# Optimal Long-term Financing under Ambiguous Volatility

by

Peter G. Hansen

Submitted to the Department of Management
in partial fulfillment of the requirements for the degree of

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## Abstract

I study a continuous-time principal-agent model with hidden action in which the principal and the agent have ambiguous beliefs about the volatility of the project cash flows. I describe a novel formulation that captures uncertainty about the underlying volatility process show how it affects the optimal contract. Ambiguity aversion generates endogenous belief heterogeneity between the principal and the agent. Under the optimal contract, the agent always trusts the benchmark probability model, while the principal forms expectations as if volatility is strictly higher and state-dependent. Additionally, I show ambiguity aversion generates asset pricing implications for the implied financial securities.

Thesis Supervisor: Andrey Malenko
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# Chapter 1

# Introduction

Models of long-term financial contracting have attracted considerable attention over the past decade. These models typically study financial frictions by examining infinite horizon incentive problems between outside investors and a firm insider who's actions are only partially observable. A common thread in this literature is that the models assume that all economic actors fully understand the model environment and that such understanding is common knowledge. This is similar to (but stronger than) an assumption of rational expectations, and has been criticized as overly restrictive in models with strategic interaction by Harsanyi (1967), Wilson (1987), Bergemann and Morris (2005), Woodford (2010), Hansen and Sargent (2012) and others.

This paper attempts to relax the assumption that economic fully understand their model environment and study the corresponding effect on financial contracting. In particular, I study a long-term contracting problem where economic actors have ambiguous beliefs about the possibly time-varying volatility of future cash flows. Motivated by the variational formulation of Maccheroni et al. (2006a), I describe a novel formulation that captures uncertainty about the underlying volatility process show how it affects the optimal contract. Ambiguity aversion leads the principal to design a contract that is robustly optimal given uncertainty about the volatility process. I adopt a continuous-time framework based on the models of DeMarzo and Sannikov (2006) and Biais et al. (2007) where a firm insider with limited liability takes a hidden action that affect project cash flows. Under the optimal contract, belief heterogeneity emerges between the principal and the agent. The agent trusts the benchmark volatility model, whereas the principal forms expectations as if volatility is strictly higher and state-dependent. As in DeMarzo and Sannikov (2006), the optimal contract can be interpreted as featuring a line of credit between the principal and the agent. I show how ambiguity aversion increases the optimal credit limit, while reducing the reliance on long-term debt. This is important since credit lines are a commonly used corporate security. Additionally, I derive asset pricing implications of volatility ambiguity under the optimal contract.

## 1.1 Related literature

My paper builds on the large literature of papers studying models of long-term financial contracting. DeMarzo and Fishman (2007), DeMarzo and Sannikov (2006), and Biais et al. (2007) show that in stationary environments with risk-neutral economic agents, the optimal long-term financial contract can be implemented by an interpretable capital structure. I build on these papers by introducing uncertainty about the volatility of the cash flow process and study how this affects the optimal contract. DeMarzo et al. (2012) and Bolton et al. (2013) examine the impact of incentive problems on investment under uncertainty. Panageas and Westerfield (2009) and Drechsler (2014) study high-water-mark contracts for delegated asset management and find, surprisingly, that despite the convexity of the compensation these contracts incentivize the asset manager to choose the optimal portfolio under CRRA utility. More recently, DeMarzo and He (2016) study incentive problems between equity investors and debtholders when the firm cannot commit to a leverage policy, and find that this produces the leverage ratchet effect documented by Admati et al. (2017). As many of these papers, I rely on the martingale approach to dynamic contracting problems developed by Sannikov (2008) and Williams (2008).

Particularly relevant are papers that take robust approaches to incentive problems, such as Bergemann and Morris (2005), Carroll (2015), and Zhu (2016). The closest paper to this one is Miao and Rivera (2016) who characterize the optimal contract in continuous time when the principal faces ambiguity about expected cash flows. As I will demonstrate, my model produces a substantially different optimal security design yet has qualitatively similar asset pricing implications. Szydlowski (2012) and Prat and Jovanovic (2014) study related problems where the principal is uncertain about the details of the agency problem. Adrian and Westerfield (2009) studies optimal contracting when the principal and the agent disagree about the underlying dynamics of the cash flow process and both learn through time. By focusing on uncertainty about second moments, my paper is similar in spirit to Wolitzky (2016).

Another literature which relates to my paper studies dynamic models of ambiguity and robustness. These models can be broadly thought of as belonging to one of three categories, namely the "recursive multiple priors" model proposed by Epstein and Schneider (2003), the "recursive smooth ambiguity" model proposed by Klibanoff et al. (2009), and the "dynamic variational preferences" model proposed by Maccheroni et al. (2006c) as a generalization of the "multiplier preferences" introduced by Hansen and Sargent (2001)[1]. My paper adds to this literature by proposing a new form of preferences that capture ambiguity or uncertainty about volatility in continuous time. To my knowledge, the only other model of volatility ambiguity is the "G-expectations" model of Peng (2007), which can be interpreted of as a recursive multiple priors model. By contrast, my approach is much closer to the variational model. In fact, I show that my preference specification can be thought of as a particular continuous-time limit of the discrete-time preferences of Maccheroni et al. (2006c).

---

[1] Note that these three categories are not mutually exclusive.

# Chapter 2

# The Model

## 2.1 Setup

I first describe a benchmark model without ambiguity, based on DeMarzo and Sannikov (2006) and Biais et al. (2007). At each instant $t$, agent chooses an effort level $a_t \in [0, 1]$. Given an effort choice, the cash-flow process $\{Y_t\}$ obeys the law of motion

$$dY_t = \mu a_t dt + \sigma dB_t \tag{2.1}$$

where $\mu, \sigma > 0$, and $B_t$ a standard Brownian motion.

The agent can derive private benefits $\lambda\mu(1-a_t)$ from the action $a_t$ where $\lambda \in (0, 1)$. Due to linearity, it is without loss of generality to take $a_t \in \{0, 1\}$. At any time $t \geq 0$ the project can be liquidated, producing a liquidation value of $L$. The principal and the agent are both assumed to be risk neutral. The principal discounts cash flows at a rate $r > 0$ while the agent discounts cash flows at a rate $\gamma > r$[1].

Our benchmark model is

**Problem 2.1.1** (benchmark model).

$$\max_{(C,\tau,a)} \mathbb{E}^{P^a} \left[ \int_0^\tau e^{-rs}(dY_s - dC_s) + e^{-r\tau} L \right] \tag{2.2}$$

---

[1] This assumption means that the agent is impatient relative to the principal, and avoids degeneracy.

*subject to*

$$\mathbb{E}^{P^a}\left[\int_0^T e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] \geq \mathbb{E}^{P^{\hat{a}}}\left[\int_0^T e^{-\gamma s}(dC_s + \lambda\mu(1-\hat{a}_s)ds)\right] \quad (2.3)$$

$$\mathbb{E}^{P^a}\left[\int_0^T e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] = W_0 \quad (2.4)$$

## 2.2 Volatility ambiguity

I introduce ambiguity aversion towards volatility in a form analogous to the variational preferences indroduced by Maccheroni et al. (2006b). Under this formulation, expectations are evaluated as if the probability laws are chosen by an adversarial nature. In particular, given an effort process $a_t$ the decision makers evaluate expectations as if cash-flow process follows

$$dY_t = \mu a_t dt + \sigma \nu_t dB_t \quad (2.5)$$

where $\nu_t$ is a progressively measurable process. Nature choses $\nu_t$ to minimize the decision maker's expected utility but pays an instantaneous cost $\psi(\nu_t)$ where $\psi(\cdot)$ is a non-negative convex function with $\psi(1) = 0$. Additionally, to avoid issues ot time inconsistency, the instantaneous cost is discounted at the same rate that the decision maker uses to discount their future cash flows. Let $Q^\nu$ denote the probability measure where

the decision maker ranks progressively measurable consumption plans $U_t$ according to

$$\mathcal{E}(U) = \inf_{\{\nu_t\}_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{U_t + \psi(\nu_t)\right\} dt\right]$$

subject to (2.5) where $\rho$ is the discount of the decision maker.

I focus on two important special cases of the instantaneous cost function $\psi(\cdot)$ :

(i) "interval uncertainty"

$$\psi(\nu) = \begin{cases} 0 & \text{if } \nu \in [\underline{\sigma}/\sigma, \overline{\sigma}/\sigma] \\ \infty & \text{otherwise} \end{cases}$$

(ii) "relative entropy"

$$\psi(\nu) = \frac{\theta}{2}\left\{\nu^2 - 1 - \log(\nu^2)\right\}$$

The cost function case (i) is a convex indicator function of the set $[\underline{\sigma}/\sigma, \overline{\sigma}/\sigma]$. It restricts the worst-case cash-flow volatility to the interval $[\underline{\sigma}, \overline{\sigma}]$. By restricting the set of unknown model parameters to a fixed compact set, this specification is analogous to the $\kappa$-ignorance specification of Chen and Epstein (2002) which can be thought of as a straightforward dynamic counterpart to the max-min expected utility of Gilboa and Schmeidler (1989). The focus on volatility ambiguity is as in Peng (2007) who dubbed the corresponding conditional expectation operator a $G$-expectation. Epstein Epstein and Ji (2013) explore the implications of this type of ambiguity in a frictionless asset pricing setting.

Case (ii) corresponds to a continuous-time limit of a relative entropy penalty. This is analogous to the robust control or "multiplier" preferences of Hansen and Sargent (2001). The parameter $\theta$ controls the level of ambiguity aversion of the decision maker. In particular, $\theta$ can be interpreted as the degree of confidence in the benchmark probability model since as $\theta \to \infty$ we recover the standard expected utility preferences with no ambiguity aversion. It is therefore natural to interpret $1/\theta$ as the degree of ambiguity aversion.

To see where the functional form in case (ii) comes from, consider two probability distributions $P$ and $\tilde{P}$ over a scalar random variable $Y$ where $Y \sim \text{Normal}(0, \sigma^2)$ under $P$ and $Y \sim \text{Normal}(0, \sigma^2\nu^2)$ under $\tilde{P}$. It is useful to work with the likelihood ratio

$$m(y, \sigma, \nu) \equiv \frac{1}{\nu}\exp\left\{-\frac{1}{2\sigma^2}\left(\frac{1}{\nu^2} - 1\right)y^2\right\}$$

so that writing $M = d\tilde{P}/dP$ we have $M = m(Y, \sigma, \nu)$. Now, computing the relative entropy directly, we have

$$\mathbb{E}_P[M \log M] = \frac{1}{2}\left\{\nu^2 - 1 - \log \nu^2\right\}$$

Thus the penalty in case (ii) corresponds to $\theta$ times the instantaneous relative entropy associated with the change of volatility $\nu$. The parameter $\theta > 0$ characterizes the decision-maker's concerns for robustness. In particular, it is natural to interpret $\frac{1}{\theta}$ as the level of ambiguity aversion. As $\theta \to \infty$ we recover standard expected utility model.

I focus primarily on case (ii). Throughout I will assume that the cost function $\psi$ is twice continuously differentiable, unless explicitly stated.

## 2.3 Interpretation as a change of probability measure

Before, $\nu_t$ was simply treated as a controlled process chosen by an adversarial nature, while the underlying Brownian motion stayed fixed. We can instead interpret this model as nature choosing the probability law of the Brownian motion. Let $P$ be the benchmark probability measure under which $B_t$ is a Brownian motion. Consider a new probability measure $Q^\nu$ under which the process $B_t^\nu$ defined by

$$dB_t^\nu = \frac{1}{\nu_t} dB_t \tag{2.6}$$

is a Brownian motion. We see that under $Q^\nu$, $B_t$ is an Itô process with zero drift and volatility $\nu_t$. The parameter $\nu_t$ can be interpreted as the local change-of-volatility between $Q^\nu$ and $P$. Note that except in the special case $\nu_t$ is identically 1, $P$ and $Q^\nu$ are mutually singular probability measures so Girsanov's theorem does not apply[2].

This formulation as a change of probability measure is non-standard. Because the set of possible probability measures are mutually singular, we cannot apply Girsanov's theorem to obtain a Radon-Nikodym derivative. Thus my approach is different from Miao and Rivera (2016) which studies optimal contracting when the principal only considers probability models which are absolutely continuous with respect to the benchmark model and therefore only considers ambiguity about the drift of the cumulative cash flow process.

---

[2] To see this, simply compute the quadratic variation of $B_t$ on any interval $[t, t+\Delta]$ for which $\nu_t$ differs from 1 on a set of positive measure.

# Chapter 3

# Optimal Contract

Assume for simplicity that only the principal is ambiguity-averse. The optimal contracting problem is given by:

**Problem 3.0.1.**

$$\sup_{(C,\tau,a)} \inf_{\nu} \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(dY_t - dC_t) + e^{-r\tau}L\right] + \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}\psi(\nu_t)dt\right] \tag{3.1}$$

*subject to* (2.3), (2.4), *and* (2.6).

Problem 3.0.1 can be thought of as a two-player, zero-sum stochastic differential game[1] between the principal and an adversarial nature. Nature chooses the time-varying change of volatility process $\nu_t$ to minimize the welfare of the agent, but choosing $\nu_t$ different from 1 has cost proportional to the instantaneous relative entropy. Next, I heuristically derive the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation for optimality.

Let $\phi_t$ be the sensitivity to $\nu_t \sigma dB_t^{\nu}$ i.e. the cash-flow shock under the probability measure $Q^{\nu}$. By the martingale representation theorem, $W_t$ satisfies

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu_t(1-a_t)dt + \phi_t\sigma\nu_t dB_t^{\nu}$$

---

[1] See Fleming and Souganidis (1989) for further discussion

## 3.1 First-Best Contract

The first-best contract is the same as the first-best contract with no ambiguity aversion in DeMarzo and Sannikov (2006). This is intuitively obvious since the first-best value function is linear, hence there are no volatility effects.

$$rF(W) = \sup_{c\geq 0,\phi} \inf_{\nu} \mu - c + \psi(\nu) + (\gamma W - c)\, F'(W) + \frac{1}{2}\phi^2\nu^2\sigma^2 F''(W). \tag{3.2}$$

It is easy to verify that at the optimum, we have $\phi = 0$ and therefore the principal's value function under the (stationary) first-best contract is

$$F(W) = \frac{\mu}{r} - \frac{\gamma}{r} W$$

which can be implemented by the principal paying the agent a constant wage of $c = \gamma W$. Of course, this can be improved if we allow time-0 lump sum transfers in which case the principal can simply give a one-time transfer of $W$ to the agent which gives

$$F(W) - W = \frac{\mu}{r}.$$

Thus with no moral hazard, volatility ambiguity produces no reduction in welfare.

## 3.2 Optimal Contract with moral hazard

It is a simple extension of lemma 3 of DeMarzo and Sannikov (2006) to show that for any change-of-volatility process $\nu_t$, the agent's incentive compatibility constraint can be written as

$$\phi_t \geq \lambda$$

The HJBI equation for the optimal contract with agency is given by

$$rF(W) = \sup_{c\geq 0, \phi\geq\lambda} \inf_{\nu} \mu - c + \frac{\theta}{2}\left\{\nu^2 - 1 - \log(\nu^2)\right\} + (\gamma W - c)\, F'(W) + \frac{1}{2}\phi^2\nu^2\sigma^2 F''(W). \tag{3.3}$$

A simple calculation shows that worst-case change of volatility $\nu$ is given by

$$\nu^2 = \frac{\theta}{\theta + \phi^2\sigma^2 F''(W)}. \tag{3.4}$$

Plugging in the our expression for $\nu^2$ the HJBI reduces to the following nonlinear HJB equation

$$rF(W) = \sup_{c\geq 0, \phi\geq\lambda} \mu - c - \frac{\theta}{2}\log(\theta) + (\gamma W - c)\, F'(W) + \frac{\theta}{2}\log(\theta + \phi^2\sigma^2 F''(W)) \tag{3.5}$$

Consider the region $[0, \overline{W})$ for which $F'(W) > -1$ so that $c = 0$ is optimal. Rearranging (3.5) gives

$$\sup_{\phi\geq\lambda} \frac{\theta}{2}\log\left(1 + \frac{\phi^2\sigma^2}{\theta}F''(W)\right) = rF(W) - \mu - \frac{\theta}{2} - \gamma W F'(W)$$

Now I apply $rF(W) - \mu \leq \gamma W$ which comes from the second-best value function being less than or equal to the first-best value function without lump-sum transfers, and $F'(W) > -1$ to obtain

$$\sup_{\phi\geq\lambda} \frac{\theta}{2}\log\left(1 + \frac{\phi^2\sigma^2}{\theta}F''(W)\right) < 0$$

which is a contradiction unless $F''(W) < 0$. This "proves"[2] that $F$ is strictly concave on $[0, \overline{W}]$.

Thus we have shown the following. On the interval $[0, \overline{W}]$, the principal's value function satisfies the ODE

$$rF(W) = \mu + \gamma W F'(W) + \frac{\theta}{2}\log\left(1 + \frac{\lambda^2\sigma^2}{\theta}F''(W)\right).$$

---

[2] Not really a proof since I assumed that the solution $F$ existed in a classical sense and was twice differentiable, but at least it's suggestive...

$F$ is strictly concave so the worst-case change of volatility given by

$$\nu^*(W)^2 = \frac{\theta}{\theta + \lambda^2\sigma^2 F''(W)} \tag{3.6}$$

is strictly greater than 1. Additionally, the incentive constraint always binds i.e.

$$\phi^*(W) = \lambda.$$

While this is the same as DeMarzo and Sannikov (2006), it stands in contrast to Miao and Rivera (2016).

The next proposition characterizes the optimal contract under the assumption that high effort is always optimal. The optimal contract with partial shirking can be described using methods similar to Zhu (2013), but such a characterization is beyond the scope of this paper.

**Proposition 3.2.1.** *Assume that* $L < \frac{\mu}{r}$ *and that implementing high effort is optimal. Assume further that there exists a unique twice differentiable solution* $F$ *to the ODE*

$$rF(W) = \mu + \gamma W F'(W) + \frac{\theta}{2}\log\left(1 + \frac{\lambda^2\sigma^2}{\theta}F''(W)\right) \tag{3.7}$$

*with boundary conditions*

$$F(0) = L,, \quad F'(\overline{W}) = -1$$

*and* $F''(W) < 0$ *for all* $W \in [0, \overline{W})$ *where* $\overline{W}$ *is defined by* $F(\overline{W}) = \frac{\mu}{r} - \frac{\gamma}{r}\overline{W}$. *Then:*

*(i) When* $W \in [0, \overline{W}]$, $F(W)$ *is the principal's value function for problem 3.0.1, the optimal cash flow sensitivity is* $\phi^*(W) = \lambda$ *and the worst case change of volatility* $\nu^*(W)$ *is given by* (3.6). *The contract delivers value* $W$ *to the agent whose continuation value* $W_t$ *evolves according to*

$$dW_t = \gamma W_t dt - dC_t^* + \phi^*(W_t)\sigma\nu^*(W_t)\; dB_t^{\nu^*}$$

*where* $dC_t^*$ *is 0 in* $[0, \overline{W})$ *and causes* $W_t$ *to reflect at* $\overline{W}$. *The contract terminates* *at time* $\tau = \inf\{t \geq 0 : W_t = 0\}$ *when the project is liquidated.*

*(ii) When* $W > \overline{W}$*, the principal's value function is* $F(W) = F(\overline{W}) - (W - \overline{W})$*. The principal immediately pays* $W - \overline{W}$ *to the agent and contracting continues with the agent's new initial value* $\overline{W}$*.*

Observe that as the degree of model confidence $\theta \to \infty$, the non-linear term $\frac{\theta}{2}\log\left(1 + \frac{\lambda^2\sigma^2}{\theta}F''(W)\right)$ converges to $\frac{1}{2}\lambda^2\sigma^2 F''(W)$ for any value of $F''(W)$. Hence (3.7) converges to the ordinary differential equation of DeMarzo and Sannikov (2006), i.e. the benchmark model with no ambiguity aversion.

**Proposition 3.2.2.** *Let* $F(\cdot)$*,* $\overline{W}$ *be defined as in proposition 3.2.1. Then high effort is optimal if and only if*

$$\min_{W \in [0,\overline{W}]} rF(W) - F'(W)(\gamma W - \lambda\mu) \geq 0.$$

The argument is simple, and is consistent with proposition 8 of DeMarzo and Sannikov (2006). Next, I show how the optimal contract changes with the level of ambiguity aversion.

**Proposition 3.2.3.** *For any promised wealth level to the agent, the principal's value function* $F(W)$ *strictly increases in* $\theta$*.*

Thus higher levels of ambiguity aversion leads to a higher payoff boundary for the agent.

This confirms the obvious intuition that the principal's value function is increasing in $\theta$ i.e. decreasing in the level of ambiguity aversion. This is illustrated in figure 1. The following proposition shows how the payoff boundary $\overline{W}$ changes with $\theta$.

**Proposition 3.2.4.** *The payoff boundary* $\overline{W}$ *is strictly decreasing in* $\theta$*.*

This result is illustrated in figure 2.

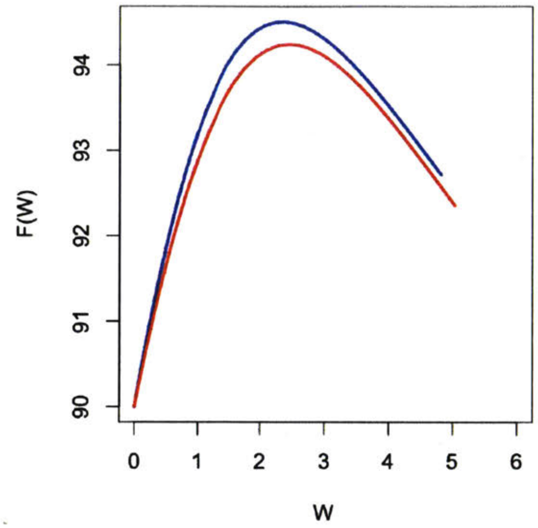

Figure 3-1: Value functions $F(W)$ for contracting problem. Value function with no ambiguity ($\theta = \infty$) shown in blue. Value function with $\theta = 5$ shown in red. Parameter values are $\mu = 10, r = 0.1, \gamma = 0.15, \lambda = 0.2, \sigma = 5, L = 90$.

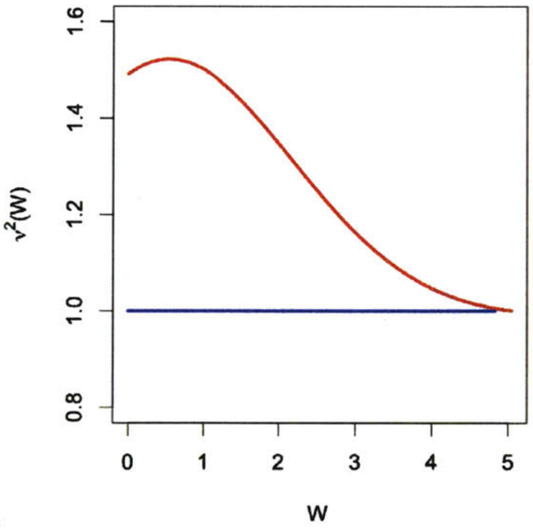

Figure 3-2: Worst case change-of-variance $\nu^2(W)$ for $\theta = 5$ shown in red. $\nu^2 = 1$ i.e. change-of-variance when $\theta = \infty$ shown in blue. Parameter values are $\mu = 10, r = 0.1, \gamma = 0.15, \lambda = 0.2, \sigma = 5, L = 90$.

## 3.3 What happens if the agent is ambiguity-averse?

Up until this point, I have assumed that only the principal was ambiguity averse. It is natural to ask what happens if the agent is ambiguity averse as well. As it turns out,

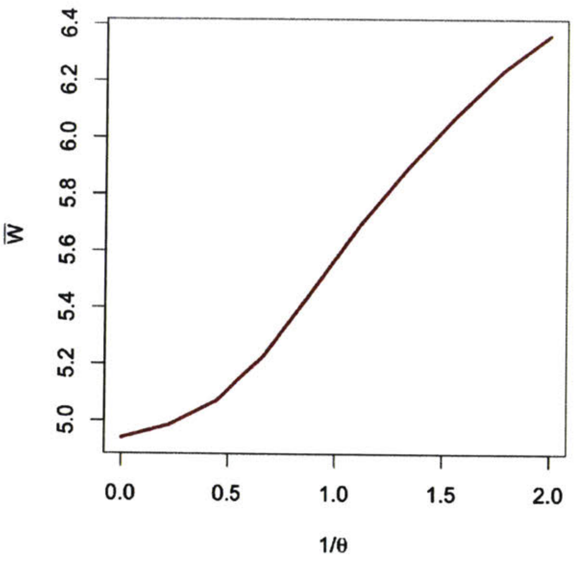

Figure 3-3: Upper payoff boundary $\overline{W}$ as a function of $1/\theta$. Parameter values are $\mu = 10, r = 0.1, \gamma = 0.15, \lambda = 0.2, \sigma = 5, L = 90$.

so long as the agent has the same form of variational ambiguity with a strictly penalty function, that the agent's ambiguity aversion will not affect the optimal contract.

**Proposition 3.3.1.** *Assume that the agent is ambiguity averse. Then the contract described in proposition 3.2.1 remains optimal. Moreover, the agent's implied worst-case belief is* $\nu(W) = 1$.

Thus even when the agent is ambiguity averse, the optimal contract is unaffected and they form expectations as if they fully trust that volatility is constant at level $\sigma$.

## 3.4 Bellman-Isaacs condition

The optimal contract characterized by proposition 3.2.1 is the solution of a particular max-min problem between the principal and nature. A natural question to ask is whether one can exchange the order of extremization and still retain the same solution. Formally, this corresponds to what is known as a Bellman-Isaacs condition. As discussed in Hansen et al. (2006), this condition is important for the interpretation of the solution to a robust control problem. In particular, it allows for an ex-post Bayesian interpretation the robust control problem.

For the robust contracting problem described in this paper, the value function is in fact globally concave, and the optimal control of nature has no binding inequality constraints, one can verify (see Fan (1953), Hansen et al. (2006)) that the Bellman-Isaacs condition is satisfied. Therefore the optimal contract described in proposition 3.2.1 is optimal in an ex-post Bayesian sense where the principal believes that volatility evolves according to (3.6), in a restricted space of contracts where changes in the agent's continuation payoff are locally linear in project cash flows. As such, it is reasonable to interpret my model as a model of endogenous belief formation about the volatility process.

This presents an important contrast to Miao and Rivera (2016). Under their optimal robust contract the Bellman-Isaacs condition is not satisfied. Thus their contract does not admit an ex-post Bayesian interpretation, and thus cannot be interpreted as an endogenous belief formation model. Hence the interpretation of their solution relies on the axiomatic approaches to variational ambiguity as in Maccheroni et al. (2006b) and Maccheroni et al. (2006c), whereas my solution can be interpreted directly as a model of endogenous belief formation.

# Chapter 4

# Implementation and Asset Pricing

## 4.1 Capital Structure

Following DeMarzo and Sannikov (2006), I show how to implement the optimal contract with equity, debt, and a credit line[1]. The implementation is as follows:

- *Equity:* The agent holds inside equity for a fraction $\lambda$ of the firm. Dividend payments are at the discretion of the agent

- *Long-term debt:* Long term debt is a consol bond that pays coupons at a rate $x = \mu - \frac{\gamma}{\lambda}\overline{W}$. If the firm ever defaults on a coupon payment, debt holders force liquidation.

- *Credit line:* The firm has a revolving credit line with credit limit $C^L = \frac{\overline{W}}{\gamma}$. Balances on the credit line are subject to an interest rate $\gamma$. The firm borrows and repays funds on the credit line at the discretion of the agent. If the balance ever exceeds $C^L$ the project is terminated.

The following characterizes how this implementation changes with the level of ambiguity aversion.

**Proposition 4.1.1.** *As the level of ambiguity aversion $1/\theta$ increases*

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[1] Alternatively, the optimal contract can be implemented using cash reserves, debt, and equity as in Biais et al. (2007)

- *The optimal credit limit strictly increases.*

- *The face value of the optimal long-term debt strictly decreases.*

Note that the fraction of equity held by the agent is determined by the incentive compatibility constraints, and does not change with $\theta$.

## 4.2 Asset Pricing Implications

I consider asset prices in a representative agent setting where the principal is the representative investor who trades debt and equity, whereas the agent is an insider who is restricted from trading in either security. We take $r$ as the risk-free rate. Then we price securities under the worst-case belief measure of the principal. This approach is analogous to those taken in Anderson et al. (2003), Biais et al. (2007), and Miao and Rivera (2016).

The value of equity is given by

$$S_t = \mathbb{E}_t^{Q^{\nu^*}} \left[ \int_t^T e^{-r(s-t)} \frac{1}{\lambda} dC_t^* \right]$$

It is straightforward to obtain that the stock price is given by $S_t = S(W_t)$ where the function $S(\cdot)$ satisfies the ODE

$$rS(W) = \gamma W S'(W) + \frac{1}{2}\lambda^2 \sigma^2 \nu^*(W)^2 S''(W)$$

with boundary conditions $S(0) = 0$ and $S'(\overline{W}) = 1$. A simple argument now shows that the equity premia is given by

$$\mathbb{E}_t \left[ \frac{dS_t}{S_t} \right] - r = -\frac{1}{2}\lambda^2 \sigma^2 \left[ \nu^*(W_t)^2 - 1 \right] \frac{S''(W_t)}{S(W_t)}$$

which I believe is strictly positive[2]. Note also that in the no-ambiguity benchmark, the equity premia is identically zero.

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[2] This would be the case if $S'' < 0$ for all $W \in [0, \overline{W}]$ which I believe is possible to show analytically

Define the credit yield spread $\Delta_t$ by

$$\int_t^\infty e^{-(r+\Delta_t)(s-t)}ds = \mathbb{E}_t^{Q^{\nu^*}}\left[\int_t^\tau e^{-r(s-t)}ds\right]$$

Which when solved yields $\Delta_t = \frac{rT_t}{1-T_t}$ where $T_t = \mathbb{E}_t^{Q^{\nu^*}}\left[e^{-r(\tau-t)}\right]$ is the the price of one unit of consumption at the time of default. $T_t = T(W_t)$ satisfies the ODE

$$rT(W) = \gamma WT'(W) + \frac{1}{2}\lambda^2\sigma^2\nu^*(W)^2T''(W)$$

with boundary conditions $T(0) = 1$ and $T'(\overline{W}) = 0$.

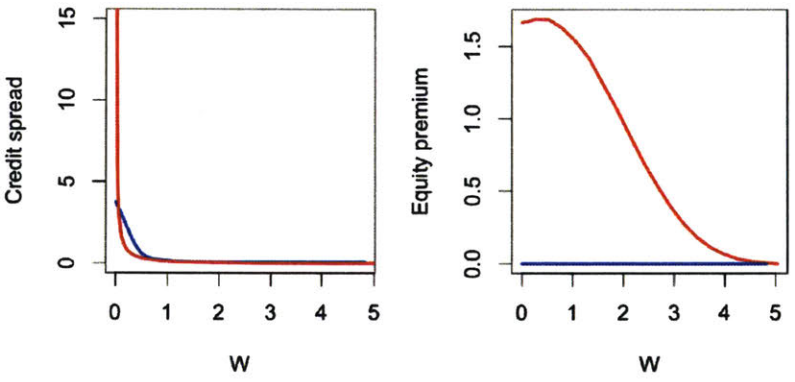

Figure 4-1: Credit yield spread and equity premium as a function of $W$. Asset prices without ambiguity ($\theta = \infty$) shown in blue. Asset prices with ambiguity ($\theta = 5$) shown in red.

## 4.3 The role of commitment

The optimal contract described is proposition 3.2.1 is not generically renegotiation-proof. For small values of $W$, the principal's value function $F$ under the optimal contract is increasing in $W$, so the principal and the agent can both be made better off by a one-off increase in the continuation value of the agent. Thus to be renegotiation-proof, the principal's value function $F(W)$ must not have positive slope. However, it is possible to modify the contract described in proposition 3.2.1 to obtain the optimal renegotiation-proof contract, which I describe in this section. Additionally, I show that the implied worst-case volatility under the optimal renegotiation-proof contract is strictly decreasing in the agent's continuation value.

Renegotiation effectively raises the minimum payoff of the agent to a point $R$ such that $F'(R) = 0$. The agent's promised value evolves on the interval $[R, \overline{W}]$ according to

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu dt + \lambda\sigma\nu_t dB_t + dP_t$$

where the processes $C$ and $P$ reflect $W_t$ at endpoints $\overline{W}$ and $R$ respectively. The project is terminated stochastically whenever $W_t$ is reflected at $R$. The probability that the project continues at time $t$ is

$$Pr(\tau \geq t) = \exp\left(-\frac{P_t}{R}\right)$$

The optimal contract can still be implemented with equity, long-term debt and a credit line, though the level of long-term debt and the length of the credit line will be different.

**Proposition 4.3.1.** *Under the optimal renegotiation-proof contract, the worst-case volatility $\nu^*(W)^2$ is strictly increasing in $W$.*

The renegotiation-proof implementation contract is in a sense a more robust implementation that the implementation described in proposition 3.2.1 in that it eliminates the incentive for the principal to renegotiate the contract with the agent. However, it still requires the principal to commit to a stochastic (unverifiable) liquidation policy. Without such commitment, there will generally be welfare loss to the principal. In particular, if the principal can only commit to deterministic liquidation policies, then the Pareto frontier is generally characterized by a solution to the same differential equation as before, but now with boundary conditions $F(0) = L$ and $F'(0) = 0$. Under this implementation, it is possible to show similar comparative statics as for the optimal contract with full commitment.

# Chapter 5

# Comparison of alternative models

## 5.1 Comparison with interval uncertainty

Consider the "interval uncertainty" formulation of ambiguity aversion. Assume that the adjustment cost function $\psi(\nu)$ faced by nature is given by

$$\psi(\nu) = \begin{cases} 0 & \text{if } \nu \in [\underline{\sigma}/\sigma, \overline{\sigma}/\sigma] \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

This is equivalent to assuming that nature is free to choose any level of volatility $\sigma_t \in [\underline{\sigma}, \overline{\sigma}]$ with no adjustment cost. This formulation of volatility ambiguity is precisely the $G$-expectation formulation of Peng (2007), and is similar to the $\kappa$-ignorance specification of Chen and Epstein (2002).

**Proposition 5.1.1.** *Consider the optimal contracting problem in which both the principal and the agent have interval uncertainty of the form* (5.1). *Assume that* $L < \frac{\mu}{r}$ *and implementing high effort is optimal. Then the optimal contract is the same as that of the optimal contract without ambiguity aversion where both the principal and the agent believe the volatility level is* $\overline{\sigma}$.

**Proposition 5.1.2.** *The payoff boundary* $\overline{W}$ *of the optimal contracting problem with interval uncertainty is strictly increasing in* $\overline{\sigma}$.

## 5.2 Comparison with drift ambiguity

This paper is closely related to Miao and Rivera (2016) who study a similar dynamic contracting problem where the principal is uncertain about the expected cash flows and is ambiguity-averse. They obtain similar asset pricing implications as I do; time-varying risk-premia that are generally higher for financially distressed firms. However there are some key differences. Firstly, the optimal contracts are quite different. In my model, the incentive compatibility constraint always binds because the principal fears inefficient liquidation and therefore does not want the agent to bear any more risk than necessary. This preserves the optimality of the simple contractual form of DeMarzo and Sannikov (2006) and Biais et al. (2007). In their model however, the principal does not like drift ambiguity, and thus in the optimal contract will sometimes force the agent to bear more cash-flow sensitivity than necessary. As a result, their incentive compatibility constraint does not bind and their optimal contract is much more challenging to interpret. Second, value function in my model is globally concave so the Bellman-Isaacs condition holds. This means that it is valid to interpret my model as a model of endogenous belief formation. This is not the case in Miao and Rivera (2016). Thirdly, my model can accommodate ambiguity aversion on the part of the agent, without any reduction in the impact of ambiguity aversion. Miao and Rivera (2016) do not model ambiguity aversion on the part of the agent, and in their framework it would produce an offsetting effect which reduces the impact of ambiguity aversion on the optimal contract.

# Chapter 6

# Empirical implications

Credit lines, also known as revolving credit facilities, are an extremely important form of firm financing. Empirically, credit lines account for more than a quarter of outstanding corporate debt of publically traded firms and an even larger fraction for smaller, non-publically traded firms.[1] To the extent that smaller firms have more ambiguous riskiness of their cash flows, this is consistent with the predictions of my model.

Under ambiguity aversion, the optimal contract generates belief heterogeneity between the principal and the agent. The agent always fully trusts their benchmark probability model while the principal endogenously believes that volatility is time-varying and strictly higher than the benchmark volatility. This is potentially related to the empirical evidence on managerial overconfidence as in Landier and Thesmar (2009) and Ben-David et al. (2013).

In terms of asset prices, my model predicts that the equity premium and credit yield spread are state-dependent and generally higher for firms closer to default. This is consistent with the literature on characteristic-based asset pricing (Daniel and Titman (1997)) as well as Friewald et al. (2014) who find that firm's equity premium and credit spread are positively correlated.

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[1] See Berger and Udell (1995), Sufi (2007) and DeMarzo and Sannikov (2006).

# Chapter 7

# Conclusion

In this paper I studied a long-term contracting problem where economic actors have ambiguous beliefs about the possibly time-varying volatility of future cash flows. In the spirit of the variational formulation of ambiguity given by Maccheroni et al. (2006a), I introduced a novel formulation of ambiguity aversion that captures uncertainty about the underlying volatility process and showed how it affects the optimal contract. Under the optimal contract, belief heterogeneity emerged endogenously between the principal and the agent. The agent trusts the benchmark volatility model, whereas the principal forms expectations as if volatility is strictly higher and state-dependent. Additionally I showed how ambiguity aversion increased reliance on a credit line under the optimal contract, and derived corresponding asset pricing implications.

I believe that the ideas developed in this paper can be applied to a variety of other settings. One possibility is to examine their effect in a q-theory model with moral hazard, similar to DeMarzo et al. (2012) or Bolton et al. (2013), and derive simultaneous implications for corporate investment and asset pricing. Another possibility would be to apply them to the problem of stress testing, where a bank regulator attempts to control the risk-taking of a bank without full confidence in a particular risk model. I hope to be able to develop these ideas in future work.

# Appendix A

# Proofs

## A.1 Proof of proposition 3.2.1

This section is not fully complete.

Write the principal's objective function as

$$\begin{aligned} J(\phi, C, a, \nu; w) &= \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(dY_t - dC_t) + e^{-r\tau}L\right] + \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}\psi(\nu_t)dt\right] \\ &= \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(\mu a_t dt - dC_t) + e^{-r\tau}L\right] + \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}\psi(\nu_t)dt\right] \end{aligned}$$

Then the principal's optimal contracting problem can be written as

$$F(w) = \sup_{(\phi,C,a)\in\Gamma(w)} \inf_{\nu\in\mathcal{H}} J(\phi, C, a, \nu; w), \ w \geq 0$$

Define a differential operator

$$\mathcal{D}^{(\phi,a,\nu)}F(W) = \mu a + F'(W)(\gamma W - \lambda\mu(1-a)) + \frac{1}{2}\phi^2\sigma^2\nu^2 F''(W) + \theta\psi(\nu)$$

Then the optimality conditions can be described as variational inequalities

$$0 = \min\left\{rF(W) - \sup_{(a,\phi)\in\Lambda} \inf_{\nu\in\mathbb{R}} \mathcal{D}^{(\phi,a,\nu)}F(W), F'(W) + 1\right\}$$

for all $W \geq 0$ and boundary conditions given in the proposition, where

$$\Gamma = \{(0, \phi) : \phi \leq \lambda\} \cap \{(1, \phi) : \phi \geq \lambda\}$$

It is easy to show that the optimal policies $(\phi^*, a^*, \nu^*)$ defined in the proposition satisfy

$$rF(W) = \sup_{(a,\phi)\in\Gamma} \inf_{\nu\in\mathbb{R}} \mathcal{D}^{(\phi,a,\nu)}F(W) = \mathcal{D}^{(\phi^*,a^*,\nu^*)}F(W)$$

for all $W \in [0, \overline{W}]$ and $F'(W) = -1$ for $W \geq \overline{W}$...

## A.2 Proof of proposition 3.2.3

*Proof.* Applying Dynkin's formula to write the value function as an integral of the differential generator and then differentiating under the integral sign and applying the envelope theorem gives

$$\frac{\partial}{\partial\theta}F(W) = \mathbb{E}\left[\int_0^{\tau} e^{-rt}\frac{1}{2}\left(\nu^*(W_t)^2 - 1 - \log(\nu^*(W_t)^2)\right)\, dt \middle| W_0 = W\right] > 0$$

□

## A.3 Proof of proposition 3.2.4

*Proof.* Differentiate the boundary condition $rF(\overline{W}) + \gamma\overline{W} = \mu$ and use the smooth pasting condition $F'(\overline{W}) = -1$ to obtain

$$r\left[\frac{\partial}{\partial\theta}F(\overline{W}) - \frac{\partial\overline{W}}{\partial\theta}\right] + \gamma\frac{\partial\overline{W}}{\partial\theta} = 0$$

which gives

$$\frac{\partial\overline{W}}{\partial\theta} = -\frac{r}{\gamma - r}\frac{\partial}{\partial\theta}F(\overline{W}) < 0$$

□

## A.4 Proof of proposition 3.3.1

*Proof.* Let $h(W)$ denote the agent's value function under the contract described in proposition 3.2.1. The HJBI equation for the agent is given by

$$\gamma h(W) = \lambda\mu(1-a) + h'(W)(\gamma W + \lambda\mu(a-1)) + \frac{1}{2}\lambda^2\sigma^2\nu^2 h''(W) + \frac{\tilde{\theta}}{2}\left\{\nu^2 - 1 - \log\nu^2\right\}$$

on $[0,\overline{W}]$ with boundary conditions $h(0) = 0$ and $h'(\overline{W}) = 1$. Now, guess and verify that $h(W) = W$ is a solution with optimal controls $\nu(W) = 1$ and $a(W) = 1$. It is easy to show that this solution must be unique. □

## A.5 Proof of proposition 4.1.1

*Proof.* This follows immediately from proposition 3.2.4 □

## A.6 Proof of proposition 4.3.1

*Proof.* Differentiating (3.7) w.r.t. $W$ we obtain

$$0 = (\gamma - r)F'(W) + \gamma W F''(W) + \frac{\theta}{2}\frac{\lambda^2\sigma^2 F'''(W)}{\theta + \lambda^2\sigma^2 F''(W)}$$

Note that the first term is negative since $\gamma > r$ and $F'(W) < 0$ on the interval $(R,\overline{W}]$. The second term is negative since $F''(W) < 0$ for $W < \overline{W}$. Thus the third term must be strictly positive. This can only happen if $F'''(W)$ is strictly positive. The result now follows from (3.6). □

## A.7 Proof of proposition 5.1.2

*Proof.* This follows immediately from proposition 5.1.1 and appendix B of DeMarzo and Sannikov (2006) □

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