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\mathbf{Z}/m -GRADED LIE ALGEBRAS AND PERVERSE SHEAVES, I

GEORGE LUSZTIG AND ZHIWEI YUN

ABSTRACT. We give a block decomposition of the equivariant derived category arising from a cyclically graded Lie algebra. This generalizes certain aspects of the generalized Springer correspondence to the graded setting.

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INTRODUCTION

0.1. Let \mathbf{k} be an algebraically closed field of characteristic $p \geq 0$. We fix an integer $m > 0$ such that $m < p$ whenever $p > 0$ and we write \mathbf{Z}/m instead of $\mathbf{Z}/m\mathbf{Z}$. For $n \in \mathbf{Z}$, let \underline{n} denote the image of n in \mathbf{Z}/m .

We also fix G , a semisimple simply connected algebraic group over \mathbf{k} and a \mathbf{Z}/m -grading $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{g}_i$ (see 0.11) for the Lie algebra \mathfrak{g} of G ; we shall assume that either $p = 0$ or that p is so large relative to G , that we can operate with \mathfrak{g} as if p was 0.

For any integer d invertible in \mathbf{k} let $\mu_d = \{z \in \mathbf{k}^*; z^d = 1\}$. The \mathbf{Z}/m -grading on \mathfrak{g} is the same as an action of μ_m on G or a homomorphism $\tilde{\vartheta} : \mu_m \rightarrow \text{Aut}(G)$. ($\tilde{\vartheta}$ induces a homomorphism $\tilde{\theta} : \mu_m \rightarrow \text{Aut}(\mathfrak{g})$ and for $i \in \mathbf{Z}/m$ we have $\mathfrak{g}_i = \{x \in \mathfrak{g}; \tilde{\theta}(z)x = z^i x \ \forall z \in \mu_m\}$.) Let $G_0 = \{g \in G; g\tilde{\vartheta}(z) = \tilde{\vartheta}(z)g \ \forall z \in \mu_m\}$, be a connected reductive subgroup of G with Lie algebra \mathfrak{g}_0 . For any $i \in \mathbf{Z}/m$, the Ad-action of G_0 on \mathfrak{g} leaves stable \mathfrak{g}_i and its closed subset $\mathfrak{g}_i^{nil} := \mathfrak{g}_i \cap \mathfrak{g}^{nil}$. (Here \mathfrak{g}^{nil} is the variety of nilpotent elements in \mathfrak{g} .)

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We are interested in studying the equivariant derived categories (see 0.11) $\mathcal{D}_{G_0}(\mathfrak{g}_i)$, $\mathcal{D}_{G_0}(\mathfrak{g}_i^{nil})$. More specifically, we would like to classify G_0 -equivariant simple perverse sheaves with support in \mathfrak{g}_i^{nil} and (in the case where $p > 0$) their Fourier-Deligne transform. The simple perverse sheaves in $\mathcal{D}_{G_0}(\mathfrak{g}_i^{nil})$ are in bijection with the pair $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is a nilpotent G_0 -orbit in \mathfrak{g}_i and \mathcal{L} is (the isomorphism class of) an irreducible G_0 -equivariant local system on \mathcal{O} . (The pair $(\mathcal{O}, \mathcal{L})$ gives rise to the simple perverse sheaf P with support equal to the closure of \mathcal{O} and with $P|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.) We denote the set of such $(\mathcal{O}, \mathcal{L})$ by $\mathcal{I}(\mathfrak{g}_i)$. This is a finite set, since the G_0 -action on \mathfrak{g}_i^{nil} has only finitely many orbits. Alternatively, if we choose $e \in \mathcal{O}$, then the local system \mathcal{L} corresponds to an irreducible representation of $\pi_0(G_0(e))$ (see 0.11), where $G_0(e)$ is the stabilizer of e under G_0 .

There are many \mathbf{Z}/m -graded Lie algebras which appear in nature.

0.2. In this subsection we assume that $m = 2$ and $\mathbf{k} = \mathbf{C}$. Then the $\mathbf{Z}/2$ -grading $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (with $\mathfrak{k} = \mathfrak{g}_0$, $\mathfrak{p} = \mathfrak{g}_1$) has been extensively studied in connection with the theory of symmetric spaces and the representation theory of real semisimple groups. In particular, the nilpotent G_0 -orbits on \mathfrak{p} are known to be in bijection with the nilpotent orbits in the Lie algebra of a real form of G determined by the $\mathbf{Z}/2$ -grading (Kostant and Sekiguchi).

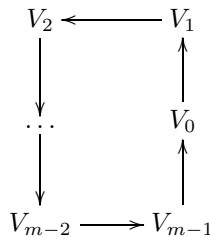
0.3. Another class of examples comes from cyclic quivers. In this subsection we assume that $m \geq 2$. We consider the simplest such example where V is a \mathbf{k} -vector space equipped with a \mathbf{Z}/m -grading $V = \bigoplus_{i \in \mathbf{Z}/m} V_i$ (see 0.11) and $G = SL(V)$ with the \mathbf{Z}/m -grading given by

$$\mathfrak{g}_i = \{T \in \mathfrak{g} = \mathfrak{sl}(V); T(V_j) \subset V_{j+i} \quad \forall j \in \mathbf{Z}/m\}.$$

In this case we have $G_0 = S(\prod_{i \in \mathbf{Z}/m} GL(V_i))$, the intersection of $SL(V)$ with the Levi subgroup $\prod_i GL(V_i)$ of a parabolic subgroup of $GL(V)$. The subspace \mathfrak{g}_1 is

(a)
$$\bigoplus_{i \in \mathbf{Z}/m} \text{Hom}(V_i, V_{i+1}).$$

We may consider a quiver Q with m vertices indexed by \mathbf{Z}/m and an arrow $i \mapsto i+1$ for each $i \in \mathbf{Z}/m$,



Then \mathfrak{g}_1 is the space of representations of Q where we put V_i at the vertex i .

More generally, if G is a classical group, then the G_0 -action on \mathfrak{g}_1 can be interpreted in terms of a cyclic quiver with some extra structure (see 9.5 for the case where G is a symplectic group).

0.4. In this subsection we forget the \mathbf{Z}/m -grading. Instead of the action of G_0 on \mathfrak{g}_i and \mathfrak{g}_i^{nil} we consider the adjoint action of G on \mathfrak{g} and on \mathfrak{g}^{nil} . Let $\mathcal{I}(\mathfrak{g})$ be the set of pairs $(\mathcal{O}, \mathcal{L})$ where \mathcal{O} is a G -orbit on \mathfrak{g}^{nil} and \mathcal{L} is an irreducible G -equivariant local

system on \mathcal{O} (up to isomorphism). From the results on the generalized Springer theory in [L1] we have a canonical decomposition

$$(a) \quad \mathcal{I}(\mathfrak{g}) = \sqcup_{(L,C)} \mathcal{I}(\mathfrak{g})_{(L,C)},$$

where (L, C) runs over the G -conjugacy classes of data L, C with L a Levi subgroup of a parabolic subgroup of G and C an L -equivariant cuspidal perverse sheaf on the nilpotent cone of the Lie algebra of L . (Actually, the results of [L1] are stated for unipotent elements in G instead of nilpotent elements in \mathfrak{g} .) We call (a) the *block decomposition* of $\mathcal{I}(\mathfrak{g})$.

Let $P(\mathfrak{g}^{nil})$ be the subcategory of $\mathcal{D}(\mathfrak{g}^{nil})$ consisting of complexes whose perverse cohomology sheaves are G -equivariant. Using (a) and [L3, (7.3.1)], we see that we have a direct sum decomposition

$$(b) \quad P(\mathfrak{g}^{nil}) = \oplus_{(L,C)} P(\mathfrak{g}^{nil})_{(L,C)},$$

where (L, C) is as in (a). We call (b) the *block decomposition* of $P(\mathfrak{g}^{nil})$. In [RR] it is shown that the following variant of (b) holds: we have a direct sum decomposition

$$(c) \quad \mathcal{D}_G(\mathfrak{g}^{nil}) = \oplus_{(L,C)} \mathcal{D}_G(\mathfrak{g}^{nil})_{(L,C)},$$

where (L, C) is as in (a). We call (c) the *block decomposition* of $\mathcal{D}_G(\mathfrak{g}^{nil})$.

In this paper we find a \mathbf{Z}/m -graded analogue of this (ungraded) block decomposition.

0.5. We fix ζ , a primitive m -th root of 1 in \mathbf{k} and we set $\vartheta = \tilde{\vartheta}(\zeta) : G \rightarrow G$, $\theta = \tilde{\theta}(\zeta) : \mathfrak{g} \rightarrow \mathfrak{g}$. Then for $i \in \mathbf{Z}/m$ we have $\mathfrak{g}_i = \{x \in \mathfrak{g}; \theta(x) = \zeta^i x\}$.

Let $\eta \in \mathbf{Z} - \{0\}$. We consider systems $(M, \mathfrak{m}_*, \tilde{C})$, where

$$M = \{g \in G; \text{Ad}(\tau)\vartheta g = g\}$$

for some semisimple element of finite order $\tau \in G_0$, $\mathfrak{m}_* = \{\mathfrak{m}_N\}_{N \in \mathbf{Z}}$ is a \mathbf{Z} -grading of the Lie algebra \mathfrak{m} of M (see 0.11) such that $\mathfrak{m}_N \subset \mathfrak{g}_N$ for all N , M_0 is the closed connected subgroup of M with Lie algebra \mathfrak{m}_0 and \tilde{C} is an M_0 -equivariant cuspidal perverse sheaf on \mathfrak{m}_η . We will review the notion of M_0 -equivariant cuspidal perverse sheaf (already defined in [L4]) on \mathfrak{m}_η in 1.2. Such a system $(M, \mathfrak{m}_*, \tilde{C})$ is said to be *admissible* if a certain technical condition involving the group of components of the center of M is satisfied (see 3.1).

Let $\underline{\Sigma}_\eta$ be the set of admissible systems up to G_0 -conjugacy. The following result is proved in 7.9.

Theorem 0.6. *There is a canonical direct sum decomposition of $\mathcal{D}_{G_0}(\mathfrak{g}_\eta^{nil})$ into full subcategories*

$$\mathcal{D}_{G_0}(\mathfrak{g}_\eta^{nil}) = \oplus_{(M, \mathfrak{m}_*, \tilde{C}) \in \underline{\Sigma}_\eta} \mathcal{D}_{G_0}(\mathfrak{g}_\eta^{nil})_{(M, \mathfrak{m}_*, \tilde{C})}$$

indexed by $\underline{\Sigma}_\eta$.

In particular, any simple perverse sheaf in $\mathcal{D}_{G_0}(\mathfrak{g}_\eta^{nil})$ belongs to a well-defined block $\mathcal{D}_{G_0}(\mathfrak{g}_\eta^{nil})_{(M, \mathfrak{m}_*, \tilde{C})}$. This gives a map

$$\Psi : \mathcal{I}(\mathfrak{g}_\eta) \rightarrow \underline{\Sigma}_\eta.$$

In fact, we will first establish the map Ψ in 3.5 and then prove the theorem in 7.9, using a key calculation in Proposition 6.4.

We also show in 3.9 and 7.8 that both the indexing set $\underline{\Sigma}_\eta$ and the blocks $\mathcal{D}_{G_\underline{0}}(\mathfrak{g}_\eta^{nil})_\xi$ (for $\xi \in \underline{\Sigma}_\eta$) only depend on the image $\underline{\eta} \in \mathbf{Z}/m$ and not on the integer η .

Note that in the case where $m = 1$, the theorem can be deduced from 0.4(a). On the other hand, for large m , a \mathbf{Z}/m -grading on \mathfrak{g} is the same as a \mathbf{Z} -grading, so that in this case the theorem can be deduced from the results of [L4]. Thus, the result about block decomposition in this paper generalizes results in [L1] and [L4].

0.7. As an explicit example, let us consider the case where $G = SL_n(\mathbf{k}), \eta = 1$. In the ungraded case, blocks are in bijection with pairs (d, χ) where d is a divisor of n and $\chi : \mu_d \rightarrow \mathbf{Q}_l^*$ is a primitive character. (See [L1].) To d we attach the subgroup $M = S(GL_d^{n/d})$ (a Levi subgroup of a parabolic subgroup) and χ gives a cuspidal perverse sheaf C_χ with support equal to the nilpotent cone of the Lie algebra of M . Now in the \mathbf{Z}/m -graded case, we have $G = SL(V), V = \bigoplus_{i \in \mathbf{Z}/m} V_i$ as in 0.3, and we identify $\mathfrak{g}_\underline{1}$ with $\bigoplus_i \text{Hom}(V_i, V_{i+1})$. In this case, the set of blocks $\underline{\Sigma}_1$ has a similar explicit description. We have a natural bijection

$$(a) \quad \underline{\Sigma}_1 \leftrightarrow \{(d, f, \chi)\} / \sim .$$

Here the right hand side is the set of equivalence classes of triples (d, f, χ) where (d, χ) is as in the ungraded case and $f : \{1, 2, \dots, n/d\} \rightarrow \mathbf{Z}/m$ is a map such that

$$(b) \quad \#\{(b, y) \in \mathbf{Z} \times \mathbf{Z}; 1 \leq b \leq n/d, 0 \leq y \leq d - 1, f(b) + \underline{y} = i\} = \dim V_i$$

for all $i \in \mathbf{Z}/m$. Two triples (d, f, χ) and (d', f', χ') are equivalent if and only if $d = d', \chi = \chi'$ and f' is obtained from f by composition with a permutation of $\{1, 2, \dots, n/d\}$.

0.8. In the ungraded case, the objects in the block $\mathcal{D}_G(\mathfrak{g}^{nil})_{(L,C)}$ are obtained from C via parabolic induction (and decomposition) through any parabolic subgroup P of G containing L as a Levi subgroup. In the \mathbf{Z}/m -graded case, a first attempt to generalize parabolic induction would be to start with a parabolic subgroup of G compatible with the \mathbf{Z}/m -grading on \mathfrak{g} , as defined in the appendix of [L5]. However, such a parabolic induction does not produce all simple perverse sheaves in $\mathcal{D}_{G_\underline{0}}(\mathfrak{g}_\eta^{nil})$. Instead, we introduce a certain induction procedure which we call *spiral induction*; see Section 4. We introduce the notion of a spiral \mathfrak{p}_* which is a sequence of subspaces $\mathfrak{p}_N \subset \mathfrak{g}_N$, one for each $N \in \mathbf{Z}$, satisfying certain conditions; see Section 2. It turns out that spirals are the correct analogues of parabolic subalgebras in the \mathbf{Z}/m -graded case. Moreover, spiral induction includes the parabolic induction defined in the appendix of [L5] as special cases. In fact there are two kinds of spiral inductions, one giving objects in $\mathcal{D}_{G_\underline{0}}(\mathfrak{g}_\eta^{nil})$ and the other giving (assuming that $p > 0$) Fourier-Deligne transforms of objects in $\mathcal{D}_{G_\underline{0}}(\mathfrak{g}_{-\underline{\eta}}^{nil})$. The latter may be viewed as an analogue of character sheaves in the \mathbf{Z}/m -graded setting.

0.9. We now discuss the contents of the various sections. Many arguments in this paper rely on results from [L4] concerning \mathbf{Z} -graded Lie algebras; in Section 1 we review some results from [L4] that we need. In Section 2 we introduce the ϵ -spirals attached to a \mathbf{Z}/m -graded Lie algebra and their splittings; the analogous concepts in the \mathbf{Z} -graded cases are the parabolic subalgebras compatible with the \mathbf{Z} -grading and their Levi subalgebras compatible with the \mathbf{Z} -grading. We also attach a canonical spiral to any element of \mathfrak{g}_η^{nil} which plays a crucial role in the arguments of this

paper. In Section 3 we introduce the admissible systems, which eventually will be used to index the blocks in $\mathcal{D}_{G_{\mathbb{Q}}}(\mathfrak{g}_{\eta}^{nil})$. In Section 4 we introduce the operation of spiral induction which is our main tool in the study of $\mathcal{D}_{G_{\mathbb{Q}}}(\mathfrak{g}_{\eta}^{nil})$. In Sections 5 and 6 we calculate explicitly the Hom space between two spiral inductions, generalizing to the \mathbf{Z}/m -graded case a result from [L4]. This is used in Section 7 to prove Theorem 0.6. In Section 8 we introduce monomial and quasi-monomial complexes on $\mathfrak{g}_{\eta}^{nil}$; we show that the monomial complexes (resp. quasi-monomial) complexes generate the appropriate Grothendieck group over $\mathbf{Q}(v)$ (resp. over $\mathbf{Z}[v, v^{-1}]$) where v is an indeterminate; this again generalizes to the \mathbf{Z}/m -graded case a result from [L4]. This result is of the same type as that which says that the plus part of a quantized enveloping algebra defined in terms of perverse sheaves is generated over $\mathbf{Q}(v)$ by monomials in the E_i and over $\mathbf{Z}[v, v^{-1}]$ by the products of divided powers of the E_i (which could be called quasi-monomials). In Section 9 we discuss the examples where $G = SL(V)$ or $G = Sp(V)$; in these cases we describe the spirals and in the case of $G = SL(V)$ we describe the blocks.

0.10. It is known that, in the ungraded case, each block of $\mathcal{D}_G(\mathfrak{g}^{nil})$ can be related to the category of representations of a certain Weyl group; if m is large, so that the \mathbf{Z}/m grading of \mathfrak{g} is a \mathbf{Z} -grading and $\mathfrak{g}_{\eta}^{nil} = \mathfrak{g}_{\eta}$, then each block of $\mathcal{D}_{G_{\mathbb{Q}}}(\mathfrak{g}_{\eta}^{nil})$ is related to the category of representations of a certain graded affine Hecke algebra with possibly unequal parameters. In fact, without assumptions on m , each block of $\mathcal{D}_{G_{\mathbb{Q}}}(\mathfrak{g}_{\eta}^{nil})$ is related to a certain graded double affine Hecke algebra (corresponding to an affine Weyl group attached to the block) with possibly unequal parameters; this will be considered in a sequel to this paper. We also plan to describe explicitly the blocks in the case where G is a classical group and relate them to cyclic quivers with extra structure. The case of the symplectic group is partially carried out in 9.5–9.7.

0.11. **Notation.** All algebraic varieties are assumed to be over \mathbf{k} ; all algebraic groups are assumed to be affine. Let l be a prime number invertible in \mathbf{k} . For any algebraic variety X we denote by $\mathcal{D}(X)$ the bounded derived category of $\bar{\mathbf{Q}}_l$ -complexes on X . Let $D : \mathcal{D}(X) \rightarrow \mathcal{D}(X)$ be Verdier duality.

For $K \in \mathcal{D}(X)$ we denote by $\mathcal{H}^n K$ the n -th cohomology sheaf of K and by $\mathcal{H}_x^n K$ the stalk of $\mathcal{H}^n K$ at $x \in X$.

If X' is a locally closed smooth irreducible subvariety of X with closure \bar{X}' and \mathcal{L} is an irreducible local system on X' we denote by $\mathcal{L}^{\sharp} \in \mathcal{D}(X)$ the intersection cohomology complex of \bar{X}' with coefficients in \mathcal{L} , extended by 0 on $X - \bar{X}'$.

If X has a given action of an algebraic group H we denote by $\mathcal{D}_H(X)$ the corresponding equivariant derived category.

If H is an algebraic group we denote by H^0 the identity component of H , by \mathcal{Z}_H the center of H . We set $\pi_0(H) = H/H^0$. Now assume that H is connected. We denote by $\mathfrak{L}H$ the Lie algebra of H and by U_H the unipotent radical of H . Let $\mathfrak{h} = \mathfrak{L}H$. If \mathfrak{h}' is a Lie subalgebra of \mathfrak{h} we write $e^{\mathfrak{h}'}$ for the closed connected subgroup of H such that $\mathfrak{L}(e^{\mathfrak{h}'}) = \mathfrak{h}'$, assuming that such a subgroup exists.

We shall often denote a collection $\{V_N; N \in \mathbf{Z}\}$ of vector spaces indexed by $N \in \mathbf{Z}$ by the symbol V_* .

If V is a \mathbf{k} -vector space, a \mathbf{Z} -grading on V is a collection of subspaces $V_* = \{V_k; k \in \mathbf{Z}\}$ such that $V = \bigoplus_{k \in \mathbf{Z}} V_k$; a \mathbf{Z}/m -grading on V is a collection of subspaces

$\{V_i; i \in \mathbf{Z}/m\}$ such that $V = \bigoplus_{i \in \mathbf{Z}/m} V_i$; a \mathbf{Q} -grading on V is a collection of subspaces $\{\kappa V; \kappa \in \mathbf{Q}\}$ such that $V = \bigoplus_{\kappa \in \mathbf{Q}} (\kappa V)$.

A \mathbf{Z} -grading for the Lie algebra \mathfrak{h} is a \mathbf{Z} -grading $\mathfrak{h}_* = \{\mathfrak{h}_k; k \in \mathbf{Z}\}$ of \mathfrak{h} as a vector space satisfying $[\mathfrak{h}_k, \mathfrak{h}_{k'}] \subset \mathfrak{h}_{k+k'}$ for all $k, k' \in \mathbf{Z}$; a \mathbf{Z}/m -grading for \mathfrak{h} is a \mathbf{Z}/m -grading $\{\mathfrak{h}_i; i \in \mathbf{Z}/m\}$ of \mathfrak{h} as a vector space satisfying $[\mathfrak{h}_i, \mathfrak{h}_{i'}] \subset \mathfrak{h}_{i+i'}$ for all $i, i' \in \mathbf{Z}/m$; a \mathbf{Q} -grading for \mathfrak{h} is a \mathbf{Q} -grading $\{\kappa \mathfrak{h}; \kappa \in \mathbf{Q}\}$ of \mathfrak{h} as a vector space satisfying $[\kappa \mathfrak{h}, \kappa' \mathfrak{h}] \subset \kappa + \kappa' \mathfrak{h}$ for all $\kappa, \kappa' \in \mathbf{Q}$.

Let Y_H be the set of homomorphisms of algebraic groups $\mathbf{k}^* \rightarrow H$. For $\lambda \in Y_H$ and $c \in \mathbf{Z}$, we define $c\lambda \in Y_H$ by $(c\lambda)(t) = \lambda(t^c)$ for $t \in \mathbf{k}^*$. We define an equivalence relation on $Y_H \times \mathbf{Z}_{>0}$ by $(\lambda, r) \sim (\lambda', r')$ whenever there exist c, c' in $\mathbf{Z}_{>0}$ such that $c\lambda = c'\lambda'$, $cr = c'r'$; the set of equivalence classes for this relation is denoted by $Y_{H, \mathbf{Q}}$. Let $\lambda/r = (1/r)\lambda$ be the equivalence class of (λ, r) . Now $\lambda \mapsto \lambda/1$ identifies Y_H with a subset of $Y_{H, \mathbf{Q}}$. For $\kappa \in \mathbf{Q}, \mu \in Y_{H, \mathbf{Q}}$ we define $\kappa\mu \in Y_{H, \mathbf{Q}}$ by $\kappa\mu = (k\lambda)/(k'r)$, where $k \in \mathbf{Z}, k' \in \mathbf{Z}_{>0}, r \in \mathbf{Z}_{>0}, \lambda \in Y_H$ are such that $\kappa = k/k', \mu = \lambda/r$; this is independent of the choices. In particular, we have $r\mu \in Y_H$ for some $r \in \mathbf{Z}_{>0}$.

Let $\lambda \in Y_H$. For $k \in \mathbf{Z}$ we set

$$\lambda \mathfrak{h} = \{x \in \mathfrak{h}; \text{Ad}(\lambda(t))x = t^k x \quad \forall t \in \mathbf{k}^*\}.$$

Note that $\{\lambda \mathfrak{h}, k \in \mathbf{Z}\}$ is a \mathbf{Z} -grading of \mathfrak{h} .

Now let $\mu \in Y_{H, \mathbf{Q}}$. For $\kappa \in \mathbf{Q}$ we set $\mu \mathfrak{h} = \frac{r}{r\kappa} \mathfrak{h}$ where $r \in \mathbf{Z}_{>0}$ is chosen so that $r\mu \in Y_H$, $r\kappa \in \mathbf{Z}$. This is well defined (independent of the choice of r). Note that $\{\mu \mathfrak{h}, \kappa \in \mathbf{Q}\}$ is a \mathbf{Q} -grading of \mathfrak{h} .

0.12. Let H be a connected algebraic group acting on an algebraic variety X and let A, B be two H -equivariant semisimple complexes on X ; let $j \in \mathbf{Z}$. We define a finite dimensional $\bar{\mathbf{Q}}_l$ -vector space $\mathbf{D}_j(X, H; A, B)$ as in [L4, 1.7]. For the purpose of this paper, we can take the following formula as the definition of $\mathbf{D}_j(X, H; A, B)$:

$$(a) \quad \mathbf{D}_j(X, H; A, B) = \text{Hom}_{\mathcal{D}_H(X)}(A, D(B)[-j])^*.$$

Let $d_j(X; A, B) = \dim \mathbf{D}_j(X, H; A, B)$, $\{A, B\} = \sum_{j \in \mathbf{Z}} d_j(X; A, B)v^{-j} \in \mathbf{N}((v))$ where v is an indeterminate.

If A, B are H -equivariant simple perverse sheaves on X , then

$$\begin{aligned} \{A, B\} &\in 1 + v\mathbf{N}[[v]] \text{ if } B \cong D(A), \\ \{A, B\} &\in v\mathbf{N}[[v]] \text{ if } B \not\cong D(A). \end{aligned}$$

(See [L4, 1.8(d)].)

For an algebraic variety X we denote by ρ_X the map $X \rightarrow (\text{point})$.

Let v be an indeterminate and let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. Let $\bar{\cdot} : \mathbf{Q}(v) \rightarrow \mathbf{Q}(v)$ be the field involution such that $\bar{v} = v^{-1}$. This restricts to a ring involution $\bar{\cdot} : \mathcal{A} \rightarrow \mathcal{A}$.

For any $\eta \in \mathbf{Z} - \{0\}$ we define $\dot{\eta} = \eta/|\eta| \in \{1, -1\}$ where $|\eta|$ is the absolute value of η .

1. RECOLLECTIONS ON \mathbf{Z} -GRADED LIE ALGEBRAS

In this section we recall notation and results from [L4] that will be used in this paper.

1.1. In this section we fix a connected reductive group H ; let $\mathfrak{h} = \mathfrak{L}H$.

Let J^H be the variety consisting of all triples $(e, h, f) \in \mathfrak{h}^3$ such that $[h, e] = 2e, [h, f] = -2f, [e, f] = h$ (then e, f are necessarily in \mathfrak{h}^{nil}). If $\phi = (e, h, f) \in J^H$, there is a unique homomorphism of algebraic groups $\tilde{\phi} : SL_2(\mathbf{k}) \rightarrow H$ such that the differential of $\tilde{\phi}$ carries $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to e, h, f respectively; we then define $\iota_\phi \in Y_H$ by $\iota_\phi(t) = \tilde{\phi} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

1.2. In the remainder of this section we assume that a \mathbf{Z} -grading \mathfrak{h}_* for \mathfrak{h} is given. Then there exists $\lambda \in Y_H$ and $r \in \mathbf{Z}_{>0}$ with $\mathfrak{h}_k = \lambda_k \mathfrak{h}$ for all $k \in \mathbf{Z}$. (It follows that $\lambda_\kappa \mathfrak{h} = 0$ for all $\kappa \in \mathbf{Q} - r\mathbf{Z}$.)

(In this paper we will often refer to results in [L4], even though, strictly speaking, in [L4] a stronger assumption on the \mathbf{Z} -grading of \mathfrak{h} is made, namely that r above can be taken to be 1. Note that the results of [L4] hold with the same proof when the stronger assumption is replaced by the present assumption.)

We have $\mathfrak{h}_k \subset \mathfrak{h}^{nil}$ for any $k \in \mathbf{Z} - \{0\}$. Note that \mathfrak{h}_0 is a Lie subalgebra of \mathfrak{h} and that $H_0 := e^{\mathfrak{h}_0} \subset H$ is well defined and it acts by the Ad-action on each \mathfrak{h}_k . If $k \neq 0$, this action has only finitely many orbits (see [L4, 3.5]); we denote by $\mathring{\mathfrak{h}}_k$ the unique open H_0 -orbit in \mathfrak{h}_k .

Let $\eta \in \mathbf{Z} - \{0\}$.

(a) We say that the \mathbf{Z} -grading \mathfrak{h}_* of \mathfrak{h} is η -rigid if there exists $\iota \in Y_H$ such that (i), (ii) below hold.

(i) ${}^t_k \mathfrak{h} = \mathfrak{h}_{\eta k/2}$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \in \mathbf{Z}$ and ${}^t_k \mathfrak{h} = 0$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \notin \mathbf{Z}$;

(ii) $\iota = \iota_\phi$ for some $\phi = (e, h, f) \in J^H$ such that $e \in \mathring{\mathfrak{h}}_\eta, h \in \mathfrak{h}_0, f \in \mathfrak{h}_{-\eta}$. It follows that $2k' \in \eta\mathbf{Z}$ whenever $\mathfrak{h}_{k'} \neq 0$. Note that ι is unique if it exists, since, by (ii), $\iota(\mathbf{k}^*)$ is contained in the derived group of H .

We show:

(b) In the setup of (a), let $\phi' = (e', h', f') \in J^H$ be such that $e' \in \mathring{\mathfrak{h}}_\eta, h' \in \mathfrak{h}_0, f' \in \mathfrak{h}_{-\eta}$. Let $\iota' = \iota_{\phi'}$. Then $\iota' = \iota$.

Let ϕ be as in (ii). Using [L4, 3.3], we can find $g_0 \in H_0$ such that $\text{Ad}(g_0)$ carries ϕ to ϕ' . It follows that $\text{Ad}(g_0)\iota(t) = \iota'(t)$ for any $t \in \mathbf{k}^*$. For $k \in \mathbf{Z}$ such that $\eta k/2 \in \mathbf{Z}$ we have

$${}^t_k \mathfrak{h} = \text{Ad}(g_0)({}^t_k \mathfrak{h}) = \text{Ad}(g_0)\mathfrak{h}_k = \mathfrak{h}_k;$$

for $k \in \mathbf{Z}$ such that $\eta k/2 \notin \mathbf{Z}$ we have

$$\begin{aligned} {}^t_k \mathfrak{h} &= \text{Ad}(g_0)({}^t_k \mathfrak{h}) = 0, \\ {}^t_{2k\eta} \mathfrak{h} &= \text{Ad}(g_0)({}^t_{2k\eta} \mathfrak{h}) = \text{Ad}(g_0)\mathfrak{h}_k = \mathfrak{h}_k. \end{aligned}$$

Thus ι' satisfies the defining properties of ι in (a). By uniqueness we have $\iota' = \iota$ as required.

Let $\mathcal{I}(\mathfrak{h}_\eta)$ be the set of all pairs $(\mathcal{O}, \mathcal{L})$ where \mathcal{O} is an H_0 -orbit in \mathfrak{h}_η and \mathcal{L} is an H_0 -equivariant irreducible local system on \mathfrak{h}_η (up to isomorphism).

Let $\mathcal{Q}(\mathfrak{h}_\eta)$ be the category of \mathbf{Q}_l -complexes on \mathfrak{h}_η which are direct sums of shifts of simple H_0 -equivariant perverse sheaves on \mathfrak{h}_η . There are up to isomorphism only finitely many such simple perverse sheaves; they form a set in bijection with $\mathcal{I}(\mathfrak{h}_\eta)$.

An H_0 -equivariant perverse sheaf A on \mathfrak{h}_η is said to be *cuspidal* if there exists a nilpotent H -orbit \mathcal{C} in \mathfrak{h} and an irreducible H -equivariant cuspidal local system \mathcal{F}

on \mathcal{C} such that $\mathring{\mathfrak{h}}_\eta \subset \mathcal{C}$ and $A|_{\mathring{\mathfrak{h}}_\eta} = \mathcal{F}|_{\mathring{\mathfrak{h}}_\eta} [\dim \mathfrak{h}_\eta]$. If such $(\mathcal{C}, \mathcal{F})$ exists, it is unique; see [L4, 4.2(c)]. Note that if A is cuspidal, then it is necessarily a simple perverse sheaf.

(c) *If there exists a cuspidal H_0 -equivariant perverse sheaf A on \mathfrak{h}_η , the grading \mathfrak{h}_* of \mathfrak{h} is necessarily η -rigid; moreover, we have $A|_{\mathfrak{h}_\eta - \mathring{\mathfrak{h}}_\eta} = 0$.*

(See [L4, 4.4(a), 4.4(b)].)

In the setup of (c), the element $\iota \in Y_H$ provided by (a) is known to satisfy

(d) $\iota_k \mathfrak{h} = 0$ unless $k \in 2\mathbf{Z}$;

we deduce that:

(e) *If $k' \in \mathbf{Z}$ and $\mathfrak{h}_{k'} \neq 0$, then $k'/\eta \in \mathbf{Z}$.*

1.3. Parabolic induction. In the setup of 1.2 assume that P is a parabolic subgroup of H with $\mathfrak{p} := \mathfrak{L}P$ satisfying $\mathfrak{p} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{p}_k$ where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$. We set $U = U_P, L = P/U, \mathfrak{u} = \mathfrak{L}U, \mathfrak{l} = \mathfrak{L}L = \mathfrak{p}/\mathfrak{u}$. We have $\mathfrak{u} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{u}_k$ where $\mathfrak{u}_k = \mathfrak{u} \cap \mathfrak{h}_k$. Setting $\mathfrak{l}_k = \mathfrak{p}_k/\mathfrak{u}_k$, we have $\mathfrak{l} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{l}_k$; this gives a \mathbf{Z} -grading of the Lie algebra \mathfrak{l} .

Now \mathfrak{p}_0 is a parabolic subalgebra of the reductive Lie algebra \mathfrak{h}_0 ; we have $\mathfrak{p}_0 = \mathfrak{L}P_0$ where P_0 is a parabolic subgroup of the connected reductive group H_0 . Let L_0 be the image of P_0 under the obvious homomorphism $P \rightarrow L$. Then $L_0 = e^{\mathfrak{l}_0} \subset L$. Now P_0 acts by the Ad-action on each \mathfrak{p}_k . Let $\pi : \mathfrak{p}_\eta \rightarrow \mathfrak{l}_\eta$ be the obvious projection. We have a diagram

$$\mathfrak{l}_\eta \xleftarrow{a} H_0 \times \mathfrak{p}_\eta \xrightarrow{b} E \xrightarrow{c} \mathfrak{h}_\eta,$$

where

$$E = \{(hP_0, z) \in H_0/P_0 \times \mathfrak{h}_\eta; \text{Ad}(h^{-1})z \in \mathfrak{p}_\eta\},$$

$$a(h, z) = \pi(\text{Ad}(h^{-1})z), b(h, z) = (hP_0, z), c(gP_0, z) = z.$$

Now a is smooth with connected fibers, b is a principal P_0 -bundle and c is proper. If $A \in \mathcal{Q}(\mathfrak{l}_\eta)$, then a^*A is a P_0 -equivariant semisimple complex on $H_0 \times \mathfrak{p}_\eta$ hence there is a well-defined semisimple complex A_1 on E such that $b^*A_1 = a^*A$. Since c is proper, the complex

$$\text{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(A) := c_!A_1$$

belongs to $\mathcal{Q}(\mathfrak{h}_\eta)$. For $B \in \mathcal{D}(\mathfrak{h}_\eta)$ we can form

$$\text{res}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(B) := \pi_!(B|_{\mathfrak{p}_\eta}) \in \mathcal{D}(\mathfrak{l}_\eta).$$

Thus we have functors $\text{res}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta} : \mathcal{D}(\mathfrak{h}_\eta) \rightarrow \mathcal{D}(\mathfrak{l}_\eta), \text{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta} : \mathcal{Q}(\mathfrak{l}_\eta) \rightarrow \mathcal{Q}(\mathfrak{h}_\eta)$.

When $\tilde{\mathfrak{l}}$ is a Levi subalgebra of \mathfrak{p} such that $\tilde{\mathfrak{l}} = \bigoplus_{k \in \mathbf{Z}} \tilde{\mathfrak{l}}_k$ with $\tilde{\mathfrak{l}}_k = \tilde{\mathfrak{l}} \cap \mathfrak{h}_k$, we will sometime consider $\text{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(A)$ with $A \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta)$ by identifying $\tilde{\mathfrak{l}}_\eta = \mathfrak{l}_\eta$ in an obvious way and A with an object in $\mathcal{Q}(\mathfrak{l}_\eta)$.

1.4. In the setup of 1.3 let S'_P be the set of Levi subgroups of P and let S_P be the set of all $M \in S'_P$ such that, setting $\mathfrak{L}M = \mathfrak{m}, \mathfrak{m}_k = \mathfrak{m} \cap \mathfrak{h}_k$, we have $\mathfrak{m} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{m}_k$, or equivalently such that $\text{Ad}(\lambda(t))\mathfrak{m} = \mathfrak{m}$ for all $t \in \mathbf{k}^*$. We have $S_P \neq \emptyset$; indeed, we can find $M \in S'_P$ such that $\lambda(k^*) \subset M$; then $M \in S_P$. Since U acts simply transitively by conjugation on S'_P , it follows that:

(a) *The unipotent group $\{u \in U; u\lambda(t) = \lambda(t)u \quad \forall t \in \mathbf{k}^*\}$ acts simply transitively by conjugation on S_P .*

1.5. **Blocks for $\mathcal{Q}(\mathfrak{h}_\eta)$.** Let $\mathfrak{M}_\eta(H)$ be the set of all systems

$$(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}),$$

where M is a Levi subgroup of some parabolic subgroup of H , $\mathfrak{m} = \mathfrak{L}M$, \mathfrak{m}_* is a \mathbf{Z} -grading of \mathfrak{m} such that $\mathfrak{m}_k = \mathfrak{m} \cap \mathfrak{h}_k$ for all k , $M_0 = e^{\mathfrak{m}_0} \subset M$ and \tilde{C} is a cuspidal M_0 -equivariant perverse sheaf on \mathfrak{m}_η (up to isomorphism). Note that H_0 acts by conjugation on $\mathfrak{M}_\eta(H)$. Let $\underline{\mathfrak{M}}_\eta(H)$ be the set of orbits for this action.

In the setup of 1.2 assume that A is a simple H_0 -equivariant perverse sheaf on \mathfrak{h}_η . By [L4, 7.5]:

(a) *There exists $P, L, L_0, \mathfrak{p}, \mathfrak{l}$ as in 1.3 and a cuspidal L_0 -equivariant perverse sheaf C on \mathfrak{l}_η such that some shift of A is a direct summand of $\text{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(C)$.*

Assume now that $P', L', L'_0, \mathfrak{p}', \mathfrak{l}'$ is another quintuple like $P, L, L_0, \mathfrak{p}, \mathfrak{l}$ and that C' is a cuspidal L'_0 -equivariant perverse sheaf on \mathfrak{l}'_η such that some shift of A is a direct summand of $\text{ind}_{\mathfrak{p}'_\eta}^{\mathfrak{h}_\eta}(C')$.

Let $M \in S_P, M' \in S_{P'}$, let $\mathfrak{L}M = \mathfrak{m} = \bigoplus_k \mathfrak{m}_k$ be as in 1.4 and let $\mathfrak{L}M' = \mathfrak{m}' = \bigoplus_k \mathfrak{m}'_k$ where $\mathfrak{m}'_k = \mathfrak{m}' \cap \mathfrak{h}_k$. Let $M_0 = e^{\mathfrak{m}_0} \subset M, M'_0 = e^{\mathfrak{m}'_0} \subset M'$. We can identify $M, M_0, \mathfrak{m}, \mathfrak{m}_k$ with $L, L_0, \mathfrak{l}, \mathfrak{l}_k$ via $P \rightarrow L$ and we can identify $M', M'_0, \mathfrak{m}', \mathfrak{m}'_k$ with $L', L'_0, \mathfrak{l}', \mathfrak{l}'_k$ via $P' \rightarrow L'$. Then C (resp. C') becomes a cuspidal M_0 -equivariant (resp. M'_0 -equivariant) perverse sheaf \tilde{C} (resp. \tilde{C}') on \mathfrak{m}_η (resp. \mathfrak{m}'_η).

Using the last sentence of [L4, 15.3], we see that there exists $h \in H_0$ such that $\text{Ad}(h)$ carries $M, M_0, \mathfrak{m}, \mathfrak{m}_k$ to $M', M'_0, \mathfrak{m}', \mathfrak{m}'_k$ and \tilde{C} to \tilde{C}' . Thus, we have:

(b) *$A \mapsto (M, M_0, \mathfrak{m}, \mathfrak{m}_k, \tilde{C})$ is a well-defined map from the set of (isomorphism classes) of simple H_0 -equivariant perverse sheaves on \mathfrak{h}_η to the set $\underline{\mathfrak{M}}_\eta(H)$.*

1.6. Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_k, \tilde{C}) \in \mathfrak{M}_\eta(H)$. We show:

(a) *There exists a parabolic subgroup P of H such that M is a Levi subgroup of P and such that, setting $\mathfrak{p} = \mathfrak{L}P$, $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$, we have $\mathfrak{p} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{p}_k$.*

Let $\mathcal{Z} = \mathcal{Z}_M^0$. Then $\mathfrak{z} = \mathfrak{L}\mathcal{Z}$ is the center of \mathfrak{m} . Since \mathfrak{m}_0 is a Levi subalgebra of a parabolic subalgebra of \mathfrak{m} , we have $\mathfrak{z} \subset \mathfrak{m}_0$ hence $\mathcal{Z} \subset M_0$. We can find $\lambda_1 \in Y_{\mathcal{Z}}$ such that the centralizer of $\lambda_1(\mathbf{k}^*)$ in H is equal to the centralizer of \mathcal{Z} in H which equals M . Let $\lambda \in Y_H, r$ be as in 1.2. Then $\lambda(\mathbf{k}^*) \subset \mathcal{Z}_{H_0}$. Now $\lambda_1(\mathbf{k}^*) \subset \mathcal{Z}$ hence $\lambda_1(\mathbf{k}^*) \subset H_0$. It follows that $\lambda_1(t)\lambda(t') = \lambda(t')\lambda_1(t)$ for any t, t' in \mathbf{k}^* . Thus we have $\mathfrak{h} = \bigoplus_{k \in \mathbf{Z}, k' \in \mathbf{Z}} (\lambda_{kr}^\lambda \mathfrak{h} \cap \lambda_{k'}^{\lambda_1} \mathfrak{h})$. Since the centralizer of $\lambda_1(\mathbf{k}^*)$ in \mathfrak{h} equals \mathfrak{m} , we have $\mathfrak{m} = \bigoplus_{k \in \mathbf{Z}} (\lambda_{kr}^\lambda \mathfrak{h} \cap \lambda_0^{\lambda_1} \mathfrak{h})$. We set

$$\mathfrak{p} = \bigoplus_{k \in \mathbf{Z}, k' \in \mathbf{Z}_{\geq 0}} (\lambda_{kr}^\lambda \mathfrak{h} \cap \lambda_{k'}^{\lambda_1} \mathfrak{h}).$$

Clearly, \mathfrak{p} is a parabolic subalgebra of \mathfrak{h} with Levi subalgebra \mathfrak{m} and such that, setting $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$, we have $\mathfrak{p} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{p}_k$. This proves (a).

1.7. To any $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{M}_\eta(H)$ we associate a simple perverse sheaf A in $\mathcal{Q}(\mathfrak{h}_\eta)$ as follows. Let \mathcal{O} be the H_0 -orbit in \mathfrak{h}_η which contains $\mathring{\mathfrak{m}}_\eta$. Let \mathcal{L}' be the irreducible M_0 -equivariant local system on $\mathring{\mathfrak{m}}_\eta$ such that $\tilde{C}|_{\mathring{\mathfrak{m}}_\eta} = \mathcal{L}'[\dim \mathfrak{m}_\eta]$. By [L4, 11.2], there is a well-defined irreducible H_0 -equivariant local system \mathcal{L} on \mathcal{O} such that $\mathcal{L}|_{\mathring{\mathfrak{m}}_\eta} = \mathcal{L}'$. By definition, A is the simple perverse sheaf on \mathfrak{h}_η such that $\text{supp } A$ is contained in the closure of \mathcal{O} and $A|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.

1.8. Assume that the \mathbf{Z} -grading \mathfrak{h}_* of \mathfrak{h} is η -rigid. A perverse sheaf A in $\mathcal{Q}(\mathfrak{h}_\eta)$ is said to be η -semicuspidal if $\text{supp } A = \mathfrak{h}_\eta$ and A is attached to some

$$(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{M}_\eta(H),$$

as in 1.7 (in particular, A is a simple perverse sheaf). In this case we have $\overset{\circ}{\mathfrak{m}}_\eta \subset \overset{\circ}{\mathfrak{h}}_\eta$; moreover,

(a) H_0 acts transitively on the set of systems $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ such that $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{M}_\eta(H)$, A is attached to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ as in 1.7, \mathfrak{p} is a parabolic subalgebra of \mathfrak{h} with Levi subalgebra \mathfrak{m} and $\mathfrak{p} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{p}_k$ where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$. (See [L4, 11.9].)

If $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is as in (a), then

(b)
$$\text{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(\tilde{C}) \cong \bigoplus_j A[-2s_j][\dim \mathfrak{m}_\eta - \dim \mathfrak{h}_\eta],$$

where $s_j \in \mathbf{N}$ are defined as follows. Choose $\phi = (e, h, f) \in J^H$ as in 1.2(ii); let $H_\phi = \{g \in H; \text{Ad}(g)(e) = e, \text{Ad}(g)(h) = h, \text{Ad}(g)(f) = f\}$, let \mathcal{B} be the variety of Borel subgroups of H_ϕ^0 ; then s_j are defined by $\rho_{\mathcal{B}!} \bar{\mathbf{Q}}_l = \bigoplus_j \bar{\mathbf{Q}}_l[-2s_j]$. (See [L4, 11.13].)

1.9. Let \mathcal{X} be the set of all systems $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{A})$ where \mathfrak{p} is a parabolic subalgebra of \mathfrak{h} with Levi subalgebra \mathfrak{m} , $\mathfrak{p} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{p}_k$, $\mathfrak{m} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{m}_k$ where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$, $\mathfrak{m}_k = \mathfrak{m} \cap \mathfrak{h}_k$, $M = e^{\mathfrak{m}}$, $M_0 = e^{\mathfrak{m}_0}$ and \tilde{A} is a simple perverse sheaf in $\mathcal{Q}(\mathfrak{m}_\eta)$ (up to isomorphism) which is η -semicuspidal. We have the following result; see [L4, 13.3].

(a) Let $A_1 \in \mathcal{Q}(\mathfrak{h}_\eta)$. There exists $C_1, C_2, \dots, C_t, C_{t+1}, \dots, C_{t+t'}$ in $\mathcal{Q}(\mathfrak{h}_\eta)$ such that

$$A_1 \oplus C_1 \oplus C_2 \oplus \dots \oplus C_t = C_{t+1} \oplus \dots \oplus C_{t+t'}$$

and each C_j is of the form $\text{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(\tilde{A})[a_j]$ for some $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{A}) \in \mathcal{X}$ (depending on j) and some $a_j \in \mathbf{Z}$.

Erratum to [L4]. In the definition of a good object in the second paragraph of [L4, 13.2], one should insert the words “shifts of” after “direct sum of” (twice).

1.10. Let $s \in \mathbf{Z} - \{0\}$. We show:

(a) the subspace $\mathfrak{h}^{(1)} := \bigoplus_{k \in s\mathbf{Z}} \mathfrak{h}_k$ of \mathfrak{h} is the Lie algebra of a well-defined connected reductive subgroup $H^{(1)}$ of H .

We can assume that $s > 0$. We shall define $e \in Z_{>0}$ as follows: if $p = 0$ we have $e = 0$; if $p > 0$ we define e by $s = s'p^e$, where $s' \in \mathbf{Z}_{>0}$ is not divisible by p . We shall argue by induction on e . (When $p = 0$ we only have to consider the case $e = 0$.) Assume first that $e = 0$.

Let \bar{H} be the adjoint group of H and let $\bar{\mathfrak{h}}$ be its Lie algebra. Then $\bar{\mathfrak{h}}$ inherits a \mathbf{Z} -grading $\bar{\mathfrak{h}} = \bigoplus_k \bar{\mathfrak{h}}_k$ from \mathfrak{h} . If we assume known that $\bar{\mathfrak{h}}^{(1)} := \bigoplus_{k \in s\mathbf{Z}} \bar{\mathfrak{h}}_k$ is the Lie algebra of a well-defined connected reductive subgroup $\bar{H}^{(1)}$ of \bar{H} , then we can take $H^{(1)}$ to be the identity component of the inverse image of $\bar{H}^{(1)}$ under the obvious map $H \rightarrow \bar{H}$. Thus we can assume that H is adjoint. Let $\lambda \in Y_H$ be such that $\lambda_k \mathfrak{h} = \mathfrak{h}_k$ for all k . Let ζ' be a primitive s -th root of 1 in \mathbf{k} . (Note that if $p > 0$, $s = s'$ is not divisible by p .) We define $\omega : H \rightarrow H$ by $\omega(g) = \text{Ad}(\lambda(\zeta'))(g)$; this is an automorphism of H . The automorphism $\omega' : \mathfrak{h} \rightarrow \mathfrak{h}$ induced by ω sends $x \in \mathfrak{h}_k$ (where $k \in \mathbf{Z}$) to $\zeta'^k x$. Hence $\omega^s = 1$ and $\mathfrak{h}^{(1)}$ is equal to $\{x \in \mathfrak{h}; \omega(x) = x\}$. Let $H^{(1)}$ be the identity component of $\{g \in H; \omega(g) = g\}$. This is a connected reductive

group with Lie algebra $\mathfrak{h}^{(1)}$. Thus (a) is proved in the case $e = 0$. We now assume that $e \geq 1$ hence $p > 0$. We can find an element $x_0 \in \mathfrak{h}$ such that $[x_0, x] = kx$ for any $k \in \mathbf{Z}$ and any $x \in \mathfrak{g}_k$. (We can take x_0 in the image of the tangent map of $\lambda : \mathbf{k}^* \rightarrow H$.) Let $\tilde{\mathfrak{h}} = \{x \in \mathfrak{h}; [x_0, x] = 0\}$. We have $\tilde{\mathfrak{h}} = \bigoplus_{k \in p\mathbf{Z}} \mathfrak{h}_k$. Let \tilde{H} be the identity component of $\{g \in H; \text{Ad}(g)x_0 = x_0\}$. Since $x_0 \in \mathfrak{h}$ is semisimple, it follows that \tilde{H} is reductive with Lie algebra $\tilde{\mathfrak{h}}$. We define a \mathbf{Z} -grading $\tilde{\mathfrak{h}} = \bigoplus_{k' \in \mathbf{Z}} \tilde{\mathfrak{h}}_{k'}$ by $\tilde{\mathfrak{h}}_{k'} = \mathfrak{h}_{pk'}$. By the induction hypothesis applied to $\tilde{H}, \tilde{\mathfrak{h}}$ we see that there is a well-defined connected reductive subgroup $\tilde{H}^{(1)}$ of \tilde{H} whose Lie algebra is $\bigoplus_{k' \in (s/p)\mathbf{Z}} \tilde{\mathfrak{h}}_{k'} = \bigoplus_{k' \in (s/p)\mathbf{Z}} \mathfrak{h}_{pk'} = \bigoplus_{k \in s\mathbf{Z}} \mathfrak{h}_k = \mathfrak{h}^{(1)}$. We can take $H^{(1)} = \tilde{H}^{(1)}$. This completes the inductive proof.

2. \mathbf{Z} \mapsto -GRADINGS AND ϵ -SPIRALS

In this section we introduce the key notion of this paper, namely a spiral. Spirals are analogues in the \mathbf{Z}/m -graded setting of parabolic subalgebras in the ungraded or \mathbf{Z} -graded setting. We also attach a canonical spiral to each nilpotent element in \mathfrak{g}_δ .

2.1. In the rest of this paper, $m \geq 1$, $G, \mathfrak{g} = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{g}_i$ are as in 0.1 and ζ, ϑ, θ are as in 0.5. Recall that for $i \in \mathbf{Z}/m$ we have $\mathfrak{g}_i = \{x \in \mathfrak{g}; \theta(x) = \zeta^i x\}$ and that $\vartheta : G \rightarrow G$ is the (semisimple) automorphism of G which induces $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$; note that $\theta(\text{Ad}(g)x) = \text{Ad}(\vartheta(g))\theta(x)$ for all $x \in \mathfrak{g}, g \in G$.

We shall fix $\delta \in \mathbf{Z}/m$.

For any semisimple automorphism $\gamma : G \rightarrow G$, we set $G^\gamma = \{g \in G; \gamma(g) = g\}$. By a theorem of Steinberg [St],

(a) G^γ is a connected reductive subgroup of G .

Now $\mathfrak{g}_\underline{0}$ is a Lie subalgebra of \mathfrak{g} . Recall that $G_\underline{0} = G^\vartheta$ and that the Ad-action of $G_\underline{0}$ on \mathfrak{g} leaves stable \mathfrak{g}_i and its closed subset $\mathfrak{g}_i^{nil} := \mathfrak{g}_i \cap \mathfrak{g}^{nil}$ for any $i \in \mathbf{Z}/m$.

Let \mathfrak{G} be the set of subgroups of G of the form $G^{\text{Ad}(\tau)\vartheta}$ for some semisimple element of finite order $\tau \in G_\underline{0}$; by (a), any group in \mathfrak{G} is a connected reductive subgroup of G . For example, we have $G_\underline{0} \in \mathfrak{G}$; hence we have $G_\underline{0} = e^{\mathfrak{g}_\underline{0}}$.

2.2. Let $\langle, \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbf{k}$ be a Killing form; it is nondegenerate and it satisfies $\langle \mathfrak{g}_i, \mathfrak{g}_j \rangle = 0$ whenever $i + j \neq \underline{0}$ in \mathbf{Z}/m . Hence for any $i \in \mathbf{Z}/m, \langle, \rangle : \mathfrak{g}_i \times \mathfrak{g}_{-i} \rightarrow \mathbf{k}$ is nondegenerate.

2.3. **The Morozov-Jacobson theorem in the \mathbf{Z}/m -graded setting.** We set $J = J^G$; see 1.1. For $x \in \mathfrak{g}^{nil}$ let $J(x) = \{(e, h, f) \in J; e = x\}$, $G(x) = \{g \in G; \text{Ad}(g)x = x\}$ and let $U = U_{G(x)^o}$. Recall the following result of Morozov-Jacobson and Kostant; see [Ko].

(a) We have $J(x) \neq \emptyset$. The U -action on $J(x)$ given by

$$u : (e, h, f) \mapsto u(e, h, f) := (e, \text{Ad}(u)h, \text{Ad}(u)f)$$

is simply transitive.

Assume now that $x \in \mathfrak{g}_\delta^{nil}$. We set

$$J_\delta(x) = \{(e, h, f) \in J(x); e = x, h \in \mathfrak{g}_\underline{0}, f \in \mathfrak{g}_{-\delta}\}.$$

We show:

(b) We have $J_\delta(x) \neq \emptyset$. The $(U \cap G_\underline{0})$ -action on $J_\delta(x)$ (restriction of the U -action in (a)) is simply transitive.

If $(e, h, f) \in J(x)$, then $(\zeta^{-\delta}e, h, \zeta^\delta f) \in J_\delta(\zeta^{-\delta}x)$ and

$$(\zeta^{-\delta}\theta(e), \theta(h), \zeta^\delta\theta(f)) \in J(\zeta^{-\delta}\theta(x)) = J(x)$$

(we use that $\theta(e) = \zeta^\delta e$). Hence $(e, h, f) \mapsto (\zeta^{-\delta}\theta(e), \theta(h), \zeta^\delta\theta(f))$ is a morphism $\theta' : J(x) \rightarrow J(x)$. Next we note that $g \mapsto \vartheta(g)$ defines a homomorphism $G(x) \rightarrow G(x)$. (If $\text{Ad}(g)x = x$, then $\theta(x) = \theta(\text{Ad}(g)x) = \text{Ad}(\vartheta(g))\theta(x)$. Since $\theta(x) = \zeta^\delta x$, we see that $\zeta^\delta x = \text{Ad}(\vartheta(g))\zeta^\delta x$ hence $x = \text{Ad}(\vartheta(g))x$ and $\vartheta(g) \in G(x)$.) This restricts to a homomorphism $\theta'' : U \rightarrow U$ with fixed point set $U^{\theta''}$. For $u \in U$, $(e, h, f) \in J(x)$ we have $\theta'(u(e, h, f)) = \theta''(u)\theta'(e, h, f)$. By (a), $J(x)$ is an affine space. Since $\theta''^m = 1$ and m is invertible in \mathbf{k} , the fixed point set $J(x)^{\theta'}$ is nonempty. Since the U -action on $J(x)$ is simply transitive, it follows that this restricts to a simply transitive action of $U^{\theta''}$ on $J(x)^{\theta'}$. We have $J(x)^{\theta'} = J_\delta(x)$ and $U^{\theta''} = U \cap G_0$. We see that (b) holds.

2.4. Let $\lambda \in Y_{G_0}$ (resp. $\mu \in Y_{G_0, \mathbf{Q}}$). Since λ (resp. μ) can be viewed as an element of Y_G (resp. $Y_{G, \mathbf{Q}}$), the decomposition $\mathfrak{g} = \bigoplus_{k \in \mathbf{Z}} (\lambda_k \mathfrak{g})$ (resp. $\mathfrak{g} = \bigoplus_{\kappa \in \mathbf{Q}} (\mu_\kappa \mathfrak{g})$) is defined as in 1.1. For $i \in \mathbf{Z}/m$ and for $k \in \mathbf{Z}$ (resp. $\kappa \in \mathbf{Q}$) we set $\lambda_k \mathfrak{g}_i = \lambda_k \mathfrak{g} \cap \mathfrak{g}_i$ (resp. $\mu_\kappa \mathfrak{g}_i = \mu_\kappa \mathfrak{g} \cap \mathfrak{g}_i$); we then have $\mathfrak{g}_i = \bigoplus_{k \in \mathbf{Z}} (\lambda_k \mathfrak{g}_i)$ (resp. $\mathfrak{g}_i = \bigoplus_{\kappa \in \mathbf{Q}} (\mu_\kappa \mathfrak{g}_i)$) for any $i \in \mathbf{Z}/m$ (we now use that $\lambda \in Y_{G_0}$ (resp. $\mu \in Y_{G_0, \mathbf{Q}}$)).

Let $s \in \mathbf{Z} - \{0\}$. We show:

(a) *The subspace $\mathfrak{g}^{(1)} := \bigoplus_{k \in s\mathbf{Z}} (\lambda_k \mathfrak{g}_{k/s})$ of \mathfrak{g} is the Lie algebra of a well-defined connected reductive subgroup $G^{(1)}$ of G .*

We apply 1.10(a) to $H = G$, $\mathfrak{h} = \mathfrak{g}$ with the \mathbf{Z} -grading $\mathfrak{g} = \bigoplus_k (\lambda_k \mathfrak{g})$. We see that there is a well-defined reductive connected subgroup $H^{(1)}$ of G whose Lie algebra is $\mathfrak{h}^{(1)} = \bigoplus_{k \in s\mathbf{Z}} (\lambda_k \mathfrak{g})$. Note that $H^{(1)}$ contains $\lambda(\mathbf{k}^*)$ and is ϑ -stable. We choose $\zeta' \in \mathbf{k}^*$ such that $\zeta'^s = \zeta$. We define $\omega : H^{(1)} \rightarrow H^{(1)}$ by $\omega(h) = \text{Ad}(\lambda(\zeta'))^{-1}\vartheta(h)$; this is an automorphism of $H^{(1)}$. The automorphism $\omega' : \mathfrak{h}^{(1)} \rightarrow \mathfrak{h}^{(1)}$ induced by ω sends $x \in \lambda_k \mathfrak{g}_i$ (where $k \in s\mathbf{Z}$, $i \in \mathbf{Z}/m$) to $\zeta'^{-k}\zeta^i x = \zeta^{i-k/s}x$. Hence $\omega'^m = 1$ and $\mathfrak{g}^{(1)}$ is equal to $\{x \in \mathfrak{h}^{(1)}; \omega'(x) = x\}$. Let $G^{(1)}$ be the identity component of $\{h \in H^{(1)}; \omega(h) = h\}$. Then $G^{(1)}$ is a connected reductive subgroup of $H^{(1)}$ with Lie algebra $\mathfrak{g}^{(1)}$. This proves (a).

Now $\lambda_0 \mathfrak{g}_0$ is a Levi subalgebra of a parabolic subalgebra of \mathfrak{g}_0 . Hence $e^{\lambda_0 \mathfrak{g}_0}$ is a well-defined subgroup of G_0 (a Levi subgroup of a parabolic subgroup of G_0). We have

(b) $e^{\lambda_0 \mathfrak{g}_0} \subset G^{(1)}$.

2.5. **The definition of ϵ -spirals.** In the rest of this section we fix $\epsilon \in \{1, -1\}$. For any $\mu \in Y_{G_0, \mathbf{Q}}$ and any $N \in \mathbf{Z}$ we set

(a)
$$\epsilon \mathfrak{p}_N^\mu = \bigoplus_{\kappa \in \mathbf{Q}; \kappa \geq N\epsilon} (\mu_\kappa \mathfrak{g}_N).$$

If $r \in \mathbf{Z}_{>0}$ is such that $\lambda := r\mu \in Y_{G_0}$ then we have

$$\epsilon \mathfrak{p}_N^\mu = \bigoplus_{k \in \mathbf{Z}; k \geq rN\epsilon} (\lambda_k \mathfrak{g}_N).$$

A collection $\{\mathfrak{p}_N; N \in \mathbf{Z}\}$ (or \mathfrak{p}_*) of subspaces of \mathfrak{g} is said to be an ϵ -spiral if there exists $\mu \in Y_{G_0, \mathbf{Q}}$ such that $\mathfrak{p}_N = \epsilon \mathfrak{p}_N^\mu$ for any $N \in \mathbf{Z}$. We then set (for $N \in \mathbf{Z}$)

$$\mathfrak{u}_N = \{x \in \mathfrak{g}_N; \langle x, \epsilon \mathfrak{p}_{-N}^\mu \rangle = 0\} = \bigoplus_{\kappa \in \mathbf{Q}; \kappa > N\epsilon} (\mu_\kappa \mathfrak{g}_N).$$

We say that $\mathfrak{u}_* = \{\mathfrak{u}_N; N \in \mathbf{Z}\}$ is the nilradical of \mathfrak{p}_* .

The following properties of $\mathfrak{p}_*, \mathfrak{u}_*$ are immediate:

$$\begin{aligned} & \dots \subset \mathfrak{p}_N \subset \mathfrak{p}_{N-\epsilon m} \subset \mathfrak{p}_{N-2\epsilon m} \subset \dots \text{ for any } N; \\ & \mathfrak{p}_N \subset \mathfrak{g}_N \text{ for any } N; \mathfrak{p}_N = 0 \text{ if } N\epsilon \gg 0; \mathfrak{p}_N = \mathfrak{g}_N \text{ if } N\epsilon \ll 0; \\ & [\mathfrak{p}_N, \mathfrak{p}_{N'}] \subset \mathfrak{p}_{N+N'} \text{ for any } N, N' \text{ in } \mathbf{Z}; \\ & \dots \subset \mathfrak{u}_N \subset \mathfrak{u}_{N-\epsilon m} \subset \mathfrak{u}_{N-2\epsilon m} \subset \dots \text{ for any } N; \\ & \mathfrak{u}_N \subset \mathfrak{p}_N \text{ for any } N; \mathfrak{u}_N = \mathfrak{g}_N \text{ if } N\epsilon \ll 0; \\ & [\mathfrak{u}_N, \mathfrak{p}_{N'}] \subset \mathfrak{u}_{N+N'} \text{ for any } N, N' \text{ in } \mathbf{Z}. \end{aligned}$$

For $N \in \mathbf{Z}$ we set $\mathfrak{l}_N = \mathfrak{p}_N/\mathfrak{u}_N$ and $\mathfrak{l} = \bigoplus_{N \in \mathbf{Z}} \mathfrak{l}_N$. We have $\mathfrak{l}_N = 0$ if $N \gg 0$ or if $N \ll 0$ hence $\dim \mathfrak{l} < \infty$; moreover, $[\cdot, \cdot] : \mathfrak{p}_N \times \mathfrak{p}_{N'} \rightarrow \mathfrak{p}_{N+N'}$ induces an operation $\mathfrak{l}_N \times \mathfrak{l}_{N'} \rightarrow \mathfrak{l}_{N+N'}$ which defines a Lie algebra structure on \mathfrak{l} .

Note that \mathfrak{p}_0 is a parabolic subalgebra of the reductive Lie algebra \mathfrak{g}_0 and $\mathfrak{u}_0 = \{x \in \mathfrak{g}_0; \langle x, \mathfrak{p}_0 \rangle = 0\}$ is the nilradical of \mathfrak{p}_0 . We set $P_0 = e^{\mathfrak{p}_0} \subset G_0$, $U_0 = e^{\mathfrak{u}_0} \subset G_0$. Then P_0 is a parabolic subgroup of G_0 and $U_0 = U_{P_0}$, so that $L_0 := P_0/U_0$ is a connected reductive group. We note that:

(b) *The Ad-action of P_0 on \mathfrak{g} leaves stable \mathfrak{p}_N and \mathfrak{u}_N for any N .*

From (b) we see that for any N there is an induced action of P_0 on $\mathfrak{l}_N = \mathfrak{p}_N/\mathfrak{u}_N$. We show:

(c) *The restriction of this action to U_0 is trivial.*

It is enough to show that the ad-action of \mathfrak{u}_0 on $\mathfrak{p}_N/\mathfrak{u}_N$ is zero. This follows from the inclusion $[\mathfrak{u}_0, \mathfrak{p}_N] \subset \mathfrak{u}_N$ which has been noted earlier.

From (b),(c) we see that for any N there is an induced action of $L_0 = P_0/U_0$ on $\mathfrak{l}_N = \mathfrak{p}_N/\mathfrak{u}_N$. We show:

(d) *if $x \in \mathfrak{p}_N$, $N\epsilon > 0$, then $x \in \mathfrak{g}_N^{nil}$.*

It is enough to show that for any $x' \in \mathfrak{g}$ we have $\text{ad}(x)^n(x') = 0$ for $n \gg 0$. We can assume that $x' \in \mathfrak{g}_i$ for some $i \in \mathbf{Z}/m$. If $N' \in \mathbf{Z}$ satisfies $\underline{N'} = i$ and $N'\epsilon \ll 0$, then $\mathfrak{p}_{N'} = \mathfrak{g}_i$; thus we have $x' \in \mathfrak{p}_{N'}$ for some N' . We have $\text{ad}(x)x' = [x, x'] \in \mathfrak{p}_{N+N'}$, $\text{ad}(x)^2(x') \in \mathfrak{p}_{2n+N'}$ and, more generally, $\text{ad}(x)^n(x') \in \mathfrak{p}_{nN+N'}$ for $n \geq 1$. If $n \gg 0$ we have $nN\epsilon + N'\epsilon \gg 0$ hence $\mathfrak{p}_{nN+N'} = 0$; thus, $\text{ad}(x)^n(x') = 0$. This proves (d).

An element $\mu \in Y_{G_0, \mathbf{Q}}$ is said to be *p-regular* if $r\mu \in Y_{G_0}$ for some $r \in \mathbf{Z}_{>0}$ such that $r \notin p\mathbf{Z}$. (This condition holds automatically if $p = 0$.) An ϵ -spiral \mathfrak{p}_* is said to be *p-regular* if $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu$ for some *p-regular* $\mu \in Y_{G_0, \mathbf{Q}}$.

2.6. Splittings of ϵ -spirals. For $\mu \in Y_{G_0, \mathbf{Q}}$ and $N \in \mathbf{Z}$ we set

$$\tilde{\mathfrak{l}}_N^\mu = \bigoplus_{\kappa \in \mathbf{Q}; \kappa = N\epsilon} (\mu_\kappa \mathfrak{g}_N) = \mu_{N\epsilon} \mathfrak{g}_N.$$

If $r \in \mathbf{Z}_{>0}$ is such that $\lambda := r\mu \in Y_{G_0}$, then we have

$$\tilde{\mathfrak{l}}_N^\mu = \lambda_{rN\epsilon} \mathfrak{g}_N.$$

A *splitting* of an ϵ -spiral \mathfrak{p}_* is a collection $\{\tilde{\mathfrak{l}}_N; N \in \mathbf{Z}\}$ (or $\tilde{\mathfrak{l}}_*$) of subspaces of \mathfrak{g} such that for some $\mu \in Y_{G_0, \mathbf{Q}}$ we have $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu$ and $\tilde{\mathfrak{l}}_N = \epsilon \tilde{\mathfrak{l}}_N^\mu$ for any $N \in \mathbf{Z}$. Let \mathfrak{u}_* be the nilradical of \mathfrak{p}_* . From the definitions we see that $\mathfrak{p}_N = \mathfrak{u}_N \oplus \tilde{\mathfrak{l}}_N$ for any N , $[\tilde{\mathfrak{l}}_N, \tilde{\mathfrak{l}}_{N'}] \subset \tilde{\mathfrak{l}}_{N+N'}$ for any N, N' and the sum $\tilde{\mathfrak{l}} := \sum_{N \in \mathbf{Z}} \tilde{\mathfrak{l}}_N$ (in \mathfrak{g}) is direct. Now $\tilde{\mathfrak{l}}$ is a Lie subalgebra of \mathfrak{g} which is \mathbf{Z} -graded by the subspaces $\tilde{\mathfrak{l}}_N$. Note that the isomorphisms $\tilde{\mathfrak{l}}_N \xrightarrow{\sim} \mathfrak{l}_N$ (restrictions of the obvious maps $\mathfrak{p}_N \rightarrow \mathfrak{l}_N$) give rise after taking \bigoplus_N to an isomorphism $\tilde{\mathfrak{l}} \xrightarrow{\sim} \mathfrak{l}$ which is compatible with the Lie algebra structures and the \mathbf{Z} -gradings.

For μ as above we can find $\lambda \in Y_{G_{\mathbb{Q}}}$ and $r \in \mathbf{Z}_{>0}$ such that $r\mu = \lambda$. Applying 2.4(a) with $s = r\epsilon$ we see that:

(a) *There is a well-defined connected reductive subgroup \tilde{L} of G whose Lie algebra is $\tilde{\mathfrak{l}}$. In particular, $\tilde{\mathfrak{l}}$ and \mathfrak{l} are reductive Lie algebras.*

Let $\tilde{L}_0 = e^{\tilde{\mathfrak{l}}_0}$. From 2.4(b) we have:

(b) $\tilde{L}_0 \subset \tilde{L}$.

We show:

(c) *Assume that we have $\tilde{\mathfrak{l}}_* = \epsilon \tilde{\mathfrak{l}}_*^\mu$, $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu$ where μ is p -regular, that is, $\mu = r\lambda$ with $\lambda \in Y_{G_{\mathbb{Q}}}$ and $r \in \mathbf{Z}_{>0}$ such that $r \notin p\mathbf{Z}$. Then there exists ζ' , a root of 1 in \mathbf{k}^* such that $\tilde{\mathfrak{l}} = \{x \in \mathfrak{g}; \text{Ad}(\lambda(\zeta')^{-1})\theta(x) = x\}$, $\tilde{L} = G^{\text{Ad}(\lambda(\zeta')^{-1})^\theta} = e^{\tilde{\mathfrak{l}}} \subset G$; note that $\tilde{L} \in \mathfrak{S}$.*

Let ζ' be a primitive root of 1 of order rm in \mathbf{k}^* such that $\zeta'^{r\epsilon} = \zeta$. We have $\mathfrak{g} = \bigoplus_{k \in \mathbf{Z}, i \in \mathbf{Z}/m} (\lambda_k^i \mathfrak{g}_i)$, $\tilde{\mathfrak{l}}_N = \lambda_{Nr\epsilon} \mathfrak{g}_N$ for all $N \in \mathbf{Z}$. For $k, N' \in \mathbf{Z}$ and $x \in \lambda_k^i \mathfrak{g}_{N'}$ we have

$$\text{Ad}(\lambda(\zeta')^{-1})(\theta(x)) = \zeta'^{-k} \zeta^{N'} x = \zeta'^{rN'\epsilon - k} x.$$

The condition that $\zeta'^{rN'\epsilon - k} = 1$ is that $rN'\epsilon - k \in rm\mathbf{Z}$ or that $k \in r\mathbf{Z}$ and $N' = k/(r\epsilon)$. We see that

$$\{x \in \mathfrak{g}; \text{Ad}(\lambda(\zeta')^{-1})(\theta(x)) = x\} = \bigoplus_{k \in r\mathbf{Z}, i \in \mathbf{Z}/m; k/(r\epsilon) = i} (\lambda_k^i \mathfrak{g}_i) = \bigoplus_{N \in \mathbf{Z}} (\lambda_{Nr\epsilon} \mathfrak{g}_N) = \tilde{\mathfrak{l}},$$

and (c) follows.

We return to the general case.

We have $\lambda(\mathbf{k}^*) \subset \tilde{L}_0$; moreover, $\text{Ad}(\lambda(t))$ acts as identity on $\tilde{\mathfrak{l}}_0 = \lambda_0 \mathfrak{g}_0 = \mathfrak{L}\tilde{L}_0$; thus, $\lambda(\mathbf{k}^*) \subset \mathfrak{Z}_{\tilde{L}_0}$. Since \mathbf{k}^* is connected, we deduce:

(d) $\lambda(\mathbf{k}^*) \subset \mathfrak{Z}_{\tilde{L}_0}^0$.

Note that:

(e) *For $t \in \mathbf{k}^*, N \in \mathbf{Z}$, $\text{Ad}(\lambda(t))$ acts on \mathfrak{l}_N as $t^{rN\epsilon}$ times identity.*

We show:

(f) *If $\tilde{\mathfrak{l}}_*$ is a splitting of an ϵ -spiral \mathfrak{p}_* , then $\tilde{\mathfrak{l}}_*$ is a splitting of an $(-\epsilon)$ -spiral.*

Let $\mu \in Y_{G_{\mathbb{Q}}, \mathbf{Q}}$ be such that $\tilde{\mathfrak{l}}_* = \epsilon \tilde{\mathfrak{l}}_*^\mu$, $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu$. Let $\mu' = (-1)\mu \in Y_{G_{\mathbb{Q}}, \mathbf{Q}}$. Then $\tilde{\mathfrak{l}}_* = -\epsilon \tilde{\mathfrak{l}}_*^{\mu'}$ is a splitting of the $(-\epsilon)$ -spiral $-\epsilon \mathfrak{p}_*^{\mu'}$.

2.7. Let \mathfrak{S} be the set of splittings of an ϵ -spiral \mathfrak{p}_* . Clearly, $\mathfrak{S} \neq \emptyset$. Let U_0 be as in 2.5. Now U_0 acts on \mathfrak{S} by $u : \tilde{\mathfrak{l}}_* \mapsto \{\text{Ad}(u)\tilde{\mathfrak{l}}_N; N \in \mathbf{Z}\}$. (We use that $\text{Ad}(u)\mathfrak{p}_N = \mathfrak{p}_N$ for any N .) We show:

(a) *This U_0 -action on \mathfrak{S} is simply transitive.*

Let \mathfrak{u}_* be the nilradical of \mathfrak{p}_* . Let $\tilde{\mathfrak{l}}_* \in \mathfrak{S}$, $\tilde{\mathfrak{l}}'_* \in \mathfrak{S}$. Since $\tilde{\mathfrak{l}}_0, \tilde{\mathfrak{l}}'_0$ are Levi subalgebras of \mathfrak{p}_0 , there is a unique $u \in U_0$ such that $\text{Ad}(u)\tilde{\mathfrak{l}}_0 = \tilde{\mathfrak{l}}'_0$. It remains to show that this u satisfies $\text{Ad}(u)\tilde{\mathfrak{l}}_N = \tilde{\mathfrak{l}}'_N$ for any N . Let $\tilde{\mathfrak{l}} = \bigoplus_N \tilde{\mathfrak{l}}_N$, $\tilde{\mathfrak{l}}' = \bigoplus_N \tilde{\mathfrak{l}}'_N$ (these are Lie subalgebras of \mathfrak{g}) and let $\tilde{L} = e^{\tilde{\mathfrak{l}}} \subset G$, $\tilde{L}' = e^{\tilde{\mathfrak{l}}'} \subset G$. Let μ, μ' in $Y_{G_{\mathbb{Q}}, \mathbf{Q}}$ be such that $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu = \epsilon \mathfrak{p}_*^{\mu'}$, $\tilde{\mathfrak{l}}_* = \epsilon \tilde{\mathfrak{l}}_*^\mu$, $\tilde{\mathfrak{l}}'_* = \epsilon \tilde{\mathfrak{l}}_*^{\mu'}$. We can find $r \in \mathbf{Z}_{>0}$ such that $\lambda := r\mu \in Y_{G_{\mathbb{Q}}}$, $\lambda' := r\mu' \in Y_{G_{\mathbb{Q}}}$. Let \tilde{L}_0 be as in 2.6 and let \tilde{L}'_0 be the analogous subgroup of \tilde{L}' . We now fix $N \in \mathbf{Z}$. The Ad-action of \tilde{L}_0 (resp. \tilde{L}'_0) on \mathfrak{g} leaves stable $\tilde{\mathfrak{l}}_N, \mathfrak{u}_N$ (resp. $\tilde{\mathfrak{l}}'_N, \mathfrak{u}_N$). Let $\tilde{L}''_0 = u\tilde{L}_0u^{-1}$, $\tilde{\mathfrak{l}}''_N = \text{Ad}(u)\tilde{\mathfrak{l}}_N$; then the Ad-action of \tilde{L}''_0 on \mathfrak{g} leaves stable $\tilde{\mathfrak{l}}''_N, \mathfrak{u}_N$. Since $\text{Ad}(u)\tilde{\mathfrak{l}}_0 = \tilde{\mathfrak{l}}'_0$, we have $u\tilde{L}_0u^{-1} = \tilde{L}'_0$ hence $\tilde{L}''_0 = \tilde{L}'_0$. Let T be a maximal torus of $\tilde{L}'_0 = \tilde{L}''_0$. Now the Ad-action of T on \mathfrak{g} leaves stable

$\tilde{l}'_N, \tilde{l}''_N, \mathbf{u}_N, \mathfrak{p}_N$. Let $\mathcal{X} = \text{Hom}(T, \mathbf{k}^*)$. For any $\alpha \in \mathcal{X}$ let

$$\mathfrak{p}_{N,\alpha} = \{x \in \mathfrak{p}_N; \text{Ad}(\tau)x = \alpha(\tau)x \quad \forall \tau \in T\}, \quad \mathbf{u}_{N,\alpha} = \mathbf{u}_N \cap \mathfrak{p}_{N,\alpha},$$

$$\tilde{l}'_{N,\alpha} = \tilde{l}'_N \cap \mathfrak{p}_{N,\alpha}, \quad \tilde{l}''_{N,\alpha} = \tilde{l}''_N \cap \mathfrak{p}_{N,\alpha}.$$

We have $\tilde{l}'_N = \bigoplus_{\alpha \in \mathcal{X}} \tilde{l}'_{N,\alpha}$, $\tilde{l}''_N = \bigoplus_{\alpha \in \mathcal{X}} \tilde{l}''_{N,\alpha}$, $\mathbf{u}_N = \bigoplus_{\alpha \in \mathcal{X}} \mathbf{u}_{N,\alpha}$. Let $\mathcal{R}' = \{\alpha \in \mathcal{X}; \tilde{l}'_{N,\alpha} \neq 0\}$, $\mathcal{R}'' = \{\alpha \in \mathcal{X}; \tilde{l}''_{N,\alpha} \neq 0\}$, $\tilde{\mathcal{R}} = \{\alpha \in \mathcal{X}; \mathbf{u}_{N,\alpha} \neq 0\}$. Since $\tilde{l}'_N, \tilde{l}''_N$ are T -stable complements of \mathbf{u}_N in \mathfrak{p}_N , the T -modules $\tilde{l}'_N, \tilde{l}''_N$ are isomorphic, hence $\mathcal{R}' = \mathcal{R}''$. Since $\lambda'(\mathbf{k}^*) \subset \mathcal{Z}_{\tilde{L}_0}^0$ (see 2.6(d)), we have $\lambda'(\mathbf{k}^*) \subset T$; hence for any $\alpha \in \mathcal{X}$ we can define $\alpha \bullet \lambda' \in \mathbf{Z}$ by $\alpha(\lambda'(t)) = t^{\alpha \bullet \lambda'}$ for all $t \in \mathbf{k}^*$.

Assume that $\alpha \in \tilde{\mathcal{R}}$. Then for any $t \in \mathbf{k}^*$, $\text{Ad}(\lambda'(t))$ acts on $\mathbf{u}_{N,\alpha}$ as multiplication by $t^{\alpha \bullet \lambda'}$ hence $\mathbf{u}_{N,\alpha} \subset \lambda'_{\alpha \bullet \lambda'} \mathfrak{g}_N$; thus $\lambda'_{\alpha \bullet \lambda'} \mathfrak{g}_N$ has a nonzero intersection with \mathbf{u}_N , so that $\alpha \bullet \lambda' > rN\epsilon$. We see that $\tilde{\mathcal{R}} \subset \{\alpha \in \mathcal{X}; \alpha \bullet \lambda' > rN\epsilon\}$. Assume now that $\alpha \in \mathcal{R}'$. Then for any $t \in \mathbf{k}^*$, $\text{Ad}(\lambda'(t))$ acts on $\tilde{l}'_{N,\alpha}$ as multiplication by $t^{\alpha \bullet \lambda'}$ hence $\tilde{l}'_{N,\alpha} \subset \lambda'_{\alpha \bullet \lambda'} \mathfrak{g}_N$; thus, $\lambda'_{\alpha \bullet \lambda'} \mathfrak{g}_N$ has a nonzero intersection with \tilde{l}'_N , so that $\alpha \bullet \lambda' = rN\epsilon$. We see that $\mathcal{R}' \subset \{\alpha \in \mathcal{X}; \alpha \bullet \lambda' = rN\epsilon\}$. It follows that $\mathcal{R}' \cap \tilde{\mathcal{R}} = \emptyset$ so that $\mathfrak{p}_{N,\alpha} = \tilde{l}'_{N,\alpha}$ for $\alpha \in \mathcal{R}'$. Since $\mathcal{R}' = \mathcal{R}''$, we have also $\mathcal{R}' \cap \tilde{\mathcal{R}} = \emptyset$, so that $\mathfrak{p}_{N,\alpha} = \tilde{l}''_{N,\alpha}$ for $\alpha \in \mathcal{R}'' = \mathcal{R}'$. Thus, for $\alpha \in \mathcal{R}' = \mathcal{R}''$ we have $\tilde{l}'_{N,\alpha} = \tilde{l}''_{N,\alpha}$ hence $\tilde{l}'_N = \tilde{l}''_N$ and $\tilde{l}'_N = \text{Ad}(u)\tilde{l}_N$. This proves (a).

For any splitting \tilde{l}_* of \mathfrak{p}_* we denote by $\tilde{L}(\tilde{l}_*)$ the connected reductive subgroup \tilde{L} of G associated to \tilde{l}_* in 2.6. The family of groups $(\tilde{L}(\tilde{l}_*))$ indexed by the various splittings \tilde{l}_* of \mathfrak{p}_* has the property that any two groups in the family are canonically isomorphic to each other; the isomorphism is provided by conjugation by a well-defined $u \in U_0$ (this follows from (a)). It follows that the groups in the family can be identified with a single connected reductive group L which is canonically isomorphic to each group in the family. Note that L is canonically attached to the ϵ -spiral \mathfrak{p}_* and that $\mathfrak{L}L = \mathfrak{l}$ canonically. Note also that L_0 in 2.5 is naturally a closed subgroup of L .

2.8. Subspirals coming from parabolics of \mathfrak{l}_* . Let \mathfrak{p}_* be an ϵ -spiral. We define $\mathbf{u}_*, \mathfrak{l}_*, \mathfrak{l}$ in terms of \mathfrak{p}_* as in 2.5. Let \mathfrak{q} be a parabolic subalgebra of \mathfrak{l} compatible with the \mathbf{Z} -grading of \mathfrak{l} that is, such that $\mathfrak{q} = \bigoplus_{N \in \mathbf{Z}} \mathfrak{q}_N$ where $\mathfrak{q}_N = \mathfrak{q} \cap \mathfrak{l}_N$. For any $N \in \mathbf{Z}$ let $\hat{\mathfrak{p}}_N$ be the inverse image of \mathfrak{q}_N under the obvious map $\mathfrak{p}_N \rightarrow \mathfrak{l}_N$. We show:

(a) $\hat{\mathfrak{p}}_*$ is an ϵ -spiral. Moreover, if \mathfrak{p}_* is p -regular then $\hat{\mathfrak{p}}_*$ is p -regular.

We can find $\mu \in Y_{G_0, \mathbf{Q}}$ such that $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu$; let $\tilde{l}_* = \epsilon \tilde{l}_*^\mu$. Let \tilde{L} be as in 2.6. Let $\tilde{\mathfrak{q}}$ be the Lie subalgebra of $\tilde{\mathfrak{l}}$ corresponding to \mathfrak{q} under the obvious isomorphism $\tilde{\mathfrak{l}} \xrightarrow{\sim} \mathfrak{l}$ and let $\tilde{\mathfrak{q}}_N = \tilde{\mathfrak{q}} \cap \tilde{\mathfrak{l}}_N$ so that $\tilde{\mathfrak{q}} = \bigoplus_{N \in \mathbf{Z}} \tilde{\mathfrak{q}}_N$. We then have $\hat{\mathfrak{p}}_N = \mathbf{u}_N \oplus \tilde{\mathfrak{q}}_N$ for all N . Let $r \in \mathbf{Z}_{>0}$ be such that $\lambda := r\mu \in Y_{G_0}$; if \mathfrak{p}_* is p -regular we assume in addition that $r \notin p\mathbf{Z}$.

From 2.6(e) we see that for $t \in \mathbf{k}^*$, $\text{Ad}(\lambda(t))$ leaves stable each $\tilde{\mathfrak{q}}_N$ hence it leaves stable $\tilde{\mathfrak{q}}$. It follows that \mathbf{k}^* acts via $t \mapsto \text{Ad}(\lambda(t))$ on the variety of Levi subalgebras of $\tilde{\mathfrak{q}}$; since this variety is isomorphic to an affine space, there exists a Levi subalgebra \mathfrak{m} of $\tilde{\mathfrak{q}}$ such that $\text{Ad}(\lambda(t))\mathfrak{m} = \mathfrak{m}$ for all $t \in \mathbf{k}^*$. Let R be the closed connected subgroup of \tilde{L} (a torus) such that $\mathfrak{L}R$ is the center of \mathfrak{m} . Since $\tilde{\mathfrak{q}}$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}$ with Levi subalgebra \mathfrak{m} , we can find $\lambda' \in Y_R$ such that,

setting for any $N' \in \mathbf{Z}$:

$$\lambda'_{N'} \tilde{\mathfrak{l}} = \{x \in \tilde{\mathfrak{l}}; \text{Ad}(\lambda'(t))x = t^{N'}x \quad \forall t \in \mathbf{k}^*\},$$

we have $\tilde{\mathfrak{q}} = \bigoplus_{N' \in \mathbf{Z}_{>0}} (\lambda'_{N'} \tilde{\mathfrak{l}})$, $\mathfrak{m} = \lambda'_0 \tilde{\mathfrak{l}}$. We have $\mathfrak{m} = \bigoplus_N \mathfrak{m}_N$ where $\mathfrak{m}_N = \mathfrak{m} \cap \tilde{\mathfrak{l}}_N$ and \mathfrak{m}_0 is a Levi subalgebra of a parabolic subalgebra of \mathfrak{m} . Hence a Cartan subalgebra of $\mathfrak{m} \cap \tilde{\mathfrak{l}}_0$ is also a Cartan subalgebra of \mathfrak{m} , so that it contains the center of \mathfrak{m} . Thus the center of \mathfrak{m} is contained in $\tilde{\mathfrak{l}}_0$, so that $R \subset \tilde{L}_0$. Since for any $t, t' \in \mathbf{k}^*$, $\lambda(t)$ is contained in $\mathcal{Z}_{\tilde{L}_0}$ and $\lambda'(t') \in \tilde{L}_0$, we have $\lambda(t)\lambda'(t') = \lambda'(t')\lambda(t)$. We can view λ' as an element of Y_{G_0} hence $\lambda'_k \mathfrak{g}_i$ is defined for $k \in \mathbf{Z}, i \in \mathbf{Z}/m$ and we have $\mathfrak{g}_i = \bigoplus_{k \in \mathbf{Z}} (\lambda'_k \mathfrak{g}_i)$ for any $i \in \mathbf{Z}/m$. We can find $a \in \mathbf{Z}_{>0}$ such that $\lambda'_k \mathfrak{g}_i = 0$ for any $i \in \mathbf{Z}/m$ and any $k \in \mathbf{Z} - [-a, a]$. Let b be an integer such that $b > 2a, b \notin p\mathbf{Z}$. We define $\lambda'' \in Y_{G_0}$ by $\lambda''(t) = \lambda(t^b)\lambda'(t) = \lambda'(t)\lambda(t^b)$ for all $t \in \mathbf{k}^*$. By definition, for $k \in \mathbf{Z}, i \in \mathbf{Z}/m$ we have:

$$\begin{aligned} \lambda''_k \mathfrak{g}_i &= \{x \in \mathfrak{g}_i; \text{Ad}(\lambda(t^b)\lambda'(t))x = t^kx \quad \forall t \in \mathbf{k}^*\} \\ &= \bigoplus_{k', k_2; k' \in b\mathbf{Z}, k_2 \in \mathbf{Z}, k' + k_2 = k} (\lambda_{k'/b} \mathfrak{g}_i \cap \lambda'_{k_2} \mathfrak{g}_i). \end{aligned}$$

When $\lambda''_k \mathfrak{g}_i \neq 0$ then $k = bk_1 + k_2$ for some $k_1 \in \mathbf{Z} \cap [-a, a], k_2 \in \mathbf{Z}$; in this case, k_1, k_2 are uniquely determined by k since $b > 2a$. Thus, we have

$$\begin{aligned} \lambda''_k \mathfrak{g}_i &= \lambda_{k_1} \mathfrak{g}_i \cap \lambda'_{k_2} \mathfrak{g}_i \text{ if } k = bk_1 + k_2 \text{ with } k_1, k_2 \text{ in } \mathbf{Z}, \\ &\lambda''_k \mathfrak{g}_i = 0, \text{ otherwise.} \end{aligned}$$

Let $\mu' = \frac{1}{br}\lambda'' \in Y_{G_0, \mathbf{Q}}$ and let $\mathfrak{p}'_* = \epsilon \mathfrak{p}^{\mu'}$. For $N \in \mathbf{Z}$ we have

$$\mathfrak{p}'_N = \bigoplus_{k_1, k_2 \in \mathbf{Z}; bk_1 + k_2 \geq Nbr\epsilon, |k_2| \leq a} (\lambda_{k_1} \mathfrak{g}_N \cap \lambda'_{k_2} \mathfrak{g}_N).$$

The only integer multiple of b in $[-a, a]$ is 0; hence the condition that $k_2 \geq b(rN\epsilon - k_1)$ (with $k_2 \in [-a, a]$) is equivalent to the condition that either $0 > b(rN\epsilon - k_1), k_2 \in [-a, a]$ or that $0 = b(rN\epsilon - k_1), k_2 \in [0, a]$. Thus, $\mathfrak{p}'_N = X \oplus X'$, where

$$\begin{aligned} X &= \bigoplus_{k_1, k_2 \in \mathbf{Z}; k_1 > rN\epsilon} (\lambda_{k_1} \mathfrak{g}_N \cap \lambda'_{k_2} \mathfrak{g}_N) = \bigoplus_{k_1 \in \mathbf{Z}; k_1 > rN\epsilon} (\lambda_{k_1} \mathfrak{g}_N) = \mathfrak{u}_N, \\ X' &= \bigoplus_{k_1, k_2 \in \mathbf{Z}; k_1 = rN\epsilon, k_2 \geq 0} (\lambda_{k_1} \mathfrak{g}_N \cap \lambda'_{k_2} \mathfrak{g}_N) = \tilde{\mathfrak{l}}_N \cap (\bigoplus_{k_2 \in \mathbf{Z}_{\geq 0}} (\lambda'_{k_2} \mathfrak{g}_N)) = \tilde{\mathfrak{l}}_N \cap \tilde{\mathfrak{q}} = \tilde{\mathfrak{q}}_N. \end{aligned}$$

Thus, we have $\mathfrak{p}'_N = \mathfrak{u}_N \oplus \tilde{\mathfrak{q}}_N = \hat{\mathfrak{p}}_N$. This proves (a).

From the computation in the previous proof we can extract the following:

(b) *the splitting $\epsilon \tilde{\mathfrak{l}}^{\mu'}$ of the ϵ -spiral $\hat{\mathfrak{p}}_* = \epsilon \mathfrak{p}^{\mu'}$ is equal to \mathfrak{m}_* .*

2.9. The spiral attached to an element $x \in \mathfrak{g}_\delta^{nil}$. *In the remainder of this paper we fix $\eta \in \mathbf{Z} - \{0\}$ such that $\underline{\eta} = \delta$.*

In this subsection we assume that $\epsilon = \eta$; see 0.12. Let $x \in \mathfrak{g}_\delta^{nil}$. We associate to x an ϵ -spiral as follows. By 2.3(b), we can find $\phi = (e, h, f) \in \mathcal{J}_\delta(x)$ such that $e = x$. Let $\iota = \iota_\phi \in Y_G$ be as in 1.1. Since the differential of ι is the linear map $\mathbf{k} \rightarrow \mathfrak{g}, z \mapsto zh \in \mathfrak{g}_0$, we have $\iota(\mathbf{k}^*) \subset G_0$ so that ι can be viewed as an element of Y_{G_0} . Then $\mathfrak{p}^*_\phi := \epsilon \mathfrak{p}^{(|\eta|/2)\iota}$ is an ϵ -spiral with splitting $\tilde{\mathfrak{l}}^*_\phi := \epsilon \tilde{\mathfrak{l}}^{(|\eta|/2)\iota}$. Note that for $N \in \mathbf{Z}$ we have

$$\mathfrak{p}^\phi_N = \bigoplus_{k \in \mathbf{Z}; k \geq 2N\epsilon} (\iota_{k/|\eta|} \mathfrak{g}_N), \quad \tilde{\mathfrak{l}}^\phi_N = \iota_{2N/\eta} \mathfrak{g}_N \text{ if } 2N/\eta \in \mathbf{Z}, \quad \tilde{\mathfrak{l}}^\phi_N = 0 \text{ if } 2N/\eta \notin \mathbf{Z}.$$

We show that:

(a) *The ϵ -spiral \mathfrak{p}^ϕ_* is p -regular; it depends only on x , not on ϕ .*

The p -regularity follows from the fact that $2 \notin p\mathbf{Z}$. We now prove the second statement of (a). By 2.3(b), another choice for ϕ must be of the form $u\phi$ where $u \in U_{G(x)^0} \cap G_0$. Let $\iota' = \iota_{u\phi}$. For $t \in \mathbf{k}^*$ we have $\iota'(t) = u\iota(t)u^{-1}$ hence $\iota'_k \mathfrak{g}_i = \text{Ad}(u)(\iota_k \mathfrak{g}_i)$ for any $k \in \mathbf{Z}, i \in \mathbf{Z}/m$. It follows that for $N \in \mathbf{Z}$ we have $\mathfrak{p}_N^{u\phi} = \text{Ad}(u)\mathfrak{p}_N^\phi$. To show that $\mathfrak{p}_N^{u\phi} = \mathfrak{p}_N^\phi$, it is enough to show that $\text{Ad}(u)\mathfrak{p}_N^\phi = \mathfrak{p}_N^\phi$. It is enough to show:

$$\text{Ad}(u)(\iota_k \mathfrak{g}) \subset \oplus_{k'; k' \geq k} (\iota_{k'} \mathfrak{g}) \text{ for any } u \in G(x), k \in \mathbf{Z}.$$

Let P be the parabolic subgroup of G such that $\mathfrak{L}P = \oplus_{k \in \mathbf{Z}; k \geq 0} (\iota_k \mathfrak{g})$. Clearly, $\text{Ad}(g)(\iota_k \mathfrak{g}) \subset \oplus_{k'; k' \geq k} (\iota_{k'} \mathfrak{g})$ for any $g \in P, k \in \mathbf{Z}$. Hence it is enough to note the known inclusion $G(x) \subset P$. This proves (a).

In view of (a) we will write \mathfrak{p}_*^x instead of \mathfrak{p}_*^ϕ , where ϕ is any element in $J_\delta(x)$; let \mathfrak{u}_*^x be the nilradical of \mathfrak{p}_*^x . Now the splitting $\tilde{\mathfrak{l}}_*^\phi$ depends in general on ϕ . We set $\tilde{\mathfrak{l}}^\phi = \oplus_{N \in \mathbf{Z}} \tilde{\mathfrak{l}}_N^\phi$; this is a \mathbf{Z} -graded Lie subalgebra of \mathfrak{g} . Let $\tilde{L}^\phi = e^{\tilde{\mathfrak{l}}^\phi} \subset G$; we have $\tilde{L}^\phi \in \mathfrak{O}$. Let $\tilde{L}_0^\phi = e^{\tilde{\mathfrak{l}}_0^\phi} \subset \tilde{L}^\phi$. We show:

(b) *We have $x \in \tilde{\mathfrak{l}}_\eta^\phi$; more precisely, x belongs to $\tilde{\mathfrak{l}}_\eta^\phi$ (the open \tilde{L}_0^ϕ -orbit on $\tilde{\mathfrak{l}}_\eta^\phi$).*

The first statement is the same as $x \in \frac{1}{2}\mathfrak{g}_\delta$; this follows from the equality $[h, x] = 2x$. The second statement can be deduced from [L4, 4.2(a)].

We set $\tilde{L}_0^\phi(x) = \tilde{L}_0^\phi \cap G(x), G_0(x) = G_0 \cap G(x)$. We show:

(c) *The inclusion $\tilde{L}_0^\phi(x) \rightarrow G_0(x)$ induces an isomorphism on the groups of components.*

Let P_0 be the parabolic subgroup of G_0 such that $\mathfrak{L}P_0 = \mathfrak{p}_0^x = \oplus_{k \in \mathbf{Z}; k \geq 0} (\iota_k \mathfrak{g}_0)$ and let $U_0 = U_{P_0}$. We set $P_0(x) = P_0 \cap G(x), U_0(x) = U_0 \cap G(x)$. Then \tilde{L}_0^ϕ is a Levi subgroup of P_0 so that $P_0 = \tilde{L}_0^\phi U_0$ (semidirect product) and $P_0(x) = \tilde{L}_0^\phi(x) U_0(x)$ (semidirect product). Since $U_0(x)$ is a connected unipotent group we see that the inclusion $\tilde{L}_0^\phi(x) \rightarrow P_0(x)$ induces an isomorphism on the groups of components. It remains to show that $P_0(x) = G_0(x)$. As we have noted in the proof of (a), we have $G(x) \subset P$ hence $G_0(x) \subset P \cap G_0$; since $P \cap G_0$ and P_0 have the same Lie algebra, namely \mathfrak{p}_0^x , they must have the same identity component; since P_0 is parabolic in G_0 , we must have $P \cap G_0 = P_0$, so that $G_0(x) \subset P_0$ and therefore $G_0(x) \subset P_0(x)$. Since the reverse inclusion is obvious, we see that $P_0(x) = G_0(x)$ and (c) is proved.

We show:

(d) *If $g \in G_0$ is such that $\text{Ad}(g^{-1})(x) \in \mathfrak{p}_\eta^x$, then $g \in P_0$.*

The assumption of (d) implies that $g \in P$. (We use [L4, 5.7] applied to the trivial \mathbf{Z} -grading of \mathfrak{g} that is, the \mathbf{Z} -grading such that in [L4, 3.1] we have $\mathfrak{g}_N = 0$ for $N \neq 0$.) Thus, we have $g \in P \cap G_0$. As in the proof of (c) we have $P \cap G_0 = P_0$ and (d) follows.

We show:

(e) *The P_0 -orbit of x in \mathfrak{p}_η^x is open dense in \mathfrak{p}_η^x .*

We argue as in [L4, 5.9]. It is enough to show that $\dim(P_0) - \dim(P_0 \cap G(x)) = \dim \mathfrak{p}_\eta^x$ or equivalently that

$$\dim \mathfrak{p}_0^x - \dim \ker(\text{ad}(x) : \mathfrak{p}_0^x \rightarrow \mathfrak{g}_\delta) = \dim \mathfrak{p}_\eta^x.$$

Since $x \in \mathfrak{p}_\eta^x$ (see (b)) and $[\mathfrak{p}_0^x, \mathfrak{p}_\eta^x] \subset \mathfrak{p}_\eta^x$, we have $\text{ad}(x)(\mathfrak{p}_0^x) \subset \mathfrak{p}_\eta^x$ so that it is enough to show that

$$\dim \ker(\text{ad}(x) : \mathfrak{p}_0^x \rightarrow \mathfrak{p}_\eta^x) = \dim \mathfrak{p}_0^x - \dim \mathfrak{p}_\eta^x,$$

or equivalently, that $\text{ad}(x) : \mathfrak{p}_0^x \rightarrow \mathfrak{p}_\eta^x$ is surjective. By the representation theory of \mathfrak{sl}_2 , the linear map

$$\text{ad}(x) : \bigoplus_{k \in \mathbf{Z}; k \geq 0} \binom{l}{k} \mathfrak{g} \rightarrow \bigoplus_{k \in \mathbf{Z}; k \geq 2} \binom{l}{k} \mathfrak{g}$$

is surjective. This restricts for any $i \in \mathbf{Z}/m$ to a (necessarily surjective) map

$$\text{ad}(x) : \bigoplus_{k \in \mathbf{Z}; k \geq 0} \binom{l}{k} \mathfrak{g}_i \rightarrow \bigoplus_{k \in \mathbf{Z}; k \geq 2} \binom{l}{k} \mathfrak{g}_{i+\delta}.$$

Taking $i = 0$ we see that $\text{ad}(x) : \mathfrak{p}_0^x \rightarrow \mathfrak{p}_\eta^x$ is surjective. This proves (e).

The assignment $x \mapsto \mathfrak{p}_*^x$ is a \mathbf{Z}/m -analogue of an assignment in the case of \mathbf{Z} -graded Lie algebras given in [L4, §5] which is in turn modelled on a construction in [KL, 7.1].

3. ADMISSIBLE SYSTEMS

In this section we introduce the set $\underline{\mathfrak{T}}_\eta$ of G_0 -conjugacy classes of admissible systems, which will be used to index the blocks in $\mathcal{D}_{G_0}(\mathfrak{g}_\delta^{nil})$. We also define a map that assigns a pair $(\mathcal{O}, \mathcal{L})$ (where \mathcal{O} is a G_0 -orbit in $\mathfrak{g}_\delta^{nil}$ and \mathcal{L} is an irreducible G_0 -equivariant local system on it) an element in $\underline{\mathfrak{T}}_\eta$.

3.1. Definition of admissible systems. We preserve the setup of 2.1.

Let \mathfrak{T}'_η be the set consisting of all systems $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$, where $M \in \mathfrak{E}$, $\mathfrak{m} = \mathfrak{L}M$, \mathfrak{m}_* is a \mathbf{Z} -grading of \mathfrak{m} , $M_0 = e^{\mathfrak{m}_0} \subset M$, \tilde{C} is a simple cuspidal M_0 -equivariant perverse sheaf on \mathfrak{m}_η (up to isomorphism).

Until the end of 3.4 we fix $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}'_\eta$. Let $\iota \in Y_M$ be associated to \tilde{C} as in 1.2(c),(a) (with M, \tilde{C} instead of H, A), so that $\binom{l}{k} \mathfrak{m} = \mathfrak{m}_{\eta k/2}$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \in \mathbf{Z}$ and $\binom{l}{k} \mathfrak{m} = 0$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \notin \mathbf{Z}$. Then we have $\mathfrak{m}_{k'} = \binom{l}{2k'/\eta} \mathfrak{m}$ for $k' \in \mathbf{Z}$ such that $2k'/\eta \in \mathbf{Z}$ and $\mathfrak{m}_{k'} = 0$ for $k' \in \mathbf{Z}$ such that $2k'/\eta \notin \mathbf{Z}$. Note that $\iota(\mathbf{k}^*)$ is contained in $\mathcal{Z}_{M_0}^0$.

The system $\dot{\xi}$ is said to be *admissible* if conditions (a),(b) below are satisfied:

- (a) we have $\mathfrak{m}_N \subset \mathfrak{g}_N$ for any $N \in \mathbf{Z}$;
- (b) there exists an element τ of finite order in the torus $\iota(\mathbf{k}^*)\mathcal{Z}_M^0$ of M_0 such that $M = G^{\text{Ad}(\tau)\vartheta}$.

We now consider the following condition on $\dot{\xi}$ which may or may not hold.

- (c) \mathfrak{m}_* is a splitting of some p -regular 1-spiral or, equivalently (see 2.6(f)), of some p -regular (-1) -spiral.

The following result will be proved in 3.2–3.4.

- (d) $\dot{\xi}$ is admissible if and only if $\dot{\xi}$ satisfies (c).

We now make some comments on the significance of condition (b). Assume that condition (a) is satisfied and that τ is any semisimple element of finite order of G_0 such that $M = G^{\text{Ad}(\tau)\vartheta}$. We show that we have automatically

$$(e) \quad \tau \in \iota(\mathbf{k}^*)\mathcal{Z}_M.$$

Note that $\vartheta(\tau) = \tau$ since $\tau \in G_0$ hence $\tau \in G^{\text{Ad}(\tau)\vartheta} = M$.

Let $N \in \mathbf{Z}$ be such that $2N/\eta \in \mathbf{Z}$. Since $\mathfrak{m}_N \subset \mathfrak{g}_N$, θ acts on \mathfrak{m}_N as ζ^N ; since $\text{Ad}(\tau)\theta$ acts as 1 on \mathfrak{m} we see that $\text{Ad}(\tau)$ acts on \mathfrak{m}_N as ζ^{-N} . On the other hand, for $t \in \mathbf{k}^*$, $\text{Ad}(\iota(t))$ acts on \mathfrak{m}_N as $t^{2N/\eta}$. Hence if $t_0 \in \mathbf{k}^*$ satisfies $t_0^{2/\eta} = \zeta^{-1}$, then we have $\text{Ad}(\iota(t_0))\text{Ad}(\tau^{-1}) = t_0^{2N/\eta}\zeta^N = \zeta^{-N}\zeta^N = 1$ on \mathfrak{m}_N . It follows that

$\text{Ad}(\iota(t_0))\text{Ad}(\tau^{-1}) = 1$ on \mathfrak{m} . Since $\iota(t_0)\tau^{-1} \in M$, we deduce that $\iota(t_0)\tau^{-1} \in \mathcal{Z}_M$ hence $\tau \in \iota(\mathbf{k}^*)\mathcal{Z}_M$, as asserted.

We see that condition (b) is a strengthening of (e) in which τ is required to lie not only in $\iota(\mathbf{k}^*)\mathcal{Z}_M$ but in its identity component.

3.2. We show:

(a) *For any element τ_0 of finite order in a torus T there exists $\lambda_0 \in Y_T$ such that $\tau_0 \in \lambda_0(\mathbf{k}^*)$.*

We can find $c \in \mathbf{Z}_{>0}$ such that $c \notin p\mathbf{Z}$ and $\tau_0^c = 1$. Let $\mu_c = \{z \in \mathbf{k}^*; z^c = 1\}$. For some $a \in \mathbf{N}$ we can identify $T = (\mathbf{k}^*)^a$ and τ_0 with $(z_1, \dots, z_a) \in (\mu_c)^a \subset T$. Now μ_c is cyclic with generator z_0 . Thus we have $z_1 = z_0^{k_1}, \dots, z_a = z_0^{k_a}$, where k_1, \dots, k_a are integers. We define $\lambda_0 \in Y_T$ by $t \mapsto (t^{k_1}, \dots, t^{k_a})$. Then $\tau_0 = \lambda_0(z_0)$, as desired.

We remark that in the proof of (a) we can assume that:

(b) $k_1 \in \mathbf{Z}_{>0}, k_1 \notin p\mathbf{Z}$.

Indeed, if $p = 0$, then $k_1 \notin p\mathbf{Z}$ is automatic. Assume now that $p > 0$. We write $k_1 = k'_1 p^e$, where $k'_1 \in \mathbf{Z} - p\mathbf{Z}, e \in \mathbf{Z}_{\geq 0}$. Define $z'_0 \in \mu_c$ by $z'_0 = z_0^{p^e}$. This is again a generator of μ_c . (Recall that $c \notin p\mathbf{Z}$.) We have $z_1 = (z'_0)^{k'_1}, z_j = (z'_0)^{k'_j}$, where $k'_j \in \mathbf{Z}_{>0}$ for $j = 2, 3, \dots, a$. Thus we can replace z_0, k_1, \dots, k_s by z'_0, k'_1, \dots, k'_s , where $k'_1 \in \mathbf{Z}_{>0}, k'_1 \notin p\mathbf{Z}$. This proves (b).

We now assume that τ as in 3.1(b) is given. We show:

(c) *There exist $f \in \mathbf{Z}_{>0}$ and $\lambda' \in Y_{\mathcal{Z}_M^0}$ such that $f \notin p\mathbf{Z}$ and such that, if $\lambda \in Y_{\iota(\mathbf{k}^*)\mathcal{Z}_M^0}$ is defined by $\lambda(t) = \iota(t^f)\lambda'(t)$ for all t , then $\tau \in \lambda(\mathbf{k}^*)$.*

If ι is identically 1, then (c) follows from (a) applied to $T = \mathcal{Z}_M^0$ (we can take $f = 1$). Assume now that ι is not identically 1. Then $\iota : \mathbf{k}^* \rightarrow M$ has finite kernel. Let $T = \mathbf{k}^* \times \mathcal{Z}_M^0$; we define $d : T \rightarrow \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ by $d(t, g) = \iota(t)g$. By definition, $\iota(\mathbf{k}^*)$ is contained in the derived subgroup of M hence it has finite intersection with \mathcal{Z}_M^0 . It follows that d has finite kernel. It is also surjective, hence we can find $\tilde{\tau} \in T$ of finite order such that $d(\tilde{\tau}) = \tau$. Using (a), we can find $\lambda_0 \in Y_T$ such that $\tilde{\tau} \in \lambda_0(\mathbf{k}^*)$; moreover, by (b), we can assume that, setting $\lambda_0(t) = (\lambda_1(t), \lambda'(t))$ with $\lambda_1 \in Y_{\mathbf{k}^*}, \lambda' \in Y_{\mathcal{Z}_M^0}$, we have $\lambda_1(t) = t^f$ for all t where $f \in \mathbf{Z}_{>0}, f \notin p\mathbf{Z}$. Let $\lambda = d\lambda_0 : \mathbf{k}^* \rightarrow \iota(\mathbf{k}^*)\mathcal{Z}_M^0$. We have $\lambda(t) = \iota(\lambda_1(t))\lambda'(t) = \iota(t^f)\lambda'(t)$ for $t \in \mathbf{k}^*$. Since $d(\tilde{\tau}) = \tau$ and $\tilde{\tau} \in \lambda_0(\mathbf{k}^*)$, we have $\tau \in \lambda(\mathbf{k}^*)$. This proves (c).

3.3. We now assume that τ as in 3.1(b) is given; let λ, λ', f be as in 3.2(c). We assume also that 3.1(a) holds. We can find $c \in \mathbf{k}^*$ of finite order such that $\lambda(c) = \tau$. (If $\tau \neq 1$, then λ is not identically 1 so it has finite kernel and any $c \in \lambda^{-1}(\tau)$ has finite order; if $\tau = 1$ we can take $c = 1$.)

Since $\lambda(\mathbf{k}^*) \subset M_0$ and $M_0 \subset G_{\underline{0}}$ (as a consequence of our assumption 3.1(a)), we can view λ as an element of $Y_{G_{\underline{0}}}$ hence $\lambda_k \mathfrak{g}_i$ is defined for any $k \in \mathbf{Z}, i \in \mathbf{Z}/m$. Since $\lambda(\mathbf{k}^*) \subset M$, we can view λ as an element of Y_M hence $\lambda_k \mathfrak{m}$ is defined for any $k \in \mathbf{Z}$.

For $t \in \mathbf{k}^*, k \in \mathbf{Z}$ such that $2k/\eta \in \mathbf{Z}$ and $x \in \mathfrak{m}_k$ we have $\text{Ad}(\lambda(t))x = \text{Ad}(\iota(t^f))\text{Ad}(\lambda'(t))x = \text{Ad}(\iota(t^f))x = t^{2kf/\eta}x$ (we use that $\lambda'(t) \in \mathcal{Z}_M^0$). Thus $\mathfrak{m}_k \subset \lambda_{2kf/\eta} \mathfrak{m}$. Recall also that $\mathfrak{m}_k \neq 0$ implies $k/\eta \in \mathbf{Z}$; see 1.2(e). Since the subspaces \mathfrak{m}_k form a direct sum decomposition of \mathfrak{m} and the subspaces $\lambda_j \mathfrak{m}$ form a

direct sum decomposition of \mathfrak{m} , it follows that:

$$(a) \quad \begin{aligned} \mathfrak{m}_k &= \lambda_{2kf/\eta} \mathfrak{m} \text{ for any } k \in \eta\mathbf{Z} \quad \text{and} \\ \lambda_j \mathfrak{m} &= 0 \text{ unless } j = 2kf/\eta \quad \text{for some } k \in \eta\mathbf{Z}. \end{aligned}$$

For $k \in \mathbf{Z}, i \in \mathbf{Z}/m$ and $x \in \lambda_k \mathfrak{g}_i$ we have

$$\text{Ad}(\tau)\theta(x) = \text{Ad}(\lambda(c))\theta(x) = \zeta^i \text{Ad}(\lambda(c))x = \zeta^i c^k x.$$

Since $\mathfrak{m} = \{x \in \mathfrak{g}; \text{Ad}(\tau)(\theta(x)) = x\}$, we see that:

$$(b) \quad \mathfrak{m} = \bigoplus_{j \in \mathbf{Z}, i \in \mathbf{Z}/m; \zeta^i c^j = 1} (\lambda_j \mathfrak{g}_i).$$

If $\lambda_j \mathfrak{g}_i$ is nonzero and contained in \mathfrak{m} then $\lambda_j \mathfrak{m}$ is nonzero hence by (a) we have $j = 2fk/\eta$ for some $k \in \mathbf{Z}$ and \mathfrak{m}_k is a nonzero subspace of \mathfrak{g}_i ; thus, by 3.1(a), we have $i = \underline{k}$ and $2k/\eta \in \mathbf{Z}$. Thus we can rewrite (b) as follows:

$$\mathfrak{m} = \bigoplus_{k \in \eta\mathbf{Z}; \zeta^k c^{2fk/\eta} = 1} (\lambda_{2fk/\eta} \mathfrak{g}_{\underline{k}}),$$

that is,

$$(c) \quad \mathfrak{m} = \bigoplus_{k \in \eta\mathbf{Z}; (\zeta^\eta c^{2f})^{k/\eta} = 1} (\lambda_{2fk/\eta} \mathfrak{g}_{\underline{k}}).$$

Assume now that $\mathfrak{m}_\eta \neq 0$. Using (a) we have $\mathfrak{m}_\eta = \lambda_f \mathfrak{m} \neq 0$. By 3.1(a) we have $\mathfrak{m}_\eta \subset \mathfrak{g}_\delta$. It follows that \mathfrak{m} has nonzero intersection with $\lambda_f \mathfrak{g}_\delta$. Now $\text{Ad}(\tau)\theta$ acts on $\lambda_f \mathfrak{g}_\delta$ as multiplication by $\zeta^\eta c^{2f}$ and it acts on \mathfrak{m} as the identity. It follows that $\zeta^\eta c^{2f} = 1$. Thus (c) can be rewritten as:

$$(d) \quad \mathfrak{m} = \bigoplus_{k \in \eta\mathbf{Z}} (\lambda_{2fk/\eta} \mathfrak{g}_{\underline{k}}).$$

Next we assume that $\mathfrak{m}_\eta = 0$. By the definition of ι (see 3.1) this implies that ι is identically 1 hence $\mathfrak{m} = \mathfrak{m}_0$. From (a) we see that $\mathfrak{m} = \lambda_0 \mathfrak{m}$, hence in (c) all summands corresponding to $k \neq 0$ are zero. Thus (d) remains true in this case. We see also that

$$\mathfrak{m}_k = \frac{|\eta|\lambda}{2fk\epsilon} \mathfrak{g}_{\underline{k}}$$

for all $k \in \mathbf{Z}$. Setting $\mu = |\eta|\lambda/(2f)$ we see that \mathfrak{p}_* is a splitting of the p -regular ϵ -spiral $\epsilon \mathfrak{p}_*^{\frac{1}{2f}|\eta|\lambda}$. We see that if ξ is admissible then it satisfies 3.1(c).

3.4. Assume now that ξ satisfies 3.1(c). Thus \mathfrak{m}_* is a splitting of an ϵ -spiral $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^\mu$ where μ is p -regular. Applying the conjugacy result 2.7(a) to the two splittings $\mathfrak{m}_*, \epsilon \tilde{\mathfrak{p}}_*^\mu$ we see that there exists a p -regular μ' such that $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^{\mu'}, \mathfrak{m}_* = \epsilon \tilde{\mathfrak{p}}_*^{\mu'}$. Thus we can find $\lambda \in Y_{G_0}, r \in \mathbf{Z}_{>0}$ such that $r \notin p\mathbf{Z}$ and

$$\mathfrak{m}_N = \lambda_{rN\epsilon} \mathfrak{g}_N$$

for any $N \in \mathbf{Z}$. In particular, 3.1(a) holds. We now show that 3.1(b) holds. From 2.6(c) we see that $M = G^{\text{Ad}(\lambda(\zeta')^{-1})^\theta}$ for some root of unity $\zeta' \in \mathbf{k}^*$. Let $\tau = \lambda(\zeta')^{-1}$. It remains to show that $\lambda(\zeta')^{-1} \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$. More generally, we show that $\lambda(t) \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ for any $t \in \mathbf{k}^*$. Now λ can be viewed as an element of Y_M hence $\lambda_k \mathfrak{m}$ is well-defined for any $k \in \mathbf{Z}$ and we have $\lambda_{rN\epsilon} \mathfrak{m} = \mathfrak{m}_N$ for any $N \in \mathbf{Z}$. Recall that for $N \in \mathbf{Z}$ we have $\mathfrak{m}_N = \lambda_{2N/\eta} \mathfrak{m}$ if $N/\eta \in \mathbf{Z}$ and $\mathfrak{m}_N = 0$ if $N/\eta \notin \mathbf{Z}$. We see that for any $N \in \eta\mathbf{Z}$ and any $t \in \mathbf{k}^*$, $\text{Ad}(\lambda(t))$ acts on \mathfrak{m}_N as $t^{rN\epsilon}$ while $\text{Ad}(\iota(t^{|\eta|}))$ acts on \mathfrak{m}_N as $t^{2N\epsilon}$. Hence $\text{Ad}(\lambda(t)^2 \iota(t)^{-r|\eta|})$ acts on \mathfrak{m}_N as 1. Since \mathfrak{m} is the sum of the subspaces \mathfrak{m}_N , we see that $\text{Ad}(\lambda(t)^2 \iota(t)^{-r|\eta|})$ acts on \mathfrak{m} as 1. It follows that $\lambda(t)^2 \iota(t)^{-r|\eta|} \in \mathcal{Z}_M$. Since $t \mapsto \lambda(t)^2 \iota(t)^{-r|\eta|}$ is a homomorphism of

the connected group \mathbf{k}^* into \mathcal{Z}_M , its image must be contained in \mathcal{Z}_M^0 . Thus, for any $t \in \mathbf{k}^*$ we have $\lambda(t)^2 \iota(t)^{-r|\eta|} \in \mathcal{Z}_M^0$ hence $\lambda(t^2) \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$. Since any $t' \in \mathbf{k}^*$ is a square, it follows that $\lambda(t') \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ for any $t' \in \mathbf{k}^*$. We see that, if $\dot{\xi}$ satisfies 3.1(c), then $\dot{\xi}$ is admissible. This completes the proof of 3.1(d).

3.5. The map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$. Let $\mathcal{I}(\mathfrak{g}_\delta)$ be the set of pairs $(\mathcal{O}, \mathcal{L})$ where \mathcal{O} is a G_0 -orbit on $\mathfrak{g}_\delta^{nil}$ and \mathcal{L} is an irreducible G_0 -equivariant local system on \mathcal{O} defined up to isomorphism. Since G_0 acts on $\mathfrak{g}_\delta^{nil}$ with finitely many orbits, see [Vi], the set $\mathcal{I}(\mathfrak{g}_\delta)$ is finite.

Let \mathfrak{T}_η be the set of all $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}'_\eta$ which are admissible (see 3.1) or equivalently (see 3.1(d)) are such that \mathfrak{m}_* is a splitting of some p -regular ϵ -spiral. The group G_0 acts in an obvious way by conjugation on \mathfrak{T}_η ; we denote by $\underline{\mathfrak{T}}_\eta$ the set of orbits, which is a finite set. We will define a map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$. Let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$. Choose $x \in \mathcal{O}$ and $\phi \in J_\delta(x)$; define $\mathfrak{u}_*^\phi, \tilde{\mathfrak{l}}_*^\phi, \tilde{\mathfrak{l}}^\phi, \tilde{L}_0^\phi, \tilde{L}_0^\phi$ as in 2.9.

Recall that $\tilde{L}^\phi \in \mathfrak{G}$. We have $x \in \tilde{\mathfrak{l}}_\eta^{\circ\phi}$ (see 2.9(b)). By 2.9(c), $\mathcal{L}_1 := \mathcal{L}|_{\tilde{\mathfrak{l}}_\eta^{\circ\phi}}$ is an

irreducible \tilde{L}_0^ϕ -equivariant local system on $\tilde{\mathfrak{l}}_\eta^{\circ\phi}$. Let A be the simple \tilde{L}_0^ϕ -equivariant perverse sheaf on $\tilde{\mathfrak{l}}_\eta^{\circ\phi}$ whose restriction to $\tilde{\mathfrak{l}}_\eta^{\circ\phi}$ is $\mathcal{L}_1[\dim \tilde{\mathfrak{l}}_\eta^{\circ\phi}]$. The map 1.5(b) associates to A an element $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ of $\mathfrak{M}_\eta(\tilde{L}^\phi)$ well defined up to conjugation by \tilde{L}_0^ϕ . Using 1.6(a) we can find a parabolic subalgebra \mathfrak{q} of $\tilde{\mathfrak{l}}^\phi$ compatible with the \mathbf{Z} -grading of $\tilde{\mathfrak{l}}^\phi$ and such that \mathfrak{m} is a Levi subalgebra of \mathfrak{q} . Setting $\mathfrak{p}'_N = \mathfrak{u}'_N + \mathfrak{q}_N$ for any $N \in \mathbf{Z}$, we see from 2.8(a) that \mathfrak{p}'_* is a p -regular ϵ -spiral and from 2.8(b) that \mathfrak{m}_* is a splitting of \mathfrak{p}'_* . We see that $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$.

We now show that the G_0 -orbit of $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is independent of the choices made. First, if x, ϕ are already chosen, then the \tilde{L}_0^ϕ -orbit of $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is well defined hence the G_0 -orbit of $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is well defined (since $\tilde{L}_0^\phi \subset G_0$). The independence of the choice of ϕ (when x is given) follows from the homogeneity of $J_\delta(x)$ under the group $U \cap G_0$ in 2.3(b). Finally, the independence of the choice of x follows from the homogeneity of \mathcal{O} under the group G_0 . Thus,

$$(\mathcal{O}, \mathcal{L}) \mapsto (G_0 - \text{orbit of } (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}))$$

is a well-defined map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$.

3.6. Let $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$. Let $\mathcal{O}_{\dot{\xi}}$ be the unique G_0 -orbit in $\mathfrak{g}_\delta^{nil}$ that contains $\mathring{\mathfrak{m}}_\eta$. Let $\dot{\xi}' = (M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, \tilde{C}') \in \mathfrak{T}_\eta$. We show:

(a) *If $\mathcal{O}_{\dot{\xi}} = \mathcal{O}_{\dot{\xi}'}$, then there exists $g \in G_0$ such that $\text{Ad}(g)$ carries $(M, M_0, \mathfrak{m}, \mathfrak{m}_*)$ to $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*)$.*

By [L4, 3.3], we can find $\phi = (e, h, f) \in J^M, \phi' = (e', h', f') \in J^{M'}$ such that:

(b) $e \in \mathring{\mathfrak{m}}_\eta, h \in \mathfrak{m}_0, f \in \mathfrak{m}_{-\eta}, e' \in \mathring{\mathfrak{m}}'_\eta, h' \in \mathfrak{m}'_0, f' \in \mathfrak{m}'_{-\eta}$.

We set $\iota = \iota_\phi \in Y_M, \iota' = \iota_{\phi'} \in Y_{M'}$. By 1.2(a),(c),(e), we have

$$(c) \quad \mathfrak{m}_k = \iota_{2k/\eta} \mathfrak{m}, \quad \mathfrak{m}'_k = \iota'_{2k/\eta} \mathfrak{m}' \text{ if } k \in \eta\mathbf{Z}, \mathfrak{m}_k = \mathfrak{m}'_k = 0 \text{ if } k \in \mathbf{Z} - \eta\mathbf{Z}.$$

By assumption, we have $e' = \text{Ad}(g_1)e$ for some $g_1 \in G_0$. Replacing the system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}, \phi)$ by its image under $\text{Ad}(g_1)$, we see that we can assume that $e = e'$. Using 3.1(a) for $\dot{\xi}$ and $\dot{\xi}'$, we can view ϕ, ϕ' as elements of J_δ^G with the

same first component. By 2.3(b), we can find $g_2 \in G_{\underline{0}}$ such that $\text{Ad}(g_2)$ carries ϕ to ϕ' . Replacing $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}, \phi)$ by its image under $\text{Ad}(g_1)$, we see that we can assume that $\phi = \phi'$ as elements of J^G . It follows that $\iota = \iota'$ as elements of Y_G .

Let

$$G_\phi = \{g \in G; \text{Ad}(g)(e) = e, \text{Ad}(g)(h) = h, \text{Ad}(g)(f) = f\}.$$

Since e, h, f are contained in \mathfrak{m} we have $\mathcal{Z}_M \subset G_\phi$. Similarly, since e, h, f are contained in \mathfrak{m}' , we have $\mathcal{Z}_{M'} \subset G_\phi$. We have also $\mathcal{Z}_M^0 \subset G_{\underline{0}}$ (since the center of \mathfrak{m} is contained in $\mathfrak{m}_0 \subset \mathfrak{g}_0$); similarly we have $\mathcal{Z}_{M'}^0 \subset G_{\underline{0}}$. Thus, \mathcal{Z}_M^0 and $\mathcal{Z}_{M'}^0$ are tori in $(G_\phi \cap G_{\underline{0}})^0$. We show that \mathcal{Z}_M^0 is a maximal torus of $(G_\phi \cap G_{\underline{0}})^0$. Indeed, assume that S is a torus of $(G_\phi \cap G_{\underline{0}})^0$ that contains \mathcal{Z}_M^0 . Since $S \subset G_\phi$, for any $s \in S$ we have $\text{Ad}(s)h = h$ hence $s\iota(t) = \iota(t)s$, that is, $\text{Ad}(\iota(t))s = s$ for $t \in \mathbf{k}^*$. Since S contains \mathcal{Z}_M^0 , for any $s \in S, z \in \mathcal{Z}_M^0$ we have $\text{Ad}(z)s = s$. Since $S \subset G_{\underline{0}}$ we have $\vartheta(s) = s$ for any $s \in S$. We see that $\text{Ad}(\iota(t))\text{Ad}(z)\vartheta(s) = s$ for any $t \in \mathbf{k}^*, z \in \mathcal{Z}_M^0, s \in S$. We can find $\tau \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ such that $M = G^{\text{Ad}(\tau)\vartheta}$. We have seen that $\text{Ad}(\tau)\vartheta(s) = s$ for $s \in S$. Thus $S \subset M$. Since $S \subset G_\phi$, we have

$$S \subset M_\phi := \{g \in M; \text{Ad}(g)(e) = e, \text{Ad}(g)(h) = h, \text{Ad}(g)(f) = f\},$$

hence $S \subset M_\phi^0$. Since e is a distinguished nilpotent element of \mathfrak{m} , we have $M_\phi^0 = \mathcal{Z}_M^0$. Thus we have $S \subset \mathcal{Z}_M^0$. By assumption, we have $\mathcal{Z}_M^0 \subset S$, hence $\mathcal{Z}_M^0 = S$. Thus \mathcal{Z}_M^0 is indeed a maximal torus of $(G_\phi \cap G_{\underline{0}})^0$, as claimed. Similarly we see that $\mathcal{Z}_{M'}^0$ is a maximal torus of $(G_\phi \cap G_{\underline{0}})^0$. Since any two maximal tori of $(G_\phi \cap G_{\underline{0}})^0$ are conjugate, we can find g_3 in $(G_\phi \cap G_{\underline{0}})^0$ such that $\text{Ad}(g_3)$ carries \mathcal{Z}_M^0 to $\mathcal{Z}_{M'}^0$. (It also carries ϕ to ϕ .)

Replacing $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}, \phi)$ by its image under $\text{Ad}(g_3)$, we see that we can assume that $\mathcal{Z}_M^0 = \mathcal{Z}_{M'}^0$ and $\phi = \phi'$.

Assume now that $e = 0$ so that $e' = 0$. By the definition of $\iota = \iota'$ we see that $\iota = \iota'$ is identically 1 hence $\mathfrak{m} = \mathfrak{m}_0, \mathfrak{m}' = \mathfrak{m}'_0$ and $G_\phi = G$. Since $e = 0$ is distinguished in \mathfrak{m} it follows that M is a torus. Hence $M = \mathcal{Z}_M^0$. Similarly $M' = \mathcal{Z}_{M'}^0$. Since $\mathcal{Z}_M^0 = \mathcal{Z}_{M'}^0$, it follows that $M = M'$. We see that (a) holds in this case.

In the remainder of the proof we assume that $e \neq 0$ hence $e' \neq 0$. Recall that $M = G^{\text{Ad}(\iota(t))\text{Ad}(z)\vartheta}, M' = G^{\text{Ad}(\iota(t'))\text{Ad}(z')\vartheta}$, for some t, t' in \mathbf{k}^* and some z, z' in $\mathcal{Z}_M^0 = \mathcal{Z}_{M'}^0$. Since $e \in \mathfrak{m}_\eta$, we have $\text{Ad}(\iota(t))\text{Ad}(z)\vartheta(e) = e$; since $\text{Ad}(z)$ acts as 1 on \mathfrak{m} , we deduce that $t^2\zeta^\eta e = e$ and since $e \neq 0$, we see that $t^2 = \zeta^{-\eta}$. Similarly, since $e \in \mathfrak{m}'_\eta$ we have $\text{Ad}(\iota(t'))\text{Ad}(z')\vartheta(e) = e$ and $t'^2 = \zeta^{-\eta}$.

We show that for any $k \in \mathbf{Z}$ we have $\mathfrak{m}_k \subset \mathfrak{m}'$. By 1.2(e) we can assume that $k \in \eta\mathbf{Z}$. Let $x \in \mathfrak{m}_k$. We must show that $\text{Ad}(\iota(t'))\text{Ad}(z')\vartheta(x) = x$. Since $\text{Ad}(z')$ acts by 1 on \mathfrak{m} , it is enough to show that $\zeta^k t'^{2k/\eta} x = x$ or that $(\zeta^\eta t'^2)^{k/\eta} x = x$. This follows from $t'^2 = \zeta^{-\eta}$.

Thus we have $\mathfrak{m}_k \subset \mathfrak{m}'$. Since this holds for any $k \in \mathbf{Z}$, we deduce that $\mathfrak{m} \subset \mathfrak{m}'$. Interchanging the roles of $\mathfrak{m}, \mathfrak{m}'$ we see that $\mathfrak{m}' \subset \mathfrak{m}$ hence $\mathfrak{m} = \mathfrak{m}'$. This implies that $M = M'$. Since $\iota = \iota'$, we see from (c) that $\mathfrak{m}_* = \mathfrak{m}'_*$. From $\mathfrak{m}_0 = \mathfrak{m}'_0$ we deduce that $M_0 = M'_0$. This completes the proof of (a).

The following result can be extracted from the proof of (a).

(d) If $\mathfrak{m}_\eta = 0$ (so that $e = 0$), then $\mathfrak{m} = \mathfrak{m}_0$ is a Cartan subalgebra of \mathfrak{g}_0 .

3.7. Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$. Let $x \in \overset{\circ}{\mathfrak{m}}_\eta$. We choose $\phi = (e, h, f) \in J^M$ such that $e = x, h \in \mathfrak{m}_0, f \in \mathfrak{m}_{-\eta}$ (see [L4, 3.3]). We can view x as an element of $\mathfrak{g}_\delta^{nil}$ and ϕ as an element of $J_\delta(x)$. We define $\tilde{\mathfrak{l}}_* = \tilde{\mathfrak{l}}_*^\phi$ as in 2.9. Recall that for $N \in \mathbf{Z}$ we have:

$$\tilde{\mathfrak{l}}_N = {}^{\iota}\!_{2N/\eta}\mathfrak{g}_N \text{ if } 2N/\eta \in \mathbf{Z}, \quad \tilde{\mathfrak{l}}_N = 0 \text{ if } 2N/\eta \notin \mathbf{Z},$$

where $\iota = \iota_\phi \in Y_G$. Let $\tilde{\mathfrak{l}} = \bigoplus_N \tilde{\mathfrak{l}}_N \subset \mathfrak{g}$ and let $\tilde{L} = e^{\tilde{\mathfrak{l}}} \subset G$. We show:

(a) \mathfrak{m} is a Levi subalgebra of a parabolic subalgebra of $\tilde{\mathfrak{l}}$ which is compatible with the \mathbf{Z} -grading of $\tilde{\mathfrak{l}}$.

We shall prove (a) without the statement of compatibility with the \mathbf{Z} -grading; then the full statement of (a) would follow from 1.6(a).

Assume first that $x = 0$. Then $h = 0$ hence ι is the constant map with image 1. It follows that $\tilde{\mathfrak{l}} = \tilde{\mathfrak{l}}_0 = \mathfrak{g}_0$ and $\mathfrak{m} = \mathfrak{m}_0$; moreover: by 3.6(d), \mathfrak{m} is a Cartan subalgebra of \mathfrak{g}_0 . Hence in this case (a) is immediate. In the rest of the proof we assume that $x \neq 0$.

Since $\overset{\circ}{\mathfrak{m}}_\eta$ carries a cuspidal local system, for any $N \in \mathbf{Z}$ such that $2N/\eta \in \mathbf{Z}$ we have $\mathfrak{m}_N = {}^{\iota}\!_{2N/\eta}\mathfrak{m}$. Since $\mathfrak{m}_N \subset \mathfrak{g}_N$, we have $\mathfrak{m}_N \subset {}^{\iota}\!_{2N/\eta}\mathfrak{g}_N$ hence $\mathfrak{m}_N \subset \tilde{\mathfrak{l}}_N$. Taking sum over all $N \in \mathbf{Z}$ such that $2N/\eta \in \mathbf{Z}$, we get $\mathfrak{m} \subset \tilde{\mathfrak{l}}$. We can find $t_0 \in \mathbf{k}^*, z \in \mathcal{Z}_M^0$, both of finite order, such that $\mathfrak{m} = \{y \in \mathfrak{g}; \text{Ad}(\iota(t_0)) \text{Ad}(z)\theta(y) = y\}$. Note that $\tilde{\mathfrak{l}}_* = \dot{\eta}\tilde{\mathfrak{l}}_*^{(|\eta|/2)\iota}$

By 2.6(c), we can find $\zeta' \in \mathbf{k}^*$ such that $\tilde{\mathfrak{l}} = \{y \in \mathfrak{g}; \text{Ad}(\iota(\zeta')^{-1})\theta(y) = y\}$. Since $\mathfrak{m} \subset \tilde{\mathfrak{l}}$, we have:

$$(b) \quad \mathfrak{m} = \{y \in \tilde{\mathfrak{l}}; \text{Ad}(\iota(t_0)) \text{Ad}(z)\theta(y) = y\} = \{y \in \tilde{\mathfrak{l}}; \text{Ad}(\iota(t_0)) \text{Ad}(z) \text{Ad}(\iota(\zeta'))y = y\}.$$

(Note that 2.6(c) is applicable since $\tilde{\mathfrak{l}}_* = \dot{\eta}\tilde{\mathfrak{l}}_*^{(|\eta|/2)\iota}$.)

Since $x \in \mathfrak{m}_\eta \subset \frac{1}{2}\mathfrak{g}$, we have $\text{Ad}(\iota(t))x = t^2x$ for any t . Taking $t = t_0^{-1}$ or $t = \zeta'$ we see that $t_0^{-2}x = \text{Ad}(\iota(t_0))^{-1}x$ and $\zeta'^2x = \text{Ad}(\iota(\zeta'))x$. Since $x \in \mathfrak{m}$ and $x \in \tilde{\mathfrak{l}}$ we have $\text{Ad}(\iota(t_0))^{-1}x = \theta(x)$ and $\text{Ad}(\iota(\zeta'))x = \theta(x)$. It follows that $t_0^{-2}x = \zeta'^2x$ so that (since $x \neq 0$) we have $t_0^{-2} = \zeta'^2$.

If $N \in \mathbf{Z}, 2N/\eta \in \mathbf{Z}$ and $y \in \tilde{\mathfrak{l}}_N$, we have $\text{Ad}(\iota(t_0\zeta'))y = (t_0\zeta')^{2N/\eta}y = y$. Since $\tilde{\mathfrak{l}} = \bigoplus_N \tilde{\mathfrak{l}}_N$ we have $\text{Ad}(\iota(t_0\zeta'))y = y$ for all $y \in \tilde{\mathfrak{l}}$. Hence (b) implies:

$$(c) \quad \mathfrak{m} = \{y \in \tilde{\mathfrak{l}}; \text{Ad}(z)y = y\}.$$

It remains to show that (c) implies (a). Since z is of finite order and $z \in \mathcal{Z}_M^0$, we can find $\lambda \in Y_{\mathcal{Z}_M^0}$ such that $z = \lambda(t_1)$ for some $t_1 \in \mathbf{k}^*$. (See 3.2(a).)

Let $\mathfrak{m}' = \{y \in \tilde{\mathfrak{l}}; \text{Ad}(\lambda(t))y = y \quad \forall t \in \mathbf{k}^*\}$. Note that \mathfrak{m}' is a Levi subalgebra of a parabolic subalgebra \mathfrak{q} of $\tilde{\mathfrak{l}}$. Since $\lambda(\mathbf{k}^*) \subset \mathcal{Z}_M^0$ we see that $\text{Ad}(\lambda(t))$ acts as 1 on \mathfrak{m} for any t hence $\mathfrak{m} \subset \mathfrak{m}'$. Now $\text{Ad}(\lambda(t_1))$ acts as 1 on \mathfrak{m}' . Since $\mathfrak{m} = \{y \in \tilde{\mathfrak{l}}; \text{Ad}(\lambda(t_1))y = y\}$ it follows that $\mathfrak{m}' = \mathfrak{m}$. Thus (a) holds.

3.8. **Primitive pairs.** Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$. Let $x \in \overset{\circ}{\mathfrak{m}}_\eta$. We can view x as an element of $\mathfrak{g}_\delta^{nil}$. We set $M_0(x) = M_0 \cap G(x), G_0(x) = G_0 \cap G(x)$. We show:

(a) *The inclusion $M_0(x) \rightarrow G_0(x)$ induces an isomorphism on the groups of components.*

Let $\phi \in J^M, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{L}$ be as in 3.7. Let $\tilde{L}_0 = e^{\tilde{\mathfrak{l}}_0} \subset \tilde{L}$. We have $x \in \overset{\circ}{\mathfrak{l}}_\eta$ (see [L4, 4.2(a)]). Let $\tilde{L}_0(x) = \tilde{L}_0 \cap G(x)$. To prove (a) it is enough to prove (i) and (ii) below.

(i) *The inclusion $M_0(x) \rightarrow \tilde{L}_0(x)$ induces an isomorphism on the groups of components.*

(ii) *The inclusion $\tilde{L}_0(x) \rightarrow G_0(x)$ induces an isomorphism on the groups of components.*

Now (i) follows from [L4, 11.2] (we use 3.7(a)) and (ii) is a special case of 2.9(c). This proves (a).

Let \mathcal{O} be the G_0 -orbit of x in $\mathfrak{g}_\delta^{nil}$. Let \mathcal{L}' be the irreducible M_0 -equivariant local system on $\overset{\circ}{\mathfrak{m}}_\eta$ such that $\tilde{C}|_{\overset{\circ}{\mathfrak{m}}_\eta} = \mathcal{L}'[\dim \mathfrak{m}_\eta]$. Let \mathcal{L} be the irreducible G_0 -equivariant local system on \mathcal{O} which corresponds to \mathcal{L}' under (a). We say that $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ is the *primitive pair* corresponding to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$; it is clearly independent of the choice of x, ϕ (we use [L4, 3.3]).

Let \mathcal{L}'' be the irreducible \tilde{L}_0 -equivariant local system on $\overset{\circ}{\tilde{\mathfrak{l}}}_\eta$ which corresponds to \mathcal{L}' under (i). Let $\mathcal{L}''^\sharp \in \mathcal{D}(\tilde{\mathfrak{l}}_\eta)$ be as in 0.11. From 1.8(b) we see that:

(b) $\text{ind}_{\mathfrak{q}_\eta}^{\tilde{\mathfrak{l}}_\eta}(\tilde{C})$ is a nonzero direct sum of shifts of \mathcal{L}''^\sharp .

Consider the map $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \mapsto (\mathcal{O}, \mathcal{L})$ (as above) from \mathfrak{T}_η to $\mathcal{I}(\mathfrak{g}_\delta)$; the image of this map is denoted by $\mathcal{I}^{prim}(\mathfrak{g}_\delta)$. From 3.6(a) and (a) we see that:

(c) *This induces a bijection $\omega : \underline{\mathfrak{T}}_\eta \xrightarrow{\sim} \mathcal{I}^{prim}(\mathfrak{g}_\delta)$.*

Using the definitions and 1.8(b), we see that:

(d) *For $\xi \in \underline{\mathfrak{T}}_\eta$ we have $\Psi(\omega(\xi)) = \xi$, where $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$ is as in 3.5.*

Combining (c) and (d), we have

(e) *the restriction of Ψ to $\mathcal{I}^{prim}(\mathfrak{g}_\delta)$ gives the inverse of ω .*

From (d) we get:

(f) *The map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$ is surjective.*

Another proof of (f) is given in 7.3.

3.9. Now let $\eta_1 \in \mathbf{Z} - \{0\}$ be such that $\underline{\eta}_1 = \delta$. We define a bijection $\mathfrak{T}'_\eta \xrightarrow{\sim} \mathfrak{T}'_{\eta_1}$ by $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \mapsto (M, M_0, \mathfrak{m}, \mathfrak{m}_{(*)}, \tilde{C})$ where $\mathfrak{m}_{(*)}$ is the new \mathbf{Z} -grading on \mathfrak{m}_* whose k -component $\mathfrak{m}_{(k)}$ is equal to $\mathfrak{m}_{k\eta/\eta_1}$ for any $k \in \eta_1\mathbf{Z}$ and is equal to 0 for any $k \in \mathbf{Z} - \eta_1\mathbf{Z}$. (This is well defined since $\mathfrak{m}_{k'} = 0$ for any $k' \in \mathbf{Z} - \eta\mathbf{Z}$; see 1.2(e).) Here we regard \tilde{C} as a simple perverse sheaf on $\mathfrak{m}_\eta = \mathfrak{m}_{(\eta_1)}$. This restricts to a bijection $\mathfrak{T}_\eta \xrightarrow{\sim} \mathfrak{T}_{\eta_1}$, which induces a bijection $\underline{\mathfrak{T}}_\eta \xrightarrow{\sim} \underline{\mathfrak{T}}_{\eta_1}$. This allows us to identify canonically all the sets \mathfrak{T}_{η_1} (for various $\eta_1 \in \mathbf{Z} - \{0\}$ such that $\underline{\eta}_1 = \delta$) with a single set \mathfrak{T}_δ and also all the sets $\underline{\mathfrak{T}}_{\eta_1}$ (for various $\eta_1 \in \mathbf{Z} - \{0\}$ such that $\underline{\eta}_1 = \delta$) with a single set $\underline{\mathfrak{T}}_\delta$. Here $\mathfrak{T}_\delta, \underline{\mathfrak{T}}_\delta$ are defined purely in terms of δ (independently of the choice of η).

An examination of the construction of the map $\Psi = \Psi_\eta : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$ (see 3.5) shows that the bijection $\underline{\mathfrak{T}}_\eta \xrightarrow{\sim} \underline{\mathfrak{T}}_{\eta_1}$ intertwines Ψ_η and Ψ_{η_1} . Therefore we have a well-defined map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\delta$.

4. SPIRAL INDUCTION

In this section we introduce the key tool in studying the block decomposition for $\mathcal{D}_{G_0}(\mathfrak{g}_\delta^{nil})$, namely the spiral induction. This is the analogue in the \mathbf{Z}/m -graded setting of the parabolic induction in the ungraded or \mathbf{Z} -graded setting.

4.1. Definition of spiral induction. In addition to $\eta \in \mathbf{Z} - \{0\}$ which has been fixed in 2.9, in this section we fix $\epsilon \in \{1, -1\}$. We denote by \mathfrak{P}^ϵ the set of all data of the form:

$$(a) \quad (\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*),$$

where \mathfrak{p}_* is an ϵ -spiral and $L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*$ are associated to \mathfrak{p}_* as in 2.5. Let

$$(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon.$$

Let $\pi : \mathfrak{p}_\eta \rightarrow \mathfrak{l}_\eta$ be the obvious projection. We have a diagram:

$$(b) \quad \mathfrak{l}_\eta \xleftarrow{a} G_{\underline{0}} \times \mathfrak{p}_\eta \xrightarrow{b} E' \xrightarrow{c} \mathfrak{g}_\delta,$$

where $E' = \{(gP_0, z) \in G_{\underline{0}}/P_0 \times \mathfrak{g}_\delta; \text{Ad}(g^{-1})z \in \mathfrak{p}_\eta\}$, $a(g, z) = \pi(\text{Ad}(g^{-1})z)$, $b(g, z) = (gP_0, z)$, $c(gP_0, z) = z$. Here a is smooth with connected fibers, b is a principal P_0 -bundle and c is proper. Now $\mathcal{Q}(\mathfrak{l}_\eta)$ is defined as in 1.2, with H, \mathfrak{h} replaced by L, \mathfrak{l} . If $A \in \mathcal{Q}(\mathfrak{l}_\eta)$, then a^*A is a P_0 -equivariant semisimple complex on $G_{\underline{0}} \times \mathfrak{p}_\eta$, hence there is a well-defined semisimple complex A_1 on E' such that $b^*A_1 = a^*A$. We can form the complex

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) = c_!A_1.$$

Since c is proper, this is a semisimple, $G_{\underline{0}}$ -equivariant complex on \mathfrak{g}_δ .

If $\tilde{\mathfrak{l}}_*$ is a splitting of \mathfrak{p}_* , we will sometimes consider ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ with $A \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta)$ by identifying $\tilde{\mathfrak{l}}_\eta$ with \mathfrak{l}_η in an obvious way and A with an object in $\mathcal{Q}(\mathfrak{l}_\eta)$.

For any $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ we have

$$(c) \quad D({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)) = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(A))[2e],$$

where e is the dimension of a fiber of a minus the dimension of a fiber of b , or equivalently

$$\begin{aligned} e &= \dim \mathfrak{g}_{\underline{0}} + \dim \mathfrak{p}_\eta - \dim \mathfrak{u}_0 - (\dim \mathfrak{p}_\eta - \dim \mathfrak{u}_\eta) - (\dim \mathfrak{p}_0 - \dim \mathfrak{u}_0) \\ &= \dim \mathfrak{u}_0 + \dim \mathfrak{u}_\eta. \end{aligned}$$

Hence, if for $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ we set

$${}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)[\dim \mathfrak{u}_0 + \dim \mathfrak{u}_\eta],$$

then

$$(d) \quad D({}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)) = {}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(A)).$$

4.2. Transitivity. We state a transitivity property of induction. In addition to the datum 4.1(a) we consider a parabolic subalgebra \mathfrak{q} of \mathfrak{l} such that $\mathfrak{q} = \bigoplus_{N \in \mathbf{Z}} \mathfrak{q}_N$ where $\mathfrak{q}_N = \mathfrak{q} \cap \mathfrak{l}_N$. For any $N \in \mathbf{Z}$ let $\hat{\mathfrak{p}}_N$ be the inverse image of \mathfrak{q}_N under the obvious map $\mathfrak{p}_N \rightarrow \mathfrak{l}_N$. Then $\hat{\mathfrak{p}}_*$ is an ϵ -spiral; see 2.8(a). Let

$$(\hat{\mathfrak{p}}_*, \hat{L}, \hat{P}_0, \hat{\mathfrak{l}}, \hat{\mathfrak{l}}_*, \hat{\mathfrak{u}}_*) \in \mathfrak{P}^\epsilon$$

be the datum analogous to 4.1(a) defined by $\hat{\mathfrak{p}}_*$. Now $\mathcal{Q}(\hat{\mathfrak{l}}_\eta)$ is defined as in 1.2, with H, \mathfrak{h} replaced by $\hat{L}, \hat{\mathfrak{l}}$. If $A \in \mathcal{Q}(\hat{\mathfrak{l}}_\eta)$, then $\text{ind}_{\mathfrak{q}_\eta}^{\hat{\mathfrak{l}}_\eta}(A) \in \mathcal{Q}(\mathfrak{l}_\eta)$ is defined as in 1.3 and we have canonically

$$(a) \quad {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}_\eta}^{\hat{\mathfrak{l}}_\eta}(A)).$$

The proof is similar to that of [L2, 4.2]; it is omitted.

4.3. In the setup of 4.1, assume that $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ is a cuspidal perverse sheaf (see 1.2). We have $A = \mathcal{L}^\sharp[\dim \mathfrak{l}_\eta]$ where \mathcal{L} is an irreducible local system on $\mathring{\mathfrak{l}}_\eta$ and $\mathcal{L}^\sharp \in \mathcal{D}(\mathfrak{l}_\eta)$ is as in 0.11. In this case we can give an alternative description of ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp)$. Let $P_0, \pi : \mathfrak{p}_\eta \rightarrow \mathfrak{l}_\eta$ be as in 4.1. Let

$$\mathring{\mathfrak{g}}_\delta = \{(gP_0, z) \in G_0/P_0 \times \mathfrak{g}_\delta; \text{Ad}(g^{-1})z \in \pi^{-1}(\mathring{\mathfrak{l}}_\eta)\}$$

be an open smooth irreducible subvariety of E' in 4.1. Let $\mathring{\mathcal{L}}$ be the local system on $\mathring{\mathfrak{g}}_\delta$ defined by $b'^*\mathring{\mathcal{L}} = a'^*\mathcal{L}$, where

$$\mathring{\mathfrak{l}}_\eta \xleftarrow{a'} G_0 \times (\pi^{-1}(\mathring{\mathfrak{l}}_\eta)) \xrightarrow{b'} \mathring{\mathfrak{g}}_\delta,$$

$a'(g, zx) = \pi(\text{Ad}(g^{-1})z), b'(g, z) = (gP_0, z)$. Let $\mathring{\mathcal{L}}^\sharp$ be the intersection cohomology complex of E' with coefficients in $\mathring{\mathcal{L}}$. From the definitions we have $a^*\mathcal{L}^\sharp = b^*\mathring{\mathcal{L}}^\sharp$ (a, b as in 4.1). We define $c' : \mathring{\mathfrak{g}}_\delta \rightarrow \mathfrak{g}_\delta$ by $c'(g, z) = z$. We show:

(a)
$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp) = c'_1 \mathring{\mathcal{L}}^\sharp.$$

Using the definitions we see that it is enough to show that the restriction of $\mathring{\mathcal{L}}^\sharp$ to $E' - \mathring{\mathfrak{g}}_\delta$ is zero. This can be deduced from 1.2(c).

4.4. Let $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$ be the subcategory of $\mathcal{D}(\mathfrak{g}_\delta)$ consisting of complexes which are direct sums of shifts of simple G_0 -equivariant perverse sheaves B on \mathfrak{g}_δ with the following property: there exists a datum $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$ as in 4.1(a) and a simple cuspidal perverse sheaf A in $\mathcal{Q}(\mathfrak{l}_\eta)$ such that some shift of B is a direct summand of ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$. We show:

(a) *If $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon$ and A' is a simple (not necessarily cuspidal) perverse sheaf in $\mathcal{Q}(\mathfrak{l}_\eta)$, then ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A') \in \mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$.*

Using [L4, 7.5] we see that some shift of A' is a direct summand of $\text{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(A)$ for some $\hat{\mathfrak{l}}, \mathfrak{q}$ as in 4.2 where A is a simple cuspidal perverse sheaf in $\mathcal{Q}(\hat{\mathfrak{l}}_\eta)$. It follows that some shift of ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')$ is a direct summand of

(b)
$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(A)).$$

It is then enough to show that the complex (b) belongs to $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$. This follows from the definitions using the transitivity property 4.2(a).

The functor

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta} : \mathcal{Q}(\mathfrak{l}_\eta) \rightarrow \mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$$

(where $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$ is as in 4.1(a)) called *spiral induction*.

Let $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$ be the abelian group generated by symbols (A) , one for each isomorphism class of objects of $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$, subject to the relations $(A) + (A') = (A \oplus A')$ (a Grothendieck group). Now $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$ is naturally an \mathcal{A} -module by $v^n(A) = (A[n])$ for any $n \in \mathbf{Z}$. We shall write A instead of (A) (in $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$). Clearly, $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$ is a free \mathcal{A} -module with a finite distinguished basis given by the various simple perverse sheaves in $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$. Now $A, B \mapsto (A : B) = \{A, D(B)\} \in \mathbf{N}((v))$ (see 0.12) defines a pairing

(c)
$$(\cdot) : \mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta) \times \mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta) \rightarrow \mathbf{Z}((v)),$$

which is \mathcal{A} -linear in the first argument, \mathcal{A} -antilinear in the second argument (for $f \mapsto \bar{f}$) and satisfies $(b_1 : b_2) = \overline{(b_2 : b_1)}$ for all b_1, b_2 in $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$.

4.5. In addition to the datum 4.1(a) we consider another datum

$$(a) \quad (\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^\epsilon$$

such that $\mathfrak{p}_N \subset \mathfrak{p}'_N$ for all $N \in \mathbf{Z}$ and $\mathfrak{p}_N = \mathfrak{p}'_N$ for $N \in \{\eta, -\eta\}$. We then have $\mathfrak{u}'_N \subset \mathfrak{u}_N$ for all $N \in \mathbf{Z}$ and $\mathfrak{u}_N = \mathfrak{u}'_N$ for $N \in \{\eta, -\eta\}$. We also have canonically $\mathfrak{l}'_N = \mathfrak{l}'_N$ for $N \in \{\eta, -\eta\}$ and $P_0 \subset P'_0$. Let $\mathcal{P} = P'_0/P_0$. Write $\rho_{\mathcal{P}!} \bar{\mathbf{Q}}_l = \bigoplus_j \bar{\mathbf{Q}}_l[-2a_j]$ where a_j are integers ≥ 0 . (Here $\rho_{\mathcal{P}!}$ is as in 0.12.) Let $A \in \mathcal{Q}(\mathfrak{l}_\eta) = \mathcal{Q}(\mathfrak{l}'_\eta)$. We show:

(b) Let $I = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$, $I' = {}^\epsilon \text{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A)$. We have $I \cong \bigoplus_j I'[-2a_j]$.

We consider the commutative diagram

$$\begin{array}{ccccccc} \mathfrak{l}_\eta & \xleftarrow{a} & G_0 \times \mathfrak{p}_\eta & \xrightarrow{b} & E' & \xrightarrow{c} & \mathfrak{g}_\delta \\ \downarrow 1 & & \downarrow 1 & & \downarrow h & & \downarrow 1 \\ \mathfrak{l}'_\eta & \xleftarrow{a'} & G_0 \times \mathfrak{p}'_\eta & \xrightarrow{b'} & \tilde{E}' & \xrightarrow{c'} & \mathfrak{g}_\delta \end{array},$$

where the upper horizontal maps are as in 4.1(b), the lower horizontal are the analogous maps when 4.1(a) is replaced by (a) and $h : E' \rightarrow \tilde{E}'$ is given by $(gP_0, z) \mapsto (gP'_0, z)$. Note that h is a P'_0/P_0 -bundle. We can find a complex A_1 (resp. A'_1) on E' (resp. \tilde{E}') such that $I = c_1 A_1$, $I' = c'_1 A'_1$. We have $A_1 = h^* A'_1$, hence

$$I = c_1 A_1 = c'_1 h_1 A'_1 = c'_1 h_1 h^* A'_1 = c'_1 (A'_1 \otimes h_1 h^* \bar{\mathbf{Q}}_l) = \bigoplus_j c'_1 A'_1[-2a_j] = \bigoplus_j I'[-2a_j].$$

This proves (b).

5. STUDY OF A PAIR OF SPIRALS

This section serves as preparation for the next one, which aims to calculate the Hom space between two spiral inductions.

5.1. In addition to $\eta \in \mathbf{Z} - \{0\}$ which has been fixed in 2.9, in this section we fix ϵ', ϵ'' in $\{1, -1\}$. Let

$$(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}, \quad (\mathfrak{p}''_*, L'', P''_0, \mathfrak{l}'', \mathfrak{l}''_*, \mathfrak{u}''_*) \in \mathfrak{P}^{\epsilon''}.$$

We show:

(a) there exists a splitting $\tilde{\mathfrak{l}}'_*$ of \mathfrak{p}'_* and a splitting $\tilde{\mathfrak{l}}''_*$ of \mathfrak{p}''_* such that, if $\tilde{L}'_0 = e^{\tilde{\mathfrak{l}}'_0} \subset G$ and $\tilde{L}''_0 = e^{\tilde{\mathfrak{l}}''_0} \subset G$, then some maximal torus \mathcal{T} of G_0 is contained in both \tilde{L}'_0 and \tilde{L}''_0 .

Let $\tilde{\mathfrak{l}}'_*$ be any splitting of \mathfrak{p}'_* and let $\tilde{\mathfrak{l}}''_*$ be any splitting of \mathfrak{p}''_* ; let $\tilde{L}'_0 = e^{\tilde{\mathfrak{l}}'_0} \subset G$, $\tilde{L}''_0 = e^{\tilde{\mathfrak{l}}''_0} \subset G$. Recall that P'_0 (resp. P''_0) is a parabolic subgroup of G_0 with Levi subgroup \tilde{L}'_0 (resp. \tilde{L}''_0); hence there exists a maximal torus \mathcal{T}_0 of G_0 contained in both P'_0, P''_0 . Let $'\tilde{L}'_0$ (resp. $'\tilde{L}''_0$) be the Levi subgroup of P'_0 (resp. P''_0) such that $\mathcal{T}_0 \subset '\tilde{L}'_0$ (resp. $\mathcal{T}_0 \subset '\tilde{L}''_0$). We can find $u' \in U_{P'_0}$, $u'' \in U_{P''_0}$ such that $\text{Ad}(u')\tilde{L}'_0 = '\tilde{L}'_0$, $\text{Ad}(u'')\tilde{L}''_0 = '\tilde{L}''_0$. Now $\{\text{Ad}(u')\tilde{\mathfrak{l}}'_N; N \in \mathbf{Z}\}$ is a splitting of $\{\text{Ad}(u')\mathfrak{p}'_N; N \in \mathbf{Z}\} = \mathfrak{p}'_*$ and $\{\text{Ad}(u'')\tilde{\mathfrak{l}}''_N; N \in \mathbf{Z}\}$ is a splitting of $\{\text{Ad}(u'')\mathfrak{p}''_N; N \in \mathbf{Z}\} = \mathfrak{p}''_*$. Note that $\text{Ad}(u')\tilde{L}'_0, \text{Ad}(u'')\tilde{L}''_0$ contain a maximal torus of G_0 ; (a) is proved.

5.2. Let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$, $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$. Let $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ be a simple cuspidal perverse sheaf. As in 4.3, we have $A = \mathcal{L}^\sharp[\dim \mathfrak{l}_\eta]$ where \mathcal{L} is an irreducible local system on $\mathring{\mathfrak{l}}_\eta$. Let

$$B = {}^{\epsilon'} \text{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp).$$

Let $\pi' : \mathfrak{p}'_\eta \rightarrow \mathfrak{l}'_\eta$ be the obvious map with kernel \mathfrak{u}'_η . We want to study the complex $K = \pi'_!(B|_{\mathfrak{p}'_\eta}) \in \mathcal{D}(\mathfrak{l}'_\eta)$. As in 4.3, let

$$\mathring{\mathfrak{g}}_\delta = G_{\underline{0}} \overset{P_0}{\times} \pi^{-1}(\mathring{\mathfrak{l}}_\eta),$$

where $\pi : \mathfrak{p}_\eta \rightarrow \mathfrak{l}_\eta$ is the obvious map; let $\dot{\mathcal{L}}$ be the local system on $\mathring{\mathfrak{g}}_\delta$ defined in terms of \mathcal{L} as in 4.3. As in 4.3, we define $c' : \mathring{\mathfrak{g}}_\delta \rightarrow \mathfrak{g}_\delta$ by $c'(g, z) = z$. Let

$$\dot{\mathfrak{p}}'_\eta = \{(gP_0, z) \in G_{\underline{0}}/P_0 \times \mathfrak{p}'_\eta; \text{Ad}(g^{-1})z \in \pi^{-1}(\mathring{\mathfrak{l}}_\eta)\}.$$

Note that $\dot{\mathfrak{p}}'_\eta$ is the closed subvariety $c'^{-1}\mathfrak{p}'_\eta$ of $\mathring{\mathfrak{g}}_\delta$. The restriction of $\dot{\mathcal{L}}$ from $\mathring{\mathfrak{g}}_\delta$ to $\dot{\mathfrak{p}}'_\eta$ is denoted again by $\dot{\mathcal{L}}$. Now c' restricts to a map $\dot{\mathfrak{p}}'_\eta \rightarrow \mathfrak{p}'_\eta$ whose composition with $\pi' : \mathfrak{p}'_\eta \rightarrow \mathfrak{l}'_\eta$ is denoted by $\sigma : \dot{\mathfrak{p}}'_\eta \rightarrow \mathfrak{l}'_\eta$. We have $\sigma : (gP_0, z) \mapsto \pi'(z)$. Using 4.3(a) and a proper base change, we see that $K = \sigma_!(\dot{\mathcal{L}})$.

We have a partition $\dot{\mathfrak{p}}'_\eta = \cup_{\Omega} \dot{\mathfrak{p}}'_{\eta, \Omega}$ into locally closed subvarieties indexed by the various (P'_0, P_0) -double cosets Ω in $G_{\underline{0}}$, where

$$\dot{\mathfrak{p}}'_{\eta, \Omega} = \{(gP_0, z) \in \Omega/P_0 \times \mathfrak{p}'_\eta; \text{Ad}(g^{-1})z \in \pi^{-1}(\mathring{\mathfrak{l}}_\eta)\}.$$

Let $\sigma_\Omega : \dot{\mathfrak{p}}'_{\eta, \Omega} \rightarrow \mathfrak{l}'_\eta$ be the restriction of σ . For any Ω we set

$$K_\Omega = \sigma_{\Omega!}(\dot{\mathcal{L}}|_{\dot{\mathfrak{p}}'_{\eta, \Omega}}) \in \mathcal{D}(\mathfrak{l}'_\eta).$$

We say that Ω is *good* if for some (or equivalently any) $g_0 \in \Omega$, the following condition holds: setting $\mathfrak{p}'_N = \text{Ad}(g_0)\mathfrak{p}_N$, $\mathfrak{u}'_N = \text{Ad}(g_0)\mathfrak{u}_N$ (for $N \in \mathbf{Z}$), the obvious inclusion

$$(\mathfrak{p}'_N \cap \text{Ad}(g_0)\mathfrak{p}_N) / (\mathfrak{p}'_N \cap \text{Ad}(g_0)\mathfrak{u}_N) \rightarrow \text{Ad}(g_0)\mathfrak{p}_N / \text{Ad}(g_0)\mathfrak{u}_N$$

is an isomorphism for any $N \in \mathbf{Z}$ that is, $\text{Ad}(g_0)\mathfrak{p}_N = (\mathfrak{p}'_N \cap \text{Ad}(g_0)\mathfrak{p}_N) + \text{Ad}(g_0)\mathfrak{u}_N$. We say that Ω is *bad* if it is not good.

Until the end of 5.4 we fix an Ω as above and we choose $g_0 \in \Omega$. Let $\mathfrak{p}''_N = \text{Ad}(g_0)\mathfrak{p}_N$; then \mathfrak{p}''_* is an ϵ'' -spiral. It determines a datum $(\mathfrak{p}''_*, L'', P''_0, \mathfrak{l}'', \mathfrak{l}''_*, \mathfrak{u}''_*) \in \mathfrak{P}^{\epsilon''}$.

By the change of variable $g = hg_0$ we may identify $\dot{\mathfrak{p}}'_{\eta, \Omega}$ with

$$\{(hP''_0, z) \in P'_0 P''_0 / P''_0 \times \mathfrak{p}'_\eta; \text{Ad}(h^{-1})z \in \text{Ad}(g_0)\pi^{-1}(\mathring{\mathfrak{l}}_\eta)\}$$

which is the same as

$$\Xi = \{(h(P'_0 \cap P''_0), z) \in P'_0 / (P'_0 \cap P''_0) \times \mathfrak{p}'_\eta; \text{Ad}(h^{-1})z \in \pi''^{-1}(\mathring{\mathfrak{l}}_\eta)\}$$

(in which $\pi'' : \mathfrak{p}''_\eta \rightarrow \mathfrak{l}''_\eta$ is the obvious map, with kernel \mathfrak{u}''_η). In these coordinates, $\sigma_\Omega : \dot{\mathfrak{p}}'_{\eta, \Omega} \rightarrow \mathfrak{l}'_\eta$ becomes $(h(P'_0 \cap P''_0), z) \mapsto \pi''(z)$.

We choose a splitting $\tilde{\mathfrak{l}}'_*$ of \mathfrak{p}'_* and a splitting $\tilde{\mathfrak{l}}''_*$ of \mathfrak{p}''_* as in 5.1(a); let $\tilde{L}'_0, \tilde{L}''_0, \mathcal{T}$ be as in 5.1(a).

Let μ', μ'' be elements of $Y_{G_{\underline{0}}, \mathbf{Q}}$ such that $\mathfrak{p}'_* = {}^{\epsilon'} \mathfrak{p}^{\mu'}$, $\tilde{\mathfrak{l}}'_* = {}^{\epsilon'} \tilde{\mathfrak{l}}^{\mu'}$, $\mathfrak{p}''_* = {}^{\epsilon''} \mathfrak{p}^{\mu''}$, $\tilde{\mathfrak{l}}''_* = {}^{\epsilon''} \tilde{\mathfrak{l}}^{\mu''}$. Let r', r'' in $\mathbf{Z}_{>0}$ be such that $\lambda' := r' \mu' \in Y_{G_{\underline{0}}}$, $\lambda'' := r'' \mu'' \in Y_{G_{\underline{0}}}$.

As in 2.6(d) we have $\lambda'(\mathbf{k}^*) \subset \mathcal{Z}_{\tilde{L}'_0}^0$, $\lambda''(\mathbf{k}^*) \subset \mathcal{Z}_{\tilde{L}''_0}^0$. Since \mathcal{T} is a maximal torus of \tilde{L}'_0 , we must have $\mathcal{Z}_{\tilde{L}'_0}^0 \subset \mathcal{T}$ hence $\lambda'(\mathbf{k}^*) \subset \mathcal{T}$. Similarly, since \mathcal{T} is a maximal torus of \tilde{L}''_0 , we have $\mathcal{Z}_{\tilde{L}''_0}^0 \subset \mathcal{T}$ hence $\lambda''(\mathbf{k}^*) \subset \mathcal{T}$. Since both $\lambda'(\mathbf{k}^*), \lambda''(\mathbf{k}^*)$ are contained in the torus \mathcal{T} , we must have $\lambda'(t')\lambda''(t'') = \lambda''(t'')\lambda'(t')$ for any t', t'' in \mathbf{k}^* . Hence, if for any k', k'' in \mathbf{Z} and $i \in \mathbf{Z}/m$ we set

$${}_{k', k''} \mathfrak{g}_i = \{x \in \mathfrak{g}_i; \text{Ad}(\lambda'(t'))x = t'^{k'}x, \text{Ad}(\lambda''(t''))x = t''^{k''}x, \quad \forall t', t'' \in \mathbf{k}^*\},$$

then we have $\mathfrak{g} = \bigoplus_{k', k'', i} ({}_{k', k''} \mathfrak{g}_i)$.

For any $N \in \mathbf{Z}$ we have a direct sum decomposition

$$(a) \quad \mathfrak{p}'_N \cap \mathfrak{p}''_N = (\tilde{l}'_N \cap \tilde{l}''_N) \oplus (\mathfrak{u}'_N \cap \tilde{l}''_N) \oplus (\tilde{l}'_N \cap \mathfrak{u}''_N) \oplus (\mathfrak{u}'_N \cap \mathfrak{u}''_N).$$

This follows immediately from the decompositions

$$\begin{aligned} \mathfrak{p}'_N \cap \mathfrak{p}''_N &= \bigoplus_{k', k''; k' \geq Nr' \epsilon', k'' \geq Nr'' \epsilon''} ({}_{k, k'} \mathfrak{g}_N), \\ \tilde{l}'_N \cap \tilde{l}''_N &= \bigoplus_{k', k''; k' = Nr' \epsilon', k'' = Nr'' \epsilon''} ({}_{k, k'} \mathfrak{g}_N), \\ \mathfrak{u}'_N \cap \tilde{l}''_N &= \bigoplus_{k', k''; k' > Nr' \epsilon', k'' = Nr'' \epsilon''} ({}_{k, k'} \mathfrak{g}_N), \\ \tilde{l}'_N \cap \mathfrak{u}''_N &= \bigoplus_{k', k''; k' = Nr' \epsilon', k'' > Nr'' \epsilon''} ({}_{k, k'} \mathfrak{g}_N), \\ \mathfrak{u}'_N \cap \mathfrak{u}''_N &= \bigoplus_{k', k''; k' > Nr' \epsilon', k'' > Nr'' \epsilon''} ({}_{k, k'} \mathfrak{g}_N). \end{aligned}$$

For $N \in \mathbf{Z}$ let \mathfrak{q}''_N be the image of $\mathfrak{p}'_N \cap \mathfrak{p}''_N$ under the obvious map $\mathfrak{p}''_N \rightarrow \mathfrak{l}''_N$; let $\mathfrak{q}'' = \bigoplus_{N \in \mathbf{Z}} (\mathfrak{q}''_N)$, a Lie subalgebra of \mathfrak{l}'' . We show:

(b) \mathfrak{q}'' is a parabolic subalgebra of \mathfrak{l}'' compatible with the \mathbf{Z} -grading of \mathfrak{l}'' .

For $N \in \mathbf{Z}$ we set $\tilde{\mathfrak{q}}''_N = \tilde{l}'_N \cap \mathfrak{p}'_N$. Let $\tilde{\mathfrak{q}}'' = \bigoplus_{N \in \mathbf{Z}} \tilde{\mathfrak{q}}''_N$, a Lie subalgebra of $\tilde{\mathfrak{l}}''$. From (a) we see that the obvious isomorphism $\tilde{\mathfrak{l}}'' \xrightarrow{\sim} \mathfrak{l}''$ carries $\tilde{\mathfrak{q}}''$ to \mathfrak{q}'' . Hence (b) follows from (c) below:

(c) $\tilde{\mathfrak{q}}''$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}''$ compatible with the \mathbf{Z} -grading of $\tilde{\mathfrak{l}}''$.

We have

$$\tilde{\mathfrak{q}}'' = \bigoplus_{k', N \in \mathbf{Z}; k' \geq Nr' \epsilon'} ({}_{k', Nr'' \epsilon''} \mathfrak{g}_N).$$

We define $\lambda_1 \in Y_{\tilde{\mathfrak{l}}''}$ by $\lambda_1(t) = \lambda'(t^{r''})\lambda''(t^{-r' \epsilon' \epsilon''})$ for all $t \in \mathbf{k}^*$. Then $\text{Ad}(\lambda_1(t))$ acts on the subspace ${}_{k', Nr'' \epsilon''} \mathfrak{g}_N$ of $\tilde{\mathfrak{l}}''$ as $t^{k'r'' - r'r''N\epsilon'}$; the last exponent of t is ≥ 0 if and only if $k' \geq r'N\epsilon'$ which is just the condition that ${}_{k', Nr'' \epsilon''} \mathfrak{g}_N$ is one of the summands in the direct sum decomposition of $\tilde{\mathfrak{q}}''$. This proves (c).

For $N \in \mathbf{Z}$ let \mathfrak{q}'_N be the image of $\mathfrak{p}'_N \cap \mathfrak{p}''_N$ under the obvious map $\mathfrak{p}'_N \rightarrow \mathfrak{l}'_N$; let $\mathfrak{q}' = \bigoplus_{N \in \mathbf{Z}} \mathfrak{q}'_N$, a Lie subalgebra of \mathfrak{l}' .

For $N \in \mathbf{Z}$ we set $\tilde{\mathfrak{q}}'_N = \tilde{l}'_N \cap \mathfrak{p}''_N$. Let $\tilde{\mathfrak{q}}' = \bigoplus_{N \in \mathbf{Z}} \tilde{\mathfrak{q}}'_N$, a Lie subalgebra of $\tilde{\mathfrak{l}}'$. The following result is proved in the same way as (b), (c).

(d) \mathfrak{q}' is a parabolic subalgebra of \mathfrak{l}' compatible with the \mathbf{Z} -grading of \mathfrak{l}' ; $\tilde{\mathfrak{q}}'$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}'$ compatible with the \mathbf{Z} -grading of $\tilde{\mathfrak{l}}'$.

We set ${}^1\tilde{\mathfrak{q}}'' = \bigoplus_N {}^1\tilde{\mathfrak{q}}''_N$, ${}^1\tilde{\mathfrak{q}}' = \bigoplus_N ({}^1\tilde{\mathfrak{q}}'_N)$, where

$${}^1\tilde{\mathfrak{q}}''_N = \bigoplus_{k' \in \mathbf{Z}; k' > Nr' \epsilon'} ({}_{k', Nr'' \epsilon''} \mathfrak{g}_N), \quad {}^1\tilde{\mathfrak{q}}'_N = \bigoplus_{k'' \in \mathbf{Z}; k'' > Nr'' \epsilon''} ({}_{Nr' \epsilon', k''} \mathfrak{g}_N).$$

The proof of (c) shows also that ${}^1\tilde{\mathfrak{q}}''$ is the nilradical of $\tilde{\mathfrak{q}}''$ and that

$$\bigoplus_{N \in \mathbf{Z}} ({}_{Nr' \epsilon', Nr'' \epsilon''} \mathfrak{g}_N)$$

is a Levi subalgebra of $\tilde{\mathfrak{q}}''$. Similarly, ${}^1\tilde{\mathfrak{q}}'$ is the nilradical of $\tilde{\mathfrak{q}}'$ and

$$\bigoplus_{N \in \mathbf{Z}} ({}_{Nr' \epsilon', Nr'' \epsilon''} \mathfrak{g}_N)$$

is a Levi subalgebra of $\tilde{\mathfrak{q}}''$. Thus,

(e) $\tilde{\mathfrak{q}}', \tilde{\mathfrak{q}}''$ have a common Levi subalgebra, namely $\oplus_{N \in \mathbf{Z}} (\mathfrak{N}_{r'\epsilon', Nr''\epsilon''} \mathfrak{g}_N)$.

5.3. In this subsection we assume that Ω is bad. Then for some N , $\tilde{\mathfrak{l}}'_N \cap \mathfrak{p}'_N$ is strictly contained in $\tilde{\mathfrak{l}}''_N$. Hence $\tilde{\mathfrak{q}}''$ is a proper parabolic subalgebra of $\tilde{\mathfrak{l}}''$ (see 5.2(c)). We will show that

(a)
$$K_\Omega = \sigma_\Omega!(\dot{\mathcal{L}}|_{\tilde{\mathfrak{p}}'_{\eta, \Omega}}) = 0 \in \mathcal{D}(\mathfrak{l}'_\eta).$$

This is equivalent to the following statement:

(b) for any $y \in \tilde{\mathfrak{l}}'_\eta$, the cohomology groups H_c^j of the variety

$$\{(h(P'_0 \cap P''_0), z) \in P'_0/(P'_0 \cap P''_0) \times \mathfrak{p}'_\eta; z - y \in \mathfrak{u}'_\eta, \text{Ad}(h^{-1})z \in \pi''^{-1}(\overset{\circ}{\mathfrak{l}}''_\eta)\}$$

with coefficients in the local system defined by $\dot{\mathcal{L}}$, are zero for all $j \in \mathbf{Z}$.

(We have identified $\tilde{\mathfrak{l}}'_\eta, \mathfrak{l}'_\eta$ via π' .) Considering the fibers of the first projection of the last variety to $P'_0/(P'_0 \cap P''_0)$, we see that it suffices to show that:

(c) for any $h \in P'_0$ and any $y \in \tilde{\mathfrak{l}}'_\eta$, the cohomology groups H_c^j of the variety

$$\{z \in \mathfrak{p}'_\eta; z - y \in \mathfrak{u}'_\eta, \text{Ad}(h^{-1})z \in \overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta\}$$

with coefficients in the local system defined by $\dot{\mathcal{L}}$, are zero for all $j \in \mathbf{Z}$.

(We have used that $\pi''^{-1}(\overset{\circ}{\mathfrak{l}}''_\eta) = \overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta$.)

If z is as in (c), then we have automatically $\text{Ad}(h^{-1})z \in \mathfrak{p}'_\eta$; since $\overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta \subset \mathfrak{p}''_\eta$, the condition that $\text{Ad}(h^{-1})z \in \overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta$ implies $\text{Ad}(h^{-1})z \in \mathfrak{p}'_\eta \cap \mathfrak{p}''_\eta$. By 5.2(a), we can then write uniquely $\text{Ad}(h^{-1})z = \gamma + \nu' + \nu'' + \mu$, where

(e)
$$\gamma \in \tilde{\mathfrak{l}}'_\eta \cap \tilde{\mathfrak{l}}''_\eta, \nu' \in \mathfrak{u}'_\eta \cap \tilde{\mathfrak{l}}''_\eta, \nu'' \in \tilde{\mathfrak{l}}'_\eta \cap \mathfrak{u}''_\eta, \mu \in \mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta.$$

The condition that $\text{Ad}(h^{-1})z \in \overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta$ can be expressed as $\gamma + \nu' \in \overset{\circ}{\mathfrak{l}}''_\eta$. The condition that $z - y \in \mathfrak{u}'_\eta$ is equivalent to $\text{Ad}(h^{-1})z - \text{Ad}(h^{-1})y \in \mathfrak{u}'_\eta$ or (if we define $y' \in \tilde{\mathfrak{l}}'_\eta$ by $\text{Ad}(h^{-1})y - y' \in \mathfrak{u}'_\eta$) to $\gamma + \nu'' = y'$. Note that y', γ, ν'' are uniquely determined by h, y . Hence the variety in (c) can be identified with

$$(\gamma + (\mathfrak{u}'_\eta \cap \tilde{\mathfrak{l}}''_\eta)) \cap \overset{\circ}{\mathfrak{l}}''_\eta \times (\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta).$$

Under this identification, the local system $\dot{\mathcal{L}}$ is the pullback of \mathcal{L} (viewed as a local system on $\overset{\circ}{\mathfrak{l}}''_\eta$) from the first factor. Now the desired vanishing of cohomology follows from the vanishing property [L4, 4.4(c)] of \mathcal{L} , since in our case $\tilde{\mathfrak{q}}'' = \oplus_N (\tilde{\mathfrak{l}}''_N \cap \mathfrak{p}'_N)$ is a proper parabolic subalgebra of $\tilde{\mathfrak{l}}''$ with nilradical $\oplus_N (\tilde{\mathfrak{l}}''_N \cap \mathfrak{u}'_N)$.

5.4. In this subsection we assume that Ω is good. Then for any N we have $\tilde{\mathfrak{l}}''_N \cap \mathfrak{p}'_N = \tilde{\mathfrak{l}}''_N$ that is, $\tilde{\mathfrak{l}}''_N \subset \mathfrak{p}'_N$. We also have $\tilde{\mathfrak{q}}'' = \tilde{\mathfrak{l}}''$. Thus $\tilde{\mathfrak{q}}''$ is reductive so it is equal to its Levi subalgebra $\oplus_{N \in \mathbf{Z}} (\mathfrak{N}_{r'\epsilon', Nr''\epsilon''} \mathfrak{g}_N)$ (see 5.2(e)) which is then equal to $\tilde{\mathfrak{l}}''$ and is also a Levi subalgebra of $\tilde{\mathfrak{q}}'$ (see 5.2(e)). Thus,

(a) $\tilde{\mathfrak{l}}''$ is a Levi subalgebra of $\tilde{\mathfrak{q}}'$.

Now $\text{Ad}(g_0)$ defines an isomorphism $\mathfrak{l} \xrightarrow{\sim} \mathfrak{l}''$. Composing this with the inverse of the obvious isomorphism $\tilde{\mathfrak{l}}'' \xrightarrow{\sim} \mathfrak{l}''$ we obtain an isomorphism of \mathbf{Z} -graded Lie

algebras $\mathfrak{l} \xrightarrow{\sim} \tilde{\mathfrak{l}}''$. Using this, we can transport \mathcal{L} (a local system on $\overset{\circ}{\mathfrak{l}}_\eta$; see 5.1) to a local system \mathcal{L}'' on $\overset{\circ}{\mathfrak{l}}''_\eta$. Let $\mathcal{L}''^\sharp \in \mathcal{D}(\tilde{\mathfrak{l}}''_\eta)$ be as in 0.11. Then

$$\mathrm{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}''_\eta}(\mathcal{L}''^\sharp) \in \mathcal{Q}(\tilde{\mathfrak{l}}'_\eta)$$

is defined as in 1.3 (we identify $\tilde{\mathfrak{l}}''$ with the reductive quotient of $\tilde{\mathfrak{q}}'$; see (a)). We now state the following result.

(b) *We have $K_\Omega = \mathrm{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}''_\eta}(\mathcal{L}''^\sharp)[-2f]$, where*

$$f = \dim \mathfrak{u}'_0 - \dim(\mathfrak{u}'_0 \cap \mathfrak{p}''_0) + \dim(\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta).$$

Let $\tilde{Q}'_0 = e^{\tilde{\mathfrak{q}}'_0} \subset \tilde{L}'_0$, a parabolic subgroup of \tilde{L}'_0 . Let

$$\Xi' = \{('h\tilde{Q}'_0, 'z) \in \tilde{L}'_0/\tilde{Q}'_0 \times \tilde{\mathfrak{l}}'_\eta; \mathrm{Ad}('h^{-1})'z \in \overset{\circ}{\mathfrak{l}}''_\eta + {}^1\tilde{\mathfrak{q}}_\eta\}.$$

Define $c'' : \Xi' \rightarrow \tilde{\mathfrak{l}}'_\eta$ by $c''('hQ'_0, 'z) = 'z$. By the argument in [L4, 6.6] (for \tilde{L}' instead of G) we have

(c)
$$\mathrm{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}''_\eta}(\mathcal{L}''^\sharp) = c''_! \hat{\mathcal{L}}'',$$

where $\hat{\mathcal{L}}''$ is a certain local system on Ξ' determined by \mathcal{L}'' and such that $\Delta^* \hat{\mathcal{L}}'' = \hat{\mathcal{L}}$ where $\Delta : \Xi \rightarrow \Xi'$ (Ξ as in 5.2) is the map induced by the canonical maps $P'_0 \rightarrow \tilde{L}'_0$ (with kernel $U_{P'_0}$) and $\mathfrak{p}'_\eta \rightarrow \tilde{\mathfrak{l}}'_\eta$ (with kernel \mathfrak{u}'_η); $\hat{\mathcal{L}}$ is the local system on Ξ considered in 5.2. We consider the following statement:

(d) *Δ is an affine space bundle with fibers of dimension f .*

Assuming that (d) holds, we have

$$K_\Omega = c''_! \Delta_! \hat{\mathcal{L}} = c''_! \hat{\mathcal{L}}'' \otimes \Delta_! \bar{\mathbf{Q}}_l = c''_! \hat{\mathcal{L}}''[-2f]$$

and we see that (b) follows from (c). It remains to prove (d).

Let $'h \in \tilde{L}'_0, 'z \in \tilde{\mathfrak{l}}'_\eta$ be such that $('hQ'_0, 'z) \in \Xi'$. Setting $h = 'hu, z = 'z + \tilde{z}$, we see that $\Delta^{-1}('hQ'_0, 'z)$ can be identified with

$$\{(u(U_{P'_0} \cap P''_0), \tilde{z}) \in (U_{P'_0}/(U_{P'_0} \cap P''_0)) \times \mathfrak{u}'_\eta; \mathrm{Ad}(u^{-1}) \mathrm{Ad}('h^{-1})('z + \tilde{z}) \in \overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta\}.$$

It suffices to show that

(e)
$$\{(u, \tilde{z}) \in U_{P'_0} \times \mathfrak{u}'_\eta; \mathrm{Ad}(u^{-1}) \mathrm{Ad}('h^{-1})('z + \tilde{z}) \in \overset{\circ}{\mathfrak{l}}''_\eta + \mathfrak{u}''_\eta\}$$

is isomorphic to $U_{P'_0} \times (\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta)$. If (u, \tilde{z}) are as in (e), we have automatically $\mathrm{Ad}(u^{-1}) \mathrm{Ad}('h^{-1})('z + \tilde{z}) \in \mathfrak{p}'_\eta$ (since $'z + \tilde{z} \in \mathfrak{p}'_\eta$ and $'hu \in P'_0$). Setting $\mathrm{Ad}('h^{-1})'z = a \in \overset{\circ}{\mathfrak{l}}''_\eta + {}^1\tilde{\mathfrak{q}}_\eta$ (where a is fixed) and $\mathrm{Ad}(u^{-1}) \mathrm{Ad}('h^{-1})\tilde{z} = \tilde{z}' \in \mathfrak{u}'_\eta$, we see that the variety (e) may be identified with the variety

(f)
$$\{(u, \tilde{z}') \in U_{P'_0} \times \mathfrak{u}'_\eta; \mathrm{Ad}(u^{-1})a + \tilde{z}' \in \overset{\circ}{\mathfrak{l}}''_\eta + (\mathfrak{p}'_\eta \cap \mathfrak{u}''_\eta)\}.$$

By 5.2(a) we can write uniquely

$$\mathrm{Ad}(u^{-1})a + \tilde{z}' = \gamma + \nu + \mu,$$

where $\gamma \in \overset{\circ}{\tilde{l}}''_\eta$, $\nu \in \tilde{l}'_\eta \cap \mathbf{u}''_\eta$, $\mu \in \mathbf{u}'_\eta \cap \mathbf{u}''_\eta$. Setting $\hat{z} = \mu - \tilde{z}$ we see that (f) can be identified with the variety of all quintuples $(u, \hat{z}, \gamma, \nu, \nu')$ in

$$U_{P'_0} \times \mathbf{u}'_\eta \times \overset{\circ}{\tilde{l}}''_\eta \times (\tilde{l}'_\eta \cap \mathbf{u}''_\eta) \times (\mathbf{u}'_\eta \cap \mathbf{u}''_\eta)$$

such that

$$(g) \quad \text{Ad}(u^{-1})a = \gamma + \nu + \hat{z}.$$

Since $a \in \tilde{l}'_\eta$, we have $\text{Ad}(u^{-1})a - a \in \mathbf{u}'_\eta$ for $u \in U_{P'_0}$. Hence in (g) we have $\gamma + \nu = a$ and $\hat{z} = \text{Ad}(u^{-1})a - a$. In particular, γ, ν are uniquely determined. Thus, our variety may be identified with $U_{P'_0} \times (\mathbf{u}'_\eta \cap \mathbf{u}''_\eta)$. This completes the proof of (d), hence that of (b).

5.5. From the results in 5.3 and 5.4 we can deduce, using the argument in [L4, 8.9] (based on [L4, 1.4]), the following result.

Proposition 5.6. *We have $K \in \mathcal{Q}(l'_\eta)$; moreover, we have (noncanonically) $K \cong \oplus_\Omega K_\Omega$, where Ω runs over good (P'_0, P_0) -double cosets in G_Ω .*

6. SPIRAL RESTRICTION

We introduce the spiral restriction functor which is adjoint to the spiral induction. The main result in this section is Proposition 6.4, which calculates the inner product $\{, \}$ (in the sense of 0.12) of two spiral inductions.

6.1. Definition of spiral restriction. In addition to $\eta \in \mathbf{Z} - \{0\}$ which has been fixed in 2.9, in this section we fix ϵ', ϵ'' in $\{1, -1\}$. Let $(\mathfrak{p}'_*, L', P'_0, l', l'_*, \mathbf{u}'_*) \in \mathfrak{P}^{\epsilon'}$. Let $\pi' : \mathfrak{p}'_\eta \rightarrow l'_\eta$ be the obvious map. For any $B \in \mathcal{D}(\mathfrak{g}_\delta)$ we set

$$\epsilon' \text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B) = \pi'_!(B|_{\mathfrak{p}'_\eta}) \in \mathcal{D}(l'_\eta).$$

We show:

(a) *If $B \in \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$, then $\epsilon' \text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B) \in \mathcal{Q}(l'_\eta)$.*

To prove this we can assume that B is in addition a simple perverse sheaf. Then, using the definition of $\mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$, we see that it is enough to prove (a) in the case where $B = \epsilon'' \text{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp)$, with $(\mathfrak{p}_*, L, P_0, l, l_*, \mathbf{u}_*) \in \mathfrak{P}^{\epsilon''}$, \mathcal{L}^\sharp as in 5.2. In this case, (a) follows from 5.6.

We thus have a functor $\epsilon' \text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta} : \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta) \rightarrow \mathcal{Q}(l'_\eta)$ called *spiral restriction*.

We have the following result.

Proposition 6.2 ((Adjunction)). *Let $C \in \mathcal{Q}(l'_\eta)$, and let $B \in \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. For any $j \in \mathbf{Z}$ we have*

$$(a) \quad d_j(l'_\eta; C, \epsilon' \text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B)) = d_{j'}(\mathfrak{g}_\delta; \epsilon' \text{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C), B),$$

where $j' = j + 2 \dim \mathbf{u}'_0$.

The proof is almost a copy of that of [L4, 9.2]. We omit it.

For $B \in \mathcal{D}(\mathfrak{g}_\delta)$ we set

$$\epsilon' \widetilde{\text{Res}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B) = \epsilon' \text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B)[\dim \mathfrak{u}'_\eta - \dim \mathfrak{u}'_0].$$

With this notation, the equality (a) can be reformulated without a shift from j to j' as follows:

$$(b) \quad d_j(\mathfrak{l}'_\eta; C, \epsilon' \widetilde{\text{Res}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B)) = d_j(\mathfrak{g}_\delta; \epsilon' \widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C), B).$$

6.3. Let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$, $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$. Let $A \in \mathcal{Q}(\mathfrak{l}_\eta)$, $A' \in \mathcal{Q}(\mathfrak{l}'_\eta)$ be cuspidal perverse sheaves. As in 4.3 we have $A = \mathcal{L}^\#[\dim \mathfrak{l}_\eta]$, $A' = \mathcal{L}'^\#[\dim \mathfrak{l}'_\eta]$ where \mathcal{L} (resp. \mathcal{L}') is a local system on \mathfrak{l}_η (resp. \mathfrak{l}'_η).

We denote by X the set of all $g \in G_0$ such that the ϵ'' -spiral $\{\text{Ad}(g)\mathfrak{p}_N; N \in \mathbf{Z}\}$ and the ϵ' -spiral \mathfrak{p}'_* have a common splitting. If $g \in X$ there is a unique isomorphism of \mathbf{Z} -graded Lie algebras $\lambda_g : \mathfrak{l} \rightarrow \mathfrak{l}'$ such that the compositions

$$\begin{aligned} \text{Ad}(g)\mathfrak{p}_N \cap \mathfrak{p}'_N &\rightarrow \mathfrak{p}'_N \rightarrow \mathfrak{l}'_N, \\ \text{Ad}(g)\mathfrak{p}_N \cap \mathfrak{p}'_N &\xrightarrow{\text{Ad}(g^{-1})} \mathfrak{p}_N \rightarrow \mathfrak{l}_N \xrightarrow{\lambda_g} \mathfrak{l}'_N \end{aligned}$$

coincide for any N (the unnamed maps are the obvious imbeddings or projections). Moreover, λ_g is induced by an isomorphism $L \rightarrow L'$. Let X' be the set of all $g \in X$ such that $\lambda_g : \mathfrak{l}_\eta \xrightarrow{\sim} \mathfrak{l}'_\eta$ carries \mathcal{L} to the dual of \mathcal{L}' . For any $g \in X'$ we set

$$\tau(g) = -\dim \frac{\mathfrak{u}'_0 + \text{Ad}(g)\mathfrak{u}_0}{\mathfrak{u}'_0 \cap \text{Ad}(g)\mathfrak{u}_0} + \dim \frac{\mathfrak{u}'_\eta + \text{Ad}(g)\mathfrak{u}_\eta}{\mathfrak{u}'_\eta \cap \text{Ad}(g)\mathfrak{u}_\eta}.$$

Note that both X and X' are unions of (P'_0, P_0) -double cosets in G_0 and that $\tau(g)$ depends only on the double coset of g . We have the following result.

Proposition 6.4. *Let*

$$\Pi = \sum_{j \in \mathbf{Z}} d_j(\mathfrak{g}_\delta; \epsilon' \widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A'), \epsilon'' \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A))v^{-j} \in \mathbf{N}((v)).$$

We have

$$\Pi = (1 - v^2)^{-r} \sum_{g_0} v^{\tau(g_0)},$$

where r is the dimension of the center of \mathfrak{l} and the sum is taken over a set of representatives g_0 for the (P'_0, P_0) -double cosets in G_0 that are contained in X' . In particular, if $\Pi \neq 0$, then $X' \neq \emptyset$.

Using 6.2, we have

$$\Pi = \sum_{j \in \mathbf{Z}} d_j(\mathfrak{l}'_\eta; A', \epsilon' \widetilde{\text{Res}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(\epsilon'' \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)))v^{-j} = \sum_{j \in \mathbf{Z}} d_{j+s}(\mathfrak{l}'_\eta; A', K)v^{-j},$$

where $s = \dim \mathfrak{u}_0 + \dim \mathfrak{u}_\eta + \dim \mathfrak{u}'_\eta - \dim \mathfrak{u}'_0 + \dim \mathfrak{l}_\eta$ and

$$K = \epsilon' \text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(\epsilon'' \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\#))$$

is as in 5.2. Using the description of K in 5.3(a), 5.4(b), 5.6, we see that

$$(a) \quad \Pi = \sum_{j \in \mathbf{Z}} \sum_g Q_j(g)v^{-j+s-2f(g)},$$

where g runs over a set of representatives for the (P'_0, P_0) -double cosets in G_0 which are good (see 5.2) and

$$Q_j(g) = d_j(\tilde{l}'_\eta; A', \text{ind}_{\tilde{q}'_\eta}^{\tilde{l}'_\eta}(\mathcal{L}''^\sharp)),$$

$$f(g) = \dim(\mathfrak{u}'_0/(\mathfrak{u}'_0 \cap \text{Ad}(g)\mathfrak{p}_0)) + \dim(\mathfrak{u}'_\eta \cap \text{Ad}(g)\mathfrak{u}_\eta);$$

the following notation is used:

\tilde{l}'_* is a certain splitting of \mathfrak{p}'_* , \tilde{l}''_* is a certain splitting of $\{\text{Ad}(g)\mathfrak{p}_N; N \in \mathbf{Z}\}$, $\tilde{q}' = \bigoplus_{N \in \mathbf{Z}} \tilde{q}'_N$ (where $\tilde{q}'_N = \tilde{l}'_N \cap \text{Ad}(g_0)\mathfrak{p}_N$) is a parabolic subalgebra of $\tilde{l}' = \bigoplus_N \tilde{l}'_N$ whose Levi subalgebra $\tilde{l}'' = \bigoplus_N \tilde{l}''_N$; A' is viewed as an object of $\mathcal{Q}(\tilde{l}'_\eta)$ via the obvious isomorphism $\tilde{l}'_\eta \rightarrow \tilde{l}''_\eta$ and $\mathcal{L}''^\sharp \in \mathcal{Q}(\tilde{l}''_\eta)$ corresponds to \mathcal{L}^\sharp via the isomorphism $\mathfrak{l}_\eta \xrightarrow{\text{Ad}(g)} \text{Ad}(g)\mathfrak{p}_\eta / \text{Ad}(g)\mathfrak{u}_\eta = \tilde{l}''_\eta$.

By the implication (a) \implies (c) in [L4, 10.6], we have $Q_j(g) = 0$ unless $\tilde{q}' = \tilde{l}'$. In this case, since \tilde{l}'' is a Levi subalgebra of \tilde{q}' , we must have $\tilde{l}' = \tilde{l}''$ so that $g \in X$. Conversely, if $g \in X$, then the (P'_0, P_0) -double coset of g is good. Indeed, let \tilde{l}'_* be a splitting of \mathfrak{p}'_* which is also a splitting for $\{\text{Ad}(g)\mathfrak{p}_N; N \in \mathbf{Z}\}$. We have

$$\text{Ad}(g)\mathfrak{p}_N = \tilde{l}'_N \oplus \text{Ad}(g)\mathfrak{u}_N \subset (\mathfrak{p}'_N \cap \text{Ad}(g)\mathfrak{p}_N) + \text{Ad}(g)\mathfrak{u}_N \subset \text{Ad}(g)\mathfrak{p}_N$$

and our claim follows. Thus the sum in (a) can be taken over a set of representatives g for the (P'_0, P_0) -double cosets in G_0 that are contained in X and for such g we have $Q_j(g) = d_j(\tilde{l}'_\eta; A', \mathcal{L}''^\sharp)$ where $\tilde{l}' = \tilde{l}''$, $\mathcal{L}''^\sharp \in \mathcal{Q}(\tilde{l}''_\eta)$ are as above. Using [L4, 15.1], we see that in the sum over g in (a) we can take $g \in X'$ and that the contribution of such g to the sum is $(1 - v^2)^{-r} v^{s-2f(g)-d}$ where $d = \dim \mathfrak{l}_\eta$. It remains to show that for g as above we have $s - 2f(g) - d = \tau(g)$. It is enough to show that:

(b) $\mathfrak{u}'_0 \cap \text{Ad}(g)\mathfrak{p}_0 = \mathfrak{u}'_0 \cap \text{Ad}(g)\mathfrak{u}_0,$

(c) $\dim(\text{Ad}(g)\mathfrak{u}_0) = \dim \mathfrak{u}'_0.$

Now (b),(c) hold since $\text{Ad}(g)\mathfrak{p}_0, \mathfrak{p}'_0$ are parabolic subalgebras of \mathfrak{g}_0 with nilradicals $\text{Ad}(g)\mathfrak{u}_0, \mathfrak{u}'_0$ and with a common Levi subalgebra. This completes the proof of the proposition.

6.5. In the special case where

$$(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) = (\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$$

and $A' \cong D(A)$, the sum $\sum_g v^{\tau(g)}$ in Proposition 6.4 is over a nonempty set of g (we have $1 \in X'$) hence the sum is nonzero and Π in 6.4 is nonzero. In particular, we see that

(a) $\epsilon'' \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) \neq 0.$

6.6. **The map ψ from simple perverse sheaves to $\underline{\mathfrak{T}}_\eta$.** Let B be a simple perverse sheaf in $\mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. We associate to B an element of $\underline{\mathfrak{T}}_\eta$ (see 3.5) as follows. We can find, $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$ and A as in 6.3 such that

$$\epsilon'' \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) \cong B[d] \oplus C,$$

where $d \in \mathbf{Z}$ and $C \in \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. Let \tilde{l}'_* be a splitting of \mathfrak{p}_* . Let $\tilde{l} = \bigoplus_N \tilde{l}'_N, \tilde{L} = e^{\tilde{l}} \subset G, \tilde{L}_0 = e^{\tilde{l}_0} \subset G$ and let \tilde{C} be the simple perverse sheaf on \tilde{l}_η corresponding to A under the obvious isomorphism $\tilde{l}_\eta \xrightarrow{\sim} \mathfrak{l}_\eta$. Then $(\tilde{L}, \tilde{L}_0, \tilde{l}, \tilde{l}_*, \tilde{C})$ is an object of

\mathfrak{T}_η and its $G_{\underline{0}}$ -orbit is independent of the choice of splitting, by 2.7(a). Now let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$, A' be as in 6.3 (with $\epsilon' = \epsilon''$) and assume that

$$\epsilon' \widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A') \cong B[d'] \oplus C',$$

where $d' \in \mathbf{Z}$ and $C' \in \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. We choose a splitting $\tilde{\mathfrak{l}}'_*$ of \mathfrak{p}'_* and we associate to it a system $(\tilde{L}', \tilde{L}'_0, \tilde{\mathfrak{l}}', \tilde{\mathfrak{l}}'_*, \tilde{C}')$ just as $(\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ was defined in terms of $\tilde{\mathfrak{l}}$; here \tilde{C}' corresponds to A' . Using 4.1(d), we see that

$$\epsilon' \widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(D(A')) \cong D(B)[-d'] \oplus D(C').$$

Let Π be as in 6.4 (with A' replaced by $D(A')$ and $\epsilon' = \epsilon''$). From the definition of Π in 6.4 we have also

$$\Pi = \{B[d] \oplus C, D(B)[-d'] \oplus D(C')\} = v^{d-d'} \text{ plus an element in } \mathbf{N}((v)).$$

(We use 0.12.) In particular we have $\Pi \neq 0$ hence X' in 6.4 is nonempty. It follows that $(\tilde{L}', \tilde{L}'_0, \tilde{\mathfrak{l}}', \tilde{\mathfrak{l}}'_*, \tilde{C}')$ and $(\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ are in the same $G_{\underline{0}}$ -orbit. This proves that $B \mapsto (\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ associates to B a well-defined element $\psi(B) \in \mathfrak{T}_\eta$.

6.7. For any $\xi \in \mathfrak{T}_\eta$ let ${}^\xi \mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ be the full subcategory of $\mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ whose objects are direct sums of shifts of simple perverse sheaves B in $\mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ such that $\psi(B) = \xi$ (see 6.6); let ${}^\xi \mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ be the (free) \mathcal{A} -submodule of $\mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ with basis given by the simple perverse sheaves B in ${}^\xi \mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$. Clearly, we have

$$\mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta) = \bigoplus_{\xi \in \mathfrak{T}_\eta} {}^\xi \mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta).$$

7. THE CATEGORIES $\mathcal{Q}(\mathfrak{g}_\delta)$, $\mathcal{Q}'(\mathfrak{g}_\delta)$

In this section we consider two categories of perverse sheaves $\mathcal{Q}(\mathfrak{g}_\delta)$, $\mathcal{Q}'(\mathfrak{g}_\delta)$ defined in terms of spiral induction; see 7.8. The simple objects in $\mathcal{Q}(\mathfrak{g}_\delta)$ are supported on $\mathfrak{g}_\delta^{nil}$, while those in $\mathcal{Q}'(\mathfrak{g}_\delta)$ have Fourier-Deligne transforms supported on $\mathfrak{g}_\delta^{nil}$. We also complete the proof of the main theorem 0.6.

7.1. Let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$. Let A_1 be the simple perverse sheaf on \mathfrak{g}_δ such that $\text{supp}(A_1)$ is the closure $\bar{\mathcal{O}}$ of \mathcal{O} in \mathfrak{g}_δ and $A_1|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.

Choose $x \in \mathcal{O}$ and $\phi \in \mathcal{J}_\delta(x)$; define $\mathfrak{p}_*^x, \tilde{\mathfrak{l}}_*^\phi, \tilde{L}^\phi, P_0$ as in 2.9. Then $\mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$ is defined in terms of $\tilde{\mathfrak{l}}_*^\phi, \tilde{L}^\phi$ and for any $A' \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$ we can consider

$$I(A') := {}^\eta \text{Ind}_{\mathfrak{p}_\eta^x}^{\mathfrak{g}_\delta}(A') \in \mathcal{Q}_\eta^\eta(\mathfrak{g}_\delta);$$

see 4.1. We show:

(a) *If $A' \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$, then the support of $I(A')$ is contained in $\bar{\mathcal{O}}$.*

Let $y \in \mathfrak{g}_\delta$ be in the support of $I(A')$. We must show that $y \in \bar{\mathcal{O}}$. From the definition of $I(A')$, there exists $g \in G_{\underline{0}}$ and $z \in \mathfrak{p}_\eta^x$ such that $\text{Ad}(g)(z) = y$. Since the support of $I(A')$ and $\bar{\mathcal{O}}$ are $G_{\underline{0}}$ -invariant we may replace y by $\text{Ad}(g^{-1})y$ hence we may assume that $y \in \mathfrak{p}_\eta^x$. Using 2.9(e), we see that \mathfrak{p}_η^x is equal to the closure of the P_0 -orbit of x in \mathfrak{p}_η^x , which is clearly contained in $\bar{\mathcal{O}}$. This proves (a).

Recall that $x \in \tilde{\mathfrak{l}}_\eta^{\circ\phi}$ (see 2.9(b)) hence $\tilde{\mathfrak{l}}_\eta^{\circ\phi} \subset \mathcal{O}$. By 2.9(c), $\mathcal{L}_1 := \mathcal{L}|_{\tilde{\mathfrak{l}}_\eta^{\circ\phi}}$ is an irreducible \tilde{L}_0^ϕ -equivariant local system on $\tilde{\mathfrak{l}}_\eta^{\circ\phi}$. Let $\mathcal{L}_1^\sharp \in \mathcal{D}(\tilde{\mathfrak{l}}_\eta^{\circ\phi})$ be as in 0.11 and let $A = \mathcal{L}_1^\sharp[\dim \tilde{\mathfrak{l}}_\eta^{\circ\phi}]$. We show:

(b) $I(\mathcal{L}_1^\sharp)|_{\mathcal{O}}$ is \mathcal{L} .

Let $E'_\mathcal{O}$ be the inverse image of \mathcal{O} under $c : E' \rightarrow \mathfrak{g}_\delta$ (where c, E' are as in 4.1 with $\mathfrak{p}_* = \mathfrak{p}_*^x, \epsilon = \dot{\eta}$). From the definitions we see that it is enough to check that the map $c_\mathcal{O} : E'_\mathcal{O} \rightarrow \mathcal{O}$ (restriction of c) is bijective on \mathbf{k} -points. Since $G_\mathcal{O}$ acts naturally on both $E'_\mathcal{O}$ and \mathcal{O} compatibly with c and the action on \mathcal{O} is transitive, it suffices to check that $c^{-1}(x)$ is a single point, namely (P_0, x) . Let $(gP_0, x) \in c^{-1}(x)$. We have $g \in G_\mathcal{O}$, $\text{Ad}(g^{-1})x \in \mathfrak{p}_\eta^x$ hence $x \in \text{Ad}(g)\mathfrak{p}_\eta^x$. From 2.9(d) we deduce that $g \in P_0$ hence $(gP_0, x) = (P_0, x)$. This proves (b).

We show:

(c) $I(\mathcal{L}_1^\sharp)$ is isomorphic to $\bigoplus_{j=1}^r A_j[t_j]$, where $t_1 = -\dim \mathcal{O}$ and for any $j \geq 2$, A_j is a simple $G_\mathcal{O}$ -equivariant perverse sheaf on \mathfrak{g}_δ with support contained in $\mathcal{O} - \mathcal{O}$ and $t_j \in \mathbf{Z}$.

This follows from the fact that $I(\mathcal{L}_1^\sharp)$ is a semisimple $G_\mathcal{O}$ -equivariant perverse sheaf on \mathfrak{g}_δ (the decomposition theorem), taking into account (a),(b).

By 1.5(a) we can find a parabolic subalgebra \mathfrak{q} of $\tilde{\mathfrak{l}}^\phi$, a Levi subalgebra \mathfrak{m} of \mathfrak{q} (with $\mathfrak{q}, \mathfrak{m}$ compatible with the \mathbf{Z} -grading of $\tilde{\mathfrak{l}}^\phi$) and a cuspidal $M_0 := e^{\mathfrak{m}_\mathcal{O}}$ -equivariant perverse sheaf C on \mathfrak{m}_η such that some shift of A is a direct summand of $\text{ind}_{\mathfrak{q}_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(C)$. From the definition we have

$$(d) \quad \Psi(\mathcal{O}, \mathcal{L}) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C) \in \underline{\mathfrak{T}}_\eta,$$

where $M = e^{\mathfrak{m}}$; see 3.5.

For any $N \in \mathbf{Z}$ let $\hat{\mathfrak{p}}_N$ be the inverse image of \mathfrak{q}_N under the obvious map $\mathfrak{p}_N \rightarrow \mathfrak{l}_N$. Then by 2.8(a), $\hat{\mathfrak{p}}_*$ is an $\dot{\eta}$ -spiral and \mathfrak{m}_* is a splitting of it, so that, by 4.2(a), we have

$$\dot{\eta} \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C) = \dot{\eta} \text{Ind}_{\hat{\mathfrak{p}}_\eta^\phi}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(C)).$$

It follows that some shift of $\dot{\eta} \text{Ind}_{\hat{\mathfrak{p}}_\eta^\phi}^{\mathfrak{g}_\delta}(A)$ is a direct summand of $\dot{\eta} \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C)$ hence, using (c), we see that some shift of A_1 is a direct summand of $\dot{\eta} \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C)$. In particular we have $A_1 \in \mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ and $\psi(A_1) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C) \in \underline{\mathfrak{T}}_\eta$; see 6.6 (with $\epsilon = \dot{\eta}$). Comparing with (d) we see that:

(e) $\psi(A_1) = \Psi(\mathcal{O}, \mathcal{L})$.

7.2. Characterization of $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ as orbital sheaves. Let A' be a semisimple $G_\mathcal{O}$ -equivariant complex on \mathfrak{g}_δ . We show:

(a) We have $A' \in \mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ if and only if $\text{supp}(A') \subset \mathfrak{g}_\delta^{nil}$.

We can assume that A' is a simple perverse sheaf. If $\text{supp}(A') \subset \mathfrak{g}_\delta^{nil}$, then we have $A' \in \mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ by the arguments in 7.1. Conversely, assume that $A' \in \mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$. We can find $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\dot{\eta}}$ and $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ such that some shift of A' is a direct summand of $B := \dot{\eta} \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(A)$. To show that $\text{supp}(A') \subset \mathfrak{g}_\delta^{nil}$ it is enough to show that $\text{supp}(B) \subset \mathfrak{g}_\delta^{nil}$ or (with c, A_1 as in 4.1 with $\epsilon = \dot{\eta}$) that $\text{supp}(c_!A_1) \subset \mathfrak{g}_\delta^{nil}$. This would follow if we can show that the image of c is contained in $\mathfrak{g}_\delta^{nil}$. By the

definition of c it is enough to show that $\mathfrak{p}_\eta \subset \mathfrak{g}_\delta^{nil}$. This follows from 2.5(d) applied with $N = \eta$.

We now restate 7.1(e) as follows.

(b) *Let A' be a simple perverse sheaf in $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ and let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ be such that $\text{supp}(A') = \bar{\mathcal{O}}$ and $A'|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$. Then $\psi(A') = \Psi(\mathcal{O}, \mathcal{L})$. (Notation of 3.5 and 6.6 with $\epsilon = \dot{\eta}$.)*

7.3. We now give another proof of the following statement (see also 3.8(f)):

(a) *The map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \rightarrow \underline{\mathfrak{T}}_\eta$ in 3.5 is surjective.*

Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, C)$ be an element of $\underline{\mathfrak{T}}_\eta$. We can find an $\dot{\eta}$ -spiral \mathfrak{p}_* such that \mathfrak{m}_* is a splitting of \mathfrak{p}_* . By 6.5(a), we have $\dot{\eta}\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C) \neq 0$, that is, there exists a simple perverse sheaf A' in $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ such that some shift of A' is a direct summand of $\dot{\eta}\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C)$. It follows that $\psi(A') = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C)$ hence, by 7.2(b), we have $\Psi(\mathcal{O}, \mathcal{L}) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C)$ where $(\mathcal{O}, \mathcal{L})$ corresponds to A' as in 7.2(b). This proves (a).

7.4. Until the end of 7.7 we assume that $p > 0$. If E, E' are finite dimensional \mathbf{k} -vector space with a given perfect bilinear pairing $E \times E' \rightarrow \mathbf{k}$, then we have the Fourier-Deligne transform functor $\Phi : \mathcal{D}(E) \rightarrow \mathcal{D}(E')$ defined in terms of a fixed nontrivial character $\mathbf{F}_p \rightarrow \bar{\mathbf{Q}}_l^*$ as in [L4, 1.9].

7.5. **Fourier transform and spiral restriction.** Let $B \in \mathcal{D}(\mathfrak{g}_\delta)$; we denote by $\Phi_{\mathfrak{g}}(B) \in \mathcal{D}(\mathfrak{g}_{-\delta})$ the Fourier-Deligne transform of B with respect to the perfect pairing $\mathfrak{g}_\delta \times \mathfrak{g}_{-\delta} \rightarrow \mathbf{k}$ defined by \langle, \rangle .

Let $\epsilon' \in \{1, -1\}$. Let $(\mathfrak{p}'_*, L', P'_0, l', l'_*, u'_*) \in \mathfrak{P}^{\epsilon'}$ and let

$$R_\eta = {}^{\epsilon'}\text{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B) \in \mathcal{D}(l'_\eta), \quad R_{-\eta} = {}^{\epsilon'}\text{Res}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(\Phi_{\mathfrak{g}}(B)) \in \mathcal{D}(l'_{-\eta}).$$

Then

(a) *$R_{-\eta}$ is the Fourier-Deligne transform of R_η with respect to the perfect pairing $l'_\eta \times l'_{-\eta} \rightarrow \mathbf{k}$ defined by \langle, \rangle .*

The proof is almost the same as that of [L4, 10.2]. We omit it.

7.6. **Fourier transform and spiral induction.** Let $\epsilon' \in \{1, -1\}$. Let

$$(\mathfrak{p}'_*, L', P'_0, l', l'_*, u'_*) \in \mathfrak{P}^{\epsilon'}.$$

Let $A \in \mathcal{D}(l'_\eta)$ be a semisimple complex; we denote by $\Phi_{l'}(A) \in \mathcal{D}(l'_{-\eta})$ the Fourier-Deligne transform of A with respect to the perfect pairing $l'_\eta \times l'_{-\eta} \rightarrow \mathbf{k}$ defined by \langle, \rangle ; note that $\Phi_{l'}(A)$ is a semisimple complex. Let

$$I_\eta = {}^{\epsilon'}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A) \in \mathcal{D}(\mathfrak{g}_\delta),$$

$$I_{-\eta} = {}^{\epsilon'}\widetilde{\text{Ind}}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(\Phi_{l'}(A)) \in \mathcal{D}(\mathfrak{g}_{-\delta}).$$

Then:

(a) *$I_{-\eta}$ is the Fourier-Deligne transform of I_η with respect to the perfect pairing $\mathfrak{g}_\delta \times \mathfrak{g}_{-\delta} \rightarrow \mathbf{k}$ defined by \langle, \rangle .*

The proof is almost the same as that of [L5, A2]. We omit it.

7.7. Characterization of $\mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$ as anti-orbital sheaves. Let $B \in \mathcal{D}(\mathfrak{g}_\delta)$ be a semisimple complex; let $B' = \Phi_{\mathfrak{g}}(B) \in \mathcal{D}(\mathfrak{g}_{-\delta})$ be its Fourier-Deligne transform, as in 7.5. Note that B' is again a semisimple complex. We show:

(a) *We have $B \in \mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$ if and only if $\text{supp}(B') \subset \mathfrak{g}_{-\delta}^{nil}$.*

We can assume that B (and hence also B') is a simple perverse sheaf.

Assume first that $B \in \mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$. We can find $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{-\dot{\eta}}$ and a cuspidal perverse sheaf C in $\mathcal{Q}(\mathfrak{l}'_\eta)$ such that some shift of B is a direct summand of ${}^{-\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C)$. Using 7.6(a) we see that some shift of B' is a direct summand of ${}^{-\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(C')$ where $C' = \Phi_{\mathfrak{l}'}(C) \in \mathcal{D}(\mathfrak{l}'_{-\eta})$ (notation of 7.6). By [L4, 10.6], C' is a cuspidal perverse sheaf in $\mathcal{Q}(\mathfrak{l}'_{-\eta})$. It follows that $B' \in \mathcal{Q}_{-\eta}^{-\dot{\eta}}(\mathfrak{g}_{-\delta})$. Using 7.2(a) (with η, δ replaced by $-\eta, -\delta$) we deduce that $\text{supp}(B') \subset \mathfrak{g}_{-\delta}^{nil}$.

Conversely, assume that B is such that $\text{supp}(B') \subset \mathfrak{g}_{-\delta}^{nil}$. Using 7.2(a), we see that $B' \in \mathcal{Q}_{-\eta}^{-\dot{\eta}}(\mathfrak{g}_{-\delta})$. We can find $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{-\dot{\eta}}$ and a cuspidal perverse sheaf C'_1 in $\mathcal{Q}(\mathfrak{l}'_{-\eta})$ such that some shift of B' is a direct summand of ${}^{-\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(C'_1)$. We can find a cuspidal perverse sheaf C_1 in $\mathcal{Q}(\mathfrak{l}'_\eta)$ such that $C'_1 = \Phi_{\mathfrak{l}'}(C)$ (we use again [L4, 10.6]). Using 7.6(a), we see that some shift of $\Phi_{\mathfrak{g}}(B)$ is a direct summand of $\Phi_{\mathfrak{g}}({}^{-\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C_1))$ hence some shift of B is a direct summand of ${}^{-\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C_1)$ so that $B \in \mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$. This completes the proof of (a).

7.8. The assumption on p in 7.4 is no longer in force. From 7.2(a) we see that $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ (hence also $\mathcal{K}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$) is independent of η as long as $\underline{\eta} = \delta$. We shall write $\mathcal{Q}(\mathfrak{g}_\delta), \mathcal{K}(\mathfrak{g}_\delta)$ instead of $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta), \mathcal{K}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$. From 7.7(a) we see that $\mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$ (hence also $\mathcal{K}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$) is independent of η as long as $\underline{\eta} = \delta$ (at least when $p > 0$, but then the same holds for $p = 0$ by standard arguments). We shall write $\mathcal{Q}'(\mathfrak{g}_\delta), \mathcal{K}'(\mathfrak{g}_\delta)$ instead of $\mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta), \mathcal{K}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$.

For $\xi \in \underline{\mathfrak{X}}_\delta$ we write ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta), {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ instead of ${}^\xi\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta), {}^\xi\mathcal{K}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ and we write ${}^\xi\mathcal{Q}'(\mathfrak{g}_\delta), {}^\xi\mathcal{K}'(\mathfrak{g}_\delta)$ instead of ${}^\xi\mathcal{Q}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta), {}^\xi\mathcal{K}_\eta^{-\dot{\eta}}(\mathfrak{g}_\delta)$. The discussion in 3.9 shows that ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta), {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ and ${}^\xi\mathcal{Q}'(\mathfrak{g}_\delta), {}^\xi\mathcal{K}'(\mathfrak{g}_\delta)$ are independent of η as long as $\underline{\eta} = \delta$.

7.9. Proof of Theorem 0.6. Let $\xi \in \underline{\mathfrak{X}}_\eta$. Let $K \in \mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})$. We say that $K \in \mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})_\xi$ if any simple perverse sheaf B which appears in a perverse cohomology sheaf of K satisfies $\psi(B) = \xi$; note that B belongs to $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$, see 7.2(a); hence $\psi(B)$ is defined as in 6.6.

Now let ξ, ξ' in $\underline{\mathfrak{X}}_\eta$ be such that $\xi \neq \xi'$. Let $K \in \mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})_\xi, K' \in \mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})_{\xi'}$. We show:

(a) $\text{Hom}_{\mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})}(K, K') = 0$.

We can assume that $K = B[n], K' = B'[n']$ where B, B' are simple perverse sheaves in $\mathcal{Q}_\eta^{\dot{\eta}}(\mathfrak{g}_\delta)$ such that $\psi(B) = \xi, \psi(B') = \xi'$ and n, n' are integers. We see that it is enough to prove (a) in the case where $K = {}^{\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A)[n], K' = {}^{\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A)[n']$ with $n, n' \in \mathbf{Z}, \mathfrak{p}_*, \mathfrak{p}'_*, A, A'$ as in 6.4, and $\epsilon' = \epsilon'' = \dot{\eta}$, since some shifts of B and B' appear as direct summands of such K and K' . By 0.12(a), we have an isomorphism

$$\text{Hom}_{\mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})}(K, K') = \mathbf{D}_0(\mathfrak{g}_\delta^{nil}, G_\mathbb{Q}; K, D(K'))^*.$$

Hence

(b) $\dim \text{Hom}_{\mathcal{D}_{G_\mathbb{Q}}(\mathfrak{g}_\delta^{nil})}(K, K') = d_{n-n'}(\mathfrak{g}_\delta^{nil}; {}^{\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A), {}^{\dot{\eta}}\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(D(A)))$.

Here we use 4.1(d). Since $\xi \neq \xi'$, the set X' defined in 6.4 for the pair $(D(A), A')$ is empty. Therefore the right side of (b) is zero by 6.4. Then (a) follows from (b). We see that Theorem 0.6 holds.

8. MONOMIAL AND QUASI-MONOMIAL OBJECTS

The results in this section are parallel to those in 1.8–1.9. They serve as preparation for the next section.

8.1. Let $\epsilon = \eta$. We denote by \mathfrak{R}^ϵ the set of all data of the form

$$(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*, A),$$

where $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon$ (see 4.1) and A is a perverse sheaf in $\mathcal{Q}(\mathfrak{l}_\eta)$ which is η -semicuspidal (as in 1.8 with H replaced by L).

8.2. An object $B \in \mathcal{Q}(\mathfrak{g}_\delta)$ is said to be η -quasi-monomial if $B \cong \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ for some $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*, A) \in \mathfrak{R}^\epsilon$; if in addition A is taken to be cuspidal, then B is said to be η -monomial. Using 1.8(b) and the transitivity property 4.2, we see that:

(a) *If $B \in \mathcal{Q}(\mathfrak{g}_\delta)$ is η -quasi-monomial, then there exists an η -monomial object $B' \in \mathcal{Q}(\mathfrak{g}_\delta)$ such that $B' \cong B[a_1] \oplus B[a_2] \oplus \dots \oplus B[a_k]$ for some sequence a_1, a_2, \dots, a_k in \mathbf{Z} , $k \geq 1$. In particular, in $\mathcal{K}(\mathfrak{g}_\delta)$ we have $(B') = (v^{a_1} + \dots + v^{a_k})(B)$.*

An object of $\mathcal{Q}(\mathfrak{g}_\delta)$ is said to be η -good if it is a direct sum of shifts of η -quasi-monomial objects.

Proposition 8.3 (8.3). *Let $B \in \mathcal{Q}(\mathfrak{g}_\delta)$. There exists η -good objects B_1, B_2 in $\mathcal{Q}(\mathfrak{g}_\delta)$ such that $B \oplus B_1 \cong B_2$.*

We can assume that B is a simple perverse sheaf. We define $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ by the requirement that $\text{supp } B$ is the closure $\bar{\mathcal{O}}$ of \mathcal{O} in \mathfrak{g}_δ and $B|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$. We prove the proposition by induction on $\dim \mathcal{O}$. Let $x \in \mathcal{O}$. We associate to x an ϵ -spiral $\mathfrak{p}_* = \mathfrak{p}_*^x$ as in 2.9; we complete it uniquely to a system $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon$. By 7.1(c), there exists $A_1 \in \mathcal{Q}(\mathfrak{l}_\eta)$ such that ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_1) \cong B[d] \oplus B'$, where $d \in \mathbf{Z}$ and $B' \in \mathcal{Q}(\mathfrak{g}_\delta)$ has support contained in $\bar{\mathcal{O}} - \mathcal{O}$. We now use 1.9(a) for L, A_1 instead of H, A_1 ; applying ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}$ to the equality in 1.9(a) we obtain

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_1) \oplus C'_1 \oplus C'_2 \oplus \dots \oplus C'_t = C'_{t+1} \oplus \dots \oplus C'_{t+t'},$$

where each C'_j is an η -quasi-monomial object with a shift (we have used the transitivity property 4.2). Thus we have

$$B[d] \oplus B' \oplus C'_1 \oplus C'_2 \oplus \dots \oplus C'_t = C'_{t+1} \oplus \dots \oplus C'_{t+t'}.$$

Now the induction hypothesis implies that B' is η -good. From this and the previous equality we see that B is η -good. The proposition is proved.

Corollary 8.4.

(a) *The \mathcal{A} -module $\mathcal{K}(\mathfrak{g}_\delta)$ is generated by the classes of η -quasi-monomial objects of $\mathcal{Q}(\mathfrak{g}_\delta)$.*

(b) *The $\mathbf{Q}(v)$ -vector space $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_\delta)$ is generated by the classes of η -monomial objects of $\mathcal{Q}(\mathfrak{g}_\delta)$.*

(a) follows immediately from 8.3; (b) follows from (a) using 8.2(a).

8.5. We show:

(a) *If B_1, B_2 are elements of $\mathcal{K}(\mathfrak{g}_\delta)$ then $\{B_1, B_2\} \in \mathbf{Q}(v)$ (notation of 4.4(c)).*

By 8.3, we can assume that B_1, B_2 are classes of η -quasi-monomial objects. By 8.2(a) we have $f_1 B_1 = B'_1, f_2 B_2 = B'_2$ where B'_1, B'_2 represent ϵ -monomial objects and f_1, f_2 are nonzero elements of \mathcal{A} . Thus, we can assume that B_1, B_2 represent η -monomial objects. In this case the result follows from 6.4.

9. EXAMPLES

In this section we consider examples where $G = SL(V)$ or $Sp(V)$. We assume that $m \geq 2$ and $\eta = 1$ hence $\delta = \underline{1}$. We write “spiral” instead of “1-spiral”. We explicitly describe the spirals and the set of blocks $\underline{\mathfrak{T}}_1$ in both cases, and describe the map Ψ in the case $G = SL(V)$.

9.1. Spirals for the cyclic quiver. We preserve the notation from 0.3. Thus we assume that $G = SL(V)$ where $V = \bigoplus_{i \in \mathbf{Z}/m} V_i$. We have an induced \mathbf{Z}/m -grading on $\mathfrak{g} = \mathfrak{sl}(V)$, so that $\mathfrak{g}_{\underline{1}}$ is the space of all maps in 0.3(a). In general, we have $\mathfrak{g}_i = \bigoplus_{j \in \mathbf{Z}/m} \text{Hom}(V_j, V_{j+i})$.

The datum $\lambda \in Y_{G_{\mathbb{Q}}, \mathbf{Q}}$ is the same as a \mathbf{Q} -grading on each V_i , i.e., $V_i = \bigoplus_{x \in \mathbf{Q}} (x V_i)$ such that $\sum_i \sum_x x \dim(x V_i) = 0$. Given such a \mathbf{Q} -grading on each V_i , the corresponding spiral $\mathfrak{p}_* = \{\mathfrak{p}_N \subset \mathfrak{g}_N\}_{N \in \mathbf{Z}}$ takes the following form:

$$\mathfrak{p}_N = \{\phi \in \mathfrak{sl}(V) \mid \phi(x V_j) \subset \bigoplus_{x' \geq x+N} (x' V_{j+N}), \quad \forall j \in \mathbf{Z}/m, x \in \mathbf{Q}\}.$$

A splitting $\mathfrak{m}_* = \{\mathfrak{m}_N \subset \mathfrak{g}_N\}_{N \in \mathbf{Z}}$ of the spiral \mathfrak{p}_* takes the form

$$\mathfrak{m}_N = \{\phi \in \mathfrak{sl}(V) \mid \phi(x V_j) \subset \bigoplus_{x' \geq x+N} (x' V_{j+N}), \quad \forall j \in \mathbf{Z}/m, x \in \mathbf{Q}\}.$$

For such a grading $x V_i$ we may introduce a quiver Q_λ as follows. Let J_λ be the finite set of pairs $(i, x) \in \mathbf{Z}/m \times \mathbf{Q}$ such that $x V_i \neq 0$. Then Q_λ has vertex set J_λ and an edge $(i, x) \rightarrow (i + 1, x + 1)$ if both (i, x) and $(i + 1, x + 1)$ are in J_λ . Then Q_λ is a disjoint union of directed chains (that is, quivers of type A with exactly one source and exactly one sink). We may identify \mathfrak{m}_1 with the representation space of the quiver Q_λ with vector space $x V_i$ on the vertex $(i, x) \in J_\lambda$.

Let B be the set of chains in Q_λ , and let $J_\lambda = \sqcup_{\beta \in B} (\beta \cdot J_\lambda)$ be the corresponding decomposition of the vertex set. Let $\beta V := \bigoplus_{(i,x) \in \beta} (x V_i)$. Then we have $V = \bigoplus_{\beta \in B} (\beta V)$. Let $M = e^{\mathfrak{m}}, M_0 = e^{\mathfrak{m}_0}$ where $\mathfrak{m} = \bigoplus_N \mathfrak{m}_N$. Then $M = S(\prod_{\beta \in B} GL(\beta V)), M_0 = S(\prod_{(i,x) \in J_\lambda} GL(x V_i))$. The center Z_M is the subgroup of M where each factor in $GL(\beta V)$ is a scalar matrix.

9.2. Admissible systems for the cyclic quiver. Let d be a divisor of $n = \dim V$. Suppose that the following hold:

- (1) Each $x V_i$ has dimension ≤ 1 .
- (2) Each connected component of the quiver Q_λ is a directed chain containing exactly d vertices.

In this case, M_0 is a maximal torus of G stabilizing each line $x V_i$ for $(i, x) \in J_\lambda$. The open M_0 -orbit $\mathring{\mathfrak{m}}_1 \subset \mathfrak{m}_1$ consists of representations of Q_λ where all arrows are nonzero (hence isomorphisms). The stabilizer of an element in $\mathring{\mathfrak{m}}_1$ under M_0 is exactly Z_M , which acts by a scalar z_β on each chain $\beta \in B$, such that $(\prod_{\beta \in B} z_\beta)^d = 1$. We see that $\pi_0(Z_M) \cong \mu_d$. For any primitive character $\chi : \mu_d \rightarrow \mathbf{Q}_l^*$, we have

a rank 1 M_0 -equivariant local system C_χ on \mathfrak{m}_1 on whose stalks Z_M acts via χ . This is a cuspidal local system because it is the restriction of the cuspidal local system on the regular nilpotent orbit of \mathfrak{m} with central character χ . Let \tilde{C}_χ be the cuspidal perverse sheaf on \mathfrak{m}_1 corresponding to C_χ . The system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$ is admissible. It is easy to see that any admissible system is of the form we just described.

Given such a grading λ , we define a function $f : B \rightarrow \mathbf{Z}/m$ such that $f(\beta) = i$ where (i, x) is the head (origin) of the chain β . Each vertex $(i, x) \in J_\lambda$ lies in a unique chain $\beta \in B$ whose head is of the form $(f(\beta), x')$. Then $x - x' = y$ is an integer between 0 and $d - 1$ and $f(\beta) + \underline{y} = i$ in \mathbf{Z}/m . This implies that $\dim V_i = \#\{x \in \mathbf{Q} \mid (i, x) \in J_\lambda\}$ is the same as the number of pairs $(\beta, y) \in B \times \{0, 1, \dots, d-1\}$ such that $f(\beta) + \underline{y} = i$. Choosing a bijection between $\{1, 2, \dots, n/d\}$ and B , the function f may be viewed as a function $\{1, 2, \dots, n/d\} \rightarrow \mathbf{Z}/m$ satisfying 0.7(b). Changing the bijection amounts to precomposing f with a permutation of $\{1, 2, \dots, n/d\}$. Summarizing the above discussion, we get a canonical bijection between $\underline{\mathfrak{T}}_1$ and the set of equivalence classes of triples (d, f, χ) as in 0.7(a).

9.3. The map Ψ for the cyclic quiver. We preserve the notation from 9.1. Let $(\mathcal{O}, \mathcal{L}) \in \chi(\mathfrak{ig}_1)$. For each element $e \in \mathcal{O}$, there exists a decomposition of V into Jordan blocks $\{ {}_\alpha W \}_{\alpha \in B_e}$ compatible with the \mathbf{Z}/m -grading in the following sense. Each Jordan block ${}_\alpha W$ is a direct sum of finitely many 1-dimensional subspaces indexed by $0, 1, \dots$, i.e., ${}_\alpha W = ({}_\alpha W_0) \oplus ({}_\alpha W_1) \oplus \dots$ such that

- (1) ${}_\alpha W_N \subset V_{h(\alpha) + \underline{N}}$ for some $h(\alpha) \in \mathbf{Z}/m$ (location of the head of the Jordan block α);
- (2) e maps ${}_\alpha W_N$ isomorphically to ${}_\alpha W_{N+1}$ whenever $N \geq 0$ and ${}_\alpha W_{N+1} \neq 0$.

The datum of $\{ {}_\alpha W \}_{\alpha \in B_e}$ as above is the equivalent to the datum of an element $\phi \in J_1(e)$; see 2.3. From this we may define a quiver Q_e whose vertex set J_e consists of pairs $(\alpha, N) \in B_e \times \mathbf{Z}_{\geq 0}$ such that ${}_\alpha W_N \neq 0$, and there is no edge $(\alpha, N) \rightarrow (\alpha, N + 1)$ if both $(\alpha, N), (\alpha, N + 1)$ are in $B_e \times \mathbf{Z}_{\geq 0}$.

Each vertex (α, N) is labelled with the element $h(\alpha) + \underline{N} \in \mathbf{Z}/m$. The isomorphism class of Q_e together with the labelling by elements in \mathbf{Z}/m is independent of the choice of e in \mathcal{O} and the choice of the Jordan block decomposition. Therefore we denote this labelled quiver by $Q_{\mathcal{O}}$, with vertex set $J_{\mathcal{O}}$ and set of chains $B_{\mathcal{O}}$.

Let $d' = \gcd\{|\alpha| \mid \alpha \in B_{\mathcal{O}}\}$ (here $|\alpha|$ is the number of vertices of the chain α). Then for any $e \in \mathcal{O}$, there is a canonical isomorphism $\pi_0(G_{\underline{0}}(e)) \cong \mu_{d'}$. The local system \mathcal{L} on \mathcal{O} corresponds to a character ρ of $\mu_{d'}$, which has order d dividing d' and a unique factorization

$$\rho : \mu_{d'} \rightarrow \mu_d \xrightarrow{\chi} \bar{\mathbf{Q}}_l^*$$

such that χ is injective (here the first map $\mu_{d'} \rightarrow \mu_d$ is given by $z \mapsto z^{d'/d}$). Now we define a new quiver $Q_{\mathcal{O}}^{[d]}$ by removing certain edges from each chain of $Q_{\mathcal{O}}$ such that each chain of $Q_{\mathcal{O}}^{[d]}$ has exactly d vertices. Let B be the set of chains of $Q_{\mathcal{O}}^{[d]}$; then B can be identified with the set $\{1, 2, \dots, n/d\}$. Define $f : \{1, 2, \dots, n/d\} \cong B \rightarrow \mathbf{Z}/m$ to be the map assigning to each $\beta \in B$ the label of its head. This way we get a triple (d, f, χ) as in 0.7(b) whose equivalence class is well-defined.

Proposition 9.4. *In the case of cyclic quivers, the map $\Psi : \mathcal{I}(\mathfrak{g}_1) \rightarrow \underline{\mathfrak{T}}_1$ sends $(\mathcal{O}, \mathcal{L})$ to the admissible system in $\underline{\mathfrak{T}}_1$ which corresponds to the equivalence class of the triple (d, f, χ) defined above under the bijection 0.7(a).*

Let $e \in \mathcal{O}$, and let $V = \bigoplus_{\alpha \in B_e} (\alpha W)$, $\alpha W = \alpha W_0 \oplus \alpha W_1 \oplus \cdots$ be a Jordan block decomposition, where $\alpha W_N \subset V_{h(\alpha) + \underline{N}}$ for $\alpha \in B_e, N \in \mathbf{Z}_{\geq 0}$. Let L be the Levi subgroup of a parabolic subgroup of G such that L stabilizes the decomposition $V = \bigoplus_{\alpha \in B_e} (\alpha W)$. Then $\mathfrak{l} = \mathfrak{L}L$ has a \mathbf{Z} -grading induced from the \mathbf{Z} -grading on each of αW . In particular, \mathfrak{l}_1 is the space of representations of the quiver Q_e . The system $(L, L_0, \mathfrak{l}, \mathfrak{l}_*)$ is the system $(\tilde{L}^\phi, \tilde{L}_0^\phi, \tilde{\mathfrak{l}}^\phi, \tilde{\mathfrak{l}}_*^\phi)$ attached to some $\phi \in J_{\underline{1}}(e)$ as in 2.9. Then e is in the open L_0 -orbit \mathfrak{l}_1 of \mathfrak{l}_1 , which is contained in the regular nilpotent orbit of \mathfrak{l} .

Let ${}_\alpha L = SL({}_\alpha W)$ be the subgroup of L which acts as identity on all blocks ${}_{\alpha'} W$ for $\alpha' \neq \alpha$. Then ${}_\alpha \mathfrak{l} = \mathfrak{L}({}_\alpha L)$ carries a \mathbf{Z} -grading compatible with that on \mathfrak{l} . For each interval $[a, b] \subset \mathbf{Z}_{\geq 0}$, let ${}_\alpha W_{[a,b]} \subset {}_\alpha W$ be the direct sum of ${}_\alpha W_N$ for $a \leq N \leq b$. We decompose ${}_\alpha W$ into $|\alpha|/d$ parts each of dimension d :

$$(a) \quad {}_\alpha W = \bigoplus_{j=1}^{|\alpha|/d} ({}_\alpha W_{[(j-1)d, jd-1]}).$$

Let ${}_\alpha M \subset {}_\alpha L$ be the subgroup stabilizing the decomposition (a). Then the Lie algebra ${}_\alpha \mathfrak{m}$ of ${}_\alpha M$ inherits a \mathbf{Z} -grading from that of ${}_\alpha \mathfrak{l}$, and the open orbit ${}_\alpha \mathring{\mathfrak{m}}_1$ carries a local system ${}_\alpha C_\chi$ corresponding to the character χ of $\mu_d \cong \pi_0(Z({}_\alpha M))$. Let ${}_\alpha \tilde{C}_\chi$ be the cuspidal perverse sheaf on ${}_\alpha \mathfrak{m}_1$ corresponding to ${}_\alpha C_\chi$. Define a parabolic subalgebra ${}_\alpha \mathfrak{q} \subset {}_\alpha \mathfrak{l}$ to be the stabilizer of the filtration ${}_\alpha W_{[|\alpha|-d, |\alpha|-1]} \subset {}_\alpha W_{[|\alpha|-2d, |\alpha|-1]} \subset \cdots \subset {}_\alpha W = {}_\alpha W_{[0, |\alpha|-1]}$. Then ${}_\alpha \mathfrak{q}$ is compatible with the \mathbf{Z} -grading on ${}_\alpha \mathfrak{l}$ and ${}_\alpha \mathfrak{m}$ is a Levi subalgebra of ${}_\alpha \mathfrak{q}$. The induction

$$\text{ind}_{{}_\alpha \mathfrak{q}_1}^{{}_\alpha \mathfrak{l}_1} ({}_\alpha \tilde{C}_\chi)$$

restricted to ${}_\alpha \mathring{\mathfrak{l}}_1$ is isomorphic to $\mathcal{L}|_{\alpha \mathring{\mathfrak{l}}_1}$, because the map c in 1.3 (applied to ${}_\alpha \mathfrak{l}, {}_\alpha \mathfrak{q}, {}_\alpha \mathfrak{m}$ in place of $\mathfrak{h}, \mathfrak{p}, \mathfrak{l}$) is an isomorphism when restricted to ${}_\alpha \mathring{\mathfrak{l}}_1$. Therefore the middle extension of $\mathcal{L}|_{\alpha \mathring{\mathfrak{l}}_1}$ to \mathfrak{l}_1 appears as a direct summand of $\text{ind}_{\alpha \mathfrak{q}_1}^{\alpha \mathfrak{l}_1} ({}_\alpha \tilde{C}_\chi)$.

Therefore, under the map defined in 1.5(b), the image of $({}_\alpha \mathring{\mathfrak{l}}_1, \mathcal{L}|_{\alpha \mathring{\mathfrak{l}}_1})$ is

$$({}_\alpha M, {}_\alpha M_0, {}_\alpha \mathfrak{m}, {}_\alpha \mathfrak{m}_1, {}_\alpha \tilde{C}_\chi).$$

Let ${}_\alpha \tilde{M} \subset GL({}_\alpha W)$ be the stabilizer of the decomposition (a). Let

$$M = S\left(\prod_{\alpha \in B_e} ({}_\alpha \tilde{M})\right) \subset L$$

with Lie algebra $\mathfrak{m} \subset \bigoplus ({}_\alpha \tilde{\mathfrak{m}})$ and the induced \mathbf{Z} -grading from each ${}_\alpha \tilde{\mathfrak{m}} = \mathfrak{L}({}_\alpha \tilde{M})$. The open M_0 -orbit on $\mathfrak{m}_1 = \bigoplus ({}_\alpha \mathfrak{m}_1)$ is $\mathring{\mathfrak{m}}_1 = \prod ({}_\alpha \mathring{\mathfrak{m}}_1)$. Let $C_\chi = \boxtimes ({}_\alpha C_\chi)$ on $\mathring{\mathfrak{m}}_1$. Let \tilde{C}_χ be the cuspidal perverse sheaf on \mathfrak{m}_1 corresponding to C_χ . By the compatibility of the assignment in 1.5(b) with direct products, in the situation $H = L$, the pair $(\mathring{\mathfrak{l}}_1, \mathcal{L}|_{\alpha \mathring{\mathfrak{l}}_1})$ maps to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$. Therefore, $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$ is the admissible system attached to $(\mathcal{O}, \mathcal{L})$ through the procedure in 2.9. By 9.2, the admissible system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$ corresponds to the triple (d, f, χ) defined in 9.3 before the statement of this proposition. This finishes the proof.

9.5. The symplectic quiver. Let V be a finite-dimensional vector space over \mathbf{k} with a nondegenerate symplectic form ω . Assume that m in 0.1 is even. Let $\tilde{\mathfrak{S}}_m = \{j; j = k/2; k = \text{an odd integer}\}$ and let \mathfrak{S}_m be the set of equivalence

classes for the relation \sim on $\tilde{\mathfrak{S}}_m$ given by $j \sim j'$ if $j - j' \in m\mathbf{Z}$. Note that the involution $j \mapsto -j$ of $\tilde{\mathfrak{S}}_m$ induces an involution of \mathfrak{S}_m denoted again by $j \mapsto -j$.

For any $N \in \mathbf{Z}$, the map $j \mapsto N + j$ of $\tilde{\mathfrak{S}}_m$ onto itself induces a map of \mathfrak{S}_m onto itself which depends only on \underline{N} and is denoted by $j \mapsto \underline{N} + j$.

The set \mathfrak{S}_m consists of m elements represented by

$$\left\{ \frac{1}{2}, \frac{3}{2}, \dots, \frac{m-1}{2}, \frac{m+1}{2}, \dots, m - \frac{1}{2} \right\}.$$

Consider a grading on V indexed by \mathfrak{S}_m :

(a)
$$V = \bigoplus_{j \in \mathfrak{S}_m} V_j,$$

such that $\omega(V_j, V_{j'}) = 0$ unless $j' = -j$ (as elements of \mathfrak{S}_m). Using the symplectic form, for $j \in \mathfrak{S}_m$ we may identify V_j with the dual of V_{-j} .

We assume that $G = Sp(V)$ and that the \mathbf{Z}/m -grading of $\mathfrak{g} = \mathfrak{sp}(V)$ is given by

(b)
$$\mathfrak{g}_i = \{ \phi \in \mathfrak{sp}(V) \mid \phi(V_j) \subset V_{i+j}, \quad \forall j \in \mathfrak{S}_m \}, \quad \forall i \in \mathbf{Z}/m.$$

In particular, an element $\phi \in \mathfrak{g}_1$ is a collection of maps $\phi_i : V_{i-\frac{1}{2}} \rightarrow V_{i+\frac{1}{2}}, i \in \mathbf{Z}/m$, which can be represented by a cyclic quiver

$$\begin{array}{ccccccc} V_{-\frac{1}{2}} & \xleftarrow{\phi_{-1}} & V_{-\frac{3}{2}} & \xleftarrow{\dots} & \dots & \xleftarrow{\phi_{\frac{m+1}{2}}} & V_{\frac{m+1}{2}} \\ \phi_0 \downarrow & & & & & & \uparrow \phi_{\frac{m}{2}} \\ V_{\frac{1}{2}} & \xrightarrow{\phi_1} & V_{\frac{3}{2}} & \xrightarrow{\dots} & \dots & \xrightarrow{\phi_{\frac{m-1}{2}}} & V_{\frac{m-1}{2}} \end{array} .$$

The condition $\phi \in \mathfrak{sp}(V)$ becomes that

(c)
$$\phi_{-i} = -\phi_i^*, \quad \forall i \in \mathbf{Z}/m.$$

Here $\phi_i^* : V_{i+\frac{1}{2}}^* \rightarrow V_{i-\frac{1}{2}}^*$ is the adjoint of ϕ_i , which can be viewed as a map $V_{-i-\frac{1}{2}} \rightarrow V_{-i+\frac{1}{2}}$ under the identifications $V_{i+\frac{1}{2}}^* \cong V_{-i-\frac{1}{2}}, V_{i-\frac{1}{2}}^* \cong V_{-i+\frac{1}{2}}$ using the symplectic pairing. In particular, for $i = 0$, $\phi_0 : V_{-\frac{1}{2}} = V_{\frac{1}{2}}^* \rightarrow V_{\frac{1}{2}}$ can be viewed as a vector $\phi_0 \in V_{\frac{1}{2}}^{\otimes 2}$. The condition (b) for $i = 0$ is equivalent to saying that $\phi_0 \in \text{Sym}^2(V_{\frac{1}{2}})$. Similarly, we may view $\phi_{\frac{m}{2}}$ as a vector in $V_{\frac{m+1}{2}}^{\otimes 2}$, and the condition (c) for $i = \frac{m}{2}$ is equivalent to saying that $\phi_{\frac{m}{2}} \in \text{Sym}^2(V_{\frac{m+1}{2}})$.

We call a representation of the quiver above in which $V_{-j} = V_j^*$, and (c) holds a *symplectic representation*. In other words, \mathfrak{g}_1 is the space of symplectic representations of the quiver above.

We have $G_0 \cong \prod_{\frac{1}{2} \leq j \leq \frac{m-1}{2}} GL(V_j)$, where $GL(V_j) \cong GL(V_{-j})$ acts diagonally on both V_j and $V_{-j} = V_{m-j}^* = V_j^*$.

9.6. Spirals for the symplectic quiver. Each element $\lambda \in Y_{G_\Omega, \mathbf{Q}}$ is the same datum as a \mathbf{Q} -grading on each $V_j, j \in \mathfrak{S}_m$, i.e., $V_j = \bigoplus_{x \in \mathbf{Q}} (x V_j)$ such that under the symplectic form $\omega, \omega(x V_j, x' V_{-j}) = 0$ unless $x + x' = 0$. Then ${}_{-x}V_{-j}$ can be identified with the dual of ${}_xV_j$ for all $(j, x) \in \mathfrak{S}_m \times \mathbf{Q}$. The spiral \mathfrak{p}_* associated to this grading is

$$\mathfrak{p}_N = \{ \phi \in \mathfrak{sp}(V) \mid \phi(x V_j) \subset \bigoplus_{x' \geq x+N} (x' V_{j+\underline{N}}), \quad \forall j \in \mathfrak{S}_m, x \in \mathbf{Q} \}.$$

A splitting \mathfrak{m}_* of the spiral \mathfrak{p}_* takes the form

$$\mathfrak{m}_N = \{\pi \in \mathfrak{sp}(V) \mid \phi(xV_j) \subset x_{+N}V_{j+N}, \quad \forall j \in \mathfrak{S}_m, x \in \mathbf{Q}\}.$$

To each such grading, we may attach a quiver Q_λ as we did for the cyclic quiver (since the symplectic quiver is a special case of a cyclic quiver). There is an involution on Q_λ sending $(j, x) \in J_\lambda$ to $(-j, -x) \in J_\lambda$. This involution stabilizes at most two chains Q'_λ and Q''_λ of Q_λ . The set of vertices of Q'_λ (possibly empty) is $J'_\lambda := \{(x, x) \mid xV_x \neq 0\} \subset J_\lambda$. The set of vertices of Q''_λ (possibly empty) is $J''_\lambda := \{(x - \frac{m}{2}, x) \mid xV_{x-\frac{m}{2}} \neq 0\} \subset J_\lambda$.

9.7. Admissible systems for the symplectic quiver. Suppose that the following hold:

- (1) For each $(j, x) \in J - (J'_\lambda \sqcup J''_\lambda)$, we have $\dim_x V_j = 1$.
- (2) The chains in Q_λ other than Q'_λ and Q''_λ all consist of a single vertex.
- (3) Let $\sharp J'_\lambda = 2a'$ for some $a' \in \mathbf{Z}_{\geq 0}$. When $a' > 0$, $(-a' + \frac{1}{2}, -a' + \frac{1}{2})$ is the head of J'_λ and $(a' - \frac{1}{2}, a' - \frac{1}{2})$ is the tail. Then $\dim_x V_x = a' + \frac{1}{2} - |x|$ for all $(x, x) \in J'_\lambda$.
- (4) Let $\sharp J''_\lambda = 2a''$ for some $a'' \in \mathbf{Z}_{\geq 0}$. When $a'' > 0$, $(-a'' - \frac{m-1}{2}, -a'' + \frac{1}{2})$ is the head of J''_λ and $(a'' - \frac{m+1}{2}, a'' - \frac{1}{2})$ is the tail. Then $\dim_x V_{x-\frac{m}{2}} = a'' + \frac{1}{2} - |x|$ for all $(x - \frac{m}{2}, x) \in J''_\lambda$.

Under these assumptions, $\mathfrak{m}_1 = \mathfrak{m}'_1 \oplus \mathfrak{m}''_1$, where \mathfrak{m}'_1 is the space of representations of the quiver Q'_λ with dimension vector $\dim_x V_x = a' + \frac{1}{2} - |x|$ and satisfying the duality condition $\psi_i = -\psi_{-i}^*$ (where $\psi_i : {}_{i-\frac{1}{2}}V_{i-\frac{1}{2}} \rightarrow {}_{i+\frac{1}{2}}V_{i+\frac{1}{2}}$) for all $i \in \{-a' + 1, \dots, a' - 1\}$. Similarly, \mathfrak{m}''_1 is the space of representations of the quiver Q''_λ with dimension vector $\dim_x V_{x-\frac{m}{2}} = a'' + \frac{1}{2} - |x|$ and satisfying the duality condition $\psi_i = -\psi_{-i}^*$. The open M_0 -orbit $\mathring{\mathfrak{m}}_1$ consists of those representations of Q'_λ and Q''_λ where each arrow has maximal rank (either injective or surjective).

Let $V' = \bigoplus_x V_x$ and $V'' = \bigoplus_x V_{x-\frac{m}{2}}$. Let $V^\dagger = \bigoplus_{(j,x) \notin J'_\lambda \cup J''_\lambda} (xV_j)$. Then we have $V = V' \oplus V'' \oplus V^\dagger$. This decomposition is preserved by M , and $M \cong Sp(V') \times Sp(V'') \times T^\dagger$, where T^\dagger is the maximal torus in $Sp(V^\dagger)$ stabilizing each line $xV_j \subset V^\dagger$. The center Z_M is isomorphic to $\{\pm 1\} \times \{\pm 1\} \times T^\dagger$ under this decomposition. The stabilizer of a point in $\mathring{\mathfrak{m}}_1$ under M_0 is exactly Z_M . Let C be the rank one local system on $\mathring{\mathfrak{m}}_1$ on whose stalks $\pi_0(Z_M)$ acts nontrivially on both factors of $\{\pm 1\}$. Then C is cuspidal because it is the restriction of the unique cuspidal local system on \mathfrak{m} . Let \tilde{C} be the cuspidal perverse sheaf on \mathfrak{m}_1 defined by C . The system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is admissible. Moreover, any admissible system is of this form. Under G_0 -conjugacy, the only invariant of an admissible system is given by the numbers a' and a'' . Since $\dim V'_j + \dim V''_j \leq \dim V_j$, we have the following inequality for all $j \in \mathfrak{S}_m$:

$$(a) \quad \begin{aligned} \dim V_j &\geq \sharp\{-a' + \frac{1}{2} \leq x \leq a' - \frac{1}{2} \mid x \equiv j \pmod{m\mathbf{Z}}\} \\ &\quad + \sharp\{-a'' + \frac{1}{2} \leq x \leq a'' - \frac{1}{2} \mid x \equiv j + \frac{m}{2} \pmod{m\mathbf{Z}}\}. \end{aligned}$$

To summarize, we have a natural bijection

$$(b) \quad \underline{\mathfrak{X}}_1 \leftrightarrow \{(a', a'') \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0} \text{ satisfying (a) for all } j \in \mathfrak{S}_m\}.$$

The map $\Psi : \mathcal{I}(\mathfrak{g}_1) \rightarrow \underline{\mathfrak{X}}_1$ for the symplectic quiver as well as other graded Lie algebras of classical type will be described in a sequel to this paper using the combinatorics of symbols.

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