Essays in Auctions and Credit Markets
by
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Abstract

The first two chapters are studies of treasury bill auctions; each paper contains a model and empirical application to the Portuguese treasury bill auction. Chapter 1 examines the effect of reservation price on government revenue in treasury bill auctions in an underdeveloped financial market. First, I present an equilibrium model of a divisible good auction with a reservation price. Next, I extend the technique of Elyakime, Laffont, Loisel, and Vuong (1993) to this divisible good model: By inverting the equilibrium bidding strategy, we can “map backwards” from an observed sample of bids to the unobserved bidder valuations. The conditional distributions which enter the inverse strategy are estimated nonparametrically; in comparison to MLE, this technique is both computationally faster and more general. Given the constructed pseudo sample of valuations, the structural model is then used to estimate what seller revenue would have been under alternative reservation price rules. Finally, I apply the technique to the Portuguese data. My conjecture is that the government could benefit from a more “active” policy that makes greater use of observable ex-ante information on the value of the bills. Simulation results suggest, however, that available gains are minimal at best.

In auctions of government securities, bidders are permitted to enter multiple price-quantity bids. Chapter 2 examines how risk-averse bidders in a common-value auction use multiple bids to limit exposure to risk: A model demonstrates that the greater the uncertainty over the value of the good, the more bidders will spread their bids. Portuguese bidding data are used to verify that bidders submit a greater number of bids and spread their bids more widely as the risk of winner’s curse increases. In particular, panel regressions show that both measures of bid-spreading increase with the volatility of market interest rates and the number of bidders, and the first measure decreases with the total quantity of bills for sale.

Chapter 3 tests the search cost hypothesis for the credit card industry. Ausubel (1991) provides evidence of excess profits for credit card issuers; later research has questioned Ausubel’s methodology, but has generally supported the conclusion that issuing credit cards was much more profitable than other banking activities over 1982–1992. Due to Ausubel, the most prominent potential explanation of excess profits is costly consumer search. This paper develops and tests a model of search by consumers who are heterogeneous with respect to default risk and desired loan size. The larger the loan, the greater the incentive for the consumer to search for a low rate, so we should observe, on average, a negative relationship between a consumer’s credit card debt and her interest rate. The magnitude of this relationship is a function of credit-worthiness: the higher the default risk, the smaller
the range of rates for which the consumer qualifies, so the weaker the association between
loan size and interest rate.

These predictions are tested using household level data. I find no evidence that con-
sumers are sorted by observable default risk characteristics such as income and home-
ownership. I find only weak evidence that large borrowers search more intensively than
small borrowers: the relevant coefficient is of the predicted sign, but small in magnitude
and statistically insignificant. I conclude that the search cost hypothesis is unlikely to
explain high credit card rates.

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To my parents
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Chapter 1

Structural Analysis of Reservation Price Policy in a Treasury Bill Auction

In many countries, including the United States, auctions of government securities form the largest recurring primary market for any security. Government interest in minimizing its cost of financing debt has motivated a number of theoretical and empirical studies which examine the revenue consequences of alternative auction rules. These studies have focused mainly on the choice between uniform and discriminatory price rules, and so far have not yielded conclusive evidence.¹

A potentially important policy instrument for treasury auctions that has not received attention is the reservation rate, which is the highest rate at which the treasury will permit a bid to be awarded. One reason for this may be that treasury auctions are perceived as highly competitive in the major markets; auctions are generally fully subscribed, and differences between high and low bids are measured in basis points (see Cammack, 1991). Under these circumstances, we might expect an optimally chosen reservation rate to offer little gain over having no reservation rate at all.² Another reason for the lack of attention is the complexity of determining the theoretically optimal reservation price in any but the simplest auction settings.

¹In the US, this debate goes back at least as far as Friedman's (1959) proposal to switch to uniform price rules, and continues today in the recent decision to experiment with uniform price rules in auctions of treasury notes. Due to the complexity of the auctions in use, this policy debate has mostly been conducted by analogy to much simpler auction settings, which may be misleading. For empirical evidence in favor of uniform price rules, see Umlauf's (1993) study of the Mexican treasury bill auction and Tenorio's (1993) study of the Zambian foreign exchange auction. Simon (1993) finds the opposite in a study of the US Treasury's experiment with a uniform price format.

²Indeed, there is no explicit reservation rate in auctions of US treasury securities. Note, however, that the Treasury reserves the right to discard any bid it considers unreasonable.
Nonetheless, the reservation rate is likely to be an important policy instrument in treasury auctions in less developed financial markets. In such markets, total bidder demand is less predictable, so there may be insufficient demand for the securities at some auctions; when the probability of undersubscription is positive, a risk-neutral bidder will want to bid at an infinitely high interest rate. In the absence of a liquid secondary market, bidders may differ considerably in their valuations for the bills, so interest rate spreads between high and low bids may be quite high. There is likely to be only a small number of regular bidders, so the treasury may be concerned about bidder collusion at the auctions. Under these circumstances, the treasury must set a reservation rate to remove the risk of having to grant bills at an arbitrarily high interest rate. Policy makers will then want to know how to best use this instrument.

This paper will demonstrate how a structural model of a treasury bill auction may be used to estimate seller revenue and quantity sold under any alternative reservation price rule that the seller may wish to consider. I will apply the method to bidding data from Portuguese treasury bill auctions. The Portugal treasury bill auction is particularly appropriate for this study because the conditions discussed in the last paragraph are all present. First, there are a small number of large bidders in these auctions; six bidders account for over half of the total volume of bids, and 12 bidders account for 80%. This suggests that the auctioneer, the Banco de Portugal, needs to be concerned about the potential for bidder collusion. Second, the secondary market for Portuguese treasury bills is illiquid, and the nearest proxy for the true value of the bills tends to be highly volatile. Consequently, the spread between the highest and lowest interest rates bid in a single auction is usually high: the average spread is 69 basis points, and it is as high as 4.94 percentage points in one auction in my sample. Third, there is great variation from auction to auction in the ratio of total quantity of bids submitted to the issue ceiling, which I call the “coverage ratio.” As can be seen in Figure 1, the coverage ratio is often less than one (i.e., there is insufficient demand to absorb supply at any price), and at other times it is as high as eight.

The models presented in Section 2 focus particularly on the role of uncertainty in bid
Figure 1:

Coverage Ratio at Auctions

91 Day Bills

182 Day Bills
quantities; using a numerical example, I show that for an expected coverage ratio similar to that observed in the sample, expected seller revenue and expected quantity sold are highly sensitive to the choice of reservation rate. In the sample, the reservation rate is clearly an important determinant of auction results: in two-thirds of the auctions, the reservation rate is the highest winning rate, and the reservation rate is strictly binding nearly half the time. Figure 2 plots the reservation rate and highest winning rate at each auction.

I briefly survey related econometric studies in Section 1. Section 2 presents two equilibrium models of bidder strategy in auctions of divisible goods; in the first model, I assume that the reservation price is public knowledge, and in the second model, I assume it is hidden. These models may be seen as extensions of multi-unit auction models, which have been studied extensively in the literature, to the important but little studied divisible good case. Section 3 describes the institutional framework for the Portuguese treasury bill auctions and Section 4 describes the data. The estimation technique is developed in Section 5. I conjecture that a reservation price rule which makes use of observable information on bidder valuations for the bills should yield superior results for the seller, and estimate what seller utility would have been under alternative rules chosen from a class of easily implemented reservation price rules. Concluding remarks are in the last section.

1 Literature on estimation of structural auction models

This paper is part of a recent trend towards the use of structural models in empirical studies of auctions. The basic idea behind these studies is straightforward: Conditional upon an equilibrium model of bidding strategy, the observed distribution of bids reveals the unobserved distribution of interest, which is the distribution of bidder valuations in an independent private values (IPV) auction or the distribution of bidder signals in a common-value auction. The traditional reliance on reduced form estimation appears to be motivated partly by the computational difficulty of calculating the equilibrium bidding strategies in structural models. In the simplest auction settings (see, e.g., Model I in this paper), such calculation typically requires numerical integration. In slightly more complex settings (see
Figure 2:

Reservation Rate and Highest Winning Rate at Auction
Model II), the equilibrium bidding strategy cannot even be expressed in closed form.

Paarsch (1992) resolves these computational problems through careful choice of functional form assumptions on the underlying unobserved distribution (i.e., of bidder valuations or bidder signals, depending on the model under consideration). The chosen distributions give rise to equilibrium bidding strategies with computationally convenient forms. Paarsch applies his method to test between the IPV and common-value paradigms in auctions of tree planting contracts in British Columbia.

Laffont and Vuoung (1993) and Laffont, Ossard and Vuoung (1991) develop an alternative method based on simulated nonlinear least squares. As a Monte Carlo method, its primary advantage is that it avoids numerical integration of bidding strategies. Consequently, there is no need to restrict the choice of distributional assumptions as in Paarsch, and a wide range of potentially complex auction models can be studied. Laffont, Ossard and Vuoung use the SNLLS method to estimate the parameters for the conditional distribution of unobserved bidder valuations in a series of auctions of greenhouse eggplants.

In contrast to the above "direct" methods of estimation, Elyakime, Laffont, Loisel, and Vuoung (1993) (hereinafter ELLV) propose an "indirect" nonparametric method. ELLV consider a first-price auction of a single (indivisible) good to a group of bidders. They prove that, under certain assumptions on the auction mechanism, a policy of an announced reservation price yields higher expected utility to a risk neutral seller than a policy of a hidden reservation price. ELLV have bidding data from a series of auctions in which a hidden reservation price was employed. Using the equilibrium bidding strategy for such an auction, they use a nonparametric technique to "map back" from the observed set of bids to retrieve estimates of the underlying bidder valuation for each bid. This pseudo sample of valuations is used to estimate the conditional distribution of bidder valuations, which they show is well approximated by a log-normal specification. Finally, ELLV estimate the gain to a policy of announced reservation price by maximum likelihood, under the assumption that bidder valuations are log-normally distributed.
2 A Divisible Good Auction with Stochastic Demands

Theoretical understanding of treasury bill and other divisible good auctions has been hampered by the complexity of auction formats in common use. In a sealed-bid auction of, say, a painting, each bidder submits a single bid which specifies a price offer; the painting is awarded to the high bidder. In contrast, bidders in a treasury bill auction are competing for shares of a divisible good. Typically, each bid must specify both a price and a quantity, and bidders are permitted to submit multiple bids; thus, there will in general be more than one winner at an auction, and each winner may be awarded identical bills at different prices. To complicate matters further, there may be an announced or unannounced reservation rate; ties at the lowest winning price may be settled by pro rata distribution; there may be a secondary market for the bills which is influenced by the outcome of the auction; in the US T-bill auction, small bidders may submit "non-competitive" bids which specify only a quantity.

Recent theoretical studies have made progress by focusing on selected features of these auctions and simplifying other aspects. Bikhchandani and Huang (1989) are interested in the informational link between the auction and secondary market; for simplicity, they assume the bills are sold in units, and that each bidder demands exactly one unit. For Back and Zender (1993), in contrast, divisibility and multiple bids are the features of interest; they assume bidders submit continuous demand schedules, as in Wilson (1979), and show that uniform price rules may be especially vulnerable to collusion; however, they restrict attention to a narrow subset of equilibria. Edsperg (1993) examines the informational link from when-issued market to auction to secondary market and the role of non-competitive bids; he also assumes continuous bidding schedules and uniform price rules.

The goal in this section is to develop a structural model for empirical analysis of the Portuguese treasury bill auction. Since the model will be used to estimate revenue under alternative reservation price rules, it must incorporate a reservation price. For these revenue estimates to be comparable to revenue in the observed sample, I follow the Portuguese auction rules in specifying discriminatory pricing. Finally, as bids specify both price and
quantity, I model the good as divisible, and allow for uncertainty over the total quantity that will be submitted as bids. For simplicity, I assume that bidder quantity demands are drawn exogenously and independently of valuations.

The model assumes independent private values for the good; i.e., that each bidder’s valuation is independent of other bidders’ valuations. Treasury bill auctions typically are modeled as common-value because bills may be immediately re-sold on a secondary market. However, in the case of Portugal there is no liquid secondary market, so the great majority of bills are held until maturity. Thus, valuations are affiliated (see Milgrom and Weber, 1982), rather than common, and are here modeled as independent for simplicity. Although I am not aware of other treasury bill auctions which lack a secondary market, the model and its IPV assumption may apply well to recent auctions of foreign exchange in Latin America and Africa.3

2.1 Model I: Announced Reservation Price

A seller auctions a supply $S$ of a divisible good to $n$ risk-neutral bidders; $n$ is exogenous and public knowledge. The bidders’ valuations per unit of the good, $v_i$, $i = 1 \ldots n$, are drawn iid from the distribution $F$, which has support $\Omega \subseteq [0, \infty)$. Bidder $i$ demands $q_i$ units of the good; the $q_i$ are distributed iid $\text{gamma}(\lambda_1, \lambda_2)$, and are independent of the $v$. Each bidder submits (at most) one bid at a price which is constrained to be greater than or equal to an announced reservation price $p_0$. The auction is discriminatory; i.e., winning bidders pay their own bid prices.

The seller intends to sell $S$ of the good, but may end up selling more or less. In the case that total demand at prices greater than or equal to the seller’s reservation price is less than or equal to $S$, then demand is satisfied, and the seller retains any leftover. The seller also agrees to “round up” quantity to satisfy the demand of the lowest winning bid; for example, if $S = 1$, the highest bid sought 0.4 units, and the next bid sought 0.8 units.

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3 Consider an auction of foreign exchange to producers who each need a fixed amount of hard currency for a planned import of machinery, and are prohibited from selling or buying foreign currency in any other market. The amount needed by each bidder is private knowledge to the bidder, and not correlated with his valuation for the foreign currency, which comes from expected profit.
then the first and second highest bids would be satisfied in full, and the remaining bids would be discarded. I introduce this last feature into the model to remove the possibility of pro rata distribution for the lowest winning bid, which would severely complicate the analysis; as long as the bidders’ quantities are unlikely to be a large fraction of the supply, this limitation in the model should not make much difference.

Let \( v_{[1]} > \ldots > v_{[n-1]} \) be the order statistics of the valuations of the bidder’s \( n - 1 \) opponents. Let \( q_{[1]}, \ldots, q_{[n-1]} \) be the corresponding quantities; note that these will be randomly ordered, because the \( q_j \) are not correlated with the \( v_j \). Let \( F_j(x) \) be the probability that \( v_{[j-1]} > x \geq v_{[j]} \). This is given by

\[
F_j(x) = \frac{(n-1)!}{(n-j)!(j-1)!} F^{n-j}(x)(1 - F(x))^{j-1}.
\] (1)

Let \( G_j(S) \) be the probability that the sum of \( j \) independent gamma(\( \lambda_1, \lambda_2 \)) random variables is less than or equal to \( S \); this is simply the cdf of a gamma distribution with parameters (\( j\lambda_1, \lambda_2 \)). Define \( G_0(S) \equiv 1 \).

Now we can find the probability that a bid at price \( b \) will win, when all other bidders use the equilibrium strategy \( B(v) \). This is the sum, \( j = 1 \ldots n \), of the joint probabilities that \( b \) comes in exactly \( j \)-th place and the quantities of the top \( j - 1 \) bids sum to less than \( S \). Since these joint probabilities are for independent events, the probability of winning with a bid \( b \) is \( H(B^{-1}(b)) \), where

\[
H(x) = \sum_{j=1}^{n} F_j(x) G_{j-1}(S).
\] (2)

Because I have fixed the rules to avoid pro rata distributions, this function is invariant with respect to the quantity \( q_i \).

The bidder’s problem is to choose \( b_i \) to maximize

\[
W(b_i) \equiv (v - b_i) q_i H(B^{-1}(b_i)).
\]

The solution follows easily using the techniques of Milgrom and Weber (1982). The first
order condition is

\[(v - b)h(B^{-1}(b))B^{-1}(b) - H(B^{-1}(b)) = 0\]

where \(h\) is the density of \(H\). In equilibrium, the bidder chooses strategy \(b = B(v)\). It follows that \(B^{-1}(b) = 1/B'(v)\), and the FOC can be re-written as

\[B'(v) = (v - B(v)) \frac{h(v)}{H(v)}. \quad (3)\]

I apply the boundary condition \(B(p_0) = p_0\), so that a bidder who is indifferent to the good at the reservation price earns no rent. For a bidder with valuation \(v_i \geq p_0\), the equilibrium bidding strategy is then

\[B(v) = v - \left(\int_{p_0}^{v} H(a)da\right)/H(v). \quad (4)\]

For any given set of parameters, this bidding equation can be solved numerically quite quickly. For reasons discussed in Section 5, it is natural to bound \(v \in [0, 1]\), so the beta distribution is a convenient assumption. For simplicity of exposition, let \(v \sim \text{beta}(\alpha, \alpha)\), so that \(v\) has mean 1/2 and variance strictly decreasing in \(\alpha\). Also, let \(q_i \sim \text{gamma}(k\mu, k)\), which has mean \(\mu\) and variance \(\mu/k\). In these distributions, \(\alpha\) and \(k\) act as precision parameters and do not affect the means. I solve for \(B\), and find that its properties accord with intuition:

- \(B\) is strictly increasing. At first it is convex, then it inflects and becomes concave. It levels out sharply for higher \(v\).
- \(\mu_1 > \mu_2 \Rightarrow \forall v > p_0, \ B(v|\mu_1) > B(v|\mu_2)\). That is, as the mean of the distribution for the \(q_i\) increases, bidders bid more aggressively.
- \(k_1 > k_2 \Rightarrow \forall v > p_0, \ B(v|k_1) > B(v|k_2)\). That is, when there is more uncertainty as to the other bidders’ quantities, bidders bid less aggressively, because there is a higher probability that demand will not exhaust supply.
- Decreasing \(\alpha\) “twists” the bidding strategy, so that it is lower for low \(v\) and higher for high \(v\). This is because a mean preserving spread in the valuations makes it more
likely that there will be valuations even lower than a low \( v \) and even higher than a high \( v \).

2.2 The importance of reservation price policy

Riley and Samuelson (1981) study the optimal reservation price rule for auctions of single indivisible good. In the case of IPV and a risk neutral bidders and seller, the optimal rule is:

\[ p_0^* = v_0 + \frac{1 - F(p_0^*)}{f(p_0^*)} \]

where \( v_0 \) is the seller's valuation for the good. As intuition suggests, the optimal rule depends on the prior distribution of bidder valuations.

In the case of Model I, the optimal reservation price is much more difficult to derive analytically, even under the assumption that all parties are risk-neutral. Furthermore, there is the practical dilemma of specifying the seller's utility as a function of both revenue raised and quantity sold. Recall that Model I assumes that a bidder's valuation is drawn independently from the bidder's quantity demand. This can be interpreted as a constant marginal valuation assumption, which is not unreasonable so long as bid quantities are fairly small relative to the overall portfolio of the bidder. However, it is less appropriate to assume that the seller has constant marginal reservation value for the good, as the amount for sale may be many times larger than the amount any one bidder will demand. In particular, for a government issuing treasury bills to close its public sector borrowing requirement, the marginal value for an unsold bill might increase fairly steeply with the amount sold. As my data do not reveal the seller's underlying utility function, the theoretical problem of optimal policy is empirically irrelevant to this study. Rather, this study will provide a way for the seller to estimate how the joint distribution of sale price and quantity depends on the reservation price. The estimated joint distribution can then be used to maximize the seller's choice of objective function.

Figure 3 demonstrates the importance of the reservation price: it shows how the expected revenue per unit and expected quantity sold vary with the reservation price. It assumes
there are ten bidders for five units of a divisible good. Bid quantities are drawn from an exponential distribution with unit mean and variance.\textsuperscript{4} Bidder valuations are drawn from a beta(2,2) distribution, which has mean 0.5 and variance 0.05. The figure shows that the slope of expected revenue per unit as a function of reservation price is approximately 0.3 at the reservation price 0.5. (The Monte Carlo technique used to create this figure is described in Appendix A.) For comparison, the figure also shows results for an auction of five indivisible units to ten bidders under the same distributional assumptions on bidder valuations. The results are similar to the divisible good case, although it is curious that the divisible good auction gives the seller higher revenue. This contrasts with the results of Wilson's (1979) result for share auctions, in which the equilibrium bidding strategy is greatly inferior for the seller to that of a unit auction. The difference appears to be due to the endogeneity of quantity demand in the Wilson model, which opens a strategic dimension that is closed to bidders in Model I, who draw quantity demands exogenously.

2.3 Model II: Stochastic Reservation Price

This model is identical to the first, except that we now assume that the reservation price is unannounced, and the bidders share a prior distribution $F_0$ on the reservation price; the support of $F_0$ is $\Omega_0 \subseteq \Omega$. To win, a bid must now not only come in above the price at which supply would be exhausted, but also come in above the reservation price. By assumption, these events are independent, so the probability that bid $b$ will win is $H(B^{-1}(b))F_0(b)$. Given that all other bidders use strategy $B$, the bidder's problem is therefore to choose $b$ to maximize

$$W(b) \equiv (v - b)qH(B^{-1}(b))F_0(b).$$

The FOC is

$$(v - b)(h(B^{-1}(b))B^{-1}(b)F_0(b) + H(B^{-1}(b))f_0(b)) - H(B^{-1}(b))F_0(b) = 0$$

\textsuperscript{4}Under this distributional assumption, the expected coverage ratio is 2, which is comparable to the mean coverage ratio of 2.19 observed in my sample. See Figure 1.
Figure 3: Effect of the Reservation Price in Model I

Expected Revenue as function of Reservation Price

- divisible good auction
- multi-unit auction

Expected Quantity Sold as function of Reservation Price

- divisible good auction
- multi-unit auction
where $H$ and $h$ are as defined in Model I and $f_0$ is the density of $F_0$. Apply the equilibrium condition $b = B(v)$ and re-arrange to get the differential equation

$$B'(v) = \frac{(v - B(v))h(v)/H(v)}{1 - (v - B(v))f_0(B(v))/F_0(B(v))},$$

the solution to which is given implicitly by

$$B(v) = v - \frac{\int_0^v H(u)F_0(B(u))du}{H(v)F_0(B(v))}$$

where $v$ is the lowest valuation in $\Omega_0$ and I apply the boundary condition that a bidder with valuation $v$ earns no rent.

In general, this equation cannot be solved explicitly for $B(v)$ and must be solved numerically. However, we can compare the solutions to Models I and II when the announced $\mu_0$ in Model I equals the expectation of $p_0$ in Model II:

- This bidding strategy must be steeper everywhere in Model II, since the numerator of equation (5) is equal to the right hand side of equation (3), but the denominator of equation (5) must be less than one.

- A bidder with valuation $E(p_0)$ must prefer a regime in which the reservation price is unannounced to one in which it is announced to be $E(p_0)$.

On the basis of several sets of numerical solutions, I conjecture that a bidder with valuation equal to the highest value in $\Omega_0$ prefers a regime in which the reservation price is announced to be $E(p_0)$ to one in which it is unannounced. If this is true in general, we can conclude that the equilibrium bidding schedules of the two model intersect once, and that this intersection occurs at some $v$ greater than $E(p_0)$ and less than the highest value in $\Omega_0$.

### 2.4 Numerical Solutions

Since my application for this model is to the sale of treasury bills, it is natural to restrict the set of valuations $\Omega$ to $[0, 1]$. A valuation of 1 corresponds to a discount price of 100, which
implies a zero interest rate; as valuation declines, the implied interest rate rises. A natural and flexible choice of cdf $F$ for this support is the beta distribution; as long its parameters $(a, b)$ satisfy $\min(a, b) > 1$, then we have the intuitively desirable property $f(0) = 0$ and $f(1) = 0$.

For similar reasons, I choose $\Omega_0 = \Omega$ and let $F_0$ be the beta$(a_0, b_0)$ cdf with $\min(a_0, b_0) > 1$. I then attempt to solve equation (5) numerically using the Runge-Kutta algorithm provided by Matlab. Successful calculation turns out to be quite sensitive to the starting values given to the algorithm, even though my chosen distribution functions are continuously differentiable.

At the boundary point $(0, 0)$, the value of $f_0(B)/F_0(B)$ is infinite and $(v - B) = 0$, so $dB/dv$ is not well-defined at the starting point. More generally, for every $v \in \Omega_0$, there is a $B \in [0, 1]$ such that the denominator is zero; this set of singularities forms a curve in $[0, 1] \times [0, 1]$ which is bounded from above by the line $B = v$. Between this curve and $B = v$, the derivative is positive and increases sharply as one moves closer to the curve. Below the curve, the derivative is negative and increases sharply in magnitude as one moves closer to the curve. The solution lies strictly in the former region, but for low values of $v$ it lies very close to the set of singularities. Because the right hand side of equation (5) is steeply sloped in this region, the algorithm has a tendency to overshoot or undershoot the equilibrium path, and thereby "loses its way."

For several sets of parameter assumptions, I have obtained solutions after a period of trial and error with starting values. Even in these cases, however, Matlab takes approximately 30 minutes to solve this equation on an IBM RS/6000. In an MLE implementation of Model II, this equation would need to be solved once for every bid in every auction in each iteration of the likelihood function. This is not computationally feasible, and is an important motivation for the nonparametric method introduced in Section 5.
2.5 Should the Seller Announce the Reservation Price?

In the standard framework of a competitive first-price sealed-bid auction of a single indivisible good, ELLV show that the seller should announce the reservation price. The proof relies on the inability of a seller with unannounced reservation price to commit to a reservation price that is ex-ante optimal but may be ex-post suboptimal. In equilibrium, the bidders know that the reservation price will be the seller’s valuation \( v_0 \) for the good, and they bid accordingly. ELLV show that there is an announced reservation price that gives the risk-neutral seller the same expected utility as a hidden reservation price of \( v_0 \); to complete the proof, they demonstrate that this announced reservation price is in general not the optimal announced reservation price.

This argument does not apply to the particular case of the Portuguese treasury bill auction, because the Portuguese government does in fact have a means of commitment to fixing its reservation price ex-ante: the division of authority between the Ministry of Finance, which sets the reservation rate, and the central bank, which administers the auction. The bidders are aware that the reservation price is set before the auction, and thus cannot be affected by the auction’s outcome. (Indeed, a similar condition applies to the auction studied by ELLV, in which the seller gives his reservation price to a third-party auctioneer before the auction. This suggests that their empirical results may greatly overstate the gain to switching to an announced reservation price.)

Therefore, I am unable to draw any theoretical conclusion here on whether the reservation should be public or hidden. Moreover, there may be factors not captured by the model which would support the present policy of not announcing the reservation price. Ashenfelter (1989) suggests that hidden reservation prices are often used in practice to thwart collusion among bidders. Tan (1991) and Vincent (1993) show that a hidden reservation price policy can be superior to an announced reservation price policy when bidders are risk-averse.
2.6 Remarks

As a model for the Portuguese treasury bill auctions, Model II has several limitations. First, it forces the bidders to make only one bid, rather than allow a schedule of bids. Multiple bid models are mathematically intractible, so the only alternative to a single bid assumption is to follow Wilson (1979) and allow bidders to submit continuous demand schedules. However, Wilson's model can be solved analytically only under restrictive circumstances which do not hold here; for example, an unannounced reservation price would complicate Wilson's model hopelessly. My model arguably is less satisfying than the Wilson framework, but does capture the divisibility of the good and the asymmetry across bidders in demand. I'm not aware of other single-bid models that do this.

A second limitation is my simplifying assumption that the seller rounds up if necessary so that all winning bids are awarded in full. This assumption would matter little if quantity demands were small relative to the supply on offer, because the probability of receiving a pro rata distribution would be small. In my data, however, the ratio of the single largest bid at an auction to the auction issue ceiling has mean 0.35 and standard deviation 0.26. Therefore, it would seem that the bidders should expect a large winning bid to be only partly awarded with a fairly high probability.

A third problem is that quantity demands may not be entirely exogenous, as bidders may demand greater quantity when they expect prices to be low. In practice, this is not likely to be a major factor in most of my sample. The spread earned on commercial lending in Portugal is quite high, so the banks generally view treasury bills as a convenient way to park excess funds rather than as an alternative to their primary business of commercial loans. Therefore, the quantities demanded will have more to do with an excess or shortage of liquidity in the system, which is exogenous for our purposes, than with expectations of winning auction prices. (In Section 5.5 I discuss a possible exception to this argument.)

Finally, bidder quantity demands in a treasury bill auction arguably should apply not to the face value of the bills, but to expenditure on the bills. That is, if the bidder has a fixed amount of cash with which to bid, then what is constrained is not the quantity (face
value) of bills that the bidder can buy, but his expenditure on bills. For example, if the bidder "draws" an amount of cash \( C_i \), then the bidder can purchase up to \( C_i/b_i \) units, which depends on what price he bids.

This last problem is likely to be much less important than it may appear at first, because large differences in winning rates at a treasury bill auction translate into small differences in the face value of bills won by a cash constrained bidder. For example, say that a bidder has $100 to spend on bills of 182 days maturity. If he wins his bid at 18%, the discount price is 91.76, so he is able to demand $108.98 in face value for his $100. If he wins at 15%, he is able to demand $107.48 in face value for his $100. In practice, two rates so far apart have very different probabilities of winning, yet the difference in quantity associated with the two bids is under 1.5%.

3 Portuguese Treasury Bill Auction Rules

The Portuguese method of auctioning treasury bills is similar to that used in the United States, particularly in that it is a discriminatory auction (i.e., each winning bidder pays his bid, so identical goods may be sold at different prices). The two most important differences are:

- Only banks that are registered in Portugal may purchase Portuguese treasury bills. The only authorized bidders are these banks and financial intermediaries (brokers) who are bidding on behalf of these banks. Consequently, neither the primary nor secondary markets for Portuguese treasury bill is open to other financial institutions, domestic or foreign, or to individuals.

- A small bidder in the US auction may submit a "non-competitive" bid, which is a quantity bid which always wins and is sold at the average price of winning "competitive" (price-quantity) bids. The Portuguese auction allows only competitive bids.
At least four working days before the issue date, the Banco de Portugal announces the maturity and issue ceiling of the bills to be made available.\footnote{I have not been able to ascertain whether the maturity and issue ceiling are chosen by the Banco de Portugal or by the Ministry of Finance.} The maturity is either 91, 182 or 364 days. Bids are due at the Banco de Portugal in sealed envelopes by 4:30 PM of the third working day before issue. Each bid consists of an interest rate and a desired quantity, in integer multiples of one million escudos (roughly $8,000). Bidders each may submit as many as six bids.\footnote{It appears that banks may exceed this limit by placing additional bids through brokers. However, banks may generally be discouraged from doing so by the ability of the Banco de Portugal to observe the re-sale of the broker’s winning bids to the bank.}

Placement takes place the morning of the second working day before issue. The bids are ordered by interest rate. Allotment starts with the lowest rate bid, and stops when the issue ceiling is reached (in which case any tie for claims to allotment are settled pro rata) or when remaining bids exceed the maximum interest rate set before the auction by the Ministry of Finance. This reservation rate is not announced to the bidders.

On the same day, bidders are informed (privately) of the face value and discount on the treasury bills allocated to them, and (publicly) of the total quantity of treasury bills sold and the weighted average interest rate of bills sold.

At the discretion of the Banco de Portugal, placement may be followed immediately by an “additional session,” in which the Banco de Portugal accepts quantity bids for treasury bills at the weighted average rate determined in the first session. Bids submitted by banks which obtained bills in the first session are satisfied first; if total demand exceeds the issue ceiling of the additional session, bills are distributed pro rata.

On the issue date, the Banco de Portugal issues the allotted treasury bills, and debits the bidders’ demand deposit accounts at the Banco de Portugal. Interest rates are converted to discount prices according to the following formula:

\[ p(r, m) = \frac{36500}{36500 + mr} \]  

(7)
where \( p \) is the discount price, \( r \) is an interest rate expressed in percentage points, and \( m \) is the maturity of the bills in days. The sale price of a winning bid is then \( q \cdot p(r, m) \), where \( q \) is the bid quantity (face value of the bill).

4 Data

My data consist of all relevant information on treasury bill auctions in Portugal between June 1988 and April 1993, and a complete record of transactions on the Portuguese interbank money market. In this section, I summarize the construction of variables and selection of a subsample of auctions for study.

For each auction (indexed by \( t \)), I have the auction date, the issue ceiling \( (S_t) \), reservation rate, maturity of the bill \( (m_t) \), and whether the auction is a primary session or an additional session. The variable \( \text{TIME}(t) \) is the number of days between January 1, 1988 and the auction date. The reservation rate, \( p_0 \), is converted from an interest rate to a discount price in \([0, 1]\) using equation (7).

The total sample has 666 sessions. I eliminate the 80 additional sessions, and create a dummy variable \( \text{HASADDSS} \) to mark primary sessions which are followed by additional sessions. It is not clear whether bidders know before an auction whether an additional session will be held. However, additional sessions were held almost exclusively in a thirteen month subperiod of my sample, and most auctions in that subperiod are followed by an additional session; therefore, in the empirical analysis below, I will assume that bidders regard \( \text{HASADDSS} \) as known and exogenous.

I eliminate all auctions for which I am missing a issue ceiling or a reservation rate. I also eliminate all 117 auctions of 364 day bills because I believe that the banks’ valuations of these instruments are determined primarily by variables not contained in my data (e.g., market prices on longer maturity government bonds). There remain 423 auctions in my sample.

For each bid submitted to an auction, I have the identity of the bidder of record (which may not be the “true” bidder, if the bid is entered through an intermediary), the interest
rate and quantity of the bid, and, for winning bids, the amount allocated to the bid at
the placement session and the sale price paid.\textsuperscript{7} I use these data to construct the following
variables:

\textbf{NBIDDERS}(t) is the number of bidders who submit bids at auction \( t \). A total of 66
bidders appear in the auctions; 12 bidders appear in a majority of auctions.

\( N_t \) is the total number of bids submitted by the bidders at auction \( t \).

\( b_{it} \) is the discount price on \textit{BIDRATE}(i, t), the rate on the \( i \)-th bid at auction \( t \).

\( q_{it} \) is the (face value) quantity of the bid.

In order to reduce the dimensionality of the regressions in Section 5, it is desirable
to suppress dependence on maturity. Therefore, for 91 day bills, I let \( b_{it} \) be the \textit{squared}
discount price, which is the price that would be paid for a 182 day bill at an equivalent
compounded rate. Similarly, I square the reservation price \( p_0 \) for auctions of 91 day bills.
Note that pooling the two maturities this way does not require a flat term structure to
be valid; I only need assume that the relationship between bidder valuations and market
benchmarks must be the same across maturities.

The most serious difficulty with applying Model II to the auction data is that bidders
generally submit multiple bids. There are two ways to deal with this:

- Form means: For each bidder at each auction, let \( q_{it} \) be the sum of quantities on all
  of bidder \( i \)'s bids at auction \( t \). Let the bid price \( b_{it} \) be the quantity-weighted mean
  bid price on all of bidder \( i \)'s bids at auction \( t \).

- Treat each bid as if it were submitted by a different bidder.

At first consideration, it would appear that the first method is more appropriate, because
it preserves the strategic idea of one bidder, one bid. The disadvantage is that estimates
of seller revenue are badly biased downward by any sort of bid averaging, since the seller

\textsuperscript{7}The last piece of information is actually redundant, but serves as a check against entry errors.
in a multiple bid auctions gets to select only the highest bids of each bidder.⁸ In my
data this bias would be quite severe, because bidders frequently submit widely dispersed
multiple bids; therefore, I abandon the first method in favor of the second. In addition to
preserving the actual distribution of bids received by the seller, the second method has other
advantages: It preserves the full size of the data, which is especially important when using
nonparametric estimation in multiple dimensions. To the extent that the bidder valuations
for treasury bills are actually downward sloping rather than constant, keeping all of the
bids lets us capture that downward sloping demand by splitting the demand into distinct
bidders, each with a different constant marginal valuation.

I also have all relevant data on every secondary market trade of a treasury bill; un-
fortunately, this is an extremely illiquid market, so unable to provide anything resembling
a continuously available market price. These data are discarded, and for the purpose of
forming a benchmark for the value of the bills, I look at longer maturity trades on the
interbank money market (IMM). The IMM is primarily used by the banks to loan and
borrow overnight money for the purpose of reserve management; turnover is on the order
of 1,500 billion escudos ($12 billion) per month for overnight loans. Nonetheless, there is a
substantial amount of trading in longer maturities comparable to the 91 and 182 day bills.
In the 86 to 96 day maturity range, monthly IMM turnover is on the order of 5 to 30 billion
escudos. Turnover in the 175 to 189 day range is typically one-third to one-half of that.
(In the over 350 day range, turnover is negligible.) Note that the recorded turnover does
not fully measure the value of the IMM as a price-setting mechanism; the banks remain
in contact with the market throughout the day, and often put in buy and sell offers as
"feelers." These offers allow banks to put bounds on the market rate in maturities that
are not actively traded, but such offers are not recorded in my data unless they lead to an
actual transaction.

⁸For example, say that two identical units are auctioned to two bidders. Each bidder submits one bid for
each unit. If one bidder bids $1 for each of the two units, and the second bidder bids $2 for each unit, then
the second bidder wins both units for $4. However, if the first bidder bids $1.50 for one unit and $0.50 for
the other, and the second bidder bids $3 for unit unit and $1 for the other, then each bidder wins one unit,
and the seller receives $4.50. In each of these two cases, the mean bids are $1 for the first bidder and $2 for
the second; the outcome of the auction, however, is quite different for both the seller and the bidders.
For each IMM trade from January 1989 onward, I observe the date on which the trade took place, the identities of lender and borrower, and the maturity, rate, and quantity of the loan. I form a 91 day benchmark rate IMM091 at \( \text{T}(t) \) by taking a weighted average of recent IMM trades. The weights are chosen to place greatest weight on trades closest in time to auction \( t \) and in maturity to 91 days: the weight on a trade \( \tau \) days before \( t \) of maturity \( m \) is \( \text{PARZEN}(\text{T}(t) - \tau, \bar{\tau}) \cdot \text{PARZEN}(\vert 91 - m \vert, \bar{m}) \), where \( \text{PARZEN}(x, b) \) is the Parzen kernel of \( x \) for bandwidth \( b \):

\[
\text{PARZEN}(x, b) = \begin{cases} 
1 - 6\left(\frac{x}{x+b+1}\right)^2 + 6\left(\frac{x}{x+b+1}\right)^3, & \text{for } x = 1, \ldots, \left\lfloor \frac{b+1}{2} \right\rfloor; \\
2\left(1 - \frac{x}{x+b+1}\right)^3, & \text{for } x = \left\lfloor \frac{b+1}{2} \right\rfloor + 1, \ldots, b. 
\end{cases}
\]

Note that trades for which maturity or time is outside the bandwidth are given zero weight. The choice of bandwidths \( \bar{m} = 18 \) and \( \bar{\tau} = 14 \) is purposely fairly wide; it moderates the influence of borrower-specific risk or lender-specific market power on rates, and also helps preserve sample size. Note that the Parzen weights decline quite steeply, so that trades near the auction date and maturity are weighted much more heavily than more distant trades.\(^9\)

I form a 182 day benchmark rate IMM182 similarly, but with wider bandwidths of \( \bar{m} = 28 \) and \( \bar{\tau} = 21 \). The correlation of IMM091 and IMM182 is 0.87. Finally, I combine these two benchmark rates to form IMMPRICE:

\[
\text{IMMPRI(t)} = \begin{cases} 
p(\text{IMM091}(t), 91)^2, & \text{if } m_t = 91; \\
p(\text{IMM182}(t), 182), & \text{if } m_t = 182. 
\end{cases}
\]

As in the construction of the bid prices and reservation prices, the 91 day benchmark price is squared in order to be directly comparable to the 182 day prices. IMM data is unavailable before January 1989, so I lose the first seven months of auctions in my sample (59 auctions). A further 28 auctions are lost either to data entry errors or when, given my choice of bandwidths, IMMPRICE cannot be constructed. This reduces my sample to 336 auctions.

\(^9\)Choice of bandwidths does not appear to have much effect on the empirical results. When I run the procedures of Section 5 with an IMMPRICE constructed with much more conservative bandwidths (\( \bar{m}_{91} = 6, \bar{\tau}_{91} = 7, \bar{m}_{182} = 12, \bar{\tau}_{182} = 10 \)), I obtain similar results.
The variable IMMVOL(t) is the standard deviation of overnight rates in the week prior to auction t. The volatility of the overnight rate varies substantially over my sample period, mostly due to changes in Banco de Portugal policy on intervention in the money market during the reserve period. Because the overnight market is by far the most liquid of all Portuguese markets, extreme volatility in this rate makes it more difficult for the banks to price longer maturity interbank money market loans and treasury bills as well. The importance of this variable is explored in greater detail in Chapter 2.

I divide my sample into three periods on the basis of the Ministry of Finance's reservation price policy. In the first regime (REGIME1), the reservation rate is fairly volatile and its changes have little correlation with changes in the interbank rate. In the second (REGIME2), the reservation rate hardly changes at all, despite high volatility on the interbank money market. In the third (REGIME3), the reservation rate changes frequently and in tandem with interbank money market rates. Although I do not have any evidence that policy changes actually occurred at the points indicated, my breakpoints are chosen quite naturally: The first and second regimes are divided by a period of a couple months in which there were no auctions. The division between the second and third regimes appears to coincide with the date on which Portugal joined the ERM (April 6, 1992); this event is the most important event in the Portuguese economy in recent years, and it is reasonable to expect that a variety of policy changes may have been made at the same time.\(^{10}\)

The divisions between the regimes are marked in Figure 4, which plots the IMM rate and the auction reservation rate over time.

There seems to have been only one change in tax policy during my sample period which would affect the relationship between the interbank money market benchmark and the treasury bill rate. Prior to May 4, 1989, interest on treasury bills were exempt from taxation; for bills issued after that date, interest is taxed at 20%. In accordance with Banco de Portugal methodology, I adjust all auction interest rates (both the reservation rate and BIDRATE) downwards by 20% for bills issued prior to the tax change.

\(^{10}\)This event was completely unexpected even by the Banco de Portugal, so it is not necessary to consider prior changes in behavior in anticipation of the change in policy.
Finally, I have all relevant information on interventions by the Banco de Portugal, including changes in credit ceiling and reserve requirement regulations, its operations on the interbank money market and its offered rate on time deposits. None of these appears to have any direct effect upon the auctions; e.g., discounting of bills and provision of time deposits to banks seem to have been quite rare. Therefore, I exclude all other information for the sake of parsimony.

5 Estimation

The method of this paper follows from the main insight in ELLV: Given an observed distribution of bids, we can invert a model of bidding strategies to back out estimates of the underlying bidder valuations. Using these constructed valuations, we can infer how bidders would have behaved under alternative auction formats or policy regimes, and thereby estimate the potential gain or loss to shifting to the alternative.

Let \( x_t^Y \) be a vector of observable independent variables at auction \( t \) that determine bidder valuations for the treasury bills. I denote by \( F_t \) the conditional distribution function \( F(\cdot|x_t^Y) \) for the valuations. Similarly, let \( x_t^0 \) be the independent variables that bidders use to form the prior distribution \( F_0 \) for the seller’s reservation price, and let \( F_{0t} \equiv F_0(\cdot|x_t^0) \). The distribution of bid quantities is conditional upon independent variables \( x_t^Q \), and is denoted \( G_t \equiv G(S_t|x_t^Q) \).

Let \( x_t \) be the collection of all unique variables in \( \{x_t^Y, x_t^0, x_t^Q\} \). The equilibrium bidding strategy at auction \( t \) as a function of the bidder’s private value \( v \) can then be denoted \( B_t(v) \equiv B(v|x_t) \). As long as \( B_t \) is strictly increasing, there is a one-to-one correspondence between an observed bid \( b \) and its underlying valuation \( v = B_t^{-1}(b) \). Thus, the observed distribution of bids, which I denote by \( Y_t \equiv Y(\cdot|x_t) \), reveals the underlying distribution of bidder valuations \( F_t \). The distribution \( Y_t \) and its density \( y_t \) are given by

\[
Y_t(b) = F_t(B_t^{-1}(b)) \quad \text{and} \quad y_t(b) = \frac{1}{B_t'(B_t^{-1}(b))} f_t(B_t^{-1}(b))
\]
Because the good for auction is a treasury bill, its value cannot be less than zero nor greater than one. As there is no pre-announced upper or lower bound to the reservation price, the same bounds must apply to the support of $F_0$. Therefore, the sets $\Omega = \Omega_0 = [0,1]$, and the support of $Y_t$ is $[0,B_t(1)]$.

In order to make use of equation (8), we need to derive the inverse of the bidding strategy. Recall from Model II that the bidding strategy $B_t(v)$ solves a first order differential equation with boundary condition $B_t(0) = 0$. Equation (5) can be written as

$$1 = (v - B_t(v)) \left( \frac{h_t(v)}{H_t(v)} \frac{1}{B_t'(v)} + \frac{f_{0t}(B_t(v))}{F_{0t}(B_t(v))} \right)$$

(9)

for all $v \in \Omega$. By writing out $H_t$ and its derivative in terms of its constituent functions, and rearranging, we can re-write the function $h_t(v)/H_t(v)$ as $\mathcal{H}_t(F_t(v))f_t(v)$, where

$$\mathcal{H}_t(z) = \frac{\sum_{j=1}^{N(t)} \frac{(N(t)-1)!}{(N(t)-j)!(j-1)!} \left[ (N(t)-j)z^{N(t)-j-1}(1-z)^{j-1} - (j-1)z^{N(t)-j}(1-z)^{j-2} \right] G_{j-1,t}}{\sum_{j=1}^{N(t)} \frac{(N(t)-1)!}{(N(t)-j)!(j-1)!} \left[ z^{N(t)-j}(1-z)^{j-1} \right] G_{j-1,t}}$$

(10)

From equation (8), we get $B_t'(v) = f_t(v)/y_t(b)$. Combining this and the definition of $\mathcal{H}_t$, and substituting $b$ for $B_t(v)$ and $Y_t(b)$ for $F_t(v)$, we re-arrange equation (9) to get

$$v = b + \frac{1}{\mathcal{H}_t(Y_t(b))y_t(b) + f_{0t}(b)/F_{0t}(b)} \equiv \xi_t(b)$$

(11)

which is the inverse of the bidding strategy $B_t(v)$.

ELLV discuss the problem of identification of $F_t$ given $Y_t$, $G_t$ and $F_{0t}$, and show that there is no problem as long as the function $\xi_t(b)$ is strictly increasing and differentiable. Although the construction of our $\xi_t$ is more complex than in their case, the argument follows similarly and need not be repeated here.

The subsections below describe an empirical implementation of equation (11). I proceed as follows:

1. Estimate $f_{0t}(b_{it})/F_{0t}(b_{it})$ for every bid $i$ at each auction $t$. 

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2. Estimate \( G_{j,t} \) for \( j = 0, \ldots, N_t - 1 \) for each auction \( t \).

3. Estimate \( Y_t(b_{it}) \) and \( y_t(b_{it}) \) for every bid \( i \) at each auction \( t \). Use the estimated \( Y_t(b_{it}) \) and \( G_{j,t} \) to estimate \( \mathcal{H}_t(Y_t(b_{it})) \) for every bid \( i \) at each auction \( t \).

4. Construct the pseudo sample of bidder valuations \( \hat{v}_{it} \) by inserting the estimated function values into equation (11). Use this pseudo sample to obtain the conditional distribution \( F'(v|x^v) \).

5. Choose an alternative set of announced reservation prices, \( \tilde{p}_0 \), and use the pseudo sample of valuations \( v_{it} \), the estimated \( G_j(S_t) \), and equation (4) to construct a pseudo sample of bids \( \hat{b}_{it} \) under the alternative scenario. Using these constructed bids, estimate what seller revenue and quantity sold would have been under the alternative scenario, and compare this to the observed revenue and quantity sold.

5.1 Conditional distribution of the reservation price

The task of this subsection is to estimate the conditional hazard rate \( f_0(b_{it}|x_t^0)/F_0(b_{it}|x_t^0) \), which for convenience I denote \( \Lambda_0(b_{it}|x_t^0) \). I assume that the reservation price adjusts stochastically to the relevant interbank money market benchmark by the following rule:

\[
p_{0t} = \text{LAG}(p_{0t}) + \beta (\text{IMMPRI}(t) - \text{LAG}(p_{0t}) + k) + \eta_t
\]

where the LAG operator refers to the reservation price at the last auction of the same maturity. The idea is that bidders expect the reservation price to converge to the IMMPRI plus some constant (which might represent the risk premium on the IMM, or any advantage or disadvantage to holding treasury bills rather than IMM loans), but that the process of convergence is not instantaneous and has stochastic component \( \eta \).

Equation (12) is estimated by OLS separately for each regime. The reduced form regression equation is

\[
p_{0t} - \text{LAG}(p_{0t}) = \beta_0 + \beta_1 (\text{IMMPRI}(t) - \text{LAG}(p_{0t})) + \eta_t.
\]
The regression results are given in Table 1. These results suggest that the reservation price is a random walk during the first two regimes, and tracks the IMM rate in the final regime. Especially in the last two regimes, the conditional variance of the reservation price is quite small; this is revealed graphically in Figure 5, which plots $p_0$ and $\hat{p}_0$ over time.

I use these estimates to construct $\hat{p}_0$ as an index variable for the bidders' expectation of $p_0$. I then estimate the hazard rate, conditional on regime $R$ and expected reservation price $\hat{p}_0$, using a nonparametric hazard rate estimator:

$$\tilde{\lambda}_0(b|\hat{p}_0, R) = \frac{1}{\delta_0} \frac{\sum_{t=1}^{T} 1(R_t = R) \phi\left(\frac{b - p_{0t}}{\delta_0}\right) \phi\left(\frac{\hat{p}_0 - p_{0t}}{\delta_{00}}\right)}{\sum_{t=1}^{T} 1(R_t = R) 1(p_{0t} \leq b) \phi\left(\frac{\hat{p}_0 - p_{0t}}{\delta_{00}}\right)}$$

(14)

where $\phi$ is the standard normal density and $\delta_0$ and $\delta_{00}$ are bandwidths. Choice of bandwidths is discussed in Appendix B.

This analysis assumes that the reservation rate at the last auction of the same maturity is known to all bidders. Although the reservation rate is not made public information after an auction, I believe this assumption is justified for the regular bidders in the majority of auctions. In nearly half the auctions, the reservation rate is strictly binding, so every bidder with at least one winning bid and at least one losing bid will have bounds on the reservation rate.\textsuperscript{11} When the reservation rate is binding at the previous auction, I observe that bidders

\textsuperscript{11}Recall that the total quantity sold is made public on the issue date; when this quantity is less than the issue ceiling, the bidders know that the reservation rate was strictly binding.
Figure 5: Reservation Price: Actual vs Predicted

Solid lines represent actual reservation price $p_0$. The plus signs represent predicted values $\hat{p}_0$. 
generally place large bids at exactly the last auction's reservation rate, which suggests that, one way or another, the bidders do form a strong prior on the last auction's reservation rate. Even when the reservation rate does not bind strictly, so that winning bidders learn only a lower bound on the reservation rate, the OLS results suggest that a bidder can still project the subsequent reservation rate with reasonable accuracy on the basis of information from earlier auctions.

Instead of this two stage procedure for estimating \( A_0(b_t|z_t^0) \), I could have used a a single nonparametric regression of the reservation price on the lag reservation price, IMMPRICE, and regime. However, the two stage procedure absorbs so much of the variance in \( p_0 \) that it is difficult to imagine any improvement from the more general technique. Furthermore, the constructed index variable \( \hat{p}_0 \) will be useful in reducing the dimensionality of the nonparametric estimation in Section 5.3.

5.2 Conditional distribution of bid quantity

Although my model allows bid quantities to be drawn from a gamma distribution, I assume the special case of an exponential distribution: \( g(q) = \lambda \exp(-\lambda q) \). This is equivalent to a gamma(1,\( \lambda \)) distribution, and implies that quantities are distributed with mean and standard deviation both equal to \( 1/\lambda \). This assumption halves the number of parameters to be estimated, and also accords well with the data; in a plot of the mean against the standard deviation of bid quantities for the auctions in my sample (figure 6), the points cluster around the 45° line.

I model the conditional distribution of bid quantities by \( \lambda_t = \exp(x_t^Q \theta) \) and then estimate \( \hat{\theta} \) by maximum likelihood.\(^{12}\) The vector \( x_t^Q \) contains a constant and the following

\[^{12}\text{The log likelihood is,} \]

\[
\ln(L) = \sum_{t=1}^{T} \sum_{i=1}^{N(t)} (x_t^Q \theta - \exp(x_t^Q \theta)q_{it}).
\]

The maximum likelihood estimator \( \hat{\theta} \) solves the first order condition:

\[
\sum_{t=1}^{T} \left( x_t^Q N_t - \exp(x_t^Q \theta) \sum_{i=1}^{N(t)} q_{it} \right) = 0.
\]
Figure 6:

Mean vs Std Dev of Bid Quantities at Auction

Log Scale
independent variables:

**TIME** I expect quantities to increase over time, due mainly to inflation (which ran at 10–15% in these years), but also possibly to the process of financial reform in Portugal, which increased bank interest in holding treasury bills.

**ln(NBIDDERS)** Note that NBIDDERS is assumed to be exogenous and known to the bidders; given the constant communication between banks on the interbank money market, it is reasonable to expect all the major players to know which of the other major players will be bidding. I expect quantities to increase with NBIDDERS, because the latter proxies for the strength of market demand for bills.

**N/NBIDDERS** I expect that quantities on each bid will be smaller when bidders are submitting more bids. The determinants of the number of bids per bidder are explored in Chapter 2, and are strong enough in predictive power to lend support to the assumption that \( N \) can be guessed reasonably well by the bidders before an auction.

**ln(S)** If the seller's supply decision is correlated with information on demand that is known to the seller but unobserved in my data, then I expect that larger supply will be associated with larger bids. Note that \( S_t \) is announced one working day before the auction, so there is no endogeneity problem here.

**IMMVOL, IMMVOL^2** When the overnight rate is volatile, I expect banks to hold back extra reserves as insurance against coming up short, and thus to reduce the size of their bids.

**HASADDSS** I expect bidders to reduce their bid quantities in the primary session when an additional session is to be held.

**MAT091** is a dummy for auctions of 91 day bills.

The results are given in Table 2. With the exception of the coefficient on HASADDSS, all of the predicted signs are observed (note that a negative sign on a coefficient implies that bid
quantities increase with that variable) and all of the coefficients are statistically significant at the 1% level. Using these results, I construct \( \hat{\lambda}_t = \exp(x_t^Q \hat{\theta}) \), and the corresponding

| Table 2: Estimation of Bid Quantity Distribution \( q_{it} \sim \text{Exponential}(x_t^Q \theta) \) |
|-----------------------------------------------|-------------------|-------------------|
| Mean†                                      | ML Estimation of \( \theta \) |
| CONSTANT                                   | 1.0000             | -4.6730           | 0.2357             |
| TIME                                       | 1.1269             | -0.5314           | 0.0248             |
| \( \ln(\text{NBIDDERS}) \)                | 2.3815             | -0.0655           | 0.0199             |
| \( N/\text{NBIDDERS} \)                    | 2.7811             | 0.2779            | 0.0140             |
| \( \ln(S) \)                               | 9.8846             | -0.2626           | 0.0220             |
| IMMVOL                                     | 1.1220             | 0.2232            | 0.0380             |
| IMMVOL\(^2\)                               | 1.8880             | -0.0446           | 0.0121             |
| HASADDSS                                   | 0.1388             | -0.0582           | 0.0334             |
| MAT091                                     | 0.4788             | 0.0496            | 0.0182             |

†: Mean of independent variable.

values of \( \hat{G}_j(S_t|x_t^Q) \).

5.3 Constructing bidder valuations

The estimation of the condition distributions \( Y_t \) and \( y_t \) is similar to that of Section 5.1, except that \( x_t \) is of higher dimension than \( x_t^0 \). The vector \( x_t \) should include all of the variables upon which bidding strategies depend; these include the number of bids submitted to the auction \( (N_t) \), and the variables which condition bidder valuations \( (\text{IMMPRICE}(t) \) and regime \( R_t \), the variables which conditional bid quantities \( (x_t^Q) \), and the variables which condition the reservation price \( (x_t^0) \). Even for very large data sets, nonparametric estimation with so many independent variables is infeasible; the problem of high dimensionality is discussed in Stoker (1991).

I resolve this problem by using \( \hat{\lambda}_t \) as an index variable for the variables in \( x_t^Q \), and using \( \hat{\rho}_{0t} \) as an index for all variables in \( x_t^0 \) except regime. That is, I am assuming that the effect of variables in \( x_t^Q \) and \( x_t^0 \) on the way in which the \( \hat{G}_j(S_t|x_t^Q) \) and \( \hat{\lambda}_0(b_{it}|x_t^0) \) determine
bidding behavior is entirely captured by their respective index variables and regime.\textsuperscript{13}

This leaves $x_t$ comprised of $\text{IMMPRICE}(t)$, $R_t$, $S_t$, $N_t$, $\lambda_t$, and $\hat{p}_{0t}$. This is still too many independent variables, so I remove two more dimensions by replacing $S_t$, $N_t$ and $\lambda_t$ with the expected coverage ratio $\hat{C}_t \equiv N_t/\hat{\lambda}_tS_t$. The assumption that $\hat{C}_t$ adequately represents the effects of $S_t$ and $\hat{\lambda}_t$ seems plausible; whether $\hat{C}_t$ also fully represents the effects of $N_t$ is less certain, but the assumption is technically necessary.

For notation simplicity in the equation below, I will henceforth write $x_t$ to mean the vector $[\text{IMMPRICE}(t) \quad \hat{p}_{0t} \quad \hat{C}_t]$. The kernel estimators are

$$y_t(b|x,R) = \frac{1}{\delta_b} \sum_{i=1}^{N(t)} \phi_3(x - x_i) \left( \frac{1}{N(t)} \sum_{i=1}^{N(t)} \phi \left( \frac{b-b_i}{\delta_b} \right) \right)$$

\hspace{1cm} (15)

$$Y_t(b|x,R) = \frac{\sum_{i=1}^{N(t)} 1(R_t = R) \phi_3(x - x_i) \left( \frac{1}{N(t)} \sum_{i=1}^{N(t)} 1(b_i \leq b) \right)}{\sum_{i=1}^{N(t)} 1(R_t = R) \phi_3(x - x_i)}$$

\hspace{1cm} (16)

where $\phi_3$ denotes the trivariate standard normal distribution. The properties of these estimators are discussed in ELLV. See Appendix B regarding choice of the bandwidth $\delta_b$ and the bandwidth matrix $\Delta_\sigma$.

I use these equations to estimate $\hat{Y}_{it} \equiv Y_t(b_{it}|x_t, R_t)$ and $\hat{y}_{it} \equiv y_t(b_{it}|x_t, R_t)$ for each bid in the sample. The pseudo sample of bidder valuations is then constructed using equation (11):

$$\hat{v}_{it} = b_{it} + \frac{1}{\mathcal{H}_t(\hat{Y}_{it})\hat{y}_{it} + \hat{\lambda}_{0t}(b_{it})}$$

\hspace{1cm} (17)

Note that the estimated $\hat{G}_j(S_t|x_t^Q)$ are used for the true probability values $G_j(S_t|x_t^Q)$ in the function $\mathcal{H}_t$.

The estimated bidder valuations can provide intuition for the scale of bidder rents, and for how bidder rents vary across auctions. For the observed sample of bids and reservation prices, the rent on a bid, conditional on winning, is estimated by $\hat{v}_{it} - b_{it}$. In Figure 7, I plot the quantity-weighted mean value of $\hat{v}_{it} - b_{it}$ for each auction.\textsuperscript{14} When compared to Figure

\hspace{1cm} \textsuperscript{13}Because I employ these index variables, rather than the entire set of independent variables, this estimator is better described as semiparametric than nonparametric.

\hspace{1cm} \textsuperscript{14}The quantity-weighted mean of $\hat{v}_{it} - b_{it}$ is indicative of the gain available to the seller from a more
4, the figure demonstrates that estimated bidder rents increase with the gap between the expected reservation rate and the IMM rate; for example, the high bidder rents around date 900 (and especially the spike at 908 which exceeds the scale of the figure) correspond to a portion of Regime 2 in Figure 4 during which the reservation rate is stable and up to two percentage points higher than prevailing IMM rates. For many auctions, there do appear to be substantial potential gains to a more aggressive reservation price policy; note that a rent of only 0.003 in discount price terms is roughly equal to a gap between valuation and bid of 70 basis points in yield terms, which is quite large. However, for most auctions, potential gains are modest: at over half the auctions, the mean \( \hat{\delta}_{it} - b_{it} \) is less than 0.00025, which corresponds to less than six basis points in yield. Six basis points would not be a small rent for a bidder in the US market (see, e.g., Cammack, 1991), but in Portugal, where holding periods are long for bills and which lacks precise market benchmarks, it seems doubtful that rents could be squeezed any smaller while maintaining present levels of total sales.

5.4 Auction results under alternative scenarios

The final task is to use the pseudo sample of bidder valuations to evaluate the performance of alternative rules for choosing the reservation price. For computational simplicity, I will assume in this section that the chosen reservation price is announced, i.e., Model I. In principle, we can create alternative scenarios using Model II as well, but the extra stochastic element would only complicate interpretation of the results.

The method of estimating auction results under the alternative price rule \( \hat{p}_0 \) is straightforward. The pseudo bids \( \hat{b}_{it} \) are generated using the pseudo sample of bidders valuations \( \hat{\delta}_{it} \) and equation (4):

\[
\hat{b}_{it} = \hat{\delta}_{it} - \left( \int_{\hat{p}_0}^{\hat{\delta}_{it}} \hat{H}(a) da \right) / \hat{H}(\hat{\delta}_{it})
\]  

(18)

aggressive reservation price policy, but there is no precise sufficient statistic for the distribution of rents among bidders in an auction; for example, an auction in which a few bidders with very high valuations earn large rents and most bidders earn modest rents may have a high mean rent, but the seller nonetheless could not raise the reservation price without large losses in total quantity sold. I have prepared similar figures for the median of \( \hat{\delta}_{it} - b_{it} \), its maximum, etc.; these variations yield qualitatively similar impressions.
Valuations - Bid Prices (Auction Means)
for all $\hat{\omega}_{it} \geq \hat{p}_{0it}$, where

$$
\hat{H}_t(v) = \sum_{j=1}^{n} \hat{F}_{jt}(v) \hat{G}_{j-1}(S_t|x_t^Q)
$$

and

$$
\hat{F}_{jt}(v) = \frac{(N_t - 1)!}{(N_t - j)!(j - 1)!} \hat{p}_t^{N_t-j}(v)(1 - \hat{F}_t(v))^{j-1}.
$$

The conditional probability $\hat{F}_t$ is given by a nonparametric estimator as in equation (16):

$$
\hat{F}(v|x) = \frac{\sum_{t=1}^{T} 1(R_t = R) \phi\left(\frac{v - \text{IMMPRICE}_t}{\delta_{\text{IMM}}}\right) \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} 1(\hat{\omega}_{it} \leq v)\right)}{\sum_{t=1}^{T} 1(R_t = R) \phi\left(\frac{\text{IMMPRICE}_t}{\delta_{\text{IMM}}}\right)}
$$

where $\delta_{\text{IMM}}$ is a chosen bandwidth (see Appendix B).

Applying both the issue ceiling $S_t$ and alternative reservation price $\hat{p}_{0it}$ to each auction, I obtain a pseudo sample of auction results; i.e., revenue and quantity of bills sold. The only remaining issue is how to compare seller utility for the pseudo sample with the results in the observed sample. In the strictest sense, we cannot do this, because we do not know the seller's preferences. As discussed earlier, it may be that the seller's marginal valuation per unit sold at an auction is not constant. Indeed, there is no reason to believe the seller's preferences even to be time-separable; for example, the seller's preferences in an auction of 91-day bills may be influenced by the desire to roll over the maturing bills issued 91 days earlier. Nonetheless, I must choose some representation of seller utility in order to be able to present results. I will assume that the seller is risk-neutral and has constant marginal valuation $v_{0t}$ for bills at auction $t$. Utility at an auction is then simply the revenue generated by the bills sold plus $v_{0t}$ times the portion of $S_t$ left unsold. Note that, in order to preserve comparability to the results observed in the sample, I impose the pro rata cutoff rule when determining the allocation of bills to the pseudo bids.

The alternative reservation price rules explored in this section are linear combinations of $p_0$ (the observed reservation price) and IMMPRICE. These seem the simplest and least arbitrary rules to express and implement; to consider a larger space of alternative rules would only create a problem similar to "data mining." Because the relationship between the interbank rate and treasury bill valuations may vary across regimes, I allow the constant
to be different for each regime. That is, I set

$$\tilde{p}_{0t} = (1 - \alpha)p_0 + \alpha(\text{IMMPRICE}(t) + \rho(R_t))$$ (19)

where $\alpha \in [0,1]$ and the $\rho$ are constants; the $\rho$ are calibration parameters set to keep the total quantity sold in each regime period in the scenario roughly equal to the amount actually sold in the observed sample.\textsuperscript{15} Calibration of the $\rho$ is a process of trial and error. As starting values, I take the mean difference between the estimated bidder valuations $\hat{v}_{it}$ and IMMPRICE($t$) for all bids in all auctions in Regime $R_t$; these values are shown in Table 3. Curiously, banks seem to have valued IMM loans more highly than treasury bills

Table 3: Mean differences between bidder valuations and IMMPRICE

<table>
<thead>
<tr>
<th>REGIME1</th>
<th>REGIME2</th>
<th>REGIME3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0079</td>
<td>0.0040</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

in the earliest period; this anomaly will be one motivation for exclusion of Regime 1 from the sample in Section 5.5.

Table 4 shows the change in seller utility and in quantity sold associated with alternative reservation price rules; these are reported as percentage changes from the results in the observed sample. I report two measures of $\Delta U$: the first values the bills at the observed reservation prices $p_0$, and the second values the bills at the IMM benchmark rate. Quantity sold is invariant to the seller's $v_{0t}$, so need be presented only once for each scenario.

In the first scenario, the reservation price is simply announced to be the unannounced rate $p_0$ actually used in the sample (i.e., $\alpha = 0$ in equation (19)). When the reservation price is announced, a bidder with $v \geq p_0$ will never bid below $p_0$; therefore, holding the choice of reservation price fixed, a seller can never sell less when announcing the reservation price than when keeping it hidden. This must hold true for each auction $t$, as well as in the

\textsuperscript{15}The assumption of linear utility seems reasonable only locally; therefore, I avoid comparison of reservation price rules which yield large differences in quantity sold.
Table 4: Auction Results: Alternative Reservation Price Rules

<table>
<thead>
<tr>
<th></th>
<th>REGIME1</th>
<th>REGIME2</th>
<th>REGIME3</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>74 auctions</td>
<td>189 auctions</td>
<td>73 auctions</td>
<td>336 auctions</td>
</tr>
<tr>
<td></td>
<td>3081 bids</td>
<td>6520 bids</td>
<td>2821 bids</td>
<td>12422 bids</td>
</tr>
<tr>
<td>$\hat{p}_0 = p_0$ ($\alpha = 0$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U$ ($v_0 = p_0$)</td>
<td>-0.10%</td>
<td>-0.02%</td>
<td>-0.02%</td>
<td>-0.04%</td>
</tr>
<tr>
<td>$\Delta U$ ($v_0$=IMMPRICE)</td>
<td>-0.03%</td>
<td>-0.03%</td>
<td>-0.02%</td>
<td>-0.03%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>8.27%</td>
<td>2.47%</td>
<td>0.69%</td>
<td>3.67%</td>
</tr>
<tr>
<td>$\hat{p}_0 =$IMMPRICE$+\rho$ ($\alpha = 1$, $\rho =[-0.0091, -0.00195, 0.0002]$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U$ ($v_0 = p_0$)</td>
<td>-0.04%</td>
<td>-0.14%</td>
<td>-0.06%</td>
<td>-0.10%</td>
</tr>
<tr>
<td>$\Delta U$ ($v_0$=IMMPRICE)</td>
<td>0.12%</td>
<td>-0.07%</td>
<td>-0.04%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>-6.03%</td>
<td>0.10%</td>
<td>3.70%</td>
<td>-0.83%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = 0.5$, $\rho =[-0.0091, -0.00195, 0.0002]$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U$ ($v_0 = p_0$)</td>
<td>-0.03%</td>
<td>-0.09%</td>
<td>-0.04%</td>
<td>-0.06%</td>
</tr>
<tr>
<td>$\Delta U$ ($v_0$=IMMPRICE)</td>
<td>0.05%</td>
<td>-0.03%</td>
<td>-0.03%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>1.54%</td>
<td>0.90%</td>
<td>0.69%</td>
<td>1.03%</td>
</tr>
<tr>
<td>$\hat{p}_0 =$IMMPRICE$+\rho$ ($\alpha = 1$, $\rho =[-0.0093, -0.0019, 0]$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U$ ($v_0 = p_0$)</td>
<td>-0.04%</td>
<td>-0.14%</td>
<td>-0.07%</td>
<td>-0.10%</td>
</tr>
<tr>
<td>$\Delta U$ ($v_0$=IMMPRICE)</td>
<td>0.11%</td>
<td>-0.07%</td>
<td>-0.05%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>-5.18%</td>
<td>0.10%</td>
<td>7.09%</td>
<td>0.05%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = A(\xi = 0.5)$, $\rho =[-0.0091, -0.00195, 0.0002]$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U$ ($v_0 = p_0$)</td>
<td>-0.03%</td>
<td>-0.11%</td>
<td>-0.04%</td>
<td>-0.07%</td>
</tr>
<tr>
<td>$\Delta U$ ($v_0$=IMMPRICE)</td>
<td>0.05%</td>
<td>-0.05%</td>
<td>-0.02%</td>
<td>-0.02%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>1.60%</td>
<td>1.26%</td>
<td>0.73%</td>
<td>1.25%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = A(\xi = 1)$, $\rho =[-0.0093, -0.0019, 0]$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U$ ($v_0 = p_0$)</td>
<td>-0.02%</td>
<td>-0.09%</td>
<td>-0.03%</td>
<td>-0.06%</td>
</tr>
<tr>
<td>$\Delta U$ ($v_0$=IMMPRICE)</td>
<td>0.06%</td>
<td>-0.02%</td>
<td>-0.02%</td>
<td>0.01%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>0.48%</td>
<td>-1.38%</td>
<td>-0.12%</td>
<td>-0.64%</td>
</tr>
</tbody>
</table>
aggregate results presented.\footnote{Results at the auction level are available upon request.}

Over the entire sample, this rule yields an increase of 3.67\% in quantity sold, and a loss of only 0.04\% in utility. It would seem then that it should not be difficult to find a rule which yields no change in quantity sold and a gain in utility. However, despite its presumed correlation with bidder valuations, IMMPRICE does not appear to provide a solution. In the next several boxes of Table 4, I provide results for \( \hat{p}_0 = \text{IMMPRICE} + \rho \) and for linear combinations of IMMPRICE and \( p_0 \). None of these yield unambiguous gains in seller utility.

In the last boxes of Table 4, I present results for rules which makes use of information on the quantity demands. The idea is to extract additional rents when the expected quantity demand is strong, and to lower the reservation price, in order to preserve sales, when demand is weak. Therefore, I set \( \hat{p}_0 \) near the maximum of \( p_0 \) and IMMPRICE when \( \hat{C} \) (the expected coverage ratio) is high, and near the minimum when \( \hat{C} \) is low. Let

\[
A(\hat{C}, p_0, \text{IMMPRICE}, \xi) \equiv \begin{cases} 
    \exp(-\xi \hat{C}), & \text{if } p_0 > \text{IMMPRICE}; \\
    1 - \exp(-\xi \hat{C}), & \text{otherwise.}
\end{cases}
\]  

(20)

In equation (19), let \( \alpha_t = A(\hat{C}_t, p_{0t}, \text{IMMPRICE}_t, \xi) \) for some chosen \( \xi \). This method too does not yield unambiguous gains.

5.5 Analysis on a reduced sample

In Section 2.6, I argue that the quantity demands of each bidder generally can be taken as exogenous. This assumption may not be valid for those auctions during Regime 2 in which the expected reservation rate was substantially lower than the IMM benchmark. In these cases, the banks may have not wished to purchase any bills at the expected reservation rate (recall that the reservation rate could nearly be taken as known during Regime 2), but felt obliged to do so by the government. Therefore, I eliminate from the sample all auctions in which the expected reservation rate (calculated by converting \( \hat{p}_0 \) from a discount price into a rate) is more than 1.5 percentage points below the IMM benchmark rate. When this
spread is high, quantity demand is unusually low: For auctions eliminated by this criterion, the mean coverage ratio is 1.3; for auctions that remain in the sample, the ratio is 2.2. Seventeen auctions, all in Regime 2, are eliminated by this criterion.

I also eliminate Regime 1 from the sample. As can be seen in Figure 2 and Table 3, banks during Regime 1 demanded higher rates on treasury bills than on interbank loans of a similar maturity. It is difficult to understand this preference; for example, I am not aware of tax rules that could be responsible. Moreover, the Portuguese banking sector has been substantially deregulated in recent years. The first regime corresponds roughly to the pre-deregulation period in which banks were implicitly taxed through binding credit ceilings which forced them to hold substantial amounts of cash. Because the incentives during this period were distorted, bidder behavior may be difficult to interpret and comparison to the later regimes may be inappropriate.

I proceed as before. The OLS estimation of \( \tilde{p}_0 \) for Regimes 2 and 3 need not be repeated. (Regressions were run separately by regime, so elimination of Regime 1 makes no difference. Also, the Regime 2 auctions eliminated in this section should not be eliminated from estimation of \( \tilde{p}_0 \); from the perspective of the bidders, these are valid auctions from which to extract information on changes in the reservation price.) However, the index variable for bid quantities (\( \tilde{\lambda} \)) needs to be re-estimated if the quantities submitted in the excluded auctions are indeed not generated by the same exogenous process. The MLE coefficients for the reduced sample, presented in Table 5, are quite similar to the full sample coefficients given in Table 2.

Finally, estimates of change in seller utility and quantity sold for alternative reservation price rules are presented in Table 6. These results are more encouraging than those of Table 4, but also do not yield unambiguous improvements. Setting \( \tilde{p}_0 = p_0 \) (i.e., simply announcing the unannounced reservation price in the sample) yields both a gain in quantity sold of 2.4% and a small positive gain in seller utility (if \( v_0 = \text{IMMPRICE} \)). There exist other alternative rules with gains in both quantity sold and utility (e.g., the sixth row in the table), but these gains do not dominate the \( \tilde{p}_0 = p_0 \) benchmark. Simply setting \( \tilde{p}_0 = p_0 + c \) also
Table 5: Estimation of Bid Quantity Distribution (Reduced Sample)

<table>
<thead>
<tr>
<th></th>
<th>Mean†</th>
<th>Coefficient</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>CONSTANT</td>
<td>1.0000</td>
<td>-3.8800</td>
<td>0.3840</td>
</tr>
<tr>
<td>TIME</td>
<td>1.3170</td>
<td>-0.3911</td>
<td>0.0321</td>
</tr>
<tr>
<td>ln(NBIDDERS)</td>
<td>2.3323</td>
<td>-0.0499</td>
<td>0.0230</td>
</tr>
<tr>
<td>N/NBIDDERS</td>
<td>2.8129</td>
<td>0.2513</td>
<td>0.0157</td>
</tr>
<tr>
<td>ln(S)</td>
<td>9.7940</td>
<td>-0.3700</td>
<td>0.0366</td>
</tr>
<tr>
<td>IMMVOL</td>
<td>1.3206</td>
<td>0.3268</td>
<td>0.0413</td>
</tr>
<tr>
<td>IMMVOL²</td>
<td>2.3221</td>
<td>-0.0610</td>
<td>0.0125</td>
</tr>
<tr>
<td>HASADDSS</td>
<td>0.1897</td>
<td>-0.0412</td>
<td>0.0341</td>
</tr>
<tr>
<td>MAT91</td>
<td>0.4625</td>
<td>0.0453</td>
<td>0.0232</td>
</tr>
</tbody>
</table>

†: Mean of independent variable.

does not improve results: Many of the estimated $\hat{v}_{it}$ are very close to the $\hat{p}_{0t}$, so even a small positive constant $c$ leads to a severe loss in quantity sold; see the second row in the table.

5.6 Obtaining standard errors

A confidence interval for estimated utility and quantity sold under a given reservation price rule may be constructed using a bootstrap method similar to the methods described in Härdle (1990, §4.2). As a single complete estimation of a reservation price scenario takes up to ten hours on an IBM RS/6000, this will be computationally feasible for (at best) one chosen scenario.

6 Conclusion

In this paper, I have derived an equilibrium model for bidding strategy in a divisible good auction where bidder quantity demands are exogenously drawn random variables. Extending the methods of ELLV to my model, I use a nonparametric estimator to construct a pseudo sample of bidder valuations from my observed sample of bids. This pseudo sample is used to estimate seller revenue under alternative reservation price rules which make fuller
Table 6: Auction Results for the Reduced Sample

<table>
<thead>
<tr>
<th></th>
<th>REGIME 2</th>
<th>REGIME 3</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>172 auctions</td>
<td>73 auctions</td>
<td>245 auctions</td>
</tr>
<tr>
<td>$\hat{p}_0 = p_0$ ($\alpha = 0$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.00%</td>
<td>-0.00%</td>
<td>-0.00%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>0.01%</td>
<td>-0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>3.00%</td>
<td>0.69%</td>
<td>2.37%</td>
</tr>
<tr>
<td>$\hat{p}_0 = p_0 + 0.0001$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>0.01%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>0.01%</td>
<td>-0.02%</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>-5.28%</td>
<td>-26.32%</td>
<td>-11.06%</td>
</tr>
<tr>
<td>$\hat{p}_0 = \text{IMMPRICE}$ ($\alpha = 1$, $\rho = [0 0]$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.04%</td>
<td>-0.04%</td>
<td>-0.04%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>0.04%</td>
<td>-0.02%</td>
<td>0.02%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>-7.75%</td>
<td>7.84%</td>
<td>-3.46%</td>
</tr>
<tr>
<td>$\hat{p}_0 = \text{IMMPRICE}+\rho$ ($\alpha = 1$, $\rho = [-0.0019 0.0003]$)</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.09%</td>
<td>-0.04%</td>
<td>-0.08%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>-0.03%</td>
<td>-0.02%</td>
<td>-0.02%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>-1.83%</td>
<td>1.76%</td>
<td>-0.84%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = 0.5$, $\rho = [-0.0019 0.0003]$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.05%</td>
<td>-0.02%</td>
<td>-0.04%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>0.00%</td>
<td>-0.01%</td>
<td>-0.00%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>0.31%</td>
<td>0.94%</td>
<td>0.48%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = 0.25$, $\rho = [-0.002 0.0002]$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.03%</td>
<td>-0.01%</td>
<td>-0.02%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>0.01%</td>
<td>-0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>2.81%</td>
<td>-0.32%</td>
<td>1.95%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = A(\xi = 0.5)$, $\rho = [-0.0019 0.0003]$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.07%</td>
<td>-0.02%</td>
<td>-0.05%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>-0.01%</td>
<td>-0.00%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>0.03%</td>
<td>0.38%</td>
<td>0.13%</td>
</tr>
<tr>
<td>$\hat{p}_0$ from Eq. (19) ($\alpha = A(\xi = 1.5)$, $\rho = [-0.0022 0.00024]$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta U (v_0 = p_0)$</td>
<td>-0.04%</td>
<td>-0.01%</td>
<td>-0.03%</td>
</tr>
<tr>
<td>$\Delta U (v_0=\text{IMMPRICE})$</td>
<td>0.02%</td>
<td>0.01%</td>
<td>0.01%</td>
</tr>
<tr>
<td>$\Delta \text{Quantity Sold}$</td>
<td>-1.56%</td>
<td>-1.03%</td>
<td>-1.42%</td>
</tr>
</tbody>
</table>
use of observable information on bidder valuations. Preliminary results do not support my conjecture that such alternative rules will be revenue superior to the rule used in the sample.

The failure to find a clearly superior reservation price rule could have a number of explanations, two of which may merit further examination. First, the interbank money market benchmark, though the best available, is not a reliable measure of the value of the treasury bills. Liquidity in the longer IMM maturities is uneven, and rates on longer trades may be idiosyncratic to the borrower. This may be especially true during Regime 2, which witnessed extremely high volatility on the IMM. It would be valuable, for other policy questions as well as this analysis, to have a better understanding of how banks determine interest rates in the absence of liquid markets. Second, the models assume that there is no collusion among the bidders. If the bidders in fact use lag reservation prices as a focal point for collusion, my estimates of the unobserved bidder valuations will be more strongly correlated with reservation price and less strongly correlated with the IMM benchmark than the true valuations. Further research is needed to determine whether we can use the estimated sample of valuations in a formal test of bidder collusion.

The model and estimation method of this paper can be adapted to evaluate other policy questions; e.g., the effect upon the joint distribution of revenue and quantity sold of changes in the issue ceiling $S$, or of increasing $N$ by permitting non-bank or foreign financial institutions to bid. It may also be applied to similar policy questions in auctions of foreign exchange currently used in several developing countries.
A Technique notes on Figure 3

In this appendix, I describe a method of calculating expected revenue and expected quantity sold as functions of the reservation price in Model I of Chapter 1. These two functions are of closed form, but nonetheless very demanding to calculate directly due to nested integrals. Monte Carlo simulation provides an alternative that is much easier to implement.

I choose a sample size $T = 1200$. For each $t = \{1, \ldots, T\}$, I draw a random sample of 10 bidder valuations from a beta(2,2) distribution and randomly draw a single reservation price from a $U[0,1]$ distribution. For each $t$, I calculate the outcome of the auction using the equilibrium bidding strategy given by equation (4).

Let $z_t$ be the revenue per unit obtained in auction $t$ and $p_{0t}$ be the reservation price. These form a sample of $z$ and $p_0$ from which to estimate the expected revenue per unit conditional on the reservation price. I use the Nadaraya-Watson kernel estimator (see Härdle, 1990, §3.1):

$$E(z|p_0) = \frac{\sum_{t=1}^{T} z_t \phi\left(\frac{p_{0t} - 0.5}{h}\right)}{\sum_{t=1}^{T} \phi\left(\frac{p_{0t} - 0.5}{h}\right)}$$

where $\phi$ is the standard normal density and $h$ is a bandwidth chosen by the standard rule of thumb (see, e.g., Härdle, 1990) to be the standard deviation of $z$ times $T^{-0.2}$. To create the top window in Figure 3, I use the random sample to estimate $E(z|p_0)$ for a vector of $p_0$ values between 0.25 and 0.75. I estimate the conditional expected quantity sold similarly.

The depicted shapes of the functions for expected revenue and expected quantity sold do depend on the assumption of a beta(2,2) distribution for the bidder's valuations. Note, however, that these functions are invariant to the choice of a distribution from which to draw reservation prices, because $p_0$ is assumed to be announced to the bidders; I use a $U[0,1]$ distribution only because it is an efficient way to create data for the nonparametric estimation of the functions.
B Selection of bandwidths in nonparametric regressions

In nonparametric regression, the degree of smoothing is controlled by the choice of bandwidth. The optimal bandwidth is the one which gives the best "signal to noise" ratio. What is best will vary from application to application, and ultimately must contain some subjective element. The usual method is to choose an "optimal" bandwidth using cross-validation or the plug-in method (see Härdle, 1990, §5.1), and then test the sensitivity of the regression locally to the choice of bandwidth.

In application to this paper, iterative methods such as cross-validation are impractical. Although the method of Section 5 is many times faster than MLE, it still requires several hours of CPU on a workstation. Therefore, I make use of the standard rule-of-thumb (see, Bierens, 1987): For a univariate $x$, let the bandwidth be the standard deviation of $x$ times the number of observations raised to the -0.2 power. For a multivariate $x$, decompose the variance-covariance matrix of $x$ into $L/L$, and set the bandwidth matrix to $L$ times the number of observations to the $-1/(k + 4)$ power, where $k$ is the dimension of $x$.

Application of this rule to the regressions in Section 5 is complicated by the presence in the regressions of the multinomial regime variable. Because I do not constrain the relationship between independent and dependent variables to be the same across regimes, the sample size is effectively divided in three. Therefore, I take the "number of observations" to be $T/3$. In Section 5.5, only the last two regimes are included, so I divide the number of observations by two.
References


Chapter 2

Multiple Bids in a Treasury Bill Auction

In the United States and other OECD countries, and in many developing countries as well, bidders at auctions of treasury securities are permitted to submit multiple bids.¹ Bidders do make use of this rule: In the Portuguese treasury bill auctions for which I have data, the median number of bids per bidder is three; a histogram of the number of bids submitted per bidder is shown in Figure 1. Despite the widespread adoption of this institutional feature and its use by bidders, for the sake of simplicity it has generally been ignored in both the theoretical and empirical literature.²

In this paper, I will use bidding data from Portugal's treasury bill auctions to demonstrate that bidders use multiple bids to control exposure to the risk of winner's curse: as the degree of uncertainty in the bidding environment increases, bidders submit a greater number of bids and spread their bids more widely. In addition to adding a new stylized fact to our limited empirical knowledge of treasury auctions, this result has a practical policy application: Bidders' use of multiple bids may be used as a gauge for the degree of uncertainty in the bidding environment. The seller cares about this uncertainty because, in common-value auctions, expected seller revenue decreases as the precision of publicly observable information on the value of the good decreases, even if bidders are risk-neutral.

¹The Joint Report on the Government Securities Market (1992, Appendix B), henceforth cited as Joint Report, explicitly mentions that Australia, Belgium and Italy permit bidders to submit multiple bids; no country is reported to prohibit multiple bids. Umlauf (1993) reports that the Mexican treasury securities auction permits multiple bids. I have confirmed the same for Portugal and Brazil.

²In a theoretical study of the interaction between primary and secondary markets for treasury securities, Bikhchandani and Huang (1989) restrict each bidder to a single bid of fixed size. In an empirical analyses of the effect of the reservation rate in Portugal's treasury bill auction, Chapter 1 of this thesis restricts each bidder to a single bid of variable size.
Figure 1:

Histograms: Number of bids submitted

Full sample

Frequent bidders only
The intuition for this theoretical result comes from viewing the auction as a mechanism design problem: as the precision of publicly observable information decreases relative to that of private information, the bidder must earn higher rent on her private information.

It might seem that the information in multiple bids would be redundant to the government, as the government ought to be able to form a direct measure of uncertainty in the bidding environment from the volatility of secondary market rates. In the United States, where markets for treasury securities are highly liquid and rates are reported continuously, such a direct measure is now available, although the data are difficult to obtain; for example, Simon (1994) uses intraday rate information from the trading screens of a primary dealer to measure volatility. Elsewhere, however, a direct measure of volatility may not be available: For simplicity, consider a country in which there is a liquid secondary market for government securities but no when-issued trading; examples include Australia, Belgium, Denmark, and Japan (Joint Report). If \( r_0 \) is the market rate for government securities of comparable maturity at the deadline hour for bids (\( t_0 \)), and \( r_1 \) is the market rate for the bills at the time auction results are announced and positions unwound in the secondary market (\( t_1 \)), then the uncertainty faced by the bidders is measured by the conditional variance \( V[r_1| r_0] \).

Unless market rates are collected at times \( t_0 \) and \( t_1 \), which are often intraday, this variance may be impossible to estimate. Bikhchandani, Edsparr and Huang (1993) point out that the period from \( t_0 \) to \( t_1 \) is distinct in the way in which information is released into the market: bidders with private information will avoid trading on that information before the auction (as it would reveal the information to the other bidders), but will want to trade on it before auction results are announced (as the auction results may reveal the information to the market).\(^3\) Therefore, a measure of market volatility based on close-of-market rates around the auction cannot simply be scaled proportionally to \( t_1 - t_0 \).

I will motivate the empirical analysis with a theoretical model of a multiple bid auction. The model predicts that as the precision of publicly observable information decreases, risk-averse bidders will spread their bids more widely and the seller will receive lower expected

\(^3\) They show in the US treasury auction that the when-issued rate change between \( t_0 \) (Monday at 1:00pm) and \( t_1 \) (Monday 3:30pm) is positively related to the information innovation in the auction.
revenue. The first result is intuitively appealing because we expect risk-averse bidders to exhibit downward sloping demand for a risky good, and multiple bids are a means of expressing a demand schedule. The second is appropriately consistent with common-value auction models for single indivisible goods, as well as empirical results for the US treasury auction by Simon (1994), Cammack (1991), and Bikhchandani, Edsparr and Huang (1993).

As far as I am aware, this is the only model in the literature of a multiple-unit auction in which bidders submit a discrete number of bids. Heretofore, the theoretical study of multiple bid auctions has followed Wilson (1979), in which bidders are assumed to submit a continuous demand schedule; see, for example, Edsparr (1993) and Back and Zender (1993). Wilson's model has important technical difficulties: First, it can be solved analytically for only a small number of special cases; even numerical solutions appear to be difficult in general. Second, the model appears to admit a continuum of equilibria in many cases, and in other cases yields multiple equilibria that are qualitatively very different. Finally, the model's equilibria can have unusual properties; e.g., the expected winning price need not be increasing in the number of bidders. It is also unclear whether a model in which bidders submit continuous demand schedules can describe well an auction in which bidders typically submit a small number of discrete bids. For example, Wilson's model avoids pro rata distribution at the stopout price, which in reality is nearly always observed at treasury security auctions and may therefore be an important consideration for the bidders. Moreover, it seems likely that the unusual properties that can emerge from the Wilson model are due to the continuity of bid schedules, and may disappear in a discrete multiple-bid framework.

The model presented here is also relevant to the policy debate on the choice between discriminatory and uniform pricing rules. To address this policy debate, I consider whether risk-neutral bidders in a discriminatory auction will submit multiple bids. Although seem-

---

4 Simon finds the markup of the average winning rate over the when-issued rate is positively related to pre-auction volatility in the bond yield. (Note that seller revenue decreases with this markup.) Cammack finds the markup is positively related to the dispersion of winning bid rates. Bikhchandani, Edsparr and Huang provide (somewhat inconclusive) evidence that bidders' expected profit increases with the dispersion of winning bid rates, which in turn is positively related to two proxies for uncertainty (the bid-ask spread in the when-issued market and the term premium).
ingly unrelated, this question is important to the debate for two reasons: First, the theoretical evidence in favor of uniform pricing depends on the assumption of risk-neutrality; if risk-neutral bidders do not submit multiple bids, and yet we observe multiple bids in practice, we may conclude that bidders are risk-averse, and thereby undercut the theoretical argument for uniform pricing. Second, recent theoretical work by Back and Zender (1993) shows that uniform price rules allow the existence of self-enforcing collusive strategies when bidders can submit multiple bids; if there is no equilibrium under discriminatory pricing in which risk-neutral bidders submit multiple bids, then discriminatory price rules do not permit this collusive equilibrium and are thus preferable. Preliminary results from my model suggest that risk-neutral bidders will not submit multiple bids; however, the results are ambiguous and require further exploration.

As far as I am aware, there is only one previous empirical study of multiple bids in the literature. Scott and Wolf (1979) propose a non-equilibrium model in which each bidder forms probability distributions over the auction stopout rate (i.e., the lowest winning rate) and post-auction market rate. The bidder, who is assumed risk-averse, selects a portfolio of multiple bids much as a risk-averse investor forms a portfolio of risky securities. Scott and Wolf use bidding data and pre-auction forecasts of the stopout rate from two primary dealers in the US treasury markets to examine how closely the dealers’ bids approximate a mean-variance efficient portfolio. Note that this model differs from standard auction analyses in that bidders ignore the effect of their own strategies on the stopout rate.

In Section 1, I develop a model of a common-value, multiple bid, discriminatory price auction with risk-averse bidders. In Section 2, I consider the same model under risk-neutrality. Empirical results from the Portuguese treasury bill auctions are presented in Section 3; both the number of bids submitted by each bidder and the dispersion of each bidder’s bids are analyzed as dependent variables. Concluding remarks follow.
1 Discriminatory auction with risk-averse bidders

The theoretical problem of multiple bids in a divisible good auction appears to be extremely difficult, as a strategy must select both price and quantity for each bid. Even if we simplify to the multiple unit case, and assume that each bid is for one unit, the problem remains intractable using the standard common-value auction framework of continuously distributed signals of the value of the good. The problem, briefly stated, is that the first order conditions of the bidder's maximization program give rise to terms of the form $S_i^{-1}(S_1(x))$, where $S_i$ is the equilibrium strategy for the $i$th bid as a function of the signal $x$. I avoid this technical difficulty by restricting the sets of possible signals and of permitted bids to be finite. The set of possible solutions for the equilibrium bidding strategy is then finite as well, so I can easily solve the model numerically.

Two bidders compete for two identical units. The true value of each unit, $v$, is unknown to the bidders at the time of the auction. The bidders share a prior distribution $F(v)$, which I assume to be the beta($\alpha\mu, \alpha(1-\mu)$) cdf; this distribution has mean $\mu$ and variance decreasing in $\alpha$. Note that the beta distribution has support $[0, 1]$, which makes it particularly suitable to represent the value of a treasury bill expressed as a discount price.

The bidders draw conditionally independent signals from the finite set $\mathcal{X} \equiv \{0, 1, \ldots, K\}$; I assume that these signals, conditional on $v$, have binomial distribution:

$$g(x|v) = \binom{K}{x} v^x (1-v)^{K-x}.$$ 

The beta distribution is conjugate for a signal drawn from a binomial distribution (see DeGroot, 1970, §9.2), so the posterior distribution on $v$, $F(v|x)$, is the beta($x+\alpha\mu, K-x+\alpha(1-\mu)$) cdf. The parameter $K$ represents the precision of private information.

I assume that bids are restricted to a finite set $\Lambda \equiv \{0, 1/\lambda, 2/\lambda, \ldots, 1\}$ for a positive integer $\lambda$.\footnote{For an equilibrium to exist, it is necessary to assume that the set of permitted bids is not dense. If there are a finite number of possible signals, then there is a nonzero probability that one's opponent has received the same signal. If the bidder can bid $\epsilon$ above the opponent's strategy for that signal, the bidder...} A consequence of this restriction is that there frequently will be ties; I as-
sume that ties are settled by randomization among the tied bids, rather than by pro rata distribution. The assumption of discrete intervals between permitted bid prices accords with practice: The Portuguese treasury bill auction restricts bid rates to be expressed as multiples of one-sixteenth of a percentage point; in the US auction, discount prices on the bids are rounded to two decimal places.

Last, I assume in this section that bidders are risk-averse. This assumption is unusual within the literature on treasury auctions; indeed, risk-neutrality has generally been assumed without comment. Branco (1993) justifies the standard assumption by noting that primary dealers, especially in the United States, are fairly large firms which should behave as if risk-neutral. However, the managers who choose the bids may be risk-averse, and may have incentive contracts which do not provide them with full insurance.

In particular, I assume that bidders have CARA utility \( U(z) = -\exp(-\gamma z) \), where \( \gamma \) is the coefficient of absolute risk aversion. Under this choice of utility function, decisions are independent of initial wealth: Given initial wealth \( W_0 \), the utility of post-auction wealth is \( U(W) = U(W_0 + \Delta W) = \exp(-\gamma W_0)U(\Delta W) \); the term \( \exp(-\gamma W_0) \) can be treated as a constant and ignored, so that utility is a function only of the change in wealth.

The bidder chooses a bidding strategy \( \hat{S} : \mathcal{X} \rightarrow \Lambda \times \Lambda \); that is, the bidding strategy maps from the set of possible signals to the set of possible pairs of bids. If the bidder receives a signal \( x \) and believes his opponent to use strategy \( S \), then the pair of bids \( (s_1, s_2) \) is chosen to maximize

\[
U(s_1, s_2 | S, x) \equiv \int_0^1 \left( U(2v - s_1 - s_2) h_2(s_1, s_2 | S, v) + U(v - s_1) h_1(s_1, s_2 | S, v) + U(0) h_0(s_1, s_2 | S, v) \right) dF(v | x) \tag{1}
\]

where \( h_i \) is the conditional probability of winning exactly \( i \) units, given \( v \) and the opponent's

---

6I have also implemented the model using pro rata settlement of ties. The results are qualitatively similar, but computational difficulties with finding equilibria (as discussed below) become more severe.
strategy $S$, and (without loss of generality) I restrict $s_1 \geq s_2$. The bidder's strategy $\bar{S}$ given the opponent's $S$ is the set of optimal pairs $(s_1(x), s_2(x))$ for each $x \in \mathcal{X}$. A strategy $S$ is an equilibrium if the optimal response to $S$ is $S$.

Let $\Delta$ be the expected difference between a bidder's high and low bids in equilibrium:

$$\Delta \equiv \sum_{x=0}^{K} (S_1(x) - S_2(x))g(x)$$

where $g(x)$, the unconditional probability of signal $x$, is given by

$$g(x) = \binom{K}{x} \frac{B(x + \alpha \mu, K - x + \alpha(1 - \mu))}{B(\alpha \mu, \alpha(1 - \mu))}$$

and the function $B$ is the Beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$; see Winkler (1972, p. 207). I conjecture that the model will show that $\Delta$ increases with the degree of risk aversion $\gamma$ and decreases with the precision parameter $\alpha$ of the prior distribution. I also conjecture that expected seller revenue, $R$, decreases with $\gamma$ and increases with $\alpha$.

The model is not difficult to implement. For a given set of parameters $\mu$, $K$ and $\lambda$, I randomly draw values for the parameters $\alpha$ and $\gamma$ and a starting guess for the equilibrium strategy $S$. I form a bidder's strategy $\bar{S}$ under the assumption that the other bidder uses $S$. If $\bar{S} = S$, then $S$ is an equilibrium. Otherwise, I set $\bar{S} = \bar{S}$, and repeat. In Appendix A, I describe how equation (1) may be simplified to avoid numerical integration.

1.1 Results

I have obtained equilibria for a large number of parameter values. In Table 1, I present a representative selection from a set of 489 equilibria (72 of which are distinct) found for $(K = 5, \lambda = 12, \mu = 0.7)$ and $\alpha$ and $\gamma$ randomly chosen from $U[0, 28]$ and $U[1, 15]$ distributions, respectively. The general pattern is clear: When the prior is very precise (high $\alpha$) or when bidders are nearly risk-neutral (low $\gamma$), we have $S_1(x) = S_2(x)$ for all signals $x$; that is, each bidder submits two identical bids. As risk aversion increases or the prior on $\nu$ becomes less precise, $\Delta$ increases. Due to the discrete nature of the model, this
pattern is often violated locally, but I have not found any gross exceptions to the rule.

To provide visual intuition for the general shape of the model’s results, it is desirable to
smooth out some of the local irregularities. I treat the set of equilibria that I have found for
\((K = 5, \lambda = 12, \mu = 0.7)\) as a data sample, and smooth using the Nadaraya-Watson kernel
estimator (see Härdle (1990, §3.1)):

\[
E(\Delta|\alpha, \gamma) = \frac{\sum_{t=1}^{T} \Delta_t \phi\left(\frac{\alpha_t - \alpha}{h_{\alpha}}\right) \phi\left(\frac{\gamma_t - \gamma}{h_{\gamma}}\right)}{\sum_{t=1}^{T} \phi\left(\frac{\alpha_t - \alpha}{h_{\alpha}}\right) \phi\left(\frac{\gamma_t - \gamma}{h_{\gamma}}\right)}
\]

where \(\phi\) is the standard normal density, \(T = 489\) is the number of equilibria I have found,
and \(h_{\alpha}\) and \(h_{\gamma}\) are bandwidths chosen by \(h_z = \sigma_z T^{-1/8}\); these bandwidths are purposely
chosen to oversmooth slightly in comparison to the standard rule-of-thumb bandwidth for
a bivariate kernel regression (see Bierens, 1987). I plot the smoothed \(\Delta\) as a function of
\((\alpha, \gamma)\) in the upper window of Figure 2. Expected revenue \(R\) may be treated similarly. In
the lower window of Figure 2, I plot smoothed \(R\) as a function of \((\alpha, \gamma)\), using the same
choice of bandwidths as in the upper window.

The figure supports my conjectures concerning \(\Delta\) and \(R\). I have also obtained results
(not reported here) for parameter sets \((K = 6, \lambda = 15, \mu = 0.8)\), \((K = 6, \lambda = 18, \mu = 0.8)\),
and \((K = 7, \lambda = 9, \mu = 0.75)\) for a variety of randomly chosen \((\alpha, \gamma)\); these results are
qualitatively similar to those in Table 1 and the figure.

1.2 Problems with existence and uniqueness

In practice, it is sometimes difficult to find equilibria. First, if there does exist a pure
strategy equilibrium, it may not be possible to get to it by tâtonnement from some starting
guesses for \(S\). Second, there may not exist a pure strategy equilibrium for some sets of
parameter values; the problem is akin to that of the simple “matching pennies” game. In
either case, the program gets stuck in a loop of the form \(S^1 \rightarrow S^2 \rightarrow \cdots \rightarrow S^t \rightarrow S^1 \rightarrow
S^2 \cdots\). This problem becomes more acute as \(K\) and \(\lambda\) increase; i.e., as the discrete model
more closely approximates a continuous one. For example, for parameters \((K = 5, \lambda = 12)\),
roughly 40% of randomly chosen \((\alpha, \gamma)\) and starting guess for \(S\) lead to equilibria; for
<table>
<thead>
<tr>
<th>Signal</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Δ</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 25.96466$</td>
<td>$\lambda S_1$</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>0.00000</td>
</tr>
<tr>
<td>$\gamma = 1.02909$</td>
<td>$\lambda S_2$</td>
<td>5</td>
<td>7</td>
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<td>7</td>
<td>7</td>
<td>0.00000</td>
</tr>
<tr>
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<td>$\lambda S_1$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
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<td>8</td>
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<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
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<td>6</td>
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<td>7</td>
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<td>7</td>
<td>7</td>
</tr>
<tr>
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<td>$\lambda S_1$</td>
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<td>6</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$\gamma = 2.23753$</td>
<td>$\lambda S_2$</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$\alpha = 19.49684$</td>
<td>$\lambda S_1$</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$\gamma = 7.05056$</td>
<td>$\lambda S_2$</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
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<td>6</td>
<td>6</td>
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<td>7</td>
</tr>
<tr>
<td>$\gamma = 11.07068$</td>
<td>$\lambda S_2$</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$\alpha = 14.99502$</td>
<td>$\lambda S_1$</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$\gamma = 9.42610$</td>
<td>$\lambda S_2$</td>
<td>4</td>
<td>4</td>
<td>6</td>
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<td>7</td>
</tr>
<tr>
<td>$\alpha = 19.53761$</td>
<td>$\lambda S_1$</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$\gamma = 13.44789$</td>
<td>$\lambda S_2$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$\alpha = 27.77026$</td>
<td>$\lambda S_1$</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>
Figure 2:

Kernel Regressions using Simulated Equilibria: CARA utility case

\[ \Delta \] (expected observed spread between a bidder's bids) and \( R \) (expected revenue), plotted as functions of risk aversion \( \gamma \) and the precision \( \alpha \) of the prior distribution of \( \nu \). Model parameters are \( (K = 5, \lambda = 12, \mu = 0.7) \).
parameters \((K = 6, \lambda = 18)\), the success rate is only 20%. A 20% success rate is not a significant problem: given enough random draws of starting values of \(S\), it typically is possible to find equilibria in all regions of the relevant space of \(\alpha\) and \(\gamma\) values. For higher \(\lambda\), however, it is quite difficult to find equilibria.

Another problem is that there exist multiple equilibria for some parameter values. This is not surprising for a discrete model, and is a concern only if there exist parameters which permit equilibria which differ substantially in \(\Delta\) or \(R\). Fortunately, it appears to be the case that when multiple equilibria exist, the equilibria are qualitatively similar. For each parameter pair \((\alpha, \gamma)\) in the set of equilibria found for \((K = 5, \lambda = 12, \mu = 0.7)\), I ran the tâtonnement program 20 times, each time with a randomly chosen initial starting value for \(S\). In many cases, more than one equilibria was found. In Table 2, I present the four equilibria found for \((\alpha = 21.3, \gamma = 13.9)\); I have not found any parameter values which yield a greater range of values for \(\Delta\). At lower levels of risk aversion, multiple equilibria exist less frequently and differ less from one another.

<table>
<thead>
<tr>
<th>Bid as function of Signal</th>
<th>(\Delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal 0 1 2 3 4 5</td>
<td></td>
</tr>
<tr>
<td>(\lambda S_1) 5 6 6 6 6 7</td>
<td>0.03154</td>
</tr>
<tr>
<td>(\lambda S_2) 5 5 5 6 6 6</td>
<td></td>
</tr>
<tr>
<td>(\lambda S_1) 4 6 6 6 6 7</td>
<td>0.04645</td>
</tr>
<tr>
<td>(\lambda S_2) 4 4 4 6 6 6</td>
<td></td>
</tr>
<tr>
<td>(\lambda S_1) 5 6 6 6 7 7</td>
<td>0.05966</td>
</tr>
<tr>
<td>(\lambda S_2) 5 5 5 6 6 6</td>
<td></td>
</tr>
<tr>
<td>(\lambda S_1) 4 6 6 6 7 7</td>
<td>0.07456</td>
</tr>
<tr>
<td>(\lambda S_2) 4 4 4 6 6 6</td>
<td></td>
</tr>
</tbody>
</table>

1.3 Should the seller permit multiple bids?

When bidders are able to submit continuous demand schedules for a divisible good, Wilson (1979) demonstrates that there may exist equilibria that are vastly revenue inferior to the equilibrium of an auction in which bidders may bid only for the entire supply. I now consider
a related question for the auction of two units: Should the seller permit the bidders to submit multiple bids, or instead restrict bidders to submitting a single price bid for the two units?

For each pair \((\alpha, \gamma)\) for which I found an equilibrium in section 1.1, I run the model with the additional restriction that \(s_1 = s_2\); selected results are in Table 3. Let \(R_1(\alpha, \gamma)\) be the expected revenue for this single-bid model. Using the same kernel regression technique described earlier, I plot in Figure 3 the smoothed expected gain to permitting bidders to submit multiple prices, \(R(\alpha, \gamma) - R_1(\alpha, \gamma)\).\(^7\)

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\gamma)</th>
<th>(\lambda S) as function of signal</th>
<th>(R_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.96466</td>
<td>1.02909</td>
<td>5 7 7 7 7 7 7 7</td>
<td>1.16664</td>
</tr>
<tr>
<td>7.67787</td>
<td>1.20931</td>
<td>3 4 5 6 7 8</td>
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<td>27.69006</td>
<td>7.16742</td>
<td>6 6 7 7 7 7</td>
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<td>5 6 6 6 6 7</td>
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<td>7.58769</td>
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<td>17.29935</td>
<td>10.54487</td>
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<td>13.64728</td>
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</tr>
<tr>
<td>3.75493</td>
<td>11.93472</td>
<td>1 2 3 3 5 6</td>
<td>0.79960</td>
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<td>5.74733</td>
<td>10.52787</td>
<td>2 3 3 4 5 6</td>
<td>0.83625</td>
</tr>
</tbody>
</table>

\[\dagger\] Model parameters are \((K = 5, \lambda = 12, \mu = 0.7)\). For comparison to the multiple-bid case, the \((\alpha, \gamma)\) are taken from the equilibria reported in Table 1; I did not find single-bid equilibria for only two of these pairs.

The answer appears to depend on the parameter values. When risk aversion is low \((\gamma\) near zero), it makes little difference, because the unrestricted equilibrium strategy converges to the single-bid equilibrium. When risk aversion is high, allowing multiple bids is revenue superior for the seller only if the precision of public information is high. When \(\alpha\) is low,

---

\(^7\)I have also run this model for parameters \((K = 6, \lambda = 18, \mu = 0.8)\), which yield qualitatively similar results.
Figure 3:

Expected Gain to Allowing Multiple Bids

$R - R_1$ (expected seller gain to allowing multiple bids), plotted as functions of risk aversion $\gamma$ and the precision $\alpha$ of the prior distribution of $v$. Model parameters are $(K = 5, \lambda = 12, \mu = 0.7)$. Sample size is 475.
the reverse holds. The intuition, I believe, is that there is a tradeoff between the gain from providing insurance to the risk-averse bidders and the loss from weakening competition between the bidders. When the public prior on \( v \) is precise, bidders cannot earn large rents on their private signals, so the gain from the insurance value of multiple bids dominates. However, when the public prior is diffuse, bidding strategies are sloped more sharply downward, which reduces the intensity of competition between the bidders (since it reduces the likelihood that one bidder will take both units); for sufficiently low \( \alpha \), this effect dominates.

Further research will be needed to understand better the revenue properties of allowing multiple bids. Research here may also shed light on the unusual properties of Wilson's (1979) model. If the pattern of Figure 3 holds more generally, it would support the practice in some "low \( \alpha \)" countries of limiting the number of bids each bidder may submit; Portugal and Italy, for example, limit each bidder to six and five bids, respectively, whereas the US places no such limit at all.

### 1.4 Extension to \( n \) bidders

An extension of the model to \( n \) bidders is desirable. In single unit auctions, the problem of the winner's curse increases with the number of bidders, because winning (i.e., having the highest signal of the \( n \) bidders) is worse news about the value \( v \) as \( n \) increases. In the multiple-bid auction, my conjecture is that bidders will spread their bids more widely as the number of bidders increases. I also conjecture that expected seller revenue will increase with \( n \), which is consistent with single-unit models but in contrast to Wilson's (1979) model.

Unfortunately, this extension is likely to be difficult because the complexity of the \( h_i \) functions increases exponentially with the number of bidders. Even for three bidders, it will be tedious to work out all of the permutations for tied bids.

### 2 Multiple bids by risk-neutral bidders

In the CARA utility case, we see that the expected spread between the bidder's two bids narrows as risk aversion decreases; i.e., as \( \gamma \to 0, \Delta \to 0 \). In this section, I examine whether
this behavior holds in the limit. That is, do there exist equilibria in which risk-neutral bidders make strategic use of multiple bids?

The question is relevant to an old and persistent policy debate on the adoption of uniform price rules. This debate began as early as Friedman's (1959) recommendation that the Treasury switch from discriminatory price rules, under which winning bidders pay their bids, to uniform price rules, under which all winning bidders pay the lowest winning discount price. Interest in auction format arose again in the wake of the Salomon scandal (see Joint Report), and led to the current experiment with uniform price auctions of US Treasury notes. The intuition which favors uniform price rules is that it reduces winner's curse, and so induces more aggressive bidding. Milgrom and Weber (1982) show that in a common-value auction of a single unit to risk-neutral bidders, second-price sealed bid is revenue superior to first-price sealed bid. For this result to be applicable to treasury bill auctions, two conditions must hold: First, the primary dealers must be risk-neutral. Milgrom and Weber (1982) show that in an independent-private-values auction with risk-averse bidders, the first-price sealed bid auction is revenue superior to the second-price auction; in a common-value auction with risk-averse bidders, the ordering is ambiguous. If the discriminatory price auction has no equilibria in which bidders submit multiple prices, then real world bidders who do submit multiple prices are risk-averse.

Second, bidders' strategic use of multiple bids must be qualitatively similar under the two auction formats. Recent evidence suggests otherwise: In a Wilson (1979) style auction model in which bidders submit continuous demand schedules, Back and Zender (1993) show that there exists a downward sloping equilibrium under uniform pricing which is

---

8In the mid-1970s, the Treasury conducted a series of uniform price auctions. Simon (1993) provides evidence that uniform price rules lowered average revenue in these auctions. However, Umlauf (1993) finds that a switch from discriminatory to uniform price rules in the Mexican cetes auction increased government revenue. Tenorio's (1993) study of the Zambian foreign exchange auction also supports the use of uniform pricing.

9Strictly speaking, this argument applies only to auctions that are pure common-value, which is clearly not the case in Portugal due to capital market imperfections. Less obviously, it may not be the case for the United States, either. When there exists a when-issued market, the bidders' idiosyncratic forward positions introduces a private values component to the auction. Regardless, there are several important markets in which there exist liquid secondary markets but no when-issued trading; for these cases, the pure common-value model seems appropriate.
revenue inferior to a flat demand schedule equilibrium under discriminatory pricing. They
do not show these equilibria are unique. If the multiple-bid model exhibits only single price
equilibria under risk-neutrality and discriminatory price rules, it suggests that Back and
Zender's discriminatory price equilibrium is unique, and thus supports their conclusion.

The model is the same as in section 1, except bidders are now assumed risk-neutral.
Under this assumption, \( U(W) = W_0 + U(\Delta W) \); initial wealth enters as constant term and
may be ignored. Therefore, the representation of utility in equation (1) is valid here as well.
In Appendix A, I derive a computationally efficient expression for \( U \) under risk-neutrality.

The model is implemented using the tâtonnement method of section 1. The problem
with existence of equilibria discussed in section 1.2 is even more acute under risk-neutrality,
especially for low values of \( \alpha \). For parameters \( (K = 5, \lambda = 12, \mu = 0.7) \), I am able to find
equilibria for only a small number of values of \( \alpha \). However, I have obtained 21 distinct
equilibria for 136 values of \( \alpha \in [2.7, 10] \) for \( (K = 7, \lambda = 9, \mu = 0.75) \), 9 distinct equilibria
for 35 values of \( \alpha \in [2.6, 10] \) for \( (K = 6, \lambda = 8, \mu = 0.7) \), and smaller numbers of distinct equilibria for several other sets of \( (K, \lambda, \mu) \).

Out of all of these equilibria, I have found only one in which bidders submit multiple
prices. This equilibrium, which I denote \( S^m \), is given in the top row in Table 4. It is
strict, in the sense that the bidder strictly prefers strategy \( S^m \) over all others when the
opponent follows \( S^m \). It exists over a range of \( \alpha \); I have checked existence for all values
of \( \alpha \in \{3.95, 4.00, \ldots, 4.20\} \). As I have only this one example, it is difficult to determine
the conditions under which multiple-price equilibria will exist. This example appears to
be somewhat special: In the same range of \( \alpha \) values, I have found three other single-price
equilibria, which are given in the last three rows of Table 4. Also, this range of \( \alpha \) contains
a special point at 4.00. For \( \alpha < 4 \), the density \( f(v) \) goes to infinity as \( v \) approaches 1; for
\( \alpha > 4 \), \( f(1) = 0 \). These observations may be a starting point for future investigation into
the risk-neutral model.

Aside from \( S^m \), there does seem to be a general rule that risk-neutral bidders will not
submit multiple bids. Even for the counter-example, the degree of bid spreading is minimum
Table 4: Multiple Equilibria under Risk-Neutrality

<table>
<thead>
<tr>
<th>Bid as function of Signal</th>
<th>Signal:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda \bar{S}_1^{m} )</td>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( \lambda \bar{S}_2^{m} )</td>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( \lambda \bar{S}_1 )</td>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
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<td>7</td>
</tr>
<tr>
<td>( \lambda \bar{S}_2 )</td>
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<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>( \lambda \bar{S}_1 )</td>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( \lambda \bar{S}_2 )</td>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Model parameters are \((K = 7, \lambda = 9, \mu = 0.75), \) and \(\alpha \approx 4\)

\((\Delta^m < 0.01)\). It seems likely that the aberration is due to the discrete nature of the model, and that for "less discrete" models (i.e., for higher values of \(\lambda\)) such aberrations should disappear. As I have not been able to obtain a substantial sample of equilibria for larger \(\lambda\), I cannot draw any stronger conclusion.

3 Evidence from the Portuguese treasury bill auction

In this section, I present evidence from Portugal’s treasury bill auctions that bidders submit more bids and spread their bids more widely when there is greater risk of winner’s curse.\(^{10}\)

To provide informal evidence and intuition for this result, I plot in Figure 4 for each auction the inter-bidder dispersion of bids against the mean intra-bidder dispersion of bids. Let \(r_{itj}\) be the rate on the \(j\)th bid submitted by bidder \(i\) at auction \(t\), and let \(\bar{r}_t\) be the quantity-weighted mean (across \(j\)) of the \(r_{itj}\). The standard deviation across bidders at auction \(t\) of the \(\bar{r}_t\), which I denote \(\Sigma_t\) and plot on the horizontal axis, is a measure of the dispersion across bidders in the bidders’ posterior distributions on the value of the bills; thus, it is a measure of the uncertainty in the bidding environment. Let \(\text{BIDWSTD}(i,t)\) be the quantity-weighted standard deviation of the \(r_{itj}\) for bidder \(i\) at auction \(t\). On the vertical axis, I plot

\(^{10}\)Auction rules and other institutional details for the Portuguese treasury bill auction are provided in Section 3 of Chapter 1.
as \( \hat{\Delta}_t \) the mean across bidders at auction \( t \) of BIDWSTD(\( i, t \)). An empirical analogue to \( \Delta \) in my model, it measures how widely, on average, the bidders at auction \( t \) spread their own bids. Clearly, there is a strong positive relationship between the \( \hat{\Delta}_t \) and \( \hat{\Sigma}_t \); the correlation is 0.58.

The evidence of Figure 4 is informal in the sense that the \( \hat{\Sigma}_t \) cannot be taken as an independent explanatory variable for \( \hat{\Delta}_t \); \( \hat{\Sigma}_t \) is an ex-post measure of uncertainty, whereas we need an ex-ante measure, and, more importantly, is determined endogenously by the equilibrium bidding strategy. In the analyses below, I will exclude \( \hat{\Sigma} \) as an explanatory variable, and instead directly measure the ex-ante uncertainty in \( \nu \) (i.e., \( 1/\alpha \) in the model) with an index of recent volatility in domestic interest rates. In the subsections below, I first describe the data, and then present the regression models and results.

### 3.1 Description of the data

The data contain a complete record of bids for every treasury bill auction between June 1988 and April 1993, and a complete record of transactions on the Portuguese interbank money market (IMM) from January 1989 to April 1993. For each auction, the data include the date of auction, the maturity of bills for auction, the issue ceiling (i.e., the quantity on offer), the unannounced reservation rate, and whether the auction was followed by an additional session. For each bid at each auction, the quantity, rate, and bidder identity are recorded, as well as the outcome of the bid.

After elimination of auctions for which the auction record is incomplete, there remain 548 auctions in sample.\(^{11} \) A total of 62 bidders appear in the sample, of which 13 bid frequently and for large amounts. In Figure 5, I plot for each bidder the number of auctions in which the bidder appears against the log of the bidder's mean total bid quantity (i.e., the average across the auctions in which the bidder appears of the total quantity of the bidder's bids); the figure shows that the 13 bidders who appear in at least 45% of the auctions are

\(^{11}\) All auctions before January 1989 were eliminated because contemporaneous IMM data was unavailable. For several other auctions, the date on which bids were submitted could not be ascertained, or the reservation rate was not recorded. In a handful of cases, the bidding record was inconsistent with published summary statistics; these appear to have been caused by data entry errors.
Figure 4:

Dispersion of Mean Bids vs Mean Own-Bid Dispersion

![Graph showing the relationship between Delta(t) and Sigma(t)]
generally the largest bidders as well. Together, these 13 frequent bidders account for 80% by volume of the bids in the sample.

As there is no secondary market, the primary source of public information on market interest rates is the interbank money market. The IMM is primarily used by the banks to loan and borrow overnight money for the purpose of reserve management; turnover is on the order of 1,500 billion escudos ($12 billion) per month for overnight loans. There is also trading on the IMM in longer maturities up to one year. In the 86 to 96 day maturity range, monthly IMM turnover is on the order of 5 to 30 billion escudos. Turnover in the 175 to 189 day range is typically one-third to one-half of that. In the over 350 day range, turnover is negligible. For 91 and 182 day bills, IMM trades are sufficiently frequent to provide plausible benchmark rates. However, trading is not heavy enough to permit construction of a measure of volatility of the longer rates. Therefore, I take as the index of market volatility the standard deviation of overnight rates in the week prior to auction. The correlation of this index with $\Sigma$ in my sample is 0.19.

The question might arise whether the volatility of overnight rates, which is driven largely by the micro-mechanics of the banks' reserve period maintenance, provides a reasonable measure of the uncertainty in longer rates. I believe it to be the index of volatility most relevant to the bidders. The overnight market is easily the most liquid financial market in Portugal, and the overnight rate is the most easily monitored. The overnight rate therefore acts as an anchor for the market, and volatility in this market makes it more difficult for the market to price longer term loans. Evidence for this is found in the volatility of 28–32 day IMM loans. This maturity is short enough to provide sufficient liquidity for the

---

12 Note that no bidder participates in more than 80% of the Portuguese auctions; primary dealers in the US, in contrast, each bid in nearly every auction. In the US, primary dealers make a business of bidding for bills at auction and selling them in the when-issued market; consequently, it is reasonable for the government to expect their regular participation. In Portugal, bills are purchased to hold in portfolio, not for resale, so participation is limited by the banks' cash positions.

13 The banks are free to buy and sell treasury bills, but transactions must be individually negotiated as there is no liquid secondary market. In practice, trades are rare, so bills are generally held by the winning bidders until maturity.

14 Note that the set of participants on the IMM is the same as the set of bidders for treasury bills, so it is reasonable to assume symmetry in the bidders' prior distributions on the value of the bills.

15 Indeed, I construct and use such benchmarks in Chapter 1.
Figure 5:

Auction Participation

Log of Avg Bid Quantity (million PTE) vs. Attendance Rate
construction of an index of volatility, but long enough that it should not be effected greatly by the micro-mechanics of the eight day reserve periods. Nonetheless, I find a correlation of 0.57 between the indexes of volatility in overnight rates and 28-32 day rates. A second reason for the importance of volatility in the overnight rate is that losing bidders will place their excess cash in the overnight market until the next auction of bills. Therefore, the overnight rate is an important component of the bidder's opportunity cost.

In Figure 6, I plot the overnight IMM rate and the index of volatility (IMMVOL) over the sample period. During the period before financial market liberalization (1989-90), the rate is quite stable. For most of 1991 and the first half of 1992 (see, especially, dates 1200-1350), the Banco de Portugal followed a policy of nonintervention in the IMM except on the first day of each reserve period; consequently, rates tended to be highly volatile, especially towards the end of each reserve periods. When the central bank relaxed this policy in the summer of 1992, the volatility on the IMM subsided. The final spike in rates occurs in September 1993 during the ERM crises.

Finally, I note that the covered interest parity rate could not provide the bidders with useful information until the final few months of the sample period. Until the end of 1993, capital controls were in place to allow the Banco de Portugal to maintain high real interest rates without causing large inflows of foreign exchange; the arbitrage needed to enforce the covered interest parity relationship was thus prevented. The head of treasury security operations at one of the major bidders confirmed to me that foreign rates were not a consideration in the bidding decision during this period.

In the absence of a secondary market for treasury bills, or even arbitrage on foreign markets that could take the place of a secondary market for informational purposes, the Portuguese treasury bill auction cannot be taken as purely common-value. Indeed, in Chap-

---

16: Turnover in this range of maturities is roughly 60 billion escudos per month, and there are almost always several trades per day.
17: I have run many of the regressions below using the 28-32 day rate volatility index instead of the overnight index. Results are quite similar to those reported. When both indexes are included, the coefficient on the overnight index is unchanged, and that on the 28-32 day index is small and insignificant.
18: It is reassuring to note that when the capital controls were lifted, IMM rates immediately aligned with the covered interest parity rate on British pounds.
Figure 6:

IMM Overnight Rate and Volatility

Overnight Rate

Regime 1  Regime 2  Regime 3

Time

IMMVOL
ter 1, I model this auction as independent-private-values. Ideally, the auction should be modelled as one of affiliated values, in the sense of Milgrom and Weber (1982). For simplicity, and to emphasize the risk of winner’s curse due to the common-value component, I model the auction here as common-value. While it is true that a private values component introduces another reason for bidders to have downward sloping demand for the bills, it does not explain why the slope varies with the quality of public information on interest rates, as this public information is tied to the common-value component of bidder valuations.

3.2 Count regressions on the number of bids submitted

In this first set of regressions, the dependent variable is \( y_{it} = \text{BIDCOUNT}(i,t) - 1 \), where \( \text{BIDCOUNT}(i,t) \) is the number of bids submitted by bidder \( i \) at auction \( t \); that is, \( y \) is the number of “multiple” bids submitted beyond one. These regressions and those in the next section will use the same set of independent variables. I list these variables here and describe their construction. A parenthetical \((i,t)\) following a variable name indicates that the variable is specific to both bidder and auction; variables that are bidder-specific are denoted with \((i)\), and those that are time-specific are denoted with \((t)\).

\( \text{LNBI}D\text{DQ}(i,t) \) is the log of the total quantity (in millions of Portuguese escudos) on the bids submitted by bidder \( i \) at auction \( t \). I predict that that the number of bids submitted will increase in \( \text{LNBI}D\text{DQ} \), because the absolute gain to using multiple bids increases with the size of the bids. As long as there is some penalty to submitting many small bids (e.g., the risk of annoying the central bank), small bidders will submit fewer bids. I take \( \text{LNBI}D\text{DQ} \) to be exogenous, because the banks bought treasury bills during this period primarily to absorb their excess liquidity, rather than as an alternative to more profitable commercial loans. The level of excess liquidity was determined by conditions outside the short-run control of the banks (e.g., shifts in account balances and lines of credit).

\( \text{MAT182}(t), \text{MAT364}(t) \) are dummy variables for 182 day bill auctions and 364 day bill auctions. As public information on interest rates comes from trade in the shortest
maturities on the IMM, we might expect that the quality of public information would decline with the maturity of the treasury bill, so the coefficients would be positive. However, it is likely that private information also is less precise for the longer maturities (i.e., lower $K$ in the model); this reduces the winner's curse, and so the coefficients would be negative.\textsuperscript{19} The signs therefore depend on whether private or public information degrades more rapidly as maturity increases, and so are ambiguous.

\textbf{REGIME1}(t), \textbf{REGIME2}(t): I divide my sample into three periods on the basis of the Ministry of Finance's reservation price policy. In the first regime (\textbf{REGIME1}), the reservation rate is fairly volatile and its changes have little correlation with changes in the interbank rate. In the second (\textbf{REGIME2}), the reservation rate rarely changes at all, despite high volatility on the interbank money market. In the third regime, the reservation rate changes frequently and in tandem with IMM rates. The divisions between the regimes are marked in Figure 7, which plots the reservation rate (and the IMM benchmark rates for 91 and 182 days) over time.\textsuperscript{20}

I show in Chapter 1 that when bidders know the current IMM benchmarks and the reservation rate of the previous auction of bills of the same maturity, the conditional variance of the reservation rate is lowest in the second regime and highest in the first regime. Therefore, I expect the coefficient on \textbf{REGIME1} to be positive and the coefficient on \textbf{REGIME2} to be negative. Note that these signs are consistent with several hypotheses: First, if changes in the reservation rate derive from changes in government interest rate targets, then frequent and unpredictable changes in the reservation rate will coincide with periods of greater uncertainty regarding future interest rates. Second, if the bidders use the expected reservation rate as a focal point for collusion, then low conditional variance facilitates coordination, which reduces the desirability of submitting multiple rates. Third, if it is desirable to follow changes in

\textsuperscript{19}The head of treasury security operations at one of the major bidders told me that he is more interested in the opinions of his competitors regarding 91 day rates than the longer rates, because the former are based on private knowledge of commercial activity, whereas the latter are merely idiosyncratic guesses about future inflation

\textsuperscript{20}My choice of breakpoints between regimes is discussed in Section 4 of Chapter 1.
the reservation rate (say, in order to predict future reservation rates), then bidders may submit multiple bids in order to place bounds ex-post on the reservation rate; the higher the conditional variance of the reservation rate, the more bidders will have to spread their bids to be likely to have at least one winning bid and one losing bid.

**NBIDDERS**\( (t) \) is the number of bidders who participate in auction \( t \). I take this as known ex-ante to the bidders, which is not unreasonable given day-to-day interaction between these bidders on the IMM. I predict a positive coefficient, as the risk of winner's curse increases with the number of bidders. The intuition is that winning a bid becomes worse news about the quality of your signal as the number of competitors increases.

**LNSUPPLY**\( (t) \) is the log of the issue ceiling for auction \( t \). I predict a negative coefficient, because a higher issue ceiling decreases the severity of the winner's curse: for a given set of bid quantities, an increase in LNSUPPLY increases the number of winning bids, so that winning contains weaker negative information on the accuracy of one's signal.

**HASADDSS**\( (t) \) is a dummy for auctions followed by an additional session. As in Section 4 of Chapter 1, I assume that bidders regard HASADDSS as known and exogenous. I predict a negative coefficient, because the ability to obtain some desired quantity in the additional session (which sells bills at a rate that incorporates information from all the bidders) reduces the risk of winner’s curse.\(^{21}\)

**IMMVOL**\( (t) \) is the standard deviation of overnight rates in the week prior to auction \( t \). This variable decreases with the precision parameter \( \alpha \) in the model; therefore, the predicted coefficient is positive.

**FREQBIDR**\( (i) \) is a dummy variable which is 1 if bidder \( i \) is among the 13 bidders who appear most regularly. I include this variable to allow for systematic differences in the quality of information or degree of risk-aversion of frequent and infrequent auction participants.

\(^{21}\)I do not necessarily advocate greater use of additional sessions. Although it has the desirable property of reducing winner's curse, it may facilitate bidder collusion in the primary auction.
Table 5 provides descriptive statistics for these variables.

<table>
<thead>
<tr>
<th>Dependent variables:</th>
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<tbody>
<tr>
<td>BIDCOUNT</td>
</tr>
<tr>
<td>BIDWSTD</td>
</tr>
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<td>INTER50</td>
</tr>
<tr>
<td>INTER70</td>
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<table>
<thead>
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<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>StdDev</th>
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<td>2.892</td>
<td>1.636</td>
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<table>
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<tr>
<th>Independent variables:</th>
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<td>LNBIDQ</td>
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<tr>
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<tr>
<td>MAT364</td>
</tr>
<tr>
<td>REGIME1</td>
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<td>REG2.ME2</td>
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<td>NBIDERS</td>
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<td>LNSUPPLY</td>
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<td>HASADDSS</td>
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<td>IMVMVol</td>
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<td>FREQBIDR</td>
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<table>
<thead>
<tr>
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<th>Type</th>
<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>StdDev</th>
</tr>
</thead>
<tbody>
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<td>10.933</td>
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<td>1.327</td>
<td></td>
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<td>1</td>
<td>0.406</td>
<td>0.492</td>
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<td>1</td>
<td>0.202</td>
<td>0.402</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>1</td>
<td>0.183</td>
<td>0.387</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>1</td>
<td>0.558</td>
<td>0.497</td>
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</tr>
<tr>
<td>t</td>
<td>0</td>
<td>32</td>
<td>12.556</td>
<td>6.752</td>
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</tr>
<tr>
<td>t</td>
<td>9.105</td>
<td>12.073</td>
<td>9.928</td>
<td>0.437</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>1</td>
<td>0.152</td>
<td>0.360</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>0.053</td>
<td>21.693</td>
<td>2.135</td>
<td>2.959</td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>0</td>
<td>0</td>
<td>0.197</td>
<td>0.401</td>
<td></td>
</tr>
</tbody>
</table>

†: The sample includes only those observations i, t for which bidder i participates at auction t.

The standard choice of model for count data is the Poisson. For a panel set, it is appropriate to allow for a multiplicative unit-specific effect; following the fixed effect Poisson model of Hausman, Hall and Griliches (1984), I assume that $y_{it} | x_{it}$ is distributed Poisson with parameter $\lambda_{it} = \phi_i \exp(x_{it} \beta)$, where $x_{it}$ is a row vector of independent variables. (Note, of course, that this $\lambda$ bears no relationship to the parameter $\lambda$ in the model of section 1.) Even allowing for bidder-specific fixed effects, the Poisson specification imposes restrictions on the conditional distribution which may not be justified; e.g., equality of mean and variance. However, Wooldridge (1990) shows that the fixed effect Poisson estimator is consistent under the much weaker moment condition

$$E[y_{it} | x_{it}, \phi_i] = \phi_i \exp(x_{it} \beta)$$
Reservation rates are marked by plus signs. The solid lines are IMM benchmark rates; construction of these benchmarks is described in Chapter 1.
and provides a method of calculating standard errors which are robust to conditional heteroskedasticity in the errors.

Results for the fixed effect Poisson specification (with robust standard errors) are reported in the second column of Table 6. The coefficients on IMMVOL, NBIDDERS, LNBIDQ, HASADDSS, REGIME2, and LNSUPPLY are all of the predicted sign and significant at conventional levels (all but REGIME2 at the 1% level). The remaining coefficients are statistically insignificant from zero.

The coefficients appear to be quite robust to specification. In the first column, I report coefficients for the more familiar (non-panel) Poisson specification. If the Poisson specification is correct, i.e., if $y_{it}|x_{it} \sim \text{Poisson}(\exp(x_{it}\beta))$, then the fixed effects Poisson estimator is consistent and the ordinary Poisson estimator is consistent and efficient. Under this null hypothesis, the Hausman specification test statistic is distributed $\chi^2(9)$.\textsuperscript{22} The value of the statistic here is 17.65, so the null is rejected at 5% significance level. Nonetheless, the coefficients which are statistically significant in the fixed effect Poisson specification are here of similar magnitude and significance level. I have also run several fixed effect negative binomial specifications (see, e.g., Hausman, Hall and Griliches (1984)), and obtained similar results, which are not reported here.

In the third column, I allow for the possibility that the effect of IMMVOL is not the same across maturities; because IMMVOL is based on the shortest maturity rates, we might expect it to be more relevant to the 91 day bill auctions than the longer maturity bill auctions. Indeed, for the fixed effect Poisson specification, I find positive coefficients on each interactive variable IMMVOL\*MATnnn, but the coefficient on IMMVOL\*MAT091 is much larger and is the only one of the three which is statistically significant using robust standard errors. The Wald test statistic for equality of the three IMMVOL\*MATnnn coefficients is 18.0, which rejects at a 1% level. The other coefficients are robust to the introduction of the interactive variables; the coefficients which are statistically significant in the first fixed

\textsuperscript{22}The statistic is formed as $H = (\hat{\beta}_{FEP} - \hat{\beta}_P)(\hat{V}_{FEP} - \hat{V}_P)^{-1}(\hat{\beta}_{FEP} - \hat{\beta}_P)$, where FEP denotes the fixed effect Poisson estimator and P denotes the ordinary Poisson estimator. Bidder-specific variables which do not appear in the fixed effect regression are excluded from $\hat{\beta}_P$ and $\hat{V}_P$. 

90
effect specification (second column) are similar in magnitude and standard errors.

A maintained assumption throughout this paper is that the decision to participate in
the auction is exogenous; the argument for this assumption is the same as for the exogeneity
of LNBIDQ. To test the robustness of the coefficients to this assumption, I run the Poisson
and fixed effect Poisson specifications on a data set which includes the bidder-auction pairs
where BIDCOUNT\((i, t) = 0\). The dependent variable is \(y_{it} = \text{BIDCOUNT}(i, t)\); that is, I
assume that the decision to submit any bids at all is governed by the same statistical process
which determines the use of multiple bids. Except for the regime dummies, the resulting
coefficients (fourth and fifth columns) are similar to those of the first two columns; in
particular, the coefficient on IMMVOL is unchanged.\(^{23}\)

3.3 Panel regressions on bid dispersion

In this set of regressions, the dependent variable is \(y_{it} \equiv 1000\cdot \text{BIDWSTD}_{it}\), where \(\text{BIDWSTD}_{it}\)
is the quantity-weighted standard deviation of the bid rates submitted by bidder \(i\) at auction
\(t\). I first consider a traditional panel model for \(y_{it}\):

\[
y_{it} = x_{it}\beta + \mu_i + \epsilon_{it}
\]

(3)

where \(x_{it}\) is a row vector of independent variables, and the \(\epsilon_{it}\) are iid random variables with
mean zero. There are two ways to regard the bidder-specific effects \(\mu_i\): In the fixed effects
model, the \(\mu_i\) are taken as bidder-specific constants; in the random effects model, the \(\mu_i\)
are zero mean random variables that are independent of the \(x_{it}\) and the \(\epsilon_{it}\).

In the first three columns of Table 7, I present the estimated \(\hat{\beta}\) for an OLS regression of
equation (3), and the fixed effects and random effects specifications.\(^{24}\) For the fixed effects
specification, the coefficients on LNBIDQ, MAT182, the regime dummies, NBIDDERS,
HASADDSS, and IMMVOL are all of the expected sign and statistically significant (all but

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\(^{23}\)In these regressions, LNBIDQ is defined as the \(\ln(1 + q_{it})\).

\(^{24}\)Note that bidder-specific dummy variable FREQBIDR is absorbed into the bidder-specific constant \(\mu_i\)
in the fixed effects model, so its coefficient is not estimated. The random effects model is estimated by GLS
as described in Hausman (1978).
Table 6: Count Regressions\textsuperscript{a}

<table>
<thead>
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<th>Specification</th>
<th>BIDCOUNT-1</th>
<th>BIDCOUNT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>POISSON</td>
<td>FE POISSON\textsuperscript{b}</td>
</tr>
<tr>
<td>LNBIDQ</td>
<td>0.232**</td>
<td>0.318**</td>
</tr>
<tr>
<td></td>
<td>(0.009)</td>
<td>(0.037)</td>
</tr>
<tr>
<td>MAT182</td>
<td>-0.019</td>
<td>-0.008</td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>MAT364</td>
<td>0.032</td>
<td>0.032</td>
</tr>
<tr>
<td></td>
<td>(0.028)</td>
<td>(0.039)</td>
</tr>
<tr>
<td>REGIME1</td>
<td>-0.067*</td>
<td>-0.021</td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td>(0.081)</td>
</tr>
<tr>
<td>REGIME2</td>
<td>-0.148**</td>
<td>-0.178*</td>
</tr>
<tr>
<td></td>
<td>(0.023)</td>
<td>(0.074)</td>
</tr>
<tr>
<td>NBIDDERS</td>
<td>0.018**</td>
<td>0.015**</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.003)</td>
</tr>
<tr>
<td>LNSUPPLY</td>
<td>-0.100**</td>
<td>-0.095**</td>
</tr>
<tr>
<td></td>
<td>(0.025)</td>
<td>(0.035)</td>
</tr>
<tr>
<td>HASADDSS</td>
<td>-0.105**</td>
<td>-0.128**</td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td>(0.045)</td>
</tr>
<tr>
<td>IMMVOL</td>
<td>0.015**</td>
<td>0.018**</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>IMMVOL*MAT091</td>
<td></td>
<td>0.042**</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.007)</td>
</tr>
<tr>
<td>IMMVOL*MAT182</td>
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<td>0.012</td>
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<td></td>
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<td>(0.007)</td>
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<tr>
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<td></td>
<td></td>
<td>(0.007)</td>
</tr>
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<td>FREQBIDR</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>(0.0238)</td>
<td></td>
</tr>
<tr>
<td>CONSTANT</td>
<td>-0.5176*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.2442)</td>
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</tr>
<tr>
<td>Observations</td>
<td>6014</td>
<td>31680</td>
</tr>
</tbody>
</table>

\textsuperscript{a} Standard errors are in parenthesis; symbols * and ** indicate statistical significance at 5% and 1% levels, respectively.

\textsuperscript{b} Wooldridge (1990) heteroskedasticity-robust standard errors reported for the fixed effect Poisson regressions.
NBIDDERS significant at the 1% level).

The coefficients from the OLS and random effects regressions appear quite similar; the coefficients on IMMVOL are very nearly identical. As the standard errors are small, however, the Hausman test for the random effects specification is rejected: The test statistic $H = (\hat{\beta}_{FE} - \hat{\beta}_{RE})(\tilde{V}_{FE} - \tilde{V}_{RE})^{-1}(\hat{\beta}_{FE} - \hat{\beta}_{RE})$ is distributed $\chi^2(9)$ under the null hypothesis that $E[\mu_i|x_{it}] = 0$;\(^{25}\) the value $H = 41.4$ allows rejection of the null at a 1% significance level. Therefore, I henceforth consider only fixed effects specifications.

In the fourth column of Table 7, I report coefficients from a panel Tobit regression, which uses the GMM method of Honoré (1992).\(^{26}\) Because the value of $y$ is always nonnegative, and equals zero whenever a bidder submits exactly one bid, the observed dependent variable should be regarded as censored at zero. That is, the true model is for a latent variable $y_{it}^* = x_{it}\beta + \mu_i + \epsilon_{it}$, but the econometrician observes $y = \max(y^*, 0)$. All of the coefficients which were reported to be statistically significant in the linear fixed effects specification are of the same sign, similar magnitude, and (with the exception of NBIDDERS) statistically significant at conventional levels.

In Table 8, I report regressions which allow for interactive IMMVOL*MAT.nnn effects. As in the regressions in the previous section, the coefficient on IMMVOL*MAT091 is larger than for the other two maturities; however, all three coefficients are statistically significant. Also as in the previous section, the other coefficients are not changed by the introduction of the interactive variables. The Wald test for the equality of the three coefficients on the IMMVOL*MAT.nnn rejects at conventional levels for the linear regressions, but do not reject for the panel Tobit specification, due to the larger standard errors for the latter set of coefficients.

Alternative measures of bid spreading may be constructed from absolute differences

\(^{25}\) Under this null hypothesis, the fixed effects (within) estimator is consistent and the random effects estimator is consistent and efficient. Note, however, that if the panel Tobit specification is correct (see below), then both the fixed effect and random effect panel least squares estimators are inconsistent, and the Hausman test on their difference is not meaningful.

\(^{26}\) I gratefully acknowledge the use of Gauss routines for the panel Tobit estimator by Bo Honoré, which he has made available in the public domain. The reported coefficients are for the quadratic loss function estimator, which is denoted $\beta_4$ in his paper.
Table 7: Panel Regressions for $y = 1000 \cdot \text{BIDWSTD}$

<table>
<thead>
<tr>
<th>Variable</th>
<th>OLS</th>
<th>FIXED</th>
<th>RANDOM</th>
<th>PANEL TOBIT</th>
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<tr>
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<td>11.723**</td>
<td>12.948**</td>
<td>12.633**</td>
<td>13.82**</td>
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<td>(2.303  )</td>
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<td>REGIME1</td>
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<td>17.493**</td>
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<tr>
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<td>(2.654  )</td>
<td>(2.594  )</td>
<td>(2.579  )</td>
<td>(11.49)</td>
</tr>
<tr>
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<td>(1.941  )</td>
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</tr>
<tr>
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<td>0.285*</td>
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<td>(0.149  )</td>
<td>(0.145  )</td>
<td>(0.144  )</td>
<td>(0.314)</td>
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<td>(1.985  )</td>
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<tr>
<td></td>
<td>(2.521  )</td>
<td>(2.393  )</td>
<td>(2.388  )</td>
<td>(5.613)</td>
</tr>
<tr>
<td>IMMVOL</td>
<td>3.003**</td>
<td>3.022**</td>
<td>3.033**</td>
<td>4.103**</td>
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<tr>
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<td>(0.361  )</td>
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<td>FREQBIDR</td>
<td>11.559**</td>
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<tr>
<td></td>
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<td>(7.258)</td>
<td></td>
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<tr>
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<td></td>
</tr>
<tr>
<td></td>
<td>(20.523 )</td>
<td></td>
<td>(19.820 )</td>
<td></td>
</tr>
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Standard errors are in parenthesis; symbols * and ** indicate statistical significance at 5% and 1% levels, respectively.
Table 8: Panel Regressions with Maturity-Specific IMMVOL

<table>
<thead>
<tr>
<th>Variable</th>
<th>OLS</th>
<th>FIXED</th>
<th>RANDOM</th>
<th>PANEL TOBIT</th>
</tr>
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<tr>
<td>LNBBIDQ</td>
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<td>12.973**</td>
<td>12.653**</td>
<td>13.86**</td>
</tr>
<tr>
<td></td>
<td>(0.708)</td>
<td>(0.953)</td>
<td>(0.909)</td>
<td>(3.735)</td>
</tr>
<tr>
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<td>-1.737</td>
<td>-1.965</td>
<td>-4.588</td>
</tr>
<tr>
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<td>(2.227)</td>
<td>(2.119)</td>
<td>(2.115)</td>
<td>(2.687)</td>
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</tr>
<tr>
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<td>(2.895)</td>
<td>(2.750)</td>
<td>(2.746)</td>
<td>(5.631)</td>
</tr>
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<td>18.316**</td>
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<tr>
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<td>(2.612)</td>
<td>(2.597)</td>
<td>(11.58)</td>
</tr>
<tr>
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<td>-12.925**</td>
<td>-16.67*</td>
</tr>
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<td>(1.941)</td>
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<td>NBIDDERS</td>
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<td>0.309*</td>
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<td>(0.150)</td>
<td>(0.145)</td>
<td>(0.144)</td>
<td>(0.3178)</td>
</tr>
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<td>(2.108)</td>
<td>(1.987)</td>
<td>(1.987)</td>
<td>(3.959)</td>
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<td>(2.395)</td>
<td>(2.390)</td>
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<td>4.744**</td>
<td>4.829**</td>
<td>5.594**</td>
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<td>(0.712)</td>
<td>(0.673)</td>
<td>(0.673)</td>
<td>(1.415)</td>
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<td>IMMVOL*MAT182</td>
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<td>2.498**</td>
<td>2.499**</td>
<td>3.786**</td>
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<td>(0.547)</td>
<td>(0.515)</td>
<td>(0.516)</td>
<td>(0.7913)</td>
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<tr>
<td>IMMVOL*MAT364</td>
<td>2.284**</td>
<td>2.4**</td>
<td>2.450**</td>
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</tr>
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<td>(0.648)</td>
<td>(0.609)</td>
<td>(0.610)</td>
<td>(0.7661)</td>
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<td>FREQBIDR</td>
<td>11.543**</td>
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<td>-38.099</td>
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<td></td>
<td>(20.607)</td>
<td></td>
<td>(19.901)</td>
<td></td>
</tr>
</tbody>
</table>

Wald test for equality of IMMVOL*MATnnn coefficients

| W    | 12.2 | 9.2  | 10.0 | 2.4 |
| p-value | 0.002| 0.010| 0.007| 0.301|

Standard errors are in parenthesis; symbols * and ** indicate statistical significance at 5% and 1% levels, respectively.
Table 9: Absolute Rate Spread: Panel Tobit Regressions

<table>
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<tr>
<th>Dependent variable:</th>
<th>1000*INTER50</th>
<th>1000*INTER70</th>
</tr>
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<td>14.86</td>
</tr>
<tr>
<td>(7.961)</td>
<td>(7.953)</td>
<td>(10.73)</td>
</tr>
<tr>
<td>MAT182</td>
<td>-13.67*</td>
<td>-10.86</td>
</tr>
<tr>
<td>(5.741)</td>
<td>(6.970)</td>
<td>(6.383)</td>
</tr>
<tr>
<td>MAT364</td>
<td>-3.065</td>
<td>0.766</td>
</tr>
<tr>
<td>(8.666)</td>
<td>(10.47)</td>
<td>(11.24)</td>
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<tr>
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<td>48.89</td>
</tr>
<tr>
<td>(27.01)</td>
<td>(27.02)</td>
<td>(30.79)</td>
</tr>
<tr>
<td>REGIME2</td>
<td>-34.19*</td>
<td>-34.50*</td>
</tr>
<tr>
<td>(15.41)</td>
<td>(15.53)</td>
<td>(21.84)</td>
</tr>
<tr>
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<td>-0.101</td>
</tr>
<tr>
<td>(0.695)</td>
<td>(0.704)</td>
<td>(0.860)</td>
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<td>5.350</td>
</tr>
<tr>
<td>(7.200)</td>
<td>(7.215)</td>
<td>(9.822)</td>
</tr>
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<tr>
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<td>(0.786)</td>
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<td>12.23**</td>
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<td>(2.467)</td>
<td></td>
<td>(3.008)</td>
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</tr>
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<td>(1.407)</td>
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<td>(1.908)</td>
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<td>8.261**</td>
</tr>
<tr>
<td>(1.524)</td>
<td></td>
<td>(2.089)</td>
</tr>
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</table>

Wald test for equality of IMMVOL*MATnnn coefficients

| W     | 0.6 | 1.3 |
| p-value | 0.74 | 0.52 |

Standard errors are in parenthesis; symbols * and ** indicate statistical significance at 5% and 1% levels, respectively.
between a bidder's high and low bids. When bidders submit several bids, it is often the case that the highest and lowest bids are for very small quantities; these bids may be primarily intended to provide bounds on the reservation rate in case of change, rather than to hedge against winner's curse. In order to place smaller weight on small outlying bids, I define \textsc{INTER50} as an interquartile difference: I order the bids submitted by bidder \( i \) at auction \( t \) by rate, and let \textsc{INTER50}(\( i, t \)) be the difference between the rate on the 75\% percentile of quantity submitted and the 25\% percentile. Similarly, \textsc{INTER70}(\( i, t \)) is defined as the difference between the rate on the 85\% percentile of quantity less the rate on the 15\% percentile. Panel Tobit regressions of \textsc{INTER50} and \textsc{INTER70} on the independent variables are reported in Table 9. Results are similar to those above in all respects, except that the coefficient on \textsc{NBIDDERS} is negative and insignificant in the \textsc{INTER50} regression. As \textsc{INTER50} and \textsc{INTER70} are highly correlated with \textsc{BIDWSTD} (correlations of 0.92 and 0.97, respectively), this similarity is expected.

4 Conclusion

Evidence is provided for the hypothesis that bidders' use of multiple bids is related to the risk of winner's curse. Under a variety of specifications, the coefficient on the index of market volatility is always positive and statistically significant. Coefficients on dummy variables that represent institutional factors effecting risk also, on the whole, take the predicted signs and are statistically significant. Finally, the regressions show that bidders' use of multiple bids increases with the number of bidders present at auction; this suggests that bidders are aware that the risk of winner's curse increases with the number of bidders.

The paper uses a model of a multiple-unit, multiple-bid auction to motivate the empirical analysis. The model takes the unusual approach of assuming signals to be drawn from a finite set and restricting bids to finite intervals. The main advantage of a discrete framework is that it makes a difficult problem tractable; these are the first theoretical results for an auction model with the realistic feature that bidders each submit a small integer number of bids. (A second advantage is that the model, like real treasury auctions, generally
produces ties at the stopout price.) However, the discrete approach limits us to numerical simulation; little can be said about the model analytically, so it is difficult to gain intuition from the mathematics or to predict whether the results are robust to distributional and other assumptions. Further research is needed especially to resolve whether or not risk-neutral bidders will submit multiple bids in a discriminatory price auction.
A. Computational notes for the model

A.1 An efficient algorithm for bidder utility

The method of locating equilibria through tâtonnement requires that bidder utility, given in equation (1), be calculated repeatedly. In this appendix, I show how that expression may be simplified under CARA utility to avoid the need for numerical integration.

Let $p_i(s|S(y))$ be the probability that the bidder wins exactly $i$ bids when bidding $s \equiv (s_1, s_2)$, conditional on the opponent receiving signal $y$ and using the strategy function $S$. Then we can write

$$h_i(s|S, v) = \sum_{y=0}^{K} p_i(s|S(y)) g(y|v).$$

Consider the last piece under the integral in equation (1):

$$U_0(s|S, x) = \int_0^1 U(0) h_0(s|S, v) dF(v|x)$$

$$= -\int_0^1 \left( \sum_{y=0}^{K} p_0(s|S(y)) g(y|v) \right) f(v|x) dv$$

$$= -\sum_{y=0}^{K} p_0(s|S(y)) \int_0^1 g(y|v) f(v|x) dv$$

$$= -\sum_{y=0}^{K} p_0(s|S(y)) g(y|x)$$

where $g(y|x)$ is the probability that the opponent’s signal is $y$ conditional on observing a signal $x$; this is given by

$$g(y|x) = \binom{K}{y} \frac{B(y + x + \alpha \mu, 2K - y - x + \alpha(1 - \mu))}{B(x + \alpha \mu, K - x + \alpha(1 - \mu))}. \quad (4)$$

The second piece under the integral in equation (1) is somewhat more complex. This can be re-written as

$$U_1(s|S, x) = \int_0^1 U(v - s_1) h_1(s|S, v) dF(v|x)$$

$$= \int_0^1 -\exp(-\gamma(v - s_1)) \sum_{y=0}^{K} p_1(s|S(y)) g(y|v) f(v|x) dv$$
\[
= -\exp(\gamma s_1) \sum_{y=0}^{K} p_1(s|S(y)) \int_{0}^{1} \exp(-\gamma v) g(y|v) f(v|x) \, dv
\]  

(5)

Denote by \( \Xi_1(y, x) \) the definite integral in the above expression. This has no closed form solution, but can be written as an infinite series of definite integrals that do have closed form solutions. Take the Taylor series expansion of the exponential function:

\[
\exp(-\gamma v) = \exp(-\gamma) \exp(\gamma(1 - v)) = \exp(-\gamma) \sum_{j=0}^{\infty} \frac{\gamma^j (1 - v)^j}{j!}.
\]

Substitute into equation (5) to get

\[
\Xi_1(y, x) \equiv \int_{0}^{1} \exp(-\gamma v) g(y|v) f(v|x) \, dv
\]

\[
= \exp(-\gamma) \int_{0}^{1} \sum_{j=0}^{\infty} \frac{\gamma^j (1 - v)^j}{j!} g(y|v) f(v|x) \, dv
\]

\[
= \exp(-\gamma) \int_{0}^{1} \sum_{j=0}^{\infty} \frac{\gamma^j (1 - v)^j}{j!} \binom{K}{y} v^y (1 - v)^{K-y} \frac{v^x + \alpha \mu - 1}{B(x + \alpha \mu, K - x + \alpha(1 - \mu))} \, dv
\]

\[
= \binom{K}{y} \frac{\exp(-\gamma)}{B(x + \alpha \mu, K - x + \alpha(1 - \mu))} \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} \int_{0}^{1} v^{y + x + \alpha \mu - 1} (1 - v)^{j + 2K - y - x + \alpha(1 - \mu) - 1} \, dv
\]

\[
= \binom{K}{y} \frac{\exp(-\gamma)}{B(x + \alpha \mu, K - x + \alpha(1 - \mu))} \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} B(y + x + \alpha \mu, j + 2K - y - x + \alpha(1 - \mu)).
\]

Finally, substitute \( \Xi_1(y, x) \) into equation (5) to obtain

\[
\mathcal{U}_1(s|S, x) = -\exp(\gamma s_1) \sum_{y=0}^{K} p_1(s|S(y)) \Xi_1(y, x).
\]

By a similar argument, the first piece under the integral in equation (1) is

\[
\mathcal{U}_2(s|S, x) = -\exp(\gamma (s_1 + s_2)) \sum_{y=0}^{K} p_2(s|S(y)) \Xi_2(y, x)
\]

where

\[
\Xi_2(y, x) \equiv \binom{K}{y} \frac{\exp(-2\gamma)}{B(x + \alpha \mu, K - x + \alpha(1 - \mu))} \sum_{j=0}^{\infty} \frac{(2\gamma)^j}{j!} B(y + x + \alpha \mu, j + 2K - y - x + \alpha(1 - \mu)).
\]

100
Finally, we re-write equation (1) as

\[ U(s|S, x) = U_2(s|S, x) + U_1(s|S, x) + U_0(s|S, x). \]

In this form, \( U \) is calculated quickly and accurately. The infinite series in the expressions for \( \Xi_1(y, x) \) and \( \Xi_2(y, x) \) do not present a problem, because these series converge monotonically and their terms diminish rapidly.\(^{27}\) Most importantly, the \( \Xi_1 \) and \( \Xi_2 \) are not functions of the bidding strategies, so need be calculated only once for each set of model parameters.

### A.2 Upper and low bounds on \( \tilde{S} \)

As \( K \) and \( \lambda \) increase, the space of functions that map \( \mathcal{X} \mapsto \Lambda \times \Lambda \) grows, and the search for the bidder’s optimal response \( \tilde{S} \) to an opponent’s function \( S \) takes corresponding longer. In this appendix, I show that lower and upper bounds may be placed on \( \tilde{S} \) to increase computational efficiency.

\( \tilde{S}_2(0) \) has a lower bound of zero, and \( \tilde{S}_1(0) \) has a lower bound of \( \tilde{S}_2(0) \). Thereafter, by the monotonicity of \( \tilde{S} \) and the restriction \( \tilde{S}_2(x) \leq \tilde{S}_1(x) \), the lower bound of \( \tilde{S}_2(x) \) is \( \tilde{S}_2(x - 1) \), and the lower bound of \( \tilde{S}_1(x) \) is \( \max(\tilde{S}_2(x), \tilde{S}_1(x - 1)) \).

I also apply a weak upper bound to \( \tilde{S} \). Neither a risk-neutral bidder nor a risk-averse bidder will ever bid above the expected value of the good, conditional on the bidder’s signal and on winning the bid. This can be bounded above by

\[ \tilde{S}_2(x) \leq \tilde{S}_1(x) \leq E[v|x, (\tilde{S}_1(x) \text{ wins})] \leq E[v|x, (y = K)] \]

where \( y \) is the opponent’s signal. Under the distributional assumptions of the model,

\(^{27}\)Monotonicity is quite important to the numerical calculation of the \( \Xi_1 \) and \( \Xi_2 \). These series have alternative expressions that use the standard Taylor series for the exponential function around zero. However, these alternative series are not monotonic and, for large \( \gamma \), contain terms that are many orders of magnitude higher in absolute value than the series sums. Consequently, severe floating point round-off error is unavoidable.
\(E[v|x, (y = K)]\) has an especially simple form. By Bayes’ theorem,

\[
E[v|x, (y = K)] = \int_0^1 v f(v|x, (y = K)) \, dv = \int_0^1 v \frac{g(x, (y = K)|v) f(v)}{g(x, (y = K))} \, dv
\]

(6)

In the denominator, the unconditional probability of conditionally independent signals \((x, K)\) is

\[
g(x, (y = K)) = \int_0^1 g(x|u) g(K|u) f(u) \, du
\]

\[
= \int_0^1 \begin{pmatrix} K \\ x \end{pmatrix} u^x(1 - u)^{K-x} \frac{u^\alpha(1-u)^{\alpha(1-\mu)-1}}{B(\alpha, \alpha(1-\mu))} \, du
\]

\[
= \begin{pmatrix} K \\ x \end{pmatrix} \frac{B(K + x + \alpha u, K - x + \alpha(1-\mu))}{B(\alpha u, \alpha(1-\mu))}.
\]

(7)

The numerator in equation (6) can similarly be re-written:

\[
v g(x, (y = K)|v) f(v) = v \begin{pmatrix} K \\ x \end{pmatrix} v^x(1 - v)^{K-x} \frac{v^\alpha(1-v)^{\alpha(1-\mu)-1}}{B(\alpha, \alpha(1-\mu))}
\]

\[
= \frac{v^{1+K+x+\alpha u-1}(1-v)^{K-x+\alpha(1-\mu)-1}}{B(\alpha u, \alpha(1-\mu))}.
\]

(8)

The conditional expectation of \(v\) is the definite integral of equation (8) divided by the expression in equation (7), which simplifies to

\[
E[v|x, (y = K)] = \frac{B(K + x + \alpha u, K - x + \alpha(1-\mu))}{B(K + x + \alpha u, K - x + \alpha(1-\mu))} = \frac{K + x + \alpha u}{2K + \alpha}.
\]

A.3 Bidder utility under risk-neutrality

Under risk-neutrality and discriminatory pricing, the bidder's utility function may be simplified from equation (1):

\[
U(s_1, s_2|S|x) = \int_0^1 ((2v - s_1 - s_2) h_2(s_1, s_2|S, v) + (v - s_1) h_1(s_1, s_2|S, v)) \, dF'(v|x)
\]

\[
= \int_0^1 \left( v(2h_2(s_1, s_2|S, v) + h_1(s_1, s_2|S, v))
-((s_1 + s_2)h_2(s_1, s_2|S, v) + s_1 h_1(s_1, s_2|S, v)) \right) \, dF(v|x)
\]
\[
= \sum_{y=0}^{K} (2p_2(s|S(y)) + p_1(s|S(y))) \left( \int_{0}^{1} v g(y|x) f(v|x) dv \right) \\
- ((s_1 + s_2)p_2(s|S(y)) + s_1p_1(s|S(y))) \left( \int_{0}^{1} g(y|x) f(v|x) dv \right)
\]

The second integral in the last expression is simply \(g(y|x)\). The first solves to

\[
\int_{0}^{1} v g(y|x) f(v|x) dv = \binom{K}{y} \frac{B(1 + y + x + \alpha \mu, 2K - y - x + \alpha(1 - \mu))}{B(x + \alpha \mu, K - x + \alpha(1 - \mu))} \\
= \frac{B(1 + y + x + \alpha \mu, 2K - y - x + \alpha(1 - \mu))}{B(y + x + \alpha \mu, 2K - y - x + \alpha(1 - \mu))} g(y|x) \\
= \frac{y + x + \alpha \mu}{2K + \alpha} g(y|x)
\]

where the second equality follows from equation (4) and the third from the definition of the B function. Finally, we get

\[
\mathcal{U}(s_1, s_2|S, x) = \sum_{y=0}^{K} g(y|x) \left( \frac{y + x + \alpha \mu}{2K + \alpha} (2p_2(s|S(y)) + p_1(s|S(y))) - ((s_1 + s_2)p_2(s|S(y)) + s_1p_1(s|S(y))) \right).
\]
References


Chapter 3

Consumer Search in the Credit Card Industry

The credit card industry has attracted attention from both politicians and economists for apparently uncompetitive performance. The failure of credit card rates to fall with other market interest rates since the mid 1980s has led on several occasions to proposed federal rate caps.\(^1\) In the academic literature, Ausubel (1991) provides evidence of excess profits for credit card issuers; later research has questioned Ausubel's methodology, but has generally supported the conclusion that issuing credit cards was much more profitable than other banking activities over 1982–1992. How could excess profits be supported? Credit card issuers compete at the national level in an extremely unconcentrated market on a seemingly undifferentiated product. There do not appear to be any significant barriers to entry, nor is there any suggestion of collusion in any form.

Due to Ausubel, the most prominent potential explanation of excess profits, at least as a temporary phenomenon, is search cost. The goal of this paper is to develop a consumer search model of the credit card industry that is rich enough to allow an effective test of the search cost hypothesis. In particular, the model will exploit observable heterogeneity among borrowers to develop empirically testable implications of consumer search. Consumer level data will be used to test the model, and provide evidence on related questions:

- How, if at all, do borrowers respond to differences across issuers in interest rates?

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\(^1\)One rate cap proposal by Senator Alphonse D'Amato in November 1991 is alleged to have precipitated a fall of 120 points in the Dow Jones Average. Many states have their own usury limits, but these have limited impact on national issuers, which are based in states without rate regulation. Other legislative efforts have focused on tighter disclosure requirements on application forms. See Evans and Schmalensee (1993) on the regulatory history of the industry.
Does this response depend on borrower characteristics such as loan size and credit worthiness?

- Is the market segmented by credit risk? That is, to what extent is price dispersion explained by sorting of high risk borrowers to high rate issuers and low risk borrowers to low rate issuers?

Section 1 will survey the theoretical and empirical literature on the credit card industry. In Section 2, I develop a search cost model for the industry. Empirical tests are presented in Section 3. Results are discussed in the Conclusion.

1 Survey of the literature

1.1 Competition and excess profits

Judging from the structure of the industry, we should expect the credit card business to be highly competitive. In 1987, there were 4000 Visa and Mastercard issuers (Ausubel 1991); by the end of 1991, there were roughly 6000 Visa issuers alone, of which at least 100 competed at the national level (Evans and Schmalensee, 1993). Visa and Mastercard do not coordinate prices among their members: each issuer sets its own interest rate and fees, as well as applicant screening and mass marketing policies. Ausubel (1991), Litan (1992), Shapiro (1992), and Evans and Schmalensee (1993) provide market share data for the largest issuers, and summary concentration statistics. Litan (1992) discusses ease of entry for new issuers. All findings are consistent with perfect competition.

Nonetheless, credit card interest rates have been high relative to other loan rates, even when higher default rates are taken into account, and quite persistent in nominal terms despite the decline over the last decade in other market rates (see, e.g., Shapiro, 1992).

\footnote{The degree of persistence in credit card rates has been a subject of contention. Ausubel (1992, 1991) and Shapiro (1992) demonstrate that card rates are quite sticky in nominal terms, but Raskovich and Froeb (1992) question whether credit card rates are stickier than other retail bank rates after correction for variation in charge-offs. Also, Evans and Schmalensee (1993) note that other components of credit card pricing, especially the annual fee, have declined substantially. As neither my model nor my data address the issue of price rigidity, this paper does not take a position on the debate.}
Ausubel (1991) finds large ex-post accounting profits for the largest issuers of credit cards in 1983–1988. As Ausubel notes, there is a simultaneity bias here: Presumably, these issuers became large because they had some competitive advantage over other issuers. Therefore, we might expect the largest issuers to be consistently profitable. The issue is whether the marginal issuer is profitable. Data are less reliable for smaller issuers, but Ausubel cites evidence from the Federal Reserve’s Functional Cost Analysis studies in the 1980s that even small issuers earn roughly twice the ordinary rate of return on bank lending. In the early 1990s, articles in the trade journal *Credit Card Management* refer to the easy profits available to issuers in the 1980s. Despite the higher level of charge-offs in the recent recession, accounting profits have remained high: In a study done on behalf of Mastercard, Litan (1992) finds a pre-tax marginal return on equity of 25% for the first half of 1991, and an average return of 35%. Shapiro (1992) cites evidence from industry studies and business and trade journals that banks earned at least twice the return on assets on their credit card portfolios as on other bank businesses in 1988–1992. However, Evans and Schmalensee (1993) point out problems with using accounting rates of return, and specifically criticize Ausubel’s methodology on several points; they conclude that “[an] upper bound for the true after-tax return on equity in [1988] was probably no more than 22 percent to 27 percent,” and has probably been lower in the 1990s.

A second measure of profitability comes from the premium paid on bundled credit card accounts. This is a better measure because it addresses ex-ante expected (economic) profits. Ausubel (1991) finds premiums of up to 27% over par in the late 1980s. However, premiums (reported in CCM) fell to 13-14% in early 1991 and the market remained weak in 1992.3

Shapiro (1992) describes the high degree of price dispersion in the credit card market; his calculations demonstrate the considerable savings available to the average borrowing household from a modest search effort. Shapiro also examines the extent to which high rates are associated with high default rates: for a sample of large issuers in 1992, the correlation

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3One might also argue that bundled accounts are worth a premium over new accounts obtained by mass mailing, because “seasoned” accounts are more likely to carry an outstanding balance and less likely to default than new accounts. This explanation, however, only begs the question: If seasoned accounts have lower default risk, why is the rent captured by the issuer rather than the customer?
between APR and charge-offs is 0.23, and OLS regression shows that a 1% higher APR is associated with 0.16% higher charge-off rate. This association is much weaker than we would expect if price dispersion were due merely to issuers' efficiency at sorting consumers into homogenous risk classes.

1.2 Theoretical explanations

Ausubel (1991, 1988) proposes that consumer search (or switching costs) can explain many of the features of the credit card industry. Search cost models can account for both excess profits and price stickiness, and are consistent with the widespread practice of special introductory deals made available only to new customers. However, Ausubel (1991) argues that search costs are unlikely to explain the magnitude of profitability in this industry: In the late 1980s, the average account had roughly $1000 in outstanding balance. Given a 25% premium on bundled accounts, the implied search cost is $250, which is implausibly high.\footnote{The 25% premium on bundled accounts lasted only briefly; by 1991, the premium had stabilized at roughly 13%. The implied search cost of $130 is somewhat more plausible, or at least permits a smaller role for adverse selection. This figure also matches a contemporaneous estimate given in CCM of $130 in marketing cost per new account; if search or switching costs do make borrowers captive, issuers should be willing to spend on marketing the net present value of the future stream of rents.}

Ausubel’s model combines search cost with a peculiar form of adverse selection. He posits a class of borrowers who do not intend to borrow on their credit cards, and so are insensitive to interest rates. However, once such a consumer obtains a credit card, she finds herself borrowing anyway. There is a second class of consumers who are bad credit risks; they do intend to borrow on their credit cards because they have no alternative source of credit. These customers will be sensitive to interest rates. Unable to tell apart these two types, banks will be reluctant to compete on interest rate, as a lower rate would disproportionately draw customers of the second type. When this adverse selection effect is superimposed on a basic model of search costs, the price stickiness of the later is compounded.

Shapiro (1992) agrees with Ausubel that consumer irrationality is largely to blame. He contends that

consumers find it difficult to understand finance charges and properly evaluate
the tradeoff between interest rates and annual fees. ... Certainly there is reason to fear that many consumers are not savvy in their credit-card behavior: a recent national survey found that 30% of households did not know the interest rate charged on their unpaid credit-card balances, and nearly half of those households with current unpaid credit-card balances have no idea how much interest they paid over the course of the past year.

In support, Shapiro provides evidence that many households fail to integrate their assets and liabilities into a single portfolio in order to minimize their cost of borrowing; many households could pay off outstanding credit card balances out of liquid assets earning lower rates of interest.

Mester (1992) takes a different approach that dispenses with both irrationality and search costs. Credit cards are viewed here as an alternative to collateralized loans. There are two types of consumers; the types differ in credit risk. In a pooling equilibrium, both types choose to borrow on credit cards. In a separating equilibrium, the low risk types take collateralized loans, and the risky types borrow on credit cards. When the banks' cost of funds is high, the low risk types pool, and when the cost of funds drop (as it did in the late 1980s) the low risk types separate out. Therefore, credit card rates will be sticky downwards, but not upwards. Note that this model does not explain excess profits.

1.3 Empirical tests of consumer search

Berlin and Mester (1991) test a specific class of models of consumer search, and find it does not provide a good fit to the pricing of credit cards in the early 1980s. As far as I am aware, this paper is the only empirical examination of search costs in the credit card industry.

Berlin and Mester derive their tests of consumer search from two closely related models by Wilde and Schwartz (RES, July 1979) and Sadanand and Wilde (RES, April 1982). These particular models are somewhat ill-suited to the credit card industry because of their restrictions on the production function: Wilde and Schartz impose capacity limits on the firms, and Sadanard and Wilde require increasing marginal costs. In the credit card industry, both of these restrictions are clearly violated; consider, for example, the large premiums paid by the largest issuers, including Citibank, in the late 1980s to acquire credit
card accounts from smaller issuers, and that the largest issuers are likely to have economies of scale in marketing, processing costs and (possibly) in cost of funds. Without capacity limits or increasing marginal costs, the equilibria which motivate Berlin and Mester's tests may not exist.

Berlin and Mester conduct three sets of tests: 5

1. Does the rate charged by low rate issuers increase with marginal cost?

2. Does the rate charged by high rate issuers increase with marginal cost?

3. Do higher priced issuers have lower sales?

The Wilde and Schwartz model predicts that the rate charged by low rate issuers monotonically increases with marginal cost. In the first set of regressions, Berlin and Mester regress the rate charged by low rate issuers on cost of funds (which they take to be the rate on CDs); contrary to the model's prediction, the estimated coefficient is significantly negative. The Sadanard and Wilde model, which is essentially a generalization of the Wilde and Schwartz model from unit demand to downward sloping demand, predicts that the highest price in the market is strictly increasing with cost. In the second set of regressions, Berlin and Mester find instead that the estimated coefficient on cost of funds is insignificantly negative. Note, however, that the Sadanard and Wilde prediction is not necessarily robust to a broader class of search models; indeed, a zero coefficient is consistent with the Wilde and Schwartz model, which predicts that the high price is invariant with respect to cost.

These two tests ignore an important "stylized fact" for bank rates in general, and credit card rates in particular: As shown by Ausubel (1992, 1991) and Raskovich and Froeb (1992), credit card rates do adjust to changes in the Treasury bill rate, but with a time lag. Other rates, such as CD rates, also adjust with a lag, but a shorter one (Ausubel 1992). There are several possible explanations for this rigidity:

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5 Berlin and Mester conduct one additional test that is intended to distinguish among three types of equilibria that may arise in the model. The results are ambiguous, but always consistent with at least one of the types of possible equilibria.
• Ausubel (1988) notes that if consumers have increasing marginal cost of search, there will be a continuum of possible equilibrium rates. In such a model, prices need not change with small changes in costs.

• Ausubel (1992) explains rigidity in bank rates with a search cost model in which consumers learn of changes to firms' marginal cost after a lag.

• In the case of credit cards, issuers advertise for new customers by distributing printed application forms, which state the card’s interest rate; this practice may give rise to "menu costs" which discourage banks from adjusting card rates to changes in the Treasury bill rate that may be short-lived.

Whatever the reasons for this lag, these tests would be more robust if lagged CD rates were included in the regression.

The final testable prediction is generic to search models with heterogenous consumers, identical firms with declining average cost curves, and homogenous goods. (Berlin and Mester do not specify a model here.) In equilibrium, firms must have identical profits. An immediate implication is that higher priced firms must have lower sales. For each period $t$, they estimate the cross-sectional equation:

$$p_i = \alpha + \beta Q_i + \epsilon_i$$

where $p_i$ is bank $i$'s rate and $Q_i$ is its outstanding balances. The search model predicts that $\beta < 0$. They find a positive $\hat{\beta}$ in each of 24 time periods (significant in 11). Note, however, that there is an endogeneity problem here: On the demand side, quantity is a downward sloping function of price. On the supply side, an issuer facing stronger demand (perhaps due to some advantage in marketing) will charge a higher price, so price is an upward sloping function of quantity. Therefore, Berlin and Mester's OLS regressions are inconsistent.

Another problem with this test is that outstanding balance is not a good measure of sales. In practice, a consumer's outstanding balance is built up and paid down slowly; until recently, it was not so easy to abandon one issuer by moving a balance to another issuer.
Therefore, a change in relative prices between firms should not strongly effect the firms' levels of outstanding balances; instead, we might expect a change in relative prices to effect the distribution among firms of new balances. A better test of the fourth prediction might thus be

\[ p_i = \alpha + \beta \cdot \Delta Q_i + \epsilon_i \]

where \( p_i \) is bank i’s rate and \( \Delta Q_i \) is its change in outstanding balances since last period.

Berlin and Mester’s work is valuable in that it contributes a number of stylized facts for the industry. However, their empirical tests take their particular choice of models too literally and ignore important institutional details of this market. I regard as open the basic question of the importance of search costs.

2 A search model with heterogenous consumers

This is a model of consumer search with heterogenous consumers, but no adverse selection. In its simple version there are only two default types (Bad and Good).

(A1) There are a continuum of consumers with unit mass in the economy, each of whom must obtain one credit card. Each consumer has a three dimensional type \((D, x, s)\).

(A2) Default risk type takes on two values. Good types have probability of default \( D_G = 0 \), and bad types have \( D_B > 0 \); a fraction \( \alpha \) of the population is Good and the remaining fraction \( 1 - \alpha \) is Bad. The consumer’s \( D \) is public knowledge; i.e., there is no asymmetric information on default risk.

(A3) The consumer needs to borrow an amount \( x \), which is drawn from a \( U[0,1] \) distribution. From the consumer’s perspective, her realization \( x \) is given exogenously, and is perfectly inelastic. \( x \) is private knowledge. There is no substitute for credit card borrowing.
(A4) The consumer has search cost $s$, which is private knowledge. The distribution, $F(s)$, in the population is exponential with parameter $\theta$; the density is given by

$$f(s) = \frac{1}{\theta} \exp(-s/\theta).$$

Note that the mean search cost is simply $\theta$.

(A5) There is a fixed (large) number $N$ of banks. Each issuer posts only one interest rate; that is, the bank cannot offer one rate for Bad types and another rate for Good types. The cost of funds to banks is $c = 0$. Banks are risk-neutral.

(A6) The consumers' rate of time preference is zero. Consumers are risk-neutral, and want to minimize the sum of search costs and expected interest payment at time 1.

At time 0, the consumer searches for a credit card issuer; each search turns up one issuer, which is chosen randomly (with replacement) from the set of issuers. Search continues, at cost $s$ per search, until the expected benefit to continued search does not justify the expected cost. Also, if the consumer is Bad, search must continue until she has found an issuer who accepts Bad types. The consumer then borrows. At time 1, the consumer repays principal plus interest (with probability $1 - D$) or defaults completely (with probability $D$).

I solve the model for a two-price equilibrium in which $k$ issuers charge a low interest rate $r_L$ and $(N - k)$ issuers charge a high interest rate $r_H$. The low price issuers are willing to lend only to the low risk (Good) consumers; this requires that $(1 + r_L)(1 - D_B) < 1$. The high price issuers are willing to take anyone. $k$ is determined endogenously to equalize profits across the two types of issuers.

2.1 Solving for a two price equilibrium

I first find the condition that governs whether or not a consumer will search. Say the consumer's best offer so far is $\hat{r}$. Since the consumer's objective is to minimize the sum of
expected interest payments and search costs, she will search if and only if

\[ x(1 - D)(1 + \hat{r}) \geq x(1 - D)(1 + E^*_D(\hat{r})) + s \]
\[ \iff s < x(1 - D)(\hat{r} - E^*_D(\hat{r})) \]  \hspace{1cm} (1)

where \( E^*_D \) is the expected best rate a type \( D \) consumer will have after only one more search, given that the best current offer is \( \hat{r} \).

For analysis of the model out of equilibrium, it is useful to have notation that generalizes to the case of \( J \) different rates, \( r_1 < r_2 < \ldots < r_J \) and cost of funds \( c \geq 0 \); this generalized notation will also be useful in the empirical section. Let \( \mu_j \) (or \( \mu(r_j) \)) be the fraction of issuers that charge \( r_j \); i.e., \( \mu \) is the probability distribution function of rates from which a consumer draws when searching. Let \( \mathcal{I}_D(r) \) be an index function that equals one iff a type \( D \) consumer is acceptable to an issuer at rate \( r \); i.e.,

\[ \mathcal{I}_D(r) = \begin{cases} 1, & \text{if } (1 + r)(1 - D) \geq 1 + c; \\ 0, & \text{otherwise.} \end{cases} \]

Then,

\[ E^*_D(\hat{r}) = \hat{r} - \sum_{j=1}^{J} \mu(r_j)\mathcal{I}_D(r_j)1_{(r_j < \hat{r})}(\hat{r} - r_j). \]  \hspace{1cm} (2)

The consumer's stop rate is the highest rate such that the consumer will search until finding that rate or better, and then stop; formally, \( r_q \) is the stop rate iff

\[ x(1 - D)(r_q - E^*_D(r_q)) < s \leq x(1 - D)(r_{q+1} - E^*_D(r_{q+1})). \]

The stop rate is analogous to stop (or "reservation," or "cutoff") price in standard models of search (see, e.g., Diamond, 1971).

Now we turn to the firms' maximization of expected profit. For the moment, hold the number of low rate issuers, \( k \), as fixed. Say that \( k - 1 \) of these low rate issuers charge a rate \( r_L \) and all \( N - k \) high rate issuers charge \( r_H. \)

\footnote{In terms of the notation already introduced, \( \mu(r_L) = (k - 1)/N, \mu(r) = 1/N \) where \( r \) is the rate charged}

Assume for now that \( \mathcal{I}_{D_L}(r_L) = 0 \), so Bad
type consumers will not be accepted by low rate issuers. Denote by $\pi^+_k(r)$ the expected profit for the $k$th low rate issuer as a function of its interest rate, restricted to $r \geq r_L$; let $\pi^-_L(r)$ be the same for interest rates $r \leq r_L$. Then at the equilibrium $r_L$, we must have

$$\left. \frac{d\pi^+_k}{dr} \right|_{r=r_L} = \left. \frac{d\pi^-_L}{dr} \right|_{r=r_L} = 0$$ (3)

First we derive $\pi^-_L(r)$. Say issuer $k$ charges $r < r_L$. A Good type consumer who happens to land first at one of the $N - k$ high rate issuers will search away if and only if $s < xA^G_2$, where

$$A^G_2 \equiv (1 - D_G)(r_H - E_{D_G}^*(r_H)) = \frac{k-1}{N}(r_H - r_L) + \frac{1}{N}(r_H - r).$$

A Good type consumer who lands (either first or after searching away from a high rate issuer) at one of the $k - 1$ rate $r_L$ issuers will search for the lowest rate issuer, issuer $k$, if and only if $s < xA^G_1$, where

$$A^G_1 \equiv (1 - D_G)(r_L - E_{D_G}^*(r_L)) = \frac{1}{N}(r_L - r).$$

As intuition suggests, we have $A^G_1 < A^G_2$; that is, the cutoff value $A^G_1$ imposes a strictly stronger restriction than does cutoff value $A^G_2$. Also note that since the support of $F$ is $\mathbb{R}^+$, we are guaranteed to have some consumers who will satisfy $s < xA^G_1$ (and thus have stop rate $r$), some with $xA^G_1 < s < xA^G_2$ (stop rate $r_L$), and some with $s > xA^G_2$ (stop rate $r_H$).

Since the profit per dollar lent at rate $r$ to a Good type is simply $r$, the total profit earned by issuer $k$ is

$$\pi^-_L(r) = \alpha r \left( \int_0^{A^G_1} \int_0^1 xdxdf(s) \right) + \frac{1}{k} \left( \int_0^{A^G_2} \int_0^1 xdxdf(s) - \int_0^{A^G_1} \int_0^1 xdxdf(s) \right) + \frac{1}{N} \left( \int_0^{A^G_2} \int_0^{\frac{s}{A^G_1}} xdxdf(s) + \int_{A^G_2}^\infty \int_0^1 xdxdf(s) \right).$$ (4)

by issuer $k$, and $\mu(r_H) = (N - k)/N$. 

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The first integral is the total borrowed by consumers with stop rate \( r \); issuer \( k \) gets all of these. The second set of terms is for the amount borrowed by consumers with stop rate \( r_L \); issuer \( k \) gets one in every \( k \) of these, since they are equally likely to end up with any low rate issuer. The last set of terms is for the amount borrowed by Good consumers who are not willing to search at all; issuer \( k \) just gets the one out of \( N \) who happens upon issuer \( k \) first.

I integrate and re-arrange terms (see Appendix A.1) to obtain
\[
\pi_L^-(r) = \frac{\alpha}{2} r \left( \frac{1}{N} + \frac{N - k}{Nk} (F(A^G) - Z(A^G)) + \frac{k-1}{k} (F(A^I) - Z(A^I)) \right),
\]
where
\[
Z(A) \equiv \int_0^A \left( \frac{A}{s} \right)^2 \frac{1}{\theta} \exp(-s/\theta) ds = \int_0^A \frac{s}{A} \frac{1}{\theta} \exp(-s/\theta) ds = A^{-2}(2\theta^2 - (A^2 + 2\theta A + 2\theta^2) \exp(-A/\theta)).
\]

The issuer maximizes equation (5) with respect to \( r \). In equilibrium, \( r_L \) is optimal, so the first order condition must be satisfied at \( r = r_L \). The resulting condition, derived in Appendix A.2, is
\[
1 - \frac{2r_L}{3\theta} \frac{k - 1}{k} + \frac{N - k}{Nk} (F(A^*) - Z(A^*)) - \frac{r_L}{N} \frac{k}{A^*} \frac{2}{k} Z(A^*) = 0
\]
where \( A^* \equiv (k/N)(r_H - r_L) \).

In Appendix A.3, I construct \( \pi_L^+(r) \), which is the profit earned by a low rate issuer who defects to \( r > r_L \), and show that the first order condition at \( r = r_L \) is identical to equation (6). This merely serves as a check on the algebra behind equation (6).

The equilibrium condition for high rate issuers is obtained similarly, but is a little more complicated, because high rate issuers accept both Good and Bad type consumers. I similarly find that the left and right derivatives of the profit function are the same at
\( r = r_H; \) the first order condition for equilibrium is

\[
\frac{\alpha}{N} \left( \left( 1 - \frac{N - 1}{N} \right) r_H \right) \frac{Z(A^*)}{N} + 1 - F(A^*) \\
+ \frac{1 - \alpha}{N - k} \left( 1 - D_B \right) \left( 1 - \frac{2}{3} \frac{N - k - 1}{N} \right) (1 + r_H)(1 - D_B) - 1) = 0. \tag{7}
\]

This expression is derived in Appendix A.4.

In equilibrium, the profit of a low rate issuer must equal the profit of a high rate issuer. This condition will determine \( k \) endogenously. Therefore, equilibrium is given by three equations ((6), (7), and \( \pi_L(r_L) = \pi_H(r_H) \)) with three unknowns \( (r_L, r_H, k) \). There appears to be no closed form solution, but it is easy to solve numerically using Matlab.\(^7\)

Finally, recall that I assumed that \( \mathcal{I}_{D_B}(r_L) = 0, \) so that low rate issuers would not accept Bad type consumers. Numerical solutions must be checked to see this separating condition does hold. I have found two conditions under which it fails: First, as the proportion of Good types \( (\alpha) \) falls, a lower proportion of issuers specialize in Good types, so the rate \( r_L \) increases; for low enough \( \alpha, r_L \) is high enough that Bad type consumers can be served profitably, and the equilibrium breaks down. Second, \( r_H \) converges towards \( r_L \) as the default rate of the Bad types \( (D_B) \) falls; for low enough \( D_B, \) the separating condition fails. For a large and “reasonable” range of the model’s parameters \( (D_B, \theta, \alpha, N) \), the sorting condition is satisfied, so the separating equilibrium does exist.

Figure 1 shows how the equilibrium rates \( (r_L, r_H) \) and proportion of low rate issuers \( (k/N) \) change as the model parameters are varied around a “base case” \( (D_B=15\%, \theta = 0.03, \alpha = 0.5, N = 500) \). The comparative statics all accord with intuition:

\( \theta: \) As mean search cost increases, it becomes more difficult for issuers to steal customers from one another; competition between issuers is thus reduced, and interest rates rise.

\( N: \) Increasing the number of issuers has only a minimal effect on interest rates. As \( N \) increases, the probability of hitting any given issuer in a single search falls. This

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\(^7\)For the sake of continuity in the solutions, I do not restrict \( k \) to be an integer, but rather allow \( k/N \) to be any real number in \([0, 1]\).
reduces the ability of an issuer to raise its rate above other issuers without losing customers, and so strengthens competition; however, it also reduces the incentive to undercut other issuers, because a prospective searcher has a smaller chance of finding the lower rate. The net effect is to reduce rates, but only slightly. Note, however, that total profit per issuer does fall with \( N \), as the same number of borrowers are divided among a greater number of issuers.

\( D_B \): As the default risk of the Bad type increases, so does the high rate \( r_H \). Less obviously, the low rate \( r_L \) increases slightly, because the high rate issuers pose weaker competition for the Good types as \( r_H \) increases.

\( \alpha \): The effect of varying the proportion of Good types is more complex. As \( \alpha \) increases, the number of issuers \( k \) serving only Good types increases. As it becomes easier for a Good type to find a low rate issuer and more difficult for a Bad type to find a high rate issuer, \( r_L \) decreases and \( r_H \) increases. This competitive effect, i.e., the relatively weak competition between issuers who serve the minority type, weakens the effect of \( \alpha \) on \( k \), so that when \( \alpha > 0.5 \), \( k/N \) increases less than one-for-one with \( \alpha \), and when \( \alpha < 0.5 \), \( k/N \) falls by less than one-for-one as \( \alpha \) falls.

Other than \( \alpha \), none of the parameters have any significant effect on the equilibrium ratio of low rate issuers to high rate issuers \( (k/N) \). I have examined similar comparative statics around other sets of parameter values, and obtained qualitatively similar results.

2.2 Relationship to Ausubel’s model

My model is technically similar to that of Ausubel (1991, 1988); the two models are based on the same search process, but extend the basic model in different directions. My model emphasizes observable consumer heterogeneity in order to show how market segmentation can arise and compound the difficulty of search for consumers. Ausubel, in contrast, emphasizes unobservable heterogeneity in order to show how adverse selection can magnify the incentive of firms in a search model to avoid competing on price. The two approaches
Figure 1: Comparative Statics

Upper left box plots equilibrium rates as function of $\theta$; other parameters are $D_B = 15\%$, $\alpha = 0.5$, $N = 50$ (solid lines) and $N = 5000$ (dashed lines). Upper right box plots equilibrium rates as function of $D_B$; other parameters are $\alpha = 0.5$, $\theta = 0.03$, $N = 500$. The bottom boxes plot equilibrium rates and proportion of issuers at low rate $r_L$ as functions of $\alpha$; other parameters are $\theta = 0.03$, $D_B = 15\%$, $N = 500$. 

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are not incompatible: Begin with my model, but assume that default risk is imperfectly observed by issuers; e.g., that an issuer mistakes a Good type consumer for Bad or Bad type for Good with some probability \( p \). Then, as long as \( p \) is not too large, I conjecture that a separating equilibrium can exist as in Section 2.1, except that some Bad types will mistakenly be accepted by low rate issuers. To introduce adverse selection, assume further that desired loan size \( x \) is positively correlated with default risk; the borrowers with the greatest incentive to search are then disproportionately Bad types. As in Ausubel’s model, an issuer who undercuts its rivals would tend to end up with the least desirable customers; I conjecture the consequent reluctance of issuers to compete on price magnifies the effects of search on equilibrium prices.\(^8\) Note that there is no need here to posit a class of irrational consumers. Ausubel’s irrational consumers should search, but do not because they mistakenly believe they will not borrow. My rational consumers do not bother to search if they do not intend to borrow enough to justify the expense.

Ausubel’s model also introduces increasing marginal cost of search; this feature delivers price stickiness in the form of a continuum of equilibria. I conjecture that this could be incorporated into my model, at the expense of tractability, with similar results.

2.3 Explaining “stylized facts” for the industry

The model provides insight into some of the “stylized facts” for the industry. First, the model explains excess profit, at least as a temporary phenomenon. If the number of banks \( N \) is not fixed, these profits will attract entry until profit per issuer equals fixed cost; this explains the rapid rate increase over the last decade in the number of Visa and Mastercard issuers. The model also predicts correctly that rates respond very little to entry.\(^9\)

Second, the model predicts partial sorting of consumers by default risk; i.e., that low rate issuers will lend only to low risk borrowers, but high rate issuers will lend to both low risk and high risk borrowers. It explains why there will be a diversity of rates in equilibrium.

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\(^{8}\)If the correlation of \( D \) and \( x \) is too large, the separating equilibrium may cease to exist.

\(^{9}\)If there is some fixed cost to entry, then entry is socially wasteful, especially if on a small scale. Excess entry has not been addressed in the academic or regulatory debate on the credit card industry.
and why charge-offs will be observed to increase with APR by less than one-for-one. The expected charge-off rate for a high rate issuer is

\[
\frac{D_B}{1 + (1 - k/N)\frac{\alpha}{1 - \alpha}(Z(A^*) + 1 - F(A^*))},
\]

which is less than \(D_B\) because of the presence of low risk borrowers, and declines with the proportion \(\alpha\) of Good types in the economy. To take a numerical example, for parameters \((\alpha = 0.9, D_B = 15\%, N = 500, \theta = 0.03)\) the difference in charge-off rates between high and low rate issuers is 48\% of the difference in interest rates. Recall, however, that Shapiro (1992) found an association only one-third as strong; I do not believe that any set of plausible parameters for the model would produce such a weak relationship between issuer interest rate and charge-offs.

Third, in my model there is clearly an advantage to a bank that is able to offer more than one rate (in violation of assumption (A5)): in equilibrium, such a bank would be able to retain all customers that searched to it.\(^{10}\) This provides intuition for why most large issuers since the mid 1980s have offered a Gold plan to credit-worthy customers, and the largest issuers have in recent years quietly introduced menus of rate and fee combinations tailored to specific groups of borrowers.\(^{11}\)

Finally, if default risk and loan size are positively correlated (as we might expect), then the model permits interest rate and total loans to be positively correlated across issuers. This offers another explanation for the positive relationship found by Berlin and Mester (1991) between an issuer’s rate and “sales.”

It should be noted that the model does not capture two potentially important aspects of

---

\(^{10}\)A formal proof is straightforward. If a low rate issuer were also able to offer \(r_H\) to Bad type customers, it would be able to retain those Bad type customers who happen upon the issuer in search. This issuer would earn the same profits on Good borrowers, so profits would strictly increase on the additional Bad borrowers. Since high rate issuers earn the same profit as low rate issuers in equilibrium, the issuer who offers both rates must also earn higher profit than the high rate issuers.

\(^{11}\)Introducing multiple rates is individually rational for the issuer, but perhaps not for the system: if every issuer offers two rates instead of one, competition increases, so rates and profits fall. Why then, were Visa and Mastercard willing to make the system-level changes necessary for their members to offer Gold cards? I conjecture that Visa and Mastercard realized that members would have begun to offer different rates to different borrowers anyway, and a distinct Gold card would allow issuers to earn an additional rent on prestige while so doing.
the credit card as a good. First, the price of a credit card includes not only interest rate, but also annual fee, grace period, and fees for missing minimum payments and other violations. There is anecdotal evidence in the trade press that customers are especially sensitive to annual fee; in contrast to the stickiness of nominal interest rates, the average annual fee fell by 35% over 1984–1992 (Evans and Schmalensee, 1993). Since the model suggests that non-revolvers should be more sensitive to differences in annual fees than revolvers, it is puzzling that card issuers choose to compete on this dimension of price. Second, issuers impose credit limits on their accounts; there is some evidence that low rate issuers, such as Arkansas banks operating under usury limits, typically extend low credit ceilings (Evans and Schmalensee, 1993). Anecdotal evidence that borrowers are sensitive to these limits may then explain why many do not seek out the lower rate cards. A positive correlation between interest rate and credit limit would also suggest that Ausubel’s adverse selection hypothesis is relevant; i.e., that low rate issuers need to use low credit ceilings to dissuade the most credit-hungry (and riskiest) borrowers.\footnote{If rates and credit limits are indeed positively correlated, it would offer yet another explanation for the positive relationship found by Berlin and Mester (1991) between an issuer’s rate and its total outstanding balances. Note, however, that Gold cards typically offer both lower interest rates and high credit limits than Standard cards.}

3 Test using consumer level data

In this section, I test an extension of the model of Section 2 using consumer level data.\footnote{Cross-sectional consumer level data, though ideal for testing my model, do not permit tests of the alternative models surveyed in Section 1.2; the implications of those models apply to time-series data, preferably at the issuer level. Therefore, a rejection of my model in the tests below would not constitute evidence in favor of any of the alternatives.}

The test exploits the model’s predicted relationship between a consumer’s loan size and credit quality (independent variables) and the interest rate the consumer is observed to pay (dependent variable). From this point onward, attention is focused entirely on consumer search behavior, i.e., the demand side of the model. Issuers’ behavior will be taken as given; one may assume that the distribution of issuer rates derives from equilibrium strategies, as in the model above, but it is not necessary for the validity of the test below. An advantage
of this approach is that it is unnecessary to have any knowledge of the distribution of default risks and loan sizes among the population of borrowers.

In the two-type model, Bad types are served exclusively by high rate issuers, whereas Good types will be found at both high rate and low rate issuers. Among the Good types, those taking larger loans will be more likely to be observed at a low rate issuer, but the relationship remains stochastic: A Good type consumer taking a large loan may also have a high search cost, and thus be unwilling to search away from a high rate. Conversely, a Good consumer taking a small loan may be observed at a low rate issuer due either to luck (i.e., the consumer happens to arrive at a low rate issuer first) or to having very low search cost. In a more general framework of many consumer types and issuer interest rates, the model's prediction may be summarized as:

There will be, on average, a non-positive relationship between loan size and interest rate. The magnitude of this relationship is affected by credit-worthiness: the higher the default risk, the smaller the range of rates for which the consumer qualifies, so the smaller (in absolute magnitude) the negative relationship between loan size and interest rate. Moreover, the relationship is made stochastic by two unobserved factors: the consumer's search cost and random sequence of issuers visited in the search process.

In principle, I could derive a structural model using an extension of the model to $J$ rates. Assume we can observe default risk $D$ directly, as well as loan size $x$ and interest rate $r$. Assume also we know the distribution of rates offered by issuers. Then it is straightforward to write a function $p^*(r|D, x, s)$, which is the probability of observing a consumer of type $(D, x, s)$ at an issuer at rate $r$. Search cost $s$ is not observed, so we can integrate $p^*$ over the distribution of $s$ to get $p(r|D, x)$. The log likelihood function for the same is then

$$\ln \mathcal{L} = \sum_{t=1}^{T} \ln p(r(t)|D_t, x_t)$$

and we can find the value for parameter $\theta$ which maximizes equation (8).

In practice, a purely structural approach is neither desirable nor feasible. It is undesirable because the relevance of the search model is itself the question of primary interest;
the structural approach provides parameter estimates conditional on the model's truth, but
does not provide a natural means of testing the model's fit. It is infeasible because the
likelihood function is relatively insensitive to the parameter of greatest interest, θ. As θ is
a distributional parameter on a variable that is integrated out (i.e., search cost s), large
changes in θ lead to only small changes in p. Furthermore, the model in its present form
does not allow for the possibility that a high risk consumer may be observed at a low rate.
This is occasionally observed in reality, so the model would need to be extended to allow
for some probability of issuer error in applicant screening; permitting this additional degree
of freedom to the model flattens the likelihood function still more. Indeed, the likelihood
function is so poorly shaped — it is not only flat, but also is not globally concave — that I
have found it impossible to estimate a structural model with my sample.

At the other extreme, it is clear that a simple linear regression will not suffice either.
If the model is correct, then the interaction between loan size and credit quality is more
important than either variable alone, and the form of the interaction, while monotonic, is
nonlinear. Therefore, the regressions below are reduced-form, but attempt to capture the
essential properties of the structural model. In this section, I first develop the reduced-form
regression equation. Next, I describe my data. Finally, I present regression results.

3.1 A reduced-form regression

Consider a consumer with default risk D who, once she obtains a card at any rate, is
unwilling to search further; this consumer may be observed ex-post at any rate between the
lowest rate for which she qualifies and the highest rate on offer, r_f. Ex-ante, the expected
rate for this consumer is

\[ r^*(D) \equiv \frac{\sum_{j=1}^J \mu_j I_D(r_j) r_j}{\sum_{j=1}^J \mu_j I_D(r_j)}. \]

If the model of Section 2 is correct, then loan size, adjusted for credit quality, will ex-
plain part of the difference between the consumer's observed rate and ex-ante expected "no
search" rate. I propose the regression equation

\[ r(t) - r^*(D_t) = \beta_0 + \beta_1 x_t \left(1 - \frac{D_t}{\hat{D}}\right)^{\lambda} + \epsilon_t \]  

(9)

where \(\beta_0\) and \(\beta_1\) are parameters to be estimated. (Note that I write \(r(t)\) for consumer \(t\)'s rate to avoid confusion with subscripted \(r_j\), which is the rate offered by issuer \(j\).) The parameter of greatest interest is \(\beta_1\), which the model predicts to be negative. The term \(x_t(1 - D_t/\hat{D})^{\lambda}\) measures the incentive to search: If loan size is zero, there is no gain to search, and this term is zero. Similarly, if \(D_t = \hat{D}\), then the consumer is qualified for only the highest rate, and there is no point to search once a \(r_j\) issuer is found. Holding fixed \(x_t > 0\), the incentive to search increases as default risk \(D_t\) approaches zero. I use \(\lambda\) to parameterize the strength of the interaction between loan size and credit quality. In principle, it could be estimated simultaneously with the \(\beta\); however, I find that it is only weakly identified against \(\beta_1\).\(^\text{14}\) Therefore, I treat \(\lambda\) as a constant, and run regressions for a number of plausible assigned values.

I have assumed thus far that the \(D_t\) are directly observable. What both issuers and econometrician actually observe is a vector \(z_t\) of consumer \(t\)'s characteristics. I model \(D_t\) as a function of a linear combination of the \(z_t\). Since the linear combination \(z_t\delta\) can be any real number, I need a continuous, strictly monotonic mapping from \((-\infty, \infty)\) to \((0, \hat{D})\). A computationally convenient choice is a scaled logistic function,

\[ D_t = D(z_t\delta) = \frac{\hat{D}}{1 + \exp(-z_t\delta)). \]

(10)

Equations (9) and (10) will be estimated by nonlinear least squares (NLLS). If the model is correct, the variance of the \(\epsilon_t\) decreases with \(D_t\), because consumers are qualified for a smaller range of interest rates as \(D_t\) increases. The NLLS estimator remains consistent under conditional heteroskedasticity, but I will need to use White (1980) heteroskedasticity-robust

\(^{14}\)A high value of \(\lambda\) and high (in magnitude) value of \(\beta_1\) is virtually indistinguishable from some pair of low values of each.
standard errors.

3.2 Data sources

All of the data used below comes from 1987, which by coincidence is likely to be the best year for the purposes of this study. By 1987, the credit card industry had matured: nine years following the Marquette deregulation, the market for credit cards was unified nationally, and consumer demand for cards was more or less saturated. Furthermore, it was still rare in 1987 for an issuer to offer more than two basic credit card packages (i.e., Gold and Standard); in more recent years, the trend has been towards tailoring combinations of interest rate and annual fee to target different market segments.\textsuperscript{15}

I have consumer level data from the 1987 Payment System Inc. (PSI) survey of household credit card use and issuer level data from RAM Research and from the Federal Reserve Interest Rate Survey (IRS). For the purposes of my study, the PSI survey contains:

- \([r]\) interest rate (APR) on each credit card;

- \([z]\) outstanding balance after most recent payment for each credit card;

- \([z]\) household income, home-ownership, non-credit card debt, credit history.

It also contains, for each card account, the annual fee on the account, the type of card (Visa or Mastercard, Gold or Standard) and whether the card has ATM access. These variables are likely to affect choice of credit card, but will not be used in this paper; I hope to make use of these data in later analysis.

There are 701 households in the PSI sample who provide data on at least one account and on the credit quality variables. The sample households appear to be more affluent than the average American household: mean income is $43,600, 82% are homeowners, none report a previous bankruptcy. The households provide data on a total of 1,158 Visa and Mastercard accounts, of which 43% are Mastercards (41% of total debt), and 9% are Gold

\textsuperscript{15}I am grateful to Robert B. McKinley of RAM Research for his comments on the development of the industry over the last ten years.
cards (19% of total debt). The mean outstanding total credit card debt per household is $940, which is slightly less than the average reported in Ausubel (1991); 42% of sample households report having no credit card debt at all, which may be somewhat higher than the percentage of non-revolvers in the population.\textsuperscript{16} Household characteristics are summarized in Table 1.

<table>
<thead>
<tr>
<th>Table 1: PSI Household Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income ($1000)</td>
</tr>
<tr>
<td>Loan size ($1000)</td>
</tr>
<tr>
<td>Other debt ($1000)</td>
</tr>
<tr>
<td>Loan size ($1000)</td>
</tr>
</tbody>
</table>

The set of interest rates offered by issuers and the probability distribution $\mu$ on that set is constructed from one of two available sources. The first is the 1987 survey by RAM Research, which includes APR and annual fee for 165 cards offered by 104 issuers; 52 of the plans are for Gold cards, and the rest Standard. The RAM survey samples the largest issuers disproportionately: Out of the top ten issuers of MasterCard and Visa cards in 1987 (as listed in Ausubel, 1991), the RAM survey includes both Standard and Gold cards for seven and Standard cards for another two. The RAM survey also appears to sample low rate issuers disproportionately. These sampling biases are desirable as an adjustment for the greater visibility of the largest issuers, through advertising and mass mailings, and the lowest rate issuers, through lists occasionally published in Fortune and other magazines.

The second source of information on issuer rates, the Federal Reserve Board’s IRS, is a quarterly survey of about 150 banks; I have the surveys conducted in February and May of 1987. Each bank reports its most common rate on several categories of consumer loans, including credit card lending; in each quarter, roughly 125 banks report a credit card

\textsuperscript{16}There are no reliable data on the percentage of non-revolvers. Canner and Fergus (1987) cite a 1983 Michigan study in which 47% of households claim to "nearly always pay in full" their credit card balances. Shapiro (1992) reports that 36.4% of households in the 1991 PSI study claim to be non-revolvers, and 23.7% claim to revolve only occasionally. However, Ausubel (1991) reports from a small survey of issuers that only one-quarter of customers avoid finance charges. According to CCM, the industry rule-of-thumb is that one-third of card accounts will be non-revolving.
rate. In contrast to the PSI, the sampling methodology of the IRS emphasizes geographic coverage; it is therefore biased in favor of the mid-sized state-chartered banks, which are generally small- to medium-sized issuers. Also, because it requests from each bank only the rate on its most common card plan, the IRS underweights Gold card rates, which tend to be lower than Standard card rates. The PSI questionnaires were mailed in April 1987, so an argument could be made for using either of the two IRS; it should make little difference, since the two distributions hardly differ.

Table 2: Interest Rates in Consumer and Issuer Data Sets

<table>
<thead>
<tr>
<th></th>
<th>RAM</th>
<th>IRS(2/87)</th>
<th>IRS(5/87)</th>
<th>PSI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>9.5</td>
<td>14.0</td>
<td>13.5</td>
<td>8.0</td>
</tr>
<tr>
<td>Maximum</td>
<td>22.2</td>
<td>22.0</td>
<td>22.0</td>
<td>25.0</td>
</tr>
<tr>
<td>Mean</td>
<td>17.2</td>
<td>18.1</td>
<td>17.9</td>
<td>17.8</td>
</tr>
<tr>
<td>Median</td>
<td>17.9</td>
<td>18.0</td>
<td>18.0</td>
<td>18.0</td>
</tr>
<tr>
<td>Mode</td>
<td>18.0</td>
<td>18.0</td>
<td>18.0</td>
<td>18.0</td>
</tr>
<tr>
<td>$\mu$(mode)</td>
<td>0.22</td>
<td>0.47</td>
<td>0.46</td>
<td>0.33</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Distribution of rates</th>
<th>RAM</th>
<th>IRS(2/87)</th>
<th>IRS(5/87)</th>
<th>PSI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r \leq 12%$</td>
<td>6.25%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>2.76%</td>
</tr>
<tr>
<td>12% &lt; $r \leq 15%$</td>
<td>13.75%</td>
<td>5.44%</td>
<td>6.46%</td>
<td>10.36%</td>
</tr>
<tr>
<td>15% &lt; $r \leq 17%$</td>
<td>18.13%</td>
<td>10.88%</td>
<td>12.92%</td>
<td>11.66%</td>
</tr>
<tr>
<td>17% &lt; $r \leq 19%$</td>
<td>41.88%</td>
<td>61.29%</td>
<td>61.31%</td>
<td>45.94%</td>
</tr>
<tr>
<td>19% &lt; $r \leq 21%$</td>
<td>17.50%</td>
<td>20.94%</td>
<td>17.74%</td>
<td>27.03%</td>
</tr>
<tr>
<td>21% &lt; $r$</td>
<td>2.50%</td>
<td>1.55%</td>
<td>1.61%</td>
<td>2.25%</td>
</tr>
</tbody>
</table>

| Sample size | 165 | 129 | 124 | 1158 |
| Number of rates | 47  | 30  | 30  | 75   |

†: Gold and Standard plans from same issuer counted distinctly. Multiple cards held by a single household in the PSI sample also counted distinctly.

Cost data comes from testimony by Visa U.S.A. Inc. before a U.S. House of Representatives committee hearing. For the first quarter of 1987, cost of funds was 7.4% and operating expenses were 5.5% of outstandings. Since it is unclear how much of the operating expense can be incorporated into the marginal cost of a credit card loan, I need to pick a reasonable number between 7.4% and 12.9%, say 9%. However imperfect this is, the alternative — to estimate $c$ as a parameter — does not appear feasible, because a small increase in $c$ is nearly indistinguishable from an increase in all consumers' $D$'s; i.e., if both are taken
as parameters, $c$ and $\delta$ are weakly identified. A similar problem forces me to choose $\tilde{D}$ exogenously. I set it to the highest value of $D$ that would qualify for the highest rate $r_J$; i.e., $\tilde{D}$ solves $(1 + r_J)(1 - \tilde{D}) = 1 + c$. I will test the sensitivity of the results to the choice of $c$.

3.3 Preparing the data

Like any consumer survey, the PSI is vulnerable to misreporting and to sampling biases. The respondent households are self-selected from those who received the PSI mailing, and appear to be more affluent than average. The reported outstanding balances on each account represent debt at a single point in time, and are not necessarily representative of the debt typically carried by the household. I make no attempt to correct for these problems. Misreporting might bias results in favor of the search cost hypothesis, while measurement error in loan size is likely to attenuate the search coefficient $\beta_1$ towards zero. I expect these biases to be relatively small, and leave an even smaller net bias. A potentially more serious problem is that the PSI survey does not contain data on credit limits; this may give rise to a bias against the search hypothesis, because the existence of credit limits, especially on low rate cards, reduces incentive to search for a lower rate.

Multiple cards per household raises another technical difficulty. As the model does not incorporate credit limits, each consumer will want to have only one credit card. Reality, of course, is inconveniently otherwise: Households in the PSI survey have an average of 1.65 Visa/Mastercard accounts for which they provide data. Even if credit limits were never binding, there are other reasons for carrying multiple cards, such as imperfect substitutability between Visa and Mastercard. I propose two alternative methods of reconciling multiple card households with the model:

- Let the household’s interest rate be the mean, weighted by outstanding balances, of interest rates on each account. (When no account has an outstanding balance, use an unweighted mean rate.) This approach best captures the borrower's decision-making process, since the borrower cares about the average rate on her portfolio of card debt,
rather than the rate on any individual card. However, this approach throws away useful information: For households with outstanding balance on at least one card, any card that has no outstanding balance is, in effect, dropped from the sample, which gives rise to a censoring bias. Furthermore, each rate held by a household — especially its lowest rate — provides information on the household's credit quality; when we take an average rate, much of this information is lost.

- Treat each card account as an independent observation. Because a household can distribute an outstanding balance among its cards in the least costly manner without incurring additional search costs, this approach will bias \( \beta_1 \) in favor of the search cost hypothesis. However, this approach is better suited to estimating the \( \delta \) coefficients, because we retain the full set of rates for which this household was accepted. Another advantage is that it preserves the largest possible sample size.

Neither method is ideal, so I will present results for both.

In the regressions below, the card's APR, expressed as a percentage, is taken as interest rate. Loan size is measured in $1000 units. The variables used to determine default risk are:

**LNINCOME** is the log\((y_t/\bar{y})\), where \( \bar{y} \) is the mean income in the sample. I predict a negative coefficient, as higher income should be associated with lower default risk.

**HOMEOWN** is a dummy for whether the household owns its primary residence. I predict a negative coefficient.

**LNHHDEBT** is the log\((HHDEBT+1)\), where HHDEBT is total household debt, including home mortgage but excluding credit card debt. I predict a positive coefficient.

### 3.4 Regression results

In the following tables, I present NLLS regression results for equations (9) and (10). The top rows show the values for \( c \) and \( \lambda \) used to obtain the coefficients in the columns below. In the regressions of Table 4, I take each credit card account as an independent observation;
in Table 5, I form an average rate paid by each household, and take the household as the unit of observation. Each table presents a set of results using the RAM survey of issuer rates and a set using the February 1987 IRS survey.

The regressions provide tests of two related hypotheses: first, that consumers are segmented by credit risk, and, second, that larger borrowers have greater incentive to search than smaller borrowers of similar credit-worthiness, so interest rate should decrease with default-risk-adjusted loan size. The first hypothesis is clearly not supported by the data: the coefficients on LNICOME, HOMEOWN, and LNHHDDEBT are not of stable sign across regressions, and all are insignificant. It appears that in 1987 low rate issuers did not have more stringent standards than high rate issuers for applicant acceptance. The weak positive association between APR and charge-offs reported for 1992 by Shaprio (1992) suggests either that low rate issuers applied more stringent credit standards with respect to credit limit (rather than acceptance), or that applicant screening became more sophisticated between 1987 and 1992; trade publications provide anecdotal evidence for at least the first conjecture.

The estimated $\hat{\beta}_1$ are consistent with the second hypothesis: in every regression but one, $\hat{\beta}_1$ is negative, as predicted. However, $\hat{\beta}_1$ is never significant, even at a 10% significance level.\textsuperscript{17} Moreover, the $\hat{\beta}_1$ are typically small in magnitude. At its largest magnitude, which is an outlier in the tables, the expected APR for a household with zero default risk declines by 1.11 percentage points for each $1000 in outstanding balance. Considering the wide range of available rates, this difference seems small. Taking any other estimate of $\hat{\beta}_1$ from the tables, or a household with higher default risk, yields a much smaller decline as outstanding balance increases.

To test the joint significance of the coefficients, I perform a Wald test on the joint hypothesis that $\beta_1 = 0$ and that household credit quality does not effect the interest rate

\textsuperscript{17} The reported standard errors are heteroskedasticity-robust, which are typically 20-50% larger than the ordinary standard errors; even using the latter, the coefficients are always insignificant.
obtained.

\[ H_0 : \beta_1 = \delta(\text{LNINCOME}) = \delta(\text{HOMEOWN}) = \delta(\text{LHHDEBT}) = 0 \]

With the exception of four regressions in Table 5, \( H_0 \) is not rejected.\(^{18}\) No regression has an \( R^2 \) greater than 1%.

Finally, I consider whether a possibly inappropriate choice of nonlinear functional form might be obscuring a simpler relationship between the variables. Table 3 shows the pairwise correlations between APR and the independent variables used above. Except for LNHHDEBT, the correlations are of the expected signs, but the magnitudes are low. Even when each card account is taken independently (which ignores the ability of households to allocate debt to their lowest rate cards without incurring additional search costs), the correlation of APR with loan size is smaller than -0.05. Correlations with the default risk characteristics are even smaller. I have also run linear regressions of APR on these variables;

**Table 3: Simple Correlations of APR with Independent Variables**

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>LNINCOME</th>
<th>HOMEOWN</th>
<th>LNHHDEBT</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Card Accounts</td>
<td>-0.048</td>
<td>-0.001</td>
<td>-0.016</td>
<td>-0.027</td>
</tr>
<tr>
<td>Household Average Rates</td>
<td>-0.031</td>
<td>-0.037</td>
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coefficients are again insignificant.

4 **Conclusion**

This paper provides a model of consumer search for the credit card industry which emphasizes consumer heterogeneity. Although other types of search models permit multiple prices in equilibrium, the dispersion of prices in this model is an intuitively appealing consequence of observable heterogeneity in loan size and credit quality. The model yields testable

\(^{18}\)It appears that the objective function may have a ridge along which the sum of squared residuals is fairly constant. In the four regressions for which the Wald statistic is large, the covariance of \( \delta(\text{LNINCOME}) \) and \( \delta(\text{HOMEOWN}) \) is fairly large and negative.
predictions for how search distributes consumers among issuers, conditional on consumer types.

So far as I am aware, this is the only model of search in the literature in which sellers do not sell to all comers, but instead impose minimum quality standards on whom they agree to serve. The model obviously may apply to other types of bank loans. It also might be applied to some quite different markets; for example, the market for colleges has similar properties. High school students search for a college based on academic prestige and price. College admissions filter the student applications much as bank credit departments do: the more prestigious the college, the higher its minimum requirements for its applicants. Colleges are willing to take students who exceed their minimum requirements, much as a high rate issuer is happy to accept a low risk consumer.

Reduced-form implications of the model are tested on consumer level data. I find no evidence that consumers are segmented by credit-worthiness. Although such segmentation is not a necessary feature of search models in general, it is necessary in my model in order to have price dispersion. The theoretical basis for the observed dispersion of interest rates therefore remains an open question. I find only a weak indication that borrowers carrying larger credit card loans search more intensively for a lower interest rate: the relevant coefficient is of the predicted sign, but small in magnitude and always insignificant. Either search costs are so high that search is prohibitively expensive for all but the largest borrowers, or there is some other barrier — possibly irrationality, as suggested by Ausubel (1991) and Shapiro (1992) — to households obtaining the lowest rate for which they qualify. I conclude that consumers may indeed engage in costly search for credit cards, but that the model’s explanatory power for the observed distribution of card rates and the distribution of consumers among those rates is slight at best.
Table 4: NLLS Regressions (All Card Accounts)

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<th>δ(CONSTANT)</th>
<th>δ(LINCOME)</th>
<th>δ(HOMEOWN)</th>
<th>δ(LINHIDE)</th>
<th>ΔR²</th>
<th>Wald Test</th>
<th>ΔR²</th>
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* Significant at the 5% level.
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**Table 5: NLLS Regressions (Household Average Rates)**

*: Significant at the 5% level.
A Solution of the model of Chapter 3

A.1 Derivation of equation (5)

Equation (4) is a sum of double integrals across loansize types and search cost types in the population. By solving the inside integral (for loansize types), I get

$$\pi_L^{-}(r) = \alpha r \left( \int_0^{A_1^G} \frac{1}{2} (1 - (s/A_1^G)^2) dF(s) \right.$$  

$$+ \frac{1}{k} \left( \int_0^{A_2^G} \frac{1}{2} (1 - (s/A_2^G)^2) dF(s) - \int_0^{A_1^Q} \frac{1}{2} (1 - (s/A_1^Q)^2) dF(s) \right)$$  

$$+ \frac{1}{N} \left( \int_0^{A_2^G} \frac{1}{2} (s/A_2^G)^2 dF(s) + \int_{A_2^G}^{\infty} \frac{1}{2} dF(s) \right) \right).$$

I substitute in the $Z$ function to obtain

$$\pi_L^{-}(r) = \alpha r \left( F(A_1^G) - Z(A_1^Q) + \frac{1}{k} \left( (F(A_2^G) - Z(A_2^G)) - (F(A_1^G) - Z(A_1^Q)) \right) + \frac{1}{N} \left( Z(A_2^G) + 1 - F(A_2^G) \right) \right),$$

and collect terms to obtain equation (5).

A.2 Derivation of equation (6)

The derivative of $\pi_L^{-}(r)$ is

$$\frac{d\pi_L^{-}}{dr} = \frac{\alpha}{2} r \left( \frac{N - k}{Nk} (f(A_2^G) - Z'(A_2^G)) \frac{dA_2^G}{dr} + \frac{k - 1}{k} (f(A_1^G) - Z'(A_1^Q)) \frac{dA_1^Q}{dr} \right)$$  

$$+ \frac{\alpha}{2} \left( \frac{1}{N} + \frac{N - k}{Nk} (F(A_2^G) - Z(A_2^G)) + \frac{k - 1}{k} (F(A_1^G) - Z(A_1^Q)) \right)$$

(11)

From the definitions for $A_1^Q$ and $A_2^Q$, we have

$$\frac{dA_1^Q}{dr} = \frac{dA_2^Q}{dr} = -\frac{1}{N}.$$
The derivative of $Z$ can be shown to equal

$$Z'(A) = f(A) - \frac{2}{A} Z(A). \quad (12)$$

Equation (11) can now be simplified to

$$\frac{d\pi^{-}}{dr} = \frac{-\alpha}{2N^r} \left( \frac{N-k}{Nk} \frac{2}{A^2} Z(A^2) + \frac{k-1}{2} \frac{Z(A^1)}{A^1} \right)$$

$$+ \frac{\alpha}{2} \left( \frac{1}{N} + \frac{N-k}{Nk} \left( F(A^2) - Z(A^2) \right) + \frac{k-1}{k} \left( F(A^1) - Z(A^1) \right) \right) \quad (13)$$

Equation (13), evaluated at $r = r_L$, is set to zero. At $r = r_L$, $A^1 = 0$ and $A^2 = A^* \equiv (k/N)(r_H - r_L)$. Repeated application of l'Hôpital's rule gives

$$Z(0) = 0 \quad (14)$$

$$\lim_{A \to 0} \frac{2Z(A)/A}{2/3\theta} = 2/3\theta. \quad (15)$$

The first order condition in equilibrium then reduces to

$$1 - \frac{2r_L k - 1}{3\theta} + \frac{N-k}{k} \left( F(A^*) - Z(A^*) \right) - \frac{r_L N-k}{N \cdot k} \frac{2}{A^*} Z(A^*) = 0.$$  

A.3 FOC for $r_L$ using right derivative of profit

Say that issuer $k$ increases its rate to $r > r_L$; assume that $I_{DB}(r) = 0$ so that this issuer will accept only Good type consumers. Issuer $k$ now ends up with two types of Good type consumers: First, there are consumers with stop rate $r_H$; such consumers must have search cost and loan size such that $s > x.A^2$, where

$$A^2 \equiv (1 - D_G)(r_H - E_G\langle r_H \rangle) = \frac{k-1}{N}(r_H - r_L) + \frac{1}{N}(r_H - r).$$
Issuer $k$ gets $1/N$ of such consumers. Second, there are consumers with stop rate $r$; such consumers must have search cost and loan size such that $x A_1^{G+} < s \leq x A_2^{G+}$, where

$$A_1^{G+} = (1 - D_G)(r - E_G(r)) = \frac{k - 1}{N}(r - r_L).$$

Issuer $k$ gets $1/k$ of these consumers. Therefore, the profit function $\pi_L^+(r)$ is

$$\pi_L^+(r) = ar \left( \frac{1}{k} \left( \int_0^{A_2^{G+}} \int_{s/A_2^{G+}}^1 x dx dF(s) - \int_0^{A_1^{G+}} \int_{s/A_1^{G+}}^1 x dx dF(s) \right) + \frac{1}{N} \left( \int_0^{A_2^{G+}} \int_{s/A_2^{G+}}^1 x dx dF(s) + \int_{A_2^{G+}}^\infty \int_{s/A_2^{G+}}^1 x dx dF(s) \right) \right)
= \frac{\alpha}{2} \left( \frac{1}{N} + \frac{N - k}{Nk} (F(A_2^{G+}) - Z(A_2^{G+}) - \frac{1}{k} (F(A_1^{G+}) - Z(A_1^{G+})) \right)$$

where the last line follows as in Appendix A.1.

I take a derivative and substitute for $Z'$ using equation (12) to get

$$\frac{d\pi_L^+}{dr} = \frac{\alpha}{2} \left( \frac{N - k}{Nk} \frac{2}{A_2^{G+}} Z(A_2^{G+}) \frac{dA_2^{G+}}{dr} - \frac{1}{k} \frac{2}{A_1^{G+}} Z(A_1^{G+}) \frac{dA_1^{G+}}{dr} \right)
+ \frac{\alpha}{2} \left( \frac{1}{N} + \frac{N - k}{Nk} (F(A_2^{G+}) - Z(A_2^{G+}) - \frac{1}{k} (F(A_1^{G+}) - Z(A_1^{G+})) \right)$$

As in Appendix A.2, I substitute

$$\frac{dA_2^{G+}}{dr} = \frac{-1}{N}, \quad \frac{dA_1^{G+}}{dr} = \frac{k - 1}{N}.$$

To evaluate the derivative at $r = r_L$, I substitute $A_1^{G+}(r_L) = 0$ and $A_2^{G+}(r_L) = A^*$, and apply equations (14) and (15). The resulting expression, set to zero, yields equation (6).

### A.4 Equilibrium condition for high rate

The equilibrium condition for the high rate issuers is obtained by the same method used for the low rate issuers. Denote by $\pi_H^+(r)$ the expected profit for the high rate issuer $N$ as a function of its interest rate, restricted to $r \geq r_H$; let $\pi_H^-(r)$ be the same for interest rates
\[ r \leq r_H. \] Then at the equilibrium \( r_L \), we must have

\[ \left. \frac{d\pi^+_H}{dr} \right|_{r=r_H} = \left. \frac{d\pi^-_H}{dr} \right|_{r=r_H} = 0. \]

First, I construct \( \pi^+_H(r) \). Assume that issuers \( k+1 \) to \( N-1 \) all offer \( r_H \), and issuer \( N \) sets \( r > r_H \). Issuer \( N \) will keep two types of consumers: First, there are Good type consumers with stop rate \( r \) who happen upon issuer \( N \) first. Such consumers make up \( \alpha/N \) of the population in expectation, and the issuer earns \( r \) per unit lent to these consumers. Second, there are Bad type consumers with stop rate \( r \) who happen upon issuer \( N \) before any of the other high rate issuers; these consumers make up \( (1-\alpha)/(N-k) \) of the population in expectation, and the issuer earns expected profit of \( (1+r)(1-D_B)-1 \) per unit lent to these consumers. These two types of consumers must have search cost and loan size that satisfy \( s > xA^{G+} \) and \( s > xA^{B+} \), respectively, where

\[
\begin{align*}
A^{G+} &\equiv (1-D_G)(r - E_G^*(r)) = \frac{k}{N}(r - r_L) + \frac{N-k-1}{N}(r - r_H) \\
A^{B+} &\equiv (1-D_B)(r - E_B^*(r)) = (1-D_B)\frac{N-k-1}{N}(r - r_H).
\end{align*}
\]

Therefore, the profit function is

\[
\pi^+_H(r) = \frac{\alpha}{N}r \left( \int_0^{A^{G+}} \int_0^{s/A^{G+}} xdx F(s) + \int_{A^{G+}}^{\infty} \int_0^1 xdx F(s) \right)
+ \frac{1-\alpha}{N-k}((1+r)(1-D_B) - 1) \left( \int_0^{A^{B+}} \int_0^{s/A^{B+}} xdx F(s) + \int_{A^{B+}}^{\infty} \int_0^1 xdx F(s) \right)
= \frac{\alpha}{2N}r(Z(A^{G+}) + 1 - F(A^{G+})) + \frac{1-\alpha}{2N-k}((1+r)(1-D_B) - 1)(Z(A^{B+}) + 1 - F(A^{B+}))
\]

Issuer \( N \) maximizes this function with respect to \( r \). Take the derivative of \( \pi^+_H(r) \), substitute for \( Z' \) using equation (12), and substitute

\[ \frac{dA^{G+}}{dr} = \frac{N-1}{N}, \quad \frac{dA^{B+}}{dr} = \frac{N-k-1}{N}(1-D_B) \]

141
to obtain
\[
\frac{d\pi^+_H}{dr} = \frac{\alpha}{2N} \left( Z(A^{G+}) + 1 - F(A^{G+}) \right) + \frac{1 - \alpha}{2N - k} (1 - D_B) \left( Z(A^{B+}) + 1 - F(A^{B+}) \right) \\
- \frac{\alpha}{2N} r^2 A^{G+} Z(A^{G+}) \frac{N - 1}{N} - \frac{1 - \alpha}{2N - k} ((1 + r)(1 - D_B) - 1) \frac{2}{A^{B+}} Z(A^{B+}) \frac{N - k - 1}{N} (1 - D_B)
\]

In equilibrium, the FOC is satisfied at \( r = r_H \). To get equation (7), substitute \( A^{G+}(r_H) = A^* \) and \( A^{B+}(r_H) = 0 \), and apply equations (14) and (15).

The construction of \( \pi_L^-(r) \) is somewhat more tedious. At \( r < r_H \), issuer \( N \) obtains four types of consumers: First, there are Bad types who have stop rate \( r \); issuer \( N \) rents all of these consumers. Second, there are Bad types who have stop rate \( r_H \); issuer \( N \) expects to get the fraction \( 1/(N - k) \) of these consumers who happen upon issuer \( N \) before any other high rate issuer. Third, there are Good types who have stop rate \( r \); issuer \( N \) expects to get the fraction \( 1/(k + 1) \) who happen upon issuer \( N \) before any low rate issuer. Finally, there are Good types who have stop rate \( r_H \); issuer \( N \) expects to get \( 1/N \) of these consumers.

The reader may confirm that the FOC for maximization of \( \pi_L^-(r) \), evaluated at \( r = r_H \), yields expression (7).
References


