The Oceans's Effect on Hurricane Intensity
by
Lars Reinhard Schade

Submitted to the Department of Earth, Atmospheric, and Planetary Sciences in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Meteorology at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY September 1994

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Abstract

We have investigated the effect of the SST reduction in the wake of a hurricane on the intensity of the hurricane using a numerical model of the coupled hurricane-ocean system. The model has two key characteristics: First, it maintains a hurricane vortex in an ambient moist neutral atmosphere and thus derives its energy entirely from a WISHE instability. This is crucial to assure that the model's sensitivity to changes in SST is realistic. Second, the model reaches a steady-state, which is essential for an objective assessment of the magnitude of the feedback effect. Both these features are unique to this coupled model study and fundamentally distinguish it from previous investigations.

A set of 1440 coupled experiments was conducted over the entire range of realistic conditions. Feedback factors as low as −70% were encountered, which demonstrates the paramount effect of the SST feedback on hurricane intensity. Using an objective fitting technique, a single analytic formula for the feedback factor was extracted from the model data. The formula is a function of six parameters: The mixed-layer depth, the hurricane translation speed, the hurricane size, the hurricane intensity at constant SST, the Coriolis parameter, and the stratification in the oceanic thermocline. The relative importance of these six parameters can be understood in physical terms from a simple scaling argument. Since the intensity of a hurricane over an ocean of uniform and fixed SST is known in terms of the boundary layer relative humidity, the SST, and the upper tropospheric temperature, the derived formula for the feedback factor effectively is a formula for the intensity of a hurricane over a mutually interacting ocean in the absence of non-axisymmetric atmospheric processes.

In the second part of this thesis, we derive a new parameterization for the turbulent entrainment at the base of the oceanic mixed-layer. The new formulation implicitly includes a transition layer which merges the mixed-layer with the upper thermocline. The key assumption of the new formulation is that the transition layer is always critical to Kelvin-Helmholtz instability, thus the name critical transition layer (CTL) closure. The new closure is shown to behave in a physically intuitive way and compares well to observations, as simulations of the SST reductions in the wake of hurricane Gloria indicate. It is also efficient to run, and easy to implement in standard mixed-layer models.

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Chapter 1

Introduction

Light winds, blue skies with a few puffy cumulus clouds, an occasional thunderstorm: That is what is commonly associated with tropical weather. Yet this typically tranquil part of the globe harbors the most violent and destructive storms we know, tropical cyclones.

Each year, about 80 tropical depressions develop into tropical storms, defined by sustained surface winds in excess of 17 ms$^{-1}$. About two thirds of these storms intensify and are labeled 'hurricane' when sustained surface winds exceed 33 ms$^{-1}$. The term ‘hurricane’ is used in the Atlantic and East-Pacific region only. The corresponding West-Pacific name is ‘typhoon’ while the generic term ‘tropical cyclone’ is used in Australia and the Indian Ocean region. In this thesis, we shall refer to all mature tropical cyclones as ‘hurricanes’. While some storms never develop much beyond minimum hurricane intensity, others intensify into extreme hurricanes such as Gilbert (1988) in which sustained winds of 88 ms$^{-1}$ and a minimum surface pressure of 888 hPa were recorded. Hurricane sizes also vary over a wide range. Tracy (1974) is an example of a very small hurricane with gale force winds confined to a radius of 50 km. In super typhoon Tip (1979), in contrast, gale force winds extended out to a radius of 1100 km. Similarly, hurricane propagation velocities differ greatly from case to case with typical values of 5–9 ms$^{-1}$.

Even though weather satellites and routine aircraft reconnaissance provide forecasters with ever more detailed real-time observations, intensity change predictions still show very little skill. While surprises like the great hurricane of 1938 no longer occur, the uncertainty of today’s hurricane forecasts necessitates large-scale evacuations of coastal regions prior to a hurricane’s predicted landfall. Such evacuations
have dramatically reduced the threat to human life but they are very costly and damaging to coastal industries, particularly to off-shore oil exploitation and to the fishing industry. In spite of this economic impact of hurricanes, the scientific interest in these impressive cyclones has been fairly limited and many aspects of hurricanes are still not well understood.

The dynamics of hurricanes involve very different scales of motion ranging from the scale of microphysical processes inside cumulus clouds and their anvils to large scale air-sea interaction and radiative processes. Very crudely, today's research on hurricanes can be divided into four parts:

- Hurricane genesis: Why do some tropical meso-scale convective clusters (MCCs) develop into tropical storms while others just dissipate? What are the crucial conditions for tropical cyclogenesis? How does the transition of an MCC into the tropical cyclone's warm-core vortex take place?

- Hurricane intensity: Why do some hurricanes become extremely intense while others remain comparatively weak? Which are the crucial processes or environmental conditions that determine a hurricane's intensity?

- Hurricane steering: What determines a hurricane's speed and direction of propagation?

- Hurricane structure: What determines a hurricane's size and its degree of axisymmetry? How do spiral bands form and what are their basic dynamics? Similarly, how does the hurricane eye form? What are the dynamics of the hurricane outflow in the upper troposphere?

A nice introduction to many of these topics is given in Anthes (1982). The listed questions just illustrate some of the current research focuses. It is obvious that the four main areas of research overlap a lot and that none of the questions can be answered in isolation without addressing many of the others.

In this thesis, we focus on the second part, the problem of hurricane intensity. More specifically, we concentrate on the role of the ocean in determining a hurricane's intensity. We thus investigate the dynamics of the interaction between the hurricane and the underlying ocean. Previous observational studies (e.g. Merrill, 1987; Evans, 1991) have shown, that there is no obvious correlation between hurricane intensity and SST, i.e. that SST alone is a poor predictor of hurricane
intensity. Rather than regarding the ocean as a fixed 'boundary condition' for hurricanes, we therefore consider the feedback of the oceanic response to a hurricane's forcing on the storm's intensity. We thus treat the coupled hurricane-ocean system as one dynamical unit with important implications for the hurricane's intensity. The results of this thesis will hopefully be incorporated into operational intensity forecast models and will, we believe, greatly improve their forecast skills.

1.1 Previous Work

The question of what determines the intensity of hurricanes has been addressed by many researchers. Riehl (1950) was one of the first to look at the energy cycle of a tropical cyclone. He already realized the importance of surface fluxes of sensible and latent heat as energy source for the cyclone and stressed the effect of the reduced surface pressure on the surface moist entropy. He stated that air moving inward toward the storm center moistens and expands at constant temperature. He also mentioned that the storm's secondary circulation could be viewed as a simple heat engine though he did not believe that parcel trajectories were actually closed. He did not consider the effect of longwave radiation as a means of energy loss from the system but rather thought that mixing processes distribute the heat gained from the ocean over a large volume of the atmosphere. Therefore, the system had to be open unlike a heat engine and could not be considered to be in true equilibrium.

Kleinschmidt (1951), in contrast, suggested that a hurricane and its environment form a dynamically closed system in which the input of heat at the sea surface is balanced by the loss of heat due to longwave radiation to space. Parcels then follow closed trajectories which form a heat engine. The kinetic energy production by this heat engine is assumed to be balanced by frictional dissipation in the boundary layer. This conceptional picture still forms the basis of our understanding of steady-state hurricane energetics.

Charney and Eliassen (1964) and Ooyama (1969) viewed the tropical atmosphere as conditionally unstable to moist convection. The energy source for the circulation then is largely contained in the environmental boundary layer and the system can be maintained advectively by convergence in the core region of the storm. In this picture, local surface fluxes close to the storm center play a negligible role compared to the advection of moisture. As demonstrated by Bjerknes
(1938), convective motions in a conditionally unstable atmosphere favor the smallest possible scale. Conditional instability of the second kind (CISK) was therefore invoked to explain the size of tropical cyclones.

Arakawa and Schubert (1974) introduced the quasi-equilibrium assumption. In this assumption the large scale destabilizing processes are balanced by convection on a very short time scale. Even though there may be local parcel instability (CAPE), this instability changes on much longer time scales than the convective adjustment time scale. Adjustment towards a quasi-equilibrium can thus be interpreted as a quasi-neutral state.

Emanuel (1988a) finally formulated an exact theory for the maximum intensity of a hurricane in a convectively neutral environment. The hurricane instability in his theory is based on the feedback effect between surface winds and surface fluxes and was labeled wind induced surface heat exchange (WISHE) instability. Energe-

Figure 1-1: Minimum central pressure of hurricanes viewed as Carnot engines. The ambient boundary layer relative humidity is assumed to be 75%. The hatched area indicates typical conditions in today's tropics. From Emanuel (1988b).
tically the hurricane is viewed as a Carnot engine transporting heat from the warm tropical ocean to the cold upper tropical troposphere and losing the kinetic energy gained from this heat transport to surface friction in the atmospheric boundary layer. Emanuel showed that the maximum hurricane intensity is a function of only a few parameters, namely the sea surface temperature (SST), a mean upper tropospheric temperature at which most of the radiation to space takes place, and the ambient boundary layer relative humidity, $H_a$. Fig. 1-1 shows the maximum hurricane intensity for $H_a = 75\%$ over an extreme range of temperatures. The predicted intensities are rather close to the intensities of the most intense hurricanes observed under corresponding conditions as pointed out by Emanuel. But most hurricanes never even come close to their potential maximum intensity as given by the Carnot theory which indicates that processes not included in the Carnot theory are acting in nature to reduce hurricane intensity. Possibly important processes include the interaction with mean tropospheric shear, interactions with upper tropospheric disturbances such as the tropical upper tropospheric trough (TUTT), internal hurricane dynamics such as eyewall replacement cycles, and the interaction with the ocean. In this thesis, we investigate the last process solely, the effect of the interaction with the ocean on hurricane intensity.

The Oceanic Response to Hurricane Forcing

Hurricanes occur only over the tropical oceans and rapidly decay after they make landfall owing to the paramount importance of surface fluxes as their energy source. From the oceanographer’s point of view, hurricanes constitute extremely fierce and rather localized surface wind forcing, and the oceanic response to hurricanes has been studied both observationally and theoretically by many researchers over the past 25 years.

One of the earliest observational studies is that of Leipper (1967) who conducted ship-based measurements of the upper ocean’s thermal structure before and after the passage of hurricane Hilda (1964) over the Gulf of Mexico. He found SST decreases in the wake of Hilda of up to $6^\circ$C.

O’Brien and Reid (1967) investigated the dynamic response of a two-layer ocean to a stationary hurricane using a numerical model. The absence of any hurricane movement created some unrealistic features in the oceanic response such as strict axisymmetry. Yet some of the basic characteristics of the oceanic response were
reproduced. A strong divergence of the mixed-layer currents was found under the storm center which resulted in strong upwelling there. When mixing between the two layers was included (O'Brien, 1967) the resulting mixed-layer density field after 24 h of integration broadly resembled some of Leipper’s data.

Geisler (1970) derived analytic expressions for the dynamic response of a two-layer ocean to hurricane forcing. The effect of mixing was not included in his linear calculations. He found the dominant feature of the baroclinic response to be a wake of inertio-gravity waves, provided the storm moves much faster than the phase speed of the fastest baroclinic waves. After the dispersion of the wake a geostrophically balanced ridge is left on the interface between the layers. Slower moving storms produce a weaker wake while the amplitude of the ridge is larger. If the phase speed of the fastest baroclinic waves exceeds the hurricane translation speed no wake is produced.

Chang and Anthes (1978) used a numerical model to study the nonlinear baroclinic response of a two-layer ocean to hurricane forcing. Mixing between the layers was parameterized following Kato and Phillips (1969) and resulted in SST decreases of 2–8°C depending on the hurricane translation speed. Both the current response and to a lesser degree the SST response were biased toward the right side of the track because the currents on that side are rotated by inertial forces into the same direction as the imposed stress vector while the opposite is the case on the left side of the track. The inclusion of nonlinear advection did not alter the response significantly, as anticipated by Geisler.

Price (1981, 1983) employed a numerical model similar to that of Chang and Anthes in a detailed analysis of the upper ocean response, its important scales, and the energy budget. Price chose to move the ocean model under the hurricane forcing, such as to view the oceanic response in a reference frame moving with the hurricane. In this reference frame, the oceanic response becomes steady. He used a parameterization of mixing based on the shear across the base of the mixed-layer (Price, 1979). His detailed description and physical explanation of the dynamics of the oceanic response to hurricane forcing are still definitive.

Cooper and Thompson (1989a,b) included bottom topography and the effect of bottom friction in a mixed-layer ocean model similar to that of Price (1981). They simulated the current response to hurricane forcing over the continental shelf and close to coastal boundaries and found reasonable agreement with observations from buoys.
In recent years, the use of research aircraft to deploy airborne expendable bathythermographs (AXBTs) and current profilers (AXCPs) provided in-situ data under extreme forcing conditions (e.g. Sanford et al., 1987, Shay et al., 1992, Price et al., 1994). Typically 30–100 probes are deployed in and ahead of the hurricane under investigation to give snapshots of the surface and upper oceanic temperature and current fields. To realistically assess the changes due to the hurricane’s forcing, a pre-hurricane survey is needed which requires the accurate prediction of the hurricane track. Another challenge is the separation of the steady and the surface wave components in the raw data which results in a typical error of $0.3 ms^{-1}$ in the retrieved mean currents. Though the data confirm the qualitative picture of the results from numerical models, they display a much more complex structure of the oceanic response. A detailed comparison of these data with model simulations is hindered by the fact that nearly all probes have been dropped on the right side of the track so that the important question of the degree of across-track asymmetry cannot be addressed adequately.

**Coupled Numerical Models**

Since hurricanes depend so heavily on the ocean as their source of energy, and because the upper ocean fields are dramatically altered by a hurricane’s wind field, Chang and Anthes (1979) proposed to consider the hurricane and the ocean together as a coupled system. They proposed that there could be a possibly important feedback mechanism due to the effect of the hurricane induced SST cooling on the hurricane’s intensity. Their numerical simulation unfortunately suffered from a number of flaws. Most importantly, the hurricane model employed a Kuo-type parameterization of convection which, as mentioned earlier, features an unrealistic reservoir of energy in the ambient environment and therefore seriously underestimates the importance of the sea surface temperature. The very coarse resolution in the hurricane model ($\Delta r = 60 km$) resulted in very weak cyclones below hurricane intensity. In addition, their model storm was stationary and the coupled integrations were carried out for only $24 h$. As a result, they found only very minor weakening of the model cyclone’s intensity due to the oceanic feedback mechanism. The inclusion of the motion of the storm into their model, Chang and Anthes anticipated, would result in an even weaker oceanic feedback effect. This negative result for the importance of the oceanic feedback mechanism discouraged further
research on the coupled behavior of the hurricane-ocean system for more than a decade.

An exception is a group of researchers at the USSR Hydrometeorological Center in Moscow who performed a number of numerical simulations of the mutual behavior of a hurricane and the tropical ocean (e.g. Sutyrin et al., 1979, Sutyrin and Khain, 1984, Khain and Ginis, 1991). Their work was unfortunately hampered by very limited computer resources which required the use of models with very coarse resolution, which in turn resulted in very weak model hurricanes. In addition, much of this work was published in Russian only and therefore had rather little influence on the western research community.

More recently, Bender et al. (1993) published results from a high-resolution 3-dimensional hurricane model coupled to a mixed-layer ocean model. Entrainment at the base of the oceanic mixed-layer is computed using the Deardorff (1983) scheme. Coupled integrations were carried out to 72 h and it was found that the feedback effect was consistently larger for slower moving storms. This set of experiments is the most complex and complete numerical simulation of the coupled system to date. But the large number of physical processes acting and interacting simultaneously and the large number of parameterizations employed hinder a simple physical interpretation of their results. In addition, the high computational cost of the model allowed a very limited number of experiments only.

A problem common to all the previous coupled simulations lies in their initial conditions. The initial stratification of the atmosphere is often taken from observations and is typically conditionally unstable to the model's representation of convection. This results in an unrealistic extremely rapid initial development of the hurricane vortex and leaves doubts about the adequacy of the energy balance in the final steady-state, if such a state is ever reached.

1.2 Motivation and Goals

The previous section already described the possibly important role of the oceanic feedback mechanism on hurricane intensity, and the need for a systematic investigation of this feedback effect is apparent. In this section, we present the motivation for the present work and its goals.

The tropical atmosphere is in a state of quasi-equilibrium with radiative destabilization in a statistical balance with convective stabilization. Hurricanes can
extract energy from this effectively neutral atmosphere due to the nonlinear feedback between surface winds and surface fluxes of sensible and latent heat. The complete feedback loop which maintains the circulation of a steady-state hurricane qualitatively looks as follows: The hurricane wind and pressure fields produce highly elevated boundary layer moist entropies in a small region close to the storm center which cause massive vertical cumulus mass fluxes there. The resulting inflow toward that region creates, due to the conservation of angular momentum, the very strong primary circulation and the corresponding surface pressure field in near gradient wind balance with the surface wind field. The subsidence compensating for the deep convection in the central region of the storm acts to suppress convection at moderate radii where the boundary layer moist entropy is already somewhat elevated. It thus contributes to the organization of the tropical cyclone and is an important part of the circulation. It is important to keep in mind that the moist entropy anomalies close to the storm center are created at the local SST and therefore are due only to the combined effect of elevated specific humidity and lowered surface pressure.

Energetically, hurricanes can be viewed as Carnot heat engines and their maximum intensity can be expressed in terms of a few parameters describing the ambient environment. Yet observed hurricanes only rarely reach their theoretical maximum intensity. This fact is illustrated in Fig. 1-2 where we have plotted the minimum possible central pressure against the observed central pressure for a large number of Atlantic hurricanes. We used hurricane data from the 'North Atlantic Tropical Cyclones Dataset', which is available e.g. over the Internet from the Colorado State University. It contains the location, the maximum wind speed, and the minimum central pressure every 6 hours for all Atlantic hurricanes since 1886. The data is meant to be best-knowledge data, i.e. all available sources of information were combined into the present data set. Particularly the older records are sparse and have been 'corrected' several times so that their reliability is questionable. We therefore chose to restrict our study to the post-1946 period when reconnaissance flights were conducted on a routine basis. Since Emanuel's (1988a) theory for the maximum intensity of hurricanes assumes a steady-state, we screen the data set for hurricanes with pressure deviations of less than 4 hPa from a 30-hour running mean of the central pressure. 78 hurricanes meet this criterion. For each of these storms, we calculate a theoretical minimum central pressure from climatological data, interpolating in time and space to the date and location of the
Figure 1-2: Comparison of the observed central pressure in Atlantic hurricanes with their theoretical minimum central pressure according to Emanuel's (1988a) theory of hurricanes. Storms from 1946 to 1990 were considered and plotted if their intensity did not change by more than 4 hPa over a 30 h period. Climatological data of SST, tropospheric sounding, and boundary layer relative humidity were used in the calculation of the theoretical maximum intensity.

Observation. SSTs are taken from Robinson et al. (1979), mixed-layer depths from Lamb (1984), and surface relative humidities from Isenmer and Hasse (1985). Temperature data on standard pressure levels from Newell et al. (1972) are used to calculate the outflow temperature, which is defined as the temperature of a pseudo-adiabatically lifted parcel at its level of neutral buoyancy. The entropy-weighted outflow temperature, $\overline{T_{out}}$, was defined by Emanuel (1988a) as

$$\overline{T_{out}} \equiv \frac{1}{s_c - s_a} \int_{s_a}^{s_c} T_{out} ds,$$

where $s_c$ and $s_a$ are the surface moist entropies at the storm center and in the ambient environment, respectively. We determine $\overline{T_{out}}$ graphically on a teplegram.
Errors of .5 °C in the SST, 3 °C in the outflow temperature, 1 % in the boundary layer relative humidity, and 3 hPa in the ambient surface pressure translate into an uncertainty of 11 hPa in the theoretical minimum pressure.

As seen in Fig. 1-2, only the most intense storms on record reach their maximum intensity as marked by the dashed line. Since the SST has been considered constant in the calculation of the maximum intensity, we hypothesize that much of the discrepancy between the observed and the maximum theoretical intensity can be ascribed to the effect of the SST changes in the wake of the storm. Various observations indicate that typical SST decreases are on the order of 1—4 °C. This could be regarded as a small effect with reference to Fig. 1-1. Yet it is important to understand that the hurricane intensity is far less sensitive to changes in the ambient SST than to local changes in SST in the vicinity of the storm center. This point is of paramount importance! An adjusted atmosphere in quasi-equilibrium over water with a uniform SST of 28 °C can support hurricanes of nearly the same intensity as an adjusted atmosphere over water with a uniform SST of 30 °C (see Fig. 1-1). Here the difference in the intensity is primarily due to the nonlinearity of the Clausius-Clapeyron equation for the saturation vapor pressure. In contrast, if the SST is reduced locally close to the storm center the boundary layer moist entropy at the reduced SST may no longer be higher than that in the environment even if the air is completely saturated. In fact, a reduction of the SST by only 2 °C would reverse the thermodynamic disequilibrium between the sea surface and the boundary layer air under typical hurricane conditions. The resulting extremely high sensitivity of the hurricane to changes in SST is the basis for a potentially large SST feedback effect on hurricane intensity. Since nearly all the SST reduction is due to entrainment of colder thermocline water into the warm mixed-layer, the realistic parameterization of this entrainment process is of fundamental importance.

The primary goal of the present study is to systematically investigate the parameter dependence of the feedback amplitude. Our strategy is to use a very simple coupled model which can be run many times and whose physics are rather transparent. Many details of the interaction between the hurricane and the ocean will intentionally not be included in the model. We instead choose to concentrate on what we consider the fundamental processes of the interaction. Chapters 2 and 3 describe the models used and the technique and the results of the parameter space exploration. The second goal of this thesis is to investigate the adequacy of
the most commonly used parameterizations of entrainment and their differences. Chapter 4 contains this part of our work, whose focus became the development of a new parameterization of the turbulent entrainment process.
Chapter 2

Even Better than the Real Thing - Models

This chapter contains a detailed description of the hurricane model and its sensitivity to all its input parameters followed by an introduction to the ocean model and finally an account of the coupling procedure. Results from default experiments are discussed to give the reader some familiarity with the typical behavior of the models and to demonstrate their strengths and limitations.

2.1 Hurricane Model

The hurricane model used in this study is a slightly modified version of the axisymmetric hurricane model of Emanuel (1989). We first lay out the physical framework of the model, then describe its numerical implementation, its typical behavior, and in the last section discuss the model’s sensitivity to all of its input parameters.

2.1.1 Physics

The hurricane model is axisymmetric, i.e. two-dimensional, with radius as the horizontal coordinate and pressure, $p$, as the vertical coordinate. Rather than using the physical radius, $r$, as the radial coordinate we employ Schubert and Hack’s (1983) potential radius, $R$, which is defined as the radius at which a parcel would attain zero tangential velocity if displaced radially while conserving angular
momentum:
\[ \frac{f}{2} R^2 = r v + \frac{f}{2} r^2, \]  
(2.1)

where \( f \) is the Coriolis parameter (assumed constant), \( r \) the physical radius (i.e. the distance from the storm center), and \( v \) the azimuthal wind speed (i.e. the velocity perpendicular to the \( R-p \) plane). This choice of radial coordinate guarantees high radial resolution in regions of high vorticity such as the eyewall and, more importantly, is the natural framework to represent convective motions along angular momentum surfaces. Upright convection in \( R-p \) space corresponds to slantwise convection in physical coordinates, so that the slantwise convection in the eyewall is simply vertical in the model’s coordinate system. In addition, radial advection in \( R-p \) space requires parcels to cross \( R \)-surfaces and thus to change their angular momentum. It is therefore due only to frictional dissipation. Since, according to (2.1), the physical location \( r \) of an \( R \)-surface directly yields the azimuthal velocity at that location the entire primary circulation (i.e. the azimuthal flow field) can be diagnosed from the position of the \( R \)-surfaces in the \( r-p \) plane.

In the vertical, the model atmosphere consists of three layers as shown in Fig. 2-1: A boundary layer of thickness \( \delta p_b \) and two layers which represent the lower and upper troposphere, each \( \delta p \) thick. The model top is a solid lid. We assume gradient wind balance in the horizontal and hydrostatic balance in the vertical direction.

In physical cylindrical coordinates, the hydrostatic form of the mass continuity
equation reads
\[ \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial p} \omega = 0, \] (2.2)
where \( u \) is the horizontal velocity and \( \omega \) is the vertical pressure velocity. We separate the vertical motions into convective updrafts in deep clouds, \( \omega_c \), and large scale vertical motions outside of clouds, \( \omega_a \), and introduce streamfunctions for the motions in the \( R-p \) plane:
\[ ru = \frac{\partial}{\partial p} (\psi - G), \] (2.3)
\[ (1-\sigma) r\omega_a = -\frac{\partial \psi}{\partial r}, \] (2.4)
\[ \sigma r\omega_c = \frac{\partial G}{\partial r}, \] (2.5)
where \( \sigma \) is the fractional coverage of deep updrafts, \( \psi \) is the mass streamfunction in the absence of deep convection, and \( G \) is the mass streamfunction of the deep convection. Radial integration of (2.5) gives
\[ G = \int_0^r \sigma \omega_c r' dr', \] (2.6)
i.e. \( G \) can also be interpreted as the radially integrated cumulus mass flux. Both \( G \) and \( \psi \) vanish at the top and at the bottom of the model domain and at the storm center (i.e. at \( R = 0 \)). Furthermore, the deep cumulus mass flux is assumed to be non-divergent in the lower tropospheric layer as discussed below.

The conserved thermodynamic variable in moist atmospheres is moist entropy, \( s \), which is related to equivalent potential temperature, \( \Theta_e \), by
\[ s = c_p \ln \Theta_e, \] (2.7)
where \( c_p \) is the heat capacity of air at constant pressure. The moist entropy is defined in the model's sub-cloud layer (\( s_B \)) and in the lower tropospheric layer (\( s_L \)). Its value in the upper tropospheric layer is assumed to be identical to \( s_B \).

The model's temperature variable is saturation moist entropy, \( s^* \), which is approximated (neglecting effects of water substance on heat capacities, etc.) by
\[ s^* = c_p \ln T + \frac{L_v r^*}{T} - R_d \ln \frac{p}{p_a}, \] (2.8)
where \( T \) is temperature, \( L_v \) the latent heat of vaporization, \( r^* \) the saturation mixing ratio, \( R_d \) the gas constant of dry air, and \( p_a \) a reference pressure chosen to be the
unperturbed sea level pressure. The saturation entropy is related to saturation equivalent potential temperature, \( \Theta^* \), by

\[
s^* = c_p \ln \Theta^*_s.
\] (2.9)

The model's convection acts to homogenize moist entropy along \( R \)-surfaces, and the background atmosphere is therefore assumed to be slantwise moist neutral. The state of slantwise moist neutrality is characterized by constant \( s^* \) along angular momentum surfaces, with \( s^* \) equal to the actual moist entropy in the boundary layer (again neglecting the effect of variable water content on density). Thus we take \( s^* \) to be uniform along \( R \)-surfaces.

Emanuel (1986) derived a thermal wind relation based on the assumptions of slantwise neutrality and hydrostatic and gradient wind balance. It may be written

\[
\frac{1}{r^2(p)} = \frac{1}{r^2(p_o)} + \frac{2}{f^2 R^3} [T(p_o) - T(p)] \frac{ds^*}{dR}. \tag{2.10}
\]

Hence the two-dimensional temperature field defines the shape of the \( R \)-surfaces given their position at the bottom of the model domain.

A relation expressing angular momentum conservation can be derived from the definition of radial velocity together with (2.3):

\[
\frac{dr}{dt} = u = \frac{1}{r} \frac{\partial}{\partial p} (\psi - G). \tag{2.11}
\]

Using the chain rule, the above can be transformed into potential radius coordinates, resulting in

\[
\frac{\partial r^2}{\partial \tau} + \frac{dR}{d\tau} \frac{\partial r^2}{\partial R} = 2 \frac{\partial}{\partial P} (\psi - G), \tag{2.12}
\]

where the new coordinates are time, \( \tau \), pressure, \( P \), and potential Radius, \( R \). Partial derivatives with respect to \( \tau \) and \( P \) hold \( R \) rather than \( r \) fixed.

We next specialize (2.12) to the three model layers. In the upper layer, we assume no friction, \( i.e. \frac{dR}{d\tau} = 0 \), because there is no solid boundary there and thus no Ekman pumping. Integration over the layer depth gives

\[
\frac{\partial r_u^2}{\partial \tau} = 2 (\psi_m - G) \tag{2.13}
\]

with

\[
r_u^2 = \int_{p_t}^{p_m} r^2 dp, \tag{2.14}
\]
where the indexes ‘t’ and ‘m’ refer to the top and bottom of the upper tropospheric layer, respectively.

In the lower tropospheric layer, we include radial diffusion of momentum. From the definition of potential energy (2.1), we have

\[
f R \frac{dR}{d\tau} = \frac{dM}{d\tau} = r D_v,
\]

(2.15)

where \( M \) is the absolute angular momentum and \( D_v \) is the frictional drag. We use an eddy viscosity formulation for \( D_v \) as in Rotunno and Emanuel (1987). This gives

\[
D_v = \frac{1}{r_o^2} \frac{\partial}{\partial r_o} \left[ r_o^3 \nu \frac{\partial}{\partial r_o} \left( \frac{v_o}{r_o} \right) \right]
= \frac{f}{2} \frac{\partial}{\partial r_o} \left[ r_o^3 \nu \frac{\partial}{\partial r_o} \left( \frac{R^2}{r_o^2} \right) \right],
\]

(2.16)

with the eddy viscosity, \( \nu \), related to the local deformation by

\[
\nu = l^2 \left| \frac{\partial v_o}{\partial r_o} - \frac{v_o}{r_o} \right|
= \frac{f}{2} l^2 r_o \left| \frac{\partial}{\partial r_o} \left( \frac{R^2}{r_o^2} \right) \right|,
\]

(2.17)

where \( l \) is the mixing length scale and the index ‘o’ refers to the top of the sub-cloud layer. We now integrate over the depth of the lower tropospheric layer assuming \( D_v \) to be constant and get

\[
\frac{\partial r_L^2}{\partial \tau} = 2 (\psi_o - \psi_m) - \delta p_L D_v
\]

(2.18)

with

\[
r_L^2 = \int_{p_m}^{p_o} r^2 dp.
\]

(2.19)

In the sub-cloud layer, we assume that the frictional term in (2.12) is large compared to the local time derivative. We therefore approximate (2.12) by a balance between inward advection and frictional destruction of angular momentum. The drag term, \( D_v \), in (2.15) is approximated by the vertical divergence of the vertical flux of angular momentum:

\[
D_v = g \frac{\partial F_v}{\partial P},
\]

(2.20)
where $g$ is the acceleration of gravity, $F_v$ is the azimuthal stress, and we assume that \( \frac{\partial}{\partial \bar{p}} \approx \frac{\partial}{\partial p} \), i.e. that there is little baroclinicity in the boundary layer. Substituting (2.20) into (2.12) and neglecting the time derivative gives

\[
\frac{\partial}{\partial \bar{p}} (\psi - G) = \frac{g}{2fR} \frac{\partial r^2}{\partial \bar{R}} \frac{\partial F_v}{\partial \bar{p}}. \tag{2.21}
\]

We again make the assumption that $R$-surfaces are nearly vertical in the boundary layer, so that \( \frac{\partial \bar{R}}{\partial \bar{p}} \approx 0 \). Then (2.21) may be integrated through the depth of the boundary layer to give

\[
\psi_v = G - \frac{g}{2fR} \frac{\partial r^2}{\partial \bar{R}} F_v, \tag{2.22}
\]

where $F_v$ is the azimuthal surface stress and we have assumed that the turbulent stress vanishes at the top of the boundary layer.

The surface stress is calculated using the aerodynamic drag law,

\[
F_v = -\rho_{\text{air}} c_D \left| v_\theta \right| v_\theta, \tag{2.23}
\]

with

\[
c_D = c_{D_b} + c_{D_l} \left| v_\theta \right|, \tag{2.24}
\]

where $\rho_{\text{air}}$ is a mean surface density of air and $c_{D_b}$ and $c_{D_l}$ are constants, the empirically determined drag coefficients. We use $c_{D_b} = 1 \times 10^{-3}$ and $c_{D_l} = 54 \times 10^{-6} \text{m}^{-1} \text{s}$ (Powell, 1980). The wind velocity used in the aerodynamic drag law represents the flow in the surface layer and is usually taken as the observed wind at the anemometer level $10 \text{ m}$ above the ground. Here ‘observed’ usually refers to a $1 \text{ min}$ average, i.e. a ‘gust-free’ average. Due to the nonlinear nature of the aerodynamic drag law, very gusty conditions will result in an under-prediction of the surface drag when ‘gust-free’ winds are used. Since our hurricane model neither resolves the surface layer spatially nor does it resolve gusts temporally, it is not clear which wind should be used in the drag law. The model only predicts the wind speed at the top of the boundary layer (typically $500 - 1000 \text{ m}$ above the ground) and we decided to follow Powell (1980) in calculating the wind at the $10 \text{ m}$ level as

\[
v_\theta = .8 \times v_b. \tag{2.25}
\]

In the following, ‘azimuthal wind’ always refers to the wind used in the aerodynamic drag law, i.e. to $10 \text{ m}$ winds deduced from the model predicted flow at the top of the sub-cloud layer using (2.25).
The saturation moist entropy, \( s^* \), is the model's central temperature variable and is directly linked to the circulation through the thermal wind relation, (2.10). We do not separate this variable into in-cloud and out-of-cloud parts, but instead use the approximation that the temperature in clouds is approximately the same as that of their environment.

The local time rate of change of the areally averaged dry entropy in the model interior is given by the sum of the vertical advection of \( s_d \) inside and outside of clouds and diabatic processes such as convective heating and radiative cooling:

\[
\frac{\partial s_d}{\partial \tau} = -\sigma \omega_c \left. \frac{\partial s_d}{\partial P} \right|_c - (1-\sigma) \omega_a \left. \frac{\partial s_d}{\partial P} \right|_a + \dot{H},
\]

where \( \left. \frac{\partial s_d}{\partial P} \right|_c \) and \( \left. \frac{\partial s_d}{\partial P} \right|_a \) are the lapse rates of dry entropy inside and outside of clouds, respectively, and \( \dot{H} \) represents the diabatic processes. Following Arakawa and Schubert (1974), we assume that most of the actual local temperature change is due to subsidence outside of clouds (compensating for convective scale updrafts inside clouds), and that heating in clouds is very nearly balanced by adiabatic cooling:

\[
\sigma \omega_c \left. \frac{\partial s_d}{\partial P} \right|_c \approx \dot{H} - \dot{H}_{\text{rad}},
\]

where \( \dot{H}_{\text{rad}} \) is the radiative cooling. As the temperature is only defined at one level in the model, we must take the static stability, \( \left. \frac{\partial s_d}{\partial P} \right|_a \), to be a constant in (2.26). Since the temperature lapse rate has already been assumed to be moist adiabatic on \( R \)-surfaces, the static stability itself will be in practice a weak function of temperature. Finally, we relate changes in saturation moist entropy, \( s^* \), at constant pressure to changes in dry entropy, \( s_d \), using an elementary relation of moist thermodynamics derived in Emanuel (1986):

\[
\delta s^*_p = \frac{\Gamma_{\text{dry}}}{\Gamma_{\text{moist}}} \delta s_d|_p,
\]

where \( \Gamma_{\text{dry}} \) and \( \Gamma_{\text{moist}} \) are the dry and moist lapse rates, respectively. The radiative cooling is parameterized as a Newtonian cooling back to the initial, unperturbed conditions:

\[
\frac{\Gamma_{\text{dry}}}{\Gamma_{\text{moist}}} \dot{H}_{\text{rad}} = -\text{rad}(s^* - s^*_i),
\]

where the index 'i' refers to the initial conditions. Then (2.26) may be rewritten

\[
\frac{\partial s^*}{\partial \tau} = -\frac{\Gamma_{\text{dry}}}{\Gamma_{\text{moist}}} \left. \frac{\partial s_d}{\partial P} \right|_m (1-\sigma) \omega_a \left. \frac{\partial s_d}{\partial P} \right|_m - \text{rad}(s^* - s^*_i).
\]
To crudely mimic the effect of clouds on the radiative cooling due to longwave radiation, the Newtonian cooling is turned off in regions where there is upward motion out of the sub-cloud layer:

\[
rad = 0 \quad \text{if} \quad \frac{\partial}{\partial R}(\psi_b - G) > 0.
\]  

(2.31)

As mentioned earlier, the conserved thermodynamic variable in moist atmospheres is the moist entropy, which is the sum of the entropies of dry air, water vapor, and condensed water (an exact definition valid in both saturated and unsaturated air is given in Emanuel, 1988a). Total entropy can be gained from the sea surface and lost by radiation to space, the two principal energetic processes in tropical cyclones. Frictional dissipation can also raise the entropy, but this is usually a small effect in the atmosphere and is therefore ignored here.

In the boundary layer, the local time rate of change of moist entropy is equal to the sum of radial advection, vertical advection, and surface fluxes of moist entropy:

\[
\frac{\partial s_B}{\partial \tau} = -\frac{dR}{d\tau} \frac{\partial s_B}{\partial R} - (1 - \sigma) \frac{\omega_{b0} + |\omega_{b0}|}{2} \frac{s_B - s_L}{\delta p_B} \\
- \sigma_s \frac{\omega_{sc} + |\omega_{sc}|}{2} \frac{s_B - s_L}{\delta p_B} + g \rho_{air} c_p |v_n| \frac{s^*_s - s_B}{\delta p_B},
\]  

(2.32)

where the index \( \omega_{sc} \) refers to shallow convection (see below for discussion). The second and the third term on the right-hand side of (2.32) are the fluxes of entropy through the top of the sub-cloud layer outside and inside of clouds, respectively; they are non-zero only when the motion is downward (hence deep convective motions do not appear in this equation). \( s^*_s \) is the saturation moist entropy at SST and surface pressure, and the aerodynamic flux formula [with \( c_p \) given by (2.24)] has been used in the surface flux term, the last term on the right-hand side of (2.32). \( s^*_s \) can be expressed as a difference from the ambient unperturbed sub-cloud layer moist entropy, \( s_{B_a} \):

\[
s^*_s = s_{B_a} + c_p \ln \left( \frac{T_s}{T_{s_a}} \right) - R_d \ln \left( \frac{p_a}{p_{B_a}} \right) \\
+ \frac{L_v v^* (T_{s_a}, p_a)}{T_{s_a}} \left[ \frac{T_{s_a}}{T_s} \frac{p_a - e^*(T_{s_a})}{p_s - e^*(T_s)} \frac{e^*(T_s)}{e^*(T_{s_a})} - H_a \right],
\]  

(2.33)

where the indices \( \ast \) and \( \ast_i \) refer to surface and initial (unperturbed) values, respectively, \( e^* \) is the saturation vapor pressure, and \( H_a \) the relative humidity in the ambient (unperturbed) environment.
In the lower tropospheric layer, the local time rate of change is given by the sum of radiative cooling and advective fluxes through the top and bottom of the layer:

\[
\frac{\partial s_L}{\partial t} = -\frac{\Gamma_{\text{maiss}}}{\Gamma_{\text{dry}}} \left[ r_{\text{rad}}(s_L - s_{L,a}) - (1 - \sigma) \frac{\omega_{\text{bm}} - |\omega_{\text{bm}}|}{2} \frac{s_L - s_B}{\delta p} \right. \\
\left. - (1 - \sigma) \frac{\omega_{\text{bo}} - |\omega_{\text{bo}}|}{2} \frac{s_B - s_L}{\delta p} - \sigma_{\text{mc}} \frac{\omega_{\text{mc}} - |\omega_{\text{mc}}|}{2} \frac{s_B - s_L}{\delta p} \right].
\]

(2.34)

In the formulation of this model, cumulus convection represents sources and sinks of mass [the $G$ terms in (2.13) and (2.22)] and of entropy [the $\omega_{\text{c}}$ terms in (2.32) and (2.34)] to the explicitly predicted environment of cumulus clouds. Momentum transport is implicitly accounted for since the convection is assumed to redistribute heat and mass along angular momentum ($R$) surfaces. Each model cloud is, in effect, a representation of the effect of a cloud integrated over its life time. The wide spectrum of convective motions in the tropical atmosphere is reduced to two idealized forms of convection, deep and shallow convection:

- Deep convection, described by $G$, acts to transport mass from the sub-cloud layer to the upper tropospheric layer. As mentioned earlier, the deep convective mass flux is assumed to be non-divergent in the lower tropospheric layer and the compensating large scale subsidence is described by the out-of-cloud streamfunction, $\psi$, so that the deep convective vertical velocity, $\omega_c$, is strictly upward. This is an idealization of the net effect of a cloud with 100% precipitation efficiency.

- Shallow convection, in contrast, has zero precipitation efficiency, i.e. all the condensate re-evaporates and there is no net heating and therefore no vertically integrated net mass flux: Shallow clouds stabilize the atmosphere by mixing entropy between the sub-cloud layer and the lower tropospheric layer. Thus, in the model, the effects of shallow clouds and of precipitating downdrafts are identical.

The convective mass flux is calculated by integrating the parcel equation for slantwise ascent, based on the buoyancy of sub-cloud layer air lifted upward along an $R$-surface, to obtain a cloud vertical velocity; this is multiplied by an assumed fractional area to get the mass flux. In the case of deep convection, we take the depth of the model atmosphere ($\delta z \approx 16 \text{ km}$) to be a typical horizontal length scale
of a cloud. We also make the assumption that there can be at most one deep cloud per grid volume, so that the fractional coverage of deep clouds becomes

$$
\sigma = \begin{cases} 
\frac{\delta z}{\delta z_c} & \text{if } \frac{\delta z}{\delta z_c} \leq 1 \\
1 & \text{otherwise.}
\end{cases} 
$$

(2.35)

For shallow clouds, updrafts and downdrafts cover the same area and we choose

$$
\sigma_\infty = \frac{1}{2}.
$$

(2.36)

The geometric slantwise velocity in convective updrafts, $U$, is given by

$$
\frac{dU}{dt} = B \frac{\partial z}{\partial l} |_R,
$$

(2.37)

where $B$ is the local parcel buoyancy and $\partial z$ and $\partial l$ are incremental distances in the vertical and along an $R$-surface, respectively. Transformation to $R$ coordinates gives

$$
\frac{\partial U}{\partial \tau} = -\frac{dR}{dt} \frac{\partial U}{\partial R} - \omega \frac{\partial U}{\partial P} + B \frac{\partial z}{\partial l} |_R,
$$

(2.38)

where $\omega$ is the upright (perpendicular to pressure surfaces) pressure velocity. The approximation $\omega \approx U \frac{\partial P}{\partial l}$ yields

$$
\frac{\partial \omega_c}{\partial \tau} = -\frac{dR}{dt} \frac{\partial \omega_c}{\partial R} - \frac{1}{2} \frac{\partial \omega_c^2}{\partial P} - B g \rho_{air} \left( \frac{\partial z}{\partial l} \right)^2 |_R,
$$

(2.39)

with

$$
B = g \frac{\Gamma_{\text{moist}}}{\Gamma_{\text{dry}}} \left| \frac{s_B - s^*}{c_p} \right|.
$$

(2.40)

The final equation for the deep convective mass flux is obtained by multiplying (2.39) by the fractional coverage of deep clouds, $\sigma$, and by then vertically integrating from the sea surface to the top of the lower tropospheric layer:

$$
\frac{\partial (\sigma \omega_c)}{\partial \tau} = -\frac{dR}{dt} \frac{\partial \sigma \omega_c}{\partial R} + \frac{(\sigma \omega_c)^2}{2 \sigma \delta p} - \sigma B g \rho_{air} \left( \frac{\partial z}{\partial l} \right)^2 |_R,
$$

(2.41)

where we have assumed $\sigma \omega_c$ to be nearly constant in the lower troposphere. The reason for using (2.41) instead of simply setting $(\sigma \omega_c)^2$ equal to twice the buoyancy
term is merely that it results in a smoother integration of the model. Simple
trigonometry yields for the slope term in (2.41)

\[
\left( \frac{\partial z}{\partial l} \right)_R = \left[ 1 + \left( \frac{\partial r}{\partial z} \right)_R \right]^{-\frac{1}{2}}.
\]  

(2.42)

From the thermal wind equation (Emanuel, 1986), we have

\[
\frac{\partial r}{\partial z} \bigg|_R = -r^3 \frac{\Gamma_{\text{moist}}}{2} \beta,
\]  

(2.43)

where \( \beta \) is a measure of the local baroclinicity and defined by

\[
\beta = -\frac{2}{f^2 R^3} \frac{ds^*}{dR}.
\]  

(2.44)

Rather than using a prognostic equation similar to (2.41) for the mass flux
of the shallow convection, we drop the time derivative in (2.41) to arrive at the
following diagnostic equation for the shallow convective mass flux:

\[
\alpha_c \omega_c = \alpha_c \left( 2 \delta p \ B_{sc} \ g \rho_{\text{air}} \right)^{\frac{1}{2}} \frac{\partial z}{\partial l} \bigg|_R,
\]  

(2.45)

with

\[
B_{sc} = g \frac{\Gamma_{\text{moist}}}{\Gamma_{\text{dry}}} \left| \frac{s_B - s^*}{c_p} \right.
\]  

(2.46)

From the gradient wind law,

\[
\frac{1}{\rho_{\text{air}}} \frac{\partial p}{\partial r} = f v + \frac{v^2}{r},
\]  

(2.47)

we derive a diagnostic equation for the surface pressure, \( P_s \), by transformation
of (2.47) into \( R \) coordinates and making use of (2.1):

\[
\frac{1}{\rho_{\text{air}}} \frac{\partial P_s}{\partial R} = \frac{f^2 R^3}{2r_o^2} - \frac{f^2}{8} \frac{\partial}{\partial R} \left( \frac{R^4}{r_o^2} + r_o^2 \right).
\]  

(2.48)

Finally, diagnostic equations for \( r_o \) and \( u_o \) are needed. Substitution of (2.10) into
(2.19) yields

\[
r_L^2 = -\frac{k_L}{\beta} \ln(1 - r_o^2 \beta \delta T_L),
\]  

(2.49)
where $\kappa_L = \frac{\delta \rho}{\delta T_L}$ is an inverse mean lower tropospheric lapse rate parameter with $\delta T_L$ being the temperature difference between the bottom and the top of the lower tropospheric layer. (2.49) can be rewritten as a diagnostic equation for $r_o^2$:

$$r_o^2 = \frac{1-e^{-\frac{\beta}{\kappa_L} \frac{R^2}{r_o^2}}}{\beta \delta T_L}, \quad (2.50)$$

and (2.1) can be rewritten as a diagnostic equation for $v_o$:

$$v_o = \frac{f}{2} \frac{R^2 - r_o^2}{r_o}. \quad (2.51)$$

Table 2.1 summarizes all the variables and constants used in the hurricane model.

### 2.1.2 Numerics

The model has five prognostic variables, $s_p, s_L, s^*, \omega_0$, and $r_L^2$, which are stepped forward in time at each time step according to (2.32), (2.34), (2.30), (2.41), and (2.18), respectively. We use a leap-frog scheme for the derivative in time with a filter to avoid time splitting.

$$
\begin{align*}
\phi^{[3]} &= \phi^{[1]} + 2 \delta t \left[ \frac{\partial \phi}{\partial t} \right]^{[2]} \\
\phi^{[1]} &= \phi^{[2]} + C_{\text{smooth}} (\phi^{[1]} - 2 \phi^{[2]} + \phi^{[3]}) \\
\phi^{[2]} &= \phi^{[3]},
\end{align*}
$$

where $\phi$ represents any one of the prognostic variables, the square-bracketed superscripts refer to time levels, and $C_{\text{smooth}}$ is a constant. All the other model variables are diagnosed at each time step from the predicted fields. The core piece of the numerical solution strategy is a Sawyer-Eliassen equation for the out-of-cloud streamfunction in the middle troposphere, $\psi_m$. It is formed by first substituting the thermal wind relation (2.10) into (2.19) and (2.14), taking the time derivative of these two equations, and then using (2.18), (2.13), and (2.30) to form an elliptic equation for $\psi_m$. It is solved iteratively with a simple over-relaxation technique. Physically, the solution to the Sawyer-Eliassen equations is that $\psi_m$ field which, when substituted into the prognostic equations for $r_L, r_o$, and $s^*$, keeps the model in thermal wind balance (2.10).
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Prognostic Variable</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_L^2$</td>
<td>pseudo volume inside $R$-surface below mid-level</td>
<td>(2.18)</td>
</tr>
<tr>
<td>$s_B$</td>
<td>moist entropy in the BL</td>
<td>(2.32)</td>
</tr>
<tr>
<td>$s_L$</td>
<td>moist entropy in the lower tropospheric layer</td>
<td>(2.34)</td>
</tr>
<tr>
<td>$s^*$</td>
<td>saturation moist entropy</td>
<td>(2.30)</td>
</tr>
<tr>
<td>$\omega_c$</td>
<td>deep convective vertical velocity</td>
<td>(2.41)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Diagnostic Variable</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>radially integrated deep cumulus mass flux</td>
<td>(2.6)</td>
</tr>
<tr>
<td>$P_s$</td>
<td>surface pressure</td>
<td>(2.48)</td>
</tr>
<tr>
<td>$r_0$</td>
<td>physical radius of $R$-surface at the top of the BL</td>
<td>(2.50)</td>
</tr>
<tr>
<td>$T_s$</td>
<td>sea surface temperature</td>
<td>external</td>
</tr>
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<td>$v_o$</td>
<td>azimuthal velocity in the BL</td>
<td>(2.51)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>baroclinicity</td>
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<tr>
<td>$\omega_{bm}$</td>
<td>ambient vertical velocity at the mid-level</td>
<td>(2.4)</td>
</tr>
<tr>
<td>$\omega_{bt}$</td>
<td>ambient vertical velocity at the top of the BL</td>
<td>(2.4)</td>
</tr>
<tr>
<td>$\omega_c$</td>
<td>shallow convective vertical velocity</td>
<td>(2.45)</td>
</tr>
<tr>
<td>$\psi_b$</td>
<td>ambient streamfunction at the top of the BL</td>
<td>(2.22)</td>
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<tr>
<td>$\psi_m$</td>
<td>ambient streamfunction at the mid-level</td>
<td>(A.11)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>fractional coverage of deep convection</td>
<td>(2.35)</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>fractional coverage of shallow convection</td>
<td>(2.36)</td>
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<table>
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<th>Symbol</th>
<th>Constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_p$</td>
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<tr>
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<td>constant drag coefficient</td>
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<tr>
<td>$c_{D1}$</td>
<td>linear drag coefficient</td>
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<tr>
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<tr>
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<td>reference density of air</td>
<td>$1.225 , \text{kg} , \text{m}^{-3}$</td>
</tr>
</tbody>
</table>

Table 2.1: Variables and constants used in the hurricane model.
A schematic diagram of the integration strategy of the hurricane model is given in Fig. 2-2.

The diffusion and radial advection terms require two boundary conditions each on \( r_o \) and \( s_B \), the Sawyer-Eliassen equation requires two boundary conditions on \( \psi_m \), and the surface pressure equation requires one boundary condition on \( P_s' \). At \( R = 0 \), symmetry yields that

\[
\frac{\partial s_B}{\partial R} = \frac{\partial \omega_c}{\partial R} = \psi_m = 0 \quad \text{at} \quad R = 0. \tag{2.53}
\]

At the outer boundary, we impose a solid wall for the out-of-cloud mass flux and require that the radial gradients of \( s_B \) and \( \omega_c \), the surface pressure perturbation, \( P_s' \), and the vorticity vanish:

\[
\frac{\partial s_B}{\partial R} = \frac{\partial \omega_c}{\partial R} = \frac{\partial}{\partial R} (R^2 - r_o^2) = P_s' = 0 \quad \text{at} \quad R = R_{outer}. \tag{2.54}
\]

The diffusion term is lagged one time step to ensure numerical stability and a staggered grid is used to improve the accuracy of radial derivatives. Fig. 2-3 is a sketch of the node arrangement. We use a conservative form of the advection terms:

\[
- \frac{dR}{dr} \frac{\partial \phi}{\partial R} = \frac{1}{r_o} \left[ \overline{\left( \psi - G \frac{\partial \phi}{\partial r_o} \right)} \right], \tag{2.55}
\]

where \( \phi \) stands for \( s_B \) and \( \omega_c \), and the overbar denotes an average between two adjacent nodes at which \( r_o \) is defined.

The hurricane model is initialized with a weak cyclonic vortex in the lower troposphere and no flow in the upper troposphere. The initial cyclone has the form

\[
v_o = \begin{cases} 
  \frac{r_o v_{max}}{r_{max}}, & \text{for } r_o \leq r_{max} \\
  \frac{r_{max} v_{max}}{r_o} \left( \frac{r_{outer}^2 - r_o^2}{r_{outer}^2 - r_{max}^2} \right), & \text{for } r_{max} < r_o < r_{outer} \\
  0, & \text{for } r_o \geq r_{outer},
\end{cases} \tag{2.56}
\]

where \( v_{max} \) and \( r_{max} \) are the maximum wind speed and the radius of maximum wind, respectively, and \( r_{outer} \) is the radius at which \( v_o = 0 \). This represents solid body rotation inside \( r_{max} \), a Rankine vortex just outside \( r_{max} \), and a linear decay to zero at \( r_{outer} \). The thermal wind equation (2.10) is then used to initialize \( s^* \). Spatially homogeneous initial values for \( s_B \) and \( s_L \) are model input parameters and the vertical velocities are initially zero.
diagnose the integrated cumulus mass flux, $G$, from (2.6)

diagnose the ambient stream function at the top of the BL, $\psi_0$, from (2.22) and (2.23)

diagnose the diffusion term in (2.18), $D_\theta$, from (2.16) and (2.17)

turn radiation on/off according to (2.31)

invert (A.11) for the ambient stream function at the mid-level, $\psi_m$

diagnose the surface pressure, $P_2$, from (2.48)

diagnose the fractional deep cloud coverage, $\sigma$, from (2.35)

diagnose the vertical velocities, $\omega_\theta$, $\omega_{km}$, and $\omega_\kappa$, from (2.4) and (2.5)

diagnose the advection term in (2.32) from (2.55)

diagnose the slope of the $R$-surfaces from (2.42) and (2.43)

predict the deep convective mass flux, $\sigma \omega_\kappa$, from (2.41) and (2.40)

diagnose the shallow convective mass flux, $\sigma_0 \omega_\kappa$, from (2.45) and (2.46)

predict the pseudo volume, $r^2$, from (2.18)

predict the moist entropy in the lower tropospheric layer, $s_L$, from (2.34)

predict the moist entropy in the BL from (2.32) and (2.33)

predict the saturation moist entropy, $s^*$, from (2.30)

advance prognostic variables with time smoother according to (2.52)

diagnose the baroclinicity parameter, $\beta$, from (2.44)

diagnose the location of the $R$-surfaces at the top of the BL, $r_0$, from (2.50)

diagnose the azimuthal wind speed at the top of the BL from (2.51)

Figure 2-2: Schematic of a time step in the hurricane model
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Initial Conditions</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{\text{max}}$</td>
<td>initial maximum azimuthal wind</td>
<td>$17 , \text{ms}^{-1}$</td>
</tr>
<tr>
<td>$r_{\text{max}}$</td>
<td>initial radius of maximum wind</td>
<td>$100 , \text{km}$</td>
</tr>
<tr>
<td>$r_{\text{outer}}$</td>
<td>outer edge of initial vortex</td>
<td>$500 , \text{km}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Ambient Conditions</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_a$</td>
<td>unperturbed sea surface temperature</td>
<td>$29 , ^\circ\text{C}$</td>
</tr>
<tr>
<td>$\mathcal{H}_a$</td>
<td>initial BL relative humidity</td>
<td>$84%$</td>
</tr>
<tr>
<td>$s_{La}$</td>
<td>ambient moist entropy in lower trop. layer</td>
<td>$s_{Ba} - \frac{J}{K \cdot \text{kg}}$</td>
</tr>
<tr>
<td>$f$</td>
<td>Coriolis parameter</td>
<td>$5 \times 10^{-5}, s^{-1}$</td>
</tr>
<tr>
<td>$p_a$</td>
<td>ambient surface pressure</td>
<td>$1013 , h\text{Pa}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Physical Parameters</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{\text{rad}}$</td>
<td>radiative relaxation parameter</td>
<td>$9 \times 10^5 , \text{s}^{-1}$</td>
</tr>
<tr>
<td>$l^2$</td>
<td>mixing length</td>
<td>$6 \times 10^{-4}\delta R^2$</td>
</tr>
<tr>
<td>$\Gamma_{\text{moist}}^{\text{dry}}</td>
<td>_{\infty}$</td>
<td>lapse rate ratio at the top of the BL</td>
</tr>
<tr>
<td>$\Gamma_{\text{moist}}^{\text{dry}}</td>
<td>_{\infty}$</td>
<td>lapse rate ratio at the mid-level</td>
</tr>
<tr>
<td>$\frac{\partial X}{\partial \Psi}$</td>
<td>dry lapse rate ratio at the mid-level</td>
<td>$-0.003 \frac{J}{K \cdot \text{kg} \cdot \text{Pa}}$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Numerical Parameters / Model Geometry</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
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<td>$\delta_H$</td>
<td>time step</td>
<td>$60 , \text{s}$</td>
</tr>
<tr>
<td>$C_{\text{smooth}}$</td>
<td>time smoothing parameter</td>
<td>$.1$</td>
</tr>
<tr>
<td>$\delta_{\psi}$</td>
<td>tolerance of $\psi_m$ inversion</td>
<td>$10^6 , \text{m}^2 \text{Pa s}^{-1}$</td>
</tr>
<tr>
<td>$N_R$</td>
<td>number of R-surfaces</td>
<td>$48$</td>
</tr>
<tr>
<td>$R_{\text{outer}}$</td>
<td>radius of computational domain</td>
<td>$2000 , \text{km}$</td>
</tr>
<tr>
<td>$\delta z$</td>
<td>height of the model top</td>
<td>$16 , \text{km}$</td>
</tr>
<tr>
<td>$\delta T_L$</td>
<td>temperature depth of the lower trop. layer</td>
<td>$22 , ^\circ\text{C}$</td>
</tr>
<tr>
<td>$\delta T_U$</td>
<td>temperature depth of the upper trop. layer</td>
<td>$72 , ^\circ\text{C}$</td>
</tr>
<tr>
<td>$\delta p_B$</td>
<td>pressure depth of the BL</td>
<td>$100 , \text{hPa}$</td>
</tr>
<tr>
<td>$\delta p$</td>
<td>pressure depth of the tropospheric layers</td>
<td>$375 , \text{hPa}$</td>
</tr>
</tbody>
</table>

Table 2.2: Default input parameters of the hurricane model.
Figure 2-3: Sketch of the numerical grid of the hurricane model

2.1.3 Default Run

In this section, we describe the behavior of the hurricane model for the set of default parameters listed in Table 2.2. The basic features of this behavior are characteristic for a wide range of input parameters. Fig. 2-4 displays a time series of the minimum central pressure and of the maximum azimuthal wind for the default run: The start-up vortex is spun down initially by surface friction while surface fluxes of sensible and latent heat increase the moist entropy in the boundary layer. After about half a day, the system starts to intensify as a result of deep convection near the center of the model. The storm now rapidly intensifies and roughly 36 h later the maximum azimuthal wind exceeds hurricane force. After about five days of integration the maximum azimuthal wind reaches a steady-state while the central pressure keeps falling for another 5 days. During this period, the only change in the vortex structure is a slow spin-up of the outer regions of the vortex without any further spin-up in the vortex core. Therefore, the central pressure still drops while the pressure gradient in the inner region of the vortex no longer increases; this is reflected in the steadiness on the maximum azimuthal wind. In this near-steady state, heating due to surface fluxes is balanced by radiative cooling and convection merely connects the two by redistributing heat vertically. The model-integrated angular momentum, in contrast, never reaches a steady-state since surface friction continuously extracts cyclonic angular momentum from the atmosphere and there is no source of cyclonic angular momentum to balance this sink. Therefore the model-integrated angular momentum becomes more and more negative (an-
Figure 2-4: Time series of the minimum central pressure (top panel) and of the maximum azimuthal wind (bottom panel) in the default run with fixed SST.

tycyclonic) as time advances. This is realized by the fact that the \( R \)-surfaces are advected out to very large radii in the upper troposphere, corresponding to very high anticyclonic azimuthal velocities. At the leading edge of the upper level outflow, a front forms after a few days of integration. These unrealistic features of the present model are a result of the axisymmetry which forbids non-axisymmetric ventilation effects for the upper level negative potential vorticity. But since we do not include vertical diffusion of momentum, the upper level anticyclone affects the lower tropospheric dynamics only through the slope term in (2.41) and (2.45) which is evaluated at the top of the sub-cloud layer. As the \( R \)-surfaces are very steep throughout most of the troposphere and flare out to large radii only at the very top of the model domain, the non-steadiness of the upper level anticyclone has hardly any effect on the lower tropospheric fields as is evident from the steadiness of these fields.

Figure 2-6 shows some of the model fields after 24 days of integration when a steady-state has been established in the lower troposphere. All panels are radial.
Figure 2-5: Time series of the radius of the computational node which carries the maximum azimuthal wind, and of the radius of gale force winds in the default run with fixed SST.

cross sections with the storm center at the left edge and radius plotted along the abscissa in hundreds of kilometers. The top left panel displays the surface pressure in hPa. The dashed line marks the ambient surface pressure. We can see that the surface pressure disturbance reaches out to about 800 km and that most of the pressure drop occurs within 100 km of the storm center. The minimum central pressure is 938 hPa. The corresponding azimuthal wind which by model design is in gradient wind balance with the surface pressure field is shown in the center left panel. Winds rapidly increase outward from the storm center and reach a maximum of 59 ms$^{-1}$ (115 knots) at a radius of 34 km outside of which they fall off initially rather steeply and further out more gradually. Tropical storm strength winds extend out to a radius of roughly 150 km. These numbers are rather realistic for an intense hurricane considering the simplicity of the model. In a close comparison to observations, the model storm's eye appears rather large and the azimuthal wind falls off rather rapidly outside the radius of maximum wind. The radial mass flow is depicted in the top right panel. The solid line represents the integral from the surface to the mid-level while the dashed line represents the integral from the surface to the top of the sub-cloud layer. Since the two lines nearly coincide virtually all the inflow occurs in the sub-cloud layer. The radial mass flows are converted into boundary layer mean radial winds and plotted in ms$^{-1}$. The radial inflow has a maximum of 45 ms$^{-1}$ at the radius of maximum azimuthal winds and sharply falls off towards smaller and larger radii. The mean vertical velocities are displayed in the center right panel. The solid line refers to ten times
Figure 2-6: Instantaneous hurricane fields after 24 days of integration.
the vertical velocity outside of clouds at the model's mid-level and the dashed line refers to the vertical velocity in deep convective updrafts. These vertical velocities are calculated by 'spreading' the corresponding mass fluxes uniformly over the grid area. Deep convection sharply peaks close to the radius of maximum azimuthal winds while the large scale subsidence shows a broad distribution and decays only very gradually outward from the storm. In the final steady-state, the parcel buoyancy in the eyewall region corresponds to a temperature anomaly of 1°C while much higher values of typically 0.5°C are simulated during the rapid intensification. The bottom left panel, finally, displays radial profiles of the three thermodynamic variables in the hurricane model. They are expressed as differences from the ambient (unperturbed) sub-cloud layer moist entropy, $s_{b,a}$, which is identical to the initial (unperturbed) saturation moist entropy at the model's mid-level, $s^*$, by virtue of the assumption of moist neutrality in the initial conditions. The dotted line (virtually identical with the solid line) represents the saturation moist entropy, $s^*$, which is seen to increase very gradually inward from a radius of 800 km and begins to rapidly rise only within a radius of 100 km from the storm center. The sub-cloud layer moist entropy, $s_b$, is shown as the solid line and is hardly distinguishable from $s^*$ which means that the model convection successfully keeps the model atmosphere very close to moist neutrality. The moist entropy in the lower tropospheric layer, $s_L$, is fairly close to its initial value throughout most of the model domain with the exception of the innermost 100 km where reduced radiative cooling and dramatically increased convective heating and moistening cause $s_L$ to approach $s_b$.

Observed hurricanes cover a wide range of sizes and intensities and can vary significantly in their radial structure. A nice overview of the observations is given in Anthes (1982). The present hurricane model reproduces the main characteristics of observed hurricanes, and the model results lie quantitatively within the range of the observations. But the model fails to produce the large variety of structures and sizes of observed hurricanes because many processes, in particular all non-axisymmetric processes, are not included in the model. Within these limitations, the hurricane model's behavior is fairly realistic which suggests that the most important dynamical processes of a hurricane are adequately included in the present model and that the constraint of axisymmetry and the imposed balance assumptions are reasonable at least to first order.

The inclusion of radiative cooling is essential in the development of a steady-
state because it is needed to balance the surface heat fluxes. The present formulation (2.31) turns radiation off in regions of upward vertical motion out of the boundary layer. Since these regions tend to be the regions with the strongest surface fluxes (which is why there is convergence in the boundary layer) the present formulation of radiative cooling forms a positive feedback mechanism for convection. This is not entirely unrealistic as shown by Craig (1994) for the case of polar lows. To demonstrate that, in our model, the development of the hurricane is primarily due to surface fluxes, i.e. a wind-surface flux feedback, we re-ran the default experiment with radiative cooling turned off everywhere. Fig. 2-7 shows time series of the central pressure and the maximum azimuthal wind speed for the default experiment (solid line) and for the experiment without radiative cooling (dotted line). The initial three days of development are very similar in the two experiments. Owing to the absence of radiative cooling in the subsidence region outside the deep convective core of the system, the outer regions of the storm warm up thus diminishing the radial temperature gradients. Therefore, the azimuthal

Figure 2-7: See Fig. 2-4. The solid line corresponds to the default experiment; the dotted line is from a run without radiative cooling and identical otherwise.
winds decay with time until deep convection breaks out at the outer edge of the model domain. At this point, the whole model basically breaks down. What appears like a weakened yet stable vortex in Fig. 2-7 turns out, upon inspection of the radial wind and temperature distribution, to be an absolutely unrealistic vortex which fills the entire model domain. The first five days of integration are still valuable in that they show the minor role played by the radiative feedback for the initial intensification. The importance of the radiative cooling for the maintenance of a steady-state hurricane is also apparent.

2.1.4 Sensitivity to the Input Parameters

To investigate the hurricane model's sensitivity to all of its input parameters, the hurricane model was run ‘solo’ with fixed SSTs so that all possible oceanic feedback effects were excluded.

For the sensitivity studies, each input parameter was modified independently from the others over the range of observed values, unless noted otherwise. The effects both on the intensity and the size (as measured by the radius of maximum winds) of the steady-state hurricane were investigated. Below, we summarize the results of the sensitivity study for each of the input parameters and emphasize the highest sensitivities.

**Initial Conditions:** The maximum azimuthal wind of the initial vortex, $v_{max}$, does not affect the final steady-state fields within the limits of accuracy of the model. But weaker initial vortices take much longer to heat the boundary layer through surface fluxes and thus prolong the initial spin-down phase. After deep convection breaks out, the spin-up phase proceeds nearly identically independent of the initial maximum wind. Since we are not interested in the dynamics of the spin-down phase, we initialize the model with a rather strong vortex ($v_{max} = 17 \, ms^{-1}$) to quickly reach a steady-state.

The geometry of the initial vortex as described by $r_{max}$ and $r_{outer}$ also has hardly any influence on the final steady-state. Over the range of tested initial geometries, the final storm intensity varies by only about 1 hPa. Similarly, the size of the final hurricane vortex is practically independent of the geometry of the initial vortex. The fact that the final steady-state vortex is independent of the initial vortex used as a trigger for development is desirable in general. Yet it poses a problem later on in Chapter 3 where we investigate the effect of the size of a
hurricane on the behavior of the coupled hurricane-ocean system.

**Ambient Conditions:** The unperturbed sea surface temperature, $T_{sa}$, influences the final hurricane intensity as predicted by the Carnot theory of hurricanes (Emanuel, 1988a). In each experiment, the model atmosphere is initially neutral to convection. The left panel of Fig. 2-8 displays the model sensitivity to the SST which is of order $-5 \frac{hPa}{K}$. The SST is set to 29°C in the default experiment.

![Graph showing sensitivity of steady-state central pressure to SST and relative humidity](image)

Figure 2-8: Sensitivity of the steady-state central pressure to the SST (left panel) and the ambient relative humidity (right panel).

Similarly, the ambient relative humidity in the boundary layer, $H_a$, has a strong influence on the final storm intensity as predicted by the Carnot theory. The model sensitivity to $H_a$ is of order $4 \frac{hPa}{\%}$ as displayed in the right panel of Fig. 2-8. This high sensitivity is a problem if one attempts to accurately predict hurricane intensity since $H_a$ is known at best to within a few percent. We choose a default value of 80%.

The moist entropy in the lower tropospheric layer in the ambient environment, $s_{la}$, represents the observed tropospheric minimum in $\Theta_e$ at around 700 hPa. The value of $s_{la}$ determines how efficiently shallow convection can cool the sub-cloud layer. The dependence of the storm intensity on $s_{la}$ is in the noise level of the model.

The hurricane model's $f$-plane is centered on the latitude $\varphi_0$. A weak depen-
dence of the steady-state intensity on $\varphi_o$ was found with larger $\varphi_o$ resulting in stronger storms ($\approx S_{\text{latitude}}^{k_{\text{rad}}}$). The default latitude is $24^\circ$N.

**Physical Parameters:** The radiative relaxation rate, $rad$, has fairly little effect on the final steady-state intensity. If $rad$ is chosen to be too small, deep convection develops inside the eye and the storm intensity fluctuates more as a result. If the radiative relaxation rate is chosen to be unrealistically large, the hurricane vortex shrinks in size and becomes squashed in the radial direction. We choose a default value of $rad = 9 \times 10^{-5} \text{ s}^{-1}$.

The square of the mixing length, $l^2$, in the eddy viscosity formulation of the radial diffusion of momentum in the sub-cloud layer affects the final storm intensity only very little over a wide range of values. Larger mixing lengths result in slightly more rapid intensification but the final steady-state is identical. The default value is $l^2 = 6 \times 10^{-4} \times \delta R^2$ where $\delta R$ is the radial resolution in $R$ space.

The ratio of the moist to the dry adiabatic lapse rate at the top of the sub-cloud layer, $\Gamma_{\text{moist}}^{\text{dry}} \bigg|_o$, and at the model's mid-level, $\Gamma_{\text{moist}}^{\text{dry}} \bigg|_m$, are known to within 5% and variations over this range do not affect the results notably. The default values are $\Gamma_{\text{moist}}^{\text{dry}} \bigg|_o = .4$ and $\Gamma_{\text{moist}}^{\text{dry}} \bigg|_m = .5$.

The partial derivative of the dry entropy with respect to pressure at the model's mid-level, $\frac{\partial S}{\partial P} \bigg|_m$, determines the rate of heating a given subsidence mass flux will cause. Values in the range of $-1 \times 10^{-3}$ to $-4 \times 10^{-3} \ J K^{-1} \text{kg}^{-1} \text{Pa}^{-1}$ result in steady-state intensities differing by less than $2 \ h\text{Pa}$. The default value is $\frac{\partial S}{\partial P} \bigg|_m = -3 \times 10^{-3} \ J K^{-1} \text{kg}^{-1} \text{Pa}^{-1}$.

The bulk aerodynamic formula is used to calculate the fluxes of heat and momentum at the ocean surface. The total drag coefficient in this formula is the sum of a constant, $c_{\text{db}}$, and a linear function of the surface wind speed with a slope given by $c_{\text{dl}}$. Both constants are known rather vaguely only and the sensitivity of the model hurricane to these two parameters was tested for $0.8 \times 10^{-3} < c_{\text{db}} < 1.3 \times 10^{-3}$ and $44 \times 10^{-6} \text{m}^{-1} \text{s} < c_{\text{dl}} < 64 \times 10^{-6} \text{m}^{-1} \text{s}$. Over this range of values the steady-state central pressures varied by $1 \ h\text{Pa}$ only as shown in Fig 2-9. Following Powell (1980), we choose default values of $c_{\text{db}} = 1 \times 10^{-3}$ and $c_{\text{dl}} = 54 \times 10^{-6} \text{m}^{-1} \text{s}$. So far we have not distinguished between the drag coefficients for heat, $c_{\text{de}}$, and for momentum, $c_{\text{du}}$. It turns out that the model is rather sensitive to the ratio of $c_{\text{de}}$ to $c_{\text{du}}$. Fig. 2-10 displays the dependence of the final steady-state intensity (left panel) and of the size of the vortex as measured by the radius of gale force winds
Figure 2-9: Contour plot of the steady-state central pressure as a function of the drag coefficients. The contour interval is .25 hPa.

(right panel) on the ratio of the drag coefficients. Higher values of \( \frac{c_{D_a}}{c_{D_u}} \) lead to larger and more intense storms. There is some hint of a saturation effect in the left panel of Fig. 2-10 for ratios larger than 1.2. The step-like increase in the vortex size in the right panel of Fig. 2-10 is an artifact of the radial discretization and only the overall trend is meaningful.

**Numerical Parameters / Model Geometry:** The time step in the hurricane model, \( \delta t \), does not affect the results provided it is small enough to guarantee numerical stability. The default value is \( \delta t = 60 \text{s} \).

The value of the factor used for the time smoothing of the leap-frog scheme (2.52), \( C_{\text{smooth}} \), does not affect the solutions notably as long as it is large enough to prevent time splitting in the leap-frog scheme. The default value is \( C_{\text{smooth}} = 0.1 \).

The elliptic equation for the streamfunction at the model’s mid-level (A.11) is solved using an iterative relaxation method. The value of \( \delta \psi \) which determines when \( \psi_m \) is considered to have converged, sensitively affects the run time of the integration but has very little effect on the steady-state intensity and size of the hurricane as long as \( \delta \psi \) is small enough. The default value is \( \delta \psi = 10^6 \text{ m}^2 \text{ Pa s}^{-1} \).

The number of radial nodes, \( N_R \), determines the radial resolution for a given domain size. The steady-state central pressure approaches a minimum of about 931 hPa with increasing resolution as shown in Fig. 2-11. Since the integration time increases rapidly with increasing resolution, economic considerations become important for the choice of \( N_R \). We choose a default value of \( N_R = 48 \) which gives
a central pressure of roughly 6 hPa above the asymptotic limit.

The domain size, $R_{\text{outer}}$, does not affect the integration provided it is large enough to contain the entire secondary circulation. A value of $R_{\text{outer}} = 2000 \text{ km}$ was found to be a reasonable choice and is used as the default value. The boundary effects visible in Fig. 2-6 vanish if the domain size is doubled. Since the hurricane itself is unaffected by the boundary effects we choose not to use the larger domain size in the interest of computational speed.

The height of the model domain, $\delta z$, is chosen to be $\delta z = 16 \text{ km}$. Values in the range from 10 to 22 km result in changes of the steady-state hurricane intensity of less than 2 hPa.

The temperature depths of the lower and the upper tropospheric layer, $\delta T_L$ and $\delta T_U$, were found to affect the steady-state intensity only as their sum. Fig. 2-12 displays this dependence of the steady-state central pressure on $\delta T_L + \delta T_U$. It is due to the dependence of the efficiency of a Carnot engine on the temperature difference between the two heat reservoirs it operates on. For realistic values of the SST and the hurricane outflow temperature the sum of $\delta T_L$ and $\delta T_U$ has to be around $95 \pm 5^\circ \text{C}$ which corresponds to a range of $\pm 7 \text{ hPa}$ in hurricane intensity. This uncertainty cannot be reduced easily because the outflow temperature is not
known precisely. It depends not only on the stratification of the environment in which the hurricane is embedded but also on the SST under the eye of the hurricane and on assumptions about the entrainment in convective updrafts. We choose default values of $\delta T_L = 23^\circ$C and $\delta T_U = 72^\circ$C.

The sensitivity to the pressure depths of the sub-cloud layer, $\delta p_B$, the lower tropospheric layer, $\delta p_L$, and the upper tropospheric layer, $\delta p_U$, was tested together. The total pressure depth of the model was held constant at 850 hPa so that two of the layer depths determined the third. No significant dependence on the ratio of the lower to the upper layer depth was found. The default values were set to $\delta p_B = 100$ hPa and $\delta p_L = \delta p_U = \delta p = 375$ hPa.

In summary, the model displays the same sensitivities the Carnot theory predicts: The key parameters for the steady-state hurricane intensity are the sea surface temperature, the outflow temperature contained in the model’s total temperature depth, and the boundary layer relative humidity. The size of the steady-state
Figure 2-12: Dependence of the steady-state central pressure on the total temperature depth of the model, $\delta T_L + \delta T_U$.

hurricane vortex was found to be rather insensitive to all of the input parameters including the initial conditions.

## 2.2 Ocean Model

The mixed-layer ocean model of Cooper and Thompson (1989) was used in this study. In this section, we describe the model’s physical framework and its numerical implementation. It has been tested and ‘verified’ by the original authors, so we did not perform comprehensive sensitivity experiments.

In the coupled experiments, the ocean model is moved under the hurricane model at a fixed speed to simulate storm motion. The ocean model’s SSTs are averaged along circles around the storm center and used as lower boundary condition in the hurricane model while the hurricane winds are interpolated to the ocean model’s nodes and used as upper boundary condition in the ocean model. A detailed description of the procedure by which the axisymmetric hurricane model
is coupled to the 3-D ocean model is given in Section 2.3.

### 2.2.1 Physics

The vertical water column is broken into four active layers as depicted in Fig. 2-13. For layer \( i \), the momentum equation is written

\[
\frac{\partial \mathbf{v}_i}{\partial t} = -f \mathbf{k} \times \mathbf{v}_i - (\nabla \cdot \mathbf{v}_i + \mathbf{v}_i \cdot \nabla)\mathbf{v}_i - h_i \mathbf{P} + \frac{\tau}{\rho_{\text{H}_{\text{O}}}} + \mathbf{W}_i, \tag{2.57}
\]

where

\[
\mathbf{v}_i = v_i h_i \tag{2.58}
\]

is the horizontal flow vector, \( v_i \) the horizontal velocity, \( h_i \) the layer depth, \( f \) the Coriolis parameter, \( \mathbf{k} \) a unit vector pointing upward, \( \mathbf{P} \) the pressure gradient force, \( \tau \) the surface wind stress, \( \rho_{\text{H}_{\text{O}}} \) a reference density of water, and \( \mathbf{W}_i \) describes the effect of the entrainment of fluid from one layer into another. The index \( i \) always refers to layer \( i \) and bold symbols indicate vector quantities.

Using the hydrostatic approximation, the horizontal pressure gradient force (vertically integrated over the layer depth) is defined as

\[
\mathbf{P} = g \left[ \sum_{j=1}^{i'} \nabla h_j - \nabla H - \sum_{j=1}^{i-1} \left( \frac{\rho_i - \rho_j}{\rho_{\text{H}_{\text{O}}}} \nabla h_j - h_j \frac{\nabla \rho_j}{\rho_{\text{H}_{\text{O}}}} \right) + \frac{h_j}{2} \frac{\nabla \rho_i}{\rho_{\text{H}_{\text{O}}}} \right] + \frac{\nabla P_s}{\rho_{\text{H}_{\text{O}}}}, \tag{2.59}
\]

where \( i' \) is the number of active layers at a given location, \( H \) is the total still water depth, and \( P_s \) is the atmospheric pressure at the sea surface. The first two terms on the right-hand side are the barotropic, i.e. depth-independent, contribution, the third term is the baroclinic component, the next two terms are the horizontal internal pressure gradients, and the last term is due to the atmospheric surface pressure gradient.

For most applications, a reduced gravity version of the pressure gradient force term is sufficiently accurate. Here we assume that the lowest model layer is at rest and that the sea surface is a solid lid which yields

\[
\mathbf{P} = g \left[ \frac{\rho_i - \rho_j}{\rho_{\text{H}_{\text{O}}}} \sum_{j=1}^{i-1} \nabla h_j + \sum_{j=1}^{i'-1} \left( \frac{\rho_i - \rho_j}{\rho_{\text{H}_{\text{O}}}} \nabla h_j - h_j \frac{\nabla \rho_j}{\rho_{\text{H}_{\text{O}}}} \right) + \frac{h_j}{2} \frac{\nabla \rho_i}{\rho_{\text{H}_{\text{O}}}} \right]. \tag{2.60}
\]

We have also neglected the atmospheric pressure gradients in (2.60) since they form a depth-independent forcing and therefore have a negligible effect on the SST.
Figure 2-13: Vertical structure of the ocean model.

The entrainment term in (2.57) describes the effect on the momentum budget of turbulent entrainment of water from the upper thermocline into the mixed-layer:

\[
\begin{align*}
W_{v_1} &= v_2 w_e \\
W_{v_2} &= -v_2 w_e \\
W_{v_i} &= 0 \quad \text{for } i > 2,
\end{align*}
\]  

(2.61)

where \( w_e \) is the so-called entrainment velocity, the velocity at which the layer interface would move under the isolated effect of entrainment. We use a critical Richardson number parameterization for \( w_e \), which was originally suggested by Pollard et al. (1973) and modified by Price (1983):

\[
\begin{align*}
w_e = \begin{cases} 
5 \times 10^{-4} R^{-4} \Delta v & \text{if } 0 \leq R \leq 1 \\
0 & \text{otherwise,}
\end{cases}
\end{align*}
\]  

(2.62)

where \( R \) is the bulk Richardson number based on the depth of the mixed-layer, \( h_1 \), and the density and velocity jump, \( \delta \rho \) and \( \delta v \), across the base of the mixed-layer:

\[
R = g \frac{\delta \rho}{\rho_{h_0}} \frac{h_1}{(\delta v)^2}.
\]  

(2.63)

For a discussion of the problem of parameterization of turbulent entrainment, see Chapter 4. Mass continuity yields for the layer depths:

\[
\frac{\partial h_i}{\partial t} = -\nabla \cdot \mathbf{V}_i + W_{h_i},
\]  

(2.64)
where the entrainment is

\[
\begin{align*}
W_{h1} &= w_e, \\
W_{h2} &= -w_e, \\
W_{hi} &= 0 \text{ for } i > 2.
\end{align*}
\] (2.65)

In the context of this model, temperature, \( T_i \), and salinity, \( S_i \), can change due to advection and, in the case of layers 1 and 2, from entrainment:

\[
\frac{\partial T_i}{\partial t} = v_i \cdot \nabla T_i + W_{Ti},
\] (2.66)

with

\[
W_{Ti} = \begin{cases} 
\frac{w_e}{h_i} (T_i' - T_i) & \text{if } i = 1 \\
\frac{w_e}{h_2} (T_2' - T_2) & \text{if } i = 2 \\
0 & \text{otherwise},
\end{cases}
\] (2.67)

and

\[
\frac{\partial S_i}{\partial t} = v_i \cdot \nabla S_i + W_{Si},
\] (2.68)

with

\[
W_{Si} = \begin{cases} 
\frac{w_e}{h_i} (S_i' - S_i) & \text{if } i = 1 \\
\frac{w_e}{h_2} (S_2' - S_2) & \text{if } i = 2 \\
0 & \text{otherwise},
\end{cases}
\] (2.69)

where \( T_2' \) and \( S_2' \) are the temperature and salinity at the top of layer 2, respectively. \( T_2' \) and \( S_2' \) are diagnosed from the layer-mean values and the values at the bottom of layer 2 by the assumption of linear profiles within each layer. The temperature and salinity at the bottom of layer 2 can only be changed advectively and are therefore constant for horizontally homogeneous initial conditions. A linear form of the equation of state for sea water is used to calculate the density field from the temperature and salinity fields:

\[
\rho_i = \rho_{H_2O} [1 + C_T (T_i - 27^\circ C) + C_S (S_i - 36)],
\] (2.70)

with \( C_T = -3.3 \times 10^{-4} \, ^\circ C^{-1}, C_S = 7.6 \times 10^{-4} \), and \( \rho_{H_2O} = 1023.5 \, kg \, m^{-3} \). In all the experiments presented in this thesis we have neglected perturbation salinities so that density anomalies are equivalent to temperature anomalies.
The surface wind stress, finally, is calculated using the aerodynamic drag law:

\[
\tau_i = \begin{cases} 
\rho_{air} \left( c_{d0} + c_{d1} |V_{\infty}| \right) |V_{\infty}| V_{10} & \text{if } i = 1 \\
0 & \text{otherwise},
\end{cases}
\]  
(2.71)

where \( V_{\infty} \) is the vector wind speed at the anemometer level 10\( m \) above the ground. It is calculated from the hurricane winds and the hurricane translation speed:

\[
V_{10} = v_{\phi} + u_{BL} + .5 u_T,
\]  
(2.72)

where \( v_{\phi} \) is the azimuthal velocity at anemometer level, \( u_{BL} \) is the mean radial velocity in the atmospheric boundary layer, and \( u_T \) is the hurricane translation velocity, which is specified. The factor .5 in (2.72) was introduced to account for the decay of the hurricane steering flow from the steering level to the surface. The model results are insensitive to the exact choice of this factor.

Table 2.3 summarizes all the model variables and Table 2.4 contains all the model input parameters along with their default values.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Prognostic Variable</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_i )</td>
<td>layer depth</td>
<td>(2.64)</td>
</tr>
<tr>
<td>( S_i )</td>
<td>salinity</td>
<td>(2.68)</td>
</tr>
<tr>
<td>( T_i )</td>
<td>temperature</td>
<td>(2.66)</td>
</tr>
<tr>
<td>( V_i )</td>
<td>horizontal flow</td>
<td>(2.57)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Diagnostic Variable</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_s )</td>
<td>atmospheric pressure at sea level</td>
<td>external</td>
</tr>
<tr>
<td>( v_i )</td>
<td>horizontal velocity</td>
<td>(2.58)</td>
</tr>
<tr>
<td>( u_b )</td>
<td>wind speed at anemometer level</td>
<td>external</td>
</tr>
<tr>
<td>( w_e )</td>
<td>entrainment velocity</td>
<td>(2.62)</td>
</tr>
<tr>
<td>( R )</td>
<td>pressure gradient force</td>
<td>(2.59), (2.60)</td>
</tr>
<tr>
<td>( \rho_i )</td>
<td>density</td>
<td>(2.70)</td>
</tr>
<tr>
<td>( \tau )</td>
<td>surface wind stress</td>
<td>(2.71)</td>
</tr>
</tbody>
</table>

Table 2.3: Variables in the ocean model.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>Coriolis parameter ($24^\circ$N)</td>
<td>$5 \times 10^{-5} s^{-1}$</td>
</tr>
<tr>
<td>$h_{1_{initial}}$</td>
<td>initial depth of layer 1</td>
<td>40 m</td>
</tr>
<tr>
<td>$h_{2_{initial}}$</td>
<td>initial depth of layer 2</td>
<td>180 m</td>
</tr>
<tr>
<td>$h_{3_{initial}}$</td>
<td>initial depth of layer 3</td>
<td>780 m</td>
</tr>
<tr>
<td>$L_p \times M_p$</td>
<td>number of nodes on grid</td>
<td>$45 \times 45$</td>
</tr>
<tr>
<td>$S_{1_{top}}$</td>
<td>initial salinity at top of layer 1</td>
<td>36.95</td>
</tr>
<tr>
<td>$S_{1_{bot}}$</td>
<td>initial salinity at bottom of layer 1</td>
<td>36.95</td>
</tr>
<tr>
<td>$S_{2_{top}}$</td>
<td>initial salinity at top of layer 2</td>
<td>36.95</td>
</tr>
<tr>
<td>$S_{2_{bot}}$</td>
<td>initial salinity at bottom of layer 2</td>
<td>36.95</td>
</tr>
<tr>
<td>$S_{3_{top}}$</td>
<td>initial salinity at top of layer 3</td>
<td>36.95</td>
</tr>
<tr>
<td>$S_{3_{bot}}$</td>
<td>initial salinity at bottom of layer 3</td>
<td>36.95</td>
</tr>
<tr>
<td>$T_{1_{top}}$</td>
<td>initial temperature at top of layer 1</td>
<td>29.0°C</td>
</tr>
<tr>
<td>$T_{1_{bot}}$</td>
<td>initial temperature at bottom of layer 1</td>
<td>29.0°C</td>
</tr>
<tr>
<td>$T_{2_{top}}$</td>
<td>initial temperature at top of layer 2</td>
<td>28.9°C</td>
</tr>
<tr>
<td>$T_{2_{bot}}$</td>
<td>initial temperature at bottom of layer 2</td>
<td>13.5°C</td>
</tr>
<tr>
<td>$T_{3_{top}}$</td>
<td>initial temperature at top of layer 3</td>
<td>13.5°C</td>
</tr>
<tr>
<td>$T_{3_{bot}}$</td>
<td>initial temperature at bottom of layer 3</td>
<td>13.5°C</td>
</tr>
<tr>
<td>$\delta t$</td>
<td>time step</td>
<td>600 s</td>
</tr>
<tr>
<td>$\delta x$</td>
<td>horizontal grid spacing</td>
<td>20 km</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_d$</td>
<td>constant drag coefficient</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>$c_{d_l}$</td>
<td>linear drag coefficient</td>
<td>$5.4 \times 10^{-5} sm^{-1}$</td>
</tr>
<tr>
<td>$C_s$</td>
<td>haline expansion coefficient</td>
<td>$7.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>$C_T$</td>
<td>thermal expansion coefficient</td>
<td>$-3.3 \times 10^{-4} ^{\circ}C^{-1}$</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration of gravity</td>
<td>$9.81 m s^{-2}$</td>
</tr>
<tr>
<td>$\rho_{air}$</td>
<td>reference air density</td>
<td>1.225 kg m$^{-3}$</td>
</tr>
<tr>
<td>$\rho_{H_2O}$</td>
<td>reference water density</td>
<td>1023.5 kg m$^{-3}$</td>
</tr>
</tbody>
</table>

Table 2.4: Input parameters and constants in the ocean model
2.2.2 Numerics

The prognostic equations (2.57), (2.64), (2.66), and (2.68) are numerically integrated using a forward-backward scheme on a C-grid (Messinger and Arakawa, 1976). The node arrangement is sketched in Fig. 2-14.

An Adams-Bashforth scheme for the advection terms in the momentum equation (2.57) avoids the weak instability which can develop in the forward-backward scheme.

The boundaries are either solid walls and fully reflective or equipped with sponge layers of variable thickness which absorb incoming waves. Either type of boundary condition can be chosen independently for the four model boundaries.

The ocean is initially at rest and an initial statically stable temperature and salinity stratification is specified. The model does not contain a representation of convection and just issues a warning message should the stratification become convectively unstable in the course of an experiment. This never happens since we do not include cooling due to surface fluxes, which would tend to destabilize a column of water, and since entrainment by itself can never cause convective instability.
2.3 Coupled Model

The coupled model is constructed from the two separate models by three coupling algorithms:

- The location (and movement) of the hurricane model relative to the ocean model needs to be prescribed as an external parameter due to the axisymmetry of the hurricane model. Rather than moving the hurricane model over the ocean we decided to move the ocean model under the hurricane. We thus view the ocean from a frame of reference moving with the hurricane in which the oceanic fields can reach a steady-state. We restrict our investigation to hurricane translation speeds higher than the phase speed of the fastest internal gravity waves (typically $1-3 \text{ms}^{-1}$) in the ocean such that no information travels into the unperturbed ocean ahead of the hurricane. We then can simply specify unperturbed ocean initial conditions at the forward (upstream) edge of the ocean which, in the reference frame co-moving with the hurricane, becomes the inflow edge of the ocean domain. The lateral boundaries of the ocean domain are equipped with sponge layers to avoid the reflection of waves propagating outward from the storm. At the backward (downstream) edge of the ocean model, which becomes the outflow edge in our frame of reference, we use the data at the most recently ‘expelled’ nodes as boundary values. Fig. 2-15 shows the coupled model geometry.

- The wind of the hurricane model is interpolated to the nodes of the ocean model and then used in (2.71) and (2.72) to calculate the momentum input into the ocean. In experiments with the full-gravity version of the ocean model, the hurricane model’s surface pressure is interpolated to the ocean model.

- The sea surface temperature at the ocean model’s nodes is interpolated onto circles around the hurricane’s eye and then averaged along these circles to give the mean SST at the intercept of the hurricane model’s $R$-surfaces with the sea surface. The hurricane thus ‘feels’ only azimuthally averaged SSTs which is a reasonable assumption in the inner regions of the storm where parcels may circle the eye before they leave the boundary layer. At larger radii, the averaging aliases the SST reduction in the wake of the storm uniformly around the storm. The introduced error is very small because, at large radii,
the narrow region of SST reduction covers only a small fraction of the circle along which the averaging is performed.

The two models interact at each time step as described above. These above three forms of interaction constitute the entire interaction between the two models, i.e. we neglect the effects of surface heat fluxes on the mixed-layer temperature and salinity fields. Therefore, in the context of this model, SST changes are entirely due to entrainment of colder thermocline water into the oceanic mixed-layer and to the much smaller effect of advection. This approximation is based on the results of model studies which include all the different forcing terms in the mixed-layer temperature budget and find the entrainment heat flux at the base of the mixed-layer to be by far the dominant term. As an example, we show results from the

\[ \text{Figure 2-15: Geometry of the coupled model. The dashed circles mark the intercept of the hurricane model's $R$-surfaces with the ocean surface. The rectangular box is the ocean model domain; the stippled regions at its sides represent the sponge regions. The general flow through the ocean model is indicated by the heavy black arrows. The hurricane track is shown as the horizontal black line.} \]

55
Figure 2-16: Comparison of the surface heat flux (left panel) and the entrainment heat flux (right panel) in a high-resolution three-dimensional coupled hurricane ocean model. From Bender et al. (1993).

modeling work of Bender et al. (1993) in Fig. 2-16. Note that the contour interval in the right panel is fifty times that in the left panel.

As mentioned in Section 2.1.4, the size of the hurricane vortex cannot be chosen freely but is selected internally by the hurricane model. Since we want to investigate, among other things, what the effect of the size of a hurricane on the coupled steady-state intensity is, we need to 'feed' wind fields from hurricanes of different sizes to the ocean model. We therefore chose to introduce a scale factor, $\gamma$, which is used to scale the physical dimensions of the hurricane and the ocean model during the exchange of information between the two models. The SST at radius $r$ is passed to the hurricane at radius $\frac{r}{\gamma}$ while the hurricane winds at radius $r$ are passed to the ocean model at radius $\gamma r$. The ocean models therefore 'feels' the winds of a hurricane of smaller size than the actual model storm if $\gamma$ is less than 1.

2.3.1 Default Run

The coupled default experiment is identical to the default hurricane-only experiment described in Section 2.1.3 except for the allowance of changes in SST. The
hurricane is moving along a straight path east to west at a speed of 6 m s\(^{-1}\) over an ocean domain extending 450 km to the north and to the south of the track. A size factor of \(\gamma = .8\) was chosen. All the other input parameters are listed in Tables 2.2 and 2.4.

The time series of the maximum azimuthal wind in the coupled default experiment is shown in Fig. 2-17 (solid line) along with the hurricane-only default experiment with fixed SST (dotted line). The general evolution is the same for

![Figure 2-17: Time series of the central pressure (top) and of the maximum azimuthal wind (bottom). The solid line marks the coupled default experiment; the dotted line is from an identical experiment with fixed SST.](image)

the two model storms but the storm with ocean interaction reaches a significantly smaller steady-state intensity. The intensity difference is entirely due to the reduction of the SST as a result of the storm's surface wind stress. We label this effect 'SST feedback' and define an objective measure of this feedback, the SST feedback factor \(F_{\text{SST}}\), as

\[
F_{\text{SST}} = \frac{(p_h - p)_{\text{interactive}}}{(p_h - p)_{\text{fixed SST}}} - 1. \tag{2.73}
\]
The SST feedback factor, $F_{\text{SST}}$, is the quantity of prime interest in the parameter space exploration in Chapter 3. It measures how strongly the ocean response to a hurricane feeds back onto its intensity. For the default run (Fig. 2-17), the feedback factor has a value of $F_{\text{SST}} = -0.41$ which means that the coupled storm reached 59% of the intensity of the uncoupled (fixed SST) storm. This is clearly a dramatic effect, and we will investigate its dependence on a set of external parameters in the next chapter of this thesis.

Figure 2-18: Day 24 snapshot of the mixed-layer currents and the change in SST. Grid points are 20 km apart. A vector of length equal to the grid spacing corresponds to a current of 1 m/s. The contour interval is 0.5 °C.

The ocean response consists primarily of two features: A narrow band of sharply decreased SSTs due to the entrainment of water from the upper thermocline into the mixed-layer and a wake of near inertial frequency internal waves. Both features are clearly visible in Fig. 2-18 which shows the mixed-layer currents and the SST change in the wake of the hurricane. The storm track is marked as the solid
line with the star at its western end representing the current hurricane position. Current vectors are plotted at each node of the ocean model (20 × 20 km grid) and the maximum speed is 1.9 ms⁻¹. The change in SST is contoured every .5°C and is negative throughout. A clear bias in the SST response to the right of the track is evident. In general, most of the SST change is confined to a narrow band of roughly 200 km width. Fig. 2-19 displays the same mixed-layer current field with the depth of the mixed-layer overlaid. The contour interval is 20 m. A pattern

![Figure 2-19: Day 24 snapshot of the mixed-layer currents and the mixed-layer depth. Grid points are 20 km apart. A vector of length equal to the grid spacing corresponds to a current of 1 m s⁻¹. The unperturbed mixed-layer is 40 m deep. The contour interval is 20 m.](image)

of upwelling and downwelling as part of the inertio-gravity waves is clearly visible and entrainment causes the mixed-layer depth change to be positive everywhere. The maximum change in the mixed-layer depth exceeds 90 m compared to an unperturbed mixed-layer depth of 40 m. This is somewhat larger than observed
amplitudes of mixed-layer deepening, owing primarily to the layered structure of the ocean model. The well-mixed first layer in the model should probably be interpreted as the sum of the mixed-layer and the transition layer in observations. We shall address this issue in more detail in Chapter 4. The entrainment velocity, again overlaid onto the mixed-layer currents for reference, is shown in Fig. 2-20. Very intense entrainment occurs in a small patch slightly to the right and directly under the storm where the shear at the base of the mixed-layer is strongest.

Figure 2-20: Day 24 snapshot of the mixed-layer currents and the entrainment velocity. Grid points are 20 km apart. A vector of length equal to the grid spacing corresponds to a current of 1 m s⁻¹. The contour interval is 1 m s⁻¹

2.4 Summary

We introduced both the hurricane and the ocean model as well as the coupling procedure and presented the results from an extensive sensitivity study of the
hurricane model. The key features of the hurricane model are:

- The hurricane model displays the same sensitivities Carnot theory predicts. It is particularly sensitive to the ambient relative humidity in the boundary layer and only weakly sensitive to the ambient SST.
- The model atmosphere is initially convectively neutral so that the model hurricane develops entirely from a WISHE instability.
- The inclusion of longwave radiation, though crude, enables the model to reach a statistically steady-state in the lower troposphere.

The ocean model is a standard mixed-layer model with a shear-based bulk Richardson number entrainment parameterization. In the coupled integrations, the hurricane wind and surface pressure field are passed on to the ocean model, which returns the SST field to the hurricane model. The coupled model successfully simulates the important characteristics of the oceanic response to hurricane forcing as observed in nature. Comparison to experiments with fixed SST show that the model hurricane is significantly weakened by the effect of SST changes due to the hurricane's winds. The SST feedback factor, $F_{\text{SST}}$, is introduced as a quantitative measure of the ocean's effect on the hurricane intensity.
Chapter 3

SST Feedback – the Big Picture

In this chapter, we describe the procedure and the results of a systematic exploration of the dependence of the SST feedback factor, $F_{\text{SST}}$, on a set of parameters believed to govern the coupled system. The goal of this work is to summarize the behavior of the coupled model in a single analytic formula for the feedback factor.

3.1 $h_o$ and $u_T$ – a Qualitative Look

It is physically intuitive that the hurricane translation speed, $u_T$, and the unperturbed mixed-layer depth, $h_o$, affect the amplitude of the SST feedback. The hurricane translation speed primarily governs the temporal extent of the forcing a column of water experiences. A secondary effect of $u_T$ is its effect on the degree of resonance between the forcing and the inertial oscillations excited in the ocean as a response to the hurricane. The degree of resonance depends on the ratio of the storm translation speed to the storm size, what could be called a 'forcing frequency', and the natural frequency of the inertial oscillations which is given by the inverse of the local pendulum day. This secondary effect of $u_T$ is small compared to the primary effect so that we expect a near linear dependence of $F_{\text{SST}}$ on $u_T$. The role of the mixed-layer depth on the SST feedback is threefold: First, $h_o$ governs the amplitude of the mixed-layer currents for a given surface momentum input. Second, the entrainment velocity is roughly inversely proportional to the mixed-layer depth (2.62). And third, the SST response for a given entrainment heat flux depends inversely on the thermal inertia of the mixed-layer, which is proportional to $h_o$. These three effects combined suggest a near third power dependence of the
Figure 3-1: Contour plot of the feedback factor, $F_{sst}$, as a function of the hurricane translation speed, $u_T$, and the initial mixed-layer depth, $h_o$. The contour interval is .1.

SST feedback factor on the mixed-layer depth. Fig. 3-1 shows the actual model results for a set of 90 experiments which only differ in the two parameters $u_T$ and $h_o$. Qualitatively, the model results confirm the above description. For deep mixed-layers and/or fast moving storms the feedback is very small with feedback factors close to zero. In the opposite extreme of slow moving storm over thin mixed-layers, the feedback factors can drop below $-50\%$ which means that the coupled hurricane achieves less than half the intensity of the uncoupled hurricane.

### 3.2 A Quantitative Approach

We shall now address the problem quantitatively. The (non-dimensional) feedback factor must be a function of a number of the input parameters in the coupled model. Not all of the input parameters will affect the feedback factor significantly and we first seek to select the relevant parameters, guided by physical intuition.
3.2.1 The Choice of Parameters

Two of the important parameters have already been mentioned above, namely the hurricane translation speed, \( u_r \), and the unperturbed mixed-layer depth, \( h_o \). The size of the hurricane affects the feedback primarily through its effect on the temporal extent of the forcing a column of water experiences. In addition, the ratio of the size of the hurricane as measured by the radius of maximum winds \((20 - 60 \text{ km})\) to the baroclinic radius of deformation in the ocean \((20 - 60 \text{ km})\) will determine the nature of the oceanic adjustment. If the forcing length scale is much larger than the radius of deformation, the oceanic response will be largely baroclinic, while a very small atmospheric forcing will result in a predominantly barotropic oceanic response. This means that the Coriolis parameter is another important parameter. Furthermore, the hurricane intensity achieved at constant SST, the 'potential intensity', could influence the amplitude of the feedback, so we include the pressure drop from the environment to the eye of the storm (for fixed SST) in the list of parameters. And, finally, the stratification in the thermocline must be of importance since it relates the mixed-layer deepening to the entrainment heat flux at the base of the mixed-layer. All the other input parameters are thought to be of secondary importance for the amplitude of the feedback. In summary, the relevant parameters are

- the unperturbed mixed-layer depth, \( h_o \),
- the hurricane translation speed, \( u_r \),
- the hurricane size as set by \( \gamma \),
- the Coriolis parameter, \( f_o \),
- the hurricane intensity as measured by \( \Delta p_t \equiv (p_t - p_o)_{\text{SST}} \),
- and the thermocline stratification, \( \Gamma \equiv \frac{\partial T}{\partial z} \).

The physical parameter space of interest is therefore 6-dimensional. Since all the parameters can be expressed in kinematic units, we can combine the six dimensional parameters into four non-dimensional parameters, according to the Buckingham \( \pi \)-Theorem. There is a large number of possible choices for the four non-dimensional parameters, and it is mathematically irrelevant which set of parameters we choose. The set of non-dimensional parameters used hereafter is
\[ R_1 = \frac{\gamma}{h_o}, \]
\[ R_2 = \frac{\Delta p}{\rho_o} u^{-2}, \]
\[ R_3 = h_o \alpha \Gamma, \]
\[ R_4 = \frac{u_r}{f_0 \gamma}; \]

where the size parameter, \( \gamma \), is always thought of as in units of length, \( \rho_o \) is a reference density of air, and \( \alpha \) is the coefficient of thermal expansion. The feedback factor, \( F_{ssr} \), now must depend on these four non-dimensional parameters solely.

### 3.2.2 The Best Fit

The functional dependence of \( F_{ssr} \) on \( [R_i] \) cannot be found without making further assumptions. When Fig. 3-1 is reproduced in double logarithmic coordinates, i.e. \( F_{ssr} \) is plotted as a function of \( \ln(h_o) \) and \( \ln(u_r) \), the contours of \( F_{ssr} \) turn out to be nearly straight parallel lines. Similarly, if any other pair of the dimensional parameters is used as axes of a two-dimensional contour plot of \( F_{ssr} \) in \( ln-ln \) co-ordinates the resulting contour lines are nearly straight. This suggests that there should be a power law dependence of \( F_{ssr} \) on \( [R_i] \) of the form

\[ F_{ssr} = \Phi \left(e^{\lambda} R_1^a R_2^b R_3^c R_4^d\right), \]

where \( \Phi \) is an unknown function. To determine the exponents \( \lambda, a, b, c, \) and \( d \), and the function \( \Phi \), we conducted a large number of experiments over the entire range of interest of the parameters. Table 3.1 lists this range in terms of the dimensional parameters together with the corresponding range of the non-dimensional parameters. We ran 1440 experiments covering the parameter range of Table 3.1. In each experiment, the coupled hurricane-ocean model was integrated for 15 days to a steady-state and the steadiness of the central pressure of the model hurricane was checked over the last three days of integration. 93 of the 1440
### Table 3.1: Explored dimensional and non-dimensional parameter ranges.

<table>
<thead>
<tr>
<th>Dimensional Parameters</th>
<th>Symbol</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>mixed-layer depth</td>
<td>( h_o )</td>
<td>( 20 \rightarrow 90 )</td>
</tr>
<tr>
<td>storm translation speed</td>
<td>( u_r )</td>
<td>( 3 \rightarrow 12 )</td>
</tr>
<tr>
<td>storm size</td>
<td>( \gamma )</td>
<td>(.4 \rightarrow 1.)</td>
</tr>
<tr>
<td>Coriolis parameter</td>
<td>( f_o )</td>
<td>( 3 \times 10^{-5} \rightarrow 6 \times 10^{-5} )</td>
</tr>
<tr>
<td>storm central pressure</td>
<td>( p_{0</td>
<td>sst} )</td>
</tr>
<tr>
<td>thermocline stratification</td>
<td>( \Gamma )</td>
<td>(.07 \rightarrow .09 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Non-dimensional Parameter (Symbol)</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_1 )</td>
<td>( 5 \times 10^{-3} \rightarrow 5 \times 10^{-2} )</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>(.2 \rightarrow 10 )</td>
</tr>
<tr>
<td>( R_3 )</td>
<td>( 5 \times 10^{-4} \rightarrow 3 \times 10^{-3} )</td>
</tr>
<tr>
<td>( R_4 )</td>
<td>( 5 \times 10^4 \rightarrow 5 \times 10^5 )</td>
</tr>
</tbody>
</table>

Experiments failed for various reasons before the regular end of the integration, mostly because of vanishing layers in the ocean model in the extreme ‘corners’ of the parameter space. We decided to exclude another 25 experiments with feedback factors smaller than \(-.7\) because they correspond to extreme forcing situations close to the collapse of the model. The remaining 1322 experiments are now the basis for an optimal fitting procedure to determine the exponents in the power law (3.2). Let \( \hat{\Phi} \) be the inverse function to \( \Phi \), then (3.2) can be rewritten as

\[
\ln \left[ \hat{\Phi} \left( F_{sst} \right) \right] = \lambda + a \ln(R_1) + b \ln(R_2) + c \ln(R_3) + d \ln(R_4).
\]  

(3.3)

This is now a linear equation in the constants \([\lambda, a, b, c, d]\). If we knew the functional form of \( \hat{\Phi} \), equation (3.3) could be used as the basis for a multi-linear least-squares regression for the set of constants. For lack of knowledge of \( \hat{\Phi} \) we choose an iterative process to determine both \( \Phi \) and the set of constants:

1. Initially assume that \( \hat{\Phi}(x) = -x \).

2. Perform a least-squares multi-linear regression to yield \([\lambda, a, b, c, d]\).

3. Plot \( F_{sst} \) against \( z \equiv e^\lambda R_1^a R_2^b R_3^c R_4^d \) and update the functional form of \( \Phi \).
Figure 3-2: $F_{	ext{sst}}$ from the coupled model runs as function of the non-dimensional parameter $z = e^\lambda R_1^a R_2^b R_3^c R_4^d$. The solid curve is the best fit function $\Phi$ (3.5).

4. Go back to step 2 if the function $\Phi$ has changed significantly during the last iteration.

Step 3 is somewhat subjective in that there are many possible choices for functional fits to the plotted data points. It is desirable to choose a function $\Phi$ which is easily invertible, and we decided on an exponential functional relationship. The iterative process converges after a few iterations and yields the following best fit constants:

$$\lambda = -.58 \; ; \; a = -.28 \; ; \; b = -.69 \; ; \; c = -.23 \; ; \; d = -.52 \quad (3.4)$$

The best fit function is

$$\Phi(z) = -.8 e^{-z} \quad \text{with} \quad z = e^\lambda R_1^a R_2^b R_3^c R_4^d. \quad (3.5)$$
Figure 3-3: Contour plot of the feedback factor, $F_{SSR}$, as a function of the hurricane translation speed, $u_t$, and the initial mixed-layer depth, $h_0$. Solid contours are from model output while dotted contours are from the analytic best fit (3.5). The contour interval is 0.1.

The standard deviation of the data points from the best fit curve is $\sigma = 0.027$. Fig. 3-2 shows all 1322 data points in a $F_{SSR}$ versus $z$ diagram with the analytical best fit curve overlaid. The fact that all the data points nicely collapse onto a curve confirms the initial assumption of the power law dependence (3.2). We can now use the analytical function $\Phi$ to predict the feedback factor rather than running the coupled model. A comparison of the analytic result to the model output is made in Fig. 3-3 which shows the model output of Fig. 3-1 with the analytic results overlaid as dotted contours. The agreement is very good overall. The analytic result slightly overpredicts the feedback amplitude in regions of very weak feedback and tends to underpredict situations of extremely strong feedback in the coupled model.

A comment on the error in the exponents seems in place here. Assuming a standard deviation of 0.03 for the model-determined feedback factor, $F_{SSR}$, we can determine individual confidence intervals for each of the constants. They are listed
<table>
<thead>
<tr>
<th>Constant</th>
<th>Best Fit Value</th>
<th>Confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>90%</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-0.58$</td>
<td>$\pm 0.19$</td>
</tr>
<tr>
<td>$a$</td>
<td>$-1.28$</td>
<td>$\pm 0.03$</td>
</tr>
<tr>
<td>$b$</td>
<td>$-0.69$</td>
<td>$\pm 0.01$</td>
</tr>
<tr>
<td>$c$</td>
<td>$-0.23$</td>
<td>$\pm 0.03$</td>
</tr>
<tr>
<td>$d$</td>
<td>$-0.52$</td>
<td>$\pm 0.02$</td>
</tr>
</tbody>
</table>

Table 3.2: Confidence intervals for the fitted constants $[\lambda, a, b, c, d]$.

in Table 3.2. The low confidence in $\lambda$ reflects the fact that we kept the function $\Phi$ fixed in this analysis. $\lambda$ should therefore be considered part of the functional fit and its low confidence is of no concern.

3.2.3 Interpretation

Now that the non-dimensional optimal fit has been found we can ‘undo’ the non-dimensionalization of the previous section to reveal the dependence of $F_{\text{SST}}$ on the dimensional parameters. (3.1), (3.4), and (3.5) yield

$$F_{\text{SST}} = -0.8 e^{-z}$$

with

$$z = 0.56 e^\lambda R_1^a R_2^b R_3^c R_4^d$$

$$= 0.56 \rho_o^{-b} \alpha^c h_o^{(c-a)} u_T^{(d-2b)} \gamma^{(a-d)} f_o^{-d} \Delta p_e^b \Gamma^c$$

$$= 4 h_o^{(1.05 \pm 0.05)} u_T^{(0.86 \pm 0.03)} \gamma^{(-0.76 \pm 0.05)} f_o^{(0.52 \pm 0.03)}$$

$$\Delta p_e^{(-0.69 \pm 0.02)} \Gamma^{(-0.23 \pm 0.04)}, \quad (3.6)$$

where the ranges of the exponents correspond to the 95% confidence intervals. (3.6) is at first sight a rather surprising result. As pointed out earlier, we would have expected a much stronger dependence on the mixed-layer depth close to a third power. Also, the thermocline stratification seems to play a minor role with an exponent of roughly minus one quarter. Yet we have to keep in mind that the feedback factor is already an exponential function of $z$ so that we should try to
interpret the relative importance of the different dimensional parameters rather than the absolute value of their exponents.

Let us try to estimate the amplitude of the SST response from a scaling argument. The initial temperature profile features a mixed-layer of constant temperature and of depth scale $h_o$ on top of a stably stratified layer with constant buoyancy frequency and temperature lapse rate $\Gamma$. Let $t$ be a time scale for the forcing period, i.e.

$$t = \frac{\gamma}{u_T},$$

where $\gamma$ is a length scale describing the storm size and $u_T$ is a typical storm translation speed. Then the SST reduction due to the storm passage will be of order

$$\Delta\text{SST} \propto \Gamma w_e t \frac{w_e t}{h_o + w_e t} \approx \Gamma \frac{w_e^2 t^2}{h_o},$$

where $w_e$ is the entrainment velocity scale. The parameterization of entrainment in the ocean model (2.62) can be written in the form

$$\frac{w_e}{u_*} \propto R_b^{-\frac{1}{2}} \quad \text{or} \quad w_e \propto \frac{u_*^2}{c},$$

where $c^2 = g \frac{\rho_o}{\rho} h_o = g \alpha \Delta T h_o$. $\Delta T$ is the temperature drop across the base of the mixed-layer and can be written in terms of the change of temperature at the top of the thermocline, $\Delta T_b$, and the change of temperature in the mixed-layer, $\Delta\text{SST}$:

$$\Delta T = \Delta T_b - \Delta\text{SST} = \Gamma w_e t \left(1 - \frac{w_e t}{2(h_o + w_e t)}\right) \approx \Gamma w_e t.$$

The entrainment velocity scale squared then becomes

$$w_e^2 = \frac{u_*^4}{c^2} = \frac{u_*^4}{g \alpha \Gamma w_e t h_o}$$

which can be solved for $w_e$:

$$w_e = \left(\frac{u_*^4}{g \alpha \Gamma t h_o}\right)^\frac{1}{3}$$

Substitution of (3.12) into (3.8) yields

$$\Delta\text{SST} \propto h_o^{-\frac{5}{3}} t^\frac{4}{3} u_*^\frac{5}{3} \Gamma^\frac{1}{3}.$$
If we identify \( u^2 \) with \( \Delta P \) and use (3.7) to express \( t \) in terms of \( \gamma \) and \( u_r \) we finally get

\[
\Delta \text{SST} \propto h_o^{-\frac{1}{3}} u_r^{-\frac{1}{3}} \gamma^{\frac{1}{3}} \Delta P^{\frac{1}{3}} \Gamma^{\frac{1}{3}}.
\] (3.14)

This dependence can now be compared to the model result (3.6). We have to keep two things in mind: Firstly, the scaling argument does not consider the effect of the inertio-gravity waves on the entrainment and thus on \( \Delta \text{SST} \) and therefore does not contain the Coriolis parameter. Secondly, the scaling argument does not include the actual feedback but rather considers the ocean as purely responding to a given hurricane forcing. The uncoupled hurricane intensity as given by \( \Delta P \) is then larger than the true intensity and the scaling argument is therefore expected to overpredict the power of \( \Delta P \) to some degree. The hurricane translation speed, \( u_r \), and the hurricane size, \( \gamma \), could enter the \( F_{\text{SST}} \) dependence at a higher power than predicted by our simple scaling argument owing to the additional effect of the temporal delay and the associated spatial separation of the SST response from the storm center. The powers of \( h_o \) and \( \Gamma \), in contrast, are strictly only oceanic parameters and should directly match the corresponding powers in the regression of the model result to within a common factor. This common factor turns out to be roughly 1.6:

\[
z^{-1.6} \propto h_o^{(-1.68 \pm .08)} \Gamma^{(0.37 \pm .06)} \iff \Delta \text{SST} \propto h_o^{(-1.67)} \Gamma^{(0.33)}
\] (3.15)

The agreement indicates that the scaling argument explains the relative dependence of \( F_{\text{SST}} \) on \( h_o \) and \( \Gamma \). It turns out that \( z^{-1.6} \) corresponds closely to the \( \Delta \text{SST} \) from the scaling argument even in the exponents of \( u_r \) and \( \gamma \):

\[
z^{-1.6} \propto u_r^{(-1.38 \pm .05)} \gamma^{(1.22 \pm .08)} \iff \Delta \text{SST} \propto u_r^{(-1.33)} \gamma^{(1.33)}
\] (3.16)

This agreement seems somewhat surprising at first but suggests an interesting aspect of the dynamics of the coupled system. It indicates that the effect of the spatial separation between the eye of the storm and the maximum SST decrease plays only a minor role in the feedback mechanism, at least in the context of the present model. This explanation is consistent with the observation that the hurricane model’s high sensitivity to changes in SST is restricted to the immediate vicinity of the eye. Observations of the radial distribution of moist entropy in the boundary layer of real hurricanes explain this very localized sensitivity: The moist entropy is significantly elevated only in the innermost few tens of kilometers.
(e.g. Hawkins and Imbembo, 1976). Our hurricane model reproduces this aspect of the boundary layer moist entropy distribution quite realistically (see Fig. 2-6) so that our interpretation of the close correspondence in (3.16) is consistent with the model dynamics. Also note that

\[ z^{-1.6} \propto \Delta P_c^{(1.1 \pm 0.03)} \iff \Delta \text{SST} \propto \Delta P_c^{(1.33)} \quad (3.17) \]

The exponent of \( \Delta P_c \) is somewhat lower in the model data than in our scaling argument which is reasonable since the feedback lowers the sensitivity of \( \Delta \text{SST} \) to \( \Delta P_c \).

The relative insensitivity of \( F_{\text{SST}} \) to \( \Gamma \) is particularly well explained by the scaling argument. Even though large thermocline gradients provide very cold water close to the base of the mixed-layer, the damping effect of the associated strong stratification on the entrainment velocity largely compensates for this effect and thus reduces the sensitivity of \( F_{\text{SST}} \) to \( \Gamma \).

The effect of the Coriolis parameter can be understood in terms of its effect on the amplitude of the inertio-gravity wave response. As pointed out by Geisler (1970), the amplitude of the vertical displacement of the base of the mixed-layer is larger for smaller \( f_0 \). The exact power at which \( f_0 \) should enter the scaling is difficult to estimate since its interaction with the entrainment will limit its importance similarly to the ‘self-limiting’ effect for \( \Gamma \). The model-produced power of roughly .5 therefore cannot be checked easily.

### 3.3 Summary

We successfully described the behavior of the coupled hurricane-ocean model by a single analytic expression for the SST feedback factor. To arrive at this expression, we first identified the governing parameters of the feedback mechanism, combined them into a set of four non-dimensional parameters, and then sampled the four-dimensional parameter space with 1440 model runs. Using an quasi-linear regression technique, we fitted a simple analytic function to the data. We then deduced the dependence on the dimensional parameters and found the mixed-layer depth, \( h_o \), the storm translation speed, \( u_r \), the storm size as given by \( \gamma \), and the potential storm intensity, \( \Delta P_c \), to be of roughly equal importance for the amplitude of the SST feedback factor. The Coriolis parameter also affects the feedback.
significantly with a stronger feedback effect at lower latitudes. The thermocline stratification, finally, turned out to play only a rather minor role.

The final analytic formula relates the pressure depression of the steady-state hurricane to the set of six dimensional parameters describing the ambient conditions. The input parameters are the mixed-layer depth, $h_o(m)$, the storm translation speed, $u_r(m s^{-1})$, the storm size, $\gamma = \frac{r_{max}}{50 \ km}$, the potential storm intensity, $\Delta p = [p_i - p_o]_{SST} (hPa)$, the Coriolis parameter, $f_0(s^{-1})$, and the thermocline stratification, $\Gamma = \frac{\partial T}{\partial z}(^o C m^{-1})$. The storm's pressure deficit in the eye, $\Delta p$, is then given as

$$\Delta p = \Delta p_c \left[1 - .8 e^{-4 h_o^{1.05} u_r^{.66} \gamma^{-0.75} f_0^{.52} \Delta p_c^{-0.65} \Gamma^{-0.23}}\right].$$

This equation can also be solved iteratively to yield the potential storm intensity in terms of the observed intensity and the set of six dimensional parameters.
Chapter 4

A Tale of Mixing – Turbulent Entrainment

We now address the problem of the parameterization of turbulent entrainment in stratified shear flows, with special application to the oceanic mixed-layer. This work was motivated by the coupled hurricane-ocean simulations described in the previous chapters, in which the entrainment of cold thermocline water into the oceanic mixed-layer was the primary cause of SST reduction in response to the hurricane's winds. We will first describe the problem of the parameterization of entrainment and critically review previous attempts and then present a new entrainment hypothesis.

4.1 The Status Quo

Entrainment is the process by which a volume of fluid filled with turbulence incorporates non-turbulent fluid at its boundaries. The speed at which the interface between the turbulent and the non-turbulent part of the fluid advances into the non-turbulent fluid is called the 'entrainment velocity'. In many geophysical flows, only part of the fluid is filled with turbulence and knowledge of the rate of entrainment is important if one wishes to predict the evolution of the flow. Examples of such situations are the oceanic and the planetary boundary layers, which are driven by a combination of shear and convection. In these examples the turbulence is confined to a layer adjacent to a boundary of the fluid and often separated from the rest of the fluid by an inversion. Cumulus clouds or convective plumes in ge-
neral are examples of turbulent flow features which penetrate into a non-turbulent environment. Again, entrainment takes place at the edges of the turbulent region and acts to increase the volume filled with turbulence.

Turbulence is usually a sub-grid scale phenomenon and its effects on the resolved flow fields must be parameterized in both numerical and analytical models. So-called Large Eddy Simulations (LES) form a partial exception: In LES the gravest and most energetic part of the turbulent spectrum is explicitly resolved while the effect of the smaller scale turbulence still must be parameterized. We shall exclude LES from the rest of this discussion because their exceptional computational cost allows model domains only a few tens of times the size of the resolved eddies. Since the fluid motions on the temporal and spatial scales of the entraining eddies are difficult to measure and chaotic in nature most of the research on entrainment has focused on the net effect of the turbulence on the mean flow rather than on the detailed characteristics of the turbulent motions. This focus is directly analogous to the task of parameterization of entrainment.

We now review the development of entrainment parameterizations over the last forty years, which was largely guided by observations from field experiments and laboratory studies. Because of the immense body of literature on this problem we choose to restrict this review to those papers which are closely related to the problem of entrainment at the base of the oceanic mixed-layer. Even with this restriction, only a small subset of the published literature can be mentioned here, the selection of which involves a certain amount of subjectivity. Our goal was to include those papers that most influenced later research as well as those papers that most helped our own understanding of the entrainment problem.

One of the earliest laboratory experiments of turbulent entrainment is due to Rouse and Dodu (1955). They produced turbulence in the upper of two homogeneous layers by means of a vertically oscillating grid and found that the interface between the two layers was sharpened as a result of the stirring, rather than becoming more diffuse as they had expected. The well-defined interface was observed to slowly descend into the non-turbulent lower layer. In addition, they reported that entrainment took place, though at different rates, for any value of the stirring intensity and the initial density jump, i.e. that there was no stability limit below which entrainment shut off.

Four years later, Ellison and Turner (1959) conducted laboratory experiments in which they pumped light fluid over a homogeneous layer of fluid of higher den-
sity and observed the thickness of the layer of less dense fluid downstream of the inlet. The surface layer initially deepened rapidly but entrainment decreased in the downstream direction and eventually shut off as the bulk Richardson number $\left( R_b \equiv \frac{g \Delta \rho h}{u^2} \right)$ increased to .8. In Ellison and Turner's definition of the bulk Richardson number, $\Delta \rho$ is the density jump at the base of the mixed-layer, $h$ is the depth of the mixed-layer, and $u$ is the mixed-layer velocity. The absence of a surface source of turbulence made their experimental setting very elegant since it eliminates the surface friction velocity as an additional velocity scale. Ellison and Turner were the first to suggest that the bulk Richardson number based on the mean shear across the base of the mixed-layer is maintained at a critical value during the entrainment process, an idea which did not receive widespread attention until two decades later.

Concerned with the height of the inversion atop a convective atmospheric boundary layer, Ball (1960) argued that the generation of TKE (in his case due to convection) was nearly balanced by the conversion of TKE into mean potential energy, thus assuming the TKE to be in steady-state. Lilly (1968) extended Ball's work and discussed different balances in the steady-state TKE budget. Ball's result corresponds to Lilly's maximum entrainment limit in which the dissipation of turbulence has been neglected. In the other extreme, the minimum entrainment limit, convective TKE generation is entirely balanced by dissipation and no entrainment takes place. Tennekes (1973) estimated that typically 80% of the convectively generated turbulence is dissipated while only the remaining 20% is used to entrain warmer air into the mixed-layer. It is important to note that a similar nearly fixed ratio of the generation of turbulence to its dissipation has not been found for largely shear generated turbulence!

Turner and Kraus (1967) and Kraus and Turner (1967) first applied a steady-state TKE closure to the problem of a mechanically and thermally driven oceanic mixed-layer. They suggested that the input of TKE at the surface of the ocean be balanced by the conversion of TKE into mean potential energy by entrainment. Their original argument was actually based on the budget of mean mechanical energy rather than TKE, yet is essentially identical to a steady-state TKE closure in the absence of a mean flow. They proposed an entrainment law of the form

$$ w_e = u_* f(R_b) \quad \text{with} \quad R_b \equiv \frac{g \Delta \rho h}{u_*^2}, $$.  

(4.1)
where the function $f$ needs to be determined experimentally. $\Delta \rho$ is again the density jump across the base of the mixed-layer, $h$ the depth of the mixed-layer, and $u_*$ the surface friction velocity. Turner (1968) conducted experiments with stirring grid-generated turbulence in a two-layer system to determine $f$. When the density difference between the two layers was due to heat alone the functional dependence on $R_\delta^*$ was close to $R_\delta^* - 1$ except for very small values of $R_\delta^*$ for which entrainment approached a finite limit. In experiments with a salinity difference between the layers, the entrainment rate was roughly proportional to $R_\delta^* - \frac{3}{2}$. The $R_\delta^* - 1$ dependence implies a rate of change of potential energy proportional to the rate of working by the stirrer while the decreased entrainment rates for salt were attributed to a slower rate of incorporation of an entrained fluid element into its surroundings by diffusion, which increases the tendency for it to return to the interface. When stirring grids were operated in both layers simultaneously, the interface between the two layers gradually moved to a central position such that the intensity of the turbulence was the same on both sides of the interface. Entrainment rates across this stationary interface were indistinguishable from those measured with only one stirrer in operation, casting doubts on the validity of the energy argument at least in the limit of a stationary interface. A year later, Turner (1969) revised the original energy argument after analyzing ocean data of storm induced mixed-layer deepening. It became clear that the total input of mechanical energy at the ocean surface exceeds by far the increase in potential energy, leaving the question of the correct entrainment law open to further studies.

In the same year, Kato and Phillips (1969) published results from laboratory experiments with an annular tank. By means of a rotating partially immersed screen, a constant stress was applied to the surface of the initially quiescent fluid of constant stratification. The development and evolution of a mixed-layer was observed for different values of the surface stress and the initial stratification. Kato and Phillips argued on dimensional grounds for an entrainment law of the form proposed by Kraus and Turner (4.1) thus neglecting the additional velocity scale due to the development of a mean flow in their experimental setup. They nevertheless found reasonable agreement with their experimental data and proposed the following entrainment law:

$$w_\varepsilon = 2.5 \frac{u_*^3}{g_\rho \Delta \rho h} \quad (4.2)$$

We shall refer to this entrainment law as the Kraus/Turner-Kato/Phillips (KTKP)
law. It was widely used for some 15 years (e.g. Chang and Anthes 1979) and has only recently been largely abandoned.

Moore and Long (1971) conducted laboratory experiments in a race track-shaped tank. Fluid could be injected and withdrawn both at the bottom and at the top of the tank so as to create an entraining shear flow in the steady-state. Moore and Long were the first to measure vertical profiles of density and momentum in a laboratory study of entrainment. They found two nearly homogeneous layers separated by a transition layer with a gradient Richardson number of order unity. Moore and Long did not comment on the significance of this result and interpreted their data as further evidence for the validity of (4.2). At the same time, Thorpe published results from several experimental studies of Kelvin-Helmholtz (KH) instability in initially non-turbulent flows (e.g. Thorpe 1971, 1973). In Thorpe’s experiments a two layer fluid system contained in a tube develops wavelike KH instability after the tube is abruptly tilted through a small angle. When the tube is returned to a horizontal position the established stratified shear flow continues to mix fluid into a widening transition layer until the gradient Richardson number reaches a value of roughly \( \frac{1}{3} \) at which point the turbulence decays.

Moore and Long’s results were consistent with those of Thorpe and led Pollard, Rhines, and Thompson (1973) (hereafter PRT) to suggest that shear instability at the base of the mixed-layer is the primary source of turbulence in the wind-driven oceanic mixed-layer. In their theory, the effect of the surface stress is to increase the mixed-layer momentum and thus to increase the shear in the transition layer, and they neglect the direct input of TKE at the surface due to breaking surface waves as comparatively small. The wind stress thus introduces turbulence only indirectly via the increase of the shear in the transition layer. The overall stability parameter then becomes the bulk Richardson number, \( R_{b_u} \), based on the velocity difference across the transition layer, \( \Delta u \), and the entrainment closure takes the form

\[
R_{b_u} = \frac{g \Delta \rho h}{(\Delta u)^2} = \text{const.} \tag{4.3}
\]

\( \Delta \rho \) and \( h \) are the density jump at the base of the mixed-layer and the depth of the mixed-layer, respectively. PRT provide three different derivations for their entrainment hypothesis, which all give a critical bulk Richardson number of order unity, but the exact critical value is left open for empirical determination. Two follow-up papers by Niiler (1975) and de Szoeke and Rhines (1975) present PRT’s
closure as a special two term balance in a more general TKE equation and provide a more thorough theoretical background for its derivation. If a surface stress is abruptly turned on at time $t = 0$ over an initially quiescent linearly stratified ocean, different balances, including those of KTKP and of PRT, are found to be valid at different stages of the entrainment process. This raises questions about the general validity of both the KTKP and the PRT closure.

Price et al. (1978) analyzed and simulated two observed cases of mixed-layer deepening due to storms. The primary goal of their study was to determine whether the relevant velocity scale in the parameterization of wind-driven mixed-layer deepening is the surface friction velocity, $u_*$, as in the KTKP law, or the mean shear across the transition layer, $\Delta u$, as suggested by PRT. Simulations employing the entrainment law (4.3) with $R_{bu} = .65$ were found to be in better agreement with the observations than simulations employing KTKP's entrainment law (4.2). To test this result with laboratory data, Price (1979) and Thompson (1979), independently from each other, re-examined the data of Kato and Phillips and those of Kantha et al. (1977). Both authors found that, after accounting for side-wall drag, all the data points fell in the vicinity of the same $R_{bu}$ in support of (4.3). Price's best estimate of the critical bulk Richardson number was .5 to .6 while Thompson deduced slightly larger values in the range .7 to 1. In a modeling study of the oceanic response to hurricane forcing, Price (1981) introduced an implementation of (4.3),

$$w_e = \begin{cases} 
5 \times 10^{-4} R_{bu}^{-4} \Delta u & \text{if } 0 \leq R_{bu} \leq 1 \\
0 & \text{otherwise.} 
\end{cases} \quad (4.4)$$

The relationship between (4.3) and (4.4) is not immediately apparent mathematically. As pointed out by Price (1981), (4.4) causes rapid entrainment when $R_{bu}$ drops below unity and thus keeps $R_{bu}$ very close to unity and effectively constant. (4.4) was widely accepted and incorporated in many models including that of Cooper and Thompson (1989) which we used in the coupled experiments described earlier in this thesis. Since this formulation goes back to the work of PRT, we shall refer to this entrainment law as the Pollard/Rhines/Thompson-Price (PRTP) law.

Since Price's and Thompson's re-analysis of some laboratory data did not conclusively render the original interpretation of the data in terms of (4.2) inappropriate, further laboratory experiments were conducted with much improved instrumentation. The work of Scranton and Lindberg (1983) put all previous results
from experiments with annular tanks into question, including those of Kato and Phillips and of Kantha et al. Scranton and Lindberg found strong secondary circulations in their annular tank, strong radial tilt of density surfaces, and a dominant dependence of flow parameters on the annular radius. Thus the basic assumption of all annular tank experiments, namely that the annular tank approximates an infinitely long straight tank and that curvature effects can be ignored, turned out not to be even closely satisfied. The only way to avoid the curvature effect was to design entrainment experiments with straight tanks, which always introduce undesired endwall effects after limited observation times. Kranenburg (1984) conducted entrainment experiments with a 30 m long straight tank surmounted by a wind tunnel. The non-dimensional entrainment rate, \( \frac{w_e}{u_*} \), was found to be roughly proportional to \( R_{b_e}^{-\frac{1}{4}} \) in agreement with Price (1979) and Thompson (1979). Measurements of the gradient Richardson number \( R_g \equiv \frac{\frac{\partial g}{\partial z}}{\left( \frac{\partial u}{\partial z} \right)^2} \) in the transition layer showed nearly constant values very close to \( \frac{1}{4} \) for a wide range of conditions.

Kranenburg’s results strongly support the view that the limiting constraint for entrainment is the supply of momentum rather than the direct supply of TKE. Yet it remains unclear over which range of conditions this result holds. As mentioned earlier, a number of regimes can be conceived each with different physical balances. It is of particular concern to investigate which balance(s) is (are) important under observed conditions in nature.

Deardorff (1983) designed an entrainment formulation with the goal of incorporating many of the different possible balances in the steady-state TKE budget. He considered the TKE budget of the transition layer and derived an iterative algorithm to solve for the degree of turbulence in this layer. He made use of two empirical relationships: A functional dependence of the entrainment velocity on the degree of turbulence from stirring grid experiments was used as well as a relationship between the ratio of the transition layer to mixed-layer depths and the degree of turbulence in the transition layer. Unfortunately, both the validity of these empirical laws and the sensitivity of the entrainment rate to the choice of these laws remain unclear. Because of its supposed generality Deardorff’s formulation was used, e.g. in the work of Bender et al. (1993). Yet a comparison between the behavior of Deardorff’s entrainment law (hereafter referred to as DD) and the PRTP formulation shows only minor differences, and the much higher computational cost of Deardorff’s parameterization seems not to be justified.
We have limited the above discussion to entrainment formulations which implicitly assume the existence of a well-mixed surface layer. There is abundant observational support for the presence of a mixed-layer which is generally thought of as primarily convectively generated in high latitudes and primarily wind-driven in the tropics. If one does not prescribe a well-mixed surface layer but rather tries to explicitly model the turbulence which is thought to cause the upper ocean to be well-mixed, equations for the turbulent fluxes must be solved on a high-resolution vertical grid. In the simplest case, the first-order closures, the second moments in the TKE equation are expressed as functions of the mean fields via an eddy viscosity $K$ in analogy to the treatment of viscous stress. The closure in these so-called $K$-models then reduces to a proper formulation for the eddy viscosities for heat and momentum. There are many possible choices; often the $K$'s are expressed as empirical functions of the local stability. The flaws of the $K$-closures led to the construction of higher-order closure models in which prognostic equations for the second or even higher moments are solved. The closure problem is thus pushed to the equations for the lowest moments not explicitly carried as variables. We do not want to discuss these models here because their complexity overshadows the underlying physics of the entrainment process while they fail to produce realistic mixed-layers. The reader is referred to Mellor and Yamada (1982) for a review of higher-order closure models. A very nice general overview of all the different turbulence closures can be found in Sorbjan (1989).

Yet another approach was chosen by Price et al. (1986), who required turbulent mixing in their model to satisfy three different stability criteria: Static stability ($\frac{\partial \theta}{\partial z} \geq 0$), bulk shear stability of the mixed-layer ($R_{b} \geq .65$), and local shear flow stability ($R_{g} \geq .25$). The second and the third criterion together describe the shear-induced mixing: The second criterion ensures the existence of a mixed-layer while the third creates a transition layer at the base of the mixed-layer. This formulation is appealing in that the resulting profiles of density and momentum feature both a mixed-layer and a transition layer while all the previously discussed models lack one of the two features. Yet the physical justification for enforcing the bulk shear stability criterion in parallel with the local shear stability criterion remains unclear. Price et al. (1994) used this formulation of turbulent mixing in a study of the ocean response to hurricane forcing and found reasonable agreement with observations. Bender et al.’s (1993) simulations of the same hurricanes gave nearly identical results when they used Deardorff’s entrainment formulation. The sensitivity of the
results to the formulation of the entrainment has not been addressed conclusively, particularly for the case of very strong forcing in hurricane conditions.

The failure of higher-order closure models to produce realistic mixed-layers and similarly the need for the bulk shear stability criterion in addition to the local shear stability criterion in Price et al.'s (1986) work leads to the question of whether the one-dimensional treatment of turbulence in all the previously mentioned studies is appropriate. Horizontal role vortices aligned with the surface wind stress, so-called Langmuir cells, are abundant in the mixed-layers of oceans and lakes. They are easily visible at the ocean surface since any sort of floating tracer such as seaweed or foam collects into lines in the convergent part of the surface flow. Their subsurface structure in nature is still somewhat uncertain but the circulation seems to be concentrated at the surface and fills the entire depth of the mixed-layer. This means that Langmuir cells might actually be the primary agent of keeping the turbulent surface layer in the ocean so extremely well-mixed. A separate question is which role Langmuir circulations play in controlling the depth of the mixed-layer. A nice review of observations and theories of Langmuir circulations is due to Leibovich (1983). The leading theory for the physical mechanism of Langmuir cells is that of Craik (1977) and Leibovich (1977b), an extension of an earlier theory by the same two authors: The basic physics is an instability mechanism in which an infinitesimal downwind jet has its vorticity, with opposite sign on the two sides of the jet, tilted by the Stokes drift of the surface waves to produce longitudinal rolls. Theses produce the surface convergence at the jet, and this is in turn reinforced due to the acceleration, by the wind stress, of the water moving towards the surface convergence. If Langmuir cells are created in a wind-wave tank (e.g. Faller and Caponi, 1978), the cells' vertical extent is typically given by the depth of the tank while the cells tend to deepen indefinitely in numerical simulations with a homogeneous ocean (e.g. Leibovich, 1977a). Garrett (personal communication) conducted numerical experiments with a linearly stratified ocean and found very rapid initial deepening of the developing Langmuir cells. But the cells did not penetrate further into the fluid after a depth of $10^{12} N$ had been reached, where $N$ is the buoyancy frequency below the mixed-layer. This corresponds to a buoyancy jump at the base of the mixed-layer that is small in comparison to realistic oceanic values, and the rapid initial deepening may just be a result of the unrealistic initial condition of a linearly stratified ocean without a capping mixed-layer. The crucial question of whether Langmuir cells directly contribute to
the turbulent erosion of the thermocline cannot be answered presently. We believe that Langmuir circulations are the primary cause of the very high homogeneity observed in the oceanic mixed-layer. A shear-driven atmospheric boundary layer is usually not nearly as well-mixed as its oceanic counterpart, possibly due to the absence of Langmuir circulations in the atmosphere. We further believe that the actual turbulent entrainment process is driven by small-scale turbulence which is either locally generated by KH instability or has been transported from the ocean surface to the base of the mixed-layer. The role of Langmuir circulations in entrainment in this picture is indirect only, through the concentration of the shear at the base of the mixed-layer and through the homogenization of turbulence throughout the mixed-layer. Consequently, the effect of Langmuir circulations is implicitly included in any model which presumes the existence of a mixed-layer. We shall follow this line of thought in the derivation of a new entrainment law in the next section.

In closing, two additional references shall be mentioned here: Turner (1981) gives a very nice (and concise) overview over the different small-scale mixing processes in the ocean with many useful references; an extensive list of references can be found in Fernando’s (1991) review paper on turbulent mixing in stratified fluids, together with a rather complete list of laboratory studies on entrainment.

4.2 The Critical Transition Layer (CTL) Entrainment Parameterization

In this section, we introduce a new entrainment parameterization for use in mixed-layer models. We do not attempt to explicitly include Langmuir circulations but rather argue that the assumption of a perfectly mixed surface layer adequately represents the net effect of Langmuir cells (see previous section).

The new formulation has two fundamental advantages over the commonly used parameterizations discussed in the previous section:

- We include the PRTP and the KTKP source terms in the TKE budget, which makes the new formulation applicable to very different entrainment regimes.

- We implicitly incorporate a transition layer, a prominent feature of observed stratified shear flows. This enables us to realistically account for the heat
and momentum contained in the transition layer.

The new formulation is based on a few physically reasonable assumptions. It is both easy to implement and efficient to run in standard mixed-layer models. It also serves to highlight some of the inconsistencies of the existing formulations.

The following is a complete list of the assumptions made in the derivation of the new entrainment parameterization:

1. **The turbulence is equilibrated.** This means that an inertial subrange has been established and that the shape of the spectral TKE density is independent of the TKE itself. Then the integral of the spectral TKE density over all scales of turbulent motion, the TKE, is proportional to the spectral TKE density at any scale, in particular at the scale of the entraining eddies. Although only rarely stated, this assumption is fundamental to any entrainment formulation which invokes the TKE budget.

2. **The TKE is in a steady-state.** Simple scaling arguments in the TKE budget strongly support this assumption for time scales larger than minutes. This assumption is common to most entrainment formulations including all the formulations discussed in Section 4.1.

3. **Horizontal turbulent fluxes can be neglected in the TKE budget.** The relative size of the horizontal gradients of density and momentum in comparison to the vertical gradients of the same quantities together with the assumption of near isotropy of the turbulence justifies this assumption. It is also common to most entrainment formulations including those discussed in Section 4.1. This assumption leads to a one-dimensional model of turbulent entrainment and thus excludes the explicit treatment of Langmuir cells. As mentioned earlier, we argue that the effect of Langmuir circulations is implicitly take into account by the assumption of a well-mixed layer at the top of the ocean (see Section 4.1).

4. **The vertical profiles of density and momentum are continuous and piecewise linear.** This assumption is distinct to our new formulation yet hardly restrictive since observed vertical profiles of mean density and momentum typically can be well approximated by continuous piecewise linear profiles.
5. **The top layer of the ocean is well-mixed.** As the name suggests, this assumption is common to all mixed-layer models. It is well supported by observations and reflects the existence of Langmuir circulations.

6. **The flow below the transition layer is non-turbulent.** This assumption is also well supported by observations and common to all mixed-layer models.

7. **The transition layer is Kelvin-Helmholtz critical.** By Kelvin-Helmholtz criticality we mean that the gradient Richardson number is equal to an adjusted or critical value, $R_c$. Kelvin-Helmholtz (KH) instability sets in at a gradient Richardson number of $R_g = \frac{1}{4}$ and ceases at $R_g = \frac{1}{2}$, and the critical value $R_c$ is expected to lie in this range. Its exact value must be determined empirically. The KH criticality assumption is justified by the short adjustment time scale for KH instability as observed e.g. by Thorpe (1978) and manifested in the absence of significantly less than critical Richardson numbers in nature. KH criticality of the transition layer is also one of the few aspects of turbulent entrainment most laboratory experiments of stratified shear flows agree upon (e.g. Thorpe, 1973; Kranenburg, 1984). In addition, recent analyses of oceanic data by Price et al. (1994) support the assumption of a KH critical transition layer.

Since this assumption is the back bone of our new entrainment parameterization we chose to name the parameterization Critical Transition Layer (CTL) parameterization.

Fig. 4-1 is a sketch of the vertical profiles of mean density and mean momentum implied by assumptions 4, 5, and 6. The derivation of the CTL entrainment law now proceeds as follows:

- Given profiles of mean density and momentum, simple geometric arguments yield that changes in these profiles are uniquely described by changes in the mean mixed-layer density and momentum, $\frac{\partial \rho}{\partial t}$ and $\frac{\partial Q}{\partial t}$, and by the movement of the layer interfaces, $w \equiv \frac{d h}{d t}$ and $w_r \equiv \frac{d}{d t}(h + H)$.

- Given the turbulent fluxes at the ocean surface, the vertical profiles of the turbulent fluxes can be expressed as functions of $w$, $w_r$, and mean quantities solely.
Figure 4-1: Vertical structure of the upper ocean under the CTL assumptions.

- The assumption of KH criticality of the transition layer yields a relationship between \( w \) and \( w_T \) which we call the Dynamic Criticality Condition (DCC). Using the DCC, we can derive expressions for the turbulent fluxes as functions of \( w_e = \frac{1}{2}(w + w_T) \) and mean quantities solely.

- We finally substitute these expressions for the turbulent fluxes into a TKE budget equation for the mixed-layer which then can be solved for the entrainment velocity \( w_e \).

In the next section, we derive the CTL entrainment law step by step. Since all of the assumptions of the CTL entrainment law and all of its physical concepts have been stated above, readers not interested in the mathematical details of the derivation may skip the next section.
4.2.1 Derivation of the CTL Entrainment Law

Under the assumption (3) of horizontal homogeneity, the budgets of heat and momentum reduce to

\[ \frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial z} \overline{\rho' w'} \]  

(4.5)

and

\[ \frac{\partial \overline{u'}}{\partial t} = - f_o \overline{k} \times \overline{u'} - \frac{\partial}{\partial z} \overline{u' w'}, \]  

(4.6)

where overbars denote Reynolds averaging. We may temporarily drop the Coriolis term in (4.6) because it does not contribute to any of the turbulent processes under consideration. Then the tendencies of density and momentum are due only to the convergence of the turbulent fluxes of density and momentum, respectively, and the turbulent fluxes therefore must be linear functions of depth in the mixed-layer and quadratic functions of depth in the transition layer in order to maintain the assumed shape of the profiles of mean density and momentum shown in Fig. 4-1. This leaves five degrees of freedom for the turbulent flux profiles and we need five constraints to uniquely determine the profiles:

1. At the ocean surface \((z = 0)\), the turbulent fluxes must match their atmospheric counterparts which are typically calculated using the aerodynamic drag formulæ. This constraint physically means that the interface region between the atmosphere and the ocean has zero heat capacity and cannot store any momentum (thus assuming that the surface wave field is equilibrated). Let \(\overline{\tau_o}\) be the surface stress vector and \(\overline{B_o}\) the surface buoyancy flux. Then the surface boundary conditions are

\[ \overline{\rho' w'} |_{o} = \frac{\rho_o}{g} \overline{B_o} \]  

(4.7)

and

\[ \overline{u' w'} |_{o} = - \frac{\overline{\tau_o}}{\rho_o}. \]  

(4.8)

2. At the bottom of the transition layer \((z = -h - H)\), turbulence goes to zero and so do the turbulent fluxes:

\[ \overline{\rho' w'} |_{-h-H} = 0 \]  

(4.9)

and

\[ \overline{u' w'} |_{-h-H} = 0. \]  

(4.10)
3. Since the mean profiles are assumed to be continuous (assumption 4), the turbulent flux profiles have to be continuous across the base of the mixed-layer as well:

$$\overline{\rho' w'}|_{-h+\delta} = \overline{\rho' w'}|_{-h-\delta}$$  \hspace{1cm} (4.11)

and

$$\overline{u' w'}|_{-h+\delta} = \overline{u' w'}|_{-h-\delta},$$  \hspace{1cm} (4.12)

where $\delta$ is an infinitesimal distance.

4. Geometric arguments yield that the density (momentum) flux convergence at the base of the transition layer is given by the vertical velocity of the base of the transition layer, $w_r$, and the mean density (momentum) profile as

$$\left( \frac{\partial}{\partial z} \overline{\rho' w'} \right)_{-h-H} = w_r \left( \frac{\Delta \rho}{H} - \frac{\rho_o}{g} N^2_o \right)$$  \hspace{1cm} (4.13)

and

$$\left( \frac{\partial}{\partial z} \overline{u' w'} \right)_{-h-H} = -w_r \frac{\Delta \bar{u}}{H}$$  \hspace{1cm} (4.14)

where $\Delta \rho$ and $\Delta \bar{u}$ are the density and momentum difference across the transition layer, respectively, and $N_o$ is the buoyancy frequency of the ocean below the transition layer.

5. The fifth constraint, finally, is an integral constraint on the total change of density and momentum in the upper ocean. Let us integrate the density tendency from a fixed depth $z = -D$ below the transition layer up to the ocean surface:

$$\int_{-D}^{0} \frac{\partial \rho}{\partial t} dz = \int_{-D}^{0} -\frac{\partial \overline{\rho' w'}}{\partial z} dz = -\overline{\rho' w'}|_o = -\frac{\rho_o}{g} B_o$$

$$= \frac{\partial}{\partial t} \int_{-D}^{0} \rho dz = \frac{\partial}{\partial t} \left[ \rho h + \frac{\rho + \bar{\rho}}{2} H + \frac{\bar{\rho} + \rho_o}{2} (D - h - H) \right]$$

$$= \frac{\partial}{\partial t} \left[ \rho \left( h + \frac{H}{2} \right) - \bar{\rho} \left( h + \frac{H}{2} \right) + \frac{\bar{\rho} + \rho_o}{2} (D + h + H) + \frac{\rho_o}{2} (D - h - H) \right]$$

$$= \left( h + \frac{H}{2} \right) \frac{\partial \rho}{\partial t} - (\bar{\rho} - \rho) \frac{\partial}{\partial t} \left( h + \frac{H}{2} \right) + \frac{1}{2} \left[ \frac{\partial \bar{\rho}}{\partial t} (D - h) + w_r (\bar{\rho} - \rho_o) \right]$$
\[ = \left( h + \frac{H}{2} \right) \frac{\partial \rho}{\partial t} - w_e \Delta \rho + \frac{1}{2} \left[ w_r \frac{\rho_o}{g} N_o^2 (D - h) - w_r (D - h - H) \frac{\rho_o}{g} N_o^2 \right] \]

\[ = (h + \frac{H}{2}) \frac{\partial \rho}{\partial t} - w_e \Delta \rho + w_r \frac{H \rho_o}{2 g} N_o^2, \]  

(4.15)

where \( \tilde{\rho} \equiv \rho + \Delta \rho \) is the density at the bottom of the transition layer. This gives for the time rate of change of the mean mixed-layer density:

\[ \frac{\partial \rho}{\partial t} = \frac{-\frac{e_o}{g} B_o + w_e \Delta \rho - w_r \frac{H}{2} \frac{e_o}{g} N_o^2}{h + \frac{H}{2}} \frac{2h}{h} = \frac{-\frac{e_o}{g} B_o + \tilde{\rho}' w'|_{-h}}{h}, \]  

(4.16)

from which we can deduce the turbulent buoyancy flux at the interface between the mixed-layer and the transition layer which we call \( \alpha_\rho \):

\[ \alpha_\rho \equiv \frac{g}{\rho_o} \tilde{\rho}' w'|_{-h} = B_o \frac{H}{2h + H} + \left( w_e g \frac{\Delta \rho}{\rho_o} - w_r \frac{H}{2} N_o^2 \right) \frac{2h}{2h + H} \]  

(4.17)

Analogously, the time rate of change of the mean mixed-layer momentum is

\[ \frac{\partial \tilde{u}}{\partial t} = \frac{\frac{\tilde{\tau}_o}{\rho_o} - w_e \Delta \tilde{u}}{h + \frac{H}{2}} = \frac{\frac{\tilde{\tau}_o}{\rho_o} + \tilde{u}' w'|_{-h}}{h}, \]  

(4.18)

with

\[ \alpha_u \equiv \tilde{u}' w'|_{-h} = -\frac{\tilde{\tau}_o}{\rho_o} \frac{H}{2h + H} - w_e \Delta \tilde{u} \frac{2h}{2h + H} \]  

(4.19)

We can now infer the complete profiles of the turbulent fluxes as

\[ \frac{g}{\rho_o} \tilde{\rho}' w' = \begin{cases} 
B_o + \frac{\tilde{\tau}_o}{h} (B_o - \alpha_\rho) & \text{for} \quad 0 \geq z > -h \\
\alpha \rho + \beta_\rho \zeta + \gamma_\rho \zeta^2 & \text{for} \quad -h \geq z > -h - H \\
0 & \text{for} \quad z \leq -h - H \end{cases} \]  

(4.20)

with

\[ \alpha_\rho = B_o \frac{H}{2h + H} + \left( w_e g \frac{\Delta \rho}{\rho_o} - w_r \frac{H}{2} N_o^2 \right) \frac{2h}{2h + H} \]

\[ \beta_\rho = \frac{-2 \alpha_\rho}{H} + w_r \left( \frac{g}{H} \frac{\Delta \rho}{\rho_o} - N_o^2 \right) \]

\[ \gamma_\rho = \frac{\alpha_\rho}{H^2} - \frac{w_r}{H} \left( \frac{g}{H} \frac{\Delta \rho}{\rho_o} - N_o^2 \right) \]

\[ \zeta \equiv -h - z \]  

(4.21)
and

\[ \overline{u'w'} = \begin{cases} 
-\frac{z_e}{\rho_o} + \frac{z}{h} \left( -\frac{z_e}{\rho_o} - \alpha_u \right) & \text{for } 0 \geq z > -h \\
\alpha_u + \beta_u \zeta + \gamma_u \zeta^2 & \text{for } -h \geq z > -h - H \\
0 & \text{for } z \leq -h - H,
\end{cases} \quad (4.22) \]

with

\[ \alpha_u = -\frac{\overline{\tau_0}}{\rho_o} \frac{H}{2h + H} - w_c \Delta \overline{u} \frac{2h}{2h + H} \]

\[ \beta_u \equiv -\frac{2 \alpha_u}{H} - w_r \frac{\Delta \overline{u}}{H} \]

\[ \gamma_u \equiv \frac{\alpha_u}{H^2} + w_r \frac{\Delta \overline{u}}{H^2} \quad (4.23) \]

Next, we derive the Dynamic Criticality Condition which links the vertical velocity of the bottom of the transition layer to the vertical velocity of its top. Let \( R_c \) be the Richardson number of the transition layer \( \left( \frac{1}{4} < R_c < \frac{1}{2} \right) \) and \( \Delta s \) the shear across the transition layer \( \left( \Delta s \equiv \sqrt{(\Delta u)^2 + (\Delta v)^2} \right) \). Then the depth of the transition layer is

\[ H = \frac{(\Delta s)^2}{g \Delta \rho} R_c. \quad (4.24) \]

The time derivative of (4.24) gives

\[ \frac{\partial H}{\partial t} = R_c \left[ \frac{2 \Delta s}{g} \frac{\partial \Delta s}{\partial t} \frac{\partial \Delta s}{\partial t} - \frac{(\Delta s)^2}{g} \frac{\partial \Delta \rho}{\partial t} \right] \]

\[ = R_c \left[ \frac{2 \Delta u}{g} \frac{\partial u}{\partial t} + 2 \Delta v \frac{\partial v}{\partial t} - \frac{(\Delta s)^2}{g} \frac{\partial \rho}{\partial t} \right] \]

\[ = w_r - w. \quad (4.25) \]

We now use (4.16) and (4.18) to eliminate the time derivatives and rearrange the terms to yield the Dynamic Criticality Condition:

\[ w_r = w \frac{h}{h + H} \left( \frac{\Delta uu^2 + \Delta vv^2}{(\Delta s)^2} - \frac{B_o}{g \Delta \rho} \right) \frac{H}{h + H} \right) \]

\[ \quad (4.26) \]
where $u_*$ and $v_*$ are the $x$ and $y$-component of the surface friction velocity, respectively, and $\chi$ is a non-dimensional stability parameter which compares the static stability of the transition layer to that below the transition layer:

$$
\chi \equiv \frac{\frac{\Delta \rho}{H}}{\frac{\Delta \rho}{H} - \frac{\partial \rho}{\partial z}} = \frac{\Delta \rho}{H} - \frac{\rho_0}{g} N_0^2.
$$

(4.27)

Since the static stability below the transition layer must be less than the static stability in the transition layer, $\chi$ must be in the range $\chi \in [0.5; 1]$.

We now turn to the budget of the turbulent kinetic energy in the mixed-layer. Let us consider the general TKE equation for a horizontally homogeneous fluid (see e.g. Sorobjan, 1989):

$$
\frac{\partial}{\partial t} \text{TKE} = -\overline{u'w'} \frac{\partial \overline{u}}{\partial z} - \frac{g}{\rho_o} \overline{p'w'} - \frac{\partial}{\partial z} \left( \overline{TKE'w'} + \frac{1}{\rho_o} \overline{p'w'} \right) - \epsilon
$$

The left hand side is equal to zero under the assumption of steadiness. We integrate the right hand side over the depth of the mixed-layer to arrive at the mixed-layer mean TKE budget. The first two terms on the right hand side represent shear and buoyancy generation of turbulence, respectively, and can be integrated directly since we know the turbulent flux profiles. The third term represent the combined effect of the turbulent transport of TKE and pressure correlations and reduces to a surface and a bottom term upon integration. Following the dimensional argument of Turner and Kraus (1967), we parameterize the surface term as proportional to the third power of the surface friction velocity while we neglect the bottom term based on scaling arguments for the turbulent flux of TKE. Physically the bottom term represents the effect of downward radiating internal gravity waves, a leak of TKE and momentum from the mixed-layer. The dissipation term, finally, is typically either neglected or parameterized as proportional to the third power of the turbulence velocity scale which turns out to be small in a scale analysis unless the constant of proportionality is chosen to be much larger than unity. We therefore neglect the dissipation term in the present first version of the CTL entrainment parameterization, while we plan to include dissipation in a future version. The integrated steady-state TKE budget then becomes

$$
0 = \mu_s^3 - B_o \frac{h(h + H)}{2h + H} - \left[ w + w_r \left( \frac{2\chi - 1}{\chi} \right) \right] \frac{h}{4h + 2H} \frac{\Delta \rho}{\rho_o} h,
$$

(4.28)
where \( s_* \) is the total surface friction velocity \( (s_0^2 \equiv u_*^2 + v_*^2) \). Using the Dynamic Criticality Condition (4.26), we can now solve for \( w_e \) and \( w_T \):

\[
\begin{align*}
    w_e &= \frac{\mu s_0^3}{c^2} \frac{2h + H}{h} \frac{(\chi + 1)h + H}{2\chi h + H} - \frac{B_0(2h + H)}{2c^2} \left( \frac{H}{2h + H} + \frac{(\chi + 1)h + H}{2\chi h + H} \right) \\
    &+ \frac{\Delta uu_*^2 + \Delta vv_*^2}{(\Delta s)^2} \frac{(1 - \chi)H}{2\chi h + H} \\
    w_T &= \left( \frac{\mu s_0^3}{c^2} \frac{2h + H}{h} - \frac{B_0(2h + H)}{2c^2} + \frac{\Delta uu_*^2 + \Delta vv_*^2}{(\Delta s)^2} \frac{H}{h} \right) \frac{2\chi h}{2\chi h + H}.
\end{align*}
\]

(4.29) \hspace{1cm} (4.30)

where \( c^2 \) is the square of the phase speed of internal gravity waves \( c^2 \equiv g \Delta \rho h \). This set of equations for \( w_e \) and \( w_T \) is the CTL entrainment law. It contains two non-dimensional constants, \( \mu \) and \( R_c \) (hidden in \( H \)), the values of which have to be determined empirically. We have used values of \( R_c = \frac{1}{4} \) and \( \mu = 1.6 \) in all experiments unless noted otherwise. The sensitivity to the choice of these two parameters will be addressed at the end of this chapter. Before we discuss the characteristics of the CTL entrainment formulation in the next section, we briefly outline the procedure by which the above equations can be embedded into a standard mixed-layer model.

The problem of implementing the CTL entrainment formulation in a standard mixed-layer is to make the CTL model geometry compatible with the mixed-layer model's geometry. Fig. 4.1 illustrates the differences: The mixed-layer model has a well-mixed layer of depth \( h_1 \) on top of a stably stratified layer of depth \( h_2 \) and buoyancy frequency \( N_o \). The CTL model, in contrast, features an additional transition layer centered about \( z = -h_1 \) and a well-mixed layer of depth \( h \leq h_1 \). To apply the CTL entrainment law we first need to diagnose the transition layer depth \( H \). Let \( \hat{\rho} \) be the density at \( z = -h_1 \) and \( \rho_1 \) and \( \rho_2 \) the mean densities of layer 1 and 2, respectively. Then the transition layer depth follows as

\[
H = \frac{\rho_1 - \hat{\rho}}{\rho_2 - \hat{\rho}} \frac{h_2}{2} + \sqrt{\left( \frac{\hat{\rho} - \rho_1}{\rho_2 - \hat{\rho}} \frac{h_2}{2} \right)^2 + \frac{(\Delta s)^2 R_c}{g \frac{\rho_2 - \hat{\rho}}{\rho_2 - \hat{\rho}} h_2}},
\]

(4.31)

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and the CTL mixed-layer depth is given as
\[ h = h_1 - \frac{H}{2}. \] (4.32)

The density drop across the transition layer is
\[ \Delta \rho = \hat{\rho} - \rho_1 + (\rho_2 - \hat{\rho}) \frac{H}{h_2}, \] (4.33)
and \( \chi \) can be diagnosed from equation (4.27). Thus the right hand sides of (4.29) and (4.30) are now given as functions of the mixed-layer model variables.

### 4.2.2 Characteristics of the CTL Entrainment Law

In this section, we discuss the behavior of the CTL entrainment law for some interesting limiting cases. Since the CTL entrainment law takes several different sources of turbulence into account it is instructive to consider each of these sources separately even though they may not be independent in nature.

- \( B_o = u_* = v_* = 0 \) In this limit, turbulence is produced only at the ocean surface by the action of the wind and breaking surface waves. If we further assume that the ocean below the transition layer is neutrally stratified \( (\chi = 1) \) the CTL entrainment law yields

\[
\begin{align*}
    w & = 2 \frac{\mu s^3_*}{c^2} \left(1 + \frac{H}{h}\right) \\
    w_e & = 2 \frac{\mu s^3_*}{c^2} \left(1 + \frac{H}{2h}\right) \\
    w_r & = 2 \frac{\mu s^3_*}{c^2}.
\end{align*}
\]

This is a modified version of the Kraus-Turner entrainment law. In the absence of shear, we recover Kraus and Turner's law with \( w = w_e = w_r \) while we find slightly enhanced entrainment and a thinning transition layer in the presence of shear.

- \( u_* = v_* = s_* = 0 \) This case corresponds to a convectively driven boundary layer. Let us again consider the special case of a fluid system consisting of two homogeneous layers \( (\chi = 1) \). Then we get
\[ w = -\frac{B_0}{c^2} (h + 2H) \]
\[ w_e = -\frac{B_0}{c^2} (h + H) \]
\[ w_T = -\frac{B_0}{c^2} h. \]

This is a slight modification of Lilly’s maximum entrainment case with shear acting to increase the entrainment velocity while the entrainment acts to thin the transition layer. In the absence of shear the three entrainment velocities are again identical (as they must be since \( H = 0 \) in that case). As pointed out by Tennekes (1973), observations are much closer to Lilly’s minimum entrainment case than to his maximum entrainment case due to the effect of dissipation of turbulence. We see that dissipation of turbulence is an order one process in the case of buoyancy generated turbulence. Since the contribution to turbulence by surface buoyancy fluxes is negligible in the case of the coupled hurricane-ocean system we proceed to neglect dissipation for the time being.

This limiting case as well as the previous one produce sharp entrainment interfaces in the absence of mean shear. This characteristic of the CTL entrainment law was one of the great surprises of the early lab experiments on entrainment (e.g. Rouse and Dodu, 1955; Turner, 1968) and is one of the few well established aspects of turbulent entrainment.

\( s_* = B_0 = 0 \) The third limit is that of entirely shear generated turbulence. The entrainment law reduces to

\[ w = \frac{\Delta uu^2 + \Delta vu^2}{(\Delta s)^2} \frac{(1 - 2\chi)2H}{2\chi h + H} \]
\[ w_e = \frac{\Delta uu^2 + \Delta vu^2}{(\Delta s)^2} \frac{(1 - \chi)H}{2\chi h + H} \]
\[ w_T = \frac{\Delta uu^2 + \Delta vu^2}{(\Delta s)^2} \frac{2\chi H}{2\chi h + H}. \]
The only structural up-down asymmetry in the system now is a possible stratification of the ocean below the transition layer. In the absence of such a stratification ($\chi = 1$) the transition layer just widens symmetrically in response to increased shear such as to keep its Richardson number critical; $w_e$ is zero and the mixed-layer density remains unchanged while the transition layer 'eats' its way into the mixed-layer. This makes intuitive sense because of the symmetry of the problem. If the ocean is stratified below the transition layer entrainment is more rapid at the bottom of the transition layer than at its top and we find positive net entrainment ($w_e > 0$). Seemingly counterintuitive, the stratification actually eases entrainment because of the smaller buoyancy fluxes required to go from a pre-existing stratification to the necessary transition layer stratification than to start with an unstratified fluid. The limiting case in which the buoyancy frequency below the transition layer equals that in the transition layer clearly demonstrates this point.

Particularly the last limiting case nicely illustrates the physical mechanism of entrainment in the CTL formulation: If the surface momentum input acts to increase the shear across the transition layer, Kelvin-Helmholtz instability rapidly adjusts the transition layer to criticality in a more or less symmetric fashion depending upon the stratification below the transition layer. Superimposed on this fast adjustment to increases in shear is an entrainment mechanism much like that described by Turner and Kraus (1967) where TKE generation at the surface of the ocean is balanced by TKE conversion into mean potential energy by entrainment.

4.2.3 Test Runs with the CTL Entrainment Law

In this section, we describe a number of experiments we have conducted with three widely used entrainment parameterizations and with the CTL entrainment law. The three standard parameterizations are

- the Kraus/Turner-Kato/Phillips parameterization (KTKP),
- the Pollard/Rhines/Thompson-Price parameterization (PRTP), and
- the Deardorff parameterization (DD).

The goal of the experiments was to investigate what the differences in behavior between the various mixed-layer entrainment parameterizations are and how big
these differences are under typical hurricane conditions. The CTL model's sensitivity to the two non-dimensional parameters, $R_c$ and $\mu$, was also tested. In addition, we wanted to test whether we could rank the different entrainment laws' performances based on some of the available ocean data.

**Idealized Hurricane Forcing**

To first order, a column of ocean experiences a moving hurricane as a strong surface wind stress which rotates either clockwise or counterclockwise depending on which side of the hurricane track the column lies on. Since the clockwise rotation of the wind stress on the right side of the track corresponds to an in-phase forcing of inertial currents while the opposite is true on the left side of the track, the current response and subsequently the SST response is much stronger to the right of the track than to the left. Thus the two sides of the track represent somewhat different entrainment regimes.

For a simple test of the behavior of the different entrainment laws on both sides of the track, we force an initially quiescent column of ocean with a zonal wind stress of constant amplitude ($u_* = .06 \, ms^{-1}$). In one experiment the stress vector rotates

![Graphs showing temperature and buoyancy frequency profiles.](image)

**Figure 4-2**: Initial ocean stratification: The left panel shows the profile of temperature and the right panel shows the corresponding buoyancy frequency (salinity is assumed constant)
clockwise to resemble the right side track while it rotates counterclockwise in an otherwise identical experiment. The rotation rate is set to the inverse of the local inertial period such as to give perfectly out-of-phase and in-phase forcing on the left and right sides, respectively. The ocean initially features a mixed-layer of 30 m depth above a density profile typical for the summertime Gulf of Mexico (Fig. 4-2). Surface buoyancy fluxes are set to zero in both sets of experiments. Fig. 4-3 shows the evolution of the zonal and meridional currents in the mixed-layer for the different entrainment parameterizations. On the left side of the track where the forcing is exactly out-of-phase with the Coriolis acceleration, no meridional currents develop and the zonal currents go back to zero after half an inertial period. Maximum currents are of order(1 m s\(^{-1}\)). On the right side of the track, in contrast, strong inertial oscillations develop with maximum currents of more than 2 m s\(^{-1}\). The resulting shear across the transition layer is shown in Fig. 4-3. While the shear increases monotonically with time on the right side of the hurricane track, it reaches a maximum after a quarter inertial period and then goes back to zero on the left of the track. One striking aspect of the current response is the similarity of the results from physically very different entrainment laws. Only the KTKP currents differ significantly from those of the three other entrainment formulations, suggesting that the current response is not a very sensitive test field for comparison of the performance of entrainment parameterizations.

The actual entrainment velocity is shown in Fig. 4-4. It is apparent that the PRTP law and the DD law behave very similarly, as mentioned much earlier. The PRTP formulation starts entrainment only after the mixed-layer has been accelerated enough to make the bulk Richardson number subcritical, and it shuts entrainment off as soon as \( R_{bu} \) exceeds its critical value. On the left side of the track the shear actually starts to decrease after a quarter inertial period, which means that \( R_{bu} \) starts to increase at that point. Thereafter PRTP ceases to entrain for the rest of the experiment, even though the surface stress is constant throughout the experiment. Qualitatively, the DD law behaves the same way, yet it turns the entrainment smoothly on and off. The KTKP law and the CTL law show fairly similar behavior as might be expected from the discussion at the end of the previous section (4.2.2). In the weak shear case (left side of track), the KTKP and CTL entrainment velocities are very similar while the shear on the right side of the track causes the CTL law to predict roughly twice as rapid entrainment as does the KTKP law. The evolution of the mixed-layer depth, which in the absence of
Figure 4-3: Evolution of zonal and meridional currents. The left panels are for a counterclockwise rotation of the surface stress vector corresponding to the left side of the hurricane track, and the right panels are for a clockwise rotation of the surface stress vector corresponding to the right side of the track. The top panels show the zonal and meridional currents, and the bottom panels show the total shear across the transition layer.
mean vertical motions is a direct integral measure of the entrainment velocity, is shown in Fig. 4-5. The three curves for the CTL experiments correspond to the top, the center, and the bottom of the transition layer, respectively. The center of the transition layer most directly compares to the other mixed-layer depths since its vertical velocity is equal to \( w_c \). The depth of the transition layer is a reflection of both the shear and the buoyancy drop across the transition layer. On the left side of the track the transition layer temporarily vanishes when the shear goes to zero, while it deepens monotonically on the right side of the track. As the shear keeps increasing on the right side, the PRTP and the DD law predict very deep mixed-layers with depths close to the combined depth of the mixed-layer and the transition layer in the CTL prediction while the CTL mixed-layer depth is much closer to that produced by the KTKP law. Fig. 4-5 finally shows the SST decrease in each of the experiments. The SST response in the PRTP and DD experiments is highly asymmetric across the track with roughly double the amplitude on the right side compared to the left side at the end of the experiments. The KTKP law, in contrast, produces by design a symmetric SST decrease. The CTL law also predicts a nearly symmetric SST response which raises questions about its validity in light of the strong asymmetry of observed SST decreases. We will address this issue shortly.
Figure 4-5: Evolution of the mixed-layer depths (top panels) and of the SST change (bottom panels). In the top panels, the three solid curves for the CTL law mark the top and the bottom of the transition layer as well as its center, the equivalent mixed-layer depth. See also Fig. 4-3.
Overall, the physically very different entrainment laws predict qualitatively rather similar oceanic responses. The SST response seems to highlight the differences best and shall be used next in connection with observational data from hurricane Gloria ’85.

Comparison to Data from Gloria ’85

Since observed SST decreases are strongly biased to the right of the hurricane track and since the different entrainment parameterizations produce very different degrees of asymmetry in the SST change, model simulations with observed winds and full ocean dynamics might be used to rank the performances of the different entrainment laws. Crucial for such a comparison is the accurate knowledge of the SST decrease which actually occurred in nature. But in-situ measurements

---

Map of the SST change deduced from satellite infrared images. Contour interval is 1°C. The solid black line is the track of Gloria; dots are 12h apart. The first dot corresponds to Gloria’s position at 0000 GMT 25 September 1985. The numbers to the left of the dots denote the minimum central pressure in hPa. The dashed diagonal line indicates where the SST section of Fig. 4-7 was taken. The stippled area at the lower right was obscured by clouds.

(From Cornillon et al., 1987)

Figure 4-6: SST cooling due to Gloria ’85

of the upper ocean’s thermal structure under hurricane conditions can only be taken with the help of air-deployed expendable bathythermographs (AXBT) and air-deployed expendable current profilers (AXCP) dropped from research aircraft.
This technique is very expensive and the probes have a high failure rate, particularly if deployed into hurricane-force winds (e.g. Sanford et al., 1987; Shay et al., 1992). Furthermore, the spatial resolution of AXBT data is usually fairly limited. In addition, a pre-hurricane survey of the upper ocean structure is needed to calculate the change in SST, which necessitates accurate predictions of the track of the hurricane. The use of satellite derived SSTs, which potentially provide high temporal and spatial resolution, tends to be limited by cloud cover particularly in the vicinity of hurricanes.

An exceptionally cloud-free data set of SSTs before and after the passage of hurricane Gloria '85 was compiled by Cornillon et al. (1987). Fig. 4-6 is taken from their paper and shows the magnitude and distribution of SST cooling. The bias of the SST cooling to the right of the track is clearly visible. We can also see

![Figure 4-7: Across-track sections of SST decrease in the wake of hurricane Gloria. The thick solid line marks the observations of Cornillon et al. (dashed line in Fig. 4-6) while the other curves represent the model simulations.](image-url)
the importance of the sub-surface thermal structure of the ocean: The biggest SST drops are observed in the slope waters north of the Gulf Stream where the mixed-layer is typically only 20 m deep, while the still more intense winds of Gloria over the Sargasso Sea (with a typically 40 m deep mixed-layer) caused a much weaker SST cooling there. Following Price et al. (1994), we used observed winds from hurricane Gloria to force the ocean model described in section 2.2 with the four different entrainment parameterizations we compared above. The ocean model is initialized with a density profile typical for the Sargasso Sea and then integrated until the ocean reaches a steady-state in a reference frame moving with the storm center. For details of the wind field and the ocean initial conditions see Price et al. (1994). Fig. 4-7 shows across-track sections of the SST decrease well behind the storm from our experiments. The thick solid line marks the satellite derived cross-section of Cornillon et al., shown as a dashed line across Gloria's track in Fig. 4-6. The most surprising aspect of Fig. 4-7 is the overall similarity between the different model results and the observations. The KTKP formulation underestimates the SST decrease on the right side of the track while the PRTP, the DD, and the CTL parameterizations show overall rather good agreement with the observed data. The CTL formulation produces a larger SST reduction on the left side of the track than the PRTP and the DD formulations but it is difficult to decide which of the entrainment formulations performs better, mainly for two reasons: First, because the observed SSTs have an absolute accuracy of about 0.5°C and a relative error (point-to-point variation) of about 0.2°C. Second, because the model is initialized with a horizontally uniform ocean, which may be why none of the model runs reproduces the local minimum in SST decrease roughly 80 km to the left of the track. It could be argued with reference to Fig. 4-6 that this minimum is a local effect and would not appear on a cross-section taken further back (south-east) across Gloria's track. We rather take a more cautious position and call the results inconclusive as far as a ranking of the different entrainment schemes is concerned. Yet one important aspect of the entrainment was demonstrated by these experiments: The ocean's three-dimensional dynamic response significantly alters the one-dimensional picture of entrainment presented in the previous section. For example, upwelling and downwelling patterns can create a strong cross-track asymmetry in the SST response even if the entrainment is forced by a near-symmetric wind stress pattern as is the case for the KTKP law.
The Choice of $R_c$ and $\mu$

As mentioned earlier, the theory of Kelvin-Helmholtz instability yields an $R_c$ of order $\frac{1}{4}$. Slightly smaller values can occur under strong shear forcing (e.g. Kranenburg, 1984) while slightly larger values still less than $\frac{1}{2}$ would be expected under weak or intermittent forcing (e.g. Price et al., 1994). To assess the sensitivity of the entrainment law to the choice of $R_c$, we have run one-dimensional experiments with the idealized hurricane forcing described above but with different values of $R_c$. The left panel of Fig. 4.2.3 shows the evolution of the SST decrease for $R_c = (0.2, 0.25, 0.3, 0.35, 0.4)$. Smaller critical Richardson numbers result in less entrainment and therefore less SST decrease, but over the range of physically meaningful

![SST Change vs Time for different Rc values](image)

Figure 4-8: Sensitivity to the choice of $R_c$ for $\mu = 1.6$ (left panel) and to the choice of $\mu$ for $R_c = \frac{1}{4}$ (right panel).

values the resulting SSTs differ by less than $0.3^\circ$C. Also compare Fig. 4.2.3 to the bottom right panel of Fig. 4-5: The CTL law remains distinct from the other three entrainment parameterizations for any reasonable choice of $R_c$. We choose $R_c = \frac{1}{4}$ for lack of conclusive observational evidence in favor of another value.

While $R_c$ is constrained to a small range of sensible values by the physics of KH instability, $\mu$ is just the constant of proportionality from a dimensional argument and thus completely unconstrained, i.e. a true tuning parameter. The right panel
of Fig. 4.2.3 shows the evolution of the SST decrease for $\mu = (1.0, 1.6, 2.5, 5.0)$ to give the reader a feel for the sensitivity of the SST response to the choice of $\mu$. We did not attempt to tune $\mu$ other than to choose a value that gives reasonable agreement in the Gloria simulations (Fig. 4-7), which is $\mu = 1.6$. A proper tuning procedure would involve a much larger observational data base and will be the subject of future work.

4.3 Summary

We first reviewed the literature on the entrainment in stratified shear flows with emphasis on work directly relevant to the entrainment at the base of the oceanic mixed-layer. Even though a large number of laboratory experiments on entrainment was conducted over the last forty years there is still no consensus on how the entrainment process can be parameterized or even on the basic physics of entrainment. The fundamental questions in TKE closures is that of the dominant forcing velocity scale. The surface friction velocity, $u_*$, represents the direct input of turbulence at the ocean surface while the shear at the base of the mixed-layer, $\Delta u$, leads to the generation of turbulence as a result of Kelvin-Helmholtz instability. As pointed out by several authors, both sources of turbulence need to be included in a general entrainment law to ensure its validity over a wide range of forcing conditions. We therefore derived a new entrainment law which considers both sources of turbulence. In the new formulation, the shear at the base of the mixed-layer is assumed to locally adjust the fluid to KH criticality on a very short time scale. A KH-critical transition layer is therefore included in the model geometry. The surface generated turbulence acts to entrain across this transition layer much like in a $u_*$-scaled entrainment law. We showed that the new formulation behaves in a physically reasonable way in the limiting cases in which the different sources of turbulence are considered in isolation. We then tested the new formulation under very idealized hurricane-like forcing and compared its behavior to that of the most commonly used entrainment laws. The different formulations displayed vastly different degrees of across-track asymmetry in SST. We therefore choose to run a more realistic test case for which the different entrainment laws were implemented in the same mixed-layer ocean model we already used in the first part of this thesis work. We specified a surface wind field resembling that of hurricane Gloria and compared a cross-track section of the observed SST changes to the corresponding
model results. Our new formulation matched the observed field very nicely but, based on this test solely, we cannot reject the popular shear-based bulk Richardson number criteria.
Chapter 5

Conclusions

In this thesis, we investigated the effect of the ocean on hurricane intensity. Previous studies show that observed hurricane intensities and sea surface temperatures are practically uncorrelated, which was often interpreted as proving that the effect of the ocean on hurricane intensity is negligible in comparison to the role of the ambient atmospheric conditions and the internal hurricane dynamics. While the importance of surface fluxes of sensible and latent heat from the ocean as the energy source for hurricanes has been recognized for many years, a consistent picture of the role of convection in maintaining a hurricane's circulation has only recently emerged. The key element was the recognition that an atmosphere in quasi-equilibrium is effectively convectively neutral. Hurricanes then rely entirely on a wind surface flux feedback (WISHE) for their energy supply, which makes them extremely sensitive to changes in SST in the source region of the deep convection close to the storm center. At the same time, observations of the upper ocean's thermal structure in the wake of hurricanes reveal dramatic reductions in SST of typically several °C. The potential for a negative feedback between hurricane and ocean is apparent.

We have investigated this SST feedback in a numerical model of the coupled hurricane-ocean system. Two features of the model hurricane's behavior are key to this investigation: First, the model maintains a hurricane vortex in an ambient moist neutral atmosphere and thus derives its energy entirely from a WISHE instability. This is crucial to assure that the model's sensitivity to changes in SST is realistic. Second, the model reaches a steady-state, which is essential for an objective assessment of the magnitude of the feedback effect. Both these features are
unique to this coupled model study and fundamentally distinguish it from previous investigations. Our strategy was to retain only the essential dynamics in the coupled model. This facilitates the interpretation of the model's behavior and makes it computationally cheap enough to be run hundreds of times, an important aspect for a systematic investigation of the parameter dependence of the SST feedback factor. We ran a set of 1440 coupled experiments over the entire range of realistic conditions, obtaining feedback factors as low as −70%. This already demonstrates the paramount importance of the SST feedback effect. Using an objective fitting technique, we extracted an analytic expression for the feedback factor as a function of six parameters from the model data. Four of the parameters were found to be of roughly equal importance for the amplitude of the SST feedback, namely the mixed-layer depth, the hurricane translation speed, the hurricane size, and the hurricane's potential intensity. The Coriolis parameter and the thermocline stratification also affect the feedback factor but turned out to be less important than the above four parameters. The relative importance of all six parameters can be understood from a simple scaling argument. Since the intensity of a hurricane over an ocean of uniform and fixed SST is known in terms of a few ambient parameters, the derived formula for the amplitude of the feedback factor effectively is a formula for the intensity of a hurricane.

Some limitations of the present study shall be mentioned here as well. We have intentionally kept a narrow focus on the role of the SST feedback on hurricane intensity. Yet there are many other effects which play a potentially important role in setting a hurricane's intensity, such as ambient tropospheric shear, interaction with upper tropospheric disturbances from the mid-latitudes, and structural changes like eyewall cycles, to name a few only. Furthermore, the oceanic response is highly asymmetric about the storm center and may thus affect the storm intensity by reducing its degree of axisymmetry, an effect which is suppressed in the strictly axisymmetric hurricane model we used. Another limitation is that of the uncertainty in the entrainment parameterization.

This uncertainty was addressed in the last part of this thesis, in which we developed a new parameterization for the turbulent entrainment at the base of the oceanic mixed-layer, preceded by a critical review of the most popular entrainment laws. The new formulation implicitly includes a transition layer which merges the mixed-layer with the upper thermocline. The key assumption of the new formulation is that the transition layer is always critical to Kelvin-Helmholtz
instability, thus the name critical transition layer (CTL) closure. A new picture of the entrainment process emerges from this formulation, in which shear acts to widen the transition layer while the surface induced turbulence acts to deepen the mixed-layer. The shear contribution to the entrainment is therefore indirect only, primarily through the thinning of the mixed-layer. The CTL parameterization’s behavior is physically intuitive and compares well to observations as preliminary tests indicate. It is also easy to implement and efficient to run in standard mixed-layer models.

The appeal of the CTL closure lies in its clear physical concept, and while the present results are very promising much further testing is needed. The inclusion of dissipation of TKE is desirable as well and will extend the range of conditions under which the closure is valid.
Appendix A

Derivation of the Sawyer-Eliassen Equation

Equations (2.18), (2.13), (2.30), and (2.10) form a set of four equations in the four unknowns \( r_L^2 \), \( r_U^2 \), \( s^* \), and \( \psi_m \), or equally in the four unknowns \( r_o^2 \), \( r_L^2 \), \( \beta \), and \( \psi_m \). Below, we solve this set of equations for \( \psi_m \).

We start with the definition of the pseudo-volume of the lower troposphere

\[
\begin{align*}
    r_L^2 & \overset{(2.19)}{=} \int_{p_m}^{p_o} r^2 dp \\
    & \approx \int_{T_m}^{T_o} r^2 \frac{\delta p_L}{\delta T_L} dT \\
    & \overset{(2.10)}{=} \kappa_L \int_{T_m}^{T_o} \frac{dT}{r_o^2 - \beta(T_o - T)} \\
    & = \frac{\kappa_L}{\beta} \left[ \ln \left( \frac{1}{r_o^2 - \beta(T_o - T)} \right) \right]_{T_m}^{T_o} \\
    & = \frac{\kappa_L}{\beta} \ln \left[ \frac{\frac{1}{r_o^2}}{\frac{1}{r_o^2} - \beta \delta T_L} \right] \\
    & = -\frac{\kappa_L}{\beta} \ln \left[ 1 - \frac{r_o^2 \beta \delta T_L}{\Omega_L} \right]. \quad (A.1)
\end{align*}
\]
Analogously, we can write for the pseudo-volume of the upper tropospheric layer

\[ r^2_U \stackrel{(2.14)}{=} \int_{p_t}^{p_m} r^2 dp \]

\[ \approx \int_{T_{\ell}}^{T_m} r^2 \frac{\delta p_U}{\delta T_U} dT \]

\[ = \kappa_U \int_{T_{\ell}}^{T_m} \frac{dT}{\frac{1}{r_o^2} - \beta (T_o - T)} \]

\[ = \frac{\kappa_U}{\beta} \ln \left( \frac{1}{\frac{1}{r_o^2} - \beta (T_o - T)} \right) \bigg|_{T_{\ell}}^{T_m} \]

\[ = \frac{\kappa_U}{\beta} \ln \left( \frac{\Omega_{L}}{\Omega_{\ell}} \right). \tag{A.2} \]

We next take the partial derivative with respect to time of (A.1)

\[ \frac{\partial}{\partial T} \left( r^2_L \right) \stackrel{(A.1)}{=} \frac{\kappa_L}{\beta^2} \frac{\partial}{\partial T} \ln \Omega_L + \frac{\kappa_L}{\beta} \frac{1}{\Omega_L} \left( \frac{\partial r_o^2}{\partial T} - \beta \frac{\delta T_L}{\Omega_L} + r_o^2 \frac{\partial}{\partial T} \frac{\delta T_L}{\Omega_L} \right) \]

\[ = \frac{\partial r_o^2}{\partial T} \left[ \frac{\delta p_L}{\Omega_L} \right] + \frac{\partial}{\partial T} \left[ \frac{\kappa_L}{\beta} \ln \Omega_L + \frac{\kappa_L}{\beta^2} \frac{1 - \Omega_L}{\Omega_L} \right] \]

\[ = \frac{\partial r_o^2}{\partial T} \left[ \frac{\delta p_L}{\Omega_L} \right] + \frac{\partial}{\partial T} \left[ - \frac{r_o^2}{\beta} + \frac{\delta p_L r_o^2}{\Omega_L \beta} \right] \]

\[ \stackrel{(2.18)}{=} 2 \left( \psi_o - \psi_m \right) - \delta p_L D_v. \tag{A.3} \]
The partial derivative with respect to time of (A.2) is

\[
\frac{\partial}{\partial \tau} \left( r_0^2 \right) = \frac{-\kappa_u \frac{\partial \beta}{\beta^2} \ln \left( \frac{\Omega_L}{\Omega} \right)}{\frac{\partial}{\partial \tau} \left( \frac{\Omega_L}{\Omega^2} \beta \Omega_L \right)} \left[ -\frac{\partial r_0^2}{\partial \tau} \beta \delta T_L - r_0^2 \frac{\partial \beta}{\partial \tau} \delta T_L \right] - \frac{\partial r_0^2}{\partial \tau} \left( \frac{\beta (\delta T_L + \delta T_U) - r_0^2 \frac{\partial \beta}{\partial \tau} (\delta T_L + \delta T_U)}{\Omega} \right)
\]

\[
= \frac{\partial r_0^2}{\partial \tau} \left[ -\kappa_u \frac{\delta T_L}{\Omega_L} + \kappa_u \frac{(\delta T_L + \delta T_U)}{\Omega} \right] + \frac{\partial \beta}{\partial \tau} \left[ -\frac{\kappa_u}{\beta^2} \ln \left( \frac{\Omega_L}{\Omega} \right) - \frac{\kappa_u}{\beta^2} \left( \frac{1 - \Omega_L}{\Omega_L} - \frac{1 - \Omega}{\Omega} \right) \right]
\]

\[
= \frac{\partial r_0^2}{\partial \tau} \left[ \frac{\delta p_u}{\Omega_L \Omega} \right] + \frac{\partial \beta}{\partial \tau} \left[ -\frac{\kappa_u}{\beta^2} \ln \left( \frac{\Omega_L}{\Omega} \right) + \frac{\delta p_u r_0^2}{\beta \Omega_L \Omega} \right].
\]

(2.13) \quad 2 \left( \psi_m - G \right) . \tag{A.4}

We next solve (A.3) for \( \frac{\delta p_L}{\Omega_L} \frac{\partial r_0^2}{\partial \tau} \):

\[
\frac{\delta p_L}{\Omega_L} \frac{\partial r_0^2}{\partial \tau} = 2 \left( \psi_0 - \psi_m \right) - \delta p_L D_v + \frac{\partial \beta}{\partial \tau} \left[ \frac{r_0^2}{\beta} - \frac{\delta p_L r_0^2}{\Omega_L \beta} \right] . \tag{A.5}
\]

Substitution of (A.5) into (A.4) gives

\[
2 \left( \psi_m - G \right) = \frac{2 \frac{\delta p_u}{\delta p_L} \left( \psi_0 - \psi_m \right) - \delta p_v D_v}{\Omega}
\]

\[
+ \frac{\partial \beta}{\partial \tau} \left[ \frac{\delta p_u}{\Omega \beta} - \frac{\kappa_u}{\beta^2} \ln \left( \frac{\Omega_L}{\Omega} \right) \right]. \tag{A.6}
\]

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The partial derivative with respect to time of (2.44) reads

\[
\frac{\partial \beta}{\partial \tau} = -\frac{2}{f^2 R^3} \frac{\partial}{\partial R} \left[ \frac{\partial s^*}{\partial \tau} \right]
\]

\[\approx\]

\[
= \frac{2}{f^2 R^3} \frac{\partial}{\partial R} \left[ \frac{\Gamma_{\text{dry}}}{\Gamma_{\text{moist}}} \right] \left( 1 - \sigma \right) \frac{\omega_a}{\partial P_m} \left[ \frac{\partial s_d}{\partial P_m} \right] + \text{rad} (s^* - s_i^*)
\]

\[\approx\]

\[
= \frac{2}{f^2 R^3} \frac{\partial}{\partial R} \left[ \frac{\Gamma_{\text{dry}}}{\Gamma_{\text{moist}}} \right] \left( -2 \frac{\partial \psi}{\partial r^2} \right) + \text{rad} (s^* - s_i^*)
\]

\[
= \frac{2}{f^2 R^3} \frac{\partial}{\partial R} \left[ Q \frac{\partial \psi_m}{\partial r^2} + \text{rad} (s^* - s_i^*) \right], \quad (A.7)
\]

with

\[
Q = -2 \frac{\Gamma_{\text{dry}}}{\Gamma_{\text{moist}}} \frac{\partial s_d}{\partial P_m}. \quad (A.8)
\]

Substitution of (A.7) into (A.6) yields

\[
2 (\psi_m - G) = \frac{2 \frac{\partial \psi}{\partial p_v} (\psi_v - \psi_m) - \delta p_v D_v}{\Omega}
\]

\[+ \left[ \frac{\delta p_v}{\delta p_L} \frac{r^2}{\Omega \beta} - \frac{\kappa_u}{\beta^2 \ln \left( \frac{\Omega_L}{\Omega} \right)} \right]
\]

\[\times \frac{2}{f^2 R^3} \frac{\partial}{\partial R} \left[ Q \frac{\partial \psi_m}{\partial r^2} + \text{rad} (s^* - s_i^*) \right]. \quad (A.9)
\]
Multiplication of (A.9) with $\frac{\Omega}{2}$ finally gives

\[
(p_m - G) \Omega = (p_b - p_m) \frac{\delta p_u}{\delta p_L} - \frac{D_v}{2} \delta p_u \\
+ \frac{1}{f^2 R^3} \left[ \frac{\delta p_u}{\delta p_L} \frac{r^2_L}{\beta} - \frac{\Omega \kappa_u}{\beta^2} \ln \left( \frac{\Omega L}{\Omega} \right) \right] \\
\times \frac{\partial}{\partial R} \left[ Q \frac{\partial p_m}{\partial r_m^2} + \text{rad} (s^*-s_i^*) \right], \tag{A.10}
\]

which can be rewritten as

\[
\left[ \Omega + \frac{\delta p_u}{\delta p_L} - \Lambda Q \frac{\partial}{\partial R} \frac{\partial}{\partial r_m^2} \right] p_m = G \Omega + p_b \frac{\delta p_u}{\delta p_L} - \frac{D_v}{2} \delta p_u \\
+ \Lambda \frac{\partial}{\partial R} \left[ \text{rad} (s^*-s_i^*) \right]. \tag{A.11}
\]

This is the final Sawyer-Eliassen equation.
References


——— and ————, 1984: Effect of the ocean-atmosphere interaction on the intensity of a moving tropical cyclone. *Izvestiya, Atmospheric and Oceanic Physica*, 20, 697-703.


