THE ANALYSIS OF A BUILDING FRAME BY MECHANICAL METHODS

By

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B.S., University of Washington
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MASTER OF SCIENCE

from the
Massachusetts Institute of Technology
1938

Signature of Author

Signature of Professor in Charge of Research

Signature of Chairman of Department Committee on Graduate Students

Signature redacted
Signature redacted
Signature redacted
Cambridge, Mass.
May 19, 1938

Mr. George W. Swett
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Mass.

Dear Sir:

In partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering, I am submitting to you herewith my thesis entitled, "THE ANALYSIS OF A BUILDING FRAME BY MECHANICAL METHODS."

Respectfully yours,

Holger P. Mittet

Holger P. Mittet
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ACKNOWLEDGEMENT

The author desires to express gratitude for the assistance from his advisor, Professor J. B. Wilbur, whose suggestions were a constant help throughout the preparation of this thesis.
SYNOPSIS

Purpose
The object of this thesis was to study the stresses in a building frame by mechanical methods.

Method
The stresses in the building frame were found by mechanical analyses of a celluloid model. The model used was a four story, symmetrical three-bay frame with rigid supports.

The analytical solution was obtained with the slope deflection equations, and the simultaneous calculator was used for solving the equations. The mechanical devices used in the analysis were the Beggs Deformeter and the M. I. T. Moment Indicator. The reactions were solved for by the deformeter; the internal stresses were found by means of both the deformeter and the indicator.

Scope
The analysis was performed for one loading. This was a one lb. wind load applied normal to the columns at the level of the top girder. Since the structure was symmetrical, the stresses in corresponding members were equal by theory, and in the mechanical solution this was used as a check.

The number of the tests was limited by the time. Many difficulties arose in the beginning of this study on account of the technique, and most of the time was
spent in overcoming errors of operation, and developing of the necessary skill.

The analysis by the Beggs Deformeter included the reactions and the stresses in the center of the girders of the outside bay. The M. I. T. Moment Indicator was used to solve for internal moments. Two types of indicator solutions were used; first, direct application of the moment indicator by which relative moments were found, the absolute values of the moments being determined by the application of the equations of statics; and, second, direct determination of the moments by means of the moment indicator used with a spring balance. Results

The results for all stresses proved to agree very closely with the theoretical solution. The moment indicator proved to be the more easily applied instrument and produced the better results.
THE BEGGS DEFORMETER

The analysis of indeterminate structures by the Beggs Deformeter is based on the same fundamental principle as is the basis of many mathematical methods, namely, Maxwell's Theorem of Reciprocal Displacement. The essential difference between the mathematical and mechanical methods of solution is that in model analysis laborious calculations of required deflections are replaced by direct measurements. The Beggs method expresses each indeterminate quantity as a function of a known load and two measurable deflections, replacing the solution of groups of simultaneous equations. Also the model analysis is self checking since the solution produces all stress components by measured deflections, and the three equations of statical equilibrium remain for checking. The model takes into account the effect of thrust and shear deformations, varying modulus of elasticity, effect of brackets, and peculiar distribution of stress around corners of frames, which are not taken into account in the approximate theoretical solutions.

Principle of the Beggs Deformeter

The Beggs Deformeter is a mechanical application of Maxwell's Law (Appendix B) which may be stated as follows:

The distortion (either angular or linear and measured in any given direction) of any point $a$ in a beam or frame due to a unit load (either angular or linear and in any direction) at any point $b$ is equal to the
distortion at the second point $b$ due to a unit load at the first point $a$, provided that at each point the loads and distortions are always measured in the same directions.

The application of this principle to a given structure is illustrated from a consideration of Fig. 1. Fig. 1a represents the center line of the model before any deformation is introduced, the loading is as shown. The reaction components--vertical, horizontal, and rotative--for the reaction at $a$, due to the load $P$ at point $b$, are to be found by the mechanical analysis.

To find the vertical component of the reaction, a small vertical deflection, $d_1$, is produced at $a$ without rotation or horizontal displacement as shown in Fig. 1b. The corresponding displacement, $d_2$, of the load point $b$ in the direction of the load is read with a microscope. The reaction is found directly from the comparison and is found to be

$$V_a = P \frac{d_2}{d_1}.$$

For the horizontal reaction at $a$ a similar procedure is followed, except that a horizontal distortion is applied at $a$, as shown in Fig. 1c. The movement at the load point in the direction of the load is again observed, and the horizontal reaction is given by

$$H_a = P \frac{d_2}{d_1}.$$

To determine the moment component of the reaction,
Fig. 1. Deflections Produced by Deformers
the support a is rotated through a small angle, defined by the value of \( d_1 \) acting at a radius of unity, as shown in Fig. 1d. The corresponding deflection, \( d_2 \), at point b in the direction of the loading is measured. The moment is given by

\[
M_a = P \frac{d_2}{d_1}
\]

In this equation \( M_a \) is the bending moment that would be produced in a structure of the scale of the model. To convert this moment to the value for the full-scale structure, it is necessary to make allowance for the scale of the model as will be shown shortly.

**Application of the Beggs Deformeter**

The function of the Beggs Deformeter, Fig. 2, is to introduce at the support, or any section of the model, a given deformation. This deformation may be a pure thrust, shear, or moment.

The gauge clamps of Fig. 2a are for the purpose of attaching the model to the table, and for producing the known distortion in the model. The deformeter consists of a stationary member, A, screwed to the table, and a movable member, B, held to A by tension springs, but separated by the plugs. This is for the case of reactions. If internal stresses are to be found the deformeter is mounted on a frictionless bearing, and both gauge clamps are attached to the member at the section where the stresses are desired.

The deformations of the model are produced by the
Fig. 2. Begg's Deformeter
distortions plugs. When the deformeter is attached to the model normal (amber) plugs are used. The distortion plugs vary from the normal plugs, and from each other by a known amount, and produce distortions in the direction of the three elements of stress—shear, thrust, and rotative. The removal of the normal plugs and the insertion of the shear plugs, flattened on a face as shown in Fig. 2b, produces a horizontal motion, $A_{\text{xx}}$, without any vertical or angular motion. Similarly the successive insertion of the small (white) and large (red) plugs of Fig. 2c produces a vertical motion, $A_{\text{yy}}$, but does not allow vertical or horizontal motion. In this manner known and controlled distortions of the three stress elements are produced in the structure.

The deformation of the load point in the direction of the loading is measured with a microscope. The direction, sense, or sign of any stress component may be noted by observation of the deformation of the model. The microscopes employed in measuring deflections are optically inverting, so that the image in the microscope moves in an opposite direction from the observed load point. Accordingly, the following practical general rule for determining the sense of any reaction or stress component has been formulated: If the image of the load point in the microscope moves in the direction of the assumed load, the reaction component acts in the same direction as the corresponding displacement of the support.
For internal reactions the model is cut at the section where the thrust, shear, and moment stress components of the structure are desired. The gauge bars are clamped to the model on either side of the cut and mounted upon a frictionless bearing of the type in Fig. 3 which allows free distortion. The distortion plugs are used in the same manner as in the case of external reaction components. The frictionless bearing is composed of two thin sheets of celluloid or glass with steel bearing balls between them.

**Calibration of the Distortion Plugs**

Since the principle of the Beggs method is that of expressing stress as the ratio $\Delta L/\Delta R$ between two deflections, it is immaterial in which units the deflections are measured so long as the same scale is used for both. For this reason it is most convenient to measure the movements in microscope units. To calibrate the thrust and shear distortion plugs the microscope is oriented to
read directly the distortion of the movable clamp. From these movements the shear and thrust distortions plugs are calibrated directly. The calibration can be checked by direct calculation from the geometry of the gauges, and the size of the distortion plugs. The size of the plugs is converted to microscope units for this calculation. Fig. 5b shows that the difference in the diameter of the plugs, \( \Delta \), acts at forty five degs to the direction of the movement. Thus the movement is \( \Delta / \cos 45 \text{ degress} \) or 1.414\( \Delta \).

The moment plugs cannot be calibrated directly. To calibrate the moment plugs, attach a rigid arm, which has several targets a given distance apart, to the movable clamp as shown in Fig. 4. Orient a microscope to read the several values of \( \Delta \) as the rotation is produced by the movable clamp. The purpose of using several targets is to obtain a good value of the rotation. From these readings the calibration constant of the moment plugs is

\[
\theta = \frac{\Delta_B - \Delta_A}{d}.
\]

If \( m \) is the number of microscope units per inch the true angle is

\[
\Delta_{zz} \text{ in radians} = \frac{\Delta_B - \Delta_A}{md} \quad \text{or} \quad m \Delta_{zz} = \theta.
\]

The use of the calibration constant in practice involves the scale of the model as shown by the following proof:

To find the moment reaction at any point \( a \) in foot
Fig. 4 Determination of Moment Constant $\theta$ for Begg's Apparatus

Fig. 5 Calculation of $\theta$
pounds due to a load of one pound at any other point \( b \), it will be necessary to find for the structure the relation

\[
Z_{ab} (\text{ft. lbs.}) = \frac{\Delta_{zb} (\text{ft.})}{\Delta_{zz} (\text{radians})}.
\]

This equation applies to the actual structure, and to apply this to the model use primes to denote model deflections, and if the structure is \( n \) times as large as the model

\[
\Delta_{zb} (\text{ft.}) = n \left[ \frac{\Delta_{zb}' (\text{micro-units})}{m} \right] \frac{1}{12}.
\]

The angular distortion at \( a \) for the structure will be the same as that for the model and, from previous equations,

\[
\Delta_{zz} (\text{radians}) = \frac{\theta}{m}.
\]

Substituting,

\[
\Delta_{ba} = \frac{n}{12} \Delta_{zb} (\text{micro-units}).
\]

Since \( n \) is the direct size ratio of structure to model in the same units, it will equal the inches of structure per inch of model, or the scale of the model in feet per inch is given by \( n/12 \). Therefore

\[
\begin{bmatrix}
\text{Ft. lbs. reaction in structure} \\
\text{Movement of load point on model in micro-units}
\end{bmatrix} = \begin{bmatrix}
\text{Scale of model} \\
\text{Calibration constant: } \theta
\end{bmatrix} \begin{bmatrix}
\text{Movement of load point in ft. per in. on model in micro-units}
\end{bmatrix}
\]
The Model

The rotation of all the elements of a member takes place about the center of gravity of the active material of the transverse section. Since the bending moments and deflections are functions of the coordinates, $x$, $y$, of the centers of rotation of the elements in flexure, the neutral axis of the model must be a scale representation of that of the structure. Therefore, the first step in making the model is to lay out the neutral axis to some convenient scale. For concrete sections the entire section is considered active.

Since the model represents the elastic properties of the structure, the transverse dimensions at any point on the model are not that of the structure to scale, but depend upon the moment of inertia of the member. The model is of a constant thickness, so the moment of inertia is proportional to the cube of the depth. Therefore, the transverse dimension is

$$\text{Depth} = K \frac{3}{\sqrt[3]{I}}$$

in which $K$ is the scale factor. This factor should be selected so the model will be flexible.

The transverse dimensions of the members are limited by the size of the model clamp and by buckling on narrow members. The maximum size member that will fit in the clamp is three-quarters of an inch. Members under one-quarter of an inch will be subject to buckling.

Celluloid is the material chiefly used in the construc-
tion of the model. The essential characteristic of the material is that it will have a constant instantaneous modulus of elasticity ($E$) within the small range of stress produced by the deformations. Celluloid models are subject to creep, but since the deformeter introduces a given distortion the creep will have no effect upon the elastic curve. The effect of creep is to lower $E$, but the rate of creep in the model will be uniform and $E$ will change uniformly.

Having determined the size of the members, the next problem is to cut them to the proper transverse dimensions. Precision is very necessary. For a one-half inch transverse dimension an error of plus or minus one-hundredth inch would produce a variation of about twelve per cent in the moment of inertia of the member. The easiest way to cut the member is with the use of a circular table saw. It is a simple matter to get the edges parallel, but the difficulty lies in obtaining the proper dimension. For this reason it is best to use the saw to cut the depth closely to that desired, and make the exact dimension by hand work. This is done with a sharp plane or a file. A vice should be used that will not allow the member to bend in the plane of the member, for if bending occurs the edges will not be parallel after cutting. A micrometer is used to measure the dimension and to obtain precision. By careful work the dimension may be cut very accurately. When the transverse dimensions are correct, cut the members to the proper lengths. This cut must be made carefully. All the particles on the end
of the member should be in a plane.

For fabricating the structure, acetone is used. The acetone will weld better if small particles of the celluloid are dropped in the solution and allowed to dissolve. Before welding the joints a full-scale drawing of the model should be made. Attach the drawing to a drawing board and place the members in their proper positions. To prevent transverse movement of the members place thumb tacks along the edges. Now, at the joint to be welded, place several drops of acetone upon the surfaces to be welded. Bring these surfaces carefully together and place weights upon the celluloid in order that no member may move. Perform the same operation upon the other joints, and allow the model to dry for one day. In this manner the entire model may be fabricated in a short time. Small pieces of paper should be placed under any joint that is welded since the acetone will adhere to the drawing as well as to the celluloid.

To find internal stresses the member must be cut. Upon completing the analysis at this section obtain a piece of the celluloid of the same width as the cut. Then place several drops of acetone upon all surfaces to be welded, put the piece of celluloid in the cut, and allow sufficient drying time.

In order to allow correct distortion place the model upon steel bearing balls on a horizontal surface. At the points the balls are used place weights upon the model to prevent buckling. The bearings should be used every
three or four inches for a very flexible model.

Applying the Deformeter to the Model

For correct application of the Beggs Deformeter, but one distortion, corresponding to one of the elements of stress, should be produced by a given set of plugs. From this it follows that the deformeter must be attached to the model normal to the member, and that the center line of the member must be located directly on the center line of the movable clamp. When the gauge is attached to the table some inclination will probably exist due to the tendency of the screws to run in the grain of the wood, and other unavoidable causes. When the clamp is at an inclination to the member shear and thrust distortions will be introduced simultaneously. This inclination must be determined.

To find the inclination insert the thrust plugs in the deformeter and measure the horizontal deflection, $\Delta_{yx}$, due to the inclination of the gauge as shown in Fig. 6.

![Fig. 6.](image)

The thrust distortion, $\Delta_{yy}$, is a known quantity. Similarly insert the shear plugs, $\Delta_{xx}$, and measure the vertical
deflection, $\Delta_{xy}$. For any load the correction to apply for thrust will be $\frac{A_{yx}}{A_{xx}}$ times the movement of the load point due to shear distortion, and the correction for shear will be $\frac{A_{xy}}{A_{yy}}$ times the movement of the load point due to thrust distortion. The sign of the correction is determined by applying the general rule for the sign, the correction being a negative factor.

No correction need be applied to the moment component so long as the model is located accurately in the center of the clamp since the movement is purely rotative, and neither vertical nor horizontal movement occurs. Large errors will occur if the member is not in the center of the clamp since comparatively large vertical deflections will be introduced.

The magnitude of the error will depend upon the magnitude of the stress element. That is, if the thrust is large a vertical movement when the shear distortion is applied will cause considerable error if the deformeter is not normal to the member.

**Errors of Deformeter Technique**

Even though the deformeter is correctly applied to the model other errors are liable to appear. These are the errors due to weakness of the tension springs of the deformeter, and slippage between the model and the clamps.

If the model is rigid the tension springs will not have sufficient strength to overcome the resistance to rotation necessary for moment distortion. Some external force must be applied to the gauges to produce the necessary
distortion. A satisfactory way of overcoming the resistance to rotation is to apply two small clamps and draw the gauge bars together. This operation must be performed carefully in order to produce the proper distortion.

When the resistance to distortion is large the frictional resistance between the model and the gauge might not be sufficient to prevent slippage. The member must be fastened rigidly to the deformeter.

Producing deformation in a given member causes distortions in the other members of the model. In many tests more than one set of deformeters is used on the model at the same time. If the deformations produced by a given deformeter cause slippage to occur in any other deformeter the elastic curve of the model will be affected. Slippage in any other deformeter occurs for the same reasons as given above; namely, that the model is not tightly attached to the deformeter, and that the tension springs do not have sufficient resistance.

Some error will occur due to the manner in which the deformeter grips the model for the case of internal stresses. In theory deformations are applied at the point the model is cut, the whole being free to deform. Actually a certain length is required to grip the model, and within this length the member is prevented from deforming.

Microscope and Readings

Improper alignment of the microscope will cause the same error as inclination of the deformeter to the member. The magnitude of this error depends upon the inclina-
tion of the microscope to the correct direction. By careful orientation of the microscope this will be corrected. Care should also be taken to have the target fall in the field of the microscope in order to prevent parallax.

Targets are made by ink spots on a white background. The spot used should not be the large ink dot, but rather a very small speck. Care must be taken not to lose the target, and it should be located by reference to a larger ink spot. Lighting should be produced by a small light attached to the microscope and shining directly on the target. Use a six or eight volt transformer off a regular one hundred and ten circuit. A large light on the model will produce temperature effects.

**Summary**

In the discussion many possible errors have been pointed out. From this one can understand the necessity of proper technique in the operation of the Beggs Defomerter. Careful application of the defomerter will give satisfactory results.
ANALYSIS BY THE BEGGS DEFORMETER

Tests by the Beggs Deformeter were made on the four story building frame of Fig. 7. The dimensions of the model are given in the accompanying table. The material used in the construction of the model was celluloid of 0.0645 inches thickness. The depth of the members was made proportional to the cube root of the moment of inertia of the full-scale sections. The neutral axis line diagram of the model was at a scale of one inch equaled two feet of the structure. Since the analysis was not for an existing structure, the transverse dimensions obtained by cutting the celluloid with a circular table saw, as previously described, were used. In order to obtain a simplified analytical solution, and to make available checks on the mechanical analysis, the model was made symmetrical. Acetone was used in the fabrication.

Calibration of the Deformeter

The calibration of the shear and thrust distortion plugs was obtained by direct observations of the movement of the movable clamp of the deformeter. The microscope used was a Spencer Microscope, number 202345. Between the dark cross-hairs of the microscope four hundred units on the vernier were measured. A single unit of the vernier equaled 0.0000375 inches.

To find the moment constant, θ, a rigid arm, with targets at one-half inch spacing, was attached to the movable gauge as shown in Fig. 4. The fixed gauge was
Fig. 7. Building Frame
## PROPERTIES OF THE MEMBERS OF FRAME

<table>
<thead>
<tr>
<th>Bar</th>
<th>Width In.</th>
<th>Relative $I$ $\text{d}^3$</th>
<th>Relative $I$ $\frac{I}{I_{\text{min.}}}$</th>
<th>Length Feet</th>
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fastened to the table. The deflections of the targets, A, produced by inserting the moment plugs were measured for the four points on the rigid arm. From the measured deflections the value of $\theta$ was calculated in microscope units per inch.

To obtain a check upon the calibration of the distortion plugs the sizes of the plugs were measured with a micrometer, and from the geometry of the deformeter the calibration constants were calculated. The values accepted as the more accurate were the measured distortions since the calculated values could not be obtained with the refinement of the microscope.

The complete set of microscope readings and the calculations of the constants are given in Appendix C. The following are the calibration constants of the Beggs Deformeter for the microscope used:

- Movement of Shear Plugs = 1360 micro units
- Movement of Thrust Plugs = 1376 " "
- Moment Constant $\theta$ = 1849 " /in.

**Analysis for Reactions**

The first tests made on the model were for the reactions. At the points A, B, C, and D of the model deformeters were mounted. The end of each of the members was fastened to the movable gauge bar, and the other gauge bar was fastened to the table. Inserting the distortion plugs in the deformeter, the shear, thrust, and rotative reaction components were introduced at the reactions. The analysis was made for a one lb. wind load on the roof.
acting normal to the columns. By orienting the microscope at the load point in the direction of the load the deformations of the load point, produced by the distortion plugs at the reactions, were measured. Each reaction component was calculated from one of the following equations:

\[ V = \frac{d_2}{d_1} \text{ lbs.} \]

\[ H = \frac{d_2}{d_1} \text{ lbs.} \]

\[ M = n\frac{d_2}{d_1} \text{ ft. lbs.} \]

in which \(d_2\) was the movement of the load point in the direction of the load produced by the distortion plugs corresponding to the given reaction component, \(d_1\) was the equivalent distortion at the reaction, \(n\) the scale of the model in feet per inch, and \(\theta\) the moment constant.

The symmetry of the structure was used in the analysis. Readings were taken for the load acting at both points R and V. By theory, neglecting the direct stress in the members, for the load acting at either side of the structure the two outside columns and the two interior columns will have reaction components of equal magnitude.

The results of the horizontal, vertical, and moment components of the reactions are shown in Fig. 8. The mechanical solution is given in Appendix C, and the analytical solution is given in Appendix A.
**CONDITIONS OF LOADING**

1. Wind load at V by Beggs
2. Wind load at R by Beggs
3. Theoretical values for wind load at V

**Fig. 8 RESULTS OF ANALYSIS FOR REACTIONS**
The two solutions by the Beggs method check closely. The largest percentage differences occur for the vertical reactions of the interior columns. This is due to the small magnitude of these reactions. The most marked difference between the mechanical and the analytical solutions is also in the vertical reactions of the interior columns. In this case the reaction by the two solutions is of opposite sign. The absolute value of the error is small since the center columns have a very small direct stress.

Probable errors may be contributed to neglecting the direct stress factor in the theoretical solution, application of the deformeter, incorrect readings of the microscope, or some error in construction of the model.

For the interior vertical reactions the error in sign must be attributed either to the model or the direct stress in the members. Special care was taken in microscope observation for these reactions, and also in the application of the deformeter. Since the value of the component was small the error of direction is not of great importance. All other results check the theoretical solution within reasonable limits. The check of statical equilibrium for the mechanical analysis indicates that the solution was correct for the model. The largest error of the equilibrium check was three percent.

**Interior Stresses by Beggs Deformeter**

Upon completion of the analysis for the reactions
the model was remounted with the reactions fixed firmly to the table, and the deformeter was applied at the center of the girders of the outside bay. Only one deformeter was used. This deformeter was applied at the center of the girder, and mounted upon a frictionless bearing. In the application of the deformeter the member was cut and the gauge bars were attached to the cut ends. The stresses at the several sections were found for the same loading as previously considered, again using the symmetry of the structure as a check.

Since the model was cut for each girder it was necessary to weld the member upon completion of the analysis. This was a distinct disadvantage and caused considerable delay because of the time necessary for drying.

Calculations of the stress components from the observed deflections of the load points are given in Appendix C, and the results are shown in Figs. 9 and 10. The results for the internal stresses were not as satisfactory as for the reactions. Using the equations of statical equilibrium to find the reaction components at D did not give results that checked the previous analysis. The value of the moment component contained a very large error. However, this may be attributed to the length of the lever arms to the upper girders. An error of one-hundredth lb. in the direct stress of the top girder would result in about a fifty per cent error in the moment at D. For this reason it would be impractical to use the laws of statics to solve for the moment. Although
Fig. 9. Results of Begg's Analysis

UNITS
Shear and thrust - lbs
Moment - ft. lbs.

Analytical Results in Brackets

T

(O. 7674)
0.76

(1.446)
1.272

V

1 lb

(0.1251)
0.1072

O

(O.0854)
0.102

(2.5072)
2.2957

P

0.245
0.218

K

(O.015)
0.005

(1.8344)
1.5495

L

0.1981
0.1757

G

(O.0165)
0.004

(1.4394)
1.2598

H

0.1652
0.1425

D

0.129

0.6434
\[ T \]  \[ \begin{array}{c} 0.2326 \\ 0.225 \end{array} \rightarrow \begin{array}{c} 0.198 \\ (0.1251) \\ 1.248 \end{array} \]

\[ O \]  \[ \begin{array}{c} 0.0854 \\ 0.0969 \end{array} \rightarrow \begin{array}{c} 0.1135 \\ (0.245) \\ 2.248 \end{array} \]

\[ K \]  \[ \begin{array}{c} 0.015 \end{array} \rightarrow \begin{array}{c} 0.204 \\ (0.1981) \\ 1.496 \end{array} \]

\[ G \]  \[ \begin{array}{c} 0.0165 \end{array} \rightarrow \begin{array}{c} 0.172 \\ (0.1605) \\ 1.322 \end{array} \]

\[ V \]  \[ \begin{array}{c} 0 \end{array} \rightarrow \begin{array}{c} 1 \text{ lb. wind load} \\ \text{at } R \end{array} \]

\[ P \]  \[ \begin{array}{c} 0 \end{array} \rightarrow \begin{array}{c} 2.5072 \\ 2248 \end{array} \]

\[ \text{UNITS} \]

\[ \text{Shear and thrust - lbs.} \]

\[ \text{Moments - ft. lbs.} \]

\[ L \]  \[ \begin{array}{c} 0 \end{array} \rightarrow \begin{array}{c} 1.322 \\ 1.322 \end{array} \]

\[ H \]  \[ \begin{array}{c} 0 \end{array} \rightarrow \begin{array}{c} 0.172 \\ 0.1494 \end{array} \]

\[ D \]  \[ \begin{array}{c} 0.6377 \end{array} \rightarrow \begin{array}{c} 0.1281 \end{array} \]

\[ \text{Fig. 10. RESULTS OF BEGGS ANALYSIS} \]
the bearing upon which the gauges were mounted allowed free movement, the microscope readings were not consistent enough to insure correct results. This condition was especially true for the top girder where the model was not sufficiently rigid. The Beggs Deformeter did not prove satisfactory for determining the internal stresses for the model.
ANALYSIS BY THE M. I. T. MOMENT INDICATOR

The theory of the moment indicator is explained in Appendix D.

Since celluloid creeps under a given load, the deformations produced by a load would not occur instantaneously, but considerable time would have to pass before the model would stop creeping. The strains due to the creep are proportional to the fiber stresses acting, and the effect of the creep corresponds to a lowering of the modulus of elasticity (E). Although E of the celluloid changes with time, the instantaneous value is constant. As a result if a given deformation is applied to the model the elastic curve will remain the same as for the condition of a constant E. For this reason in the application of the moment indicator a given deformation of the load point in the direction of the load was applied.

Analysis for Column Moments

In the first tests on the model by the moment indicator (Fig. 11) the instrument was applied to the columns of the second story across the entire frame. The equation for the bending moment at any section,
\[
M = \frac{6EI\phi}{L^2},
\]
is a function of the moment of inertia, \(E\) of the member, and the span of the moment indicator. By the use of the indicator the relative values of the moment for two sections of the member may be found. From statics, the shear at the story times the distance between the sections at which the relative moments were found equals the sum of the moments at the given sections. Therefore, to solve the moments at the given sections the indicator was attached to each column symmetrically and the relative moments at the sections were found. The sum of these moments was equal to the shear in the story (one lb.) times the span of the indicator expressed in the scale of the structure. Since the relative values of the moments and the sum of the moments were known, the moments were calculated directly.

The indicator was attached to the model by two pins on a transverse section. This caused some of the deformation measured at points \(a\) and \(b\) of Fig. 11 to be shear distortion. Correction for the shear distortion is given by the expression

\[
\Delta = 0.0636d^3(\delta_a + \delta_b)
\]
in which \(d\) is the depth of the section and \(\Delta\) is the correction for shear distortion which is to be subtracted from each reading. The proof of this is given in Appendix D.

From the moments at the measured sections the end moments of the columns were calculated as shown in Appendix D.
Theoretical Results in Brackets
Moments: Ft. lbs.
Signs: Ordinates plotted on compression edge

Fig. 12 Moments by Moment Indicator
The results are given in Fig. 12.

The greatest discrepancy occurred for the end moments of the exterior columns. The model was not originally constructed for use with the moment indicator, and the outside columns were too narrow for accurate results. Also the material was not sufficiently thick. The interior columns were of greater depth and as a result better results were obtained. Even though the model was not well adapted to the M. I. T. Moment Indicator, the largest error of the end moments was six per cent, considering the theoretical correct.

The same tests as just described were performed with a moment indicator of the type of Fig. 13. This indicator attached to the neutral axis of the member and a correction of shear was unnecessary. The indicator has a magnifying device and the analysis is made for but one section for a given setup. This indicator was constructed for rigid models and was not well adapted to the model of the building frame.

The calculations of the moments from the readings of the indicator are given in Appendix D, and Fig. 14 shows the results obtained. Again the maximum error was of
Theoretical Results in Brackets
Moments: Ft lbs
Signs: Ordinates plotted on compression edge

Fig. 14  Moments by Moment Indicator
six per cent in magnitude.

**M. I. T. Moment Indicator with Spring Balance**

For the moments in the girders there were no equations of statics available to convert the relative moments to absolute values. This was overcome by the use of the spring balance which was developed in the M. I. T. Structural Analysis Laboratory. The theory and calibration of the spring balance is given in Appendix D. The calibration was performed upon a cantilever beam. By the use of the calibration constant developed the moment at any section was measured directly from the distortions measured on the spring balance and the moment indicator. The spring balance was made out of the same material as the model in order to eliminate the creep factor.

The tests made on the model by the moment indicator with the spring balance were for the outside bay, the same as the solution by the Beggs Method. The moment indicator used was the one illustrated by the diagramatic sketch in Fig. 13. Moments at two sections on each girder were found in the analysis, and from these the end moments were calculated as given in Appendix D. The results are shown in Fig. 15. These results are much more satisfactory than those by the Beggs method. The largest discrepancy was in the top girder. The load was applied at the top girder, and the method of application of the load did not allow the frame to distort freely in this region. Also the members were very flexible.
Fig 15. Moments by Moment Indicator

(Theoretical results in brackets)

Moments: Ft lbs.
Signs: Ordinates plotted on compression edge
and the top girder was distorted by the weld from the Beggs analysis. These conditions were a decided handicap in the moment indicator solution.

The values of the end moments agreed very closely with the analytical solution. This close agreement indicates that the Beggs solution was in considerable error.
CONCLUSIONS

Comparison of the Moment Indicator and the Beggs Deformeter

The model used in this study was designed for analysis by the Beggs Deformeter, and the analysis by the moment indicator was made as a supplementary study. The conditions for the moment indicator were not ideal. First, the material of which the model was made was too thin for the indicator; second, the depth of the sections was limited by the limitations of the deformeter and some of the sections were very flexible; and third, the analysis was first completed by the Beggs method which involved cutting the members, this causing discrepancies in the model.

Even with these difficulties the M. I. T. Moment Indicator gave more accurate results for the internal stresses. The Beggs Deformeter did not give satisfaction for these stresses.

The best use of the Beggs method is for external reactions. Using the proper technique of operation the deformeter presents a reliable method to obtain external reactions. All three elements of stress are found from one setup. For internal stresses the three elements of stress are also found from one setup, but the member must be cut and the time for welding makes the method impractical if the time of the analysis is of importance. The deformeter may be applied to a member of any shape and with a variable moment of inertia.
The moment indicator presents a method of analysis that is also simple and direct. For any one setup the moment is the only stress element that may be calculated, but the other elements may be calculated from the moments. Stress analysis of a structure would involve finding the moments at all sections, and it is a simple matter to find the shear and thrust at any section from the moments and the laws of statical equilibrium. The indicator may be applied only to straight sections, and if the moment of inertia varies calculations will be necessary to find the moments.

Results

The mechanical analysis presented results that were in very close agreement with the theoretical slope deflection analysis. The major discrepancies that were present have been accounted for in the body of this paper.

The use of the mechanical analysis is not of advantage for structures that may be easily analyzed by the classical methods of approach, but its use would be warranted in structures difficult of theoretical analysis.
BIBLIOGRAPHY


McCullough, C. B. and Thayer, E. S. Elastic Arch Bridges. John Wiley and Sons, Inc.


Analysis of Frame by Slope Deflection

\[ M_{ab} = 2EK_{ab}(2\theta_a + \theta_b - 3\Psi) \]

\[ \Psi_1 = \text{Sidesway of 1st floor} \]

\[ \Psi_2 = \text{" 2nd "} \]

\[ \Psi_3 = \text{" 3rd "} \]

\[ \Psi_4 = \text{" 4th "} \]

\[ \sum M_{ab} = 0 \]

(E) \[ (2K_{ae} + 2K_{ef} + 2K_{ei})(2\theta_e) + 8K_{ef}(2\theta_e) + 8K_{ei}(2\theta_i) - K_{ae}(6E\psi_1) - K_{ei}(6E\psi_2) = 0 \]

(3.5856)(2\theta_e) + 1.0958(2\theta_e) + 0.258(2\theta_e) - 0.439(6E\psi_1) - 0.258(6E\psi_2) = 0 \quad \text{Eq. 1} \]

(I) \[ (2K_{ie} + 2K_{in} + 2K_{im})(2\theta_i) + K_{in}(2\theta_i) + K_{im}(2\theta_m) - K_{ie}(6E\psi_2) - K_{im}(6E\psi_3) = 0 \]

3.019(2\theta_i) + 0.258(2\theta_e) + 1.0958(2\theta_e) + 0.1557(2\theta_m) - 0.258(6E\psi_2) - 0.1557(6E\psi_3) = 0 \quad \text{Eq. 2} \]

(M) \[ (2K_{mi} + 2K_{mr} + 2K_{mr})(2\theta_m) + K_{mi}(2\theta_i) + K_{mn}(2\theta_n) + K_{mr}(2\theta_r) - K_{mi}(6E\psi_3) - K_{mr}(6E\psi_4) = 0 \]

2.6568(2\theta_m) + 0.1557(2\theta_i) + 1.0958(2\theta_n) + 0.0769(2\theta_r) - 0.1557(6E\psi_3) - 0.0769(6E\psi_4) = 0 \quad \text{Eq. 3} \]

(R) \[ (2K_{rm} + 2K_{rs})(2\theta_i) + K_{rm}(2\theta_i) + K_{rs}(2\theta_s) - K_{rm}(6E\psi_4) = 0 \]

0.6272(2\theta_i) + 0.0769(2\theta_s) + 0.2367(2\theta_s) - 0.0769(6E\psi_4) = 0 \quad \text{Eq. 4} \]

(F) \[ (2K_{fe} + 2K_{fb} + 2K_{fg} + 2K_{fj})(2\theta_f) + K_{fe}(2\theta_e) + K_{fb}(2\theta_g) + K_{fg}(2\theta_f) - K_{fj}(6E\psi_4) = 0 \]

11.6574(2\theta_f) + 1.0958(2\theta_e) + 0.9662(2\theta_g) - 2.128(6E\psi_4) - 0.9662(6E\psi_2) = 0 \quad \text{Eq. 5} \]

(J) \[ (2K_{n} + 2K_{nj} + 2K_{nk} + 2K_{nr})(2\theta_n) + K_{nj}(2\theta_j) + K_{nk}(2\theta_k) + K_{nr}(2\theta_r) - K_{nj}(6E\psi_3) - K_{nr}(6E\psi_4) = 0 \]

8.1544(2\theta_n) + 1.0958(2\theta_j) + 0.9662(2\theta_k) + 0.3715(2\theta_r) - 0.9662(6E\psi_2) - 0.3715(6E\psi_4) = 0 \quad \text{Eq. 6} \]

(N) \[ (2K_{nm} + 2K_{kn} + 2K_{kn} + 2K_{ns})(2\theta_n) + K_{nm}(2\theta_m) + K_{kn}(2\theta_n) + K_{ns}(2\theta_s) - K_{nm}(6E\psi_3) - K_{ns}(6E\psi_4) = 0 \]

6.3758(2\theta_n) + 1.0958(2\theta_m) + 0.3715(2\theta_n) + 0.0769(2\theta_s) - 0.3715(6E\psi_3) - 0.0769(6E\psi_4) = 0 \quad \text{Eq. 7} \]

(S) \[ (2K_{sr} + 2K_{sn} + 2K_{st})(2\theta_s) + K_{sr}(2\theta_r) + K_{sn}(2\theta_n) + K_{st}(2\theta_t) - K_{sn}(6E\psi_4) = 0 \]

1.3373(2\theta_s) + 0.2367(2\theta_r) + 0.0769(2\theta_t) - 0.0769(6E\psi_4) = 0 \quad \text{Eq. 8} \]
Shear Equations

\( P(\text{story height}) + 2\ \text{End Moments} = 0 \)

1st floor
\[ 11 + 2[2E\Theta_{AE}(\Theta_E - 3\psi + 2\Theta_E - 3\psi)] + 2KE_{BF}(3\Theta_F - 6\psi) = 0 \]
\[ 11 + 2.634(2E\Theta_E) - 1.756(6E\psi) + 12.738(2E\Theta_F) - 8.492(6E\psi) = 0 \]  
Eq. 9

2nd floor
\[ 13 + 2[2E\Theta_{EI}(3\Theta_E + 3\Theta_f - 6\psi)] + 2E\Theta_{ES}(3\Theta_F + 3\Theta_f - 6\psi) = 0 \]
\[ 13 + 1.548(2E\Theta_E) + 1.548(2E\Theta_F) + 5.792(2E\Theta_I) + 5.792(2E\Theta_S) - 4.8968(6E\psi) = 0 \]  
Eq. 10

3rd floor
\[ 13 + 2[2E\Theta_{IM}(3\Theta_E + 3\Theta_m - 6\psi)] + 2E\Theta_{IN}(3\Theta_F + 3\Theta_n - 6\psi) = 0 \]
\[ 13 + 0.9342(2E\Theta_E) + 0.9342(2E\Theta_M) + 2.229(2E\Theta_I) + 2.229(2E\Theta_N) - 21088(6E\psi) = 0 \]  
Eq. 11

4th floor
\[ 13 + 2[2E\Theta_{MR}(3\Theta_E + 3\Theta_r - 6\psi)] + 2E\Theta_{NS}(3\Theta_N + 3\Theta_r - 6\psi) = 0 \]
\[ 13 + 0.9228(2E\Theta_E) + 0.9228(2E\Theta_M) + 0.4614(2E\Theta_N) + 0.4614(2E\Theta_S) - 0.6152(6E\psi) = 0 \]  
Eq. 12

Results

\[ 2E\Theta_E = +0.337 \]
\[ 2E\Theta_F = +0.6396 \]
\[ 2E\Theta_I = +0.4685 \]
\[ 2E\Theta_J = +0.737 \]
\[ 2E\Theta_M = +0.858 \]
\[ 2E\Theta_N = +0.572 \]
\[ 2E\Theta_R = +2.586 \]
\[ 2E\Theta_S = +0.937 \]
\[ 6E\psi_1 = +1.955 \]
\[ 6E\psi_2 = +4.539 \]
\[ 6E\psi_3 = +8.136 \]
\[ 6E\psi_4 = +24.826 \]
BENDING MOMENTS IN THE MEMBERS

Clockwise moments are positive

\[ M_A = 0.439 \left( 0.337 - 1.955 \right) = -0.7103 \]
\[ M_B = 2.123 \left( 0.6396 - 1.955 \right) = -2.7926 \]
\[ M_{EA} = 0.439 \left( 0.574 - 1.955 \right) = -0.5623 \]
\[ M_{EF} = 1.0958 \left( 0.674 + 0.6396 \right) = +1.4394 \]
\[ M_{EI} = 0.258 \left( 0.674 + 0.4685 - 4.539 \right) = -0.8763 \]
\[ M_{FE} = 1.0958 \left( 1.2792 + 0.337 \right) = +1.7710 \]
\[ M_{FB} = 2.123 \left( 1.2792 - 1.955 \right) = -1.4347 \]
\[ M_{FG} = 1.0958 \left( 1.2792 + 0.6396 \right) = +2.1026 \]
\[ M_{FJ} = 0.9662 \left( 1.2792 + 0.737 - 4.539 \right) = +2.4375 \]
\[ M_{IE} = 0.258 \left( 0.937 + 0.337 - 4.539 \right) = -0.8424 \]
\[ M_{IJ} = 1.0958 \left( 0.937 + 0.737 \right) = +1.8344 \]
\[ M_{IM} = 0.1557 \left( 0.937 + 0.858 - 8.136 \right) = -0.9873 \]
\[ M_{JI} = 1.0958 \left( 0.4685 + 1.474 \right) = +2.1286 \]
\[ M_{JF} = 0.9662 \left( 1.474 + 0.6396 - 4.539 \right) = -2.3434 \]
\[ M_{JK} = 1.0958 \left( 1.474 + 0.737 \right) = +2.4228 \]
\[ M_{JN} = 0.3715 \left( 1.474 + 0.572 - 8.136 \right) = -2.2624 \]
\[ M_{MI} = 0.1557 \left( 1.716 + 0.4685 - 8.136 \right) = -0.9266 \]
\[ M_{MN} = 1.0958 \left( 1.716 + 0.572 \right) = +2.5072 \]
\[ M_{MR} = 0.0769 \left( 1.716 + 2.586 - 24.826 \right) = -1.5783 \]
\[ M_{NM} = 1.0958 \left( 1.144 + 0.858 \right) = +2.1938 \]
\[ M_{NJ} = 0.3715 \left( 1.144 + 0.737 - 8.136 \right) = -2.3237 \]
\[ M_{NO} = 1.0958 \left( 1.144 + 0.572 \right) = +1.8804 \]
\[ M_{NS} = 0.0769 \left( 1.144 + 0.937 - 24.826 \right) = -1.7491 \]
\[ M_{RM} = 0.0769 \left( 5.172 + 0.858 - 24.826 \right) = -1.4454 \]
\[ M_{RS} = 0.2367 \left( 5.172 + 0.937 \right) = +1.4460 \]
\[ M_{SR} = 0.2367 \left( 1.874 + 2.586 \right) = +1.0557 \]
\[ M_{SN} = 0.0769 \left( 1.874 + 0.572 - 24.826 \right) = -1.7210 \]
\[ M_{ST} = 0.2367 \left( 1.874 + 0.937 \right) = +0.6654 \]

**DIRECT STRESS IN COLUMNS**

- is tension
- is compression

**Member**

\[ \frac{M_{RM} + M_{RS}}{20} = \frac{1.446 + 1.0557}{20} = +0.12509 \]
\[ \frac{M_{NM} + M_{MN}}{20} = 0.12509 + \frac{2.5072 + 2.1938}{20} = +0.3601 \]
\[ \frac{M_{IJ} + M_{JI}}{20} = 0.3601 + \frac{1.8344 + 2.1286}{20} = +0.5582 \]
\[ \frac{M_{EF} + M_{FE}}{20} = 0.5582 + \frac{1.4394 + 1.771}{20} = +0.7187 \]
\[ \frac{M_{ST} + M_{TS} - M_{RS} - M_{SR}}{20} ; \quad \text{By symmetry } M_{ST} - M_{TS} \]
\[ = \frac{M_{SR} - M_{RS}}{20} = 0.06654 - 0.12509 = -0.05855 \]

Similar expressions are obtained for the other interior columns

\[ NJ = -0.05855 + 0.18804 - 0.23505 = -0.10556 \]
\[ JF = -0.10556 + 0.24228 - 0.19815 = -0.06143 \]
\[ FB = -0.06143 + 0.21026 - 0.16052 = -0.01169 \]

**SHEAR IN COLUMNS**

<table>
<thead>
<tr>
<th>Member</th>
<th>( \frac{M_{RM} + M_{MR}}{13} )</th>
<th>( \frac{1.445 + 1.5783}{13} )</th>
<th>=</th>
<th>0.2326</th>
</tr>
</thead>
<tbody>
<tr>
<td>RM</td>
<td>0.9266 + 0.9873</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.1472</td>
</tr>
<tr>
<td>MI</td>
<td>0.8424 + 0.8763</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.1322</td>
</tr>
<tr>
<td>EI</td>
<td>0.5623 + 0.7103</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.1157</td>
</tr>
<tr>
<td>EA</td>
<td>( \frac{1.7210 + 1.7491}{13} )</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.2669</td>
</tr>
<tr>
<td>SN</td>
<td>( \frac{2.3237 + 2.2624}{13} )</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.3528</td>
</tr>
<tr>
<td>NJ</td>
<td>( \frac{2.3434 + 2.4375}{13} )</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.3678</td>
</tr>
<tr>
<td>JF</td>
<td>( \frac{1.4347 + 2.7926}{13} )</td>
<td>( \frac{1.445 + 1.5783}{13} )</td>
<td>=</td>
<td>0.3843</td>
</tr>
</tbody>
</table>
APPENDIX B

MAXWELL'S LAW APPLIED TO THE BEGGS DEFORMETER
MAXWELL'S LAW APPLIED TO THE BEGGS DEFORMETER

If, in the simple beam span (Fig. 16), a unit vertical load be applied at any point \( a \) producing moments \( m_a \) throughout the length of the beam, the vertical displacement of any other point \( b \) is given by the expression

\[ \delta_{ba} = \frac{ds}{m_b m_a E I} \]

where the term \( \delta_{ba} \) is deflection of \( b \) due to a unit load at \( a \).

\( m_a \) - the moment at any section of the beam due solely to the unit load at point \( a \), and

\( m_b \) - the corresponding moment due solely to a unit auxiliary load applied at point \( b \).

In the subscript notation used for the deflection the first subscript defines the point at which the deflection is to be measured, and the second subscript the position of the load.

If, now, the above load be removed and the beam reloaded with a unit vertical load at point \( b \), the vertical displacement of point \( a \) is given by the expression

\[ \delta_{ab} = \frac{ds}{m_a m_b E I} \]
Unit Load

\[ \delta_{ba} = \sum m_b m_a \frac{ds}{EI} \]

Unit Load

\[ \delta_{ab} = \sum m_b m_a \frac{ds}{EI} \]

Unit Moment

\[ \delta_{ba} = \sum (l.0) m_b \frac{ds}{EI} \]

Unit Load

\[ \theta_{ab} = \sum m_b (l.0) \frac{ds}{EI} \]

Fig. 46.
The second term of the first equation is clearly identical with the second term of the second equation, and therefore

\[ \delta_{ba} = \delta_{ab}. \]

This expression is Maxwell's Law of Reciprocal Displacements. Expressed in words this equation becomes:

The vertical displacement of any point \( a \) in a beam or frame due to a unit vertical load at any other point \( b \) is equal to the corresponding vertical displacement of the second point \( b \) for a unit vertical load applied at the first point \( a \).

If a reaction existed at \( a \) the deflection of \( a \) due to a unit load at \( b \) would be 0. This can be expressed by

\[ \delta_{ab} = R_a \delta_{aa} = 0 \]

from which

\[ R_a = \frac{\delta_{ab}}{\delta_{aa}}. \]

By Maxwell's Law as proven above

\[ \delta_{ab} = \delta_{ba}. \]

Then

\[ R_a = \frac{\delta_{ba}}{\delta_{aa}}. \]

The Beggs Deformeter introduces a fixed distortion at \( a \) by an unknown force \( Q \). This creates a movement at \( a \) equal to \( Q\delta_{aa} \) and a movement at \( b \) equal to \( Q\delta_{ba} \).
However, the value of \( Q \) is a constant for any instant, and the value of \( R_a \) depends only upon the ratio of the deflections, and the ratio remains unchanged.

The principle also applies to rotative reactions as well as linear. From Fig. 16,

\[
\sigma_{ab} = (1.0) \frac{m_b}{EI} = \sigma_{ba},
\]

or expressed in words this becomes:

The angular displacement of any point \( a \) in a beam or frame due to a unit vertical load at any other point \( b \) is equal to the vertical displacement of the second point \( b \) due to a unit moment couple applied at the first point \( a \).

In this case the distortions introduced by the Beggs Deformeter are also produced by a force \( Q \), but this expression appears in all deflection terms. Again, by the use of Maxwell's Law, the moment is found from the distortion, and it is equal to

\[
m_a = \frac{\sigma_{ba}}{\sigma_{aa}}
\]

in which \( \sigma_{aa} \) is the angular distortion at \( a \), and \( \sigma_{ba} \) is the deflection of point \( b \) due to the angular distortion at \( a \).
APPENDIX C

BEGGS DEFORMETER ANALYSIS
Calibration of Begg's Apparatus

### THRUST

<table>
<thead>
<tr>
<th>Clamp Reaction</th>
<th>Readings</th>
<th>clamp readings</th>
<th>Reaction</th>
<th>White</th>
<th>Red</th>
<th>White</th>
<th>Red</th>
<th>White</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
</tr>
<tr>
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<td></td>
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<td></td>
<td></td>
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<td></td>
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### SHEAR

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<th>Right</th>
<th>Left</th>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td>8.462</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>White</td>
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<td></td>
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</tbody>
</table>

**Moment Constant, \( \Theta \)**

<table>
<thead>
<tr>
<th>Clamp Reaction</th>
<th>Movement of pt.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>W-R</td>
<td></td>
<td>8.463</td>
<td>15.190</td>
<td>13.400</td>
<td>7.045</td>
</tr>
<tr>
<td>R-W</td>
<td></td>
<td>11.273</td>
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<td>6.244</td>
<td>16.243</td>
</tr>
<tr>
<td>W-R</td>
<td></td>
<td>8.462</td>
<td>15.194</td>
<td>13.393</td>
<td>7.045</td>
</tr>
<tr>
<td>Average Difference</td>
<td></td>
<td>1017</td>
<td>1950</td>
<td>2952</td>
<td>3798</td>
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</tbody>
</table>

The method of calibration for \( \Theta \) is shown in Fig. 4.

\[ \Theta = 1849 \text{ microscope units/inch} \]
Calibration of Begg's Apparatus

Size of Plugs: White = 0.3069 in.
Red = 0.3428 in.
Shear = 0.3428 in.
0.3069 in.

Unit of Microscope = 0.0000375 in.
Scale of Model: 1 in. = 2'0"

Calculation of Moment Constant $\theta$

See Fig. 5.

\[ \Delta = 0.3428 - 0.3069 = 0.0359 \text{ in.} \]

\[ 1.414 \Delta = 0.05076 \text{ in.} \]

\[ \theta = \frac{0.05076}{0.75} = 0.06768 \]

\[ \theta = \frac{0.06768}{0.0000375} = 1805 \text{ mic. units/in.} \]

Calculation of Movement of Shear and Thrust Plugs

The shear movement equals the thrust movement

Movement = $1.414 \Delta = 0.05076 \text{ in.}$

\[ " \frac{0.05076}{0.0000375} = 1354 \text{ microscope units} \]

<table>
<thead>
<tr>
<th></th>
<th>Calculated</th>
<th>Measured</th>
</tr>
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<tbody>
<tr>
<td>MOVEMENT OF SHEAR PLUGS</td>
<td>1354 \text{ micro units}</td>
<td>1360</td>
</tr>
<tr>
<td>MOVEMENT OF THRUST PLUGS</td>
<td>1354 \text{ &quot; &quot;}</td>
<td>1376</td>
</tr>
<tr>
<td>MOMENT CONSTANT $\theta$</td>
<td>1805 \text{ &quot; \text{/in.} }</td>
<td>1849</td>
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</tbody>
</table>
# REACTIONS BY BEGG'S METHOD

**Unit Wind Load at Pt. V**

<table>
<thead>
<tr>
<th>Clamp Reaction</th>
<th>Reaction</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
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<td>11.453</td>
<td>11.166</td>
<td>11.166</td>
<td>10.096</td>
</tr>
<tr>
<td>W</td>
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<td>9.257</td>
<td>11.147</td>
<td>11.212</td>
<td>12.246</td>
</tr>
<tr>
<td>R</td>
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<td>11.446</td>
<td>11.169</td>
<td>11.164</td>
<td>10.086</td>
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<tr>
<td>Average Difference</td>
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<td>17</td>
<td>49</td>
<td>957</td>
</tr>
<tr>
<td>Correction</td>
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<td>5</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Corrected Difference</td>
<td></td>
<td>990</td>
<td>22</td>
<td>47</td>
<td>957</td>
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<tr>
<td>Reaction</td>
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<td>0.016</td>
<td>0.0342</td>
<td>0.696</td>
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<td></td>
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<td>-0.01169</td>
<td>-0.01169</td>
<td>0.7187</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Horiz. Reaction</th>
<th>Reaction</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
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</thead>
<tbody>
<tr>
<td>Left</td>
<td></td>
<td>10.917</td>
<td>10.275</td>
<td>10.198</td>
<td>11.191</td>
</tr>
<tr>
<td>Right</td>
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<td>10.359</td>
<td>11.301</td>
<td>11.327</td>
<td>11.345</td>
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<tr>
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<td>10.278</td>
<td>10.200</td>
<td>11.197</td>
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<tr>
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<td>11.350</td>
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<tr>
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<td>10.205</td>
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<td>11.203</td>
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<td>524</td>
<td>150</td>
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<tr>
<td>Correction</td>
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<td>0</td>
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</tr>
<tr>
<td>Reaction</td>
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<td>0.1102</td>
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<td>0.1157</td>
<td>0.3843</td>
<td>0.3843</td>
<td>0.1157</td>
</tr>
</tbody>
</table>
## Unit Wind Load at Pt. V

<table>
<thead>
<tr>
<th>Clamp Reaction</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>W-R</td>
<td>9.259</td>
<td>7.348</td>
<td>7.370</td>
<td>10.311</td>
</tr>
</tbody>
</table>

| Average Difference | 763  | 2664 | 2620 | 727  |

| Moment )          | 0.825 | 2.885 | 2.835 | 0.781 |

| Analytical        | 0.7103 | 2.7926 | 2.7926 | 0.7103 |

### Check by Laws of Statics

\[
\begin{align*}
\sum V &= 0.735 = 0.7232 \\
\sum H &= 0.9929 = 1.0 \\
\sum M_A &= 49.774 = 50.0 \\
\sum M_D &= 50.262 = 50.0
\end{align*}
\]
### Unit Wind Load at Pt. R

<table>
<thead>
<tr>
<th>Clamp Reaction</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
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<td>10.217</td>
<td>11.197</td>
<td>12.230</td>
</tr>
<tr>
<td>R</td>
<td>9.075</td>
<td>10.211</td>
<td>11.245</td>
<td>14.374</td>
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<tr>
<td>W</td>
<td>11.267</td>
<td>10.219</td>
<td>11.195</td>
<td>12.236</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>992</td>
<td>8</td>
<td>47</td>
<td>940</td>
</tr>
<tr>
<td><strong>Difference</strong></td>
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<td>5</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td><strong>Corrected</strong></td>
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<td>45</td>
<td>940</td>
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<td>-0.01169</td>
<td>-0.01169</td>
<td>0.7187</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Horiz. Reaction</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
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<td>10.080</td>
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<td>13.145</td>
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<td>11.191</td>
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<td>10.086</td>
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<td>135</td>
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<td>0</td>
<td>0</td>
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<tr>
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Unit Wind Load at Pt. R

<table>
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<tr>
<th>Clamp</th>
<th>Reaction</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-W</td>
<td>9.220</td>
<td>7.251</td>
<td>10.239</td>
<td>12.229</td>
<td></td>
</tr>
<tr>
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<td>2661</td>
<td>727</td>
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<td>2.7926</td>
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</table>

Check by Laws of Statics

\[ \Sigma V = 0.729 - 0.714 \]

\[ \Sigma H = 0.969 = 1.0 \]

\[ \Sigma M_A = 49.27 = 50.0 \]

\[ \Sigma M_D = 50.35 - 50.0 \]
## Unit Wind Load at Pt. V

<table>
<thead>
<tr>
<th>Clamp Reaction</th>
<th>Reaction Center of Member</th>
<th>TV</th>
<th>DP</th>
<th>KL</th>
<th>GH</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
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<td>8.224</td>
<td>10.280</td>
<td>7.200</td>
<td>5.279</td>
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<td>W</td>
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<td>0.102</td>
<td>0.005</td>
<td>0.0004</td>
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### Thrust

<table>
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<th>3</th>
</tr>
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<tr>
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</tr>
<tr>
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<td>6.364</td>
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<td>6.366</td>
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<td>7.206</td>
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<td>297</td>
<td>239</td>
</tr>
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<td>0.1757</td>
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### Shear

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<th>5</th>
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</thead>
<tbody>
<tr>
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<td>7.097</td>
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<td>7.096</td>
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<td>R-W</td>
<td>11.250</td>
<td>10.210</td>
<td>7.193</td>
</tr>
<tr>
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<td>192</td>
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<td>+0.1157</td>
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### Moment

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<td>Red-White</td>
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<td>+0.1157</td>
<td>-0.2075</td>
</tr>
</tbody>
</table>

See Fig. 9.
<table>
<thead>
<tr>
<th>Reaction Center of Member</th>
<th>Clamp Reaction</th>
<th>TV</th>
<th>OF</th>
<th>KL</th>
<th>GH</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>W</td>
<td>10.226</td>
<td>6.257</td>
<td>9.211</td>
<td>5.238</td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>10.236</td>
<td>6.253</td>
<td>9.212</td>
<td>5.240</td>
</tr>
<tr>
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<td>310</td>
<td>133</td>
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<td>4</td>
</tr>
<tr>
<td><strong>Thrust</strong></td>
<td></td>
<td>0.225 Comp 0.0969 Ton.</td>
<td>0.0</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

| **Shear**                | Left           | 10.337 | 6.247 | 9.207 | 5.236 |
|                          | Right          | 10.194 | 5.353 | 8.378 | 5.032 |
|                          | L              | 10.342 | 6.238 | 9.211 | 5.233 |
|                          | R              | 10.199 | 5.353 | 8.382 | 5.030 |
|                          | L              | 10.335 | 6.247 | 9.214 | 5.235 |
| **Average Diff.**        |                | 143   | 230  | 231  | 203 |
| **Shear**                |                | 0.105  | 0.2135 | 0.170 | 0.1494 |

| **Moment**               | Red-White      | 12.219 | 5.233 | 9.188 | 5.208 |
|                          | R-W            | 12.410 | 5.388 | 9.001 | 5.048 |
|                          | W-R            | 12.228 | 5.283 | 9.190 | 5.204 |
|                          | R-W            | 12.405 | 5.390 | 9.003 | 5.045 |
|                          | W-R            | 12.227 | 5.285 | 9.194 | 5.206 |
| **Average Diff.**        |                | 183   | 105  | 189  | 159 |
| **Moment**               |                | -0.198 | -0.1135 | +0.204 | +0.172 |
APPENDIX D

ANALYSIS BY M. I. T. MOMENT INDICATOR
THEORY OF M.I.T. MOMENT INDICATOR

The M.I.T. moment indicator is based on the principle of the slope deflection equations, and is explained by reference to Fig. 17. From Fig. 17b

\[ bb' = d + \frac{L}{3} \theta_A + \frac{2L}{3} \theta_B \]

\[ bb' - d = \text{relative moment of points } b \text{ and } b' = b \]

\[ \frac{L}{3} \theta_A + \frac{2L}{3} \theta_B = \frac{L}{3} (2\theta_B + \theta_A) \]

also

\[ aa' = d + \frac{2L}{3} \theta_A + \frac{L}{3} \theta_B \]

\[ aa' - d = \text{relative movement of points } a \text{ and } a' = a \]

By the slope deflection theorem

\[ M_{AB} = \frac{2EI}{L} (2\theta_A + \theta_B) = \frac{6EIa}{L^2} \]

\[ M_{BA} = \frac{2EI}{L} (2\theta_B + \theta_A) = \frac{6EIt}{L^2} \]

By reading points on the deformeter, a and b may be determined, and if E is known the moments may be determined.
Fig. 17a. Indicator before distortion

Fig. 17b. Indicator after distortion
End Moments in the Columns of the Second Floor

The moment indicator used was attached on a transverse axis. This produced shear distortion in the indicator. Refering to Fig. 18, the correction factor is found by the following:

\[
\frac{1}{L} = \frac{4}{96''}
\]

The moments at sections 1 and 2 are given by

\[
M_1 = \frac{6EI\delta_1}{L^2}
\]

and

\[
M_2 = \frac{6EI\delta_2}{L^2}
\]

Then the shear is

\[
S = \frac{M_1 + M_2}{L} = \frac{6EI}{L^2}(\delta_1 + \delta_2).
\]

The deflection due to shear is equal to

\[
\frac{KSL}{A'G}
\]

in which

\[
K = 1.2, \quad A' = A = bd, \quad \text{and} \quad G = 0.4E.
\]

Also

\[
I = \frac{bd^3}{12}
\]
By substitution, the equation for the deflection due to shear becomes

\[ \Delta = \frac{1.5d^2(\delta + \delta_2)}{L^2} \]

The span of the indicator used was 4.86 inches. Then the correction to subtract from the observed readings on the moment indicator is \(0.0636d^2(\delta + \delta_2)\).

### Corrections

<table>
<thead>
<tr>
<th>Member</th>
<th>(d)</th>
<th>(d^2)</th>
<th>(\delta + \delta_2)</th>
<th>Correction (\Delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EI</td>
<td>0.401</td>
<td>0.161</td>
<td>580</td>
<td>6</td>
</tr>
<tr>
<td>FJ</td>
<td>0.623</td>
<td>0.389</td>
<td>446</td>
<td>11</td>
</tr>
<tr>
<td>GK</td>
<td>0.623</td>
<td>0.389</td>
<td>458</td>
<td>11</td>
</tr>
<tr>
<td>HL</td>
<td>0.401</td>
<td>0.161</td>
<td>627</td>
<td>6</td>
</tr>
</tbody>
</table>

The indicator was attached to read two points on the column. These moments are indicated by primes.

<table>
<thead>
<tr>
<th>Mom</th>
<th>(\delta)</th>
<th>Correction</th>
<th>(\delta)</th>
<th>K</th>
<th>Relative Mom</th>
<th>Moment'</th>
</tr>
</thead>
<tbody>
<tr>
<td>EI'</td>
<td>293</td>
<td>6</td>
<td>287</td>
<td>0.258</td>
<td>74.1</td>
<td>1</td>
</tr>
<tr>
<td>IE'</td>
<td>287</td>
<td>6</td>
<td>281</td>
<td></td>
<td>72.5</td>
<td>0.978</td>
</tr>
<tr>
<td>FJ'</td>
<td>236</td>
<td>11</td>
<td>225</td>
<td>0.9662</td>
<td>217.5</td>
<td>2.93</td>
</tr>
<tr>
<td>JF'</td>
<td>210</td>
<td>11</td>
<td>199</td>
<td></td>
<td>192.2</td>
<td>2.595</td>
</tr>
<tr>
<td>GK'</td>
<td>238</td>
<td>11</td>
<td>227</td>
<td></td>
<td>219</td>
<td>2.955</td>
</tr>
<tr>
<td>KG'</td>
<td>220</td>
<td>11</td>
<td>209</td>
<td></td>
<td>202</td>
<td>2.725</td>
</tr>
<tr>
<td>HL'</td>
<td>323</td>
<td>6</td>
<td>317</td>
<td>0.258</td>
<td>81.9</td>
<td>1.103</td>
</tr>
<tr>
<td>LH'</td>
<td>304</td>
<td>6</td>
<td>293</td>
<td></td>
<td>76.9</td>
<td>1.037</td>
</tr>
</tbody>
</table>
In the table $K$ is the rigidity ratio, $I/L$, of the members.

Having the moments at two points on the columns, the end moments were calculated by the use of Fig. 19.

![Fig. 19]

The shear in the column is found, and

$$x = \frac{M}{\text{shear}}.$$  

Then the end moments are found by

$$\text{End } M = \frac{(1.64 + x)(M')}{x} = \left(\frac{1.64}{x} + 1\right)(M')$$

The $M'$ of the equation is the moment found at the section on the beam corresponding to the given end moment.
Check by Moment Indicator without Shear Correction

The end moments of the columns of the second floor were found by the use of the indicator of the type in Fig. 13. This attached on the longitudinal axis of the member. Fig. 20 is used to transverse the measured moments to the end moments.
From Fig. 20, \( M = \frac{(2 + x)(M)}{x} \)

End Moment = \( \left( \frac{2}{x} + 1 \right)(M) \)

<table>
<thead>
<tr>
<th>MOM</th>
<th>( f )</th>
<th>K</th>
<th>Relative Mom.</th>
<th>( M' )</th>
<th>( x )</th>
<th>( \frac{2}{x} + 1 )</th>
<th>Moment lb. ft.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EI</td>
<td>584</td>
<td>0.258</td>
<td>150.7</td>
<td>1.0</td>
<td>0.5669</td>
<td>4.53</td>
<td>1.442</td>
</tr>
<tr>
<td>IE</td>
<td>578</td>
<td>&quot;</td>
<td>149.1</td>
<td>0.989</td>
<td>0.5607</td>
<td>4.47</td>
<td>1.447</td>
</tr>
<tr>
<td>FJ</td>
<td>465</td>
<td>0.9662</td>
<td>449.3</td>
<td>2.981</td>
<td>1.6901</td>
<td>4.63</td>
<td>1.432</td>
</tr>
<tr>
<td>JF</td>
<td>439</td>
<td>&quot;</td>
<td>424.2</td>
<td>2.814</td>
<td>1.5954</td>
<td>4.37</td>
<td>1.457</td>
</tr>
<tr>
<td>GK</td>
<td>476</td>
<td>&quot;</td>
<td>459.9</td>
<td>3.052</td>
<td>1.7304</td>
<td>4.63</td>
<td>1.432</td>
</tr>
<tr>
<td>KG</td>
<td>449</td>
<td>&quot;</td>
<td>433.8</td>
<td>2.878</td>
<td>1.6317</td>
<td>4.37</td>
<td>1.451</td>
</tr>
<tr>
<td>HL</td>
<td>660</td>
<td>0.258</td>
<td>170.3</td>
<td>1.130</td>
<td>0.6407</td>
<td>4.71</td>
<td>1.425</td>
</tr>
<tr>
<td>LH</td>
<td>602</td>
<td>&quot;</td>
<td>155.3</td>
<td>1.030</td>
<td>0.5840</td>
<td>4.29</td>
<td>1.467</td>
</tr>
</tbody>
</table>

**Analysis by Moment Indicator with Spring Balance**

The moment indicator solution with the spring balance was calibrated by the use of Fig. 21. The calibration is shown in the following:

\[
e_2 = K_2 \frac{P}{E}
\]

\( K_2 \) = constant of spring balance

Then \( E = \frac{K_2 P}{e_2} \)
The moment is given by \[ M = \frac{6EIe_1}{L^2} \]

\[ M = Pd \]

Substituting: \[ Pd = \frac{6IK_2Pe_1}{L^2e_2} \]

Therefore \[ K_2 = \frac{dL^2e_2}{6Ie_1} \]

Fig. 21

From the test on the calibrating cantilever

\[ e_1 = 801 \]
\[ e_2 = 126 \]

Then \[ K_2 = \frac{(7.6)(4.5)^2(126)}{(6)(0.0645)(0.512)^3(801)} = 5593 \]

\[ K_2 \] may also be calculated by theory from the equation

\[ K_2 = \frac{3AI^2}{2th^3} \]

Then \[ K_2 = \frac{(3)(7.55)(3.4)^2}{(2)(0.0645)(0.45)^3} = 7360 \]
This constant was applied to the model by the following:

\[
M = \frac{6EIe_1}{L^2}
\]

\[
E = K_2 \frac{P}{e_2}
\]

Then

\[
M = \frac{6K_2PIe_1}{L^2e_2} = K_3' \frac{e_1}{e_2} P
\]

\[L = \text{span of indicator. } \quad L^2 = 20.25\]

\[
K_3 = \frac{K_2P6}{L^2 e_2} = \frac{(5593)(6)(1)}{(20.25)(591)} = 2.8037
\]

In the above equation \(e_2\) is the movement of the points \(E\) of Fig. 21. Since the only load used was a unit wind load at the top floor, this remained a constant for all readings, and was combined in the constant \(K_3\). Then the moment at any section was evolved from the following:

\[
M = K_3 e_1 I = 2.8037 e_1 I
\]

\(I = \text{Moment of Inertia of member}\)

\(e_1 = \text{Reading of the moment indicator}\)

**Moments in Outside Row of Girders**

The indicator was set to read moments at two points on the girder as indicated in Fig. 22. By the use of the figure, and the distortion measured on the indicator the moments were calculated.
Fig. 22.

\[ M_a = \left( \frac{5.5 + x_a}{x_a} \right) (M'_a) \]

\[ M_b = \left( \frac{5.5 + x_b}{x_b} \right) (M'_b) \]

<table>
<thead>
<tr>
<th>Moment</th>
<th>e</th>
<th>I</th>
<th>M'</th>
<th>x</th>
<th>5.5 + x</th>
<th>End Mom.</th>
</tr>
</thead>
<tbody>
<tr>
<td>GH</td>
<td>129</td>
<td>0.002267</td>
<td>0.6199</td>
<td>5.34</td>
<td>10.84</td>
<td>1.664</td>
</tr>
<tr>
<td>HG</td>
<td>89</td>
<td>&quot;</td>
<td>0.5657</td>
<td>3.66</td>
<td>9.16</td>
<td>1.416</td>
</tr>
<tr>
<td>KL</td>
<td>175</td>
<td>&quot;</td>
<td>1.1123</td>
<td>5.38</td>
<td>10.88</td>
<td>2.249</td>
</tr>
<tr>
<td>LK</td>
<td>118</td>
<td>&quot;</td>
<td>0.750</td>
<td>3.62</td>
<td>9.12</td>
<td>1.889</td>
</tr>
<tr>
<td>OP</td>
<td>142</td>
<td>&quot;</td>
<td>0.9025</td>
<td>4.00</td>
<td>9.50</td>
<td>2.1434</td>
</tr>
<tr>
<td>PO</td>
<td>177</td>
<td>&quot;</td>
<td>1.1250</td>
<td>5.00</td>
<td>10.50</td>
<td>2.465</td>
</tr>
<tr>
<td>TV</td>
<td>212</td>
<td>0.0004898</td>
<td>0.2911</td>
<td>2.60</td>
<td>8.10</td>
<td>0.9068</td>
</tr>
<tr>
<td>VT</td>
<td>519</td>
<td>&quot;</td>
<td>0.7127</td>
<td>6.40</td>
<td>11.90</td>
<td>1.325</td>
</tr>
</tbody>
</table>