Self-Excited Vibrations of a Spinning Disk

by

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Abstract

The existence of instabilities of spinning flexible disks limits the performance of practical devices such as circular saws, magnetic and optical computer memory storage disks, bladed disks of turbomachinery, and automobile disk brakes. Such instabilities take place when agents that are interacting with the disk, such as the air-field or the load system, and under some precise operating conditions, channel the kinetic energy of the disk into unstable transverse vibratory modes.

In this thesis we study the dynamics of a flexible spinning disk interacting with an adjacent air-field and with a localized mass-spring-dashpot load system. Most destabilizing mechanisms start to manifest themselves above the lowest critical speed of the disk. The shear loading from the air-film is studied and it is found that, under specific operating regimes, the shear loading could lower the disk’s critical speeds and thus reduce the useable range of operation. The shear loading is also destabilizing at all speeds of the disk.

The qualitative nature of the self-excited vibrations induced by a load system are then analyzed for a traveling string. The string has essentially the same dynamics as the spinning ndisk. There, the mechanisms of instability due to each individual element of the load system are obtained and the means by which energy is channeled into unstable modes are explained.

Lastly, we give an experimental evidence of the existence of such self-excited waves with emphasis on a rather rare wave phenomena. It is that the disk supports solitary waves which are the manifestation of a truly nonlinear phenomenon.

Thesis Supervisor: Stephen H. Crandall
Title: Professor Emeritus
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Chapter 1

Introduction

The existence of instabilities of spinning flexible disks limits the performance of practical devices such as circular saws, magnetic and optical computer memory disks, bladed disks of turbomachinery, and automobile disk brakes. Such instabilities take place when agents such as the air-film or the load system, under some precise operating conditions, channel the kinetic energy of the spinning disk into transverse vibratory modes of the disk. This energy is of course supplied by the motor that is driving the disk. The disk in itself cannot "create" energy. In most applications, the spinning disk dynamics are governed by the same equation. The differences however are due to the relative importance of the various forces present in that equation. For instance, in the floppy disk situation the membrane stresses are of the same order of importance as the flexural rigidity and the load system applies a transverse force on the spinning disk. In an automobile disk brake however, the effect of the centrifugally induced membrane stresses is negligible when compared to that of the flexural rigidity and the brake-pads apply forces that act tangentially, along the plane of the disk. In this thesis we study the dynamics of a flexible spinning disk interacting with an adjacent fluid field and with a localized load system such as a saw guide or computer diskette's read-write head. The goal is to identify and understand the physical nature of the various instability mechanisms using simplified models whenever possible. We concentrate our study here on the common floppy diskette. The results can be extended and analogies drawn for other dynamic systems of the same family.
1.1 The Common Floppy Diskette

The performance of a floppy disk is measured in terms of how fast the read/write head can transfer data between the computer memory and the spinning disk. There are two key factors that directly affect the diskette's performance. The first one is latency—the time it takes the read/write head to find a specific location on the disk, and the second is the disk's spin rate. To reduce latency the mass of the disk, and thus its thickness, must be minimized. With the current standards, floppy disks operate at speeds that are well below the first critical. However, with the modern trend towards faster and thinner disks, the next generation of floppy disks could bring the operating speed closer to the lowest critical speed. Thus, making floppy disks more prone to instabilities.

![Floppy Disk Assembly](image)

Figure 1-1: Floppy Disk Assembly

Before we can understand how a disk can become unstable, we need to know the dynamics of the disk and the agents it interacts with. For this, we take as an example the very common floppy diskette shown in figure 1-1. The main constituents are a flexible disk mounted on a rigid spindle. The assembly is then put together between two rigid covers (casing) having each a slot for access by the two read/write electromagnetic heads.

When the disk is spun, a self-lubricating air-film is established between the disk and each of the rigid covers. The two air-films prevent the rubbing of the disk against
each of the rigid covers\textsuperscript{1} and any damaging scratches. In modern designs, the diskette is “double-sided” and there are two heads, protruding through two slots as shown in figure 1-2, that are in contact with the disk.

![Diagram of a flexible disk with read/write heads and air-films](image)

Figure 1-2: Diskette with Two Read/Write Heads

Under proper (or rather improper) operating conditions, the interaction of the flexible disk with the surrounding air-films and the read/write heads may give rise to self-excited traveling waves in the disk. The process of data transfer between the read/write electromagnetic head and the floppy disk it rides on would be made very unreliable if such traveling waves took place and were sustained by some mechanism. The transverse motion resulting from such traveling waves would cause the air gap between the read/write head and the spinning disk to vary during the data transfer. Such operating conditions must of course be avoided. The knowledge of the frequencies associated with these traveling waves, and thus of the natural frequencies of the spinning disk, is of primary importance for the design of the next generation of high speed floppy drives. The designer must ensure that the floppy disk operates away from any resonances and provide sufficient damping to stabilize the disk’s operation.

1.2 Wave Kinematics in a Spinning Flexible Disk

Because of its flexibility, the spinning disk can support traveling waves in the circumferential direction. The gyroscopic nature of the spinning disk, along with the centrifugally induced membrane stresses, leads to natural frequencies that change as a function of the disk’s spin rate $\Omega$. To better visualize this phenomenon, let us first

\textsuperscript{1}In the actual floppy diskette, there is a piece of soft cloth between the flexible disk and each of the rigid covers.
consider a non-rotating disk clamped at its inner edge and free at its outer edge as shown in figure 1-3.

Figure 1-3: Spinning Flexible Disk with a Traveling Wave

Any given traveling wave in such a disk can be characterized by the number $n$ of its nodal diameter and the number $m$ of its nodal circles. The wave that is shown in figure 1-3 has ten nodal diameters and one nodal circle—clamp at the inner edge, and is denoted wave $(10, 1)$. There are in fact two such waves, one traveling forward and the other traveling backward. Let us denote by $\pm \omega_n$ the angular frequencies corresponding to these waves. The frequency $\omega_n$ is measured here with respect to the disk. When the disk is spun at a small rate $\Omega$, a stationary observer sees the forward wave move faster in the forward direction and the backward wave move slower in the backward direction. The corresponding frequencies, as seen by our stationary observer and the read/write heads of the disk, suffer a Doppler shift and become now

\[
\omega_n^{bak} = -\omega_n + n\Omega \\
\omega_n^{fwd} = +\omega_n + n\Omega
\]  

At very high rotation speeds $\Omega$, both the forward and the backward wave will be seen to travel forward by our stationary observer. The value of $\Omega$ for which $\omega_n^{bak}$ vanishes is called a critical speed. At this speed $\Omega_c$, the phase speed of the backward wave goes through zero and changes its sign\(^2\). It is at this critical spin rate $\Omega_c$ that the read/write

\(^2\)This is more than just a kinematic condition. At a critical speed, the disk has zero effective bending rigidity and thus, it cannot support any transverse load.
heads can induce self-excited vibrations of the spinning disk. The surrounding air-film causes instabilities at higher values of the rotation speed $\Omega$. Because the disk has several possible mode shapes, it has accordingly several critical speeds—one for each mode. It is usually the lowest critical speed that is of importance since current spinning disks applications are all subcritical.

![Campbell Diagram](image.png)

Figure 1-4: Natural Frequencies of the Mode with Two Nodal Diameters.

The linear dependency of the disk's frequencies upon the spin rate $\Omega$ would lead us to believe that if we graphed them in terms of the spin rate, we would get perfectly straight lines. This is however not the case for our spinning disk. The centrifugally induced in-plane stresses will cause an increase in the disk’s effective stiffness, resulting in an increase in the disk based frequency $\omega_n$. This causes the natural frequencies of our spinning disk to have the “non-linear” dependence shown in the Campbell diagram of figure 1-4. Such a diagram and has very valuable information on the disk’s dynamics. It is very useful in identifying the critical speeds, where the descending branch reflects off, and the regions of unstable operation of our spinning disk.
1.3 Major Contributions of this Thesis

The research reported in this dissertation encompasses several aspects pertaining to the stability of spinning disks. There are three major contributions and they are:

- Effects of the shear loading on the dynamics of the spinning disk.
- Physical explanations of several divergence and flutter instabilities.
- Experimental evidence of solitary waves in the disk.

When a disk is spinning in a fluid medium, the viscous drag from the fluid induces in-plane shear stresses in the disk. When these stresses are of appreciable magnitude—relatively speaking, they could cause buckling of the spinning disk. Even if the buckling condition is never reached, the shear loading reduces the effective bending stiffness of the disk. This lowers the critical speeds of the disk and makes it more prone to instabilities.

Instabilities of rotating disks have been reported in the literature for quite some time now. However, and because of several mathematical difficulties associated with the equation of motion of the disk, the majority of the work to date is mainly numerical. In this thesis physical explanations of the instability mechanisms are given for a simplified model in which the rotating disk is replaced by a rotating loop of stretched string. These explanations constitute the second major contribution of this thesis.

Finally, the third contribution reports on the existence of solitary waves in the disk. These are the manifestation of a truly nonlinear phenomenon. The practical implications of such finding are not clear yet and we give however a qualitative physical explanation for such behavior.

1.4 Thesis Contents

This first chapter—the one you are reading now, presented an introduction to the subject on rotating flexible disks along with some relevant kinematics as well as an overview of the contents of this thesis. In the second chapter, we present the equations governing the transverse motion of a flexible disk that is spinning against
a rigid baseplate and the pressure in the air-film between the two. The mathematical difficulties are discussed and an extensive literature review is presented. We then specialize our study to a disk that is spinning between two plates and obtain the natural frequencies, critical speeds, and mode shapes for a spinning disk by means of the finite difference technique. The third chapter covers the treatment on the fluid-induced shear loading and the resulting effects on the dynamics of the disk. In chapter four, we investigate the instabilities induced by the load-system for a close cousin of our disk. It is a spinning string and it has essentially the same type of instabilities as the full disk. We give a thorough presentation of the kinematics of the spinning string along with the mechanisms governing the various instabilities. In chapter five, we report on some interesting experimental findings and we describe the experimental facility used in this dissertation. Closing this thesis is chapter six where a summary of the main results is presented along with suggestions for future work.
Chapter 2

Flexible Spinning Disk Interacting with an Air-Field

In this chapter we present and discuss the equation of motion of a flexible disk that is spinning against a rigid stationary plate along with the loading from the ensuing air-film and the reaction from a mass-spring-dashpot load system. A comprehensive literature review is presented that highlights major historical contributions and describes the mathematical difficulties associated with solving the spinning disk's equation. The interaction of the disk with the air that is above it is also discussed and illustrated with the experimental findings of fellow researchers. The study is then specialized to a disk that is spinning between two rigid plates as a model for the common floppy diskette. For this relatively yet not so simple problem, we study the free vibrations of the disk. By assuming periodic solutions in the circumferential direction, the partial differential equation governing the transverse motion of the disk is transformed to an ordinary differential equation that is then solved with the finite difference technique. The natural frequencies and critical speeds of the disk are calculated and presented for the disk alone—without the pressure loading from the air-film. Internal damping in the disk is then introduced and we show that it is always stabilizing. The pressure loading is not treated in this presentation since several researchers have undertaken such study. The effects of the mass-spring-dashpot load system are deferred to chapter 5 where a simpler model is studied.
2.1 Mathematical Model of the Spinning Disk

Let us consider the flexible disk shown in figure 2-1 spinning at a rate $\Omega$ next to a rigid back-up plate. The read/write electro-magnetic head in the floppy drive is modeled here as a mass-spring-dashpot load system that rides on and always remains in contact with the spinning disk. The disk's rotation generates a pressure gradient in the air-film in the radial direction and causes an outward fluid flow—in essence, the disk behaves as a very inefficient centrifugal pump. The transverse deflection of the spinning disk is thus coupled to the pressure in the air-film. The equation governing small transverse displacements $w(r, \theta, t)$ of the spinning disk is a bending plate equation with centrifugal stiffening driven by the pressure $p(r, \theta, t)$ in the air-film, the reaction $f_L (r, \theta, t)$ from the load system, and the gravity force $\varepsilon_d h_d g$ acting in the negative $z$-direction.

$$
\varepsilon_d h_d \frac{D^2 w}{Dt^2} + D_b \left(1 + \eta \frac{D}{Dt} \right) \nabla^4 w + c \frac{\partial w}{\partial t} \ldots
$$

$$
- \frac{h_d}{r} \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial w}{\partial r} \right) - \frac{h_d}{r^2} \frac{\partial^2 w}{\partial \theta^2} = p + f_L - (\varepsilon_d h_d) g
$$

(2.1)

Figure 2-1: Spinning Disk next to a Back-Up Plate with Load System

With respect to stationary polar coordinates $(r, \theta)$ the deflection $w(r, \theta, t)$ of the disk is governed by the partial differential equation
where $D/Dt$ is the convective derivative and $\nabla$ is the two-dimensional differential operator expressed in polar coordinates

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\] (2.2)

The disk has Young’s modulus $E$, Poisson’s ratio $\nu$, mass density $\rho_d$, and uniform thickness $h_d$. With these parameters, the bending stiffness $D_b$ of the disk is

\[
D_b = \frac{E h_d^3}{12 (1 - \nu^2)}
\] (2.3)

External and internal damping are modeled by uniformly distributed mechanisms with parameters $c$ and $\eta$. The external damping is velocity dependent with respect to the stationary frame. The internal damping is strain-rate dependent for an element of fixed identity. The internal damping is an inherent property of the disk’s material and the external damping is due to the air loading on the free surface of the disk.

The last two terms on the left hand side of equation 2.1 are stiffening terms introduced by the in-plane membrane stresses $\sigma_\theta$ and $\sigma_r$. These stresses are induced by centrifugal forces in the disk caused by the rotation rate $\Omega$. For a disk of inner radius $a$ and outer radius $b$, the stresses $\sigma_r$ and $\sigma_\theta$ are given by Adams [2]

\[
\sigma_r = \frac{\rho_d \Omega^2}{8} \left[ (3 + \nu) (b^2 - r^2) \left( 1 + \frac{1 - \nu}{3 + \nu} \frac{ka^2}{r^2} \right) \right]
\] (2.4)

\[
\sigma_\theta = \frac{\rho_d \Omega^2}{8} \left[ (1 + \nu) (a^2 + kb^2) - (1 + 3\nu) r^2 - (1 - \nu) \frac{ka^2 b^2}{r^2} \right]
\] (2.5)

where we have used the dimensionless parameter $k$ defined by

\[
k = \frac{(3 + \nu) b^2 - (1 + \nu) a^2}{(1 + \nu) b^2 + (1 - \nu) a^2}
\] (2.6)

The boundary conditions for equation 2.1 are those of a clamped edge at the hub i.e., both the displacement and the slope must vanish

\[
[w]_{r=a} = 0
\] (2.7)

\[
\left[ \frac{\partial w}{\partial r} \right]_{r=a} = 0
\] (2.8)
and those of a free edge i.e., vanishing bending moment and vanishing shear force, at the outer radius

$$\left[ \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \left( \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \right]_{r=b} = 0 \quad (2.9)$$

$$\left[ \frac{\partial}{\partial r} \left( \nabla^2 w \right) + \frac{1 - \nu}{r^2} \frac{\partial^2}{\partial \theta^2} \left( \frac{\partial w}{\partial r} - \frac{w}{r} \right) \right]_{r=b} = 0 \quad (2.10)$$

Under laminar flow conditions\(^1\), the pressure in the air-film that is between the spinning disk and the back-up plate is governed by a Reynolds's lubrication\(^2\) equation [44] with an added term due to centrifugal loading:

$$\frac{\partial}{\partial r} \left( rh^3 \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( h^3 \frac{\partial p}{\partial \theta} \right) = 12\mu_a r \left( \frac{Dh}{Dt} - \frac{\Omega \partial h}{\partial \theta} \right) + \frac{3}{10} \rho_a \Omega^2 \frac{\partial}{\partial r} \left( r^2 h^3 \right) \quad (2.11)$$

where the quantities \(\rho_a\) and \(\mu_a\) are respectively the mass density and the kinematic viscosity of the air-film, taken to be incompressible. The thickness \(h(r, \theta, t)\) of the air-film between the disk and the back-up plate is given by

$$h(r, \theta, t) = h_a + w(r, \theta, t) \quad (2.12)$$

where \(h_a\) is the thickness of the air-film at the hub, \(r = a\). The pressure distribution inside the fluid film must satisfy the boundary conditions of atmospheric pressure at both the inner and the outer edges of the spinning disk

$$p(r = a) = 0 \quad \text{and} \quad p(r = b) = 0 \quad (2.13)$$

The reaction of the load system located at \(r = r_0\) and \(\theta = \theta_0\) is modeled as a point force. The rotary parameters of the load system and the in-plane friction force\(^3\) between the head and the spinning disk are ignored.

$$f_L = -\delta (r - r_0) \delta (\theta - \theta_0) \left[ m_L \frac{\partial^2 w}{\partial t^2} + c_L \frac{\partial w}{\partial t} + k_L \right] w(r, \theta, t) \quad (2.14)$$

with the quantities \(m_L\), \(c_L\), and \(k_L\) being respectively the mass, dashpot constant, and stiffness of the load system.

---

\(^1\)Small values of the Reynolds number \(Re = \nu^2 \Omega / \mu_a < 3 \times 10^8\). (See [48, page 106])

\(^2\)Obtained by ignoring the temporal inertia term in the full Navier–Stokes equation.

\(^3\)Chen and Bogy [15] found that the contribution of this force is only second order.
The solution of the disk's equation 2.1, including the loading from the air-film 2.11 and the reaction 2.14 of the load system, is not a trivial task. The complications come from the centrifugal stiffening terms in the disk's equation, the nonlinear dependence on $h$ —the gap height, and thus on the transverse displacement $w$ of the disk in the air-film's equation, and finally the singularity introduced by the load system.

To put our contribution i.e., this dissertation and the many years it has taken to complete, in perspective, let us review the work of fellow researchers on the fascinating subject that is the dynamics of spinning flexible disks.

2.2 Literature Review

The first recorded investigation on spinning disks was the work of Lamb and Southwell [34]. The complication they encountered, with the computational tools of their time, comes from the presence of both flexural rigidity and centrifugal stiffening effects. They obtained lower bounds to the disk's natural frequencies by summing the individual contributions of the bending rigidity and that of the centrifugal loading, taken separately. In a subsequent paper, Southwell [51] studied the free vibrations of a circular disk, and the stiffening effects introduced by rotation.

Shortly after, Wilfred Campbell—who was with General Electric at that time, published an extensive study [9] on the vibrations of turbine disk wheels. Among the many aspects he discussed, we report here his showing, experimentally, of the existence of traveling waves in the spinning disk. He also displayed various mode shapes using Chladni's technique and introduced what he called at that time the "Frequency-Speed" diagram. This became later the "Campbell Diagram" for obvious reasons.

In 1957, Tobias and Arnold [55, 56] found that imperfect disks (whose properties vary with the $\theta$-direction), exhibit unstable behavior because of non-linear mode coupling. Their work was the first to report instabilities of rotating disks. Up until the work of Tobias, whose renowned contributions are in the field of machine-tools' chatter, the practical applications were for large circular saws in the timber industry and for bladed turbine disks. Spinning disks did not escape the space age, the work
of Eversman [24] and Simmonds [50] dealt with applications such as large rotating satellite antennas and solar sails. The application to computer disks did not come until later on with the work of Bulkeley [8], Pearson [41] and several others at IBM. The diskette era started in the mid 60's and several researchers began looking into the dynamics of flexible spinning disks. We give here an account, not necessarily chronological, of some of the major contributions and key contributors to this field. We begin with the work of Pelech and Shapiro [42] here at MIT. They calculated the steady state deflection of a disk spinning on an air-film against a rigid stationary plate. They transformed the formulation of the coupled problem to an initial value problem with three first order differential equations that they integrated numerically. They solved the fluid-structure interaction problem, and found that the air-film always exerts a pulling force on the disk, bringing it closer to the rigid plate as shown on figure 2-2. They also measured the disk's steady state deflection for various operating conditions and obtained good agreement with their numerical results.

![Diagram](image)

Figure 2-2: Steady State Deflection of Disk Spinning against ONE Baseplate

Adams [1, 2] confirmed the result of [42]. He used an iterative approach to solve a boundary value problem he obtained from the coupled nonlinear ODE's governing the steady state configuration of the disk. In his technique, he begins by guessing an initial shape for the pressure distribution and calculates the resulting disk deflection. He then uses the deflected shape he obtained to solve for the resulting pressure from the air film's equation. This pressure is compared with the initial guess and the difference is used to improve the guess iteratively until convergence is reached. In an attempt to eliminate the air-film's equation, he further modeled the air-film as an
elastic foundation and obtained the following expression for the foundation’s stiffness

\[ K = -\frac{\delta p}{\delta w} = \frac{18\mu_a Q}{\pi} \left\{ \frac{\int_a^b \left( \frac{dr}{r_h^3} \right) \int_a^b \left( \frac{dr}{r_h^3} \right) - \int_a^b \left( \frac{dr}{r_h^3} \right) \int_a^b \left( \frac{dr}{r_h^3} \right)}{\int_a^b \left( \frac{dr}{r_h^3} \right)} \right\} \] (2.15)

where \( Q \) is here the total outflow rate of air in the radial direction. The details of obtaining this result however were not given. In addition, we tried using this result to simplify the disk equation and replace the pressure loading by a simple stiffness term. We obtained \( K = 0 \) for a disk spinning between two rigid plates i.e., with uniform air-gap thickness. We concluded that the expression given by Adams has very strong limitations. Furthermore, because of the circulatory forces in the air-film, the stiffness parameter should be complex valued, not purely real as reported here.

Of related interest, we mention the work of Licari and King \[36\], Benson and Bogy \[4, 5\], Carpino and Domoto \[11, 12\], and Hosaka and Nishida \[29\] who all studied the steady state characteristics of a spinning disk with an air-film and/or a load system. Ono and Maeno \[39\] studied, both theoretically and experimentally, the vibration of a spinning disk with a point contact head (not a load system) which is a model for an extremely stiff load system. They found good agreement of measured and predicted natural frequencies.

The potential of the mass-spring-dashpot load system to render the operation of a spinning disk unstable was first reported by Iwan and Moeller \[32\]. Using Galerkin’s technique, they found that any given mode has two unstable regions. The first one (due to the spring of the load system) occurred right above the critical speed for the corresponding mode and extended over a finite range of the disk’s spin rate. The second region, which they called “Terminal Instability Region” was due to the mass of the load system. They also found that the addition of the dashpot of the load system makes the disk unstable for all speeds above the lowest critical.

The work of Chen and Bogy \[17, 16, 13, 14\] at the University of California, Berkeley, encompasses several aspects of computer disks and circular saws. The bulk of the work they published is mainly numerical and focuses on the effects of various models for the interaction between the disk and the load system, the sensitivity of the natural frequencies due to various modeling parameters, and the calculation of natural
frequencies and critical speeds for the various models. They also studied the effect of
the in-plane friction force between the load system and the disk in [15]. They found
that such a force has no effect (to first order) on the stability of the disk. In their
analysis, they accounted for the in-plane component of the force by a change in the
in-plane stresses field \((\sigma_r, \sigma_\theta, \sigma_\phi)\). For the vertical component –obtained by multiply-
ing the friction force with the local slope, they treated it as a transverse loading to
the spinning disk. It is not clear whether this formulation represents the “buckling”
effect of the friction force. Such effect is proportional to the second derivative\(^4\) of the
local displacement.

To gain further insight into the destabilizing effects of the load system, Ono, Chen,
and Bogy [38] included the rotary counterparts of the mass, damping and stiffness of
the load system as well and again, studied the effects on the stability of a spinning
disk. They displayed the results of several calculations but did very little to explain
the physical aspects of the unstable behavior of the disk. They found that the rotary
parameters introduce the same type of instabilities as their translational counterparts.
The rotary characteristics become important at high frequencies and have significant
effects on the higher modes (short waves).

Since the work of Iwan and Moeller [32], several researchers investigated the unstable
behavior of the disk and most of the work was numerical. This is all well. From a
design perspective we want to stay away from the instability regions. How the disk
becomes unstable is somewhat of secondary importance. With very few exceptions,
none of the above cited papers dealt with the physics of the various instabilities. This
is partly understandable, the mathematics of the disk are so involved that no simple
physical explanation can be easily deduced. Furthermore, the instabilities are caused
by the load system and the air-film, the disk alone is never unstable. Investigating the
instabilities always requires solving for the air-film’s equation and the load system.
The first one introduces nonlinearities, the second singularities, and both increase the
mathematical difficulties by an order of magnitude. In a recent paper, Shen and Mote

\(^{4}\)This is analogous to the Bernoulli-Euler column whose governing equation of transverse motion
is given by: \(\sigma \frac{\partial^4 w}{\partial t^4} + EI \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^4 w}{\partial x^4} = 0\)
[49] report that for a stationary disk with a rotating load system, the disk becomes unstable when the out-of-plane centrifugal forces overcome the elastic restoring forces. In the unstable range, modes occur in conjugate pairs, one has a growing amplitude and the other has a decaying amplitude. The explanation given by Shen and Mote does not explain the decaying mode solution.

The interest in spinning disks here at MIT was revived with the work of Hosaka and Crandall [28] who studied the self-excited vibrations of a flexible disk rotating on an air-film above a flat surface and without a load system. They solved the coupled disk and air-film equations based on both the finite difference technique and a single mode analysis. They got very good agreements of the two techniques in calculating for the disk’s critical speeds. One of their key results is that for any given mode, the air-film is destabilizing for rotation speeds larger than twice the critical speed for that mode. They also gave an elegant physical explanation for the air induced instability: the mechanism is essentially parallel to that of oil whip in journal bearings [19].

In the rotordynamics literature, it has been known for quite a while now that the self-excited vibrations, if they are to take place, always manifest themselves above\(^5\) the lowest critical speed [25, page 1.73]. From the designer’s perspective, as long as the shaft’s rpm is lower than the first critical, things are guaranteed to be smooth and mellow. Supercritical operation on the other hand requires careful attention to ensure that all potential instabilities are controlled. Our spinning disk is no exception to the rotordynamics family, even though it has its own particularities, and several researchers were concerned primarily with the precise determination of the disk’s critical speeds and the lowest amongst the set. We mention here the work of Adams [3], Greenberg [26], and Chonan [18] amongst others. A recent publication by Yano and Kotera [57, 58] reports an application where there exists regions of instability below the first critical. They studied a disk with a radial in-plane load as a model for railway wheels. We know (now) of two mechanisms that, if taken alone without any

\(^5\)"Above" is to be taken here in a wide sense, we do not mean exactly above the critical speed, but rather anywhere above it
other effects, may make the disk unstable at speeds below the critical\textsuperscript{6}. The first one is the air-film's shear loading (presented in chapter 4) and the second is the in-plane friction force between the read/write head and the disk. These two situations are in fact similar but extreme cases. They both act in the tangential direction. The first one in concentrated and the second is distributed uniformly on the surface of the disk.

2.3 Air Loading on the Spinning Disk

The work to date on the coupling of the disk's motion with that of the surrounding air field has focused on the loading from the air-film that is between the spinning disk and a rigid backup plate. Most probably, this is a consequence of the original work by Pelech and Shapiro [42]. The effect of the air that is above the spinning disk has received very little attention. The closest treatment we found was the work by Schlichting [48, page 102]. He considered the flow field due to the rotation of a rigid disk in a relatively viscous (Low Reynold's Number) semi-infinite fluid medium.

![Diagram of fluid flow](image)

**Figure 2-3: Flow induced by the Rotation of a Disk**

The distribution of the three components of the fluid velocity Schlichting obtained are shown in figure 2-3 along with the flow streamlines. The distribution of the circumferential component is rather obvious. The radial velocity is due to the centripetal

\textsuperscript{6}That at which the phase speed of a disk based backward wave vanishes and then changes sign.
acceleration caused by the circumferential flow and finally, the axial inflow towards the disk is there to compensate for the radial outflow and thus, to satisfy the mass conservation requirement. All these three flows give rise to loadings in their respective direction. The circumferential and the radial flows both apply shear stresses on the disk. The axial flow results in a normal pressure loading which, in contrast with the air-film treated in [42], is always pushing on the disk.

The treatment and results obtained by Schlichting however, break down when the rigid disk spins at relatively high speeds. The flow is no longer laminar and the determination of the air loading necessitates solving for the whole three dimensional flow above the disk\textsuperscript{7}. An investigation by Kobayashi, Kohama, and Takamadate [33] reports that in the transition regime\textsuperscript{8}, there exists Spiral Vortices in the boundary layer above a spinning rigid disk. The pattern Kobayashi and his co-workers reported is sketched in figure 2-4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{flow_regimes_and_vortices.png}
\caption{Flow Regimes and Spiral Vortices Observed by Kobayashi et. al.}
\end{figure}

Hansen and Hunston [27] reported experiments with a rigid disk coated with a com-

\textsuperscript{7}This is a good subject for another PhD thesis.
\textsuperscript{8}When the inertia and viscous forces are of the same order of magnitude
pliant material\textsuperscript{9} and spinning in an enclosure filled with a viscous glycerol solution. Hansen and Hunston observed that when the disk is spinning, a pattern of flow-generated waves forms on the compliant surface of the disk. The waves were confined to the outer periphery of the disk and took place only above a certain critical speed which was a function of the properties of both the disk's coating and the fluid in the enclosure. They further found that such waves exist for laminar as well as for turbulent flow regimes, provided adequate properties of the disk and the fluid are met. D'Angelo and Mote [30, 31] reported recently an interesting series of experiments where they proved that the three dimensional air flow around the disk is destabilizing. Their experimental apparatus, shown schematically in figure 2-5, consists of a flexible steel disk with its drive unit all encased in a vacuum chamber. A camera and a Moiré imaging system were used to record the disk's deflections.

![Diagram of experimental apparatus](image)

Figure 2-5: Experimental Apparatus used in D'Angelo and Mote's Experiments

By spinning the disk in the vacuum chamber they observed that the disk was always stable and they could not see any traveling waves at all. They then performed the same experiment in open air and observed indeed that the interaction with the air gives rise to traveling waves in the disk. They believe that the waves are due to

\textsuperscript{9}This is not our spinning flexible disk yet. The waves here can be thought of as "surface waves" in contrast with the waves in our spinning disk, which are truly "body" waves.
aeroelastic flutter\textsuperscript{10}. By varying the density of the gas\textsuperscript{11} inside the vacuum chamber, they also observed that the onset speed of instability increases with lighter gases. D'Angelo and Mote also reported a rather interesting observation. It is that of a lock-in phenomenon for some ranges of the spin rate $\Omega$. This is the situation where the observed frequencies of the disk stay constant over a finite range of the disk's spin rate. D'Angelo and Mote believe the mechanism to be somewhat analogous to that of vortex shedding [6] of tubes in a crosswise flow.

2.4 Disk Spinning between Two Rigid Plates

When the disk is spinning next to only one plate, the air being pumped outwards by the rotation of the disk provides a pulling force that brings the disk closer to the back up plate. The disk will thus depart from its perfectly flat initial shape to accommodate for this pulling force. The resulting steady state deflection and pressure distribution are difficult to obtain. Even though the problem is mathematically challenging, it has very little practical significance. The second complication for the disk with one baseplate is the loading from the air that is above the disk. As we have seen in the preceding section, such effect has not been studied and its importance is not known. Finally, the third complication comes from the dependence of the in-plane stresses in the disk upon the transverse deflection as in the von Kármán plate theory.

![Diagram of Disk Spinning between Two Rigid Plates]

Figure 2-6: Flexible Disk Spinning between Two Rigid Plates

\textsuperscript{10}Like the waves on a flag during a windy day
\textsuperscript{11}By sealing the chamber and filling it with Helium
All these three aspects make a careful solution of the disk an overwhelming numerical task the benefits of which are certainly not evident to those of us interested in understanding the physical nature of the self excited vibrations of the spinning disk. In this thesis we depart from the one baseplate physical model and confine our spinning disk between two rigid plates as shown in figure 2-6. The reason as we have said earlier is twofold: the disk must have a perfectly flat steady state configuration and even more important, the actual floppy disk, which is the raison d'être of this dissertation, is always confined between two rigid walls. We will ensure that the gap spacing is identical on both sides of the disk. To solve for the problem of a disk with two baseplates then, the strategy adopted here is

- Determine the steady state pressure distribution in the air-film.
- Obtain the linearized equations governing the disk vibration.
- Assume periodic solutions and formulate the eigenvalue problem.
- Obtain natural frequencies, mode shapes, and critical speeds using the finite difference technique.

We begin by solving for the steady state pressure distribution in the air-film.

### 2.5 Steady State Solution of the Spinning Disk

The steady state solution is obtained by setting the time and \( \theta \) derivatives to zero in equations 2.1 and 2.11. When the disk is confined between two plates however, the steady state deflection is uniformly zero and we are left with solving for the pressure distribution only. The laminar flow in the air-film between the disk and each of the baseplates is now governed by the equation\(^{12}\)

\[
\frac{d}{dr} \left( r h^3 \frac{dp}{dr} \right) = \frac{3}{10} \theta_a \Omega^2 \frac{d}{dr} \left( r^2 h^3 \right)
\]

\(^{12}\)We will note that because the pressure \( p \) is now a function of the radial coordinate only, we have changed the partial differential operator to the ordinary differential operator.

35
If we replace the air-film thickness \( h \) by the now constant gap thickness \( h_a \), we obtain the simplified equation

\[
\frac{\partial^2 p_o}{\partial r^2} + \frac{1}{r} \frac{\partial p_o}{\partial r} = \frac{3}{5} q_a \Omega^2
\]  

(2.17)

The solution \( p_o(r) \) to the above differential equation is

\[
p_o(r) = \frac{3}{20} q_a \Omega^2 \left[ r^2 + C + D \ln(r/b) \right] \]  

(2.18)

Using the boundary conditions of vanishing pressure at both the inner radius \( r = a \) and the outer radius \( r = b \), to solve for the constants of integration, we obtain

\[
p_o(r) = \frac{3}{20} q_a \Omega^2 \left[ r^2 - b^2 + \frac{b^2 - a^2}{\ln(a/b)} \ln(r/b) \right] \]  

(2.19)

or, in dimensionless form

\[
\frac{p_o(r)}{\varepsilon_a \Omega^2 b^2} = \frac{3}{20} \left[ \left( \frac{r}{b} \right)^2 - 1 + \frac{1 - (a/b)^2}{\ln(a/b)} \ln(r/b) \right]
\]  

(2.20)

The normalized pressure distribution is graphed in figure 2-7 for \( a/b = 0.2 \).

![Steady-State Pressure in Air-Film](image)

Figure 2-7: Steady State Pressure Distribution in the Air-Film

We can see that the pressure is negative everywhere and thus, as we stated earlier, the air-film is always "pulling" the spinning disk closer to the baseplate. The disk however does not deflect because, when it is confined between two plates, the pulling forces from the two air-films balance each other.
2.6 Linearized Equations of Motion

The addition of a second plate introduces minor changes to the spinning disk’s equation 2.1. We need to drop the external damping term and double\(^{13}\) the pressure loading on the right hand side of the disk’s equation. The resulting equation when both the load system and the gravity loading are removed takes now the form

\[
\rho_d h_d \frac{D^2 w}{Dt^2} + D_b \left(1 + \eta \frac{D}{Dt}\right) \nabla^4 w - \frac{h_d}{r} \frac{\partial}{\partial r} \left(\sigma_r r \frac{\partial w}{\partial r}\right) - \frac{h_d}{r^2} \sigma_\theta \frac{\partial^2 w}{\partial \theta^2} = 2p
\]  
(2.21)

The equation governing the air-film between the spinning disk and one rigid plate remains unchanged. However, and because the steady state deflection of the disk is uniformly zero, the air-film equation can now be easily linearized. We will then get a set of coupled linear partial differential equations that are much easier to solve. Let us rewrite the air-film’s equation 2.11 as

\[
\frac{\partial}{\partial r} \left( r h^2 \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( h^3 \frac{\partial p}{\partial \theta} \right) = 12 \mu_a r \left( \frac{\partial h}{\partial t} + \frac{\Omega}{2} \frac{\partial h}{\partial \theta} \right) + \frac{3}{10} \varepsilon_a \Omega^2 \frac{\partial}{\partial r} \left( r^2 h^3 \right)
\]  
(2.22)

To linearize this equation, let us express the air-film thickness \(h\) and the pressure \(p\) inside the air-film as the sum of their steady state values \((h_a, p_o)\) and the corresponding first order perturbations \((w, p_1)\).

\[
h = h_a + w \quad \text{and} \quad p = p_o + p_1
\]  
(2.23)

Upon inserting these expressions in equation 2.22, using the linear approximation \((1 + \epsilon)^m \approx 1 + m\epsilon\), grouping the terms with the same order of magnitude, and few pages of algebra with MathCad, we obtain the linear equation

\[
\frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial p_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p_1}{\partial \theta^2} + 3 \frac{\partial p_o}{\partial r} \frac{\partial w}{\partial r} = 12 \frac{\mu_a}{h_a^3} \left( \frac{\partial w}{\partial t} + \frac{\Omega}{2} \frac{\partial w}{\partial \theta} \right) + \frac{9}{10} \varepsilon_a \Omega^2 \frac{r}{h_a} \frac{\partial w}{\partial r}
\]  
(2.24)

Substituting for the steady state pressure \(p_o\) from equation 2.19, and dropping the subscript 1, the above equation becomes

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \left[ \frac{9}{20} \frac{\Omega^2 \varepsilon_a b^2 - a^2}{h_a \ln (a/b)} \right] \frac{\partial w}{\partial r} - 12 \frac{\mu_a}{h_a^3} \left( \frac{\partial w}{\partial t} + \frac{\Omega}{2} \frac{\partial w}{\partial \theta} \right) = 0
\]  
(2.25)

\(^{13}\)This is correct only if the gap height \(h_a\) at the hub is the same on both sides of the disk. The factor of two simply means that the disk is subjected to two pressures acting in parallel. This is drawn by analogy from the case of a lumped mass attached between two identical springs acting on both sides of the mass. The two springs are in effect in parallel and their contributions add.
2.7 Free Vibration Solutions of the Spinning Disk

One type of solutions the unloaded spinning disk can sustain are traveling dispersive flexural waves in the $\theta$-direction. The phase speed, with respect to the rotating disk, for a wave with a given wavelength will increase with the spin rate $\Omega$ of the disk because of centrifugal stiffening. For any given wavelength there will always be a forward and a backward wave, relative to the rotating disk. As the disk spin rate $\Omega$ increases, a stationary observer would see the forward wave move faster in the positive $\theta$-direction and the backward wave move slower in the negative $\theta$-direction. Eventually, with even higher rotation rates $\Omega$ both the backward and forward wave (with respect to the disk) of a given mode will be transported forward (as seen by our stationary observer). The frequencies corresponding to the forward and backward waves change with the spin rate $\Omega$ as a result of two mechanisms acting simultaneously. These are the stiffening introduced by $(\sigma_r, \sigma_\theta)$ and the Doppler shift due to the rotation of the disk. For the forward wave, the two mechanisms act together, and for the backward wave, they work against each other. There is a critical value $\Omega_{\text{critical}}$ (one for each mode) at which the backward wave of a given mode\textsuperscript{14} appears steady for a stationary observer. As we will see later on, these critical speeds are where several destabilizing mechanisms start to manifest themselves. The knowledge of such critical speeds, and the lowest amongst them, is very important from a design perspective. The operation of the disk should be such that all the critical speeds are above\textsuperscript{15} the disk's running speed.

For the safe operation of our spinning disk, not only do we need to make sure that the spin rate of the disk stays away from the critical speeds, but also we need to ensure none of the disk's resonances are excited. This task is a bit difficult to accomplish in practice because the disk has several natural frequencies, and these frequencies change with the spin rate of the disk. Fortunately, real disks always have some dissipation mechanisms and the resonances are not as problematic as the self-excited unstable

\textsuperscript{14}This is true for all but the umbrella mode, which has no nodal diameters and is thus a pure vibratory mode.

\textsuperscript{15}In the rotordynamics literature, this is referred to as subcritical operation.
vibrations which take place, among other locations, right after the critical speeds. The natural frequencies and the critical speeds of our spinning disk are obtained by solving an eigenvalue problem associated with the disk-air-film system. It is an eigenvalue problem because the natural frequencies and critical speeds are characteristics of the disk itself and they are thus unaffected by any forcing mechanisms.

### 2.8 Eigenvalue Problem for the Spinning Disk

Because we are looking for harmonic waves solutions, let us apply separation of variables and use the periodic solutions for the disk’s transverse displacement $w$

$$w(r, \theta, t) = w_n(r) e^{in\theta} e^{st}$$

where the parameter $n$ is the number of nodal diameters of the mode of interest and $s = \sigma + i\omega$ is the Laplace complex frequency of transverse motion of the disk as seen by a stationary observer. Upon inserting this solution in the partial differential equations 2.21, we obtain an ordinary differential equation

$$A_1 \frac{d^4 w_n}{dr^4} + A_2 \frac{d^3 w_n}{dr^3} + A_3 \frac{d^2 w_n}{dr^2} + A_4 \frac{dw_n}{dr} + A_5 w_n = 0$$

(2.27)

with the four boundary conditions

$$[w_n]_{r=a} = 0$$

(2.28)

$$\left[ \frac{dw_n}{dr} \right]_{r=a} = 0$$

(2.29)

$$\left[ \frac{d^2 w_n}{dr^2} + \frac{\nu}{r} \frac{dw_n}{dr} - \frac{n^2}{r^2} w_n \right]_{r=b} = 0$$

(2.30)

$$\left[ \frac{d^3 w_n}{dr^3} + \frac{1}{r} \frac{d^2 w_n}{dr^2} + \frac{n^2 (\nu - 2) - 1}{r^2} \frac{dw_n}{dr} + \frac{n^2 (3 - \nu)}{r^3} w_n \right]_{r=b} = 0$$

(2.31)

The five nonconstant coefficients of the disk’s ODE are given by

$$A_1 = [1 + \eta (s + i n \Omega)] D_b$$

(2.32)
\[ A_2 = \left[ 1 + \eta (s + i n \Omega) \right] D_b \frac{2}{r^2} \]  
(2.33)

\[ A_3 = -\left[ 1 + \eta (s + i n \Omega) \right] D_b \frac{1 + 2n^2}{r^2} - \frac{h_d \sigma_r}{r} \]  
(2.34)

\[ A_4 = \left[ 1 + \eta (s + i n \Omega) \right] D_b \frac{1 + 2n^2}{r^3} - \frac{h_d}{r} \sigma_r \frac{b^2 - 3r^2}{b^2 - r^2} \]  
(2.35)

\[ A_5 = \phi_d h_d (s + i n \Omega)^2 + \left[ 1 + \eta (s + i n \Omega) \right] D_b \frac{n^2}{r^4} \left( n^2 - 4 \right) + \frac{n^2}{r^2} h_d \sigma_\theta \]  
(2.36)

The four boundary conditions of the disk, when using the periodic solution \( w(r, \theta, t) = w_n(r)e^{i n \theta}e^{i \omega t} \), take the simplified form

\[ w_n(r = a) = 0 \]  
(2.37)

\[ \frac{dw_n}{dr}(r = a) = 0 \]  
(2.38)

\[ \left[ \frac{d^2 w_n}{dr^2} + \frac{\nu}{r} \frac{dw_n}{dr} - \frac{n^2}{r^2} w_n \right]_{r = b} = 0 \]  
(2.39)

\[ \left[ \frac{d^3 w_n}{dr^3} + \frac{1}{r} \frac{d^2 w_n}{dr^2} + \frac{n^2(\nu - 2) - 1}{r^2} \frac{dw_n}{dr} + \frac{n^2(3 - \nu)}{r^3} w_n \right]_{r = b} = 0 \]  
(2.40)

We have reduced a partial differential equation to an ordinary differential equation that is much easier to handle. The closed form solution of this equation is however difficult to obtain and the finite difference technique seems very appropriate to use here.

### 2.9 Finite Difference Solution

The ODE we obtained for our spinning disk, along with the four boundary conditions, constitute a boundary value problem that is very suitable to solve with the finite difference technique. Because of the various nature of the forces present in the spinning disk, the coefficients of the associated ODE contain several powers of both the eigenvalue \( s \) and the disk’s spin rate \( \Omega \). The finite difference formulation will consequently give rise to several matrices accounting for the various types of forces and the corresponding powers of \( s \) and \( \Omega \). To use the finite difference technique and
transform the formulation of our spinning disk to a matrix eigenvalue problem, we
begin by dividing the radius of the disk in \( N \) intervals of length \( h \) each as shown on
figure 2-8.

![Diagram of disk discretization](image)

**LEGEND**
- ○ — Nodes where ODE applies
- ● — Nodes where Boundary Conditions apply
- ○ — Additional Nodes needed for finite-difference scheme

Figure 2-8: Discretization of the Disk for The Finite Difference Technique

To transform the disk's ODE to a set of difference equations, let us use the central
difference scheme proposed by Crandall [19, Page 246] and shown in figure 2-9.

\[
\begin{align*}
\frac{\partial^2 w}{\partial x^2} & = \frac{1}{h^2} \left\{ \begin{array}{cccc}
1 & -4 & 6 & -4 & 1 \\
1 & -2 & 0 & -2 & 1 \\
\end{array} \right\} + O(h^3) \\
\frac{\partial^2 w}{\partial x^2} & = \frac{1}{2h^2} \left\{ \begin{array}{cccc}
1 & -2 & 1 \\
-1 & 0 & 1 \\
\end{array} \right\} + O(h^3) \\
\frac{\partial w}{\partial x} & = \frac{1}{2h} \left\{ \begin{array}{c}
1 \\
-1 \\
\end{array} \right\} + O(h^3)
\end{align*}
\]

Figure 2-9: Finite Difference approximation of Derivatives

Applying this scheme at the interior points of the discretized disk, we obtain for each
one of these points a difference equation of the form

\[
A_{i-2} w_{i-2} + A_{i-1} w_{i-1} + A_i w_i + A_{i+1} w_{i+1} + A_{i+2} w_{i+2} = 0
\]  
(2.41)
where the five new coefficients are not to be confused with those of the disk’s ODE. The coefficients of the ODE have numeral subscripts whereas the coefficients of the difference equation have subscripts in terms of the node number \( i \) at which the difference equation is applied. The coefficients of the difference equation are obtained from those of the ODE thru the following relation

\[
\begin{align*}
\begin{bmatrix}
A_{i-2} \\
A_{i-1} \\
A_i \\
A_{i+1} \\
A_{i+2}
\end{bmatrix} &= 
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-4 & 2 & 1 & -1 & 0 \\
6 & 0 & -2 & 0 & 1 \\
-4 & -2 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix} 
\begin{bmatrix}
A_1/h^4 \\
A_2/2h^3 \\
A_3/h^2 \\
A_4/2h \\
A_5
\end{bmatrix} 
\end{align*}
\]

(2.42)

By dividing the radius of the disk in \( N \) intervals (\( N + 1 \) nodes) as shown on figure 2-8 and applying equation 2.41 at each of the internal points of the disk, we end up with \( N - 1 \) equations and \( N + 1 \) unknown nodal displacements \( w_i \) of the disk. The boundary conditions and the finite difference scheme require us to solve for three additional nodes, one for the inner boundary condition \( (w_{-1}) \) and two for the outer boundary condition \( (w_{N+1} \text{ and } w_{N+2}) \). We then have to solve for a total of \( N+4 \) nodal points. The four boundary conditions give us four additional equations to bring the total number of equations to \( N + 3 \). We are missing one equation that we can obtain by just applying the ODE one more time\(^{16} \) at the outer edge (point \( N \)) of the disk. The finite difference representation of the boundary conditions at the inner and outer radii of the disk are given by

\[
w_0 = 0
\]

\[
\left[ \frac{-1}{2h} \right] w_{-1} + \left[ \frac{1}{2h} \right] w_1 = 0
\]

\[
\left[ \frac{1}{h^2} - \frac{\nu}{2h^2} \right] w_{N-1} + \left[ \frac{-2}{h^2} + \nu \frac{n^2}{h^2} \right] w_N + \left[ \frac{1}{h^2} + \frac{\nu}{2h^2} \right] w_{N+1} = 0
\]

(2.45)

\[
\left[ \frac{-1}{2h^2} \right] w_{N-2} + \left[ \frac{1}{h^2} + \frac{1}{h^2} - \frac{n^2(\nu - 2) - 1}{2h^2} \right] w_{N-1} + \left[ \frac{-2}{h^2} + \frac{n^2(3 + \nu)}{h^2} \right] w_N + \cdots
\]

\[
\left[ \frac{-1}{h^2} + \frac{1}{h^2} + \frac{n^2(\nu - 2) - 1}{2h^2} \right] w_{N+1} + \left[ \frac{1}{2h^2} \right] w_{N+2} = 0
\]

\( ^{16} \)Suggested by Prof. Crandall
The equations obtained from applying the difference equation in the interior points of the disk, along with the four boundary conditions given above, are put together in matrix form. The resulting matrix equation has the structure shown in figure 2-10.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\vdots
\end{bmatrix}
\]

**Figure 2-10: Finite Difference Matrix Formulation**

The above matrix can be decomposed into constituent matrices according to the various types of forces present in the disk. The reason being twofold: The matrix eigenvalue problem is not clear from the matrix shown in figure 2-10; and since we need to calculate the eigenvalues for several values of the spin rate, we need to write the matrix formulation in the somewhat computationally efficient form

\[
\left( s^2[M] + s[D + \Omega G] + [K + \Omega C + \Omega^2 E] \right) \{w_i\} = 0
\]  

(2.47)

To build the six matrices of the above eigenvalue problem, let us begin by splitting the coefficients of the original disk’s ODE according to powers of the spin rate $\Omega$ and the eigenvalue $s$.

\[
A_1 = sA_1^D + A_1^K + \Omega A_1^C \\
A_2 = sA_2^D + A_2^K + \Omega A_2^C \\
A_3 = sA_3^D + A_3^K + \Omega A_3^C + \Omega^2 A_3^E \\
A_4 = sA_4^D + A_4^K + \Omega A_4^C + \Omega^2 A_4^E \\
A_5 = s^2 A_5^D + sA_5^D + s\Omega A_5^C + A_5^K + \Omega A_5^C + \Omega^2 A_5^E
\]  

(2.48)

where we have omitted the subcoefficients that are zero. The superscripts of the various coefficients indicate which of the six matrices the specific coefficient contributes to. Next, the greatest task of all, we need to write each of the twenty new coefficients
(RHS of equation 2.48) we just introduced. We begin with the coefficient contributing to the Mass matrix

\[
A^M_5 = \varepsilon_d h_d
\]  \hspace{1cm} (2.49)

The Damping matrix depends upon five coefficients, these are

\[
\begin{align*}
A^D_1 &= \eta D_b \\
A^D_2 &= \eta D_b \frac{n^2}{r^4} \\
A^D_3 &= -\eta D_b \frac{n^2}{r^4} \\
A^D_4 &= \eta D_b \frac{1+2n^2}{r^4} \\
A^D_5 &= \eta D_b \frac{n^2(n^2-4)}{r^4}
\end{align*}
\]  \hspace{1cm} (2.50)

The coefficient for the Gyroscopic matrix is

\[
A^G_5 = 2i\varepsilon_d h_d
\]  \hspace{1cm} (2.51)

There are five coefficients that contribute to the Stiffness matrix

\[
\begin{align*}
A^K_1 &= D_b \\
A^K_2 &= D_b \frac{n^2}{r^4} \\
A^K_3 &= -D_b \frac{1+2n^2}{r^4} \\
A^K_4 &= D_b \frac{1+2n^2}{r^4} \\
A^K_5 &= D_b \frac{n^2(n^2-4)}{r^4}
\end{align*}
\]  \hspace{1cm} (2.52)

and five again for the Circulatory matrix

\[
\begin{align*}
A^C_1 &= i\varepsilon D_b \\
A^C_2 &= i\varepsilon D_b \frac{n^2}{r^4} \\
A^C_3 &= -i\varepsilon D_b \frac{1+2n^2}{r^4} \\
A^C_4 &= i\varepsilon D_b \frac{1+2n^2}{r^4} \\
A^C_5 &= i\varepsilon D_b \frac{n^2(n^2-4)}{r^4}
\end{align*}
\]  \hspace{1cm} (2.53)

Finally, for the matrix of Centrifugal forces, there are three coefficients

\[
\begin{align*}
A^E_3 &= -h_d \frac{\varepsilon_3}{r^3} \\
A^E_4 &= -h_d \left( \frac{\varepsilon_3}{r} \right) \\
A^E_5 &= n^2 h_d \left( -\varepsilon_d + \frac{1}{r^2} \frac{\varepsilon_3}{r^3} \right)
\end{align*}
\]  \hspace{1cm} (2.54)

For any given matrix, the number of coefficients we computed above corresponds to the bandwidth of that specific matrix. Hence, we'll note that the mass M and gyroscopic G matrices are both diagonal, the centrifugal matrix E is three-banded and the damping D, circulatory C, and stiffness K matrices are all five-banded. All six matrices are very sparse and their non-zero entries are shown in figure 2-11.
Figure 2-11: Non-Zero Elements of the Six Matrices M, D, G, K, C, E

The dark circles correspond to entries obtained from applying the difference equation at the interior points of the disk and the gray boxes are from applying the boundary conditions. We will note that the new set of coefficients we introduced on the previous page are not exactly the entries of the corresponding matrices, they are coefficients of the Continuous problem. The coefficients of the corresponding discrete problem are obtained by applying a transformation to the subcoefficients of the continuous ODE. Performing the transformations then, we obtain for the mass matrix

\[
\{M_i\} = [1] \{A^M_S\} \quad (2.55)
\]

The five elements of the i'th row of the damping matrix are obtained from

\[
\begin{align*}
D_{i-2} &= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A^D_1 / h^4 \\ A^D_2 / 2h^3 \\ A^D_3 / h^2 \\ A^D_4 / 2h \\ A^D_5 \end{bmatrix} \\
D_{i-1} &= \begin{bmatrix} -4 & 2 & 1 & -1 & 0 \end{bmatrix} \\
D_i &= \begin{bmatrix} 6 & 0 & -2 & 0 & 1 \end{bmatrix} \\
D_{i+1} &= \begin{bmatrix} -4 & -2 & 1 & 1 & 0 \end{bmatrix} \\
D_{i+2} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\quad (2.56)
\]
The gyroscopic matrix is also diagonal and its elements are calculated from

\[ \{G_i\} = [1] \{A_{5}^G\} \]  

(2.57)

For the stiffness matrix, the five coefficients are obtained from

\[
\begin{bmatrix}
K_{i-2} & K_{i-1} & K_i & K_{i+1} & K_{i+2}
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-4 & 2 & 1 & -1 & 0 \\
6 & 0 & -2 & 0 & 1 \\
-4 & -2 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_{1}^K/h^4 \\
A_{2}^K/2h^3 \\
A_{3}^K/h^2 \\
A_{4}^K/2h \\
A_{5}^K \\
\end{bmatrix}
\]  

(2.58)

Similarly, the entries of the circulatory matrix are

\[
\begin{bmatrix}
C_{i-2} & C_{i-1} & C_i & C_{i+1} & C_{i+2}
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-4 & 2 & 1 & -1 & 0 \\
6 & 0 & -2 & 0 & 1 \\
-4 & -2 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_{1}^C/h^4 \\
A_{2}^C/2h^3 \\
A_{3}^C/h^2 \\
A_{4}^C/2h \\
A_{5}^C \\
\end{bmatrix}
\]  

(2.59)

and finally, the entries of the centrifugal matrix are obtained from

\[
\begin{bmatrix}
E_{i-1} & E_i & E_{i+1}
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 \\
-2 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_{3}^E/h^2 \\
A_{4}^E/2h \\
A_{5}^E \\
\end{bmatrix}
\]  

(2.60)

After we have built all six matrices of our eigenvalue problem, we can put them together in the standard form given by equation 2.47. This form of the eigenvalue problem, with the various matrices as shown in figure 2-11 is not ready yet for numerical implementation. The reason being that all but the stiffness matrix are not full rank\textsuperscript{17}. This is because we included the boundary conditions in the matrix eigenvalue formulation. For safe numerical implementation, we need then to reduce the size of the matrices by removing the nodal displacements \(w_{-1}, w_0, w_{N+1}\), and \(w_{N+2}\). This

\textsuperscript{17}This could lead to numerical inaccuracies while solving the eigenvalue problem
operation is done using the four boundary conditions and is represented schematically in figure 2-12. The reduced order component matrices $M$, $D$, $G$, $K$, $C$, and $E$ are now full rank and the eigenvalue routine in Matlab is quite happy now. The solution of the quadratic eigenvalue problem 2.47 requires rearranging it into a first order problem

$$\begin{bmatrix}
  s \begin{bmatrix}
  D + \Omega G & M \\
  -M & 0
  \end{bmatrix} + \begin{bmatrix}
  K + \Omega C + \Omega^2 E & 0 \\
  0 & M
  \end{bmatrix}
\end{bmatrix} \begin{bmatrix}
  w_i \\
  sw_i
  \end{bmatrix} = 0$$

(2.61)

All the steps presented in this section are implemented in a Matlab program whose listing is given in appendix A.

2.10 Natural Frequencies and Critical Speeds

With all the analytical treatment presented in the previous sections, it is now time to present the results of the implementation of the finite difference technique. Figure 2-13 shows the natural frequencies of an undamped disk spinning in vacuum i.e., with no interaction with any external agents. We are showing the modes with $n = 0, 1, 2, 3$ nodal diameters. We note here that, because of the absence of non-conservative mechanisms, the eigenvalues of the disk are purely imaginary. This corresponds to purely oscillatory frequencies. The parameters given correspond to those of the disk we used in the experimental part of this thesis and are organized in table 5.1.

The critical speeds of the disk, those where the backward waves reverse their direction of propagation and get dragged forward, are calculated from setting the natural frequency to zero in the eigenvalue problem 2.47. Because of the absence of damping,
Figure 2-13: Campbell Diagram for a Flexible Disk Spinning in Vacuum. Disk Parameters are:
\[ a = 10 \text{ [mm]}, \ b = 140 \text{ [mm]}, \ h_d = 38 \text{ [\mu m]}, \ \rho_d = 1100 \text{ [kg/m}^3], \ E = 1.47 \times 10^8 \text{ [N/m}^2], \ \nu = 0.3 \text{ [-]} \]
the circulatory matrix $[C]$ is also zero and the result is an eigenvalue problem where this time, it is the spin rate $\Omega$ that we need to solve for

$$\left(\Omega^2[E] + [K]\right) \{w_i\} = 0$$

(2.62)

The critical speeds obtained from the above calculation are shown in figure 2-14.

![Critical Speeds of the Disk in Vacuum](image)

**Figure 2-14: Critical Speeds for a Flexible Disk Spinning in Vacuum**

We have ordered the modes according to the number $n$ of nodal diameters. The fact that different modes have different critical speeds is a direct consequence of dispersion, an inherent property of bending waves. Figure 2-14 shows that it is the mode with $n = 5$ nodal diameters and $m = 0$ nodal circles that has the lowest critical speed. Figure 2-14 also shows that for any mode number, there are several possible critical speeds. These correspond to modes with different numbers $m$ of nodal circles. Those are higher frequency modes. It is clear that the maximum number of nodal circles we can solve for is controlled by the number $N$ of grid points in the finite difference formulation. The inclusion of internal damping through the parameter $\eta$ causes all the frequencies of the disk to have a negative real component as shown in figure 2-15. This corresponds to positive damping. Thus, internal damping in a spinning disk
Figure 2.15: Campbell Diagram for a Flexible Disk Spinning in Vacuum, and with Internal Damping.

\( a = 10 \text{[mm]}, b = 140 \text{[mm]}, a_2 = 38 \text{[\mu m]}, \nu = 0.3 \), \( \eta = 0.02 \).
is always stabilizing\(^8\) and tends to reduce the amplitude of traveling waves in the disk. We note also that the internal damping has no effect, to first order, on both the oscillating component of the natural frequencies and the critical speeds of the disk. The effects of the pressure loading from the air-film have been treated by several researchers. We mention here the work of Hosaka and Crandall [28] who found that the pressure from the air-film increases the disk's critical speeds and introduces non-conservative circulatory forces. These forces provide positive damping for all forward waves. For any backward wave however, the damping is positive (negative) when the disk's spin rate is lower (higher) than twice that wave's critical speed.

### 2.11 Mode Shapes of the Spinning Disk

To complete the picture, let us now show the mode shapes for our spinning disk and their dependence upon the number of nodal diameters \(n\), and on the disk's spin rate \(\Omega\). Figure 2-16 shows the modes shapes\(^9\) with 2, 5, 10, and 16 nodal diameters for a spin rate of \(\Omega = 1200\). Not surprising at all, the modes shapes are confined to the outer periphery of the disk. This effect gets even more pronounced as the number of nodal diameters increases. For a circular nonrotating plate i.e., with only the flexural stiffness present, the mode shapes are given by Bessel functions

\[
w_n(r) = A \cdot J_n(r) + B \cdot Y_n(r) + C \cdot I_n(r) + D \cdot K_n(r)
\]

(2.63)

where the four constants \(A, B, C,\) and \(D\) are to be evaluated from the boundary conditions for the circular plate. The "original" Bessel functions \(J_n\) and \(Y_n\) are oscillatory whereas the modified functions \(I_n\) and \(K_n\) resemble real exponentials, one is decaying the other is diverging. We have all four Bessel functions here because the ODE we obtained from the disk's equation, is fourth order\(^{20}\). The subject on Bessel functions is certainly beyond the scope of this thesis and books literally, have been written on

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\(^{8}\)This is in contrast with the internal damping in rotating shafts, which is always destabilizing.

\(^{9}\)The air-film loading is not included in the calculations that gave those figures

\(^{20}\)Analogous to the four exponentials - two real and two complex, that make up the general solution of the vibrating beam's equation.
Figure 2.16: Disk's Mode Shapes with \( n=2,5,10, \) and 16 Nodal Diameters and Zero Nodal Circle.
the subject. Let us just say here that not all four functions always contribute to the mode shapes of the disk. Those mode shapes that have *only one* nodal circle —clamp at the inner radius of the disk, are made up of the $I_n$ and $K_n$ functions. Modes of the disk that have more than one nodal circle are made up of all four\textsuperscript{21} Bessel functions.

For our situation of the disk clamped at the inner radius and free at the outer edge, it is the $I_n$ function that dominates the modes with one nodal circle. Such a function stays almost at zero for small values of its argument and then takes off very rapidly. The higher the order $n$, the more pronounced this effect is. This is why the mode shapes of the disk are confined at the outer radius. In fact, an asymptotic analysis [34] shows that the mode shapes of the disk can be approximated by

$$w_n(r) \sim \left(\frac{r}{b}\right)^n$$  \hspace{1cm} (2.64)

We can clearly see here that the higher the number of nodal diameters, the more pronounced will the mode shape amplitudes be near the outer edge of the disk.

![Figure 2-17: Dependence of a Typical Mode of the Disk upon the Spin Rate $\Omega$](image)

The in-plane stresses in the disk increase with the *square* of the disk's spin rate $\Omega$. Thus, the contribution of the centrifugal loading to the effective bending stiffness is

\textsuperscript{21}Depending on the boundary conditions of the disk obviously.
more pronounced at high spin rates of the disk. Because the in-plane stresses \((\sigma_r, \sigma_\theta)\) are maximum at the center of the disk and decrease as we approach the outer edge, the center of the disk will “gain” more stiffness as the disk’s rpm increases. This implies that large values of the disk’s spin rate accentuate the localization effect of the mode shapes. This is illustrated in figure 2-17 for the mode shape with three nodal diameters. There, large values of the spin rate \(\Omega\) tend to flatten the inside of the disk and cause the mode shape to be more pronounced at the outer edge.

2.12 Conclusions

We have presented the equation governing the transverse motion of a flexible disk spinning between two rigid plates. We used the finite difference technique to solve the boundary value problem of our spinning disk and obtained the natural frequencies, mode shapes, and critical speeds. The calculations were done for a disk without the pressure loading from the air-films. We also presented an extensive literature review with an emphasis on the effects of the air loading on our spinning disk.
Chapter 3

Fluid Induced Shear Loading on the Spinning Disk

3.1 Introduction

When the load system is present, the onset speed of instability for our spinning disk is controlled by the lowest critical speed of a given mode. The knowledge of the disk's critical speeds and the mechanisms that control them is thus of primary importance. At a critical speed, the backward wave with respect to the disk is being dragged forward such that its phase speed, as seen by a stationary observer, vanishes. From a dynamic balance point of view, this frozen wave results from the exact cancellation of the out-of-plane centrifugal forces and the elastic restoring forces provided by both the bending stiffness and the in-plane tensioning stresses induced by the centrifugal loading. In the previous chapter we have seen that the transverse pressure loading from the air-film raises the critical speeds of the disk. The viscosity effects of the air-film however, give rise to wall shear stresses acting on the surface of the disk in both the radial and the circumferential directions. It is not apriori clear what the net effect of this shear loading on the spinning disk is. With a Poiseuille flow in the radial direction, the wall shear stresses add to the centrifugal forces in the disk and thus increase the disk's bending stiffness. In the circumferential direction however, the shear stresses will cause the disk to buckle if the shear load exceeds some critical
value. Even if buckling is never reached, the circumferential component of the shear stresses will *decrease* the effective bending stiffness of the disk, resulting in a *lowering* of the disk's critical speeds. This *shear loading* needs to be investigated and its effects on the critical speeds in particular and the stability aspects in general, determined. Will the overall effect of this shear loading increase or decrease the critical speeds of the disk? It is this question, amongst others, that we need to answer in this chapter.

![Diagram](image)

*Figure 3-1: Transverse and In-Plane Loading from the Air-Film*

The *transverse* pressure loading from the air-film is perceived by our spinning disk as a stationary elastic foundation plus a convective damping field. The shear loading on the other hand applies forces *along the plane* of the disk and it is not yet clear what they are equivalent to in terms of springs and dashpots. To investigate the effects of this shear loading and determine the consequences, let us then organize our plan of action as follows:

- Calculate the distribution of wall shear stresses acting on a spinning disk.
- Obtain the resulting distribution of in-plane stresses in the disk.
- Include the shear loading effects in the disk's equation of motion.
- Formulate the buckling problem for a stationary disk under rotating shear flow and investigate the effects of the shear loading.
- For a rotating disk, investigate the effects of the shear loading. Namely calculate the natural frequencies, critical speeds, and mode shapes.
3.2 Shear Flow Loading on the Spinning Disk

Our first task is to solve the fluid mechanics problem for our disk and obtain the wall shear stresses our spinning disk is subjected to. For this, let us consider the physical model shown in figure 3-2 for a flexible disk spinning in a "bath" of viscous fluid\(^1\) between two rigid plates.

![Figure 3-2: Flexible Disk Spinning Between Two Rigid Plates](image)

The thickness of the air-film is much exaggerated in the above figure. For the common computer diskette the aspect ratio is such that \(b/h_a \sim 100\). With this realistic assumption for the air gap \(h_a\) and the standard rotation rate (3600 [rpm]) of current floppy drives, there will be a \textit{laminar} shear flow [42] in both the radial and circumferential directions. The circumferential component \(v_\theta\) of the fluid velocity is controlled by the rotation rate \(\Omega\) of the spinning disk. The resulting \textit{Couette} flow in the circumferential direction gives rise to wall shear stresses whose magnitude is

\[
\tau_{\text{wall}} = \mu_a \frac{v_\theta}{h_a} = \mu_a \frac{\Omega r}{h_a}
\]  

(3.1)

This is the wall shear stress that is applied by only one air-film. For a disk that is spinning between two plates, this contribution needs to be doubled. Furthermore, when the disk deflects by an amount of \(w\), the wall shear stresses on both sides of the disk will be different. On one side of the disk the wall shear increases and on the other it decreases. Their sum will however remain constant (to first order) for small values of \(w/h_a\).

\(^1\)It is understood here that it is the air that we are referring to as a "fluid".
To obtain the radial component of the wall shear stress, we need first to solve for the radial component \( v_r \) of the fluid velocity. From [42], we have a relationship between the pressure gradient given by equation 2.19 and precisely the component \( v_r \) of the radial velocity

\[
 v_r (r, z) = \frac{h_a^2}{2\mu_a} \left\{ \left( \frac{z}{h_a} \right)^2 - \frac{z}{h_a} \right\} \frac{dp}{dr} - \frac{\rho_a \Omega^2}{6} r \left[ \left( \frac{z}{h_a} \right)^4 - \frac{z}{h_a} \right]
\]  

(3.2)

We will note here that the radial velocity profile is not parabolic. The reason is because the centrifugal forces that are driving the flow are not uniformly distributed across the air-film's thickness. With the knowledge of the velocity distribution, the radial component \( \tau_{r^\text{wall}} \) of the wall shear stress is then readily obtained

\[
 \tau_{r^\text{wall}} = \mu_a \frac{\partial v_r}{\partial z} \bigg|_{z = h_a} = \frac{h_a}{2} \left( \frac{dp}{dr} - \frac{\rho_a \Omega^2}{6} r \right) = \frac{\rho_a h_a \Omega}{10} \left[ \frac{7}{2} - \frac{3}{r^2 \ln(\alpha/b)} \right]
\]  

(3.3)

Here too we need to double this contribution because our disk is confined between two air-films. To account for the effects of the wall shear stresses in the equation of motion of our spinning disk we could use either of two options. In the first one, the wall shear stresses are treated as forcing mechanisms. In the second approach, the one we opted for, the wall shear stresses are accounted for through the resulting in-plane shear stresses in the disk.

### 3.3 In-Plane Stresses in the Rotating Disk

As we have said earlier, the wall shear stress loading on the spinning disk has two components. The circumferential component \( \tau_{\theta^\text{wall}} \) induces in-plane shear stresses \( \sigma_{r\theta} \) in the spinning disk. The radial component \( \tau_{r^\text{wall}} \) however will add to the centrifugal loading but will not contribute to the in-plane shear. For a differential element \( h_d r dr d\theta \) of the spinning disk subjected to both the centrifugal and the shear loading, the equations of differential equilibrium are given by [54]

\[
 \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_{\theta}}{r} + R = 0
\]  

(3.4)

\[
 \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial r} + 2 \frac{\sigma_{r\theta}}{r} + S = 0
\]  

(3.5)
The various stresses and loading terms are shown in figure 3-3 along with the differential element \( h_d r dr d\theta \) of our spinning disk.

![Differential Element of Disk with Stresses and Loadings](image)

**Figure 3-3: Differential Element of Disk with Stresses and Loadings**

The quantities \( R \) and \( S \) are external forces (per unit volume) acting in the radial and circumferential directions respectively. They are clearly due to both the centrifugal forces and the fluid induced wall shear stresses. They are given by

\[
R = \varrho_d r \Omega^2 + 2 \frac{r_{wall}}{h_d} \tag{3.6}
\]

\[
S = 2 \frac{\tau_{\theta wall}}{h_d} \tag{3.7}
\]

Because the loading is axisymmetric, the stresses field is uniform in the circumferential direction. Dropping the \( \theta \) derivatives then and substituting for the wall shear stresses \( \tau_{r wall} \) and \( \tau_{\theta wall} \), equations 3.4 and 3.5 become

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + K_s \left( \frac{\varrho_d r \Omega^2}{r} \right) = 0 \tag{3.8}
\]

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + 2 \frac{\sigma_{r\theta}}{r} + 2 \frac{\mu_a}{h_a h_d} \Omega r = 0 \tag{3.9}
\]

where we have introduced the factor \( K_s \) to account for the additional radial force induced by the radial component \( \tau_{r wall} \) of the wall shear stress

\[
K_s = \left\{ 1 + \frac{1}{5} \frac{\varrho_a h_a}{\varrho_d h_d} \left[ \frac{7}{2} - \frac{3 b^2 - a^2}{r^2 \ln(a/b)} \right] \right\} \tag{3.10}
\]
The correction factor $K_*$ for the disk parameters given in table 5.1, is graphed in figure 3-4 as a function of the disk radius.

![Correction Factor for Parameters of Table 2.1](image)

**Figure 3-4: Correction Factor $K_*$ for parameters of table 2.1.**

We note here that the correction factor is largest near the inner radius of the disk. This is in fact expected because the radial shear loading is proportional to the radial component $v_r$ of the fluid velocity which decreases with $r$ because of the mass conservation requirement.

In the previous chapter, we gave expressions 2.4 and 2.5 for the centrifugally induced in-plane stresses $\sigma_r$ and $\sigma_\theta$. Those expressions are no longer valid in the presence of the additional wall shear stresses imposed by the fluid viscosity. It is clear that it is only the radial component of the wall shear stress that does contributes an additional radial load. One way to get the stresses $\sigma_r$ and $\sigma_\theta$ is to solve the equations of differential equilibrium 3.8 and 3.9. Since the stresses are induced by the centrifugal loading, and the correction factor $K_*$ is nearly equal to one, a much simpler way of getting$^2$ these stresses when the fluid loading is included is simply to multiply the original expressions for $\sigma_r$ and $\sigma_\theta$ by the correction factor $K_*$. Accounting for the

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$^2$This is not mathematically rigorous and would be correct only if $K_*$ does not change with the radius $r$. 

60
fluid shear loading then, the in-plane tensile stresses given by equations 2.4 and 2.5 now become

\[ \sigma_r = K_s \left\{ \frac{2\Omega^2}{8} \left[ (3 + \nu)(b^2 - r^2) \left( 1 + \frac{1 - \nu}{3 + \nu} \frac{a^2}{r^2} \right) \right] \right\} \]  
(3.11)

\[ \sigma_\theta = K_s \left\{ \frac{2\Omega^2}{8} \left[ (1 + \nu)(a^2 + kb^2) - (1 + 3\nu)r^2 - (1 - \nu) \frac{ka^2b^2}{r^2} \right] \right\} \]  
(3.12)

This is for the normal stresses. We still have to calculate the in-plane shear stress \( \sigma_{r\theta} \) induced by the circumferential component of the wall shear stresses. For this, we could certainly integrate equation 3.9 and obtain the distribution of the \( \sigma_{r\theta}(r) \) component in the disk. An alternative and elegant way for calculating \( \sigma_{r\theta}(r) \) is through the use of the concept of shear flow. The idea is rather simple and is based on the extension of a global equilibrium requirement: the torque needed to spin the disk and supplied by the drive mechanism is balanced by the circumferential wall shear stresses \( \tau_{wall} \) acting on the surface of the disk. Extending this equilibrium requirement, let us consider an annular portion of the disk extending from a radius \( r > a \) to the disk's outer radius \( b \) as shown in figure 3-5.

![Figure 3-5: Cylindrical Cross-Section of Spinning Disk](image)

The torque resulting from integrating the in-plane shear stresses \( \sigma_{r\theta} \) on the cylindrical cross section of area \( 2\pi rh_d \) must balance the torque due to the wall shear stresses,
acting on the two sides of the disk, between the radius \( r \) of the cylindrical cross section
and the disk's outer radius \( b \). Putting this statement mathematically then, we obtain

\[
 r \times 2\pi rh_d \times \sigma_{r\theta}(r) = 2 \int_{\xi=r}^{b} \xi \times 2\pi \xi \times \tau_{\theta\theta}^{\text{wall}}(\xi) d\xi 
\]  
(3.13)

where the factor of two in front of the integral is because there are two air-films.
Solving the above equation for the shear stress \( \sigma_{r\theta} \), we find

\[
\sigma_{r\theta}(r) = \frac{1}{2} \frac{\mu_a \Omega}{h_a h_d} \left( \frac{b^4 - r^4}{r^2} \right) 
\]  
(3.14)

Now that we have all three components of the in-plane stresses in the disk, we can
compare the shear stress to the two normal stresses. We have that the in-plane shear
stresses increase linearly with the spin rate \( \Omega \) whereas the tensile stresses increase with
\( \Omega^2 \). This observation leads us to conclude that the shear stresses may be important
only at small spin rates \( \Omega \). Deciding on whether or not to include the fluid induced
shear stresses is thus problem dependent and we thus keep them for the most general
situation.

### 3.4 Disk's Equation Including the Shear Loading

To introduce the effects of the in-plane shear stresses resulting from the air-film's
shear loading, let us take equation 2.1 and for the sake of simplicity, ignore all the
damping and loading terms

\[
\varrho_d h_d \frac{D^2 w}{Dt^2} + D_b \nabla^4 w - \frac{h_d}{r} \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial w}{\partial r} \right) - \frac{h_d}{r^2} \sigma_{\theta \theta} \frac{\partial^2 w}{\partial \theta^2} = 0 
\]  
(3.15)

The above equation can be rewritten as

\[
\varrho_d h_d \frac{D^2 w}{Dt^2} + D_b \nabla^4 w = \frac{1}{r} \frac{\partial}{\partial r} \left( r N_r \frac{\partial w}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( N_\theta \frac{1}{r} \frac{\partial w}{\partial \theta} \right) 
\]  
(3.16)

where \( N_r \) and \( N_\theta \) are in-plane (tension) forces per unit length, acting on a small
element in the radial and circumferential directions respectively. The interpretation
of the two terms on the right hand side is now clearer. These are restoring forces due
to non-uniform in-plane tension\footnote{By analogy with the equation $\rho \frac{\partial^2 W}{\partial z^2} = \frac{\partial}{\partial z} \left( T \frac{\partial W}{\partial z} \right)$ for a string with space dependent tension force.} forces which are given by

$$N_r (r) = h_d \sigma_r \quad \text{and} \quad N_\theta (r) = h_d \sigma_\theta$$ \hspace{1cm} (3.17)

If we now include the effects of the in-plane shear stresses, the right hand side of the
disk's equation will have the two additional terms [7, pages 208–209]

$$\frac{1}{r} \frac{\partial}{\partial r} \left( N_r \frac{\partial w}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( N_\theta \frac{\partial w}{\partial r} \right)$$ \hspace{1cm} (3.18)

where we have introduced the in-plane shearing force $N_\theta$ defined as

$$N_\theta (r) = h_d \sigma_\theta$$ \hspace{1cm} (3.19)

If we now put everything together, we get the spinning disk's equation including the
effects of air-film's shear loading

$$\nu_d h_d \frac{D^2 w}{Dt^2} + D_b \nabla^4 w - \frac{1}{r} \frac{\partial}{\partial r} \left( r N_r \frac{\partial w}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( N_\theta \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \ldots
- \frac{1}{r} \frac{\partial}{\partial r} \left( N_\theta \frac{\partial w}{\partial \theta} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( N_\theta \frac{\partial w}{\partial r} \right) = 0$$ \hspace{1cm} (3.20)

We note here that the introduction of the in-plane shear stresses gives rise to first
order derivatives in the variable $\theta$ in the disk's equation. This will lead to complex
valued coefficients once we transform the disk's equation to an ODE by assuming
periodic solutions. The mode shapes will thus be complex and the interpretation
here is simply that the nodal "diameters" are no longer straight lines.

### 3.5 Eigenvalue Problem of the Spinning Disk

To solve for the natural frequencies, critical speeds, and mode shapes of our spinning
disk, when it is spinning between two plates and including the shear loading, let us
assume the periodic solution given by equation 2.26. Upon inserting this solution into the two terms of equation 3.20 that are due to the shear loading, we obtain

$$-\frac{1}{r} \frac{\partial}{\partial r} \left( N_{r\theta} \frac{\partial w}{\partial \theta} \right) = -\frac{\ln}{r} \left( \frac{dN_{r\theta}}{dr}w_n(r) + N_{r\theta} \frac{dw}{dr} \right) e^{i\omega t} \quad (3.21)$$

$$-\frac{1}{r} \frac{\partial}{\partial \theta} \left( N_{r\theta} \frac{\partial w}{\partial r} \right) = -\frac{\ln}{r} \left( N_{r\theta} \frac{dw_n}{dr} \right) e^{i\omega t} \quad (3.22)$$

The coefficients $A_4$ and $A_5$ of the ODE for our disk need then to be augmented by

$$A_4 \rightarrow A_4 - 2\ln h_d \frac{\sigma_{r\theta}}{r} \quad \text{and} \quad A_5 \rightarrow A_5 - \ln h_d \frac{1}{r} \frac{d\sigma_{r\theta}}{dr} \quad (3.23)$$

The change reflects the addition of the circulatory forces due to viscous drag. We note here that this formulation for the effects of the wall shear stresses is independent upon the pressure loading from the air-film. Thus, we can study the isolated effects of the shear loading upon the dynamics of our spinning disk. Before we do that, and to verify that we have properly accounted for the wall shear stresses, let us see what happens to a stationary disk under the action of a rotating shear flow.

### 3.6 Stationary Disk with Rotating Shear Flow

Let us consider the buckling problem of a stationary disk subjected to tangential wall shear stresses induced by the rotation of two neighboring rigid plates.

![Figure 3-6: Fixed Disk between Two Rotating Rigid Plates](image)
The two rigid plates are mounted in proximity to the disk and are spun at the same rate with a set of gears. As the two plates are spinning, a Couette shear flow develops in the circumferential direction between each of the plates and the fixed disk. This shear flow gives rise to wall shear stresses that act on both sides of the disk. There will also be radial shear stresses applied on the stationary disk but their magnitude is relatively small and we will ignore them. The static equilibrium equation for the disk is now given by

\[ D_b \nabla^4 w - \frac{1}{r} \frac{\partial}{\partial r} \left( N_r \frac{\partial w}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( N_{\theta} \frac{\partial w}{\partial \theta} \right) = 0 \]  

(3.24)

The buckling condition is reached when the above equation has a solution other than the trivial case \( w = 0 \). This simply means that the stiffness operator must vanish at the critical buckling condition. We have in fact an eigenvalue problem where the magnitude of the loading \( (N_{\theta}) \) is the eigenvalue and the shape of the disk (the number of nodal diameters and nodal circles) is the eigenfunction. At the onset speed of buckling, the shape of the disk is given by

\[ w (r, \theta) = w_n (r) e^{in\theta} \]  

(3.25)

The finite difference formulation for this case gives an eigenvalue problem of the form

\[ ([K] + \Omega[C]) \{w_i\} = 0 \]  

(3.26)

Our task is now to search for values of the spin rate \( \Omega \) that are solutions of the above equation. Such values are the results of solving the above eigenvalue problem, they are shown in figure 3-7 as critical buckling speeds in terms of the number \( n \) of nodal diameters of the disk. The disk parameters are the ones given in table 5.1. When we double the thickness \( h_d \) of our stationary disk, we obtain the critical speeds shown in figure 3-8. Increasing the disk thickness is thus one way of pushing the critical buckling speeds higher and giving a wider operating range to our disk. We note once more that for any given number of nodal diameters, there exists several possible critical buckling speeds. These correspond to modes with different numbers of nodal circles. The first mode to buckle is the one with only one nodal diameter. To buckle modes
Figure 3-7: Critical Buckling Speeds for a Stationary Disk confined between Two Rotating Plates and Subjected to a Rotating Shear Flow.

Figure 3-8: Same as Above with however a disk with TWICE the thickness $h_d$. 
with higher \( n \), it is necessary to spin the rigid plates faster. This of course makes sense. The "modal stiffness" of a disk increases with the mode number. The higher the number of nodal lines and nodal diameters, the more shear loading is required to buckle a specific mode. Now that we have shown that the shear loading reduces the effective bending stiffness of a stationary disk, let us go back to our rotating disk and see what happens there, when the shear loading is introduced.

3.7 Natural Frequencies and Critical Speeds

We now solve the eigenvalue problem associated with equation 3.20. That equation takes into account only the shear loading of the air-films interacting with our spinning disk. After assuming periodic solutions in \( w \), we obtain an eigenvalue problem similar to that given by equation 2.47. The only difference is that the circulatory matrix \( C \) is modified to account for the shear loading. The natural frequencies of such a system are displayed in figure 3-9. Comparing these graphs with those shown in figure 2-13, it seems that there is no change in the critical speeds. We note in figure 3-9 that the frequencies of the disk are now complex valued. The shear loading thus introduces non-conservative destabilizing forces. When the internal damping is now introduced, along with the shear loading, we obtain the Campbell diagram of figure 3-10. There we can clearly see that the internal damping has "stabilized" some of the modes. We have to indicate here that the reason there are so many "branches" in the graph for the growth/decay component is because our finite difference implementation solves for all possible modes of the disk. There are fewer branches in the graph for the natural frequencies simply because the other branches are outside of the range displayed.

From the theory of plate buckling\(^4\), let us recall that not all the modes offer the same resistance to buckling loads. Some modes are thus "softer" than others. After experimenting with several combinations of nodal diameters \( n \) and nodal circles \( m \) we found that the mode with \((m, n) = (1, 10)\) clearly shows the buckling effect of the shear loading. Figure 3-11 shows the Campbell diagram for the modes with \( n = 10 \)

\(^4\)See Brush and Almroth [7] for example.
Figure 3-9: Campbell Diagram for a Disk with the Fluid Shear Loading. Disk Parameters are:

\[a = 10 \text{[mm]}, \quad b = 140 \text{[mm]}, \quad h_d = 38 \text{[\mu m]}, \quad \rho_d = 1100 \text{[kg/m}^3], \quad E = 1.47 \times 10^8 \text{[N/m}^2], \quad \nu = 0.3 [-].\]

Air-Film's Parameters are: \(h_a = 1 \text{[mm]}, \quad \rho_a = 1.29 \text{[kg/m}^3], \quad \mu_a = 1.81 \times 10^{-5} \text{[kg/m} \cdot \text{s}].\)
Figure 3-10: Campbell Diagram for a Disk with both the Shear Loading and Internal Damping.

\[ a = 10 \text{[mm]}, \quad b = 140 \text{[mm]}, \quad h_d = 38 \text{[\mu m]}, \quad \rho_d = 1100 \text{[kg/m}^3], \quad E = 1.47 \times 10^8 \text{[N/m}^2], \quad \nu = 0.3 [-], \quad \eta = 0.02 [-]. \]

Air-Film's Parameters are: \[ h_a = 1 \text{[mm]}, \quad \rho_a = 1.29 \text{[kg/m}^3], \quad \mu_a = 1.81 \times 10^{-5} \text{[kg/m} \cdot \text{s}]. \]
Figure 3-11: Campbell Diagram for the Modes with $n = 10$ Nodal Diameters. NO Shear Loading.

$a = 10 \,[mm], \ b = 140 \,[mm], \ h_d = 38 \,[\mu m], \ \rho_d = 1100 \,[kg/m^3], \ E = 1.47 \times 10^8 \,[N/m^2], \ \nu = 0.3 \,[-].$

Air-Film’s Parameters are: $h_o = 1 \,[mm], \ \rho_o = 1.29 \,[kg/m^3], \ \mu_o = 1.81 \times 10^{-5} \,[kg/m \cdot s]$. 
Figure 3-12: Campbell Diagram for the Modes with \( n = 10 \) Nodal Diameters. WITH Shear Loading.

\[ a = 10 \text{ [mm]}, \ b = 140 \text{ [mm]}, \ h_d = 38 \text{ [\mu m]}, \ \varepsilon_d = 1100 \text{ [kg/m}^3], \ E = 1.47 \times 10^8 \text{ [N/m}^2], \ \nu = 0.3 [-]. \]

Air-Film’s Parameters are: \( h_a = 1 \text{ [mm]}, \ \rho_a = 1.29 \text{ [kg/m}^3], \ \mu_a = 1.81 \times 10^{-5} \text{ [kg/m} \cdot \text{s}]. \)
nodal diameters, and without the shear loading. The critical speed of the (1, 10) mode is there around 77 [rpm]. When the shear loading is included, and as can be seen in figure 3-12, the critical speed of the (1, 10) mode has decreased to 73 [rpm]. This is certainly not a considerable variation (from an engineering perspective at least) but it clearly shows the expected behavior.

3.8 Mode Shapes with the Shear Loading

In the absence of fluid shear loading and all other nonconservative mechanisms, the mode shapes of the spinning disk are characterized by perfectly straight nodal diameters. In our model of the disk, there are two mechanisms which could cause the nodal diameters to depart from a straight line. The first mechanism was proposed by Hosaka and Crandall [28] and is due to the non-proportional nature of the damping mechanisms. For any given mode, the direction in which the skewing of the nodal diameters occurs depends on whether that mode's critical speed is below or above the disk's spin rate. The second mechanism, which we observed experimentally is due to the presence of in-plane shear stresses induced by the fluid shear loading.

![Spinning Disk](image)

*Fixed Baseplate*

Figure 3-13: Disk Deflection with the Fluid Shear Loading

Figure 3-13 shows a possible deflection of the spinning disk with the shear loading. In contrast with the mechanism described in [28], the warping here is always in the direction opposite that of the spin rate. The warping of the nodal diameters is not always as pronounced as that shown in figure 3-13. In fact, increasing the disk's rpm
will straighten the nodal lines. The “straightening” of the disk’s nodal diameters is caused by the in-plane tensile stresses which, for large values of the spin rate, overcome the in-plane shear stresses. It is true that the shear stress $\sigma_{r\theta}$ increases with the spin rate $\Omega$, but on the other hand, the tensile stresses $(\sigma_r, \sigma_\theta)$ increase with the square of the disk’s spin rate.

3.9 Destabilizing Effects of the Shear Loading

From our eigenvalue analysis, we found that the shear loading introduces “non-conservative” forces that are responsible for the growing and the decaying solutions. To understand the mechanisms by which these forces introduce such unsteady solutions, let us consider a “developed” annular segment of the disk as shown in figure 3-14. In the absence of waves in the disk i.e., when the disk is perfectly flat, there are still wall shear stresses that apply a viscous drag on the disk. The motor that is driving the disk must supply a torque to balance the effect of the viscous drag and thus spin the disk. In the full disk, the torque is transmitted from one annular segment to the other through the in-plane shear stress $\sigma_{r\theta}$.

![Figure 3-14: Developed Annular Segment of the Spinning Disk](image)

In the straightened version shown in the above figure, the driving torque supplied by the disk drive is represented by a difference between the two tension forces acting at the two ends of the annular segment.

$$\Delta T = T_2 - T_1 = 2\tau_w \times L \times B$$  \hspace{1cm} (3.27)
where $B$ is the width of the annular segment. With the presence of traveling waves however, and depending on whether the wave is traveling forward or backward with respect to the ring's segment, the difference $\Delta T$ will correspondingly either increase or decrease. The consequence is that the magnitude of the torque that is supplied by the motor of the disk drive will not be the same for the two traveling waves. It is this difference that reflects the fact that one of the waves has positive damping whereas the other one has negative damping. The amplitude of one wave will thus decay with time whereas that of the other wave will increase with time.

3.10 Conclusions

To close this study of the effects of the fluid shear loading on the dynamics of our spinning disk, let us summarize the main findings. We have determined that the air-film exerts wall shear stresses on the rotating disk in both the radial and circumferential directions. The distribution of wall shear stresses and the resulting in-plane stresses in the disk were calculated. The equation of motion of the spinning disk was then augmented to account for such effects. By treating a stationary disk with a rotating shear flow we found that the shear loading effectively reduces the bending rigidity of the disk. For a rotating disk, the effect is not as pronounced because the centrifugally induced membrane stresses dominate the scene as they tend to stiffen the disk. The importance of the shear loading is thus problem dependent. The mode shapes of the disk are effected by such loading, the nodal lines get skewed. We have also found that the shear loading, when considered alone -without any other loading or dissipative mechanism, is destabilizing at all speeds of rotation of the disk.
Chapter 4

Traveling String Interacting with a Complete Load System

4.1 Introduction

In this chapter we introduce a simplified model which provides insight into the qualitative nature of the instability mechanisms of the disk interacting with the load system. We emphasize here that our analysis focuses on those instabilities that are due to the load system alone. The spinning disk is replaced by a traveling string without centrifugal stiffening and the load system is modeled —as in chapter 2, by a single-degree-of-freedom mass-spring-dashpot system. This simplified model has instability mechanisms which are similar to those of the full head-disk system, but because the analysis is simplified it is much easier to identify the agent causing the instabilities. We begin our analysis by presenting the string's equation along with some relevant derivations. We then present three different solution techniques —two exact and one approximate, and show that the results are essentially identical. Galerkin's method —our approximate technique, is then used throughout the remaining part of this chapter to obtain the natural frequencies of the string, identify the regimes of unstable operation, investigate the sensitivity of the stability regions upon the load system's parameters, and finally explain the physical aspects of the destabilizing effects of each of the three elements of the load system.
4.2 Mathematical Model and Solution Techniques

Let us consider the string\(^1\) of figure 4-1 with linear mass density \( \rho \) and subjected to a constant tension \( T \). The string is wrapped around a circle of radius \( R \) and spun with a rate \( \Omega = \nu/R \). The load system consists of a single-degree-of-freedom mass-spring-dashpot that is connected to the string at \( x = 0 \) through frictionless rollers. To keep our model as simple as possible, we will ignore the bending rigidity of the string and the stiffening introduced by the centrifugally induced hoop stresses.

![Diagram](image)

Figure 4-1: Traveling String with Complete Load System

In a stationary frame of reference, the equation of motion governing small transverse displacements \( w(x, t) \) of our traveling string-load system is

\[
\rho \left( \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial x} \right)^2 w - T \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < L = 2\pi R \tag{4.1}
\]

with the continuity and force balance boundary conditions

\[
w(0, t) = w(L, t) \tag{4.2}
\]

\[
(T - \rho \nu^2) \left[ \frac{\partial w}{\partial x} (0, t) - \frac{\partial w}{\partial x} (L, t) \right] = f_L(t) = F_L e^{i\omega t} \tag{4.3}
\]

\(^1\)Schajer [47] solved this problem with only the spring \( k_L \) of the load system. His analysis was for subcritical operation only.
where the force applied by the load-system on the string is given by

\[ f_L = \left[ m_L \frac{\partial^2}{\partial t^2} + c_L \frac{\partial}{\partial t} + k_L \right] w(0,t) \]  

(4.4)

The continuity boundary condition is rather trivial, the string must have the same displacement at both ends. To obtain the force balance boundary condition let us consider a small control volume extending from \(-dx\) to \(+dx\) as shown on figure 4-2.

![Figure 4-2: String with Control Volume to Compute the Force Balance BC](image)

Newton's second law for a control volume is given by Panton [40, page 129]

\[ \frac{d}{dt} \int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v} (\vec{v} \cdot \vec{n}) dA = \vec{F}_{CV} \]  

(4.5)

This equation says that the rate of change of the linear momentum (first term) in a control volume \(CV\) plus the momentum flux (second term) through the control surface \(CS\) is balanced by the force (right hand side) acting on this control volume \(CV\). Specializing equation 4.5 to figure 4-2, and considering only the vertical direction we obtain the contribution of each of the three terms

\[ \frac{d}{dt} \int_{CV} \rho \vec{v} dV = \frac{d}{dt} \left[ \rho v dx \left( \tan \alpha_1 + \tan \alpha_2 \right) \right] \]  

(4.6)

\[ \int_{CS} \rho \vec{v} (\vec{v} \cdot \vec{n}) dA = \rho v^2 (-\tan \alpha_1 + \tan \alpha_2) \]  

(4.7)

\[ \vec{F}_{CV} = T (-\tan \alpha_1 + \tan \alpha_2) - f_L \]  

(4.8)
where we have used the small angles $\alpha_1$ and $\alpha_2$ given by

$$\tan \alpha_1 = \frac{\partial w}{\partial x}(L, t) \quad \text{and} \quad \tan \alpha_2 = \frac{\partial w}{\partial x}(0, t) \quad (4.9)$$

Grouping all terms and taking the limit as the size of the control volume shrinks to zero ($dx \to 0$) we will get the force balance boundary condition given by equation 4.3. This boundary condition introduces major mathematical difficulties.

When the load system is removed and the spin rate set to zero ($\nu = 0$), the string supports an infinite number of pairs of waves each with wavelength $\lambda_n = 2\pi R/n$. The integer $n$ represents here the number of “full wavelengths” in the circular string and corresponds to the number of nodal diameters of the original full disk. For each pair of waves, one will be traveling forward and the other backward with respect to the string. The frequencies corresponding to both the forward and backward waves with wavelength $\lambda_n$, as seen by a stationary observer, are given by

$$\omega_n = \pm \frac{2\pi}{\lambda_n} c_0 = \pm \frac{nc_0}{R} \quad (4.10)$$

where $c_0 = \sqrt{T/\rho}$ is the wave speed, as measured by an observer that is moving with the string. When the string is made to spin with a rate $\Omega = \nu/R$ however, the two frequencies suffer a Doppler shift and become

$$\omega_n^{wd} = \frac{n}{R}(+c_0 + \nu) \quad \text{and} \quad \omega_n^{bk} = \frac{n}{R}(-c_0 + \nu) \quad (4.11)$$

The two frequencies vary linearly with the transport speed $\nu$. In contrast with the full disk, the phase speed $c_\phi = c_0$ is here constant\footnote{In the full disk the dependence $\omega = \omega(\nu)$ is less trivial owing to centrifugal stiffening forces.}. To get the complete picture, let us now solve for the natural frequencies of our vibratory system for the more general case when the load system is present. For this, and because of the singularity introduced by the load system, we investigate three possible solution techniques.

- Galerkin’s technique: Assumed Modes solution.
- Transfer Matrix technique: Forward + Backward waves.
- Continuous Green’s Function: Forward + Backward waves.
The first technique is an approximate method that has proven extremely powerful in a large number of applications. The second and third techniques yield exact solutions and the cost there is heavy algebraic manipulation that leads to transcendental functions of the frequency variable $\omega$ and the transport speed $v$. For simplicity we will consider only the stiffness $k_L$ of the load system in our derivations. The effects of the mass $m_L$ and the dashpot $c_L$ can be introduced ad-hoc in the final results.

4.2.1 Galerkin’s Technique

The singularity introduced by the load system into the string’s equation makes it necessary to seek alternate solution techniques. The use of Galerkin’s technique is here appropriate and computationally appealing. To begin, let us rewrite the equation of motion of our spinning string and this time include the force boundary condition in it and divide all terms by the linear density $\rho$ of the string

$$\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right]^2 w - c_0^2 \frac{\partial^2 w}{\partial x^2} + \frac{k_L}{\rho} w \delta (x) = 0 \quad \text{for} \quad 0 \leq x \leq L = 2\pi R \quad (4.12)$$

A key step in Galerkin’s technique is the choice of the trial solution. For our traveling string, a reasonable choice is given by the three-parameter\(^3\) standing waves solution

$$w (x, t) = [A_0 + B_n \sin (nx/R) + A_n \cos (nx/R)] e^{st} \quad (4.13)$$

We switch here to the complex Laplace frequency $s = \sigma + i\omega$, which is more convenient for the derivations that follow. This solution is not restrictive to standing waves. The amplitudes $A_n$ and $B_n$ are in general complex-valued and thus, the above solution describes standing as well as traveling waves. Inserting the above solution into the equation of motion 4.12, we obtain the residual

$$\Phi = [s^2]A_0 + [2s(n\Omega)][B_n \cos(nx/R) - A_n \sin(nx/R)] +$$
$$[s^2 + \omega_n^2 - (n\Omega)^2][B_n \sin(nx/R) + A_n \cos(nx/R)] +$$
$$[2\pi \omega_n^2 \delta (x)][A_0 + B_n \sin(nx/R) + A_n \cos(nx/R)] \quad (4.14)$$

\(^3\)Note that we include here the rigid body translation $A_0$. This degree of freedom, which corresponds to the “umbrella” mode of the full disk, is sometimes omitted in the literature.
where we have used the quantity \( \omega_L \), the ratio between the spring stiffness \( k_L \) and the total mass of the string \( 2\pi R \rho \), as a convenient measure of the load stiffness \( k_L \)

\[
\omega_L^2 = \frac{k_L}{2\pi R \rho}
\] (4.15)

The residual \( \Phi \) given by equation 4.14 vanishes when the trial solution is the exact solution. Because we want our trial solution to approximate as well as possible the exact solution, we make the residual orthogonal to the components in our trial solution. Mathematically, this statement is written as

\[
\int_{-\pi R}^{\pi R} \Phi dx = 0, \quad \int_{-\pi R}^{\pi R} \Phi \sin(nx/R) dx = 0, \quad \int_{-\pi R}^{\pi R} \Phi \cos(nx/R) dx = 0
\] (4.16)

Performing the above calculations and putting the result in matrix form, we get

\[
\begin{bmatrix}
    s^2 + \omega_L^2 & 0 & \omega_L^2 \\
    0 & \frac{1}{2}[s^2 - (n\Omega)^2 + \omega_n^2] & -s(n\Omega) \\
    \omega_L^2 & s(n\Omega) & \frac{1}{2}[s^2 - (n\Omega)^2 + \omega_n^2] + \omega_L^2
\end{bmatrix}
\begin{bmatrix}
    A_0 \\
    B_n \\
    A_n
\end{bmatrix} = 0
\] (4.17)

The above result is a matrix eigenvalue problem for the complex frequency \( s \) and the mode shape \( \{A_0, B_n, A_n\}^T \) and can be written in the more compact form

\[
(s^2[M] + s[G] + [K + \Omega^2 E]) \{A_0, B_n, A_n\}^T = 0
\] (4.18)

The four matrices reflect the various types of forces present in our spinning string. We have presented the underlying idea behind Galerkin’s technique with a three-parameter trial shape \( w(x, t) \). Conceptually, we could try any number of sine and cosine components. In this case, our trial shape becomes in some sense a truncated Fourier series. To compare the results from Galerkin’s technique against those of the exact solutions in the next two sections, let us use the trial shape

\[
w(x, t) = \left\{ A_0 + \sum_{n=1}^{3} [B_n \sin(nx/R) + A_n \cos(nx/R)] \right\} e^{st}
\] (4.19)

We now have a total of seven parameters that lead to square matrices that are all of size \( 7 \times 7 \). The complex frequencies \( s \) resulting from solving the eigenvalue problem
corresponding to this seven-parameter solution are displayed as Campbell diagrams in figures 4-3 and 4-4 for the cases without the load system then with only the spring $k_L$ of the load system respectively. We have used there the three dimensionless variables.

$$\beta = \frac{v}{c_0}, \quad \alpha_L = \frac{k_L R}{\pi T} = 2\frac{\omega_L^2}{\omega_1^2}, \quad z = \frac{sR}{c_0} = \frac{s}{\omega_1} \quad (4.20)$$

Examining the diagrams of figure 4-3 we see that the natural frequencies for the string alone are imaginary for all values of the transport speed parameter $\beta$ and thus they correspond to pure and steady oscillations. The various "branches" in the diagram correspond in fact to the expressions given by equation 4.11. When the stiffness of the load system is introduced through the parameter $\alpha_L$, there appears regions of the transport speed parameter $\beta$, in figure 4-4, where the frequencies of our spinning string are now complex valued. These regions correspond to unsteady solutions -one decaying and the other growing, of the string and occur at intersections of the various branches in the Campbell diagram for the imaginary part \(\Im\{z\}\) of the string's frequencies. As more terms are taken in the trial shape for Galerkin's technique, there will be more of these instability regions because of the many more intersections between the various branches. One important observation here is that all the regions of instability -where \(\Re\{z\} \neq 0\), occur for speeds that are larger than the critical i.e., for $\beta > 1$. Below the critical speed, the various branches repel each other and lead to two distinct frequencies -for each intersection in the original diagram of figure 4-3 where there is no load system. Above the critical speed, the story is however not that simple because at some of the intersections the frequencies remain purely imaginary whereas at other intersections they become complex valued. There is certainly a pattern here. When the two intersecting branches correspond to two waves that are traveling in the same direction with respect to the string, the coupling at the intersection is conservative and the corresponding two frequencies remain pure imaginary. However, when the two waves are traveling in opposite directions with respect to the string, the frequencies when the spring $k_L$ is introduced become complex-valued. This is certainly the kinematics part of the story only, the implications are presented further in this chapter. Lastly, and for a complete load system consisting of both the mass $m_L$ and
Figure 4-3: Campbell Diagrams for the Imaginary and Real Parts of the Frequencies of a Spinning String WITHOUT the Load System – Obtained by Galerkin's Technique with a Seven-Parameter Trial Shape.
Figure 4-4: Campbell Diagrams for the Imaginary and Real Parts of the Frequencies of a Spinning String WITH the Spring $k_L$ of the Load System — Obtained by Galerkin’s Technique with a Seven-Parameter Trial Shape.
the dashpot $c_L$ in addition to the spring $k_L$, we need to substitute the combined effect of all three elements of the load system in place of the stiffness $k_L$

$$k_L \rightarrow k_L + i\omega c_L - \omega^2 m_L$$  \hspace{1cm} (4.21)

We will come back to Galerkin’s technique and use it extensively. For now let us see how the results we obtained with Galerkin’s technique compare against those of the exact solution. We begin with the transfer matrix technique.

### 4.2.2 Transfer Matrix Solution

Since we are interested in harmonic waves, let us look for solutions of the form

$$w(x, t) = Ae^{i(\kappa x + \omega t)}$$  \hspace{1cm} (4.22)

When this solution is inserted into the string's equation 4.1, we get the dispersion relation between the frequency $\omega$ and the wavenumber $\kappa$

$$-(\omega + \kappa u)^2 + \kappa^2 c_0^2 = 0 \quad \Rightarrow \quad \omega = \kappa (\pm c_0 + u)$$  \hspace{1cm} (4.23)

We note here that this is the same result as that given by equation 4.11 but written in a form that is more appropriate here. There are two possible wavenumbers solutions $\kappa_f$ and $\kappa_b$, they correspond to the forward and backward waves respectively

$$\kappa_f = \frac{\omega}{c_0 + u} \quad \text{and} \quad \kappa_b = \frac{\omega}{c_0 - u}$$  \hspace{1cm} (4.24)

With these two wavenumbers, the displacement $w(x, t)$ and the slope $\psi(x, t)$ of our traveling string take now the form

$$w(x, t) = [A_f e^{-i\kappa_f x} + A_b e^{i\kappa_b x}] e^{i\omega t} = W(x) e^{i\omega t}$$  \hspace{1cm} (4.25)

$$\psi(x, t) = \frac{\partial w}{\partial x} = [-i\kappa_f A_f e^{-i\kappa_f x} + i\kappa_b A_b e^{i\kappa_b x}] e^{i\omega t} = \Psi(x) e^{i\omega t}$$  \hspace{1cm} (4.26)

where $A_f$ and $A_b$ are the amplitudes of the forward and backward waves accordingly. The displacement amplitude $W(x)$ and the slope $\Psi(x)$ can now be put in matrix form

$$\begin{bmatrix} W(x) \\ \Psi(x) \end{bmatrix} = \begin{bmatrix} e^{-i\kappa_f x} & e^{i\kappa_b x} \\ -i\kappa_f e^{-i\kappa_f x} & i\kappa_b e^{i\kappa_b x} \end{bmatrix} \begin{bmatrix} A_f \\ A_b \end{bmatrix}$$  \hspace{1cm} (4.27)
Specializing the above result for \( x = 0 \), we can solve for the amplitudes \( A_f \) and \( A_b \)

\[
\begin{align*}
\begin{bmatrix}
A_f \\
A_b
\end{bmatrix} = \frac{1}{\kappa_b + \kappa_f} \begin{bmatrix}
\kappa_b & i \\
\kappa_f & -i
\end{bmatrix} \begin{bmatrix}
W_0 \\
\Psi_0
\end{bmatrix}
\end{align*}
\]  
(4.28)

Combining equations 4.27 and 4.28, we can solve for the amplitude functions \( W(x) \) and \( \Psi(x) \) in terms of the same quantities when evaluated at \( x = 0 \)

\[
\begin{align*}
\begin{bmatrix}
W(x) \\
\Psi(x)
\end{bmatrix} = \frac{1}{\kappa_b + \kappa_f} \begin{bmatrix}
\kappa_b e^{-i\kappa_f x} + \kappa_f e^{i\kappa_b x} & i e^{-i\kappa_f x} - i e^{i\kappa_b x} \\
-i \kappa_b \kappa_f (e^{-i\kappa_f x} - e^{i\kappa_b x}) & \kappa_f e^{-i\kappa_f x} + \kappa_b e^{i\kappa_b x}
\end{bmatrix} \begin{bmatrix}
W_0 \\
\Psi_0
\end{bmatrix}
\end{align*}
\]  
(4.29)

The above result can be written in the more compact form

\[
\begin{align*}
\begin{bmatrix}
W(x) \\
\Psi(x)
\end{bmatrix} = \begin{bmatrix}
T_{11}(x) & T_{12}(x) \\
T_{21}(x) & T_{22}(x)
\end{bmatrix} \begin{bmatrix}
W_0 \\
\Psi_0
\end{bmatrix}
\end{align*}
\]  
(4.30)

The above form of the result is in essence the idea behind the Transfer Matrix Method\(^4\). This representation permits us to express the field variables \((W, \Psi)\) at any location \( x \) in terms of the same field variables at \( x = 0 \). What is nice about this technique is that it is very powerful in modeling the dynamics of systems that are made of several cascaded elements. Some applications where this technique excels are stepped rotors and duct acoustics. Back to our traveling string problem, and after some algebra with Mathcad, the four elements of the transmission matrix are

\[
\begin{align*}
T_{11}(x) &= \frac{1}{2c_o} [(c_o + v)e^{-i\kappa_f x} + (c_o - v)e^{i\kappa_b x}] \\
T_{12}(x) &= \frac{c_o^2 - v^2}{2c_o \omega} [i e^{-i\kappa_f x} - i e^{i\kappa_b x}] \\
T_{21}(x) &= \frac{\omega}{2c_o} [-i e^{-i\kappa_f x} + i e^{i\kappa_b x}] \\
T_{22}(x) &= \frac{1}{2c_o} [(c_o - v)e^{-i\kappa_f x} + (c_o + v)e^{i\kappa_b x}]
\end{align*}
\]  
(4.31-4.34)

If we now evaluate the field variables \( W(x) \) and \( \Psi(x) \) at \( x = L \), we obtain

\[
\begin{align*}
\begin{bmatrix}
W(L) \\
\Psi(L)
\end{bmatrix} = \begin{bmatrix}
T_{11}(L) & T_{12}(L) \\
T_{21}(L) & T_{22}(L)
\end{bmatrix} \begin{bmatrix}
W_0 \\
\Psi_0
\end{bmatrix}
\end{align*}
\]  
(4.35)

\(^4\)See Lecture Notes for MIT course "16.56 Noise Control Engineering" by K. Uno Ingard.
On the other hand, the two boundary conditions of our string, as given by equations 4.2 and 4.3 can be put in matrix form

\[
\begin{pmatrix}
W(L) \\
\Psi(L)
\end{pmatrix} =
\begin{bmatrix}
1 & 0 \\
-k_L/(T - \theta v^2) & 1
\end{bmatrix}
\begin{pmatrix}
W_0 \\
\Psi_0
\end{pmatrix}
\] (4.36)

Taking the difference of the above two results, we obtain the eigenvalue problem

\[
\begin{bmatrix}
T_{11}(L) - 1 & T_{12}(L) \\
T_{21}(L) + k_L/(T - \theta v^2) & T_{22}(L) - 1
\end{bmatrix}
\begin{pmatrix}
W_0 \\
\Psi_0
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\] (4.37)

In this formulation all the entries of the matrix and the eigenvector are in general complex-valued. We have to be careful here with the interpretation. Let us transform this eigenvalue problem to a formulation in terms of real quantities only. For this, let us rewrite equation 4.37 as

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{pmatrix}
W_0 \\
\Psi_0
\end{pmatrix} = 0
\] (4.38)

Next, write all quantities in terms of their real and imaginary components

\[
\begin{bmatrix}
A_R + iA_I & B_R + iB_I \\
C_R + iC_I & D_R + iD_I
\end{bmatrix}
\begin{pmatrix}
\Re\{W_0\} + i\Im\{W_0\} \\
\Re\{\Psi_0\} + i\Im\{\Psi_0\}
\end{pmatrix} = 0
\] (4.39)

The above two complex equations can be written as four real equations

\[
\begin{bmatrix}
A_R & -A_I & B_R & -B_I \\
A_I & A_R & B_I & B_R \\
C_R & -C_I & D_R & -D_I \\
C_I & C_R & D_I & D_R
\end{bmatrix}
\begin{pmatrix}
\Re\{W_0\} \\
\Im\{W_0\} \\
\Re\{\Psi_0\} \\
\Im\{\Psi_0\}
\end{pmatrix} = 0
\] (4.40)

The determinant \(D(\omega, v)\) of the fourth order matrix in this eigenvalue problem is now purely real and can be thought of as a surface above the \(\omega - v\) plane. The surfaces for the cases of the traveling string without and then with the stiffness \(k_L\) of the load system are shown as three dimensional projections in figure 4-5. The contour plots of such surfaces near the plane \(D = 0\) are shown in figure 4-6 for the same values of the load system's stiffness \(k_L\). These contour plots give a better indication of the
Figure 4.5: Surfaces \( D(\omega, v) \) of the Transfer Matrix Solution for \( k_z = 0 \) and \( k_z \neq 0 \).
Figure 4.6: Contour Plots of the Surfaces $D(\omega, v)$ shown on the Previous Page.
location of the roots $\omega = \omega(v)$ and are very appropriate to compare with the results from Galerkin’s technique. We notice that, for the case $k_L \neq 0$, there are regions of the speed $v$ for which there is no intersection of $D(\omega, v)$ with the plane $D = 0$. This simply means that the “ridges” in figure 4-5 fall below the plane $D = 0$ and a complex-valued $\omega$ is needed to “push” those ridges up and make them “touch” the plane $D = 0$. These regions –where the solution of $D = 0$ requires the frequencies $\omega$ to be complex-valued, are regions where unstable operation takes place.

### 4.2.3 Continuous Green’s Function Solution

The idea here is to solve for the response of the spinning string in terms of a point excitation force acting at the origin $x = 0$. The result is a Green’s function. Further, the response of the spring $k_L$ of the load system is also determined. The Green’s function is then evaluated at the attachment point of the spring and the result equated to the spring’s response. Figure 4-7 is helpful throughout the derivations that follow.

![Figure 4-7: Subcomponents Used for the Continuous Solution Approach](image)

From equation 4.25, we have an expression for the general solution $W(x)$

$$W(x) = A_f e^{-i\kappa_f x} + A_b e^{i\kappa_b x}$$  \hspace{1cm} (4.41)

Upon inserting this solution in the two boundary conditions 4.2 and 4.3, we obtain

$$\begin{bmatrix} 1 - e^{-i\kappa_f L} & 1 - e^{i\kappa_b L} \\ -i\kappa_f (1 - e^{-i\kappa_f L}) & i\kappa_b (1 - e^{i\kappa_b L}) \end{bmatrix} \begin{bmatrix} A_f \\ A_b \end{bmatrix} = \frac{1}{T - gu^2} \begin{bmatrix} 0 \\ F_L \end{bmatrix}$$  \hspace{1cm} (4.42)

We can then solve for the amplitudes $A_f$ and $A_b$ of the forward and backward waves

$$\begin{bmatrix} A_f \\ A_b \end{bmatrix} = \frac{F_L}{\Delta (T - gu^2)} \begin{bmatrix} -1 + e^{i\kappa_b L} \\ 1 - e^{-i\kappa_f L} \end{bmatrix}$$  \hspace{1cm} (4.43)
with the determinant $\Delta$ being given by

$$\Delta = i \frac{2 \omega c_0}{c_0^2 - \nu^2} \left(1 - e^{-i \kappa f L}\right) \left(1 - e^{i \kappa s L}\right)$$

(4.44)

Substituting for the amplitudes $A_f$ and $A_s$ into our general solution $W(x)$ and evaluating the result at the attachment point $x = 0$ of the load system, we obtain

$$W(0) = i \frac{F_L}{2 \omega \rho c_0} \frac{e^{-i \kappa f L} - e^{i \kappa s L}}{(1 - e^{-i \kappa f L})(1 - e^{i \kappa s L})}$$

(4.45)

On the other hand, the reaction due to the spring $k_L$ of the load system is

$$F_L = -k_L W(0) \quad \Rightarrow \quad W(0) = -\frac{F_L}{k_L}$$

(4.46)

Taking the difference of the above two results, dividing by the reaction $F_L$ from the load system and rearranging, we finally obtain

$$i \frac{2 \omega \rho c_0}{e^{-i \kappa f L} - e^{i \kappa s L}} \left(1 - e^{-i \kappa f L}\right) \left(1 - e^{i \kappa s L}\right) - k_L = 0$$

(4.47)

The natural frequencies of our traveling string and spring system are then those values of $\omega$ which make the function $G(\omega, \nu) = 0$

$$G(\omega, \nu) = \frac{2 \omega \rho c_0}{e^{-i \kappa f L} - e^{i \kappa s L}} \left(1 - e^{-i \kappa f L}\right) \left(1 - e^{i \kappa s L}\right) - k_L$$

(4.48)

Similarly to transfer matrix result, $G(\omega, \nu)$ can be thought of as a surface above the $\omega - \nu$ plane. The intersection of this surface with the plane $G = 0$ gives us the eigenvalues $\omega = \omega(\nu)$. In contrast with the result of the transfer matrix technique, the function $G(\omega, \nu)$ is here complex valued. Contour plots of the surface $\Re\{G(\omega, \nu)\}$, for the values of $k_L = 0$ then $k_L \neq 0$, near the plane $G = 0$ are shown in figure 4-8. These contours indicate that the roots $\omega = \omega(\nu)$ are the same as those obtained from the transfer matrix solution and shown in figure 4-6. This is of course expected since the two techniques are variants of one another.

Comparing the Campbell diagrams of figure 4-3 to those in figures 4-8 and 4-6, we can see that there is very good agreement between the two exact techniques and the approximate Galerkin's technique. We further compared the mode shapes (eigenvectors) for several roots $\omega$ and there again, the agreement is quite good. The only
Figure 4.8: Contour Plots of the Surfaces $G(\omega, \nu)$ shown on the Previous Page
difference is that both the continuous and transfer matrix results show a slope discontinuity at the origin $x = 0$ of the string. In contrast, the solution obtained from Galerkin's technique is smooth. Conceptually, as more modes are taken into the trial solution, the shape obtained from Galerkin's technique approaches that of the exact solutions. For the remaining part of this chapter, Galerkin's technique is used as a universal tool for solving for the string's natural frequencies and mode shapes.

4.3 Roadmap of what is Yet to Come

In the sections ahead, we address several aspects pertinent to our spinning string. We begin our investigation with the effects the spring $k_L$ of the load system has on the dynamics of the string and investigate the consequences of including the rigid body transverse translation. The solutions with and then without the rigid body translation are compared and stability maps are obtained. We then discuss the effects of adding a modulated elastic foundation and show that any departure from symmetry can lead to self-excited vibrations of the traveling string. We then come back and study extensively the string with and then without the rigid body translation. In addition to the Campbell diagrams we also give the mode shapes of the string and elaborate on the kinematics of the problem. The mass of the load system is then incorporated and there again, we present several results and use alternate ways to describe the kinematics of the problem. The dashpot of the load system is finally introduced and specific results presented. The three components of the load system are then put together and a brief comparative study of their effects is presented. Throughout our presentation we focus our attention on the one-nodal diameter mode because there, Galerkin's technique has a very nice interpretation. Whenever relevant we make extensions to higher modes –those having several nodal diameters, and discuss specific aspects. The dynamics of the mode that has only one nodal diameter are then explained by a rigid gyroscopic ring that is interacting with a load system. Using a straightened version of our string we then illustrate how the energy stored in the axial motion of the string can be channeled into transverse vibratory modes.
4.4 Two- versus Three-Parameter Solution

During the early stages of this research, we looked at the effects of including the rigid body mode in our trial solution. We find that the character of the solution, when the rigid body mode is included, is affected only when the stiffness of the load system is present. For the case where the mass and/or the dashpot - but not the spring, of the load system is/are present, including the rigid body mode does not affect the solution. The reason for us including the rigid body mode is that in the literature on spinning disks, the coupling between flexural modes of the disk and the rigid body modes is seldom considered. There are treatments for flexible disks and then there are treatments for rigid disks. A combination of the two situations didn't seem very realistic if not practical. Large circular saws in the timber industry have long ago been observed to undergo severe lateral vibrations during the cutting process. A technique that has proven quite reliable to effectively reduce those vibrations is to have the saw sliding-free on its drive shaft, and guided by a rigid support system very close to the workpiece as shown in figure 4-9. In order for the vibrating saw to remain "motionless" at the rigid support location, the rigid body translation mode as well as the umbrella mode must be included in the formulation of such a dynamic system.

![Circular Saw Diagram]

Figure 4-9: Reducing the Vibration of a Circular Saw by Adding a Rigid Support

With this in mind, let us now go back to our string and compare the two solutions, namely that with and then without the rigid body translation. We can use the results we obtained in the section on Galerkin's technique and specialize them to the case $n = 1$. We begin with the case when the rigid body translation of the string is allowed.
The shape $w(\theta, t)$, when the rigid body translation is included, is given by

$$w(\theta, t) = [A_0 + B_1 \sin \theta + A_1 \cos \theta] e^{zt}, \quad \theta = x/R \quad (4.49)$$

Figure 4-10 shows clearly what the three parameters $(A_0, B_1, A_1)$ correspond to. $A_0$ is the amplitude of the rigid body translation whereas $(B_1, A_1)$ are the amplitudes of two truly "flexural" modes. Later on in this chapter, we will see that $B_1$ and $A_1$ can be interpreted as rotations about the two orthogonal axes $\theta = 0$ and $\theta = \pi/2$. Proper combination of the $B_1$ and $A_1$ parameters gives traveling waves.

Figure 4-10: Spinning String with Three "Degrees-of-Freedom" $A_0$, $B_1$, and $A_1$.

Upon applying Galerkin's technique with our three-parameter trial shape, we obtain

$$\begin{bmatrix}
2x^2 + \alpha_L & 0 & \alpha_L \\
0 & z^2 + 1 - \beta^2 & -2\beta z \\
\alpha_L & 2\beta z & z^2 + 1 - \beta^2 + \alpha_L
\end{bmatrix}
\begin{bmatrix}
A_0 \\
B_1 \\
A_1
\end{bmatrix} = 0$$

(4.50)

where we have used the dimensionless variables defined by equation 4.20. We note here that the load system -parameter $\alpha_L$, manifests itself at entries coupling the rigid body mode with amplitude $A_0$ and the cosine standing wave with amplitude $A_1$. There is no coupling between the load system and the sine standing wave with amplitude $B_1$ because $\sin \theta$ vanishes at the point of application of the load. The normalized natural frequencies $z$ of this eigenvalue problem are graphed in figure 4-11 for the case $\alpha_L = 0$. The monotonically increasing line corresponds to the forward wave in the string, and the one that decreases, reaches zero, then increases corresponds to the backward wave (with respect to the string). The thick horizontal line at zero
frequency corresponds to the rigid body mode of the string. In fact, when the string is not restrained from vertical motion, it can translate vertically as a rigid body thus, with zero oscillating frequency. The abscissa where the branch corresponding to the backward wave reflects off the $\beta$-axis is precisely the critical speed for the mode being considered. We note here that the critical speed occurs when the transport speed $v$ matches exactly the wave speed $c_o$, i.e., when $\beta = 1$. For the string, this critical speed is the same for all the modes (wavelengths) because the string is nondispersive—waves of different wavelengths travel with the same phase speed. From a stability point of view, we can see that when the load system is absent, the frequencies of the string are always pure imaginary. The string is thus stable for all operating speeds.

![Diagram](image)

Figure 4-11: Campbell Diagram for the Three-Parameter Solution with $\alpha_L = 0$

When the spring of the load system is introduced through the parameter $\alpha_L = 1$, there exists a range of transport speeds $v = \beta c_o$, as shown in figure 4-12, for which the frequencies are complex-valued. In this speed range, the string has two unsteady solutions, one is decaying and the other is growing. The decaying solution is very well behaved. The growing solution however is the one that corresponds to self-excited vibrations of the string. Because the oscillating frequency component $\Im\{z\}$ is nonzero inside the unstable range, such instability is referred to as flutter. Structural flutter occurs when two modes of a vibrating system coalesce and thus have the same

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\(^5\)See Mote [43]
oscillating frequency. This is certainly the case for our string where the backward wave couples strongly with the rigid body translation mode.

Figure 4-12: Campbell Diagram for the Three-Parameter Solution with \( \alpha_L = 1 \)

Figure 4-13: Campbell Diagram for the Two-Parameter Solution with \( \alpha_L = 0 \)

Let us now see what happens when the \( A_0 \) degree of freedom is eliminated. The eigenvalue problem for this situation is obtained from that in equation 4.50 by deleting the first row and the first column. The corresponding characteristic polynomial is

\[
\Delta(z) = z^4 + z^2 \left[ 2 \left( 1 + \beta^2 \right) + \alpha_L \right] + \left( 1 - \beta^2 \right) \left( 1 - \beta^2 + \alpha_L \right)
\]

(4.51)
The roots of this characteristic polynomial—the frequencies \( z \) which we are seeking, are graphed in figure 4-13 for \( \alpha_L = 0 \). We note there the similarity to figure 4-11. We still have the branches corresponding to the forward and backward waves. The only difference is the absence of the thick line at zero-frequency which corresponds to the rigid body degree of freedom. If we now include the stiffness of the load system, the frequency loci changes to that shown in figure 4-14. A *divergence* region—Instability with zero oscillation frequency, opens up right above the critical speed.

![Diagram](image)

Figure 4-14: Campbell Diagram for the Two-Parameter Solution with \( \alpha_L = 1 \)

Another observation—that is also true for the three-parameters solution, is that besides introducing unstable regions, the spring has changed the frequencies of the string. This can be seen by overlapping figures 4-14 and 4-13. The reason being that the spring makes the string “stiffer” and thus increases the wave speed \( c_o \). This in turns causes the observed change in frequencies. We have to note that it is Galerkin’s technique that is responsible for “spreading” the stiffness \( k_L \) of the load system over the whole length of the string. We also have to note, and this is extremely important, that even though Galerkin’s technique provides in some sense an “average” solution, it does retain the particularities of the problem. The solution obtained from Galerkin’s technique still knows that there is a discrete spring at the origin \( \theta = 0 \). This can be seen in the eigenvalue problem of equation 4.50 where the spring parameter \( \alpha_L \) couples the rigid body mode to the cosine standing wave.
Depending on whether the solution includes or not the rigid body mode, the instability region is correspondingly of flutter or divergence type. This is definitely a fundamental difference. The immediate conclusion we should retain from this comparative study is that the rigid body mode should definitely be included in a careful study of such rotordynamics systems. The reason being that the means by which such instabilities are controlled may vary considerably depending, precisely, on the type of instability at hand. For both solutions—with and without the rigid body translation $A_0$, the width of the unstable region varies depending upon the strength of the spring $k_L$. Figure 4-15 shows such instability regions for both trial shapes in the $\beta - \alpha_L$ plane. We can see there that for any given value of the load stiffness parameter $\alpha_L$, the flutter region is much wider than the divergence region.

![Divergence Region for 2D Solution](image1)

![Flutter Region for 3D Solution](image2)

Figure 4-15: Stability Regions for String without and with the $A_0$ Translation.

We can also see that the onset speed of instability predicted by both solutions is not the same. In the limiting case of an infinitely stiff load system, stability of the string in never regained at high transport speeds. Looking carefully at the results of figure 4-6, it seems that the exact solution has only flutter instabilities. There are no divergence regions at all. While this is true for our non-dispersive string interacting with a load stiffness $k_L$, it is certainly not the general case. In the next section we present cases where the string can have both types of instability.
It might seem a bit odd though that a conservative element such as the spring\textsuperscript{6} can cause our traveling string to undergo self-excited vibrations. This is not the correct train of thought to look at this problem though. We have to view the spring as an element that is “breaking” the symmetry of the perfectly axisymmetric string. In fact, in the rotordynamics literature, it has been known for some time now that any departure from symmetry — be it of the stiffness or inertia properties, may introduce regions of unstable operation. Typical examples are shafts with unequal bending rigidity along two orthogonal directions and two-blade propellers. The work of Crandall [20] and Ehrich [25, Chap. 1] present good overviews of this subject matter.

### 4.5 String with a Non-Uniform Foundation

To understand how asymmetric stiffness properties of the string lead to unstable behavior, we studied various models where our traveling string sits on a non-uniform foundation. The list includes amongst others a partial elastic foundation — extending over a fraction of the string circumference, a sinusoidally modulated foundation, and a foundation with “stepped” variations in its stiffness. The two cases we choose for illustration here — and which we find quite interesting, are the first for a sinusoidally modulated foundation, and the second for a partial uniform foundation. The equation of motion of the string, when it is sitting on a sinusoidally modulated elastic foundation, is given by

\[
\rho \left[ \frac{\partial}{\partial t} + \frac{v}{R} \frac{\partial}{\partial \theta} \right]^2 w - \frac{T}{R^2} \frac{\partial^2 w}{\partial \theta^2} + k(1 + \epsilon \cos \theta)w = 0
\]  

(4.52)

where \( k \) is an average foundation stiffness (per unit length) and the parameter \( \epsilon \) is here a modulation coefficient. We will restrict our study to \( 0 \leq \epsilon \leq 1 \). The amount of “assymetry” can be controlled precisely through the parameter \( \epsilon \).

\textsuperscript{6}The whole system consisting of the spinning string and the load system is however not conservative. In the unstable range, the motor that is spinning the string supplies energy to the growing solution and takes energy from the decaying solution. The energy has to come from somewhere and the first law of thermodynamics has to hold.
Upon using Galerkin’s technique with the three-parameter solution, we obtain

\[
\begin{bmatrix}
2z^2 + 1 & 0 & \epsilon/2 \\
0 & z^2 + 1 - \beta^2 & -2\beta z \\
\epsilon/2 & 2\beta z & z^2 + 1 - \beta^2
\end{bmatrix}
\begin{bmatrix}
A_0 \\
B_1 \\
A_1
\end{bmatrix} = 0
\] (4.53)

The foundation’s modulation is reflected by the two off-diagonal terms \(\epsilon/2\) which couple the bounce mode to the cosine standing wave. We note here that we have changed our definition of the wave speed \(c_o\) to account for the additional stiffness introduced by the elastic foundation

\[
c_o^2 = \frac{T + kR^2}{\theta}
\] (4.54)

The dimensionless quantities \(\beta\) and \(z\) are still as defined by equations 4.20 with however the above expression for the wave speed \(c_o\). The frequencies \(\zeta\) resulting from solving the above eigenvalue problem, when \(\epsilon = 0\), are displayed in figure 4-16.

![Campbell Diagram for a Spinning String Sitting on a Uniform Elastic Foundation (\(\epsilon = 0\)).](image)

Figure 4-16: Campbell Diagram for a Spinning String Sitting on a Uniform Elastic Foundation (\(\epsilon = 0\)).

This is the axissymmetric case and evidently, there are no unstable regions. The Campbell diagram clearly shows the branches corresponding to the two waves and the bounce mode. The three modes are uncoupled. If we now modulate the foundation by setting \(\epsilon = 1\), we obtain the Campbell diagram of figure 4-17. There are now two
regions of unstable operation. The first is of the divergence type and the second is of the flutter type. We note here that the onset speed of instability is below the critical speed. Thus, the latter is by no means an absolute stability boundary.

Figure 4-17: Campbell Diagram for a Spinning String Sitting on a Sinusoidally Modulated Elastic Foundation with Modulation Parameter $\epsilon = 1$.

The widths of the two regions of instability evidently increase with higher values of the modulation coefficient $\epsilon$ as clearly indicated in the stability map of figure 4-18.

Figure 4-18: Stability Map in the $\epsilon - \beta$ plane for a Spinning String Sitting on a Sinusoidally Modulated Elastic Foundation.

We have chosen the simplest example to show that a departure from symmetry leads to unstable operation. The results presented here can be extended to other cases.
4.6 Two-Parameter Solution with the Spring $k_L$

The mode shapes at some key values in the string’s Campbell diagram are shown in figure 4-20. Because it “sees” the spring $k_L$, the cosine standing wave at zero transport speed has a higher frequency than that of the sine standing wave. We note here that, although they look like rigid body tilting modes, the shapes are truly “flexural” modes and it is the tension $T$ in the string that acts as a restoring force. The other modes shown in figure 4-20 are for non-zero transport speeds. Before we can explain the mechanics for those modes we need to understand the effects of the various forces present in the string’s equation. For this, let us take equation 4.1, and expand the total derivative inertia term. The result is

$$\frac{\varepsilon}{\tau^2} \frac{\partial^2 w}{\partial \tau^2} + 2\nu \frac{\partial w}{\partial \tau} \frac{\partial w}{\partial \xi} + \frac{\rho \omega^2}{\partial^2 \xi^2} - T \frac{\partial^2 w}{\partial \xi^2} = 0$$

(4.55)

where we have labeled the various terms. The effect of the inertia and the elastic restoring forces is rather clear. The two terms of interest here are those corresponding to the centrifugal and Coriolis forces. Let us begin with the easy one i.e., the centrifugal force, and consider the segment of the string shown in figure 4-19.

![Figure 4-19: Centrifugal and Coriolis Forces acting on an Element of the String.](image)

To make things a bit clearer, we can think of our traveling string as a hollow pipe that is stationary. The spinning of the string is replaced here by water that is flowing in the pipe at a uniform velocity $v$. When the water goes through the curb, it pushes the glass pipe so as to increase the displacement $w$. Thus, centrifugal forces tend to increase the displacement of the string. In a sense, these centrifugal forces are
Figure 4-20: Mode Shapes of the Spinning String with the Load Stiffness $k_2$. Two-Parameter Solution.
destabilizing at all speeds \( v \). We are all accustomed to seeing the centrifugal force as the ratio of \( v^2 \) to a radius. Here the second derivative of \( w \) with respect to \( x \) is in fact the local curvature –inverse of the radius of curvature, of the string.

Let us now move to the Coriolis force. From equation 4.55 we have that the Coriolis force is proportional to a mixed derivative with respect to both time \( t \) and space \( x \). Looking at this expression carefully we can interpret it as follows:

\[
2qv \frac{\partial}{\partial t} \frac{\partial w}{\partial x} = 2qv \frac{\partial}{\partial t} \left( \frac{\text{Local Slope}}{\omega_c} \right) = 2qv \left( \frac{\text{Angular Velocity}}{\omega_c} \right) = \rho (2v \times \omega_c)
\]  

(4.56)

We have now the familiar form of the Coriolis force which is proportional to twice the cross product between the local velocity and the local angular velocity. We have to stress that the Coriolis and the centrifugal forces can be seen –or rather felt, only from a stationary reference frame. An observer moving with the rotating frame will see the string as stationary and will not “feel” any of the two speed-dependent forces. When the equation of motion is written in the rotating frame, the total derivative operator is replaced by an ordinary differential operator and thus the effects of the transport speed \( v \) –both the centrifugal and Coriolis forces, will not be present.

Now, let us go back to the mode shapes of our spinning string and figure 4-20. The mode shape at \( \beta = 1 \) indicates clearly what happens at a critical speed: A “frozen” wave in space yet the string is spinning. At this speed, the centrifugal forces are balancing exactly the elastic restoring forces. This is the reason why we have a fixed shape in space. At the upper limit of the unstable range, there is another “frozen wave”. The spring is pulling down on the string yet the string does not want to move. To understand the mechanics of this peculiar situation, let us look at all the forces the string is subjected to. The differential equilibrium in this situation can be obtained by dropping the temporal derivatives\(^7\) in the string’s equation, leaving

\[
qv^2 \frac{\partial^2 w}{\partial \theta^2} - T \frac{\partial^2 w}{\partial \theta^2} = -k_L w R \delta(\theta)
\]

(4.57)

The only forces left then are the centrifugal forces, the elastic restoring forces caused

\(^7\)Since the shape of the string is frozen in space, nothing changes with time.
by the tension $T$ in the string, and finally the reaction from the spring $k_L$. Figure 4-21 shows such forces acting on a differential element $dx = Rd\theta$ of the string.

![Diagram of forces acting on a differential element of the string.]

Figure 4-21: Forces acting on a Differential Element of the frozen string.

The centrifugal forces tend to increase the "tipping" of the string while the elastic forces and the spring force tend to reduce it. To understand how this is possible, let us calculate the torque due to all distributed forces about the axis $\theta = \pi/2$ and equate it with that due to the reaction from the spring $k_L$. For this, let us multiply both sides of equation 4.57 by the length $Rd\theta$ of the differential element, then by the lever arm $R\cos\theta$ and finally integrate over the length of the string

$$
\int_{-\pi}^{+\pi} \left[ (\varphi v^2 - T) \frac{\partial^2 w}{\partial \theta^2} \right] \times R \cos \theta \times Rd\theta = -\int_{-\pi}^{\pi} [k_L wR\delta(\theta)] \times R \cos \theta \times Rd\theta \quad (4.58)
$$

Substituting for the shape $w = \cos \theta$ of the frozen string, performing the above integration and using $T = \varphi c_0^2$ and $k_L = \pi \alpha_L T/R$, we end up with the relationship

$$
\beta = \sqrt{1 + \alpha_L} \quad (4.59)
$$

This result simply says that the frozen wave we observed at the upper limit of the divergence region can take place only when the above condition is satisfied. This condition is nothing else than the equation defining the upper limit of the unstable range and can be obtained by setting $z = 0$ in the characteristic equation 4.51 from which the frequencies in the campbell diagram of figure 4-14 for this problem were obtained. In closing, everything checks out and the frozen shape at the upper limit of the unstable range is in perfect "static" equilibrium. The implications of this result...
are even more dramatic and they go deeper than just for this single situation. In general, Galerkin's technique ensures that the equation of motion is satisfied in an "average" sense. The differential equilibrium is usually not satisfied at every single point in the interval $\theta \in [-\pi, \pi]$. Here however, and for the one nodal diameter mode only, applying Galerkin's technique is equivalent to calculating the balance about the two orthogonal axes $\theta = 0, \pi/2$ of the torques due to all forces acting on the string. This is definitely a key interpretation and it deserves to be framed.

\begin{center}
\begin{tabular}{|c|}
\hline
Galerkin's Technique $\equiv$ Torque Balance about axes $\theta = 0, \pi/2$ \hline
\end{tabular}
\end{center}

We found that at the lower limit of the unstable range the nodal line of the mode is at an angle $\theta = 0$ whereas at the upper limit it moves to $\theta = \pi/2$. Inside the unstable range, and for any given $\beta$, there are two possible situations $\theta = \pm \theta_0$ - where $\theta_0$ is in the range $\theta_0 \in [0, \pi/2]$, owing to the existence of a diverging as well as a decaying mode. The mode shapes inside the unstable range ($1 < \beta < \sqrt{1+\alpha}$) are shown in figure 4-22. The angle $\theta_0$ depends upon the transport speed parameter $\beta$.

![Mode Shapes for the Diverging and the Decaying Solutions.](image)

Figure 4-22: Mode Shapes for the Diverging and the Decaying Solutions.

We can see from figure 4-22 that the difference between the two solutions is rather subtle—the spring in both situations is pushing on the string to reduce the wave's amplitude. What is then so fundamentally different between the two geometric configurations? Before we can answer this question let us check to see that the two
configurations are in equilibrium and thus, that they do not violate the laws of dynamics. For this, let us rearrange the string’s equation of motion and write it as

\[
\frac{\partial^2 w}{\partial t^2} = -2T \frac{\partial}{\partial t} \frac{\partial w}{\partial \theta} - \left( T - \rho u^2 \right) \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} - k_L \frac{\delta(\theta)}{R}
\]  

(4.60)

To understand the effect these forces have on the string, let us take a transport speed exactly in the middle of the unstable range – so that the nodal lines are exactly at \( \theta = \pm \pi/4 \), and calculate then plot the distributions of these forces. Exactly in the middle of the unstable range, the shape of the string is described by

\[
w(\theta, t) = \cos(\theta \pm \pi/4)e^{\alpha t}
\]  

(4.61)

The various forces are then calculated and properly normalized. The results are shown in figures 4-23 and 4-24 for the decaying and the diverging mode respectively. Careful examination of figures 4-23 and 4-24 clearly shows how the torques due to all the distributed forces add up and balance the torque due to the reaction of the spring of the load system. From a d'Alembert point of view, the systems are in “Dynamic equilibrium” and explain the observed behavior. Back to figure 4-20 and at large values of the transport speed parameter \( \beta \), there are two modes wobbling one faster and the other slower than the rotation rate of the string. These correspond to the forward and backward waves respectively.

All the discussion we presented up until now dealt with modes that have only one nodal diameter. The results we presented can be very well extended to “higher modes” with however one exception. It is that Galerkin's technique – for the higher modes, retains its conventional interpretation of an “average” solution. The torque balance interpretation that we gave for the modes with only one nodal diameter no longer holds for higher modes. The interpretation is certainly different but the mathematics are the same and we should certainly expect the dynamics of the higher modes to be analogous to those of the one nodal diameter solution. The mode shapes for the case with \( n = 2 \) nodal diameters are shown in figure 4-25 along with the Campbell diagram for \( \alpha_L = 1 \). The shapes shown there are certainly analogous to those shown
Figure 4-23: Distribution of Forces for the Decaying Mode, inside the Unstable Range

Figure 4-24: Distribution of Forces for the Diverging Mode, inside the Unstable Range
Figure 4-25: Mode Shapes of the Spinning String with the Load Stiffness $k_L$ — Two-Parameter Solution with $n = 2$
in figure 4-20. For the mode at the upper limit of the unstable range, the force from the spring balances the difference between the centrifugal and the elastic forces. The two mode shapes inside the divergence region are shown below.

![Mode Shapes](image)

Figure 4-26: Mode Shapes for the Diverging and the Decaying Solutions for the case with $n = 2$ Nodal Diameters.

Throughout the unstable range, all modes share something in common. It is that the phase speed vanishes and the result is a wave — if we can still call it a wave, that is no longer propagating. At the lower limit of the unstable range, the wave has zero phase speed as a result of the exact cancellation of the out-of-plane centrifugal forces and the elastic restoring forces due to the tension $T$ in the string. At speeds higher than the critical, the centrifugal forces overcome those due to the tension $T$ but the reaction from the spring provides an additional restoring force that “helps” the tension $T$. In some sense then, all the speeds inside the unstable range are “critical” speeds since the centrifugal and elastic forces — those due to the tension $T$ and that due to the spring $k_L$, are balancing each other.
4.7 Three-Parameter Solution with the Spring $k_L$

As we did for the two-parameter solution, let us now gain a better understanding of the kinematics of our string when this time, the rigid body is included. Figure 4-28 shows the mode shapes for some key values of $(\beta, z)$ in the Campbell diagram. We begin with the mode at $\beta = 0$ and $z = 1$. This is a standing sine oscillation. Like in the two-parameter situation, it is the restoring forces from the tension in the string that cause this mode to "rock" back and forth. The other two modes at zero transport speed result from the interaction of the cosine standing wave and the rigid body translation. There are two situations depending on whether the standing wave and the translation are in phase or out of phase. These modes are analogous to those of the rigid rod in figure 4-27, that is sitting on two springs with unequal stiffnesses.

![Mode Shapes of a Rigid Rod Sitting on Two Unequal Springs.](image)

At precisely the critical speed $\beta = 1$, any combination of the three parameters $(A_0, B_1, A_1)$ of the mode shape —provided $A_0 = -A_1$, is a possible configuration. At large values of the transport speed parameter $\beta$, there is a rigid body mode with bounce frequency\(^8\) $\omega_L$ —the same $\omega_L$ defined in equation 4.15. Strictly speaking, this mode shape has also contributions from both $B_1$ and $A_1$. These are however negligibly small owing to the gyroscopic "stiffening" of the string at high spin rates. Still at large values of the transport speed, the two branches at 45 degrees correspond to the two waves in our string. One is traveling backward with respect to the string and the other forward. When viewed by a stationary observer however, both waves

\(^8\)When $\alpha_L = 1$, the "dimensionless" bounce frequency is exactly $z = 1/\sqrt{2}$. This agrees with the reading from the Campbell diagram in figure 4-28.
Figure 4-28: Mode Shapes of the Spinning String with the Load Stiffness $k_L$ – Three-Parameter Solution.
appear to be traveling forward. Inside the unstable flutter range, the story however is much complicated. There, the kinematics involve a combination of the backward wave—traveling forward, and the rigid body translation. The visualisation of the string's kinematics can be made much clearer if we write the displacement \( w(\theta, t) \) in terms of, not standing, but rather traveling waves

\[
w(\theta, t) = [A_0 + C_1 e^{i\theta} + C_2 e^{-i\theta}] e^{\epsilon t}
\]  \hspace{1cm} (4.62)

where the new constants we just introduced are related to those of equation 4.49 through the matrix transformation

\[
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
-i & 1 \\
i & 1
\end{bmatrix} \begin{bmatrix}
B_1 \\
A_1
\end{bmatrix}
\]  \hspace{1cm} (4.63)

With this transformation, and after calculating how the three mode shape components \( \{A_0, C_1, C_2\} \) vary as a function of the transport speed \( \beta \), we find that \( C_1 \) and \( C_2 \) are in general complex-valued. This is not surprising at all but we need a representation that is a bit easier to visualise. After all, we want as clear a picture as possible of the kinematics of the string. For this, let us write the constants \( C_1 \) and \( C_2 \) in terms of their magnitude and phase as

\[
C_1 = |C_1| \cdot e^{i\phi_1} \hspace{1cm} \text{and} \hspace{1cm} C_2 = |C_2| \cdot e^{i\phi_2}
\]  \hspace{1cm} (4.64)

The dependences of the five quantities \( \{A_0, |C_1|, |C_2|, \phi_1, \phi_2\} \) in terms of the transport speed parameter \( \beta \) are shown in figures 4-30, 4-31, and 4-32 for all three modes of the string. The components \( \{A_0, C_1, C_2\} \) are normalized such that the "length" of the resulting eigenvector is unity—the mode shapes are thus orthonormal. Also shown are the Campbell diagrams where we have emphasized the branch of the frequency to which the five parameters in each figure correspond to. There is obviously a tremendous amount of information in those figures and it is from them that we obtained the various mode shapes displayed on figure 4-28. The information we extract from those figures helps us now understand better the kinematics of the string in the unstable range. The first observation we make from those figures is that, above the critical
speed, the parameter $C_1$ is very small. This of course makes sense since both waves in
the string are traveling forward. Making use of this observation, and for supercritical
operation only, we can now write

$$w(\theta, t) \approx [A_0 + C_2 \cdot e^{-i\theta}]e^{st}$$  \hspace{1cm} (4.65)

We now have only two "degrees-of-freedom" to keep track of. This certainly simplifies
the kinematics tremendously. In the unstable range, the region we are most interested
in, the motion of the string results results from the coupling of two basic composite
motions. These are a bounce motion and a traveling wave which — for the case with
only one nodal diameter $n = 1$, can be thought of as a precession.

![Composite Bounce and Precession motions for the Spinning string.](image)

Figure 4-29: Composite Bounce and Precession motions for the Spinning string.

We still do not know the difference between the decaying and the growing solutions.
From figures 4-31 and 4-32, we can clearly see that, inside the unstable range, the
amplitude components $A_0$ and $|C_2|$, at any given transport speed $v = \beta c_0$, are the
same for the two unsteady solutions. The difference lies however in the phase $\phi_2$.
Specifically, we can see that

- $\phi_2$ is positive for the decaying solution.
- $\phi_2$ is negative for the growing solution.

Whenever phase information is introduced, we need to be precise about its meaning.
It is the phase measured with respect to what? Since here we forced the component $A_0$
Figure 4-30: Components of the FIRST Mode as a function of the Transport Speed.
Figure 4-31: Components of the SECOND Mode as a function of the Transport Speed.
Figure 4-32: Components of the THIRD Mode as a function of the Transport Speed.
to be purely real, the phase $\phi_2$ is measured then with respect to $A_0$. The phase $\phi_2$ can be interpreted as either spatial or temporal. We retain here the spatial interpretation because it is more practical. We then rewrite the solution $w(\theta, t)$ of our string as

$$w(\theta, t) = [A_0 + |C_2| \cdot e^{i(\phi_2 - \theta)}]e^{st} \quad (4.66)$$

To understand the meaning of the phase $\phi_2$ let us first consider the case $\phi_2 = 0$. This is the situation where, at the attachment point of the spring $k_L$, the two composite motions reach maximum excursion at the same instant in time. When however the phase is different from zero, the two motions reach their peak excursion at different instants during the cycle of vibration. The phase in this situation can be interpreted as the angle that is between the “nodal line” of the wave and the line defined by $\theta = 0$ when, precisely, the bounce mode is at its top most position. A clear picture is obtained by looking at the two composite motions at time $t = 0$ – where the bounce mode is exactly at maximum excursion. In this situation, the shape of the string is described by

$$w(\theta, 0) = A_0 + |C_2| \cdot \cos(\phi_2 - \theta) \quad (4.67)$$

The case at the lower limit ($\beta \approx 1.3$) of the unstable range is shown in figure 4-33. The two motions are there 180 degrees out of phase and this can also be seen in the graph for $\phi_2$ in figure 4-30. At the upper limit of the unstable range, the two composite motions are in phase and the corresponding picture at $t = 0$ is shown in figure 4-34. At the limits of the unstable range, the forward wave is either in phase or at 180 degrees out of phase with the bounce motion. As we might have guessed by now, in the unstable range there is a phase $\theta = \pm \phi_2$, $\phi_2 \in [0, \pi]$, between the two motions, depending on whether it is the decaying or the growing solution, the phase will be accordingly $+\phi_2$ or $-\phi_2$. Snapshots of the string at time $t = 0$ are shown in figures 4-35 and 4-36 for the decaying and the growing solutions respectively. We have chosen the value $\beta = 2.1$ for the transport speed parameter because the phase is there nearly 90 degrees. As time progresses all three components – for any of the solutions, will oscillate each in its own pattern. The two waves will travel in their
Figure 4-33: "Snapshot" at time $t = 0$ of the Bounce and Precession Components at the LOWER Limit of the Unstable Range, $\beta \approx 1.3$.

Figure 4-34: "Snapshot" at time $t = 0$ of the Bounce and Precession Components at the UPPER Limit of the Unstable Range, $\beta \approx 2.4$.

Figure 4-35: "Snapshot" at time $t = 0$ of the Bounce and Precession Components for the DECAYING Solution at $\beta = 2.1$.

Figure 4-36: "Snapshot" at time $t = 0$ of the Bounce and Precession Components for the GROWING Solution at $\beta = 2.1$. 

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respective directions and the bounce mode will just move the string up and down. These motions take place all at the same frequency such that the phase information is preserved. The amplitudes of all three motions will increase, decrease, or remain with constant amplitude in time, depending on whether it is the growing solution, the decaying solution, or a steady solution that is outside the unstable range.

The results presented here can too be extended to higher modes – with more than one nodal diameter, of the string. This however is a rather obvious task and we move now to other elements of the load system.
4.8 Effects of the Mass $m_L$ of the Load System

In order to improve our model of the load system, let us now introduce the mass $m_L$ and see how it affects the dynamics of our spinning string. We note here that the mass does not couple the traveling waves of the string to the rigid body mode. Thus, we will consider only the two-parameter solution. The eigenvalue problem, when the mass $m_L$ alone is included, is then given by

$$
\begin{bmatrix}
  z^2 + 1 - \beta^2 & -2\beta z \\
  2\beta z & (1 + 2\mu_L)z^2 + 1 - \beta^2
\end{bmatrix}
\begin{bmatrix}
  B_1 \\
  A_1
\end{bmatrix} = 0
$$

(4.68)

where we have used the mass ratio $\mu_L = m_L/2\pi R_0$ as a measure of the mass $m_L$ of the load system. The Campbell diagrams corresponding to this eigenvalue problem are shown in figure 4-37 as for a mass ratio $\mu_L = 0.5$. We see there that a terminal instability region appears far above the critical speed. This instability region results from the nonconservative coupling between the two waves in the string. This result makes sense from the perspective that the mass becomes important at high frequencies $z$ — the system is in a mass controlled regime.

![Campbell Diagram](image)

Figure 4-37: Campbell Diagram for the Two-Parameter Solution with $\mu_L = 0.5$

What is interesting to note here is that, in the unstable range, both waves are traveling backward with respect to the string. How we arrived at this result is rather simple. In the Campbell diagram for the oscillation frequencies, the 45 degrees line where
$\Im\{z\} = \beta$ corresponds to a wave that is moving exactly at the same rate as the string. Branches that are below that line correspond to waves that are slower than the string whereas branches that are above the 45 degrees line correspond to waves that are moving faster than the string. The idea parallels that, in rotordynamics, of sub-synchronous, synchronous, and sup-synchronous whirls.

By further increasing the value for the load mass parameter to $\mu_L = 1$, we can see in figure 4-38 that the onset speed of instability has decreased. This was not the case for the spring $k_L$ where the onset speed –at least for the two-parameter solution, was fixed at exactly the critical speed.

![Figure 4-38: Campbell Diagram for the Two-Parameter Solution with $\mu_L = 1$](image)

We can also see that, at any given speed $\beta$, the frequencies in the unstable range have further decreased. In addition, as we further increase the mass ratio $\mu_L$, the increase in the oscillation frequency $\Im\{z\}$ with the speed parameter $\beta$ becomes less and less pronounced. There seem to be a special significance to the value $\mu_L = 4$. For this "special" value, the Campbell diagrams are shown in figure 4-39. In the unstable range, the string enters a state of "lock-in" where further increases of the transport speed only affect the growth/decay rate $\Re\{z\}$ but not the oscillation frequency $\Im\{z\}$. This is a rather surprising result, I do not know of any linear system that exhibits a lock-in phenomenon. Lock-ins are usually associated with vortex shedding in fluid mechanics and mechanical systems with dry friction. These two are very non-linear
mechanisms. This is yet another puzzle we have unraveled.

Figure 4-39: Campbell Diagram for the Two-Parameter Solution with $\mu_L = 4$

This lock-in is definitely not a limiting case because what happens when the mass ratio exceeds the special value of $\mu_L = 4$ is even more dramatic. Figure 4-40 shows the Campbell diagrams for the case with $\mu_L = 5$.

Figure 4-40: Campbell Diagram for the Two-Parameter Solution with $\mu_L = 5$

In Figure 4-40, we can see that depending upon the transport speed parameter $\beta$, the string can have either a flutter or a divergence instability. Furthermore, and in the flutter range, increasing the speed of the string reduces the oscillation frequency $\Im \{\omega\}$ of the two forward traveling waves. In addition, the two unstable regions are
contiguous. We note that here the onset of instability and the borderline between the two unstable regions can be predicted very easily. At the corresponding two speeds, the branches in the Campbell diagram coalesce and give a double root. All we have to do then is to solve for the double root condition from the characteristic polynomial associated with the eigenvalue problem given in equation 4.68. The resulting stability map is shown in figure 4-41.

![Figure 4-41: Regions of Instability for the String with the Mass $m_L$.](image)

The next step is to see how the mass $m_L$ of the load system affects the mode shapes of the traveling string. These are shown in figure 4-42 for some key values in the Campbell diagram. In contrast with the case where we have the stiffness $k_L$ of the load system, the frequency of the standing cosine wave—at zero transport speed, is here lower than that of the standing sine wave. At the critical speed $\beta = 1$, there is no unstable region as we had for the spring and any combination of the sine and cosine components is a possible mode shape. In a sense, there is a double root at the critical speed and thus, the mode shape is no longer unique. At high values of the transport speed, we have the two forward waves but this time one has its amplitude decaying with time and the other has it growing with time.

By choosing our solution to be the sum of $\sin \theta$ and $\cos \theta$, we have used two standing waves to describe the motion of our string. That choice is very convenient for all the algebraic manipulations involved in applying Galerkin’s technique. To better visualise
Figure 4.42: Mode Shapes of the Spinning String with the Load Mass $m_L$. -Two-Parameter Solution.
the kinematics of our string, we need however to use the more vivid representation

\[ w(\theta, t) = [D_1 \cdot \cos \theta + D_2 \cdot e^{-i\theta}]e^{st} \]  
(4.69)

where the two constants we introduced are related to those of the standing sine and cosine waves through a matrix transformation

\[
\begin{pmatrix}
D_1 \\
D_2
\end{pmatrix}
= 
\begin{bmatrix}
-i & 1 \\
0 & i
\end{bmatrix}
\begin{pmatrix}
B_1 \\
A_1
\end{pmatrix}
\]  
(4.70)

We force here the amplitude \( D_2 \) of the forward wave to be purely real. The amplitude \( D_1 \) of the standing cosine wave will thus be –in general, complex valued. Like any complex number, the coefficient \( D_1 \) can be written in the form

\[ D_1 = |D_1| \cdot e^{i\phi} \]  
(4.71)

The real part of the solution \( w(\theta, t) \), when we use \( s = \sigma + i\omega \), takes now the form

\[ w(\theta, t) = [|D_1| \cos(\omega t + \varphi) \cdot \cos \theta + D_2 \cos(\theta - \omega t)]e^{st} \]  
(4.72)

It is clear from the above formulation that \( D_1 \) is the amplitude of a pure "rocking" motion whereas \( D_2 \) is the amplitude a forward traveling wave –a precession motion in some sense. These two composite motions are illustrated in figure 4-43.

---

Figure 4-43: Composite Rocking and Precession motions for the wobbling string.
The relative amplitudes of the two composite motions depend on the spin rate of the string and the phase $\varphi$ on whether the motion is stable, decaying, or growing.

- $\varphi = 0$ for stable backward waves.
- $\varphi = \pi$ for stable forward waves.
- $0 < \varphi < +\pi$ for the decaying solution in the unstable regime.
- $-\pi < \varphi < 0$ for the growing solution in the unstable regime.

Before we can illustrate the kinematic implications of these results, let us look at our string "from above" and track the trajectory that the $z$-axis makes on a plane $P$ located at a distance $R$ from the string.

![Diagram of orbit traced by the z-axis of the wobbling string](image)

**Figure 4-44:** Orbit Traced by the $z$-axis of the Wobbling String.

When the load system is absent, the motions of the forward and backward waves can be thought of as precessions with constant tip angle. In this situation, the $z$-axis
Figure 4-45: Orbits of the String's z-axis for Several Roots on the Campbell Diagram. Here, the load system consists of only the mass $m_L$. The little circles in the orbit plots are starting points i.e., the position of the z-axis at the initial time $t = 0$. 
would trace perfect circles on the \( P \)-plane. The introduction of the the load system however—and for the two-parameter solution only, modulates both the precession rate and the precession’s “cone angle”. This causes the trajectories on the \( P \)-plane to depart from perfect circles. The orbits for various roots on the Campbell diagram for the string with the mass \( m_L \) of the load system are shown in figure 4-45. Those orbits are very descriptive of the kinematics of the spinning string. We can see clearly what the information given by the phase angle \( \phi \) we defined earlier corresponds to.

This “orbits” representation of the string’s kinematics makes sense only for the mode with only one nodal diameter. For the higher modes however, we need an alternate representation. A clear picture of the kinematics can be obtained by considering a straightened version of the string and looking at the evolution of a traveling wave in the \( x - t \) plane. Figure 4-46 shows “snapshots”—taken every 1/16th of the oscillation period, of a forward stable wave when the mass \( m_L \) of the load system is present. We can clearly see there that the amplitude of the wave is modulated by the presence of the mass \( m_L \). The modulation of the phase speed can be seen by looking at figure 4-46 at grazing incidence with the plane of the paper and along the oblique dashed lines. These are lines of constant phase speed and the “nodes” of the wave “oscillate” about these oblique lines.

For the forward and stable wave shown in figure 4-46, the peak-to-peak wave-amplitude reaches a minimum precisely when the mass \( m_L \) is on a crest of the wave, and it attains a minimum when the mass is on a “node” of the traveling wave. For the backward (disk-based) wave however, the kinematics depend on whether the wave is sub- or super-critical. Below the critical speed \( \beta = 1 \), the peak-to-peak amplitude is maximum when the mass \( m_L \) is on a crest of the wave and minimum when the mass is on a node. Above the critical speed, the kinematics are the same as those of the (disk-based) forward wave.

All of this is for stable operating regimes only. In the unstable range, the amplitude of the wave reaches a minimum when the mass is either after or before the wave crest depending on whether it is the wave with a decaying or growing amplitude. These observations can be deduced from the various orbits shown in figure 4-45.

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Figure 4-46: Evolution of the Forward Wave in the $x - t$ plane for the String with the Mass of the Load System. $\mu_L = 4$ and $\beta = 0.5$
4.9 Effects of the Dashpot $c_L$ of the Load System

We have postponed the effects of the dashpot $c_L$ of the load system but now is the time to see what its effects on the dynamics of the string are. When it is only the dashpot $c_L$ that is present, the eigenvalue problem for our traveling string is given by

$$\begin{bmatrix} z^2 + 1 - \beta^2 & -2\beta z \\ 2\beta z & z^2 + 2\eta_L z + 1 - \beta^2 \end{bmatrix} \begin{bmatrix} B_1 \\ A_1 \end{bmatrix} = 0 \quad (4.73)$$

We study here only the two-parameter solution because –like the mass $m_L$, the dashpot does not couple the rigid body mode to the two traveling waves. The dimensionless parameter $\eta_L$ is here a measure of the dashpot constant $c_L$. The Campbell diagram corresponding to this eigenvalue problem is shown in figure 4-47 for $\eta_L = 0.5$.

![Figure 4-47: Campbell Diagram for $\eta_L = 0.5$.](image)

We can see from figure 4-47 that the forward wave has positive damping at all speeds and thus, for this wave, the dashpot is always stabilizing. Further, and for this same wave, we can also see that the faster the speed $\beta$ the higher the damping value. For the backward wave however, the damping decreases with increasing values of the transport speed. When the backward wave reverses its direction of propagation, it becomes unstable. At zero transport speed we still have the sine and cosine standing waves. The presence of the dashpot reduces to second order only, the frequency of oscillation of the cosine wave as in $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ for a simple harmonic oscillator.
We can also see that there is a speed near $\beta = 0.3$ where the two waves in the string have the same damping value $\Re\{z\}$ and the same oscillation frequency $\Im\{z\}$. There is really nothing dramatic happening with the dashpot $c_L$. The speed at which the backward wave goes unstable is always the critical speed $\beta = 1$, and this, irrespective of the strength of the dashpot. Different values of the dashpot constant $c_L$ simply give different damping values to the frequencies of the two waves.

4.10 Putting Everything Together

We have presented treatments on how each of the elements of the load system introduces instabilities of our traveling string. In real world applications however, all effects are present simultaneously. The Campbell diagram when both the spring $k_L$ and the mass $m_L$ —but not the dashpot $c_L$, are included is shown in figure 4-48.

![Campbell Diagram](image)

Figure 4-48: Campbell Diagram for $\alpha_L = 1$ and $\mu_L = 0.5$.

As expected, there are two regions of instability introduced one by the spring $k_L$ and the other by the mass $m_L$. We note in this situation that there is a range of transport speeds —between the two unstable regions, where supercritical stable operation is still possible. At zero transport speed, the frequencies of the sine and cosine standing waves are identical. This is a consequence of our choosing values for the parameters $\alpha_L$ and $\mu_L$ that make the natural frequency of the load system coincide with that of
the standing cosine wave. When we add the dashpot $c_L$ to obtain a *complete* load system, we get the Campbell diagram shown in figure 4-49.

![Campbell Diagram](image)

Figure 4-49: Campbell Diagram for $\alpha_L = 1$, $\mu_L = 0.5$, and $\eta_L = 0.5$, for the Traveling String with the Two-Parameter Solution.

Here, and because of the dashpot $c_L$, the string is unstable for all speeds above the critical. Let us digress for a moment and see what happens when we use the three-parameter solution with the same load system parameters.

![Campbell Diagram](image)

Figure 4-50: Campbell Diagram for $\alpha_L = 1$, $\mu_L = 0.5$, and $\eta_L = 0.5$, for the Traveling String with the Three-Parameter Solution.

From figure 4-50 we can see that the onset speed of instability is no longer at the critical speed $\beta = 1$. The three parameter solution –this is true for the string only since
the exact solution has only flutter instabilities, gives thus a bigger margin for operation of the string. This aspect is presented in [22]. Going back to our two-parameter solution, we have that the characteristic equation for the eigenvalue problem, when the dashpot is included, has both even as well as odd powers of the frequency variable \( s \). This results in frequencies \( z \) that are complex valued for all transport speeds \( v \).

![Complex s-plane](image)

Figure 4-51: Root Loci in the Complex \( s \)-plane for the String with a Complete Load System. Left: \( \eta_L = 0 \) and Right: \( \eta_L = 0.5 \). In both cases, \( \alpha_L = 1 \) and \( \mu_L = 0.5 \)

The spring and the mass, being conservative elements, share a property in common. It is that they both affect primarily the frequencies of oscillation of our string. If we think of the root locus in the complex \( s \)-plane, both the mass and the spring cause the roots to migrate along the \( i\omega \)-axis. Instability occurs when two migrating roots approach one another, collide, then veer at 90 degrees. One root moves then along the negative (positive) \( \sigma \)-axis and corresponds to the decaying (growing) solution. The characteristic equation of the associated eigenvalue problem has only even powers of the frequency variable \( s \). As a consequence, the root-locus in the complex \( s \)-plane will be symmetric with respect to both the \( j\omega \)- and the \( \sigma \)-axes as shown in figure 4-51. The dashpot \( c_L \) however, affects mainly the decay/growth rate of the vibration of the string and this is reflected by roots \( s \) that migrate along the \( \sigma \)-axis of the \( s \)-plane. The root locus for this situation is shown in figure 4-51 along with that when the dashpot \( c_L \) is not present.
4.11 Gyroscopic Ring with a Load System

Let us take the rigid ring, restrained from vertical motion and shown in figure 4-52, whose position is determined by the set of Euler angles $\phi$, $\vartheta$, and $\psi$

Figure 4-52: Spinning Rigid Ring with three degrees of freedom $\phi$, $\vartheta$, and $\psi$.

Let us recall here that the position of the ring is completely defined by the body-fixed frame $123$ relative to the space-fixed frame $XYZ$. A set of three basic rotations and two intermediate frames are needed to go from the $XYZ$ frame to the $123$ frame.

$$XYZ \xrightarrow{\phi} abc \xrightarrow{\vartheta} xyz \xrightarrow{\psi} 123$$

(4.74)

In the $xyz$-frame$^9$, the angular velocity vector and the inertia tensor are

$$\vec{\omega} = \begin{Bmatrix} \dot{\vartheta} \\ \dot{\phi} \sin \vartheta \\ \dot{\psi} + \dot{\phi} \cos \vartheta \end{Bmatrix}, \quad I = \begin{bmatrix} \frac{1}{2}mR^2 \\ \frac{1}{2}mR^2 \\ mR^2 \end{bmatrix}$$

(4.75)

To avoid the singularity of Euler angles near $\vartheta = 0$, let us use

$$\vartheta = \frac{\pi}{2} + \xi_1 \quad \text{and} \quad \phi = \frac{\pi}{2} + \xi_2$$

(4.76)

$^9$We use here the same notation as that in Crandall [23, pp 224-246].
We then have the relationships
\[ \sin \theta = \cos \xi_1 , \quad \cos \vartheta = -\sin \xi_1 , \quad \dot{\theta} = \dot{\xi}_1 , \quad \dot{\phi} = \dot{\xi}_2 \] (4.77)

With these new coordinates, the angular velocity vector becomes
\[
\vec{\omega} = \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \cos \xi_1 \\ \psi - \dot{\xi}_2 \sin \xi_1 \end{pmatrix} \approx \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \psi - \dot{\xi}_2 \xi_1 \end{pmatrix}
\] (4.78)

The angular momentum \( \vec{H} \) of the rigid ring, when the relative spin \( \psi \) is constant, is
\[
\vec{H} = I \vec{\omega} = \frac{1}{2} m R^2 \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ 2 (\psi - \dot{\xi}_2 \xi_1) \end{pmatrix} = \frac{1}{2} m R^2 \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ 2n \end{pmatrix}
\] (4.79)

where \( n = \psi - \dot{\xi}_2 \xi_1 \) is here the absolute spin. The torque \( \vec{\tau} \) needed to sustain a given motion of the rigid ring is obtained from
\[
\vec{\tau} = \left( \frac{\partial \vec{H}}{\partial t} \right)_{rel} + \vec{\Omega}_{frame} \times \vec{H} \quad \text{where} \quad \vec{\Omega}_{frame} = \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ -\dot{\xi}_2 \xi_1 \end{pmatrix}
\] (4.80)

where, after some algebra, we obtain the three components of the torque vector
\[
\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \frac{1}{2} m R^2 \begin{pmatrix} \ddot{\xi}_1 + \dot{\xi}_2 \left(2n + \dot{\xi}_2 \xi_1 \right) \\ \ddot{\xi}_2 - \dot{\xi}_1 \left(2n + \dot{\xi}_2 \xi_1 \right) \\ -2 \left(\dot{\xi}_2 \xi_1 + \dot{\xi}_1 \dot{\xi}_2 \right) \end{pmatrix} \approx \frac{1}{2} m R^2 \begin{pmatrix} \ddot{\xi}_1 + 2n \dot{\xi}_2 \\ \ddot{\xi}_2 - 2n \dot{\xi}_1 \\ 0 \end{pmatrix}
\] (4.81)

The third component of the torque vector is here zero since, in the absence of any dissipative mechanism, the spin rate of the ring is constant and no work is needed to keep the ring spinning. We can then ignore the third component \( \tau_3 \) and rewrite the above result in the familiar form
\[
\begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{pmatrix} + n \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}
\] (4.82)
where we have used the three moments of inertia \( I_1 = I_2 = \frac{1}{2} m R^2 \) and \( I_3 = m R^2 \). When there is nothing attached to the ring, the torque vector is identically zero. If however we attach a complete load system to our spinning ring, we then have

\[
\begin{pmatrix}
\tau_1 \\
\tau_2
\end{pmatrix} = R^2 \begin{pmatrix}
0 \\
-mL\dot{\xi}_1 - cL\dot{\xi}_1 - kL\xi_1
\end{pmatrix}
\]

(4.83)

where we have multiplied by the factor \( R^2 \) because the quantities \( \xi_1 \) and \( \xi_2 \) are “angular” coordinates. Upon inserting the above expression into equation 4.82 for our rigid ring, and assuming periodic solutions of the form

\[
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} = \begin{pmatrix}
\Xi_1 \\
\Xi_2
\end{pmatrix} e^{st}
\]

(4.84)

we obtain the eigenvalue problem

\[
\begin{pmatrix}
s^2 I_1 & sn I_3 \\
-sn I_3 & s^2 (I_2 + mL R^2) + scL R^2 + kL R^2
\end{pmatrix} \begin{pmatrix}
\Xi_1 \\
\Xi_2
\end{pmatrix} = 0
\]

(4.85)

The frequencies \( s \) resulting from solving this eigenvalue problem all have \( \Re \{s\} \leq 0 \). This simply means that our rigid spinning ring is always stable. What is then so different between this rigid ring and our flexible spinning string? The difference lies precisely in the flexibility of the string. Such flexibility allows the string to have two types of forces that the rigid ring simply cannot have. These are the elastic restoring forces due to the constant tension \( T \) and the out-of-plane centrifugal forces proportional to \( \rho v^2 \). The combination of these two forces can be thought of as a speed-dependent distributed elastic foundation. To further compare the rigid ring to the flexible string, we can relate the cardan coordinates \( \xi_1 \) and \( \xi_2 \) to the amplitudes \( B_1 \) and \( A_1 \) of the standing sine and cosine waves. These relationships are

\[
\xi_1(t) = \frac{-B_1 e^{st}}{R} \quad \text{and} \quad \xi_2(t) = \frac{-A_1 e^{st}}{R}
\]

(4.86)

Let us now go back to our traveling string and rewrite its equation of motion as

\[
\rho \frac{\partial^2 w}{\partial t^2} + 2\rho \frac{v}{R} \frac{\partial}{\partial t} \frac{\partial w}{\partial \theta} = (T - \rho v^2) \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\delta(\theta)}{R} \left[ mL \frac{\partial^2 w}{\partial t^2} + kL w \right]
\]

(4.87)
The left hand side of the previous equation can be interpreted as the inertia of the string and the right hand side can be interpreted as external forces. Upon applying Galerkin's technique with a two-parameter solution, we get a torque balance

$$
\begin{bmatrix}
  z^2 & -2\beta z \\
  2\beta z & z^2
\end{bmatrix}
\begin{cases}
  B_1 \\
  A_1
\end{cases}
= -
\begin{bmatrix}
  1 - \beta^2 & 0 \\
  0 & 2\mu_L z^2 + 1 - \beta^2 + \alpha_L
\end{bmatrix}
\begin{cases}
  B_1 \\
  A_1
\end{cases}
$$

(4.88)

On one side we have the inertia "torques" of our spinning string whereas on the other we have the "external" torques applied to it. We note here that even though the above equation is a torque balance --our interpretation of Galerkin's technique for the mode with only one nodal diameter, the two sides do not have the correct dimension because we have been working in dimensionless variables. Using our analogy with the spinning rigid ring we presented on the previous pages, we can then write

$$
\begin{cases}
  \tau_1 \\
  \tau_2
\end{cases}
= -\pi T
\begin{bmatrix}
  1 - \beta^2 & 0 \\
  0 & 2\mu_L z^2 + 1 - \beta^2 + \alpha_L
\end{bmatrix}
\begin{cases}
  B_1 \\
  A_1
\end{cases}
$$

(4.89)

We have obviously skipped few steps here and brought in the factor $\pi T$, with $T$ being the tension force in the string, to make the right hand side have the correct dimension. The two torques $(\tau_1, \tau_2)$ are resolved in a stationary frame. Next, let us resolve them in a frame that is "welded" to the nodal line of our wobbling string.

Figure 4-53: Torques $\{\tau_1, \tau_2\}$ in the Fixed and $\{\tau_1, \tau_2\}$ in the Rotating frames.
The torques $T_1$ and $T_2$ can be obtained from $\tau_1$ and $\tau_2$ through a rotation matrix

$$\begin{pmatrix}
T_1 \\
T_2
\end{pmatrix} = 
\begin{bmatrix}
\cos \theta_0 & \sin \theta_0 \\
-\sin \theta_0 & \cos \theta_0
\end{bmatrix}
\begin{pmatrix}
\tau_1 \\
\tau_2
\end{pmatrix}$$

(4.90)

where the position $\theta = \theta_0$ of the nodal line is obtained from solving

$$w(\theta_0, t) = \left[\Re\{B e^{i\omega t}\}\right] \cdot \sin \theta_0 + \left[\Re\{A e^{i\omega t}\}\right] \cdot \cos \theta_0 = 0$$

(4.91)

from which, and after using $A = A_R + iA_I$ and $B = B_R + iB_I$, we obtain

$$\theta_0 = \arctan \left\{ \frac{A_R \cos \omega t - A_I \sin \omega t}{B_R \cos \omega t - B_I \sin \omega t} \right\}$$

(4.92)

Because the string is spinning, it has an angular momentum $\vec{H}$ along the $z$-axis. By virtue of the angular momentum theorem, the applied torques will change the direction of the vector $\vec{H}$ and with it, the position of our spinning string.

![Figure 4-54: String with Angular Momentum $\vec{H}$ and applied torques $T_1$ and $T_2$.](image)

From the above figure we can clearly see that the precession of the string is controlled by the torque $T_1$,

- $T_1 > 0$ forward traveling wave
- $T_1 = 0$ standing wave
- $T_1 < 0$ backward traveling wave
whereas the \( \text{growth/decay} \) of the string's amplitude is controlled by the torque \( T_2 \):

\[
\begin{align*}
T_2 > 0 & \quad \text{decaying wave amplitude} \\
T_2 = 0 & \quad \text{constant wave amplitude} \\
T_2 < 0 & \quad \text{growing wave amplitude}
\end{align*}
\]

In the absence of the load system, the motion of the string is a pure precession with constant tip angle and constant precession rate. In this case the torque \( T_1 \) is constant over time. When the load system is introduced, the precession rate and the precession angle –respectively \( \dot{\phi} \) and \( \theta \) in figure 4-52, become modulated. The torques \( T_1 \) and \( T_2 \) will thus oscillate about some mean values. It is these mean values that now tell us whether the motion is a forward, backward, or standing wave, and with growing, decaying, or steady amplitude.

To close this section, let us now illustrate the above results with some specific cases of our spinning string with the mass \( m_L \) and the stiffness \( k_L \) of the load system. The torques \( T_1 \) and \( T_2 \) are displayed in dimensionless form in figure 4-56 for some key-values in the Campbell diagram for the string with the stiffness \( k_L \) of the load system and in figure 4-55 for the string with the mass \( m_L \) of the load system. The graphs in those figures clearly agree with our predictions.
Figure 4-55: Angle $\theta_0$ of the Nodal Line and Torques $T_1$ and $T_2$ for the Spinning String with the Stiffness $k_L$ of the Load System— with $\beta = 0.5$. 

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Figure 4-56: Angle $\theta_0$ of the Nodal Line and Torques $T_1$ and $T_2$ for the Spinning String with the Mass $m_L$ of the Load System – for $\beta = 3$. 
4.12 Mechanisms of Instability

We have focused our presentation on understanding the kinematics of the string for speeds that are inside the various unstable ranges. We still have a piece of unfinished business before we move to the mechanisms of the various instabilities. We have to understand the mechanics of the stable modes i.e., those for speeds that are outside the unstable regions. Let us consider just one monochromatic wave and remove the load system. As it is traveling in the smooth and continuous string, such wave keeps both its amplitude and its phase speed constant. When the load system is present however, such wave sees both its amplitude and phase speed modulated\(^\text{10}\) by the reaction from the load system. It is true that in formulating the solution of our string we have used simple monochromatic waves with constant amplitude. The trick however is that any pair of waves—with the same wavelength, and traveling in opposite direction with respect to the string, can be written as a single wave with modulated phase speed and amplitude. In fact we have already encountered such behavior of our string. The orbits for the string with the load mass \(m_L\) shown in figure 4-45 clearly indicate that the amplitude of the wobbling mode is modulated. The position \(\theta_0\) of the nodal line as shown in figure 4-56 clearly shows that the phase speed of the corresponding wave is modulated. This was the kinematics part of the story, and we move now to physical explanations on how the kinetic energy stored in the string gets channeled into transverse vibratory modes. Clearly, this energy must be supplied by some “source”. The first law of thermodynamics has to hold. In our situation, it is the drive mechanism that is spinning the string that supplies\(^\text{11}\) the energy.

A possible construction of a drive system for our spinning circular string would be similar to that of a bicycle wheel and is shown in figure 4-57. There, it is the spokes that transmit the torque that is available at the hub to the rim of the wheel. It is clear from that figure that the torque \(\tau\) supplied by the drive system must balance,

\(^{10}\)In some sense, this is the manifestation of Green's law which says that, when a wave is traveling in a medium whose properties are changing with space, the amplitude and the speed of propagation of such wave vary to “accomodate” such changes.

\(^{11}\)For the decaying modes, the drive mechanism reverses its role and behaves as a “sink” rather than a source of energy.
or rather overcome, the horizontal reaction from the load system.

![Figure 4-57: Bicycle Wheel](image)

A better account of the drive mechanism can be visualised by considering an infinitely long “straightened” version of our traveling string and then taking the string segment enclosed by a control volume CV. The torque supplied by the drive system is now represented by the difference between the two tensions $T_1$ and $T_2$ at the ends of the control volume.

![Figure 4-58: Segment of an Infinitely Long Traveling String](image)

When there are waves traveling in the string, the presence of the load system causes the two tension forces, and consequently their difference, to vary in both magnitude and direction with time. The difference between the two tensions, when multiplied by the radius $R$ of the string, is precisely the torque that is needed from the motor driving the string. To understand this modulation concept let us consider a traveling
wave whose transverse displacement is described by

\[ w(x,t) = A \cos(\kappa x - \omega t) \quad , \quad \omega = \kappa(\pm c_0 + v) \] (4.93)

The displacement \( w(0, t) \) at the attachment point of the load system causes the latter to push on the string with a vertical force \( f_L(t) \). Because of the frictionless rollers, this reaction in turns gives rise to a "horizontal" force that, by virtue of equilibrium in the horizontal direction, must balance the difference between the two tensions at the two ends of the control volume

\[ \Delta F = T_2 - T_1 = f_L(t) \cdot \frac{\partial w}{\partial x}(0, t) \] (4.94)

We have to note here that the momentum "influx" at the left end of the control volume is balanced by a momentum "outflux" at the right end of the same \( CV \). Thus, the net contribution from the \( \kappa v^2 \) term is zero. Upon inserting the shape \( w(x, t) \) given by 4.93 into the above result, we obtain

\[ \Delta F = f_L(t) \cdot \kappa A \cos \omega t \] (4.95)

From this result we can clearly see that the "horizontal" reaction \( \Delta F \) from the load system oscillates during the cycle of vibration of the string. For stable operating regimes, this force spends half of its time pushing on the wave in the positive \( x \)-direction and the other half pushing on it in the negative \( x \)-direction. During one half of the cycle the load system will be opposing the drive system and in the other half it will be helping it. This is the mechanism that is responsible for modulating the phase speed of the wave. The "vertical" reaction of the load system is the one responsible for modulating the amplitude of the wave. The force \( \Delta F \), when multiplied by the transport velocity \( v \) of the string, is precisely the instantaneous power \( \Pi_{\text{drive}}(t) \) that is supplied/taken by the drive system to/from the traveling string.

\[ \Pi_{\text{drive}} = \text{Velocity} \times \text{Force} = v(T_2 - T_1) = v \left( f_L \cdot \frac{\partial w}{\partial x} \right)_{x=0} \] (4.96)

On the other hand, the power that is input by the load system into the wave is

\[ \Pi_{\text{Load System}} = \text{Force} \times \text{Local Velocity} = f_L(t) \times \left( \frac{Dw}{Dt} \right)_{x=0} \] (4.97)
During stable operation, and in the absence of the dashpot \( c_L \), the two power quantities average to zero over one cycle of vibration i.e., there is no net power flow between the string and each of the load and the drive systems. Energy just travels back and forth like in an undamped simple harmonic oscillator. For the unsteady modes however, the situation depends on whether we are dealing with a divergence or a flutter type of instability. We consider hereafter four distinct cases. These are

- Divergence due to the spring \( k_L \) with the two-parameter solution.
- Flutter due to the spring \( k_L \) with the three-parameter solution.
- Flutter due to the mass \( m_L \) with the two-parameter solution.
- Instability due to the dashpot \( c_L \) with the two-parameter solution.

We have to note here that the nature of the dashpot induced instability is fundamentally different from those introduced by the mass and the spring of the load system. The reason being that the dashpot is a non-conservative element and thus, it always dissipates energy whereas the mass and the spring simply store it then return it.

### 4.12.1 Diverging and Decaying Modes with the Spring \( k_L \)

Now that we fully understand the kinematics of the string and the power flow concepts, let us specialize our study to the decaying and the diverging modes of our traveling string when it is interacting with the spring \( k_L \) of the load system. This case is the simplest because, in the unstable range, the oscillation frequency is zero. The understanding of the mechanics is however crucial here because we will build on this case to later explain the flutter instability of the three-parameter solution and then that of the two-parameters with the load mass.

Going back to figure 4-22 we have to note a key difference between the mode that is diverging and the one that is decaying. It is that at the attachment point of the spring, the displacement and the slope of the string have the same sign for the decaying solution and opposite signs for the diverging solution. To understand the implications of this observation let us take our segment of the string with a control
volume and specialize it to the two modes we have at hand. We begin with the diverging mode. The forces acting on the string are shown in figure 4-59.

Figure 4-59: Force Balance for the Diverging Mode

From the above figure, we can see that the horizontal reaction $\Delta F$ of the spring is always pushing against the drive mechanism and thus, the latter must do work to overcome this force. The power $\Pi_{drive}$ that is input by the drive mechanism is

$$\Pi_{\text{drive}} = v(T_2 - T_1) = v \left( -k_L w \cdot \frac{\partial w}{\partial x} \right)_{x=0} > 0 \quad (4.98)$$

Since the slope and the displacement have opposite signs, their product is always negative. This results in a positive power flow being transferred from the drive system to the string. To accommodate this incoming energy, the amplitude of the wave in the string must increase and this is in essence the mechanism of the divergence instability.

Figure 4-60: Force Balance for the Decaying Mode

The forces acting on the string, for the decaying mode, are shown in figure 4-60. We can see there that this time we have $T_1 > T_2$. Consequently, the force $\Delta F$ is helping the drive system. The power flow calculation in this situation leads to $\Pi_{\text{drive}} < 0$ because the product between the displacement and the slope of the string at $x = 0$ is
always positive. This means that energy stored in the string will be traveling to the drive system. Here, the drive system absorbs the energy—both kinetic and potential—that is available in the wave until it drains it completely. The amplitude of the wave must then decrease and this is the mechanism for the decaying solution.

4.12.2 Flutter Modes with the Spring $k_L$

Inside the unstable range, we have seen that the motion of the string is dominated by two components, the “rigid body” bounce motion and the forward wave. Considering each one of these motions separately, we obtain that none of them can drain energy from the drive system. For the traveling wave, and as we have seen earlier, energy flows back and forth between the drive system and this wave without any net power flow. For the bounce motion however, and because the slope of the string is zero everywhere—including at the attachment point of the load system, the two tensions at the two ends of the control volume $CV$ are always equal and thus, there can never be an energy flow between the drive system and this bounce motion.

When the bounce mode and the traveling wave are either in phase or at 180 degrees out of phase, the horizontal reaction from the spring $k_L$ still spends half of its time “speeding-up” the wave and the other half “slowing” it down. In the unstable range however, the two composite motions—bounce mode and traveling wave, combine in such a way as to make the horizontal component $\Delta F$ of the spring’s reaction spend more time opposing the drive mechanism and less time helping it for the growing flutter solution. The situation is reversed for the decaying flutter mode. To verify this statement let us calculate the horizontal reaction from the string $\Delta F$ and see how it changes with time. When using the three-parameter standing waves solution given by equation 4.49, and for the case with only one nodal diameter, we obtain

$$\Delta F = -k_L w(0, t) \frac{\partial w}{\partial x}(0, t) = \Re \left\{ -k_L \cdot (A_0 + A_1) \cdot e^{i\omega t} \right\} \times \Re \left\{ B_1 \cdot e^{j\theta t} \right\} \quad (4.99)$$

Upon using $k_L = \alpha_L \pi T/R$ and setting the radius $R = 1$ for convenience, we obtain the dimensionless form

$$\frac{\Delta F}{T} = \alpha_L \pi \cdot \Re \left\{ (A_0 + A_1) \cdot e^{i\omega t} \right\} \times \Re \left\{ B_1 \cdot e^{j\theta t} \right\} \quad (4.100)$$

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Figure 4-61: Time Histories of the Horizontal Component $\Delta F$, induced by the Reaction $f_L$ of the Load System, for some key values in the string's Campbell diagram. Three-parameter solution with the Spring of the Load System. $\alpha_L = 1$. 

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This force is graphed in figure 4-61 for various interesting cases. We have to note that, for clarity, we show only the oscillatory component, without the exponentially growing/decaying envelope. The power that is supplied/taken by the drive system to/from the wave is proportional to the force $\Delta F$. The graphs shown in figure 4-61 represent thus the instantaneous power that is flowing between the drive system and the string. A measure of the amount of energy $W_{drive}$ that has been channeled to (or from) the string is given by the area under the time history of the force $\Delta F = T_2 - T_1$.

$$W_{drive}(t) = \int_0^t \Pi_{drive}(t) \cdot dt = \int_0^t v \cdot (T_2 - T_1) \cdot dt$$  \hspace{1cm} (4.101)

The key argument resides then in how the two composite motions couple so as to give a net power flow from or to the drive system, depending on wether it is the growing or the decaying flutter mode accordingly.

### 4.12.3 Flutter Modes with the Mass $m_L$

The mechanism is here similar to that of the three-parameter solution with the spring $k_L$. Here, and in the unstable range, we have seen that the motion can be decomposed into a forward wave plus a cosine standing wave. This cosine wave has always zero slope at the attachment point of the load system. Thus, if taken alone, the rocking motion can never be unstable. This motion plays the role of the bounce mode in the previous subsection. The combination of the forward wave with this standing wave motion, with the proper phase lag/delay, is the necessary ingredient that allows the horizontal reaction from the load system to spend unequal times “speeding-up” then “slowing-down” the traveling wave in the string. This, as we have seen is the necessary ingredient that makes power flow possible between the drive system and the traveling wave. The power that is supplied/taken by the drive system is

$$\Pi_{drive} = v \left( -m_L \frac{\partial^2 w}{\partial t^2} \cdot \frac{\partial w}{\partial x} \right)_{x=0} = vm_L \cdot \Re \left\{ s^2 A_1 \cdot e^{st} \right\} \times \Re \left\{ B_1 \cdot e^{st} \right\}$$  \hspace{1cm} (4.102)

whereas the power introduced by the mass $m_L$, which is also the rate of work done by the reaction from the mass on the wave, is given by

$$\Pi_{mass} = f_L \cdot \frac{Dw}{Dt}(0, t) = \left[ -m_L \frac{\partial^2 w}{\partial t^2} \cdot \left( \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial x} \right) \right]_{x=0}$$  \hspace{1cm} (4.103)
Figure 4-62: Time Histories of the Powers $\Pi_{\text{drive}}$ and $\Pi_{\text{Mass}}$ for the String with the Mass $m_L$ of the Load System and for some key values in the string's Campbell diagram. Two-parameter solution with $\mu_L = 0.5$. 
The two instantaneous powers are graphed in figure 4-62 for some key roots in the Campbell diagram for our string—for both stable and unstable operating regimes. Once again, we have removed the growth/decay envelope $e^{2\sigma t}$ simply for clarity. There, we can clearly see that the average power $\Pi_{\text{drive}}$ over time is zero for stable operation, positive for the growing mode, and negative for the decaying mode. The power $\Pi_{\text{Mass}}$ that is delivered by the mass of the load system has always a zero mean because the mass can neither absorb nor create energy. It simply stores it in the form of kinetic energy then gives it back when needed.

4.12.4 Instability due to the Dashpot $c_L$

We have seen in a prior section that the instability introduced by the dashpot of the load system is fundamentally different from those introduced by both the spring and the mass of the load system. The mechanism of instability is too very different. For the case with the conservative elements—mass $m_L$ and spring $k_L$, unstable operation results from the nonconservative coupling between two modes. For the case of the dashpot however, no coupling is necessary and a single mode—or wave, becomes unstable. To understand the mechanism by which the kinetic energy stored in the axial motion of the string is transferred to/from the wave with a growing/decaying amplitude, let us one more time consider a straightened string with a dashpot $c_L$ attached at the origin $x = 0$.

![Diagram of traveling string with dashpot $c_L$](image)

Figure 4-63: Traveling String with Dashpot $c_L$ of the Load System.

Since we need only a “single mode”, let us use the displacement $w$ given by

$$w(x, t) = A \sin(\kappa x - \omega t), \quad \omega = \kappa c_{\phi} = \kappa(\pm c_0 + v)$$  \hspace{1cm} (4.104)
We can now write the power that is supplied by the drive system

\[ \Pi_{\text{drive}} = v(T_2 - T_1) = v \left( -c_L \frac{\partial w}{\partial t} \cdot \frac{\partial w}{\partial x} \right)_{x=0} \quad (4.105) \]

Performing the calculation and averaging over one complete period, we obtain

\[ \langle \Pi_{\text{drive}} \rangle = \frac{1}{2} \kappa^2 A^2 c_L \cdot v c_\phi \quad (4.106) \]

Clearly, whether the power is supplied/taken by the drive mechanism to/from the string-dashpot system depends on the sign of the phase speed \( c_\phi \). This argument alone is not sufficient because the total power that is supplied by the drive mechanism is partitioned in two. One fraction goes to the wave and the other is dissipated in the dashpot. The dashpot always "eats up" energy whether the motion is steady, growing or decaying. It is a "non-discriminating" sink and the power it dissipates is

\[ \langle \Pi_{\text{dashpot}} \rangle = \left\langle \left( c_L \frac{\partial w}{\partial t} \cdot \frac{Dw}{Dt} \right)_{x=0} \right\rangle = \frac{1}{2} \kappa^2 A^2 c_L \cdot (c_\phi - v)c_\phi \quad (4.107) \]

We have both the power that is supplied, taken by the drive mechanism and that which is dissipated by the dashpot. The difference \( \Pi_{\text{wave}} = \Pi_{\text{drive}} - \Pi_{\text{dashpot}} \) is the net power that either goes into or is drained from the wave.

\[ \langle \Pi_{\text{wave}} \rangle = \frac{1}{2} \kappa^2 A^2 c_L \cdot (2v - c_\phi)c_\phi \quad (4.108) \]

Figure 4-64 shows all three power quantities, averaged over time and in dimensionless form, for both the forward and the backward (disk-based) waves and this, as a function of the transport speed parameter \( \beta = v/c_\phi \). The normalizations are done by dividing all three power quantities by the expression \( \kappa^2 A^2 c_L c_\phi^2 \).

Let us now analyze the results shown in figure 4-64. At zero transport speed, there is no energy supplied by the drive system \((v = 0)\) and all the energy that is available in the waves is dissipated entirely in the dashpot. At the critical speed all three energies are zero for the backward wave. This of course makes sense since the wave has zero phase speed and the dashpot does not move at all. We can clearly see that the energy dissipated in the dashpot increases monotonically for the forward wave. This agrees with the Campbell diagram of figure 4-47 where, for the forward wave, the decay rate
increases with the transport speed \( v \). These results are in accordance with those of Crandall [21] where the preceding discussion was adapted from.

One peculiar result is that, for the supercritical backward wave, the energy dissipated by the dashpot is negative i.e., the dashpot seems to be delivering energy rather than burning it. This does not make physical sense and we have to be careful with the interpretation. In fact it is the drive system that is delivering the energy. The kinematics of the problem make it impossible for the dashpot to burn all that incoming energy. The dashpot still burns a fraction of the incoming energy and channels the remaining part into the backward wave. This is the mechanism by which energy is transferred from the drive system to the wave, and through the dashpot.

![Graph of Powers for the Forward and Backward Waves](image)

Figure 4-64: Averaged Powers supplied by the Drive System \( \Pi_{\text{drive}} \), Dissipated in the Dashpot \( \Pi_{\text{dashpot}} \), and Introduced in the Wave \( \Pi_{\text{wave}} \) for the forward (left) and backward (right) waves.

### 4.13 Conclusion

The self-excited vibrations that result from the interaction of a flexible spinning disk with a mass-spring-dashpot load system were studied here for a traveling string—a close cousin of the full disk. Because the mathematics of the string are less involved than those of the full disk, we were able to obtain the exact solution for the string's equation and compare it with that obtained from Galerkin's technique. We thus an-
swered the question of whether such instabilities are genuine or rather an artifact of the approximate solution obtained by Galerkin's technique. An extensive study of the effects of each of the elements of the load system has on the dynamics of the string was undertaken and specific results were presented. We also showed that any departure from symmetry —of the conservative inertia and elastic properties, introduces regions of unstable operation. The kinematics of the string were also studied and several alternate representations were given that illustrate vividly the behavior of our traveling string under both stable as well as unstable operating conditions. We also studied the dynamics of a rigid gyroscopic ring interacting with a complete load system and drawn analogies to the flexible string. The thorough understanding of the string's kinematics allowed us then to identify then explain the physical mechanisms responsible for transferring the kinetic energy stored in the axial motion of the string into transverse flexural unsteady vibratory modes.
Chapter 5

Experimental Results

The treatment we have presented so far focused on improving the model for our spinning disk and explaining the mechanisms of the various instabilities that exist in the disk when it interacts with the air surrounding it and with the load system. In this chapter we present some interesting experimental observations that we obtained for a disk spinning against one baseplate and then between two rigid plates. In our experiments there is no external forcing mechanism to enhance the various possible vibratory modes. Instead, we report on those modes that are self-excited and thus unstable. Let us not forget that this thesis deals with the instabilities of flexible spinning disk. Work on how the natural frequencies predicted by various models compare with those observed experimentally have been reported by several authors. All the self-excited waves we observed were traveling backward with respect to the disk and forward with respect to a stationary observer. This is in agreement with all of our analytical results which globally say that any given mode can become unstable only for speeds above its critical speed. One of the key results here is for the disk spinning against one baseplate. We found that there is some critical spin rate above which the disk supports solitary waves. These are the manifestation of a truly non-linear mechanism and we give in this chapter a qualitative explanation of the observed solitons. Before we present the experimental results per say, we give first a description of the experimental apparatus used here along with the various measurement techniques.
Figure 5-1: General View of the Experimental Setup and Instrumentation

Figure 5-2: Close-Up View of the Disk with the Baseplate and the Fotonic Sensors
5.1 Experimental Apparatus

The "Spinning Disk Station" (SDS) used throughout our experiments consists of a stepper motor mounted on a massive base, itself mounted on a rigid frame. An overall view of the SDS along with some of the instrumentation is shown in figure 5-1 and a close-up view of the disk with the non-contact fotonic sensors is shown in figure 5-2. The key components of the SDS are shown schematically in figure 5-3.

![Diagram of SDS components](image)

**Figure 5-3: Detail of Key Components of the SDS**

To the motor casing is attached a structure to support a rigid plate against which our disk is spun. The rigid plate rests on three micrometric screws that allow for precise setting of the air gap $h_a$ between the disk and the rigid plate. The spindle of the motor extends through the rigid plate and is equipped with a clamping attachment that can accommodate disks with various parameters, but however the same inner diameter of $d = 2a = 25 \text{[mm]}$. The speed of the motor is controlled by an integrated pulse generator. The available range extends from zero to $3600 \text{[rpm]}$. The frequency
delivered by the pulse generator is controlled by a built-in VCO\textsuperscript{1} that offers several possible settings on the SDS's control panel, depending on which regime of operation is desired. The setting we found most practical was to use an external function generator and send a square wave \textit{TTL} signal to the "External Clock" input on the control panel of the SDS. For measurements of the disk's transverse displacement, an XYZ precise positioning system for the non-contact fotonic sensor is also available though it was not used in this research. The reason is that we are interested in the frequency content of the sensor signal and not in absolute amplitudes.

For experiments with the disk spinning between \textit{two} plates, we used a "plexiglas" plate that rests on the bottom plate with three adjustable screws. The plexiglas has several drillings to accommodate the fotonic sensor at various locations. The three screws permit us to change the spacing between the two plates and thus the thickness of the air-film on both sides of the spinning disk. The layout of the disk and the two plates is essentially that shown in figure 3-2. Along with the fancy electronic instrumentation, we also used a high speed digital camera to record the rapidly traveling waves and then play them back at a lower frame rate to extract useful information.

5.2 Experimental Procedures

What we are after in these experiments is to detect self-excited waves in the spinning disk and verify if their kinematics are in accordance with the analytical predictions of chapters 2 and 3. Let us recall that for the disk interacting with the air-film, the critical speed for a given mode is around\textsuperscript{2} twice the critical speed of that mode. The waves we are interested in are self-excited, and thus no forcing apparatus is needed. The forcing mechanism is the negative damping that arises from the interaction of the disk with the air-film. To detect these waves and determine which direction they are traveling, we initially used two fotonic sensors. We obtained the frequency from the spectrum of one of the sensors and the direction of propagation from the cross-

\textsuperscript{1}Voltage Controlled Oscillator

\textsuperscript{2}Without internal damping and the fluid shear loading, the mode becomes unstable at \textit{exactly} twice its critical speed.
correlation between the two sensors and the knowledge of the wavelength of the wave being observed. We obtained the wavelength by aiming a strobe light at the disk and setting it to the frequency given by the sensor in order to freeze the wave in space and then simply count the number of full waves in the frozen disk. Care here has to be taken not to compute the spectrum of the sensor’s signal while the strobe is on. The reason is that the reading of the fotonic sensor depends on how much light is reflected from the disk’s surface and under no circumstances we want to corrupt the reading. Later on we found that the use of a high speed camera and a single sensor greatly simplified the data gathering. We kept the sensor to obtain a precise reading of the frequency.

5.3 Obtaining the Disk’s Parameters

One of the difficult aspect we were faced with during the early stages of the experimental investigation was the selection of a disk material that exhibits the expected behavior. The obvious choice was to use an actual floppy diskette. The problem there is that we are limited by the maximum speed of operation of the SDS which is far below the disk’s first critical speed. The obvious driving parameter is the thickness $h_d$ of the disk. By making the disk thinner we could lower the critical speeds and thus be able to observe supercritical self-excited waves on the spinning disk. We experimented for some time with a VCR-foil disk—the same material that is used in VCR tapes. The problem there is that the material gets wrinkled after a single run and was therefore not a suitable candidate. We also tried several other candidates such as aluminum foil and plain paper with however little success. We finally opted for the material used in typical plastic shopping bags. Such material has properties that lead to critical speeds that are well below 3600 [rpm], and in contrast with the VCR-foil, does not wrinkle as easily. We then undertook the task of measuring the material properties. We began with Young’s modulus that we obtained from a standard tensile test. We performed the test with samples cut in different directions so as to obtain an average value for $E$. The reason being that most sheet plastic materials are obtained
from rolling mills and thus it is very likely that the material is non-isotropic. Figure 5-4 shows the stress-strain curves (mean ± one deviation) we obtained.

Figure 5-4: Stress-Strain Curve for the Disk's Material.

To determine the thickness of the disk, we stacked twenty sheets of disk material, measured the total thickness with a micrometric dial gauge and divided the reading by 20. This reduces the uncertainty of the measuring device. For the mass density \( \rho \), we simply weighted a known volume of disk material and then divided the mass by such volume. The parameters we obtained are organized in Table 5.1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Designation</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner Radius</td>
<td>( a )</td>
<td>10</td>
<td>( mm )</td>
</tr>
<tr>
<td>Outer Radius</td>
<td>( b )</td>
<td>140</td>
<td>( mm )</td>
</tr>
<tr>
<td>Disk Thickness</td>
<td>( h_d )</td>
<td>38</td>
<td>( \mu m )</td>
</tr>
<tr>
<td>Mass Density</td>
<td>( \rho_d )</td>
<td>1100</td>
<td>( kg/m^3 )</td>
</tr>
<tr>
<td>Young's Modulus</td>
<td>( E )</td>
<td>( 1.47 \times 10^8 )</td>
<td>( N/m^2 )</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td>( \nu )</td>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>Air-Film Thickness</td>
<td>( h_a )</td>
<td>1</td>
<td>( mm )</td>
</tr>
<tr>
<td>Air Density</td>
<td>( \rho_a )</td>
<td>1.29</td>
<td>( kg/m^3 )</td>
</tr>
<tr>
<td>Air Viscosity</td>
<td>( \mu_a )</td>
<td>( 1.81 \times 10^{-5} )</td>
<td>( kg/m \cdot s )</td>
</tr>
</tbody>
</table>

Table 5.1: Parameter Values for the Spinning Disk and the Air-films
Table 5.1 also lists other parameters that we fixed for operating our spinning disk. With this set of parameters, two different sets of experiments were performed. The first set was for the disk spinning against only one baseplate, and the second for the disk spinning between two rigid plates.

5.4 Disk Spinning against ONE Plate

With the parameters given in table 5.1, we spun the disk at several values of $\Omega$ and recorded the disk’s shape with a high speed digital camera. For moderately low values of the spin rate, the disk supports harmonic waves that are traveling in the forward direction with a rate that is smaller than the spin rate. Figure 5-5 shows such situation. We note there that the wave is purely harmonic i.e., it has only one wavenumber component. When the spin rate of the disk is increased to $1500[\text{rpm}]$, the transverse displacement looks like that in figure 5-6. We note there that the pattern is no longer coherent. For even higher values of the disk’s spin rate, the harmonic wave disappears completely and the disk now has solitary waves. What is interesting to note here is that, at any given speed, the number of solitons in the disk can be changed by slightly perturbing the disk. This is done by gently touching the disk or by placing then removing the top plexiglas plate. In a sense, all configurations are stable and the number of solitons depends only on the initial conditions.

For a disk with the parameters given in table 5.1, we did obtain all “mode shapes” from two to nine solitary waves. The first six modes are shown in figures 5-7 to 5-12. We also observed that the solitons, like the harmonic waves, travel forward but however at a rate much smaller than that of the harmonic waves. The next logical course of action was to measure the angular speed with which these solitons travel as a function of both the disk’s spin rate and the number of solitons in the disk. We measured the frequency with a fotonic sensor and the results are presented in figure 5-13. The results in that figure clearly indicate that the angular speeds of the solitons do not depend upon the spin rate. This is a lock-in phenomenon and is certainly due to the fluid-structure interaction. Processing further the data in figure 5-13, let us divide the
Figure 5-5: Spinning Disk with Harmonic Wave, Spin Rate 1000 \([\text{rpm}]\). Both the Disk and the Wave are rotating in the counter-clockwise direction.

Figure 5-6: Spinning Disk with Non-coherent Wave, Spin Rate 1500 \([\text{rpm}]\). Both the Disk and the Wave are rotating in the counter-clockwise direction.
Figure 5-7: Disk with Two Solitary Waves, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.

Figure 5-8: Disk with Three Solitary Waves, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.
Figure 5-9: Disk with Four Solitary Waves, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.

Figure 5-10: Disk with Five Solitary Waves, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.
Figure 5-11: Disk with Six Solitary Waves, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.

Figure 5-12: Disk with Seven Solitary Waves, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.
Figure 5-13: Frequencies of Solitary Waves as seen by a Stationary Observer.

Figure 5-14: “Normalized” Frequencies of Solitary Waves.
frequency for each mode by the number of solitons in that mode. We thus obtain the angular speed for each of the modes. The results for such calculation are displayed in figure 5-14. Yet another interesting result, the rate at which the solitons travel in the circumferential direction is the same no matter how many solitons there are. For harmonic waves, we have seen that there is a strong dependence between the phase speed of such waves and its corresponding wavenumber—a consequence of dispersion. We also found that the onset speed at which the solitons appear in the spinning disk depends upon the air gap thickness $h_a$. Large values of $h_a$ lower the onset speed whereas small values decrease it. In fact, for the relatively small thickness $h_a = 0.1 [mm]$, we were not able to obtain any solitons at all throughout the entire speed range of the SDS. If we look back at figure 2-2, we have that, in the steady state configuration, the disk is not perfectly plane. In fact, the faster the disk spins and the more it is pulled towards the fixed base plate. Thus, the higher the air gap at the hub $h_a$ and the more the disk deflects in the steady state configuration. Under adequate conditions, the outer circumferential “rings” of the disk get compressed when the disk pulled towards the baseplate. Rather than sustaining compressive in plane stresses, the disk “buckles” and localized bumps—solitary waves appear. These bumps resemble the ones we get when we put a large carpet on the floor. Furthermore, in the “buckled” configuration—with the solitary waves, there are no stresses in the those regions of the disk. The centrifugally induced stresses are balanced by the ones due to the compression of the disk’s outer rings. This partly explains why the frequencies of the solitons—as shown in figures 5-13 and 5-14, do not change with the disk’s spin rate. This theory needs certainly to be verified and we leave that to a further study.

5.5 Disk Spinning between TWO Plates

When the plexiglas plate is installed above the disk, the solitary waves disappear and we now have only harmonic waves. At most spin rates however, the patterns observed consist always of the superposition of several harmonic waves—the signal picked up by the fotonic sensor has several spectral components. At the high spin
rate of 3300 [rpm], we did obtain a single wave as shown in figure 5-15.

Figure 5-15: Disk Spinning between Two Plates, Spin Rate 3300 [rpm]. Both the Disk and the Wave are rotating in the counter-clockwise direction.

At this speed of 3300 [rpm], the noise generated by the spinning disk resembles a "whistling" tone. This clearly confirms the single mode pattern. One last observation is that the phase speed of this harmonic wave is much higher than that of the solitary waves that are established by simply removing the top plexiglas plate.

5.6 Conclusion

Traveling self-excited waves have long ago been predicted but published reports showing experimental evidence of their existence are numbered. In this chapter we tried to fill this gap and show indeed that such waves can be observed experimentally. We also reported on the existence of solitary waves in the disk. Among the peculiarities associated with such waves, and in contrast with harmonic waves, is that their propagation speed is quasi-independent upon the spin rate of the disk, and that different modes propagate with the same phase speed.
Chapter 6

Conclusions and Recommendations

The work presented in this dissertation encompasses several aspects associated with the dynamics of spinning flexible disks. Now is the time to highlight our major contributions and suggest areas where future investigation is needed. The author sees this thesis as bringing three distinct contributions. These are the account of the fluid induced shear loading, the physical explanation of the destabilizing mechanisms introduced by the load system, and finally the experimental reporting of self-excited harmonic and especially solitary waves.

6.1 Fluid Induced Shear Loading

When we started investigating the effect of the shear loading we thought that it would reduce the effective bending rigidity of the disk and with it the disk's critical speeds. Our numerical results show that some of the critical speeds are in effect lowered. However, these critical speeds are much higher than the first lowest and thus their relevance, from a stability point of view, is somewhat secondary. The relative importance of the shear stresses in the disk depends on the thickness of the fluid layer the disk is in contact with. If the thickness of the air-film is very large (the fluid is then governed by the full Navier–Stokes equation) the shear loading may be
more important than the pressure loading. The importance of the shear loading is thus problem dependent. For the common floppy diskette the shear loading has little effect on the instability mechanisms. We also found that the shear loading – when taken alone, is destabilizing at all speeds of operation of the disk. This is definitely a key result since, up to now, all other destabilizing mechanisms start to manifest themselves at speeds higher than the first critical.

6.2 Physics of the Destabilizing Mechanisms

We used a close cousin of the disk, a traveling string, to explain the destabilizing effects caused by each of the components of the mass-spring-dashpot load system. We thus greatly simplified the mathematical aspects of the spinning disk problem and were able to obtain qualitative explanations on how the energy stored in the rotation of the disk can be channeled into transverse vibratory modes. Because of its relatively simple dynamics, the traveling string allowed us to use alternate kinematic representations that enhanced our understanding of the string's dynamics and the mechanisms behind the various instability mechanisms. The instabilities we obtained can be divided into two distinct groups. The first one is triggered by nonconservative mode coupling and can be induced by both the mass and the spring of the load system. The second is controlled by the kinematics of a single traveling wave that is interacting with the dashpot of the load system. The instabilities of the first type could be of divergence as well as flutter type. The dashpot however causes only flutter-type of instabilities. Both the mass and the spring can cause either type of instability and we gave physical explanations for several possible cases.

6.3 Experimentally Observed Solitary Waves

The results here are purely experimental and we have yet to establish the theory that will be able to predict such waves. Nevertheless, our experimental investigation reports the non-dependence of the speed of propagation of such waves on both the
disk's spin rate and the mode number. Those waves are due to the asymmetry associated with the disk when it is spinning against one plate. When the disk is confined between two plates however, there are no solitons and the only waves there are of harmonic nature.

6.4 Suggestions for Future Work

We have certainly answered several questions in this dissertation. Along with that we also turned some stones and there are few issues we think it is necessary to get to the bottom of. One such issue is to investigate the importance of the shear loading against that of the pressure loading from the air-film. There may be applications where the shear loading is more important than the pressure loading. In such applications, the geometry and the parameters of the fluid field would be different from those of the computer floppy diskette. The second item on the agenda would be to augment our finite difference model of the disk to incorporate the pressure loading from the air-films and compare the predicted natural frequencies with those obtained experimentally. This is in a sense a model validation issue and will tell us how adequate our model of the disk is. The third item is to work out the theoretical aspects governing the solitary waves in the disk. This would require a great deal of numerical studies since, for the large deformations of the disk with the solitons, the classical disk equation needs to be modified to account for such large deformations. The vonKarman plate formulation would be the starting point in such investigation.
Appendix A

Matlab Program Listing for the Disk's Finite Difference Solution

The two Matlab programs included here are an implementation of the finite difference solution described in chapter two. They were used to generate all of the Campbell Diagrams given in chapters one, two and three.

The main program is finiteww.m and it is the one that should be run in Matlab. The program matrixww.m generates the system matrices of equation 2.47 and is called by finiteww.m. Throughout the lines of code, we have referenced several of the equations given in chapters two and three whenever necessary.

We will also note that the two programs take full advantage of the "vector-based" operations of Matlab and thus, many of the vector quantities in chapters two and three become matrices in the program listings.
% function FINITEWW.M
% Calculates Natural Frequencies of a Spinning Flexible Disk with the Finite
% Difference Technique and Displays the results in the form of Campbell Diagrams
% Needs the function MATRIXWW.M

clc, clc, clear, drawnow  % Clean up windows and variables in workspace

% Define and Initialize Parameters of Disk and those of Air—Film

global Ey nu a b rho_d h_d eta rho_a h_a mu_a

Ey = 1.47e8;  % Young's Modulus of DISK
nu = 0.333;  % Poisson's ratio of DISK
a = 10e-3;  % Inner radius of DISK
b = 140e-3;  % Outer radius of DISK
rho_d = 1100;  % Mass density of DISK
h_d = 76e-6;  % Thickness of DISK
eta = 0.02;  % Internal damping in DISK
rho_a = 1.29;  % Mass density of AIR
h_a = 1e-3;  % Thickness of AIR—FILM
mu_a = 1.81e-5;  % Kinematic viscosity of AIR

% Define and Initialize Computational Variables

global N

N = 10;  % Number of Points along radius of DISK
nmax = 5;  % Maximum number of Nodal Diameters
RPMmin = 0;  % Minimum RPM for Campbell Diagram
RPMmax = 100;  % Maximum RPM for Campbell Diagram
RPMpts = 200;  % Number of points along RPM axis

% Generate RPM points for which calculation is to be done

Wrpm = linspace(RPMmin, RPMmax, RPMpts);

% Reserve array to store frequencies

ZZ = zeros(2*N, RPMpts);
for n_counter = 0:nmax

[M,D,G,K,C,E,Z]=matrixww(n_counter);

for ii = 1:RPMpts
    disp([n_counter ii])
    Spin=(pi/30)*Wrpm(ii);
    KALL = [K+Spin*C+Spin*Spin*E,Z;Z,M];
    MALL = [D+Spin*G,M,-M,Z];
    ZZ(:,ii) = eig(KALL,MALL);
end

subplot(121), plot(Wrpm, abs(imag(ZZ)), '. '), hold on, ...
axis('square'), axis([0 10 0 .6]), drawnow

subplot(122), plot(Wrpm, real(-ZZ), '. '), hold on, ...
axis('square'), axis([0 10 -1 1]), drawnow
end

subplot(121), title('Campbell Diagram'), ... 
xlabel('Spin Rate [rpm]'), ylabel('Natural Frequencies [rad/s]')

subplot(122), title('Campbell Diagram'), ... 
xlabel('Spin Rate [rpm]'), ylabel('Growth/Decay Rate [rad/s]')
function [M,D,G,K,C,E,Z] = matrixww(n)

% function MATRIXWW.M
% Calculates and returns all six matrices (M,D,G,K,C,E) that constitute
% the eigenvalue problem of equation 2.47
% Arguments: Takes only the number "n" of nodal diameters. No need to pass
% the disk's parameters since they are declared as "global".
% This function is called by FINITEWW.M

global Ey nu a b rho_d h_d eta rho_a h_a mu_a

% Calculate the step size, the r-coordinates of the grid points
% and calculate the bending rigidity "Db" of the disk

h = (b-a)/N;  % Step size for discretization
r = linspace((a+h),b,N);  % Divide radius of disk in N points
Db = Ey*h_d^3/(12*(1-nu^2));  % Bending rigidity of disk

% Calculate In–Plane Stresses and their Derivatives
% See equations 2.4, 2.5, 2.6, and 3.14

k = ((3+nu)+(1+nu)*((b/a)^2))/((1+nu)+(1-nu)*(b/a)^2);
S_r = (rho_d/8)*((3+nu)*((b^2-r^2).*(1+((1-nu))/(3+nu)))*(k*a^2)./(r^2));
S_t = (rho_d/8)*((1+nu)*((a^2+k*b^2)-(1+3*nu)*r^2-(1-nu)*(k*a^2*b^2)./(r^2));
S_r = (mu_a/(2*(h_a*h_d)))*((b^4 - r^4)./(r^2));
dS_r = -(rho_d/4)*((3+nu)*r+(1-nu)*((k*a^2*b^2)./(r^3));
dS_r = -(mu_a/(h_a+h_d)))*((b^4 + r^4)./(r^3));

% Calculate subcoefficients of Disk's ODE. See equations 2.48 to 2.54

A5M = rho_d*h_d*ones(size(r));  % for the MASS matrix
A5G = 2*i*n*A5M;  % for the GYROSCOPIC matrix
A1K = Db*ones(size(r));  % for the STIFFNESS matrix
A2K = Db*(2/r);
A3K = -Db*(1+2*n^2)]./(r^2);
A4K = -A3K./r;
A5K = Db*n*n*(n*n-4)./(r^2);
A1C = i*n*eta*A1K; % for the CIRCULATORY matrix
A2C = i*n*eta*A2K;
A3C = i*n*eta*A3K;
A4C = i*n*eta*A4K - 2*i*n*h_d*S_r./r;
A5C = i*n*eta*A5K - i*n*h_d*S_r./r;

A3E = -h_d*S_r; % for the CENTRIFUGAL matrix
A4E = -h_d*(dS_r + S_r./r);
A5E = n*n*h_d*(-rho_d + S_t./(r.^2));

% Define transformation matrices to be used in equations 2.56 to 2.60
T55 = [ 1 -1 0 0 0; -4 2 1 -1 0; 6 0 -2 0 1; -4 -2 1 1 0; 1 1 0 0 0 ];
T33 = [ 1 -1 0 ; -2 0 1 ; 1 1 0 ];

% Form column vectors on RHS of equations 2.55 to 2.60
M1band = A5M;
G1band = A5G;
K5band = [A1K/(h.^4) A2K/(2*h.^3) A3K/(h.^2) A4K/(2*h) A5K]*T55';
C5band = [A1C/(h.^4) A2C/(2*h.^3) A3C/(h.^2) A4C/(2*h) A5C]*T55';
E3band = [A3E/(h.^2) A4E/(2*h) A5E]*T33';

% Initialize all matrices, except the damping matrix D (see below).

M = zeros(N+4);
G = zeros(N+4);
K = zeros(N+4);
C = zeros(N+4);
E = zeros(N+4);

% Build matrices of figure 2–11 according to equations 2.55 to 2.60 ...

for NN=1:N
    M(2+NN,2+NN) = M1band(NN);
    G(2+NN,2+NN) = G1band(NN);
    K(2+NN,NN:(NN+4)) = K5band(NN,:);
    C(2+NN,NN:(NN+4)) = C5band(NN,:);
    E(2+NN,(NN+1):(NN+3)) = E3band(NN,:);
end

% ... and build the DAMPING matrix as a multiple of the STIFFNESS matrix

D = eta*K;
\[
\begin{align*}
D(3,3) &= D(3,3) + D(3,1); \\
K(3,3) &= K(3,3) + K(3,1); \\
C(3,3) &= C(3,3) + C(3,1);
\end{align*}
\]

\[
\begin{align*}
M &= M(3:(N+4),3:(N+4)); \\
G &= G(3:(N+4),3:(N+4)); \\
D &= D(3:(N+4),3:(N+4)); \\
K &= K(3:(N+4),3:(N+4)); \\
C &= C(3:(N+4),3:(N+4)); \\
E &= E(3:(N+4),3:(N+4));
\end{align*}
\]

\[
\begin{align*}
% To eliminate the nodal points at (N+1) and (N+2), begin by \\
% rewriting equations 2.45 and 2.46 in the form
% \[ BC1 \cdot W(N-1) + BC2 \cdot W(N) + BC3 \cdot W(N+1) = 0 \]
% \[ BC4 \cdot W(N-2) + BC5 \cdot W(N-1) + BC6 \cdot W(N) + BC7 \cdot W(N+1) + BC8 \cdot W(N+2) = 0 \]
% where the coefficients BC1 to BC8 are defined as:
BC1 &= 1/(h*h) - nu/(2*b*h); \\
BC2 &= -2/(h*h) + nu*n*n/(b*b); \\
BC3 &= 1/(h*h) + nu/(2*b*h); \\
BC4 &= -1/(2*h*h*h*h); \\
BC5 &= 1/(h*h*h*h) + 1/(b*h*h*h) - (n*n*(nu-2)-1)/(2*b*b*h*h); \\
BC6 &= -2/(b*h*h*h) + n*n*(3+nu)/(b*b*b); \\
BC7 &= -1/(h*h*h*h) + 1/(b*h*h*h) + (n*n*(nu-2)-1)/(2*b*b*h*h); \\
BC8 &= 1/(2*h*h*h*h);
\end{align*}
\]

\[
\begin{align*}
% Solve for W(N+1) and W(N+2) in terms of W(N-2), W(N-1), W(N)
% \[ W(N+1) = X1 \cdot W(N-1) + X2 \cdot W(N) \]
% \[ W(N+2) = X3 \cdot W(N-2) + X4 \cdot W(N-1) + X5 \cdot W(N) \]
% where X1 to X5 are given by:
X1 &= -(BC1/BC3); \\
X2 &= -(BC2/BC3); \\
X3 &= -(BC4/BC8); \\
X4 &= (BC7*BC1/BC3 - BC5)/BC8; \\
X5 &= (BC7*BC2/BC3 - BC6)/BC8;
\end{align*}
\]
% Now, modify W(N-2), W(N-1), and W(N) by implementing the above constraints ...

K(N-1,N-1) = K(N-1,N-1) + K(N-1,N+1)*X1;
K(N-1,N)  = K(N-1,N)  + K(N-1,N+1)*X2;
K(N,N-2)  = K(N,N-2)  + K(N,N+2)*X3;
K(N,N-1)  = K(N,N-1)  + K(N,N+1)*X1 + K(N,N+2)*X4;
K(N,N)   = K(N,N)   + K(N,N+1)*X2 + K(N,N+2)*X5;

D(N-1,N-1) = D(N-1,N-1) + D(N-1,N+1)*X1;
D(N-1,N)  = D(N-1,N)  + D(N-1,N+1)*X2;
D(N,N-2)  = D(N,N-2)  + D(N,N+2)*X3;
D(N,N-1)  = D(N,N-1)  + D(N,N+1)*X1 + D(N,N+2)*X4;
D(N,N)   = D(N,N)   + D(N,N+1)*X2 + D(N,N+2)*X5;

C(N-1,N-1) = C(N-1,N-1) + C(N-1,N+1)*X1;
C(N-1,N)  = C(N-1,N)  + C(N-1,N+1)*X2;
C(N,N-2)  = C(N,N-2)  + C(N,N+2)*X3;
C(N,N-1)  = C(N,N-1)  + C(N,N+1)*X1 + C(N,N+2)*X4;
C(N,N)   = C(N,N)   + C(N,N+1)*X2 + C(N,N+2)*X5;

E(N,N-1) = E(N,N-1) + E(N,N+1)*X1;
E(N,N)  = E(N,N)  + E(N,N+1)*X2;

% ... and resize all SIX matrices by eliminating rows (N+1) & (N+2)
% and columns (N+1) & (N+2)

M = M(1:N,1:N);
D = D(1:N,1:N);
G = G(1:N,1:N); % All six matrices have now N rows and N columns
K = K(1:N,1:N);
C = C(1:N,1:N);
E = E(1:N,1:N);

Z = zeros(size(K)); % Create matrix of ZEROS, to be used in equation 2.61
Bibliography


