[ \lambda^k C^i_1(\lambda^j) = 0 \]

The Theory of Invariants (see Appendix A) predicts that for isotropic turbulence the correlation tensor \( R^{ij} \) and the correlation spectrum tensor \( \phi^{ij} \) defined by:

\[ R^{ij}(\varepsilon^k, t) = R_1(\varepsilon^k, t) \varepsilon^i \varepsilon^j + R_2(\varepsilon^k, t) \delta^{ij} \]
\[ + R_3(\varepsilon^k, t) \varepsilon^{ijk} \lambda^k \]

must be of the form:

\[ R^{ij}(\varepsilon^k, t) = R_1(\varepsilon^k, t) \varepsilon^i \varepsilon^j + R_2(\varepsilon^k, t) \delta^{ij} \]
\[ + R_3(\varepsilon^k, t) \varepsilon^{ijk} \lambda^k \]

and:

\[ \phi^{ij}(\lambda^k, t) = \phi_1(\mu, t) \lambda^i \lambda^j + \phi_2(\mu, t) \delta^{ij} \]
\[ + \phi_3(\mu, t) \varepsilon^{ijk} \lambda^k \]

Conservation of mass further restricts these forms to be:

\[ R^{ij} = -\frac{1}{2} \frac{\delta f}{\delta r} \varepsilon^i \varepsilon^j + (f + \frac{\tau R}{2} \frac{\delta f}{\delta r}) \delta^{ij} + R_3 \varepsilon^{ijk} \lambda^k \]

In terms of the \( K_1^{ij} \), \( R^{ij} \) may be calculated as follows:

\[ R^{ij}(\varepsilon^k, t) = E \left\{ \int_{-\infty}^{\infty} K_1^{ij}(x^k + x_1^k, t) \, dr(s_1, \omega) \right\} \int_{-\infty}^{\infty} K_1^{ij}(x^k + \varepsilon^k + x_1^k, t) \]
\[ \cdot dr(s_1, \omega) \right\} \]
\[ = \int_{-\infty}^{\infty} K_1^{ij}(x^k, t) K_1^{ij}(x^k + \varepsilon^k, t) \, dx \]
From this result $\Phi^{ij}$ may be calculated as follows:

$$R^{ij} = \int_{-\infty}^{\infty} k_1(-\lambda, t) \Phi^{ij}_{1}(x, t) d\lambda$$

$$= \left(\frac{1}{2\pi}\right)^{3} \int_{-\infty}^{\infty} A^{i}_{1}(\lambda, t) e^{i\lambda} d\lambda$$

$$= \left(\frac{1}{2\pi}\right)^{3} \int_{-\infty}^{\infty} A^{i}_{1}(\lambda, t) \Phi^{ij}_{-\lambda} d\lambda$$

Therefore:

$$\Phi^{ij}_{1}(\lambda, t) = A^{i}_{1}(\lambda, t) A^{j}_{1}(\lambda, t)$$

$$= C^{i}_{1}(\lambda) C^{j}_{1}(\lambda) \exp(-2\mu^2 t/B_e)$$

Since $\Phi^{ij}$ is the outer product of two vectors, it must necessarily be of rank one; i.e. all $2 \times 2$ submatrices of the $3 \times 3$ matrix $\Phi^{ij}$ must have zero determinant. The result of this restriction is:

$$\mu^2 (\Phi_1)^2 + (\Phi_3)^2 = 0$$

thus:

$$\Phi_3 = \pm j\mu \Phi_1$$

Thus, the reflectional symmetry usually associated with isotropic turbulence by assuming $\Phi_3 = 0$ is not possible. The ambiguous sign in equation 4.2.46 is merely a reminder that the alternator $e^{ijk}$ changes sign on reflection of the coordinate axes and therefore either sign of $\Phi_3$ is possible. The sign will be taken positive for the remainder of this work, i.e.

$$\Phi_3 = j\mu \Phi_1$$
Thus the correlation spectrum tensor is of the form:

$$\phi_{ij}(\mu, \nu, t) = [\mu^2 \phi_{ij}^1 - \lambda^1 \phi_{ij}^1 + \mu \nu \phi_{ij}^1 \phi_{ij}^1] F(\mu) \exp(-2\mu^2 t/R_e)$$

where $F(\mu)$ is related to $C^i(-\lambda^1)C^j(\lambda^1)$. The asymptotic form of the decay is dependent on the behavior of $F(\mu)$ around $\mu = 0$, and the initial conditions are needed to specify $F(\mu)$, so the problem is not completely stated until the initial conditions are given. However, a few additional restrictions may be placed on $F(\mu)$.

First, the energy spectrum tensor:

$$\phi^{kk} = 2\mu^2 F(\mu) \exp(-2\mu^2 t/R_e)$$

is the transform of:

$$R^{kk} = E \left\{ \epsilon^{k}(x^1) \epsilon^{k}(x^1 + \epsilon^1) \right\}$$

The correlation $R^{kk}$ is a function which is smooth everywhere, has its maximum at $\epsilon^1 = 0$, and decays uniformly to zero as $r \to \infty$. Thus $F(\mu)$ should be analytic everywhere except (perhaps) at $\mu = 0$ or $\mu = -\infty$, and therefore $F(\mu)$ may be expanded in a Laurent series about zero as:

$$F(\mu) = \sum_{n=-\infty}^{\infty} b_n \mu^n$$

Secondly, Cramer's Theorem* states: "The necessary and sufficient conditions that $R^{ij}$ be the correlation tensor of a continuous, stationary, vector random process is that $R^{ij}$ be expressible in the form:

$$R^{ij} = (\frac{1}{2\pi})^3 \int_3 \int_3 \int_3 e^{i \lambda^1 \mu^1 \lambda^2 \mu^2 \lambda^3 \mu^3} d\tau_\lambda$$

*See Batchelor, p. 25.
where $\phi^{ij}$ is a positive (not necessarily definite) tensor such that:

$$
\int_3^\infty \int_{-\infty}^\infty \left| \phi^{ij} \right| d\tau \lambda
$$

is bounded. Since $\phi^{ij}$ is positive:

$$
\nu^i \nu^j \phi^{ij} = \left[ \mu^2 (\nu^k \nu^k) - (\nu^k \nu^k) \right]^2 F(\mu) \exp(-2\mu^2 t/R_e) \geq 0
$$

for any real $\nu^i$, and therefore:

$$
F(\mu) \geq 0 \text{ for all real } \mu.
$$

The condition that:

$$
\int_3^\infty \int_{-\infty}^\infty \left| \phi^{ij} \right| d\tau \lambda
$$

be bounded is equivalent to the condition that:

$$
\int_0^\infty \mu^4 F(\mu) \exp(-2\mu^2 t/R_e) d\mu
$$

be bounded. Thus, the Laurent series may extend only to the $\mu^{-4}$ term, i.e.:

$$
F(\mu) = \sum_{-4}^{\infty} b^n \mu^n
$$

and furthermore $b_{-4}$ must be positive.

At this point investigators of isotropic turbulence have customarily assumed that at least the first few moments of $R^{ij}$, e.g.

$$
L^{ijkl} = \int_3^\infty \int_{\infty}^\infty R^{ij} \epsilon^k \epsilon^l d\tau \tau
$$

were bounded for all time. In an important paper Batchelor and Proudman showed that such an assumption may be false when particular types of pressure terms are present. In the present case, however,
the pressure terms are not of such a character, and the physical problems giving rise to isotropic turbulence seem to allow integrals such as $L^i_{jk}k^l$ (above) to exist. This is equivalent to stating that the first few derivatives of $F(\mu)$ with respect to $\mu$ exist, but if only $F(\mu)$ as $\mu \to 0$ is bounded, then equation 4.2.58 becomes:

$$F(\mu) = \sum_{n=0}^{\infty} b_n \mu^n$$

With this result, the decay of the energy for isotropic turbulence is given by:

$$\frac{1}{2} \int E \{ u^i(x^k, t) u^i(x^k, t) \} = \frac{1}{2} R^{kk}(r=0, t)$$

$$= \left( \frac{1}{2\pi} \right)^{3/2} \int_{-\infty}^{\infty} \mu^2 F(\mu) \exp(-2\mu^2 t/R_e) \, d\tau$$

$$\xrightarrow{t \to \infty} \left( \frac{1}{2\pi} \right)^{3/2} \frac{R_e^{5/2}}{b_o} \int_{0}^{\infty} \mu^2 e^{-\mu^2} d\mu$$

$$= \frac{3}{16\pi^{3/2}} \frac{R_e^{5/2}}{b_o}$$

Thus the energy decays as $t^{-5/2}$, a well-known result of experiment.

The total correlation tensor may thus be calculated but the results are rather complicated and give no additional information. The term associated with the alternator, viz:

$$e^{ijk} \mu^k \Phi_{ij} \exp(-2\mu^2 t/R_e)$$

has an interesting asymptotic form as $t \to 0$. This term contributes to $R^{ij}$ a quantity proportional to:

*The deciding factor is the exponential singularity of $\Phi_{ij}$ as $\mu \to \infty$. In the present case the singularity is of the form $e^{-\beta \mu}$ which does no harm. For other decaying flows, e.g. the decay of axisymmetric homogeneous turbulence behind a wire gauze, the dependence may be of the form $e^{-\beta \mu}$ which is the source of trouble mentioned by Batchelor and Townsend. See references 1 and 21.
\[ e_{ijk} \left( \frac{Re}{2t} \right)^3 \]

This term decays as \( t^{-3} \), while the energy decays as \( t^{-5/2} \), which explains why good results may be obtained by ignoring the term in \( R^{ij} \) associated with the alternator. In fact in the limit as \( \frac{r^2}{t} \to 0 \), the correlation tensor takes the form

\[ R^{ij}(\epsilon^k, t) \xrightarrow{\frac{r^2}{t} \to 0} (\text{const}) \cdot \delta^{ij} \left( \frac{Re}{2t} \right)^{5/2} \exp \left( -\frac{Re}{8} \cdot \frac{r^2}{t} \right) \]

3. The Decay of an Initially Isotropic Turbulent Field

in the Presence of a Uniform D.C. Magnetic Field

Suppose an isotropic turbulent field is established in a conducting fluid at time \( t = 0 \), such that the correlation spectrum tensor \(^*\) at \( t = 0 \) is of the form:

\[ \varphi^{ij} = [\mu^2 \delta^{ij} - \lambda^{i}\lambda^{j}] + j\mu e^{ijk}k \exp(-2\mu^2T_0/R_e) \]

4.3.1

Suppose, furthermore, that there is a uniform DC magnetic field in the \( i = 2 \) direction, and that the fluid is in such a state of decay that the leading terms in the homogeneous polynomial expansion of \( u^1, p, b^1, e^1 \) and \( j^1 \) are sufficient. Under these assumptions the pertinent variables are written:

\[ u^1(x^k, t) = \int_{-\infty}^{\infty} K^1(x^k + x^k_1, t)dr(s_1, \omega) \]

4.3.2

\[ b^1(x^k, t) = \delta^1_2 \]

4.3.4

\[ p(x^k, t) = \int_{-\infty}^{\infty} W^1(x^k + x^k_1, t)dr(s_1, \omega) \]

4.3.5

\[ e^1(x^k, t) = \int_{-\infty}^{\infty} E^1(x^k + x^k_1, t)dr(s_1, \omega) \]

4.3.6

\(^*\)For definitions of the terms and justification of some of the assumptions of the first few paragraphs, see Section 4.2 and Appendix A1.
and the corresponding equations are:

\[
\frac{\partial \mathbf{K}_1^{\perp}}{\partial t} - \frac{1}{Re} \nabla^2 \mathbf{K}_1^{\perp} + \frac{\partial \mathbf{W}_1^{\perp}}{\partial x} = \frac{M^2}{Re} e^{ijk} \mathbf{J}_1^{\perp} \delta_{2k}^{\perp}
\]

\[
\frac{\partial \mathbf{J}_1^{\perp}}{\partial x} = 0
\]

4.3.7
4.3.8
4.3.9
4.3.10

where \( M \) is the Hartmann number.

The \( \mathbf{E}_1^{\perp} \) are induced electric fields and depend for their generation on induced magnetic fields. The induced magnetic fields will be small provided that the Magnetic Reynolds number, \( R_m = \sigma u L V \) is small. Assuming that induced fields are negligible (which is quite reasonable for many physical situations) the equations may be reduced to two equations involving only \( \mathbf{K}_1^{\perp} \), viz:

\[
\frac{\partial \mathbf{K}_1^{\perp}}{\partial t} - \frac{1}{Re} \nabla^2 \mathbf{K}_1^{\perp} + \frac{\partial \mathbf{W}_1^{\perp}}{\partial x} = - \frac{M^2}{Re} (\mathbf{K}_1^{\perp} - \delta_{2k}^{\perp} \delta_{k}^{\perp})
\]

4.3.11

and:

\[
\frac{\partial \mathbf{K}_1^{\perp}}{\partial x} = 0
\]

4.3.12

In any case, the analysis can be extended to include induced effects.

The solution to equations 4.3.11 and 4.3.12 is most easily effected by employing the three dimensional Fourier transform as follows:

\[
\mathbf{K}_1^{\perp}(x^k, t) = \left( \frac{1}{2\pi} \right)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{A}_1^{\perp}(\lambda^k, \mathbf{k}, t) e^{i\lambda^k x^k} d\lambda^k
\]

4.3.13
and: \[ W_1(x^k, t) = \left(\frac{1}{2\pi}\right)^3 \int_3 \int_\Omega_1(\lambda^k, t)e^{j\lambda^k x^k} d\lambda \]

The equations defining $A_1$ then become:

\[ \frac{\partial A_1}{\partial t} + \frac{\mu^2}{Re} A_1 + j\lambda^1 \Omega_1 = - \frac{M^2}{Re} (A_1 - \delta^1 A_1) \]

and:

\[ j\lambda^1 A_1 = 0 \]

where again: \[ \mu^2 = \lambda^k \lambda^k \]

Thus:

\[ \frac{\partial A^2_1}{\partial t} + \frac{\mu^2}{Re} A^2_1 + j\lambda^2 \Omega_1 = 0 \]

and:

\[ \mu^2 \Omega_1 = - \frac{M^2}{Re} j\lambda^2 A^2_1 \]

Therefore:

\[ \frac{\partial A^2_1}{\partial t} + \frac{1}{Re} (\mu^2 + M^2 (\lambda^2 (\frac{2}{\mu}))) A^2_1 = 0 \]

The general solution for $A^2_1$ and $\Omega_1$ is therefore:

\[ A^2_1 = C_1(\lambda^k) \exp \left[ - \frac{1}{Re} (\mu^2 + M^2 (\lambda^2 (\frac{2}{\mu}))) t \right] \]

\[ \Omega_1 = - \frac{1}{\mu^2} \frac{M^2}{Re} C_1(\lambda^k) \exp \left[ - \frac{1}{Re} (\mu^2 + M^2 (\lambda^2 (\frac{2}{\mu}))) t \right] \]

Thus, for $i = 1$ or $3$, $A^i_1$ satisfies:

\[ \frac{\partial A^i_1}{\partial t} + \left(\frac{\mu^2 + M^2}{Re}\right) A^i_1 + j\lambda^i [- \frac{1}{\mu^2} \frac{M^2}{Re} C_1(\lambda^k) \exp \left[ - \frac{1}{Re} (\mu^2 + M^2 (\lambda^2 (\frac{2}{\mu}))) t \right]] = 0 \]

The general solution for $A^i_1$ for $i = 1$ or $3$ is therefore:
\[ A_1^1 = N_1(\lambda^k) e^{i2j\lambda^j} \exp\left[- \frac{1}{Re} (\mu^2 + \lambda^2) t \right] \]

\[ C_1(\lambda^k) \frac{\lambda^i \lambda^j}{\beta^2} \exp\left[- \frac{1}{Re} (\mu^2 + \lambda^2 \left(\frac{\lambda^2}{\mu} \right)^2) t \right] \]

where: \[ \beta^2 = \mu^2 - (\lambda^2)^2 = (\lambda^1)^2 + (\lambda^3)^2 \]

The Theory of Invariants (see Appendix A1) predicts that for homogeneous, axisymmetric turbulence which is independent of the axial coordinate, the correlation spectrum tensor must be of the form:

\[ \phi^{ij}(\lambda^k, t) = \lambda^1 \lambda^j \phi_1(\mu, t) + \delta^{ij} \phi_2(\mu, t) \]

\[ + [\lambda^2 (\lambda^1 \delta^{ij} + \lambda^j \delta^{1i}) - \mu^2 \delta^{ij}] \phi_3(\mu, t) \]

\[ + [(\lambda^2 \lambda^1 - \delta^{ij} \lambda^2 \mu^2) e^{jil2} + (\lambda^2 \lambda^j - \delta^{ij} \mu^2) e^{ilj2}] \lambda^k \phi_4(\mu, t) \]

\[ + e^{ijk} \lambda^k \phi_5(\mu, t) \]

and conservation of mass requires that:

\[ \mu^2 \phi_1(\mu, t) + \phi_2(\mu, t) + (\lambda^2)^2 \phi_3(\mu, t) = 0 \]

A straightforward but lengthy calculation involving the above definition of \( \phi^{ij} \) and the conditions that:

\[ \phi^{ij}(\lambda^k, t) = A_1^{ij}(-\lambda^k, t) A_1^{j}(\lambda^k, t) \]

and:

\[ \phi^{ij}(\lambda^k, 0) = [\mu^2 \delta^{ij} - \lambda^1 \lambda^j + j \mu e^{ijk} \lambda^k] \exp(-2 \mu^2 T_0 / Re) \]

will uniquely define the \( \phi_1, \phi_2, \ldots, \phi_5 \) to be:

\[ \phi_1 = \left[ - \frac{\mu^2}{\beta^2} \exp(-2M^2 t / Re) + \left(\frac{\lambda^2}{\beta^2} \right)^2 \exp(-2(M^2 / \mu^2) (\lambda^2 / \mu)^2 t) \right] \exp(-2(\mu^2 / Re) (T_0 + t)) \]

\[ \phi_2 = \mu^2 \exp(-2M^2 t / Re) \exp(-2\mu^2 (T_0 + t) / Re) \]
\[
\phi_3 = \left[ + \frac{\mu}{\beta^2} \exp(-2(M^2 t/R_e)) - \frac{\mu}{\beta^2} \exp(-2(M^2/R_e)(\lambda^2/\mu) t) - \lambda^2(\mu) t \exp(-2 \mu^2 (T_o + t)/R_e) \right]
\]
\[4.3.30c\]
\[
\phi_4 = 0
\]
\[4.3.30d\]
\[
\phi_5 = j \exp(-2(M^2/R_e)(\lambda^2/\mu) t) \exp(-2 \mu^2 (T_o + t)/R_e)
\]
\[4.3.30e\]

The energy spectrum function:
\[
\phi = \frac{1}{2} \phi^{kk}
\]
\[4.3.31\]
is therefore:
\[
\phi = \frac{1}{2} \mu^2 \left[ \exp(-2 M^2 t/R_e) + \exp(-2(M^2/R_e)(\lambda^2/\mu) t) \right] \exp(-2 \mu^2 (T_o + t)/R_e)
\]
\[4.3.32\]

Thus (converting the \( \lambda^k \) to a spherical coordinate system):
\[
E \left\{ u^1(x^k, t) \cdot u^1(x^k, t) \right\}
\]
\[4.3.33\]
\[
= \left( \frac{1}{2\pi} \right)^2 \int_0^\infty \int_0^\pi \mu^4 \left\{ \exp(-2(M^2 t/R_e)) \exp(-2 \mu^2 (T_o + t)/R_e) \right\}
\]
\[
\cdot \left\{ 1 + \exp(-2(M^2 t/R_e)(\cos^2 \phi - 1) t) \right\} \sin \phi \ d\phi
\]
\[
= \frac{3}{32\pi^{3/2}} \left( \frac{R_e}{2(T_o + t)} \right)^{5/2} \exp(-2(M^2 t/R_e))
\]
\[
\cdot \int_0^\pi \left\{ \pi \exp(-2(M^2 t/R_e)) + \int_0^\pi \exp(-2(M^2 t/R_e)(\cos^2 \phi) \sin \phi \ d\phi \right\}
\]
\[4.3.34\]

Now, the integral:
\[
L(\alpha) = \int_0^\pi \exp(-\alpha(\cos^2 \phi)) \sin \phi \ d\phi
\]
\[4.3.35\]
can be evaluated in terms of the error function by substituting
\[u = -\cos \phi\] in which case:
\[
L(\alpha) = \frac{1}{4} \left( \frac{\pi}{\alpha} \right)^{1/2} \text{erf}(\alpha^{1/2})
\]
\[4.3.36\]
where:
\[
\text{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x e^{-z^2} \ dz
\]
Therefore:
\[
\int_{0}^{\pi} \exp(-2(M^2 t/R_e) \cos^2 \phi) \sin \phi \, d\phi = \left(\frac{\pi}{2 \frac{M^2}{R_e}}\right)^{1/2} \text{erf} \left(2 \frac{M^2}{R_e} t\right)^{1/2}
\]

The error function \(\text{erf}(x)\) has the properties that:
\[
\frac{\text{erf}(x)}{x} \xrightarrow{x \to 0} \frac{2}{\pi^{1/2}} \quad 4.3.38
\]
\[
\frac{\text{erf}(x)}{x} \xrightarrow{x \to \infty} 1 \quad 4.3.39
\]

Thus for \((2(M^2 t/R_e))\) near zero, the energy decay law is approximately:
\[
E \left\{ u^1 u^1 \right\} \xrightarrow{} (\text{const.}) \cdot (T_0 + t)^{-5/2} \quad 4.3.40
\]
which is the isotropic decay law. But as \((2 \frac{M^2}{R_e} t)\) tends to infinity:
\[
E \left\{ u^1 u^1 \right\} \xrightarrow{} (\text{const}) \cdot (T_0 + t)^{-5/2} \cdot t^{-1/2} \xrightarrow{} (\text{const}) t^{-3} \quad 4.3.41
\]
which is a power law (not exponential as might be expected) but a somewhat faster decay than for isotropic turbulence.

The decay law for the \(u^2\) component is given by:
\[
E \left\{ u^2 u^2 \right\} = \left(\frac{2}{\pi}\right)^2 \int_{0}^{2\pi} \int_{0}^{\infty} \frac{R_e}{2(T_0 + t)} \exp(-2 \mu^2 (T_0 + t)/R_e) \exp(-2(M^2 t/R_e) \cos^2 \phi) \sin^3 \phi \, d\mu \, d\phi
\]
\[
= \frac{3}{32\pi^{3/2}} \left\{ \frac{R_e}{2(T_0 + t)} \right\} \cdot \left(\frac{2}{2 \frac{M^2}{R_e}}\right)^{5/2} \cdot \left[ \left(\frac{t}{(2 \frac{M^2}{R_e})}\right)^{1/2} \cdot \left(1 - \frac{1}{2(2 \frac{M^2}{R_e})}\right) \right] \text{erf}\left(2 \frac{M^2}{R_e} t\right)^{1/2}
\]
\[
+ \frac{1}{2(2 \frac{M^2}{R_e} t)} \exp(-2 \frac{M^2 t}{R_e}) \quad 4.3.42
\]
This decay law behaves in much the same way as does the decay law for $E \left\{ u^1 u^1 \right\}$. However, the decay of $E \left\{ u^1 u^1 + u^3 u^3 \right\}$ given by:

$$E \left\{ u^1 u^1 + u^3 u^3 \right\} = E \left\{ u^1 u^1 - u^2 u^2 \right\} = E \left\{ u^1 u^1 \right\} - E \left\{ u^2 u^2 \right\}$$

$$= \frac{3}{32\pi^{3/2}} \left[ \frac{R_e}{2(T_0 + t)} \right]^{5/2} \left[ \frac{\pi}{2\frac{M^2}{R_e} t} \right]^{1/2} \cdot \frac{1}{2(2\frac{M^2}{R_e} t)} \text{erf} \left( \frac{2 \frac{M^2}{R_e}}{t} \right)$$

$$- \frac{1}{2(2\frac{M^2}{R_e} t)} \exp(-2(M^2 t/R_e))$$

behaves asymptotically like:

$$\text{(const.)} \; t^{-4}$$

Thus, an initially isotropic turbulent field placed in a uniform d.c. magnetic field will decay into a homogeneous axisymmetric field with a time constant of $2(M^2/R_e)$. If the initial energy distribution decays according to the law $(T_0 + t)^{-5/2}$, then the final energy distribution will decay according to the law $t^{-3}$, somewhat faster than isotropic but not exponential. Furthermore as $t \to \infty$ almost all of the energy left is contained in the velocity component in the direction of the applied magnetic field and the energy in the other two components decays as $t^{-4}$. 

4.3.44
Most technically interesting turbulent flow situations may be termed Turbulent Shear Flows; i.e. the turbulence is essentially generated and maintained by the mean shear stresses in the flow. By and large, these flows are characterized by boundary conditions (and initial conditions) which are non-random in nature.

For such flow situations, it is to be expected that some essentially nonlinear, gain-limiting interaction is present which prevents the decay or unbounded growth of the turbulent quantities. Thus, although the random quantities may be sensibly small compared with the mean quantities, perturbation (or small signal) methods will not generally be applicable. In particular, the velocities may not be expanded in a convergent series of Homogeneous Polynomial Functionals because, as indicated in the preceding chapter, such an expansion is in reality a generalized perturbation method. It is therefore to be expected that, for turbulent shear flows, the more general (but more difficult) Orthogonal Polynomial Functionals must be used.

Perhaps the simplest turbulent shear flows of technical interest are the channel flows in which the mean flow is essentially in one direction—down the channel. Of the channel flows, the simplest example is Plane Poiseuille Flow. This flow has received a great amount of attention both experimentally* and theoretically;** furthermore, this is the only technically interesting flow which has received any significant attention for the magneto-fluid dynamic case.***

* See Nikuradse, 17 and Laufer. 10
** See Townsend, 29 and Pai. 18, 19
*** See Harris. 8
The following three chapters are concerned with solution of the problem of Turbulent Plane Poiseuille Flow. The work seems to fall naturally into three parts, and thus (because of the length of the presentation) is presented in three chapters. The main body of the work is concerned with the ordinary fluid dynamics of the problem but, where it seemed appropriate, the magneto-fluid dynamic problem was treated also.

Chapter V is concerned with some of the more general features of the problem—especially the assumptions involved in the expansion of the velocities and pressure, and development of the equations of motion. The development of the equations of motion centers on the development of the nonlinear terms, and thus this development is done carefully—for two reasons: 1. the nonlinear terms are all-important to turbulent shear flows; and 2. the analysis is complicated but similar arguments will hold for other shear flows (e.g. circular Poiseuille Flow).

Chapter VI is concerned with relating the work of Chapter V to the "physics" of the problem. Much can be said about this development in view of currently known experimental and theoretical results. Symmetry conditions are considered in detail, and the magneto-fluid dynamic equations in the limit as $R_m \to 0$ are developed.

Chapter VII is concerned with approximate solutions to the problem. In this chapter a great amount of information is included which might logically be omitted. This information (whose significance we cannot measure) is included on the grounds that it may be of use to other researchers. The computed results are compared with existing experimental data and it is found that the approximations are accurate to about 6½%.
CHAPTER V

Turbulent Channel Flows

1. Application of Orthogonal Polynomial Functionals to Plane Poiseuille Flow

Plane Poiseuille Flow is taken here to mean the flow of a Newtonian, incompressible fluid with scalar conductivity between infinite parallel plates as shown in Fig. 5.1.

\[ y, x^2 \quad \text{Flow} \quad x, x^1 \quad \text{B}_o \quad y = +L_o \]

\[ z, x^3 \quad \text{y} \quad -L_o \]

Figure 5.1

In this flow situation the mean velocity and pressure drop are in the \( x \)-direction with no variation in the \( z \)-direction, and a uniform magnetic field, if applied, is in the \( y \)-direction. Furthermore, if a \( B \) field is applied, then the mean current (I) across any \((x, y)\) plane is zero (thus only the "flow damper" case is considered). All average quantities are functions of \( y \) only, and thus the length of scale is taken as \( L_o \). The velocity scale \( V_o \) is taken as the mean velocity at \( y = 0 \) (although other choices, such as flow rate velocity or the "friction velocity"* are possible). Then the Reynolds number, Magnetic Reynolds Number, Energy Density Ratio, and Hartmann Number are given respectively by:

* See Chapter VI, Section 1.
\[
R_e = \rho \frac{L_0 V_0}{\eta} \\
R_m = \mu_0 \sigma L_0 V_0 \\
R_h = \frac{B_0^2}{\mu_0 \rho v_0} \\
M = L_0 B_0 \left( \frac{\sigma}{\eta} \right)^{1/2}
\]

where \( \rho, \eta, \mu_0, \) and \( \sigma \) are, respectively, the mass density, dynamic viscosity, free space permeability, and electrical conductivity.

The normalized dynamic equations of motion are then:

\[
\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} = -\frac{1}{R_e} \nabla^2 u^i + \frac{\partial p}{\partial x^i} = F^i
\]

and the kinematic equations relating the body force \( F^i \) to \( u^i \), \( \epsilon^i \), and \( b^i \) are then:

\[
F^i = \frac{M_0^2}{R_e} e^{ijk} j^j b^k
\]

\[
\epsilon^i = e^i + e^{ijk} u^j b^k
\]

\[
e^{ijk} \frac{\partial}{\partial x^j} e^k = -\frac{\partial}{\partial t} b^i
\]

\[
e^{ijk} \frac{\partial}{\partial x^j} b^k = R_m j^i
\]

\[
\frac{\partial b^i}{\partial x^i} = 0
\]

where \( u^i \), \( F^i \), \( \epsilon^i \), \( b^i \), and \( j^i \) are, respectively, the velocity, body force, electric intensity, magnetic induction, and electrical current density vectors.
For the moment, consider the ordinary hydrodynamic case, i.e. $B_0 = 0$ and therefore all electrical effects are zero (the analysis will later be extended to the general case). Then the equations of motion become simply:

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} - \frac{1}{Re} \nabla^2 u^i + \frac{\partial p}{\partial x^i} = 0$$  \hspace{1cm} 5.1.12

$$\frac{\partial u^i}{\partial x^i} = 0$$  \hspace{1cm} 5.1.13

The velocities (and pressure gradients) are to be stationary random functions of $x$, $z$, and $t$, hence it is appropriate to expand the velocities in terms of orthogonal polynomial functionals as follows:

$$u^i(x,y,z,t) = \sum_0^\infty G_v \{ K^i_\nu(x+x_1, z+z_1, t+t_1; \ldots, x+x_\nu, z+z_\nu, t+t_\nu; y) \}, \omega$$ \hspace{1cm} 5.1.14

and similarly the pressure gradients are expanded as:

$$\frac{\partial p}{\partial x^i} = \sum_0^\infty G_v \{ \frac{\partial W}{\partial x^i}, \omega \}$$ \hspace{1cm} 5.1.15

The only assumption implied by the above two expansions is that the expanded functions are strictly stationary random functions of $x, z, and t$, i.e. the functions are stationary on $x, z, and t$, and have bounded second moments. Energy considerations make the assumption of bounded second moments quite reasonable, and the

---

*It is formally necessary to write the pressure gradient in terms of functionals rather than the pressure itself. This is because the functional expansion implies that $\partial W_0/\partial x^i$ is dependent on $y$ only whereas $W_0 = E\{p\}$ is a linear function of $x$ also. The notation:

$$p = \sum_0^\infty G_v \{ W_v, \omega \}$$

is however convenient, and will be used with the understanding that $W_0$ will contain a linear term in $x$.}
very statement of the problem implies the stationary properties mentioned. However, these expansions are to represent solutions to a differential equation, and thus it is necessary to assume that (at least) the first few time and space derivations of the functional expansions exist and may be represented by derivatives of the Kernels of the functionals, e.g.:

$$\frac{\partial u^i}{\partial t} = \sum_v G^i_v \left[ \frac{\partial K^i_v}{\partial t}, \omega \right]$$  \hspace{1cm} 5.1.16

This appears to be an entirely reasonable assumption because of the smooth (on a small scale) properties of the velocities (see Chapter II, Section 1) and the homogeneous boundary conditions (see Chapter II, Section 7).

2. Consequences of the Law of Continuity of Mass

The continuity of mass equation for an incompressible fluid requires:

$$\frac{\partial u^i}{\partial x^i} = 0$$  \hspace{1cm} 5.2.1

which, in terms of the functional expansion becomes:

$$\sum_v G^i_v \left[ \frac{\partial K^i_v}{\partial x^i}, \omega \right] = 0$$  \hspace{1cm} 5.2.2

Since the $G_v$ operators are essentially unique, it follows that every term in the sum must be zero and therefore:

$$\frac{\partial K^i_v}{\partial x^i} = 0 \quad \text{for } v = 0,1,...$$  \hspace{1cm} 5.2.3

However, it is instructive here to make use of a function $Q\mu$ (similar to the $Q\mu$ defined by 3.4.29) defined as the symmetric part of:
Thus, if:
\[ f(x, y, z, t) = \sum_0^\infty G_v[F_v(x+x_1, z+z_1, t+t_1; \ldots; x+x_v, z+z_v, t+t_v; y); \omega] \]

then:
\[ E \left\{ G_\mu [Q_\mu; \omega] \cdot f \right\} = E \left\{ G_\mu [Q_\mu; \omega] \cdot \sum_0^\infty G_v[F_v, \omega] \right\} \]

\[ = E \left\{ G_\mu [Q_\mu; \omega] \cdot G_\mu [F_\mu; \omega] \right\} \]

\[ = E \left\{ G_\mu [Q_\mu; \omega] \cdot G_\mu [F_\mu; \omega] \right\} \]

\[ = E \left\{ G_\mu [Q_\mu; \omega] \cdot G_\mu [F_\mu; \omega] \right\} \]

\[ = \mu! \int_\infty^{\infty} 3\mu \int Q_\mu(\xi_1, \eta_1, \tau_1; \ldots; \xi_\mu, \eta_\mu, \tau_\mu) \]

\[ = F_\mu(x+x_1, z+z_1, t+t_1; \ldots; x+x_\mu, z+z_\mu, t+t_\mu; y) \]

\[ = \frac{\mu}{\mu!} \int_{j=1}^{\infty} d\xi_j d\eta_j d\tau_j \]

Hence: \[ F_\mu(x+x_1, z+z_1, t+t_1; \ldots; x+x_\mu, z+z_\mu, t+t_\mu; y) \]

\[ = \lim_{h \to 0} \frac{1}{\mu} \int E \left\{ G_\mu [Q_\mu; \omega] \cdot f \right\} \]

a result comparable to equation 3.4.31.

With this result, equation 5.2.2 implies that:
\[ \frac{\partial k_v}{\partial x} = 0 \; ; \; v = 0, 1, \ldots \]
because:

\[ 0 = \mathbf{E} \left\{ G_\mu [Q_\mu, \omega] \cdot \frac{\partial u_i}{\partial x^i} \right\} \]
\[ = \mathbf{E} \left\{ G_\mu [Q_\mu, \omega] \cdot \sum_\nu G_\nu \left[ \frac{\partial K_i^1}{\partial x^i}, \omega \right] \right\} \]
\[ = \mathbf{E} \left\{ G_\mu [Q_\mu, \omega] \cdot G_\mu \left[ \frac{\partial K_i^1}{\partial x^i}, \omega \right] \right\} \]
\[ = \mu' \frac{\partial K_i^1}{\partial x^i} = 0 \]

hence:

\[ \frac{\partial K_i^1}{\partial x^i} = 0 \]

as above.

3. Consequences of the Momentum Balance Laws

The momentum balance equations for an incompressible, Newtonian fluid require:

\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x^j} - \frac{1}{\mathbf{R}} \nu^2 u_i + \frac{\partial p}{\partial x^i} = 0 \]

Application of the expansions of the velocities and pressure gradients to equation 5.3.1 is complicated by the nonlinear term \( u_j \frac{\partial u_i}{\partial x^j} \). It is to be expected that the non-linear term will greatly complicate the resulting equations, but the matter is somewhat simplified by the following definitions:

Let:

\[ \gamma_{ij} = \sum_{\nu} \sum_{\nu} G_\nu [K_\nu^i, \omega] = u_i u_j \]
\[ = \left\{ \sum_{\nu} G_\nu [K_\nu^i, \omega] \right\} \cdot \left\{ \sum_{\mu} G_\mu [K_\mu^j, \omega] \right\} \]

5.3.2
\[ i = \sum_{v}^{\infty} G_{v} [L_{v}', \omega] = \frac{\partial}{\partial x^{i}} \gamma^{ij} = \sum_{v}^{\infty} G_{v} \left[ \frac{\partial \gamma^{ij}}{\partial x^{j}} \right], \omega \]  

Then the equations of motion become:

\[ \frac{\partial u^{i}}{\partial t} + L_{v}^{i} - \frac{1}{Re} \nabla^{2} u^{i} + \frac{\partial p}{\partial x^{i}} = 0 \]  

from which it follows immediately that:

\[ K_{v}^{i} + L_{v}^{i} - \frac{1}{Re} \nabla^{2} K_{v}^{i} + \frac{\partial W_{v}}{\partial x^{i}} = 0 ; \quad v = 1, 2, 3, \ldots \]  

The problem now is to determine the \[ \gamma^{ij}_{v} \] in terms of the \[ K_{v}^{i} \] from the equation:

\[ \gamma^{ij}_{v} = \sum_{v}^{\infty} G_{v} [\gamma^{ij}_{v}, \omega] = u^{i} u^{j} \left\{ \sum_{v}^{\infty} G_{v} [K^{i}_{v}, \omega] \right\} \left\{ \sum_{v}^{\infty} G_{v} [K^{j}_{v}, \omega] \right\} \]  

From equation 3.4.48, the \[ G_{v} \] functionals are defined by:

\[ G_{v} [K^{i}_{v}, \omega] = \sum_{n=v}^{\infty} \frac{(-1)^{n} v^{!}}{n^{v} (v-2n)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ds_{1} s_{2} \ldots s_{v-2n} \]  

\[ \cdot K^{i}_{v}(s_{1}, s_{2}, \ldots, s_{n}, \sigma_{n}; \lambda_{1}, \lambda_{2}, \ldots, \lambda_{v-2n}) \]  

where, for simplicity, the \( \sigma_{k} \) and \( s_{k} \) variables in \( K^{i}_{v} \) represent the triplets \( (x_{k}, x_{k}', z_{k}, z_{k}') \) and the differential \( ds_{k} \) represents \( (dx_{k}, dz_{k}, dt_{k}) \). In addition, it was shown in Chapter III, Section 4 that:

\[ E \left\{ G_{v} [A_{v}, \omega] \cdot G_{\mu} [B_{\mu}, \omega] \right\} = 0 \quad \mu \neq v \]  

and:

\[ E \left\{ G_{\mu} [A_{\mu}, \omega] \cdot G_{\mu} [B_{\mu}, \omega] \right\} \]  

\[ = \mu! \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{\mu}(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{\mu}) \cdot B_{\mu}(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{\mu}) \frac{1}{\mu!} ds_{1} \]

\[ \begin{cases} \frac{n}{2} & \text{n even} \\ \frac{n-1}{2} & \text{n odd} \end{cases} \]
Thus:
\[
E \{ \gamma^{ij} \} = \Gamma^{ij}_o = E \{ u^{i} u^{j} \}
\]
\[
= \sum_{0}^{\infty} \int_{0}^{\infty} \int K^{i}_{\nu}(\sigma_1, \sigma_2, \ldots, \sigma_{\nu}) K^{j}_{\nu}(\sigma_1, \sigma_2, \ldots, \sigma_{\nu}) \frac{v^\nu}{j^1} d\sigma_j
\]
This important equation defines the \( u^i u^j \) correlation tensor and therefore \( u^j \frac{\partial}{\partial x^j} u^i \) and is of great importance in determining \( \Gamma^{ij}_o \), the mean velocity.

The higher order \( \Gamma^{ij}_\nu \) are considerably more difficult to determine. To simplify matters, consider the problem of determining the \( C_\lambda \) in the equation:

\[
\Sigma_\lambda G_{\lambda}[C_{\lambda}, \omega] = G_{\nu}[A_{\nu}, \omega] \cdot G_{\mu}[B_{\mu}, \omega]
\]
where, not only the \( C_{\lambda} \), but also the range of the summation is to be determined. From the form of the functional operator defined by equation 5.3.7, it is clear that the product of two \( G \) functions (of orders \( \mu \) and \( \nu \)) is expressible as a sum of functionals of order no greater than \( (\mu + \nu) \). In fact, if \( (\mu + \nu) \) is even (odd) the product of \( G_{\mu} \) and \( G_{\nu} \) is expressible in terms of even (odd) order functionals of order no greater than \( (\mu + \nu) \). Thus, equation 5.3.11 may be written:

\[
\left[ \frac{\mu + \nu}{2} \right] \sigma^{\lambda} G_{\mu+\nu-2\lambda}[C_{\mu+\nu-2\lambda}, \omega] = G_{\nu}[A_{\nu}, \omega] \cdot G_{\mu}[B_{\mu}, \omega]
\]

Now suppose equation 5.3.12 is multiplied by \( G_{\sigma}[F_{\sigma}, \omega] \) and the expectation taken as shown in equation 5.3.13:

\[
E \left\{ G_{\sigma}[F_{\sigma}, \omega] \cdot \sum_{0}^{\infty} \sigma^{\lambda} G_{\mu+\nu-2\lambda}[C_{\mu+\nu-2\lambda}, \omega] \right\} = E \left\{ G_{\sigma}[F_{\sigma}, \omega] \cdot G_{\nu}[A_{\nu}, \omega] \cdot G_{\mu}[B_{\mu}, \omega] \right\}
\]
Because of the orthogonality of the \( G \) functionals the left hand side is zero unless:

\[
\sigma = \mu + \nu - 2\lambda \quad \text{for some} \; \lambda = 1, 2, \ldots, \left[ \frac{\mu + \nu}{2} \right]
\]
Furthermore, since the product of any two functionals is a sum of functionals or order no greater than the sum of the orders of the original two functionals, the right hand side is zero unless:

\[ \sigma \leq \mu + \nu \]  
\[ \nu \leq \sigma + \mu \]  
\[ \mu \leq \sigma + \nu \]

and simultaneously. Equation 5.3.15a is automatically satisfied whenever equation 5.3.14 is satisfied, but equation 5.3.15b and equation 5.3.15c imply that for a non-zero expectation, \( \sigma \) must be in the range:

\[ |\mu - \nu| \leq \sigma \leq \mu + \nu \]

Thus, equations 5.3.12 and 5.3.13 may be rewritten with the sum reduced as follows:

\[ \frac{\lambda_m}{\lambda} \sum_{\sigma} G_{\mu + \nu - 2\lambda}(C_{\mu + \nu - 2\lambda}, \omega) = G_{\nu}(A_{\nu}, \omega) \cdot G_{\mu}(B_{\mu}, \omega) \]

and:

\[ E\left\{ G_{\sigma}(F_{\sigma}, \omega) \cdot \sum_{\sigma} G_{\mu + \nu - 2\lambda}(C_{\mu + \nu - 2\lambda}, \omega) \right\} = E\left\{ G_{\sigma}(F_{\sigma}, \omega) \cdot G_{\nu}(A_{\nu}, \omega) \cdot G_{\mu}(B_{\mu}, \omega) \right\} \]

where:

\[ \lambda_m = \frac{\mu + \nu - |\mu - \nu|}{2} \]

Now if the \( F_{\sigma} \) in equation 5.3.16 is replaced by the \( Q_{\sigma} \) defined in equation 5.2.4, then equation 5.3.16 may be rewritten as:

\[ \lambda! \cdot C_{\lambda} = \lim_{h \to 0} E\left\{ G_{\lambda}(Q_{\lambda}, \omega) \cdot G_{\nu}(A_{\nu}, \omega) \cdot G_{\mu}(B_{\mu}, \omega) \right\} \]

where \( \sigma \) has been replaced by \( \lambda \). By employing the formula for the \( G \) functionals (equation 5.3.7), equation 5.3.17 may be written:
\[ \lambda! C_\lambda = \lim_{h \to 0} \left\{ \begin{array}{l} \int \lambda \int Q_\lambda \prod_{j=1}^{\lambda} \frac{\lambda}{\lambda_1} \text{dr}(\sigma_j, \omega) + \text{lower order terms} \\ \cdot \left[ \int \mathcal{V} \int A_\nu \prod_{j=1}^{\nu} \text{dr}(s_j, \omega) + \text{lower order terms} \right] \\ \cdot \left[ \int \mu \int B_\mu \prod_{j=1}^{\mu} \text{dr}(t_j, \omega) + \text{lower order terms} \right] \end{array} \right\} \]

The rule for evaluating an expression such as the right hand side of equation 5.3.18 is to take the \( \sigma, s, \) and \( t \) variables, associate by pairs in all possible ways, integrate over the pairs, and add up the results (see Chapter III, Section 3). However, if two of the \( \sigma \)'s were associated together, then the result would be equivalent to a lower order term in the expansion of \( G_\lambda[Q_\lambda] \) and the result would be zero. For example, if exactly one pair of \( \sigma \)'s were associated in the leading term of \( G_\lambda[Q_\lambda] \) the result would be:

\[ \frac{\lambda!}{2! (\lambda-2)!} \int_{\lambda-2}^{\lambda-2} \prod_{j=1}^{\lambda-2} \text{dr}(\sigma_j, \omega) \int dp \quad Q_\lambda(\sigma_1, \ldots, \sigma_{\lambda-2}, p, p) \]

where the coefficient \( \frac{\lambda!}{2! (\lambda-2)!} \) comes from the number of ways of selecting two variables from \( \lambda \) variables. However, the second term in the expansion of \( G_\lambda[Q_\lambda] \) is:

\[ \frac{-\lambda!}{(\lambda-2)!} \int_{\lambda-2}^{\lambda-2} \prod_{j=1}^{\lambda-2} \text{dr}(\sigma_j, \omega) \int dp \quad Q_\lambda(\sigma_1, \ldots, \sigma_{\lambda-2}, p, p) \]

which exactly cancels the results of equation 5.3.19. In fact, it is generally true, that the result of any pairing of \( \sigma \)'s will be cancelled by lower order terms, and therefore all lower order terms are dropped.

Thus in equation 5.3.18, the modified rule for evaluating the right hand side is: take all the \( \sigma, s, \) and \( t \) variables, associate
in pairs in all ways possible keeping no two $\sigma$'s (or $s$'s or $t$'s) paired, integrate over the pairs, and add up the results. To perform such an evaluation, a certain number (say $k$) of the $\sigma$'s will be associated with $k$ of the $s$'s, and $\lambda-k$ of the $\sigma$'s will be associated with $\lambda-k$ of the $t$'s, then the remaining $\nu-k$ of the $s$'s will be associated with $\mu-\lambda+k$ of the $t$'s. Schematically, this association is:

$$Q_{\lambda}(\sigma_1 \ldots \sigma_k; s_{k+1} \ldots s_{\lambda}) \cdot A_{\nu}(s_1 \ldots s_k; t_{k+1} \ldots t_{\lambda}) \cdot B_{\mu}(t_1 \ldots t_{\lambda-k}; t_{\lambda-k+1} \ldots t_{\mu})$$

For the pairing to be done properly, $k$ must satisfy:

$$\nu-k = \mu-\lambda+k$$

i.e.:

$$k = \frac{\lambda-\mu+\nu}{2}$$

Thus equation 5.3.18 becomes:

$$\lambda! C^\lambda_{\lambda} = \lim_{h \to 0} \sum_{k=1}^{\lambda} \int_{\lambda-k}^{\lambda-k} \int_{\nu-k}^{\nu-k} dt_m$$

where the sum extends over all possible ways of forming the pairs and integrating. However, since the $Q_{\lambda}$, $A_{\nu}$, and $B_{\mu}$ are symmetrical in their respective arguments, all terms in the sum are identical, and the result is some coefficient times the integral in equation 5.3.23. The coefficient is obtained as follows: there are $\binom{\lambda}{k}$ ways of selecting $k$ variable from $Q_{\lambda}$ and $\binom{\nu}{\nu-k}$ ways of associating these $k$ variables with $k$ variables in $A_{\nu}$; there are then $\binom{\mu}{\mu-\lambda+k}$ ways of associating the remaining $\lambda-k$ variables in $Q_{\lambda}$ with $\lambda-k$ variables in $B_{\mu}$; finally there are $(\nu-k)!$ ways of associating the remaining $\nu-k$ variables in $A_{\nu}$ with the $\nu-k$ variables remaining in $B_{\mu}$. Thus
the coefficient replacing the sum in equation 5.3.23 is:
\[
\frac{\lambda! \cdot \nu! \cdot \mu!}{k!(\lambda-k)!(\nu-k)!(\mu-k+\lambda+k)!} \cdot (v-k)!
\]
which can be written:
\[
\frac{\lambda! \cdot \mu! \cdot \nu!}{(\frac{\mu+\lambda-\nu}{2})! \cdot (\frac{\mu+\nu-\lambda}{2})! \cdot (\frac{\lambda+\nu-\mu}{2})!}
\]
With this result, equation 5.3.23 becomes:
\[
C_{\lambda}(\sigma_1, \ldots, \sigma_{\lambda}) = \frac{\mu! \cdot \nu!}{(\frac{\mu+\lambda-\nu}{2})! \cdot (\frac{\mu+\nu-\lambda}{2})! \cdot (\frac{\lambda+\nu-\mu}{2})!} \int \frac{\mu+\nu-\lambda}{2} \prod_{j=1}^{\mu+\nu-\lambda} ds_j
\]
\[
\left\langle A_\nu(\sigma_1, \ldots, \sigma_{\lambda-\mu+\nu} \frac{\lambda-\mu+\nu}{2}, s_1 \ldots s_{\frac{\mu+\nu-\lambda}{2}}) \right\rangle
\]
\[
\left\langle B_\mu(\sigma_{\frac{\lambda-\mu+\nu}{2}+1} \ldots \sigma_{\lambda} \frac{\mu+\nu-\lambda}{2}, s_1 \ldots s_{\frac{\mu+\nu-\lambda}{2}}) \right\rangle
\]
where as before \(\langle \ldots \rangle_\sigma\) means "the symmetrical part on the \(\sigma\) variable." Therefore:
\[
C_{\mu+\nu-2\lambda}(\sigma_1 \ldots \sigma_{\mu+\nu-2\lambda}) = \frac{\mu! \cdot \nu!}{\lambda!(\mu-\lambda)! \cdot (\nu-\lambda)!} \cdot \int \frac{\lambda}{\mu+\nu-\lambda} \prod_{j=1}^{\lambda} ds_j
\]
\[
\left\langle A_\nu(\sigma_1 \ldots \sigma_{\lambda-\nu-\lambda} \frac{\lambda-\mu+\nu}{2}, s_1 \ldots s_{\frac{\mu+\nu-\lambda}{2}}) \right\rangle \cdot B_\mu(\sigma_{\lambda-\nu-\lambda+1} \ldots \sigma_{\mu+\nu-2\lambda+1} \frac{\mu+\nu-\lambda}{2}, s_1 \ldots s_{\frac{\mu+\nu-\lambda}{2}}) \right\rangle_\sigma
\]
Combining all the above results, the expansion for \(\gamma^{ij}\) (see equation 5.3.6) may be written:
\[
\gamma^{ij} = \sum_{\lambda=0}^{\infty} \Sigma_{\nu}^{i} G_{\nu}^{i}[K_{\mu}^{i}, \omega] = \left\{ \sum_{\lambda=0}^{\infty} G_{\nu}^{i}[K_{\mu}^{i}, \omega] \right\} \left\{ \sum_{\lambda=0}^{\infty} G_{\nu}^{j}[K_{\mu}^{j}, \omega] \right\}
\]
\[
= \Sigma_{\mu, \nu}^{\infty} \Sigma_{\lambda}^{\mu+\nu-2\lambda}[C_{\mu+\nu-2\lambda}^{ij}, \omega]
\]
where:
\[
C_{\mu+v-2\lambda}^{i,j}(\sigma_1, \ldots, \sigma_{\mu+v-2\lambda}) = \frac{\mu!v!}{\lambda!(\mu-\lambda)!(v-\lambda)!} \int \lambda \int \prod_{j=1}^{\lambda} ds_j
\]
\[
\left< K^i_\mu(\sigma_1, \ldots, \sigma_{v-\lambda}; s_1 \ldots s_\lambda), K^j_{\nu}(\sigma_{v-\lambda+1}, \ldots, \sigma_{\mu+v-2\lambda}; s_1 \ldots s_\lambda) \right>_\sigma
\]  

Equation 5.3.26 may be resumed and inverted to yield:
\[
\Gamma^{i,j}_\lambda(\sigma_1 \ldots \sigma_\lambda) = \sum_{\lambda}^{\infty} \sum_{\ell}^{\infty} \frac{(\lambda+k-\ell)!}{(\ell-\ell)!} \frac{(k+\ell)!}{(\ell)!} \int k \int \prod_{j=1}^{k} ds_j
\]
\[
\left< K^{i}_{\lambda+k-\ell}(\sigma_1, \ldots, \sigma_{\ell-\ell}; s_1 \ldots s_k), K^{j}_{\ell+k}(\sigma_{\ell-\ell+1}, \ldots, \sigma_{\lambda}; s_1 \ldots s_k) \right>_\sigma
\]

where: for \( \lambda = 0 \), the formula is to be interpreted as:

\[
\Gamma^{i,j}_0 = \sum_{\lambda}^{\infty} \sum_{\ell}^{\infty} \frac{k!}{\ell!} \int \prod_{j=1}^{k} ds_j
\]
\[
\left< K^{i}_{k}(s_1 \ldots s_k), K^{j}_{k}(s_1 \ldots s_k) \right>
\]

and: for \( \lambda = 1 \), the formula is to be interpreted as:

\[
\Gamma^{i,j}_1(\sigma_1) = \sum_{\lambda}^{\infty} \sum_{\ell}^{\infty} \frac{k!}{\ell!} \int \prod_{j=1}^{k} ds_j \left\{ (k+1)! K^{i}_{k+1}(\sigma_1; s_1 \ldots s_k) K^{j}_{k}(s_1 \ldots s_k) \right\}
\]

\[
+(k+1)! K^{i}_{k}(s_1 \ldots s_k) K^{j}_{k+1}(\sigma_1, s_1 \ldots s_k)
\]

The above formulae are really quite complicated as was anticipated. However, a careful formulation was desirable because the \( \Gamma^{i,j}_\nu \) contain a great deal of information about turbulent flows.

4. The Magneto-Fluid Dynamic Equations Expanded in Terms of Orthogonal Polynomial Functionals

The treatment of the magneto-fluid dynamic case for Plane Poiseuille Flow follows closely the analysis of the last two sections. The equations of motion (5.1.5 through 5.1.11) contain the variables \( u^i, p, F^i, j^i, b^i, \) and \( e^i \). These variables possess both a mean and a random part and are therefore expanded as:
\[
\begin{align*}
\begin{pmatrix}
  u^i \\
p \\
F^i \\
j^i \\
b^i \\
e^i
\end{pmatrix}
&= \sum_{o \nu} G_{\nu} \left\{ \begin{pmatrix}
  K^i_{\nu} \\
W^i_{\nu} \\
F^i_{\nu} \\
j^i_{\nu} \\
B^i_{\nu} \\
E^i_{\nu}
\end{pmatrix}, \omega \right\} \\
\text{which are comparable with equation 5.1.14 and 5.1.15.}
\end{align*}
\]

As in the preceding section, matters are simplified if the nonlinear terms \(j^i b^j\) and \(u^i b^j\) are written:

\[
\begin{align*}
j^i b^j \triangleq \lambda^{ij} &= \sum_{o \nu} G_{\nu} [\Lambda^{ij}_{\nu}, \omega] \\
u^i b^j &= \gamma^{ij} = \sum_{o \nu} G_{\nu} [Z^{ij}_{\nu}, \omega]
\end{align*}
\]

Then, by the results of the last section, \(\Lambda^{ij}_{\nu}\), and \(Z^{ij}_{\nu}\) may be written:

\[
\begin{align*}
\Lambda^{ij}_{\nu} (\sigma_1 \ldots \sigma_v) &= \sum_{o k} \sum_{o \ell} (v+k-\ell)!(\ell+k)! \cdot \int k \int \frac{1}{j} ds_j \\
&\quad \cdot \left\langle j^{i}_{v+k-\ell} (\sigma_1 \ldots \sigma_{v-\ell} ; s_1 \ldots s_k) \cdot B^{j}_{k+\ell} (\sigma_{v-\ell+1} \ldots \sigma_v ; s_1 \ldots s_k) \right\rangle_{\sigma} \\
\text{and:}
\end{align*}
\]

\[
\begin{align*}
Z^{ij}_{\nu} (\sigma_1 \ldots \sigma_v) &= \sum_{o k} \sum_{o \ell} (v+k-\ell)!(\ell+k)! \cdot \int k \int \frac{1}{j} ds_j \\
&\quad \cdot \left\langle k^{i}_{v+k-\ell} (\sigma_1 \ldots \sigma_{v-\ell} ; s_1 \ldots s_k) \cdot B^{j}_{k+\ell} (\sigma_{v-\ell+1} \ldots \sigma_v ; s_1 \ldots s_k) \right\rangle_{\sigma}
\end{align*}
\]

where the same conventions for special cases of the sum (e.g., \(v=0, \ell = 0\), etc.) apply as in equation 5.3.28.

With these results, it follows immediately from the equations of motion that:
\[
\begin{align*}
\frac{\partial K^i_v}{\partial t} + \frac{\partial}{\partial x^j} \Gamma^i_{v}^{\ j} - \frac{1}{R_e} \nabla^2 K^i_v + \frac{\partial W_v}{\partial x^i} &= F^i_v \\
F^i_v &= \frac{M^2}{R_e} e^{\sum_{j<k} \epsilon_{ijk}} \\
J^i_v &= E^i_v + e^{\sum_{j<k} \epsilon_{ijk}} Z^j_v \\
e^{ijk} \frac{\partial E^k_v}{\partial x^j} &= - \frac{\partial B^i_v}{\partial t} \\
e^{ijk} \frac{\partial B^k_v}{\partial x^j} &= R_m J^i_v
\end{align*}
\]
and:
\[
\frac{\partial K^k_v}{\partial x^k} = 0 ; \quad \frac{\partial B^k_v}{\partial x^k} = 0
\]

If, in addition:
\[
b^{i} b^{j} \Delta b^{ij} = \sum_{o}^{\infty} G_v[B^{ij}_v, o]
\]
then:
\[
B^i_v(\sigma_1 \ldots \sigma_v) = \sum_{o}^{\infty} \sum_{k}^{\infty} \frac{(v+k)!(k)!(v-k)!}{(v+k-k)!(k)!} \int \int \int \int \int ds_j
\]

With this definition, the equations of motion may be combined to give (compare with equations 2.6.1 and 2.6.2):
\[
\begin{align*}
\frac{\partial K^i_v}{\partial t} + \frac{\partial}{\partial x^j} \Gamma^i_v - \frac{1}{R_e} \nabla^2 K^i_v + \frac{\partial W_v}{\partial x^i} &= \frac{M^2}{R_e R_m} \left\{ \frac{\partial}{\partial x^j} B^i_v - \frac{1}{2} \frac{\partial}{\partial x^k} B^{k} \right\} \\
\frac{\partial B^i_v}{\partial t} + \frac{\partial}{\partial x^j} (Z^j_v - Z^i_v) - \frac{1}{R_m} \nabla^2 B^i_v &= 0
\end{align*}
\]
and:
\[
\frac{\partial K^k_v}{\partial x^k} = 0 ; \quad \frac{\partial B^k_v}{\partial x^k} = 0
\]
These equations comprise a complete and consistent set of eight equations in the seven variable $k_{v}^{i}$, $b_{v}^{i}$, and $w_{v}$ (the equation $\frac{\partial b_{v}^{i}}{\partial x} = 0$ is actually redundant), and, subject to the boundary conditions, completely specify the problem.
CHAPTER VI

Analysis of Plane Poiseuille Flow

1. The Averaged Equation

In Chapter V, the normalized velocities and pressure existing in Plane Poiseuille Flow were expanded in terms of Orthogonal Polynomial Functionals as follows (see equations 5.1.14 and 5.1.15):

\[ u_i(x, y, z, t) = \sum_{o} G_v^o [K_v^i(x+x_1, z+z_1, t+t_1; \ldots ; x+x_v, z+z_v, t+t_v; y); \omega] \]

and:

\[ p(x, y, z, t) = \sum_{o} G_v^o [W_v(x+x_1, z+z_1, t+t_1; \ldots ; x+x_v, z+z_v, t+t_v; y); \omega] \]

and similarly the nonlinear term \( u^i u^j \) was expanded as (see equation 5.3.2):

\[ u^i u^j = \gamma^i j = \sum_{o} G_v^o [\Gamma_v^i j, \omega] \]

In terms of these definitions, the equations of motion in the absence of electrical effects are (see equations 5.2.3 and 5.3.5):

\[ \frac{\partial K_v^i}{\partial t} + \frac{\partial}{\partial x_j} (\Gamma_v^i j - \frac{1}{Re} \nu K_v^i j) + \frac{\partial W_v^i}{\partial x_j} = 0 \]

and:

\[ \frac{\partial K_v^i}{\partial x_i} = 0 \]

The equations for \( \nu = 0 \) are simply the Navier-Stokes equations averaged over the ensemble. In fact:

\[ \bar{u}^i = R_o^i \]

\[ \bar{u}^i \bar{u}^j = \gamma^i j = R_o^i j \]

and:

\[ \bar{p} = W_o \]

where: \( \bar{f} = \mathbb{E} \{ f \} \)
Thus, since the mean velocity is totally $x$-directed, and, since all averages of velocities and pressure gradients must be functions of $y$ only:

\[ K_o^1 = \text{function of } y \text{ only} \]  
\[ K_o^2 = 0 \]  
\[ K_o^3 = 0 \]

6.1.8a  
6.1.8b  
6.1.8c

and the continuity equation is satisfied identically by $K_o^1$. The momentum balance equations for $v=0$ then become:

\[ \frac{d}{dy} \Gamma_o^{12} - \frac{1}{Re} \frac{d^2}{dy^2} K_o^1 + \frac{\partial W_o}{\partial x} = 0 \]  
6.1.9

\[ \frac{d}{dy} \Gamma_o^{22} + \frac{\partial W_o}{\partial y} = 0 \]  
6.1.10

and:

\[ \frac{d}{dy} \Gamma_o^{32} = 0 \]  
6.1.11

In more familiar terms:

\[ \frac{d}{dy} \frac{u_x u_y}{y} - \frac{1}{Re} \frac{d^2}{dy^2} \frac{u_x}{u_x} + \frac{\partial \nu}{\partial y} = 0 \]  
6.1.12

\[ \frac{d}{dy} \frac{u_y u_y}{y} + \frac{\partial \nu}{\partial y} = 0 \]  
6.1.13

and:

\[ \frac{d}{dy} \frac{u_y u_z}{y} = 0 \]  
6.1.14

Differentiation of equation 6.1.10 with respect to $x$ yields:

\[ \frac{\partial}{\partial x} \frac{\partial W_o}{\partial y} = \frac{\partial}{\partial y} \frac{\partial W_o}{\partial x} = 0 \]  
6.1.15

from which it follows that:

\[ \frac{\partial W_o}{\partial x} = \text{constant} = \Delta - f \]  
6.1.16

*As before, a bar (e.g. $\bar{u}_x$) stands for an average quantity and a prime (e.g. $u'_x$) stands for the random part with zero mean; thus, $u_x = \bar{u}_x + u'_x$ and $\bar{u}' = 0$. 
The constant $f$ is known as the "friction factor" because it is related to the wall shear stress $\tau$ by:

$$ f = \frac{\tau}{\rho v_0^2} $$

The quantity $u^* = \left(\frac{\tau}{\rho}\right)^{1/2}$ is known as the "friction velocity" and therefore:

$$ \frac{u^*}{v_0} = f^{1/2} $$

Equation 6.1.14 may be integrated to give:

$$ u \frac{u}{v} = \text{constant} $$

but since the velocities are zero at $y = \pm 1$, it follows that:

$$ u \frac{u}{v} = 0 $$

Furthermore, since by symmetry $u_x$ is an even function of $y$, and $\frac{\partial p}{\partial x}$ is a constant ($-f$), it follows from equation 6.1.12 that $u_x u_y$ is an odd function of $y$ -- plus (perhaps) a constant, but boundary conditions require the constant to be zero. Thus, equation 6.1.12, which may now be written:

$$ \frac{d}{dy} u_x u_y - \frac{1}{R_e} \frac{d^2}{dy^2} u_x - f = 0 $$

may be integrated to give:

$$ u_x u_y - \frac{1}{R_e} \frac{d}{dy} u_x - fy = 0 $$

(where the integration constant is zero as a result of the boundary conditions). This equation may be rewritten in the form:

$$ \frac{d}{dy} u_x = R_e (fy - u_x u_y) $$

At this point a look at experimental results is worthwhile. Laufer\textsuperscript{10,11} has made some measurements of Plane Poiseuille Flow for a variety of Reynolds Numbers. Typical of his results are those for $R_e = 12300.0$, a moderately large Reynolds number. He
found that for this Reynolds Number:

$$R_e = 22.12$$

and Pai\textsuperscript{19} found that a formula which fits the measured data with great precision is:

$$\bar{u}_x = 1 - (0.3293)y^2 - (0.6707)y^3$$

This approximate velocity is shown in Fig. 6.1.

Over the center part of the channel (say, $0 < y < 0.85$) the derivative of the mean velocity is approximately:

$$\frac{d}{dy} \bar{u}_x \approx -(0.6586)y$$

and since $R_e$ is so large (22.12) compared to this, it follows that over the same part of the channel:

$$\frac{\overline{u_x u_y}}{\bar{u}_x} = f_y$$

In addition, it should be true that

$$\frac{\overline{u_x u_y}}{\bar{u}_x} \leq f_y$$

for otherwise, $\frac{d\bar{u}_x}{dy}$ becomes positive, an unreasonable physical phenomenon. In fact, it appears that:

$$\frac{\overline{u_x u_y}}{\bar{u}_x} = \frac{R_e}{0.6586}$$

over the greater part of the channel. Near the walls, however, $\overline{u_x u_y}$ must go to zero. In fact, since:

$$\bar{u}_x = \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = \pm 1$$

the $\overline{u_x u_y}$ correlation should have (at least) a triple order zero at $y = \pm 1$--provided the functional representation of $u^i$ is "sufficiently" convergent and summable. (Note that Pai's formula, equation 6.1.25, does not satisfy this requirement, and in fact, his formula is somewhat inaccurate near the walls).
Figure 6.1

Pai's Empirical Curve for Laufer's Data, \( R_e = 12300.0 \)
Thus, the $u_x u_y$ correlation should be a straight line over the central part of the channel with a slope of somewhat less than $f$. Near the walls, the function should go to zero very rapidly, approaching zero at $y = \pm 1$ to third order. This concept of the $u_x u_y$ correlation corresponds exactly to the experimental evidence (see Figs. 8 and 26 of Laufer\textsuperscript{11}).

Now suppose that the $u_x u_y$ correlation is written:

$$u_x u_y = \frac{\beta}{R_e} (y-g(y))$$ \hspace{1cm} 6.1.31

where $g(y)$ is an odd function of $y$ which is very small compared to $y$ for $(0 < y < 0.85)$, but which rapidly rises in the interval $(0.85 < y < 1)$ until at $y = 1$,

$$g = \frac{dg}{dy} = 1, \frac{d^2g}{dy^2} = 0 \text{ at } y = \pm 1 \hspace{1cm} 6.1.32$$

then the mean velocity may be written:

$$\bar{u}_x = \frac{R e f - \beta}{2} (1-y^2) + \beta \int g(\xi) d\xi$$ \hspace{1cm} 6.1.33

but: $\bar{u}_x(0) = 1$, hence:

$$1 = \frac{R e f - \beta}{2} + \beta \Delta$$ \hspace{1cm} 6.1.34

where:

$$\Delta = \int_0^1 g(\xi) d\xi$$ \hspace{1cm} 6.1.35

Thus:

$$f = \frac{\beta + 2(1-\beta \Delta)}{R e}$$ \hspace{1cm} 6.1.36

and:

$$\frac{R e f - \beta}{2} = 1 - \beta \Delta$$ \hspace{1cm} 6.1.37

and finally:

$$K_0^1 = 1 - (1-\beta \Delta) y^2 - \beta \int_0^y g(\xi) d\xi$$ \hspace{1cm} 6.1.38

There is an interesting relationship between this last form of the mean velocity and von Kármán's empirical "velocity defect
law." This law states*: "The difference between the physical 
velocity and its value at center channel divided by \( u^* \) (the 
friction velocity) is a universal function of \( y/L \) independent 
of \( R_e \)." In terms of the normalized equations used above, this 
law states:

\[
\frac{V_0}{u^*} (K_o^1(y) - 1) = \frac{K_o^1}{f^{1/2}} = \phi(y) \text{ over } (0 < y < 0.85) \tag{6.1.39}
\]

where \( \phi \) is some universal function of \( y \). Literally, this law 
would require that:

\[
\frac{1 - \delta \Delta}{f^{1/2}} = \text{constant} \tag{6.1.40}
\]

More likely, the "constant" is some very slowly varying function 
of \( R_e \).

Finally, since \( \bar{u}_x \bar{u}_y \) is very close to \( f_y \) over the center of 
the channel, one is led to speculate that the magnitude of the 
normalized random velocities is on the order \( f^{1/2} \), i.e. the 
physical velocities are of order \( u^* \). The measurements of Laufer 
have verified this speculation.

2. The Higher Order Equations

The random part of the velocities and pressure is the sum of 
all terms with \( v \geq 1 \) in the expansions in terms of Orthogonal 
Polynomial Functionals, i.e.,

since: \( \bar{u}^i = G_o [K_o^i, \omega] = K_o^i \tag{6.2.1} \)

then: \( u'^i = u^i - \bar{u}^i = \sum_{1}^{\infty} G_v [K_v^i, \omega] \tag{6.2.2} \)

*See Harris, \(^8\) p. 9.
The equations satisfied by the random velocity are:

\[ \frac{\partial K_v^i}{\partial t} + \frac{\partial}{\partial x} \Gamma_{ij}^v - \frac{1}{Re} \nu^2 K_v^i + \frac{\partial W_v}{\partial x^i} = 0 \quad v = 1, 2, \ldots \quad 6.2.4 \]

\[ \frac{\partial K_v^i}{\partial x^i} = 0 \quad 6.2.5 \]

or in terms of more familiar quantities:

\[ \frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x} \left( u_i' u_j' + u_j' u_i' \right) + \frac{\partial}{\partial x} \left( u_i' u_j' - \overline{u_i' u_j'} \right) = 0 \quad 6.2.6 \]

\[ - \frac{1}{Re} \nu^2 u_i' + \frac{\partial}{\partial x} = 0 \]

\[ \frac{\partial u_k'}{\partial x^k} = 0 \quad 6.2.7 \]

Now, experimental evidence confirms the speculation of the last section that \( u_i' \) is of order \( f^{1/2} \). For Laufer's data at \( Re = 12300.0 \), \( f^{1/2} = 0.042 \) so that \( u_i' \) is but 4\% of \( \overline{u_i} \).

Generally, it is found that \( u_i' \) is less than 10\% of \( \overline{u_i} \), so that one is tempted to drop the products of \( u_i' u_j' \) from equation 6.2.6, and solve the (quasi-linear) problem which results. Unfortunately, these are precisely the terms which form the gain limiting phenomena that keeps the turbulence stable, so that these terms may not be dropped from an analytic solution. However, variational methods may still be employed where the \( u_i' u_j' \) terms have been partially neglected (see Chapter VII).

The kinetic energy of the flow must be bounded (in fact, the orthogonal functional representation is not valid if \( u_i' \) is not square integrable) and this implies the convergence of the sum:
\[
E \left\{ u^k u^k \right\} = E \left\{ \gamma^{kk} \right\} = i_{k0}^{kk}
\]
\[
= \sum_{v} \int_{0}^{\infty} v! \int v \int K_{v}^{k}(\sigma_1, \ldots, \sigma_v) \cdot K_{v}^{k}(\sigma_1, \ldots, \sigma_v) \prod_{j=1}^{v} d\sigma_j
\]

For the sum to converge, the addends must tend to zero as \( v \) tends to infinity, so that:
\[
v! \, K_{v}^{k} \, K_{v}^{k} \triangleq v! \int_{v} \int [K_{v}^{k}(\sigma_1, \ldots, \sigma_v)]^2 \prod_{j=1}^{v} d\sigma_j \quad \xrightarrow{v \to \infty} 0
\]
and since the sum is absolutely convergent (all terms are positive) somewhat tighter bounds are required, so it appears that the \( K_{v}^{k} \) are of order at least smaller than \((v!)^{-1/2}\) for large \( v \). In fact, convergence of all the sums in equation 5.3.28 would indicate that the bounds are more severe. Long experience with this problem leads the present writer to the speculation that \( K_{v}^{k} \) are approximately of order \((v!)^{-1}\) for large \( v \), but it should be emphasized that this is sheer speculation—no convincing arguments have been found either for or against this speculation.

The expansion of \( \Gamma_{v}^{ij} \) given in equation 5.3.28 is:
\[
\Gamma_{v}^{ij}(\sigma_1, \ldots, \sigma_v) = \sum_{k} \sum_{\ell} \frac{(v+k-l)! \cdot (k+l)!}{(v-l)! \cdot (k)!(\ell)!} \int k \prod_{j=1}^{k} ds_j \cdot \left( K_{v+k-l}^{i}(\sigma_1, \ldots, \sigma_{v-l}; s_1, \ldots, s_k) \cdot K_{k+l}^{j}(\sigma_{v-l+1}, \ldots, \sigma_v; s_1, \ldots, s_k) \right)
\]
The term \( K_{v}^{i} \) is involved in the sum only if either \( v+k-l \) or \( k+l \) (or both) are zero, so that
\[
\Gamma_{v}^{ij} = K_{v}^{i} K_{v}^{j} + K_{v}^{j} K_{v}^{i} + \theta_{v}^{ij}
\]
for \( v > 0 \)
where \( \theta_{v}^{ij} \) depends only on \( K_{v} \) for \( v \) greater than zero. The expansion for \( \theta_{v}^{ij} \) is therefore:
\[ \theta_{ij}^{1}(\sigma_1, \ldots, \sigma_v) = \left( \sum_{l=0}^{v-1} \frac{1}{l!} \left( \sum_{k=0}^{v} \frac{v-k-1)!}{s_k} \right) \cdot \int_{\frac{k}{j}}^{k} ds \right) \] 6.2.12

\[ \langle K_{i}^{1}(\sigma_1, \ldots, \sigma_v; s_1, \ldots, s_k) \cdot K_{j}^{1}(\sigma_{v-1}+1, \ldots, \sigma_v; s_1, \ldots, s_k) \rangle \sigma \]

for \( v \geq 1 \)

Thus the equations for \( v \geq 1 \) become:

\[ \frac{\partial K_{i}^{1}}{\partial t} + K_{i}^{1} \frac{\partial K_{i}^{1}}{\partial x} + \delta_{i}^{1} K_{i}^{1} \frac{\partial K_{i}^{1}}{\partial y} - \frac{1}{Re} \nabla^2 K_{i}^{1} + \frac{\partial W_{i}}{\partial x} = \frac{\partial \sigma_{v}^{1}}{\partial x} \] 6.2.13

The terms on the right hand side of this equation should be of order (at least) \( f^{1/2} = 0.042 \) times the order of each of the terms on the left, because the \( K_{i}^{1} \) (for \( v \geq 1 \)) are of order \((f/v!)^{1/2}\) (at least), while the terms on the right are integrated products of the \( K_{i}^{1} \) (for \( v > 1 \)). Thus the terms on the left must very nearly cancel, with the difference being balanced by the terms on the right. Moreover, the term \(-\frac{1}{Re} \nabla^2 K_{i}^{1}\) can be large only near the walls, so it is to be expected that the \( K_{i}^{1} \) have a behavior at the walls rather different from their behavior in the central part of the channel. Laufer's data suggests that this is indeed the case, and this type of behavior is comparable with the behavior of \( \overline{u_x} \) and \( \overline{u_x u_y} \) described in the preceding section.

It is interesting to note that the partial differential equations (6.2.13) are quasi-linear not only in the customary sense (the \( \nabla^2 K_{i}^{1} \) term enters linearly), but also in the following sense: the dependent variables (the \( K_{i}^{1} \)) are functions of the independent variables (\( \sigma_1, \ldots, \sigma_v; y \)) where each \( \sigma \) represents the triplet \( \sigma_k = (x_k+\xi, z_k+\eta, t_k+t) \); and in equation 6.2.13 there exist no products of dependent variables as functions of the same two independent variables, except \( y \). Thus in any product nonlinearity, a particular variable (say \( \sigma_k \)) will appear in one
and only one factor. This property of equation 6.2.13 suggests that transform techniques may be applied to the equation with less difficulty than might be expected since convolutions will not appear.

3. Symmetry Conditions

Symmetry properties of the fluid flow situation offer some interesting results. Harris has shown* that any averaged product of the form:

$$T_{ij}^{ij} = \frac{a^i b^j}{x}$$

(where both $a^i$ and $b^j$ are one of $u^i$, $\frac{\partial x}{\partial x}$, $j^i$, $b^i$, or $e^i$) must be either an even or an odd function of $y$ and that the symmetry may be determined by the following rule:

Assign to $T_{ij}^{ij}$ one mark for each of the following conditions met:

1. $i = 2$ or $3$
2. $j = 2$ or $3$
3. $a^i$ is not $u^i$ or $\frac{\partial y}{\partial x}$
4. $b^j$ is not $u^j$ or $\frac{\partial y}{\partial x}$

The function is then even or odd depending on whether the number of marks is even or odd, respectively. Thus the quantity $\frac{u_x u_y}{x}$ receives one mark and is therefore odd in $y$; whereas, the quantity $\frac{u_x b_y}{x}$ receives two marks and is therefore even in $y$.

This process can be extended to averages of the form:

$$T_{kijn}^{ij} = [\left(\frac{\partial}{\partial x}\right)^m a^i][\left(\frac{\partial}{\partial x}\right)^n b^j]$$

*See Harris, p. 19 et seq.
The symmetry for this case is determined by forming the number of marks as above and adding m more if \( k = 2 \) or 3, and n more if \( \ell = 2 \) or 3. The symmetry is then determined by the parity of the number of marks assigned.

To determine the consequences of this symmetry, consider the quantity:

\[
\overline{u_x u_y} = \sum_{j=1}^{\infty} \int \int \int K_{v}^1(\sigma_1 \ldots \sigma_y; y) K_{v}^2(\sigma_1 \ldots \sigma_y; y) \prod_{j=1}^{v} d\sigma_j
\]

or in an abbreviated notation:

\[
\overline{u_x u_y} = \sum_{j=1}^{\infty} K_{v}^1 K_{v}^2
\]

This quantity \( \overline{u_x u_y} \) is an odd function of \( y \), and the various averages of derivatives of \( u_x \) with derivatives of \( u_y \) obey the symmetry conditions imposed by equation 6.3.2. Moreover, the other velocity correlations have averages with the following symmetry:

\[
\overline{u_x u_z} \quad \text{is odd,} \quad 6.3.5a
\]

\[
\overline{u_y u_z} \quad \text{is even,} \quad 6.3.5b
\]

and \( \overline{u_x u_y}, \overline{u_y u_z}, \overline{u_z u_z} \) are even.

\[
\overline{u_x u_y} = \sum_{j=1}^{\infty} \int \int \int K_{v}^1 K_{v}^2 \quad (\text{odd}) \quad 6.3.6
\]

\[
\overline{u_x u_z} = \sum_{j=1}^{\infty} \int \int \int K_{v}^1 K_{v}^3 \quad (\text{odd}) \quad 6.3.7
\]

and

\[
\overline{u_y u_z} = \sum_{j=1}^{\infty} \int \int \int K_{v}^2 K_{v}^3 \quad (\text{even}) \quad 6.3.8
\]

must possess the same symmetry as the sum.
If this is the case, then the functions:

\[
\begin{align*}
K_v^1 \left( \frac{\partial}{\partial z} \right)^{m-1} K_v^2, & \quad K_v^1 \left( \frac{\partial}{\partial y} \right)^{m-1} K_v^2, & \quad K_v^1 \left( \frac{\partial}{\partial z} \right)^{m-1} K_v^3, & \quad 6.3.9a \\
K_v^1 \left( \frac{\partial}{\partial y} \right)^{m-1} K_v^3, & \quad K_v^2 \left( \frac{\partial}{\partial z} \right)^m K_v^3, & \quad K_v^2 \left( \frac{\partial}{\partial y} \right)^m K_v^3, & \quad 6.3.9d
\end{align*}
\]

must be even or odd depending on whether \( m \) is even or odd, respectively. Thus if a certain polarity \( p_v \) (+ for even, - for odd on \( y \)) is associated with the even and odd parts of the \( \hat{K}_v \) on the \( z \) parameter, then about the only reasonable way the assignment of polarities can be made is as follows:

let:

\[
\begin{align*}
K_v^1 (+z) + K_v^1 (-z) & \rightarrow -p_v & \quad 6.3.10a \\
K_v^1 (+z) - K_v^1 (-z) & \rightarrow +p_v & \quad 6.3.10b \\
K_v^2 (+z) + K_v^2 (-z) & \rightarrow +p_v & \quad 6.3.10c \\
K_v^2 (+z) - K_v^2 (-z) & \rightarrow -p_v & \quad 6.3.10d \\
K_v^3 (+z) + K_v^3 (-z) & \rightarrow +p_v & \quad 6.3.10e \\
K_v^3 (+z) - K_v^3 (-z) & \rightarrow -p_v & \quad 6.3.10f
\end{align*}
\]

In words, the above assignment of polarities would read: "If the even part of \( K_v^2 \) on \( z \) is even (odd) on \( y \), then the odd part of \( K_v^2 \) on \( z \) is odd (even) on \( y \), the even part of the \( K_v^1 \) on \( z \) is odd (even) on \( y \), ... etc."

If the above assignment of symmetry conditions on the \( \hat{K}_v \) is made, then all of the conditions on the \( T_{k,lmn}^{ij} \) of equation 6.3.2 are met identically. If the above symmetry conditions

*The notation \( K_v (-z) \) is intended to mean, \( K_v (x+x_v, -z-z_v, t+t_v; \ldots; x+x_v, -z-z_v, t+t_v) \), etc.
were not made, then there would be an extremely large class of auxiliary conditions imposed on the $K^i_v$, so much so that no other set of symmetry conditions on the $K^i_v$ have been found consistent with the physical problem and the equations of motion.

Equations 6.3.10 do not specify which sign (+ or −) is to be associated with each $p_v$ since the symmetry conditions will allow either. The equations of motion, however, definitely fix these polarities as follows: for $v$ even $p_v$ is always − (minus); while for $v$ odd, the $p_v$ are either all + (plus) or all − (minus); provided the sums involved (in $\Gamma^i_\nu$ or $\Theta^i_\nu$) consist of terms with the same symmetry as the sum. (This is the same hypothesis as was supposed concerning equations 6.3.6-8) For an explanation of this statement, consider the equations of motion in the form:

$$\frac{\partial K^i_v}{\partial t} + K^i_v \frac{\partial K^i_v}{\partial x} + \delta^i_1 K^2_v \frac{\partial K^1_v}{\partial y} - \frac{1}{R_e} v^2 K^1_i + \frac{\partial W_v}{\partial x} = \phi^i_v$$

6.3.11

where:

$$\phi^i_v(\sigma_1, \ldots, \sigma_v) = \left\{ \sum_{k=0}^{\infty} \Sigma_{\nu, \sigma} \sum_{(k=0)}^{v-1} \right\} \left( \frac{(v+k-\nu)!}{(v-\nu)!} \right)^k \int_{\nu}^{k} ds \jmath \left( \frac{(v+k-\nu)!}{(v-\nu)!} \right)^k \sum_{j=1}^{v-1} j \sigma^j$$

6.3.12

First, the leftmost four terms of the equation are all of similar polarity, viz. the same polarity as $K^i_v$. Thus, both $\phi^i_v$ and $\frac{\partial W_v}{\partial x}$ must have the same polarity as $K^i_v$. For the pressure, this requires that:

$$W_v(\nu+z) + W_v(-z) \rightarrow -p_v$$

6.3.13a

$$W_v(\nu+z) - W_v(-z) \rightarrow +p_v$$

6.3.13b

For the $\phi^i_v$, each term in the sum (equation 6.3.12) consists of the integral of a product such as:
\[ k_j^1 \frac{\partial k_i^1}{\partial x_j} = k_1^1 \frac{\partial}{\partial x} k_i^1 + k_2^1 \frac{\partial}{\partial y} k_i^1 + k_3^1 \frac{\partial}{\partial z} k_i^1 \]

where for brevity, \( \mu = v+k-\xi \), and \( \lambda = k+\xi \). Consider the component of this term for \( i = 1 \). The integrand

\[ k_j^1 \frac{\partial k_i^1}{\partial x_j} \]

may be replaced by:

\[
\begin{align*}
(1) & \quad K_{e} \frac{\partial}{\partial x} K_i^1 \\
(2) & \quad K_{o} \frac{\partial}{\partial x} K_i^1 \\
(3) & \quad K_{e} \frac{\partial}{\partial y} K_i^1 \\
(4) & \quad K_{o} \frac{\partial}{\partial y} K_i^1 \\
(5) & \quad K_{e} \frac{\partial}{\partial z} K_i^1 \\
(6) & \quad K_{o} \frac{\partial}{\partial z} K_i^1 \\
\end{align*}
\]

where for brevity:

\[ k_{ve}^1 = \frac{1}{2}[K_v^1(x_1,z_1,t_1;\ldots;x_v,z_v,t_v) + K_v^1(x_1,-z_1,t_1;\ldots;x_v,-z_v,t_v)] \]

and:

\[ k_{vo}^1 = \frac{1}{2}[K_v^1(x_1,z_1,t_1;\ldots;x_v,z_v,t_v) - K_v^1(x_1,-z_1,t_1;\ldots;x_v,-z_v,t_v)] \]

The result of integrating the first two terms of equation 6.3.15 and taking the symmetrical part on the remaining variables is an even function of \( z \), and the same is true of the pairs of terms (5,6) and (11,12). Similarly, pairs (3,4), (7,8), and (9,10) yield odd functions of \( z \). Furthermore, it is true that, if the \( k_i^1 \) and \( k_i^1 \) have symmetry properties as specified in equations 6.3.10, then the contribution of the expanded terms to \( \phi_{ve}^1 \) will be of polarity \( p_{v} p_{\lambda} \) while the contribution to \( \phi_{vo}^1 \) will be of polarity \( -p_{v} p_{\lambda} \), as may be verified by simply checking all terms. Similarly, it is not difficult to show that the contribution of

\[ k_j^1 \frac{\partial k_i^1}{\partial x_j} \]

to \( \phi_{v}^1 \) is of the form:
\[ \phi_{ve}^1 \rightarrow +p_\mu p_\lambda \]
\[ \phi_{vo}^1 \rightarrow -p_\mu p_\lambda \]
\[ \phi_{ve}^2 \rightarrow -p_\mu p_\lambda \]
\[ \phi_{vo}^2 \rightarrow +p_\mu p_\lambda \]
\[ \phi_{ve}^3 \rightarrow -p_\mu p_\lambda \]
\[ \phi_{vo}^3 \rightarrow +p_\mu p_\lambda \]

The requirement that \( \phi_v^1 \) have polarities congruent to those of \( k_v^1 \) is satisfied if and only if:

\[ -p_v \equiv p_\mu p_\lambda \]

However, if \( v \) is even, then \( k^{1}_{\mu} \frac{\partial k^{1}_{\lambda}}{\partial x^{j}} \) cannot contribute to \( \phi_v^1 \) unless either, both \( \mu \) and \( \lambda \) are even, or both \( \mu \) and \( \lambda \) are odd; whereas, if \( v \) is odd, then either, \( \mu \) is even and \( \lambda \) is odd, or \( \mu \) is odd and \( \lambda \) is even. Since \( p_v \) is -, it follows that \( p_v \) is - for all \( v \) even. Then since all \( p_v \) for even \( v \) are -, it follows that for \( v \) odd, the \( p_v \) are either all +, or all -.

In summary, symmetry conditions indicate very strongly that the \( k_v^1 \) must satisfy the following symmetry requirements:

\[ k_{ve}^{1} \rightarrow -p_v \]
\[ k_{vo}^{1} \rightarrow +p_v \]
\[ k_{ve}^{2} \rightarrow +p_v \]
\[ k_{vo}^{2} \rightarrow -p_v \]
\[ k_{ve}^{3} \rightarrow +p_v \]
\[ k_{vo}^{3} \rightarrow -p_v \]

where:

\[ k_{ve}^{1} = \frac{1}{2}[k_{v}^{1}(x_1, z_1, t_1; \ldots; x_v, z_v, t_v) + k_{v}^{1}(x_1, -z_1, t_1; \ldots; x_v, z_v, t_v)] \]
\[ k_{vo}^{1} = \frac{1}{2}[k_{v}^{1}(x_1, z_1, t_1; \ldots; x_v, z_v, t_v) - k_{v}^{1}(x_1, -z_1, t_1; \ldots; x_v, -z_v, t_v)] \]
and a $p_v$ of + stands for an even function of $y$, whereas a $p_v$ of - stands for an odd function of $y$. Beyond this, the equations of motion require that $p_v$ be - for $v$ even, and for $v$ odd the $p_v$ are either all - or all +. The above listed conditions, although not proven, seem well established because of the numerous (infinite in fact) symmetry conditions they cause to be satisfied. The sign of the $p_v$ for odd $v$ has not been (and cannot be) specified by symmetry conditions. Some numerical approximations and calculations of the next chapter indicate that $p_v$ is + for odd $v$, but this hypothesis is not so strongly based as the other symmetry conditions listed above.

4. The Magnetofluid Dynamic Equations

The problem of Plane Poiseuille Flow becomes considerably more complicated with the application of a magnetic field. Not only are the equations twice in number with seven variables instead of four, but also there is a lack of experimental evidence by which hypotheses may be tested. In addition, little theoretical work has been done on the problem; the only comprehensive treatment is given by Harris.\(^8\) q.v. Thus, any results obtained here are much less firmly based than for the ordinary hydrodynamic case. From the point of view of the researcher in fluid dynamics this is not too important because the magnetic field adds nothing essentially new to the problem. However, from the point of view of the researcher in magneto-fluid dynamics, the lack of information is unfortunate because turbulence will drastically alter the characteristics of any fluid device.

If (as in Chapter V, Section 6) the velocities, pressure, and magnetic field components are expanded as:

$$u^i = \sum_{v=0}^{\infty} G_v^i [K_v^i, \omega]$$  \hspace{1cm} 6.4.1

$$p = \sum_{v=0}^{\infty} G_v [W_v, \omega]$$  \hspace{1cm} 6.4.2
\[
b^i = \sum_0^\infty G_v[B^i_v, \omega] \quad 6.4.3
\]

and the following definitions are made:

\[
\gamma^{ij} = u^{i} u^{j} = \sum_0^\infty G_v[I^{ij}_v, \omega] \quad 6.4.4
\]

\[
y^{ij} = u^{i} b^{j} = \sum_0^\infty G_v[Z^{ij}_v, \omega] \quad 6.4.5
\]

\[
b^{ij} = b^{i} b^{j} = \sum_0^\infty G_v[B^{ij}_v, \omega] \quad 6.4.6
\]

then the equations of motion may be written:

\[
\frac{\partial K^i_v}{\partial t} + \frac{\partial}{\partial x^j} \gamma^{ij} - \frac{1}{R_e} \nabla^2 K^i_v + \frac{\partial W_v}{\partial x^i} = \frac{M^2}{R_e R_m} \left\{ \frac{\partial}{\partial x^j} B^{ij}_v - \frac{1}{2} \frac{\partial}{\partial x^i} B^{kk}_v \right\} \quad 6.4.7
\]

\[
\frac{\partial B^i_v}{\partial t} + \frac{\partial}{\partial x^j} (Z^{ij}_v - Z^{ij}_v) - \frac{1}{R_m} \nabla^2 B^i_v = 0 \quad 6.4.8
\]

and:

\[
\frac{\partial k_v}{\partial t} = 0 \quad ; \quad \frac{\partial B^k_v}{\partial x^i} = 0 \quad 6.4.9 \quad 6.4.10
\]

These comprise a set of eight equations in the seven variables \(K^i_v, B^i_v\) and \(W_v\) (the equation \(\partial B^k_v / \partial x^i = 0\) is redundant). It may be noted that equation 6.4.7 contains two gradients of scalars, viz., \(\partial W_v / \partial x^i\) and \((\partial / \partial x^i)(B^{kk}_v / 2)\), and thus it will sometimes be convenient to combine these two terms by defining:

\[
W^*_v = W_v + \frac{1}{2} B^{kk}_v \quad 6.4.11
\]

The mean velocity is totally \(x\) directed, while the mean magnetic field has a constant component in the \(y\) direction and an induced component in the \(x\) direction. Thus, since all electrical quantities are normalized with respect to \(B_0 = b^2\), the averaged equations of motion (i.e., \(v = 0\)) are:
\[
\frac{d}{dy} \gamma_{o}^{12} - \frac{1}{Re} \frac{d^2}{dy^2} \gamma_{o}^{1} + \frac{\partial W_{o}}{\partial x} = \frac{M^2}{Re m} \left[ \frac{d}{dy} B_{o}^{12} \right]
\]

6.4.12

\[
\frac{d}{dy} \gamma_{o}^{22} + \frac{\partial W_{o}}{\partial y} = \frac{M^2}{Re m} \left[ \frac{d}{dy} B_{o}^{22} - \frac{1}{2} \frac{d}{dy} B_{o}^{kk} \right]
\]

6.4.13

\[
\frac{d}{dy} \gamma_{o}^{32} = \frac{M^2}{Re m} \frac{d}{dy} B_{o}^{32}
\]

6.4.14

\[
\frac{d}{dy} [z_{o}^{21} - z_{o}^{12}] - \frac{1}{Rm} \frac{d^2}{dy^2} B_{o}^{1} = 0
\]

6.4.15

\[
\frac{d}{dy} [z_{o}^{22} - z_{o}^{22}] = 0
\]

6.4.16

\[
\frac{d}{dy} [z_{o}^{23} - z_{o}^{32}] = 0
\]

6.4.17

or in more familiar terms:

\[
\frac{d}{dy} \frac{u_{x} u_{y}}{y} = \frac{1}{Re} \frac{d^2}{dy^2} \frac{u_{x}}{y} + \frac{\partial p}{\partial y} = \frac{M^2}{Re m} \left[ \frac{d}{dy} \frac{b_{x}}{y} + \frac{d}{dy} \frac{b_{y} b_{y}'}{y} \right]
\]

6.4.18

\[
\frac{d}{dy} \frac{u_{x} u_{y}}{y z} + \frac{\partial p}{\partial y} = \frac{M^2}{Re m} \frac{d}{dy} \left[ - \frac{1}{2} (b_{x} b_{y} - b_{y} b_{y}' + b_{z} b_{z}') \right]
\]

6.4.19

\[
\frac{d}{dy} \frac{u_{x} u_{y}'}{y z} = \frac{M^2}{Re m} \frac{d}{dy} \frac{b_{y}' b_{y}'}{y z}
\]

6.4.20

\[
\frac{1}{Rm} \frac{d^2}{dy^2} \frac{b_{x}}{x} + \frac{d}{dy} \frac{u_{x}}{y} - \frac{\partial}{\partial y} \left[ \frac{u_{x} b_{y}'}{y} - \frac{b_{y} b_{y}'}{y} \right] = 0
\]

6.4.21

\[
\frac{d}{dy} \left[ \frac{u_{y} b_{y}'}{y z} - \frac{b_{y} u_{y}'}{y z} \right] = 0
\]

6.4.22

By employing various boundary and symmetry conditions and combining equations, the following equations result:

\[
\frac{\partial p}{\partial x} = \text{constant} \Delta f; \text{ the friction factor},
\]

6.4.23

\[
\frac{u_{x} u_{y}'}{y z} - \frac{M^2}{Re m} \frac{b_{y} b_{y}'}{y z} = 0
\]

6.4.24

\[
\frac{u_{x} b_{y}'}{y z} - \frac{b_{y} u_{y}'}{y z} = 0
\]

6.4.25
\[
\frac{d^2}{dy^2} - M^2 \bar{u}_x = -Re C_a + Re \frac{d}{dy} \left[ \bar{u}_x' \bar{u}_y' - R_h \bar{b}_x' \bar{b}_y' \right] + M^2 \left[ \bar{u}_x' \bar{u}_y' - \bar{u}_y' \bar{u}_x' \right]
\]

where:
\[
\frac{db_x}{dy}igr|_{y=1} = 6.4.28
\]

For the flow damper case, the boundary conditions are:
\[
u_x = b_x = 0 \text{ at } y = \pm 1.
\]

At this point, it would be convenient to simplify the problem as much as possible to make it more amenable to analysis, and yet not oversimplify so much that there is no correspondence with existing experiments. Following Harris, the mean velocity equation may be simplified to:
\[
\frac{d^2}{dy^2} - M^2 \bar{u}_x = -Re C_a + Re \frac{d}{dy} \bar{u}_x' \bar{u}_y
\]

provided:
\[
\bar{u}_x' \bar{b}_y' - \bar{b}_x' \bar{u}_y' \ll \bar{u}_x
\]

and:
\[
R_h \bar{b}_x' \bar{b}_y' = \frac{M^2}{Re R_m} \bar{b}_x' \bar{b}_y' \ll \bar{u}_x' \bar{u}_y
\]

Now, from the equations:
\[
e^{ijk} \frac{\partial}{\partial x^j} b^k = R_m j^i = R_m (e^i + e^{ijk} u^j b^k)
\]

and:
\[
e^{ijk} \frac{\partial}{\partial x^j} e^k = - \frac{\partial b^i}{\partial t}
\]

one is tempted to deduce that for small \( R_m \) the induced magnetic field is of order \( R_m \); in fact, the laminar solution for \( b_x \) is:
which suggests that for large $M$ (essentially large applied magnetic field), the induced magnetic field is of order $R_m / M$. Moreover, for ordinary turbulent flow, $u'$ is of order $f^{1/2}$, so that whereas one expects $b_x$ is of order $R_m$ (or $R_m / M$), $b'$ is of order $f^{1/2} R_m$ (or $f^{1/2} R_m / M$). Thus, it is to be expected that equation 6.4.30 and equation 6.4.31, respectively, will be satisfied if:

$$\frac{R_f}{M^2} \ll 1 \quad \text{and} \quad \frac{\frac{M^2 R_m}{R_e}}{\frac{R_m}{R_e}} \ll 1$$

for low applied fields, or:

$$\frac{\frac{R_f}{M^2}}{\frac{R_e}{M}} \ll 1 \quad \text{and} \quad \frac{\frac{M^2 R_m}{R_e}}{\frac{R_m}{R_e}} \ll 1$$

for high fields. For Murgatroyd's experiments $R_m$ about $3 \cdot 10^{-4}$ and $\frac{M^2 R_m}{R_e}$ never exceeded $3 \cdot 10^{-3}$, but for large $M$, $\frac{M^2 R_m}{R_e}$ should be replaced by $\frac{R_m}{R_e}$ which is a constant $\gamma (\frac{\mu}{\rho})$ whose value for mercury is $1.7 \cdot 10^{-7}$. Thus the replacement of equation 6.4.26 by equation 6.4.29 seems to be a reasonable approximation for mercury and the experiments of Murgatroyd. For flows of NaK, however, these approximations may be very poor and would require further justification.

Unfortunately, since equation 6.4.29 may not be integrated once (as was equation 6.1.21), and since no detailed flow measurements have been made, not as much can be said about $K^{1}_{0} = \overline{u'_x}$ or $\Gamma^{12} = \overline{u'_x u'_y}$ for the magneto-fluid dynamic case as was said for the zero field case. However, some information can be obtained from equation 6.4.29. First, over the central part of the channel $(d^2/dy^2)(\overline{u_x})$ never exceeds a magnitude of 0.6 for
the zero field case, so one can safely take the magnitude of
\[ \frac{d^2 u_x}{dy^2} \] as order 1.0 for the magneto-fluid case. Thus, the
center channel force balance implied by equation 6.4.29
changes its character as \( M \) ranges over (say) 0-10; so that by
the time \( M = 10 \), the pressure and Reynolds stresses \( \frac{d}{dy} \bar{u} u' u' \)
are balanced largely by electromagnetic forces, while viscous
forces are negligible.

Second, if \( M \) is or order 10 or so, then over the center of
the channel:

\[ \bar{u}_x = \frac{R}{M^2} \left[ C_a - \frac{d}{dy} \bar{u}' u' \right] \]  \hspace{1cm} 6.4.37

Thus, if \( \bar{u}_x \) is to be a monotone decreasing function of \( y \), with
\( u_x(0) = 1 \), and \( u_x(1) = 0 \); then:

\[ C_a - \frac{d}{dy} \bar{u}' u' > 0 \]  \hspace{1cm} 6.4.38

and:

\[ \frac{d^2}{dy^2} \bar{u}' u' > 0 \]  \hspace{1cm} 6.4.39

over the central part of the channel.

The dynamic equations satisfied by the random part of the
velocity are:

\[ \frac{\partial u'_i}{\partial t} + \frac{\partial}{\partial x_j} \left( u'_i u'_j + u'_i u'_j + u'_i u'_j - \bar{u}' u' \right) - \frac{1}{Re} \nabla^2 u'_i + \frac{\partial p'_i}{\partial x} \]  \hspace{1cm} 6.4.40

\[ = \frac{M^2}{Re} \left\{ e_{ij} k [e_{b} k] - e_{b} k \right\} \]

\[ - u'_b b + u'_b k + b'_b k - b'_b k \]

Now, the justification of the approximations which allowed
equation 6.4.29 to be written in place of equation 6.4.26, is
that the inequalities listed in equations 6.4.30 and 6.4.31 cause
all induced fields to be negligible compared to the applied field.
The same reasoning allows the random driving force to be approxi-
mated by:
\[
F^i = - \frac{M^2}{R_e} \left( u^i - \delta^i_2 u^2 \right) \]

In fact, with the driving force approximated in this way, the total dynamic set of equations in terms of the \( K^i_\nu \) may be written:

\[
\frac{\partial K^i_\nu}{\partial t} + \frac{\partial}{\partial x^j} \Gamma^i_{\nu j} - \frac{1}{R_e} \nabla^2 K^i_\nu + \frac{\partial W_\nu}{\partial x^i} = - \frac{M^2}{R_e} (K^i_\nu - \delta^i_2 K_2) \]

\[
\frac{\partial K^k_\nu}{\partial x^k} = 0
\]

Thus by suitable approximations (which correspond well to the experiments of Murgatroyd) the kinematic equations have been eliminated and a resultant set of four equations in four unknowns have been obtained.

If the approximations which led to the neglect of the induced magnetic field are no longer valid, then the total system of equations must be considered, and the symmetry and boundary conditions on the induced magnetic field become important.

For the symmetry conditions, the same arguments which lead to the characterization of the \( K^i_\nu \) specified by equation 6.3.20 lead to the following characterization of the magnetic field:

\[
\begin{align*}
B^1_{ve} & \rightarrow +p_v \\
B^1_{vo} & \rightarrow -p_v \\
B^2_{ve} & \rightarrow +p_v \\
B^2_{vo} & \rightarrow -p_v \\
B^3_{ve} & \rightarrow +p_v \\
B^3_{vo} & \rightarrow -p_v
\end{align*}
\]

where the symbolism is the same as in equation 6.3.20. Thus for example, since \( p_o \) is- as was shown in the last section, the
induced mean field in the x(i=1) direction is an odd function of y.

The boundary conditions impose more of a problem. For the mean part of the induced field, the flow damper case requires that $B_o^1 = 0$ at $y = \pm 1$. However, the conditions on the random part of the field must be matching conditions on the fields (both e and b) inside and outside the channel. Generally, the space outside is filled with a moderately good conductor which has a high permeability—a magnet pole piece of iron. The solution for the fields outside the channel may be found directly from ordinary electromagnetic theory and then boundary conditions at $y = \pm 1$ would be matched with the fields in the fluid.
CHAPTER VII

Approximate Solutions for Turbulent Plane Poiseuille Flow

1. Approximate Methods

If the velocities and pressure existing in turbulent Plane Poiseuille Flow are expanded in terms of Orthogonal Polynomial Functionals as:

\[ u^i = \sum_{\nu}^{\infty} G_{\nu}^{i} [K_{\nu}^i, \omega] \]  
\[ p = \sum_{\nu}^{\infty} G_{\nu} [W_{\nu}, \omega] \]

then the resulting equations of motion are:

\[ \frac{\partial K_{\nu}^i}{\partial t} + \frac{\partial}{\partial x_j} \Gamma_{\nu}^{ij} - \frac{1}{\text{Re}} \nabla^2 K_{\nu}^i + \frac{\partial W_{\nu}}{\partial x_i} = 0 \]

and:

\[ \frac{\partial K_{\nu}^i}{\partial x_i} = 0 \]

where:

\[ \Gamma_{\nu}^{ij}(\sigma_1, \ldots, \sigma_\nu) = \sum_{\epsilon=1}^{\infty} \frac{\nu!}{\sigma_\nu!} \frac{(\nu+k-\epsilon)!}{(k+\epsilon)!} \int_{\sigma_{\nu+k+\epsilon}}^{\infty} ds_i \int_{\sigma_{\nu+k+\epsilon}}^{s_j} ds_j \]

(see Chapter VI, Sections 1 and 2)

These equations form an extremely complicated, infinite set of completely coupled, nonlinear, partial differential equations. Thus, it is appropriate to seek approximate solutions to these equations.

The types of approximations and approximate methods to be used here are aimed at producing physically reasonable velocities and pressures, employing whatever information may be obtained
from experiment or the "physics" of the problem. However, no numbers obtained from experiment should be used directly (e.g. one should not force the friction factor to be the measured value, etc.) for then the methods become semi-empirical, if not outright curve-fitting.

Generally, the velocities will be approximated in some sense and the approximate methods will optimize the approximation in some sense. Of the numerous approximate methods which have been usefully employed, three will be (or have been) used in this study of turbulence, viz., Perturbation methods, Iteration methods, and Variational methods.

**Perturbation Methods** or "small signal" methods were used in the study of Decaying Turbulent Flows (Chapter IV) with some success. However, the essential nonlinearity of turbulent shear flows (see Chapter V) indicates that perturbation methods are not generally valid for the study of Plane Poiseuille Flow, but some modified perturbation methods (analogous to the Feenberg or the Fredholm methods) may perhaps give more usable results (see Chapter VIII).

**Iteration Methods** are systematic ways of improving approximations. Generally, iteration methods are associated with either perturbation methods or variational methods, because there is the problem of picking the initial approximation to be improved. One way of constructing a sequence \( \{ u_n^i \} \) of approximate velocities (which hopefully converges) by use of the equations of motion is as follows:

\[
\frac{\partial u_n^i}{\partial t} - \frac{1}{Re} \nabla^2 u_n^i + \frac{\partial p_n}{\partial x^i} = - \frac{\partial}{\partial x^j} u_{n-1}^i u_{n-1}^j
\]

and:

\[
\frac{\partial u_n^i}{\partial x} = 0
\]
There are many variations to these equations (e.g., one might require that the mean equation is satisfied by each $u^i_n$) but all such methods have some common properties. For example, there is always a question of the convergence of the sequence, and there is a certain degree of arbitrariness in the initial approximation. Thus, if the iteration method is to be used, it will probably be best employed in conjunction with a variational technique.

**Variational Techniques** are based on the formulation of some scalar measure of the "goodness" of an approximation. Unfortunately, there seems to be no "natural" measure of the goodness of an approximation (as one can find for (say) the Laplace Equation). However, if the approximate velocities are required to satisfy continuity of mass ($\frac{\partial u^i}{\partial x^i} = 0$) and to approximate solutions of the momentum balance equations, i.e.:

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^j} u^i u^j - \frac{1}{Re} \nu^2 u^i + \frac{\partial p}{\partial x^i} = c^i = 0$$

then $c^i$ is zero if and only if the "approximations" are true solutions. Thus, some scalar measure of the magnitude of $c^i$, for example:

$$E \left\{ \int_{-1}^{+1} dy \ |c^i|^2 \right\}$$

will form an appropriate measure of the "goodness" of the approximation. Presumably, the approximation will contain some parameters which are to be varied for an optimum. The optimum solution is then that set of parameters which minimizes the chosen measure of the $c^i$.

Another approximate method which will be frequently employed in this chapter is related to the Describing Function methods known to electrical engineers. This method is often employed
when a given function is to be approximated by a sum of orthogonal functions. Suppose, for example, that $\psi$ is a solution of some nonlinear equation, i.e.:

$$N[\psi] = 0$$

If now $\psi$ is expanded in terms of some orthogonal set of functions (say $\{\phi_n\}$) and only a finite number of terms are retained, e.g.:

$$\psi = \sum_{n=0}^{m} a_n \phi_n$$

then, $m+1$ appropriate equations for the $\phi_n$ would be:

$$\phi_k^* N[\sum_{n=0}^{m} a_n \phi_n] = 0 \quad k = 0,1,\ldots,m$$

Such a method will be applied frequently in this chapter when functions are expanded in terms of an orthogonal set of functions (or functionals).

2. A Class of Approximate Velocities

A. Finite Sum of Orthogonal Functionals

Since the velocities and pressure may be expanded in a convergent sum of $G$ functionals, they may be approximated as closely as desired (in an l.i.m. sense) by a finite sum of the functionals. It can be shown, however, that the velocities cannot be represented exactly by a finite sum of $G$ functionals, so that such a truncated representation is in fact always an approximation. Nevertheless, it is appropriate to approximate

*The proof of this statement is something as follows: Any finite sum of $G$ functionals is absolutely summable and convergent and therefore may be resummed in terms of Homogeneous Polynomial Functionals. But no steady solution exists in terms of Homogeneous Polynomial Functionals, so that the $G$ series may not be resummed in terms of homogeneous functionals and therefore the $G$ series is infinite.
the velocities as a finite sum of $G$ functionals as:

$$u^i = \sum_{\nu}^{\mu} G_{\nu}^{[K^i_{\nu}, \omega]}$$

Then the nonlinear term $u^{i}_{u^{j}}$ is expressible as:

$$u^{i}_{u^{j}} = \gamma^{i}_{j} = \sum_{\nu}^{2\mu} G_{\nu}^{[\Gamma^{ij}_{\nu}, \omega]}$$

Now, if it is required that the continuity equation $\frac{\partial u^k}{\partial x^k} = 0$ is satisfied identically, * then the momentum balance equations are not identically satisfied and therefore:

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x^j} u^i u^j - \frac{1}{Re} \nabla^2 u^i + \frac{\partial P}{\partial x^i} = C^i = 0$$

or in terms of the $K^i_{\nu}$:

$$\frac{\partial K^i_{\nu}}{\partial t} + \frac{\partial}{\partial x^j} K^i_{\nu} j - \frac{1}{Re} \nabla^2 K^i_{\nu} + \frac{\partial W}{\partial x^i} = C^i_{\nu} = 0; \nu = 0, 1, \ldots, 2\mu$$

Since there are $\mu+1$ terms in the velocity approximation, it is appropriate to consider only the first $\mu+1$ equations. Furthermore, while $C^i_{\nu}$ would ordinarily be of order 1, the $C^i_{\nu}$ for $\nu > 1$ will ordinarily be of order $f^{1/2}$. Thus the error may be considerably reduced by taking:

$$C^i_{\nu} = 0$$

i.e., by defining $K^i_{\nu}$ (and $W_{\nu}$) in terms of the $\frac{\partial}{\partial x^j} \Gamma^{ij}_{\nu}$ (which depend only on $K^i_{\nu}$ for $\nu \geq 1$).

### B. A Two-Dimensional Model

The total equations of motion in terms of the approximations described above may be written:

$$\frac{\partial K^i}{\partial t} + \frac{\partial}{\partial x^j} K^{ij} - \frac{1}{Re} \nabla^2 K^i + \frac{\partial W^i}{\partial x^i} = C^i; \nu = 0, 1, \ldots, \mu$$

*If the divergence of $u^i$ is not zero, then there is some question about the stress law; see Chapter II, Section 2.*
where, using the index notation summation convention, all indices (and indexed summations) are taken to run from 1 to 3. Furthermore, direct measurements of the physical velocities indicate that all three velocities are present and are functions of all three coordinates as well as time, and therefore turbulence in Plane Poiseuille Flow is indeed a three-dimensional phenomenon.

Suppose, however, that the indices in equation 7.2.6 run only over the values of 1 and 2 (this assumption is also to apply to \( \nabla^2 = \frac{\partial^2}{\partial x^k \partial x^k} \)). The result is a two-dimensional mathematical model of turbulence. It is to be noted that such an approximation is purely a mathematical approximation—there exist no two-dimensional turbulent flows.

The reason for making such an approximation is the gross simplifications it introduces; thus, a two-dimensional incompressible velocity field can be represented in terms of one scalar stream function, i.e.:

\[
\mathbf{u}^i = e^{ij3} \frac{\partial}{\partial x^j} s
\]  

where:

\[
s = \sum_v^\mu G_v [S_v, \omega]
\]

because in this case:

\[
\frac{\partial \mathbf{u}^i}{\partial x^i} = e^{ij3} \frac{\partial^2}{\partial x^k \partial x^j} s = 0
\]

In view of the considerable simplification introduced, the justification for the two-dimensional approximation might well come a posteriori, i.e. the results seem to approximate physical reality very well (see Section 5 of this chapter). For an a priori justification, the following arguments may be given:

1. There are no average physical quantities (except current \( j^1 \) which may be eliminated) in the z or \( i = 3 \) direction, and no average quantities depend
on $z$, so that in a sense, $z$ is the least important coordinate.

2. The resulting equations in index form look very much like the full set of equations, so that it would seem that the solutions would also exhibit similar properties.

3. Experience with other equations in mathematical physics shows that good results may be obtained by a similar reduction in dimensionality--some examples are the Laplace and the Helmholtz equations whose solutions in two- and three-dimensions are very similar.

If the two-dimensional model is taken as an appropriate approximation with the velocities expressed as:

$$u^i = e^{ij3} \frac{\partial}{\partial x^j} s$$

then the equations of motion become:

$$\frac{\partial}{\partial t} e^{j3} \frac{\partial s}{\partial x^j} + \frac{\partial}{\partial x^j} e^{j3} \frac{\partial s}{\partial x^j} = e^{jm3} \frac{\partial s}{\partial x^m} - \frac{1}{Re} v^2 e^{j3} \frac{\partial s}{\partial x^j} + \frac{\partial p}{\partial x^j} = 0$$

Moreover, if the operator $e^{in3} \frac{\partial}{\partial x^n}$ is applied to this equation, the result is:

$$\frac{\partial}{\partial t} v^2 s + e^{km3} \frac{\partial s}{\partial x^m} (\frac{\partial}{\partial x^k} v^2 s) - \frac{1}{Re} v^4 s = 0$$

or:

$$\frac{\partial}{\partial t} v^2 s + u^k \frac{\partial}{\partial x^k} v^2 s - \frac{1}{Re} v^4 s = 0$$

Now, since:

$$u^1 = \frac{\partial s}{\partial y}, \quad u^2 = -\frac{\partial s}{\partial x}$$
and the boundary conditions on \( u^1 \) are:

\[
\dot{u}^1 - u^2 = \frac{\partial u^2}{\partial y} = 0 \quad \text{at} \quad y = \pm 1
\]

it follows that:

\[
s = \frac{\partial s}{\partial y} = 0 \quad \text{at} \quad y = \pm 1
\]

Thus, equation 7.2.13 is a fourth order non-linear partial differential equation with a consistent, homogeneous set of boundary conditions.

If the stream function \( s \) is approximated in terms of the \( G \) functionals as:

\[
s = \sum_{\nu} G_{\nu} [S_{\nu}, \omega]
\]

then the non-linear term may be written:

\[
e^{\lambda m_3} \frac{\partial s}{\partial x} \left( \frac{\partial}{\partial x} \nabla^2 s \right) = \eta = \sum_{\nu} G_{\nu} \left[ \frac{K}{\varphi} \frac{\partial}{\partial x} \nabla^2 S_{\nu} - \frac{d^2 K_0}{\partial y^2} \frac{\partial S_{\nu}}{\partial x} + N_{\nu} \right] \omega
\]

\[
\frac{2 \mu}{\mu + 1} + \sum_{\nu} G_{\nu} [N_{\nu}, \omega]
\]

where:

\[
N_{\nu} = \left\{ \sum_{k=1}^{v-1} \frac{\varphi}{\varphi} \sum_{\ell=1}^{v-1} \frac{v-1}{(v-k-\ell)!(k+\ell)!} \int k \int \frac{k}{j} ds_{j} \right\}
\]

\[
\left\langle [\frac{\partial}{\partial x} \nabla^2 S_{k+\ell} (\sigma_{v-\ell+1}, \ldots, \sigma_{v} \sigma_{s_1}, \ldots, s_k) \right]\right\rangle_{\sigma} \quad \text{for} \quad v = 1, 2, \ldots, 2 \mu
\]

If only the first \( \mu + 1 \) equations yielded are considered significant, then the equations of motion become:

\[
\frac{d^2}{dy^2} K_0 = -R_f + R_e \frac{d}{dy} \gamma_0^{12}
\]
and:
\[
\frac{\partial}{\partial t} \nabla^2 S_v + K^1_0 \nabla^2 \frac{\partial S_v}{\partial x} - \frac{d^2 K^1_0}{dy^2} \frac{\partial S_v}{\partial x} + N_v - \frac{1}{Re} \nabla^4 S_v = C_v \approx 0
\]
\[\text{for } v = 1, \ldots, \mu \]

where \( C_v \) is the error.

The corresponding equations for the magneto-fluid dynamic case follow in a completely analogous manner. If equations 6.4.39 are considered valid, then the equations of motion in terms of the two-dimensional stream function become:
\[
[\frac{d}{dy} - M^2]K^1_o = - Re C_a + Re \frac{d}{dy} \Gamma^1_0
\]
\[\text{for } v = 1, \ldots, \mu \]

and:
\[
\frac{\partial}{\partial t} \nabla^2 S_v + K^1_0 \nabla^2 \frac{\partial S_v}{\partial x} - \frac{d^2 K^1_0}{dy^2} \frac{\partial S_v}{\partial x} + N_v - \frac{1}{Re} \nabla^4 S_v + \lambda \frac{\partial^2 S_v}{\partial y^2} = C_v \approx 0
\]

where:
\[\lambda = \frac{M^2}{Re}\]

and the \( \Gamma^1_0 \), \( S_v \), and \( N_v \) are as defined above.

C. Expansion of the Stream Function

Even with the approximations listed thus far, the equations of motion (equations 7.2.21) are very complicated and thus it will be convenient to expand the solutions (the stream function \( S_v \)) on the independent variables \( x \) and \( t \). Since the independent variables range over the interval \((-\infty, \infty)\), there are only two reasonable ways of making such an expansion--Fourier Transform, or Hermite Functions. The Fourier Transform method may perhaps allow a simplification of the problem since convolutions do not appear, * but it seems more appropriate at this point to expand the \( S_v \) in terms of Hermite Functions.

---

*See Chapter VI, Section 2.
Let the Hermite Functions be defined as:
\[ h_n(\xi) = (-\frac{1}{2})^n \frac{d^n}{d\xi^n} e^{-\xi^2}, \quad n = 0,1,\ldots \]
These functions are orthogonal over the interval \((-\infty, \infty)\). The generating function is:
\[ e^{-\xi^2+2\xi \tau - \tau^2/2} = \sum_{n=0}^{\infty} \frac{(2\tau)^n}{n!} h_n(\xi) \]
and the normalizing integral is:
\[ \int_{-\infty}^{\infty} h_n(\xi) h_m(\xi) d\xi = \pi^{1/2} \frac{n!}{2^n} \]
In terms of the Hermite Functions, the \( S_v \) may be expanded as:
\[ S_v(x_1,t_1,\ldots,x_v,t_v;y) = \sum_{m,n} \phi_{m,n}(y) \prod_{j=1}^{v} h_m(x_j) h_n(t_j) \]
where the sum extends over all the indices \((n_1,\ldots,n_v;m_1,\ldots,m_v)\)
and the \( \phi(y) \) are integer functions of the index variables.
Consider, for example, the case for \( v = 1 \):
\[ S_1(x_1,t_1;y) = \sum_{m,n} \phi_{m,n}(y) h_m(x_1) h_n(t_1) \]
If the \( S_v \) are to be approximated by such a sum (i.e. the sum
does not run to \( \infty \)) then the \( \phi(y) \) must satisfy the same boundary
conditions as the \( S_v \) does; thus:
\[ \phi_{m,n}(y) = \frac{d}{dy} \phi_{m,n}(y) = 0 \quad \text{at} \quad y = \pm 1 \]
Furthermore, in terms of the stream function \( s \), the correlation
\( u_x u_y \) is given by:
\[ \frac{\partial u_x}{\partial y} = -\frac{\partial s}{\partial x} \cdot \frac{\partial s}{\partial y} \]
and a straightforward computation shows that the contribution
of \( S_1 \) to \( u_x u_y \) is:
\[ \sum_{m=0}^{M-1} \sum_{n=0}^{N} \frac{n!}{2^{m+1}} \cdot (m+1)! \cdot \left\{ \phi_{m,n}(y) \frac{d}{dy} \phi_{m+1,n} - \phi_{m+1,n} \frac{d}{dy} \phi_{m,n} \right\} \]
where M and N are the upper limits on the sum in equation 7.2.29 for the subscripts m and n respectively. The interesting thing here is the form of the dependence on y, viz:

\[
\phi \frac{d}{dy} \psi - \psi \frac{d}{dy} \phi
\]  

It can be shown that the higher order \( S_y \) contribute to the \( \overline{u_x u_y} \) correlation sums of terms of the same form. Such a dependence on y can be expressed as:

\[
\phi \frac{d}{dy} \psi - \psi \frac{d}{dy} \phi = (\psi)^2 \frac{d}{dy} (\phi) = - (\phi)^2 \frac{d}{dy} (\psi)
\]

Since all of the \( \phi \) in equation 7.2.29 must have a double order zero at \( y = \pm 1 \), it follows that the \( \overline{u_x u_y} \) correlation must have a fourth order zero at \( y = \pm 1 \). This is characteristic of the two-dimensional approximation and is probably an error, since the three-dimensional equations would not necessarily require a fourth order zero at the walls; more likely, the \( \overline{u_x u_y} \) correlation has a third order zero at the walls (see Chapter VI, Section 1). The principle effect of this error is that the predicted mean velocity will be larger than the measured values near the walls.

Some additional information about the \( \phi^y \) can be obtained from the symmetry conditions and the form of the \( \overline{u_x u_y} \) correlation. The symmetry conditions (see Chapter VI, Section 3) imply that for \( v \) even, all \( \phi^y \) are odd functions of \( y \). Thus, a Taylor series expansion around \( y = 0 \) for the even ordered \( \phi^y \) has the form

\[
\phi = a_1 y + a_3 y^3 + a_5 y^5 + \ldots
\]

and:

\[
\psi = b_1 y + b_3 y^3 + b_5 y^5 + \ldots
\]

so that the characteristic form of \( \overline{u_x u_y} \) is:

\[
\phi \frac{d}{dy} \psi - \psi \frac{d}{dy} \phi = 0 \cdot y + 2(a_1 b_3 - b_1 a_3) y^3 + 4(a_1 b_5 - b_1 a_5) y^5 + \ldots
\]
The important point here is that, whatever the coefficients $a_n$ and $b_n$ are, the linear term is missing from equation 7.2.36. Yet, the most characteristic feature of the $u_x u_y$ correlation is the very linear behavior over the central part of the channel (see Chapter VI, Section 1). Thus if the $\phi^\nu$ for $\nu$ odd were taken as odd functions of $y$, the $u_x u_y$ correlation would be totally missing its characteristic linear term near $y = 0$. It is therefore necessary to take $\phi^\nu$ for $\nu$ odd as even functions of $y$, for in this event:

$$\phi = a_0 + a_2 y^2 + a_4 y^4 + \ldots$$
$$\psi = b_0 + b_2 y^2 + b_4 y^4 + \ldots$$

and the characteristic form of $u_x u_y$ is:

$$\phi \frac{d}{dy} \psi - \psi \frac{d}{dy} \phi = 2(a_0 b_2 - b_0 a_2) y + 4(a_0 b_4 - b_0 a_4) y^3 + \ldots$$

Comparison of this result with the three-dimensional case makes it plausible to suppose that the $p_\nu$ of equations 6.3.18 are of the form:

$$p_\nu = \begin{cases} 
+ & \nu \text{ odd} \\
- & \nu \text{ even}
\end{cases}$$

but the argument is not altogether convincing.

3. A Simple Velocity Approximation

A. Assumption

Perhaps the simplest reasonable velocity approximation in terms of Orthogonal Polynomial Functionals may be obtained by the following set of assumptions:

1. $u^1 = G_0[K^i_0, \omega] + G_1[K^i_1, \omega]$

2. $K^i_1 = e^{ij} 3 \frac{\partial}{\partial x^j} S_1$; and $S_1$ is independent of $z = x^3$
3. \( S_1 = \left[ \frac{2}{\pi \Re} \right]^{1/2} \sum_{m} \sum_{n} \phi_{mn}(y)h_m(x)h_n(t) \)

and: \( \phi_{mn} = \frac{d}{dy} \phi_{mn} = 0 \) at \( y = \pm 1 \)

\( \phi_{mn} \) are even functions of \( y \)

4. \( \frac{d^2}{dy} \frac{K_0}{\Re} = -\Re f + \Re \frac{d}{dy} \int_{-\infty}^{\infty} f(x) \, dx \, dt \quad K_0^1 = K_0^2 \)

and: \( K_0^1 = 0 \) at \( y = \pm 1 \); \( K_0^2 = 1 \) at \( y = 0 \)

\( \frac{d}{dy} \frac{K_0}{\Re} = -\Re f \) at \( y = +1 \)

5. The appropriate equations to consider for the \( \phi_{mn} \) are:

\[
\left[ \frac{2}{\pi \Re} \right] \int_{-\infty}^{\infty} \int dx \, dt \, h_m(x)h_n(t)
\]

\[
\left\{ \frac{\partial}{\partial t} \nabla^2 S_1 + K_0 \nabla^2 \frac{\partial S_1}{\partial x} - \frac{d^2 K_0^1}{dy^2} \frac{\partial S_1}{\partial x} - \frac{1}{\Re} \nabla^4 S_1 \right\} = \pi \frac{m! n!}{2^{m+n}} \phi_{mn}(y)
\]

for \( m = 0,1; n = 0,1 \)

The first assumption is that the velocities are approximated by only the first two terms of the \( G \) series. This is the minimum number of terms which will allow the velocity to have both a mean and a random part. Since the random part is composed entirely of a \( G_1 \) functional, the probability distribution for the approximate velocity is Gaussian. The experimental evidence gathered by Laufer* indicates that while the velocity is certainly not entirely Gaussian, that part which is Gaussian (i.e. that part represented by the \( G_1 \) functional) is the dominant part.

The second assumption listed is essentially the two-dimensional model of turbulent Plane Poiseuille Flow described in the last section (q.v.).

*Especially the calculation of the "flatness factor," \( \frac{u_4}{(u_2)^2} \), see Laufer 10,11.
The third assumption listed is that the $S_1$ stream function is approximated by the first two terms in its expansion in terms of Hermite Functions. At least two terms (one of even order and one of odd order) in the expansion on $x$ are needed if the $u_x u_y$ correlation is to be non-zero, and at least two terms (again one even, one odd) are needed if the resolvent equations are to have terms corresponding to the time derivative in the original equations. In the interest of simplicity, a minimum number of terms are taken. The boundary conditions on the $\phi_{mn}$ come directly from the boundary conditions on the stream function $S_1$, and the symmetry conditions on the $\phi_{mn}$ come from the requirement that $\frac{d}{dy} u_x u_y \neq 0$ at $y = 0$.

The fourth assumption listed is that the averaged equations of motion hold exactly. Thus, $K_o^1$ is defined in terms of the $K_1^1$ (and similarly the $y$ dependence of the pressure gradient could be calculated if this were of interest). The conditions on $K_o^1$ at $y = 0$ and $y = 1$ are simply a result of the boundary and normalizing conditions, and $f$ (the friction factor) is then defined in terms of $K_o^1$ consistent with these conditions.

The fifth assumption actually consists of two separate assumptions. First, since the approximate velocity consists of the first two terms in the $G$ series expansion, this approximation will not identically satisfy the equations of motion, i.e.

$$\frac{\partial u_i^1}{\partial t} + \frac{\partial}{\partial x} u_i^1 u_j^1 - \frac{1}{Re} \nabla^2 u_i^1 + \frac{\partial p_i^1}{\partial x} = c_i^1 = 0$$

7.3.1

and $c_i^1$ is expressible as:

$$c_i^1 = G_o^1[G_0^1, \omega] + G_1^1[G_1^1, \omega] + G_2^1[G_2^1, \omega]$$

7.3.2

The assumption involved here is that, since only the $G_o^1$ and $G_1^1$ terms appear in $u_i^1$, only the first two terms in the expansion of $c_i^1$ are significant. However, since the averaged equations are
satisfied identically, $C_0^1 = 0$. Thus, the significant part of the error is contained in the term $C_1^1$. In terms of the stream function $S_1$, the pertinent equation is then:

$$\frac{\partial}{\partial t} \nabla^2 S_1 + K_0^1 \nabla^2 \frac{\partial S_1}{\partial x} = \frac{d^2}{dy^2} \frac{\partial S_1}{\partial x} - \frac{1}{Re} \nabla^4 S_1 = e^{ij3} \frac{\partial C_1^1}{\partial x} = D_1$$ \hspace{1cm} 7.3.3

The second part of the assumption is much like the first. The error $D_1$ in the above equation may be expanded as:

$$D_1 = \left[ \frac{2}{Re} \right]^{1/2} \sum_{m,n} d_{m,n} (y) h_m(x) h_n(y)$$ \hspace{1cm} 7.3.4

where the sum is finite. Since, in the expansion of $S_1$ in terms of Hermite Functions on both $x$ and $t$, only the $h_0$ and $h_1$ terms are retained; it follows that the significant $d_{m,n}$ in the expansion of $D_1$ are those for $m = 0,1$ and $n = 0,1$.

The above approximations are subject to some question. If such approximations were to be used in the analysis of a device, then considerably more justification would be required. However, for a first investigation, the assumptions involved do not seem unreasonable.

**B. The Approximate Equations**

In terms of the expansion of $S_1$ described above, the $u_x u_y$ correlation becomes:

$$\frac{u_x u_y}{\Gamma_o} = \frac{2}{Re} \frac{\pi}{2} \left[ (\phi_{o,0} \frac{d}{dy} \phi_{1,0} - \phi_{1,0} \frac{d}{dy} \phi_{o,0}) + \frac{1}{2} (\phi_{o,1} \frac{d}{dy} \phi_{1,1} - \phi_{1,1} \frac{d}{dy} \phi_{o,1}) \right]$$ \hspace{1cm} 7.3.5

and the equations for the $d_{m,n}(y)$ defined in equation 7.3.4 are:

$$\frac{1}{2}[D^2 - \frac{1}{2}]\phi_{o,1} - \frac{1}{Re} [D^4 - D^2 + \frac{3}{4}] \phi_{o,o} + \frac{1}{2}[K_0^1(D^2 - \frac{3}{2}) \phi_{1,0} - \phi_{1,1} \frac{d}{dy} \phi_{o,1}] = d_{o,o}$$ \hspace{1cm} 7.3.6a
\[-\left[ \frac{D^2}{2} - \frac{1}{2} \right] \phi_{o,0} - \frac{1}{Re} \left[ D^4 - D^2 + \frac{3}{4} \right] \phi_{o,1} + \frac{1}{2} \left[ K^1_o (D^2 - \frac{3}{2}) \phi_{1,1} - \phi_{o,0} D^2 K^1_o \right] = d_{o,1} \]  
7.3.6b

\[ \frac{1}{2} \left[ D^2 - \frac{3}{2} \right] \phi_{1,1} - \frac{1}{Re} \left[ D^4 - 3D^2 + \frac{15}{4} \right] \phi_{1,0} - \left[ K^1_o (D^2 - \frac{3}{2}) \phi_{o,0} - \phi_{o,0} D^2 K^1_o \right] = d_{1,0} \]  
7.3.6c

\[-\left[ D^2 - \frac{3}{2} \right] \phi_{1,0} - \frac{1}{Re} \left[ D^4 - 3D^2 + \frac{15}{4} \right] \phi_{1,1} - \left[ K^1_o (D^2 - \frac{3}{2}) \phi_{o,1} - \phi_{o,1} D^2 K^1_o \right] = d_{1,1} \]  
7.3.6d

where:
\[ D = \frac{d}{dy} \]  
7.3.7

Equations 7.3.6 yield some interesting information about the behavior of the \( \phi_{m,n} \) near \( y = \pm 1 \). Since the terms involving \( K^1_o \) go to zero at the walls, the behavior near the walls is characterized approximately by the homogeneous solution of:

\[
\begin{bmatrix}
\frac{1}{Re} (D^4 - D^2 + \frac{3}{4}) & -\frac{1}{2} (D^2 - \frac{1}{2}) \\
(D^2 - \frac{3}{2}) & \frac{1}{Re} (D^4 - D^2 + \frac{3}{4})
\end{bmatrix}
\begin{bmatrix}
\phi_{o,0} \\
\phi_{o,1}
\end{bmatrix} = 0
\]  
7.3.8

and:

\[
\begin{bmatrix}
\frac{1}{Re} (D^2 - 3D^2 + \frac{15}{4}) & -\frac{1}{2} (D^2 - \frac{3}{2}) \\
(D^2 - \frac{3}{2}) & \frac{1}{Re} (D^4 - 3D^2 + \frac{15}{4})
\end{bmatrix}
\begin{bmatrix}
\phi_{1,0} \\
\phi_{1,1}
\end{bmatrix} = 0
\]  
7.3.9

The above matrix operators can be factored approximately (with an accuracy on the order of \( \frac{1}{Re} \)) to give:

\[
\begin{bmatrix}
D^2 - \frac{1}{2} & 0 \\
0 & D^2 - \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{Re} (D^2 - \frac{1}{2}) & -\frac{1}{2} \\
\frac{1}{Re} (D^2 - \frac{1}{2}) & \frac{1}{Re} (D^2 - \frac{1}{2})
\end{bmatrix}
\begin{bmatrix}
\phi_{o,0} \\
\phi_{o,1}
\end{bmatrix} = 0
\]  
7.3.10

and:
\[
\begin{bmatrix}
D^2 - \frac{3}{2} & 0 \\
0 & D^2 - \frac{3}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{Re}(D^2 - \frac{3}{2}) & -\frac{1}{2} \\
1 & \frac{1}{Re}(D^2 - \frac{3}{2})
\end{bmatrix}
\begin{bmatrix}
\phi_{1,0} \\
\phi_{1,1}
\end{bmatrix} = 0
\]

Thus, the behavior of \(\phi_{0,0}\) and \(\phi_{0,1}\) near \(y = +1\) is characterized by the forms:

\[
cosh(\sigma_0 y) \text{ and } \exp(\gamma_0(1-y)) \cdot \begin{cases}
\cos \\
\sin
\end{cases}
(\omega_0(1-y))
\]

where:

\[
\sigma_0 = \left(\frac{1}{2}\right)^{1/2}
\]

\[
\gamma_0 = \left\{\frac{1}{2}\frac{Re^2}{2} + \frac{1}{4}\right\}^{1/2}
\]

\[
\omega_0 = \left\{\frac{1}{2}\frac{Re^2}{2} + \frac{1}{4}\right\}^{1/2}
\]

and the behavior of \(\phi_{1,0}\) and \(\phi_{1,1}\) near \(y = \pm 1\) is characterized by the forms:

\[
cosh(\sigma_1 y) \text{ and } \exp(\gamma_1(1-y)) \cdot \begin{cases}
\cos \\
\sin
\end{cases}
(\omega_1(1-y))
\]

where:

\[
\sigma_1 = \left(\frac{3}{2}\right)^{1/2}
\]

\[
\gamma_1 = \left\{\frac{1}{2}\frac{Re^2}{2} + \frac{9}{4}\right\}^{1/2}
\]

\[
\omega_1 = \left\{\frac{1}{2}\frac{Re^2}{2} + \frac{9}{4}\right\}^{1/2}
\]

For \(Re = 12300.0\), the exponential constants are:

\[
\gamma_0 \approx 65.946 \ 620 \quad \gamma_1 \approx 65.950 \ 411
\]

\[
\omega_0 \approx 65.942 \ 829 \quad \omega_1 \approx 65.939 \ 038
\]

Thus, the behavior of the \(\phi_{m,n}\) near \(y = \pm 1\) is characterized by an exponential behavior which decays very rapidly as the distance from the wall is increased.
C. Report on an Iteration Method

It is conceivable that equations 7.3.6 with \( d_{m,n} \) set to zero might have a non-trivial solution, because the equations are non-linear, i.e. \( K_o^1 \) depends on the \( \phi_{m,n} \). In order to see if indeed there is a non-trivial solution to equations 7.3.6 and, if one did exist, to calculate it, a numerical iteration technique was established for the equations and a long series of computations were performed on the IBM 7090 EDPM at the M.I.T. Computation Center. The equations used were:

\[
\frac{1}{2}( D^2 - \frac{1}{2} ) \phi_{o,1} - \frac{1}{Re} ( D^2 - \frac{1}{2} )^2 \phi_{o,1} = -\frac{1}{2} [ K_o^1 ( D^2 - \frac{3}{2} ) \phi_{1,0} - \phi_{1,0} D^2 K_o^1 ]
\]

\[
-( D^2 - \frac{1}{2} ) \phi_{o,0} - \frac{1}{Re} ( D^2 - \frac{1}{2} )^2 \phi_{o,1} = -\frac{1}{2} [ K_o^1 ( D^2 - \frac{3}{2} ) \phi_{1,0} - \phi_{1,0} D^2 K_o^1 ]
\]

\[
\frac{1}{2}( D^2 - \frac{3}{2} ) \phi_{1,1} - \frac{1}{Re} ( D^2 - \frac{3}{2} )^2 \phi_{1,0} = [ K_o^1 ( D^2 - \frac{3}{2} ) \phi_{1,0} - \phi_{1,0} D^2 K_o^1 ]
\]

\[
-( D^2 - \frac{3}{2} ) \phi_{1,0} - \frac{1}{Re} ( D^2 - \frac{3}{2} )^2 \phi_{1,1} = [ K_o^1 ( D^2 - \frac{3}{2} ) \phi_{1,0} - \phi_{1,0} D^2 K_o^1 ]
\]

\[
DK_o^1 = -Re_{efy} + Re \Gamma_o^{12}
\]

\[
Re^{12} = ( \phi_{o,0} D \phi_{1,0} - \phi_{1,0} D \phi_{o,0} ) + \frac{1}{2} ( \phi_{o,1} D \phi_{1,1} - \phi_{1,1} D \phi_{o,0} )
\]

The iteration was performed as follows: Starting with a given set of \( \phi_{m,n} \) the \( K_o^1 \) was calculated according to equations 7.3.18 and 7.3.19. Then the non-linear terms on the right of equations 7.3.17 were calculated, and from this an "improved" set of \( \phi_{m,n} \) (the \( \phi_{m,n} \)) were calculated. This process was continued until convergence was achieved.

Indeed convergence was achieved in every case. For every set of initial \( \phi_{m,n} \) tried, the sequence of \( \phi_{m,n} \) generated tended uniformly and quickly to zero. Thus, the iteration method failed--not a totally unexpected result, considering the crude nature of the approximations.
Considerably more success was had with variational methods, but a discussion of these results is deferred to Section 6 of this chapter.

D. The Magneto-Fluid Dynamic Case

A simple approximate velocity for the magneto-fluid dynamic case of turbulent Plane Poiseuille Flow by the following set of assumptions:

1. \( u = G_0 \left[ K_0^1, \omega \right] + G_1 \left[ K_1^1, \omega \right] \)

2. \( K_1^1 = e^{ij3} \frac{\partial}{\partial x_j} S_1 \), and \( S_1 \) is independent of \( z = x^3 \)

3. \( S_1 = \left[ \frac{2}{\pi R e} \right] \sum_m \sum_n \phi_{mn} (y) h_m (x) h_n (t) \)

and: \( \phi_{mn} = \frac{d}{dy} \phi_{mn} = 0 \), at \( y = \pm 1 \)

\( \phi_{mn} \) are even functions of \( y \).

4. \( \left[ -\frac{d^2}{dy^2} - M^2 \right] K_0^1 = -R e C_a + R e \frac{d}{dy} \int_2^\infty \int dx \ dt \ K_1^1 K_1^2 \)

and: \( C_a \) is such that \( K_0^1 (0) = 1; K_0^1 (\pm 1) = 0 \)

then: \( -R e f = \left. \frac{dK_0}{dy} \right|_{y = \pm 1} \)

5. The appropriate equations to consider for the \( \phi_{mn} \) are:

\[ \left\{ \frac{\partial}{\partial t} \sqrt{2} S_1 + K_0^1 \sqrt{2} \frac{\partial S_1}{\partial x} - \frac{d^2 K_0^1}{dy^2} \frac{\partial S_1}{\partial x} - \frac{1}{R e} \sqrt{4} S_1 + \lambda \frac{\partial^2 S_1}{\partial y^2} \right\} \]

\[ = \pi \frac{m!n!}{2^m n} \ m \ (y) \ \text{for} \ m = 0, 1; \ n = 0, 1; \ \lambda = \frac{M^2}{R e} \]

These assumptions are completely analogous to those described above for the ordinary hydrodynamics case (see Subsection A of this section) and in fact reduce to that case if \( M \), (the
Hartman number) is zero.

The equations for the \( d_{mn}(y) \) resulting from assumption 5a are as follows:

\[
\frac{1}{2}[D^2 - \frac{1}{2}]\phi_{o,0} - \frac{1}{Re_0}[D^4 - D^2 + \frac{3}{4}]\phi_{o,0} + \lambda D^2 \phi_{o,1} + \frac{1}{2}[K_0^1(D^2 - \frac{3}{2})]\phi_{1,0} - \phi_{1,1}D^2K_0^1 = d_{o,0} \tag{7.3.20a}
\]

\[
-D^2 - \frac{1}{2}]\phi_{o,0} - \frac{1}{Re_0}[D^4 - D^2 + \frac{3}{4}]\phi_{o,1} + \lambda D^2 \phi_{o,1} - \frac{1}{2}[K_0^1(D^2 - \frac{3}{2})]\phi_{1,1}D^2K_0^1 = d_{o,1} \tag{7.3.20b}
\]

\[
\frac{1}{2}[D^2 - \frac{3}{2}]\phi_{1,0} - \frac{1}{Re_0}[D^4 - 3D^2 + \frac{15}{4}]\phi_{1,0} + \lambda D^2 \phi_{1,0} - \frac{1}{2}[K_0^1(D^2 - \frac{3}{2})]\phi_{1,0}D^2K_0^1 = d_{1,0} \tag{7.3.20c}
\]

\[
-D^2 - \frac{3}{2}]\phi_{1,1} - \frac{1}{Re_0}[D^4 - 3D^2 + \frac{15}{4}]\phi_{1,1} + \lambda D^2 \phi_{1,1} - \frac{1}{2}[K_0^1(D^2 - \frac{3}{2})]\phi_{1,0}D^2K_0^1 = d_{1,1} \tag{7.3.20d}
\]

where: \( D = \frac{d}{dy} \)

As for the ordinary hydrodynamic case, the behavior of the \( \phi_{mn} \) near the walls is characterized approximately by the homogeneous solutions of:

\[
\begin{align*}
\left\{ \begin{array}{c}
\frac{1}{Re} (D^4 - D^2 + \frac{3}{4}) - \lambda D^2 \\
(D^2 - \frac{1}{2})
\end{array} \right\} \phi_{o,0} &= 0 \\
\frac{1}{Re} (D^4 - D^2 + \frac{3}{4}) - \lambda D^2 \\
(D^2 - \frac{1}{2})
\end{align*} \tag{7.3.22}
\]

and:

\[
\begin{align*}
\left\{ \begin{array}{c}
\frac{1}{Re} (D^4 - 3D^2 + \frac{15}{4}) - \lambda D^2 \\
(D^2 - \frac{3}{2})
\end{array} \right\} \phi_{1,0} &= 0 \\
\frac{1}{Re} (D^4 - 3D^2 + \frac{15}{4}) - \lambda D^2 \\
(D^2 - \frac{3}{2})
\end{align*} \tag{7.3.23}
\]

The above matrix operators can be factored approximately (with an accuracy on the order of \( \frac{1}{Re} \)) to give:
\[
\left\{ \begin{align*}
\left[ (D^2 - \theta_m) + \lambda q_m \right] - \frac{1}{2} q_m \\
q_m \\
\left[ (D^2 - \theta_m) + \lambda q_m \right]
\end{align*} \right\} \\
\left\{ \begin{align*}
\frac{1}{Re} [(D^2 - \theta_m) - M^2 - \lambda q_m] - \frac{1}{2} \left[ 1 - \frac{q_m}{Re} \right] \\
[1 - \frac{q_m}{Re}] \frac{1}{Re} \left[ (D^2 - \theta_m) - M^2 - \lambda q_m \right]
\end{align*} \right\} \phi_{m, 0} = 0
\]

where:

\[
q_m(\lambda) = 2\theta_m \frac{\lambda}{1 + 2\lambda^2}; \quad \lambda = \frac{M^2}{Re}
\]

and:

\[
\phi_0 = \frac{1}{2}, \quad \phi_1 = \frac{3}{2}
\]

Thus, the behavior of \(\phi_{m, 0}\) and \(\phi_{m, 1}\) near \(y = +1\) is characterized by the forms:

\[
\cosh(\sigma_m y) \cos(\tau_m y), \sinh(\sigma_m y) \sin(\tau_m y),
\]

and:

\[
\exp(\gamma_m (1-y)) \cdot \left\{ \begin{align*}
\cos(\omega_m (1-y)) \\
\sin(\omega_m (1-y))
\end{align*} \right\}
\]

where the \(\sigma_m\), \(\tau_m\), \(\gamma_m\), and \(\omega_m\) may be calculated from:

\[
(\sigma_m)^2 - (\tau_m)^2 = \theta_m - \lambda q_m
\]

\[
2\sigma_m \tau_m = \frac{\sqrt{2}}{2} q_m
\]

\[
(\gamma_m)^2 - (\omega_m)^2 = \theta_m + M^2 + \lambda q_m
\]

\[
2\gamma_m \omega_m = \frac{\sqrt{2}}{2} [R_e - q_m]
\]

For comparison, a short table of the values of these exponential constants for \(m = 0\), \(R_e = 12300.0\), as a function of \(M^2\) is given in Table 7.1.
<table>
<thead>
<tr>
<th>$M^2$</th>
<th>$\lambda$</th>
<th>$q(\lambda)$</th>
<th>$\sigma_0$</th>
<th>$\tau_0$</th>
<th>$\gamma_0$</th>
<th>$\omega_0$</th>
</tr>
</thead>
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<tr>
<td>0.0</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>65.946620</td>
<td>65.942829</td>
</tr>
<tr>
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<td>0.000100</td>
<td>0.707107</td>
<td>0.00050</td>
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</tr>
<tr>
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<td>0.0010</td>
<td>0.000998</td>
<td>0.707106</td>
<td>0.00499</td>
<td>65.991561</td>
<td>65.897921</td>
</tr>
<tr>
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<td>0.0100</td>
<td>0.009804</td>
<td>0.707054</td>
<td>0.004902</td>
<td>66.412850</td>
<td>65.479894</td>
</tr>
<tr>
<td>1230.0</td>
<td>0.1000</td>
<td>0.083333</td>
<td>0.647101</td>
<td>0.045530</td>
<td>70.768911</td>
<td>61.449352</td>
</tr>
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<td>1.0000</td>
<td>0.333333</td>
<td>0.477149</td>
<td>0.246990</td>
<td>116.971150</td>
<td>37.177500</td>
</tr>
</tbody>
</table>

Table 7.1

Examination of this table reveals that the behavior of the $\phi_{o,n}$ near the wall is essentially unchanged until extremely large values of $M^2$ are reached. The center channel behavior is altered much before this however, because the mean velocity is considerably changed for values of $M$ as low as 2.0 or 3.0.

4. An Improved Velocity Approximation

There are a number of ways of improving the simple velocity approximation discussed in the last section. Three such improvements are: 1. approximate the velocity as a sum of the first three $G$ functionals rather than the first two; 2. treat the full three-dimensional problem rather than the two-dimensional model; and, 3. in the expansion of the $K^1_v$ include more than just the first two Hermite Functions.

It is felt that all of the above improvements (and probably others too) should be made and the appropriate calculations performed. Such calculations will be extremely complicated and certainly cannot be done by hand—indeed, such computations performed on a modern, large-scale digital machine (e.g. the 7090) would be severely limited in efficiency by available memory size.

Experience with the simple approximation presented in the last section indicates that the most limiting factor is the small number of terms retained in the expansion of $S_1$ in terms of Hermite
Functions on \( x \) and \( t \). Thus, a somewhat improved velocity approximation may be obtained by modifying this approximation of the last section (Subsection A) to the extent of retaining the first four terms of the expansion of \( S_1 \) in terms of Hermite Functions, i.e.:

\[
S_1 = \left[ \frac{2}{\pi R_e} \right]^{1/2} 3 \sum_{m=0}^{3} \sum_{n=0}^{3} \phi_{mn}(y)h_m(x)h_n(t)
\]

7.4.1

Thus, the number of \( \phi_{mn} \) is 16, quadruple the number in the simple approximation.

To simplify notation a bit, let:

\[
\psi_m(y,t) = \sum_{n=0}^{3} \phi_{mn}(y)h_n(t)
\]

then, the \( \bar{u}_{xy} \) correlation becomes:

\[
\bar{u}_{xy} = \frac{12}{R_e} \int_0^{\infty} \left\{ \psi_0 D\psi_1 - \psi_1 D\psi_0 \right\} + \frac{3}{2} [\psi_2 D\psi_3 - \psi_3 D\psi_2] \, dt
\]

7.4.3

where:

\[
D = \frac{\partial}{\partial y}
\]

7.4.4

Moreover, if:

\[
D_1 = \Sigma_{m,n} d_{mn}(y)h_m(x)h_n(t) \equiv \Sigma \Delta_m(y,t)h_m(x)
\]

(see Subsection A of the preceding section for the definition and discussion of \( D_1 \)), then the equations for the \( \Delta_m \) may be written in the compact notation:

\[
\frac{\partial}{\partial t}[B_e \psi_e] - \frac{1}{R_e}[A_e \psi_e] + H_e = \Delta_e
\]

7.4.6a

\[
\frac{\partial}{\partial t}[B_o \psi_o] - \frac{1}{R_e}[A_o \psi_o] + H_o = \Delta_o
\]

7.4.6b

where:

\[
\psi_e = \begin{bmatrix} \psi_0 \\ \psi_2 \end{bmatrix}, \quad \psi_o = \begin{bmatrix} \psi_1 \\ \psi_3 \end{bmatrix}
\]

7.4.7a

7.4.7b
\[
[B_e] = \begin{bmatrix}
D^2 - \frac{1}{2} & \frac{1}{2} \\
1 & D^2 - \frac{5}{2}
\end{bmatrix} \\
[A_e] = \begin{bmatrix}
(D^4 - D^2 + \frac{3}{4}) & (D^2 - \frac{3}{2}) \\
2(D^2 - \frac{3}{2}) & (D^4 - 5D^2 + \frac{39}{4})
\end{bmatrix}
\]
\[
[B_o] = \begin{bmatrix}
D^2 - \frac{3}{2} & \frac{3}{2} \\
1 & D^2 - \frac{7}{2}
\end{bmatrix} \\
[A_o] = \begin{bmatrix}
(D^4 - 3D^2 + \frac{15}{4}) & 3(D^2 - \frac{5}{2}) \\
2(D^2 - \frac{5}{2}) & (D^4 - 7D^2 + \frac{75}{4})
\end{bmatrix}
\]
\[
H_e = (K_o) \cdot \begin{bmatrix}
\frac{1}{2}[(D^2 - \frac{3}{2}) \psi_1 + \frac{3}{2} \psi_3] \\
3[(D^2 - \frac{9}{2}) \psi_3 - \frac{2}{3}(D^2 - 3) \psi_1]
\end{bmatrix}
\]
\[
H_o = -(K_o) \cdot \begin{bmatrix}
[(D^2 - \frac{3}{2}) \psi_1 - (D^2 - 3) \psi_2] \\
(D^2 - \frac{9}{2}) \psi_2 + \psi_0
\end{bmatrix}
\]
\[
\frac{1}{2}[\psi_1] \\
\frac{3}{2}[\psi_3 - \frac{2}{3} \psi_1]
\]

\[
H_e = (K_o) \cdot \begin{bmatrix}
\frac{1}{2}[\psi_1] \\
\frac{3}{2}[\psi_3 - \frac{2}{3} \psi_1]
\end{bmatrix}
\]
\[
H_o = -(K_o) \cdot \begin{bmatrix}
[(D^2 - \frac{3}{2}) \psi_1 - (D^2 - 3) \psi_2] \\
(D^2 - \frac{9}{2}) \psi_2 + \psi_0
\end{bmatrix}
\]

\[
[A_e] = [B_e]^2 + \begin{bmatrix}
0 & 0 \\
0 & 3
\end{bmatrix}
\]
\[
[A_o] = [B_o]^2 + \begin{bmatrix}
0 & 0 \\
0 & 5
\end{bmatrix}
\]

and:
\[
\Delta_e = \begin{bmatrix}
\Delta_0 \\
\Delta_1 \\
\Delta_2
\end{bmatrix} \\
\Delta_o = \begin{bmatrix}
\Delta_0 \\
\Delta_1 \\
\Delta_3
\end{bmatrix}
\]

It is a very convenient piece of luck that:
\[
[A_e] = [B_e]^2 + \begin{bmatrix}
0 & 0 \\
0 & 3
\end{bmatrix}
\]
and:
\[
[A_o] = [B_o]^2 + \begin{bmatrix}
0 & 0 \\
0 & 5
\end{bmatrix}
\]

so that equations 7.4.6 may be closely approximated by:
\[
[B_e] \left\{ \frac{\partial}{\partial t} - \frac{1}{R_e} [B_e] \right\} \psi_e + H_e = \Delta_e
\]
and:
\[
[B_o] \left\{ \frac{\partial}{\partial t} - \frac{1}{R_e} [B_o] \right\} \psi_o + H_o = \Delta_o
\]
Thus, the behavior of the $\phi_{un}$ is characterized by the homogeneous solutions of:

$$\begin{bmatrix} [B_e] \end{bmatrix} \left\{ \frac{\partial}{\partial t} - \frac{1}{Re} [B_e] \right\} \psi_e = 0$$ \hspace{1cm} 7.4.14a

and:

$$\begin{bmatrix} [B_o] \end{bmatrix} \left\{ \frac{\partial}{\partial t} - \frac{1}{Re} [B_o] \right\} \psi_o = 0$$ \hspace{1cm} 7.4.14b

The calculations will be carried through for the $\psi_e$ (similar results hold for $\psi_o$) to illustrate how matrix operator methods can considerably reduce the amount of labor involved here:

$$\pi_n(y) = \begin{bmatrix} \phi_{o,n} \\ \phi_{2,n} \end{bmatrix}$$ \hspace{1cm} 7.4.15

then:

$$\psi_e = \sum_{n=0}^{3} \pi_n h_n(t)$$ \hspace{1cm} 7.4.16

and the appropriate four equations for characterizing the behavior of the $\pi_n$ near the walls are:

$$\int_{-\infty}^{\infty} dt \ h_n(t) \cdot \left\{ \begin{bmatrix} [B_e] \end{bmatrix} \left\{ \frac{\partial}{\partial t} - \frac{1}{Re} [B_e] \right\} \psi_e \right\} = 0; \ n = 0, 1, 2, 3$$ \hspace{1cm} 7.4.17

which may be written:

$$\begin{bmatrix} -\frac{1}{Re} [B_e] & \frac{1}{2} [I] & 0 & 0 \\ -[I] & -\frac{1}{Re} [B_e] & [I] & 0 \\ 0 & -[I] & -\frac{1}{Re} [B_e] & \frac{3}{2} [I] \\ 0 & 0 & -[I] & -\frac{1}{Re} [B_e] \end{bmatrix} \begin{bmatrix} [B_e] \pi_o \\ [B_e] \pi_1 \\ [B_e] \pi_2 \\ [B_e] \pi_3 \end{bmatrix} = 0$$ \hspace{1cm} 7.4.18

where: $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ \hspace{1cm} 7.4.19

Thus the solutions for the $\pi_n$ satisfy one of the following two equations:
\[
[B_e] \pi_n = 0
\]

or:

\[
\begin{bmatrix}
-\frac{1}{Re} [B_e] & \frac{1}{2} [I] & 0 & 0 & \pi_0 \\
-[I] & -\frac{1}{Re} [B_e] & [I] & 0 & \pi_1 \\
0 & -[I] & -\frac{1}{Re} [B_e] & \frac{3}{2} [I] & \pi_2 \\
0 & 0 & -[I] & -\frac{1}{Re} [B_e] & \pi_3
\end{bmatrix} = 0
\]

7.4.21

The first equation immediately yields the characteristic forms:

\[
\cosh(\sigma y) ; \sigma = \frac{3 \pm \sqrt{6}}{2}
\]

7.4.22

The solutions to the second equation may be obtained as follows:

Let:

\[
[C] = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
1 & -\frac{5}{2} \\
D^2 & 0 \\
0 & D^2
\end{bmatrix}
\]

7.4.23

then:

\[
[B_e] = [C] + [C]
\]

7.4.24

Then, let \( e \) be an eigenvector of \([C]\) with corresponding eigenvalue \( \mu \). The possible values of \( \mu \) are:

\[
\mu = -3 \pm \sqrt{6}
\]

7.4.25

Then, to find the characteristic form of the \( \pi_n \), let:

\[
\pi_n = e^{\beta y} \cdot e
\]

7.4.26

and:

\[
\alpha = -\frac{\beta^2 + \mu}{Re}
\]

7.4.27

Thus, if \( \pi_n \) is a solution of equation 7.4.21, then the determinant:
must be zero, and therefore:
\[ \alpha^4 + 3\alpha^2 + \frac{3}{4} = 0 \] 7.4.29

For \( R_e = 12300.0 \), the primary solutions for \( \beta \) are then found to be:

\[
\begin{align*}
(80.332 & \, 455) + i(80.330 & \, 742) \\
(80.340 & \, 079) + i(80.323 & \, 120) \\
(142.490 & \, 362) + i(142.489 & \, 396) \\
(142.494 & \, 659) + i(142.485 & \, 098)
\end{align*}
\] 7.4.30a

and the other solutions for \( \beta \) are found by noting that if \( \beta \) is a solution, then so are \( \beta^* \) (conjugate), \(-\beta\), and \(-\beta^*\). Since the \( \beta \) are grouped in pairs with the members of each pair being very close in numerical value, the characteristic forms of the homogeneous solutions for the \( \phi_{mn} \) are approximately:

\[
\text{cosh}(\sigma y), \quad \text{exp}(\gamma(1-y)) \left\{ \frac{\sin(\omega(1-y))}{\cos(\omega(1-y))} \right\}
\] 7.4.31a, b

where:
\[
\begin{align*}
\sigma & \approx 2.725 \quad \text{or} \quad 0.275 \\
\gamma & \approx 80.3 \quad \text{or} \quad 142.5 \\
\omega & \approx 80.3 \quad \text{or} \quad 142.5
\end{align*}
\] 7.4.32a, b, c, d, e, f

Thus, the exponential behavior of the \( \phi_{mn} \) near \( y = +1 \) is characterized by constants somewhat higher than those of the simple velocity approximation.

5. Results of Numerical Computations

The approximate velocities described in the last two sections have been the subject of a great amount of numerical computation. For the most part, variational methods were used, the single
exception being an iteration technique (see Section 3C) which
did not work.

Formulation of an effective variational technique requires
the definition of a suitable error criterion as well as an
approximation to the $\phi_{mn}$ appearing in:

$$S_1 = \frac{2}{\pi R^2_e} \sum_{m} \sum_{n} \phi_{mn}(y) h_m(x) h_n(t)$$

A. The Error

For the sake of discussion, consider the simple velocity
approximation described in Section 3B. The appropriate
equations* for the $\phi_{mn}$ (equations 7.3.6) are:

$$\frac{1}{2} [D^2 - \frac{1}{2}] \phi_{0,0} - \frac{1}{Re_d} [D^2 - \frac{1}{2}]^2 \phi_{0,0} + \frac{1}{2} [K_0^1 (D^2 - \frac{3}{2}) \phi_{1,0} + \phi_{1,0} D^2 K_0^1] = d_{o,o}$$

7.5.1a

$$\frac{1}{2} [D^2 - \frac{3}{2}] \phi_{1,1} - \frac{1}{Re_d} [D^2 - \frac{3}{2}]^2 \phi_{1,1} + \frac{1}{2} [K_0^1 (D^2 - \frac{3}{2}) \phi_{1,1} - \phi_{1,1} D^2 K_0^1] = d_{1,1}$$

7.5.1b

$$\frac{1}{2} [D^2 - \frac{3}{2}] \phi_{1,0} - \frac{1}{Re_d} [D^2 - \frac{3}{2}]^2 \phi_{1,0} + \frac{1}{2} [K_0^1 (D^2 - \frac{3}{2}) \phi_{1,0} - \phi_{1,0} D^2 K_0^1] = d_{1,0}$$

7.5.1c

and, changing notation a bit, the equations for $K_0^1$ are:

let:

$$\beta(r) = \frac{1}{Re_d} r^2 = [\phi_{0,0} D \psi_{0,0} - \phi_{1,0} D \phi_{0,0}] + \frac{1}{2} [\phi_{0,1} D \psi_{1,1} - \phi_{1,1} D \phi_{0,1}]$$

7.5.2

*The terms involving the Reynolds number, e.g. $\frac{1}{Re_d} [D^2 - \frac{1}{2}]^2 \phi_{0,0}$ in
equations 7.5.1 have been approximated (to order $\frac{1}{Re_d}$) for
simplicity. See Section 3B.
where:
\[
\beta = R \frac{d}{dy} \Gamma_0 \bigg|_{y=0}^{12} ; \text{i.e. } \frac{dr}{dy} \bigg|_{y=0} = 1
\]

then:
\[
D_{10} = -R_f y + \beta r(y)
\]

Now the most obvious way of formulating an error criterion for these equations (i.e. the first way tried) is as follows:

let:
\[
e \sim \int_{-1}^{+1} dy \ E \left\{ G_1(D_1) \cdot G_1[D_1] \right\}
\]

where:
\[
D_1 = \sum_{m}^1 \sum_{n}^1 d_{m,n}(y) h_m(x) h_n(t)
\]

thus:
\[
e = \int_{-1}^{+1} dy \left\{ (d_{0,0})^2 + \frac{1}{2}(d_{0,1})^2 + \frac{1}{2}(d_{1,0})^2 + \frac{3}{4}(d_{1,1})^2 \right\}
\]

The error is then \( e \), and this is an appropriate measure of the goodness of the velocity approximation to the extent that it measures how closely the velocities fit the equations. This measure of the error has two unsatisfactory features however. First, the solution \( \phi_{mn} = 0 \) will always cause \( e \) to be zero, its minimum value. Thus, when minimizing \( e \) numerically, it might be difficult to distinguish sequences converging to a turbulent (i.e. non-zero) solution from sequences converging to the laminar solution. Secondly, the high order derivatives existing in the definitions of the \( d_{mn} \) will cause the behavior of the error in the \( \phi_{mn} \) near the walls (\( y = \pm 1 \)) to dominate. Since it is felt that the approximation of the \( \phi_{mn} \) will be more accurate over the central part of the channel, dominance of the error near the walls may lead to spurious results.

To eliminate the above-mentioned difficulties the error may be reformulated as follows:
Redefine the $d_{mn}$ as follows:

\[
\frac{1}{2} \phi_{0,0} - \frac{1}{R_e} [D^2 - \frac{1}{2}] \phi_{0,0} + [D^2 - \frac{1}{2}]^{-1} \left\{ \frac{1}{2} [K^1_0(D^2 - \frac{3}{2}) \phi_{1,0} - \frac{1}{2} - \phi_{1,1} D^2 K^1_0] \right\} = d_{0,0}
\]

\[
-\phi_{0,0} - \frac{1}{R_e} [D^2 - \frac{1}{2}] \phi_{0,1} + [D^2 - \frac{1}{2}]^{-1} \left\{ \frac{1}{2} [K^1_0(D^2 - \frac{3}{2}) \phi_{1,1} - \phi_{1,0} D^2 K^1_0] \right\} = d_{0,1}
\]

\[
\frac{1}{2} \phi_{1,1} - \frac{1}{R_e} [D^2 - \frac{3}{2}] \phi_{1,0} + [D^2 - \frac{3}{2}]^{-1} \left\{ -[K^1_0(D^2 - \frac{3}{2}) \phi_{0,0} - \phi_{0,1} D^2 K^1_0] \right\} = d_{1,0}
\]

\[
-\phi_{1,0} - \frac{1}{R_e} [D^2 - \frac{3}{2}] \phi_{1,1} + [D^2 - \frac{3}{2}]^{-1} \left\{ -[K^1_0(D^2 - \frac{3}{2}) \phi_{0,1} - \phi_{0,0} D^2 K^1_0] \right\} = d_{1,1}
\]

and define $e$ as:

\[
e = \frac{1}{\beta} \int_{-1}^{+1} dy \left\{ (d_{0,0} - c_{0,0} \cosh(\frac{1}{2} y))^2 + \frac{1}{2} (d_{0,1} - c_{0,1} \cosh(\frac{1}{2} y))^2 + \frac{1}{2} (d_{1,0} - c_{1,0} \cosh(\frac{3}{2} y))^2 + \frac{3}{4} (d_{1,1} - c_{1,1} \cosh(\frac{3}{2} y))^2 \right\}
\]

where, as above:

\[
\beta = R \left. \frac{d\Gamma_0}{dy} \right|_{y=0}
\]

The error $e$ is then to be minimized with respect to the parameters $c_{mn}$ as well as any parameters appearing in the $\phi_{mn}$. The homogeneous solutions (i.e. the terms involving the $c_{mn}$) are introduced to avoid the difficulty involved in specifying the homogeneous solutions to the singular operator $[D^2 - \theta]^{-1}$.

This error criterion seems to be considerably superior to the first one formulated. The inverse operator (essentially $(\nabla^2)^{-1}$) removes the dominance of the wall error, and division of the error by $\beta$ effectively eliminates the laminar solution from consideration.

Analogous procedures are used for formulating the error for both the magneto-fluid dynamic case (for the simple velocity) and
the ordinary fluid dynamic case for the improved velocity.

These procedures are summarized below:

For the magneto-fluid dynamic case (see Section 3D):

\[
\begin{aligned}
&\left[\frac{1}{Re}\left(D^2 - \frac{1}{2}\right) - \frac{M^2}{2} - \lambda q_o \right] - \frac{1}{2}\left[1 - \frac{q_o}{Re}\right] \phi_{o,0} \\
&\left[1 - \frac{q_o}{Re}\right] \frac{1}{Re}\left[D^2 - \frac{1}{2}\right] - \frac{M^2}{2} - \lambda q_o \phi_{o,1} \\
+ &\left[\left(D^2 - \frac{1}{2}\right) + \lambda q_o \right] - \frac{1}{2} q_o \\
&\left[\left(D^2 - \frac{1}{2}\right) + \lambda q_o \right]^{\frac{1}{2}} &\left[K_o^1\left(D^2 - \frac{3}{2}\right)\phi_{1,0} - \phi_{1,0} D^2 K_o^1\right] \\
&\left[\left(D^2 - \frac{1}{2}\right) + \lambda q_o \right]^{\frac{1}{2}} &\left[\frac{1}{2} K_o^1\left(D^2 - \frac{3}{2}\right)\phi_{1,1} - \phi_{1,1} D^2 K_o^1\right] \\
&d_{o,0} &d_{o,1}
\end{aligned}
\]

and similarly for \(d_{1,0}\) and \(d_{1,1}\) where all terms are defined in Section 3D, q.v. The error is then:

\[
e \sim \frac{1}{\beta} \int_{-1}^{+1} dy E \left\{ G_1[D_1] \cdot G_1[D_1] \right\}
\]

where:

\[
D_1 = \Sigma_{m} \Sigma_{n} (d_{mn}^h - d_{mn}^h) h_m(x) h_n(t)
\]

and \(d_{mn}^h\) is the homogeneous solution of the inverse operator and \(e\) is to be minimized with respect to the amplitude of the homogeneous solution as well as the free parameters appearing in the \(\phi_{mn}\).

For the improved velocity approximation (see Section 4):

Let:

\[
\frac{\partial}{\partial t} \psi_e - \frac{1}{Re}[B_e] \psi_e + [B_e]^{-1} H_e = [B_e]^{-1} \Delta_e \Delta d_e
\]

\[
\frac{\partial}{\partial t} \psi_o - \frac{1}{Re}[B_o] \psi_o + [B_o]^{-1} H_o = [B_o]^{-1} \Delta_o \Delta d_o
\]
where the terms are defined in Section 4, q.v. The error is then:

$$e \sim \frac{1}{\beta} \int_{D_1}^{+1} dy \left\{ G_1(D_1) \cdot G_1(D_1) \right\}$$

where $D_1$ is formed from the $d_e$ and $d_o$ in a manner similar to that above, and $e$ is to be minimized with respect to the homogeneous solution of $[\beta_e]^{-1}$ and $[\beta_o]^{-1}$ as well as the free parameters in the $\phi_{mn}$.

B. Approximation of the $\phi_{mn}$

In order for the variational techniques described above to be usefully employed, it will be necessary to approximate the $\phi_{mn}$ in terms of some fixed functional form with a few free parameters to vary.

A great deal of information is at hand about the $\phi_{mn}$, and in fact, there are only a limited number of forms that they may take. First, from symmetry conditions the $\phi_{mn}$ must be even functions of $y$, in fact, for the $u_x u_y$ correlation to be linear (see Section 1C) it appears that the $\phi_{mn}$ are closely approximated by the form:

$$\phi \approx a + by^2$$

over the central part of the channel. Near the edge of the channel the characteristic behavior of the $\phi_{mn}$ is exponential with a large (50-100) decay constant. Thus, one reasonable approximation to the $\phi_{mn}$ is:

$$\phi_{mn} = [a_{mn} + b_{mn} \frac{y^2}{2}]q(y)$$

where $q(y)$ is an even function of $y$ which is nearly unity over the central part of the channel, but has a second order zero at $y = \pm 1$. A convenient and appropriate choice of $q(y)$ is:

$$q(y) = \left[ \frac{\cosh \gamma - \cosh \gamma y}{\cosh \gamma - 1} \right]^2$$
but since $\gamma$ is of order 50-100, $q(y)$ may be closely approximated over the half-interval $0 \leq y \leq 1$ by:

$$q(y) = [1 - \exp(-\gamma(1-y))]^2.$$  \hspace{1cm} 7.5.19

In terms of these approximate $\phi_{mn}$ the $u_x u_y$ correlation defined by:

$$\begin{align*}
\Re u_x u_y &= \Re \phi_{12}^0 \\
&= \beta r(y) = \left( \phi_{0,0} D\phi_{1,0} - \phi_{1,0} D\phi_{0,0} \right) \\
&\quad + \frac{1}{2} \left( \phi_{0,1} D\phi_{1,1} - \phi_{1,1} D\phi_{0,1} \right)
\end{align*}$$  \hspace{1cm} 7.5.20

may be written:

$$r(y) = y [1 - \exp(-\gamma(1-y))]^4$$  \hspace{1cm} 7.5.21

and:

$$\beta = \left( a_0 b_1, o - a_1, o b_0, o \right) + \frac{1}{2} \left( a_0 b_1, 1 - a_1, 1 b_0, 1 \right).$$  \hspace{1cm} 7.5.22

so that:

$$\overline{u_x u_y} = \gamma_{12}^0 = \frac{\beta}{\Re} r(y) = \frac{\beta}{\Re} y [1 - \exp(-\gamma(1-y))]^4$$  \hspace{1cm} 7.5.23

Then the mean velocity becomes (see Chapter VI, Section 1)

$$K_0^1 = 1 - (1 - \beta \Delta) y^2 - \beta g(y)$$  \hspace{1cm} 7.5.24

and the friction factor is:

$$f = \frac{\beta + 2(1 - \beta \Delta)}{\Re}$$  \hspace{1cm} 7.5.25

where:

$$g(y) = - \int [r(\xi) - y] d\xi$$  \hspace{1cm} 7.5.26a

$$z = \exp(-\gamma(1-y))$$  \hspace{1cm} 7.5.26b

$$\Delta = \frac{25}{12} \frac{1}{\gamma} - \frac{415}{144} \frac{1}{\gamma^2}$$  \hspace{1cm} 7.5.26c

The above approximation seems to yield reasonable answers (as far as calculations have thus far proceeded) but the form of the $\phi_{mn}$ may be criticized on two grounds:
1. The exponential behavior of \( q(y) \) near \( y = 1 \) is dominated by \( \exp(-2\gamma(1-y)) \) rather than \( \exp(-\gamma(1-y)) \);

2. The characteristic forms indicated by the calculation in Section 3B are not simple exponentials only, but exponentials multiplied by sinusoids with approximately the same argument. Thus, a more appropriate approximation might be:

\[
\phi_{mn} = a_{mn} + \frac{b_{mn}}{2} y^2 + \exp(-\gamma(1-y)) [a_{mn}' \cos(\omega(1-y)) + b_{mn}' \sin(\omega(1-y))] 
\]

or perhaps:

\[
\phi_{mn} = (a_{mn} + \frac{b_{mn}}{2} y^2) 1 + \exp(-\gamma(1-y)) [a_{mn}' \cos(\omega(1-y)) + b_{mn}' \sin(\omega(1-y))] 
\]

where the \( \gamma, \omega, a_{mn}', \) and \( b_{mn}' \) are chosen to satisfy the homogeneous equations of Section 3B closely near \( y = \mp 1 \). These approximations were considered in detail with totally negative results. The trouble comes from the sinusoidal terms which oscillate rapidly near \( y = 1 \), and cause the function:

\[
D^1 = -Rfy + \beta r(y) 
\]

to change sign near the walls. This difficulty is probably due to the small number of terms taken in the expansions, and may well disappear when better approximations are formulated.

An approximation to \( q(y) \) which causes the behavior near the wall to be more nearly correct is:

\[
q(y) = [1-(1 + \gamma(1-y))\exp(-\gamma(1-y))] 
\]

and calculations were performed using this approximation.

Approximate forms for the magneto-fluid dynamic case are more difficult to come by. Examination of the semi-empirical
results obtained by Harris\textsuperscript{8} indicates however, that a suitable form for the $\phi_{mn}$ (at least for $\frac{M}{Re}$ small) is:

$$\phi_{mn} = (a_{mn} + b_{mn} \frac{\cosh \frac{\ell y - y}{\ell}}{\ell}) q(y)$$

7.5.31

where $q(y)$ has the same form as above. The parameter $\ell$, according to the results of Harris, should be a single valued monotonically increasing function of $\frac{M^2}{Re^{1/2}}$ and should tend to zero as $\frac{M^2}{Re^{1/2}}$ tends to zero. If the $\phi_{mn}$ are approximated as above, then the equation for the mean velocity becomes:

$$[D^2-M^2]K_0 = -Re C_a + D \left\{ \beta r(y) \right\}$$

7.5.32

where:

$$r(y) = \frac{\sinh \frac{\ell y}{\ell}}{\ell} q^2(y)$$

7.5.33

and, as above:

$$\beta = (a_0,0,b_1,0,-a_1,0,b_0,0) + \frac{1}{2}(a_0,1,b_1,1,-a_1,1,b_0,1)$$

7.5.34

Calculations were performed using the above approximation.

C. The Calculated Velocities

An extensive set of computations were performed using the approximations and approximate methods described above. Of these results, five sets of computation results are presented here. These calculations may be catalogued as follows: The first three sets of calculations use the simple velocity approximation with:

$$\phi_{mn} = (a_{mn} + b_{mn} \frac{\cosh \frac{\ell y - y}{\ell}}{\ell})(1-\exp(-\gamma(1-y)))^2$$

7.5.35

The differences in the calculations are a matter of the values of $Re$, $M$, $\ell$ and whether or not $\gamma$ was varied for minimum error. Table 7.2 summarizes these first three sets of calculations:
<table>
<thead>
<tr>
<th>Calc. No.</th>
<th>$R_e$</th>
<th>$M$</th>
<th>$l$</th>
<th>$\gamma$ varied</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12300.0</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>24600.0</td>
<td>0</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>3</td>
<td>12300.0</td>
<td>10</td>
<td>0.1</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 7.2

The fourth and fifth sets of calculations were made at a Reynolds number of 12300.0 and a zero Hartmann number. In both of these sets of calculations, the form of the $\phi_{mn}$ was:

$$
\phi_{mn} = (a_{mn} + \frac{b_{mn}}{2} y^{2}) (1 - (1 + \gamma(1-y)) \exp(-\gamma(1-y)))
$$

The difference in the two sets of calculations is that for the fourth set the simple velocity approximation was used, whereas for the fifth set the improved velocity approximation was used.

In all of these calculations, one additional improvement was made in the error formulation. The variable $x$ was replaced by $\xi = x - C_T t$, i.e. a moving coordinate system was used. The only effect on the equations defining $e$ is that $K^1_o$ is replaced by $K^1_o - C_T$. The result of this improvement is to lessen the effect of the terms involving $K^1_o$. The error in all cases was minimized on $C_T$ as well as the other parameters, and it is to be expected the $C_T$ will be of order 1.

The first set of calculations yielded the following results when the error was minimized with respect to variations in the parameters $a_{mn}$, $b_{mn}$ and $\gamma$.

$$
\begin{align*}
\gamma &= 60.177025 \\
C_T &= 0.9285332 \\
\beta &= 15.560520 \\
f &= 1.3420791 \cdot 10^{-3} \\
a_{o,0} &= 2.0996812 \\
a_{1,0} &= -0.2281481 \\
a_{1,1} &= -2.9020526 \\
b_{o,0} &= 5.6881612 \\
b_{1,0} &= 5.1528166 \\
b_{1,1} &= 5.3461414
\end{align*}
$$
Under these conditions, the normalized error defined by:
\[
e_n = \frac{\pi R}{2} e
\]
is:
\[
e_n = 0.832577 \cdot 10^{-2}
\]
The minimum value of \( e_n \) here is about an order of magnitude lower than the value for typical starting parameters.

The calculated mean velocity is shown graphically in Figure 7.1. It may be observed that the predicted velocity is somewhat lower than the observed (the upper curve is Pai’s\(^{19}\) empirical formula for Laufer’s\(^{10}\) data at \( R_e = 12300.0\)), the maximum error being about \( 9 \frac{1}{2} \% \). The calculated friction factor may be compared with the measured value of:
\[
f_{\text{obs}} = 1.798 \cdot 10^{-2}
\]
The error in the calculated friction factor is about 25.4%.

Using the same error formulation as above, calculations were performed (the second set) with a Reynolds number of 24600.0. The results of this calculation are shown below:

\[
\begin{align*}
\gamma & = 89.849160 \\
C_T & = 0.9480895 \\
\beta & = 23.636278 \\
f & = 0.99816798 \cdot 10^{-3} \\
\end{align*}
\]
\[
\begin{align*}
a_{0,0} & = 2.5762542 \\
a_{0,1} & = 6.0680793 \\
\end{align*}
\]
\[
\begin{align*}
a_{1,0} & = -0.2455243 \\
a_{1,1} & = -0.0161298 \\
b_{0,0} & = 6.9112762 \\
b_{0,1} & = 0.2675770 \\
\end{align*}
\]
\[
\begin{align*}
b_{0,1} & = 7.0221164 \\
b_{1,1} & = -4.8610994 \\
\end{align*}
\]
and \( e_n = 0.7236457 \cdot 10^{-2} \)

In Figure 7.2 the calculated results for \( R_e = 24600.0 \) and \( R_e = 12300.0 \) are compared. As expected, the curve for the higher Reynolds number

*Figures for this chapter are on pp. 148a through 148d.
is slightly flatter in the center and has a somewhat higher derivative near the wall. Also, as expected the friction factor decreases, but the amount of decrease seems to be much to great—the error in the friction factor is estimated (on the basis of the semi-empirical theory, see Harris \(^8\)) to be about 40%; the observed value is about \(1.6 \cdot 10^{-3}\).

Again, using the same error formulation as for the first set, calculations were performed (the third set) with

\[
\begin{align*}
R_e &= 12300.0 \\
M &= 10.0 \\
\gamma &= 75.0 \\
\ell &= 0.1 \\
\end{align*}
\]

These calculations were performed with both \(\gamma\) and \(\ell\) held constant and were done for the purpose of seeing if such approximations were reasonable. The parameters \(\gamma\) and \(\ell\) were not varied simply because of the computational time involved—keeping these parameters constant avoided the calculation of \(\exp(x)\) about 500,000 times. The results of this calculation are shown below.

\[
\begin{align*}
\gamma &= 75.000000 \\
C_T &= 1.000000 \\
\beta &= 16.246195 \\
f &= 1.8315631 \cdot 10^{-3} \\
a_{0,0} &= 0.1613010 \\
a_{1,0} &= -1.7064355 \\
a_{0,1} &= 4.7229864 \\
a_{1,1} &= -0.2530655 \\
b_{0,0} &= 2.5245220 \\
b_{1,0} &= -0.0105002 \\
b_{0,1} &= -5.9600354 \\
b_{1,1} &= 2.6880850 \\
\text{and} \\
e_n &= 0.1315925
\end{align*}
\]
It should be noticed that the error here is an order of magnitude greater than the first two calculations and thus little reliance should be placed on the precise numerical values calculated. However, the trends which the results show are informative, and seem correct. In Figure 7.3, the calculated velocity is compared with the laminar profile for \( M = 10 \), and the turbulent profile (predicted above) with \( M = 0 \). As expected, the turbulent magneto-fluid dynamic profile is flatter in the center and has a higher gradient near the wall than either of the other two profiles. The calculated values of \( \frac{M}{Re^{1/2}} \) and \( \frac{M^2}{Re^{1/2}} \) are:

\[
\frac{M}{Re^{1/2}} = 0.190 \cdot 10^{-1} ; \quad \frac{M^2}{Re^{1/2}} = 0.190
\]

Thus, according to the analysis of Harris, the mean velocity profile should be little affected over the center of the channel. The calculation presented above (see Figure 7.3) indicates however that the center channel behavior of the mean velocity is very much affected. In fact, the present analysis seems to indicate that the center channel behavior of \( K^1_o \) is characterized more by the parameter \( M \) than \( \frac{M^2}{Re^{1/2}} \). Further experimentation is needed to resolve this question as well as many other poorly understood features of magneto-fluid dynamic flows.

Since the approximations seem to be predicting the mean velocity much better than the friction factor, it was decided that the approximate velocities should be improved systematically before a series of calculations at various Reynolds numbers was performed.

Thus, the fourth set of calculations was performed using the simple velocity approximation and the following form for the \( \phi_{mn} \):
\[ \phi_{mn} = (a_{mn} + \frac{b_{mn}}{2} y^2)(1-(1 + \gamma(1-y)) \exp(-\gamma(1-y))) \]

This form of the \( \phi_{mn} \) is felt to be a better approximation than the form used in the preceding examples because of its more reasonable behavior near the wall. The results of this calculation are shown below:

\[
\begin{align*}
\gamma &= 93.259924 \\
C_T &= 0.9985888 \\
f &= 1.2036702 \cdot 10^{-3} \\
a_{0,0} &= 0.7915341 \\
a_{1,0} &= 3.1823278 \\
b_{0,0} &= -2.3706252 \\
b_{1,0} &= 2.0957354 \\
a_{0,1} &= 4.4407199 \\
a_{1,1} &= -1.7238145 \\
b_{0,1} &= -2.4267150 \\
b_{1,1} &= 2.9199985 \\
e_n &= 0.73962185 \cdot 10^{-2}
\end{align*}
\]

The parameter \( \gamma \) was fixed at the magnitude of the complex exponential parameter characterizing the behavior of the equations near the walls (see Section 3B), and \( \gamma \) was not varied for minimum (because of the machine time involved calculating exponentials).

The predicted mean velocity is shown in Fig. 7.1 along with the first calculated curve and the observed velocity. It may be observed that, although \( e_n \) is less, this approximation is considerably poorer than the first, being in error by about 20% at the worst point. Furthermore, the predicted friction factor is not as close as the first; the error here is about 33%. Nevertheless, it is felt that the form of the \( \phi_{mn} \) used here is inherently more reasonable than the first, and that the difficulty comes from an insufficient number of terms in the expansion of \( S_1 \) in terms of Hermite Functions.
Thus, the fifth set of calculations was performed using the improved velocity approximation described in Section 4 (q.v.), and the following form for the $\phi_{mn}$:

$$\phi_{mn} = (a_{mn} + \frac{b_{mn}}{2} y^2)(1-(1+\gamma(1-y))\exp(-\gamma(1-y)))$$ 7.5.39

Minimization of the error function yielded the following results:

$$\gamma = 93.259924 \quad \beta = 19.891030$$
$$c_T = 1.0073687 \quad f = 1.6861996 \cdot 10^{-3}$$

<table>
<thead>
<tr>
<th>$a_{0,0}$</th>
<th>$b_{0,0}$</th>
<th>$a_{1,0}$</th>
<th>$b_{1,0}$</th>
<th>$a_{2,0}$</th>
<th>$b_{2,0}$</th>
<th>$a_{3,0}$</th>
<th>$b_{3,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9518468</td>
<td>-2.0027466</td>
<td>1.2623613</td>
<td>-1.2123622</td>
<td>0.5084020</td>
<td>-0.9142674</td>
<td>0.9941959</td>
<td>-1.1437314</td>
</tr>
<tr>
<td>$a_{0,1}$</td>
<td>$b_{0,1}$</td>
<td>$a_{1,1}$</td>
<td>$b_{1,1}$</td>
<td>$a_{2,1}$</td>
<td>$b_{2,1}$</td>
<td>$a_{3,1}$</td>
<td>$b_{3,1}$</td>
</tr>
<tr>
<td>1.9172332</td>
<td>1.7009008</td>
<td>-0.0982317</td>
<td>1.0254978</td>
<td>0.8180051</td>
<td>0.6732667</td>
<td>-0.1106314</td>
<td>1.4259378</td>
</tr>
<tr>
<td>$a_{0,2}$</td>
<td>$b_{0,2}$</td>
<td>$a_{1,2}$</td>
<td>$b_{1,2}$</td>
<td>$a_{2,2}$</td>
<td>$b_{2,2}$</td>
<td>$a_{3,2}$</td>
<td>$b_{3,2}$</td>
</tr>
<tr>
<td>-0.2532874</td>
<td>1.9774477</td>
<td>-1.4988113</td>
<td>0.8587383</td>
<td>-0.0545774</td>
<td>0.7755887</td>
<td>-1.2316289</td>
<td>1.0396803</td>
</tr>
<tr>
<td>$a_{0,3}$</td>
<td>$b_{0,3}$</td>
<td>$a_{1,3}$</td>
<td>$b_{1,3}$</td>
<td>$a_{2,3}$</td>
<td>$b_{2,3}$</td>
<td>$a_{3,3}$</td>
<td>$b_{3,3}$</td>
</tr>
<tr>
<td>-0.7020122</td>
<td>-1.9089613</td>
<td>0.1003496</td>
<td>-1.0206763</td>
<td>-0.3332980</td>
<td>-0.8601413</td>
<td>0.1570128</td>
<td>-1.3572282</td>
</tr>
</tbody>
</table>

and $e_n = 0.46863848 \cdot 10^{-2}$
Again, for simplicity $\gamma$ was not varied. The mean velocity profile predicted by this method is shown in Figure 7.1. The improvement over the fourth calculation is significant. Both the mean velocity and the friction factor are off (low) by about $6 \frac{1}{2} \%$.

The behavior of the calculated mean velocity near the wall deserves some explanation. The calculated mean velocity and Pai's empirical formula are shown in Figure 7.4 for the region $0.9 \leq y \leq 1.0$. Although it may not be apparent from the graph, the curves actually cross twice; once at $y = 0.94$ and once at $y = 0.99$. Furthermore, it may be observed that the calculated velocity is more linear near the wall and breaks more sharply (at $y = 0.97$) than does Pai's curve. These are several reasons for these differences. First, Pai's curve is somewhat in error, due to the fact that the $u_x u_y$ correlation predicted from his empirical formula:

$$u_x = 1 - 0.3293 y^2 - 0.6707 y^{32}$$  \hspace{1cm} 7.5.40

is:

$$\frac{u_x u_y}{R_e} = \frac{(21.4624)}{y(1-y^{30})}$$  \hspace{1cm} 7.5.41

This form has only a first order zero at $y = 1$, whereas there should be a triple order zero there (see Section 2C). A comparison of Pai's curve with Laufer's experimental data* indicates that the true velocity should be more linear near the walls and should break at about $y = 0.98$. Secondly, the velocity calculated here is based on the two-dimensional model which inherently gives a fourth order zero at $y = 1$, where there

---

*See Figure 2(a) of Pai. 19
should be only a third order zero. Furthermore (and perhaps more important), the calculations were based on $\gamma \approx 93.3$, a value which came from considerations of the behavior of the simple velocity approximation. Calculations presented above (see Section 4) for the improved velocity approximation indicate that a more appropriate choice for $\gamma$ is 80.3 since the form:

$$1-(1 + \gamma(1-y))\exp(-\gamma(1-y))$$  

may be considered to be an approximation of:

$$1-(\cos(\omega(1-y)) + \sin(\omega(1-y)))\exp(-\gamma(1-y))$$  

provided $\gamma = \omega$, as is indeed the case.

6. Discussion

This very long chapter has attempted to trace a rather narrow path through a maze of mathematics to obtain reasonable approximations to the solution of turbulent Plane Poiseuille Flow. The approximations and approximate methods used here may be constructively criticized on a number of grounds, and in fact we would agree with most of the criticism. Realizing some of the faults in the approximations, we have made a conscious effort to show how the approximations may be systematically improved, and in some cases, the method of improvement has been demonstrated. Our principle justification for the cruder approximations is, a priori, we are looking for solutions to a very difficult problem; a posteriori, reasonable solutions have apparently been found.
Figure 7.1
Comparison of Various Velocity Profiles for $R_e = 12300.0$
Figure 7.2
Comparison of Calculated Turbulent Profiles.
Figure 7.3
Comparison of Various Laminar and Turbulent Profiles for $M = 0$ or 10
Figure 7.4
Detail of Pai’s Curve and Calculation No. 5
Near the Wall. $R_e = 12300.00$
CHAPTER VIII

Conclusions and Suggestions for Further Study

1. Concluding Remarks

The aim of this work has been to develop a rational analysis of turbulent flow problems. A great deal is known about turbulence, a great many measurements have been made and the field has been covered by many able researchers. However, analysis has heretofore been hampered by lack of an appropriate method of characterizing random solutions or nonlinear, partial differential equations. The Calculus of Random Functionals (as developed in Chapter III) seems to provide a suitable method of analysis and Chapters IV, V, VI and VII are devoted to exploiting the possibilities of solution to otherwise well understood problems in turbulence.

The careful reader will note the absence of a number of items considered by most other researchers in the field. Thus, for decaying turbulent flows, only the simplest problems are considered, and these only under least interesting circumstances (the final period of decay). Moreover, for the channel flows, a number of interesting features are left completely undiscovered, e.g. the spectrum of turbulence and transport processes. This is in keeping with the objectives of this thesis; i.e. we are interested primarily in developing analytic and computational methods for solving turbulent flow problems. Thus, we are content (at least to start with) to solve only the simple problems and these only for the simplest physical quantities.

The most satisfying results of this work are the results of the calculations for the three turbulent flows considered. The results of the calculations for decaying turbulent flows seem altogether reasonable on physical grounds, and the mean velocity and friction factor for Plane Poiseuille Flow have been predicted to within $6 \frac{1}{2}$%.
The philosophy underlying the methods of analysis employed here deserves some comment. There are few problems in modern mathematical physics which can be completely solved analytically. Many more problems can be solved using approximate (e.g. perturbation, variational, etc.) analytic techniques, but the vast majority of problems seem to yield only to numerical analysis. Thus, numerical analysis—especially in conjunction with a modern digital machine—seems to be one of the most important tools available to the research worker of today. Thus, the scornful attitude with which science has traditionally viewed numerical analysis must vanish. The research workers in quantum mechanics have long realized this, and the results of this work (as well as others, e.g. Penhune\textsuperscript{20}) indicate that many problems in continuum physics are more easily solved by the use of numerical methods.

2. Theoretical Extensions

A great number of extensions and improvements in the theory and development presented here are possible.

For the Calculus of Random functionals, several extensions seem possible. For example, only two types of functional series have been used: Homogeneous Polynomial Functionals and Orthogonal Polynomial Functionals. The homogeneous functionals offer the advantage of simplicity over the more general orthogonal functionals, but convergence and summability are problems. Since homogeneous functionals correspond closely to perturbation methods, perhaps the functional series can be resummed in a manner analogous to the Feenberg or Fredholm resummation\textsuperscript{*} of perturbation series. The Fredholm method seems particularly appropriate.

\textsuperscript{*}See Morse and Feshback,\textsuperscript{15} Chapter 9.
Furthermore, various sets of functionals which are orthogonal with respect to some weighting function can be constructed and may prove more useful than the G series developed here. In line with this, it may prove more useful to approximate the velocities in terms of functionals in some other way than by just truncating the series. Thus, for example, a velocity approximation of the form:

$$u^i = \cosh(G_1[\phi_1, \omega]) \sum_v G_v[K_v^i, \omega]$$

might be more appropriate for some problems.

Much more can be done in analyzing decaying turbulent flows. In particular, regions of decay in which more than just the first terms suffice are very interesting both theoretically and practically (especially with regard to plasma devices). The study of such problems will require a close look at what can be said about initial and/or boundary conditions.

The study of channel flows (or any other shear flow for that matter) can be extended in many ways. In particular, a considerably improved approximation of the type discussed in section 3 of Chapter VII can be made by assuming:

$$u^i = G_0[K_0^i, \omega] + G_1[K_1^i, \omega] + G_2[K_2^i, \omega]$$

$$K_1^i = e^{ijk} \frac{\partial}{\partial x^j} s_1^k$$

$$K_2^i = e^{ijk} \frac{\partial}{\partial x^j} s_2^k$$

$$s_1^k = \sum_{\alpha \beta \mu \nu} \phi_{\alpha \beta \mu \nu}^k (y) h_\alpha (x) h_\beta (z) h_\nu (t)$$

$$s_2^k = \left\langle \sum_{\alpha \beta \mu \nu} \phi_{\alpha \beta \mu \nu}^k (y) h_\alpha (x_1) h_\beta (x_2) h_\mu (z_1) h_\nu (z_2) h_\alpha (t_1) h_\beta (t_2) \right\rangle_{1,2}$$
where:
\[ \phi^k_{\ell mn}(y) \] is\[
\begin{cases} 
\text{even if } k=1,3 \text{ and } m = 0,2 \text{ or } k=2, m=1,3 \\
\text{odd otherwise}
\end{cases}
\]
and:
\[ \psi^k_{\ell mn}(y) \] is\[
\begin{cases} 
\text{odd if } k=1,3 \text{ and } m=0,2 \text{ or } k=2, m=1,3 \\
\text{even otherwise}
\end{cases}
\]
The even functions (of \( y \)) might then be of the form:
\[ (a + \frac{b}{2} y^2) q(y) \]
and the odd ones of the form:
\[ cy q(y) \]
where \( q(y) \) is some function (or class of function) very nearly equal to one over the central portion of the channel but having a second order zero at the walls. Such an approximation eliminates most of the improper behavior of the simpler approximations examined (e.g. the fourth order zero in \( \overline{u_x u_y} \) at \( y = 1 \), and the zero in \( u_x \) at \( y = 0 \)) and might reasonably be expected to give better than 1% answers. However, there are over 300 free parameters (\( a' \)s, \( b' \)s, and \( c' \)s in addition to those in \( q(y) \)) to vary and the error formulation is very difficult. Thus, a considerable amount of work should be put into carefully formulating the \( \phi' \)s and \( \psi' \)s.

In addition, a more sophisticated approximation such as described above should yield good results for the magneto-fluid dynamic case.

3. Suggestions for Experimental Study

All experimental work to date has been aimed at measuring specific averaged physical quantities. Furthermore, all such measurements (including processes for multiplication and averaging) have been done entirely with analog equipment.
It would seem, however, that much more useful results could be obtained by digitally recording the random fluctuations and then performing all calculations on a digital machine. There are at least three important reasons for doing this. First, this method would be more accurate and more precise, since digital recording methods are much better than most transducers (e.g. hot wires), whereas analog correlation machines are of questionable accuracy. Second, time correlation measurements--at present unfeasible because of delay problems--could be made with a minimum of effort. Finally, with suitable programming, it may well be possible to actually determine the kernels of the G series operators.*

In addition, there is still a very great need for experimental work in magneto-fluid dynamic flows.

*See Weiner, Chapters 10-13.
APPENDIX A1

Correlation Tensors and the Theory of Invariants

A correlation tensor is a statistical average (i.e. an ensemble average, represented by $\mathbb{E}\{\cdot\}$) of any random tensor found in a physical situation at any number of points in space and time. These correlation tensors are found to be rather useful in the study of those turbulent flows which possess a high degree of symmetry and homogeneity, because in such cases the general form of the tensors can be broken down into a reasonably simple expression. The discussion of a few of these tensors and the restrictions placed on them by the theory of invariants and Conservation of Mass (in its incompressible form) is the subject of this appendix.

Several terms such as "isotropic", "axisymmetric", and "homogeneous" are used to describe correlation tensors. The term "isotropic" will be taken to mean that the tensor is invariant under all rotations of the coordinate system, i.e. there are no preferred directions. The term "axisymmetric" will be taken to mean that the tensor is invariant under all rotations about some preferred axis, and the only preferred direction is in the direction of this axis. The term "homogeneous" has the implication that the correlation tensor is invariant to some class of displacements, but the class of displacements is dependent on the type of symmetry involved—in fact, the type of symmetry quite often specifies the degree of homogeneity, e.g. "isotropic" tensors are generally homogeneous in all space dimensions and the term "isotropic" is commonly used to imply this. On the other hand, the term "homogeneous axisymmetric" is generally taken to mean that the tensor is homogeneous in those space dimensions in a plane
perpendicular to the preferred direction, but the tensor may be a function of the coordinate in the preferred direction. If such a tensor were indeed independent of the coordinate in the preferred direction, a separate assumption should be stated.

Of the several correlation tensors possible in a study of random magneto-fluid dynamics, two will be found particularly useful in the present work. These are:

1. The instantaneous Pressure-Velocity correlation:
\[ L^i(x^k, \xi^k, t) = E \{ p(x^k, t) u^i(x^k + \xi^k, t) \} \quad \text{Al.1.1} \]

and 2. The instantaneous Double-Velocity correlation:
\[ R^{ij}(x^k, \xi^k, t) = E \{ u^i(x^k, t) u^j(x^k + \xi^k, t) \} \quad \text{Al.1.2} \]

The correlation spectrum tensors (the Fourier transforms of \( L^i \) and \( R^{ij} \)) are also found useful, and in some cases a hybrid tensor (e.g. \( R^{ij} \) transformed on less than three space variables) is found useful.

1. Consequences of the Theory of Invariants

The Theory of Invariants as developed by Robertson\textsuperscript{25} and Chandrasekhar\textsuperscript{4} deals with a method of satisfying the restricted form that correlation tensors must take as a result of isotropic, axisymmetric, or planar symmetry. For the purposes of discussion, consider the configuration of vectors shown in Fig. Al.1.

If \( a^i \) and \( b^j \) are unit vectors independent of position, then the scalar
\[ R = a^i b^j R^{ij} \quad \text{Al.1.3} \]
is a true tensor invariant possessing all the properties of symmetry and homogeneity of the flow.
For Isotropic Turbulence which is homogeneous in all space variables, and is invariant under rigid body rotations of $a^i$, $b^i$, and $\xi^i$, $R$ must be a function of the three vectors $a^i$, $b^i$, $\xi^i$ only, i.e.

$$R = R(a^i, b^i, \xi^i)$$

Moreover, a scalar invariant can be a function only of the invariants of its arguments. These invariants are $a^k b^k$, $a^k \xi^k$, $b^k \xi^k$, $\xi^k \xi^k$, and $a^i b^j \xi^k e_{ijk}$. Thus:

$$R = R(a^k b^k, a^k \xi^k, b^k \xi^k, \xi^k \xi^k, a^i b^j \xi^k e_{ijk})$$

However, the scalar $R$ must be homogeneous and linear in the $a^i$ and $b^j$ so that the form of $R$ is restricted to:

$$R = a^i b^j [R_1(r,t)\xi^i \xi^j + R_2(r,t)\delta^{ij} + R_3(r,t)e^{ijk}\xi^k]$$

where:

$$r^2 = \xi^k \xi^k$$

Since $a^i$ and $b^i$ are arbitrary unit vectors, the tensor quotient law implies that:

$$R^{ij} = R_1(r,t)\xi^i \xi^j + R_2(r,t)\delta^{ij} + R_3(r,t)e^{ijk}\xi^k$$
In a similar fashion it may be demonstrated that the instantaneous pressure-velocity correlation $L^i$ for isotropic turbulence must be of the form:

$$L^i = L_1(r, t) \xi_i$$ \hspace{1cm} \text{Al.1.9}

For Axisymmetric Turbulence which has the preferred axis in the direction of $\lambda^i$, all average quantities must be invariant under all rigid body rotations of $a^i, b^i, \lambda^i$ and $\xi^i$ in the coordinate system. The invariants of these four vectors are $a^i b^k, a^i \lambda^k, a^i \xi^k, b^i \lambda^k, b^i \xi^k, \lambda^i \xi^k, a^i b^j \lambda^k ijk, a^i b^j \lambda^k ijk, a^i \lambda^j \xi^k ijk$ and $b^i \lambda^j \xi^k ijk$. The scalar $R$ is to be a function of these invariants and homogeneous in $a^i$ and $b^i$, thus the form of $R^{ij}$ must be:

$$R^{ij} = R_1 \xi^i \xi^j + R_2 \delta^{ij} + R_3 [\xi^i \lambda^j \xi^j \lambda^i] + R_4 \lambda^i \lambda^j$$ \hspace{1cm} \text{Al.1.10}

$$+ R_5 [\xi^i e_j \xi^m + \xi^j e_i \xi^m] \xi^i \lambda^m + R_6 [\lambda^i e_j \xi^m + \lambda^j e_i \xi^m] \xi^i \lambda^m$$

$$+ R_7 [\xi^i \lambda^j \lambda^i + \xi^j \lambda^i \lambda^i] + R_8 e_i \lambda^j \lambda^k + R_9 e_i \lambda^j \lambda^k$$

$$+ R_{10} [\xi^i \lambda^j \lambda^i] + R_{11} [\lambda^i \lambda^j \lambda^k \lambda^k] \xi^i \lambda^m + R_{12} (e_i \lambda^j \lambda^m \xi^i \lambda^m) (e_j \lambda^i \lambda^m \xi^i \lambda^m)$$

where: \hspace{1cm} $R_n = R_n (\rho, \lambda^i \xi^k, \lambda^i k, k, \lambda^i)$ \hspace{1cm} \text{Al.1.11}

and: \hspace{1cm} $\rho^2 = \xi^i \xi^k - (\lambda^i \xi^k) \lambda^i$ \hspace{1cm} \text{Al.1.12}

and: \hspace{1cm} $\lambda^i k, k = 1$ \hspace{1cm} \text{Al.1.13}

Fortunately not all of these terms are needed for it is not difficult to show that the terms involving $R_{10}, R_{11}$, and $R_{12}$ can be written in terms of lower order $R_n$.

In a similar fashion it may be demonstrated that the instantaneous pressure-velocity correlation for axisymmetric turbulence must be of the form:

$$L^i = L_1 \xi^i + L_2 \lambda^i + L_3 e^{ij} k \lambda^i$$ \hspace{1cm} \text{Al.1.14}
2. Further Development of Isotropic Tensors

The correlation spectrum tensors for Isotropic Turbulence will be defined by:

$$R^{ij}(\xi^k, t) = \left(\frac{1}{2\pi}\right)^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^{ij}(\mathbf{x}^k, t)e^{i\mathbf{x}^k \cdot \xi^k} d\tau_k$$  \hspace{1cm} A1.2.1

and:

$$L^{i}(\xi^k, t) = \left(\frac{1}{2\pi}\right)^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \wedge^{i}(\mathbf{x}^k, t)e^{i\mathbf{x}^k \cdot \xi^k} d\tau_k$$  \hspace{1cm} A1.2.2

where $d\tau_k$ is a volume differential in $\mathbf{x}^k$ space.

In the last section, it was shown that the tensors $R^{ij}$ and $L^{i}$ for Isotropic Turbulence may be written:

$$L^{i}(\xi^k, t) = L_{1}(r, t)\xi^i \Delta \frac{\partial \pi_{1}(r, t)}{\partial \xi^i}$$  \hspace{1cm} A1.2.3

and:

$$R^{ij}(\xi^k, t) = R_{1}(r, t)\xi^i \xi^j + R_{2}(r, t)\delta^{ij} + R_{3}(r, t)e^{ijk}\xi^k$$  \hspace{1cm} A1.2.4

$$\Delta \frac{\partial \pi_{1}(r, t)}{\partial \xi^i \partial \xi^j} + \pi_{2}(r, t)\delta^{ij} + e^{ijk}\frac{\partial \pi_{3}(r, t)}{\partial \xi^k}$$

from which it follows that the transforms may be written:

$$\wedge^{i}(\mathbf{x}^k, t) = \wedge_{1}(\mu, t)\mathbf{x}^i$$  \hspace{1cm} A1.2.5

and:

$$\phi^{ij}(\mathbf{x}^k, t) = \phi_{1}(\mu, t)\mathbf{x}^i \mathbf{x}^j + \phi_{2}(\mu, t)\delta^{ij} + \phi_{3}(\mu, t)e^{ijk}\mathbf{x}^k$$  \hspace{1cm} A1.2.6

where:

$$\langle \mu \rangle^2 = \mathbf{x}^k \mathbf{x}^k$$  \hspace{1cm} A1.2.7

For an incompressible fluid, Continuity of Mass requires (c.f. equation 2.3.1):

$$\frac{\partial u^i}{\partial x^i} = 0$$  \hspace{1cm} A1.2.8

and therefore:

$$\frac{\partial L^{i}}{\partial \xi^i} = 0$$  \hspace{1cm} A1.2.9
and: \[
\frac{\partial R^{ij}}{\partial x^i} - \frac{\partial R^{ij}}{\partial x^j} = 0
\]
from which it follows that:
\[
x^1 \Lambda^1 - \mu^2 \Lambda_1 = 0
\]
and:
\[
x^1 \phi^{ij} = \phi_1 \mu^2 x^1 + \phi_2 x^1 = 0
\]
and:
\[
x^1 \phi^{ij} = \phi_1 \mu^2 x^1 + \phi_2 x^1 = 0
\]
Thus, the instantaneous pressure-velocity correlation is zero, \( i.e. L^1 = 0, \Lambda^1 = 0 \). This means that in isotropic turbulence, any pressure fluctuations are completely uncorrelated with the velocity fluctuations.

In addition, the form of the transform of the instantaneous double velocity correlation may be simplified to:
\[
\phi^{ij}(x, t) = \phi_1(\mu, t)[x^1 x^1 - \mu^2 \delta^{ij}] + \phi_2(\mu, t) \epsilon^{ijk} x^k
\]
and \( R^{ij} \) may be written as:
\[
R^{ij}(\xi^k, t) = \left[ - \frac{1}{2r} \frac{\partial f(r, t)}{\partial r} \xi^1 \xi^j + (f(r, t) + \frac{r}{2} \frac{\partial f(r, t)}{\partial r}) \delta^{ij} + R_3(r, t) \epsilon^{ijk} \xi^k \right]
\]
In this form \( f(r, t) \) and \( R_3(r, t) \) may be given simple physical interpretations. For example, \( f(r, t) \) is the correlation for velocities parallel to each other and parallel to the separation, \( i.e. \) if \( \lambda^k \) is a unit direction vector, then:
\[
R_p(r, t) = \lambda^1 \lambda^j R^{ij}(r \lambda^k, t) = - \frac{r}{2} \frac{\partial f(r, t)}{\partial r} + f(r, t) + \frac{r}{2} \frac{\partial f(r, t)}{\partial r} + 0
\]
hence: \( R_p(r, t) = f(r, t) \)
3. Further Development of Axisymmetric Tensors

In Section 1 of this appendix, Axisymmetric Tensors were developed assuming homogeneity only in the planes normal to the preferred axis. For example, the tensor $R_n^{ij}$ was expressed in terms of scalars of the form:

$$R_n = R_n (\rho, \lambda^k \xi^k, \lambda^k \chi^k, t)$$

where:

$$\rho^2 = x^2 - (\lambda^k \xi^k)^2 = \xi^k \xi^k - (\lambda^k \xi^k)^2$$

For this work, however, a restricted form will be developed in which the tensors are completely independent of the $x^k$, i.e. $R_n$ is independent of $\lambda^k \xi^k$. In this event it is appropriate to consider the transforms of $R_n^{ij}$ and $L^i$ defined above as:

$$R_n^{ij} (\xi^k, \lambda^k, t) = \left( \frac{1}{2\pi} \right)^3 \int_\infty^\infty \int \phi^{ij} (\xi^k, \lambda^k, t) e^{\xi^k \lambda^k} d\tau^k$$

and:

$$L^i (\xi^k, \lambda^k, t) = \left( \frac{1}{2\pi} \right)^3 \int_\infty^\infty \int \phi^i (\xi^k, \lambda^k, t) e^{\xi^k \lambda^k} d\tau^k$$

In Section 1 of this chapter, it was shown that the tensors $R_n^{ij}$ and $L^i$ for Axisymmetric Turbulence may be written:

$$L^i (\xi^k, \lambda^k, t) = L_1 \xi^i + L_2 \lambda^i + L_3 e^{ijk} \lambda^j \lambda^k$$

$$= \frac{\partial L_1}{\partial \xi^i} + L_2 \lambda^i + e^{ijk} \lambda^j \lambda^k$$

*The general formulation is given in Reference 21.*
and:

\[ R_{ij}^{*}(\xi^k, \lambda^k, t) = R_1^{ij} \xi^i \lambda^j + R_2^{ij} \delta^{ij} + R_3^{ij} (\xi^i \lambda^j + \xi^j \lambda^i) + R_4^{ij} \lambda^i \lambda^j \]  
\[ + R_5^{ij} (\xi^i e^{jlm} + \xi^j e^{ilm}) \xi^l \lambda^m + R_6^{ij} (\lambda^i e^{jlm} + \lambda^j e^{ilm}) \xi^l \lambda^m \]  
\[ + R_7^{ij} (\xi^i \lambda^j - \xi^j \lambda^i) + R_8^{ij} e^{ijk} \xi^k + R_9^{ij} e^{ijk} \lambda^k \]  
\[ A_{\frac{\partial^2 R_1}{\partial x^i \partial x^j}} + \tau_2 \delta^{ij} + (\lambda^i \frac{\partial}{\partial x^j} + \lambda^j \frac{\partial}{\partial x^i}) \tau_3 + \tau_4 \lambda^i \lambda^j \]  
\[ + (\lambda^i \frac{\partial}{\partial x^j} - \lambda^j \frac{\partial}{\partial x^i}) \tau_7 + e^{ijk} \xi^k \frac{\partial}{\partial x^l} \tau_8 + e^{ijk} \lambda^k \tau_9 \]  

where the \( L_n \), \( f_n \), and \( \tau_n \) are functions of \( r \), \( \lambda^k \xi^k \), and \( t \) only. From this it follows that the \( \phi^{ij} \) and \( \Lambda^{ij} \) may be written:

\[ \Lambda^{ij} = \Lambda_{1}^{ij} + \Lambda_{2}^{ij} + \Lambda_{3}^{ij} e^{ijk} \lambda^l \lambda^k \]  

and:

\[ \phi^{ij} = \phi_{1}^{ij} \xi^i \lambda^j + \phi_{2}^{ij} \delta^{ij} + \phi_{3}^{ij} (\xi^i \lambda^j + \xi^j \lambda^i) + \phi_{4}^{ij} \lambda^i \lambda^j \]  
\[ + \phi_{5}^{ij} (\xi^i e^{jlm} + \xi^j e^{ilm}) \lambda^l \lambda^m + \phi_{6}^{ij} (\lambda^i e^{jlm} + \lambda^j e^{ilm}) \lambda^l \lambda^m \]  
\[ + \phi_{7}^{ij} (\xi^i \lambda^j \lambda^i) + \phi_{8}^{ij} e^{ijk} \xi^k + \phi_{9}^{ij} e^{ijk} \lambda^k \]  

where the \( \Lambda_n \) and \( \phi_n \) are functions of \( \mu, \lambda^k \xi^k \) and \( t \) only.

As for the Isotropic case, continuity of mass for an incompressible fluid requires:

\[ \frac{\partial u_i}{\partial x^j} = 0 \]  

and therefore:

\[ \frac{\partial L_i}{\partial t} = 0 \]
and:
\[
\frac{\partial \mathbf{R}^{ij}}{\partial \xi^i} = \frac{\partial \mathbf{R}^{ij}}{\partial \xi^j} = 0
\]

Thus \( \Lambda^i \) must satisfy:
\[
\Lambda^i = \Lambda^j_1 \mu^2 + \Lambda^j_2 \mu^2 \lambda^k \kappa^k = 0
\]

Thus \( \Lambda^i \) may be written:
\[
\Lambda^i = (\lambda^k \kappa^k + \mu^2 \lambda^i) \Lambda^j_1 + e^{ijk} \lambda^k \Lambda^j_2
\]

where the \( \Lambda_n \) have been renumbered. Furthermore, \( \phi^{ij} \) must satisfy:
\[
\mathbf{x}^i \phi^{ij} = \mathbf{x}^j [\mu^2 \phi_1 + \phi_2 + \lambda^k \kappa^k (\phi_3 - \phi_7)]
\]
\[
+ \lambda^j [\mu^2 \phi_3 + \lambda^k \kappa^k \phi_4 + \mu^2 \phi_7]
\]
\[
+ e^{ijk} \lambda^m \mu^m [\mu^2 \phi_5 + \lambda^k \kappa^k \phi_6 - \phi_9]
\]

and:
\[
\mathbf{x}^j \phi^{ij} = \mathbf{x}^i [\mu^2 \phi_1 + \phi_2 + \lambda^k \kappa^k (\phi_3 + \phi_7)]
\]
\[
+ \lambda^i [\mu^2 \phi_3 + \lambda^k \kappa^k \phi_4 - \mu^2 \phi_7]
\]
\[
+ e^{ijm} \lambda^m [\mu^2 \phi_5 + \lambda^k \kappa^k \phi_6 + \phi_9]
\]

It follows that:
\[
\mu^2 \phi_1 + \phi_2 + \lambda^k \kappa^k (\phi_3 - \phi_7) = 0
\]
\[
\mu^2 \phi_1 + \phi_2 + \lambda^k \kappa^k (\phi_3 + \phi_7) = 0
\]
\[
\mu^2 (\phi_3 + \phi_7) + \lambda^k \kappa^k \phi_4 = 0
\]
\[
\mu^2 (\phi_3 - \phi_7) + \lambda^k \kappa^k \phi_4 = 0
\]
\[
\mu^2 \phi_5 + \lambda^k \mu \kappa \phi_6 + \phi_9 = 0
\]

and:
\[
\mu^2 \phi_5 + \lambda^k \mu \kappa \phi_6 + \phi_9 = 0
\]
From which it is clear that:

$$\Phi_7 = \Phi_9 = 0$$

Then, renumbering the terms, $\phi^{ij}$ may be written:

$$\phi^{ij}(x^k, \lambda^k, t) = x^i x^j \phi_1 + \delta^{ij} \phi_2 + \mu^2 \lambda^i \lambda^j \phi_3$$

$$+ [\lambda^k x^i \lambda^j + x^j \lambda^i] \phi_4$$

$$+ [(\lambda^k x^i - x^i \lambda^2) e^{ilm} + (\lambda^k x^j - x^j \lambda^2) e^{ilm}] x^k x^m \phi_4$$

$$+ e^{ijk} x^k \phi_5$$

with the constraint:

$$\mu^2 \phi_1 + \phi_2 + \lambda^k \phi_3 = 0$$

where the $\phi_n$ are functions of $\mu$, $\lambda^k$, and $t$. 

163.

Al. 3.17

Al. 3.18

Al. 3.19
The approximate methods of solution presented in Chapter VII are certainly much too complicated to be carried out either analytically or by hand computation. Even if the problem were much simpler, one would hesitate to perform extensive calculations when the methods employed stand a good (a priori) chance of failing. Modern digital machines, however, allow such unproven methods to be tried with a minimum of effort—in fact, this work could not possibly have been completed without the aid of such a machine.

Thus, all of the approximate methods used in this work were performed numerically, using the IBM 7090 EDPM with programs written for the most part in FORTRAN\*⁴. However, the numerical work associated with most fluid dynamics problems is rather difficult because the solutions are characterized by very large gradients near the walls, but smooth behavior over the center of the channel. Thus close point spacing is required near the walls, but if close point spacing is maintained across the channel then an inordinate number of points is required. Methods of avoiding this problem, along with an illustrative example are presented in Section 1.

Another considerable problem is the minimization of a function of a large number of variables (34 in the case of the fifth set of calculations discussed in Chapter VII). Some of the methods found useful in this work are presented in Section 2.

*In this appendix, superscripted numerals refer to references listed at the end of this appendix.
Since programming has all of the features of an experimental science, some of the difficulties which may be encountered in programming more complicated problems (as discussed in Section 5 of Chapter VIII) are discussed in Section 3.

1. Some Numerical Methods

A. An Example

To illustrate some of the numerical methods employed in this work, consider the equations:

\[
\begin{bmatrix}
\frac{1}{Re} (D^2 - \frac{1}{2})^2 & - \frac{1}{2} (D^2 - \frac{1}{2}) & \phi_0 & d_0 \\
(D^2 - \frac{1}{2}) & \frac{1}{Re} (D^2 - \frac{1}{2})^2 & \phi_1 & d_1 \\
\end{bmatrix}
\]

A2.1.1

where \( Re = 12300.0 \). These are very similar to equations 7.3.8. Both the drives \( d \) and the responses \( \phi \) are to be even functions of the independent variable \( y \) (\( D = \frac{d}{dy} \)) and the boundary conditions on the \( \phi \) are:

\[
\phi_{o,1} = \frac{d\phi}{dy} = 0 \quad ; \quad y = \pm 1
\]

A2.1.2

The even homogeneous solutions of equations A2.1.1 are characterized by the forms (see 7.3.12):

\[
\cosh(\sigma y), \cosh(\gamma y) \cos(\omega y), \text{ and } \sinh(\gamma y) \sin(\omega y)
\]

A2.1.3

where:

\[
\sigma = 0.707107 \\
\gamma = 65.946620 \\
\omega = 65.942829
\]

The numerical solution of A2.1.1 is complicated by two things. First, the solutions near \( y = \pm 1 \) are characterized by very
quickly decaying exponentials, but over the central part of
the channel, the solutions are much smoother (assuming of
course that the drives are smooth).

Second, almost any stepwise numerical integration
procedure concocted for equations A2.1.1 will be unstable
because of the very large exponentials involved in the
homogeneous solution. Suppose, for example that a formula
is devised for integrating from the center out. Then if an
error of ε is made (say by round off) near y = 0, this error
is bound to be propagated by the rising homogeneous solution
so that at y = 1, the error has grown to order εeγ. However,
eγ is of order 10^{28} and present-day digital machines have a
precision of about 10^{-8} (10^{-16} in double precision) so that
such errors are clearly intolerable. There are several ways
out of this difficulty; one could, for example, invert the
(n x n where n is the number of points) matrix instead of
stepwise integrating. On the other hand, one can try to
separate the equations in such a way that rising exponentials
do not appear. It is the latter course of action which will
be pursued below.

B. Geometric Point Spacing

The first problem then is to find a way of avoiding an
excessively large number of integration points. If the
independent variable y were made discrete by the usual linear
point spacing, i.e.:

\[ y_n = nh \ ; \ n = 0,1,\ldots,N_0 \]

\[ h = \frac{1}{N_0} \]

Then \( N_0 \) would have to be very large (500-1000) to accommodate
the rapid behavior near \( y = 1 \). Now it is not necessary to pick
a linear point spacing; in fact, integration procedures can be
 devised which make use of most any set of points $y_i$ such that:

$$0 = y_0 < y_1 < y_2 < \ldots < y_n = 1$$  \hspace{1cm} (A2.1.5)

but the linear point spacing has the advantage that only a
very small number of constants (those used in the integration
formulae) need be calculated and saved. Thus for example, the
$6^{th}$ order, second derivative formula: *

$$f_{n+1} - 2f_n + f_{n-1} = \frac{h^2}{12} (f''_{n+1} + 10 f''_n + f''_{n-1}) + O(h^6)$$  \hspace{1cm} (A2.1.6)

is complete except for the specification of $h$.

Fortunately, a nonlinear point spacing formula can be found
which retains most of the advantages of the linear formula.

Let:  \hspace{1cm} 0.95 < \beta < 1  \hspace{1cm} (A2.1.7)
and:  \hspace{1cm} \rho = 1/\beta  \hspace{1cm} (A2.1.8)

Then define $y_n$ as:

$$y_n = \frac{1-\beta^n}{1-\rho \beta^n} ; \ n = 0,1,\ldots,N_0$$  \hspace{1cm} (A2.1.9)

and let:

$$\frac{h_{n+1}}{h_n} = \beta^n \frac{\rho-1}{\rho^n}$$  \hspace{1cm} (A2.1.10)

Such a set of points might well be termed "geometric" or
"exponential" because:

$$\frac{h_{n+1}}{h_n} = \beta \quad \text{for all } n$$  \hspace{1cm} (A2.1.11)

*See Hildebrand, p. 223.
Thus, any approximate numerical formula—whether differentiating, integrating, smoothing, or whatever—is similar (in a geometrical sense) under all translations along \( n \) in the range \( 0 \leq n \leq N_0 \).

For example, the linear formula in equation A2.1.6 may be written:

\[
f_{n+1} - f_n = f_{n-1} - f_n + \frac{h^2}{12} (f''_{n+1} + 10f''_{n} + f''_{n-1}) + O(h^6) \tag{A2.1.12}
\]

The corresponding geometric formula is:

\[
f_{n+1} - f_n = \beta (f_{n} - f_{n-1}) + (h_n)^2 \left[ \frac{1}{12} \rho - \frac{\rho^2}{12} \right] f''_{n+1} + \frac{1+4\rho + 4\rho^2 + \rho^3}{12\rho} f''_{n} + \frac{\rho^2 + \rho - 1}{12\rho} f''_{n-1} + O((1-\beta)h^5_n) + O(h^6)
\tag{A2.1.13}
\]

Thus, all of the coefficients (save \( h_n \)) may be calculated once and for all and stored somewhere. The only table of numbers required is the \( h_n \); but since several formulae of various types are used, and since all require only the table of \( h_n \), the formulae are simple to use.

The problem of calculating the coefficients of a given formula is usually difficult (those of equation A2.1.13 are exceptionally easy), but it is generally not difficult to set up matrix equations which these coefficients must satisfy. The matrices behave well and standard matrix inversion subroutines for a digital machine handle the problem nicely, so there is no problem.

Typical values for the geometric formula of equations A2.1.7 through A2.1.10 are as follows:

for \( N_0 = 100 \), take\* \( \beta = 0.97070313 \), then \( \rho = 1.03018108 \)

\*This value of \( \beta \) is chosen for its simple binary form; \( \beta = 0.7618 \)
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The above values have been tried many times and seem to be appropriate for the problem at hand.

C. Factorization of the Matrix Operator

The second problem is that of avoiding excessive growth of propagated roundoff error. The trick here is to factor the matrix operator of equation A2.1.1 into two parts—one characterized by exponentials of the form $e^{-\gamma y}$ and the other characterized by exponentials of the form $e^{+\gamma y}$. Then the integration can be done in two steps—the matrix involving $e^{-\gamma y}$ can be integrated stepwise from the center outward, the matrix involving $e^{+\gamma y}$ can be integrated from the walls inward. Since the solutions are even in $y$ (similar statements apply if the solutions are odd in $y$) the functions need be integrated only over one half ($0 \leq y \leq 1$) of the channel.

The matrix operator of equation A2.1.1 can be factored to yield:

$$[L] = [A][B][C]$$  

A2.1.14
where:  

\[
\begin{bmatrix}
\frac{1}{Re} (D^2 - \frac{1}{2})^2 & - \frac{1}{2} (D^2 - \frac{1}{2}) \\
(D^2 - \frac{1}{2}) & \frac{1}{Re} (D^2 - \frac{1}{2})^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{Re} (D-\gamma) \\
\frac{\sqrt{Re}}{2\gamma}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{\sqrt{Re}} (D + \gamma) \\
- \frac{\sqrt{Re}}{2\gamma}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{\sqrt{Re}} (D^2 - \frac{1}{2}) \\
0
\end{bmatrix}
\]

and:  

\[
\gamma = \left[ \frac{1}{4} + \frac{Re}{2} (\frac{1}{2} + \frac{1}{4Re}) \right]^{1/2}
\]

It seems to be a fortunate characteristic of the type of expansion used (employing Hermite Functions, see Chapter VII) that the matrix operators can be factored in this way and that the factors have real coefficients. Another fortunate feature of this factorization is that the matrices commute.

2. Minimization Procedures

The efficient use of the variational methods described in Chapter VII requires the minimization of an error \( e \) with respect to a large set of variables \( \{ a_n \} \). If only the \( G_0 \) and \( G_1 \) terms

*The only variables considered here are the various amplitudes of the parts of the approximating functions.
are retained in the approximate velocity then the error is of the form:

\[ e = \frac{f_1(a_n) + \beta f_2(a_n) + \beta^2 f_3(a_n)}{\beta} \]  

A2.2.1

where \( f_1, f_2, f_3 \) and \( \beta \) are homogeneous quadratic functions of the \( \{a_n\} \). If more than the first two functionals \( (G_0 \text{ and } G_1) \) are retained, the error is of the form:

\[ e = \frac{[f_1(a_n) + g_1(a_n) + h_1(a_n)] + \beta [f_2(a_n) + g_2(a_n)] + \beta^2 [f_3(a_n)]}{\beta} \]  

A2.2.2

where \( f_1, f_2, f_3, \) and \( \beta \) are homogeneous second order functions of the \( \{a_n\} \); \( g_1 \) and \( g_2 \) are homogeneous third order functions, and \( h_1 \) is a fourth order homogeneous function.

Now since \( e \) is a complicated function of the \( \{a_n\} \), it is impractical to compute derivatives of the \( e \) analytically. Thus a Newton-Raphson method* such as the algorithm:

1. \[ \Sigma_m \frac{\partial^2 e}{\partial a_n \partial a_m} \delta a_m = \frac{\partial e}{\partial a_n} \]

A2.2.3a

2. \[ a_n \leftarrow a_n - \delta a_n \]

A2.2.3b

3. go to 1. until convergence

A2.2.3c

is not particularly suitable since it would require an excessive number of calculations of \( e \).

Furthermore, minimization methods which sequentially step the \( a_n \) have been found to be unsatisfactory. While such methods eventually converge, the convergence seems to be highly oscillatory and not at all rapid.

*For a discussion of this method, see Hildebrand, 2 Chapter 10.
Another method which was used with some success is based on calculating the gradient of $e$: i.e.

$$ v_n = \frac{\delta e}{\delta a_n} $$

Having calculated the vector $v_n$, the error was written as a function of the single parameter $\varepsilon$ as:

$$ e(a_n + \varepsilon \cdot v_n) $$

The error was then minimized on the single parameter $\varepsilon$ (holding $v_n$ constant) by a Newton-Raphson method. This method yielded better results than the single step method outlined above, but the convergence was still rather oscillatory. It was noted that the parameter $\beta$ oscillated almost directly with the vector $v_n$, and this led to the speculation that a modified gradient, normal to the gradient of $\beta$, would be more suitable. Thus the following minimization scheme was tested:

$$ v_n = \frac{\delta e}{\delta a_n} $$

$$ b_n = \frac{\delta \beta}{\delta a_n} $$

$$ v'_n = v_n - \frac{\sum_{m} v_m b_m}{\sum_{m} b_m b_m} \cdot b_n $$

and:

$$ e = e(a_n + \varepsilon \cdot v'_n) $$

The error $e$ was then minimized with respect to the single parameter $\varepsilon$. Then the error was minimized with respect to $\beta$ as follows:

Since: $$ \beta(\lambda a_n) = \lambda^2 \beta(a_n) $$

Minimize: $$ e(\lambda a_n) $$

with respect to $\lambda$. Thus, a two stage minimization process was employed: first with respect to $\varepsilon$ as in equations A2.2.6 and
then with respect to \( \lambda \) as in equations A2.2.7. This minimization process worked quite well, and in every case tried converged quickly and without oscillation.

3. Notes on Programming

Almost all of the programs written in conjunction with this work were written in FORTRAN,\(^4\) a rather versatile and efficient algorithm decoder written primarily for the IBM 700-7000 series of Electronic Data Processing Machines. Occasionally, a small section of coding was done in the more basic FAP\(^3,5\) language--especially coding for simple arithmetic functions such as:

\[
\frac{\sinh(\theta y)}{\theta} \quad \text{or} \quad \frac{\cosh(\theta y) - 1}{\theta^2}
\]

The total program for the variational problems\(^*\) was divided into four major subprograms along with a number of minor subroutines for input, output, arithmetic functions, etc. The most important single subprogram (named VERROR) had the task of calculating all of the velocity components, inverting a number of matrix operators, forming the error and minimizing the error with respect to the homogeneous solutions present. This subprogram required about three seconds to execute on the IBM 7090. The comparable program for the approximation described in Section 2 of Chapter VIII will execute in a time estimated at about 20 to 30 seconds for the 7090. On the improved 7094, these times are estimated to be 1.5 and 10 to 15 seconds, respectively.

The other important programs were: a "(MAIN)" program to read in data, print results, etc.; a \$SETUP subprogram to setup important integration constants, the geometric point spacing,

\*See Chapter VII.
etc.; and a minimization program named MIN which, using the results of VERROR, minimized the error with respect to the available free parameters.

The minimization technique described in Section 2 of this appendix requires on the order of n calls to VERROR for minimization along one gradient, where n is the number of free parameters. For the approximation described in Section 2 of Chapter VIII there are about 360 free parameters. Thus, if ten gradients are required and VERROR takes about ten seconds per calculation, then about 360.10.10/3600 = 10 hours of machine time on the 7094 would be required for the minimization of the error for one value of $R_e$ and one value of M! At an economic value of $550 per hour, such an amount of machine time is out of the question. A very careful optimization of subprogram VERROR might reduce its execution time to five seconds, and an improved minimization method might require only five gradients, but even so execution time for each value of $R_e$ and M would be on the order of 2.5 hours. Thus, such a problem is feasible only on a machine of much larger computing capacity; e.g. the IBM 7030 ("STRETCH"), which for such a program would be (at least) an order of magnitude faster--15 minutes of machine time for each of (say) 20 to 100 points--begins to sound reasonable.

In addition to problems of execution speed, "debugging" such an iterative--variational program is very difficult, mainly because many errors do not appear until subprogram VERROR has been executed several times. However, by the time an error is detected, VERROR has been called so many times that the "evidence" has been wiped out. Break-point dumping is no real solution because the dumps must necessarily be selective and (inevitably) the important quantities will be forgotten.
The ideal solution seems to be a real-time, online man-machine interaction system; e.g. the "time-sharing" system proposed by Corbató, Daggett, and Daley.¹

Thus, future work on turbulence problems* seems to hinge critically on the availability of a machine with three important features:

1. An algorithm decoder considerably improved over the present capabilities of FORTRAN.

2. A large scale digital machine at least an order of magnitude faster than the 7090-7094 machines and preferably with a memory capacity on the order of $10^5$ words.

3. A time-sharing system for man-machine interaction.

The writers of FORTRAN are doubtless aware of its limitations and are taking steps to eliminate them, so that 1. will probably come to pass within the year. In addition, current proposals for time-sharing systems and the associated "hardware" would adequately cover 2. and 3.

*At least as formulated in this work. There is no question that much better ways will eventually be formulated.


5. IBM: "FORTRAN Assembly Program (FAP) for the IBM 709/7090," form J28-6098-1, IBM, New York.
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