Essays on Inequality, Interest Rates and Macroeconomic Policies

by

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Abstract

This thesis consists of three chapters on inequality, interest rates and macroeconomic policies. The first chapter explores the macroeconomic consequences of the recent rise in permanent income inequality. First, I show that in many common macroeconomic models, consumption is a linear function of permanent (labor) income. This implies that macroeconomic aggregates are neutral with respect to shifts in the distribution of permanent income. Motivated by this neutrality result, I develop novel approaches to test for linearity in U.S. household panel data. The estimates suggest an elasticity of 0.7, soundly rejecting linearity. I quantify the effects of this deviation from neutrality using a novel non-homothetic precautionary-savings model. In the model, the rise in U.S. permanent labor income inequality since the 1970s caused: (a) a decline in real interest rates of around 1%; (b) an increase in the wealth-to-GDP ratio of around 30%; (c) wealth inequality to rise almost as rapidly as it did in the data.

The second chapter, joint with Sebastián Fanelli, develops a theory of foreign exchange interventions in a small open economy with limited capital mobility between home and foreign bond markets. Due to limited capital mobility, the central bank can implement nonzero bond spreads by managing its portfolio. Crucially, spreads are inherently costly as they allow foreign intermediaries to make carry-trade profits. Optimal interventions balance these costs with terms of trade benefits. We show that they lean against the wind of global capital flows to avoid excessive currency appreciation. Due to the convexity of the costs, interventions should be small and spread out, relying on credible promises (forward guidance) of future interventions. By contrast, excessive smoothing of the exchange rate path may create large spreads, inviting costly speculation. Finally, in a multi-country extension of our model, we find that the decentralized equilibrium features too much reserve accumulation and too low world interest rates, highlighting the importance of policy coordination.

The third chapter, joint with Iván Werning, reconsiders the well-known Chamley-Judd result, according to which capital should not be taxed in the long run. For the main model in Judd (1985), we prove that the long-run tax on capital is positive and significant, whenever the intertemporal elasticity of substitution is below one. The main model in Chamley (1986) imposes an upper bound on capital taxes. We provide conditions under which these constraints bind forever, implying positive long run taxes. When this is not the case, the long-run tax may be zero. However, if preferences are recursive and discounting is locally non-constant (e.g., not additively separable over time), a zero long-run capital tax limit must be accompanied by zero private wealth (zero tax base) or by zero labor taxes (first best). Finally, we explain why the equivalence of a positive capital tax with ever increasing consumption taxes does not provide a firm rationale against capital taxation.
Acknowledgments

I am indebted to many people who supported me along the journey that is a Ph.D., which culminated in this thesis.

I could not have hoped for a better main advisor than Iván Werning. I got to know Iván when I asked to work for him as a research assistant (RA) in the summer after my second year at MIT. He immediately provided me with work, on a topic I had only known cursorily from class: the models behind the Chamley-Judd result. In joint work, the third chapter of this thesis, we realized how the models had implications for capital taxes that had been unnoticed for 30 years, in some situations implying the polar opposite of the famous zero-tax prediction. Even outside this project, Iván has been a constant source of tremendously helpful advice and inspiration, with an unparalleled availability and devotion to his students (even right after his youngest son’s birth!). I am grateful for having the opportunity to learn from him and watch him work for so many years.

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One of the most positive surprises during grad school was Alp Şimşek’s return to MIT. Alp is an incredibly devoted and engaging teacher and among us students quickly became one of the most popular macro faculty to ask for advice. Alp’s comments are always deep and fascinatingly forward-looking in that he is able to spot possible stepping stones far in advance. It has been an absolute pleasure and honor to interact with him here, and he will always serve as a junior faculty role model for me.

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Chapter 1

Consumption, Savings, and the Distribution of Permanent Income

Rising inequality in the permanent component of labor income, henceforth *permanent income*, has been a major force behind the secular increase in U.S. labor income inequality. This paper explores the macroeconomic consequences of this rise. First, I show that in many common macroeconomic models—including models with precautionary savings motives—consumption is a *linear* function of permanent income. This implies that macroeconomic aggregates are neutral with respect to shifts in the distribution of permanent income. Motivated by this neutrality result, I develop novel approaches to test for linearity in U.S. household panel data, which consistently estimate the elasticity of consumption to permanent income in common precautionary savings models. The estimates suggest an elasticity of 0.7, soundly rejecting linearity. To quantify the effects of this deviation from neutrality, I extend a canonical precautionary savings model to include non-homothetic preferences across periods, capturing the idea that permanent-income rich households save disproportionately more than their poor counterparts. The model suggests that the U.S. economy is far from neutral. In the model, the rise in U.S. permanent labor income inequality since the 1970s caused: (a) a decline in real interest rates of around 1%; (b) an increase in the wealth-to-GDP ratio of around 30%;

I am indebted to Iván Werning, Jonathan Parker, Robert Townsend, and Alp Simsek for their continuous guidance and support throughout this project. I am very grateful to Daron Acemoglu, Marios Angeletos, Adrien Auclert, Sebastián Fanelli, Ernest Liu, Matthew Rognlie, and Nathan Zorzi for numerous helpful conversations. I also would like to thank Martin Beraja, Vivek Bhattacharya, Ricardo Caballero, Arnaud Costinot, Emily Gallagher, Daniel Greenwald, Adam Guren, Greg Howard, Chen Lian, Christopher Palmer, Andrés Sarto, and Olivier Wang for many useful comments, and the seminar participants at the MIT macro and finance lunches. I thank the Macro-Financial Modeling Group for financial support. All errors are my own.
(c) wealth inequality to rise almost as rapidly as it did in the data.

1.1 Introduction

U.S. labor income inequality has increased substantially over the past few decades (Katz and Murphy, 1992; Autor et al., 2008), with the top 10% now earning over 35% of all labor income (Piketty and Saez, 2003). A significant share of this increase appears to be the result of rising dispersion in the fixed-effect component of labor income, which captures the returns to skill or ability and which I henceforth refer to as *permanent income*.\(^1\) Indeed, Guvenen et al. (2017) argue that “newer cohorts enter with much higher inequality than older cohorts, which is the main force behind rising income inequality” (p. 38).\(^2\)

According to many well-known macroeconomic models, shifts in the distribution of permanent income are predicted to be entirely or approximately *neutral*: macroeconomic aggregates, such as consumption, wealth, and interest rates, are independent of permanent income inequality since consumption is a *linear* function of permanent income. While this neutrality result holds almost by construction in models adhering to the permanent income hypothesis (Friedman, 1957), it is much broader: even canonical precautionary-savings models (Aiyagari, 1994; Carroll, 1997; Gourinchas and Parker, 2002), which are widely known to generate concave consumption functions in current income or liquid assets (Zeldes, 1989; Carroll and Kimball, 1996), predict a linear consumption function in permanent income, and are therefore neutral.\(^3\)

In this chapter, I challenge the existing neutrality paradigm, both empirically and quantitatively. I have two main findings. First, I propose ways to consistently estimate the permanent income elasticity of consumption; I find estimates around 0.7, significantly below 1, indicating a concave consumption function in permanent income. Second, I incorporate non-homothetic preferences into a canonical precautionary-savings model to match this elasticity and study the quantitative implications: the model suggests that the increase in permanent income inequality since 1970 has pushed equilibrium interest rates down by around 1% through the present day, and is expected to lower interest rates by another 1% going forward (despite

---

\(^1\)In the terminology of this paper, permanent income refers each individual’s fixed effect in log labor income and does not include returns to capital.

\(^2\)Complementing this view, Sabelhaus and Song (2010) and Guvenen et al. (2014) provide evidence that both transitory and persistent shock variances have declined in recent decades; Kopczuk et al. (2010) and DeBacker et al. (2013) argue that either the variance of persistent shocks or the dispersion in fixed effects has increased. See also Figure 1.A.1 in Appendix 1.A.

\(^3\)There are exceptions to this, including Hubbard et al. (1994, 1995) and De Nardi (2004). See the discussion below.
assuming stable inequality going forward).

The first contribution of this chapter is to propose new ways to test the linearity of consumption in permanent income, building on previous work by Friedman (1957), Mayer (1972), and Dynan et al. (2004), among others. What distinguishes my work is the use of a large household panel data set—the Panel Study of Income Dynamics (PSID)—which since 1999 includes measures of both total consumption expenditure and income. I estimate a log-linear relationship between consumption and permanent income, which I demonstrate is a good fit to the data. This yields the permanent income elasticity of consumption, \( \phi \), which is equal to 1 under the null hypothesis that consumption is a linear function of permanent income.

The key challenge in estimating \( \phi \) is that permanent income is not directly observable and must be distinguished from income shocks, especially persistent ones. This is important since consumption is naturally smoothed in response to income shocks, so that ignoring income shocks results in attenuation bias in \( \phi \). I propose two novel solutions to this challenge, which depend on the autocovariance structure of persistent income shocks. If persistent income shocks follow an AR(1) process, \( \phi \) is identified and can be consistently estimated by instrumenting log current income with future quasi-differenced log incomes. If persistent income shocks follow a random walk, \( \phi \) is partially identified, and one can estimate an upper bound using initial incomes when entering the labor market as an instrument. Both approaches suggest that \( \phi \) is around 0.7, statistically and economically significantly below 1. I supplement these tests with a number of extensions and robustness checks that all yield similar results. Among these are specifications that include proxies for preference or rate-of-return heterogeneity, that deal with private and public transfers, and that are based on different measures of consumption expenditure.

This finding is best interpreted as follows: if working-age household A always earns twice as much in after-tax income as working-age household B, household A will not spend 100% more, but rather only 70% more. One may wonder whether this constitutes a significant source of non-neutrality. As I illustrate with a simple back-of-the-envelope calculation that assumes consumption \( c \) to be a power function of income \( y \), \( c \sim y^{\phi} \), the difference is sizable: the shift in the U.S. income distribution between 1980 and 2014 implies translates into a reduction in aggregate consumption by approximately 4%.

The second contribution of this chapter is to investigate the implications of a concave consumption function in permanent income quantitatively. To this end, I build a non-neutral version of a quantitative life-cycle model with idiosyncratic income risk and incomplete markets.
in the tradition of Deaton (1991), Huggett (1996) and Gourinchas and Parker (2002). Aside from standard sources of non-neutrality, such as nonlinear tax-and-transfer and social security systems (Hubbard et al., 1994, 1995; Scholz et al., 2006), the key elements of my model are two kinds of non-homothetic preferences. The first element follows the seminal work of De Nardi (2004) and assume that bequests are treated as a luxury good. The second key element concerns “life-cycle” non-homothetic preferences, over consumption across periods, which turn out to be the most important source of non-neutrality in the model. Such non-homothetic preferences capture the idea that permanently richer agents save a larger fraction of their income, either for bequests or for other expenses later in life, which a poorer household cannot afford to do. While the model does not require enumeration of what those expenses are, one may think of college tuition payments for kids, expensive medical treatments later in life, or charitable giving. I calibrate the strength of the life-cycle non-homotheticity to match the elasticity $\phi$ when estimating the same regressions on artificial panel data simulated from the model. Importantly, I find that models without life-cycle non-homotheticity, even non-neutral ones, cannot rationalize the empirical magnitude of $\phi$. Although I do not target any specific moments of the wealth distribution, the model matches the (highly unequal) wealth distribution of 2014 quite well—except at the very top, for the fractiles within the top 1%.

I use the calibrated economy as a laboratory to study the implications of rising permanent income inequality. In partial equilibrium (PE), keeping the interest rate fixed, I find that a shift from a steady state with the level of permanent income inequality of 1970 to that of 2014 shows a considerable increase in aggregate wealth, of just above 130% of GDP. As a point of comparison, the U.S. net foreign asset position has declined by “only” 18% of GDP since the 1997-98 Asian financial crisis, which is often attributed to the recent “global savings glut”. This is a first hint that rising income inequality may not have been neutral in the past few decades.

I then simulate the general equilibrium transitional dynamics from the 1970 steady state

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4See also Imrohoroglu et al. (1995), Rios-Rull (1996) and Carroll (1997) among others.
5The idea of non-homothetic savings behavior goes back at least to Fisher (1930) and Keynes (1936). Fisher (1930) noted that if one person’s income is simply a scaled version of another person’s in all periods, then “the smaller the income, the higher the preference for present over future” consumption. Keynes (1936) argued that as long as one’s “primary needs” are not satisfied, consumption is “usually a stronger motive than the motives towards [wealth] accumulation”.
6In that sense, my calibration shares similarities with indirect inference (Gourieroux et al., 1993; Smith, 1993, 2008; Guvenen and Smith, 2014), in that one of the IV regressions is treated as “auxiliary model” along which the model is matched to the data.
7It is well known that a substantial fraction of the top 1% and especially the top 0.1% are business owners, who are not modeled here, which could explain the deviation. See Quadrini and Rios-Rull (1997) and Cagetti and De Nardi (2006) for models of entrepreneurship and the wealth distribution.
to recent levels of permanent income inequality, which I assume to remain constant after 2014.\(^8\) This exercise allows the model to speak directly to the forces behind three important recent macroeconomic trends: (a) the decline in real (natural) interest rates since the 1980s (Laubach and Williams, 2003, 2015), (b) the rising private wealth to GDP ratio (Piketty and Zucman, 2015), and (c) the large and rapid increase in U.S. wealth inequality (Saez and Zucman, 2016).

Regarding the first macroeconomic trend, I find that the real interest rate declines by around 1% through 2017, which explains approximately one-third of the decline in the U.S. natural rate since the 1980s. Interestingly, despite the absence of any further increases in income inequality, the model predicts the interest rate will continue to decline, eventually falling by another 1%. The reason for this result is intuitive: in the model, the generation entering the labor market today is the first to experience the highest level of permanent income inequality for their entire working lives. In particular, this means the most able or skilled workers entering today will amass much larger fortunes over their lifetimes than previous generations. This effect causes a large and predictable decline in interest rates going forward.

Concerning the second trend, the endogenous interest rate response limits the rise in aggregate wealth to around 30% of GDP through 2017 (again, roughly one-third of the rise in the data), with an eventual total increase of 55%.

Finally, the model offers an explanation for the size and speed of the increase in the top 10% wealth share and around two-thirds of the increase in the top 1% wealth share.\(^9\) In sum, the model suggests that rising permanent labor income inequality alone can account for a significant share of three major macroeconomic trends.

Literature. This chapter combines several strands of a vast literature at the intersection of inequality, consumption dynamics, and macroeconomics.\(^10\)

First, this chapter contributes to the large empirical literature that tests the permanent income hypothesis (PIH), starting with Friedman (1957) himself. One can group the predictions

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\(^8\)The transitional dynamics are computationally non-trivial since the model has a large number of idiosyncratic states, as well as endogenous bequest distributions over which agents have rational expectations. I overcome these difficulties by improving existing algorithms along a number of margins (see Appendix 1.F). The improvements were developed jointly with Adrien Auclert and Matt Rognlie.

\(^9\)The dynamics of the wealth distribution have recently been investigated theoretically by Gabaix et al. (2017), and numerically by Hubner et al. (2016), Kaymak and Poschke (2016), and Aoki and Nirei (2017) in incomplete-markets models.

\(^10\)For recent surveys and books, see among others Bertola et al. (2005); Krusell and Smith (2006); Heathcote et al. (2009); Guvenen (2011); Quadrini and Rios-Rull (2015); De Nardi et al. (2015); Piketty and Zucman (2015); De Nardi and Fella (2016); Attanasio and Pistaferri (2016); Piketty (2017); Benhabib and Bisin (2017).
of the PIH into two conceptually-distinct categories: predictions about changes in consumption in response to predictable or unpredictable, transitory or permanent, income changes; and predictions about the level of consumption in relation to the level of the permanent component of income. Throughout the 1950s and 1960s, many economists viewed the second prediction as the "most controversial aspect of the permanent income theory" (Mayer, 1972, p.34) and consequently it received relatively more attention. Partly due to data quality issues, however, the evidence remained inconclusive, and the focus of empirical work on the PIH subsequently shifted almost entirely to testing the first set of predictions.

The main exception to this is the work of Dynan et al. (2004), henceforth DSZ. Their paper computes savings rates, either as consumption-based measures \((Y - C)/Y\) (CEX) or as wealth difference based measures \(\Delta A/Y\) (SCF, PSID), and documents two main facts. First, savings rates increase across current income quintiles. Second, savings rates still increase in income quintiles if income is instrumented by lagged or future income or education. My empirical exercise follows their lead, innovating along several dimensions. First, I focus solely on consumption and not on wealth differences, which are problematic because it is generally difficult to disentangle ex-ante savings behavior from ex-post returns or transfers. Second, I show that the relationship between log consumption and log income is roughly log-linear, which allows a focus on a single elasticity parameter \(\phi\). Finally, and most importantly, I use a panel data set with consumption and income (the PSID since 1999), which allows for the development of two new instruments under mild assumptions on the income process. I show that the two instruments either estimate \(\phi\) consistently or estimate an upper bound for \(\phi\) consistently in canonical precautionary savings models. This improves upon simple instruments such as lagged or future income (which lead to downward-biased results under the assumptions of a canonical neutral model), or education (which could be correlated with preferences, income profiles, etc). In fact, my approach allows for additional proxies to try to control for heterogeneity in preferences and returns.

The second main contribution of this chapter is the analysis of rising permanent income inequality in a non-neutral, incomplete-markets economy. Here, I combine elements from two literatures, one on non-neutral economies and one on rising income inequality. Seminal work on non-neutral economies, by Hubbard et al. (1994, 1995) and De Nardi (2004), argues that a

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11 See e.g. Friedman (1957), Mayer (1966), Evans (1969), and Mayer (1972).

12 This empirical work started using Euler equation-based tests (Hall, 1978; Flavin, 1981; Hall and Mishkin, 1982) and now also includes well-identified empirical studies (Johnson et al., 2006; Parker et al., 2013).

13 For similar approaches see Bozio et al. (2013) for the UK and Alan et al. (2015) for Canada. Gustman and Steinmeier (1999) and Venti and Wise (2000) propose a look at the relationship between retirement wealth and lifetime income. As part of my analysis in Section 1.5, I show that this relationship is not well suited to inform the degree of non-neutrality in the presence of income shocks.
realistic social safety net and non-homothetic bequest motives, respectively, can significantly increase wealth inequality compared to a homothetic model, albeit by not quite as much as the data.\textsuperscript{14} This chapter follows their lead and argues that one needs (considerably) more non-neutrality than what the existing model elements generate. I therefore include non-homothetic preferences over consumption within the life-cycle as well, which, when calibrated to match the empirical evidence, generate a wealth distribution that fits the recent U.S. distribution relatively well.

In addition, this chapter builds on a recent literature that studies the quantitative effects of income inequality in (mostly) neutral economies. Here, Auclert and Rogulie (2017) show how greater inequality has aggregate effects that crucially depend on the types of incomes (transitory, persistent or permanent) that become more unequal. They also consider implications for economies at the zero lower bound, where there can be a feedback loop between aggregate demand and endogenous income risk. Heathcote et al. (2010) investigate the human capital investment and family labor supply implications of rising income inequality. Kaymak and Poschke (2016) and Hubmer et al. (2016) consider the effects of rising income inequality on wealth inequality. Favilukis (2013) studies the joint implications of greater income risk, relaxed borrowing constraints and an increase stock market participation rate. Krueger and Perri (2006) argue that insurance against shocks may improve with greater within-group income risk.\textsuperscript{15} The key distinguishing feature of this chapter is that I focus on rising permanent income inequality, arguably among the most important drivers of rising income inequality in the U.S..\textsuperscript{16} And, to study its consequences, it is important to model an economy that gets the degree of non-neutrality right.

The interest in modeling non-homothetic consumption-savings behavior is shared by a number of earlier, mostly deterministic, papers. Uzawa (1968) was the earliest well-known efforts to do this using recursive Koopmans (1960) utility and Epstein and Heynes (1983) and Epstein (1987) subsequently extended this work. Lucas and Stokey (1984) used such preferences to study many-agent neoclassical growth models without degenerate wealth distributions (see also Obstfeld (1990)). Interestingly, however, these papers end up focusing on the opposite of the empirical case, namely economies where richer agents save less than

\textsuperscript{14}See also Huggett and Ventura (2000), Scholz et al. (2006) and De Nardi and Yang (2014). Kumhof et al. (2015) study the interaction of income inequality, debt and defaults in a two-agent economy with non-homothetic preferences.

\textsuperscript{15}Holm (2017) studies the differential effects of monetary policy during times with increased persistent income risk.

\textsuperscript{16}See Guvenen et al. (2017) for direct evidence on the importance of permanent income inequality, in line with Sabelhaus and Song (2010), who document a reduction in transitory and persistent income volatility. Suggestive evidence also appears in DeBacker et al. (2013) and Kopczuk et al. (2010).
poorer agents, since this is precisely the case in which multi-agent deterministic infinite-horizon models have a stationary distribution.\textsuperscript{17} In contrast to these papers, my economy admits a non-degenerate stationary wealth distribution despite richer agents saving more, which is possible because my model also includes borrowing constraints and idiosyncratic risk.

There is a long tradition of studying and identifying the degree to which consumption is insured from changes in incomes. Of particular relevance and inspiration to my research are Blundell et al. (2008) and Kaplan and Violante (2010).\textsuperscript{18} In a landmark result, Blundell et al. (2008) describe a way in which one can estimate the degrees to which consumption responds to persistent or transitory income shocks. They use this method to show that household consumption “under-reacts” to permanent income shocks. Kaplan and Violante (2010) extend the framework to allow for shocks with persistence less than unity and show that those shocks generally lead to larger degrees of under-reaction due to self-insurance. Compared to these papers, the focus in this chapter is on the relationship between the level of consumption and the level of permanent income, rather than on changes. I show in extensions of both the empirical analysis and my model that the degree of partial insurance is largely orthogonal to the curvature in consumption as a function of permanent income. However, this chapter very much shares the spirit of these papers in that they identify important moments in similar panel data on consumption and income and use them to inform microfounded consumption-savings models.

**Layout.** I begin in Section 1.2 by demonstrating in a stylized, two-agent framework how a concave consumption function in permanent income can be modeled and what its likely effects are. Section 1.3 introduces a canonical precautionary-savings model and explains under what assumptions this model is neutral with respect to changes in the permanent income distribution. I test for neutrality in Section 1.4. Extending the canonical model, Section 1.5 then relaxes the neutrality assumptions—mainly by introducing non-homothetic preferences—and highlights the main properties of the non-homothetic model. The effects of rising income inequality in partial and general equilibrium are simulated in Section 1.6. Section 1.7 concludes and discusses potential avenues for future research. The appendix

\textsuperscript{17}In a few more theoretically-oriented papers, however, richer agents do save more. Cole et al. (1992, 1995, 1998), Robson (1992), Ray and Robson (2012) (implicitly or explicitly) model utility over one’s wealth rank. Carroll (2000) models utility over wealth directly. In Becker and Mulligan (1997) agents can invest in raising their discount factor. Recent models of savings behavior with disagreements between current and future selves can generate similar patterns, see e.g. Harris and Laibson (2001) and Cao and Werning (2016).

Finally, a number of papers studies the non-homothetic preferences at the intersection of macroeconomics and development (see, e.g. Moav (2002); Galor and Moav (2004)).

\textsuperscript{18}For other papers in this literature see e.g. Cochrane (1991), Townsend (1994), Attanasio and Davis (1996), Attanasio and Pavoni (2011), Blundell et al. (2016), Arellano et al. (2017).
contains all proofs, as well as additional empirical and quantitative results.

1.2 Permanent Income Inequality in a Two-Agent Model

I begin by studying a stylized two-agent OLG framework to illustrate the main effects of rising permanent income inequality in neutral and non-neutral models. For this purpose, Section 1.2.1 introduces the utility maximization problem of a single dynasty, exemplifying how the consumption function can be concave in permanent income. Section 1.2.2 then combines two such dynasties and studies the partial equilibrium implications of greater permanent income inequality. Section 1.2.3 closes the economy by adding a standard neoclassical supply side and characterizes the general equilibrium implications of rising inequality. All figures in this section are constructed using a standard calibration which I explain in Appendix 1.B.19

1.2.1 Concave consumption functions

Consider a dynasty of 1-period lived generations that earn a constant stream of wage incomes $w > 0$ and can save in risk-free bonds paying a constant interest rate $R > 1$. They face the following decision problem: each period $t = 0, 1, 2, \ldots$ the current generation solves

$$\max_{c_t, a_{t+1}} u(c_t) + \beta U(a_{t+1})$$  \hspace{1cm} (1.1)

$$c_t + R^{-1} a_{t+1} \leq a_t + w.$$ \hspace{1cm} (1.2)

Here, $a_t$ denotes the value of financial wealth held by the dynasty at the beginning of period $t$, $a_t + w$ can be regarded as the dynasty’s “cash on hand”, $c_t$ denotes the consumption choice in period $t$, and $a_{t+1}$ is the bequest left to the subsequent generation. Observe that in this model, $w$ is the dynasty’s permanent income level, where I use the term permanent income, as explained in the introduction, to denote the fixed-effect component of labor income. In fact, this model is so stylized that there is no other component of labor income, that is, no income shocks, no life-cycle earnings profile, and so on.

The choice of the two utility functions is critical for this model: the flow utility $u$ and the joy-of-giving utility $U$. For simplicity, I assume that both have a constant elasticity,

$$u(c) = \frac{(c/z)^{1-\sigma} - 1}{1 - \sigma} \quad U(a) = \frac{(a/z)^{1-\Sigma} - 1}{1 - \Sigma},$$ \hspace{1cm} (1.3)

19Atkinson (1971) and Benhabib et al. (2011) study related OLG economies with a non-homothetic bequest motive.
**Figure 1-1:** Stylized model: Consumption and savings schedules.

![Graph of consumption and savings schedules](image)

**Note.** This figure shows consumption and asset choices in the stylized model. Panel (a) shows the optimal consumption policy as a function of cash on hand. Panel (b) shows the asset position after 20 years as a function of permanent income. “Non-homothetic” refers to a model where saving has an income elasticity greater than 1. “Homothetic” refers to a model where saving has an income elasticity of 1.

but the two (inverse) elasticities $\sigma, \Sigma > 0$ are allowed to differ. In this formulation $z > 0$ is a normalization parameter that allows the model to retain aggregate scale invariance.\(^{20}\) Heterogeneity in the inverse elasticities $\sigma, \Sigma$ represents the single deviation from a standard homothetic consumption-savings model. It allows the model to capture the fact that richer dynasties may have a greater propensity to save in the following way: the utility maximization problem (1.1) can be thought of as a simple decision problem between two goods, consumption $c_t$ and savings $a_{t+1}$. In this decision problem when saving is a “luxury good”—that is, its income elasticity is greater than one—a richer dynasty decides to save a larger fraction of its wealth.\(^{21}\) With utilities as power functions, this is the case if $\Sigma < \sigma$, so that the utility over savings (bequests) is more linear than the utility over consumption. This can also be seen from the Euler equation,

$$
c_t/z = (\beta R)^{-1/\sigma} (a_{t+1}/z)^{\Sigma/\sigma},$$

which shows that consumption $c_t$ adjusts by less than savings $a_{t+1}$ when $\Sigma/\sigma < 1$.\(^{22}\)

Figure 1-1 illustrates two key outcomes of the utility maximization problem (1.1). Panel

\(^{20}\)In a model with growth, it would be natural to assume $z$ grows at the same speed as the economy. This captures the idea that for savings behavior it is not the absolute level of one’s income that matters, but the income relative to the aggregate economy.

\(^{21}\)See also Strotz (1955) and Blinder (1975) for early deterministic life-cycle models with non-homothetic utility over bequests. SeeAndreoni (1989) for further work on joy-of-giving preferences and Abel and Warshawsky (1988) for the relationship with perfect altruism.

\(^{22}\)It is worth emphasizing that this model generates a consumption function that is concave in permanent (labor) income, which, as I argue in Section 1.3 is not the case in a canonical precautionary-savings model, where consumption is concave in current income or assets, but not permanent income.
(a) shows the optimal short-run consumption choice $c_t$ as a function of cash on hand. I call it “short-run” as it takes current assets as given. Panel (b) shows the optimal long-run asset position as a function of permanent income $w$, where long-run means after 20 years. In both panels the agent starts with the average wealth and income position in the economy. The panels show two cases: the homothetic case, where $\Sigma = \sigma$, and the non-homothetic case, where $\Sigma < \sigma$ and savings are treated as a luxury good. While optimal short-run consumption and long-run savings schedules are both linear in the homothetic case, consumption is concave and savings convex in the non-homothetic case.

As a side note, the consumption schedule turns out to be well approximated by a simple power function $c_t = k(a_t + w)^\phi$ for large values of cash on hand, where the exponent is given by the ratio of the elasticities, $\phi = \Sigma / \sigma$. The elasticity $\phi$ takes a central role in this chapter, as it succinctly characterizes the degree of concavity in consumption as a function of permanent income.

1.2.2 Partial equilibrium effects of greater inequality

Having introduced the decision problem of a single dynasty, I now describe the effects of shifts in income inequality between two dynasties. Thus, assume an economy is populated by two dynasties, both with the exact same preferences (1.1). The only difference between these dynasties is their permanent income level: one dynasty, the “rich” $r$, is assumed to have a strictly greater permanent (labor) income than the other, the “poor” $p$, that is, $w_r > w_p$. Assume that the population share of the rich dynasty is $\mu \in (0, 1)$. The share of labor income earned by the rich dynasty, $\gamma = \mu w^r / (\mu w^r + (1 - \mu) w^p)$, will serve as the measure of inequality in this economy. The economy is more unequal the further away $\gamma$ is from $\mu$. In all figures below, I take $\mu$ to be 1%. Denote by $W = w^r + w^p$ total labor income, which is assumed to be constant in this subsection, so that inequality $\gamma$ uniquely defines $w^r$ and $w^p$.

Imagine that the economy is initially perfectly equal, $\gamma = \mu$, and consider an unanticipated increase in inequality, $\gamma > \mu$. Figure 1-2 shows what happens to short-run consumption and long-run savings in this scenario. Given the curves in Figure 1-1 the result is unsurprising, yet powerful: in the homothetic model, where $\Sigma = \sigma$, nothing happens to either consumption or savings. This is a direct consequence of the linearity in Figure 1-1 and makes this economy a simple example of an economy where the permanent income distribution is neutral. In the non-homothetic economy, aggregate consumption falls on impact, and long-run savings rise.
1.2.3 General equilibrium effects of greater inequality

The results in the previous subsection raise the question of what happens in general equilibrium. Closing the model requires to specify the supply side of this economy, which is assumed to be given by a Cobb-Douglas aggregate production function,

\[ Y = F(K, L^r, L^p) = AK^\alpha (L^r)^{(1-\alpha)\gamma}(L^p)^{(1-\alpha)(1-\gamma)}, \]

where \( A > 0, K \) denotes capital (assumed to depreciate at rate \( \delta > 0 \)), and \( L^p \) and \( L^r \) denote labor supplied by the poor and rich dynasties.

Again the same experiment is conducted: Starting at perfect equality, \( \gamma = \mu \), what happens in this economy when \( \gamma \) is increased? The six panels in Figure 1-3 show the long-run outcomes for the homothetic and the non-homothetic economies. As anticipated, the homothetic economy is neutral, so that none of the aggregate quantities \( K, C, Y \) or interest rates are affected. Wealth inequality increases, but only at the same rate as income inequality, reflecting the proportionality of assets and income in the model. Similarly, consumption inequality increases at the same rate, as well.

By contrast, in the non-homothetic economy, the capital stock and output increase with greater inequality, while interest rates fall. Interestingly, wealth inequality rises faster than one-for-one with inequality, but this does not translate into greater consumption inequality: the level of consumption inequality is similar to that in the homothetic model. This is mainly due to the endogenous interest rate decrease, which reduces the rich dynasty’s capital income.
Figure 1-3: Stylized model: Neutrality and non-neutrality in general equilibrium.
1.2.4 Takeaway for the rest of this chapter

The results in this section show that homothetic preferences tend to induce a linear consumption function in permanent income, while non-homothetic preferences induce a concave consumption function. This allows models endowed with the former to be neutral and models endowed with the latter to be non-neutral. These ideas foreshadow the rest of this chapter: the general model in Section 1.3 nests the stylized homothetic model and proves a general neutrality result; the concavity parameter \( \phi \) is estimated in the data in Section 1.4; then, I extend the general model to include non-homothetic preferences to match \( \phi \) in Section 1.5. Finally, Section 1.6 investigates the partial and general equilibrium properties of rising income inequality.

1.3 General Model and Neutrality Result

In the previous section I demonstrated that a simple homothetic dynastic economy is neutral with respect to changes in the permanent income distribution: all aggregates are invariant in partial and general equilibrium, while measures of inequality change linearly in permanent income inequality. This section generalizes these results and proves that they carry over to a large class of models. The general model introduced in this section also lays the foundation for what is to come: the conditions identified here motivate the empirical analysis in Section 1.4, and the quantitative model in Section 1.5 is a version of the general model.

1.3.1 Setup

Time is discrete, \( t \in \{0, 1, \ldots\} \), and there is no aggregate risk. The model is an overlapping generations (OLG) version of an Aiyagari (1994) model. It allows for an endogenous bequest distribution which agents receive at the time of their parents’ death.\(^2\) I focus on the steady state of the economy.

Birth, death and skills. The economy is populated by a continuum of mass 1 of agents at all times, each of whom is assigned a permanent type in a finite set \( S \subseteq \mathbb{N} \). One can think of a permanent type \( s \in S \) as innate skill or ability.\(^2\) Agents with skill \( s \) are endowed with on average a single efficiency unit of skill \( s \) and make up a constant share \( \bar{\pi}_s \in [0, 1] \) of the population. To allow for overlapping generations, I assume that there is a constant inflow

\(^2\) Endogenous bequests are important in a realistic model of wealth inequality. See e.g. Castaneda et al. (2003), De Nardi (2004), Benhabib et al. (2011).

\(^2\) This model abstracts from endogenous investment into human capital. See Heathcote et al. (2010) for a model along those lines.
and outflow of agents at rate $\delta \geq 0$, where zero is included. An agent’s age is indexed by $k \in \mathbb{N}$. With an OLG structure, $\delta > 0$, each agent has a single offspring that is born at fixed parental age $k_{\text{born}} > 0$, and dies with certainty at age $K_{\text{death}} \in \mathbb{N} \cup \{\infty\}$. Henceforth I assign all agents ever to live in this economy the unique label $i \in [0, \infty)$.

**Production.** There is a single consumption good, which is produced using a neoclassical aggregate production function $Y = F(K, \{L_s\}_{s \in S})$ from $K$ units of capital and $L_s$ efficiency units of skill $s$. I assume that $F$ is Cobb-Douglas, that is, $F = AK^\alpha \prod_{s} L_s^{(1-\alpha)\gamma_s}$, where $\gamma_s > 0$ is the labor income share of skill $s$. I denote by $w_s$ the price of an efficiency unit of skill $s$ and by $r$ the real interest rate, so that in equilibrium, $Y = (r + \delta)K + \sum_{s \in S} w_s L_s$. The main comparative statics exercise in this section will be a shift in the distribution of labor income shares $\{\gamma_s\}$, which induces a shift in the distribution of skill prices $\{w_s\}$, since $w_s = \gamma_s Y / L_s$.

**Government.** There is a government that levies a constant tax rate $\tau^b \in [0, 1]$ on any bequests (relevant only in the OLG case, if $\delta > 0$) and applies to all agents a possibly age-dependent income tax function $T_k(y_{\text{pre}})$, where $k \geq 1$ is an agent’s age, and $y_{\text{pre}}$ an agent’s pre-tax income. I allow for age-dependence to nest the case where the government provides a social security and pension system, in which case $T_k$ would be negative for retired individuals. The government holds a level of government debt $B$ and chooses its spending $G$ to balance its budget.

**Agents.** In the life-cycle case, an agent is born at some date $t_0$ with zero asset holdings and with some skill $s \in S$. In the infinite-horizon case, agents are already alive at date $t = 0$. The agent faces idiosyncratic shocks captured by a Markov chain $z_t \in \mathcal{Z}$ with transition probabilities $\pi_{zz'}$ from state $z$ to state $z'$, initialized at date $t_0$ with a fixed initial distribution $\{\pi_z\}$. The idiosyncratic shocks determine the agent’s stochastic endowment of efficiency units of skill $s$, which is given by a function $\Theta_{t-t_0}(z_t)$ at time $t$. The agent’s income is then $y_{t\text{pre}} = \Theta_{t-t_0}(z_t)$ before taxes and $y_t = y_{t\text{pre}} - T_{t-t_0}(y_{t\text{pre}})$ after taxes. I assume the function $\Theta_k(z)$ is normalized such that it averages to 1 when averaged over the whole population of agents and over all idiosyncratic states. In the life-cycle case, an agent dies after age $k$ with probability $\delta_k \in [0, 1]$. In case of death, the agent is allowed to derive utility over bequests. I denote by $u_k(c)$ the agent’s possibly age-dependent, per-period utility over the consumption good, and by $U(a)$ the utility from bequeathing asset position $a$.

**Bequests.** It is assumed that each agent of skill $s$ has an offspring with skill $s'$, where $s'$ is randomly drawn from a transition matrix $P_{ss'}$. The process for skills is assumed to be independent of $\{z_t\}$. Bequests are not necessarily received at the beginning of life, so it is

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25The results in this section generalize to arbitrary neoclassical production functions and arbitrary shifts in the distribution of labor income.
important to specify each agent’s belief about the distribution of bequests they may receive later on. I assume that \( \varphi \in \{0, 1\} \) is an indicator for whether an agent has already received a bequest and that \( v(\cdot|s, k, \varphi) \) denotes the probability distribution over bequest sizes to be received next period conditional on age \( k \), skill \( s \), and indicator \( \varphi \). Formally, \( v(\cdot|s, k, \varphi) \) is defined over the product space of bequests and bequest indicators, \( \mathbb{R}_+ \times \{0, 1\} \), together with the Borel \( \sigma \)-algebra.

**Agent’s optimization problem.** Taken together, an agent born at date \( t_0 \) with skill \( s \) solves the following optimization problem,

\[
V_{k,s}(a, z, \varphi) = \max_u u_k(c) + \beta(1 - \delta_k)\mathbb{E}_{z', \varphi} V_{k+1,s}(a' + b', z', \varphi') + \beta \delta_k U(a') \tag{1.5}
\]

\[
c + \frac{1}{1 + r} a' \leq a + \Theta_k(z)w_s - T_k(\Theta_k(z)w_s)
\]

\[
(b', \varphi') \sim v(\cdot|s, k, \varphi)
\]

\[
a' \geq 0.
\]

### 1.3.2 Equilibrium

Denote the state space by \( S = S \times \{1, \ldots, K_{\text{death}}\} \times \mathbb{R}_+ \times \mathcal{Z} \times \{0, 1\} \) endowed with the Borel \( \sigma \)-algebra \( B_S \) on \( S \). I define a steady state equilibrium as follows.

**Definition 1** (Steady-state equilibrium). A steady state equilibrium in the benchmark economy is a vector of aggregate quantities \( \{Y, K, L_s\} \), a probability distribution \( \mu \) defined over \( (S, B_S) \) and a measure of bequests \( \chi \) defined over \( S \times \{1, \ldots, K_{\text{death}}\} \times \mathbb{R}_+ \) with the Borel \( \sigma \)-algebra, a set of policy functions \( \{c_{k,s}(a, z, \varphi), a_{k,s}(a, z, \varphi)\} \), a set of prices \( \{r, w_s\} \) such that: (a) the policy functions solve the optimization problem (1.5), where the conditional bequest distribution \( v(\cdot|s, k, \varphi) \) is given by

\[
v(B, \varphi'|s, k, \varphi) = \begin{cases} 1\{0,1\}(B, \varphi') & \text{if } \varphi = 1 \\ (1 - \delta_{k+k_{\text{born}}})1\{0,0\}(B, \varphi') + \frac{1}{\mu_s} \sum_{s'} P_{ss'} \chi(s', k + k_{\text{born}}, B) & \text{if } \varphi = 0 \end{cases} \tag{1.6}
\]

where \( B \subset \mathbb{R}_+ \) is measurable and the notation \( 1_X \) denotes the indicator function for a given set \( X \), (b) the representative firm maximizes profits \( F(K, L_s) - (r + \delta)K - \sum w_s L_s \), (c) the government budget constraint

\[
G + rB \leq \int_{(s,k,a,z,\varphi)} T_k(\Theta_k(z_k)w_s) d\mu + \tau^b \int_{(s,k,b)} bd\chi
\]
is satisfied, (d) the goods market clears, \( Y = \delta K + \int c_{k,s}(a, z, \varnothing) d\mu \), (e) all markets for efficiency units of each skill clear, \( L_s = \overline{\mu} \), (f) the asset market clears,

\[
\frac{1}{1 + r} A = \frac{1}{1 + r} \int_{(s,k,a,z,\varnothing)} a d\mu = B + K,
\]

(g) the bequest distribution is consistent with the distribution over states, \( \chi(s,k,(1 - \tau^b)A, z, \varnothing) = \delta_k \mu(s,k,A,z,\varnothing) \), where \( A \subset \mathbb{R}_+ \) measurable, and (h) aggregate flows and bequests are consistent

\[
\mu(s,k+1,A,z',\varnothing) = \sum_{\varphi} \int (b',z') s.t. \varphi' = \varnothing \int (s,k,a,z,\varnothing) s.t. a_{k,s}(a,z,\varnothing) + b' \in A \Pi_{z'z} d\mu(\cdot | s,k,\varnothing)
\]

\[
\mu(s,1,A,z,\varnothing) = \pi \overline{\mu} 1_{[0]}(\varnothing) 1_{[0]}(A).
\]

Similar to Section 1.2, I now show three sets of results in this economy: first, that consumption functions are linear in permanent income; second, consumption and wealth inequality move one-to-one with income inequality; and third, the aggregate economy in partial and general equilibrium is unaffected by changes in permanent income inequality.

### 1.3.3 Assumptions for neutrality

To state the results, I formally introduce three necessary assumptions to obtain neutrality. Each of these is relaxed in Section 1.5 to explore their respective roles in generating a concave consumption function. The first assumption is that utility functions over consumption and bequests each have a constant elasticity; and moreover, that elasticities are the same and not age-dependent.

**Assumption 1** (Homothetic utility functions). (i) The per-period utility function \( u_k(c) \) is homogeneous with a constant elasticity of intertemporal substitution, that is, \( u_k(c) = c^{\frac{1-\sigma}{1-\sigma}} \) for some \( \sigma > 0 \). (ii) The bequest utility function \( U(a) \) is homogeneous with the same elasticity, that is, \( U(a) = \kappa \frac{a^{1-\sigma}}{1-\sigma} \) for some parameter \( \kappa \geq 0 \).

This assumption is the reason I refer to this benchmark economy as *homothetic*. The second assumption is that the income tax schedule is linear.

**Assumption 2** (Linear tax schedule). The income tax function is linear in pre-tax income, that is, \( T_k(y^{pre}) = \tau_k y^{pre} \) for some \( \tau_k \in \mathbb{R} \), for each \( k \in \{1, \ldots, K_{death} \} \).

This assumption restricts both income taxes and any social security payments to be entirely linear. As I discuss below, however, richer, progressive tax-and-transfer schedules
can still be allowed without breaking the linearity result below. The final assumption for the neutrality results is that bequests play no redistributational role, that is, no rich person leaves any wealth to a less-skilled offspring.

Assumption 3 (Perfect skill persistence). One of the following three assumptions is satisfied: (i) Skills are perfectly persistent, that is, the transition matrix $P_{ss'}$ is the identity, $P_{ss'} = 1$ if $s = s'$ and $P_{ss'} = 0$ otherwise; (ii) There are no bequests, that is, the model is an infinite-horizon economy or a perfectly deterministic life cycle model without bequest utility; (iii) Bequests are perfectly taxed, $T_b = 1$.

A commonly-used fourth alternative, not modeled here, is the assumption of a perfect annuities market (and no preferences for bequests). In addition to those three economic assumptions, I make a fourth technical one to rule out boundary cases with ill-defined equilibrium wealth distributions.

Assumption 4 (Unique wealth distribution given $r$). Given any interest rate $r > 0$ and permanent incomes $\{w_s\}$, there exists at most a single wealth distribution $\mu$ for which (a) and (g) of Definition 1 can be satisfied.

This assumption essentially rules out the special case of no income risk and an infinite horizon, where it is well known that there does not exist a unique wealth distribution. Note that it still allows for multiple steady-state equilibria to exist (as in Acikgöz (2017)) as long as each equilibrium interest rate is associated with a unique wealth distribution.

Having made these important but common assumptions, I can now characterize the micro implications of steady state equilibria in this economy.

1.3.4 Linearity and aggregate neutrality

I start by showing a helpful auxiliary result which states that all steady state policy functions and asset distributions scale with permanent income.

Lemma 1. Under Assumptions 1–4, for any measurable set $(s, k, A, z, \varphi) \subseteq S$, any state $(s, k, a, z, \varphi) \in S$ and any skill $s' \in S$ it holds in any equilibrium that

$$
\mu(s, k, A, z, \varphi) = \frac{\mu_s}{\mu_{s'}} \times \mu \left( s', k, A \frac{w_{s'}}{w_s}, z, \varphi \right) \quad \text{and} \quad \chi(s, k, A) = \frac{\mu_s}{\mu_{s'}} \times \chi \left( s', k, A \frac{w_{s'}}{w_s} \right)
$$

$$
c_{k,s}(a, z, \varphi) = \frac{w_s}{w_{s'}} \times c_{k,s'} \left( a \frac{w_{s'}}{w_s}, z, \varphi \right) \quad \text{and} \quad a_{k,s}(a, z, \varphi) = \frac{w_s}{w_{s'}} \times a_{k,s'} \left( a \frac{w_{s'}}{w_s}, z, \varphi \right).
$$
Lemma 1 has two crucial implications: distributions (over assets and bequests) and policy functions (for consumption and assets) scale in permanent income. For instance, fix an age \( k \), an income state \( z \), and a bequest indicator \( \varphi \). Lemma 1 shows that an agent with skill \( s \) and asset position \( a \) consumes exactly \( w_s / w_{s'} \) times as much as an agent with skill \( s' \) and asset position \( a w_{s'}/w_s \). An interesting implication of this is that the distribution of MPCs is the same for each skill \( s \).\(^{26}\) This is an immediate consequence of differentiating the equation for \( c_{k,s}(a, z, \varphi) \) in Lemma 1 with respect to \( a \). I state and prove this result formally in Appendix 1.C.2.

Lemma 1 can be used to derive testable predictions based on micro-level consumption behavior. Proposition 1 does this for the relationship between individual consumption and permanent income.

**Proposition 1** (Linear consumption function). Under Assumptions 1-4, in any equilibrium, each agent \( i \) with age \( k \) has a linear consumption function in permanent income, that is, in logs

\[
\log c_{ik} = \text{const}_k + \log w_{s(i)} + \epsilon_{ik},
\]

where \( \text{const}_k \in \mathbb{R} \) and \( \mathbb{E}[\epsilon_{ik}|k, s] = 0 \). Moreover, the agent’s after-tax income process satisfies

\[
\log y_{ik} = \text{const}_k + \log w_{s(i)} + \tilde{\epsilon}_{ik},
\]

where \( \text{const}_k \in \mathbb{R} \), and \( \mathbb{E}[\tilde{\epsilon}_{ik}|k, s] = 0 \).

Proposition 1 motivates a simple log-linear specification that I use as a basis for testing linearity in Section 1.4. The next result directly follows from Lemma 1 as well.

**Proposition 2** (Consumption and wealth inequality under linearity). Under Assumptions 1-4, in any equilibrium, the variances of log consumption and log wealth move one-to-one with the variance of log permanent income,

\[
\begin{align*}
\text{Var}_{i,k} \log c_{ik} &= \text{const} + \text{Var}_s \log w_s \\
\text{Var}_{i,k} \log (a_{ik} + y_{ik}) &= \text{const} + \text{Var}_s \log w_s,
\end{align*}
\]

for all \( t \), where the constants are independent of the distribution of permanent incomes \( \{\log w_{s(i)}\} \).

Despite its simplicity, this is a striking result, especially in light of the recent U.S. experience. While there is still some debate about how much consumption inequality rose compared

\(^{26}\)There is some evidence that MPCs decrease with education (not conditioning on assets \( a \)), see Jappelli and Pistaferri (2006, 2014).
to income inequality (Attanasio and Pistaferri, 2016), there is clear evidence that wealth inequality has significantly outpaced income inequality in recent decades (Piketty and Saez, 2003; Saez and Zucman, 2016).

So far, I have focused on the micro predictions of the model, which are entirely independent of the supply side of the economy. I now turn to the macro predictions. To do this, I consider shifts in the distribution of labor income $\{\gamma_s\}$. This leads to the following general equilibrium result.

**Proposition 3 (Neutrality).** Suppose Assumptions 1-4 hold. Then, aggregate consumption and savings are linear functions of the average permanent income $E_s w_s$,

$$C = \kappa_C \times E_s w_s \quad \text{and} \quad A = \kappa_A \times E_s w_s$$

where $\kappa_A, \kappa_C > 0$ are two constants that do not depend on $\{\gamma_s\}$. It follows that any redistribution of permanent incomes through a change in labor income shares $\{\gamma_s\}$ leaves all aggregate quantities unchanged. This means that the distribution of permanent incomes is irrelevant for aggregate consumption, savings, investment, tax revenues, bequests, asset prices, and the interest rate.

The intuition behind this result is straightforward given the discussion of the previous subsection. Any individual’s consumption is linear in permanent incomes $w_s$, so the distribution of $w_s$ is irrelevant for aggregate consumption and savings. Therefore, all aggregate quantities are unchanged in general equilibrium.

**1.3.5 Discussion**

These results are an example of an exact aggregation result. In essence, the Engel curves for consumption in different time periods are linear in permanent income (and symmetric across agents). This allows one to treat the economy as if there were only a single skill type earning the average permanent income. The focus on permanent incomes—that is, individual fixed effects—distinguishes this result from previous aggregation results: in Constantinides and Duffie (1996) there are only permanent shocks (here income shocks are very general), and there is no trade in equilibrium (whereas here there is). The approximate aggregation result in Krusell and Smith (1998) is about the asymptotic linearity of the consumption function out of assets for large levels of assets, not the linearity of consumption as a function of permanent income—in fact, in the above economy, consumption can be an arbitrarily-curved function of assets and still be linear in permanent income.
The linearity result has been stated in a fairly general way, but not as general as possible. Similar results hold with progressive tax systems, endogenous labor supply, habit formation, aggregate risk, or non-separable preferences (e.g. Epstein-Zin preferences).

There are a few limitations to the linearity result, however. First and foremost, this is a long-run result. If changes in the distribution of skill prices hit currently-living generations in mid-life, rather than only affecting new cohorts, there will be a period of adjustment to the new set of skill prices. In Section 1.6 below, I explore this effect quantitatively and find it to be negligible, even for large changes in the income distribution, such as a sudden movement from the U.S. distribution in 1970 to the U.S. distribution in 2014.

Second, the result also no longer applies when initial assets or borrowing constraints are nonzero and do not scale with an agent’s permanent income level \( w' \). Again, I explored these departures numerically, and they have only marginal effects on the validity of the linearity proposition. I discuss other limitations and how they affect my tests of the proportionality hypothesis in Section 1.4.4.

Finally, I assume joy-of-giving preferences. If, instead, one assumes altruistic preferences, and relaxes Assumption 3, introducing imperfect skill persistence, the result no longer holds exactly. In that case, altruistic preferences introduce two reasons for non-neutrality: first, parents treat bequests as a luxury good since the higher their own permanent income, the relatively lower their child’s permanent income is expected to be, inducing parents to save more. In an economy with a joy-of-giving utility \( U \), this would correspond to bequests being treated as a luxury good, and will be an integral part of the quantitative model in Section 1.5. Second, altruism also allows rising permanent income inequality to directly affect savings behavior due to greater precautionary savings. This second feature cannot be informed by the empirical exercise in this chapter, and is not a feature of models with an ad-hoc joy-of-giving utility \( U \).

---

27 This works as long as post-tax incomes are a power function of pre-tax incomes, which holds relatively well in U.S. data. See, e.g. Benabou (2000, 2002) and Heathcote et al. (2017), as well as Appendix 1.D.7 of this chapter.

28 Especially for borrowing constraints, it seems natural that they would scale in one’s permanent income level.

29 For evidence on the strength of this channel, see the recent work by Boar (2017). For changes in income inequality in an infinite-horizon economy, which can be thought of agents linked by altruistic bequest motives, see Auclert and Rognlie (2017). For quantitative OLG models with altruism, see Castaneda et al. (2003); Cagetti and De Nardi (2006, 2009).
1.4 Testing for Neutrality

I introduced a general, neutral model in the previous section. In this section I test for neutrality in the data. Inspired by Proposition 1, the goal is to estimate the following system of equations,

$$\log c_{it} = X_{it}'\beta + \phi \log w_{s(i)} + \epsilon_{it} \quad (1.10a)$$

$$\log y_{it} = \hat{X}_{it}'\tilde{\beta} + \log w_{s(i)} + \eta_{it} + \psi_{it} + \nu_{it}. \quad (1.10b)$$

Here, $c_{it}$, $y_{it}$, $w_{s(i)}$ denote current consumption, current income and permanent income as before, $X_{it}$, $\hat{X}_{it}$ are sets of controls, $\eta_{it}$ is a persistent income shock, $\psi_{it}$ is a transitory income shock, and $\nu_{it}$ is measurement error. The key question is whether $\phi$, the permanent income elasticity of consumption, is equal to 1 or not.

To answer this question, I pursue three separate approaches: an OLS approach, where each household’s permanent income level is computed as a symmetric average over log residualized incomes (Section 1.4.2); and two IV approaches, where each household’s current income level is instrumented with two different predictors of permanent income (Section 1.4.3). In Section 1.4.4 I discuss at length various potential weaknesses of my empirical designs and provide robustness checks (see also Appendix 1.D.1). Finally, I discuss the economic significance of the estimated value of $\phi$ using a simple partial equilibrium consumption function framework in Section 1.4.5.

It is important to stress that my analysis is not about assessing the relationship between changes in consumption and changes in permanent or persistent income. Instead, my analysis focuses on the dependence of the consumption level on a given level of permanent income.

Throughout this section, I will denote by $\hat{y}_{it} \equiv \log y_{it} - \hat{X}_{it}'\tilde{\beta}$ log income residuals after partialing out observable controls $\hat{X}_{it}$; by $\hat{w}_{i} \equiv \log w_{s(i)}$ log permanent income residuals of agent $i$; and by $\hat{c}_{it} \equiv \log c_{it}$ log consumption.

1.4.1 Data description

Overview. I use data from the 1999 – 2013 waves of the Panel Study of Income Dynamics (PSID). The PSID began in 1968 and is currently the longest running longitudinal household survey in the world. Its initial sample consisted of 5,000 households, of which 3,000 (the

\[30\] All variables will be formally defined below.

\[31\] Temporary responses to income shocks and the degree of insurance against such shocks are the subject of extensive research, recent papers include Arellano et al. (2017), Blundell et al. (2008), Blundell et al. (2016), Guvenen and Smith (2014), Heathcote et al. (2014), and Kaplan and Violante (2010), among many others.
“Survey Research Center”, or SRC, sample) were chosen as a representative sample of the U.S. population at the time. Since 1968, researchers have followed both these households and their children's “split-off” households. The survey was conducted annually until 1996, and biennially since 1997. It is known for having comparatively low attrition rates and relatively high response rates (Becketti et al., 1988; Andreski et al., 2014).

Consumption expenditure data. Until 1997, the PSID only collected information on specific consumption categories (food, housing and childcare). Since 1999, however, the PSID now collects a much wider set of consumption expenditure data that captures around 70% of the expenditures surveyed in the Consumer Expenditure Survey (CEX) and in the U.S. National Income and Product Accounts (NIPA). These new categories include expenditures on food, housing, mortgages and rents, utilities, transportation and vehicles, education and health care. The largest categories missing in the revised consumption survey are home repairs and maintenance, household furnishing, and clothing. Those were added in a further update in 2005. Since 2005, the PSID consumption data captures almost all categories of the CEX (Andreski et al., 2014).

In the analysis below, I use the longer-running but slightly less comprehensive consumption expenditure data as my baseline measure. As I will show in Section 1.4.4 below, moving to the more comprehensive (since 2005) consumption measure has only a minor effect on my results. In my baseline measure, I include all available expenditure categories, including durable goods. Since mortgage payments are the sum of imputed rents and accumulation of housing wealth, which is a form of saving, I replace them by imputed rent, as computed by the PSID. I include (non-housing) durable goods since, under analogous assumptions to the ones in Section 1.3.4, durable goods purchases would scale linearly in permanent income. Still, I also show results for non-durable consumption expenditure below.

Income data. As the income variable in my baseline regressions, I use post-tax household labor income. This is the right income concept to use for my exercise since, as the aforementioned literature on various channels of partial insurance convincingly argues, there exist various channels which may be operative, at least temporarily, in response to changes in permanent incomes. Among the most important such channels are the tax-and-transfer system and the labor supply of family members, both of which are accounted for by using post-tax household labor income. The income measure consists of labor income of all family members

---

32 Unlike the CEX, however, the post-1999 PSID consumption data does not suffer from a downward trend relative to the PCE (see, e.g. Blundell et al. (2016)).

33 My results are very similar if computing imputed rents as 6% of the house price, as do Blundell et al. (2016) and Poterba and Sinai (2008). When using the comprehensive post-2005 consumption measure, I exclude home repairs and maintenance costs since these are investments.
minus taxes (computed using NBER’s TAXSIM program). I discuss below robustness with respect to alternative income measures—most prominently, with respect to using an after-tax total income measure which includes all forms of capital income as well as private and public transfers.

Sample Selection. My baseline sample includes all PSID waves from 1999 to 2013, and consists of all households whose head is between 30 and 65 years old. I exclude households without a single non-missing consumption and income observation, as well as extreme observations, with income below 5% of the yearly average income. The PSID waves prior to 1999, when no broad consumption measure is available in the dataset, will only be used for their income data. My baseline sample consists of 5,881 distinct households with at least one observation. I discuss several alternative sample choices in Section 1.4.4. Throughout, I use PSID’s post-1999 longitudinal sample weights.

Controls. In my benchmark specifications, I use as controls $X_{it}$ in the income equation the household head’s five-year age bracket, dummies for household size, and year dummies; and as controls $X_{it}$ in the consumption equation the same controls plus a location dummy to capture heterogeneity in living costs. The results are robust to several other sets of controls, see Section 1.4.4.

1.4.2 A first look at the data

Motivated by Proposition 1, I start by showing results for specifications in which log permanent income $\hat{w}_t$ is proxied for by a simple income average. Even if these specifications turn out to be biased under the neutrality assumptions of the benchmark model (see Section 1.4.3), they are intuitive and set the stage for the more formal econometric investigation in the next section. Consecutive observation periods $t, t+1$ are two years apart in the PSID sample.

I use symmetrically-averaged income residuals as proxies for permanent income, constructed as

$$\bar{y}^T_{it} = \frac{1}{T} \sum_{\tau=-(T-1)/2}^{(T-1)/2} \hat{y}_{i,t+\tau},$$

where $T$ is the odd number of incomes being averaged. When $T = 1$, $\bar{y}^T_{it}$ is equal to current income residuals $\hat{y}_{it}$. When $T > 1$, it is an average of $T$ income observations over $2T - 1$

---

34 The location dummy is constructed as the interaction of an urban-rural dummy and dummies for the nine Census divisions.

35 See Kopczuk et al. (2010) for similar symmetrically-averaged income measures.
Note. The graph shows consumption and average income in logs for the baseline sample of PSID households. To construct it, log consumption is regressed on controls (year, age, household size, location) and 50 bins for average log income residuals. Log income residuals are obtained by partialing out year, age, and household size dummies and then averaged over a symmetric 9-year interval for each household \((T = 5)\). The blue line is the estimated linear relationship with slope \(\phi\), the red line is the 45° line.

years, due to the biennial nature of the sample. The OLS specification to test (1.8) is then

\[
\hat{c}_{it} = X'_{it}\beta + \phi \bar{y}_{it}^T + \epsilon_{it}. \tag{1.11}
\]

For long averages, \(T \to \infty\), under a suitable law of large numbers for the income processes \(\eta_{it}, \psi_{it}\), the average income residuals \(\bar{y}_{it}^T\) are measuring \(\bar{w}_i\) without noise. In that case, the neutral model in Section 1.3.4 would correspond to \(\phi = 1\), implying a linear consumption function in permanent incomes. Since \(T\) is finite, however, one can generally expect elasticities \(\phi\) below 1, even if the assumptions of the neutral model are satisfied (see Section 1.4.3). An alternative way to construct a proxy for permanent income is as income fixed effects. In this case, household \(i\)'s permanent income proxy is an average over all of \(i\)'s observed income residuals \(\bar{y}_{it}\).

Results. To avoid relying on functional form assumptions, Figure 1-4 shows the results of a non-parametric version of (1.11) spanning 9 years \((T = 5)\). Specifically, it shows the results of a regression of \(\hat{c}_{it}\) on controls and 50 bins of \(\bar{y}_{it}^T\). Two observations are immediate: the relationship is almost exactly linear in logs, and its slope is significantly below 1. I further investigate this in Table 1.1 using the linear specification (1.11). Columns 1–3 show the
Table 1.1: OLS specifications with various proxies for permanent income.

<table>
<thead>
<tr>
<th>log household consumption</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>log income</td>
<td>0.396</td>
<td>0.547</td>
<td>0.645</td>
<td>0.561</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>Year FE, Age FE</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Hh.size FE</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Location FE</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>24994</td>
<td>7979</td>
<td>2050</td>
<td>2644</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.47</td>
<td>0.56</td>
<td>0.60</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Note. This table shows results from OLS regressions with various proxies of permanent income as regressors. In column 1, the regressor is current log income (residuals), in column 2 (3) it is log incomes averaged over 9 (17) years. Column 4 shows the results from a fixed effects regression, where log consumption fixed effects are regressed against log income fixed effects. Standard errors are corrected for heteroskedasticity and clustered by household.

results of specification (1.11) for different values of $T$. It is evident that longer averages push up the estimated $\phi$, likely by reducing the downward bias. For the largest $T$ shown, $T = 9$ income observations are averaged across 17 years—around half of an entire work-life. Even then, the estimated elasticity $\phi$ is around 0.65—far from 1. Column 4 shows results from a specification where (1.11) is implemented using fixed effects, regressing each household’s consumption fixed effect on its income fixed effect, for the subsample of households with at least 5 non-missing income and consumption observations. The estimated $\phi$ is similar to the estimate for $T = 5$.

1.4.3 Econometric approach

I now investigate the possible biases in the OLS specifications more formally and propose solutions. The economy consists of a set of households $i \in I$, of which each $i$ enters the labor market at time $t_0(i)$, with initial age $k = 1$, and is observed until age $k = K_{death} > 1$. As
before, their consumption and income processes are governed by,

\[
\begin{align*}
\dot{c}_{it} &= \phi \hat{w}_{it} + X'_{it} \beta + \epsilon_{it} & (1.12a) \\
\dot{y}_{it} &= \hat{w}_{it} + \eta_{it} + \psi_{it} + \nu_{it}, & (1.12b)
\end{align*}
\]

where \( t \in t_0(i) + \{0, 1, \ldots, K_{\text{death}} - 1\} \). I make the following baseline assumptions on the model (1.12), all of which are satisfied in the neutral model of Section (1.3). First, all random variables in (1.12) are iid across households \( i \). Second, measurement error \( \nu_{it} \) is iid over time and uncorrelated with consumption, \( \text{Cov}(\epsilon_{it}, \nu_{it}) = 0 \). Third, permanent income \( \hat{w}_{it} \) and the controls \( X_{it} \) are uncorrelated with the income shocks \( \eta_{it}, \psi_{it} \), measurement error \( \nu_{it} \) and the consumption error term \( \epsilon_{it} \). Fourth, future transitory income shocks \( \psi_{it+r} \) are uncorrelated with current consumption, that is, \( \text{Cov}(\epsilon_{it}, \psi_{it+r}) = 0 \) for \( r > 0 \); and past transitory income shocks \( \psi_{it-r} \) are positively correlated with current consumption, that is, \( \text{Cov}(\epsilon_{it}, \psi_{it+r}) \geq 0 \) for \( r \leq 0 \) (positive income shocks in the past only raise consumption going forward, all else equal).

The critical assumptions are the third and fourth: the third requires permanent incomes \( \hat{w}_{it} \) to be uncorrelated with \( \epsilon_{it} \), ruling out the presence of unobserved heterogeneity in savings preferences that could be correlated with \( \hat{w}_{it} \). And the fourth assumption requires the unforecastability of transitory income shocks by the agent. Both assumptions are discussed in Section 1.4.4.

The two biases of OLS regressions. Having introduced these assumptions, it is possible to investigate the biases of OLS regressions. As an example, consider a simple OLS regression of \( \dot{c}_{it} \) on current income \( \dot{y}_{it} \) (and controls), corresponding to \( T = 1 \) in the previous section (column 1 of Table 1.1). It is straightforward to show that

\[
\begin{align*}
\text{plim}_{N \to \infty} \hat{\phi}_{\text{OLS}} &= \phi - \frac{\text{Cov}(\epsilon_{it}, \eta_{it} + \psi_{it})}{\text{Var}(\eta_{it} + \psi_{it})} \frac{\text{Var}(\eta_{it} + \psi_{it})}{\text{Var}(\dot{y}_{it})} \\
\text{plim}_{N \to \infty} \hat{\phi}_{\text{OLS}} &= -\frac{\text{Var}(\nu_{it})}{\text{Var}(\dot{y}_{it})} < 0 \text{ (consumption smoothing bias)} < 0 \text{ (attenuation bias)}
\end{align*}
\]

There are two possible biases in OLS: the first is what one may call “consumption smoothing bias”, since it is nonzero precisely when the agent’s consumption reaction to income shocks—captured by the slope coefficient \( \text{Cov}(\epsilon_{it}, \eta_{it} + \psi_{it})/\text{Var}(\eta_{it} + \psi_{it}) \)—is different from the reaction to permanent income—captured by \( \phi \). In most reasonable models of consumption behavior, the former is less than the latter due to consumption smoothing, inducing a natural downward bias in the OLS estimate. The second bias is standard attenuation bias due to the presence of measurement error in income.
It turns out that the consumption smoothing bias is rather hard to overcome. Simple instruments, such as future or lagged incomes are able to eliminate attenuation through measurement error, but due to the presence of persistent income shocks \( \eta_{it} \), the consumption smoothing bias remains.\(^{36}\) I now propose novel IV strategies that eliminate both biases, using additional assumptions on the autocorrelation structure of the persistent income shocks \( \eta_{it} \).

**ARMA process for \( \eta_{it} \).** Assume \( \eta_{it} \) follows a (stationary) ARMA\((p, q)\) process, that is, one can express the process as

\[
a(L)\eta_{it} = b(L)\epsilon^q_{it},
\]

where \( a \) is a polynomial of order \( p \), \( b \) is a polynomial of order \( q \), and \( L \) denotes the lag operator. One implication of stationarity is that \( a(1) \neq 0 \). Again, assume that the agent cannot foresee future innovations \( \epsilon_{it}^q \), that is, \( \text{Cov}(\epsilon_{it+\tau}^q, \epsilon_{it}) = 0 \) for \( \tau > 0 \) (see Section 1.4.4 for a discussion). A standard example of such a process is an AR\((1)\) process with persistence parameter \( \rho < 1 \), which case \( a(L) = 1 - \rho L \) and \( b(L) = 1 \).

Define the process

\[
z_{it} = a(L)\dot{\eta}_{it}.
\]

By construction, \( z_{it} \) is independent of realizations of the persistent shock \( \eta_{it} \) that lie more than \( q \) periods in the past. Indeed, one can express \( z_{it} \) as

\[
z_{it} = a(1)\dot{\psi}_i + a(L)(\dot{\psi}_i + \nu_i) + b(L)\epsilon^q_{it}.
\]

This shows that since \( a(1) \neq 0 \), \( z_{it} \) is correlated with \( \dot{\psi}_i \), yet uncorrelated with \( \epsilon_{it-\tau} \) for \( \tau > \max\{p, q\} \). Thus, any future \( z_{it+\tau} \) with \( \tau > \max\{p, q\} \) is a valid instrument for current income \( \dot{\eta}_{it} \) in a regression of consumption \( \dot{c}_{it} \) on current income \( \dot{y}_{it} \). When \( \eta_{it} \) is an AR\((1)\) process, the instrument is simply given by quasi-differenced future incomes,

\[
z_{it+\tau} = \dot{y}_{it+\tau} - \rho \dot{y}_{it+\tau-1}.
\]

Intuitively, this IV strategy combines two ideas. Using future incomes, rather than lagged incomes, is helpful because in a world without persistent income shocks, this would yield consistent estimates. The reason for this is that the actual realizations of future income shocks is not known to the agent in that case, so the only source of correlation between current and future incomes is the permanent component \( \dot{\psi}_i \). When there is a persistent income shock, however, it needs to be differenced out first, which is precisely the role of quasi-differencing.

\(^{36}\)For instance, some of the specifications Mayer (1972) and Dynan et al. (2004) used instruments along these lines.
Together, quasi-differenced future incomes are a valid instrument for \( \hat{w}_t \) under the ARMA assumptions above.

**Random walk \( \eta_{it} \).** A downside of this approach is that it is only consistent for non-unit-root processes. Indeed, if \( a(1) \) were equal to zero, the instrument \( z_{it} \) would be independent of \( \hat{w}_t \) altogether. A common formulation of the persistent shock \( \eta_{it} \), however, is as a random walk.\(^{37}\)

I consider this case now. In particular, assume that

\[
\eta_{it} = \eta_{i(t-1) + \epsilon_{it}}.
\]

Let \( z_i \) be the agent’s initial labor income when entering the labor market at time \( t_0(i) \),

\[
z_i \equiv \hat{y}_{i,t_0(i)} = \hat{w}_i + \eta_{i,t_0(i)} + \psi_{i,t_0(i)} + \nu_{i,t_0(i)}.
\]

For an arbitrary process \( \eta_{it} \), this variable is not an exogenous instrument and not even one whose asymptotic bias can be signed. However, when \( \eta_{it} \) follows a random walk, the initial persistent draw \( \eta_{i,t_0(i)} \) is indistinguishable from permanent income \( \hat{w}_i \) for the agent, and thus can be set to zero without loss of generality. This then means that the only endogenous variable in \( z_i \) is \( \psi_{i,t_0(i)} \), which I argued above is positively correlated with future errors in the consumption equation, \( \epsilon_{it} \). Therefore, if \( \eta_{it} \) follows a random walk, an IV strategy with instrument \( z_i \) provides an asymptotic upper bound of \( \phi \) which can then also be used to test neutrality.\(^{38}\)

**Results of the IV approaches.** To operationalize the approach for a stationary \( \eta_{it} \), I model \( \eta_{it} \) as an AR(1) process with annual persistence \( \rho \). I use all periods for which at least three future incomes \( z_{i,t+i} \) are observable and use all available instruments for each observation. The persistence parameter \( \rho \) is estimated in the data at a similar horizon (15 years) as the maximum span of the instruments \( z_{i,t+i} \) I use in the estimation (see Appendix 1.D.4 for details). This procedure yields a moderate annual persistence of \( \rho = 0.90 \).\(^{39}\) Columns 1–6 of Table 1.2 show the results of a two-stage-least-squares estimation for various choices of \( \rho \) around the estimated level of \( \rho = 0.9 \) as well as \( \rho = 0 \).

Overall, a consistent picture emerges. While estimates do increase with larger choices of

\(^{37}\)See the recent survey by Meghir and Pistaferri (2010). The method proposed here can be extended to include the case where the transitory shock \( \psi_{it} \) follows an MA process, rather than being iid over time (MaCurdy, 1982; Abowd and Card, 1989).

\(^{38}\)In my simulations in Section 1.5 I will show that this upper bound is generally very tight in models with random walk income processes and provides a close upper bound even in models without random walk income processes.

\(^{39}\)This persistence estimate lies between the typical estimates of 0.95 – 1.00 for “restricted income profile” income processes and of 0.80 – 0.85 for “heterogeneous income profile” estimates.
Table 1.2: IV specifications for time-averages and group-averages approach.

<table>
<thead>
<tr>
<th>log household consumption</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = 0 )</td>
<td>0.599</td>
<td>0.675</td>
<td>0.686</td>
<td>0.699</td>
<td>0.717</td>
<td>0.741</td>
<td>0.732</td>
</tr>
<tr>
<td>( \rho = 0.88 )</td>
<td>(0.02)</td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.05)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>( \rho = 0.89 )</td>
<td>0.686</td>
<td>0.669</td>
<td>0.717</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
<td>0.732</td>
</tr>
<tr>
<td>( \rho = 0.9 )</td>
<td>0.699</td>
<td>0.717</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
</tr>
<tr>
<td>( \rho = 0.91 )</td>
<td>0.717</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
<td>0.732</td>
</tr>
<tr>
<td>( \rho = 0.92 )</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
<td>0.732</td>
<td>0.741</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year FE, Age FE</th>
<th>Yes</th>
<th>Yes</th>
<th>Yes</th>
<th>Yes</th>
<th>Yes</th>
<th>Yes</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hh.size FE</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Location FE</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

| Observations    | 7158 | 7158 | 7158 | 7158 | 7158 | 7158 | 21642 |
| 1st stage \( F \) | 537.6 | 59.2 | 49.5 | 40.2 | 31.5 | 23.4 | 101.5 |
| R-squared       | 0.48 | 0.45 | 0.44 | 0.44 | 0.43 | 0.41 | 0.41 |

Note. Columns 1-6 show IV results with \( \rho \)-differenced future incomes as instruments, for various choices of \( \rho \). Column 7 shows results when the autocovariances of log income residuals are used without assuming a parametric form for income shocks. Standard errors are corrected for heteroskedasticity and clustered by household.

Persistences \( \rho \) (which they are expected to—see the discussion on misspecification of \( \rho \) in the next section), they fall between 0.60 and 0.75. Importantly, for all specifications, the F-statistics are above 10, suggesting that there is no weak instruments problem for those values of \( \rho \).\(^{40}\)

In column 7 of Table 1.2, I show the results of the second IV approach using as IV the household’s labor income at the head’s age of 25. The result is consistent with this estimate providing an upper bound, even if the underlying income process may not exactly be a random walk.

1.4.4 Discussion and robustness

There are a variety of concerns one might have about the specifications in Section 1.4.3. In this subsection, I address a few of the most important. Appendix 1.D.1 executes a number of additional robustness checks, including among others: a specification that controls for cash on hand; group-level specifications, similar to those in the seminal work of Attanasio and Davis (1996); and specifications using the imputed consumption measure in Blundell et al. (2008). The regression results for each of the following robustness exercises appear in Table 1.3. To facilitate comparison with my previous results, the first row shows baseline estimates taken from Table 1.1, column 3 and Table 1.2, columns 4 and 7.

\(^{40}\) Values of \( \rho \) beyond around 0.94 (annual), however, do generate significant weak instrument problems.
### Table 1.3: Robustness checks.

<table>
<thead>
<tr>
<th></th>
<th>OLS with $T = 9$</th>
<th>IV with $\rho = 0.90$</th>
<th>IV with initial income</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Baseline</td>
<td>0.64</td>
<td>0.70</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>2. Education and preference controls</td>
<td>0.58</td>
<td>0.69</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.07)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>3. Controlling for positive business wealth</td>
<td>0.55</td>
<td>0.68</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>4. HIP – education specific trends</td>
<td>0.59</td>
<td>0.68</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.06)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>5. Comprehensive consumption</td>
<td>0.66</td>
<td>0.71</td>
<td>0.77</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>6. Education as IV</td>
<td></td>
<td></td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.03)</td>
<td></td>
</tr>
<tr>
<td>7. Total post-tax income</td>
<td>0.69</td>
<td>0.71</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>8. Nondurable consumption</td>
<td>0.58</td>
<td>0.59</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>9. Aguiar-Bils relative expenditures</td>
<td>0.62</td>
<td>0.72</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.10)</td>
<td>(0.15)</td>
</tr>
</tbody>
</table>

**Note.** This table lists OLS and IV estimates for 8 different specifications. Row 1 shows the baseline specifications. Row 2 adds controls for several proxies for preferences (education, race, sex, and self-reported bequest intentions). Row 3 adds a dummy for positive business wealth. Row 4 controls for education-specific household income trends. Row 5 uses the comprehensive post-2005 consumption measure. Row 6 uses education as instrument for current income residuals. Row 7 uses total post-tax household income as income measure. Row 8 uses non-durable consumption as consumption measure. Row 9 uses data on relative expenditures on luxuries vs necessities to achieve robustness to non-classical measurement error as in Aguiar Bils (2015). All specifications control for year, age, household size and location dummies. All IV specifications have first stage F statistics above 10. Standard errors are corrected for heteroskedasticity and clustered by household.

**Heterogeneous time or bequest preferences.** One of the most immediate concerns one may have about the specifications in Section 1.4.3 is endogeneity through preference heterogeneity that is correlated with permanent income levels. For instance, better-educated workers may be more “financially responsible” and save more conditional on a given level of permanent income. Since educated workers usually earn higher incomes, this could lead to a downward bias in $\tilde{\phi}^{IV}$.

To address these concerns, I rerun the baseline OLS specification from Table 1.1, column 2 as well as the IV specification from Table 1.2, column 3, with additional controls that proxy for factors influencing savings preferences: the household head’s education, race, sex.

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41The results are very similar for the other OLS and IV specifications.
Additionally, I include the household head’s reported bequest intention, which was elicited in the 2007 wave of the PSID. While these are merely proxies for preferences, one can expect that if preference heterogeneity is an important confounding factor in my specifications, some of it should be picked up by my controls.

The second row in Table 1.3 shows that both the OLS and IV estimates are very similar to the baseline estimates in magnitude, and the IV estimates differ insignificantly from the baseline estimates. These results seem counterintuitive at first. It is worth reiterating that they hold conditional on permanent income: unconditionally, households with a college-educated household head do save significantly more. However, as my results suggest, this is almost entirely driven by the fact that such households can “afford” to save due to higher earnings, and not because they are more frugal than their less educated counterparts.

Observe that even if unobserved preference heterogeneity were confounding my estimates, this would not resurrect the neutrality result in Section 3. I demonstrate this point in Appendix 1.D.9 in a setting where unobserved heterogeneity in preferences is the sole cause of \( \hat{\phi} < 1 \).

**Heterogeneous returns on wealth.** A recent literature powerfully demonstrates that returns on wealth are very heterogeneous in the population and generally increase with wealth (Fagereng et al., 2016). One relative advantage of my specification based on consumption expenditure (rather than wealth differences or wealth-to-income ratios) is that its results cannot be “mechanically” driven by heterogeneity in returns that is correlated with income. The effect of such heterogeneity on my results generally depends on the elasticity of intertemporal substitution (EIS): if the EIS is below 1, as is typically assumed in precautionary savings models, it may well be that high-return agents consume more out of their labor income, not less.

Since return heterogeneity is hard to disentangle from noise in the PSID, I use as a simple proxy whether a household has positive business wealth or not. The results appear in the third row of Table 1.3 and fall slightly below the baseline numbers.

**Partial insurance.** One may wonder how the presence of partial insurance arrangements affect my results. First, notice that many insurance channels, such as government tax and

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42 The question asked was: “Some people think that leaving an estate or inheritance to their children or other relatives is very important, while others do not. Would you say this is very important, quite important, not important, or not at all important?” Answers were given on a numeric scale from 1 (very important) to 4 (not at all important).

43 These results do not imply that preferences over the type of consumption are entirely independent of education and the other aforementioned proxies. For instance, compared to households headed by a high school dropout, I find that college-educated households tend to spend less on consumption categories such as transportation or housing, but spend significantly more on education, health and food (for a given level of permanent income).
transfers as well as total family labor supply, are already taken into account in the post-tax income measure that I use. Still, there may be informal insurance among households that I cannot observe. In a recent paper, Guvenen and Smith (2014) present an explicit model of partial insurance, where an agent hit by an innovation \( c' \) to the persistent shock \( \eta \) receives an unobserved transfer \(-\theta e\) that mitigates the shock, \( \theta \in (0, 1) \). This does not affect the two IV strategies, since neither relies on any specifics of the income shock other than the persistence parameter.

**Heterogeneous income profiles and advance information.** A recent literature argues that income age profiles may have heterogeneous slopes (Guvenen, 2007, 2009; Guvenen and Smith, 2014; Guvenen et al., 2016), and more generally, agents may have advance information about future income realizations (Primiceri and van Rens, 2009). The presence of these features do not affect my initial income IV results when \( \eta \) follows a random walk. When \( \eta \) follows an AR(1), this is likely to bias \( \hat{p} \) upwards: when an agent expects greater future income growth, then, all else equal, the consumption \( c \) will be higher.

A straightforward way in which the robustness of my results with respect to heterogeneous income profiles can be explored is to allow in the controls \( X \) and \( X \) for heterogeneity in income profiles based on observables. This is exactly what I do in the fourth row of Table 1.3 by adding an interaction of an age trend with education to \( X \). The results are shifted downward—as one may have expected—but not by a whole lot, at least for this rough measure of heterogeneous income profiles. In fact, the IV estimate is insignificantly different from the baseline estimate.

**Misspecification of \( \rho \).** A similar logic carries over to the case where the econometrician specifies the wrong \( \rho \) in the construction of the AR(1) IV. For instance, if the persistence parameter chosen by the econometrician \( \rho \) is too large relative to the true persistence, say \( \rho^* \), the IV estimate \( \hat{p} \) tends to be biased upwards. In that case, the \( \rho \)-difference \( \hat{y} - \hat{y}^{\rho \rho} \) contains the term \( -\left( \rho - \rho^* \right) \eta \), which is positively correlated with the error term \( \epsilon \), as agents with high income shocks \( \eta \) tend to save more and consume less out of current income. The bias goes in the opposite direction if \( \rho > \rho^* \). This explains why the estimates in Table 1.2 tend to be smaller for smaller \( \rho \)’s.

**Comprehensive consumption.** As mentioned in Section 1.4.1 when describing the dataset, the PSID introduced a somewhat more comprehensive consumption measure in 2005 by adding six consumption categories. Since my OLS and AR(1) IV specifications rely on future income data, the consumption data that is effectively used spans the years 1999–2005 and thus barely overlaps with the new measure. To have a meaningful comparison between the two consumption measures, I impute the missing comprehensive measure before 2005 by
estimating a set of demand equations for each of the new consumption categories on post-2005 data (see Appendix 1.D.3 for details). The fifth row of Table 1.3 shows the results. The OLS and the AR(1) IV estimates are close to their baseline counterparts, and the estimate for the initial income IV is somewhat larger.

*Education as instrument.* Row 6 of Table 1.3 shows IV results when education is used as the instrument. This is a common instrument in the previous literature (Dynan et al., 2004; Bozio et al., 2013). The estimate for \( \phi \) is slightly larger than that of my baseline IV specification.

*Total post-tax income.* My specifications so far used a measure of post-tax labor income, not including capital income. In row 7 of Table 1.3, I use a total post-tax, post-transfer income measure, which includes capital income. The results are similar.

*Non-durable consumption.* Row 8 of Table 1.3 shows results for nondurable consumption only (defined as the sum of food expenditure, rent, property taxes, home insurance expenditure, utilities, transportation, education, childcare and health-related expenditures). The somewhat lower estimates suggest that including durable consumption is important since permanent-income richer households seem to spend a disproportionate amount on those expenditures.

*Non-classical measurement error.* In a recent paper, Aguiar and Bils (2015) argue that there is non-classical measurement error in the interview survey of the Consumer Expenditure Survey (CEX), in the sense that richer respondents are more likely to underreport their spending. Aguiar and Bils (2015) show that this matters for the measurement of consumption inequality. There are two reasons why this may be less of a concern for this chapter: Compared to the CEX, there is no evidence of a downward trend in the coverage of the PSID consumption expenditure data (Blundell et al., 2016); and, in the present study, the estimate for \( \phi \) is only affected if consumption is relatively more underreported than income.

Nevertheless, I use several strategies to deal with this issue. First, I present estimates using three different consumption measures: the post-1999 PSID measure, the comprehensive post-2005 measure, and the BPP imputed measure (see Appendix 1.D.1). Second, I re-estimated the three main specifications using the PSID's post-1999 wealth data, finding a permanent income elasticity of wealth of approximately 2 – 3 (depending on the exact specification and measure of wealth). This is significantly greater than 1 (neutrality) and is in line with the analogous estimate of 2 in the non-homothetic model of Section 1.5. And finally, I applied the method proposed by Aguiar and Bils (2015) to this chapter. Using their estimates for expenditure elasticities, I back out total expenditure measures along the income distribution from the ratios of luxury expenditures to expenditures on necessities (for

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44See Blundell et al. (2008) or Boar (2017) for recent examples that employ similar imputation strategies.
details, see Appendix 1.D.2). The resulting estimates are shown in row 9 of Table 1.3 and are comparable with the baseline estimates.

1.4.5 Is the degree of concavity economically significant?

The evidence in the previous sections suggests that the permanent income sensitivity of consumption $\phi$ is around 0.70, and significantly below 1 in a statistical sense. This raises the obvious question of whether an estimate of 0.70 is also economically significantly different from 1. While this question is central to my quantitative general equilibrium analysis in Section 1.5, a simple “back of the envelope” calculation—inspired by the simple model in Section 1.2—already informs this question.

To operationalize this approach, suppose there is a continuum of one-period-lived households $i \in [0, 1]$, with permanent income levels $y_i$ and an ad-hoc consumption function $C(y_i) = \kappa y_i^{\phi}$ with some $\kappa > 0$. I now characterize how aggregate consumption $C = \int C(y_i) d\nu_i$ depends on the distribution of permanent incomes $\{y_i\}$, holding aggregate income $Y = \int y_i d\nu_i$ constant.

Let $F(y)$ and $f(y)$ denote the initial cdf and pdf of the distribution of $\{y_i\}$ and $G(y)$ and $g(y)$ the new cdf and pdf, assumed to have the same mean as $F(y)$. The object of interest is the percentage change in consumption,

$$\frac{\Delta C}{C} = \frac{\int_0^\infty (g(y) - f(y)) C(y) dy}{\int_0^\infty C(y) f(y) dy}. \tag{1.13}$$

I evaluate expression (1.13) using two different data sets on the income distribution since 1980. The first is the PSID dataset used above. Here I take the income distribution to be the \pm 4 year average $\bar{y}_it$ of residual incomes $\hat{y}_it$ (net of year, age, and household size dummies), where all income observations within a nine-year window are included.\footnote{This includes 9 income observations until 1997, and 5 income observations during the biennial survey years thereafter.} The second dataset is the distribution of post-tax national income by equal-split adults as computed by Piketty et al. (2016). For details on the calculation of $\Delta C/C$ see Appendix 1.D.10.

Figure 1-5 plots the percentage annual consumption decline relative to 1980 using the two datasets. Using the PSID data I find an annual consumption shortfall due to the rise in inequality of around 3% in 2013. Using the administrative-level data, the shortfall is almost twice as large, and has reached around 5.5% in recent years. Despite the difference in magnitudes, both are on a steady downward path, with no sign of slowing down. Interestingly, 5% is the estimate obtained by Alan Krueger using an entirely different (but similarly ad-hoc)
Figure 1-5: Annual consumption shortfall due to rising income inequality according to ad-hoc model.

Note. The figure shows simulated shortfalls in aggregate consumption based on an ad-hoc consumption function with income sensitivity $\phi = 0.7$. As income data, the dashed line uses post-tax income residuals from the PSID, averaged within households in a ±4 year window; the solid line uses the post-tax income distribution from Piketty et al. (2016).

approach in a speech as Chair of the Council of Economic Advisors (Krueger, 2012). As another simple check, one can explicitly compute $\Delta C/C$ when $F$ and $G$ are Pareto distributions. As I illustrate in Appendix 1.D.8, using the measured decline in the Pareto tail coefficient since 1980, this yields a decline of 5.4%, almost identical to the one in Figure 1-5.46

1.4.6 Taking stock

The evidence strongly rejects the linearity of consumption in permanent income. Indeed, the elasticity $\phi$ seems closer to 0.7 than it is to 1. There are many reasons that could be behind this, which generally fall into one of two categories: heterogeneity in transfers or bequests received and nonlinearity in Engel curves over consumption at different times.

Two cases where the first category can generate $\phi$'s smaller than one are: a concave social security system; and if it were the case that poorer agents receive larger private transfers or bequests so that these households can consume a larger fraction of their measured permanent income. While the first case follows almost mechanically, and will be a core ingredient in the quantitative model, the second seems fairly implausible. In fact, it most likely goes the opposite way in the data, as it is usually richer children that inherit disproportionately larger estates.

Among explanations involving nonlinear Engel curves two of the most prominent are:

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46Interestingly, even if all the variation in the data were driven by unobserved preference heterogeneity, the impact of rising income inequality is still very large. See Appendix 1.D.9.
non-homotheticities in consumption and bequests; and differential income risks faced by the rich vs. the poor, or their respective children. Here, non-homotheticities broadly encompass all reasons why higher-income households have a larger marginal preference for consumption or bequests later in life. This includes many things poor agents generally cannot afford, for instance, spending on college education (or gifts) for kids or grandkids, charitable giving, or expensive medical treatments. These types of expenditures are typically back-loaded and occur late in one’s work life or in retirement.\textsuperscript{47} Rising income risk with permanent incomes could also be a powerful force that allows richer agents to save more. Recent evidence by Guvenen et al. (2016) presents a mixed picture of this issue: on the one hand, second-moment income risk generally declines with income (except at the very top of the income distribution), while left-skewness generally increases with income (except at the very top). Finally, it could also be the case that households save to insure their children against income risk, as documented in a recent paper by Boar (2017). Since an offspring’s income risk in levels is likely to scale with his or her income level, this explanation is, for the purposes of this chapter, similar to bequests being treated as luxury goods by parents.

I present evidence for the nonlinearity in Engel curves (the second channel) in Appendix 1.D.6: there, I show that estimates of the elasticity $\phi$ generally increase in age, pointing towards a more back-loaded consumption pattern; and that transfers to children—mostly to support educational expenses and home purchases—are very skewed towards high permanent income households and occur late in life, with more than half of all transfers occurring after the household head turns 70 years (conditional on survival).

1.5 A Non-Homothetic Life-Cycle Economy

I now investigate the quantitative implications of my empirical results. To do so, I modify the neutrality framework of Section 1.2 to allow for several forces pushing for a concave consumption function in permanent income. I study the steady state implications of the model in this section and simulate historical transitional dynamics in Section 1.6.

At the outset of this quantitative investigation it is by no means obvious in what ways the neutrality model of Section 1.2 should be modified in order to be consistent with the type of consumption and savings pattern that I documented. Indeed, several forces are discussed in Section 1.4.6 that could plausibly push the estimate $\phi$ into the direction observed in the data. One way to account for these forces is to model all of them separately. Yet, that quickly

\textsuperscript{47}Another channel in this category is a dependence of mortality on income which also induces richer households to save more for consumption late in life.
pushes the limits of what is currently feasible computationally and what can be calibrated independently. I therefore take a more narrow approach, in which I modify the model in as few ways as possible that still do justice to the kinds of forces at play. I focus mainly on two: a social safety net that is concave in lifetime incomes and non-homothetic preferences over consumption and bequests. As I will discuss, especially the latter is important in generating the significant deviations from the linearity of the consumption function that I found in the data.

This section proceeds as follows. In Section 1.5.1, I introduce the non-homothetic life cycle model, which then is calibrated in Section 1.5.2. In Section 1.5.3, I show the calibration results and discuss the importance of the non-homothetic savings motive. I compare my empirical estimation of $\phi$ in the non-homothetic life cycle model with a variety of other precautionary-savings models in Section 1.5.4.

Throughout this section, I will compare all results from my non-homothetic life cycle model with an (almost) neutral homothetic model that I introduce below as well. In Appendix 1.E, I provide details of the model computations used for this section, as well as additional results.

### 1.5.1 Model

The quantitative model is a version of the (general equilibrium) life-cycle precautionary-savings model introduced in Section 1.3.1, with the following modifications.

**Timing.** Each period or age corresponds to one year.

**Agents.** In the model, agents are "born" at the model age $k = 1$, corresponding to a biological age of 25 when agents enter the labor market. They have an offspring that enters the labor market at model age $k = 25$ (biological age 50). They retire at model age $K^{ret} \equiv 40$ (biological age 65), and die with certainty at model age $K_{death} \equiv 65$ (biological age 90). After retiring, all the way to the certain death, agents face a positive mortality rate $\delta_k$ from age $k$ to $k + 1$. Their pre-tax income process is stochastic subject to idiosyncratic productivity shocks and given by

$$\log y_t^{pre} = \theta_{t-t_0} + \eta_t + \psi_t$$

where $\eta_t$ is an AR(1) process with persistence $\rho$ and a standard deviation of its innovation of $\sigma_\eta$; and $\psi_t$ is iid over time, with standard deviation $\sigma_\psi$. Two initial conditions are imposed for agents. First, I assume that agents start initially with zero assets and a persistent income.

---

48 It turns out not to matter for the positive predictions of the model whether some part of what is picked up as non-homothetic preferences here is in reality induced by policies, rather than preferences, such as college tuition subsidies that decline with income. Both generate the same non-homothetic savings behavior.

49 The assumption of a zero death probability before 65 is made for computational simplicity.
shock $\eta_0$ drawn from a normal distribution with variance $\sigma^2_\eta$. Second, I assume agents inherit their parents' skill with probability $P_{\text{inherit}} \in [0, 1]$ and with probability $1 - P_{\text{inherit}}$ are assigned a random skill according to their population shares.

**Government.** The government levies income taxes and makes social security payments. In particular, retired agents earn social security payments $T_{\text{socsec}}(\bar{y})$ which are modeled with a piecewise linear schedule according to the Old Age and Survivor Insurance component of the Social Security system (see also Huggett and Ventura (2000) or De Nardi and Yang (2014))

$$T_{\text{socsec}}(\bar{y}, W) = 0.9 \min(\bar{y}, 0.2W) + 0.32 \left( \min(\bar{y}, 1.24W) - 0.2W \right)^+ + 0.15 \left( \min(\bar{y}, 2.47W) - 1.24W \right)^+.$$ 

Here, $W$ is the average labor income, and $\bar{y}$ is a measure of an individual agent’s lifetime income. Since keeping track of an agent’s actual lifetime income is computationally costly—it adds an additional state variable—I predict each agent’s lifetime income based on that agent’s last working-age income.

All agents pay income taxes according to a progressive income tax system. As in Benabou (2000) and Heathcote et al. (2017), I assume that pre-tax earnings $y^{\text{pre}}$ are taxed according to

$$T^{\text{inctax}}(y^{\text{pre}}) = y^{\text{pre}} - T^{\text{inctax}}(y^{\text{pre}})^{1-\lambda}, \quad (1.14)$$

where $T^{\text{inctax}} > 0$ is a constant and $\lambda \geq 0$ is a tax progressivity parameter: a larger $\lambda$ corresponds to more progressive income taxation. In addition, following Hubbard et al. (1995), even households with very low incomes receive basic assistance and basic health care. I capture this by a government-provided income floor of $\bar{y}$. After-tax incomes are then

$$y_t = \max\{\bar{y}, y_t^{\text{pre}} - T^{\text{inctax}}(y_t^{\text{pre}})\}.$$ 

Finally, there is a proportional tax $\tau^{\text{cap}} > 0$ on capital income. Henceforth, $r$ denotes the after-tax interest rate, $r = (1 - \tau^{\text{cap}}) r^{\text{pre}}$.

**Preferences.** I allow households to have non-homothetic preferences, both over consumption and bequests, breaking Assumption 1. In particular, I assume preferences over consumption at age $k$ are given by

$$u_k(c) = \frac{(c/z)^{1-\sigma_k}}{1 - \sigma_k}, \quad (1.15)$$

where $\sigma_k > 0$ is an age-dependent elasticity that is constant and equal to $\sigma$ during retirement,

---

50 Since agents accumulate assets as they grow older, one could easily regard this economy as one where agents are “born” at age 30, rather than at age 25, with a non-trivial initial asset position (that was accumulated over the 5 previous years).
Preferences over bequests are given by
\[ U(a) = \kappa \frac{((a + \bar{a})/z)^{1-\sigma}}{1 - \sigma}, \]

where \( \kappa > 0 \) and \( \bar{a} > 0 \). These two assumptions constitute the most important deviations from the neutrality result of Section 1.3.4 and therefore merit an extensive discussion. Both assumptions seek to capture some of the core forces behind the empirical findings in Section 1.4.

**Age-dependent elasticities \( \sigma_k \).** The key idea behind these preferences is to change income elasticities of spending across periods. To achieve this in the most straightforward and transparent way, I use “addilog” preferences that were pioneered by Houthakker (1960) among others. In a static setup, Houthakker (1960) shows that when the utility function is iso-elastic and additively separable, with power \( 1 - \sigma_k \) on good \( k \), the income elasticities \( \varepsilon_k \) are inversely proportional to \( \sigma_k \), that is, \( \varepsilon_k \sim \sigma_k^{-1} \). Therefore, goods with low elasticity \( \sigma_k \) have a high income elasticity, and are most attractive if an agent has a sufficiently large income. Moreover, in a two-good setting, the income elasticity is equal to the ratio \( \sigma_k/\sigma_{k+1} \) for high incomes (see also Section 1.2).

I apply this logic to an intertemporal context and assume that \( \sigma_k \) is lower for higher ages, capturing the fact that higher income agents seem to spend relatively more in the future, in accordance with the evidence in Section 1.4 and Appendix 1.D.6. Moreover, I assume that the ratios \( \sigma_k/\sigma_{k+1} \) are constant until retirement, with a constant that will be determined by the calibration.

These preferences have two additional implications for household behavior, aside from the changes in income elasticities they induce. First, the preferences change attitudes towards risk both over the life cycle and as a function of permanent income, pushing down risk aversion at higher ages and at higher income or wealth levels. This turns out to be a feature, not a bug: since homothetic models are well-known to predict increasing curvature in the value function with age, the non-homothetic model is able to generate curvatures much more in line with the evidence, e.g. from the age structure in portfolio allocation.\(^{51}\) Second, the preferences imply an elasticity of intertemporal substitution (EIS) that rises with income or wealth, in line with estimates by Blundell et al. (1994) who find that the EIS increases in permanent income (see Attanasio and Browning (1995) for similar findings).

**Non-homothetic bequest motive.** The second, more standard source of non-homotheticity

---

\(^{51}\)Ameriks and Zeldes (2004) show that the risky share in one’s portfolio does not decline with age. Such a decline would be predicted by a canonical homothetic life-cycle economy, since human capital acts like a bond position and declines over one’s life. An important paper explaining this fact is Wachter and Yogo (2010), who incorporate intratemporal addilog preferences over two goods (a necessity and a luxury) into a canonical life-cycle economy.
is in the form of bequests. Bequest utilities are allowed to have a different intercept than that of consumption. In particular, when \( \alpha \) is relatively large, bequests are treated as a luxury good. Thus richer agents choose to save in order to leave bequests, while poorer agents do not, or less frequently so. The idea to incorporate such preferences in a canonical life-cycle economy goes back to the seminal work of De Nardi (2004).

Finally, analogous to the preference specification in Section 1.2, I assume there is a normalization constant \( z > 0 \) in both \( u_k(c) \) and \( U(a) \).

Why do I allow for both sources of non-homothetic consumption-savings behavior, and not merely focus on non-homotheticity in bequests? As I argue below using simulations, the reason is that non-homotheticity in bequests by itself cannot quantitatively account for the concavity \( \phi \) I document in Section 1.4, without implying implausibly large bequests. This is because bequests are estimated to be in the vicinity of 5\% of GDP (see Section 1.5.2 below), limiting their quantitative role.\(^{52}\)

*Net foreign assets and net exports.* Finally, I allow agents to hold foreign assets. In particular, I incorporate the net foreign asset position in a way that allows the U.S. to earn a larger return on its assets compared to its liabilities. In particular, this means that a net foreign asset position of \( NFA \) implies a net income stream that is larger than \( \frac{r}{1+r}NFA \). I call the difference “external excess return”. The balance of payments at the steady state is then given by

\[
0 = \text{ExtExcessReturn} + NX + \frac{r}{1+r}NFA.
\]

While this part of the model is not crucial for any results, it captures returns to U.S. wealth holdings more accurately.

### 1.5.2 Calibration

Wherever possible, I calibrate the model to the U.S. economy in 2014. Even though one of my main experiments will be a comparison with 1970, it is important to calibrate the economy at roughly the same time during which I estimated the concavity parameter \( \phi \).\(^{53}\)

*Birth death skills.* I assume there are \( S = 3 \) skills with population shares of \( \bar{\pi}_s \) of 0.90, 0.09, and 0.01, capturing the bottom 90\%, the next 9\% and the top 1\%. This choice is motivated by recent increases in incomes going to top income groups (Piketty and Saez, 2003), but the

\(^{52}\)The age pattern I find in Appendix 1.D.6 also suggests that there must be an additional force for non-homotheticity in the economy, since an economy that only has a non-homothetic bequest motive would generate a permanent income elasticity \( \phi \) that declines with age.

\(^{53}\)As it turns out, however, the concavity parameter \( \phi \) is quite stable over time. I estimated \( \phi \) in the 1970 steady state from which the transitional dynamics in Section 1.6 begin and found estimates that were around 0.65 – 0.7.
results are similar using different skill groups.\textsuperscript{54} I calibrate mortality rates \( \{ \delta_k \} \) by age to the data from the Center for Disease Control for 2011.\textsuperscript{55}

Production. The gross capital share \( \alpha \) is computed for the U.S. non-financial corporate sector for 2014. to avoid issues regarding the treatment of mixed incomes.\textsuperscript{56} This yields \( \alpha = 36.7\% \). The calibration of income shares \( \gamma_s \) is explained below. Depreciation is set to a value of \( \delta = 0.07 \) in order to match the capital-output ratio in 2014 of \( K/Y = 3.05 \), using a post-tax interest rate of \( r = 3\% \) (see below). I use \( A \) to normalize GDP \( Y \) to 1. I adjust the normalization when changing the distribution of income \( \{ \gamma_s \} \) to focus on the pure redistributional effects without mechanical changes in \( Y \).

Government. Government debt \( B \) is set equal to 73\% of GDP to match the 2014 ratio of total federal debt held by the public to GDP, which excludes bond positions held by the social security trust fund and other government entities (not the Federal Reserve). It is difficult to calibrate estate taxes, since, as is well known, there exist numerous (legal and illegal) ways in which the estate tax burden can be reduced. I therefore choose to follow the literature and assume an intermediate value of \( \tau^{\text{eq}} = 0.10 \) as in De Nardi (2004). I set the income floor \( y \) to be 30\% of average household labor income, corresponding to around $10,000 per adult in 2014 dollars. This is a conservative estimate following Guvenen et al. (2016). Other choices yield almost identical results. I estimate the income tax progressivity \( \lambda \) from PSID data on pre-tax and post-tax incomes, produced by NBER’s TAXSIM program (see details in Appendix 1.D.7). For 2013, this yields \( \lambda = 0.159 \). The average income tax rate \( \tau^{\text{inctax}} \) is set to match the sum of personal tax receipts, employers’ contributions to government social insurance as a fraction of total labor income and fraction \( 0.5(1 - \alpha) \) of tax income from production and imports.\textsuperscript{57} Together this results in an average income tax of \( \tau^{\text{inctax}} = 30\% \). Capital taxes are set to cover the remaining fraction of total government receipts, giving approximately \( \tau^{\text{cap}} = 40\% \). Government spending \( G \) is set to be the residual in the government budget constraint. In the calibration, this gives a value of \( G/Y = 14\% \).

Idiosyncratic productivity process. I determine the process for the idiosyncratic productivity

\textsuperscript{54} While the PSID does cover some very high incomes (the highest pre-tax income in survey year 2013 is over $6m) the very top shares (top 1\%, and especially top 0.1\%) are underrepresented. The results here should therefore be understood conditional on the top 1\% being still described by the same relationship between consumption and income as the bottom 99\%. A model without the 1\% generates similar aggregate predictions but less wealth inequality at the top.

\textsuperscript{55} To avoid a somewhat larger mass of agents dying right at age 90 compared to other ages, I smooth mortality rates over the last ten years of the life (this has no effect on the results).

\textsuperscript{56} In particular, I compute the gross corporate labor share as the compensation of employees divided by the gross value added net of taxes of the U.S. non-financial corporate sector using data from the BEA’s National Income and Product Accounts. See also Rognlie (2015) and Barkai (2016).

\textsuperscript{57} Around 50\% of the taxes levied on production and imports is property tax income and is counted towards capital taxation. The rest is split according to capital and labor income ratios.
Table 1.4: Elasticity of per-period utility function $u$ by age.

<table>
<thead>
<tr>
<th>Age group</th>
<th>25 – 44 years</th>
<th>45 – 64 years</th>
<th>65+ years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average $\sigma$</td>
<td>7.2</td>
<td>3.6</td>
<td>1.6</td>
</tr>
</tbody>
</table>

*Note.* This table shows the age profile of elasticities $\sigma_k$, averaged within three age groups. The declining age profile in $\sigma_k$ captures the empirical fact that consumption early in one's life-cycle has a permanent income elasticity below 1, implying that permanently richer agents save relatively more.

process on the pre-1997 annual sample of working-age PSID households (see Appendix 1.D.4 for details) by estimating

$$\hat{y}_{it}^{\text{pre}} = \theta_{l-\to(i)} + \hat{w}_i + \eta_{it} + \psi_{it} + \nu_{it}.$$ 

Here, $\hat{y}_{it}^{\text{pre}}$ is the log of pre-tax household labor income; I model the age efficiency profile $\theta_k$ as a cubic polynomial in age; $\hat{w}_i$ is assumed to follow a normal distribution; and $\nu_{it}$ is measurement error. Since $\nu_{it}$ and $\psi_{it}$ are indistinguishable, I follow Heathcote et al. (2010) and assume the variance of $\nu_{it}$ is equal to $\sigma^2 = 0.02$.58 This yields the following results. The persistence of $\eta_{it}$ is found to be $\rho = 0.90$, the variance of the innovation to $\eta_{it}$ is equal to $0.026$, and the variance of the transitory shock $\psi_{it}$ is given by $0.052$. The labor income shares $\gamma_{it}$ are calibrated to match the bottom 90%, the next 9% and top 1% income shares using updated data from Piketty and Saez (2003). The inheritance probability of parental skill, $p_{\text{inherit}}$, is calibrated to match the slope between between parental and child income ranks measured in Chetty et al. (2014). This gives $p_{\text{inherit}} = 0.35$.

Preferences. As in the discussion in Section 1.2.1, I choose a simple parametric form for the age profile in consumption elasticities, namely a simple exponential decay, $\sigma_k/\sigma_{k+1} = \sigma^{\text{slope}} > 0$, during one’s working life and flat thereafter.59 In addition to the slope, I pick the median elasticity $\bar{\sigma}$. I choose a standard parameter, $\bar{\sigma} = 2.5$. I calibrate jointly $\{\beta, \sigma^{\text{slope}}, \kappa, a\}$ to match the following four moments: (1) an (after-tax) real interest rate of $r = 3\%$, which one should understand as the “total rate of return” in the economy; (2) a bequest flow over GDP of 5%—an intermediate value between the recent estimate in Alvaredo et al. (2017) of 8% and the estimate of 2% (see, e.g., Hendricks (2001)); (3) a 30% share of households with bequests below 6.25% of average income (De Nardi, 2004); (4) an estimate of the permanent income elasticity of consumption that matches column 5 of Table 1.2. This last moment is matched

58I also include this measurement error term in any simulated income data below.

59I experimented with several other parametric choices and, to the extent that $\sigma_k$ is downward sloping during one’s working or entire life with a flexible slope parameter, the qualitative and quantitative results are similar.
using Monte-Carlo simulations from the model (see Appendix 1.E.1 for details). It is this 
last set of moments that ultimately determines the life-cycle non-homotheticity parameter 
$\sigma^{slopes}$. The parameters are found to be: $\beta = 0.89$, $\sigma^{slopes} = 0.94$, $\kappa = 16$, $\sigma = 1.6$. The 
parameter $z$, which is irrelevant in the homothetic case ($\sigma^{slopes} = 1$), is set to 30% of average 
household income, or $21,000 in 2014 dollars, which can be thought of as a “minimum” level 
of consumption below which agents barely save.\footnote{60} Table 1.4 shows the age-dependence of $\sigma_k$ 
and average levels of $\sigma_k$ for three stages in one’s life cycle.

\textit{Interest rate $r$.} I choose a level of $r = 3\%$ for the after-tax interest rate (before taxes this 
corresponds to $r^{pre} \approx 5\%$), giving rise to a private wealth to GDP ratio of around 4.2, in line 
with recent U.S. levels of household net worth over GDP (Piketty and Zucman, 2014).

\textit{NFA.} The net foreign asset position, relative to GDP, in 2014 is $-27.4\%$. Net exports 
relative to GDP are given by $-2.9\%$.

Table 1.5 summarizes the calibration.

\textbf{Homothetic benchmark.} In addition to the non-homothetic life-cycle model I described, 
I also calibrate a “homothetic version” of this economy. For that, I set $\sigma_k = \bar{\sigma}$ and $\sigma = 0$ to 
eliminate all forms of non-homothetic savings behavior. I then re-calibrate $\beta$ and $\kappa$ to still 
match a post-tax real interest rate of 3\% and the same total bequest flow relative to GDP 
of 5\%. This gives $\beta = 0.99$, and $\kappa = 1.3$. Moreover I assume social security payments to be 
a linear function of (projected) lifetime incomes, with a slope of 20\% chosen to match the 
sum of social security expenses in the non-homothetic economy. This homothetic model is 
therefore very close to the neutrality benchmark in Section 1.3.\footnote{61}

1.5.3 Calibration results

I explore the calibrated non-homothetic and homothetic models in a number of dimensions.\footnote{62}

\textit{Estimating $\phi$ in simulated data.} I simulate artificial data from the non-homothetic and 
homothetic models and estimate the exact same regressions as those designed to test linearity 
in Section 1.4 (see Appendix 1.E.2 for details). Table 1.6 shows the results, alongside the

\footnote{60}It turns out that numerically, $z$ is fairly aligned with $\beta$, so different values of $z$ mainly shift the discount 
factor around but do not materially affect any other results. 
\footnote{61}It is not exactly neutral since there is imperfect skill persistence and the income tax schedule is progressive, 
that is, Assumptions 2 and 3 are violated. 
\footnote{62}Additional calibration results can be found in Appendix 1.E.3. There, I discuss model implications for: 
risk aversion over the life cycle; MPCs out of current income; life cycle profiles for consumption, income and 
wealth; and the joint distribution of labor income and wealth.
Table 1.5: Calibrated parameters of the baseline non-homothetic life cycle model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birth, death, skills</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>Number of permanent types</td>
<td>3</td>
<td>see text</td>
</tr>
<tr>
<td>${\bar{\mu}_s}$</td>
<td>Population shares by type</td>
<td>0.9, 0.09, 0.01</td>
<td>see text</td>
</tr>
<tr>
<td>${\delta_k}$</td>
<td>Mortality rates by age</td>
<td></td>
<td>CDC, 2011</td>
</tr>
<tr>
<td>Production</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Capital share</td>
<td>0.37</td>
<td>NIPA, 2014</td>
</tr>
<tr>
<td>${\gamma_s}$</td>
<td>Labor income shares</td>
<td>0.65, 0.24, 0.11</td>
<td>Piketty and Saez (2003), updated to 2014</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Depreciation</td>
<td>0.07</td>
<td>match $K/Y = 3.05$ (NIPA, 2014)</td>
</tr>
<tr>
<td>$A$</td>
<td>Total factor productivity</td>
<td>0.66</td>
<td>normalize $Y = 1$</td>
</tr>
<tr>
<td>Government</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B/Y$</td>
<td>Federal debt held by the public / GDP</td>
<td>0.73</td>
<td>NIPA, 2014</td>
</tr>
<tr>
<td>$\tau^b$</td>
<td>Bequest tax</td>
<td>0.10</td>
<td>see text</td>
</tr>
<tr>
<td>$y$</td>
<td>Income floor</td>
<td>0.30W</td>
<td>literature</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Income tax progressivity</td>
<td>0.16</td>
<td>PSID, 2013</td>
</tr>
<tr>
<td>$\tau^{inc\text{tax}}$</td>
<td>Average income tax</td>
<td>0.30</td>
<td>NIPA, see text</td>
</tr>
<tr>
<td>$\tau^{cap}$</td>
<td>Capital tax</td>
<td>0.40</td>
<td>NIPA, see text</td>
</tr>
<tr>
<td>$G/Y$</td>
<td>Government spending / GDP</td>
<td>0.14</td>
<td>gov. budget constraint</td>
</tr>
<tr>
<td>Productivities</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>Income shock persistence</td>
<td>0.90</td>
<td>PSID</td>
</tr>
<tr>
<td>$\sigma^2_\mu$</td>
<td>Var. of innovations to persistent shock</td>
<td>0.028</td>
<td>PSID</td>
</tr>
<tr>
<td>$\sigma^2_\sigma$</td>
<td>Var. of transitory income shocks</td>
<td>0.055</td>
<td>PSID</td>
</tr>
<tr>
<td>$\sigma^2_\nu$</td>
<td>Var. of measurement error in incomes</td>
<td>0.02</td>
<td>literature, see text</td>
</tr>
<tr>
<td>$\pi_{\text{ inherit}}$</td>
<td>Prob. of intergen. skill transmission</td>
<td>0.35</td>
<td>Chetty et al (2014)</td>
</tr>
<tr>
<td>Preferences</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z$</td>
<td>Scale term in utility function</td>
<td>0.30</td>
<td>30% of average income</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>0.89</td>
<td>match interest rate $r = 0.03$</td>
</tr>
<tr>
<td>$\sigma^{\text{slope}}$</td>
<td>Ratio of elasticities $\sigma_{k+1}/\sigma_k$</td>
<td>0.94</td>
<td>match $\phi = 0.699$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Weight on bequest motive</td>
<td>16.06</td>
<td>match bequests / GDP = 0.05</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Intercept in bequest utility</td>
<td>1.62</td>
<td>30% share with beq. $\leq 6.25%$ avg. income</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Elast. of intertemp. substitution, median age</td>
<td>2.5</td>
<td>literature</td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$NFA/Y$</td>
<td>Net foreign asset position over GDP</td>
<td>-0.27</td>
<td>2011 US NFA, Lane and Milesi-Ferretti (2007)</td>
</tr>
<tr>
<td>$NX/Y$</td>
<td>Net exports over GDP</td>
<td>-0.029</td>
<td>US. net exports (NIPA 2014)</td>
</tr>
</tbody>
</table>
Table 1.6: Estimating $\phi$ in simulated data from the non-homothetic lifecycle model.

<table>
<thead>
<tr>
<th></th>
<th>OLS</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 0$</td>
<td>$T = 4$</td>
</tr>
<tr>
<td>Data</td>
<td>0.40</td>
<td>0.64</td>
</tr>
<tr>
<td>Non-homothetic</td>
<td>0.53</td>
<td>0.71</td>
</tr>
<tr>
<td>Homothetic</td>
<td>0.68</td>
<td>0.89</td>
</tr>
</tbody>
</table>

*Note.* This table compares regressions testing the linearity of consumption in permanent labor income in the data and in two models: A non-homothetic and a homothetic life-cycle model. The first two columns are results from an OLS regression of log consumption on log income residuals, averaged across $T$ observations. Columns 3–5 are IV results with quasi-differenced future incomes as instruments; Column 6 shows IV results with initial incomes as instruments.

results I found in Section 1.4. In columns 1–2, it is visible that in both the data and the two models, the estimate without income averaging ($T = 0$) is significantly attenuated relative to the ones with income averaging, as expected from the discussion in Section 1.4.3. Interestingly, the $T = 4$ estimate for the homothetic model is already significantly different from the other two, 0.89 compared to 0.64 and 0.71 respectively, but still significantly below 1.

Three IV specifications for stationary $\eta_\ell$ appear in columns 3–5, where—just as in Section 1.4.3—I estimate the regressions with quasi-differenced future incomes as instruments, treating $\rho$ as a parameter chosen by the econometrician. The columns show a relatively good match between the regression results in the non-homothetic economy and the data. This is partly by construction: I calibrated the non-homothetic model to match the data estimate at the assumed persistence $\rho = 0.90$. The results in the homothetic economy are very close to 1. This is reassuring and a confirmation that the assumptions in Section 1.4.3 are indeed reasonable in the context of a life-cycle precautionary savings model. The result with the initial income IV specification (Column 6) is similar: as explained in Section 1.4.3, there is an upward bias in the estimated elasticity $\phi$, which is visible in both models. The non-homothetic model’s estimate is relatively close to the data, while the distance to the homothetic model is considerable.

Wealth inequality. The fact that richer people save relatively more naturally generates more wealth inequality than would occur in a neutral economy. But how much more? Figure 1-6 shows the Lorenz curve for wealth or, in other words, what fraction of wealth the bottom $x$ percent share of the population holds, where $x$ is anywhere between 0% and 100%. As is visible, the non-homotheticity generates a significant amount of wealth inequality that matches that in the data quite successfully overall, despite wealth inequality not having been
Figure 1-6: Lorenz curve for wealth.

![Lorenz curve for wealth](image)

**Note.** The Lorenz curve shows how much wealth the bottom $x$ percent of the population hold, where $x$ varies along the horizontal axis. The closer the plot is to the 45° line, the more equal the distribution is. The more this plot is pushed towards the bottom right corner, the more unequal the distribution is. The data on wealth inequality is from Saez and Zucman (2016).

Table 1.7: Top wealth shares.

<table>
<thead>
<tr>
<th>Share of wealth in top</th>
<th>0.1%</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
<th>20%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data (Saez-Zucman, 2016)</td>
<td>20.5%</td>
<td>38.6%</td>
<td>60.6%</td>
<td>73.0%</td>
<td>86.4%</td>
<td>100.1%</td>
</tr>
<tr>
<td>Non-homothetic model</td>
<td>10.1%</td>
<td>38.8%</td>
<td>67.8%</td>
<td>78.1%</td>
<td>87.0%</td>
<td>97.3%</td>
</tr>
<tr>
<td>Homothetic model</td>
<td>2.5%</td>
<td>11.8%</td>
<td>26.0%</td>
<td>37.5%</td>
<td>55.0%</td>
<td>87.0%</td>
</tr>
</tbody>
</table>

**Note.** This table shows the plot in Figure 1-6 in numbers.

a calibration target. Table 1.7 confirms this but highlights that the mechanism does not seem to capture the very top. This is partly due to the fact that there is no top 0.1% skill group in the model, but may also reflect the need for other forces to explain tail inequality, such as entrepreneurship and more generally return heterogeneity, as in the models of Quadrini (2000), Cagetti and De Nardi (2006), and Benhabib et al. (2015, 2017), among others.

**Consumption profiles.** To illustrate the non-homotheticity, Figure 1-7 plots expenditure shares over the life cycle. The shares are constructed as the fraction of present value of consumption at a given age $k$, $(1 + r)^{-k}c_{k,s}$, in the total present value of income. For simplicity, Figure 1-7 shows the shares relative to the shares of the bottom 90% skill group. This forcefully illustrates the non-homotheticity. While in the homothetic model (Panel (b)), agents of all skills choose the exact same expenditure shares (i.e. all skill groups are on the
Figure 1-7: Expenditure shares (of present value income) by skill group.

(a) Non-homothetic model.  
(b) Homothetic model.

Note. This graph shows age profiles of expenditure shares for each skill group in two models. The expenditure shares are computed as fractions of present value of income. The graph is normalized by subtracting the expenditure shares of the bottom 90% skill group. The plot illustrates that early consumption has a lower permanent income elasticity than late consumption in the non-homothetic model. See Appendix 1.E.3 for regular current value consumption profiles.

same linear Engel curve), this is not the case in the non-homothetic model (Panel (a)): there, agents with higher permanent incomes choose to shift their present-value consumption profiles considerably more towards higher ages.

1.5.4 Comparison with alternative models

I showed in Table 1.6 that the non-homothetic model matches the empirical evidence in Section 1.4 relatively successfully, especially when compared to the homothetic benchmark economy. Yet, one may still wonder how other economies fare under the same test. For example, what happens if the persistent shocks are more persistent than an AR(1) with $\rho = 0.90$, as was assumed in the two models of the previous section? What if there is partial insurance against income shocks? What if agents are subject to shocks to their discount factors—a common model ingredient that is generally used to increase wealth concentration (Krusell and Smith, 1997; Hubmer et al., 2016)? What if the income process is entirely different, e.g. one with kurtosis, inspired by those of Guvenen et al. (2016), or one with an extremely productive state, as in Castaneda et al. (2003) and Kindermann and Krueger (2017)?
This subsection answers these questions by comparing the main IV specifications—the AR(1) IV with quasi-differencing ($\rho = 0.90$) and the initial income IV—across a wide variety of other precautionary savings models. To this end, I computed a number of extensions to the homothetic benchmark economy which are designed to capture certain additional model features. All of these alternative models are calibrated to match the same post-tax steady state interest rate of $r = 3\%$, and their model parameters are set to standard values—wherever possible, to the same parameter values that are used in the non-homothetic model. I then simulated data from these models and estimated the same specifications as in the data. Details on the estimation on model-simulated data can be found in Appendix 1.E.2, and details on the model extensions are in Appendix 1.E.4.

Table 1.8 shows the results. The first three rows in that table are the data, the non-homothetic economy and the homothetic economy. The alternative models considered in Table 1.8 split into three blocks, starting with models with alternative preference or transfer assumptions, then models with income processes that do not satisfy the assumptions of the econometric model in Section 1.4.3, and a few other models. The columns represent the various specifications: the first column shows the true $\phi$ parameter in the model, which, reassuringly, exactly coincides with the AR(1) IV as long as that is the income process being used. If the true $\phi$ is equal to 1, the model is exactly neutral, as in Section 1.3.4. The second and third columns show the two IV specifications. I added two more specifications that have been considered by the literature and that are discussed below.

**Alternative preferences or transfers.** The first model in group 2 is an entirely neutral economy, with $\phi = 1.00$ and serves as the framework for further extensions. The second model in group 2 extends this model by including the concave actual social security schedule $T^{socsec}$. One can see that the AR(1) IV estimate of $\phi$ shrinks somewhat, although only by around 0.05. The third model in group 2 is a model with a luxury bequest motive only (i.e. $U_k = const$), where $\kappa, b$ are calibrated as before to match the total bequest flow in the economy and a 30% share of bequests below 6.25% of average income (see De Nardi (2004)). Again, while this helps to push down the stationary IV estimate of $\phi$, it does so modestly, by around 0.05.63

The initial income IV estimates are similar.

**Alternative income processes.** Group 3 in Table 1.8 considers alternative income processes. The first model considers an AR(1) persistent component $\eta_t$ with persistence $\rho = 0.95$ the innovation variance $\sigma^2_\eta$ of which is calibrated to match the same overall income dispersion as

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63This is not to say this motive is not important, however. Indeed, it generates significant improvements in matching wealth inequality. However, it appears to do so mostly by slowing old-age dissaving, rather than creating savings rate dispersion among the younger or middle-aged agents.
Table 1.8: Comparison across models.

<table>
<thead>
<tr>
<th></th>
<th>true φ</th>
<th>IV, ρ = 0.9</th>
<th>IV, ini</th>
<th>BPP</th>
<th>Ret. wealth slope</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Data and main models</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>0.70</td>
<td>0.70</td>
<td>0.73</td>
<td></td>
<td>0.18</td>
</tr>
<tr>
<td>Non-homothetic</td>
<td>0.70</td>
<td>0.72</td>
<td>0.36</td>
<td>0.18</td>
<td></td>
</tr>
<tr>
<td>Homothetic</td>
<td>0.99</td>
<td>0.99</td>
<td>1.00</td>
<td>0.38</td>
<td>0.18</td>
</tr>
<tr>
<td><strong>2. Alternative preferences or transfers</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Homothetic w/ out bequests</td>
<td>1.00</td>
<td>1.00</td>
<td>1.02</td>
<td>0.36</td>
<td>0.24</td>
</tr>
<tr>
<td>Homothetic w/ social security</td>
<td>0.95</td>
<td>0.95</td>
<td>0.96</td>
<td>0.33</td>
<td>0.20</td>
</tr>
<tr>
<td>Homothetic u but luxury bequests</td>
<td>0.95</td>
<td>0.95</td>
<td>0.96</td>
<td>0.33</td>
<td>0.20</td>
</tr>
<tr>
<td><strong>3. Alternative income process</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AR(1) with ρ = 0.95</td>
<td>1.00</td>
<td>0.93</td>
<td>1.01</td>
<td>0.52</td>
<td>0.24</td>
</tr>
<tr>
<td>Permanent-transitory</td>
<td>1.00</td>
<td>0.92</td>
<td>1.01</td>
<td>0.65</td>
<td>0.24</td>
</tr>
<tr>
<td>Heavy-tailed</td>
<td>1.00</td>
<td>0.90</td>
<td>1.00</td>
<td>0.38</td>
<td>0.20</td>
</tr>
<tr>
<td>Extreme productivity state</td>
<td>1.00</td>
<td>0.91</td>
<td>1.00</td>
<td>0.55</td>
<td>0.24</td>
</tr>
<tr>
<td>Heterogeneous income profiles</td>
<td>1.00</td>
<td>1.10</td>
<td>1.03</td>
<td>0.26</td>
<td>0.24</td>
</tr>
<tr>
<td><strong>4. Other</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Partial insurance</td>
<td>1.00</td>
<td>1.00</td>
<td>1.04</td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td>Random discount factors</td>
<td>1.00</td>
<td>1.00</td>
<td>1.12</td>
<td>0.37</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Note. This table compares the two main IV tests of the linearity of consumption in permanent income, across a wide range of models. It estimates the regressions on Monte-Carlo simulated data. The models are recalibrated to match the same equilibrium interest rate. See text and Appendix 1.E.4 for details.

the benchmark economies with ρ = 0.90. The second and third models consider permanent-transitory income shocks and heavy-tailed income shocks. For the former, I use the parameters in Kaplan and Violante (2010), and for the latter I adapt the income process in Guvenen et al. (2016). Both push down the stationary IV estimate, but not below 0.90. The initial income IV in those models is consistently above 1. The fourth model is a simple adaption of the “extreme productivity state” income process of Kindermann and Krueger (2017) (see Castaneda et al. (2003) for the seminal work in this regard). The AR(1) IV estimate is somewhat lower. Finally, the last model in this group considers the role of heterogeneity in income profiles in a homothetic economy, calibrated to match a life-cycle increase within-cohort log income dispersion of 0.2. This tends to push up the AR(1) IV estimates, but leaves the initial income IV unchanged. It shall be noted, however, that some models with heterogeneous income profiles include a learning component (Guvenen, 2007; Guvenen and Smith, 2014), which may push these estimates down somewhat.

Partial insurance economy. The first model in group 4 is a partial insurance economy,
where an innovation $\epsilon_{it}^q$ to $\eta_t$ is mitigated by a transfer $-\theta \epsilon_{it}^q$ (Guvenen and Smith, 2014). In line with my reasoning in Section 1.4.4, the AR(1) IV estimate is equal to 1. In addition, even the OLS specification with $T = 4$ barely changes compared to the homothetic benchmark economy. This underscores that the issue of partial insurance against income shocks is largely separate from the concavity in consumption as a function of permanent income.

**Discount factor shocks.** In the last model in Table 1.8, I simulate an economy with discount factor shocks. In particular, I adapt the parametrization of these shocks from Hubmer et al. (2016). While this technique is extremely useful in matching wealth inequality, it is not well-suited to match the estimated concavity in the consumption function. Indeed, the IV estimate is at 0.99, very close to 1. Why is this? While there are very persistent savers in this economy, the discount factor shocks are assumed to be entirely independent of one’s income or consumption choices. This is where my model departs: by modeling non-homotheticity in consumption decisions, this essentially induces an “effective” discount factor that depends positively on consumption. Thus, non-homotheticity positively aligns income and savings decisions a lot more than a random-$\beta$ environment would.

**BPP regressions.** For all models, I also compute the partial insurance coefficients with respect to persistent shocks as in Blundell et al. (2008), henceforth BPP. One can see that they are generally relatively low, indicating a substantial amount of self-insurance with respect to permanent shocks. Interestingly, introducing a non-homothetic savings motive leaves the BPP coefficient almost identical, despite changing the permanent income elasticity parameter $\phi$.

**Retirement wealth slope.** In an effort to inform the relationship between savings and permanent income, some authors have pointed towards the lifetime income-slope of the ratio of retirement wealth and lifetime income (see e.g. Gustman and Steinmeier (1999) and Venti and Wise (2000)). Putting aside any general issues with wealth-based savings statistics one may have, the last column computes these slopes in the models. Interestingly, in all models the slope is positive, around 0.15 – 0.25 (in units of average income), even in entirely neutral models. One reason for this is that agents that receive positive income shocks late in the life cycle will not just earn relatively more due to high life cycle productivity, but also save more out of the additional income increase. This generates a positive correlation between lifetime incomes and savings when entering retirement. Interestingly, the non-homothetic economy does not generate an outstandingly large slope here, in fact, the slope is similar to the other economies, suggesting that this slope parameter, while intrinsically interesting by itself, may

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64 The presence of capital gains, receipt of bequests, or heterogeneity in returns on wealth makes them relatively unattractive from the perspective of the consumption approach of this chapter
not be well-suited to inform the linearity of consumption as a function of permanent income.

1.6 The Non-Neutral Effects of Rising Permanent Income Inequality

The previous section demonstrates the ability of the non-homothetic model to capture the empirical degree of concavity of consumption as a function of permanent income. This section explores the quantitative implications of rising permanent income inequality in the non-homothetic economy, in two steps: first, in Section 1.6.1, I keep the interest rate fixed and compute the partial equilibrium implications of a rise in permanent income inequality that mirrors what the U.S. has experienced since 1970. I do this mainly by comparing the 1970 and 2014 steady states with each other. In Section 1.6.2, I compute the general equilibrium transitional dynamics induced by the shift in the distribution of permanent income.

1.6.1 Comparing steady states in partial equilibrium

To operationalize this partial equilibrium experiment, I fix interest rates at their 2014 level but assume the distribution of labor income shares \( \{\gamma_s\} \) matches the 1970 income distribution (Piketty and Saez, 2003).\(^{65}\) Then, I assume from one day to the next that labor income shares \( \{\gamma_s\} \) jump back to their 2014 levels. I focus on two outcomes in this subsection: the immediate shortfall in consumption—to allow for a comparison with the ad-hoc exercise in Section 1.4.5—as well as the long-run accumulation of wealth that is implied by the new steady state with greater inequality. I compare the results with the response in the homothetic economy (which is almost neutral in the long-run).

**Short-run effect on consumption.** I calculate the percentage decrease in aggregate consumption after the unanticipated shift in income inequality. In the ad-hoc exercise in Section 1.4.5, the consumption response was around \(-4\%\). Panel (a) of Figure 1-8 shows these results. In the non-homothetic model, initial consumption drops by around \(-2.5\%\) in total, somewhat below the ad-hoc estimate. In the homothetic model, the number is also negative, around \(-0.1\%\), since as discussed in Section 1.3, the homothetic model is not neutral in the short-run.

**Long-run effect on savings.** After the economy settled to the new steady state with higher income inequality, the non-homothetic economy has accumulated a whopping 130\% of GDP.
Figure 1-8: The partial equilibrium effects of a rising income inequality.

(a) Short-run drop in aggregate consumption.

(b) Long-run rise in aggregate savings.

Note. Panel (a) shows the simulated short-run drop in consumption in the non-homothetic and homothetic economies after income inequality permanently and unexpectedly shifts from 1970 to 2014 levels. Panel (b) shows the simulated long-run (steady-state) rise in accumulated wealth after such an experiment in both economies. Both panels are in partial equilibrium, with a fixed interest rate. “Savings glut” refers to the increase in savings by foreigners in U.S. assets since 1999.

in additional wealth, whereas the homothetic economy has not accumulated any extra wealth. As rough order of magnitude: the U.S. net foreign asset position declined by 18% of GDP since the end of the Asian crisis in 1999. This reflects greater savings by foreigners in U.S. assets and is sometimes referred to as evidence of a “global savings glut”. Remarkably, the savings-glut driven increase in aggregate savings pales in comparison with the effects of rising income inequality.⁶⁶

Why is the long run so much larger than the short run in the non-homothetic economy? To understand this, notice that the consumption adjustment in Panel (a) of Figure 1-8 is very persistent: permanently richer agents have greater savings rates, so an increase in their incomes snowballs into a large pile of additional savings.

1.6.2 General equilibrium transitional dynamics

The partial equilibrium experiment already suggests that the rise in permanent income inequality may have had a sizable effect on the U.S. aggregate economy. In particular, it raises two questions: how large are the effects in general equilibrium? And how long does it take for the effects to show? This section answers these questions by considering the general equilibrium (GE) transitional dynamics.

⁶⁶The observed decline in the U.S. net foreign asset position is a general equilibrium outcome. Still, through the lens of this model, this is the right comparison: since I assume the NFA is interest inelastic in this model, any PE decline in the NFA is exactly equal to the eventual GE decline. Therefore, the 18% of GDP can be regarded as a PE decline in the NFA.
This exercise requires to overcome significant computational challenges, however. The state space $S$ of the non-homothetic economy studied so far has 1.3 million idiosyncratic states; in addition to that, it is not only the interest rate that is endogenous along the transition path, but also the entire distribution of bequests—which matters in an economy where bequests are not received at birth, but rather in mid-life. I deal with these challenges by reducing the state space of the model to 400k states, and by improving existing algorithms along a number of dimensions. I discuss these improvements in Appendices 1.E and 1.F and formally define a non-stationary equilibrium in Appendix 1.E.

To simulate the transitional dynamics, I initiate the model in a steady state with the 1970 income distribution, and then feed in an exogenous path for labor income shares $\{\gamma_{s,t}\}$, that matches the actual one (see Figure 1-9). I assume that the values for $\gamma_{s,t}$ remain constant after 2014, and that agents have perfect foresight over the entire path. Finally, I assume that all agents hold all assets in equal proportions to their wealth.\(^6\)\(^7\)

The transitional dynamics exercise in this section is designed to speak directly to the origins behind three important recent macroeconomic trends: (a) the decline in real interest rates since the 1980s (Laubach and Williams, 2003, 2015); (b) the rising private wealth to GDP ratios (Piketty and Zucman, 2015); (c) the large and rapid increase in U.S. wealth

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\(^6\)While assets are interchangeable in this economy, they are subject to different valuation changes upon realization of a shock, such as rising income inequality.
Figure 1-10: Transitional dynamics from rising permanent income inequality.

Note. The figures show the response of a non-homothetic and a homothetic economy to an increase in permanent income inequality. The responses are expressed relative to 1970. The bottom plots also show the evolution wealth inequality in the data from Saez and Zucman (2016) (right vertical axis, with same scale).

inequality (Saez and Zucman, 2016).

Figure 1-10 depicts the model-implied transitional dynamics for these three outcomes. First, the real interest rate declines by around 1% through 2017, explaining approximately one third of the decline in the U.S. natural rate since the 1980s. Interestingly, despite the absence of any further increases in income inequality, the model predicts the interest rate will continue to decline, eventually falling by another 1%. The reason for this result is intuitive: in the model, the generation entering the labor market today is the first to experience the highest level of permanent income inequality for their entire working lives. In particular, this means the most able or skilled workers—one may think of recent computer science graduates—entering today will amass much larger fortunes over their lifetimes than previous
generations. This effect causes a large and predictable decline in interest rates going forward.

Second, the endogenous interest rate response limits the rise in aggregate wealth to around 30% of GDP through 2017 (again roughly one third of the rise in the data), with an eventual total increase of 55%. Here, the increase in wealth is accommodated by an increase in the quantity of capital since the model works with a standard neoclassical supply side. In a robustness exercise, I simulated the transitional dynamics for a Lucas-tree economy with a fixed capital stock $K$, in which any increase in wealth is necessarily accommodated by the price of capital. The resulting increase in aggregate wealth was even stronger in this case, around 50% of GDP through 2017.

Finally, the model explains almost the entire size and speed of the increase in the top 10% wealth share and around half of the increase in the top 1% wealth share. In sum, the model suggests that rising permanent labor income inequality alone can already account for a significant share of these three macroeconomic trends.

1.7 Conclusion

The ultimate goal of this chapter is to study the implications of rising inequality in permanent income for macroeconomic aggregates, as well as for the wealth distribution. Motivated by a “neutral” model in which the distribution of permanent income is irrelevant, I propose new ways to test the proportionality of consumption and permanent income in the data, finding an elasticity of $\phi \approx 0.7$. I show that a quantitative model that incorporates non-homotheticities in life-cycle spending can match this estimate, while standard models cannot. This has important consequences: the recent rise in permanent income inequality has pushed interest rates down and generated a rise in wealth inequality almost as rapid as that in the data.

There are several avenues related to this project that are well worth exploring in future research. First, in the quantitative model of Section 1.5, non-homothetic preferences drive almost the entire observed concave relationship between consumption and permanent income. This allowed me to conduct counterfactual analyses with the right overall level of concavity, but avoided the question of which forces at a microeconomic level account for how much of the observed concavity. Knowing the sources of concavity, however, is important, especially when the likely consequences of targeted policies are to be informed by the model.

Second, the interaction between technology and income inequality deserves further study. In particular, by way of lowering equilibrium interest rates, income inequality can indirectly induce investment into new technologies such as automation. This, in turn, may change the labor income distribution and the capital share going forward. Such a feedback loop could
make shocks to income inequality very persistent, or even permanent, in terms of convergence to a new steady state with high income and wealth inequality and low interest rates.

Finally, the non-homothetic savings behavior studied in this chapter has implications beyond income inequality, for instance for connection between private debt and aggregate debt. In a canonical, homothetic, infinite horizon economy, higher levels of private debt, e.g. induced by more relaxed borrowing constraints, do not affect steady state aggregate demand: while higher debt entails larger debt payments, forcing borrowers to cut back on consumption, savers perfectly balance this shortfall. This is because both borrowers and savers react to such a permanent transfer in the same way. In contrast, in a non-homothetic economy, savers do not perfectly balance the shortfall. Private debt overhang therefore weighs negatively on aggregate demand. As long as interest rates are not against the zero lower bound, this necessitates a reduction in interest rates, which stimulates more private debt. In Straub (2017) I explore this nexus of private debt, aggregate demand and interest rates.
Appendix
Figure 1.A.1: The importance of rising permanent income inequality.

(a) Inequality among 50-55 year old men.

(b) Rise of initial inequality.

Note. Panel (a) shows the evolution income inequality at the peak of the life cycle, for 50-55 year old men. Panel (b) decomposes this trend into inequality that is already present for 30-35 year old men, and a residual. The former is inequality in initial incomes, while the latter can be regarded as changes to the life-cycle increase in income inequality. The two measures of inequality plotted here are the standard deviation as well as the log(P90/P10) ratio, which is divided by 2 to make it roughly comparable to the standard deviation. The initial year is the first for which all measures are available in the data of Guvenen et al. (2017).

Appendix

1.A The importance of permanent income inequality

In a recent paper, Guvenen et al. (2017) investigate the dynamics of inequality in lifetime incomes, and inequality by age and cohort. In particular, they show that most of the increase in income inequality is due to rising inequality in initial incomes. Together with other evidence that rising inequality was not due to transitory shocks (Kopczuk et al., 2010; Sabelhaus and Song, 2010), this points to a rise in permanent income inequality, or—which turns out to be quite similar—a rise in the initial variance of the persistent component of income.

In this section, I use their data on male earnings from the Continuous Work History Subsample of the U.S. Social Security Administration’s Master Earnings File. Figure 1.A.1 shows two measures of income inequality, the standard deviation and the log(P90/P10) ratio. It decomposes the rise in income inequality at the life-cycle earnings peak (50-55 year olds), which is shown in Panel (a), into two pieces, which are shown in Panel (b): inequality that

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68I thank Fatih Guvenen for posting data on inequality statistics by age and cohort online at https://fguvenendotcom.files.wordpress.com/2017/03/gksw2017_figuredata_v1.xlsx.
was already present for 30-35 year olds (blue), and a residual (red). The figure illustrates that until the 1990s, both types of inequality increased, even though initial inequality increased more rapidly. Since the 1990s, however, initial inequality seemed to have accounted for more than the observed rise in inequality at ages 50-55.

While this evidence will almost surely not be the last word on the importance of permanent income inequality, it does suggest, however, that permanent factors were an important driver of income inequality in recent decades, worthy of a thorough investigation.

1.B Calibration of the simple model

Calibration. To generate the plots, I calibrate the equality steady state of the two models (homothetic and non-homothetic) as follows. As mentioned in Section 1.2, the share of rich agents is given by $\mu = 1\%$. The capital share is taken to be $\alpha = 0.33$, the interest rate $r = 0.05$, and depreciation $\delta = 0.06$, giving a capital stock of $K = 3$. The curvature of the per-period utility function is $\sigma = 1.5$, which is also equal to $\Sigma$ in the homothetic economy. In the non-homothetic economy, $\Sigma = 0.7\sigma$, foreshadowing my empirical results, and the scaling parameter $z$ is chosen so that the economy is in equilibrium. This procedure implies $\beta = 7.2$ and $z = 3.2$. (Notice that $\beta$ has the role of what usually is $\beta/(1 - \beta)$, which explains why it is so large.)

Figure 1-1. Combining the Euler equation (1.4) with the budget constraint (1.2), the consumption policy function $c(a + w)$ can be found by solving the implicit equation

$$c/z + R^{-1} (\beta R)^{1/\Sigma} (c/z)^{\sigma/\Sigma} = (a + w)/z.$$  

The savings schedule after 20 years, $a_{20}(w)$, starting at the perfect equality steady state where per head assets are equal to $K$, can be found by iterating the asset policy function $a(a + w) = R(a + w - c(a + w))$. The steady state savings schedule $a_{\infty}(w)$ can be found as the solution to the Euler equation, after replacing $c$ by its steady state value of $c = (1 - R^{-1})a + w$,

$$a = z (\beta R)^{1/\Sigma} \left( \frac{(1 - R^{-1})a + w}{z} \right)^{\sigma/\Sigma}. \quad (1.16)$$  

Figure 1-2. At the equality steady state, per-head wealth levels of poor and rich are equal, $a^r = a^p \equiv a = RK$. Total PE consumption after a shift in income inequality is then given by $C(\gamma) = \mu c(a + \gamma W/\mu) + (1 - \mu) c(a + (1 - \gamma) W/(1 - \mu))$, where $\gamma \in [\mu, 1]$. Aggregate wealth after 20 years is constructed as the sum of both types' asset positions after 20 years, which in
turn are obtained by iterating the asset policy function given a distribution of wages. Figure 1-3. In general equilibrium, the interest rate \( R \) is endogenously determined, as the value of total assets in the economy, \( R^{-1}A \), must equal the capital stock \( K \). \( K \) is determined by

\[
\frac{Y}{\alpha K} = R - 1 + \delta.
\]

This equation and (1.16) jointly describe steady state assets \( A \) and the steady state interest rate \( R \). In principle, this system can have multiple solutions, but in this calibration, there is a unique solution. The calculation of the other quantities in Figure 1-3 is standard.

### 1.C Omitted proofs

#### 1.C.1 Proof of Lemma 1

The key idea behind the proof of this result is that the dynamic programming problem (1.5) is separate for each skill \( s \), by Assumption 3. Take two skills \( s, s' \). I will argue by contradiction that if the scaling properties in Lemma 1 do not hold for \( s \) and \( s' \), one can construct an alternative wealth distribution \( \tilde{\mu} \), contradicting Assumption 4.

Assume that the scaling property of the bequest distribution \( \chi \) in Lemma 1 does not hold, that is, there exist two skills \( s, s' \), an age \( k_0 \), and a measurable set \( A_0 \subset \mathbb{R}^+ \) so that

\[
\chi(s, k_0, A_0) \neq \frac{\bar{\mu}_s}{\bar{\mu}_{s'}} \times \chi \left( s', k_0, A_0 \frac{w_{s'}}{w_s} \right).
\]

Fix \( s, s' \) and define a new bequest distribution \( \tilde{\chi} \) that is equal to \( \chi \) for all skills except \( s' \), where we define

\[
\tilde{\chi}(s', k, A) \equiv \frac{\bar{\mu}_{s'}}{\bar{\mu}_s} \chi \left( s', k, A \frac{w_s}{w_{s'}} \right)
\]

for any age \( k \) and measurable set \( A \subset \mathbb{R}^+ \). Similarly, define the anticipated bequest distribution \( \tilde{\nu} \) that corresponds to bequest distribution \( \tilde{\chi} \) as in (1.6). Notice that for any measurable set \( A \subset \mathbb{R}^+ \) it holds that

\[
\tilde{\nu}(A, \varphi'|s', k, \varphi) = \tilde{\nu} \left( \frac{w_s}{w_{s'}} A, \varphi'|s, k, \varphi \right)
\]

due to Assumption 3, according to which either the skill transition matrix is the identity, \( P = I \), or there are no bequests.

Next, consider the dynamic programming problem (1.5) with anticipated bequest distribu-
tion \( \hat{v} \). Denote the corresponding value function by \( \hat{V} \). It must be that

\[
\hat{V}_{k,s}(a, z, \varphi) = \left( \frac{w_{s'}}{w_s} \right)^{1-\sigma} \hat{V}_{k,s} \left( \frac{w_{s'}}{w_s} a, z, \varphi \right)
\]

for the following reasons: if \( \hat{V} \) were not the (unique) solution to the convex programming problem (1.5), one could easily use (1.18) to construct a second solution. \( \hat{V} \) solves (1.5) due to Assumptions 1 and 2, the fact that the agent starts with zero assets, and equation (1.17). Building on (1.18), it is immediate that the unique policy functions satisfy

\[
\tilde{c}_{k,s}(a, z, \varphi) = \frac{w_{s'}}{w_s} c_{k,s} \left( a \frac{w_s}{w_{s'}}, z, \varphi \right)
\]

and

\[
\tilde{a}_{k,s}(a, z, \varphi) = \frac{w_{s'}}{w_s} \tilde{a}_{k,s} \left( a \frac{w_s}{w_{s'}}, z, \varphi \right).
\]

Finally, constructing \( \bar{\mu} \) as in (1.7) given \( \tilde{v} \) and \( \tilde{a} \) yields

\[
\bar{\mu}(s', k, A, z, \varphi) = \frac{\bar{w}_{s'}}{\bar{w}_s} \mu \left( s, k, A \frac{w_s}{w_{s'}}, z, \varphi \right).
\]

This proves that, if the scaling property for the bequest distribution \( \chi \) does not hold, one can construct a (different) wealth distribution \( \bar{\mu} \) which satisfies (a) and (g) of Definition 1, for the same interest rate \( r \) and the same skill prices \( \{w_s\} \). This contradicts Assumption 4. Therefore, the scaling property must hold for \( \chi \). Following the same steps as above, however, also establishes similar scaling properties of \( \chi, v, V, a, c \) and \( \mu \). This concludes our proof of Lemma 1.

### 1.C.2 MPCs in the neutral economy

Define the **MPC out of current income** of an agent in state \((s, k, a, z, \varphi) \in S\) as

\[
\text{MPC}_{k,s}(a, z, \varphi) = \frac{\partial}{\partial a} c_{k,s}(a, z, \varphi).
\]

As it turns out, the distribution of MPCs is the same across skill groups.

**Corollary 1** (MPCs in a neutral economy). **Under Assumptions 1–4, for any state** \((s, k, a, z, \varphi) \in S\) **it holds that**

\[
\text{MPC}_{k,s}(a, z, \varphi) = \text{MPC}_{k,s'} \left( \frac{w_{s'}}{w_s} a, z, \varphi \right).
\]

(1.19)
In particular, the distribution of MPCs is the same within all skill groups.

Proof. Equation (1.19) is a direct consequence of the definition of MPCs and Lemma 1,

\[
\text{MPC}_{k,s}(a, z, \varphi) = \frac{\partial}{\partial a} c_{k,s}(a, z, \varphi)
\]

\[
= \frac{w_s}{w_{s'}} \frac{\partial}{\partial a} c_{k,s'} \left( a \frac{w_{s'}}{w_s}, z, \varphi \right) = \text{MPC}_{k,s'} \left( \frac{w_{s'}}{w_s} a, z, \varphi \right).
\]

To prove the claim on the distribution of MPCs, define first the following conditional distribution

\[
\mu(k, a, z, \varphi|s) \equiv \frac{1}{\mu_s} \mu(s, k, a, z, \varphi).
\]

Notice that by Lemma 1,

\[
\mu(k, A, z, \varphi|s) = \mu \left( k, A \frac{w_{s'}}{w_s}, z, \varphi|s' \right). \tag{1.20}
\]

The claim is that the distribution of the random variable \((k, a, z, \varphi) \mapsto \text{MPC}_{k,s}(a, z, \varphi)\) under probability distribution \(\mu(k, a, z, \varphi|s)\) is the same for each skill \(s \in S\). This immediately follows from the combination of (1.19) and (1.20).

The result in Corollary 1 is interesting because it documents that without conditioning on cash-on-hand \(a\), the distribution of MPCs is unaffected by permanent income. Aside from potential endogeneity issues, this is at odds with the evidence in Jappelli and Pistaferri (2006, 2014), which documents that MPCs tend to decline in education.

1.C.3 Proof of Proposition 1

From Lemma 1 it follows that

\[
\log c_{k,s}(aw_s, z, \varphi) - \log w_s = \log c_{k,s'}(aw_{s'}, z, \varphi) - \log w_{s'} \tag{1.21}
\]

Define the conditional distribution given age \(k\) and skill \(s\) as

\[
\mu(A, z, \varphi|s, k) \equiv \frac{\mu(s, k, A, z, \varphi)}{\mu(s, k, \mathbb{R}^+, \mathbb{Z}, \{0, 1\})}.
\]

As direct consequence of the scaling property of \(\mu\),

\[
\mu(w_s A, z, \varphi|s, k) = \mu(w_{s'} A, z, \varphi|s', k) \tag{1.22}
\]
Using that notation, define

\[ \text{const}_k = \int (\log c_{k,s}(aw_s, z, \varphi) - \log w_s) \, d\mu(\cdot|s, k) \]

which is well-defined by (1.22). By definition of \text{const}_k, one can then write

\[ \log c_{k,s}(aw_s, z, \varphi) = \text{const}_k + \log w_s + \epsilon(k, a, z, \varphi). \]

Translating the behavior into the associated stochastic process, one can then write

\[ \log c_{it} = \text{const}_{k(i,t)} + \log w_{s(i)} + \epsilon_{it} \]

where by construction of \( \epsilon(k, a, z, \varphi) \), \( \epsilon_{it} \) has zero mean conditional on age \( k \) and skill \( s \).

Equation (1.9) follows by definition of income as

\[ \log y^{pre}_{k,s}(z) = \log \Theta_k(z) + \log w_s \]

where again \( \log \Theta_k(z) \) can be further decomposed into an average and a mean zero term. This gives (1.9).

1.C.4 Proof of Proposition 2

Using (1.21), one can decompose

\[ \log c_{k,s}(a, z, \varphi) = \log w_s + C \left( k, \frac{a}{w_s}, z, \varphi \right). \tag{1.23} \]

By the law of total variance, the variance of the left hand side under probability measure \( \mu \) can be computed as

\[ \text{Var}_\mu [\log c_{k,s}(a, z, \varphi)] = \mathbb{E}_\pi [\text{Var}_\mu(\cdot|s) [\log c_{k,s}(a, z, \varphi)]] + \text{Var}_\mu [\mathbb{E}_\mu(\cdot|s) [\log c_{k,s}(a, z, \varphi)]] \].

The first term does not involve \( s \) since the distribution of \( C \left( k, \frac{a}{w_s}, z, \varphi \right) \) under \( \mu(\cdot|s) \) is the same for all \( s \). Define

\[ \text{const} = \mathbb{E}_\pi [\text{Var}_\mu(\cdot|s) [\log c_{k,s}(a, z, \varphi)]] \].

For the same reason, one can write

\[ \mathbb{E}_\mu(\cdot|s) [\log c_{k,s}(a, z, \varphi)] = \log w_s + \text{const}' \]
with some other constant \( \text{const}' \). Together,

\[
\text{Var}_\mu [\log c_{k,s}(a, z, \varphi)] = \text{const} + \text{Var}_\mu [\log w_s].
\]

A similar argument shows the result for log wealth (cash on hand), concluding our proof of Proposition 2.

1.C.5 Proof of Proposition 3

Denote by \( C(\{w_s\}) \) aggregate consumption as function of skill prices \( \{w_s\} \). Using the law of iterated expectations, one can express \( C \) as

\[
C(\{w_s\}) = \mathbb{E}_\mu \left[ \mathbb{E}_{\omega(\cdot|s)} c_{k,s}(a, z, \varphi) \right].
\]

With decomposition (1.23), this becomes

\[
C(\{w_s\}) = \mathbb{E}_\mu \left[ w_s \mathbb{E}_{\omega(\cdot|s)} \exp C \left( k, \frac{a}{w_s}, z, \varphi \right) \right]
\]

where again \( \mathbb{E}_{\omega(\cdot|s)} \exp C \left( k, \frac{a}{w_s}, z, \varphi \right) \equiv \kappa_C \), which is independent of \( w_s \). Therefore,

\[
C(\{w_s\}) = \kappa_C \sum_s \bar{\mu}_s w_s
\]

and similarly for total assets as function of skill prices \( A(\{w_s\}) \), and total bequests \( A^{\text{beg}} \equiv \int_{(s,k,b)} b d\chi \).

To show that a change in labor income shares \( \{\gamma_s\} \) leaves all aggregate quantities unchanged, notice that

\[
\bar{\mu}_s w_s = (1 - \alpha)\gamma_s Y
\]

and so \( C = \kappa_C (1 - \alpha)Y \) and \( A = \kappa_A (1 - \alpha)Y \).

These derivations show that neither goods market clearing, nor asset market clearing, nor the government budget constraint are affected by a change in labor income shares \( \{\gamma_s\} \). Therefore, there exists an equilibrium where also the aggregate interest rate and the aggregate capital stock stay the same.
Table 1.D.1: More robustness checks.

<table>
<thead>
<tr>
<th></th>
<th>OLS with $T = 9$</th>
<th>IV with $p = 0.90$</th>
<th>IV with initial income</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Baseline</td>
<td>0.64</td>
<td>0.70</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>2. Incl. household heads &gt; 65 years</td>
<td>0.68</td>
<td>0.71</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>3. Controlling for liquid assets</td>
<td>0.63</td>
<td>0.74</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>4. Households with original sample heads</td>
<td>0.61</td>
<td>0.66</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.07)</td>
<td>(0.06)</td>
</tr>
</tbody>
</table>

Note. This table lists OLS and IV estimates for 5 different specifications. Row 1 shows the baseline specifications. Row 2 shows results for households with heads of all ages. Row 3 controls for a cubic in liquid assets relative to income. Row 4 runs the baseline OLS and IV specifications on the subsample with only the households headed by original PSID sample heads. All specifications control for year, age, household size and location dummies. All IV specifications have first stage F statistics above 10. Standard errors are corrected for heteroskedasticity and clustered by household.

1.D Additional empirical results

1.D.1 Additional robustness exercises

This section provides additional OLS and IV specifications at the household level, as well as a set of group-level specifications and specifications that use the imputed consumption measure by Blundell et al. (2008).

Additional household-level specifications. All results are shown in Table 1.D.1. As comparison, the first row contains the baseline estimates from Table 1.3. The second row shows the baseline specification, only on the larger sample of households whose head is between 30 and 80 years old, rather than between 30 and 65 years old. The results are similar as in the baseline case. The third row adds a cubic in the ratio of liquid assets to income as controls to $X_{it}$. I measure liquid assets as the sum of a household's wealth in cash and stocks, net of credit card, medical, legal and other debts as well as net of loans from relatives. The estimates are somewhat larger but insignificantly different. The fourth row re-estimates the two baseline specifications on the subsample of households whose heads are original PSID sample heads, rather households headed by heads that joined the PSID by way of marriage. Hryshko and Manovskii (2017) argue that these groups of households differ in terms of their

---

89Since labor income is ill-defined for most households above 65, I use total household income, including income from capital for this regression.
Table 1.D.2: Group-level specifications.

<table>
<thead>
<tr>
<th>log group consumption</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OLS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td></td>
<td></td>
</tr>
<tr>
<td>log group income</td>
<td>0.642</td>
<td>0.720</td>
</tr>
<tr>
<td>(0.04)</td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>Year FE</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Age FE</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>256</td>
<td>104</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.73</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Note. This Table lists OLS and IV specifications at the group level. Groups are constructed using the interaction of 5-year birth cohorts and education dummies. Column 1 shows simple OLS estimates of log group consumption on log group income residuals. Column 2 shows IV estimates using future income as instrument. Standard errors are corrected for heteroskedasticity.

income shocks. The results here show that they do not differ significantly in terms of the concavity of their consumption function.

**Group-level specifications.** In Table 1.D.2, I show estimates for a number of group-level specifications. Following the seminal work of Attanasio and Davis (1996), I define groups as the interaction of 5-year birth cohorts and education groups of household heads (no high school, high school, less than four years of college, four years of college). For columns 1–2, I use a group-averaged version of (1.11). Specifically, let \( \{I_g\} \) be a set of mutually exclusive groups of households and define for each group their average log consumption, \( \bar{c}_{gt} \equiv |I_g|^{-1} \sum_{i \in I_g} \hat{c}_{it} \), and their average income residuals, \( \bar{y}_{gt} \equiv |I_g|^{-1} \sum_{i \in I_g} \hat{y}_{it} \). I estimate

\[
\bar{c}_{gt} = X'_{gt} \beta + \phi \bar{y}_{gt} + \epsilon_{gt}. \tag{1.24}
\]

The IV approach in column 2 uses 2-year ahead group income \( \bar{y}_{gt+1} \) as instrument for \( \bar{y}_{gt} \). The results are close to the ones found in the household-level approaches in Section 1.4.

**Imputed consumption measure by Blundell et al. (2008).** In an important paper, Blundell et al. (2008) estimate a demand equation for food consumption expenditure in data from the CEX and use it to impute various measures of consumption expenditure for the PSID
Table 1.D.3: OLS and IV estimates using the imputed consumption data from Blundell et al. (2008).

<table>
<thead>
<tr>
<th>OLS</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>$T = 1$</td>
</tr>
<tr>
<td>log Income</td>
<td>0.555</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
</tr>
<tr>
<td>Year FE, Age FE</td>
<td>Yes</td>
</tr>
<tr>
<td>Hh.size FE</td>
<td>Yes</td>
</tr>
<tr>
<td>Location FE</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>703</td>
</tr>
<tr>
<td>1st stage $F$</td>
<td>39.0</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.38</td>
</tr>
</tbody>
</table>

Note. This Table lists OLS and IV estimates when using the imputed consumption measure from Blundell et al. (2008). Columns 1 and 2 show OLS estimates for the case of no income averaging ($T = 1$) and averaging over $T = 9$ income observations. Columns 3-7 show the estimates from IV specifications with various assumed income shock persistences $p$. Column 8 shows estimates with age 25 income as instrument. Standard errors are clustered by household and corrected for heteroskedasticity.

from 1980 to 1992. Among the measures of consumption expenditure imputed is total consumption expenditure, which I will be using below. The results and conclusions are very similar with the other measures the authors impute, namely a measure of nondurable consumption expenditure and measures that include the service flow of durables rather than the expenditure.

This consumption data is especially helpful in the context of this paper, for at least three reasons: first, the data is from an entirely different source, namely the CEX, not the PSID, which I have been using for my analysis. Second, similar to the comprehensive post-2005 consumption measure in the PSID, the imputed consumption data includes total expenditure across all categories. And third, the imputed data is from an entirely different time—from 1980 to 1992—whereas my analysis had to be restricted to the time after 1999 due to data limitations.

In Table 1.D.3 I reestimate my main OLS and IV specifications. First, while the estimates are somewhat more noisy, it is reassuring that they all lie around 0.7, thus confirming my previous results. In fact, the estimates in columns 1-7 of Table 1.D.3 are generally around 0.05 – 0.15 lower than their counterparts using the PSID consumption data after 1999 (Tables 1.1 and 1.2). The upper bound estimate using the initial income IV is around 0.05

70I thank the authors for making their data available online.
larger than its post-1999 counterpart and considerably noisier.

1.D.2 Robustness to non-classical measurement error

In a recent paper, Aguiar and Bils (2015) argue that there is non-classical measurement error in the interview survey of the Consumer Expenditure Survey (CEX), in the sense that richer respondents are more likely to underreport their spending. Aguiar and Bils (2015) show that this matters for the measurement of consumption inequality. In this paper, the estimate for $\phi$ can be affected by such measurement error whenever high-income households underreport their spending relatively more compared to low-income households, and relatively more compared to their income. To test this concern, I propose an estimation strategy similar to the one proposed in Aguiar and Bils (2015).

Denote by $c_{igt}$ the observed expenditure of household $i$ on good $g$ at time $t$, and by $c_{it}$ its total observed expenditure across all goods. Denote by $c^*_{igt}$ and $c^*_{it}$ the true expenditures. As before, assume that true total expenditures are governed by

$$\log c^*_{it} = \phi \log w_i + X_{it} \beta + \epsilon_{it}.$$  

In addition, I follow Aguiar and Bils (2015) in assuming: that true good-level expenditures can be represented by a log-linear demand system,\footnote{This is not possible globally, see the discussion in Aguiar and Bils (2015).}

$$\log c^*_{igt} = \gamma_g \log c^*_{it} + \tilde{X}_{it} \tilde{\beta}_g + \epsilon_{igt},$$

where time fixed effects are part of $\tilde{X}_{it}$; and that measurement error at the level of goods, $\log c_{igt} - \log c^*_{igt}$, is orthogonal to true total expenditures $\log c^*_{it}$. In other words, when high-income households underreport their spending, they do so proportionally across categories. In the following, fix some good $g_0$ (it is irrelevant which good).

Under these three assumptions, one obtains that

$$\frac{\log c_{igt} - \log c^*_{igt}}{\gamma_g - \gamma_{g_0}} = \phi \log w_i + X_{it} \beta + \tilde{X}_{it} \tilde{\beta}_g + \tilde{\epsilon}_{igt}$$

with $\tilde{\beta}_g = (\beta_g - \beta_{g_0}) / (\gamma_g - \gamma_{g_0})$ and $\tilde{\epsilon}_{igt} = (\epsilon_{igt} - \epsilon_{igt}) / (\gamma_g - \gamma_{g_0})$. To implement this regression, I use the estimated expenditure elasticities in Aguiar and Bils (2015, Table 2, column 1) for the following categories which are (more or less) comparable across the CEX and the PSID: housing, food at home, transportation (incl. other transport), utilities,
health expenditures, food away from home, education, child care. As controls $\tilde{X}_{it}$, I use time dummies, 5-year age brackets and household size dummies. The results are in row 9 of Table 1.3.

1.D.3 Imputation procedure for the comprehensive consumption measure

I use the following approach to impute the comprehensive consumption measure to years before survey year 2005 (calendar year 2004). Denote by $c_{it}$ household $i$'s total consumption in year $t$ according to the reduced 70% measure, and by $c_{it}$ household $i$'s consumption according to the comprehensive post-2005 measure. Label by $k = 1, \ldots, 6$ the six new consumption categories introduced in 2005, and by $c_{it}^k$ household $i$'s consumption in those categories. Thus, for all years $t \geq 2005$,

$$c_{it} = c_{it} + \sum_{k=1}^{6} c_{it}^k.$$ 

I impute the value of $c_{it}^k$ for every $k$ separately. In particular, for a given $k$, I model $c_{it}^k$ as

$$\log c_{it}^k = P_t \eta_{it}^k + \gamma^k \log c_{it} + X_{it} \beta^k + e_{it}$$

where $X_{it}$ includes a variety of household controls, such as a cubic polynomial in age, and dummies for 5-year cohort bracket, household size, race, and sex; $P_t$ is a vector of log prices, one for each consumption category (including those already in $c_{it}$). To avoid measurement error in $\log c_{it}$, I instrument $\log c_{it}$ using future consumption $\log c_{it+2}$. Having estimated $\eta_{it}^k, \gamma^k, \beta^k$ I compute predictions $\tilde{c}_{it}^k$ for each household $i$, consumption category $k$ and time $t < 2005$. The imputed comprehensive measure of consumption expenditure is then

$$\tilde{c}_{it} \equiv \begin{cases} c_{it} + \sum_{k=1}^{6} \tilde{c}_{it}^k & t < 2005 \\ c_{it} & t \geq 2005 \end{cases}.$$ 

1.D.4 Estimation of the income process

In this section, I explain the estimation strategy I use to estimate the income process used in this paper. Throughout, I measure "income" as pre-tax labor income, as measured in the PSID. As sample, I use all years in the PSID from survey year 1981, the first year without significant income top-coding, to survey year 1997, which is the last year for which there is annual income information. I include all households between ages 30 and 55, to avoid issues
due to early retirement, and exclude households with pre-tax labor income below 25\% of any given year's average pre-tax labor income (for the survey year 2013, this threshold amounts to a household income of approximately \$15k). To avoid small sample bias, the sample is restricted to households for whom there are at least 15 not necessarily consecutive years of income information. This leaves me with a sample of 2,817 households.

**Estimating the income process.** Consider the following standard model for log incomes

\[ \log y_{it} = f(X_{it} \beta) + \alpha_i + \eta_{it} + \psi_{it} + \nu_{it}, \]

where \( f(X_{it}, \beta) \) are set of controls, \( \alpha_i \) are income fixed effects, \( \eta_{it} \) is an AR(1) process whose persistence is \( \rho_\eta \) and whose innovations have variance \( \sigma_{\eta}^2 \), \( \psi_{it} \) is a transitory income shock with variance \( \sigma_{\psi}^2 \), and \( \nu_{it} \) is a measurement error term with variance \( \sigma_{\nu}^2 \). I take the controls \( f(X_{it}, \beta) \) to be a cubic polynomial in age, dummies for household size, and year dummies. Measurement error cannot be distinguished from the transitory income shock. I follow Heathcote et al. (2010) and assume that \( \sigma_{\nu}^2 = 0.02 \). This leaves me with four parameters to estimate: \( \sigma_{\alpha}^2, \rho_\eta, \sigma_{\eta}^2, \) and \( \sigma_{\psi}^2 \).

I employ a stationary minimum distance estimation (MDE) procedure that is standard by now and was first developed in Chamberlain (1984).\(^2\) First in the procedure, I residualize incomes by partialing out the demographic and life-cycle controls \( f(X_{it}, \beta) \). Denote the income residual for individual \( i \) at age \( k \) by \( \hat{y}_{ik} \), where the lowest age is normalized to \( k = 1 \), so that the maximum age is \( K = 26 \). The autocovariances of \( \hat{y}_{ik} \) are then given by

\[ \text{Cov}(\hat{y}_{ik}, \hat{y}_{ik+s}) = \sigma_{\alpha}^2 + \sigma_{\eta}^2 \sum_{j=0}^{k-1} \rho^{2j} + 1_{\{s>0\}} \left( \sigma_{\psi}^2 + \sigma_{\nu}^2 \right), \]

for any \( s \geq 0 \). As mentioned before, \( \sigma_{\psi}^2 \) is set exogenously. I use 15 time periods for estimation, so \( s \) ranges between 0 and 14. Implicit in this formulation is the assumption that the initial variance of the persistent income process is also given by \( \sigma_{\eta}^2 \). Denote the right hand side by \( g_{k,k+s}(\sigma_{\alpha}^2, \rho, \sigma_{\eta}^2, \sigma_{\psi}^2) \), and symmetrically set \( g_{k+s,k} \). Define the empirical covariance matrix of income by

\[ G_{k,k'} = \frac{1}{|\mathcal{L}_{k,k+s}|} \sum_{i \in \mathcal{L}_{k,k+s}} \hat{y}_{i,k} \hat{y}_{i,k'}, \quad k, k' \in \{1, \ldots, K\}. \]

The MDE estimator minimizes the distance between \( g \) and \( G \). To implement it, stack all \( K(K + 1)/2 \) unique values of \( G_{k,k'} - g_{k,k'} \) into a vector, denoted by \( \mathbf{G}(\sigma_{\alpha}^2, \rho, \sigma_{\eta}^2, \sigma_{\psi}^2) \). The

\(^2\)See Meghir and Pistaferri (2010) for a recent survey over the literature.
Table 1.D.4: Estimated AR(1) + iid process.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma^2_\alpha$</th>
<th>$\sigma^2_\eta$</th>
<th>$\sigma^2_{\psi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.904</td>
<td>0.131</td>
<td>0.026</td>
<td>0.052</td>
</tr>
<tr>
<td>(0.064)</td>
<td>(0.017)</td>
<td>(0.011)</td>
<td>(0.015)</td>
</tr>
</tbody>
</table>

Note. This table shows the estimated parameters of a process for log income residuals. The process consists of a permanent component, an AR(1) component and a transitory shock. Standard errors are block-bootstrapped with 500 iterations and clustered at the household level.

minimum distance estimates of $(\sigma^2_\alpha, \rho, \sigma_\eta, \sigma^2_{\psi})$ are then the solution to

$$\min_{\sigma^2_\alpha, \rho, \sigma_\eta, \sigma^2_{\psi}} G(\sigma^2_\alpha, \rho, \sigma_\eta, \sigma^2_{\psi})W G(\sigma^2_\alpha, \rho, \sigma_\eta, \sigma^2_{\psi})$$

where $W$ is a weighting matrix. I use an identity weighting matrix, which was shown to be less prone to small sample bias by the simulations in Altonji and Segal (1996). This procedure yields consistent and asymptotically normal estimates for $(\sigma^2_\alpha, \rho, \sigma_\eta, \sigma^2_{\psi})$, whose asymptotic standard errors I compute using a block-bootstrap with 500 iterations that is clustered at the household level. Table 1.D.4 shows the results.

Alternative way to estimate the persistence. The most relevant parameter in this estimation is the persistence $\rho$. In the same stationary setup as the one used above, one can identify $\rho$ directly as follows. Define $m_s \equiv \mathbb{E}[\hat{y}_{ik} \hat{y}_{ik+s}]$ where the expectation includes $k$, and $s > 0$. Then, for any $s > 0$,

$$\log(m_{s+1} - m_s) = \text{const} + s \log \rho. \quad (1.25)$$

I used this simple linear relationship to confirm that in a variety of other settings and samples, the persistence parameter $\rho$ estimated this way lies around 0.90 or below.

Needless to say, there is an active debate about the "right" income process. The simple estimation strategy in (1.25) can be viewed as being focused entirely on estimating the $\rho$ that best matches the relative magnitudes of the moments $m_s$. As is well-known, and as I point out in Section 1.D.6, an estimated $\rho$ around 0.90 fails to generate the almost linear shape of the income dispersion over the life-cycle.

1.D.5 Scatter plots for the IV specifications

To show that there is also no particular source of nonlinearity that drives the two IV results, this section shows the non-parametric results for the AR(1) IV specification ($\rho = 0.9$), as well as the initial income IV specification.
Figure 1.D.1: Non-parametric estimates for the IV second stages.

(a) AR(1) IV with $\rho = 0.9$. 
(b) Initial income IV.

Note. This figure shows the results of a regression of log consumption on controls and predicted permanent income bins, which are constructed using a first-stage regression of income on controls and the instrument(s). Panel (a) does this for the AR(1) IV with persistence parameter $\rho = 0.9$ and panel (b) does this for the initial income IV.

To this end, I estimate a set of predicted permanent incomes $\hat{w}_{it}^p$ by projecting current income residuals $\hat{y}_{it}$ on the controls $X_{it}$ in the consumption equation as well as the set of instruments used—$\{z_{i,t+r}\}_{r>1}$ for the AR(1) IV and $z_i$ for the initial income IV. I then run a non-parametric regression of log consumption $\hat{c}_{it}$ on controls $X_{it}$ and dummies for 20 bins of the predicted permanent incomes $\hat{w}_{it}^p$. The results for the two IV specifications are shown in Figure 1.D.1. No obvious nonlinearity or outliers drive the result.

1.D.6 Evidence on the mechanism

One may wonder what happens with the additionally saved funds of permanently richer households. This section provides evidence for two channels: future consumption and intergenerational transfers to children. As I explain in Section 1.4.6, however, there are several other plausible channels.

Future consumption. To investigate whether permanent-income richer households increase their future consumption relatively more than permanently poorer households, I re-run the IV estimation in Section 1.4.3 by 5-year age groups. Since the initial income IV specification does not require any future labor incomes, one can re-estimate it for every 5-year age group from 30 to 65 years. The orange triangles in Figure 1.D.2(a) shows the results. There is a 

73 See the definitions in Section 1.4.3.
significant increase in the elasticity, with later ages consuming relatively more.

*Intergenerational transfers.* In 2013, the PSID surveyed households about the size and type of intergenerational transfers they received from their parents, or made to their children. Using log total transfers made as left-hand-side variable instead of \( \tilde{c}_{it} \) in the initial income IV regressions one finds that transfers (those made during one’s work-life) have a permanent income elasticity of around 1.8 and therefore are a key reason why permanently richer agents save. At what ages are transfers given, and what are they used for? Figure 1.D.2(b) shows the average annual transfer given by age, and for three types of transfers: school-related transfers, such as college tuition payments, house purchase related transfers, and other types of transfers. It is clearly visible that the transfers are sizable and happen late in life (and therefore need to be saved towards).

*Comparison with the model.* The blue circles in Panel (a) show the results when replicating the initial IV regressions by 5-year age bins on simulated data from the quantitative non-homothetic model of Section 1.5. It can be seen that in the model, the regressions predict a greater slope in age than what seems to be in the data. Part of that could be explained by the presence of other expenses (such as the ones in Panel (b)), which are not part of consumption expenditure in the data, but are treated as such by the model.

---

**Figure 1.D.2:** Evidence on the mechanism: Why do rich people save more?

(a) Future consumption: \( \tilde{\phi}^{IV} \) by age.

(b) Transfers to children, by type and age. (USD)

*Note.* Panel (a) shows the estimated permanent income elasticity of consumption using the initial income instrument, by 5-year age groups (blue squares, with grey error bars). In black are the estimates from the quantitative model of Section 1.5. Panel (b) shows transfers to children, averaged by age, for three types of transfers.
1.D.7 Estimation of the income tax progressivity

My life cycle model in Section 1.5 requires an estimate of tax progressivity. To this end, I follow Benabou (2000) and Heathcote et al. (2017) and assume in a given year $t$, total post-tax incomes are a power function of pre-tax incomes,

$$y_{\text{posttax}} = \tau_t \alpha_{\text{inctax}} (y_{\text{pretax}})^{1-\lambda_t}$$

(1.26)

where $\tau_t \alpha_{\text{inctax}}$ is a constant and $\lambda_t \in [0, 1]$ is the tax progressivity parameter (see (1.14)). Clearly, the closer $\lambda_t$ is to 1, the more extremely high incomes are taxed. On its face, the power function seems like an arbitrary assumption, yet as shown in Figure 1.D.3 the fit is remarkable (see also Heathcote et al. (2017)): Figure 1.D.3 is a binned scatter plot for the year 2011, plotting log $y_{\text{posttax}}$, as computed using NBER’s TAXSIM program, against log $y_{\text{pretax}}$ in 20 bins. The sample as in Section 1.4.1, only that I restrict it to households whose head is less than 65 years old (i.e. working age). I further exclude households with very low pre-tax household incomes, namely less than 50% of the average pre-tax income in that year.\(^7^4\)

As part of their data preparation, Heathcote et al. (2017) adjust pre-tax earnings by adding employers’ shares (50%) of social security and medicare taxes. To be consistent with their adjustment, I scale log $y_{\text{pretax}}$ equivalently in all years so that my estimate for $\lambda$ over the same survey years 2001–2007 as theirs yields the same estimate of $\lambda = 0.181$. In the model of Section 1.5 I use the estimated elasticity for 2013, which is $\lambda = 0.16$.

1.D.8 Analytical example for the economic significance of the concavity

To illustrate the economic significance of the estimated income elasticity of consumption, $\phi = 0.7$, I derive a simple example for the case where the income distribution follows an exact Pareto distribution. Denote by $F(y)$ the cdf and by $f(y)$ the pdf of a Pareto distribution with tail parameter $\xi > 0$ and lower bound $\underline{y} > 0$. Since the mean of the income distribution is kept constant under comparative statics, aggregate consumption is given by

$$C = \int_{\underline{y}}^{\infty} f(y)(y/\bar{y})^\phi dy.$$  

\(^7^4\)Both in the data and in my model in Section 1.5, there are other support mechanisms that are active for lower incomes and distort the power law (1.26) upwards, pushing $y_{\text{posttax}}$ above the predicted value from the power law.
Figure 1.D.3: Estimating income tax progressivities.

Note. The figure shows a binned scatter plot with 20 bins of log total post-tax household incomes against total log total pre-tax household incomes for PSID households with a working-age head in 2011. Household incomes below 50% of the average income have been excluded. The blue line is the linear fit, the red line is the 45° line.

Using that $f(y) = \xi y^{-\xi-1}y^\xi$ this can be shown to yield

$$C = \left(\frac{\xi - 1}{\xi}\right)^{\phi} \frac{\xi}{\xi - \phi}.$$  

(1.27)

As I explain in Section 1.D.10 below, I estimate Pareto tails using post-tax income data from Piketty et al. (2016) and the PSID. For example, for the data from Piketty et al. (2016), I find a decline in the Pareto tail parameter from $\xi \approx 2.17$ in 1980 to $\xi \approx 1.66$ at the end of the sample period. This yields a percentage decline in $C$ of $\Delta C/C \approx 5.4\%$, very much in line with Figure 1-5.

1.D.9 Ad-hoc model with pure preference heterogeneity

In this section, I consider the extreme case where the entire variation in Section 1.4 is driven by unobserved preference heterogeneity. I argue—using the ad-hoc model of Section 1.4.5—that even in this case, under plausible assumptions, the effects of rising income inequality are significant. Preference heterogeneity is therefore no guarantee for neutrality.

Thus, suppose the empirical relationship between consumption and permanent income is entirely driven by unobserved preference heterogeneity and not by a concave consumption function in permanent income. In particular, each agent $i$ has a consumption function $C_i(y) = k_i y$ with idiosyncratic shifter $k_i > 0$. So, higher $i$ agents not only have larger incomes but are also more patient, that is, $k_i$ is declining in $i$. To generate the empirical relationship
Note. This figure is the analog of Figure 1-5 for the case of an ad-hoc linear consumption function with preference heterogeneity. As income data, the dashed line uses post-tax income residuals, averaged within households in a ±4 year window; the solid line uses the post-tax income distribution from Piketty et al. (2016).

between \( C_i(y_i) \) and \( y_i \), one would need \( k_i = k y_i^{\phi-1} \) for some common \( k > 0 \). Using the same notation for cdfs \( F \) and \( G \) as before, the change in consumption is now given by

\[
\frac{\Delta C}{C_0} = \int_0^\infty (G^{-1}(F(y)) \cdot y^{-1} - 1) C(y)f(y)dy / \int_0^\infty C(y)f(y)dy.
\]

How large is this quantitatively? Figure 1.D.4 illustrates this for the same two income datasets that were used in Section 1.4.5.\(^75\) It is evident that the assumption of pure preference heterogeneity does not necessitate zero effects from changes in the income distribution. In fact, as Figure 1.D.4 illustrates, the effects may be even larger.

1.D.10 Construction of Figures 1-5 and 1.D.4

Figures 1-5 and 1.D.4 show \( \Delta C/C \) for the concave permanent consumption function case and the pure preference heterogeneity case. \( \Delta C/C \) is computed by computing a measure of consumption \( C \) for each year that holds aggregate income constant. Then, \( \Delta C/C \) is just the percentage change of the consumption measure in any given year relative to 1980.

\(^75\) For details on the calculation of \( \Delta C/C \) see Appendix 1.D.10.
for $C$ is then given by
\[ C = \int_0^\infty f(y)\left(y/\bar{y}\right)\phi \, dy. \]

Since I use data on income shares to compute $C$, I now explain how to rewrite this expression entirely in terms of top income shares
\[ \bar{s}(q) \equiv \bar{y}^{-1} \int_{F(q)}^\infty f(y)\,dy \]
which is a number in $[0,1]$ for each quantile $q \in [0,1]$. I call it “top income share” because it is the income share of all individuals in the top $1-q$ of the population, or equivalently, one minus the Lorenz curve for income. From its definition (1.28) it follows that $\bar{s}'(q) = -\bar{y}^{-1}F^{-1}(q)$ and thus $C$ can be expressed as
\[ C = \int_{0}^{1} (-\bar{s}'(q)) \phi \, dq. \]

For a distribution with Pareto tail with tail parameter $\xi$ and lower bound $y$, $F(y) = 1-(y/y_0)^{-\xi}$ for large $y$, and
\[ \bar{s}(q) = \frac{1}{\bar{y}} \frac{\xi}{\xi-1} y^\xi \left(\frac{y}{1-q}\right)^{-1/\xi} = \frac{\xi}{\xi-1} \frac{y}{\bar{y}} (1-q)^{(\xi-1)/\xi} \]
for $q$ close to 1. Notice that $\bar{y}$ is only equal to $\frac{\xi}{\xi-1} y$ if $F$ is an exact Pareto.

In the data, I only observe $\bar{s}(q)$ on an equally spaced grid with spacing $\Delta q = 0.01$. Denote the grid points by $q_i$ for $i = 0, \ldots, 100$. I approximate $C$ by numerically approximating the integral until a cutoff quantile $q_I$, $I \in \{0, \ldots, 100\}$ and approximating the tail after $q_I$ by a Pareto distribution, such that
\[ C \approx \sum_{i=0}^{I-1} \left( \frac{\bar{s}(q_i) - \bar{s}(q_{i+1})}{\Delta q} \right)^\phi \Delta q + \left( \frac{y}{\bar{y}} \right)^\phi \frac{(1-q_I)^{1-\phi/\xi} - 1}{\phi/\xi}, \]
where the second term is the Pareto tail integral. I set $I = 90$, but the result is very robust to other reasonable choices of $I$. I estimate $\xi$ and $y/\bar{y}$ by regressing $\log \bar{s}(q_i)$ on $\log(1-q_i)$ for $i = I, I+1, \ldots, 100$, which according to (1.29) recovers $(\xi - 1)/\xi$ as slope parameter and $\log \left(\xi/(\xi - 1)y/\bar{y}\right)$ as intercept.

Figure 1.D.4. In the case with pure preference heterogeneity, the expression for $C$ depends both on the income distribution of the year for which $C$ is being calculated and the income distribution that was present at the time $\phi$ was being measured. I denote by $F(y)$ the income
distribution at the time of measurement of \( \phi \) and by \( G(y) \) the income distribution in the year for which \( C \) is being calculated. As before, I observe the top income shares \( \bar{s}_F(q) \) and \( \bar{s}_G(q) \) for both distributions, I measure their respective Pareto tail coefficients \( \xi_F, \xi_G \) and ratios \( \bar{y}_F/\bar{y}_F, \bar{y}_G/\bar{y}_G \) as outlined above, and I pick a threshold quantile \( q_t \).

In this case, \( C \) can be written as

\[
C = \int_0^1 (-\bar{s}''_F(q))^{\phi-1} (-\bar{s}''_G(q)) dq
\]

which is approximately

\[
C \approx \sum_{i=0}^{I-1} \left( \frac{\bar{s}_F(q_i) - \bar{s}_F(q_{i+1})}{\Delta q} \right)^{\phi-1} \left( \bar{s}_G(q_i) - \bar{s}_G(q_{i+1}) \right) + \left( \frac{\bar{y}_F}{\bar{y}_F} \right)^{\phi-1} \left( \frac{\bar{y}_G}{\bar{y}_G} \right) \frac{1 - q_t}{1 + (1 - \phi)/\xi_F - 1/\xi_G}.
\]

I choose \( I = 90 \) as before and assume the year where \( \phi \) was measured in the data is the midpoint of the sample in which I measured it, which is 2007. This concludes the construction of Figure 1.D.4.

1.E Computational Appendix

In this section I explain the methods that were used to simulate the models in Section 1.5. I start by laying out the definition and computation of the steady state of the non-homothetic model in Section 1.E.1. Then, Section 1.E.2 provides details on the regression analyses on model-simulated data that are shown in Table 1.6. In Section 1.E.3, I discuss a number of additional model implications. Finally, Section 1.E.4 goes over the details of the alternative models mentioned in Table 1.8.

1.E.1 Simulation of the model of Section 1.5.1

I first set up the household maximization problem, then I explain how I solve it. Here, I allow for non-stationary environments so as to nest the transitional dynamics exercise of Section 1.6.

Household maximization

Given parameters and given an interest rate path \( \{r_t\} \), a household solves the following dynamic programming problem.
\[
V_{k,s,t}(a,z,\varphi) = \max_{\{c,a'\}} u_k(c) + \beta(1-\delta_k)E_{z,\varphi}V_{k+1,s,t+1}(a' + b', z, \varphi') + \beta\delta_k U(a')
\] (1.30)

\[
c + \frac{1}{1+r_t}a' \leq a + y_{k,s,t}(z)
\]

\[
(b', \varphi') \sim v_t(\cdot|s,k,\varphi)
\]

\[
a' \geq 0.
\]

Here, \(V_{k,s,t}(a,z,\varphi)\) is the agent's value function at time \(t\), \(w_{s,t}\) is the agent's skill price at time \(t\), \(v_t(\cdot|s,k,\varphi)\) is the endogenous distribution of bequests at time \(t\), \(b'\) is the random bequest which is received at the parent's death, and the agent's post-tax, post-transfer income is given by

\[
y_{k,s,t}(z) = \max \left\{ y, \Theta_k(z)w_{s,t} - T_t^{\text{inctax}}(\Theta_k(z)w_{s,t}) \right\}
\]

before retirement, \(k \leq K_{ret}\), and by

\[
y_{k,s,t}(z) = \max \left\{ y, T^{\text{socsec}}(\overline{y}_{s,t}(z), W_t) - T_t^{\text{inctax}}(T^{\text{socsec}}(\overline{y}_{s,t}(z), W_t)) \right\}
\]

after retirement; \(\overline{y}_{s,t}(z)\) is the predicted average pre-tax income, conditional on ending up in state \(z\) when moving into retirement,

\[
\overline{y}_{s,t}(z) = \frac{1}{K_{ret}} \sum_{k=1}^{K_{ret}} E[\Theta_k(z)w_{s,t}|z_{K_{ret}} = z].
\]

**Government.** The government levies a time-dependent nonlinear income tax schedule,

\[
T_t^{\text{inctax}}(y^{pre}) = y^{pre} - \tau_t^{\text{inctax}}(y^{pre})^{1-\lambda_t},
\]

where, given a path for tax progressivity \(\{\lambda_t\}\), \(\{\tau_t^{\text{inctax}}\}\) is chosen to yield the same aggregate tax income in each period.

**Production.** The production function is allowed to be time-dependent,

\[
Y_t = F(t, K_t, \{L_s\}_{s \in S}) = AK_t^\alpha \prod_s (L_s/\overline{L}_s)^{(1-\alpha)\gamma_{s,t}},
\]

where I normalized the skill endowments by group to 1 since in \(L_s\) is inelastically supplied at...
I assume that the representative firm maximizes profits subject to adjustment costs $\zeta(\cdot)$,

$$J_t(K_-) = \max_{d, K, I, s, t} \left\{ d + \frac{1}{1 + r_t} J_{t+1}(K) \right\}$$

$$d = F_t(K_-, \{L_s\}) - \sum_s w_{s,t} L_s - (I + \zeta(I/K_- - \delta)K_-)$$

$$K = K_- - \delta K_- + I$$

The skill prices are given by

$$w_{s,t} = \partial F(t, K_t, \{L_s\})/\partial L_s = (1 - \alpha) \gamma_{s,t} Y_t$$

and the average wage is given by $W_t = \sum_s \bar{\mu}_s w_{s,t}$.

**Adjustment costs.** I choose a standard quadratic adjustment cost function $\zeta(x) = \frac{1}{40} \frac{x^2}{2}$ as for instance in Auclert and Rognlie (2017).

**External excess returns.** As explained above, there can be non-trivial external excess returns in the model, which need to be included in total wealth held by domestic agents. I define this recursively as

$$J_{ext}^t = ExtExcessReturn_t + \frac{1}{1 + r_t} J_{valuation}^t,$$

where excess returns follow from the balance of payments

$$0 = ExtExcessReturn_t + NX + \frac{r_t}{1 + r_t} NFA.$$

**Definition of equilibrium**

**Definition 2.** A *competitive equilibrium* consists of a path of aggregate quantities $\{Y_t, K_t, I_t\}$, paths of distributions $\{\mu_t\}$ of agents and $\{\chi_t\}$ of bequests, both defined over the state space $S$, paths for policy functions $\{c_{k,s,t}(a, z, \varphi), a_{k,s,t}(a, z, \varphi)\}$, paths of prices $\{r_t, w_{s,t}\}$ such that:

(a) each agent solves the optimization problem (1.30) given $\{r_t, w_{s,t}\}$, where the conditional bequest distribution $v_t(\cdot|s, k, \varphi)$ is given by

$$v_t(B, \varphi'|s, k, \varphi) = \begin{cases} 1_{\{0,1\}}(B, \varphi') & \text{if } \varphi = 1 \\ (1 - \delta_{k+k_{born}})1_{\{0,0\}}(B, \varphi') + \sum_{s'} P_{ss'} \chi_t(s', k + k_{born}, B) & \text{if } \varphi = 0 \end{cases}$$
(b) the representative firm sets \( \{K_t, I_t\} \) to solve the profit-maximization problem (1.31) given \( \{r_t, w_{s,t}\} \), (c) the government sets government spending \( G_t \) according to its budget constraint

\[
G_t = \int T^\text{inc-tax}_t (\Theta_k(z_k) w_s) d\mu_t(s, k, a, z, \varphi) + r^b \int b\nu_t(s, k, b) - r_B,
\]

\[
G_t = \int_{(s,k,a,z)} T_k(\Theta_k(z_k) w_s,t) d\mu_t + r^b \int_{(s,k,a,z)} b\chi_t - r_B
\]

(d) the goods market clears,

\[
Y_t = I_t + \int c_{k,s,t}(a, z, \varphi) d\mu_t + N_X,
\]

(e) all markets for efficiency units of each skill clear, \( L_s = \bar{\mu}_s \), (f) the asset market clears,

\[
A_t \equiv \int a d\mu_t = (1 + r_t)B + J_t(K_t) + J^\text{ext}_t + NFA,
\]

(g) the bequest distribution is consistent with the distribution over states, \( \chi_t(s, k, A, z, \varphi) = \delta_k \mu_t(s, k, A, z, \varphi) \), where \( A \subset \mathbb{R}_+ \) measurable, and (h) aggregate flows and bequests are consistent

\[
\mu_{t+1}(s, k + 1, A, z', \tilde{\varphi}) = \sum_{\varphi} \int_{(b', \varphi')} \int_{(s,k,a,z,\varphi)} \int_{a_{k,s,t}(a,z,\varphi) + b' \in A} \prod_{z,z'} d\mu_t d\nu_t(|s, k, \varphi).
\]

**Computing the household's optimal decisions**

I use a version of the method of endogenous grid points, modified to allow for the receipt of bequests. Since this method is fairly standard, I do not explain the basics and instead refer the interested reader to background materials by Carroll (2005).

At age \( T \), the household solves a simple maximization problem between consumption and bequests, which I solve to find the consumption policy \( c_{K\text{death}, s,T}(a, z, \varphi) \). I then iterate backwards using the Euler equation and the method of endogenous grid points,

\[
\hat{c}_{k,s,t}(a', z, \varphi)^{-\sigma} = \beta(1 + r_t) \sum_{z'} \Pi_{zz'} \int_{(b', \varphi')} \left( (1 - \delta_k)c_{k+1,s,t+1}(a + b', z, \varphi')^{-\sigma} + \delta_k U'(a') \right) d\nu_t(b', \varphi'|s, k, \varphi)
\]

where the new policy function is then computed by inverting the budget constraint,

\[
c_{k,s,t}(a', z, \varphi) + \frac{1}{1 + r_t} a' = a + y_{k,s,t}(z)
\]

95
to obtain the asset policy function $a' = a_{k,s,t}(a, z, \varphi)$, and the consumption policy function 

$$c_{k,s,t}(a, z, \varphi) = a + y_{k,s,t}(z) - \frac{1}{1 + r_t} a_{k,s,t}(a, z, \varphi).$$

After solving for the consumption and savings policy functions at all ages and times, I iterate forward the savings policy functions to compute the sequence of distributions across all idiosyncratic states $\{\mu_t(s, k, a, z, \varphi)\}$. Finally, I compute the endogenous bequest distributions $\{v_t(b', \varphi'|s, k, \varphi)\}_t$. I iterate over this entire process until the endogenous bequest distributions have converged.

In this type of framework with stochastic bequests, a key time factor in the simulation is the convolution of a given bequest distribution with a given asset distribution. To ease this issue, I implemented my own non-uniform fast Fourier transform (FFT) algorithm that, different from conventional FFT algorithms still allows me to work on non-uniform grids, as long as those grids are piecewise uniform. This modification sped up the steady state computation considerably.

**Solving for the general equilibrium steady state**

In my steady state analysis, I implement this method with 151 asset states. I discretized the persistent part of the income process using the Rouwenhorst (1995) method on 11 states, and the transitory shock on 3 states. In addition to 65 age states, 3 skill groups, and a number of inheritance states to keep track whether a household has made an inheritance already, this leaves me with 8712 idiosyncratic states, and $\approx 1.3$ million asset–idiosyncratic state pairs. My implementation of the above algorithm allows me to solve the economy given a bequest distribution (and given an interest rate $r$) in approximately 20 seconds. It takes 60 seconds to iterate until the bequest distribution converges (tolerance $10^{-6}$). All times are measured on a 2009 MacBook Pro. The code was implemented in Matlab and C.

To solve for the general equilibrium steady state in this economy, I compute the household’s maximization problem, and aggregate the agents’ wealth levels to get total asset demand $A_{\text{demand}}$. The total asset supply $A_{\text{supply}}$ in the economy splits into four parts: the net foreign asset position $NFA$, government bonds $B$, financial wealth in equities $v = (F_K K - \delta K)/r$, and financial wealth in claims on external excess returns. I use Matlab’s `fzero` command to solve for the equilibrium interest rate $r$ that equalizes $A_{\text{demand}}$ and $A_{\text{supply}}$.

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76The Tauchen (1986) method is generally less precise than the Rouwenhorst (1995) method for persistent income processes (Kopecky and Suen, 2010).
1.E.2 Estimation of regressions on model-simulated data

In Table 1.6 I show regressions that were performed on simulated data from the model. To implement those regressions, I closely follow my empirical analyses in Section 1.4. I construct the same measure of post-tax labor income,

\[ y_{it}^{postax} = y_{it}^{pre} - \frac{y_{it}^{pre}}{y_{it}} T_{intax}(y_{it}^{pre}), \]

that is, I subtract from pre-tax labor income the share of taxes accounted for by labor income, rather than transfers \( y \). In practice, \( y^{pre} = y \) for almost all agents, except when an agent’s income is below the minimum income threshold, below which the agents receives income transfers. The sample of agents I focus on is the same as in the data, namely agents between ages 30 and 65. The data I use to run specifications is a panel of 75,000 agents whose income draws are determined by Monte-Carlo simulations and whose behavior is given by the consumption policy functions implied by the model.\(^7\) After simulating the data, all income observations are multiplied by the measurement error term \( \exp\{\nu_{it}\} \), and then residualized by partialing out age effects.

I implement the following specifications:

**OLS specifications.** Average income residuals are computed as symmetric averages of \( T \) observations, spaced out over \( 2T - 1 \) years to mimic the biennial nature of the PSID sample I use.

**AR(1) IV specifications.** As in Section 1.4, I use quasi-differenced future incomes as instruments, and I include all observations with at least three such instruments. Again, I pretend the panel was biennial, as is the relevant subsample of the PSID that I use.

**Initial income IV specification.** The initial income IV specification is constructed using initial income at age 25 as instrument. For comparability for the previous two specifications, I also restrict the consumption data to be between years 30 and 57 (as I do in the data).

In addition to those three specifications, I also run additional specifications for comparison across models in Table 1.8.

**BPP specification.** I replicate a (simplified) version of the approach in Blundell et al. (2008), henceforth BPP, as follows. Conditional on simulated data on log labor income residuals \( \hat{y}_{it} \) and log consumption residuals \( \hat{c}_{it} \) (both after partialing out age dummies), BPP

\(^7\)I also experimented with many iterations of smaller samples to ensure there are no small sample biases.
estimate the specification

\[ \hat{y}_{it} = \eta_{it} + \psi_{it} \]

\[ \eta_{it} = \eta_{it-1} + \epsilon^\eta_{it} \]

\[ \psi_{it} = \epsilon^\psi_{it} + \theta \epsilon^\psi_{it-1} \]

\[ \Delta \hat{c}_{it} = \phi^{BPP} \epsilon^\eta_{it} + \chi \epsilon^\psi_{it} + \xi_{it}. \]

Here, \( \{\epsilon^\psi_{it}, \epsilon^\eta_{it}, \xi_{it}\} \) are iid shocks, mutually independent of each other. In words, BPP assume a permanent-transitory income process, with the transitory component being modeled as MA(1) process, and evaluate how much of the permanent income innovation \( \epsilon^\eta_{it} \) is “passed through” to consumption; this is captured by the coefficient \( \phi^{BPP} \); and how much of the transitory shock \( \psi_{it} \) is passed through; this is captured by \( \chi \). Here, I focus only on \( \phi^{BPP} \).

Following the reasoning in BPP’s Appendix C and extending the logic to a MA(1) process for transitory income shocks, \( \phi^{BPP} \) is identified in the above model as the IV estimate of a regression of \( \Delta \hat{c}_{it} \) on \( \Delta \hat{y}_{it} \), using \( \hat{y}_{it+2} - \hat{y}_{it-3} \) as instrument.

Retirement wealth slope. Finally, I implement a specification based on retirement wealth that has been used to test for savings non-homotheticity before, see e.g. Gustman and Steinmeier (1999); Venti and Wise (2000). To this end, I measure each agent’s asset position at retirement \( a^\text{ret}_i \), and compute each agent’s (annualized) lifetime earnings,

\[ y^P V_i = \sum_{k=1}^{K_{ret}} R^{-k} y_{ik} \quad y^P V,a_i = \frac{1}{\sum_{k=1}^{K_{ret}} R^{-k} y^P V_i}. \]

Here, \( y_{ik} \) denotes agent \( i \)’s post-tax and post-transfer income, \( y_{ik} = \max\{y, \text{Inc}_i(y^\text{pre}_t)\} \), at age \( k \). Finally, I estimate

\[ \frac{a^\text{ret}_i}{y^P V_i} = \alpha + \gamma y^P V,a_i + v_{it} \]

and report \( \gamma \) in Table 1.8.

1.E.3 Additional steady state results

The schedule of elasticities \( \sigma_k \)

Table 1.E.1 shows the age-dependent elasticities \( \sigma_k \) for each age \( k \).

---

\( ^{78} \)The notation was adjusted to match the one used in this paper. Since the economy in which I estimate this specification is stationary, I only consider the case in which \( \phi \) and \( \chi \) as well as all shock variances are constant.
Table 1.E.1: The age profile of elasticities.

<table>
<thead>
<tr>
<th>Age</th>
<th>$\sigma_k$</th>
<th>Age</th>
<th>$\sigma_k$</th>
<th>Age</th>
<th>$\sigma_k$</th>
<th>Age</th>
<th>$\sigma_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>11.7204</td>
<td>45</td>
<td>3.7369</td>
<td>65</td>
<td>1.6402</td>
<td>85</td>
<td>1.2581</td>
</tr>
<tr>
<td>26</td>
<td>11.0686</td>
<td>46</td>
<td>3.5389</td>
<td>66</td>
<td>1.5972</td>
<td>86</td>
<td>1.2581</td>
</tr>
<tr>
<td>27</td>
<td>10.453</td>
<td>47</td>
<td>3.3561</td>
<td>67</td>
<td>1.5576</td>
<td>87</td>
<td>1.2581</td>
</tr>
<tr>
<td>28</td>
<td>9.8716</td>
<td>48</td>
<td>3.1872</td>
<td>68</td>
<td>1.5211</td>
<td>88</td>
<td>1.2581</td>
</tr>
<tr>
<td>29</td>
<td>9.3226</td>
<td>49</td>
<td>3.031</td>
<td>69</td>
<td>1.4875</td>
<td>89</td>
<td>1.2581</td>
</tr>
<tr>
<td>30</td>
<td>8.8041</td>
<td>50</td>
<td>2.8865</td>
<td>70</td>
<td>1.4567</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>8.3144</td>
<td>51</td>
<td>2.7527</td>
<td>71</td>
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<tr>
<td>32</td>
<td>7.852</td>
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<td>2.6288</td>
<td>72</td>
<td>1.4028</td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>7.4153</td>
<td>53</td>
<td>2.514</td>
<td>73</td>
<td>1.3795</td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>7.0029</td>
<td>54</td>
<td>2.4075</td>
<td>74</td>
<td>1.3585</td>
<td></td>
<td></td>
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<tr>
<td>35</td>
<td>6.6134</td>
<td>55</td>
<td>2.3088</td>
<td>75</td>
<td>1.3397</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>6.2456</td>
<td>56</td>
<td>2.2172</td>
<td>76</td>
<td>1.3229</td>
<td></td>
<td></td>
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<tr>
<td>37</td>
<td>5.8982</td>
<td>57</td>
<td>2.1323</td>
<td>77</td>
<td>1.3082</td>
<td></td>
<td></td>
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<tr>
<td>38</td>
<td>5.5702</td>
<td>58</td>
<td>2.0534</td>
<td>78</td>
<td>1.2955</td>
<td></td>
<td></td>
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<tr>
<td>39</td>
<td>5.2604</td>
<td>59</td>
<td>1.9802</td>
<td>79</td>
<td>1.2847</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>4.9678</td>
<td>60</td>
<td>1.9123</td>
<td>80</td>
<td>1.2758</td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>4.6915</td>
<td>61</td>
<td>1.8493</td>
<td>81</td>
<td>1.2687</td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>4.4306</td>
<td>62</td>
<td>1.7909</td>
<td>82</td>
<td>1.2634</td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>4.1842</td>
<td>63</td>
<td>1.7367</td>
<td>83</td>
<td>1.2599</td>
<td></td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>3.9514</td>
<td>64</td>
<td>1.6866</td>
<td>84</td>
<td>1.2581</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. This table shows the age profile of elasticities $\sigma_k$. 
Risk aversion over the life cycle

To elucidate the implications of non-homothetic preferences for risk aversion, and how it changes over the life cycle, I compute for each age the average curvature of the value function. Let \( \overline{\mu}_k(s, a, z, \varphi) \) be the distribution over \((s, a, z, \varphi)\) conditional on age \(k\). Then, define the average risk aversion at age \(k\) as

\[
\Sigma_k \equiv - \int \frac{a \partial_a \partial_a V_{k,s}(a, z, \varphi)}{\partial_a V_{k,s}(a, z, \varphi)} d\overline{\mu}_k(s, a, z, \varphi).
\]

Figure 1.E.1 shows the path of risk aversions \( \Sigma_k \) for the homothetic and the non-homothetic economies.

As can be seen, risk aversion in the homothetic economy increases considerably over the life cycle. If risky assets were available in this economy, this would suggest that young agents have by far the largest shares of risky assets in their portfolios, while older agents hold mostly risk-less assets. The intuition for this is generally that labor income can be viewed as "bond" with a limited amount of risk, which young agents are well endowed with, much more than older agents.

As shown in Ameriks and Zeldes (2004), this is not the case in the data. Younger agents tend to own roughly equally safe portfolios as their older agents. Interestingly, this is exactly what the non-homothetic model generates: there, \( \Sigma_k \) increases only mildly with age, rationalizing why portfolios do not become more risky over the life-cycle.\(^79\)

The distribution of MPCs

As Section 1.C.2 illustrates, in a neutral model, the distribution of MPCs is the same within different skill groups. In the data, it seems to be the case that MPCs actually (unconditionally) decline in measures of permanent income, such as education Jappelli and Pistaferri (2006, 2014).

Interestingly, this is precisely what the non-homothetic economy implies. Figure 1.E.2 shows MPCs at the median income state (when \( \eta = \psi = 0 \)), averaged over all asset states and ages, for the bottom 90\% (solid) and the top 1\% (dashed). The colors represent the homothetic economy (red) and the non-homothetic economy (blue).

\(^79\)See Wachter and Yogo (2010) for an important contribution that makes this point in a non-homothetic life-cycle economy.
Figure 1.E.1: Risk aversion over the life cycle.

Note. The figure shows the age profiles of risk aversion, measured as average curvature in the value function, in the homothetic (red, dashed) and non-homothetic (blue, solid) models.

Figure 1.E.2: Average MPCs by skill at the median income state.

Note. The figure shows the MPC at the median income state, averaged over all asset states and ages, for the bottom 90% (solid) and the top 1% (dashed). The colors represent the homothetic economy (red) and the non-homothetic economy (blue).
Table 1.E.2: Work-life savings rates by permanent income.

<table>
<thead>
<tr>
<th></th>
<th>Savings rate in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>top 1%</td>
</tr>
<tr>
<td>Non-homothetic model</td>
<td>57%</td>
</tr>
<tr>
<td>Homothetic model</td>
<td>17%</td>
</tr>
</tbody>
</table>

Note. This table shows savings rates by permanent income, for the non-homothetic and the homothetic economies. The savings rates are defined as \((y + ra - c)/(y + ra)\) and are averaged over the entire work-life (25–64 years) and all idiosyncratic income and asset states.

In the homothetic economy, MPCs of the top 1% lie above those of the bottom 90% conditional on assets, and would be exactly equal unconditionally, since the top 1% tend to have larger asset positions. In the non-homothetic economy, MPCs of the bottom 90% are higher than their homothetic counterparts, for small asset positions, while the MPCs of the top 1% are lower.

This illustrates that non-homothetic preferences increase the spread in the distribution of MPCs, and can rationalize why high permanent income agents may have lower MPCs (both unconditionally or conditional on assets).

Savings rates by permanent income group

One way to look at the influence of non-homothetic preferences on agents’ behavior is through savings rates. Table 1.E.2 shows average work-life (25–64 years) savings rates, defined as \(\frac{y + ra - c}{y + ra}\), for each of the three skill groups. Vast differences are visible in the non-homothetic model: the average savings rate in the top 1% is 57%, while the next 9% are able to save 30%. The bottom 90% only save on average 1%. The homothetic economy, by contrast, is almost neutral\(^8\) and essentially shows no difference in savings behavior by permanent income.

Joint distribution of labor and capital income

The reason behind the success in generating wealth inequality in this model is purely non-homothetic savings behavior. A way to gauge whether this channel is too strong in the model relative to the data is to study implications for the joint distribution of income and wealth, which was recently highlighted in Aaberge et al. (2013) and Alvaredo et al. (2013).

\(^8\)Recall that it does allow for bequests and imperfect skill transmission, and therefore is not entirely neutral.
In Table 1.E.3, I compute the fraction of agents in the top 1% of the labor income distribution that are also in the top 1% of the wealth distribution (and vice versa), and compare it to the data in Aaberge et al. (2013). It is clear that top incomes and top wealth levels are too aligned relative to the data, but not by very much. In fact, the data is for the year 2000, and it could well be that the income and wealth distributions have become even more aligned in recent years. In addition, there is of course a lot more sources of variation in the data than in the model—for instance heterogeneity in returns on wealth—that could explain why the income-wealth alignment is somewhat lower in the data.

**Lorenz curves**

The Lorenz curves for pre-tax incomes, consumption and wealth are shown and compared to the data in Figure 1.E.3, panel (a) (see Appendix 1.E.3 for details). To construct the Lorenz curve for pre-tax income in the data, I add back the estimated cubic age profile to the residualized incomes of the 2011 and 2013 waves of the PSID. This gives me a distribution of incomes that is comparable across years and households of various sizes but does not strip out the age efficiency profile. The curves align fairly closely with pre-tax income in the model.

The data for the empirical Lorenz curve for wealth is taken from World Top Income Database for the U.S. in 2014 (Saez and Zucman, 2016). Wealth is here measured as net personal wealth, that is assets (including housing) net of debts, capitalizing capital income data from income tax returns. In the model, I use the wealth distribution, i.e. the distribution of $a_i$'s, of all households (of all ages). Here, as stressed in Section 1.5.3, the model is successful in matching the overall distribution of wealth, especially compared to the homothetic model, which is shown as the dotted red line.

The data for the Lorenz curve for consumption is computed in the same way as the Lorenz curve for income, from the PSID. Strikingly, it shows a much smaller difference between the non-homothetic and homothetic economies—much like in the simple model of Section 1.2.

---

**Table 1.E.3: Joint distribution of income and wealth.**

<table>
<thead>
<tr>
<th></th>
<th>Fraction of top 1% (labor income)</th>
<th>Fraction of 1% (wealth)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in top 1% (wealth)</td>
<td>in top 1% (labor income)</td>
</tr>
<tr>
<td>Data for 2000 (Alvaredo et al., 2015)</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>Model (calibrated to 2014)</td>
<td>0.34</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Note. This table shows the joint distribution of top income and top wealth shares, both in the non-homothetic baseline economy and in the data.

---

81 In fact, this is exactly what happens during the model’s transitional dynamics.
The non-homothetic model does, however, predict larger consumption inequality at the top end compared to the data. One reason for this could be that certain expenses (such as for kids’ education, see Figure 1.D.2) are counted as expenditure in the model, but are not in the data.

Life cycle profiles

Panels (b) and (c) of Figure 1.E.3 show the life cycle profiles of post-tax income, consumption and wealth in the two models, for the 25th, 50th, 75th and 90th percentiles. While the income profiles are the same during the working life, the retirement system in the homothetic case is a simple linear rule for social security. Comparing the life cycle profiles of consumption, the key difference is that consumption profiles are generally higher for the lower percentiles (25th and 50th) and shifted more towards higher ages for the higher percentiles (75th and 90th). This behavior is at the heart of the non-homothetic life cycle model: richer agents tend to save more out of their income, which limits their consumption expenditure early on and increases both consumption expenditure and bequests later in life.

The consequences for wealth accumulation and within-cohort wealth dispersion are especially striking: The top wealth percentiles increase much more rapidly during the work life, and fall less rapidly during retirement. Again, this is the product of two key model ingredients. First, due to the non-homotheticity in preferences, the wealth of the rich increases more rapidly during the work-life. Second, due to the non-homotheticity in bequests, this additional wealth is dissaved much more slowly, which feeds back into larger bequests for the children of rich parents. Since disproportionately many of these children are high-skilled themselves, this again implies a steeper increase in wealth during the work-life of an average high-skilled agent.

Construction of the life cycle profiles. I explain the computation of the life cycle plots in Figure 1.E.3 by percentiles using the age profile of wealth as an example. For every age $k$, I compute the asset distribution of agents at that age. Call its quantile function $Q_k(p)$. Since the distribution is discrete, I approximate $Q_k(p)$ for each age $k$ locally around a given percentile $p_0$ with a log-normal distribution, $Q_k^{lognormal}(p)$. Since I do not consider the very top percentiles, the log-normal distribution fits very well. For a given percentile $p_0$, the plot shows the age profile of the fitted log-normal percentiles $Q_k^{lognormal}(p_0)$. The dollar figures were computed using an average household income of $70,000 (USD in 2014).
Figure 1.E.3: Key life-cycle characteristics of the non-homothetic and homothetic model.

(a) Both economies: Lorenz curves for pre-tax income, consumption, and wealth.

(b) Non-homothetic economy: Life-cycle profiles of pre-tax income, consumption, and wealth.

(c) Homothetic economy: Life-cycle profiles of pre-tax income, consumption, and wealth.

Note. This figure shows key characteristics of two economies: a non-homothetic economy, where rich households save disproportionately more than poor households in relation to their incomes, and a standard homothetic economy, where the savings rates are equal. See Appendix 1.E.3 for details.
Life cycle dispersion

While the plots in Figure 1.E.3 are instructive to illustrate the non-homotheticity, there is a second more standard way to illustrate the degree to which the dispersion in consumption and income increases over the work-life. This way of looking at life-cycle consumption and savings behavior goes back to Deaton and Paxson (1994) who used it to think about the degree to which agents have access to informal insurance arrangements. More recently, age profiles of dispersion in income and consumption have been examined by Storesletten et al. (2004), Guvenen (2007) and Huggett et al. (2011), among many others.

There are two common ways to match both the rise in income dispersion and consumption dispersion: either a persistent AR(1) income process, see e.g. Storesletten et al. (2004); or an income process with less persistence but ex-ante heterogeneous income profiles (HIP), which, together with learning about the slope, allows to match both life-cycle dispersion plots (Guvenen, 2007).

The baseline non-homothetic model introduced in Section 1.5 falls in neither of those two categories. The income process has a persistence parameter between these two categories, but without heterogeneity in income profiles. I now explore the dispersion profiles in the baseline non-homothetic economy, as well as in an economy that in addition has heterogeneity in income profiles (modeled as in Guvenen (2007, 2009)), to better capture the increase in dispersion over the life cycle.82

Figure 1.E.4 shows the age profiles of income and consumption dispersions in both the baseline non-homothetic economy and the HIP extension. It can be seen that the baseline economy predicts too small an increase in the income dispersion and about the right increase in consumption dispersion. Also, the consumption dispersion is convex in age, rather than approximately linear, which is likely driven by the simplistic assumption that $\sigma_k$ falls exponentially in age. The HIP extension improves the fit for income dispersions, since by construction, it matches the overall increase in income dispersion. It also leads to greater consumption dispersion, but not by a lot.

Data construction. The income dispersion profile was computed using PSID waves from survey year 1969 to survey year 2013. I follow the construction in Huggett et al. (2011). Income is pre-tax labor income, and the sample consists of all male-headed households whose real income (in 1968 USD) is between $1,500 and $1.5m, and who supplied between 520 and 5820 hours of work in the respective year. The consumption dispersion profile was computed

---

82 The addition of heterogeneity in income profiles slightly pushes up the estimates of the permanent income elasticity of consumption shown in Section 1.5.3, by around 0.03. (The HIP model could be re-calibrated to match the empirical estimate for $\phi$.)
Figure 1.E.4: Age profiles of income and consumption dispersion.

Note. This figure shows the income and consumption dispersion over the work-life (normalized to 0 at age 25) for two models: the baseline non-homothetic model and an extension with heterogeneous income profiles that is designed to match the increase in income dispersion in the data. The data on the income dispersion comes from the PSID and on the consumption dispersion comes from the CEX. Both were constructed using cohort fixed effects following Deaton and Paxson (1994).

using data from the CEX that was downloaded from Fabrizio Perri’s website (see the data appendix to Krueger and Perri (2006)). I use total consumption expenditure and exclude all observations with consumption expenditure below 5% of the average.

1.E.4 Background on the models used for comparison in Section 1.5.4

I compare the non-homothetic and homothetic life cycle models to a variety of other life cycle and infinite horizon models in Table 1.8. Here, I explain for each model how it was calibrated and list its parameters. Generally, my goal was to use parameters as standard as possible, and as consistent across models as possible. After parameter choices were made, the discount factor was calibrated in all models to yield the same post-tax equilibrium interest rate of 3%. This is important to avoid, for instance, that agents are constantly against their borrowing constraint (e.g. if $\beta$ is too low relative to $r$), which would make finding $\phi = 1$ unsurprising. All model parameters are listed in Table 1.E.4.

All alternative models are extensions of the homothetic framework that was introduced at the end of Section (1.5.2). In addition, to ensure that the model is perfectly neutral, I assume the government levies a 100% tax rate on bequests.83 This entirely neutral framework is the

83 One could have also introduced a perfect market for annuities to ensure neutrality. I expect the results to be very similar.
Table 1.E.4: Calibrated parameters the comparison models.

<table>
<thead>
<tr>
<th>Model Type</th>
<th>Parameter 1</th>
<th>Parameter 2</th>
<th>Parameter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-homothetic</td>
<td>0.886</td>
<td>16.059</td>
<td>1.616</td>
</tr>
<tr>
<td>Homothetic</td>
<td>0.990</td>
<td>1.340</td>
<td></td>
</tr>
<tr>
<td>Homothetic w/ out bequests</td>
<td>0.997</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Homothetic w/ social security</td>
<td>0.998</td>
<td>0.392</td>
<td></td>
</tr>
<tr>
<td>Homothetic w/ but luxury bequests</td>
<td>0.999</td>
<td>19.967</td>
<td>9.208</td>
</tr>
<tr>
<td>AR(1) with $\rho = 0.95$</td>
<td>0.998</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Permanent-transitory</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heavy-tailed</td>
<td>0.991</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extreme productivity state</td>
<td>0.975</td>
<td></td>
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<tr>
<td>Heterogeneous income profiles</td>
<td>0.998</td>
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<td></td>
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<tr>
<td>Partial insurance</td>
<td>1.003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Random discount factors</td>
<td>0.956</td>
<td></td>
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</tr>
</tbody>
</table>
model underlying the first model in group 2 in Table 1.8.

Alternative preferences or transfers. In the second model in group 2, the government uses a realistic social security schedule $T^{socsec}(y, W)$, given by the one used in Section 1.5.2. The third model in group 2 extends the framework by a non-homothetic bequest motive, and calibrates $\kappa$ and $a$ to match both the total bequest flow (relative to GDP) in the economy, as well as the share of agents with bequests below 6.25% of average income.

Alternative income processes. The first model in group 3 is an extension of the neutral framework where the persistence of the income process is changed from $\rho = 0.90$ to $\rho = 0.95$, keeping the stationary variance $\sigma_\eta^2/(1 - \rho^2)$ of the persistence component the same.

The second model is an economy with a permanent-transitory income shock, that is, with $\rho = 1$. I use the parameters in Kaplan and Violante (2010), which are $\sigma_\eta^2 = 0.01$, $\sigma_\psi^2 = 0.05 - \sigma_\psi^2$, $\sigma_\alpha^2 = 0.15$.

The third model is a model with heavy-tailed income shocks. Here, I incorporate some elements of the income process in Guvenen et al. (2016), and assume that the income process is given by

$$\hat{y}_{it} = \eta_{1t} + \eta_{2t} + \psi_{it},$$

where $\eta_{lt}$ is an AR(1) for $l = 1, 2$, with innovations that are mixed normals,

$$\eta_{lt} - \rho \eta_{lt-1} \sim \begin{cases} -p^l \mu^l & \text{with prob. } 1 - p^l \\ N((1 - p^l) \mu^l, (\sigma^l)^2) & \text{with prob. } p^l \end{cases}.$$

The initial standard deviations of $\eta_{lt}$ are given by $\sigma^l_{ini}$. I take the parameters $\rho^l, \sigma^l, \mu^l, \sigma^l_{ini}$ directly from Guvenen et al. (2016), Column 3 of Table II. Since I do not incorporate the entire (more complicated) income process in Guvenen et al. (2016), I rescale the innovation $\eta_{lt} - \rho \eta_{lt-1}$ to match the increase in the life cycle variance of 0.6 in Guvenen et al. (2016). This implies $\sigma^{l=2} = 0.20$ and $\mu^{l=2} = -0.24$. Finally, I compute $p^l$ as an average over the age-dependent probabilities in Table III in Guvenen et al. (2016).

The fourth model is a simple adaption of the extreme-productivity state in Kindermann and Krueger (2017). In particular, I assume that: (a) there is no transitory component, $\sigma_\psi^2 = 0$; (b) the persistent component $\eta_{lt}$ is given by a 7-state Markov chain, with transition
matrix

\[
\Pi = \begin{bmatrix}
0.957899 & 0.028954 & 0.000328 & 0.000002 & 0 & 0.012817 & 0 \\
0.007239 & 0.958063 & 0.021717 & 0.00164 & 0 & 0.012817 & 0 \\
0.000055 & 0.014478 & 0.958118 & 0.014478 & 0.000055 & 0.012817 & 0 \\
0 & 0.000164 & 0.021717 & 0.958063 & 0.007239 & 0.012817 & 0 \\
0 & 0.000002 & 0.000328 & 0.028954 & 0.957899 & 0.012817 & 0 \\
0 & 0 & 0.028087 & 0 & 0 & 0.969688 & 0.002225 \\
0 & 0 & 0 & 0 & 0 & 0.267852 & 0.732148
\end{bmatrix}
\]

and productivity levels

\[
\eta = \begin{bmatrix}
0.2112 & 0.4595 & 1 & 2.1761 & 4.7353 & 7.3949 & 1284.3139
\end{bmatrix}
\]

that corresponds to the transition matrix of educated agents in Kindermann and Krueger (2017). In particular, there is an extreme productivity state (state 7) in which agents earn a large multiple of the earnings in all other states. The persistent component \( \alpha_i \) has 4 states, \( \pm \sigma \pm p \) where \( \sigma^2 = 0.1517 \) and \( p \) (the college premium) is equal to \( \log(1.8) \) (Krueger and Ludwig, 2016).

For the fifth model in group 5, the income process is changed to allow for heterogeneity in ex-ante known income profiles,

\[
\tilde{y}_{ik} = \alpha_i + \beta_i(k - 1) + \eta_{it} + \psi_{it}
\]

where \( k \geq 1 \) is the agent’s age, and \( \beta_i \) is drawn from a normal distribution with variance \( \sigma^2_{\beta} \). I assume \( \rho = 0.82 \) as in Guvenen (2009) and \( \sigma^2_{\beta} = 0.00012 \) to match the life cycle increase in the variance of income of 0.3.

Other models. The first model in group 4 is a partial insurance economy, where, following Guvenen and Smith (2014) agents partially insure themselves against innovations to \( \eta_{it} \), so that actual income \( \log y_{it}^{actual} \) differs from observed income \( \log y_{it} \) and is given by

\[
\log y_{it}^{actual} = \log y_{it} + \theta \{ \mathbb{E}_{t-1}[\log y_{it}] - \log y_{it} \}.
\]

For example, if an agent experiences an unexpected decline in income, the term in curly brackets is positive. Thus, if, say, \( \theta = 0.5 \), that agent would receive a transfer that dampens the impact of this shock by 50%. \( \theta \) is estimated by Guvenen and Smith (2014) to be equal to

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84 The results are similar if the transition matrix for uneducated agents is being used here.
0.451 and this is the parameter I use in my simulation.

The second model in group 4 is a random discount factor economy. Random discount factors are commonly used in infinite horizon economies (interpretable as dynastic economies), so I consider them in an infinite horizon economy as well. This means, I assume that there is no death risk, \( \delta_k = 0 \) for all \( k \), no life cycle earnings structure, \( \Theta_k(z) = 1 \) for all \( k \), and no retirement, \( K_{ret} = K_{death} = \infty \) (and therefore also no social security). I introduce shocks to discount factors by allowing the discount factor between periods \( t \) and \( t+1 \), denoted by \( \beta_t \), to evolve according to the AR(1) process

\[
\beta_t = \rho^\beta \beta_{t-1} + (1 - \rho^\beta) \mu^\beta + \sigma^\beta \epsilon_t^\beta,
\]

where I pick \( \rho^\beta = 0.992 \) and \( \sigma^\beta = 0.0019 \) as in the recent paper by Hubmer et al. (2016). To estimate the regression on an infinite horizon economy, I simulate the model from some time \( t \) onwards, and pretend that all agents had age \( k = 1 \) at that time.

1.F Simulation of the transitional dynamics in Section 1.6

The transitional dynamics are computed as a (non-stationary) equilibrium, defined in Section 1.E.1. As defined there, the equilibrium has perfect foresight.

**Simulation.** I simulate the transitional dynamics starting at a steady state with the 1970 income distribution, for \( T = 400 \) years into the future. After 2014, the labor income shares that are fed into the model are assumed to remain constant (see Figure 1-8). To avoid memory problems when simulating long transitional dynamics, it is necessary to reduce the state space, down from 1.3 million states. I achieve this by reducing the income states to 10 (5 for the persistent shock \( \times 2 \) for the transitory shock), so that in total there are 400k states in this reduced version of the model. I verified that the steady state predictions of the reduced version are comparable to the larger model.

**Algorithm.** The computation of an equilibrium candidate given an interest rate path \( \{r_t\} \) is as follows:

1. Assume a path for the measure of bequests \( \{\chi_t\} \).

2. Given \( \{\chi_t\} \), iterate backwards in time using the Euler equation to find the paths for policy functions \( \{c_{k,s,t}(a,z,\varphi), a_{k,s,t}(a,z,\varphi)\} \).

3. Given the policy functions \( \{a_{k,s,t}(a,z,\varphi)\} \), iterate forward in time to compute the path of equilibrium distributions \( \{\mu_t\} \) as well as the path of bequest measures \( \{\chi_t\} \).
4. Repeat Steps 1–3 until the path of bequest measures \( \{ \chi_t \} \) converged.

Each such simulation produces a path for the aggregate asset imbalance,

\[
\delta A_t = -A_t + (1 + r_t)B + J_t(K_t) + J_t^{ext} + NFA.
\]

Below, I use the notation \( \delta A \) for the vector of asset imbalances \( \{ \delta A_t \} \) and \( r \) for the vector of interest rates \( \{ r_t \} \). I use the following algorithm to simulate the transitional dynamics equilibrium:\(^5\)

1. **Compute an approximated Jacobian \( J \), whose columns are denoted by (\( J_t \))**: 
   
   (a) Compute the "revaluation impulse" \( J^{\text{reval}} \) as the slope of the response \( \delta A \) that is obtained if the initial value of capital \( J_0(K_0) + J_0^{\text{ext}} \) is increased by a small amount. This step is important because any future interest rate change affects the value of current non-bond assets from capital, and from trading external assets.

   (b) Denote by \( e_t \) the \( t \)-th unit vector. Compute \( J_t \equiv \frac{1}{\Delta r} \delta A_t (r^{\text{final}} + \Delta r \times e_t) \) for a small step \( \Delta r \) for a small number of times \( t = t_1, \ldots t_i \). I use \( i = 8 \) and \( \{ t_i \} = \{ 1, 2, 3, 4, 5, 10, 55, 100 \} \). Store the "revaluation effect" that each such simulation produces, that is, store \( \delta v_t \equiv \frac{\Delta (J_0(K_0) + J_0^{\text{ext}})}{\Delta r_t} \), and compute the "pure" response (without revaluation) \( J_t^{\text{pure}} \equiv J_t - \delta v_t J^{\text{reval}} \).

   (c) For any time \( t \) that is not in \( \{ t_1, \ldots, t_i \} \), compute \( J_t^{\text{pure}} \) as follows. Here, \((J_t^{\text{pure}})_s\) denotes the \( s \)-th element of the vector.

   i. Extrapolate \( J_t^{\text{pure}} \) assuming a log-linear decay at both sides. After this step, \((J_t^{\text{pure}})_s\) is treated as a two-sided vector, where \((J_t^{\text{pure}})_s\) is well defined for any \( s \in \mathbb{Z} \). Extrapolate \( J^{\text{reval}} \) in the same way.

   ii. When \( t < t_i \) and \( t \) lies between \( t_\ell < t \leq t_{\ell'} \) with \( t_\ell, t_{\ell'} \in \{ t_1, \ldots t_i \} \), interpolate \( J_t^{\text{pure}} \) horizontally, that is, for all \( s \in \mathbb{Z} \),

   \[
   (J_t^{\text{pure}})_{t+s} \equiv \exp \left\{ \frac{t_{\ell'} - t}{t_{\ell'} - t_\ell} \log(J_{t_\ell}^{\text{pure}})_{t_\ell+s} + \frac{t - t_\ell}{t_{\ell'} - t_\ell} \log(J_{t_{\ell'}}^{\text{pure}})_{t_{\ell'}+s} \right\}.
   \]

   iii. When \( t > t_i \), shift \( J_t^{\text{pure}} \) horizontally,

   \[
   (J_t^{\text{pure}})_{t+s} \equiv (J_{t_0}^{\text{pure}})_{t+s}.
   \]

\(^5\)The algorithms presented here were jointly developed with Adrien Auclert and Matt Rognlie.
(d) Compute the columns of the approximated Jacobian as

\[ J_t = J_{t}^{pure} + \delta v_t \times J_{t}^{eval} \]

which gives a vector \( J_t \in \mathbb{R}^T \) for each \( t \in \{0, \ldots, T-1\} \).

2. Use a nonlinear version of Krylov subspace method \( GMRES \) to solve for the equilibrium interest rate \( r \), starting with guess \( r^{(0)} = r^{final} \):

(a) Given guess \( r^{(n)} \), \( n \geq 0 \), evaluate \( \delta A(r^{(n)}) \).

(b) Compute \( r^{(n+1)} = r^{(n)} - J^{-1} \cdot \delta A(r^{(n)}) \).

(c) Compute \( r^{(n+1)} = \sum_{m=1}^{n+1} \lambda_m \tilde{r}^{(m)} \) where \( \lambda_m = 1 \) and the weights \( \{\lambda_m\} \) minimize the norm

\[ \| \sum_{m=0}^{n} \lambda_{m+1} \delta A(r^{(m)}) \|. \]

(d) Go back to step 2(a) until \( \| \delta A(r^{(n)}) \|_{\infty} \) is sufficiently small.

This procedure takes around 80 minutes to converge on a 2009 MacBook Pro, with a tolerance of \( 10^{-7} \) for \( \| \delta A(r^{(n)}) \|_{\infty} \).
1. G References


- and _ , "Measuring the Natural Rate of Interest Redux," 2015


Chapter 2

A Theory of Foreign Exchange Interventions

This chapter develops a theory of foreign exchange interventions in a small open economy with limited capital mobility. Home and foreign bond markets are segmented and intermediaries are limited in their capacity to arbitrage across markets. As a result, the central bank can implement nonzero spreads by managing its portfolio. Crucially, spreads are inherently costly, over and above the standard costs from distorting households’ consumption profiles. The extra term is given by the carry-trade profits of foreign intermediaries, is convex in the spread—as more foreign intermediaries become active carry traders—and increasing in the openness of the capital account—as foreign intermediaries find it easier to take larger positions. Optimal interventions balance these costs with terms of trade benefits. We show that they lean against the wind of global capital flows to avoid excessive currency appreciation. Due to the convexity of the costs, interventions should be small and spread out, relying on credible promises (forward guidance) of future interventions. By contrast, excessive smoothing of the exchange rate path may create large spreads, inviting costly speculation. Finally, in a multi-country extension of our model, we find that the decentralized equilibrium features too much reserve accumulation and too low world interest rates, highlighting the importance of policy coordination.

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2.1 Introduction

Foreign exchange interventions are among the most important macroeconomic policy tools, yet among the least understood. Countries mainly use them for two purposes: To manage their exchange rate, when relying on monetary policy alone is infeasible or undesired, and to accumulate reserves as insurance against sudden stops. Both roles are of crucial importance. There is ample evidence that many countries intervene to dampen exchange rate volatility, slow down exchange rate adjustments or lean against capital flows. And reserve accumulation has gone so far that now $12 trillion, or 80% of US GDP, are being saved by the world’s central banks. The effect of this reserve hoarding on global imbalances, world interest rates and exchange rates can hardly be overstated.

In light of the popularity of foreign exchange interventions among policymakers, it might almost come as a surprise that there is relatively little guidance from theory on how they should be conducted. Although there has been important progress in recent years (see Cavallino, 2015 and Liu and Spiegel, 2015), the answer to many important questions remains incomplete. When are foreign exchange interventions desirable? How costly are they and what is the right, welfare-relevant way to measure these costs? How should interventions be designed to maximize their effectiveness, and how does that depend on the specific goal of the intervention? What are the implications of the increasingly common usage of interventions for the world economy? Should countries coordinate their interventions?

In this chapter, we propose a tractable and microfounded framework that speaks to all these questions. We base our analysis on a canonical real small open economy model augmented with limited capital mobility, in which the country faces endowment and interest rates shocks. As is well-known in this type of model, the economy has market power as exporter of its endowment, generating a desire for terms-of-trade management. In the model’s financial markets, domestic and foreign intermediaries can arbitrage between domestic and foreign bond markets, but arbitrage is limited due to a fixed transaction cost and position limits. Under these conditions, a portfolio balance channel emerges: changes in the portfolio of the central bank induce short-lived interest rate spreads—that is, exchange rate adjusted or “UIP” spreads—between domestic and foreign bonds, as in Kouri (1976), Branson and

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1 This holds for emerging markets and advanced economies alike. For emerging markets, see for example the “fear of floating” literature around Calvo and Reinhart (2002), Levy-Yeyati and Sturzenegger (2005a) and McKinnon and Schnabl (2004), among many others. Among advanced economies, recent interventions have for example been conducted by Denmark, Switzerland, or the Czech Republic to depreciate their respective currencies. See Section 2.2 below for more examples.

2 See e.g. Costinot et al. (2014), or Farhi and Werning (2012, 2013).

3 UIP is short for the uncovered interest parity condition which is satisfied if the expected excess return from investing in local bonds is exactly zero.
Henderson (1985), or more recently Gabaix and Maggiori (2015). We analyze this model through the lens of the small open economy’s central bank as social planner and ask: How should it optimally manage its holdings of foreign bonds?

Our first contribution is to show that this central bank planning problem can be entirely framed in terms of the UIP spreads that the central bank’s portfolio choice generates. The logic is straightforward: If a country seeks to depreciate its exchange rate, it sells home bonds and purchases foreign ones, generating a positive UIP spread. Crucial to our analysis, UIP spreads are inherently costly, over and above the standard costs from distorting (domestic) households’ consumption profiles. The reason is intuitive: UIP spreads invite foreign intermediaries to take profitable carry trade positions, and hence the country as a whole is losing money at an amount equal to the carry trade profits made by foreign intermediaries. These additional costs are naturally convex in the level of the spread—as more foreign intermediaries become active carry traders when spreads are higher—and increasing in the openness of the capital account—as foreign intermediaries then find it easier to take larger positions. It is worth stressing that our costs are the welfare-relevant costs identified by our model and apply to the whole country; as such, our costs can, and often will, be different from the central banks’ own quasi-fiscal cost of holding reserves, which do not include costs incurred by the country’s private economy from nonzero UIP spreads.

The formulation of the planning problem in terms of UIP spreads highlights an interesting connection with the recent literature on optimal capital controls (see, for example, Farhi and Werning, 2013). In this literature, the planner typically chooses proportional taxes on capital flows, which also manifest themselves as UIP spreads. The crucial difference to this literature is that in our model, the planner faces an extra cost from nonzero UIP spreads, coming from the carry-trading activities of foreign intermediaries. Indeed, we show that in the limit of zero private capital mobility (financial autarky), our planning problem becomes essentially analogous to one of optimal capital controls. This seems to suggest that capital controls and foreign exchange interventions are substitutes. Yet, this is not the case: Precisely in the limit of zero capital mobility, capital controls are meaningless. Instead, as we explain below, our theory suggests that capital controls and foreign exchange interventions are complements: the former enhance the effectiveness of the latter.

Our second contribution is to fully characterize the optimal policy. We find that it should lean against the wind of capital inflows by implementing positive UIP spreads, and thus depreciate the exchange rate (vice versa when capital flows reverse direction). When faced with positive endowment or wealth shocks, e.g. productivity boosts in the export sector or

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4For a recent study on the quasi-fiscal cost of reserve holdings, see Adler and Mano (2016).
natural resource discoveries, the central bank should intervene to avert any excessive exchange rate movements and replicate the frictionless equilibrium (zero UIP spreads).

For intermediate degrees of capital mobility, the additional cost term delivers several new insights about the optimal design of foreign exchange interventions. First, interventions should be smooth. Since nonzero spreads lead to costly carry trade activities and households are forward-looking, it is undesirable to adjust the spread immediately after a shock hits. Second, the convexity of the cost term generates a desire for spreading out interventions over time. In particular, this includes promising future interventions, a form of “forward guidance” of foreign exchange interventions. As a consequence of this, interventions should be highly inertial, possibly lasting significantly longer than the shock itself. This, however, leads to natural and novel type of time inconsistency: After the shock has subsided the central bank would like to revert to zero UIP spreads, if allowed to reoptimize. Thus, central bank credibility turns out to be an essential input into a successful conduct of intervention policies. To sum up, we find that foreign exchange interventions should be small, frequent, persistent and credible. None of these properties can be obtained in the special case with no private capital mobility.

Our lessons are not limited to the canonical model in which management of a country’s terms-of-trade is the underlying motive. We present three extensions that each exhibit a different rationale to conduct foreign exchange interventions. In the first extension, we explain how an economy with an exchange rate peg and sticky prices in the domestic good optimally intervenes to smooth the path of output gaps.\textsuperscript{5} We show that in this case the planner still leans against the wind of capital inflows but for a different reason: By accumulating reserves it is able to raise the domestic interest rate and shift household spending to the future in order to avoid excessive consumption of the domestic good in the present. The qualitative properties of interventions—size, frequency, persistence and credibility—go through unaltered. In our second extension, we allow for taste shocks in intermediaries’ demand for home bonds. This allows us to capture, albeit in reduced form, aspects such as liquidity premia or heterogeneous beliefs. Analyzing this case is particularly important given that many central banks allegedly intervene when the exchange rate moves “away from fundamentals”. In such a scenario, there is an incentive for the home country to behave as a monopolist in the supply of its own bond, optimally selling, but not fully accommodating, foreigners’ demand for the home bond.

\textsuperscript{5}Countries that face(d) this kind of problem are for example advanced economies during the Bretton Woods era, or European countries that peg to the Euro such as Denmark or the Czech Republic. Even Switzerland can be counted into this category during the time they had in place an exchange rate floor, which was effectively a peg, from 2011 to 2015. Our analysis abstracts from an effective lower bound on policy rates and instead stresses that foreign exchange interventions can help a pegging country regain some control over its interest rate more generally. For an analysis with a binding zero lower bound see Amador et al. (2016).
Unlike before, the central bank can now make profits, behaving as a speculator in the sense of Friedman (1953). Aside from this profit opportunity, we show that the dynamic properties of optimal interventions are still qualitatively the same as before. Finally, as our third extension, we present an economy which is pursuing a “managed float” policy in which the exchange rate is required to follow a smooth path. We show that the slow exchange rate adjustments at the core of this kind of policy—which seems to be very common among EMEs—may cause significant costs by creating large UIP spreads, which invite foreign intermediaries to enter carry trades against the central bank.

Our third contribution is to characterize the positive and normative consequences of widespread foreign exchange interventions for the international monetary system. We embed our baseline model in a world composed of two continua of small open economies: a continuum of “emerging market economies” (EMEs), which are subject to limited capital mobility as before; and a continuum of “advanced economies” (AEs), which have perfect capital mobility. We hit AEs with a savings shock to capture recent trends like population aging or debt overhang. We show that in response to the capital inflows from AEs, EMEs engage in “reserve wars”, as each EME tries to manipulate its terms-of-trade in its favor—an effort which turns out to be self-defeating in the world equilibrium. Interestingly, such behavior by EMEs leads to public flows flowing upstream and private flows flowing downstream, which is what the evidence in Gourinchas and Jeanne (2013) and Aguiar and Amador (2011) suggests. In addition, such reserve wars depress the world interest rate, making the incentive for other EMEs to intervene even stronger. We explain that for reasonable calibrations, both AEs and EMEs would be better off if interventions were ruled out altogether. In fact, the model suggests that even a “unilateral” move by EMEs to coordinate their interventions would significantly reduce their volumes, possibly all the way to zero, again with welfare gains for both AEs and EMEs.

**Literature** This chapter is part of a nascent literature that incorporates a portfolio balance channel into a general equilibrium framework to study foreign exchange interventions. Like this chapter, Cavallino (2015) and Liu and Spiegel (2015) embed imperfect intermediation across borders within a standard New Keynesian model like the one considered in this chapter, and study optimal policy. Cavallino (2015) studies the optimal response to nonfundamental capital flow shocks. He solves the optimal policy analytically and shows interventions lean-against-the-wind. By contrast, we also study the response to fundamental shocks. While the optimal policy still leans-against-the-wind, we show that the spread on the return between home and foreign-currency bonds is smooth, even when the shock itself is not, which is not true
for nonfundamental shocks. In addition, we show that limited financial integration benefits
the planner for fundamental shocks, giving rise to a complementarity with capital controls.
Liu and Spiegel (2015) study numerically the jointly optimal response of taxes on financial
assets, foreign exchange interventions, and monetary policy to fundamental shocks, and also
find interventions lean against the wind. We focus on a real model where the only tool is
foreign exchange interventions and characterize the solution tightly in this environment. None
of these papers study the solution without commitment, an economy without terms-of-trade
manipulation and fixed exchange rates, or coordination of interventions in a multi-country
setting. In other related work, Chang and Velasco (2016) build a model with borrowing
constraints on the financial sector and show that foreign exchange interventions may be
useful when those contraints are binding. Amador et al. (2016) consider an environment
similar to ours to study foreign exchange interventions in the zero lower bound. They find
interventions could be useful to mitigate the recession. Benes et al. (2013), Blanchard et al.
(2015), Devereux and Yetman (2013) and Ostry et al. (2012) study the effects of interventions
but lack a fully microfounded model. Gabaix and Maggiori (2015) study the effect of small
foreign exchange interventions, but lack a fully microfounded model.6

As we mentioned in the introduction, this chapter is related to the burgeoning literature
on optimal capital controls.7 In an environment similar to ours but with perfect financial
markets, Farhi and Werning (2012, 2013) find that optimal capital controls are used to lean
against the wind after interest rate shocks while they are not used against endowment shocks.
In our baseline model, we derive an analogous result for foreign exchange interventions but
show that the additional costs associated with foreign exchange interventions are crucial for
the optimal policy. In a model with two large economies, Costinot et al. (2014) show that a
country might want to use capital controls to depress the international interest rate if the
endowment is growing over time. This is related to some of our results in Section 2.6.

There is a large literature documenting EMEs' reserve accumulation in the past decades.8
The main goal of these papers has been to quantify the contribution of different potential
explanations, such as building buffers against sudden stops or “neo-mercantilist” strategies of
real exchange rate undervaluation. However, most papers do not tackle the question of how

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6Devereux and Yetman (2013) has a microfounded new Keynesian model but their modeling of capital
immobility is ad hoc, which makes the results somewhat harder to interpret. In particular, the model does
not predict a cost of sterilization, which is a feature of microfounded models.
7See, among many others, Bianchi (2011); Magud et al. (2011); Farhi and Werning (2012, 2013); Heathcote
8See, for instance, Aizenman and Lee (2007); Alfaro and Kanczuk (2009); Benigno and Fornaro (2012);
Bianchi et al. (2012); Hur and Kondo (2014); Jeanne and Rancière (2011); Jeanne (2012); Korinek and Serven
(2010).
the country actually manages to manipulate the net foreign asset position of the country to
achieve this objectives. Closest to us in this literature is Jeanne (2012), which emphasizes
the interaction of public and private flows. That paper allows domestic households to access
foreign markets subject to a transaction cost, which allows the planner to costlessly manipulate
the UIP within certain limits. However, since capital is otherwise perfectly mobile, there is no
region in which the planner balances “costs” and “benefits” of foreign exchange interventions.

Finally, our study of a world equilibrium with reserve accumulation in Section 2.6 is
related to Obstfeld (2011), who emphasizes the dangers of currency wars through reserve
accumulation and its consequences for the global interest rate. Models of low global interest
rates are also put forth by Coeurdacier et al. (2015) and Caballero and Farhi (2015). In our
case, lower interest rates are a consequence of reserve accumulation and an increasing share
of EMEs in world markets (see Section 2.6).

2.2 Background on Foreign Exchange Interventions

Foreign exchange interventions are defined as changes in a central bank’s holdings of reserve
assets, where reserve assets are defined as “those external assets that are readily available
to and controlled by, the monetary authorities for meeting balance of payments financing
needs, for intervention in exchange markets to affect the currency exchange rate, and for
other related purposes” (IMF, 2011). In this section, we briefly describe the history of such
interventions, followed by a discussion of the most commonly cited benefits and costs of
interventions. Finally, we summarize the debate on the optimal way of trading off costs and
benefits. Throughout this section, we will both draw on existing papers as well as establish
new facts. A reader mainly interested in our theoretical analysis may skip this section.

2.2.1 A brief history

Foreign exchange interventions have been a key part of the international monetary system in
the last century. During the times of the gold standard, and after its collapse, during the
Bretton Woods system, interventions have routinely been used to “break” the trilemma and
generate some degree of monetary independence (Bordo et al., 2015). After the collapse of
Bretton Woods, exchange rates were allowed to float, with one of the promises being that
this would reduce the need of large scale foreign exchange interventions. Yet, this promise
soon turned out to be false as advanced economies continued regular, and often coordinated,
interventions until the turn of the century. For example, as the United States struggled
with a strong dollar, the five most important central banks negotiated the “Plaza Accord”
in September 1985, following which they engaged in massive interventions to depreciate the dollar. In the 1980s and 1990s, European countries’ central banks intervened to limit exchange rate volatility in the context of the European Exchange Rate Mechanism, a fixed exchange rate system which preceded the introduction of the Euro. Furthermore, Japan was intervening heavily until 2004, mostly in order to moderate the appreciation of the Yen. After that, however, interventions by advanced economies did begin to fall out of favor—at least until a few years ago, when, faced with the limitations of monetary policy at very low interest rates, some European countries, such as Switzerland or Denmark, have resorted once again to foreign exchange interventions in order to achieve their policy objectives.

In contrast with the recent decline among advanced economies, interventions have become a very important policy tool for many emerging market economies (EMEs). Since the famous “sudden stop” episodes of the 1990s, one of the main objectives of EMEs’ interventions has been to build a “war chest” of reserves to insure against sudden stops. As a result, EMEs often engage in reserve-accumulation strategies when their current level of reserves is perceived to be inadequate. Furthermore, unlike the early interventions by advanced economies after the collapse of Bretton Woods, these reserve accumulation programs are conducted unilaterally by each EME central bank and generally tend to rely on comparisons with peers, raising concerns about coordination failures and amplification of reserve hoarding behavior IMF (2011). In addition, Obstfeld (2011) emphasizes the spillovers excessive reserve accumulation may have on the world equilibrium through the world interest rate. We address these issues.
in Section 2.6.

This type of policies has drastically altered the landscape of the international monetary system. Figure 2-1 plots in Panel (a) the reserve holdings of EMEs and AEs, relative to world GDP, and in Panel (b) EMEs' and AEs' shares of world GDP. Before the 1990s, most of global reserves were owned by AEs—as one would expect given their dominant share of world GDP. Since then, however, the rate of reserve accumulation by EMEs has been staggering. As of 2011, EMEs' reserves increased to 9% of world GDP, twice those held by AEs, despite the fact that EMEs only account for half as much GDP (Panel (b)).

2.2.2 Benefits and costs of interventions

Over the years, policymakers have intervened based on a wide variety of different reasons. In this subsection, we briefly summarize those reasons and the costs associated with foreign exchange interventions.

Benefits. The policy debate around foreign exchange interventions has identified the following three broad reasons why a country might engage in foreign exchange interventions: (i) exchange rate management, either to reduce exchange rate volatility, to smooth out exchange rate adjustments over time, or to improve the terms of trade; (ii) reserve accumulation, to insure against future sudden stops; (iii) regaining some monetary independence despite fixed exchange rates. Our baseline model in Section 2.3 features a canonical terms of trade management motive, finding that the central bank leans against the wind.

Actual policy implementation in the data seems to share this property with the optimal policy. Chang (2007) documents that many EME central banks have indeed been leaning against the wind of private capital flows—even those countries that appear to follow a policy of inflation targeting. Figure 2-2 shows a similar picture. Panel (a) plots the time series of quarterly net private flows into a large sample of 50 EMEs (blue, dashed) and flows into reserves (red, solid), aggregated over the countries in our sample. The comovement is striking, and not driven by aggregation: 49 of 50 in our sample show a similarly strong positive comovement, with an average correlation of 0.51. It is particularly interesting that there is not just a strong positive correlation between the two lines, but their volatilities are also of

A sudden stop is a quick reversal in capital flows that leads to a shortage of international liquidity.

For example, in 2011 Chile started a reserve accumulation program to raise its reserves from 13.3% to 17% of GDP within a single year.

The lion share of this increase of course came from China, which held reserves worth 4% of world GDP in 2011. Nevertheless, even without China, EME reserves over world GDP still rose eightfold between 1990 and 2011, compared to twelvefold when China is included.
similar size. Figure 2-2(b) shows the two volatilities in a scatter plot across the same set of countries. While volatilities do not necessarily exactly line up along the 45° line (dashed), the volatility of reserves is significant at around one half of the volatility of net private flows (red, solid). By comparison, the volatility of reserves in the US is only 3.5% of the volatility of net private capital flows.

In Section 2.5 below, we also discuss how our model can be used to analyze different motives for interventions than to manage the terms of trade.

Costs. There are many ways economists have been measuring the costs associated with foreign exchange interventions. Our model takes a particularly practical stance on this issue: The relevant economic costs are transfers that the country implicitly pays to foreign arbitrageurs running successful carry trades against central bank interventions. Critical to this story is that there is an empirical connection between UIP spreads and foreign exchange interventions. To provide evidence that there is, we combine data from Lustig et al. (2011) with the IMF’s Balance of Payments Statistics and run the following OLS regression,

\[ UIP_{spread_{it}} = \alpha_i + \delta_t + \beta ResFlows_{it} + X_{it}\gamma + \epsilon_{it} \]

\[ 12 \text{ For example, Adler and Mano (2016) measures the costs as quasi-fiscal costs the central bank incurs when it invests in low-yielding assets, selling high-yielding ones.} \]

\[ 13 \text{ We thank Adrien Verdelhan for sharing his data with us.} \]
Table 2.1: Do increases in reserves coincide with positive wedges in the uncovered interest parity condition (UIP)? Quarterly reserve flows are taken from the IMF’s Balance of Payments statistics, merged with quarterly UIP wedges and interest rate spreads computed based on data from Lustig et al. (2011). Details on the table can be found in Appendix 2.A.3. Note: Standard errors, corrected for heteroskedasticity, are in brackets. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

where $\alpha_t$ and $\delta_t$ are country and time fixed effects; and $X_{it}$ is a set of controls, in which we include past reserve flows as well as the past interest rate spread (which is well known to predict excess returns, see e.g. Lustig et al. (2011)). The outcomes, in Table 2.1, show a strong significant relationship between reserve purchases and positive UIP spreads, that is, expected excess returns from investing in the intervening country’s bond market.

### 2.2.3 Empirical evidence on effectiveness

Convincing empirical evidence about the effectiveness of foreign exchange interventions is very rare. This is mainly due to a simple simultaneity problem: Interventions influence exchange rates but also respond to shocks to exchange rates. One of the more solid papers that attempts to go around this issue is Kearns and Rigobon (2005), exploiting a “natural experiment”, in which Japan and Australia changed their foreign exchange policy for arguably exogenous reasons. They show that in both countries, interventions seem to have a significant effect on the exchange rate, with most of the effect occurring during the day of the intervention.

It is worth pointing out that, despite the shortage of convincing formal evidence on interventions, policymakers’ own experiences with them seem to suggest that they are indeed
effective. For example, a survey by the BIS shows that 90% of the respondents believe were their foreign exchange interventions were completely or partially successful.

One of the least controversial points regarding the effectiveness of interventions, is that the effectiveness should be tightly linked to the (in)ability of capital to flow freely across borders. This is important for our theory since the assumption of limited capital mobility is at the core of our analysis. It is important to notice that such limited capital mobility need not necessarily come from limits to arbitrage in the financial sector: Canales-Kriljenko (2003) documents that central banks in EMEs are large players in their foreign exchange markets, partially due to consciously taken policy measures designed to inhibit free cross-border capital flows.\footnote{In a sample of 90 countries, he finds that 36% have surrender requirements, 90% have some form of position limits, 50% prohibit usage of foreign currency for some domestic transactions and in 45% both legs of foreign exchange transactions are settled at the central bank. Mohanty and Berger (2013) confirms this observation in a more recent survey.}

As already previewed in the introduction, this points to a complementarity between capital controls and macroprudential measures on the one side and foreign exchange interventions on the other.\footnote{Even in the case of advanced economies, this seems to be the case. Kearns and Rigobon (2005) find that Australian interventions create more “bang for the buck” due to a smaller foreign exchange market. And the central bank of New Zealand stated that interventions are more “likely to be effective” when “there is a relative absence of capital flows that might offset the intervention”.}

### 2.2.4 Implementation

There are roughly three important degrees of freedom in the implementation of foreign exchange interventions: Frequency vs. size, rules vs. discretion, and exchange rate rules vs. quantity-based rules.\footnote{Arguably, whether to intervene in spot or forward markets is another degree of freedom. However, most, i.e. 70-80%, of the interventions are carried out in spot markets, since forward markets are generally more illiquid. There are some notable exceptions, such as Brazil.} We discuss each of them in turn.

**Frequency vs size.** Countries are divided as to the optimal size and frequency of interventions. For example, Kearns and Rigobon (2005) document that Australia and Japan abandoned their small and frequent interventions in favor of large and infrequent ones in an effort to maximize their impact on the exchange rate. In contrast, other countries have had a very persistent presence in foreign exchange markets. Adler and Tovar (2011) document that Brazil and Uruguay intervened two-thirds of the days between 2004 and 2010.

**Rules vs discretion.** Another source of debate refers to whether interventions should be secret or public information. In a well-known survey of EME central banks, Canales-Kriljenko...
documents that about one half of the respondents carry out their interventions in secret. Among advanced economies, public interventions are more common. However, the historically largest intervener—the Bank of Japan—has frequently favored secret interventions. Note, however the line separating secret and public interventions becomes blurred in shallower markets, where the central bank’s presence hardly goes undetected.

Exchange rate rules vs quantity-based rules. In addition, even if one agrees that transparency and predictability are desirable, there is heterogeneity among countries regarding the kind of rules that they implement. On the one hand, some countries follow quantity-based rules. For example, Chile’s reserve accumulation program of 2011 consisted of buying USD 12 billion in pre-announced daily amounts at an average of USD 50 million per day. On the other hand, some countries follow exchange-rate based rules, mostly aimed at smoothing the exchange rate path. For example, Colombia had a rule that authorized the central bank to auction put options up to a specific amount whenever the exchange rate fell more than 5% below its average of the previous 20 days.

Perspective of our model. Our model provides simple yet powerful guidance on these questions. If the goal is to minimize the aforementioned costs coming from foreign arbitrageurs running carry trades (and hence speculating) against central bank interventions, then interventions should be frequent but small in size and pre-announced. Moreover, following a quantity-based rule is generally found to be closer to the optimal policy than an exchange-rate based rule that guarantees smooth exchange rate movements.

Frequent but small interventions are powerful due to two reasons: First, they span over a significant time period, so they are likely to affect interest rates for longer, amplifying the initial response of economic agents, especially when interventions are pre-announced. Second, the relatively small size of any specific intervention is associated with relatively minor UIP spreads and hence limits the room for foreign arbitrageurs to take advantage of the central bank action. Exchange rate rules are found to do the exact opposite: As we show in Section 2.5, by slowing down the exchange rate adjustment, central banks invite foreign arbitrageurs to take bets and trade against the central banks’ interventions.

\[^\text{17}^\text{In an empirical setting, our model would predict that the exchange rate should respond the moment the policy is announced. Interestingly, Tapia and Tokman (2004) finds that the announcement of intervention in Chile in 2001 had a large and significant effect on the exchange rate.}\]
2.3 Baseline Model

In this section, we present our baseline model. The model is a real small open economy (SOE) model in continuous time. The model is stylized as we strive to focus on the two essential model ingredients. These are on the one hand a finitely elastic foreign demand for home bonds that allows the home central bank to change home interest rates via a portfolio balance channel;\(^{18}\) and on the other a terms-of-trade management motive, which gives the central bank a reason for such interventions.\(^{19}\) We first describe the model and then discuss the equilibrium dynamics without interventions. Optimal interventions are then characterized in great detail in Section 2.4.

2.3.1 Model setup

There are four agents in our model: Domestic households and a domestic central bank, and, foreign and domestic intermediaries. In line with our SOE assumption we also introduce an export demand curve of foreign households. There are two goods markets (a “home good” and a “foreign good”) as well as two asset markets (“home bonds” and “foreign bonds”). Throughout, we use “home” and “domestic”, as well as “UIP spread” and “UIP wedge” interchangeably. We start by describing domestic households and the two goods markets.\(^{20}\)

**Households.** There is a continuum of households in the home country, maximizing a common utility function \(\int_{0}^{\infty} e^{-\rho t} \log(c_t) dt\), with \(c_t\) being a consumption bundle defined as \(c_t = \kappa c_{Ht}^{1-\alpha} c_{Ft}^{\alpha}\). Here, \(c_{Ht}\) and \(c_{Ft}\) denote home’s consumption of home and foreign goods, respectively, and \(\kappa \equiv (1-\alpha)^{-(1-\alpha)}\alpha^{-\alpha} > 0\) is a positive normalization constant. Throughout our analysis, we normalize the foreign good’s price to 1 and refer to that numeraire as “dollars”. The relative price of the home good is denoted by \(p_t\). The per-period dollar budget constraint of the household is given by

\[
\dot{b}_{Ht} = p_t y_{Ht} + y_{Ft} - p_t c_{Ht} - c_{Ft} + r_t b_{Ht} + t_t + \pi_t, \tag{2.1}
\]

where \(y_{Ht}\) is home’s endowment of the home good, \(y_{Ft}\) is home’s endowment of the foreign good, \(b_{Ht}\) is the households’ position in home bonds, \(t_t\) are transfers from the central bank,

\(^{18}\)This is similar to papers by Lahiri and Végh (2003), Gabai and Maggiori (2015), and Liu and Spiegel (2015) among others.

\(^{19}\)This is similar to the recent literature on capital controls by Fachi and Werning (2012, 2013) and Costinot et al. (2014), among others, which is based on the framework by Gali and Monacelli (2005).

\(^{20}\)This presumes trade taxes are infeasible, as is standard in this literature, so terms-of-trade management is a second best tool. See, e.g. Costinot et al. (2014).
and \( \pi_t \) are profits from domestic financial intermediaries. Both \( t_t \) and \( \pi_t \) are specified below. We denote by \( q_t \equiv p_t^{-(1-\alpha)} \) the country’s real exchange rate, following the convention that high values correspond to depreciated exchange rates. Here, domestic households are only allowed to trade home bonds with a real interest rate of \( r_t \). Later, we introduce financial intermediaries, who may access both home and foreign bond markets. We wish to stress that in this environment, domestic households’ own a nontrivial share of the home good and exhibit home bias in their preferences. These two assumptions are essential in generating the terms of trade management motive in our environment.

Maximizing utility subject to this budget constraint yields the following Euler equation,

\[
\frac{\dot{c}_t}{c_t} = r_t - \rho + \frac{q_t}{q_t}.
\]

Finally, home’s total dollar expenditure is given by \( q_t^{-1}c_t = p_t c_{Ht} + c_{Ft} \), which henceforth we denote by \( \theta_t \equiv q_t^{-1}c_t \) due to its prominent role the analysis to come. The optimal demand for home and foreign goods is then

\[
c_{Ht} = (1 - \alpha) \frac{\theta_t}{p_t} \tag{2.3}
\]

\[
c_{Ft} = \alpha \theta_t.
\]

By symmetry, foreign’s demand for home goods is

\[
c_{Ht}^* = \alpha \frac{c_t^*}{p_t} \tag{2.4}
\]

where in the following we assume foreign’s consumption \( c_t^* \) to be equal to 1.

**Foreign intermediaries.** There are two types of intermediaries in our model: Foreign intermediaries and domestic intermediaries. Both types of intermediaries will behave similarly but they differ in their ownership structure, which will play a key role in our analysis. We describe foreign intermediaries in detail in this section and their domestic counterparts in the next.

The key ingredient in our model that makes foreign exchange interventions effective is a finite elasticity of demand for home bonds. As a result, a change in the portfolio of the central bank has an effect on the expected return of domestic assets \( r_t \) relative to their foreign counterpart \( r_t^* \), i.e. the UIP wedge.\(^{21}\) Backus and Kehoe (1989) pointed out that these

\[^{21}\text{Since our model is deterministic, there is no difference between realized and expected returns. Hence, any}\]
portfolio balance effects are muted in general equilibrium in a frictionless world in which Ricardian equivalence holds, as any actions by the central bank would be perfectly undone by the private sector. We break away from this result by modeling limited asset market participation, in the spirit of Bacchetta and Van Wincoop (2010) and Gabaix and Maggiori (2015). In particular, we assume that there exists a continuum of intermediaries owned by foreigners, labeled by \( j \in [0, \infty) \), which can trade in both foreign and domestic bond markets. Foreign intermediaries' investment decisions are subject to three important restrictions.

First, each intermediary is subject to a net open position limit \( X > 0 \). Second, we follow Alvarez et al. (2009) in assuming that intermediaries face heterogeneous participation costs. In particular, each intermediary \( j \) active in the domestic bond market at time \( t \) is obliged to pay a participation cost of exactly \( j \).

Putting these two ingredients together, intermediary \( j \) optimally invests an amount \( x_{jt} \), solving

\[
\max_{x_{jt} \in [-X, X]} x_{jt} (r_t - r_t^*) - 1_{\{x_{jt} \neq 0\}} j.
\]

Intermediary \( j \)'s present value of net profits conditional on investing is \( X |r_t - r_t^*| \), so investing is optimal for all intermediaries \( j \in [0, \overline{j}] \) with the marginal intermediary \( \overline{j} \) given by \( \overline{j} = X |r_t - r_t^*| \). This gives an aggregate investment volume of

\[
b^I_t = \overline{j} X \cdot \text{sign} (r_t - r_t^*).
\]

Defining \( \Gamma_F \equiv (X^2)^{-1} \) and substituting \( \overline{j} \), we obtain

\[
b^I_t = \frac{1}{\Gamma_F} [r_t - r_t^*]. \tag{2.5}
\]

Equation (2.5) embodies that the foreign intermediaries' demand for home bonds has a finite elasticity to the return spread. This equation is crucial to our analysis because it implies that changes in \( b^I_t \), as for example induced by foreign exchange interventions, can indeed affect home interest rates. The key parameter in (2.5) is the inverse demand elasticity \( \Gamma_F \).

UIP violation would also lead to a violation of the covered interest parity (CIP). In reality, foreign exchange interventions deal with assets of different risk characteristics. We deal with this in an extension with risk premium shocks in Section 2.5.

\(^{22}\)It is worth noting that, as discussed in Section 2.2, many emerging market central banks in fact do impose position limits on intermediaries' investments as a form of capital controls, hence artificially decreasing \( X \).

\(^{23}\)It is straightforward to relax this assumption of linear costs to a more general monotonic function \( f(i) \). We show in Appendix 2.E.2 that assuming linear costs is inessential for our results.

\(^{24}\)Strictly speaking, \( \Gamma_F \) is only a semi-elasticity. For simplicity, we abuse the terminology and call it an "elasticity" in the remainder of this chapter.
When $\Gamma_F$ is large, e.g. if position limits $X$ are very small, intermediation is obstructed as evidenced by both, small levels of $b_{Ht}$ and a small sensitivity of $b_{Ht}$ to the interest rate spread. The case where $\Gamma_F = \infty$ corresponds to financial autarky. In this case, $b_{Ht} = 0$, which shuts down any sort of private financial intermediation in this baseline model. When $\Gamma_F$ is small, e.g. if position limits $X$ are very high, this leads to an equilibrium with less imperfect intermediation. In fact, if position limits were infinite, $\Gamma_F$ would equal to zero and we would recover the infinite elasticity, $r_t = r_t^*$. We call this case the frictionless economy.

**Home intermediaries.** In addition to foreign intermediaries, we shall also assume there is a similar continuum of home intermediaries generating an analogous bond demand schedule,

$$b_{Ht}^H = \frac{1}{\Gamma_H} [r_t - r_t^*].$$

There are two main differences between domestic and foreign intermediaries. First, domestic intermediaries are allowed to have a different elasticity $\Gamma_H$, which can be anywhere in $(0, \infty]$. Second, home intermediaries are owned by the representative household at home, whereas foreign intermediaries are owned by foreign households. To compute intermediaries’ profits we need to take a stance on the way transaction costs are being paid. To keep the model tractable, we assume that domestic intermediaries pay the transaction costs as transfers to each other, ultimately reaching domestic households through profits. This means, domestic intermediaries’ total profits are just given by their total revenues, and so households receive the following stream of per-period profits (in dollars),

$$\pi_t = b_{Ht}^H(r_t - r_t^*).$$

**Central bank.** The home central bank is the home country’s social planner in our model. It chooses a foreign exchange intervention (FXI) policy $\{b_{Gt}, b_{Gt}^*, t_t\}$ consisting of home bond investments $b_{Gt}$, foreign bond investments $b_{Gt}^*$, and transfers $t_t$ to home households, subject to the central bank budget constraint

$$b_{Gt} + b_{Gt}^* = r_t b_{Gt} + r_t^* b_{Gt}^* - t_t.$$  

\(^{25}\)It would make absolutely no difference if transaction costs would directly flow to households, it would just require an additional term in the representative household’s budget constraint.  

\(^{26}\)Note we implicitly assumed that the relevant interest rate for marginal changes of reserves is $r_t^*$. One might argue that negative levels of $b_{Gt}^*$ should be associated with a different, higher interest rate. In reality however, reserves are (almost) always positive and so marginal changes in reserves are associated with the foreign interest rate on savings, $r_t^*$.  

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The central bank’s FXI policy must also ensure that the country satisfies a no-Ponzi condition,

$$\lim_{t \to \infty} e^{-\int_0^t r_s ds} nfa_t = 0 \quad (2.9)$$

where $nfa_t \equiv b_{Ht} + b_{Gt}^* + b_{Gt}$ is the net foreign asset position of the country. Note that in this economy, it is without loss to set $b_{Gt}^* + b_{Gt} = 0$ due to the availability of transfers between the central bank and households.

**Competitive equilibrium.** The model is closed with a goods market clearing condition,

$$c_{Ht} + c_{Ht}^* = y_{Ht} \quad (2.10)$$

and a bond market clearing condition,

$$b_{Ht} + b_{It} + b_{Ht}^* + b_{Gt} = 0. \quad (2.11)$$

We can now formally define a competitive equilibrium in this environment.

**Definition 3.** Given initial debt positions $(b_{H0}, b_{I0}, b_{H0}^*, b_{G0}, b_{G0})$, paths for shocks $\{y_{Ht}, y_{Ft}, r_t\}$, and a central bank FXI policy $\{b_{Gt}, b_{Gt}^*, t_t\}$, an allocation $\{c_t, c_{Ht}, c_{Fl}, c_{Ht}^*, b_{Ht}, b_{It}, b_{Ht}^*, \pi_t\}$ together with prices $\{q_t, r_t\}$ is a competitive equilibrium iff they solve (2.1)-(2.11).

Next, we characterize the competitive equilibrium, with the goal to derive “implementability conditions” that describe the set of competitive equilibria that can be attained through different FXI policies. Substituting consumption demands (2.3) and (2.4) into the goods market clearing condition (2.10) gives us an expression for the dollar value of the endowment of home goods,

$$q_t^{1/(1-\alpha)} y_{Ht} = (1 - \alpha)\theta_t + \alpha. \quad (2.12)$$

Using the households’ dollar budget constraint (2.1), we then obtain

$$b_{Ht} = \alpha(1 - \theta_t) + y_{Ft} + r_t b_{Ht}^* + t_t + \pi_t. \quad (2.13)$$

Here, the policy variable $t_t$, can be eliminated after adding the central bank’s budget constraint (2.8), which allows us to rewrite the households’ budget constraint as a country-wide budget constraint,

$$nfa_t = \alpha(1 - \theta_t) + y_{Ft} + (r_t - r_t^*)(b_{Ht} + b_{Gt}) + r_t^* nfa_t + \pi_t + \frac{\pi_t}{b_{Ht}^*(r_t - r_t^*)} . \quad (2.14)$$
In this equation, policy variable \( b_{Gt} \) can be expressed as \( -b_{Ht} - b_{It} - b_{Ht}^I \) using home bond market clearing (2.11), where intermediaries’ bond demand \( b_{It} \) is given by (2.5). Then, the country-wide budget constraint (2.14) simplifies to

\[
\text{nfat} = \alpha(1 - \theta_t) + y_{Ft} + r_t^* \text{nfat} - \frac{1}{\Gamma_F} (r_t - r_t^*)^2.
\]  

(2.15)

Up to the last term, equation (2.15) is nothing more than a standard open economy budget constraint. It implies that home’s net foreign asset position improves if the trade balance (net exports) \( \alpha(1 - \theta_t) + y_{Ft} \) is large, or interest income \( r_t^* \text{nfat} \) from existing foreign assets is large. The last term, however, is new. It captures the costs the country incurs if the interest rate spread \( r_t - r_t^* \), which is the same as a UIP deviation in our context, is different from zero.

Why does the country face costs from UIP deviations? Suppose the spread \( r_t - r_t^* \) is positive. This invites foreign intermediaries to come in and take a position \( b_{It} = \frac{1}{\Gamma_F} (r_t - r_t^*) \) in the domestic bond market, taking home revenues

\[
b_{It} \cdot (r_t - r_t^*) = \frac{1}{\Gamma_F} (r_t - r_t^*)^2.
\]  

(2.16)

These carry trades represent economic costs to home as they are paying a premium to foreign investors over and above the world interest rate \( r_t^* \). Naturally, the costs increase when foreign intermediaries become more elastic to the UIP wedge, that is, when \( \Gamma_F \) is lower. In that case, for a given UIP wedge, intermediaries take larger positions, generating larger costs. Vice versa, if there are no active foreign intermediaries, \( \Gamma_F = \infty \), the country does not incur any costs.

Noticeably, the costs in (2.16) are independent of the degree of domestic intermediation \( \Gamma_H \). The reason is straightforward: While domestic intermediaries, similar to foreign ones, take a position that is proportional to the UIP wedge, their revenues do not leave the country and are instead rebated to domestic households.\(^{27}\)

Next we study the set of equilibria that are implementable by choosing a given path of foreign exchange interventions. For this result and the remainder of the chapter, we introduce as notation for the UIP wedge \( \tau_t \equiv r_t - r_t^* \).

Rewriting the budget constraint (2.15) in present value terms we obtain the following

\(^{27}\)Even though the assumption that domestic intermediaries’ revenues entirely enter the representative household’s budget constraint seems like a strong one, the model can easily cope with less extreme situations where only a fraction of those revenues fall to agents that enter the government’s welfare considerations. In that case, one would merely relabel “domestic” and “foreign” intermediaries as “those who enter the government’s welfare considerations” and “those who do not”.

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Proposition 4 (Implementability conditions.). Suppose $\Gamma_F > 0$ and $\Gamma_H > 0$. Let $\theta_t = q_t^{-1}c_t$ be the dollar value of home consumption and $\tau_t = r_t - r_t^*$ be the "wedge" in the uncovered interest parity (UIP) condition. Then, given an initial net foreign asset position $nfa_0$ and shocks $\{y_{Ht}, y_{Ft}, r^*_t\}$, the paths $\{c_t\}$ and $\{q_t, \tau_t\}$ are part of a competitive equilibrium iff the corresponding $\{\theta_t, \tau_t\}$ solve the following two conditions: The Euler equation,

$$\frac{\dot{\theta}_t}{\theta_t} = r_t^* + \tau_t - \rho$$  \hspace{1cm} (2.17a)

and the country-wide present value budget constraint,

$$\int_0^\infty e^{-\int_0^t r_s^* ds} \left[ \alpha (\theta_t - 1) + y_{Ft} + \frac{1}{\Gamma_F} \tau_t^2 \right] dt = nfa_0.$$ \hspace{1cm} (2.17b)

Proposition 4 gives us a simple characterization of the set of competitive equilibria as it is commonly used in models of optimal Ramsey taxation (see, e.g., Lucas and Stokey, 1983 or Chari and Kehoe, 1999). A key difference with this literature, however, is that the planner in our model does not choose a path of taxes, but rather an FXI policy as defined above. The importance of Proposition 4 is that it shows that setting FXI policies—which are paths of asset positions—in fact is equivalent to setting wedges $\tau_t$ in the UIP condition—which behave like taxes.

As a side remark, we would like to stress that in addition to the costs $-r_t\tau_t$ coming from foreign intermediation, setting a path of nonzero UIP wedges $\tau_t$ is, of course, already "costly" in that it distorts the consumption choices of domestic households. This will be the reason a planner in our economy only cares to deviate from $\tau_t = 0$ if there is an additional reason, like managing the terms of trade, that makes such deviations beneficial. There, the costs $\frac{1}{\Gamma_F} \tau_t^2$ coming from foreign intermediation will be an additional resource cost that the country incurs, and that, as it turns out, critically changes the optimal policy.

A simple corollary of Proposition 4 is that the set of implementable allocations is independent of the degree of domestic intermediation.

Corollary 2. In Proposition 4 the set of implementable allocations is independent of the degree of domestic intermediation $\Gamma_H$ (as long as $\Gamma_H > 0$).

We next set up the planning problem of choosing the optimal FXI policy.
2.3.2 Planning problem

We think of the central bank as the home economy’s social planner. Thus, the central bank maximizes the welfare of domestic households across all competitive equilibria it can possibly implement using foreign exchange interventions. Domestic households' utility is given by $\int_0^\infty e^{-\rho t} \log ct \, dt$. Since the dollar value of home consumption, $\theta_t = q_{t-1} c_t$, is slightly more convenient to use, we express utility in terms of $\theta_t$ and state the planning problem as:

$$\max_{\{\theta_t, \tau_t\}} \int_0^\infty e^{-\rho t} \{\log \theta_t - (1 - \alpha) \log ((1 - \alpha) \theta_t + \alpha)\} \, dt$$

subject to the two implementability conditions (2.17a) and (2.17b).

In the planning problem (2.18), the freedom of setting different FXI policies is completely embodied in the choice of the UIP wedge $\tau_t$. When the central bank desires to raise consumption in period $t$ relative to the next, it lowers $\tau_t$. Such a policy would then be implemented by selling reserves and purchasing home bonds, which, due to a finitely elastic foreign demand function, affects the domestic interest rate $r_t$ and thus $\tau_t$.

One possibility for the central bank in this baseline model is to set $\tau_t = 0$ in all periods, in which case it implements an allocation that would prevail if $\Gamma_F$ were equal to zero, that is, it “undoes” the imperfect intermediation friction. This is clearly possible in this model since the central bank can freely access both bond markets, and thus it may always create the right kind of bond supply to ensure that $r_t = r^*_t$, in which case foreign intermediaries’ positions $b_{ft}$ are zero. We wish to emphasize that, however, there is no simple relationship between $\tau_t$ and the country’s reserve position $b^*_G$ in this baseline model. In particular, $\tau_t = 0$ does not necessarily correspond to zero reserves, and the relationship between $\tau_t$ and reserves more generally depends on $\Gamma_F$ and $\Gamma_H$.\footnote{Still, however, it is clear that $\tau_t = 0$ implies that foreign intermediaries are indifferent to holding home bonds versus foreign bonds.}

As a benchmark, we now characterize the first best allocation. We define this to be the optimum to the planning problem (2.18) subject only to the resource constraint (2.17b).\footnote{For ease of notation, we abbreviate the planner’s per-period objective as $V(\theta) \equiv \log \theta - (1 - \alpha) \log ((1 - \alpha) \theta + \alpha)$.}

**Lemma 2** (First best.). When only the resource constraint (2.17b) is binding, the optimal

\footnote{See Appendix 2.B.2 for a derivation.}

\footnote{Even though $\Gamma_H$ does not enter the planning problem directly, it turns out to matter for the reserve accumulation policy that implements the optimal paths for $\theta_t$ and $\tau_t$.}

\footnote{There are other ways to define “first best” here. For instance, one could allow the planner to set optimal tariffs on exports of the home good. In that case, however, the SOE can extract an unlimited amount of resources from the rest of the world, so this alternative definition is rather meaningless.}
(first best) allocation \( \{\theta_t, \tau_t\} \) and the corresponding shadow resource cost \( \lambda \) satisfy (i) \( \tau_t = 0 \), (ii) the implicit equation,
\[
\frac{e^{-\rho t} \psi'(\theta_t)}{\text{marg utility at time } t} = e^{\int_0^t r^*_s ds} \lambda \alpha^{\int_0^t r^*_s ds} \lambda \alpha
\]
for each \( t \geq 0 \), and (iii) the resource constraint (2.17b).

Lemma 2 states the obvious: Absent any incentive compatibility conditions, the planner equates the marginal utility of (dollar) consumption in any period \( t \) to the corresponding resource cost. Still, Lemma 2 will prove to be a useful benchmark later on.

A general advantage of writing the planning problem in terms of the UIP spread \( \tau_t \) is that it it provides us with a convenient link to the large literature on optimal capital controls. We now explore this link.

### 2.3.3 Connection to literature on capital controls

Our planning problem (2.18) is related to a recent literature on capital controls (see, e.g., Bianchi (2011); Farhi and Werning (2012, 2013); Heathcote and Perri (2014); Jeanne (2012)). In that literature, capital controls are typically modeled as a proportional tax on capital flows, which directly induces a spread in the uncovered interest parity equation, i.e. a spread between \( r_t \) and \( r^*_t \)—just like in our model, where foreign exchange interventions induce such as spread. The key difference with our framework is the additional cost term \( \frac{1}{\Gamma_F} \tau_t^2 \) in the resource constraint, capturing net losses from foreign intermediation.

To better understand the connection to the literature on capital controls, consider a model with the same real structure as this chapter but frictionless financial markets. Suppose the planner is allowed to pick a path for taxes on capital inflows, which immediately show up as UIP wedges \( \tau \). Then, the planning problem in that economy,\(^{31}\) casted in terms of UIP wedges \( \tau \), is exactly the same as our planning problem, except for the extra resource costs \( \frac{1}{\Gamma_F} \tau_t^2 \). Indeed, our planning problem and the one described above become formally equivalent when \( \Gamma_F = \infty \). As will become clear in Section 2.4, the new term will shape the response of optimal policy and deliver several new insights.

However, we would like to stress that the economic interpretation is very different. Capital flow taxes are powerful to the extent that they shape the response of the private sector. Therefore, they are ineffective in the absence of private intermediation, which is precisely when FXI policies are most effective. The converse is true when private intermediation is

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\(^{31}\)Farhi and Werning (2013) analyze a version of this planning problem that includes labor.
frictionless. More generally, the two tools can be viewed as complements: capital controls may effectively put sand in the wheels of private intermediation, which increases $\Gamma_F$ and thereby relaxes the planner’s FXI problem.\footnote{This may explain why policymakers often put taxes on both inflows and outflows (Fernández et al., 2015) or put in place position limits on foreign exchange positions (BIS, 2005, 2013).}

\subsection{2.3.4 Zero-reserves and zero UIP wedge allocations}

Before we move on to study how the central bank should optimally use foreign exchange interventions, we analyze two specific implementable allocations: the “laissez-faire” equilibrium, where the central bank keeps reserves at zero, and a $\tau_t = 0$ economy, where the central bank undoes the financial friction. These two allocations will help gauge the optimal interventions we study in the next section. In both cases, we characterize the response of the economy to three perfect foresight shocks: one to the world interest rate $r^*_t$, one to the endowment of the home good $y_{Ht}$, and one to the endowment of the foreign good $y_{Ft}$\footnote{Since this is a deterministic economy, we refer to “shocks” as the deterministic response of an economy previously in steady state to a change in the deterministic path of a parameter such as $r^*_t$, $y_{Ht}$ or $y_{Ft}$.}. We choose the signs of the shocks so that they imply a positive response of home consumption. To be consistent with our exercise in Section 2.4 below, we use the same simple calibration for the graphs shown in this section. In order to avoid repetition, we refer the reader to Section 2.4.1 for details on calibrated parameter values and a discussion of our calibration strategy.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2-1.png}
\caption{Response of the laissez-faire economy (blue, solid) and the $\tau_t = 0$ economy (red, dashed) to a negative $r^*$ shock in a deterministic model economy calibrated to Brazil, with $\Gamma_F = 10$. The shock lowers $r^*$ by 2\% for $t \in [0, 3]$.}
\end{figure}
Negative shock to the world interest rate $r^*_t$. Figure 2-1 illustrates the effects of a negative, 3-period-long shock to $r^*_t$ on the two allocations. First, consider the laissez-faire allocation, the solid blue line. There, the shock increases foreigners’ demand for home bonds, which pushes down $r_t$, but less than one-for-one with the shock to $r^*_t$ due to the finite demand elasticity. This means, $\tau_t$ rises over time (Panel (a)). The lower rate $r_t$ then has the following consequences. It leads to higher domestic consumption $c_t$, an appreciated real exchange rate (Panel (b)) and a worsening of the net foreign asset position. As shown in Panel (c), reserves are zero by design.

The response of the $\tau_t = 0$ economy (dashed red line) is quite different from that. First, as a direct consequence of $\tau_t = 0$, interest parity holds, $r_t = r^*_t$. Therefore, the shock to $r^*_t$ is passed through to $r_t$ one-for-one. This leads to a more pronounced uptick in consumption and hence a more pronounced real exchange rate appreciation (Panel (a)), compared to the laissez-faire economy. Finally, note that since zero spreads reduce intermediation profits to zero, the central bank must do all the intermediation, selling reserves to reflect the country’s desire to borrow (Panel (c)).

Why does the central bank need to intervene in the same direction as the shock in order to achieve $\tau_t = 0$? Notice that any friction inhibiting the free flow of international capital always exhibits the following two properties: First, there is the “elasticity” property, discussed in detail above: By taking a certain foreign exchange position, the central bank has the ability to influence domestic real interest rates and exchange rates. But there is also an “underreaction” property: In response to shocks to the foreign interest rate $r^*_t$, the level of domestic interest rates underreacts and moves less than one for one, giving rise to a positive UIP spread $\tau_t > 0$ (Panel (a)). The real exchange rate underreacts as well, since the expected UIP spreads are all positive. It is precisely this “underreaction” aspect of limited capital mobility that seems to suggest reserve decumulation is necessary to achieve $\tau_t = 0$ after a negative $r^*_t$ shock.

Seemingly contrary to our model’s prediction, policymakers believe that domestic real interest rates $r_t$ are more likely to overreact rather than underreact to global liquidity shocks, absent any intervention. However, in practice, global liquidity shocks do not only affect $r^*$ but also the risk and liquidity properties of local bonds vis-a-vis foreign bonds. At the end of Section 2.5.2 we discuss how shocks with these features that comove with $r^*$ shocks might eliminate any “underreaction”, without affecting the “elasticity” aspect. For expositional

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34To check the prediction of our model we also conducted a (preliminary) analysis of the response of various countries’ UIP spreads to identified US monetary policy shocks. We could not find any statistically significant evidence for a nonzero response of UIP wedges.

35In a recent paper Engel (2016) makes a related point, arguing that multiple comoving shocks are necessary to explain the covariance between interest rate differentials and UIP spreads.
clarity, in Section 2.4's plots we show the reserve purchases or sales the central bank needs to make at the optimum relative to any interventions it might have to conduct (if any) to achieve \( \tau_t = 0 \). This avoids confusion as to whether paths for reserves are determined by the "underreaction" aspect or rather by the optimal policy itself.

**Shocks to endowments** \( y_{Ht} \) and \( y_{Ft} \). Assuming \( r^*_t = \rho \), it follows immediately that—for any path \( \{y_{Ht}, y_{Ft}\} \)—a constant path of dollar consumptions \( \{\theta_t\} \) achieves the first-best outcome described in Lemma 2. Note that when \( \theta_t \) is constant, equation (2.12) directly links higher values for \( y_{Ht} \) to a depreciated real exchange rate in a way that exactly compensates \( y_{Ht} \), leaving the exported value in dollars, and hence the current account, constant. This is a direct consequence of assuming Cole-Obstfeld preferences. This implies that reacting to \( y_{Ht} \) does not actually require any action by the central bank.

In contrast, \( y_{Ft} \) shocks require an active portfolio management by the central bank. After a positive \( y_{Ft} \) shock, the central bank needs to accumulate reserves in order to save on behalf of households, undoing the financial friction. In the laissez-faire equilibrium, households can only save by paying a premium, which lets dollar expenditure become procyclical. In other words, the laissez-faire equilibrium has inefficient real exchange rate fluctuations in response to wealth shocks \( \{y_{Ft}\} \).

As we have just seen, for \( y_{Ht} \) and \( y_{Ft} \) shocks, the first best is the optimal policy, as there is no terms-of-trade management motive in this case. This is not the case for world interest rate shocks \( r^*_t \), in response to which we know study the optimal foreign exchange interventions.

### 2.4 Optimal foreign exchange interventions

In this section, we present our main results about the normative behavior of foreign exchange interventions. The fundamental trade-off that determines the optimal use of foreign exchange intervention in our model is between two forces. The first is the desire to minimize the additional cost term (2.16). The second is a terms-of-trade management motive, which appears when the economy is subject to \( r^*_t \) shocks. More generally, however, our analysis should carry over to any sort of "macroeconomic stabilization" motive, which might lead the central bank to use foreign exchange interventions as a second best tool to influence the exchange rate or home interest rates. We explore this trade-off in three steps in this section. In Section 2.4.1 we briefly introduce the calibration underlying the plots shown in this section and the previous one. Section 2.4.2 characterizes the optimal intervention policy without any motive to manage the terms-of-trade. In this case, the planner would like best not to
distort private consumption decisions. And in a final step in Section 2.4.3, we include our terms-of-trade management motive and study the optimal policy under the full trade-off. Before this, we briefly introduce the calibration underlying the plots in this section and the previous one.

2.4.1 Calibration

For our illustrative simulations, we calibrate the model parameters to Brazil. For the discount rate we pick $\rho = 0.075$, corresponding to the average 5yr treasury yield from 2000–2015 plus the average J.P. Morgan EMBI+Brazil return over the same time period. For the openness $\alpha$ of the economy, we choose $\alpha = 0.15$ matching a 15% imports to GDP ratio in 2013. We normalize $y_H$ to 1 and $y_F$ to 0. Our results do not really depend on the initial net foreign asset position, so for simplicity, we set it to zero at the beginning of our figures. To get an idea of the relative size of domestic compared to foreign intermediation, we note that Brazil’s domestic banks operated balance sheets roughly five times the size of foreign banks’ subsidiaries in Brazil. Interpreting balance sheet size as rough proxy for portfolio constraints (corresponding to $X$ in our microfoundation), this leads us to calibrate $\Gamma_H/\Gamma_F = 5^2$.

Unfortunately, there is no easy way to calibrate $\Gamma_F$. Therefore, we provide three values for $\Gamma_F$, $\Gamma_F \in \{1, 10, \infty\}$, wherever it does not clutter up figures; elsewhere we use a single value, mostly $\Gamma_F = 10$, as illustration. Notice that when varying $\Gamma_F$, we also vary $\Gamma_H$ according to our calibration of the ratio $\Gamma_H/\Gamma_F$. As will become clear below, choices for $\Gamma_F$ of 1 or 10 approximately imply that a real exchange rate depreciation of 1% for one year requires a peak accumulation of reserves relative to GDP of 1.5% for $\Gamma_F = 10$, and of 7% for $\Gamma_F = 1$. This seems to be in the ballpark of empirical estimates. Kearns and Rigobon (2005) use structural breaks in the intervention policies of two advanced economies, Japan and Australia, to identify the effectiveness of interventions. Converting their findings into this context reveals that the same 1% depreciation requires a reserve accumulation of 4% over GDP for both economies. Equivalent numbers for emerging market economies are most certainly lower than these. De Gregorio (2013) mentions that practitioners in those countries often use a reserve accumulation of 1-2% (over GDP) as benchmark.

Finally, we consider in our plots $r^*$ shocks that temporarily lower $r^*$ by 2% for 3 years before they return to the steady state value of 7.5%.
2.4.2 Optimal interventions without terms-of-trade management

We can shut off the macro stabilization motive in our planning problem (2.18) in two ways. First, we can set $\alpha = 1$. In this case, the country loses home bias for its own good and hence the central bank loses the ability to influence the price of its own good by reallocating consumption over time. Thus, there is no longer a motive to manage the terms-of-trade. Second, it turns out that there is also no such motive when the world interest rate is constant and equal to home’s discount factor, i.e. $r^*_t = \rho$ at all times. We now investigate both of these cases, showing that in both of them the optimal UIP wedge $\tau_t$ is equal to zero. After that, we discuss the implications for reserves $b^*_Gt$ and actual interventions.

**Proposition 5.** Suppose $\alpha = 1$ or $r^*_t = \rho$ for all $t \geq 0$. Then, the optimal allocation coincides with the first best. In particular, $\tau_t = 0$ at all times $t$.

Proposition 5 identifies two cases for which the planner sees no need for nonzero UIP wedges and hence chooses not to distort the economy. The arguments behind the two cases are distinct. When $\alpha = 1$, the home economy has no home bias and therefore its consumption is unable to affect the real exchange rate. Therefore, interventions are completely ineffective in this case. When $r^*_t = \rho$ at all times, even if $\alpha$ is possibly less than 1 or there are endowment shocks $\{y_{Ht}\}$, home’s consumption $\theta_t$ is constant over time, and so are home’s exports. Thus, there is no reason to manipulate the terms of trade over time.

In the next subsection, we explore deviations from this “neutrality” result. In particular, when $\alpha < 1$ and the economy faces $r^*_t$ shocks, it turns out that the central bank has a macroeconomic stabilization motive and generally finds it optimal to implement nonzero UIP wedges $\tau_t$. For the remainder of Section 2.4, we set $\alpha < 1$.

2.4.3 Optimal interventions with terms-of-trade management

In this subsection, we study the solution to the planning problem (2.18) in the presence of a motive to manage the terms of trade. This motive is also at the core of many papers on capital controls (see e.g. Costinot et al., 2014 or Farhi and Werning, 2012, 2013). In Section 2.5.1 below, we study an alternative motive for intervention, based on fixed exchange rates and sticky prices.

We focus on $r^*$ shocks since we have already seen in the previous section that time-varying $\{r^*_t\}$ is crucial for an intervention motive. Specifically, we refer to paths $\{r^*_t\}$ such that $r^*_t > \rho$ for all $t \in [0, T)$ and $r^*_t = \rho$ thereafter as *positive interest rate shocks*; and to paths $\{r^*_t\}$ such that $r^*_t < \rho$ for all $t \in [0, T)$ and $r^*_t = \rho$ thereafter as *negative interest rate shocks*. We assume that $\{r^*_t\}$ is integrable throughout this section. As before, we first characterize the
optimal foreign exchange intervention policy in terms of the path of induced UIP wedges \{\tau_t\}. Subsequently, we discuss the implications for reserves and exchange rates, relative to the “undoing” benchmark, \(\tau_t = 0\) for all \(t\) (see the discussion in Section 2.3.4).

**The benchmark of financial autarky, \(\Gamma_F = \infty\)**

We begin the analysis by studying the special case of our model in which the private sector is in financial autarky (\(\Gamma_F = \infty\)). This is useful to isolate the motive for intervention by the central bank.

**Proposition 6.** Suppose \(\Gamma_F = \infty\). Then, the optimal intervention after a positive interest rate shock hits is to set \(\tau_t < 0\) for any \(t \in [0, T)\) and \(\tau_t = 0\) thereafter. In particular, the central bank only intervenes during the time of the interest rate shock. Analogously, \(\tau_t > 0\) for \(t \in [0, T)\) and \(\tau_t = 0\) thereafter in response to a negative interest rate shock.

Similar to Costinot et al. (2014) our model embeds a terms-of-trade management motive: Individual agents do not internalize the effect of their consumption decisions on the price of the exported good. This effect is nonzero as a result of the assumption of home bias. To fix ideas, suppose the foreign interest rate is temporarily low at time \(t\), \(r^*_t < \rho\). Since this implies that exports are relatively low at time \(t\) (borrowing against future income), the planner would like to lower the export price, or equivalently depreciate the real exchange rate. Setting a positive UIP wedge, \(\tau_t > 0\) then reduces current consumption, which in turn, achieves the desired real exchange rate depreciation.

This is also what we see in Figure 2-1, which shows the economy’s reaction to the simple negative interest rate shock described in our calibration in Section 2.4.1. The red line shows the response in case of financial autarky, \(\Gamma_F = \infty\). It is evident that the UIP wedge jumps up as the shock hits and back down to zero as the shock fades (Panel (b)), thereby reallocating domestic demand into the future and depreciating the real exchange rate (Panel (c)). The economy executes this intervention by accumulating additional reserves during the period of the shock (Panel (d)). Such a policy is often referred to as “leaning against the wind” of international capital flows.

**Intervention smoothing**

Compared to the special case of \(\Gamma_F = \infty\), studying optimal policy with intermediate degrees of capital mobility delivers three key new insights. Given that setting wedges is costly, one may expect that the optimal policy would lie somewhere between the \(\Gamma_F = 0\) solution, that is
Figure 2-1: Optimal intervention after a negative $r^*$ shock in a deterministic model economy calibrated to Brazil, for various degrees of capital market imperfection $\Gamma_F$. The shock lowers $r^*$ by 2% for $t \in [0, 3]$. The results are relative to the zero UIP wedge economy ($\tau_t = 0$).

$\tau_t = 0$, and the $\Gamma_F = \infty$ solution characterized in Proposition 6. The following result shows this intuition is fundamentally wrong.

**Proposition 7 (Smoothing).** Suppose $\Gamma_F \in (0, \infty)$. Then, at the optimum, $\tau_t$ is continuous in $t \in (0, \infty)$, with $\tau_0 = 0$.

Proposition 7 highlights a property of the model that is only present for intermediate degrees of capital mobility: the central bank chooses a smooth path for $\tau_t$.\(^{36}\) Contrast this with the $\Gamma_F = \infty$ solution: There, $\tau_t$ jumps whenever $r^*$ jumps. The reason for the “smoothing” result is very natural: With $\Gamma_F \in (0, \infty)$, each deviation of $\tau_t$ away from zero incurs convex costs (2.16). Hence, it is optimal to spread out interventions over time, optimally making interventions small and long lived, rather than large and short lived.

This result follows from a helpful lemma. To state the lemma, we introduce

$$T_t = \frac{e^{-\int_0^t \tau_u du} \lambda}{\text{marg resource cost at time } t} - \frac{e^{-\rho t} V'(\theta_t)}{\text{marg utility at time } t}$$

as the deviation of time $t$ (dollar) consumption $\theta_t$ from first best levels. $T_t > 0$ whenever consumption $\theta_t$ is too large relative to first best, and $T_t < 0$ whenever $\theta_t$ is too small relative to first best.\(^{37}\) Here, $V(\theta)$ is the planner’s per-period objective. Using this notation, the lemma can be stated.

\(^{36}\)This is reminiscent of a number of “tax smoothing” results in the optimal taxation literature, spawned by Barro (1979).

\(^{37}\)Notice that this is not a mathematically rigorous statement since $\lambda$ here is not the same as in the first best problem. $T_t$ still turns out to be a very useful object.
Lemma 3. Suppose $\Gamma_F \in (0, \infty)$. Let $V(\theta)$ be the planner’s per period objective, as defined before Lemma 2. Then, under the optimal foreign exchange intervention policy, the interest rate spread $\tau_t$ satisfies the following first order condition

$$e^{-\int_0^t r^*_s ds} \lambda \frac{2}{\Gamma_F} \tau_t = \int_0^t T_s ds.$$  \hspace{1cm} (2.19)

Lemma 3 is a straightforward consequence of the first order conditions of the planning problem and immediately implies Proposition 7. First, since the integrals of $r^*_s$ and $T_s$ are continuous functions, it follows that $\tau_t$ is continuous. And second, since the right hand side of (2.19) is zero at $t = 0$, it must be that $\tau_0 = 0$.

The first order condition (2.19) has a useful intuition. Suppose the planner increased $\tau_t$ by a marginal unit. Increased carry trades by foreign intermediaries would then consume an extra $\frac{2}{\Gamma_F}$ of the economy’s resources, valued at shadow price $\lambda$. This is captured as the marginal cost term on the left hand side of (2.19). However, such an intervention would also lower $\theta_s$ in all previous periods $s \leq t$, due to the forward looking nature of the Euler equation. In each such period $s$, it saves one unit of resources and increases marginal utility, whose joint effect on utility is precisely captured by $T_s$. This explains the right hand side of (2.19).

Panel (b) of Figure 2-1 illustrates our “intervention smoothing” result. Even though the path for the interest rate shock $\{r^*_t\}$ is discontinuous, and in stark contrast with the optimal UIP wedges when $\Gamma_F = \infty$ (the red line), for finite positive values of $\Gamma_F$ the optimal UIP wedges are continuous and start at zero. Their sign is the same as the one for $\Gamma_F = \infty$, so here again, the planner leans against the wind and accumulates reserves (Panel (c)) to depreciate the real exchange rate (Panel (a)). It is worth pointing out that the reason for why the lowest value for $\Gamma_F$, $\Gamma_F = 1$, is associated with the largest accumulation of reserves comes from the fact that we calibrate the ratio $\Gamma_H/\Gamma_F$ to the data, and so lower values for $\Gamma_F$, capturing more foreign intermediation, automatically lead to lower values for the $\Gamma_H$ as well, capturing more domestic intermediation. This leads to two countervailing forces: Lower $\Gamma_F$ pushes for less aggressive and smoother interventions, while lower $\Gamma_H$ means the central bank needs to accumulate more reserves to achieve a given spread $\tau_t$.

The fact that in Panel (b) of Figure 2-1, optimal UIP wedges for $\Gamma_F < \infty$ are still positive well beyond the end of the shock at $t = 3$ is the subject of our next subsection.
Forward guidance

The smoothness of the intervention $\tau_t$ has two interesting indirect consequences: The first one, described in this subsection, could be described as “FXI forward guidance”. Since interventions are smoothed out over time, the planner in fact has an interest in promising to keep intervening—that is, creating nonzero wedges $\tau_t \neq 0$—even at times $t > T$, after the shock subsided. We formalize this in the following result.

**Proposition 8.** Suppose $\Gamma_F \in (0, \infty)$. Then, after a positive interest rate shock, $\tau_t < 0$ at all times $t$ (including $t > T$). Analogously, after a negative interest rate shock, $\tau_t > 0$ at all times $t$ (including $t > T$).

To see the intuition behind this result, consider the first order condition (2.19). While the marginal cost of a marginal intervention at time $t$ is born at that current time due to increased private carry trade activity, the marginal benefits $T_s$ of influencing future interest rates accrue at all times $s \leq t$ before $t$. In that sense, the logic is analogous to forward guidance in a New Keynesian model (see, e.g., Eggertsson and Woodford 2003 or Werning, 2011), where marginal benefits of low rates after the zero lower bound stops binding also propagate back in time through the Euler equation.

This result also speaks to the ongoing debate over whether the likely channel through which foreign exchange interventions work is a portfolio balance channel or some kind of signaling channel. While the core of our model consists of a portfolio balance channel—foreign intermediaries only imperfectly react to the interest rate spread $r - r^*$, well in the spirit of the old portfolio balance literature (see, e.g. Kouri, 1976, Branson and Henderson (1985) or Kenen (1987))—our microfoundations and rational expectations weave a natural signaling channel into our model. Since future interventions are effective through a portfolio balance channel in the future and agents are forward-looking, signaling future interventions has the power to affect agents’ actions today. In Panel (b) of Figure 2-1, the forward guidance aspect of optimal interventions is clearly visible. For finite, positive $\Gamma_F$, the optimal UIP wedges are not only smooth over time, but also stretch well into the future, beyond the period of the shock.

Time inconsistency

Clearly, in contrast with the direct effect through current portfolio choices, the effectiveness of signaling future interventions critically depends on the credibility of the central bank. This naturally opens the door to problems of credibility and time inconsistency. We formulate this in the next proposition.
Proposition 9. Suppose $\Gamma_F \in (0, \infty)$. The optimal policy is time-inconsistent and re-optimization at any time $t_0 \geq T$ yields $\tau_t = 0$ for all $t > t_0$. Moreover, a planner without any commitment power can only achieve the no-intervention outcome, $\tau_t = 0$ at all times $t$.

The argument behind the first part of Proposition 9 is quite straightforward. We already saw above that in an environment without shocks (see Lemma 5) the optimal UIP wedge is zero at all times. The time after a shock has faded, that is, $t > T$ for the interest rate shocks we have focused on in this section, is precisely such a time of no more shocks. Yet the optimal policy as described in Proposition 8 requires nonzero UIP wedges in all periods. The planning problem thus is time inconsistent.

The second part is more involved. Even if a planner without any commitment power will set $\tau_t = 0$ for all $t > T$, why would he do so during the time of the shock as well? The answer to this question lies in the fact that even during the time of the shock, interventions derive their effect from affecting earlier consumption decisions, see e.g. the first order condition in (2.19). Yet, since those consumption decisions are in the past, a planner without any commitment power does not take them into account and instead chooses as optimal policy $\tau_t = 0$.

The time inconsistency issue raises the question of how much the effectiveness of interventions depends on the credible signaling of future interventions. We explore this question in our simulations in Figure 2-2. Here we compare a full commitment (FC) policy with $\Gamma = 10$ to a limited commitment (LC) policy, where the central bank can only credibly commit to interventions until the shock fades at time $T = 3$. We see in Panel (b) that the LC planner uses the limited commitment to promise strong interventions during the time of the shock, to make up for the interventions the FC planner promises after $t = T$. While the increased interventions until $t = T$ are costly to the LC planner, they do achieve almost exactly the same extent of real exchange rate depreciation as the FC planner. Panel (d) shows that the LC planner accumulates more reserves than the FC counterpart until $t = T$. At that time, however, when the UIP wedge $\tau_t$ drops to zero and home as well as foreign intermediaries close their carry trade positions, the LC planner balances this by repatriating a large fraction of the accumulated reserves.

We should note that the type of time inconsistency is different from the standard time inconsistency coming from the tendency to depreciate one's (real or nominal) exchange rate when foreigners hold domestic local currency bonds. Since we analyzed our model in terms of domestic bonds measured in dollars, this type of time inconsistency does not appear in our setup and hence does not get entangled with the novel type of time inconsistency that we describe in Proposition 9. Of course, were we to denote initial positions in terms of the local
price index, the more standard time inconsistency would re-emerge.

In sum, Sections 2.4.3 and 2.4.3 highlight that foreign exchange interventions are more powerful when they are coupled with signaling and when the central bank has at least some amount of commitment power.

2.5 Extensions

In this section, we present and discuss three extensions to our baseline model. In a first one, in Section 2.5.1, we introduce a model with an alternative motive for interventions. There, the central bank faces constraints on exchange rate movements—to make it stark we assume a fixed exchange rate—and yet seeks to use foreign exchange interventions to stabilize the output gap. We find that all our previous analytical results go through in this economy. In Section 2.5.2 we enrich our model somewhat to include liquidity and risk premia, at least in some abstract form. We show that when a country sells “safe haven” bonds, it might actually earn money from foreign exchange interventions, rather than pay for them. Finally, Section 2.5.3 presents an economy which is pursuing a “managed float” policy in which the real exchange rate is required to follow a smooth path. We show that this kind of policy, which resonates well with many exchange-rate based rules of EMEs, may significantly backfire, inviting costly speculation.

In Appendix 2.E we provide two additional extensions, one on whether the time inconsistency of our baseline model can be fixed by the planner itself using assets of multiple
maturities a la Lucas and Stokey (1983); and the other generalizing intermediaries’ asset demand to nonlinear demand schedules.

2.5.1 Sticky prices and fixed exchange rates

So far, our analysis focused on foreign exchange interventions driven by a terms-of-trade management motive. One possible interpretation of that model is that there is a monetary authority in the background choosing the nominal interest rate to close the output gap at all times. To see this, consider a simple extension in which the home good is actually produced with a simple technology \( y_{Ht} = n_t \), and the household experiences disutility of labor given by \( v(n_t) \), so preferences are given by

\[
\int_0^\infty e^{-nt} \{\log(c_t) - v(n_t)\} dt. \tag{2.20}
\]

In addition, to have a meaningful monetary policy problem, assume the home currency price of the home good is fixed at \( p_H = 1 \) and the nominal exchange rate is given by \( e_t \). 38 This implies that the dollar value of \( y_{Ht} \) is given by

\[
e^{-1}_{t-1} y_{Ht} = (1 - \alpha) \theta_t + \alpha c^*, \tag{2.21}
\]

where we re-introduced \( c^* \) from Section 2.3.1. Compare this to the flexible price allocation: There, output is given as \( y_{Ht}^f = n_t^f \), pinned down jointly with the flexible price exchange rate \( e_t = e_t^f \) by combining (2.21) with

\[
v'(n_t^f) = \theta_t^{-1} (e_t^f)^{-1}. \]

Thus in our previous analysis, we can imagine that monetary policy is implementing \( e_t = e_t^f \) at all times. 39 In this sense, the output gap objective takes priority over the real exchange rate objective. In this subsection, we explore the polar opposite: We assume that—for some unmodeled reason—the monetary authority has some exchange rate objective \( e_t \). To make it stark, we assume a fixed exchange rate regime, \( e_t \equiv e \), and ask: How can the planner...
use foreign exchange interventions to regain some monetary independence and mitigate the impact on the domestic economy? Examples of interventions of this sort arguably include recent interventions by Euro neighbors like Denmark, Switzerland, or the Czech Republic, which try to fend off appreciations and at the same time avoid being pushed into the zero, or effective, lower bound for interest rates.

In a first step, we ask which allocations can be implemented by central bank policies. Fortunately, it is straightforward to show that, in fact, when stated in terms of \( \{\theta_t, r_t\} \), the same implementability conditions as in Proposition 4 continue to hold in this economy.\(^{40}\) The reason is that sticky prices let labor supply and the home endowment depend on \( \theta_t \), yet neither enters the implementability conditions. They do, however, enter the objective function. Replacing labor with (2.21) in the utility function (2.20) and following the same steps as before, we find the planning problem to be

\[
\max_{\{\theta_t, r_t\}} \int_0^\infty e^{-\rho t} \{ \log \theta_t - (1 - \alpha) \log ((1 - \alpha) \theta_t + \alpha c^*) - v ((1 - \alpha) \theta_t + \alpha c^*) \} \, dt \quad (2.22)
\]

subject to (2.17a) and (2.17b).

The crucial difference to our previous planning problem is the objective function. The reason for this is that the rationale for intervening has changed.\(^ {41}\) Before, the central bank was intervening to manage the country’s terms of trade. Now a second rationale emerges: Regaining monetary independence despite the fixed exchange rate. Suppose the world interest rate decreases temporarily. The flexible exchange rate response would be to let the currency appreciate today and depreciate in the future. Since this is impossible with a fixed exchange rate, the economy experiences a boom and a subsequent recession. In this situation, by accumulating reserves and hence generating a positive UIP spread, the planner is able to shift expenditure into the future. This mitigates both the boom and the subsequent recession.

To see whether this “leaning against the wind” property, as well as our other results in Section 2.4, carry over to this fixed exchange rate environment, notice that the planning problem (2.22) is almost unchanged: It still involves a strictly concave per period objective function and the maximization is subject the exact same constraints. Therefore, the results in Lemmas 2 and 3 and Propositions 5–9 carry over to this alternative environment, one for one. In particular, optimal interventions in this fixed exchange rate environment are still small, frequent, persistent and credible. This example illustrates that while the reason for

\(^{40}\)Strictly speaking, with \( c^* \neq 1 \), the term \( \alpha(\theta_t - 1) \) in (2.17b) needs to be replaced by \( \alpha(\theta_t - c^*) \) but everything else is unchanged.

\(^{41}\)To see this most clearly, one may set \( c^* = 0 \). In that case, there is no “terms-of-trade management” motive and, yet, the planner would like to intervene to stabilize the output gap.
intervening may differ across applications, the way interventions are implemented does not. In this sense, the results of Section 2.4 are robust.

2.5.2 Safe havens

While the fact that our analysis is deterministic has several advantages, in particular in terms of clarity and tractability, it lacks risk or “safety” premia. In recent years, some advanced economies, most notably Switzerland, have conducted foreign exchange interventions in a setting where domestic bond yields are typically lower than the world interest rate, despite the interventions. This raises the question of whether these interventions are costly to the central bank at all, or if they might actually benefit from them.\(^4\) Here, by cost we mean our cost term \(1/\Gamma_F \tau^2\) which can never be negative (i.e. a profit) in our analysis without risk premia so far. In this subsection, we add a simple modification to our existing model that seeks to capture the key implications of risk premia for foreign exchange interventions, without giving up our tractable deterministic framework.\(^4\)

The reason why risk is important to study in our context is that agents in the domestic economy might value bonds using a different stochastic discount factor than foreign intermediaries.\(^4\) To capture this idea in our deterministic model, we now explore what happens if intermediaries perceive an additional benefit of \(\xi_t\) for each additional unit of the domestic bond held,\(^4\)

\[ b_{It}^H = \frac{1}{\Gamma_H} (r_t + \xi_t - r^*_t) \quad \text{and} \quad b_{It}^F = \frac{1}{\Gamma_F} (r_t + \xi_t - r^*_t). \]

\(^4\) Of course, our analysis cannot speak to the political circumstances that engulf such large-scale interventions. These were among the reasons that led the Swiss central bank to stop its interventions in January 2015, causing a large appreciation of the Swiss Franc and a corresponding valuation loss on the SNB’s foreign exchange holdings. See, e.g., http://www.wsj.com/articles/swiss-national-bank-scRAPs-minimum-exchange-rate-1421315392 and http://www.wsj.com/articles/swiss-national-bank-reports-23-5-billion-loss-in-2015-1457099558.

\(^4\) We also would like to highlight a different but highly complementary perspective by Amador et al. (2016) to understanding Swiss interventions, using CIP rather than UIP violations: If one interprets the risk of the Swiss National Bank (SNB) abandoning the peg as an exogenous event that is outside the SNB’s control, then the costs of interventions are indeed characterized by CIP violations. Due to expected appreciation in the forward markets at the time, these “CIP costs” were positive for the SNB. Notice that of course, such CIP costs only realize eventually, after the peg is abandoned. To judge, however, whether such a peg is costly if kept permanently (i.e. under perfect commitment, irrespective of beliefs in the forward market), it is advisable to consider the actual, UIP related, costs the country incurs instead. We argue in this subsection that for Switzerland, these “UIP costs” may well have been negative.

\(^4\) Risks that symmetrically affect all bond market participants (intermediaries and the central bank) can be captured by movements in \(r^*_t\) in our model.

\(^4\) A simple microfoundation of these benefits would be a latent risk that materializes with some Poisson intensity \(\lambda \to 0\) and intermediaries that are Knightian to different degrees. See Caballero and Farhi (2015) for a microfoundation of safety along those lines in a deterministic economy.
These equations only affect the previous intermediary bond demands (2.5) and (2.6) but leave our definition of competitive equilibrium otherwise unchanged. In particular, intermediary profits are still given by \( b_j'(t_t - r_t^*) \) for intermediary \( j = H, F \). While it turns out that, as before, domestic intermediaries’ bond demand does not enter the implementability conditions, the new foreign intermediaries’ demand function changes the costs of UIP wedges \( \tau_t = r_t - r_t^* \) to

\[
\tau_t b_H^F = \frac{1}{\Gamma_F} \tau_t^2 + \frac{1}{\Gamma_F} \xi_t \tau_t = \frac{1}{\Gamma_F} \left( \tau_t + \frac{\xi_t}{2} \right)^2 + \frac{1}{\Gamma_F} \frac{\xi_t^2}{4}.
\]

In this environment, it is evident that not all interventions are costly. If—as one could argue applies to safe havens like Denmark or Switzerland—the domestic bond is considered a particularly safe asset to investors, captured by a positive \( \xi_t \), then intervening in the foreign exchange market to generate positive UIP wedges \( \tau_t \) can leave the country with a net profit. The intuition for this is straightforward: In the case of \( \xi_t > 0 \), the country is the sole producer of an asset that outside investors value higher than the asset’s producer (the home country). Thus, it can supply the market with the asset and charge a premium in form of a positive \( \tau \) for it. This generates profits.

Apart from the fact that interventions can profitable, (2.23) also reveals that the myopically optimal UIP wedge \( \tau_t \) is no longer zero, and instead equal to \(-\xi_t/2\). That is, the planner now seeks to smooth out \( \tau_t + \xi_t/2 \) over time, rather than \( \tau_t \), since those are the deviations from the myopically optimal policy. This is essentially the key difference to our previous analysis. Figure 2-1 illustrates the optimal intervention in response to a \( \xi_t \) shock, again relative to the non-intervention economy. It can be seen that qualitatively, the plots look very similar to their counterparts in Figure 2-1.

Intermediary preference shocks are also useful in a different way. In Section 2.3.4 we argued that limited capital mobility plays two roles in our analysis. On the one hand, they determine the elasticity with which reserve flows are able to affect domestic interest rates. On the other, they cause underreaction of domestic interest rates in the laissez-faire competitive equilibrium in response to shocks to the world interest rate \( r_t^* \). The preference shocks \( \xi_t \) can help disentangle the elasticity role—which we are ultimately interested in—from the underreaction. To provide a clean argument, consider a situation where \( \xi_t \) only affects home intermediaries. When \( \xi_t \) increases as \( r_t^* \) falls, or in other words intermediaries associate a smaller risk premium with domestic bonds, it is possible that changes in \( r_t^* \) are passed through one-for-one to changes in \( r_t \) (rather than less than one-for-one). In this case, both the equilibrium UIP spread and equilibrium reserve holdings are exactly zero. In this interpretation, since home intermediaries’ demands do not affect the planning problem, Figures 2-1 and 2-2 represent
the actual holdings of reserves.

2.5.3 Smooth exchange rates vs. smooth UIP wedges

There is ample evidence by now that some emerging market policymakers, especially East-Asian ones, seem to conduct policies aimed at smoothing exchange rates, at short to medium horizons.\footnote{See the voluminous “fear of floating” literature, e.g., Calvo and Reinhart (2002), Levy-Yeyati and Sturzenegger (2005b) or McKinnon and Schnabl (2004).} This is sometimes referred to as a “managed float” and should not be confused with the kind of “intervention smoothing” policy that we found to be optimal in our model: Here, UIP wedges $\tau_t$ are smooth, while exchange rates jump initially, albeit by less than without intervention if the optimal policy is employed. This is the case even though an exchange rate that is smooth at $t = 0$ would certainly lie in the space of implementable allocations. This raises the natural question of why our optimal policy problem did not select such a “managed float” allocation.

To explore this question, we simulate the optimal policy under an additional ad-hoc “smooth exchange rate” constraint, namely that the initial real exchange rate $q_0$ be the same as in the steady state, $q_0 = q_{ss}$.\footnote{To be clear, by “smooth exchange rate” we mean continuity of the exchange rate, not any kind of differentiability. Furthermore, note we can interpret this a smooth nominal exchange rate requirement, together with the commitment to pick a flex-price allocation, as explained in 2.5.1.} Since the real exchange rate in the model moves one-to-one with dollar consumption $\theta_t$, this is equivalent to adding the constraint

$$\theta_0 = \theta_{ss} = 1 + \alpha^{-1} r^* nfa_0.$$ (2.24)
We thus solve our planning problem (2.18) subject to (2.17a), (2.17b) and (2.24). It is worth stressing that when $\Gamma_F \in (0, \infty)$ this additional constraint does not make the problem trivial, in the sense that it is impossible to approximate the solution of our original problem arbitrarily closely. The reason for this comes from the fact that this would require infinitely large, and infinitely costly, UIP wedges $\tau_t$ for $t$ close to zero.

We plot the optimal managed float intervention in Figure 2-2 and compare it to our unconstrained optimal policy, computed without (2.24) as constraint. Panel (b) shows the two economies’ real exchange rate paths. It is clearly visible that while the unconstrained optimal policy features a sharp and sudden exchange rate appreciation (solid), the managed float policy generates a much slower appreciation that stretches even beyond the first year of the shock (dashed). As a consequence of the slow and predictable appreciation, the country must bear the cost of an excessive UIP spread (Panel (a)), inviting a significant amount of carry trade activity both by foreign and by domestic intermediaries. To implement this policy, the domestic central bank acts as a “shock absorber”: it accumulates reserves at a rapid pace initially, and then slowly decumulates them over the years.

The reader may wonder why the central bank trying to implement a peg in Section 2.5.1 did not encounter such heightened carry-trade activity in the beginning of the intervention period. The reason for this is that in Section 2.5.1 we allowed for a non-zero output gap while in this subsection we assumed the output gap is zero at all times, i.e. the central bank is choosing one among the flex-price allocations. Thus, this subsection emphasizes that
smoothing the exchange rate path may be very costly if domestic objectives are perceived by agents to be more important than external objectives. Put differently, a policymaker seeking to implement a smooth exchange rate path may avoid speculation by foregoing some domestic stability, as in Section 2.5.1.

2.6 Reserve wars

So far, we have analyzed the optimal policy of a small open economy (SOE) against a passive rest of the world. Now, we explore the strategic interaction among different SOEs’ foreign exchange interventions and their effects on the rest of the world. For this purpose, we consider a world with two kinds of countries: Advanced economies (AE), which are assumed to have frictionless financial markets, and emerging markets (EME), which are SOEs like the one described in Section 2.3. For ease of exposition, we focus on a two period version of our model.

The structure of this section is as follows. In Subsection 2.6.1, we describe the model. Then, we characterize the world equilibrium in a decentralized setting where each EME central bank chooses its own foreign exchange policy taking as given the actions of the other EMEs. Finally, we consider the world equilibrium where foreign exchange policies are set by a single “EME cooperative planner”, who can be interpreted as a stand-in for closer central bank cooperation.

2.6.1 Setup

The world consists of an interval \([0, 2]\) of small open economies and exists for two periods \(t = 0, 1\). Economies \(i \in [0, 1]\) are “advanced economies” (AE) while economies \(i \in (1, 2]\) are “emerging market economies” (EME). Economies within each of the two regions are identical in all respects. A typical SOE \(H \in [0, 2]\) has the same preferences as the SOE in Section 2.3 except for being two-period lived, that is, \(U = \ln(C_0) + (1 + \rho_j)^{-1} \ln(C_1)\) with \(C_t = \kappa c_{Ht}^{1-\alpha} c_{Ft}^\alpha\), where the discount factor \(\rho_j\) is allowed to depend on the region \(j = AE, EME\). Here, \(c_{Ht}\) denotes consumption of \(H\)'s own good, and \(c_{Ft}\) denotes consumption of a common foreign good. We assume that the foreign good is a composite good given by \(c_{Ft} = c_{AEt} + c_{EMEt}\), where \(c_{AEt}\) and \(c_{EMEt}\) are themselves aggregates of varieties produced in each region,

\[
\ln(c_{AEt}) = \int_{i \in [0,1]} \ln(c_{it}) di \quad \text{and} \quad \ln(c_{EMEt}) = \int_{i \in (1,2]} \ln(c_{it}) di.
\]

This market structure captures that EMEs compete more with one another than with AEs in world good markets and allows us to obtain clean benchmark results. We normalize the price
of the composite foreign good to 1 in each period. We assume that each SOE is endowed with its own good: EMEs with $\chi \in [0, 1]$ units of their own good and AEs with $1 - \chi$ units of their own good. Henceforth, we will exploit symmetry and label with a star “*” variables from a typical AE, and without stars variables from a typical EME.

EMEs are characterized by imperfect financial mobility. Residents in any given EME are only allowed to trade a bond in the jurisdiction of their own country, paying gross interest rate $1 + r$ (as before in units of the foreign good which we call “dollars”). In addition, there is a continuum of intermediaries located in AEs who can freely access each of the AEs’ bond markets, which pay a common interest rate $r^*$ in units of the foreign good $c_{Ft}$. They can trade in each of the EMEs’ bond markets but only up to a position limit $X\sqrt{\chi}$, and only after paying an idiosyncratic transaction cost. Following the same steps as in Section 2.3, this leads to a finitely elastic demand function for each EME bond, $b_t = \chi \Gamma_t^{-1} (r - r^*)$. As in the previous sections, central banks are assumed to have perfect access to AEs’ bond markets as well as their own bond market, but not other EMEs’ bond markets. We define $\theta_t \equiv \chi^{-1} P_t C_t$ and $\theta^*_t \equiv (1 - \chi)^{-1} C^*_t$ as the (normalized) dollar consumptions of EMEs and AEs. As before, the UIP wedge is defined as $\tau = \frac{1 + r}{1 + r^*} - 1$.

We next characterize and define a competitive equilibrium in this economy. The home market clearing for a typical EME is

$$\alpha \chi \theta_t + \alpha (1 - \chi) \theta^*_t = \chi C_{EMEt} + (1 - \chi) C_{AEt}. \quad (2.27)$$

In what follows, we focus on a symmetric equilibrium with positive consumption of both aggregate goods, $P_{EMEt} = P_{AEt} = P_{Ft} = 1$. By symmetry we mean that every EME central

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48 We normalize $X$ by $\sqrt{\chi}$ to make sure the limit where EMEs are small, $\chi \to 0$, is well-defined.
49 If central banks could access other EMEs bond markets, they would behave like arbitrageurs, exploiting opportunities generated by other central banks. This may generate another motive for central bank coordination, independent of the one we focus on. In any event, our results would go through as long as trading in other EME bonds by central banks is costly.
50 This will always occur in equilibrium if discount factors are not too different. Note, however, that
bank finds it optimal to carry out the same foreign exchange policy, implying that

\[ p_{Ht} = p_{Ht}^* = 1. \]

Symmetry allows us to simplify (2.25), (2.26) and (2.27), yielding

\[(1 - \alpha)\theta_t + C_{EME_t} = (1 - \alpha)\theta_t^* + C_{AEt} = 1. \tag{2.28} \]

and

\[\chi \theta_t + (1 - \chi)\theta_t^* = 1. \tag{2.29} \]

In each SOE, the optimal solution of the consumers' problem implies the following Euler equations,

\[\theta_1 = \frac{(1 + r^*)}{1 + \rho_{EME}}(1 + \tau) \theta_0 \quad \text{and} \quad \theta_1^* = \frac{1 + r^*}{1 + \rho_{AE}}\theta_0^*. \tag{2.30} \]

Finally, consolidating the consumer's budget constraint with the central bank's yields the country-wide budget constraint,

\[\alpha \theta_0 - C_{EME0} + \frac{1}{1 + r^*}(\alpha \theta_1 - C_{EME1}) + \frac{1}{\Gamma_F} \tau^2 = 0, \]

which using (2.28) implies

\[(\theta_0 - 1) + \frac{1}{1 + r^*}(\theta_1 - 1) + \frac{1}{\Gamma_F} \tau^2 = 0. \tag{2.31} \]

For simplicity, we take initial net foreign asset positions to be zero.

We are ready to formally define a symmetric world competitive equilibrium in this economy. For ease of exposition and brevity, we state the foreign exchange policy directly in terms of \(\tau\).

**Definition 4.** A symmetric world competitive equilibrium given an EME central bank foreign exchange policy \(\tau\), is an allocation \(\{\theta_t, \theta_t^*, C_{EME_t}, C_{AE_t}\}_{t=0,1}\) together with an interest rate \(r^*\), such that equations (2.28) – (2.31) hold.

This defines a competitive equilibrium given a (symmetric) set of EME foreign exchange policies. We now characterize the competitive equilibrium that occurs if EME central banks play a Nash equilibrium in their choice of foreign exchange policies. For our numerical analysis would still be true if one of the aggregate goods had negative consumption, in the interpretation that then extra supply of that good would be created, using some of the other aggregate good as inputs.
illustrations, we stick with our baseline calibration and assume that $\Gamma_F = 10$ and $\alpha = 0.15$. Interpreting one period as the equivalent of 3 years, so that our 2-period model captures the same kind of shock as before, we set $1 + \rho_{EME} = 1.075^3$. We re-calibrate $\rho_{AE}$ as we vary $\chi$ such that that the annualized response of the world interest rate when $\tau = 0$ is 5.5%. All results below will be stated in annualized terms.

**Noncooperative world equilibrium.** The typical EME central bank maximizes its own welfare taking two objects as given: the world interest rate $r^*$ and foreign expenditure levels $\{C_{EMEt}\}$. Proceeding exactly like in equation (2.18), we find that the problem of an individual EME central bank is

$$\max \sum_{t=0}^{1} (1 + \rho_{EME})^{-t} \{\ln(\theta_t) - (1 - \alpha) \ln((1 - \alpha)\theta_t + C_{EMEt})\}$$

subject to

$$\alpha\theta_0 - C_{EME0} + \frac{1}{1 + r^*}(\alpha\theta_1 - C_{EME1}) + \frac{1}{\Gamma_F} \left(\frac{\theta_1(1 + \rho_{EME})}{\theta_0(1 + r^*)} - 1\right)^2 = 0. \tag{2.33}$$

This is a two-period version of the problem analyzed in Section 2.4, where we had $C_{EMEt} = \alpha$ and $\chi = 1$. In (2.33) we also substituted out $\tau = \frac{\theta(1+\rho_{EME})}{\theta(1+r^*)} - 1$. The intuition behind this problem is the same as before: When the interest rate $r^*$ is lower than the EME discount rate $\rho_{EME}$, or foreign expenditure $C_{EMEt}$ is higher at $t = 0$ than at $t = 1$, the EME central bank optimally accumulates reserves and implements a positive UIP wedge $\tau$. It is worth noting that the IC constraint (2.33) is a non-linear function of $\theta_0$ and $\theta_1$ for any $\Gamma_F \in (0, \infty)$.

In Appendix 2.F.1 we prove that it is without loss to relax (2.33) as inequality and that this inequality constraint describes a convex, bounded set in $(\theta_0, \theta_1) \in \mathbb{R}_+^2$. Thus, whenever $C_{EMEt} \geq 0$ for all $t$ (to ensure the concavity of the objective) this is a well-behaved concave maximization problem with a convex constraint.

We next characterize the central bank Nash equilibrium, in which each central bank solves (2.32) taking $\{C_{EMEt}\}$ and $r^*$ as given, but in equilibrium $\{C_{EMEt}\}$ is pinned down by (2.28), and $r^*$ is pinned down by (2.29) and (2.30). To simplify the proofs, we assume that EMEs are small in the following formal result ($\chi = 0$). For $\chi > 0$, we verified numerically that the proposition still holds.

**Proposition 10** (Reserve wars.). Assume imperfect capital mobility, $\Gamma_F \in (0, \infty)$, and that emerging markets are small, $\chi = 0$. If emerging market central banks choose their foreign exchange policy in a non-cooperative way, then it holds that:
1. There exists a unique Nash equilibrium.

2. In the Nash equilibrium, emerging markets accumulate reserves and the UIP wedge $\tau$ is strictly positive.

3. Compared to a no-intervention world, capital flows more upstream towards the advanced economies, driven by reserves, while private capital flows more downstream towards emerging markets.

4. Welfare of emerging markets is lower than without intervention.

Proposition 10 describes four key properties of the Nash equilibrium. We discuss them in turn. First, the existence of a unique Nash equilibrium follows because for $\chi = 0$, foreign exchange interventions in the model are strategic substitutes: When EMEs choose to accumulate reserves, they depreciate their real exchange rates leading to higher total consumption of the EME good $C_{EME,0}$ in the first period. Because the EME good is a Cobb-Douglas aggregate of all EMEs, this also raises the first period demand for any single EME, hence calling for a more appreciated real exchange rate to exert monopoly power. Notice that, by contrast, for larger values of $\chi$, a force for complementarity emerges: Then, more reserve accumulation lowers world real interest rates even more, causing even stronger capital inflows into every single EME, and raising their desire to counter that with more reserve accumulation.

In the unique Nash equilibrium, the UIP wedge $\tau$ is strictly positive since advanced economies are attempting to save, $\rho_{AE} < \rho_{EME}$, causing private capital to flow into EMEs.
The reserve accumulation by EMEs then pushes public funds upstream, while the positive UIP wedge lets intermediaries take larger downstream positions. This explains parts 2 and 3 of Proposition 10. We illustrate these outcomes in Figure 2-1 as function of the overall size of EMEs $\chi$ (red, solid line). Panel (a) shows that the equilibrium UIP wedges are positive throughout, and rise with $\chi$ as the feedback loop through lower world interest rates kicks in (Panel (b)). Analogously, reserves are positive and increase with $\chi$ relative to GDP, as shown in Panel (c).

In general equilibrium, interventions are self-defeating: Even if all EMEs accumulate reserves to depreciate their $t = 0$ real exchange rate, this does not happen. Foreign demand is infinitely elastic if $\chi = 0$, fixing the real exchange rate at 1, as can be inferred from equation (2.28). This means that, in contrast to their intended purpose, interventions cause welfare losses for emerging markets. In addition, if $\chi > 0$, interventions might also reduce the welfare of advanced economies by depressing the world interest rate. Put together, these kinds of noncooperative reserve wars can cause welfare losses for all countries. We illustrate this in Figure 2-2 as function of the overall size of EMEs $\chi$ (red, solid line). As Panel (a) shows, welfare of EMEs suffers due to the competitive devaluations up until EMEs are so large that their effect on the world interest rate compensates for the welfare losses associated with the devaluations. In Panel (b) we see that welfare of AEs rises ever so slightly for small $\chi$ due to intermediary profits from carry trades against emerging markets and then rapidly falls below zero as AEs try to save in an increasingly low interest rate environment.
Cooperation of emerging market central banks  The self-defeating nature of interventions suggest that there may be gains from policy coordination among EME central banks. This is what we consider next. The world equilibrium can now be regarded as the outcome of a planning problem in which a single “EME planner” maximizes the objective (2.32) subject to the IC constraint (2.33), but now takes into account the endogeneity of \( \{C_{EME_t}\} \) and \( r^* \), coming from equilibrium conditions (2.28), (2.29) and (2.30). In the case where \( \chi = 0 \), we can prove the following result, standing in stark contrast with the noncooperative outcome.

Proposition 11 (Central bank cooperation.). Assume imperfect capital mobility, \( \Gamma_F \in (0, \infty) \), and that emerging markets are small in total, \( \chi = 0 \). If emerging market central banks cooperate, then it is optimal for emerging markets not to accumulate any reserves, implying a zero UIP wedge \( \tau = 0 \).

We illustrate the contrast between the Nash equilibrium outcome (red line, solid) and the cooperative solution (blue line, dashed) in Figures 2-1 and 2-2 as function of the overall size of EMEs \( \chi \). Figure 2-1 compares the equilibrium UIP wedge, the world interest rate and total reserves position as fraction of GDP. When \( \chi = 0 \), internalizing that competitive devaluations are self-defeating and with no possibility of manipulating the world interest rate in its favor, the cooperative planner sets \( \tau = 0 \). As \( \chi \) increases, the planner boosts savings in an attempt to lower the interest rate thereby increasing \( \tau \). Panel (b) shows that the cooperative planner believes that EMEs’ reserve wars let the world interest rate fall too low in the Nash equilibrium. Put differently, the Nash equilibrium has reserve over-accumulation even from the point of view of the cooperative planner (Panel (c)).

Figure 2-2 shows the welfare of EMEs and AEs under policy cooperation. Naturally, the welfare of EMEs is now always larger than with a \( \tau = 0 \) policy or with the Nash equilibrium policy, since both are feasible policies. Interestingly, although the cooperative policy is still of the “beggar-thy-neighbor” type—EMEs manipulate the interest rate in their favor at the expense of AEs—AEs are better off under this partial degree of cooperation than in the Nash equilibrium for reasonable values of \( \chi \). In other words, even partial cooperation can make everyone better-off with respect to decentralized foreign exchange interventions.\(^{51}\) Finally, note that given that international transfers are infeasible, a \( \tau \) lying between this “partial” cooperative solution and \( \tau = 0 \) may be an indirect way of transferring resources to AEs for a global planner. It should be noted, however, that we abstracted from heterogeneity in initial NFA positions, which is important in reality to assess the consequences of world interest rate movements.

\(^{51}\)The only exception is when \( \chi \) is small enough that foreigners are better-off as a result of carry-trade profits.
2.7 Conclusion

Foreign exchange interventions are one of the most important policy tools for many countries around the world. Yet, many debates regarding their usefulness and the best implementation design persist. We believe this is partly due to the lack of a unified framework to analyze the optimal design jointly with the macroeconomic rationales behind interventions. In this chapter, we provided such a framework. At the core of our model lies the assumption of limited capital mobility, which gives rise to a general equilibrium portfolio balance channel. We showed that interventions essentially manage a path of UIP spreads, and that each nonzero UIP spread represents a cost to the economy, coming from foreign intermediaries’ carry trade activity. These costs, which are naturally convex as larger spreads invite further speculation, lie at the heart of our optimal policy design. In a nutshell, they make it optimal to spread out interventions.

Our findings pick a clear side in the debate. Interventions should be small and frequent to avoid inviting significant speculation. Furthermore, they should be highly inertial and pre-announced to maximize the impact on the contemporaneous exchange rate. Interventions were found to be more powerful if the monetary authority is more credible, as this allows it to spread out interventions even further into future, minimizing the overall cost of generating an exchange rate response today. Finally, we also showed that the optimal policy is better approximated by a quantity rule rather than a smooth exchange rate rule. In the case of the latter, speculative costs may become prohibitively costly if the monetary authority tries to close the output gap at the same time.

Our unified framework allowed us to derive these “micro” features of optimal interventions and at the same time to analyze the macroeconomic motives for interventions. We found that interventions lean against the wind after global interest rate shocks—either for a terms-of-trade manipulation motive or a “output gap stabilization” motive, and serve a market-making role after large commodity shocks. In addition, since our framework is embedded in a standard macroeconomic model, we also used the model to tackle the important question about the degree to which intervention policies should be coordinated across countries. We made the point that coordination is essential to avoid wasteful competitive devaluations and reserve over-accumulation. Such reserve over-accumulation was shown to have important amplification effects on the fall of the world interest rate, hurting advanced economies. As a result, committing to replicate a world with free capital mobility led to a strict Pareto improvement over the Nash equilibrium.

We believe there are several avenues for future research. Using a richer model with a realistic calibration seems necessary for a more serious quantification of the importance of
the channels stressed in this chapter. For example, one may add the friction of limit capital
mobility to a medium-scale version of a standard New Keynesian dynamic stochastic general
equilibrium model and estimate it. In addition, it may be interesting to use such a structural
model to back out an estimation of the foreign exchange intervention reaction function and
compare it to the model-predicted optimal policy.

In addition, there is still much progress to be made even from a purely theoretical side.
For example, this chapter assumes that there is Ricardian Equivalence between the central
bank and domestic households. Yet this is certainly a strong assumption. More realistically,
in a model with heterogeneous households and/or firms one could imagine that interventions
have important redistributive effects that could both amplify or mitigate the effectiveness of
interventions.
Appendix

2.A Data used in Section 2.2

2.A.1 Time series of reserve holdings

The two plots in Figure 2-1 show data from the 2011 revision of Lane and Milesi-Ferretti (2007). On the left, we aggregate paths for reserves over world GDP by the identifier “income_class”, which categorizes countries into low, middle, and high income. On the left, we show the shares of world GDP by “income_class”.

2.A.2 Time series and scatter plots of reserve flows

To construct Figure 2-2, we use IMF’s quarterly Balance of Payments statistics from 1990:1 to 2008:4 and restrict the sample to only include emerging markets, that is, countries with “income_group” identifier of 2 or 3. We also keep Israel and South Korea since they only recently became recognized as advanced economies. The final sample is an unbalanced panel of 50 emerging markets. We focus on three variables: quarterly reserve flows (FX reserve flows “BFRAFX”), quarterly private capital flows (net financial account minus reserve assets, “BF” minus “BFRA”), (annualized) trend GDP.

For Figure 2-2(a), we then aggregate reserve flows and private capital flows across countries and plot their ratios with trend GDP. For Figure 2-2(b) we compute ratios over (trend) GDP by country and plot the standard deviations of reserve flows over GDP vs. the standard deviations of private capital flows over GDP.
2.A.3 Reserve flows and UIP wedges

To construct Table 2.1, we first create a quarterly version of the UIP wedge data from Lustig et al. (2011) by summing their monthly excess return measure over the months of each quarter. We also use their data on interest rate differentials, averaged for each quarter. This data is then merged the IMF’s quarterly Balance of Payments statistics (this time across all countries and available times). We use the same measure of reserve flows over trend GDP as described in Appendix 2.A.2.

2.B Proofs for Section 2.3

2.B.1 Implementability conditions

This section proves Proposition 4. It requires two directions. We start by showing that (2.17a) and (2.17b) are necessarily satisfied if \{c_t, q_t, r_t\} belong to a competitive equilibrium with interest rate shocks \{r^*_t\}. The paragraph below Definition 3 already showed that the flow version (2.15) of the present value budget constraint (2.17b) holds along a competitive equilibrium. (2.17a) follows directly from the Euler equation (2.2) and the definitions of \theta_t and \tau_t.

Now, consider the reverse direction: Given paths \{\theta_t, \tau_t\}, \{r^*_t, y_{Ht}\}, and an initial net foreign asset position nfa_0 that satisfy (2.17a) and (2.17b), can we always find a competitive equilibrium consisting of initial debt positions (b_{H0}, b_{I0}, b^H_{G0}, b^C_{G0}), a central bank FXI policy \{b_{G0}, b^*_t\} and an allocation \{c_t, c_{Ht}, c_{Ft}, c^*_t, b^*_H\}, \{b_{Ht}, b_{It}, b^*_H\} with prices \{q_t, r_t\} such that (2.1)-(2.11) hold?

We first construct the equilibrium objects and then check optimality conditions. We can take the initial debt positions to be b_{H0} = nfa_0. Moreover, we define for any T > 0

\[ b^I t = \int_T^\infty e^{-\int_T^s r^*_s ds} \left[ \alpha(\theta_t - 1) + \frac{1}{\Gamma_F r^*_t} \right] dt, \]

and thus construct b_{It} = \frac{1}{\Gamma_F r_t} \tau_t, b^H_{It} = \frac{1}{\Gamma_H r_t} \tau_t, b^*_G = -b_{Gt} = b^H_{It} + b_{It} + b^H_{It}, and \pi_t = b^H_{It}(r_t - r^*_t) for each t ≥ 0. Transfers are defined to be t_t = r_t b_{Gt} + r^*_t b^*_G. We let the real exchange rate be defined by \[ q_t^{1/(1-\alpha)} y_{Ht} = (1-\alpha)q_t + \alpha \] and let consumption paths be \[ c_{Ft} = \alpha q_t \] and \[ c_{Ht} = (1-\alpha)q_t^{\alpha/(1-\alpha)} c_t \] and \[ c^*_H = \alpha q_t^{1/(1-\alpha)}. \] This concludes our construction of a candidate equilibrium. We move on to checking the equilibrium conditions.

The Euler equation (2.2) is equivalent to (2.17a). Equations (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), and (2.11) hold by construction. It is straightforward to check that the home good
market clears—that is, equation (2.10) holds—given our definitions for \( c_{Ht} \) and \( c_{*Ht} \). Finally, reversing the steps in equations (2.12)-(2.15) shows that the differential (flow) version of the (2.34) (which is exactly (2.15)) implies the budget constraint (2.1).

2.B.2 Simplifying the planner’s objective

In this section we derive the simplified objective function used in (2.18) from the original per period utility \( \log c_t \). Using (2.12), we can express

\[
c_t = q_t \theta_t = \left( \frac{y_{Ht}}{(1 - \alpha)\theta_t + \alpha} \right)^{1-\alpha} \theta_t
\]

which then yields a per period utility of

\[
\log c_t = \log \theta_t - (1 - \alpha) \log ((1 - \alpha)\theta_t + \alpha) + (1 - \alpha) \log y_{Ht}.
\]

The \( y_{Ht} \) term in this expression is exogenous so it is without loss for our planning problem to drop it. This gives us the objective in (2.18).

2.B.3 First best

Here, we prove Lemma 2 that characterizes the first best allocation, that is, the optimal allocation when the Euler implementability condition (2.17a) does not bind. In that case the first order conditions with respect to \( \theta_t \) and \( \tau_t \) of the planning problem (2.18) read

\[
\tau_t = 0 \quad \text{and} \quad e^{-\rho t} V'(\theta_t) = e^{-\rho^* t} e^{s \lambda},
\]

using the notation \( V'(\theta) \) for the planner’s per period objective, as described in Lemma 2, and calling \( \lambda \) the multiplier on the resource constraint (2.17b). Due to the Inada conditions of this problem, the first order conditions are necessary. This proves Lemma 2.

2.C Proofs for Section 2.4

2.C.1 Model without macro stabilization motive

Here, we provide a proof to Proposition 5. Consider first the case where \( \alpha = 1 \). Let \( \{\theta_t\}, \lambda \) be the first best path of dollar consumption and the corresponding first best shadow value of resources, respectively. We now show that the (2.17a) does not bind, that is, it is satisfied for
the first best $\theta_t$ with $\tau_t = 0$. Lemma 2 describes $\theta_t$ as the solution to the first order condition

$$e^{-\rho t}V'(\theta_t) = e^{-\int_0^t r^*_s ds} \lambda \alpha, \quad (2.35)$$

where, with $\alpha = 1$, $V'(\theta_t) = 1/\theta_t$. Log-differentiating this first order condition yields $\dot{\theta}_t/\theta_t = r^*_t - \rho$ which is exactly what we needed to show.

Now consider the case where $\alpha$ is allowed to be less than 1, but $r^*_t = \rho$. Again, let $\{\theta_t\}, \lambda$ be as in the first best allocation, satisfying (2.35). Since $r^*_t$ is constant, this implies that $\theta_t$ is constant too, trivially satisfying the Euler equation $\dot{\theta}_t/\theta_t = r^*_t - \rho = 0$. This concludes our proof of Proposition 5.

2.C.2 Financial autarky, $\Gamma_F = \infty$

Before we prove Proposition 6, we first derive very generally, for any $\Gamma_F > 0$, the (necessary) first order conditions to the planning problem (2.18) in the following lemma, a version of Lemma 3.

**Lemma 4.** Suppose $\Gamma_F \in (0, \infty)$. Let $V(\theta)$ be the planner's per period objective, as defined before Lemma 2, and let $\tilde{T}_t(\lambda, \theta_t)$ be defined by $^52$

$$\tilde{T}_t(\lambda, \theta_t) = \lambda - e^{\int_0^t (r^*_s - \rho) ds} V'(\theta_t).$$

Then, under the optimal foreign exchange intervention policy, the interest rate spread $\tau_t$ and the (dollar) consumption $\theta_t$ satisfy the following first order condition

$$\dot{\tau}_t = r^*_t \tau_t + \frac{\Gamma_F}{2 \lambda} \tilde{T}_t(\lambda, \theta_t), \quad (2.36a)$$

$$\frac{\dot{\theta}_t}{\theta_t} = r^*_t - \rho + \tau_t \quad (2.36b)$$

where the derivative of $\tau_t$ exists whenever $r^*_t$ does not jump. When $\Gamma_F = \infty$, $\tilde{T}_t(\lambda, \theta_t) = 0$, instead of (2.36a).

**Proof.** The current value Hamiltonian of the planning problem (2.18) with $\Gamma_F \in (0, \infty)$ is given by

$$H(\theta, \tau, \lambda, \mu, t) = e^{-\int_0^t (\rho - r^*_s) ds} V(\theta) - \lambda \alpha (\theta - 1) - \lambda \frac{1}{\Gamma_F} \tau^2 + \mu (r^* + \tau - \rho).$$

$^52\tilde{T}_t$ is the current value equivalent of $T_t$, defined before Lemma 3.
This is an optimal control problem with a subsidiary condition, as in Gelfand and Fomin (1963). The state variable is $\theta$ and has a free initial value $\theta_0$. $\theta$ has costate $\mu$. As before, $\lambda$ is the multiplier on the resource constraint (2.17b), and $\tau$ is the control variable. Notice that when $\lambda > 0$, this is a strictly concave problem with a unique maximizer $\{\theta_t, \tau_t\}$ and unique multipliers $\{\mu_t\}, \lambda$, satisfying the first order conditions. And indeed $\lambda$ has to be positive, or else shadow cost to scaling $\theta_t$ up indefinitely (which would not violate the Euler equation (2.17a)) is zero or even negative, which is not resource feasible according to (2.17b).

The first order condition for $\tau$ is simply $\mu_t = \frac{2}{\Gamma_F} \tau_t$, and the costate equation for $\theta$ is $\dot{\mu}_t = r^*_t \mu_t + \tilde{T}_t(\lambda, \theta_t)$. Putting these two equations together yields (2.36a). Equation (2.36b) is just the Euler equation (2.17a).

When $\Gamma_F = \infty$, the Hamiltonian is

$$H(\theta, \tau, \lambda, \mu, t) = e^{-\int_0^t (\rho - r^*_s)ds} V(\theta) - \lambda \alpha (\theta - 1) + \mu (r^* + \tau - \rho)$$

implying that the costate $\mu_t$ is equal to zero at all times, $\mu_t = 0$ and hence that $\tilde{T}_t(\lambda, \theta_t) = 0$. 

With Lemma 4 under the belt, we now approach Proposition 6. For concreteness, we prove the proposition for positive interest rate shocks, that is, we assume that there is some $T > 0$ such that $r^*_t > \rho$ for all $t \in [0, T]$ and $r^*_t = \rho$ thereafter. We show that, at the optimum, $\tau_t < 0$ for all $t \in [0, T)$ and $\tau_t = 0$ for $t \geq T$, with a jump at $t = T$.

Define as an auxiliary object, $\hat{\theta}_t = e^{\int_0^t (\rho - r^*_s)du} \theta_t$. The necessary conditions now become

$$\hat{\theta}_t / \hat{\theta}_t = \tau_t$$

and

$$0 = \tilde{T}_t(\lambda, e^{\int_0^t (\rho - r^*_s)du} \theta_t) = \lambda - \frac{1}{\hat{\theta}_t} + \frac{(1 - \alpha)^2}{(1 - \alpha) \hat{\theta}_t + \alpha e^{\int_0^t (\rho - r^*_s)du}},$$

which can be simplified to

$$\frac{\lambda}{\alpha} \hat{\theta}_t = 1 + \frac{1}{\hat{\theta}_t e^{\int_0^t (\rho - r^*_s)du} + \frac{\alpha}{1 - \alpha}}.$$  

(2.37)

For any $t < T$, this equation has a unique solution for $\hat{\theta}_t$ that is strictly decreasing in $t$. Therefore, $\tau_t < 0$ for $t < T$. For $t \geq T$, there is no explicit time dependence in (2.37) and $\hat{\theta}_t$ is constant, which means $\tau_t = 0$ for $t \geq T$. At $t = T$, $\tau_t$ jumps since the only time varying element of (2.37), $\int_0^t (\rho - r^*_u)du$, has a kink at $t = T$. This concludes our proof of Proposition 6.

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53 It is also possible to prove directly that the constraint set of the planning problem is convex, analogous to the steps in Appendix 2.F.1.
2.C.3 Intervention smoothing

Lemma 3 is a special case of Lemma 4 in the previous subsection. In particular, (2.19) is an integral version of (2.36a). Moreover, as explained in the text below Lemma 3, Proposition 7 is a direct consequence of Lemma 3.

2.C.4 Forward guidance

This section contains the proof of Proposition 2.4.3. As in Section 2.C.2, we will only provide a proof for positive interest rate shocks \(\{r_t^*\}\). The case of negative interest rate shocks is analogous. We start with the first order conditions from Lemma 4, (2.36a) and (2.36b), for the case where \(\Gamma_F \in (0, \infty)\). Again, as in Section 2.C.2, we work in terms of \(\hat{\tau}_t\), leading to the first order conditions

\[
\dot{\tau}_t = r_t^* \tau_t + \frac{\Gamma_F}{2\lambda} \tilde{T}_t(\lambda, e^{\int_0^T (r_s^* - \rho) du} \hat{\theta}_t)
\]

This is a 2-dimension system of ODEs, with initial condition \(\tau_0 = 0\) and the terminal condition that \((\tau_t, \hat{\theta}_t)\) converge to the unique steady state given by \(\tau^{ss} = 0\) and

\[
\tilde{T}_T(\lambda, e^{\int_0^T (r_s^* - \rho) du} \hat{\theta}^{ss}) = 0.
\]

Since the system is stationary after \(t = T\), the state at \(t = T\), \((\tau_T, \hat{\theta}_T)\), has to lie on the stationary system’s stable arm. Figure 2.C.1 illustrates the phase diagram and its stable arm. The green line that then merges into the red line depicts the shape of the optimal trajectory that we are trying to determine mathematically.

To do this, it turns out to be helpful to define the path \(\{\hat{\theta}_t^\infty\}\) as the solution to \(\tilde{T}_t(\lambda, e^{\int_0^T (r_s^* - \rho) du} \hat{\theta}_t^\infty) = 0\), for all \(t \geq 0\).\(^{54}\) In a first step, we show that it can never be that \(\tau_t \geq 0\) and \(\hat{\theta}_t > \hat{\theta}_t^\infty\) for any \(t > 0\). In Figure 2.C.1, this would be a state \((\tau_t, \hat{\theta}_t)\) that lies to the top right of the time-\(T\) \(\tau\)–locus. In such a case, for any \(s > t\), both \(\hat{\theta}_s^\infty\) and \(\tau_s\) are positive and bounded away from zero, and hence the state \((\hat{\theta}_t, \tau_t)\) would diverge to \(\infty\). Translating the divergence back to \(\theta_t\), this would mean that the growth rate of \(\theta_t\), \(\dot{\theta}_t / \theta_t = r_t^* - \rho + \tau_t\), diverges to infinity, violating the resource constraint (2.17b).

\(^{54}\) We call this \(\hat{\theta}_t^\infty\) since, given a certain \(\lambda\), it corresponds to the optimal path for \(\hat{\theta}_t\) when \(\Gamma_F = \infty\). This is also why \(\hat{\theta}_t^\infty\) is well-defined, continuous and piece-wise differentiable for each \(t\).
Second, consider the possibility that for some $t > 0$,

$$(\hat{\theta}_t, \tau_t) \in \{ (\theta, \tau) \mid \tau \geq 0 \text{ and } \hat{\theta} \leq \hat{\theta}_t^\infty \} \equiv \Theta_t.$$  

Given $\hat{\theta}_t^\infty$ is decreasing in $t$, if $(\hat{\theta}_t, \tau_t) \in \Theta_t$, then $(\hat{\theta}_s, \tau_s) \in \Theta_s$ for any $s < t$ as well. In particular $(\hat{\theta}_t, \tau_t) \in \Theta_0$. Given no path satisfying the ODEs can ever enter $\Theta_0$ (that is, $\Theta_0$, is a "source" in the vector field sense), it must hold that $(\hat{\theta}_0, \tau_0) \in \text{int}\Theta_0$ (the interior of $\Theta_0$). This contradicts the fact that $\tau_0 = 0$. Together, these two steps prove that $\tau_t \geq 0$ is impossible for any $t > 0$. This concludes the proof of the proposition.

2.C.5 Time inconsistency

This section proves the results claimed in Proposition 9. When there is re-optimization at any time $t_0 \geq T$, there is no more interest rate shock to $r^*$ after $t_0$. Thus, Proposition 5 applies to this re-optimization problem, making it optimal to set $\tau_t = 0$ for all $t \geq t_0$.

Consider now the case of no commitment at all. For ease of notation, call $b_t$ Home’s net foreign asset position $\text{nfa}_t$ and let $v(t, b_t)$ be the no-commitment planner’s current time-$t$ value function with current net foreign asset position $b_t$. The Hamilton-Jacobi-Bellman equation for the time-$t$ planner is then

$$r_t^* v(t, b) = \max_{\tau, \theta} V(\theta) + v_t(t, b) + v_b(t, b) \left( \alpha(1 - \theta) + r_t^* b - \frac{1}{\Gamma_F} \tau^2 \right).$$

Figure 2.C.1: Describing the optimal policy in the state space for $(\tau, \theta)$. 
The first order condition for $\tau$ immediately yields the desired result, $\tau_t = 0$.

2.D Proofs for Section 2.5

2.D.1 Fixed exchange rate economy

In this section, we list the results we proved in the home-bias economy and how their proof can be modified to also apply to the fixed exchange rate economy. To do this, define the per period objective as follows,

$$V(\theta) \equiv \log \theta_t - (1 - \alpha) \log((1 - \alpha)\theta_t + \alpha c^*) - v((1 - \alpha)\theta_t + \alpha c^*).$$

Further, we assume that the multiplier on the resource constraint is positive, $\lambda > 0$.

Lemma 2 and Propositions 5—9 go through word by word as in the baseline model, given our assumption of a positive multiplier $\lambda$.

2.E Additional extensions

2.E.1 Lucas-Stokey and long term bonds

In a seminal contribution, Lucas and Stokey (1983) show how a planner without commitment power can achieve the full-commitment solution by essentially implementing a specific composition of household asset holdings across certain asset classes. Even though the asset classes are linearly dependent and hence redundant, they react different to policy changes, which lets a planner today select asset classes that would “punish” future planners upon deviations.

In our setup, we allow all agents—households, intermediaries and the central bank—to trade Arrow securities in addition to the existing short-term bonds. As in Lucas and Stokey (1983) the central bank can now control its asset composition across all asset classes. Crucially, however, this will not pin down the country’s asset composition since households are free to trade with intermediaries in whatever asset classes they like. Therefore, in our model, the central bank cannot implement a specific country’s asset composition. In this section, we sketch out a natural version of our economy with multiple types of bonds and show that they do not help the planner overcome his time consistency problem. As in the main body of

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55Compared to our baseline model, the per period objective $V(\theta)$ is no longer monotonically increasing in the fixed exchange rate economy. This means, under certain, arguably contrived, circumstances, it can occur that more resources hurt the economy. To focus on the “normal” case, we assume that $\lambda > 0$, which is always the case if the fixed exchange rate $\tau$ is sufficiently low or the initial labor wedge sufficiently close to zero.
our paper, we restrict our attention to the case where domestic bonds are also measured in dollars. This lets us focus on our novel type of time inconsistency, different from the standard exchange rate based time inconsistency.\footnote{Here, we mean the tendency of a re-optimizing planner with external local currency liabilities to depreciate the currency.}

Let $\pi_{t,s}$ denote the state price density for an international dollar payment at time $s$, measured at time $t$. That is, if a country promises a stream of payments $x_{t,s}$ in the foreign bond market at time $t$, and has no other external assets or liabilities, its time-$t$ net foreign asset position is $nfa_t = \int_t^\infty \pi_{t,s} x_{t,s} ds$. Similarly, $\pi_{t,s}$ denotes the state price density for a promised payment in the domestic bond market at time $s$, measured at time $t$. Note that $\pi_{t,s} = e^{\int_t^s r_u du}$ and $\pi_{t,s} = e^{\int_t^s r_u du}$. To see the link to our previous short-term bonds, a single outstanding domestic short-term bond promises payments $x_{t,s} = r_s$ so that its value is exactly 1.

We allow all domestic and foreign bond market participants to trade all Arrow securities with each other, including short-term bonds. We choose the following notation: For each possible short term bond position in our previous notation $X_t = b_{Ht}, b_{It}, b_{Gt}$, we allow positions in Arrow securities denoted by $b_{Ht,s}, b_{It,s}, b_{Gt,s}, b_{Gt,s}$. In this notation, domestic bond market clearing requires that

$$b_{Ht,s} + b_{It,s} + b_{Gt,s} = 0$$

in addition to the short term bond market clearing condition (2.11). The main effect of the different Arrow securities in our model is that it creates an indeterminacy in intermediaries and households' positions. Using a natural extension of the intermediaries' maximization problem to this setup gives intermediary demand functions

$$b_{It} + \int_t^\infty \pi_{t,s} b_{It,s} ds = \frac{1}{\Gamma_F} (r_t - r^*_t) \quad \text{and} \quad b_{Ht} + \int_t^\infty \pi_{t,s} b_{Ht,s} ds = \frac{1}{\Gamma_H} (r_t - r^*_t).$$

Evidently, the demand functions do not pin down intermediary positions uniquely.

In this economy, the country's time $t$ implementability conditions are given by (2.17a) and

$$\int_t^\infty e^{\int_t^s r_u du} \left[ \alpha (\theta_s - 1) + \frac{1}{\Gamma_F} \right] ds = nfa_t,$$

where the net foreign asset position is now given by

$$nfa_t = b_{Ht} + b_{Gt} + b_{Gt} + \int_t^\infty \pi_{t,s} (b_{Ht,s} + b_{Gt,s}) ds + \int_t^\infty \pi_{t,s} b_{Gt,s} ds.$$
Without loss of generality due to Ricardian equivalence between the central bank and domestic households, we let the central bank choose foreign and domestic bond portfolios whose values sum to zero. Then,

\[ n_{f_{a,t}} = b_{H,t} + \int_t^{\infty} \pi_{t,s} b_{H,t,s} ds. \]

Here, \( \pi_{t,s} \) is affected by central bank policies. So in order to fix a time inconsistency problem, the central bank would have to be able to control the positions \( \{b_{H,t}, b_{H,t,s}\} \). Yet, for any change in the composition of the central bank position \( \{b_{G,t}, b_{G,t,s}\} \) (leaving the total value of the position unchanged), there exist changes in intermediary portfolio compositions such that the household positions \( \{b_{H,t}, b_{H,t,s}\} \) as well as the equilibrium quantities and prices \( \{c_t\} \) and \( \{q_t, r_t\} \) do not change. This makes it impossible for the central bank to commit themselves to future interventions through this mechanism.

### 2.E.2 Nonlinear intermediary demands

In our baseline model, intermediaries’ demand functions for domestic bonds are linear functions of the UIP wedge \( \tau_t \). In this extensions, we explore the implications of nonlinear demand schedules

\[ b_{H,t} = g^F(\tau_t) \quad \text{and} \quad b_{H,t}'' = g^H(\tau_t) \]

(2.38)

where \( g^F, g^H \) are strictly increasing and differentiable functions defined for all reals, with \( g^k(0) = 0 \) and \( g^{k'}(0) = \frac{1}{t_k} \), for \( k = F, H \). Such general demand schedules can be microfounded the same way we microfounded the linear demand schedules in Section 2.3.1, just with a more general transaction function: one where \( f(j) \) is strictly increasing and continuous, with \( f(0) = 0 \). Using the nonlinear schedules (2.38), instead of the linear demands (2.5) and (2.7), Definition 3 defines the right notion of competitive equilibrium.

The key difference to the linear demand model is that now costs are no longer exactly quadratic, rather only locally so, given by

\[ b_{H,t} \tau_t = g^F(\tau_t) \tau_t. \]

The new resource constraint implementability condition is then given by

\[ \int_0^{\infty} e^{-\int_0^{\tau} r_s ds} [\alpha(\theta_t - 1) + \tau_t g^F(\tau_t)] dt = n_{f_{a,0}}. \]

To ease notation in this subsection, we introduce \( G(\tau) \equiv g^F(\tau) \tau \). To ensure the associated planning problem with such an implementability condition is well defined, we further assume
that \( G(\tau) = \tau g^F(\tau) \) is strictly convex in \( \tau \). Since \( \tau g^F(\tau) \) is always locally convex around \( \tau = 0 \), this essentially restricts \( g^F(\tau) \) to not become too flat for large positive or negative values of \( \tau \).

Next, we argue that under these conditions, despite nonlinear demand functions, our key results remain true. Lemma 2 and Proposition 5 and their proofs go through without any changes. Lemma 4 in the appendix still holds when (2.36a) is replaced by

\[
\hat{\tau}_t = r^*_t G'(\tau_t) + \frac{1}{G''(\tau_t)} \hat{T}_t(\lambda, \theta_t). \tag{2.39}
\]

Proposition 6 trivially holds as well since it treats the special case where \( \Gamma_F = \infty \), that is translated into this notation, \( g^F(\tau) = 0 \) for all \( \tau \). The first order condition in Lemma 3 is still the integral equivalent of (2.39) and reads

\[
e^{-\int_0^t \tau^* sds} \lambda G'(\tau_t) = \int_0^t \tau ds,
\]

with the exact same interpretation as in Lemma 3. Based on this lemma, Proposition 7 carries over as well. We speculate that Proposition 8 goes through unchanged (there is no intuitive reason why not) but we have not worked out the formal proof. The partial commitment part of Proposition 9 holds conditional on Proposition 8 being correct, while the no-commitment part can be proven using the same argument as before.

2.F Proofs for Section 2.6

2.F.1 Convexity of the planning problem

To show that planning problem (2.32) is well-behaved, first note that the objective is strictly concave and strictly increasing in \((\theta_0, \theta_1)\) as before. Therefore, if the constraint set were bounded and convex, the unique maximizer would necessarily lie on the constraint set's boundary. We now prove that the inequality version of (2.33),

\[
B(\theta_0, \theta_1) \equiv (\alpha \theta_0 - C_{EME0}) + \frac{1}{1 + r^*} (\alpha \theta_1 - C_{EME1}) + \frac{1}{\Gamma_F} \left( \frac{\theta_1 (1 + \rho_{EME})}{\theta_0 (1 + r^*)} - 1 \right)^2 \leq 0.
\]

is indeed bounded and convex, thus validating our relaxation.

Boundedness is straightforward as the intermediary cost term \( 1/\Gamma_F(...)^2 \) is always bounded from below by zero. For convexity, consider two points \( \theta = (\theta_0, \theta_1) \) and \( \theta' = (\theta'_0, \theta'_1) \) in \( \mathbb{R}^2_{++} \),
and choose $\lambda \in [0, 1]$. Define $\theta^\lambda = \lambda \theta + (1 - \lambda)\theta'$. We now prove that $\lambda \mapsto B(\theta^\lambda)$ is a quasi-convex function, implying that $B(\theta^\lambda) \leq \max\{B(\theta), B(\theta')\} = 0$ as desired.

$B(\theta^\lambda)$ consists of two linear terms and a third, non-linear term. The nonlinear term is

$$\frac{1}{\Gamma_F} \left( \frac{(1 + \rho_{\text{EME}})}{(1 + \rho_{\text{EME}})^2} \right) \left( \frac{(1 - \lambda)\theta_0 + (1 - \lambda)\theta_0' - 1}{(1 - \lambda)\theta_0 + (1 - \lambda)\theta_0'} \right)^2$$

which is a composition of a convex function $f(z) = \frac{1}{\Gamma_F} \left( \frac{(1 + \rho_{\text{EME}})}{(1 + \rho_{\text{EME}})^2} (z - 1)^2 \right)$ and a function $g(\lambda) = \frac{\lambda \theta_1 + (1 - \lambda)\theta_1'}{\lambda \theta_0 + (1 - \lambda)\theta_0'}$. Here, $g$ is either strictly monotone or constant. Either way, the composition $f(g(\lambda))$ is quasiconvex, implying the result. Therefore, the constraint set $B(\theta_0, \theta_1) \leq 0$ is convex.

2.F.2 Characterizing the Nash equilibrium

Part 1

This subsection provides a proof for Proposition 10. We start with part 1 and rewrite the planning problem in a slightly more convenient form, by defining $\kappa_t = \frac{(1 + \rho_{\text{EME}})}{(1 + \rho_{\text{EME}})^t}$, $\hat{\theta}_t \equiv \kappa_t \theta_t$, and $\hat{C}_{\text{EME}} t \equiv \kappa_t C_{\text{EME}} t$. We also abbreviate $\rho = \rho_{\text{EME}}$ and $\hat{C} = \hat{C}_{\text{EME}}$ and let the per period objective be defined as

$$V(\hat{\theta}, \hat{C}) \equiv \log \hat{\theta} - (1 - \alpha) \log((1 - \alpha)\hat{\theta} + \hat{C})$$

and the constraint function be defined as

$$B(\hat{\theta}_t, \hat{\theta}_1, \hat{C}_0, \hat{C}_1) \equiv \sum_{t=0}^{1} (1 + \rho)^{-t} \left( \alpha \hat{\theta}_t - C_t \right) + \frac{1}{\Gamma_F} \left( \frac{\hat{\theta}_t}{\hat{\theta}_0} - 1 \right)^2.$$

As before, $\hat{\theta}$ and $\hat{C}$ are complements in the utility function, that is $V_{\hat{\theta}, \hat{C}} > 0$. Using this notation, (2.32) becomes

$$\max \sum_{t=0}^{1} (1 + \rho)^{-t} \left\{ V(\hat{\theta}_t, \hat{C}_t) - \alpha \log \kappa_t \right\}$$

subject to

$$B(\hat{\theta}_0, \hat{\theta}_1, \hat{C}_0, \hat{C}_1) \leq 0,$$

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where we relaxed the constraint as inequality, applying the result from Appendix 2.F.1. In that section we also mentioned that (for any $\tilde{C}_t > 0$) the optimum choice for $(\tilde{\theta}_0, \tilde{\theta}_1)$ lies on a downward sloping, convex arc of the constraint set. On that arc, it holds that the marginal rate of substitution, \( MRS(\tilde{\theta}_0, \tilde{\theta}_1) \equiv \frac{\partial \tilde{B}}{\partial \tilde{\theta}_1} \), is weakly increasing in $\tilde{\theta}_0$ and weakly decreasing in $\tilde{\theta}_1$. Notice that $MRS(\tilde{\theta}_0, \tilde{\theta}_1)$ is independent of \{\tilde{C}_t\}. In a (pure strategy) Nash equilibrium, the demands $\tilde{C}_t$ are then determined by

\[
\tilde{C}_t = \kappa_t - (1 - \alpha)\tilde{\theta}_t \tag{2.41}
\]

and the equilibrium interest rate is pinned down by AEs' discount rate, $r^* = \rho_{AE}$ since $\chi = 0$. As a third equilibrium condition, we need the resource constraint to hold, which, having substituted in (2.41) becomes

\[
\tilde{B}(\tilde{\theta}_0, \tilde{\theta}_1) \equiv \sum_{t=0}^{1}(1 + \rho)^{-t} \left( \tilde{\theta}_t - \kappa_t \right) + \frac{1}{\Gamma_F} \left( \frac{\tilde{\theta}_1}{\tilde{\theta}_0} - 1 \right)^2 \leq 0. \tag{2.42}
\]

The Euler equation of (2.40) reads

\[
V_\theta(\tilde{\theta}_0, \tilde{C}_0) = MRS(\tilde{\theta}_0, \tilde{\theta}_1)V_\theta(\tilde{\theta}_1, \tilde{C}_1)
\]

and substituting in the Nash equilibrium condition (2.41) we find

\[
V_\theta(\tilde{\theta}_0, 1 - (1 - \alpha)\tilde{\theta}_0) = MRS(\tilde{\theta}_0, \tilde{\theta}_1)V_\theta(\tilde{\theta}_1, \kappa_1 - (1 - \alpha)\tilde{\theta}_1). \tag{2.43}
\]

Using the properties of $V$ and $MRS$ mentioned above, it is immediate that the left hand side of this equation strictly decreases in $\tilde{\theta}_0$, while the right hand side increases in $\tilde{\theta}_0$ and falls with $\tilde{\theta}_1$. This implicit equation thus describes a strictly increasing, continuous function $\tilde{\theta}_1 = h(\tilde{\theta}_0)$, defined for any $\tilde{\theta}_0 > 0$. Notice that the solution to (2.43) must necessarily generate a positive $MRS(\tilde{\theta}_0, h(\tilde{\theta}_0))$, and hence positive derivatives $B_{\tilde{\theta}_0}$ and $B_{\tilde{\theta}_1}$.$^{57}$ It is easy to see that the positivity of the two derivatives immediately implies that $h(\tilde{\theta}_0)/\tilde{\theta}_0 \rightarrow 1$ as $\tilde{\theta}_0 \rightarrow 0$. A Nash equilibrium can be found as a solution to (2.42) with $\tilde{\theta}_1 = h(\tilde{\theta}_0)$ substituted in.

Before we show existence and uniqueness of such a solution, we establish a few helpful auxiliary results. First, note that $\tilde{B}(\tilde{\theta}_0, h(\tilde{\theta}_0))$ is strictly increasing in $\tilde{\theta}_0$. Suppose it were not: Then there has to be a point $(\tilde{\theta}_0, h(\tilde{\theta}_0))$ where either $\tilde{B}_{\tilde{\theta}_0}$ (which is larger than $B_{\tilde{\theta}_0}$ or

$^{57}$Notice that it can never be the case that both $B_{\tilde{\theta}_0}$ and $B_{\tilde{\theta}_1}$ are negative since in that case, there would have to exist some positive \{\tilde{C}_t\} such that the budget set $B \leq 0$ does not include any points close to zero. Yet, by scaling down any point $(\tilde{\theta}_0, \tilde{\theta}_1)$ in such a budget set, $(0,0)$ can always be approximated arbitrarily closely.
\( \bar{B}_{\hat{\theta}_1} \), (which is larger than \( B_{\hat{\theta}_1} \)) is negative, contradicting the positivity of \( MRS(\hat{\theta}_0, h(\hat{\theta}_0)) \). Second, note that \( \bar{B}(\hat{\theta}_0, h(\hat{\theta}_0)) \) approaches \( \infty \) as \( \hat{\theta}_0 \to \infty \) and approaches a negative number as \( \hat{\theta}_0 \to 0 \). The former is straightforward since the quadratic cost term is bounded below by zero. The latter follows because \( h(\hat{\theta}_0)/\hat{\theta}_0 \to 1 \) as \( \hat{\theta}_0 \to 0 \), as explained above, and so the cost term vanishes as \( \hat{\theta}_0 \to 0 \). The intermediate value theorem then proves that there exists a unique number \( \hat{\theta}_0 \) such that \( \bar{B}(\hat{\theta}_0, h(\hat{\theta}_0)) = 0 \). This proves the existence and uniqueness of the symmetric world equilibrium as defined in Definition 4.

**Part 2**

To show that \( \tau \) is positive, we now argue that \( h(\hat{\theta}_0) > \hat{\theta}_0 \) for all \( \hat{\theta}_0 \). Suppose this were not the case. Then, the right hand side of (2.43) can be bounded from above by

\[
MRS(\hat{\theta}_0, \hat{\theta}_1) V(\hat{\theta}_1, \kappa_1 - (1 - \alpha)\hat{\theta}_1) \leq MRS(\hat{\theta}_0, \hat{\theta}_0) V(\hat{\theta}_0, \kappa_1 - (1 - \alpha)\hat{\theta}_0) < V(\hat{\theta}_0, 1 - (1 - \alpha)\hat{\theta}_0).
\]

This contradicts (2.43). Therefore, \( h(\hat{\theta}_0) > \hat{\theta}_0 \) and hence \( \tau > 0 \) in the Nash equilibrium.

**Part 3**

Compared to a no-intervention world, the EME consumption profile \( (\hat{\theta}_0, \hat{\theta}_1) \) is tilted more towards the future, and hence EME’s net foreign assets are larger after \( t = 0 \). Positivity of the UIP wedge, \( \tau > 0 \), however, means that private capital must flow more downstream, implying that the increase in net foreign assets is driven by rise of EME reserves.

**Part 4**

This follows from Proposition 11 below.

**2. F. 3 Characterizing the cooperative outcome**

This section proves Proposition 11. Utilizing the notation introduced in Appendix 2.F.2 above, the cooperative planning problem can be written as

\[
\max \sum_{t=0}^{1} (1 + \rho)^{-t} \left\{ \log \hat{\theta}_t - \alpha \log \kappa_t \right\}
\]

subject to the total EME resource constraint (2.42) which takes into account the endogeneity of \( \{C_{EME_t}\} \). To analyze this problem, we guess and verify that it is without loss to relax the
constraint (2.42) by setting $\Gamma_F = \infty$. In that case, the solution is simply given by

$$\hat{\theta}_0 = \hat{\theta}_1 = \frac{1 + (1 + \rho)\kappa_1}{2 + \rho}.$$  

This "relaxed" features that $\hat{\theta}_1/\hat{\theta}_0 = 1$, that is, $\tau = 0$. Therefore, relaxing (2.42) by setting $\Gamma_F$ to $\infty$ was without loss of generality, and we found the solution.
2.G References


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Chapter 3

Positive Long Run Capital Taxation: Chamley-Judd Revisited

According to the Chamley-Judd result, capital should not be taxed in the long run. In this chapter, we overturn this conclusion, showing that it does not follow from the very models used to derive it. For the main model in Judd (1985), we prove that the long run tax on capital is positive and significant, whenever the intertemporal elasticity of substitution is below one. For higher elasticities, the tax converges to zero but may do so at a slow rate, after centuries of high tax rates. The main model in Chamley (1986) imposes an upper bound on capital taxes. We provide conditions under which these constraints bind forever, implying positive long run taxes. When this is not the case, the long-run tax may be zero. However, if preferences are recursive and discounting is locally non-constant (e.g., not additively separable over time), a zero long-run capital tax limit must be accompanied by zero private wealth (zero tax base) or by zero labor taxes (first best). Finally, we explain why the equivalence of a positive capital tax with ever increasing consumption taxes does not provide a firm rationale against capital taxation.

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3.1 Introduction

One of the most startling results in optimal tax theory is the famous finding by Chamley (1986) and Judd (1985). Although working in somewhat different settings, their conclusions were strikingly similar: capital should go untaxed in any steady state. This implication, dubbed the Chamley-Judd result, is commonly interpreted as applying in the long run, taking convergence to a steady state for granted.\(^1\) The takeaway is that taxes on capital should be zero, at least eventually.

Economic reasoning sometimes holds its surprises. The Chamley-Judd result was not anticipated by economists' intuitions, despite a large body of work at the time on the incidence of capital taxation and on optimal tax theory more generally. It represented a major watershed from a theoretical standpoint. One may even say that the result is puzzling, as witnessed by the fact that economists have continued to take turns putting forth various intuitions to interpret it, none definitive nor universally accepted.

Interpretation aside, a crucial issue is the result's applicability. Many have questioned the model's assumptions, especially that of infinitely-lived agents (e.g. Banks and Diamond, 2010). Still others have set up alternative models, searching for different conclusions. These efforts notwithstanding, opponents and proponents alike acknowledge Chamley-Judd as one of the most important benchmarks in the optimal tax literature.

Here we question the Chamley-Judd results directly on their own ground and argue that, even within the logic of these models, a zero long-run tax result does not follow. For both the models in Chamley (1986) and Judd (1985), we provide results showing a positive long-run tax when the intertemporal elasticity of substitution is less than or equal to one. We conclude that these models do not actually provide an unambiguous argument against long-run capital taxation. We discuss what went wrong with the original results, their interpretations and proofs.

Before summarizing our results in greater detail, it is useful to briefly recall the setups in Chamley (1986) and Judd (1985), where in the latter case we will specifically focus on the model in Judd (1985, Section 3).\(^2\)

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\(^1\)To quote from a few examples, Judd (2002): "[...] setting \(\tau_k\) equal to zero in the long run [...] various results arguing for zero long-run taxation of capital; see Judd (1985, 1999) for formal statements and analyses." Atkeson et al. (1999): "By formally describing and extending Chamley's (1986) result [...] This approach has produced a substantive lesson for policymakers: In the long run, in a broad class of environments, the optimal tax on capital income is zero." Phelan and Stacchetti (2001): "A celebrated result of Chamley (1986) and Judd (1985) states that with full commitment, the optimal capital tax rate converges to zero in the steady state." Saez (2013): "The influential studies by Chamley (1986) and Judd (1985) show that, in the long-run, optimal linear capital income tax should be zero."

\(^2\)Judd (1985) also provides extensions to the model in Judd (1985, Section 3) that generally bring the
Start with the similarities. Both papers assume infinitely-lived agents and take as given an initial stock of capital. Taxes are basically restricted to proportional taxes on capital and labor—lump-sum taxes are either ruled out or severely limited. To prevent expropriatory capital levies, the tax rate on capital is constrained by an upper bound. Turning to differences, Chamley (1986) focused on a representative agent and assumed perfect financial markets, with unconstrained government debt. Judd (1985) emphasizes heterogeneity and redistribution in a two-class economy, with workers and capitalists. In addition, the model in Judd (1985) features financial market imperfections: workers do not save and the government balances its budget, i.e. debt is restricted to zero. As emphasized by Judd (1985), it is most remarkable that a zero long-run tax result obtains despite the restriction to budget balance. Although extreme, imperfections of this kind may capture relevant aspects of reality, such as the limited participation in financial markets, the skewed distributions of wealth and a host of difficulties governments may face managing their debts or assets.

We begin with the model in Judd (1985) and focus on situations where desired redistribution runs from capitalists to workers. Working with an isoelastic utility over consumption for capitalists, \( U(C) = \frac{C^{1-\sigma}}{1-\sigma} \), we establish that when the intertemporal elasticity of substitution (IES) is below one, \( \sigma > 1 \), taxes rise and converge towards a positive limit tax, instead of declining towards zero. This limit tax is significant, driving capital to its lowest feasible level. Indeed, with zero government spending the lowest feasible capital stock is zero and the limit tax rate on wealth goes to 100%. The long-run tax is not only not zero, it is far from that.

The economic intuition we provide for this result is based on the anticipatory savings effects of future tax rates. When the IES is less than one, any anticipated increase in taxes leads to higher savings today, since the substitution effect is relatively small and dominated by the income effect. When the day comes, higher tax rates do eventually lower capital, but if the tax increase is sufficiently far off in the future, then the increased savings generate a setup somewhat closer to that in Chamley (1986). In particular, Judd (1985, Section 4-5) allow workers to save, capitalists to work and considers non-constant discounting a la Uzawa (1968). However, throughout the formal analysis in Judd (1985) the government is assumed to run a balanced budget, i.e. no government bonds are allowed. Interpolating our results for Judd (1985, Section 3) and Chamley (1986), we believe similar conclusions apply for these variant models in Judd (1985, Section 4-5).

Consumption taxes (Coleman II, 2000) and dividend taxes with capital expenditure (investment) deductions (Abel, 2007) can mimic initial wealth expropriation. Both are disallowed.

Because of the presence of financial restrictions and imperfections, the model in Judd (1985) does not fit the standard Arrow-Debreu framework, nor the optimal tax theory developed around it such as Diamond and Mirrlees (1971).

Another issue may arise on the other end. Without constraints on debt, capitalists may become highly indebted or not own the capital they manage. The idea that investment requires “skin in the game” is popular in the finance literature and macroeconomic models with financial frictions (see Brunnermeier et al., 2012; Gertler and Kiyotaki, 2010, for surveys).
higher capital stock over a lengthy transition. This is desirable, since it increases wages and
tax revenue. To exploit such anticipatory effects, the optimum involves an increasing path
for capital tax rates. This explains why we find positive tax rates that rise over time and
converge to a positive value, rather than falling towards zero.

When the IES is above one, $\sigma < 1$, we verify numerically that the solution converges to
the zero-tax steady state.\textsuperscript{6} This also relies on anticipatory savings effects, working in reverse.
However, we show that this convergence may be very slow, potentially taking centuries for
wealth taxes to drop below 1%. Indeed, the speed of convergence is not bounded away from
zero in the neighborhood of a unitary IES, $\sigma = 1$. Thus, even for those cases where the
long-run tax on capital is zero, this property provides a misleading summary of the model’s
tax prescriptions.

We confirm our intuition based on anticipatory effects by generalizing our results for the
Judd (1985) economy to a setting with arbitrary savings behavior of capitalists. Within this
more general environment we also derive an inverse elasticity formula for the steady state tax
rate, closely related to one in Piketty and Saez (2013). However, our derivation stresses that
the validity of this formula requires sufficiently fast convergence to an interior steady state, a
condition that we show fails in important cases.

We then turn to the representative agent Ramsey model studied by Chamley (1986). As is
well appreciated, in this setting upper bounds on the capital tax rate are imposed to prevent
expropriatory levels of taxation. We provide two sets of results.

Our first set of results show that in cases where the tax rate does converge to zero, there
are other implications of the model, hitherto unnoticed. These implications undermine the
usual interpretation against capital taxation. Specifically, if the optimum converges to a
steady state where the bounds on tax rates are slack, we show that the tax is indeed zero.
However, for recursive non-additive utility, we also show that this zero-tax steady state is
necessarily accompanied by either zero private wealth—in which case the tax base is zero—or
a zero tax on labor income—in which case the first best is achieved. This suggests that zero
taxes on capital are attained only after taxes have obliterated private wealth or allowed the
government to proceed without any distortionary taxation. Needless to say, these are not
the scenarios typically envisioned when interpreting zero long-run tax results. Away from
additive utility, the model simply does not justify a steady state with a positive tax on labor,
a zero tax on capital and positive private wealth.

Our second set of results show that the tax rate may not converge to zero. In particular,

\textsuperscript{6}We complement these numerical results by proving a local convergence result around the zero-tax steady
state when $\sigma < 1$. 

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we show that the upper bounds imposed on the tax rate may bind forever, implying a positive long-run tax on capital. We prove that this is guaranteed if the IES is below one and debt is high enough. Importantly, the debt level required is below the peak of the Laffer curve, so this result is not driven by budgetary necessity: the planner chooses to tax capital indefinitely, but is not compelled to do so. Intuitively, higher debt leads to higher labor taxes, making capital taxation attractive to ease the labor tax burden. However, because the tax rate on capital is capped, the only way to expand capital taxation is to prolong the time spent at the bound. At some point, for high enough debt, indefinite taxation becomes optimal.

All of these results run counter to what is certainly by now established wisdom, cemented by a significant follow-up literature, extending and interpreting long-run zero tax results. In particular, Judd (1999) presents an argument against positive capital taxation without requiring convergence to a steady state, using a representative agent model without financial market imperfections close to the one in Chamley (1986). However, as we explain, these arguments fail because they invoke assumptions on endogenous multipliers that may be violated at the optimum. We also explain why the intuition offered in that paper, based on the observation that a positive capital tax is equivalent to a rising tax on consumption, does not provide a valid rationale against indefinite capital taxation.

To conclude, we present a hybrid model that combines heterogeneity and redistribution as in Judd (1985), but allows for government debt as in Chamley (1986). Capital taxation turns out to be especially potent in this setting: whenever the IES is less than one, the optimal policy sets the tax rate at the upper bound forever. This suggests that positive long-run capital taxation should be expected for a wide range of models that are descendants of Chamley (1986) and Judd (1985).

**Literature.** Aside from a long literature finding different kinds of zero capital tax results, this chapter is part of a strand of papers that find positive long-run capital taxes can be optimal. Almost all of these papers find this result by modifying the environment, moving  

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7 For papers with exogenous (or no) growth see e.g. Chamley (1980, 1986); Judd (1985, 1999); Atkeson et al. (1999); Chari and Kehoe (1999). For papers with endogenous growth see e.g. Lucas (1990); Jones et al. (1993, 1997). For results on capital taxation fluctuating around zero in RBC models see e.g. Zhu (1992); Judd (1993); Chari et al. (1994). For zero capital tax results with heterogeneous agents, see e.g. Kocherlakota (2004, 2005); Werning (2007); Greulich et al. (2016). Papers that stated and sometimes even proved a correct variant of the Chamley result, where bounds are only imposed in the initial period.

8 For models using setups close to Chamley (1986) and Judd (1985) see Lansing (1999); Reinborn (2002 and 2013); Bassetto and Benhabib (2006); Piketty and Saez (2013). For results on capital taxation in OLG models, see e.g. Erosa and Gervais (2002). For models with social weights on future periods/generations, see e.g. Bonis and Spataro (2005); Farhi and Werning (2010, 2013). For results with limited commitment, see e.g. Chari and Kehoe (1990); Stokey (1991); Farhi et al. (2012). For models with incomplete markets and idiosyncratic risk, see e.g. Aiyagari (1995); Conesa et al. (2009).
away from the setups in Chamley (1986) and Judd (1985), with the following exceptions: Lansing (1999) considered a special case of the setup in Judd (1985, Section 3) and found positive long-run capital taxes are possible (see our discussion in Section 3.2); Reinhorn (2002 and 2013) clarified the discrepancy with Judd (1985, Section 3); and Bassetto and Benhabib (2006) found a positive long-run tax result in a setup that is a hybrid between the models of Chamley (1986) and Judd (1985), somewhat similar to our hybrid model in Section 3.4.

Finally, several authors stated a variant of the Chamley (1986) economy in which capital tax bounds are only imposed in the initial period and are therefore never positive indefinitely.\(^9\)

### 3.2 Capitalists and Workers

We start with the two-class economy without government debt laid out in Judd (1985). Time is indefinite and discrete, with periods labeled by \( t = 0, 1, 2, \ldots \).\(^{10}\) There are two types of agents, workers and capitalists. Capitalists save and derive all their income from the returns to capital. Workers supply one unit of labor inelastically and live hand to mouth, consuming their entire wage income plus transfers. The government taxes the returns to capital to pay for transfers targeted to workers.

**Preferences.** Both capitalists and workers discount the future with a common discount factor \( \beta < 1 \). Workers have a constant labor endowment \( n = 1 \); capitalists do not work. Consumption by workers will be denoted by lowercase \( c \), consumption by capitalists by uppercase \( C \). Capitalists have utility

\[
\sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{with} \quad U(C) = \frac{C^{1-\sigma}}{1-\sigma}
\]

for \( \sigma > 0 \) and \( \sigma \neq 1 \), and \( U(C) = \log C \) for \( \sigma = 1 \). Here \( 1/\sigma \) denotes the (constant) intertemporal elasticity of substitution (IES). Workers have utility

\[
\sum_{t=0}^{\infty} \beta^t u(c_t)
\]

where \( u \) is increasing, concave, continuously differentiable and \( \lim_{c \to 0} u'(c) = \infty \).

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\(^9\)See e.g. Chari and Kehoe (1999), Werning (2007) and Sargent and Ljungqvist (2004).

\(^{10}\)Judd (1985) formulates the model in continuous time, but this difference is immaterial. As usual, the continuous-time model can be thought of as a limit of the discrete time one as the length of each period shrinks to zero.
Technology. Output is obtained from capital and labor using a neoclassical constant returns production function \( F(k_t, n_t) \) satisfying standard conditions.\(^{11} \) Capital depreciates at rate \( \delta > 0. \) In equilibrium \( n_t = 1, \) so define \( f(k) = F(k, 1). \) The government consumes a constant flow of goods \( g \geq 0. \) We normalize both populations to unity and abstract from technological progress and population growth. The resource constraint in period \( t \) is then
\[
C_t + G_t + G_{t+1} \leq f(k_t) + (1 - \delta)k_t.
\]
There is some given positive level of initial capital, \( k_0 > 0. \)

Markets and Taxes. Markets are perfectly competitive, with labor being paid wage \( w_t^* = F_a(k_t, n_t) \) and the before-tax return on capital being given by
\[
R_t^* = f'(k_t) + 1 - \delta.
\]
The after-tax return equals \( R_t \) and can be parameterized as either
\[
R_t = (1 - \tau_t)(R_t^* - 1) + 1 \quad \text{or} \quad R_t = (1 - \tau_t)R_t^*.
\]
where \( \tau_t \) is the tax rate on the net return to wealth and \( \tau_t \) the tax rate on the gross return to wealth, or wealth tax for short. Whether we consider a tax on net returns or on gross returns is irrelevant and a matter of convention. We say that capital is taxed whenever \( R_t < R_t^* \) and subsidized whenever \( R_t > R_t^*. \)

Capitalist and Worker Behavior. Capitalists solve
\[
\max \left\{ C_t, a_{t+1} \right\} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.} \quad C_t + a_{t+1} = R_t a_t \quad \text{and} \quad a_{t+1} \geq 0,
\]
for some given initial wealth \( a_0. \) The associated Euler equation and transversality conditions,
\[
U'(C_t) = \beta R_{t+1} U'(C_{t+1}) \quad \text{and} \quad \beta^t U'(C_t)a_{t+1} \rightarrow 0,
\]
are necessary and sufficient for optimality.

\(^{11}\)We assume that \( F \) is increasing and strictly concave in each argument, continuously differentiable, and satisfying the standard Inada conditions \( F_k(k, 1) \rightarrow \infty \) as \( k \rightarrow 0 \) and \( F_k(k, 1) \rightarrow 0 \) as \( k \rightarrow \infty. \) Moreover assume that capital is essential for production, that is, \( F(0, n) = 0 \) for all \( n. \)
Workers live hand to mouth, their consumption equals their disposable income

\[ c_t = w_t^* + T_t = f(k_t) - f'(k_t)k_t + T_t, \]

which uses the fact that \( F_n = F - F_k k \). Here \( T_t \in \mathbb{R} \) represent government lump-sum transfers (when positive) or taxes (when negative) to workers.\(^\text{12}\)

**Government Budget Constraint.** As in Judd (1985), the government cannot issue bonds and runs a balanced budget. This implies that total wealth equals the capital stock \( a_t = k_t \) and that the government budget constraint is

\[ g + T_t = (R_t^* - R_t) k_t. \]

**Planning Problem.** Using the Euler equation to substitute out \( R_t \), the planning problem can be written as\(^\text{13}\)

\[
\max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t (u(c_t) + \gamma U(C_t)), \tag{3.1a}
\]

subject to

\[
c_t + C_t + g + k_{t+1} = f(k_t) + (1 - \delta)k_t, \tag{3.1b}
\]
\[
\beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t, \tag{3.1c}
\]
\[
\beta^t U'(C_t)k_{t+1} \rightarrow 0. \tag{3.1d}
\]

The government maximizes a weighted sum of utilities with weight \( \gamma \) on capitalists. By varying \( \gamma \) one can trace out points on the constrained Pareto frontier and characterize their associated policies. We often focus on the case with no weight on capitalists, \( \gamma = 0 \), to ensure that desired redistribution runs from capitalists towards workers. Equation (3.1b) is the resource constraint. Equation (3.1c) combines the capitalists' first-order condition and budget constraint and (3.1d) imposes the transversality condition; together conditions (3.1c) and (3.1d) ensure the optimality of the capitalists' saving decision.

\(^\text{12}\)Equivalently, one can set up the model without lump-sum transfers/taxes to workers, but allowing for a proportional tax or subsidy on labor income. Such a tax perfectly targets workers without creating any distortions, since labor supply is perfectly inelastic in the model.

\(^\text{13}\)Judd (1985) includes upper bounds on the taxation of capital, which we have omitted because they do not play any important role. As we shall see, positive long run taxation is possible even without these constraints; adding them would only reinforce this conclusion. Upper bounds on taxation play a more crucial role in Chamley (1986).
The necessary first-order conditions are

\begin{align*}
\mu_0 &= 0, \\
\lambda_t &= u'(c_t), \\
\mu_{t+1} &= \mu_t \left( \frac{\sigma - 1}{\sigma \kappa_t + 1} + 1 \right) + \frac{1}{\beta \sigma \kappa_t + 1} v_t (1 - \gamma v_t), \\
\frac{u'(c_{t+1})}{u'(c_t)} (f'(k_{t+1}) + 1 - \delta) &= \frac{1}{\beta} + v_t (\mu_{t+1} - \mu_t),
\end{align*}

where \( \kappa_t \equiv k_t / C_{t-1} \), \( v_t \equiv U'(C_t) / u'(c_t) \) and the multipliers on constraints (3.1b) and (3.1c) are \( \beta^t \lambda_t \) and \( \beta^t \mu_t \), respectively.\(^14\)

### 3.2.1 Previous Steady State Results

Judd (1985, pg. 72, Theorem 2) provided a zero-tax result, which we adjust in the following statement to stress the need for the steady state to be interior and for multipliers to converge.

**Theorem 1** (Judd, 1985). Suppose quantities and multipliers converge to an interior steady state, i.e. \( c_t, C_t, k_{t+1} \) converge to positive values, and \( \mu_t \) converges. Then the tax on capital is zero in the limit: \( T_t = 1 - R_t / R^*_t \to 0 \).

The proof is immediate: from equation (3.2d) we obtain \( R^*_t \to 1 / \beta \), while the capitalists’ Euler equation requires that \( R_t \to 1 / \beta \). The simplicity of the argument follows from strong assumptions placed on endogenous outcomes. This raises obvious concerns. By adopting assumptions that are close relatives of the conclusions, one may wonder if anything of use has been shown, rather than assumed. We elaborate on a similar point in Section 3.3.3.

In our rendering of Theorem 1, the requirement that the steady state be interior is important: otherwise, if \( c_t \to 0 \) one cannot guarantee that \( u'(c_{t+1}) / u'(c_t) \to 1 \) in equation (3.2d). Likewise, even if the allocation converges to an interior steady state but \( \mu_t \) does not converge, then \( v_t (\mu_{t+1} - \mu_t) \) may not vanish in equation (3.2d). Thus, the two situations that prevent the theorem’s application are: (i) non-convergence to an interior steady state; or (ii) non-convergence of \( \mu_{t+1} - \mu_t \) to zero. In general, one expects that (i) implies (ii). The literature has provided an example of (ii) where the allocation does converge to an interior steady state.

**Theorem 2.** (Lansing, 1999; Reinhorn, 2002 and 2013) Assume \( \sigma = 1 \). Suppose the allocation converges to an interior steady state, so that \( c_t, C_t \) and \( k_{t+1} \) converge to strictly positive values.

\(^{14}\)We chose the sign of \( \lambda_t \) in the conventional way and the sign of \( \mu_t \) such that the term in the current value Lagrangian is given by \( \mu_t (\beta U'(C_t)(C_t + k_{t+1}) - U'(C_{t-1})k_t) \).
Then,
\[ T_t \to \frac{1 - \beta}{1 + \gamma v \beta / (1 - \gamma v)}, \]
where \( v = \lim v_t \) and the multiplier \( \mu_t \) in the system of first-order conditions (3.2c) does not converge. This implies a positive long-run tax on capital if redistribution towards workers is desirable, \( 1 - \gamma v > 0 \).

The result follows easily by combining (3.2c) and (3.2d) for the case with \( \sigma = 1 \) and comparing it to the capitalist’s Euler equation, which requires \( R_t = \frac{1}{\beta} \) at a steady state. Lansing (1999) first presented the logarithmic case as a counterexample to Judd (1985). Reinhorn (2002 and 2013) correctly clarified that in the logarithmic case the Lagrange multipliers explode, explaining the difference in results.\(^{15}\)

Lansing (1999) depicts the result for \( \sigma = 1 \) as a knife-edge case: “the standard approach to solving the dynamic optimal tax problem yields the wrong answer in this (knife-edge) case [...]” (from the Abstract, page 423) and “The counterexample turns out to be a knife-edge result. Any small change in the capitalists’ intertemporal elasticity of substitution away from one (the log case) will create anticipation effects [...] As capitalists’ intertemporal elasticity of substitution in consumption crosses one, the trajectory of the optimal capital tax in this model undergoes an abrupt change.” (page 427) Lansing (1999) suggests that whenever \( \sigma \neq 1 \) the long-run tax on capital is zero. We shall show that this is not the case.

### 3.2.2 Main Result: Positive Long-Run Taxation

**Logarithmic Utility.** Before studying \( \sigma > 1 \), our main case of interest, it is useful to review the special case with logarithmic utility, \( \sigma = 1 \). We assume \( \gamma = 0 \) to guarantee that desired redistribution runs from capitalists to workers.

When \( U(C) = \log C \) capitalists save at a constant rate \( s > 0 \),
\[ C_t = (1 - s)R_t k_t \quad \text{and} \quad k_{t+1} = sR_t k_t.\]

Although \( s = \beta \) with logarithmic preferences, nothing we will derive depends on this fact, so we can interpret \( s \) as a free parameter that is potentially divorced from \( \beta \).\(^{16}\)

\(^{15}\) Lansing (1999) suggests a technical difficulty with the argument in Judd (1985) that is specific to \( \sigma = 1 \). Indeed, at \( \sigma = 1 \) one degree of freedom is lost in the planning problem, since \( C_{t-1} \) must be proportional to \( k_t \). However, since equations (3.2) can still satisfied by the optimal allocation for some sequence of multipliers, we believe the issue can be framed exactly as Reinhorn (2002 and 2013) did, emphasizing the non convergence of multipliers.

\(^{16}\) This could capture different discount factors between capitalists and workers or an ad hoc behavioral assumption of constant savings, as in the standard Solow growth model. We pursue this line of thought in.
The planning problem becomes

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad c_t + \frac{1}{s} k_{t+1} + g = f(k_t) + (1 - \delta)k_t,$$

with $k_0$ given. This amounts to an optimal neoclassical growth problem, where the price of capital equals $\frac{1}{s} > 1$ instead of the actual unit cost. The difference arises from the fact that capitalists consume a fraction $1 - s$. The government and workers must save indirectly through capitalists, entrusting them with resources today by holding back on current taxation, so as to extract more tomorrow. From their perspective, technology appears less productive because capitalists feed off a fraction of the investment. Lower saving rates $s$ increase this inefficiency.17

Since the planning problem is equivalent to a standard optimal growth problem, we know that there exists a unique interior steady state and that it is globally stable. The modified golden rule at this steady state is $\beta s R^* = 1$. A steady state also requires $s R k = k$, or simply $s R = 1$. Putting these conditions together gives $R/R^* = \beta < 1$.

**Proposition 12.** Suppose $\gamma = 0$ and that capitalists have logarithmic utility, $U(C) = \log C$. Then the solution to the planning problem converges monotonically to a unique steady state with a positive tax on capital given by $T = 1 - \beta$.

This proposition echoes the result in Lansing (1999), as summarized by Theorem 2, but also establishes the convergence to the steady state. Interestingly, the long-run tax rate depends only on $\beta$, not on the savings rate $s$ or other parameters.

Although Lansing (1999) and the subsequent literature interpreted this result as a knife-edge counterexample, we will argue that this is not the case, that positive long run taxes are not special to logarithmic utility. One way to proceed would be to exploit continuity of the planning problem with respect to $\sigma$ to establish that for any fixed time $t$, the tax rate $T_t(\sigma)$ converges as $\sigma \to 1$ to the tax rate obtained in the logarithmic case (which we know is positive for large $t$). While this is enough to dispel the notion that the logarithmic utility case is irrelevant for $\sigma \neq 1$, it has its limitations. As we shall see, the convergence is not uniform and one cannot invert the order of limits: $\lim_{t \to \infty} \lim_{\sigma \to 1} T_t(\sigma)$ does not equal $\lim_{\sigma \to 1} \lim_{t \to \infty} T_t(\sigma)$. Therefore, arguing by continuity does not help characterize the long run

Section 3.2.3 below.

17This kind of wedge in rates of return is similar to that found in countless models where there are financial frictions between “experts” able to produce capital investments and “savers”. Often, these models are set up with a moral hazard problem, whereby some fraction of the investment returns must be kept by experts, as “skin in the game” to ensure good behavior.
tax rate \( \lim_{t \to \infty} T_t(\sigma) \) as a function of \( \sigma \). We proceed by tackling the problem with \( \sigma \neq 1 \) directly.

**Positive Long-Run Taxation: IES \(<\ 1.** We now consider the case with \( \sigma > 1 \) so that the intertemporal elasticity of substitution \( \frac{1}{\sigma} \) is below unity. We continue to focus on the situation where no weight is placed on capitalists, \( \gamma = 0 \). Section 3.2.4 shows that the same results apply for other value of \( \gamma \), as long as redistribution from capitalists to workers is desired.

Towards a contradiction, suppose the allocation were to converge to an interior steady state \( k_t \to k \), \( C_t \to C \), \( c_t \to c \) with \( k, C, c > 0 \). This implies that \( \kappa_t \) and \( v_t \) also converge to positive values, \( \kappa \) and \( v \). Combining equations (3.2c) and (3.2d) and taking the limit for the allocation, we obtain

\[
\frac{1}{\beta} + v(\mu - \mu_{t-1}) = \frac{1}{\beta} + \mu_t \frac{\sigma - 1}{\sigma \kappa} v + \frac{1}{\beta \sigma \kappa}.
\]

Since \( \sigma > 1 \), this means that \( \mu_t \) must converge to

\[
\mu = \frac{1}{(\sigma - 1)\beta v} < 0.
\]  

Now consider whether \( \mu_t \to \mu < 0 \) is possible. From the first-order condition (3.2a) we have \( \mu_0 = 0 \). Also, from equation (3.2c), whenever \( \mu_t \geq 0 \) then \( \mu_{t+1} \geq 0 \). It follows that \( \mu_t \geq 0 \) for all \( t = 0, 1, \ldots \), a contradiction to \( \mu_t \to \mu < 0 \). This proves that the solution cannot converge to any interior steady state, including the zero-tax steady state.

**Proposition 13.** If \( \sigma > 1 \) and \( \gamma = 0 \), no solution to the planning problem converges to the zero-tax steady state, or any other interior steady state.

It follows that if the optimal allocation converges, then either \( k_t \to 0 \), \( C_t \to 0 \) or \( c_t \to 0 \). With positive spending \( g > 0 \), \( k_t \to 0 \) is not feasible; this also rules out \( C_t \to 0 \), since capitalists cannot be starved while owning positive wealth.

Thus, provided the solution converges, \( c_t \to 0 \). This in turn implies that either \( k_t \to k_g \) or \( k_t \to k^g \) where \( k_g < k^g \) are the two solutions to \( \frac{1}{\beta} k + g = f(k) + (1 - \delta)k \), using the fact that (3.1c) implies \( C = \frac{1 - \beta}{\beta} k \) at any steady state. We next show that the solution does indeed converge, and that it does so towards the lowest sustainable value of capital, \( k_g \), so that the

\[\text{footnote 18: Notice that this reasoning only relies on the necessity of the first-order conditions. Therefore, it also applies when the planning problem (3.1a) is non-convex.}\]

\[\text{footnote 19: Here we assume that government spending } g \text{ is feasible, that is, } g < \max_k \{f(k) + (1 - \delta)k - \frac{1}{\beta}k\}.\]
long-run tax on capital is strictly positive. The proof uses the fact that $\mu_t \to \infty$ and $c_t \to 0$, as argued above, but requires many other steps detailed in the appendix.

**Proposition 14.** If $\sigma > 1$ and $\gamma = 0$ then any solution to the planning problem converges to $c_t \to 0$, $k_t \to k_g$, $C_t \to \frac{1-\beta}{\beta}k_g$, with a positive limit tax on wealth: $T_t = 1 - \frac{R_t}{R^*_t} \to T^g > 0$. The limit tax $T^g$ is decreasing in spending $g$, with $T^g \to 1$ as $g \to 0$.

The zero-tax interpretation of Judd (1985) is invalidated here because the allocation does not converge to an interior steady state and multipliers do not converge. According to our result, the tax rate not only does not converge to zero, it reaches a sizable level. Perhaps counterintuitively, the long-run tax on capital, $T^g$, is inversely related to the level of government spending, since $k_g$ is increasing with spending $g$. This underscores that long-run capital taxation is not driven by budgetary necessity.

As the proposition shows, optimal taxes may reach very high levels. Up to this point, we have placed no limits on tax rates. It may be of interest to consider a situation where the planner is further constrained by an upper bound on the tax rate for net returns ($\tau$) or gross wealth ($T$), perhaps due to evasion or political economy considerations. If these bounds are sufficiently tight to be binding, it is natural to conjecture that the optimum converges to these bounds, and to an interior steady state allocation with a positive limit for worker consumption, $\lim_{t \to \infty} c_t > 0$.

**Solution for IES near 1.** Figure 3-1 displays the time path for the capital stock and the tax rate on wealth, $T_t = 1 - R_t/R^*_t$, for a range of $\sigma$ that straddles the logarithmic $\sigma = 1$ case. We set $\beta = 0.95$, $\delta = 0.1$, $f(k) = k^\alpha$ with $\alpha = 0.3$ and $u(c) = U(c)$. Spending $g$ is chosen so that $\frac{g}{f(k)} = 20\%$ at the zero-tax steady state. The initial value of capital, $k_0$, is set at the zero-tax steady state. Our numerical method is based on a recursive formulation of the problem described in the appendix.

To clarify the magnitudes of the tax on wealth, $T_t$, consider an example: If $R^* = 1.04$ so that the before-tax net return is $4\%$, then a tax on wealth of $1\%$ represents a $25\%$ tax on the net return; a wealth tax of $4\%$ represents a tax rate of $100\%$ on net returns, and so on.

A few things stand out in Figure 3-1. First, the results confirm what we showed theoretically in Proposition 14, that for $\sigma > 1$ capital converges to $k_g = 0.0126$. In the figure this convergence is monotone \textsuperscript{20}, taking around 200 years for $\sigma = 1.25$. The asymptotic tax rate is very high, approximately $T^g = 1 - R/R^* = 85\%$, lying outside the figure’s range, and, since

\textsuperscript{20}This depends on the level of initial capital. For lower levels of capital the path first rises then falls.
**Figure 3-1:** Optimal time paths for capital (left) and wealth taxes (right).

*Note.* This figure shows the optimal time paths of capital $k_t$ (left panel) and wealth taxes $T_t$ (right panel) for various values of the inverse IES $\sigma$.

The after-tax return equals $R = 1/\beta$ in the long run, this implies that the before-tax return $R^* = f'(k_g) + 1 - \delta$ is exorbitant.

Second, for $\sigma < 1$, the path for capital is non-monotonic\(^{21}\) and eventually converges to the zero-tax steady state. However, the convergence is relatively slow, especially for values of $\sigma$ near 1. This makes sense, since, by continuity, for any period $t$, the solution should converge to that of the logarithmic utility case as $\sigma \to 1$, with positive taxation as described in Proposition 12. By implication, for $\sigma < 1$ the rate of convergence to the zero-tax steady state must be zero as $\sigma \uparrow 1$. To further punctuate this point, Figure 3-2 shows the number of years it takes for the tax on wealth to drop below 1% as a function of $\sigma \in (\frac{1}{2}, 1)$. As $\sigma$ rises, it takes longer and longer and as $\sigma \uparrow 1$ it takes an eternity.

The logarithmic case leaves other imprints on the solutions for $\sigma \neq 1$. Returning to Figure 3-1, for both $\sigma < 1$ and $\sigma > 1$ we see that over the first 20-30 years, the path approaches the steady state of the logarithmic utility case, associated with a tax rate around $T = 1 - \beta = 5\%$. The speed at which this takes place is relatively quick, which is explained by the fact that for $\sigma = 1$ it is driven by the standard rate of convergence in the neoclassical growth model. The solution path then transitions much more slowly either upwards or downwards, depending on whether $\sigma > 1$ or $\sigma < 1$.

**Intuition: Anticipatory Effects of Future Taxes on Current Savings.** Why does the optimal tax eventually rise for $\sigma > 1$ and fall for $\sigma < 1$? Why are the dynamics relatively slow for $\sigma$ near 1?

\(^{21}\)This is possible because the state variable has two dimensions, $(k_t, C_{t-1})$. At the optimum, for the same capital $k$, consumption $C$ is initially higher on the way down than it is on the way up.
To address these normative questions it helps to back up and review the following positive exercise. Start from a constant tax on wealth and imagine an unexpected announcement of higher future taxes on capital. How do capitalists react today? There are substitution and income effects pulling in opposite directions. When \( \sigma > 1 \), the substitution effect is muted compared to the income effect, and capitalists lower their consumption to match the drop in future consumption. As a result, capital rises in the short run and falls in the long run.\(^{22}\) When instead \( \sigma < 1 \), the substitution effect is stronger and capitalists increase current consumption. In the logarithmic case, \( \sigma = 1 \), the two effects cancel out, so that current consumption and savings are unaffected.

Returning to the normative questions, lowering capitalists' consumption and increasing capital is desirable for workers. When \( \sigma < 1 \), this can be accomplished by promising lower tax rates in the future. This explains why a declining path for taxes is optimal. In contrast, when \( \sigma > 1 \), the same is accomplished by promising higher tax rates in the future; explaining the increasing path for taxes. These incentives are absent in the logarithmic case, when \( \sigma = 1 \), explaining why the tax rate converges to a constant.

When \( \sigma < 1 \) the rate of convergence to the zero-tax steady state is also driven by these anticipatory effects. With \( \sigma \) near 1, the potency of these effects is small, explaining why the rate of convergence is low and indeed becomes vanishingly small as \( \sigma \uparrow 1 \).

In contrast to previous intuitions offered for zero long-run tax results, the intuition we provide for our results—zero and nonzero long-run taxes alike, depending on \( \sigma \)—is not about

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\(^{22}\)It is important to note that \( \sigma > 1 \) does not imply that the supply for savings "bends backward". Indeed, as a positive exercise, if taxes are raised permanently within the model, then capital falls over time to a lower steady state for any value of \( \sigma \), including \( \sigma > 1 \). Higher values of \( \sigma \) imply a less elastic response over any finite time horizon, and thus a slower convergence to the lower capital stock. The case with \( \sigma > 1 \) is widely considered more plausible empirically.
the desired level for the tax. Instead, we provide a rationale for the desired slope in the path for the tax: an upward path when \( \sigma > 1 \) and a downward path when \( \sigma < 1 \). The conclusions for the optimal long-run tax then follow from these desired slopes, rather than the other way around.

Our intuition based on slopes has an interesting implication for the effects of limited commitment in this economy. Since the planner promises higher future taxation when \( \sigma > 1 \), renegotiation by the planner might lead to lower rather than higher capital taxes. This is the polar opposite of the conventional wisdom, according to which limited commitment leads to higher capital taxation.

3.2.3 General Savings Functions and Inverse Elasticity Formula

The intuition suggests that the essential ingredient for positive long run capital taxation in the model of Judd (1985, Section 3) is that capitalists’ savings decrease in future interest rates. To make this point even more transparently, we now modify the model and assume capitalists behave according to a general “ad-hoc” savings rule,

\[
k_{t+1} = S(R_t k_t; R_{t+1}, R_{t+2}, \ldots),
\]

where \( S(I_t; R_{t+1}, R_{t+2}, \ldots) \in [0, I_t] \) is a continuously differentiable function taking as arguments current wealth \( I_t = R_t k_t \geq 0 \) and future interest rates \( \{R_{t+1}, R_{t+2}, \ldots\} \in \mathbb{R}_+^N \). We assume that savings increase with income, \( S_I > 0 \). This savings function encompasses the case where capitalists maximize an additively separable utility function, as in Judd (1985), but is more general. For example, the savings function can be derived from the maximization of a recursive utility function, or even represent behavior that cannot be captured by optimization, such as hyperbolic discounting or self-control and temptation.

Again, we focus on the case \( \gamma = 0 \). The planning problem is then

\[
\max_{\{c_t, R_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t),
\]

subject to

\[
c_t + R_t k_t + g = f(k_t) + (1 - \delta)k_t,
\]

\[
k_{t+1} = S(R_t k_t; R_{t+1}, R_{t+2}, \ldots),
\]

with \( k_0 \) given.
We can show that, consistent with the intuition spelled out above, long-run capital taxes are positive whenever savings decrease in future interest rates.

**Proposition 15.** Suppose $\gamma = 0$ and assume the savings function is decreasing in future rates, so that $S_R(I; R_1, R_2, \ldots) \leq 0$ for all $t = 1, 2, \ldots$ and all arguments $\{I, R_1, R_2, \ldots\}$. If the optimum converges to an interior steady state in $c, k,$ and $R$, and at the steady state $\beta R S_I \neq 1$, then the limit tax rate is positive and $\beta R S_I < 1$.

This generalizes Proposition 13, since the case with iso-elastic utility and IES less than one is a special case satisfying the hypothesis of the proposition. Once again, the intuition here is that the planner exploits anticipatory effects by raising tax rates over time to increase present savings.

The result requires $\beta R S_I < 1$ at the steady state, which is satisfied when savings are linear in income, since then $S_I R = 1$ at a steady state. Note that savings are linear in income in the isoelastic utility case. More generally, $RS_I < 1$ is natural, as it ensures local stability for capital given a fixed steady-state return, i.e. the dynamics implied by the recursion $k_{t+1} = S(R k_t, R, R, \ldots)$ for fixed $R$.

**Inverse Elasticity Formula.** There is a long tradition relating optimal tax rates to elasticities. In the context of our general savings model, spelled out above, we derive the following “inverse elasticity rule”

$$T = 1 - \frac{R}{R^*} = \frac{1 - \beta R S_I}{1 + \sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{S,t}}$$

where $\epsilon_{S,t} = \frac{R}{S} \frac{\partial S}{\partial R} (R_0 k_0; R_1, R_2, \ldots)$ denotes the elasticity of savings with respect to future interest rates evaluated at the steady state in $c, k,$ and $R$. Although the right hand side is endogenous, equation (3.4) is often interpreted as a formula for the tax rate. Our inverse elasticity formula is closely related to a condition derived by Piketty and Saez (2013, see their Section 3.3, equation 16).

We wish to make two points about our formula. First, note that the relevant elasticity in this formula is not related to the response of savings to current, transitory or permanent, changes in interest rates. Instead, the formula involves a sum of elasticities of savings with respect to future changes in interest rates. Thus, it involves the anticipatory effects discussed

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23Their formula is derived under the special assumptions of additively separable utility, an exogenously fixed international interest rate and an exogenous wage. None of this is important, however. The two formulas remain different because of slightly different elasticity definitions; ours is based on partial derivatives of the primitive savings function $S$ with respect to a single interest rate change, while theirs is based on the implicit total derivative of the capital stock sequence with respect to a permanent change in the interest rate.
above. Indeed, the variation behind our formula changes the after-tax interest rate at a single future date $T$, and then takes the limit as $T \to \infty$. For any finite $T$, the term $\sum_{t=1}^{T} \beta^{-t+1} \epsilon_{s,t}$ represents the sum of the anticipatory effects on capitalists' savings behavior in periods 0 up to $T - 1$; while $\sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{s,t}$ captures the limit as $T \to \infty$. It is important to keep in mind that, precisely because it is anticipatory effects that matter, the relevant elasticities are negative in standard cases, e.g. with additive utility and IES below one.

Second, the derivation we provide in the appendix requires convergence to an interior steady state as well as additional conditions (somewhat cumbersome to state) to allow a change in the order of limits and obtain the simple expression $\sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{s,t}$. These latter conditions seem especially hard to guarantee ex ante, with assumptions on primitives, since they may involve the endogenous speed of convergence to the presumed interior steady state.\textsuperscript{24} As we have shown, in this model one cannot take these properties for granted, neither the convergence to an interior steady state (Proposition 14) nor the additional conditions. Indeed, Proposition 15 already supplies counterexamples to the applicability of the inverse elasticity formula.

Corollary. Under the conditions of Proposition 15, the inverse elasticity formula (3.4) cannot hold if $1 + \sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{s,t} < 0$.

This result provides conditions under which the formula (3.4) cannot characterize the long run tax rate. Whenever the discounted sum of elasticities with respect to future rates, $\sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{s,t}$, is negative and less than $-1$, the formula implies a negative limit tax rate. Yet, under the same conditions as in Proposition 15, this is not possible since this result shows that if convergence takes place, the tax rate is positive.

The case with additive and iso-elastic utility is an extreme example where the sum of elasticities $\sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{s,t}$ diverges. As it turns out, in this case $\beta^{-t} \epsilon_{s,t} = -\frac{\sigma-1}{\sigma} \frac{1-\beta}{\beta}$ at a steady state and the sum of elasticities diverges. It equals $+\infty$ if the IES is greater than one, or $-\infty$ if the IES is less than one.\textsuperscript{25} In both cases, formula (3.4) suggests a zero steady state tax rate. Piketty and Saez (2013) use this to argue that this explains the Chamley-Judd result of a zero long-run tax. However, as we have shown, when the IES is less than one the limit tax rate is not zero. This counterexample to the applicability of the inverse elasticity formula (3.4) assumes additive utility and, thus, an infinite sum of elasticities. However, the problem may also arise for non-additive preferences or with ad hoc saving functions. Indeed,

\textsuperscript{24}Unfortunately, one cannot ignore transitions by choice of a suitable initial condition. For example, even in the additive utility case with $\sigma < 1$ and even if we start at the zero capital tax steady state, capital does not stay at this level forever. Instead, capital first falls and then rises back up at a potentially slow rate.

\textsuperscript{25}Proposition 23 in the appendix shows that the infinite sum $\sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{s,t}$ also diverges for general recursive, non-additive preferences.
the conditions for the corollary may be met in cases where the sum of elasticities is finite, as long as its value is sufficiently negative.

It should be noted that our corollary provides sufficient conditions for the formula to fail, but other counterexamples may exist outside its realm. Suggestive of this is the fact that when the denominator is positive but small the formula may yield tax rates above 100%, which seems nonsensical, requiring $R < 0$. More generally, very large tax rates may be inconsistent with the fact that steady state capital must remain above $k_g > 0$.

To summarize, the inverse elasticity formula (3.4) fails in important cases, providing misleading answers for the long run tax rate. This highlights the need for caution in the application of steady state inverse elasticity rules.

### 3.2.4 Redistribution Towards Capitalists

In the present model, a desire to redistribute towards workers, away from capitalists, is a prerequisite to create a motive for positive wealth taxation. Proposition 14 assumes no weight on capitalists, $\gamma = 0$, to ensure that desired redistribution runs in this direction. When $\gamma > 0$ the same results obtain as long as the desire for redistribution continues to run from capitalists towards workers. In contrast, when $\gamma$ is high enough the desired redistribution flips from workers to capitalists. When this occurs, the optimum naturally involves negative tax rates, to benefit capitalists.

We verify these points numerically. Figure 3-3 illustrates the situation by fixing $\sigma = 1.25$ and varying the weight $\gamma$. Since initial capital is set at the zero-tax steady state, $k^*$, the direction of desired redistribution flips exactly at $\gamma^* = u'(c^*)/U'(C^*)$. At this value of $\gamma$, the planner is indifferent between redistributing towards workers or capitalists at the zero-tax steady state $(k^*, c^*, C^*)$. When $\sigma > 1$ and $\gamma > \gamma^*$ the solution converges to the highest sustainable capital $k^g$, the highest solution to $\frac{1}{\delta} k + g = f(k) + (1 - \delta)k$, rather than $k_g$, the lowest solution to the same equation.

A deeper understanding of the dynamics can be grasped by noting that the planning problem is recursive in the state variable $(k_t, C_{t-1})$. It is then possible to study the dynamics for this state variable locally, around the zero tax steady state, by linearizing the first-order conditions (3.2). We do so for a continuous-time version of the model, to ensure that our

---

26 Rather than displaying $\gamma$ in the legend for Figure 3-3, we perform a transformation that makes it more easily interpretable: we report the proportional change in consumption for capitalists that would be desired at the steady state, e.g. $-0.4$ represents that the planner’s ideal allocation of the zero-tax output would feature a 40% reduction in the consumption of capitalists, relative to the steady state value $C = \frac{1-\delta}{\delta} k$. The case $\gamma = \gamma^*$ corresponds to 0 in this transformation.
Proposition 16. For a continuous-time version of the model,

1. if $\sigma > 1$, the zero-tax steady state is locally saddle-path stable;
2. if $\sigma < 1$ and $\gamma \leq \gamma^*$, the zero-tax steady state is locally stable;
3. if $\sigma < 1$ and $\gamma > \gamma^*$, the zero-tax steady state may be locally stable or unstable and the dynamics may feature cycles.

The first two points confirm our theoretical and numerical observations. For $\sigma > 1$ the solution is saddle-path stable, explaining why it does not converge to the zero-tax steady state—except for the knife-edged cases where there is no desire for redistribution, in which case the tax rate is zero throughout. For $\sigma < 1$ the solution converges to the zero tax steady state whenever redistribution towards workers is desirable. This lends theoretical support to our numerical findings for $\sigma < 1$, discussed earlier and illustrated in Figure 3-1.

The third point raises a distinct possibility which is not our focus: the system may become unstable or feature cyclical dynamics. This is consistent with Kemp et al. (1993), who also studied the linearized system around the zero-tax steady state. They reported the potential for local instability and cycles, applying the Hopf Bifurcation Theorem. Proposition 16 clarifies that a necessary condition for this dynamic behavior is $\sigma < 1$ and $\gamma > \gamma^*$. The latter condition is equivalent to a desire to redistribute away from workers towards capitalists. We have instead focused on low values of $\gamma$ that ensure that desired redistribution runs from...
capitalists to workers. For this reason, our results are completely distinct to those in Kemp et al. (1993).

3.3 Representative Agent Ramsey

In the previous section we worked with the two-class model without government debt in Judd (1985, Section 3). Chamley (1986), in contrast, studied a representative agent Ramsey model with unconstrained government debt; Judd (1999) adopted the same assumptions. This section presents results for such representative agent frameworks.

We first consider situations where the upper bounds on capital taxation do not bind in the long run. We then prove that these bounds may, in fact, bind indefinitely.

3.3.1 First Best or Zero Taxation of Zero Wealth?

In this subsection, we first review the discrete-time model and zero capital tax steady state result in Chamley (1986, Section 1) and then present a new result. We show that if the economy settles down to a steady state where the bounds on the capital tax are not binding, then the tax on capital must be zero. This result holds for general recursive preferences that, unlike time-additive utility, allow the rate of impatience to vary. Non-additive utility constituted an important element in Chamley (1986, Section 1), to ensure that zero-tax results were not driven by an “infinite long-run elasticity of savings”\(^\text{27}\). However, we also show that other implications emerge away from additive utility. In particular, if the economy converges to a zero-tax steady state there are two possibilities. Either private wealth has been wiped out, in which case nothing remains to be taxed, or the tax on labor also falls to zero, in which case capital income and labor income are treated symmetrically. These implications paint a very different picture, one that is not favorable to the usual interpretation of zero capital tax results.

Preferences. We write the representative agent’s utility as \(V(U_0, U_1, \ldots)\) with per period utility \(U_t = U(c_t, n_t)\) depending on consumption \(c_t\) and labor supply \(n_t\). Assume that utility

\(^{27}\)At any steady state with additive utility one must have \(R = 1/\beta\) for a fixed parameter \(\beta \in (0, 1)\). This is true regardless of the wealth or consumption level. In this sense, the supply of savings is infinitely elastic at this rate of interest.
\( \mathcal{V} \) is increasing in every argument and satisfies a Koopmans (1960) recursion

\[
V_t = W(U_t, V_{t+1}) \tag{3.5a}
\]
\[
V_t = \mathcal{V}(U_t, U_{t+1}, \ldots) \tag{3.5b}
\]
\[
U_t = U(c_t, n_t) \tag{3.5c}
\]

Here \( W(U, V') \) is an aggregator function. We assume that both \( U(c, n) \) and \( W(U, V') \) are twice continuously differentiable, with \( W_U, W_V, U_c > 0 \) and \( U_n < 0 \). Consumption and leisure are taken to be normal goods,

\[
\frac{U_{cc}}{U_c} - \frac{U_{nc}}{U_n} \leq 0 \quad \text{and} \quad \frac{U_{cn}}{U_c} - \frac{U_{nn}}{U_n} \leq 0,
\]

with at least one strict inequality.

Regarding the aggregator function, the additively separable utility case amounts to the particular linear choice \( W(U, V') = U + \beta V' \) with \( \beta \in (0, 1) \). Nonlinear aggregators allow local discounting to vary with \( U \) and \( V' \), as in Koopmans (1960), Uzawa (1968) and Lucas and Stokey (1984). Of particular interest is how the discount factor varies across potential steady states. Define \( \bar{U}(V) \) as the solution to \( V = W(\bar{U}(V), V) \) and let \( \bar{\beta}(V) \equiv W_V(\bar{U}(V), V) \) denote the steady state discount factor. It will prove useful below to note that the strict monotonicity of \( \mathcal{V} \) immediately implies that \( \bar{\beta}(V) \in (0, 1) \) at any steady state with utility \( V \).

**Technology.** The economy is subject to the sequence of resource constraints

\[
c_t + k_{t+1} + g_t \leq F(k_t, n_t) + (1 - \delta)k_t \quad t = 0, 1, \ldots \tag{3.6}
\]

where \( F \) is a concave, differentiable and constant returns to scale production function taking as inputs labor \( n_t \) and capital \( k_t \), and the parameter \( \delta \in [0, 1] \) is the depreciation rate of capital. The sequence for government consumption, \( \{g_t\} \), is given exogenously.

**Markets and Taxes.** Labor and capital markets are perfectly competitive, yielding before tax wages and rates of return given by \( w^*_t = F_n(k_t, n_t) \) and \( R^*_t = F_k(k_t, n_t) + 1 - \delta \).

---

\(^{28}\)A positive marginal change \( dU \) in the constant per period utility stream increases steady state utility by some constant \( d\mathcal{V} \). By virtue of (3.5a) this implies \( d\mathcal{V} = W_U dU + W_V dV \), which yields a contradiction unless \( W_V < 1 \).
The agent maximizes utility subject to the sequence of budget constraints

\[ \begin{align*}
    c_0 + a_1 & \leq w_0 n_0 + R_0 k_0 + R_0^b b_0, \\
    c_t + a_{t+1} & \leq w_t n_t + R_t a_t \quad t = 1, 2, \ldots,
\end{align*} \]

and the No Ponzi condition \( \frac{a_{t+1}}{R_t R_{t+1} \ldots R_0} \to 0 \). The agent takes as given the after-tax wage \( w_t \) and the after-tax gross rates of return, \( R_t \). Total assets \( a_t = k_t + b_t \) are composed of capital \( k_t \) and government debt \( b_t \); with perfect foresight, both must yield the same return in equilibrium for all \( t = 1, 2, \ldots \), so only total wealth matters for the agent; this is not true for the initial period, where we allow possibly different returns on capital and debt. The after-tax wage and return relate to their before-tax counterparts by \( w_t = (1 - \tau_t^a) w_t^* \) and \( R_t = (1 - \tau_t)(R_t^* - 1) + 1 \) (here it is more convenient to work with a tax rate on net returns than on gross returns).

Importantly, we follow Chamley (1986) and allow for an indirect constraint on the capital tax rate given by \( R_t \geq 1 \). For positive before-tax interest rates \( R^* - 1 \) this is precisely equivalent to assuming \( \tau_t \leq 1 \). As is well understood, without constraints on capital taxation the solution involves extraordinarily high initial capital taxation, typically complete expropriation, unless the first best is achieved first. Taxing initial capital mimics the missing lump-sum tax, which has no distortionary effects. We note that our main result in this section, Proposition 17, does not depend on the specific form of the capital tax constraint.

**Planning problem.** The implementability condition for this economy is

\[ \sum_{t=0}^{\infty} (V_{ct} c_t + V_{nt} n_t) = V_{c0} \left( R_0 k_0 + R_0^b b_0 \right), \quad (3.7) \]

whose derivation is standard. In the additive separable utility case \( V_{ct} = \beta^t U_{ct} \) and \( V_{nt} = \beta^t U_{nt} \) and expression (3.7) reduces to the standard implementability condition popularized by Lucas and Stokey (1983) and Chari et al. (1994). Given \( R_0 \) and \( R_0^b \), any allocation satisfying the implementability condition and the resource constraint (3.6) can be sustained as a competitive equilibrium for some sequence of prices and taxes.\(^{30}\)

\(^{29}\)When \( R^* - 1 \) is negative, however, an upper bound directly imposed on taxes \( \tau_t \) allows arbitrarily low after-tax interest rates \( R_t \).

\(^{30}\)The argument is identical to that in Lucas and Stokey (1983) and Chari et al. (1994).
To enforce the constraints on the taxation of capital in periods \( t = 1, 2, \ldots \) we impose

\[
V_{ct} = R_{t+1} V_{ct+1}, \tag{3.8a}
\]

\[
R_t \geq 1. \tag{3.8b}
\]

The planning problem maximizes \( V(U_0, U_1, \ldots) \) subject to (3.6), (3.7) and (3.8). In addition, we take \( R_0^b \) as given. The constraint \( R_t \geq 1 \) may or may not bind forever. In this subsection we are interested in situations where the constraint does not bind asymptotically, i.e. it is slack after some date \( T < \infty \). In the next subsection we discuss the possibility of the constraint binding forever.

Chamley (1986) provided the following result—slightly adjusted here to make explicit the need for the steady state to be interior, for multipliers to converge and for the bounds on taxation to be asymptotically slack.

**Theorem 3** (Chamley, 1986, Theorem 1). Suppose the optimum converges to an interior steady state where the constraints on capital taxation are asymptotically slack. Let \( \tilde{\Lambda}_t = V_{ct} \Lambda_t \) denote the multiplier on the resource constraint (3.6) in period \( t \). Suppose further that the multiplier \( \Lambda_t \) converges to an interior point \( \Lambda_t \to \Lambda > 0 \). Then the tax on capital converges to zero \( \frac{R_t}{R^*_t} \to 1 \).

The proof is straightforward. Consider a sufficiently late period \( t \), so that the bounds on the capital tax rate are no longer binding. Then the first-order condition for \( k_{t+1} \) includes only terms from the resource constraint (3.6) and is simply \( \tilde{\Lambda}_t = \tilde{\Lambda}_{t+1} R_{t+1}^* \). Equivalently, using that \( \tilde{\Lambda}_t = V_{ct} \Lambda_t \) we have

\[
V_{ct} \Lambda_t = V_{ct+1} \Lambda_{t+1} R_{t+1}^*.
\]

On the other hand the representative agent’s Euler equation (3.8a) is

\[
V_{ct} = V_{ct+1} R_{t+1}.
\]

The result follows from combining these last two equations.

With the specific constraint \( R_t \geq 1 \) on capital taxation assumed here and in Chamley (1986), there would be no need to require the constraints on capital taxation not to bind. The reason is that in this case the constraints imposed by (3.8) do not involve \( k_{t+1} \), so the argument above goes through unchanged. In fact, this is essentially the form that Theorem 1 in Chamley (1986) takes, although the assumption of converging multipliers is not stated explicitly, but imposed within the proof. We chose to explicitly assume the capital tax constraints to be no longer binding to allow a broader applicability of the theorem to situations without the
specific constraints in (3.8).\textsuperscript{31}

The main result of this subsection is stated in the next proposition. Relative to Theorem 3, we make no assumptions on multipliers and prove that the steady-state tax rate is zero. More importantly, we derive new implications of reaching an interior steady state.

**Proposition 17.** Suppose the optimal allocation converges to an interior steady state and assume the bounds on capital tax rates are asymptotically slack. Then the tax on capital is asymptotically zero. In addition, if the discount factor is locally non-constant at the steady state, so that $\beta'(V) \neq 0$, then either

1. private wealth converges to zero, $a_t \to 0$; or

2. the allocation converges to the first-best, with a zero tax rate on labor.

This result shows that at any interior steady state where the bounds on capital taxes do not bind, the tax on capital is zero; this much basically echoes Chamley (1986), or our rendering in Theorem 3. However, as long as the rate of impatience is not locally constant, so that $\beta'(V) \neq 0$, the proposition also shows that this zero tax result comes with other implications. There are two possibilities. In the first possibility, the capital income tax base has been driven to zero—perhaps as a result of heavy taxation along the transition. In the second possibility, the government has accumulated enough wealth—perhaps aided by heavy taxation of wealth along the transition—to finance itself without taxes, so the economy attains the first best. Thus, capital taxes are zero, but the same is true for labor taxes.

To sum up, if the economy converges to an interior steady state, then either both labor and capital are treated symmetrically or there remains no wealth to be taxed. Both of these implications do not sit well with the usual interpretation of the zero capital tax result. To be sure, in the special (but commonly adopted) case of additive separable utility one can justify the usual interpretation where private wealth is spared from taxation and labor bears the entire burden. However, this is no longer possible when the rate of impatience is not constant. In this sense, the usual interpretation describes a knife edged situation.

### 3.3.2 Long Run Capital Taxes Binding at Upper Bound

We now show that the bounds on capital tax rates may bind forever, contradicting a claim by Chamley (1986). This claim has been echoed throughout the literature, e.g. by Judd (1999), Atkeson et al. (1999) and others.

\textsuperscript{31}Note that as long as the multiplier $\Lambda_t$ converges, one does not even need to assume the allocation converges to arrive at the zero-tax conclusion. This is essentially the argument used by Judd (1999). However, the problem is that one cannot guarantee that the multiplier converges. We shall discuss this in subsection 3.3.3.
For our present purposes, and following Chamley (1986) and Judd (1999), it is convenient to work with a continuous-time version of the model and restrict attention to additively separable preferences,\(^{32}\)

\[
\int_0^\infty e^{-\rho t} U(c_t, n_t) dt.
\]

\[
U(c, n) = u(c) - v(n) \quad \text{with} \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad v(n) = \frac{n^{1+\zeta}}{1+\zeta},
\]

where \(\sigma, \zeta > 0\). Following Chamley (1986), we adopt an iso-elastic utility function over consumption; this is important to ensure the bang-bang nature of the solution. We also assume iso-elastic disutility from labor, but we believe similar results to ours can be shown for arbitrary convex disutility functions \(v(n)\). The resource constraint is

\[
c_t + k_t + g = f(k_t, n_t) - \delta k_t,
\]

where \(f\) has constant returns to scale with \(f(0, n) = f(k, 0) = 0\), is differentiable and strictly concave in each argument, and satisfies the usual Inada conditions. For simplicity, government consumption is taken to be constant at \(g > 0\). We denote the before-tax net interest rate by \(r_t^* = f_k(k_t, n_t) - \delta\). The implementability condition is now

\[
\int_0^\infty e^{-\rho t} (u'(c_t)c_t - v'(n_t)n_t) = u'(c_0)a_0,
\]

where \(a_0 = k_0 + b_0\) denotes initial private wealth, consisting of capital \(k_0\) and government bonds \(b_0\). Since unrestricted subsidies to capital act as lump-sum tax when initial private wealth is negative, we focus on the case where initial private wealth is positive, \(a_0 > 0\).\(^{33}\) To enforce bounds on capital taxation we follow Chamley (1986) and impose

\[
\dot{\theta}_t = \theta_t(\rho - r_t), \quad (3.12a)
\]

\[
r_t \geq 0, \quad (3.12b)
\]

where \(\theta_t = u'(c_t)\) denotes the marginal utility of consumption, and \(r_t\) denotes the after-tax interest rate. Whenever the before-tax return on capital \(r_t^* = f_k(k_t, n_t) - \delta\) is positive, constraint (3.12b) corresponds to a capital tax constraint \(\tau_t = 1 - r_t/r_t^* \leq \bar{r}\) with \(\bar{r} \equiv 1\). The

\(^{32}\)Continuous time allowed Chamley (1986) to exploit the bang-bang nature of the optimal solution. Since we focus on cases where this is not the case it is less crucial for our results. However, we prefer to keep the analyses comparable.

\(^{33}\)Observe that, in Proposition 18, \(a_0 > 0\) is always satisfied if \(b_0 \in [b, \bar{b}]\) for \(\sigma > 1\), so the focus on \(a_0 > 0\) does not affect our main result.
planning problem maximizes (3.9a) subject to (3.10), (3.11) and (3.12).

Chanley (1986, Theorem 2, pg. 615) formulated the following claim regarding the path for capital tax rates.\(^{34}\)

**Claim.** There exists a time \(T\) with the following three properties:

1. for \(t < T\), the constraint (3.12b) is binding, i.e. \(r_t = 0\) and \(\tau_t = 1\);

2. for \(t > T\) capital income is untaxed, i.e. \(r_t = r^*_t\) and \(\tau_t = 0\);

3. \(T < \infty\).

At a crucial juncture in the proof of this claim, Chamley (1986) states in support of part 3 that “The constraint \(r_t \geq 0\) cannot be binding forever (the marginal utility of private consumption [...] would grow to infinity [...] which is absurd).”\(^{35}\) Our next result shows that there is nothing absurd about this within the logic of the model and that, quite to the contrary, part 3 of the above claim is incorrect: indefinite taxation, \(T = \infty\), may be optimal.

Our result depends on the tax burden the government needs to impose on the agent. As measure of the tax burden, we use initial government debt \(b_0\). As with a regular, static “Laffer curve” there exists a maximum burden of taxes agents can finance, here given by a threshold level for initial government debt, \(\bar{b}\). When \(b_0 > \bar{b}\), no feasible allocation exists, while there are always feasible allocations if \(b_0 < \bar{b}\). It would seem natural that at the maximum of the Laffer curve, that is, when \(b_0 = \bar{b}\), capital taxation can be at its upper bound indefinitely. This is indeed the case, as we characterize in the following proposition. Crucially, however, capital taxes can even be at their upper bound indefinitely when \(b_0 < \bar{b}\):

**Proposition 18.** Suppose preferences are given by (3.9). Fix any initial capital stock \(k_0 > 0\) and assume initial private wealth \(k_0 + b_0\) is positive.

Then, the bang-bang property holds, that is, at any optimum of the planning problem, there exists a time \(T \in [0, \infty]\), such that capital taxes \(\tau_t\) are binding at their upper bound \(\bar{r} = 1\) until \(t = T\) and are zero thereafter. Whenever the economy is not at its first best, \(T\) is positive. Moreover:

A. For \(\sigma > 1\):

(a) If \(b_0 = \bar{b}\), the unique optimum exhibits \(T = \infty\) and there is no feasible allocation with \(T < \infty\).

\(^{34}\)Similar claims are made in Atkeson et al. (1999), Judd (1999) and many other papers.

\(^{35}\)It is worth pointing out, however, that although Chamley (1986) claims \(T < \infty\) it never states that \(T\) is small. Indeed, it cautious to the possibility that it is quite large saying “the length of the period with capital income taxation at the 100 per cent rate can be significant.”
(b) If $b_0 \in [\underline{b}, \overline{b})$, the unique optimum exhibits $T = \infty$ but there are feasible allocations with $T < \infty$.

(c) If $b_0 < \underline{b}$, $T < \infty$, any optimum exhibits $T < \infty$.

B. For $\sigma = 1$:

(a) If $b_0 = \overline{b}$, the unique optimum exhibits $T = \infty$ and there is no feasible allocation with $T < \infty$.

(b) If $b_0 < \overline{b}$, any optimum exhibits $T < \infty$.

C. For $\sigma < 1$: Any optimum exhibits $T < \infty$.

Proposition 18 offers a full characterization of the optimal capital tax policy in this economy. First, we prove a bang-bang property of capital taxes, according to which capital taxes are binding at their upper bound, $\tau_t = 1$, until some time $T$ and drop to zero thereafter. It turns out that previous proofs of the bang-bang property (see, e.g., Chamley (1986) or Atkeson et al. (1999)) heavily relied on the false premise that capital taxes cannot be positive forever. We provide a new proof that avoids this issue.

Using the bang-bang property of capital taxes, we then characterize optimal capital taxes, distinguishing by the position of $\sigma$ relative to 1. For $\sigma > 1$, we prove that it is optimal to tax capital indefinitely for a positive-measure interval of $b_0$. Crucially, for $b_0 < \overline{b}$ indefinite taxation is not driven by budgetary need—there are feasible plans with $T < \infty$; however, the plan with $T = \infty$ is simply better. This is illustrated in Figure 3-1 with a qualitative plot of the set of states $(k_0, b_0)$ for which indefinite capital taxation is optimal if $\sigma > 1$. By contrast, for $\sigma < 1$ we show that at any optimum, $T < \infty$, so $T = \infty$ is never optimal. The case $\sigma = 1$ lies in between, in that $T = \infty$ is optimal only if $b_0 = \overline{b}$.
The basic idea behind our proof of part A of Proposition 18 is simple. To illustrate it, let \( \lambda_t \) denote the multiplier on the resource constraint (3.10) at time \( t \) and \( \mu \) be the multiplier on the IC constraint (3.11). Both can be proven to be non-negative. Using this notation, if the period \( T \) of positive capital taxation is finite, the first order condition for consumption \( c_t \) after time \( T \) reads

\[
\lambda_t = (1 - \mu(\sigma - 1))u'(c_t),
\]

which requires \( \mu \leq 1/(\sigma - 1) \). Yet, as initial government debt \( b_0 \) becomes large, \( b_0 > b \), so does \( \mu \), to the point where it crosses \( 1/(\sigma - 1) \), making it impossible for finite capital taxation to be optimal. Therefore, a sufficiently large burden of taxation due to high \( b_0 \), coupled with an intertemporal elasticity \( \sigma^{-1} \) less than \( 1 \) points to indefinite capital taxation. To make this approach watertight, we specifically construct allocations with \( T = \infty \) and show that they satisfy the first order conditions whenever \( b_0 \geq b \). Since, as we show, the planning problem can be recast into a concave maximization problem, the first order conditions (together with transversality conditions) are sufficient for an optimum.

Our next result assumes \( g = 0 \) and constructs the solution for a set of initial conditions that allow us to guess and verify its form.

**Proposition 19.** Suppose that preferences are given by (3.9) with \( \sigma > 1 \), and that \( g = 0 \). There exist \( k < k \) and \( b_0(k_0) \) such that: for any \( k_0 \in (k, k] \) and initial debt \( b_0(k_0) \) the optimum satisfies \( \tau_t = 1 \) for all \( t \geq 0 \) and \( c_t, k_t, n_t \to 0 \) exponentially with constant \( n_t/k_t \) and \( c_t/k_t \).

Under the conditions stated in the proposition the solution converges to zero in a homogeneous, constant growth rate fashion. This explicit example illustrates that convergence takes place, but not to an interior steady state. It turns out that this latter property is more general: at least with additively separable utility, whenever indefinite taxation of capital is optimal, \( T = \infty \), no interior steady state exists, even if capital taxes are constrained by tax bounds \( \bar{r} < 1 \), that is, if we impose \( r_t \geq r^*_t (1 - \bar{r}) \).

To see why this is the case consider first the case with \( \bar{r} = 1 \). Then the after tax interest rate is zero whenever the bound is binding. Since the agent discounts the future positively this prevents a steady state. In contrast, when \( \bar{r} < 1 \) the before-tax interest rate may be positive and the after tax interest rate equal to the discount rate, \( (1 - \bar{r})r^* = \rho \), the condition for constant consumption. This suggests the possibility of a steady state. However, we must also verify whether labor, in addition to consumption, remains constant. This, in turn, requires a constant labor tax. Yet, one can show that under the assumptions of Proposition 18, but allowing \( \bar{r} < 1 \), we must have

\[
\partial_t r^n_t = (1 - r^n_t)\tau_t r_t^*,
\]
implying that the labor tax strictly rises over time whenever the capital tax is positive, $\tau_t > 0$. This rules out an interior steady state. Intuitively, the capital tax inevitably distorts the path for consumption, but the optimum attempts to undo the intertemporal distortion in labor by varying the tax on labor. We conjecture that the imposition of an upper bound on labor taxes solves the problem of an ever-increasing path for labor taxes, leading to the existence of interior steady states with positive capital taxation.

3.3.3 Revisiting Judd (1999)

Up to this point we have focused on the Chamley-Judd zero-tax results. A follow-up literature has offered both extensions and interpretations. One notable case doing both is Judd (1999). This paper is related to Chamley (1986) in that it studies a representative agent economy with perfect financial markets and unrestricted government bonds. It also allows for other state variables, such as human capital, and in that sense builds on Judd (1985, Section 5) and Jones et al. (1993). At its core, Judd (1999) provides a zero capital tax result without requiring the allocation to converge to a steady state. The paper also offers a connection between capital taxation and rising consumption taxes to provide an intuition for zero-tax results. Let us consider each of these two points in turn.

Bounded Multipliers and Zero Average Capital Taxes. Abstracting away from some of the additional ingredients in Judd (1999), the essence of the main result in Judd (1999) can be restated using our continuous-time setup from Section 3.3.2. With $\bar{r} = 1$, the planning problem maximizes (3.9a) subject to (3.10), (3.11), (3.12a), and (3.12b). Let $\Lambda_t = \theta_t \lambda_t$ denote the co-state for capital, that is, the current value multiplier on equation (3.10), satisfying

$$\dot{\Lambda}_t = \rho \Lambda_t - \tau_t^* \Lambda_t.$$ 

Using that $\dot{\Lambda}_t / \Lambda_t = \hat{\theta}_t / \theta_t + \Lambda_t / \Lambda_t$ and $\hat{\theta}_t / \theta_t = \rho - \tau_t$ we obtain

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = r_t - \tau_t^*.$$ 

If $\Lambda_t$ converges then $r_t - \tau_t^* \to 0$. Thus, the Chamley (1986) steady state result actually follows by postulating the convergence of $\Lambda_t$, without assuming convergence of the allocation. Judd (1999, pg. 13, Theorem 6) goes down this route, but assumes that the endogenous multiplier $\Lambda_t$ remains in a bounded interval, instead of assuming that it converges.

**Theorem 4 (Judd, 1999).** Let $\theta_t \lambda_t$ denote the (current value) co-state for capital in equation (3.10) and assume

$$\Lambda_t \in [\bar{\Lambda}, \bar{\Lambda}],$$

220
for $0 < A < \bar{A} < \infty$. Then the cumulative distortion up to $t$ is bounded,

$$\log \left( \frac{A_0}{\Lambda} \right) \leq \int_0^t (r_s - r^*_s)ds \leq \log \left( \frac{A_0}{\bar{\Lambda}} \right),$$

and the average distortion converges to zero,

$$\frac{1}{t} \int_0^t (r_s - r^*_s)ds \to 0.$$

In particular, under the conditions of this theorem, the optimum cannot converge to a steady state with a positive tax on capital. More generally, the condition requires departures of $r_t$ from $r^*_t$ to average zero.

Note that our proof proceeded without any optimality condition except the one for capital $k_t$. In particular, we did not invoke first-order conditions for the interest rate $r_t$ nor for the tax rate on capital $\tau_t$. Naturally, this poses two questions. Do the bounds on $A_t$ essentially assume the result? And are the bounds on $A_t$ consistent with an optimum?

Regarding the first question, we can say the following. The multiplier $e^{-\rho t} \hat{A}_t$ represents the planner’s (time 0) social marginal value of resources at time $t$. Thus,

$$\text{MRS}_{t,t+s}^{\text{Social}} = e^{-\rho s} \frac{\hat{A}_{t+s}}{\Lambda_t} = e^{-\int_0^s \tau_{t+s} d\bar{s}},$$

represents the marginal rate of substitution between $t$ and $t+s$, which, given the assumption $\bar{\tau} = 1$, is equated to the marginal rate of transformation. The private agent’s marginal rate of substitution is

$$\text{MRS}_{t,t+s}^{\text{Private}} = e^{-\rho s} \frac{\theta_{t+s}}{\theta_t} = e^{-\int_0^s \tau_{t+s} d\bar{s}},$$

where $\theta_t$ represents marginal utility. It follows, by definition, that

$$\text{MRS}_{t,t+s}^{\text{Social}} = \frac{\Lambda_{t+s}}{\Lambda_t} \cdot \text{MRS}_{t,t+s}^{\text{Private}}.$$

This expression shows that the rate of growth in $\Lambda_t$ is, by definition, equal to the wedge between social and private marginal rates of substitution. Thus, the wedge $\frac{\Lambda_{t+s}}{\Lambda_t} = e^{\int_0^s (r_{t+s} - \tau_{t+s}) d\bar{s}}$ is the only source of nonzero taxes. Whenever $\Lambda_t$ is constant, social and private MRSs coincide.

36 The result is somewhat sensitive to the assumption that $\bar{\tau} = 1$; when $\bar{\tau} \neq 1$ and technology is nonlinear, the co-state equation acquires other terms, associated with the bounds on capital taxation.

37 In this continuous time optimal control formulation, the costate equation for capital is the counterpart to the first-order condition with respect to capital in a discrete time formulation. Indeed, the same result can be easily formulated in a discrete time setting.
and the intertemporal wedge is zero, \( r_t = r_t^* \); if \( \Lambda_t \) is enclosed in a bounded interval, the same conclusion holds on average.

These calculations afford an answer to the first question posed above: assuming the (average) rate of growth of \( \Lambda_t \) is zero is tantamount to assuming the (average) zero long-run tax conclusion. We already have an answer to the second question, whether the bounds are consistent with an optimum, since Proposition 18 showed that indefinite taxation may be optimal.

**Corollary.** *At the optimum described in Proposition 18 we have that \( \Lambda_t \rightarrow 0 \) as \( t \rightarrow \infty \). Thus, in this case the assumption on the endogenous multiplier \( \Lambda_t \) adopted in Judd (1999) is violated.*

There is no guarantee that the endogenous object \( \Lambda_t \) remains bounded away from zero, as assumed by Judd (1999), making Theorem 4 inapplicable.

**Exploding Consumption Taxes.** Judd (1999) also offers an intuitive interpretation for the Chamley-Judd result based on the observation that an indefinite tax on capital is equivalent to an ever-increasing tax on consumption. This casts indefinite taxation of capital as a villain, since rising and unbounded taxes on consumption appear to contradict standard commodity tax principles, as enunciated by Diamond and Mirrlees (1971), Atkinson and Stiglitz (1972) and others.

The equivalence between capital taxation and a rising path for consumption taxes is useful. It explains why prolonging capital taxation comes at an efficiency cost, since it distorts the consumption path. If the marginal cost of this distortion were increasing in \( T \) and approached infinity as \( T \rightarrow \infty \) this would give a strong economic rationale against indefinite taxation of capital. We now show that this is not the case: the marginal cost remains bounded, even as \( T \rightarrow \infty \). This explains why a corner solution with \( T = \infty \) may be optimal.

We proceed with a constructive argument and assume, for simplicity, that technology is linear, so that \( f(k,n) = \delta k = r^*k + w^*n \) for fixed parameters \( r^*, w^* > 0 \).

**Proposition 20.** *Suppose utility is given by (3.9), with \( \sigma > 1 \). Suppose technology is linear. Then the solution to the planning problem can be obtained by solving to the following static problem:*

\[
\begin{align*}
\max_{T,c,n} & \quad u(c) - v(n), \\
\text{s.t.} & \quad (1 + \psi(T))c + G = k_0 + \omega n, \\
& \quad u'(c)c - v'(n)n = (1 - \tau(T))u'(c)\alpha_0,
\end{align*}
\]
where $\omega > 0$ is proportional to $w^*$; $G$ is the present value of government consumption; and, $c$ and $n$ are measures of lifetime consumption and labor supply, respectively. The functions $\psi$ and $\tau$ are increasing with $\psi(0) = \tau(0) = 0$; $\psi$ is bounded away from infinity and $\tau$ is bounded away from $1$. Moreover, the marginal trade-off between costs ($\psi$) and benefits ($\tau$) from extending capital taxation

$$\frac{d\psi}{d\tau} = \frac{\psi'(T)}{\tau'(T)}$$

is bounded away from infinity.

Given $c$, $n$ and $T$ we can compute the paths for consumption $c_t$ and labor $n_t$. Behind the scenes, the static problem solves the dynamic problem. In particular, it optimizes over the path for labor taxes. In this static representation, $1 + \psi(T)$ is akin to a production cost of consumption and $\tau(T)$ to a non-distortionary capital levy. On the one hand, higher $T$ increases the efficiency cost from the consumption path. On the other hand, it increases revenue in proportion to the level of initial capital. Prolonging capital taxation requires trading off these costs and benefits.

Importantly, despite the connection between capital taxation and an ever increasing, unbounded tax on consumption, the proposition shows that the tradeoff between costs and benefits is bounded, $\frac{d\psi}{d\tau} < \infty$, even as $T \to \infty$. In other words, indefinite taxation does not come at an infinite marginal cost and helps explain why this may be optimal.

Should we be surprised that these results contradict commodity tax principles, as enunciated by Diamond and Mirrlees (1971), Atkinson and Stiglitz (1972) and others? No, not at all. As general as these frameworks may be, they do not consider upper bounds on taxation, the crucial ingredient in Chamley (1986) and Judd (1999). Their guiding principles are, therefore, ill adapted to these settings. In particular, formulas based on local elasticities do not apply, without further modification.

Effectively, a bound on capital taxation restricts the path for the consumption tax to lie below a straight line going through the origin. In the short run, the consumption tax is constrained to be near zero; to compensate, it is optimal to set higher consumption taxes in the future. As a result, it may be optimal to set consumption taxes as high as possible at all times. This is equivalent to indefinite capital taxation.

### 3.4 A Hybrid: Redistribution and Debt

Throughout this chapter we have strived to stay on target and remain faithful to the original models supporting the Chamley-Judd result. This is important so that our own results are
easily comparable to those in Judd (1985) and Chamley (1986). However, many contributions since then offer modifications and extensions of the original Chamley-Judd models and results. In this section we depart briefly from our main focus to show that our results transcend their original boundaries and are relevant to this broader literature.

To make this point with a relevant example, we consider a hybrid model, with redistribution between capitalists and workers as in Judd (1985), but sharing the essential feature in Chamley (1986) of unrestricted government debt. It is very simple to modify the model in Section 3.2 in this way. We add bonds to the wealth of capitalists \( a_t = k_t + b_t \), modifying equation (3.1c) to

\[
\beta U'(C_t)(C_t + k_{t+1} + b_{t+1}) = U'(C_{t-1})(k_t + b_t)
\]

and the transversality condition to \( \beta' U'(C_t)(k_{t+1} + b_{t+1}) \rightarrow 0 \). Together, these two conditions imply a present value implementability condition, which with \( U(C) = C^{1-\sigma}/(1 - \sigma) \) and initial returns on capital and bonds of \( R_0 \) and \( R_b^0 \) is given by

\[
(1 - \sigma) \sum_{t=0}^{\infty} \beta^t U(C_t) = U'(C_0)(R_0k_0 + R_b^0b_0).
\]

### Anticipated Confiscatory Taxation

For \( \sigma > 1 \) the left hand side in equation (3.14) is decreasing in \( C_t \) and the right hand side is decreasing in \( C_0 \). In particular, the values of \( C_t \), for all \( t = 0, 1, \ldots \), can be set infinitesimally small without violating (3.14). Since (3.14) is strictly speaking not defined for \( C_t = 0 \), the problem without weight on capitalists (\( \gamma = 0 \)) has a supremum that can only be approximated as \( C_t \rightarrow 0 \). Given \( \sigma > 1 \), this limit can be implemented by making \( R_t \) infinitesimally small in some period \( t \geq 1 \), or, equivalently, setting the wealth tax (i.e. tax on gross returns) \( T_t \) in that period arbitrarily close to 100%. This same logic applies if the tax is temporarily restricted for periods \( t \leq T - 1 \) for some given \( T \), but is unrestricted in period \( T \).

### Proposition 21

Consider the two-class model from Section 3.2 but with unrestricted government bonds. Suppose \( \sigma > 1 \) and \( \gamma = 0 \). If capital taxation is unrestricted in at least one period, then the optimum (a supremum) features a wealth tax \( T_t \rightarrow 100\% \) in some period \( t \) and \( C_t \rightarrow 0 \) for all \( t = 0, 1, \ldots \).

This result exemplifies how extreme the tax on capital may be without bounds. In addition to this result, even when \( \sigma < 1 \), if no constraints are imposed on taxation except at \( t = 0 \), then in the continuous time limit as the length of time periods shrinks to zero, taxation tends to infinity. This point was also raised in Chamley (1986) for the representative agent Ramsey model, and served as a motivation for imposing a stationary constraint, \( R_t \geq 1 \).
Long Run Taxation with Constraints. We now impose upper bounds on capital taxation and show that these constraints may bind forever, just as in Section 3.3.2.

**Proposition 22.** Consider the two-class model from Section 3.2 but with unrestricted government bonds. Suppose $\sigma > 1$ and $\gamma = 0$. If capital taxation is restricted by the constraint $R_t \geq 1$, then at the optimum $R_t = 1$ in all periods $t$, i.e. capital should be taxed indefinitely.

Intuitively, $\sigma > 1$ is enough to ensure indefinite taxation of capital in this model because $\gamma = 0$ makes it optimal to tax capitalists as much as possible. Similar results hold for positive but low enough levels of $\gamma$, so that redistribution from capitalists to workers is desired. The results also hold for less restrictive constraints than $R_t \geq 1$.

Proposition 22 assumes that transfers are perfectly targeted to workers and capitalists do not work. However, indefinite taxation, $T = \infty$, is also possible when these assumptions are relaxed, so that capitalists work and receive equal transfers.

We have also maintained the assumption from Judd (1985) that workers do not save. In a political economy context, Bassetto and Benhabib (2006) study a situation where all agents save (in our context, both workers and capitalists) and are taxed linearly at the same rate. Indeed, they report the possibility that indefinite taxation is optimal for the median voter.

Overall, these results suggest that indefinite taxation can be optimal in a range of models that are descendants of Chamley-Judd, with a wide range of assumptions regarding the environment, heterogeneity, social objectives and policy instruments.

### 3.5 Conclusions

This study revisited two closely related models and results, Chamley (1986) and Judd (1985). Our findings contradict well-established results and their standard interpretations. We showed that, provided the intertemporal elasticity of substitution (IES) is less than one, the long run tax on capital can actually be positive. Empirically, an IES below one is considered most plausible.

Why were the proper conclusions missed by Judd (1985), Chamley (1986) and many others? Among other things, these papers assume that the endogenous multipliers associated with the planning problem converge. Although this seems natural, we have shown that this is not necessarily true at the optimum. In fact, on closer examination it is evident that presuming the convergence of multipliers is equivalent to the assumption that the intertemporal rates of substitution of the planner and the agent are equal. This then implies that no intertemporal distortion or tax is required. Consequently, analyses based on these assumptions amount to little more than assuming zero long-run taxes.
In quantitative evaluations, it may well be the case that one finds a zero long-run tax on capital, e.g. for the model in Judd (1985) one may set $\sigma < 1$, and in Chamley (1986) the bounds may not bind forever if debt is low enough.\footnote{Any quantitative exercise could also evaluate the welfare gains from different policies. For example, even when $T < \infty$ is optimal, the optimal value of $T$ may be very high and indefinite taxation, $T = \infty$, may closely approximate the optimum. One can also compare various non-optimal simple policies, such as never taxing capital versus always taxing capital at a fixed rate.} In this chapter we refrain from making any such claim, one way or another. We confined our attention to the original theoretical zero-tax results, widely perceived as delivering ironclad conclusions that are independent of parameter values or initial conditions. Based on our results, we have found little basis for such an interpretation.
Appendix

3.A Recursive Formulation of (3.1a)

In our numerical simulations, we use a recursive representation of the Judd (1985) economy. The two constraints in the planning problem feature the variables \(C_{t-1}, k_t, C_t, k_{t+1}\) and \(c_t\). This suggests a recursive formulation with \((k_t, C_{t-1})\) as the state and \(c_t\) as a control. The associated Bellman equation is then

\[
V(k, C_-) = \max_{c \geq 0, (k', C) \in A} \{u(c) + \gamma U(C) + \beta V(k', C)\}
\]

\(c + C + k' + g = f(k) + (1 - \delta)k\)

\(\beta U''(C)(C + k') = U'(C_-)k\)

\(c, C, k' \geq 0.\)

Here, \(A\) is the feasible set, that is, states \((k_0, C_{-1})\) such that there exists a sequence \(\{k_{t+1}, C_t\}\) satisfying all the constraints in (3.1) including the transversality condition. At \(t = 0\), capital \(k_0\) is given, so there is no need to impose \(\beta U'(C_0)(C_0 + k_1) = U'(C_{-1})k_0\). Thus, the planner maximizes \(V(k_0, C_{-1})\) with respect to \(C_{-1}\). If \(V\) is differentiable, the first order condition is

\[V_C(k_0, C_{-1}) = 0.\]

Since one can show that \(\mu_t = V_C(k_t, C_{t-1})U''(C_{t-1})k_t\), this is akin to the condition \(\mu_0 = 0\) in equation (3.2a). \(^{39}\)

\(^{39}\)Alternatively, we may impose that \(R_0\) is taken as given, with \(R_0 = R_0^*\) for example, to exclude an initial capital tax. In that case the planner solves

\[
\max_{k_1, c_0, C_0} \{u(c_0) + \gamma U(C_0) + \beta V(k_1, C_0)\}
\]
3.B Proof of Proposition 14

The proof of Proposition 14 consists of three parts. In the first part, we provide a few definitions that are necessary for the proof. In particular, we define the feasible set of states. In the second part, we characterize the feasible set of states geometrically. The proofs for the results in that part are somewhat cumbersome and lengthy, so they are relegated to the end of this section to ensure greater readability. Finally, in the third part, we use our geometric results to prove Proposition 14. Readers interested only in the main steps of the proof are advised to jump straight to the third part.

3.B.1 Definitions

For the proof of Proposition 14 we make a number of definitions, designed to simplify the exposition. A state \((k, C)\) as in the recursive statement (3.15) of problem (3.1a) will sometimes be abbreviated by \(z\), and a set of states by \(Z\). The total state space is denoted by \(Z_{all} \subset \mathbb{R}^2\) and is defined below. It will prove useful at times to express the set of constraints in (3.15) as

\[
\begin{align*}
  k' &= x - C - \left(\frac{\beta x}{k}\right)^{1/\sigma} \\
  C &= C - \left(\frac{\beta x}{k}\right)^{1/\sigma} \\
  C^{\sigma/(\sigma-1)} \left(\frac{\beta}{k}\right)^{1/(\sigma-1)} &\leq x \leq f(k) + (1 - \delta)k - g,
\end{align*}
\]

where \(x = k' + C\) replaces \(c = f(k) + (1 - \delta)k - g - x\) as control. In the last equation, the first inequality ensures non-negativity of \(k'\) while the second inequality is merely the resource constraint. Substituting out \(x\), we can also write the law of motion for capital as

\[
k' = \frac{1}{\beta} k \frac{C^\sigma}{C_0} - C,
\]

which we will be using below.

The whole set of future states \(z'\) which can follow a given state \(z = (k, C)\) is denoted by \(\Gamma(z)\), which can be the empty set. We will call a path \(\{z_t\}\) feasible if (a) \(z_{t+1} \in \Gamma(z_t)\) for all \(t \geq 0\), which precludes \(\Gamma(z_t)\) from being empty; and (b) if the transversality condition holds

subject to

\[
\begin{align*}
  C_0 + k_1 &= R_0 k_0 \\
  c_0 + C_0 + k_1 &= f(k_0) + (1 - \delta)k_0 \\
  c_0, C_0, k_1 &\geq 0.
\end{align*}
\]

This alternative gives rise to similar results.
along the path, \( \beta' C_t^{-\sigma} k_{t+1} \rightarrow 0 \). Similarly, a state \( z \) will be called \textit{feasible}, if there exists a feasible (infinite) path \( \{ z_t \} \) starting at \( z_0 = z \). In this case, \( z \) is \textit{generated by} \( \{ z_t \} \). Because \( z_1 \in \Gamma(z) \), we also say \( z \) is \textit{generated by} \( z_1 \). A \textit{steady state} \( z = (k, C_-) \in \mathbb{R}_+^2 \), is defined to be a state with \( C_- = (1 - \beta)/\beta k \). For very low and high capital levels \( k \), steady states turn out to be infeasible, but all others are \textit{self-generating}, \( z \in \Gamma(z) \), as we argue below. Similarly, a set \( Z \) is called \textit{self-generating} if every \( z \in Z \) is generated by a sequence in \( Z \). Denote by \( Z^* \) (= \( A \) in the notation above) the set of all feasible states. An integral part of the proof will be to characterize \( Z^* \).

It will be important to specify between which capital stocks the economy is moving. For this purpose, define \( k_g \) and \( k^g > k_g \) to be the two roots to the equation

\[
\begin{align*}
    k &= f(k) + (1 - \delta)k - g - \frac{1 - \beta}{\beta}k. \\
    \equiv f'(k)
\end{align*}
\]

Equation \( (3.17) \) was derived from the resource constraint, demanding that capitalists' consumption is at the steady state level of \( C = \frac{1 - \beta}{\beta} k \) and workers' consumption is equal to zero. Equation \( (3.17) \) need not have two solutions, not even a single one, in which case government consumption is unsustainably high for any capital stock. Such values for \( g \) are uninteresting and therefore ruled out. Corresponding to \( k_g \) and \( k^g \), we define \( C_g \equiv (1 - \beta)/\beta k_g \) and \( C^g \equiv (1 - \beta)/\beta k^g \) as the respective steady state consumption of capitalists. The steady states \((k_g, C_g)\) and \((k^g, C^g)\) represent the lowest and highest feasible steady states, respectively. The reason for this is that the steady state resource constraint \( (3.17) \) is violated for any \( k \not\in [k_g, k^g] \).

As in the Neoclassical Growth Model, the set of feasible states of this model is easily seen to allow for arbitrarily large capital stocks. This is why we cap the state space for high values of capital, and we take the total state space to be \( Z_{\text{all}} = [0, \bar{k}] \times \mathbb{R}_+ \) for states \((k, C_-)\), where \( \bar{k} \equiv \max\{k_{\max}, k_0\} \) and \( k = k_{\max} \) solves \( k = f(k) + (1 - \delta)k - g \). This way, the set of capital stocks that are resource feasible given an initial capital stock of \( k_0 \) must necessarily lie in the interval \([0, \bar{k}]\), so the restriction for \( \bar{k} \) is without loss of generality for any given initial capital stock \( k_0 \). Note that with this state space, the set of feasible states \( Z^* \) is also capped at \( \bar{k} \) in its \( k \)-component.

We now characterize the geometry of the set of feasible states \( Z^* \). The results derived there are essential for the actual proof of Proposition 14 in Section 3.B.3.
Figure 3.B.1: The state space of the Judd (1985) planning problem.

Note. This figure shows the two-dimensional state space of the Judd (1985) model. The entire state space is denoted by $Z_{\text{all}}$, which includes the feasible set $Z^*$ (between the two red curves), and all sets $Z_i$ (separated by the blue curves). The point $(k^*, C^*)$ is the zero-tax steady state. Showing that this is the qualitative shape of the feasible set $Z^*$ is an integral part of the proof of Proposition 14.

3. B.2 Geometry of $Z^*$

For better guidance through this section, we refer the reader to figure 3.B.1, which shows the typical shape of $Z^*$. The main results in this section are characterizations of the bottom and top boundaries of $Z^*$. We proceed by splitting up the state space, $Z_{\text{all}} = [0, \bar{k}] \times \mathbb{R}_+$, into four pieces and characterizing the feasible states in each of the four pieces.

Define

$$w_g(k) = \begin{cases} \frac{1-\beta}{\beta} k & \text{for } 0 \leq k \leq k_g \\ C_g \left( \frac{k}{k_g} \right)^{1/\sigma} & \text{for } k_g \leq k \leq \bar{k} \end{cases}$$

and

$$w^g(k) = \begin{cases} \frac{1-\beta}{\beta} k & \text{for } 0 \leq k \leq k^g \\ C^g \left( \frac{k}{k^g} \right)^{1/\sigma} & \text{for } k^g \leq k \leq \bar{k}, \end{cases}$$

and split up the state space as follows (see figure 3.B.1)

$$Z_{\text{all}} = \left\{ k < k_g, C_- \geq \frac{1-\beta}{\beta} k \right\} \cup \left\{ C_- < w_g(k) \right\} \cup \left\{ k \geq k_g, w_g(k) \leq C_- \leq w^g(k) \right\} \cup \left\{ k \geq k_g, C_- \geq w^g(k) \right\}.$$
Lemma 5 characterizes the feasible states in sets $Z_1$ and $Z_2$.

**Lemma 5.** $Z^* \cap Z_1 = Z^* \cap Z_2 = \emptyset$. All states with $k < k_g$ or $C_- < w_g(k)$ are infeasible.

**Proof.** See Subsection 3.B.4. \(\square\)

In particular, Lemma 5 shows that all states with $C_- < w_g(k)$ are infeasible. Lemma 6 below complements this result stating that all states with $w_g(k) \leq C_- \leq \bar{w}(k)$ (and $k \geq k_g$) in fact are feasible, that is, lie in $Z^*$. This means, $\{C_- = w_g(k), k \geq k_g\}$ constitutes the lower boundary of the feasible set $Z^*$.

**Lemma 6.** $Z_3 \subseteq Z^*$, or equivalently, all states with $w_g(k) \leq C_- \leq \bar{w}(k)$ and $k \geq k_g$ are feasible and generated by a feasible steady state. Moreover, states on the boundary $\{C_- = w_g(k), k > k_g\}$ can only be generated by a single feasible state, $(k_g, C_g)$. Thus, there is only a single “feasible” control for those states, $c > 0$.

**Proof.** See Subsection 3.B.4. \(\square\)

Lemma 6 finishes the characterization of all feasible states with $C_- \leq \bar{w}(k)$. What remains is a characterization of feasible states with $C_- > \bar{w}(k)$, or in terms of the $k - C_-$ diagram of Figure 3.B.1, the characterization of the red top boundary. This boundary is inherently more difficult than the bottom boundary because it involves states that are not merely one step away from a steady state. Rather, paths might not reach a steady state at all in finite time. The goal of the next set of lemmas is an iterative construction to show that the boundary takes the form of an increasing function $\bar{w}(k)$ such that states with $C_- > \bar{w}(k)$ are feasible if and only if $C_- \leq \bar{w}(k)$.

For this purpose, we need to make a number of new definitions: Let $\psi(k, C_-) \equiv (k+C_-)/C_-$. Applying the $\psi$ function to the successor $(k', C)$ of a state $(k, C_-)$ and using the IC constraint (3.1c) gives $\psi(k', C) = \beta^{-1}k/C_-^\alpha$, a number that is independent of the control $x$. Hence, for every state $(k, C_-)$ there exists an iso-$\psi$ curve containing all its potential successor states.

In some situations it will be convenient to abbreviate the laws of motion for capitalists’ consumption and capital, equations (3.16a) and (3.16b), as $k'(x, k, C_-)$ and $C(x, k, C_-)$.

Finally, define an operator $T$ on the space of continuous, increasing functions $v : [k_g, k] \to \mathbb{R}_+$, as,

$$Tv(k) = \sup \{C_- \mid \exists x \in (0, F(k)) : v(k'(x, k, C_-)) \geq C(x, k, C_-)\},$$  \hspace{1cm} (3.18)

where recall that $F(k) = f(k) + (1 - \delta)k - g$, as in (3.17). The operator is designed to extend a candidate top boundary of the set of feasible states by one iteration. To make this formal, let $Z^{(i)}$ be the set of states with $C_- \geq \bar{w}(k)$ which are $i$ steps away from reaching $C_- = \bar{w}(k)$. For example, $Z^{(0)} = \{C_- = \bar{w}(k)\}$. Lemma 7 proves some basic properties of the operator $T$. \(231\)
Lemma 7. $T$ maps the space of continuous, strictly increasing functions $v : [k_g, \bar{k}] \to \mathbb{R}^+$ with $\psi(k, v(k))$ strictly decreasing in $k$ and $v(k_g) = C_g$, $v(k^g) = C^g$, into itself.

Proof. See Subsection 3.B.4. \hfill \Box

Lemma 8 uses the operator $T$ to describe the sets $Z^{(i)}$.

Lemma 8. $Z^{(i)} = \{w^g(k) \leq C_- \leq T^i w^g(k)\}$. In particular $T^i w^g(k) \geq T^j w^g(k) \geq w^g(k)$ for $i \geq j$.

Proof. See Subsection 3.B.4. \hfill \Box

The next two lemmas characterize the limit function $\bar{w}(k)$, whose graph will describe the top boundary of the set of feasible states.

Lemma 9. There exists a continuous limit function $\bar{w}(k) \equiv \lim_{i \to \infty} T^i w^g(k) = T \bar{w}(k)$, with $\bar{w}(k_g) = C_g$ and $\bar{w}(k^g) = C^g$. All states with $C_- = \bar{w}(k)$ are feasible, but only with policy $c = 0$.

Proof. See Subsection 3.B.4. \hfill \Box

Lemma 10. No state with $C_- > \bar{w}(k)$ (and $k_g < k \leq \bar{k}$) is feasible.

Proof. See Subsection 3.B.4. \hfill \Box

Finally, Lemma 11 shows an auxiliary result which is both used in the proof of Lemma 10 and in Lemma 13 below.

Lemma 11. Let $\{k_{t+1}, C_t\}$ be a path starting at $(k_0, C_{-1})$ with controls $c_t = 0$. Let $k_g < k_0 \leq \bar{k}$. Then:

1. If $C_{-1} = \bar{w}(k_0)$, $(k_{t+1}, C_t) \to (k^g, C^g)$.

2. If $C_{-1} > \bar{w}(k_0)$, $(k_{t+1}, C_t) \not\to (k^g, C^g)$.

Proof. See Subsection 3.B.4. \hfill \Box
3.B.3 Proof of Proposition 14

Armed with the results from Section 3.B.2 we now prove Proposition 14 in a series of intermediate results. For all statements in this section, we consider an economy with an initial capital stock of $k_0 \in [k_g, \bar{k}]$. We call a path $\{k_{t+1}, C_t\}$ optimal path, if the initial $C_{t-1}$ was optimized over given the initial capital stock $k_0$. Analogously, we call a path $\{k_{t+1}, C_t\}$ locally optimal path, if the initial $C_{t-1}$ was not optimized over but rather taken as given at a certain level, respecting the constraint that $(k_0, C_{t-1})$ be feasible. If $\{k_{t+1}, C_t\}$ is a locally optimal path, with control $c_{t+1}$ at some point $\{k_{t+1}, C_t\}$ we say this control is optimal at $\{k_{t+1}, C_t\}$.

Notice that along both optimal and locally optimal paths, first order conditions are necessary, as long as paths are interior; they need not be sufficient, in the sense that there could be multiple optima that satisfy our characterization below.

The first lemma proves that the multiplier on the capitalists' IC constraint explodes along an optimal path, and at the same time, workers' consumption drops to zero.

**Lemma 12.** Along any optimal path, $c_t \to 0$.

**Proof.** Let $\{k_{t+1}, C_t\}$ be the optimal path. Suppose first the optimal path hits the boundary of the feasible set $Z^*$ at some finite time. Given that no path can hit the $k = \bar{k}$ boundary after $t = 0$, and given Lemma 6 this means the path hits the top boundary—the graph of $\bar{w}$—after finite time. Lemma 9 showed that along that boundary, the control is necessarily zero, $c = 0$.

Now suppose the optimal path is interior at all times. In that case, the first order conditions are necessary. Using the notation from problem (3.1a) the necessary first order conditions are equations (3.2a)-(3.2d). In particular, the one for $\mu_t$ states

$$
\mu_{t+1} = \mu_t \left( \frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \kappa_{t+1} \varphi_t}.
$$

From Lemma 5 we know that $\kappa_{t+1} = k_{t+1}/C_t$ is bounded away from $\infty$. Since $\mu_0 = 0$ by (3.2a) and $\sigma > 1$, it follows that $\mu_t \geq 0$ and $\mu_t \to \infty$. To show that $c_t \to 0$, suppose to the contrary that $c_t \neq 0$. In this case, there exists $c > 0$ and an infinite sequence of indices $(t_s)$ such that $c_{t_s} \geq c$ for all $s$. Along these indices, the FOC for capital (3.2d) implies

$$
\frac{u'(c_{t_s})(f''(k_{t_s}) + (1 - \delta))}{\leq u'(c_{t_s})} = \frac{1}{\beta} \frac{u'(c_{t_{s-1}})}{\geq 0} \cdot \frac{U'(c_{t_{s-1}})}{\text{bounded away from } 0} \cdot \frac{\mu_{t_s} - \mu_{t_{s-1}}}{\geq \text{const} \cdot \mu_{t_{s-1}} \to \infty},
$$

and so $k_{t_s} \to 0$ for $s \to \infty$, which is impossible within the feasible set $Z^*$ because it violates $k \geq k_g$ (see Lemma 5). This proves that also for interior optimal paths, $c_t \to 0$. $\square$
Lemma 12 is important because it shows that workers’ consumption drops to zero. Together with the following lemma, this gives us a crucial geometric restriction of where an optimal path goes in the long run.

**Lemma 13.** The set of states where \( c = 0 \) is an optimal control is the top boundary, the graph of \( \bar{w} \). It follows that an optimal path approaches either \((k_g, C_g)\) or \((k^9, C^9)\).

**Proof.** First, we show that any state in the interior of \( Z^* \) can be generated by a path with positive controls \( c > 0 \). Any state in the interior of \( Z^* \) is element of some \( Z^{(i)} \), \( i < \infty \), and can thus reach the set \( \{ C_\leq w^g(k) \} \setminus \{(k_g, C_g), (k^9, C^9)\} \) in finite time. From there, at most two steps are necessary to reach a interior steady state \((k_{ss}, C_{ss})\) with \( k_g < k_{ss} < k^9 \) and hence positive consumption \( c_{ss} > 0 \). Note that such an interior steady state can be reached without leaving the interior of the feasible set, since by Lemmas 6 and 11, hitting the upper or lower boundary once means convergence to a non-interior steady state.\(^{40}\) This proves that any state in the interior is generated by such an interior path, with positive controls \( c > 0 \).

Now take an interior state \((k_0, C_{-1})\). We prove that any optimal control at that state is positive. Suppose to the contrary, \( c_0 = 0 \) is an optimal control at \((k_0, C_{-1})\). This means, \((k_0, C_{-1})\) is generated by a locally optimal path \( \{k_{t+1}, C_t\} \), where \((k_1, C_0)\) is precisely linked to \((k_0, C_{-1})\) using control \( c_0 = 0 \), or equivalently, \( x_0 = F(k_0) \). Since \((k_0, C_{-1})\) is interior, any state \((k'(\bar{x}_0, k_0, C_{-1}), C(\bar{x}_0, k_0, C_{-1}))\) with slightly positive controls, that is, \( \bar{x}_0 < F(k_0) \), has to be feasible too. Therefore, we find the following first order necessary condition for local optimality of \( c_0 \),\(^{41}\)

\[
\frac{u'(c_1)}{u'(c_0)} \left( f'(k_1) + 1 - \delta \right) \geq \frac{1}{\beta} + u_0(\mu_1 - \mu_0),
\]

where the inequality is there due to the (implicit) boundary condition \( c_0 \geq 0 \). This condition can only be satisfied if \( c_1 = 0 \) as well. We can iterate this logic: If \((k_1, C_0)\) is interior, it must be that \( c_2 = 0 \) is optimal at \((k_2, C_1)\). If \((k_1, C_0)\) is not interior, then it must be on the top boundary of \( Z^* \), that is, on the graph of \( \bar{w} \),\(^{42}\) where it has policy \( c = 0 \) forever after. This proves, by induction, that if any interior state \((k, C_{-})\) has \( c = 0 \) as an optimal policy, any locally optimal path starting at \((k, C_{-})\) with \( c = 0 \) as initial optimal policy must have \( c = 0 \) forever, yielding utility \( u(0)/(1 - \beta) \). This, however, contradicts local optimality of such a path: We showed above that any interior state \((k_0, C_{-1})\) is generated by a path with strictly positive controls. Therefore, any optimal control at an interior state \((k_0, C_{-1})\) is positive.

\(^{40}\)Note that hitting the right boundary at \( k = \bar{k} \) (other than with \( k_0 \)) is of course not feasible due to depreciation.

\(^{41}\)A locally optimal path still satisfies the first order conditions \((3.2b)-(3.2d)\), just not \((3.2a)\) which comes from the optimal choice of \( C_{-1} \).

\(^{42}\)On the lower boundary of \( Z^* \) (excluding \((k_g, C_g)\)), a policy of \( c = 0 \) would not be feasible, see Lemma 6.
Finally, notice that states \((k, C_-), k > k_g\), along the bottom boundary of \(Z^*\) only admit a single feasible control, which is positive (see Lemma 6). Thus, by Lemma 9, the set where \(c = 0\) is an optimal control is precisely the top boundary \(\{(k, C_-) | k \in [k_g, \bar{k}], C_- = \bar{w}(k)\}\). It follows that an optimal path either hits the boundary of \(Z^*\) at some point, in which case it converges either to \((k_g, C_g)\) or \((k^9, C^9)\) (by Lemma 11), or it remains interior forever and thus (by Lemma 12) approaches the set \(\{c = 0\}\) of all states where \(c = 0\) is an optimal control, that is, the graph of \(\bar{w}\). Then it must share the same limiting behavior as states in the set \(\{c = 0\}\). By virtue of Lemma 11, it can then either converge to \((k_g, C_g)\) or \((k^9, C^9)\). □

**Lemma 14.** If an optimal path \(\{k_{t+1}, C_t\}\) converges to \((k^9, C^9)\), then the value function \(V\) is locally decreasing in \(C\) at each point \((k_{t+1}, C_t)\), for all \(t > T\), with \(T\) large enough.

**Proof.** Let \(x_t \equiv F(k_t) - c_t\) and consider the following variation: Suppose that at a point \(T, (k_{T+1}, C_T)\) is not at the lower boundary (in which case it cannot converge to \((k^9, C^9)\) anyway) and that \(c_t < F(k_t) - F'(k_t)k_t\) for all \(t \geq T\). For simplicity, call this \(T = -1\). Do the perturbation \(\hat{C}_{t-1} \equiv C_{t-1} - \epsilon, \hat{k}_0 = k_0\), but keep the controls \(c_t\) at their optimal level for \((k_0, C_{t-1})\), that is \(\hat{c}_t = c_t\). Denote the perturbed capital stock and capitalists’ consumption by \(k_{t+1} = k_{t+1} + dk_{t+1}\) and \(\hat{C}_t = C_t + dC_t\). Then the control \(x\) changes by \(dx_t = F'_t(dk_t)\) to first order. We want to show that \(dk_{t+1} > 0\) and \(dC_t < 0\) for all \(t \geq 0\), knowing that \(dC_{t-1} = -\epsilon\) and \(dk_0 = 0\).

From the constraints we find,

\[
dk_{t+1} = \frac{F'(k_t)dk_t}{C_{t-1}^0} - \frac{C_t}{C_{t-1}^0} dC_{t-1} + \frac{1}{\sigma x_t} \frac{C_t F(k_t) - F'(k_t)k_t - c_t}{k_t} dk_t > 0
\]

\[
dC_t = \frac{C_t}{C_{t-1}^0} dC_{t-1} - \frac{1}{\sigma x_t} \frac{C_t F(k_t) - F'(k_t)k_t - c_t}{k_t} dk_t < 0.
\]

---

43By the Maximum Theorem, the control \(c\) is upper hemicontinuous in the state, so its graph is closed. Hence, if along a path \(\{k_{t+1}, C_t\}\) it holds that \(c_t \to 0\), then \(\{k_{t+1}, C_t\}\) necessarily approximates the set \(\{c = 0\}\), in the sense that the distance between \(\{k_{t+1}, C_t\}\) and the set shrinks to zero (or else you could take a subsequence \(\{k_{n_t+1}, C_{n_t}\}\) in the graph of \(c\) whose limit is not in the graph, contradicting the graph being closed).

44The formal reason for this is as follows: Suppose the optimal path \(\{k_{t+1}, C_t\}\) did not share the limiting behavior of the set \(\{c = 0\}\), that is, suppose the path had a convergent subsequence \(\{k_{n_t+1}, C_{n_t}\} \to \{k^*, C^*\} \in \{c = 0\} \setminus \{(k_g, C_g), (k^9, C^9)\}\). Suppose \(k^* \in (k_g, k^9)\), the case \(k^* > k^9\) is analogous. Because \(\bar{w}(k^*) > \frac{1-\beta}{\beta} k^*\), \(h(k^*_{n_t+1}, C_{n_t})\) is eventually strictly decreasing in \(t\) (see logic around equation (3.30)) and converges to \(h(k^*, C^*)\). But convergence of \(h(k^*_{n_t+1}, C_{n_t})\) implies \(C^* = \frac{1-\beta}{\beta} k^*\)—a contradiction.

Such a finite \(T > 0\) exists for two reasons: (a) because \(c_t \to 0\); and (b) because \(F(k) - F'(k)k\) which is positive in a neighborhood around \(k = k^9\) since \(k^9\) was defined by \(F(k^9) = k^9/\beta\) and \(F'(k^9) < 1/\beta\).
Using matrix notation, this local law of motion can be written as

\[
\begin{pmatrix}
dk_{t+1} \\
dC_t
\end{pmatrix} = \begin{pmatrix}
a_t + b_t & -d_t \\
-b_t & d_t
\end{pmatrix} \begin{pmatrix}
dk_t \\
dC_{t-1}
\end{pmatrix},
\]

with \(a_t = F''(k_t), d_t = C_t/C_{t-1}, b_t = \frac{1}{\sigma x_t} \frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}.\) Close to \((k^g, C^g),\) this matrix has \(d \approx 1.\) Suppose for one moment that \(a\) was zero; the fact that \(a > 0\) only works in favor of the following argument. With \(a = 0,\) the matrix has a single nontrivial eigenvalue of \(b + d,\) which exceeds 1 strictly in the limit, and the associated eigenspace is spanned by \((1, -1).\) The trivial eigenvalue’s eigenspace is spanned by \((d, b).\) Notice that the latter eigenvector is not collinear with the initial perturbation \((0, -1),\) implying that \(dk_{\infty} > 0\) and \(dC_{\infty} < 0.\) Hence, \(k_{\infty} > k^g\) and \(C_{\infty} < C^g.\)

But notice that to the bottom right of \((k^g, C^g),\) the new point is interior, which implies a continuation value strictly larger than \(u(0)/(1 - \beta)\) (see proof of Lemma 13). More formally, this means there must exist a time \(T' > 0\) for which the continuation value of \((k_{T'+1}, C_{T'})\) is strictly dominated by the one for \((k_T, C_T),\) that is, \(V(k_{T'+1}, C_{T'}) < V(k_{T'+1}, C_{T'}).\) Because all controls were equal up until time \(T',\) this implies that \(V(k_{T+1}, C_T) < V(k_{T+1}, C_T - \epsilon)\) for \(\epsilon\) small (Recall that we had set \(T = -1\) during the proof). Thus, the value function must increase if \(C_T\) is lowered, for a path starting at \((k_{T+1}, C_T),\) for large enough \(T.\) This proves that the value function is locally decreasing in \(C\) at that point. \(\square\)

And finally, Lemma 15 proves Proposition 14.

**Lemma 15.** An optimal path converges to \((k_g, C_g).\)

**Proof.** By Lemma 13 it is sufficient to prove that an optimal path does not converge to \((k^g, C^g).\) Suppose the contrary held and there was an optimal path converging to \((k^g, C^g).\) By Lemma 14, this means that the value function is locally decreasing around the optimal path \((k_{t+1}, C_t)\) for \(t \geq T,\) with \(T > 0\) sufficiently large. Consider the following feasible variation for \(t = -1, 0, \ldots, T, \tilde{C}_t = C_t(1 - d\epsilon_t), \tilde{k}_{t+1} = k_{t+1}, \tilde{x}_t = x_t - C_t d\epsilon_t\) where\(^{46}\)

\[
d\epsilon_t = \left( 1 - \frac{1}{\sigma x_t} \right)^{-1} d\epsilon_{t-1}. \tag{3.19}
\]

Observe that (3.19) is precisely the relation which ensures that the variation satisfies all the constraints of the system (in particular (3.16b) of which (3.19) is the linearized version). Workers’ consumption increases with this variation by \(dc_t = C_t d\epsilon_t > 0.\) Therefore, the value

\(^{46}\)Notice that \(x_t = C_t + k_{t+1} \geq C_t\) by definition of \(x_t,\) and \(\sigma > 1.\) Hence this expression is well defined.
of this path changes by
\[ dV = \sum_{t=0}^{T} \beta^t u'(c_t)dc_t + \beta^{T+1} (V(k_{T+1}, C_T - C_Tdt) - V(k_{T+1}, C_T)) > 0, \]
which is contradicting the optimality of \( \{k_{t+1}, C_t\} \). An optimal path converges to \((k_g, C_g)\). □

3.3.4 Proofs of Auxiliary Lemmas

Proof of Lemma 5

Proof. Focus on \( Z_1 \) first and consider a state \((k_1, C_0) \in Z_1\), that is, \( k_1 < k_g \) and \( C_0 \geq \frac{1-\beta}{\beta} k_1 \).
Suppose \((k_1, C_0)\) was feasible, and as such generated by a path of states \( \{(k_{t+1}, C_t)\}_{t \geq 0} \), each of which compatible with (3.16a)-(3.16c). We now show by induction the claim that \((k_{t+1}, C_t) \in Z_1 \) and \( k_{t+1} \leq F(k_t) \) for any \( t \geq 0 \). This will lead to a contradiction since \( \beta F(k) \) is a concave and increasing function with \( \beta F(0) < 0 \) and smallest fixed point \( \beta F(k_g) = k_g \). Thus, any sequence of capital stocks \( \{k_{t+1}\} \) satisfying \( k_{t+1} \leq F(k_t) \), starting at any \( k_1 < k_g \), necessarily drops below zero in finite time, contradicting feasibility.

Pick a point \((k_t, C_{t-1})\) of the sequence and assume \((k_t, C_{t-1}) \in Z_1\). Then, \( x_{t+1} = k_{t+1} + C_t \leq F(k_t) \) by (3.16c), and so
\[
k_{t+1} = x_{t+1} - C_t \left( \frac{\beta x_{t+1}}{k_t} \right)^{1/\sigma} \leq \beta x_{t+1} \left( \frac{1}{\beta} - \frac{C_t}{k_t} \right) \leq \beta x_{t+1} \leq \beta F(k_t), \tag{3.20}
\]
where in the first inequality we used the fact that \( \beta x_{t+1}/k_t \leq \beta F(k_t)/k_t < 1 \) which holds since \( k_t < k_g \); and in the second inequality we used that \( C_t \geq \frac{1-\beta}{\beta} k_t \). Building on (3.20), the fact that \( k_{t+1} \leq \beta x_{t+1} \) proves that
\[
C_t = x_{t+1} - k_{t+1} \geq \frac{1-\beta}{\beta} k_{t+1}. \tag{3.21}
\]

To sum up, this implies that \( k_{t+1} \leq \beta F(k_t) < k_g \) and that \( C_t \geq \frac{1-\beta}{\beta} k_{t+1} \), so \( (k_{t+1}, C_t) \in Z_1 \). Moreover, \( k_{t+1} \leq \beta F(k_t) \). This proves the aforementioned claim and hence the desired contradiction. No state in \( Z_1 \) is feasible.

Now consider a state \((k_1, C_0) \in Z_2\). Again, suppose it was generated by a path of feasible states \( \{(k_{t+1}, C_t)\}\). Define \( h(k, C_-) = k/C_-^\sigma \) for any state \((k, C_-)\). The proof idea is to show the claim that \((k_{t+1}, C_t) \in Z_2 \) for all \( t \) and that \( h(k_{t+1}, C_t) \) is strictly increasing and diverges
to $+\infty$. Since $k_{t+1}$ is bounded from above by $\bar{k}$, this will mean that $C_t \to 0$. Moreover, $k_{t+1}$ is bounded away from zero since feasibility requires $\beta F(k) \geq 0$ and $\beta F(k)$ turns negative for $k$ sufficiently close to zero. Lemma 16 below proves that this combination of convergence of $C_t$ to zero and $k_{t+1}$ bounded away from zero violates the transversality condition.

We now prove the aforementioned claim by induction. Take a state $(k_t, C_{t-1}) \in Z_2$ from the sequence. By construction of $Z_2$, it holds that $C_{t-1} < w_g(k_t)$, or in particular, $(C_{t-1}/C_g)^\sigma < k_t/k_g$.\(^{47}\) Notice that if the next state in the sequence, $(k_{t+1}, C_t)$, satisfied $C_t \geq \frac{1-\beta}{\beta} k_{t+1}$, we must have $(k_{t+1}, C_t) \in Z_1$ which is infeasible according to the above.\(^{48}\) Therefore, $C_t < \frac{1-\beta}{\beta} k_{t+1}$. Then,

$$h(k_{t+1}, C_t) = \frac{k_{t+1}}{C_t^\sigma} \leq \frac{k_t}{C_{t-1}^\sigma} \frac{k_{t+1}}{\beta (k_{t+1} + C_t) > h(k_t, C_{t-1}), \quad (3.22)}$$

which, together with $C_t < \frac{1-\beta}{\beta} k_{t+1}$ implies that both $(k_{t+1}, C_t) \in Z_2$ and $h(k_{t+1}, C_t)$ is strictly increasing in $t$. To show that $h(k_{t+1}, C_t)$ diverges to $+\infty$, suppose it were the case that $h(k_{t+1}, C_t)$ converged to some $H > 0$. Using (3.22), convergence of $h(k_{t+1}, C_t)$ would imply that $k_{t+1}/(\beta (k_{t+1} + C_t)) \to 1$, or equivalently that $k_{t+1}/C_t \to \beta/(1-\beta)$. Since $k_{t+1}$ is bounded away from zero (see argument in previous paragraph), this can only be the case if $(k_{t+1}, C_t)$ converges to a feasible steady state,\(^{49}\) that is some $\left(k, \frac{1-\beta}{\beta} k\right)$ with $k_g \leq k \leq k_g$. However, as $(k_{t+1}, C_t) \in Z_2$ for any $t$, it is the case that $(C_t/C_g)^\sigma < k_{t+1}/k_g$, or,

$$h(k_{t+1}, C_t) > h(k_g, C_g) = \sup_{k_g \leq k \leq k_g} h(k, (1-\beta)/\beta k),$$

where the equality follows because $k/((1-\beta)/\beta k)^\sigma$ is decreasing in $k$. This shows that $h(k_{t+1}, C_t) \to \infty$ and hence completes the proof by contradiction. No state in $Z_2$ is feasible.\(\Box\)

**Lemma 16.** Suppose that $C_t \to 0$ and $k_{t+1}$ bounded away from zero for a given path of states $(k_{t+1}, C_t)$. Then, this path is not feasible.

**Proof.** Suppose the path $(k_{t+1}, C_t)$ is feasible. In particular, this necessitates that the IC

\(^{47}\)This inequality even holds if $k_t < k_g$ because there, $C_g(k_t/k_g)^{1/\sigma} > (1-\beta)/\beta k_t$. To see this recall that $C_g = (1-\beta)/\beta k_g$ and so $C_g(k_t/k_g)^{1/\sigma}/(1-\beta)/(\beta k_t) = (k_t/k_g)^{1/\sigma - 1} > 1$, where we used $\sigma > 1$.

\(^{48}\)Note that if $C_t \geq (1-\beta)/\beta k_{t+1}$, then $k_{t+1} < k_g$. The reason is as follows: The constraints (3.16a) and (3.16b) can be rewritten as $k_{t+1} = (C_t/C_{t-1})^\sigma k_t/\beta - C_t$. Because $(C_{t-1}/C_g)^\sigma < k_t/k_g$, this implies that $k_{t+1} > (C_t/C_g)^\sigma k_g/\beta - C_t$. Note that the right hand side of this inequality is increasing in $C_t$ as long as it is positive (which is the only interesting case here). Substituting in $C_t \geq (1-\beta)/\beta k_{t+1}$, this gives $k_{t+1} > (k_{t+1}/k_g)^\sigma k_g/\beta - (1-\beta)/\beta k_{t+1}$. Rearranging, $k_{t+1}/k_g > (k_{t+1}/k_g)^\sigma$, a condition which can only be satisfied if $k_{t+1}/k_g < 1$ (recall that $\sigma > 1$).

\(^{49}\)Notice that, if $k_{t+1}/C_t^\sigma \to H > 0$ and $k_{t+1}/C_t \to \beta/(1-\beta)$ then convergence of $k_{t+1}$ and $C_{t+1}$ themselves immediately follows.
condition \( \beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t \) and the transversality condition \( \beta'U'(C_t)k_{t+1} \to 0 \) hold. We back out (after tax) interest rates from the allocation as \( R_t \equiv U'(C_{t-1})/(\beta U'(C_t)) \).

Thus we can recover the capitalists’ per period budget constraint \( C_t + k_{t+1} = R_t k_t \), and, using the transversality condition, also present value budget constraints starting at any given time \( t_0 \geq 0 \),

\[
\sum_{t=t_0}^{\infty} \frac{1}{R_{t_0,t}} C_t = R_{t_0}k_{t_0}, \tag{3.23}
\]

where we denote \( R_{t_0,t} \equiv R_{t_0+1} \cdots R_t \). Also, by construction of \( R_t \), consumption can be expressed as

\[
C_t = \beta^{(t-t_0)/\sigma} \left( \frac{R_{t_0,t}}{C_t} \right)^{1/\sigma} C_{t_0}. \tag{3.24}
\]

Define \( K \equiv \inf_t k_{t+1} > 0 \) and \( \bar{K} \equiv \sup_t k_{t+1} > 0 \). Using the per period budget constraints, we then have

\[
R_t = \frac{C_t + k_{t+1}}{k_t} \geq \frac{k_{t+1}}{k_t} \tag{3.25}
\]

and similarly,

\[
\frac{1}{R_{t_0,t}} \geq \frac{k_{t+1}}{k_{t_0+1}} \geq \frac{K}{\bar{K}}.
\]

Combining (3.23), (3.24) and (3.25), we find

\[
k_{t_0} = \frac{1}{R_{t_0}} \sum_{t=t_0}^{\infty} \beta^{(t-t_0)/\sigma} \left( \frac{R_{t_0,t}}{C_t} \right)^{1/\sigma} C_{t_0} \leq C_{t_0} \left( \frac{K/\bar{K}}{1 - \beta^{1/\sigma}} \right)^{(2-1/\sigma)}.
\]

Since \( t_0 \) was arbitrary, this implies that \( k_1 \to 0 \), leading to the desired contradiction. Thus, the path \((k_{t+1}, C_t)\) cannot be feasible.

**Proof of Lemma 6**

**Proof.** Consider a state \((k, C_-)\) with \( w_g(k) \leq C_- \leq w_g(k) \) and \( k \geq k_g \). In particular, \( C_- \leq (1 - \beta)/\beta k_g \), \((C_-/C_g)^\sigma \geq k/k_g \) and \((C_-/C_g)^\sigma \leq k/k_g \).\(^{50}\) The idea of the proof is to show that in fact such a state can be generated by a steady state \((k_{ss}, C_{ss})\) (with \( C_{ss} = (1 - \beta)/\beta k_{ss} \) and \( k_g \leq k_{ss} \leq k^g \)). By definition of \( k_g \) and \( k^g \), such a steady state is always self-generating.

Guess that the right steady state has \( k_{ss} = (\beta C_-/(1 - \beta))^{\sigma/(\sigma-1)} k^{-1/(\sigma-1)} \) and \( C_{ss} = (1 - \beta)/\beta k_{ss} \). It is straightforward to check that this steady state can be attained with control \( x = (C_{ss}/C_-)^\sigma k/\beta \). This steady state is self-generating because \( k_g \leq k_{ss} \leq k^g \), which follows from \((C_-/C_g)^\sigma \geq k/k_g \) and \((C_-/C_g)^\sigma \leq k/k_g \). Finally, the control \( x \) is resource-feasible.

\(^{50}\)These inequalities hold for all \( k \geq k_g \). The proofs are analogous to the proofs in footnotes 47 and 51.
because $C_- (1 - \beta)/\beta k$ and thus,

$$x = \frac{1}{\beta} \left[ \left( \frac{\beta}{1 - \beta} \frac{C_-}{k} \right)^{\sigma/(\sigma - 1)} \right] \leq \frac{k}{\beta} \leq f(k) + (1 - \delta)k - g,$$

where the latter inequality follows from the fact that $k_g \leq k \leq k^g$ and the definition of $k_g$ and $k^g$. This concludes the proof that all states with $w_g(k) \leq C_- \leq w_g(k)$ and $k \geq k_g$ are feasible.

Now regard a state on the boundary $\{C_- = w_g(k), k > k_g\}$, so we also have that $C_- < (1 - \beta)/\beta k$.\footnote{This holds because $C_- = w_g(k) = C_g(k/k_g)^{1/\sigma}$ and thus $C_- / ((1 - \beta)/\beta k) = (k/k_g)^{1/\sigma - 1} < 1$.} Such a state is generated by $(k_{ss}, C_{ss}) = (k_g, C_g)$. Moreover, the unique control which moves $(k, C_-)$ to $(k_g, C_g)$ is $x < k/\beta \leq f(k) + (1 - \delta)k - g$, or in terms of $c$, \( c > 0 \).

To show that $(k_g, C_g)$ is in fact the only feasible state generating $(k, C_-)$, let $(k', C)$ be a state generating $(k, C_-)$. If $k' < k_g$, then $(k', C)$ is not feasible by Lemma 5, and if $k' = k_g$ only $(k_g, C_g)$ generates $(k, C_-)$. Suppose $k' > k_g$. Then, $C < (1 - \beta)/\beta k'$,\footnote{This holds because by the IC constraint (3.1c), $\beta(k' + C)/C_g = k_g/C_g^\sigma$ or equivalently $(k' + C)/C = 1/(1 - \beta) \cdot C_g^\sigma \cdot (k'/k_g)^\sigma$. Thus, letting $\kappa = k'/C$, $(\kappa + 1)\kappa^\sigma = (1 - \beta)^{-1} \cdot (\beta/(1 - \beta))^\sigma \cdot (k'/k_g)^\sigma$. Since the right hand side is increasing in $\kappa$, the fact that $k' > k_g$ tells us that $\kappa > \beta/(1 - \beta)$, which is what we set out to show.} and so we can recycle equation (3.22) to see $h(k', C) > h(k, C_-)$. Because $h(k, C_-) = h(k_g, C_g)$ however, this implies that $h(k', C) > h(k_g, C_g)$, or put differently, $C < w_g(k')$. Again by Lemma 5 such a $(k', C)$ is not feasible. Therefore, the only state that can generate a state on the boundary $\{C_- = w_g(k), k > k_g\}$ is $(k_g, C_g)$, and the associated unique control involves positive $c$.

Proof of Lemma 7

Proof. Let $\mathcal{V} (\hat{\mathcal{V}})$ be the space of all continuous, weakly (strictly) increasing functions $v : [k_g, \tilde{k}] \rightarrow \mathbb{R}_+$ with $\psi(k, v(k))$ weakly (strictly) decreasing in $k$, and $v(k_g) = C_g$, $v(k^g) = C^g$. For these functions, $T$ is well-defined since for small values of $C_-$, $k'(F(k), k, C_-)$ tends to $F(k) \in (k_g, \tilde{k})$. Moreover, the supremum in (3.18) is attained for all $k \in [k_g, \tilde{k}]$ since the set of $C_-$ in (3.18) is closed and bounded. We next show that (a) instead of considering all possible controls $x$, it is sufficient to consider $x = F(k)$; and (b) instead of looking for $C_-$ that satisfy the inequality in (3.18), it suffices to look for solutions to the corresponding relation with equality. This will allow us to write

$$T v(k) = \max \{C_- | v(k'(F(k), k, C_-)) = C(F(k), k, C_-)\},$$

(3.26)
The formal arguments behind these two steps are:

(a) Fix \( k \in [k_g, \bar{k}] \) and \( v \in \mathcal{V} \). Suppose the supremum in (3.18) is attained by \( C_- \), with control \( x_0 < F(k) \). Define \( \Phi_{v,k,C_-} : [0, F(k)] \rightarrow \mathbb{R} \) by

\[
\Phi_{v,k,C_-}(x) = \psi(k'(x, k, C_-), C(x, k, C_-)) - \psi(k'(x, k, C_-), v(k'(x, k, C_-)))
\]

and notice that \( v(k'(x_0, k, C_-)) \geq C(x_0, k, C_-) \) is equivalent to \( \Phi_{v,k,C_-}(x_0) \geq 0 \). Since \( \Phi_{v,k,C_-}(x) \) is weakly increasing in \( x \) due to \( v \in \mathcal{V} \), \( \Phi_{v,k,C_-}(F(k)) \geq \Phi_{v,k,C_-}(x_0) \) and so \( v(k'(F(k), k, C_-)) \geq C(F(k), k, C_-) \). Therefore, focusing on controls \( x = F(k) \) is without loss in (3.18).

(b) Now argue that equality (rather than inequality) is without loss in (3.18). Suppose the supremum were attained by \( C_- \) with control \( x = F(k) \) and strict inequality, \( v(k'(F(k), k, C_-)) > C(F(k), k, C_-) \). Since both sides of this inequality are continuous in \( C_- \), it follows that slightly increasing \( C_- \) still satisfies the inequality and hence \( C_- \) could not have attained the supremum in the first place.

Notice also that the equation \( v(k'(F(k), k, C_-)) = C(F(k), k, C_-) \) can never have more than one solution since raising \( C_- \) weakly decreases the left hand side and strictly increases the right hand side.

Now we argue that \( T \) maps \( \mathcal{V} \) into \( \hat{\mathcal{V}} \). Take \( v \in \mathcal{V} \). To show \( Tv \) is continuous and strictly increasing, define first the auxiliary function \( \Psi_v : [k_g, \bar{k}] \times \mathbb{R}^+ \rightarrow \mathbb{R} \) by

\[
\Psi_v : (k, C_-) = \psi(k'(F(k), k, C_-), C(F(k), k, C_-)) \times \text{ increasing in } k \text{ and decreasing in } C_-
\]

\[
- \psi(k'(F(k), k, C_-), v(k'(F(k), k, C_-))) \times \text{ decreasing in } k \text{ and increasing in } C_-.
\]

The function \( \Psi_v \) is continuous and consists of two terms: The first term is equal to \( \beta^{-1}k/C_-^\sigma \), using the definition of \( \psi \), and hence strictly increasing in \( k \) and strictly decreasing in \( C_- \). For the second term, recall that

\[
k'(F(k), k, C_-) = F(k) \left( 1 - C_- \left( \frac{\beta}{kF(k)^{\sigma-1}} \right)^{1/\sigma} \right)
\]

is strictly increasing in \( k \) and strictly decreasing in \( C_- \), and \( v \) is such that \( \psi(k, v(k)) \) is weakly decreasing in \( k \). Thus, the second term is weakly decreasing in \( k \) and weakly increasing in \( C_- \).
Putting both terms together gives us that \( \Psi_v(k, C_-) \) is continuous, strictly increasing in \( k \), and strictly decreasing in \( C_- \). We can rewrite \( T_v \) as

\[
T_v(k) = C_- \quad \text{where } C_- \text{ is the unique number with } \Psi_v(k, C_-) = 0.
\]

Since \( \Psi_v \) is continuous, strictly increasing in \( k \), strictly decreasing in \( C_- \) and admits a unique solution \( C_- = T_v(k) \) to the equation \( \Psi_v(k, C_-) = 0 \), it follows that \( T_v(k) \) is continuous and strictly increasing.\(^{53}\)

To prove that \( k \mapsto \psi(k, T_v(k)) \) is strictly decreasing, pick \( k_1 < k_2 \) in \([k_g, \bar{k}]\). Suppose \( \psi(k_1, T_v(k_1)) \leq \psi(k_2, T_v(k_2)) \). Since \( T_v(k) \) is strictly increasing, it follows that

\[
\frac{k_1}{T_v(k_1)^\sigma} - \frac{k_2}{T_v(k_2)^\sigma} < \frac{k_1}{T_v(k_1)^\sigma} + T_v(k_1)^{1-\sigma} - \frac{k_2}{T_v(k_2)^\sigma} - T_v(k_2)^{1-\sigma} \leq 0.
\]

Defining \( k'_1 \equiv k'(F(k_1), k_1, T_v(k_1)) \) and \( C_i \equiv C(F(k_i), k_i, T_v(k_i)) \), we find

\[
\psi(k'_1, C_1) = \beta^{-1} \frac{k_1}{T_v(k_1)^\sigma} < \beta^{-1} \frac{k_2}{T_v(k_2)^\sigma} = \psi(k'_2, C_2).
\]

This, however, implies that \( T_v(k_2) \) cannot have been optimal: Pick an alternative consumption level \( C_{2,-} \) as \( C_{2,-} = T_v(k_1)/(k_2/k_1)^{1/\sigma} \), which exceeds \( T_v(k_2) \) by (3.28). Moreover, pick the policy \( x_2 \equiv F(k_1) \), which is feasible, \( x_2 \leq F(k_2) \). Since \( k_1/T_v(k_1)^\sigma = k_2/C_{2,-}^\sigma \) by construction, it follows that \( (k'(x_2, k_2, C_{2,-}), C(x_2, k_2, C_{2,-})) = (k'_1, C_1) \), which lies on the graph of \( v \). Hence \( T_v(k_2) \) cannot have been optimal and so \( \psi(k, T_v(k)) \) is decreasing in \( k \).

Finally, we prove that \( T_v(k_g) = C_g \). Note that \( k'(F(k_g), k_g, C_g) = k_g \) and \( C(F(k_g), k_g, C_g) = C_g \). Because \( k'(F(k_g), k_g, C_g) \) is strictly decreasing in \( C_- \) and so \( k'(F(k_g), k_g, C_-) < k_g \) for \( C_- > C_g \) (for \( k < k_g \), \( v(k) \) is not even defined), this implies that \( T_v(k_g) = C_g \), concluding the proof that \( T(V) \subset \mathcal{V} \).

Proof of Lemma 8

Proof. Note that any state \((k, C_-)\) reaches the space \( \{C_- \leq v(k)\} \) in one step if and only if \( C_- \leq T_v(k) \) (provided that \( v \) satisfies the regularity properties in Lemma 7). Thus, by iteration, \( Z^{(i)} = \{w^g(k) \leq C_- \leq T^iw^g(k)\} \). Because \( Z^{(i)} \supseteq Z^{(j)} \) for \( i \geq j \), it holds that

\[^{53}\text{This is a fact that holds more generally: If } I_1, I_2 \subset \mathbb{R} \text{ are intervals and } f : I_1 \times I_2 \to \mathbb{R} \text{ is continuous, strictly increasing in } x, \text{ and strictly decreasing in } y \text{ with the property that for each } x \text{ there exists a unique } y^*(x) \text{ s.t. } f(x, y^*(x)) = 0, \text{ then } y^*(x) \text{ must be continuous and strictly increasing in } x.\]
Proof of Lemma 9

Proof. The existence of the limit \( \lim_{i \to \infty} T^i w^g(k) \) is straightforward for every \( k \) (monotone sequence, bounded above because for large values of \( C_- \), \( k'(F(k), k, C_-) < k_g \) for any \( k \)). It can easily be verified that \( w^g \in \mathcal{V} \). Thus, using Lemma 7, \( \bar{w} \) must be weakly increasing, \( \bar{w}(k_g) = C_g \), \( \bar{w}(k^g) = C^g \), and \( \psi(k, \bar{w}(k)) \) must be weakly decreasing. To show \( \bar{w} \in \mathcal{V} \), suppose now that \( \bar{w} \) were not continuous. Then, there would have to be two arbitrarily close values of \( k, k_1 < k_2 \) with a significant gap between \( T^N w^g(k_1) \) and \( T^N w^g(k_2) > T^N w^g(k_1) \) for some large \( N \). Since \( k'(\ldots) \) and \( C(\ldots) \) are both continuous, \( k_1 \) and \( k_2 \) can be chosen sufficiently close so that

\[
k'_i \equiv k'(F(k_1), k_1, T^N w^g(k_1)) > k'(F(k_2), k_2, T^N w^g(k_2)) \equiv k'_2,
\]

yet the inequality is reversed for \( C(\ldots) \), \( C_1 \equiv C(F(k_1), k_1, T^N w^g(k_1)) < C(F(k_2), k_2, T^N w^g(k_2)) \equiv C_2 \). However, this contradicts the definition of \( T^N w^g \) since both pairs \((k'_1, C_1)\) and \((k'_2, C_2)\) have to lie on the graph of the same increasing function \( T^N w^g \) but the latter is to the top left of the former. Therefore, \( \bar{w} \) is continuous and \( \bar{w} \in \mathcal{V} \).

Applying Dini's Theorem, the convergence of \( T^n w^g \) to \( \bar{w} \) is also uniform, and by interchanging limits we find that

\[
\bar{w}(k'(F(k), k, \bar{w}(k))) = \lim_{n \to \infty} T^n w^g(k'(F(k), k, T^{n+1} w^g(k))) = \lim_{n \to \infty} C(F(k), k, T^{n+1} w^g(k)) = C(F(k), k, \bar{w}(k)),
\]

and thus, by the representation of \( T \) in (3.26), \( \bar{w} = T \bar{w} \). This also means that \( \bar{w} \in \mathcal{V} \), so \( \bar{w} \) is strictly increasing and \( \psi(k, \bar{w}(k)) \) strictly decreasing. Hence, for any given \( k \), the only feasible policy at point \((k, \bar{w}(k))\) is \( x = F(k) \) (or equivalently \( c = 0 \)) since for any feasible policy \( x, \Phi_{\bar{w}, k, \bar{w}(k)}(x) \) from (3.27) needs to be non-negative; but by \( \bar{w} \in \mathcal{V} \) and (3.29), \( \Phi_{\bar{w}, k, \bar{w}(k)} \) is strictly increasing with \( \Phi_{\bar{w}, k, \bar{w}(k)}(F(k)) = 0 \), so \( x = F(k) \) is the only feasible policy. \( \square \)

\[^{54}\text{A subtlety here is that } Z^{(i)} \supseteq Z^{(j)} \text{ only holds because states in the set } \{C_- = w^g(k)\} \text{ is "self-generating", that is, if a path hits the set } \{C_- = w^g(k)\} \text{ after } j \text{ steps, it can stay in that set forever. In particular, it can hit the set after } i \geq j \text{ steps as well. This explains why } Z^{(i)} \supseteq Z^{(j)}.\]
**Proof of Lemma 10**

*Proof.* Define $h$ as before, $h(k', C) \equiv k'/C'$. Fix a state $(k_1, C_0)$ with $C_0 > \bar{w}(k_1)$. First, consider the case $C_0 \geq (1 - \beta)/\beta k_1$ and suppose it were generated by a feasible path \{(k_{t+1}, C_t)\}. As an intermediate result we now establish that $C_t > (1 - \beta)/\beta k_{t+1}$ along such a path. We do this by distinguishing the following two cases:

(a) If $k_{t+1} \leq k^g$, this follows directly from $C_t > \bar{w}(k_{t+1}) \geq (1 - \beta)/\beta k_{t+1}$. The former inequality holds by construction of $\bar{w}$, the latter by Lemma 8.

(b) If instead $k_{t+1} > k^g$, it must be the case that $k_{s+1} > k^g$ for all $s < t$ as well. But then, using that $x_t \leq F(k_t) < k_t/\beta$ for $k_t > k^g$,

$$\frac{k_{t+1}}{C_t} = \frac{x_t}{C_t} - 1 < \frac{k_t/\beta}{C_{t-1}} - 1 = \frac{\beta}{1 - \beta}.$$ 

We use our intermediate result as follows (still for the case $C_0 \geq (1 - \beta)/\beta k_1$). Consider

$$h(k_{t+1}, C_t) = \frac{k_{t+1}}{C_t^\sigma} = \frac{k_t}{C_{t-1}^\sigma} \frac{k_{t+1}}{\beta(k_{t+1} + C_t)} < h(k_t, C_{t-1}).$$ (3.30)

If $h(k_{t+1}, C_t)$ converges to zero, then either $k_{t+1} \to 0$ or $C_t \to \infty$ (in which case $k_{t+1} \to 0$ by the law of motion for capital and the fact that $k_t \leq \bar{k}$). Such a path is not feasible because then $F(k_{t+1})$ drops below zero in finite time (see the proof of Lemma 5 for a similar argument). Hence, suppose $h(k_{t+1}, C_t) \to h > 0$. Then, $k_{t+1}/(\beta(k_{t+1} + C_t)) \to 1$, so the path must approximate the steady state line described by \{(k, C_-) \mid C_- = (1 - \beta)/\beta k\}. Because $C_t > \bar{w}(k_{t+1})$ along the path, $(k_{t+1}, C_t)$ must be converging to $(k^g, C^g)$.

Next we show that this convergence is still true if we take $c_t$ to be zero. Suppose there were times with $c_t > 0$. Then, define a new path \{($k_{t+1}, \dot{C}_t$)\}, starting at the same initial

\textsuperscript{55}If it were violated, $C_0 \leq T^i \bar{w}(k_1) = \bar{w}(k_1)$ by construction of $\bar{w}$. This would contradict our assumption that $C_0 > \bar{w}(k_1)$.

\textsuperscript{56}The reason for this is that for any state $(k, C_-)$ with $k \leq k^g$ and $C_- > \bar{w}(k)$ we have that $k' \equiv k'(x, k, C_-) \leq k^g$ for any control $x \leq F(k)$. First, if $\psi(k', C) \geq \psi(k^g, C^g)$, then the curve \{(k'(x, k, C_-), C(x, k, C_-)), x > 0\} and the graph of $\bar{w}$ necessarily intersect at a state $\bar{k}$ with capital less than $k^g$. The intersection is unique since $\psi(k, \bar{w}(k))$ is strictly increasing. Since $C_- > \bar{w}(k)$ it cannot be that $\bar{k} = k'(x, k, C_-)$ for a feasible $x \leq F(k)$ and therefore, any $k'(x, k, C_-)$ with a feasible $x \leq F(k)$ is necessarily less than $\bar{k} \leq k^g$. Second, if $\psi(k', C) < \psi(k^g, C^g)$, that is, $k/C_- < k^g/(C^g)$, then $k' \leq F(k) - C_- \left(\frac{\beta F(k)}{k}\right)^{1/\sigma} < F(k^g) - C^g \left(\frac{\beta F(k^g)}{k^g}\right)^{1/\sigma} = k^g.$
state \((k_1, C_0)\) but with controls \(c_t = 0\). Observe that

\[
\begin{aligned}
    h(k_{t+1}, C_t) &= \psi(k_{t+1}, C_t) - C_t^{1-\sigma} = \beta^{-1} h(k_t, C_{t-1}) - h(k_t, C_{t-1})^{(1-1)/\sigma} (\beta F(k_t))^{-(1-1)/\sigma} \\
    \dot{k}_{t+1} &= F(k_t) - \left( \frac{\beta F(k_t)}{h(k_t, C_{t-1})} \right)^{1/\sigma},
\end{aligned}
\]

where the first equation is increasing in \(h(k_t, C_{t-1})\) for the relevant parameters for which \(h(k_{t+1}, C_t) \geq 0\), and similarly the second equation is increasing in \(F(k_t)\) if \(\dot{k}_{t+1} \geq 0\). By induction over \(t\), if \(h(k_t, C_{t-1}) \geq h(k_t, C_{t-1})\) and \(k_t \geq k_t\) (induction hypothesis), then, because \(F(k_t) \geq x_t\),

\[
\begin{aligned}
    h(k_{t+1}, C_t) &\geq \beta^{-1} h(k_t, C_{t-1}) - h(k_t, C_{t-1})^{(1-1)/\sigma} (\beta x_t)^{-(1-1)/\sigma} = h(k_{t+1}, C_t) \\
    \dot{k}_{t+1} &\geq F(k_t) - \left( \frac{\beta F(k_t)}{h(k_t, C_{t-1})} \right)^{1/\sigma},
\end{aligned}
\]

confirming that \(\dot{k}_t \geq k_t\) and \(h(k_t, C_{t-1}) \geq h(k_t, C_{t-1})\) for all \(t\). Given that \(h(k_{t+1}, C_t) \to h > 0\), either \((\dot{k}_{t+1}, C_t) \to (k^g, C^g)\) as well, or \(\{ (\dot{k}_{t+1}, C_t) \}\) converges to some steady state between \(k_g\) and \(k^g\). The latter cannot be because of \(\dot{C}_t > \bar{w}(\dot{k}_{t+1})\) along the path. But the former is precluded by Lemma 11 below. This provides a contradiction, proving that a state \((k_1, C_0)\) with \(C_0 > \bar{w}(k_1)\) and \(C_0 > (1-\beta)/\beta k_1\) cannot be feasible.

Now, consider the case \(C_0 < (1-\beta)/\beta k_1\). Due to \(C_0 > \bar{w}(k_1)\), this can only be the case if \(k_1 > k^g\). Again, suppose \((k_1, C_0)\) were generated by a feasible path \(\{(k_{t+1}, C_t)\}\). Given the first half of this proof, if at any point \((k_{t+1}, C_t)\) lies above the steady state line, we have the desired contradiction. Therefore, suppose \(C_t < (1-\beta)/\beta k_{t+1}\) for all \(t\). In that case,

\[
    h(k_{t+1}, C_t) = \frac{k_{t+1}}{C^g_t} = \frac{k_t}{C^g_{t-1}} \frac{k_{t+1}}{\beta (k_{t+1} + C_t)} > h(k_t, C_{t-1}).
\]

Note that \(h(k_{t+1}, C_t)\) is bounded from above, for example by \(h(k_g, C_g)\) (because all states below the steady state line with \(h\) equal to \(h(k_g, C_g)\) are below the graph of \(w^g\) and thus below \(\bar{w}\) as well). So, \(h(k_{t+1}, C_t)\) converges and \(k_{t+1}/(\beta (k_{t+1} + C_t)) \to 1\). The state approximates the steady state line. Because the only feasible steady state with below the steady state line but above the graph of \(\bar{w}\) is \((k^g, C^g)\) it follows that \((k_{t+1}, C_t) \to (k^g, C^g)\). Following the same steps as before, it can be shown that without loss of generality, controls \(c_t\) can be taken to be zero along the path. By Lemma 11 below this is a contradiction, concluding our proof that no state \((k_1, C_0)\) with \(C_0 > \bar{w}(k_1)\) is feasible.  

\[\square\]
Proof of Lemma 11

Proof. We prove each of the results in turn.

1. Notice that \( c = 0 \) takes any state on the graph of \( \bar{w} \) to another state on the graph of \( \bar{w} \) (because \( T\bar{w} = \bar{w} \)). Suppose \( k_1 < k^g \) (the case \( k_1 > k^g \) is analogous). Then, no future capital stock \( k_{t+1} \) can exceed \( k^g \). Because if it did, there would have to be a capital stock \( k \in (k_g, k^g) \) with \( k'(F(k), k, \bar{w}(k)) = k^g \), by continuity of \( k \mapsto k'(F(k), k, \bar{w}(k)) \). But this is impossible by definition of \( k^g \). Thus, along the path, \( C_t > (1 - \beta)/\beta k_{t+1} \) and so \( h(k_{t+1}, C_t) \) is decreasing. As \( h(k_g, C_g) > h(k, \bar{w}(k)) \) for all \( k > k_g \), this means \((k_{t+1}, C_t) \to (k^g, C^g)\).

2. For simplicity, focus on the case \( k_0 < k^g \). Again, the case \( k_0 > k^g \) is completely analogous. Suppose \((k_{t+1}, C_t)\) were converging to \((k^g, C^g)\). Note that at \( k^g \), \( F(k)/k \) is decreasing. Thus, there exists a time \( T > 0 \) for which the capital stock \( k_T \) is sufficiently close to \( k^g \) that \( F(k)/k \) is decreasing for all \( k \) in a neighborhood of \( k^g \) which includes \( \{k_t\}_{t \geq T} \). Let \( \{\hat{k}_{t+1}, \hat{C}_t\} \) denote the path with \( c_t = 0 \), starting from \((k_T, \bar{w}(k_T))\). We already know that \( \{\hat{k}_{t+1}, \hat{C}_t\} \) does indeed converge to \((k^g, C^g)\) from the first part of this proof. Also, observe that both \((k_{t+1}, C_t)\) and \((\hat{k}_{t+1}, \hat{C}_t)\) have controls \( c_t = 0 \) here, unlike in the proof of Lemma 10.

In the remainder of this proof, we denote the “zero control c = 0” laws of motion for capital and capitalists’ consumption by \( L_k(k, C_-) \equiv k'(F(k), k, C_-) \) and \( L_C(k, C_-) \equiv C(F(k), k, C_-) \) (only for this proof). Since \( F(k)/k \) is locally decreasing, it follows that \( dL_k/dk > 0 \), \( dL_C/dC_- < 0 \) and \( dL_C/dC_- > 0 \). This implies that because \( C_{T-1} > \bar{w}(k_T) \) (which must hold or else \( C_0 \leq \bar{w}(k_1) \) by construction of \( \bar{w} \)), \( C_t > \hat{C}_t \) and \( k_{t+1} > \hat{k}_{t+1} \) for all \( t \geq T \). Moreover, borrowing from equation (3.22), we know that

\[
h(k_{t+1}, C_t) = h(k_t, C_{t-1}) \left( \frac{1}{\beta} - \left( \frac{1}{h(k_t, C_{t-1})} \right)^{1/\sigma} \left( \frac{1}{\beta F(k_t)} \right)^{1-1/\sigma} \right),
\]

57 By definition of \( k^g \), \( F(k^g) = k^g + C^g \), and so, \( F(k) < k^g + C^g \) for \( k < k^g \).
58 Note that \( \bar{w}(k) > w_g(k) \) and \( h(k, w_g(k)) = \text{const} \), see Lemmas 5 and 6 above.
59 This holds because \( F'(k^g) < 1/\beta \) and \( F(k^g) = 1/\beta k^g \), and so \( dF(k)/k < 0 \).
which implies that by induction \( h(k_{t+1}, C_t) \leq h(\hat{k}_{t+1}, \hat{C}_t) \), that is,

\[
\log h(k_{t+T}, C_{t+T-1}) = \log h(k_T, C_{T-1}) + \sum_{s=0}^{t-1} \log \left( \frac{1}{\beta} - \frac{1}{h(k_{T+s}, C_{T+s-1})} \right)^{1/\sigma} = \log h(\hat{k}_{t+T}, \hat{C}_{t+T-1}) + \log h(k_T, C_{T-1}) - \log h(\hat{k}_T, \hat{C}_{T-1}).
\]

As \( t \to \infty \), this equation yields

\[
\log h(k^g, C^g) \leq \log h(k^g, C^g) + \log h(k_T, C_{T-1}) - \log h(\hat{k}_T, \hat{C}_{T-1}),
\]

which is a contradiction. Therefore, \((k_{t+1}, C_t) \not\in (k^g, C^g)\).

\[\square\]

3.C Numerical Method

To solve the Bellman equation (3.15) we must first compute the feasible set \( Z^* \). We restrict the range of capital to a closed interval \([k, \bar{k}]\) with \( k \geq k_g \). This leads us to seek a subset \( Z^{*k} \subset Z^* \) of the feasible set \( Z^* \). We compute this set numerically as follows.

Start with the set \( Z^*_0 \) defined by \( C_- = \frac{1-\beta}{\beta} k \) and \( k \in [k, \bar{k}] \). This set is self generating and thus \( Z^*_0 \subset Z^{*k} \). We define an operator that finds all points \((k, C_-)\) for which one can find \( c, K', C \) satisfying the constraints of the Bellman equation and \((k', C) \in Z^*_0 \). This gives a set \( Z^*_1 \) with \( Z^*_0 \subset Z^*_1 \). Iterating on this procedure we obtain \( Z^*_0, Z^*_1, Z^*_2, \ldots \) and we stop when the sets do not grow much. We then solve the Bellman equation by value function iteration. We start with a guess for \( V_0 \) that uses a feasible policy to evaluate utility. This ensures that our guess is below the true value function. Iterating on the Bellman equation then leads to a monotone sequence \( V_0, V_1, \ldots \) and we stop when iteration \( n \) yields a \( V_n \) that is sufficiently close to \( V_{n-1} \). Our procedure uses a grid that is defined on a transformation of \((k, C_-)\) that maps \( Z^* \) into a rectangle. We linearly interpolate between grid points.

The code was programmed in Matlab and executed with parallel 'parfor' commands, to improve speed and allow denser grids, on a cluster of 64-128 workers. Grid density was adjusted until no noticeable difference in the optimal paths were observed.
3.D Proof of Proposition 15

As in Appendix 3.B we use the notation that \( F(k) = f(k) + (1 - \delta)k \). The derivatives of \( S \) evaluated at some time \( \tau \) are denoted by \( S_{t,\tau} = \frac{\partial S}{\partial \tau} \) and \( S_{t,R_t} = \frac{\partial S}{\partial R_t} \), for \( t > \tau \).

Define the following object,

\[
\omega_{\tau} = \frac{dW_{\tau}}{dk_{\tau+1}} = \sum_{\tau' \geq \tau + 1} \beta^{\tau'-\tau} u'(c_{\tau'}) (F'(k_{\tau'}) - R_{\tau'}) \left( \prod_{s=\tau+1}^{\tau'-1} S_{t,s} R_s \right),
\]

which corresponds to the response in welfare \( W_{\tau} \), measured in units of period \( \tau \) utility, of a change in savings by an infinitesimal unit between periods \( \tau \) and \( \tau + 1 \). Now consider the effect of a one-time change in the capital tax, effectively changing \( R_t \) to \( R_t + dR \) in period \( t \). This has three types of effects on total welfare: It changes savings behavior in all periods \( \tau < t \) through the effect of \( R_t \) on \( S_t \). It changes capitalists’ income in period \( t \) through the effect of \( R_t \) on \( R_t k_t \). And finally it changes workers’ income in period \( t \) directly through the effect of \( R_t \) on \( F(k_t) - R_t k_t \). Summing up these three effects, one obtains a total effect of

\[
dW = \sum_{\tau=0}^{t-1} \beta^{\tau-t} \omega_{\tau} S_{t,R_t} dR + \omega_t S_{t,k_t} dR - u'(c_t) k_t dR.
\]

The total effect needs to net out to zero along the optimal path, that is,

\[
\omega_t S_{t,I,t} - u'(c_t) = -\frac{1}{k_t} \sum_{\tau=0}^{t-1} \beta^{\tau-t} \omega_{\tau} S_{t,R_t}.
\]

By optimization over the initial interest rate \( R_0 \), we find the condition

\[
\omega_0 S_{I,0} k_0 - u'(c_0) k_0 = 0.
\]

This shows that \( S_{I,0} > 0 \) and so \( \omega_0 \in (0, \infty) \). By their definition (3.31), the \( \omega_{\tau} \) satisfy the recursion

\[
\omega_{\tau} = u'(c_{\tau+1}) (F'(k_{\tau+1}) - R_{\tau+1}) + \beta S_{t,\tau+1} R_{\tau+1} \omega_{\tau+1}.
\]

Since it is easy to see that \( R_{\tau+1} > 0 \) for all \( \tau \),\(^6\) it follows that \( \omega_{\tau} \) is finite for all \( \tau \). Then, due

\(^6\)Otherwise capital would be zero forever after due to \( S(0,\ldots) = 0 \), a contradiction to the allocation converging to an interior steady state.
to the recursive nature of (3.32), if $\omega_\tau > 0$ for $\tau < t$,

$$\omega_t S_{I,t} - u'(c_t) = -\frac{1}{k_t} \sum_{r=0}^{t-1} \beta^{\tau-r} \omega_r S_{I,r} \geq 0.$$ 

In particular, using the initial condition (3.33), this proves by induction that

$$\omega_t S_{I,t} - u'(c_t) \geq 0 \quad \text{for all } t > 0. \quad (3.34)$$

Now suppose the economy were converging to an interior steady state with non-positive limit tax (either zero or negative), that is, $\Delta_t = F'(k_t) - R_t$ converges to a non-positive number, $c_t \to c > 0$ and $S_{I,t}R_t \to S_IR > 0$. It is immediate by (3.31) that if $\Delta_t$ converges to a negative number, then $\omega_t$ must eventually become negative—contradicting (3.34). Hence suppose $\Delta_t \to 0$. Distinguish two cases.

**Case I:** Suppose first that $\beta S_{I}R > 1$. Thus, $\prod_{s=1}^{\tau'} (\beta S_{I,s}R_s)$ is unbounded and diverges to $\infty$. Then, because $\omega_0$ is finite, we have that the partial sums in the expression for $\omega_0$ coming from (3.31) have to converge to zero,

$$\bar{\omega}_\tau = \sum_{r' \geq \tau + 1} \beta u'(c_{r'}) (F'(k_{r'}) - R_{r'}) \prod_{s=1}^{\tau'} (\beta S_{I,s}R_s) \to 0, \quad \text{as } \tau \to \infty.$$ 

Hence,

$$\omega_\tau = \left( \prod_{s=1}^{\tau} (\beta S_{I,s}R_s) \right)^{-1} \bar{\omega}_\tau \to 0,$$

contradicting the fact that $\omega_t$ is bounded away from zero by $u'(c)/S_I$. Therefore, $\beta S_{I}R > 1$ is not compatible with any interior steady state. (This argument does not use the fact that we focus on $\Delta_t \to 0$.)

**Case II:** Suppose $\beta S_{I}R < 1$. In this case, we show convergence of $\omega_\tau$ to zero directly. Fix $\epsilon > 0$. Let $\tau$ be large enough such that $\beta S_{I,s}R_s < b$ for some $b < 1$ and that $|u'(c_{r'}) \Delta_{r'}| < \epsilon (1 - b)$. Then,

$$|\omega_\tau| \leq \sum_{r' = r + 1}^{\tau} \epsilon (1 - b) b^{r'-1-r} = \epsilon.$$ 

Again, this contradicts the fact that $\omega_t$ is bounded away from zero by $u'(c)/S_I$.

This concludes our proof, establishing that the capital tax $\mathcal{T}_t = \Delta_t / F'(k_t)$ must converge.
to a positive number at the interior steady state.

3.E Derivation of the Inverse Elasticity Rule (3.4) and Proof of the Corollary

**Derivation of the Inverse Elasticity Rule.** In this section, we continue using the notation and results of Section 3.D. Consider equation (3.32). Because $\beta S_I R < 1$, $\omega_t$ converges to

$$\omega = \frac{\beta}{1 - \beta S_I R} (F'(k) - R) u'(c).$$

We make the additional convergence assumption

$$\sum_{\tau=1}^t \beta^{-\tau} \omega_{t-\tau} k_{t-\tau} \epsilon_{S_{t-\tau},R_t} \to \sum_{\tau=1}^\infty \beta^{-\tau} \epsilon_{S_t,R_t} \in [-\infty, \infty], \quad \text{as } t \to \infty,$$

which amounts to first taking the limit of the summands as $t \to \infty$, and then taking the limit of the series, instead of considering both limits simultaneously. Under this order of limits assumption, we can characterize the limit of equation (3.32) as $t \to \infty$,

$$S_{I,t} - \frac{u'(c_t)}{\omega_t} \to S_I - \frac{u'(c)}{\omega}.$$

Distinguish two cases according to whether $\omega = 0$ or $\omega \neq 0$. First, if $\omega = 0$, or equivalently the limit tax $T$ is zero, then (3.36) reveals that $\sum_{\tau=1}^\infty \beta^{-\tau} \epsilon_{S_t,R_t}$ is either plus or minus infinity. Therefore, the inverse elasticity formula holds in this case as both sides of (3.4) converge to zero.

Second, if $\omega \neq 0$, then by taking the limit of (3.32) as $t \to \infty$ and using the condition (3.35), we find

$$S_I - \frac{u'(c)}{\omega} = - \sum_{\tau=1}^\infty \beta^{-\tau} \epsilon_{S_t,R_t},$$

which can be rewritten as

$$\frac{\beta S_I R}{1 - \beta S_I R} (F'(k) - R) - R = - \frac{1}{1 - \beta S_I R} (F'(k) - R) \sum_{\tau=1}^\infty \beta^{-\tau+1} \epsilon_{S_t,R_t}.$$
Note that $F'(k) - R = \frac{T}{1-T} R$. Therefore, we can rearrange the condition to

$$\frac{\beta S_t R}{1 - \beta S_t R} - \frac{1 - T}{T} = -\frac{1}{1 - \beta S_t R} \sum_{\tau=1}^{\infty} \beta^{-\tau+1} \epsilon_{S,\tau}$$

$$\Rightarrow \ T = \frac{1 - \beta R S_t}{1 + \sum_{\tau=1}^{\infty} \beta^{-\tau+1} \epsilon_{S,\tau}}.$$ 

This is precisely the inverse elasticity formula (3.4).

**Proof of the Corollary.** Notice that by Proposition 15 the limit tax rate is positive, $T > 0$, conditional on convergence to an interior steady state. If now the inverse elasticity formula implies a negative tax rate, then either the regularity condition for the inverse elasticity rule is not satisfied or the allocation does not converge to an interior steady state.

### 3.F Infinite Sum of Elasticities with Recursive Utility

In this section, we prove the result that the infinite sum $\sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau}$ does not converge for any recursive utility function that is locally non-additive.

More specifically, we consider the capitalist’s optimization problem as in Section 3.2.3, just with recursive preferences as in Section 3.3.1, with $U = c$. In particular, the capitalist’s utility is characterized by the recursion $V_t = W(C_t, V_{t+1})$, assuming $W$ is twice continuously differentiable and strictly increasing in both arguments. Analogous to our analysis in Section 3.3.1, we define $\bar{\beta}(c) \equiv W_V(c, V(c))$ as the steady state discount factor along a constant consumption stream yielding steady state utility $V(c) = W(c, V(c))$.

Any such recursive utility function naturally yields an optimal savings function $a_{t+1} = S(R_t a_t, R_{t+1}, \ldots)$. Fix now constant interest rates $R$ and a steady state of the capitalist’s optimization problem $(a, c, V)$. Let $\beta = W_V(c, V(c)) = \bar{\beta}(c)$ the discount factor in that specific steady state. Define $\epsilon_{S,\tau} = \frac{1}{a} R \frac{\partial \log S}{\partial \log R_{t+\tau}}$. The following proposition characterizes the behavior of the infinite sum $\sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau}$.

**Proposition 23.** Suppose capitalists have recursive preferences represented by (3.5a) (see Section 3.3.1, with $U = c$). Then, if the discount factor is locally non-constant, $\bar{\beta}(c) \neq 0$, the series $\sum_{\tau=1}^{T} \beta^{-\tau} \epsilon_{S,\tau}$ does not have a finite limit as $T \to \infty$.

**Proof.** We first compute the elasticities $\epsilon_{S,\tau}$ and then prove that the infinite sum does not have a finite limit. To compute $\epsilon_{S,\tau}$, we consider an agent with the recursive preferences introduced above, who is at a steady state $(a, c, V)$ given a constant interest rates $R$. Note
that because utility is strictly increasing in a permanent increase in consumption at the steady
state, we have $\beta = W_V \in (0, 1)$.

The conditions for optimality are then,

$$V_t = W \left( R_t a_t - a_{t+1}, V_{t+1} \right)$$

$$W_C \left( R_t a_t - a_{t+1}, V_{t+1} \right) = R_{t+1} W_V \left( R_t a_t - a_{t+1}, V_{t+1} \right) W_C \left( R_{t+1} a_{t+1} - a_{t+2}, V_{t+2} \right).$$

The first equation is the recursion for utility $V_t$ and the second equation is the Euler equation.
In particular, note that the latter implies that $\beta R = 1$ at the steady state. Linearizing these
equations around the steady state (denoted without time subscripts) yields,

$$W_V \, dV_{t+1} = -W_C R \, da_t + W_C \, da_{t+1} + dV_t - W_C a \, dR_t \tag{3.37}$$

and

$$(RW_C W_V - RW_{CC} - WC) \, da_{t+1} + WC \, da_{t+2} - (W_V W_C + WC a) \, dR_{t+1}$$

$$+ (W_{CV} - RW_C W_{VV}) \, dV_{t+1} - W_{CV} \, dV_{t+2}$$

$$= (R^2 W_C W_V - W_{CC} R) \, da_t + (RW_C W_{VC} a - WC a) \, dR_t, \tag{3.38}$$

where all derivatives are evaluated at the steady state ($\left( R - 1 \right) a, V$). To save on notation,
we define $\omega \equiv W_V - \beta W_{CC} / W_C \in \mathbb{R}$, which is a term that will appear multiple times below.
We solve (3.37) and (3.38) by the method of undetermined coefficients, guessing

$$da_{t+1} = \omega \lambda \, da_t + a \sum_{\tau=0}^{\infty} \beta^\tau \theta_{t+\tau} \, dR_{t+\tau} \tag{3.39a}$$

$$dV_t = W_C R \, da_t + (WC a) \sum_{\tau=0}^{\infty} \beta^\tau \, dR_{t+\tau}. \tag{3.39b}$$

The form of equation (3.39b) is what is required by the Envelope condition. We are left to find
$\lambda$ and the sequence $\{ \theta_t \}$, where $\theta_{t+\tau} = \beta^{-\tau} \epsilon_{S,\tau}$, for $\tau \geq 1$, is exactly the sequence of elasticities
we are looking for. Substituting the guesses (3.39a) and (3.39b) into (3.38), we obtain an
expression featuring $da_t$, $da_{t+1}$, $da_{t+2}$ and $dR_{t+\tau}$ for $\tau = 0, 1, \ldots$. Setting the coefficient on
$da_t$ to zero gives a quadratic for $\lambda$,

$$\omega^2 \lambda^2 + \left( -(1 + R)\omega + (R - 1)\beta'(0) \right) \lambda + R = 0. \tag{3.40}$$
Note that the solution of this equation can never be zero, i.e. \( \lambda \neq 0 \). Also, if \( \overline{\beta}'(c) = 0 \), the term in parentheses simplifies to \(- (1 + R) \omega \) and the solutions are just \( \lambda = \omega^{-1} \) and \( \lambda = \omega^{-1} R \).

Setting the coefficient on \( dR_t \) to zero gives

\[
\theta_0 = \beta \omega \lambda.
\]

Similarly for \( dR_{t+1} \) we find after various simplifications,

\[
\theta_1 = \omega \lambda (\theta_0 - 1) + \lambda \left( \beta^2 a^{-1} + (1 - \beta) \overline{\beta}'(c) \right)
\]

and for \( dR_{t+\tau} \) after some more simplifications

\[
\theta_\tau = \omega \lambda \theta_{\tau-1} + \lambda (1 - \beta) \overline{\beta}'(c),
\]

for \( \tau = 2, 3, \ldots \). The result then follows from this expression: If \( \overline{\beta}'(c) \neq 0 \), the sum \( \sum_{\tau=1}^{T} \beta^{-\tau} \epsilon_{s,\tau} = \sum_{\tau=1}^{T} \theta_{\tau} \) cannot converge. To see this, consider

\[
\sum_{\tau=1}^{T} \theta_{\tau} = \theta_1 + \sum_{\tau=2}^{T} \theta_{\tau} = \theta_1 + \sum_{\tau=1}^{T-1} \omega \lambda \theta_{\tau} + \sum_{\tau=2}^{T} \lambda (1 - \beta) \overline{\beta}'(c).
\]

If the left hand side of this equation converged to some limit \( \Theta \in \mathbb{R} \), the right hand side of this equation would diverge since the last sum diverges (while all other terms would remain finite). Therefore, \( \sum_{\tau=1}^{T} \beta^{-\tau} \epsilon_{s,\tau} \) cannot converge to a finite limit.

\[\square\]

### 3.G Linearized Dynamics and Proof of Proposition 16

A natural way to prove Proposition 16 would be to linearize our first order conditions in (3.2), and to solve forward for the multipliers \( \mu_t \) and \( \lambda_t \) using transversality conditions, arriving at an approximate law of motion of the form

\[
\begin{pmatrix}
  k_{t+1} \\
  C_t
\end{pmatrix}
- \begin{pmatrix}
  k_t \\
  C_{t-1}
\end{pmatrix} = J \begin{pmatrix}
  k_t - k^* \\
  C_{t-1} - C^*
\end{pmatrix}.
\]

To maximize similarity with Kemp et al. (1993), however, we do not take that route; rather we start with the continuous time problem, derive its first order conditions and linearize them.
around the zero tax steady state. The problem in continuous time is

\[
\max \int_0^\infty e^{-\rho t} \left( u(c_t) + \gamma U(C_t) \right) dt \\
\text{s.t. } c_t + C_t + g + \dot{k}_t = f(k_t) - \delta k_t \\
\dot{C}_t = \frac{C_t}{\sigma} \left( \frac{f(k_t)}{k_t} - \delta - \frac{c_t}{k_t} - \rho \right).
\]

Let \( p_t \) and \( q_t \) denote the costates corresponding respectively to the states \( k_t \) and \( C_t \). The FOCs are,

\[
\begin{align*}
\dot{u}'(c_t) &= p_t c_t + q_t \frac{1}{\sigma} \frac{C_t}{k_t} \\
\dot{p}_t &= \rho p_t - p_t \left( f'(k_t) - \delta \right) + q_t \frac{1}{k_t} - q_t \frac{C_t}{k_t} \left( f'(k_t) - \delta \right) \\
\dot{q}_t &= \rho q_t - \gamma U'(C_t) - q_t \frac{1}{\sigma} \left( \frac{f(k_t)}{k_t} - \delta - \frac{c_t}{k_t} - \rho \right).
\end{align*}
\]

Just like Kemp et al. (1993), we require the two transversality conditions to hold,

\[
\lim_{t \to \infty} e^{-\rho t} q_t C_t = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{-\rho t} p_t k_t = 0.
\]

Denote the 4-dimensional state of this dynamic system by \( x_t = (k_t, C_t, p_t, q_t) \) and its unique positive steady state (the zero-tax steady state) by \( x^* = (k^*, C^*, p^*, q^*) \). The linearized system is,

\[
\dot{x}_t = J(x_t - x^*),
\]

where \( J \) is a \( 4 \times 4 \) matrix with determinant

\[
\det J = (1 - \sigma) \frac{f''(k^*) u'(c^*) \rho^2}{u''(c^*) \sigma^2} > 0.
\]

Its eigenvalues can be written as,

\[
\lambda_{1-4} = \frac{\rho}{2} \pm \left[ \left( \frac{\rho}{2} \right)^2 - \frac{\chi}{2} \pm \frac{1}{2} \left( \chi^2 - 4 \det J \right)^{1/2} \right]^{1/2},
\]

with

\[
\chi = \frac{\rho}{\sigma} \left( \frac{u'(c^*) - \gamma U'(C^*)}{u''(c^*) k^*} - \frac{f''(k^*) u'(c^*)}{u''(c^*)} \right).
\]
There are two "±" signs in (3.45). In the remainder, we number eigenvalues according to those two signs in (3.45): $\lambda_1$ has $++$, $\lambda_2$ has $+-$, $\lambda_3$ has $-+$, and $\lambda_4$ has $--$. For convenience, define $\gamma^*$ by $\gamma^* = u'(c^*)/U'(C^*)$.

In general, a solution $x_t$ to the linearized FOCs (3.43) can load on all four eigenvalues. However, taking the two transversality conditions into account restricts the system to only load on eigenvalues with $\text{Re}(\lambda_i) \leq \rho/2$. In Lemma 17 below, we show that this means the solution loads on eigenvalues $\lambda_3$ and $\lambda_4$.

**Lemma 17.** The eigenvalues in (3.45) can be shown to satisfy the following properties.

1. It is always the case that
   \[
   \text{Re} \lambda_1 \geq \text{Re} \lambda_2 \geq \rho/2 \geq \text{Re} \lambda_4 \geq \text{Re} \lambda_3.
   \]

2. If $\sigma > 1$, then $\det J < 0$, implying that
   \[
   \text{Re} \lambda_1 = \lambda_1 > \rho > \text{Re} \lambda_2 \geq \rho/2 \geq \text{Re} \lambda_4 > 0 > \lambda_3 = \text{Re} \lambda_3.
   \]  
   (3.47)

   In particular, there is a exactly one negative eigenvalue. The system is saddle-path stable.

3. If $\sigma < 1$ and $\gamma \leq \gamma^*$, then $\det J > 0$ and $\delta < 0$, implying that
   \[
   \text{Re} \lambda_1, \text{Re} \lambda_2 > \rho > 0 > \text{Re} \lambda_4, \text{Re} \lambda_3.
   \]  
   (3.48)

   In particular, there exist exactly two eigenvalues with negative real part. The system is locally stable.

4. If $\sigma < 1$ and $\gamma > \gamma^*$, the system may either be locally stable, or locally unstable (all eigenvalues having positive real parts).

**Proof.** We follow the convention that the square root of a complex number $a$ is defined as the *unique* number $b$ that satisfies $b^2 = a$ and has nonnegative real part (if $\text{Re}(b) = 0$ we also require $\text{Im}(b) \geq 0$). Hence, the set of all square roots of $a$ is given by $\{\pm b\}$. We prove the results in turn.

1. First, observe the following fact: Given a real number $x$ and a complex number $b$ with
nonnegative real part, it holds that \( \text{Re} (\sqrt{x + b}) \geq \text{Re} (\sqrt{x - b}) \).\(^{61}\) From there, it is straightforward to see that \( \text{Re} \lambda_1 \geq \text{Re} \lambda_2 \) and \( \text{Re} \lambda_4 \geq \text{Re} \lambda_3 \). Finally \( \text{Re} \lambda_2 \geq \rho/2 \geq \text{Re} \lambda_4 \) holds according to our convention of square roots having nonnegative real parts.

2. The negativity of \( \det J \) follows immediately from (3.44). This implies
\[
-\frac{\delta}{2} + \frac{1}{2} (\delta^2 - 4 \det J)^{1/2} > 0 > -\frac{\delta}{2} - \frac{1}{2} (\delta^2 - 4 \det J)^{1/2},
\]
and so (3.47) holds, using monotonicity of \( \text{Re} \sqrt{x} \) for real numbers \( x \).

3. The signs of \( \det J \) and \( \delta \) follow immediately from (3.44) and (3.46). In this case,
\[
-\delta/2 \pm 1/2 \text{Re} (\delta^2 - 4 \det J)^{1/2} > 0 \text{ proving (3.48).}
\]

4. This is a simple consequence of the fact that if \( \det J > 0 \), then either
\[
-\delta/2 \pm 1/2 \text{Re} (\delta^2 - 4 \det J)^{1/2} > 0, \text{ or } -\delta/2 \pm 1/2 \text{Re} (\delta^2 - 4 \det J)^{1/2} < 0,
\]
where under the latter condition the system is locally unstable. \( \square \)

### 3.H Proof of Proposition 17

In this proof, we first exploit the recursiveness of the utility \( V \) to recast the IC constraint (3.7) entirely in terms of \( V_t \) and \( W(U, V') \). Then, using the first order conditions, we are able to characterize the long-run steady state. Throughout this section, we denote by \( X_{zt} \) the derivative of quantity \( X \) with respect to \( z \), evaluated at time \( t \). To save on notation, we define \( f(k, n) \equiv F(k, n) + (1 - \delta)k \).

Let \( \beta_t \equiv \prod_{s=0}^{t-1} W_{Vs} \). Using the definition of the aggregator in (3.3) this implies that \( V_{ct} = \beta_t W_{Ut} U_{ct} \) and \( V_{nt} = \beta_t W_{Ut} U_{nt} \). Thus the IC constraint (3.7) can be rewritten as
\[
\sum_{t=0}^{\infty} \beta_t W_{Ut} (U_{ct} c_t + U_{nt} n_t) = W_{U_0} U_{c_0} \left( R_0 k_0 + R_0^b p_0 \right), \quad (3.49)
\]

\(^{61}\)To prove this, let \( \bar{b} \) denote the complex conjugate of \( b \) and note that \( \text{Re} (\sqrt{x + b}) \) is monotonic in the real number \( x \). Then, \( \text{Re} \left( \sqrt{x + b} \right) = \text{Re} \left( \sqrt{x + \bar{b}} \right) = \text{Re} \left( \sqrt{x - b + (b + \bar{b})} \right) \geq \text{Re} \left( \sqrt{x - \bar{b}} \right) \) where \( \bar{b} + b = 2 \text{Re}(b) \geq 0 \) and monotonicity are used.
and the planning problem becomes

$$\max_{\{V_t, c_t, n_t, R_0\}} V_0$$

s.t. $V_t = W(U(c_t, n_t), V_{t+1})$  \( (3.50) \)

RC (3.6), IC (3.49), $R_t \geq 1$.

To state the first order conditions, define for each $t \geq 0$, $A_{t+1} \equiv \frac{1}{\beta_{t+1}} \frac{\partial}{\partial V_{t+1}} \sum_{s=t}^{\infty} \beta_s W_U s(U c_s c_s + U n_s n_s)$ and $B_t \equiv \frac{1}{\beta_t} \sum_{s=t}^{\infty} \frac{\partial W_U s}{\partial V_{t+1}} (U c_s c_s + U n_s n_s)$. Let $\beta_t \nu_t$ be the present value multiplier on the Koopmans constraint (3.50), $\beta_t \lambda_t$ the present value multiplier on the resource constraint (3.6), and $\mu$ the multiplier on the IC constraint (3.49). As stated in the proposition, we assume that the capital tax bound $R_t \geq 1$ is not binding eventually, say from period $T$ onwards. The first order conditions for $V_{t+1}, c_t, n_t,$ and $k_{t+1}$ (in that order) are for each $t \geq T$ given by

$$-\nu_t + \nu_{t+1} + \mu A_{t+1} = 0$$

$$-\nu_t W_{Ut} U_c + \mu W_{Ut} (U c_t + U c_t c_t + U n_c n_t) + \mu B_t U_{c_t} = \lambda_t$$

$$\nu_t W_{Ut} U_n - \mu W_{Ut} (U n_t + U n_t c_t + U n_n n_t) - \mu B_t U_{n_t} = \lambda_t f_{n_t}$$

$$-\lambda_t + \lambda_{t+1} W_{V_t} f_{k_{t+1}} = 0.$$ 

Suppose the allocation converges to an interior steady state in $c, k,$ and $n$. Then $U_t$ and $V_t$ converge, as well as their first and second derivatives (when evaluated at $c_t, k_t,$ and $n_t$). Similarly, the representative agent’s assets $a_t$ converge to a value $a$, which can be characterized using a time $t + 1$ version of the IC constraint,

$$a = \lim_{t \to \infty} a_{t+1} = \lim_{t \to \infty} (W_{Ut} U_{c_t} + \beta_{t+1} R_{t+1})^{-1} \sum_{s=t+1}^{\infty} \beta_s W_U s(U c_s c_s + U n_s n_s)$$

$$= ((1 - \beta) U c R)^{-1} (U c c_c + U n n_n),$$

where $\bar{\beta} \equiv \bar{\beta}(V) = W_V \in (0, 1)$ (see footnote 28). Using this expression, we see that $A_{t+1}$, which can be written as,

$$A_{t+1} = \frac{W_{UV} t}{W_{Vt}} (U c_t c_t + U n_t n_t) + \frac{W_{VV} t}{W_{Vt}} \beta_{t+1}^{-1} \sum_{s=t+1}^{\infty} \beta_s W_U s(U c_s c_s + U n_s n_s),$$. 

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converges as well, to some limit $A$,

$$A_{t+1} \to \frac{\beta_U}{\beta} (U_c c + U_n n) + \frac{\beta_V}{\beta} W_U U_c R a$$

$$= \left( \frac{1 - \beta}{W_U} \beta_U + \beta_V \right) \frac{1}{\beta} W_U U_c R a = \frac{\bar{\beta}'(V)}{\beta} W_U U_c R a \equiv A. \quad (3.51)$$

where we defined $\beta_X \equiv W_{V,X}$ and $X = U, V$. Similarly, we can show that $B_t$ converges to some finite value $B$. Taking the limits of quantities in the first order conditions above, we thus find a system of equations for multipliers $\nu_t, \mu, \lambda_t$,

$$-\nu_t + \nu_{t+1} + \mu A = 0 \quad (3.52a)$$

$$-\nu_t + \mu \left( 1 + \frac{U_{cc}}{U_c} + \frac{U_{nc} n}{U_c} \right) + \mu \frac{B}{W_U} = \lambda_t \frac{1}{W_U U_c} \quad (3.52b)$$

$$-\nu_t + \mu \left( 1 + \frac{U_{nn}}{U_n} + \frac{U_{mn} n}{U_n} \right) + \mu \frac{B}{W_U} = -\lambda_t \frac{f_n}{W_U U_n} \quad (3.52c)$$

$$-\lambda_t + \lambda_{t+1} \beta f_k = 0. \quad (3.52d)$$

Substituting out $\lambda_t$ from (3.52d) using (3.52a) and (3.52b), we find

$$\beta f_k - 1 = \frac{\lambda_t}{\lambda_{t+1}} - 1 = -\frac{W_U U_c}{\lambda_{t+1}} \mu A. \quad (3.53)$$

We now move to the two main results of this section. First, we show that steady state capital taxes are zero. Second, we show that steady state labor taxes are also zero, unless $\bar{\beta}'(V) = 0$, when preferences are locally additive separable.

**Lemma 18.** At an interior steady state, capital taxes are zero, i.e. $\beta f_k = 1$.

**Proof.** If $A = 0$ or $\mu = 0$ the result is immediate from (3.53). Suppose instead that $A \neq 0$ and $\mu \neq 0$. Then, (3.52a) implies that $\nu_t$ and hence $\lambda_t$ diverges to $+\infty$ or $-\infty$. Then again, $\beta f_k = 1$ follows from (3.53).

We move to our second result.

**Lemma 19.** At an interior steady state, labor taxes are zero, i.e. $\tau^n = 1 + \frac{U_n}{U_c f_n} = 0$ if $\bar{\beta}'(V) \neq 0$ and $a > 0$.

---

62 Notice that $\lambda_t \to 0$ requires $\mu = 0$ by (3.54), so the optimal allocation is first best to begin with, implying $\beta f_k = 1$. 

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Proof. By combining equations (3.52b) and (3.52c) we find an expression for \( \tau^n \),

\[
\lambda_t \tau^n = \frac{W_{it} U_n}{f_n} \left( \frac{U_{cc} U_n - U_{cn} U_{cn} - U_{nn} U_n}{U_c U_n} \right),
\]

(3.54)

Note that by normality of consumption and labor the term in brackets is negative, \( \frac{U_{cc} U_n - U_{cn} U_{cn} - U_{nn} U_n}{U_c U_n} < 0 \). It is immediate from (3.54) that \( \tau^n = 0 \) if \( \lambda_t \) diverges to either \( +\infty \) or \( -\infty \).\(^{63}\) Suppose \( \lambda_t \rightarrow \lambda \in \mathbb{R} \). We distinguish whether \( \mu = 0 \) or \( \mu \neq 0 \). If \( \mu = 0 \), the economy was first best to start with, and the labor tax must be zero at any date, including at the steady state. If \( \mu \neq 0 \), convergence of \( \lambda_t \) (equivalent to convergence of \( \nu_t \)) necessitates that \( A = 0 \), using (3.52a). But then (3.51) contradicts our assumptions that preferences are not locally additively separable, \( \beta'(V) \neq 0 \), and steady state assets are positive \( a > 0 \). \( \square \)

3.1 Proof of Proposition 18

In this section, we prove Proposition 18. The proof is organized as follows. In Section 3.1.1 we introduce the planning problem, derive and discuss the first order conditions, and define the largest feasible level of initial government debt \( \tilde{b} \). Section 3.1.2 then focuses on parts A and B (i) of Proposition 18. Finally, Section 3.1.3 proves the bang-bang property and parts B (ii) and C of Proposition 18.

3.1.1 Planning problem and first order conditions

As in the statement of the proposition, we fix some positive initial level of capital \( k_0 > 0 \). The problem under scrutiny is

\[
V(b_0) \equiv \max_{\{c_t, n_t, k_t, r_t\}} \int_0^{\infty} e^{-\rho t} (u(c_t) - v(n_t)) dt
\]

(3.55a)

\[
\dot{c}_t = c_t \frac{1}{\sigma} (r_t - \rho) \quad (3.55b)
\]

\[
c_t + g + \dot{k}_t = f(k_t, n_t) - \delta k_t \quad (3.55c)
\]

\[
\int_0^{\infty} e^{-\rho t} (u'(c_t)c_t - v'(n_t)n_t) dt \geq u'(c_0)(k_0 + b_0) \quad (3.55d)
\]

\( c_t > 0, n_t \geq 0, k_t \geq 0, r_t \geq 0 \)

where recall that \( u(c) = c^{1-\sigma}/(1 - \sigma) \) and \( v(n) = n^{1+\zeta}/(1 + \zeta) \), \( \zeta > 0 \). In the entire analysis in this section, we write value functions such as \( V(b_0) \) without explicit reference to \( k_0 \) since \( ^{63}\)Since \( A_t \rightarrow A \neq 0 \) and \( \mu \) is constant over time, \( \nu_t \) and thus also \( \lambda_t \) have a well-defined limit in \( [-\infty, \infty] \).
we treat \(k_0\) as fixed. The current-value Hamiltonian of this optimal control problem with
subsidiary condition \((3.55d)\) (see, e.g., Gelfand and Fomin, 2000) can be written as

\[
H(c, k; n, r; \lambda, \eta, \mu) = \Phi_v u(c) - \Phi_v v(n) + \eta c \frac{1}{\sigma} (r - \rho) + \lambda (f(k, n) - \delta k - c - g),
\]

where we defined \(\Phi_v \equiv 1 + \mu(1 + \zeta)\) and \(\Phi_u \equiv 1 + \mu(1 - \sigma)\) with \(\mu\) being the multiplier on the
IC constraint; and where we denoted the costates of consumption and capital by \(\eta_t\) and \(\lambda_t\),
respectively. Notice that \(\eta_t \leq 0\) or else \(r_t = \infty\) were optimal, violating the resource constraint.

Problem \((3.55a)\) implies the following first order conditions for the controls \(\{n_t, r_t\}\),

\[
\Phi_v u'(n_t) = \lambda_t f_n(k_t, n_t) \tag{3.57a}
\]

\[
r_t = \begin{cases} 
0 & \text{if } \eta_t < 0 \\
\in [0, \infty) & \text{if } \eta_t = 0,
\end{cases} \tag{3.57b}
\]

the following laws of motion for the costates,

\[
\dot{\eta}_t = \rho \eta_t + \lambda_t - \Phi_u u'(c_t) \tag{3.57c}
\]

\[
\dot{\lambda}_t = (\rho - r_t^*) \lambda_t \tag{3.57d}
\]

and the following optimality condition for the initial state of consumption \(c_0\),

\[
\eta_0 = -\mu \sigma c_0^{\sigma - 1} (k_0 + b_0). \tag{3.57e}
\]

In equation \((3.57d)\) we defined the before-tax return on capital as \(r_t^* = f_k(k_t, n_t) - \delta\). The condi-
tions \((3.57a)-(3.57e)\), together with the constraints \((3.55b)-(3.55d)\) and the two transversality
conditions

\[
\lim_{t \to 0} e^{-\rho t} \lambda_t k_t = 0 \tag{3.57f}
\]

\[
\lim_{t \to 0} e^{-\rho t} \eta_t c_t = 0 \tag{3.57g}
\]

are sufficient for an optimum if we are able to establish that the planning problem \((3.55a)\) is a
concave maximization problem, or can be transformed into one using variable transformations.

The first order conditions \((3.57a)-(3.57e)\) (though not the transversality conditions \((3.57f)\)
and \((3.57g)\)) are necessary at an optimum since interiority is ensured by the imposition of
Inada conditions; that is, with the exception when that optimum is also maximizing the
subsidiary constraint, which is the IC constraint in our case (see Gelfand and Fomin, 2000).
More specifically, the above first order conditions are not necessary when the optimum to (3.55a) achieves the supremum in

$$
\bar{b} = \sup_{\{c_t, n_t, k_t\}} \frac{1}{u'(c_0)} \int_0^\infty e^{-pt} (u'(c_t)c_t - v'(n_t)n_t) \, dt - k_0
$$

subject to the two other constraints (3.55b) and (3.55c). We deliberately formulated (3.58) in a way to define $\bar{b}$ as the highest level of $b_0$ for which there can possibly exist a feasible allocation. Notice that $\bar{b} \in [-\infty, \infty]$, allowing for $\bar{b} = -\infty$ if no feasible allocation exists at all (which might happen if $g$ is very large), and $\bar{b} = \infty$ if there exists a feasible allocation for any value of $b_0$.

Since in the case that $b_0 = \bar{b}$ the supremum in (3.58) is attained, there are still necessary first order conditions the allocation satisfies, namely the ones corresponding to (3.58). These are exactly the same as (3.57a)-(3.57e) after substituting $\mu \eta_t$ for $\eta_t$ and $\mu \lambda_t$ for $\lambda_t$, and then dividing by $\mu$ and setting $\mu = \infty$. This replaces $\Phi_u$ by $(1 - \sigma)$ and $\Phi_v$ by $(1 + \zeta)$ in (3.57a)-(3.57c), leaves (3.57d) unchanged and alters (3.57e) to $\eta_0 = -\sigma c_0^{\sigma-1}(k_0 + b_0)$.

One additional remark about the setup in (3.55a) is in place. We stated an inequality IC constraint (3.55d), corresponding to a non-negative multiplier $\mu$. This is without loss of generality in our setup, since at any optimum, $\mu$ will indeed be non-negative: From the first order condition (3.57e), we see that our assumption of positive initial private wealth, $k_0 + b_0 > 0$, together with the non-positivity of $\eta_0$ means that $\mu \geq 0$.

### 3.1.2 Proof of parts A and B (i)

Our proof in this subsection proceeds in three steps. First, we characterize the space of solutions to a restricted planning problem, in which the length $T$ of capital taxation is restricted to be infinity. Then we use these insights to prove that $T = \infty$ is in fact optimal in the unrestricted planning problem for levels of initial debt $b_0 \in [b, \bar{b}]$ (with non-empty interior if $\sigma > 1$). Finally, we show that for all $b_0 < \bar{b}$ there are feasible policies with $T < \infty$. Throughout, we assume that $\sigma \geq 1$, as is assumed in parts A and B (i) of Proposition 18.

1st step: The restricted problem. We start by studying a restricted planning problem, where we restrict ourselves to the case of indefinite capital taxation (at its upper bound). Effectively, this implies that $r_t = 0$ for all $t$ and the path of $c_t$ is entirely characterized by $c_0$ and (3.55b). To characterize this restricted problem, it will prove useful to define the minimum discounted sum of labor disutilities, henceforth effective disutility from labor, needed...
to sustain this path \( \{c_t\} \) as

\[
\tilde{v}(c_0) = \min_{\{n_t, k_t\}} \int_0^\infty e^{-\rho t} v(n_t) dt
\]

\[
\text{s.t. } c_0 e^{-\rho t} + g + \dot{k}_t = f(k_t, n_t) - \delta k_t.
\]

We prove important properties of the effective disutility \( \tilde{v} \) and the optimal control problem (3.59a) in Lemma 20.

**Lemma 20.** The function \( \tilde{v} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is strictly convex, strictly increasing and continuously differentiable at any \( c_0 \in \mathbb{R}_{++} \). It satisfies \( \tilde{v}(0) > 0 \). Moreover, for any \( c_0 \geq 0 \), there exists a unique solution \( \{n_t^\infty, k_t^\infty\} \) and a costate of capital \( \{\lambda_t^\infty\} \). Upon defining \( c_t^\infty = c_0 e^{-\rho t} \) it holds that, \( \{c_t^\infty, n_t^\infty, k_t^\infty, \lambda_t^\infty\} \rightarrow (c^\infty, n^\infty, k^\infty, \lambda^\infty) \), where

\[
c^\infty = 0
\]
\[
f(k^\infty, n^\infty) = \delta k^\infty + g
\]
\[
f_k(k^\infty, n^\infty) = \rho + \delta
\]
\[
v'(n^\infty) = \lambda^\infty f_n(k^\infty, n^\infty).
\]

**Proof.** The proof has 4 steps: First, we prove existence and uniqueness of the solution to a “bounded” version of the optimal control problem (3.59a) with bounds on \( n_t \) and \( k_t \). Second, we characterize the optimal paths \( \{k_t^\infty, n_t^\infty\} \) of this problem. Third, we show that increasing the bounds on \( k_t \) and \( n_t \) makes the bounded problem equivalent to (3.59a). Finally, we establish that the claimed properties of \( \tilde{v} \).

**First step.** For this step, relax the constraint (3.59b) to be an inequality “\( \leq \)” and introduce upper bounds \( \overline{k} > 0 \) and \( \overline{n} > 0 \) on \( k \) and \( n \). Using the definition of \( k^\infty \) and \( n^\infty \) in (3.60a)-(3.60b), pick \( \overline{k} > \max\{k_0, k^\infty\} \) and pick \( \overline{n} \) large enough so that \( \overline{n} > n^\infty \) and so that
\( \dot{k}_0 > 0 \) is feasible at time \( t = 0. \) This means the problem is given by

\[
\bar{v}(c_0) = \min_{\{n_t, k_t\}} \int_0^\infty e^{-\rho t} v(n_t) dt
\]

s.t. \( c_0 e^{-\rho t} + g + \dot{k}_t \leq f(k_t, n_t) - \delta k_t \)

\( k_t \in [0, \bar{k}], \quad n_t \in [0, \bar{n}]. \)

This problem is clearly a strictly convex minimization problem (strictly convex objective and a convex constraint), even without bounds on \( k \) and \( n, \) and therefore at most admits a single solution. A straightforward application of Seierstad and Sydsaeter (1987, Section 3.7, Theorem 15) to the optimal control problem \((3.61a)\) reveals that there always exist paths \( \{n_t^\infty, k_t^\infty\} \) that attain the minimum in \((3.61a).\)

Second step. We now study the long-run properties of the solution to the problem \((3.61a)\). Before we dive into the details, we note that \( k^\infty > 0 \) and \( n^\infty > 0 \) are uniquely determined by \((3.60a)\) and \((3.60b)\) due to the Inada properties of \( f_k(\cdot, n) \) and the fact that \( f/k \geq f_k. \) \( \lambda^\infty \) follows from \((3.60c)\). At each point where \( k_t < \bar{k} \) and \( n_t < \bar{n}, \) the necessary first order conditions corresponding to \((3.61a)\) are given by

\[
\dot{v}'(n_t) = \lambda_t f_n(k_t, n_t)
\]

\[
\dot{\lambda}_t = \lambda_t (\rho - r_t^*),
\]

where \( \lambda_t \) denotes the costate of \( k_t. \) Notice that \( n_t \) is continuous, as an immediate consequence of \((3.62a)\) and of the fact that both \( k_t \) and \( \lambda_t \) are continuous. Also note that \((3.62a)\) implies \( \lambda_t \geq 0, \) meaning our relaxation of the resource constraint \((3.59b)\) to an inequality was without loss of generality. Using the resource constraint \((3.59b)\) and \((3.62a)-(3.62b),\) we can derive an ODE system entirely in terms of \( n_t \) and \( k_t, \) consisting of the resource constraint \((3.59b)\) itself and of

\[
(\zeta + \alpha_t) \frac{\dot{n}_t}{n_t} = \rho + (1 - \alpha_t) \delta - \alpha_t \gamma + c_t \frac{g + c_t}{k_t},
\]

where \( \alpha_t = \alpha(k_t/n_t) = \frac{\partial \log f_n}{\partial \log (k_t/n_t)}. \) We can also abbreviate the ODEs as \( \dot{k} = \dot{k}(k, n, c_t) \) and

---

\( ^{64} k_0 > 0 \) iff \( f(k_0, \bar{n}) - \delta k_0 - g - c_0 > 0. \)

\( ^{65} \) This relies on our choice of \( \bar{n} \) which ensures that \( \dot{k}_0 > 0, \) so even for low values of \( k_0 \) there exist admissible paths \( \{n_t, k_t\}. \)

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Figure 3.I.1: Phase diagram characterizing the solution to the restricted problem (3.59a).

\[ \dot{n} = \dot{n}(k, n, c_t) \]. Define the two sets

\[ A_t \equiv \{ (k, n) | \dot{n}(k, n, c_t) > 0, \dot{k}(k, n, c_t) > 0 \} \]
\[ B_t \equiv \{ (k, n) | \dot{n}(k, n, c_t) < 0, \dot{k}(k, n, c_t) < 0 \} \].

To illustrate these sets, note that for large \( t \), \( c_t \approx 0 \), we can draw the phase diagram that corresponds to the ODE system. This is done in Figure 3.1.1 for the Cobb-Douglas case where \( a_t = \text{const.} \). In that figure, \( A_t \) is the top right area, while \( B_t \) is the bottom left area.

We now argue that the state \( (k_t, n_t) \) can never be in \( A_t \) for any \( t \), and never be in \( B_t \) for large \( t \). If for any \( t \), \( (k_t, n_t) \in A_t \), \( n_t \) can be lowered to achieve \( \dot{k}_t = 0 \) at all times, clearly improving the objective. If there does not exist a time \( s \) such that \( (k_t, n_t) \notin B_t \) for \( t > s \), then it must be that asymptotically \( (k_t, n_t) \in B_t \) for all sufficiently large \( t \). But in that case, \( k_t \to 0 \), contradicting feasibility (since government spending is positive, \( g > 0 \)). Therefore, it must be that \( (k_t^\infty, n_t^\infty) \to (k^\infty, n^\infty) \).

Note that the optimal costate \( \lambda_t^\infty \) can be computed using the first order condition for labor, (3.62a). Due to the steady state convergence of the system, the transversality condition \( \lim_{t \to \infty} e^{-\rho t} \lambda_t^\infty k_t^\infty = 0 \) naturally holds.

Third step. We now show that there exists a sufficiently large bound \( \overline{n} \) such that the solutions of the problem without bounds, (3.59a) and the problem with bounds (3.61a) coincide. This is the case if there exists a \( \overline{n} \) such that \( n_t^\infty < \overline{n} \) at the optimum at all times \( t \). Assume the contrary held, that is, no matter how large \( \overline{n} \) is, at the corresponding optimal path, which we denote by \( (k_t^\infty(\overline{n}), n_t^\infty(\overline{n})) \) to emphasize the dependence on \( \overline{n} \), there exist times \( t \) where \( n_t^\infty(\overline{n}) = \overline{n} \). Since \( n_t^\infty(\overline{n}) \) can never approach \( \overline{n} \) from below (this would require \( (k_t, n_t) \in A_t \)), it must be that there exists a time \( s > 0 \) such that \( n_t^\infty(\overline{n}) = \overline{n} \) for any \( t \in [0, s] \) and any arbitrarily large \( \overline{n} \). It is straightforward to see that this lets \( k_s^\infty(\overline{n}) \) grow unboundedly.
large, in particular leading to \((k^\infty_\pi, n^\infty_\pi) \in A_s\) — a contradiction. This completes our
proof that problem (3.59a) admits a unique solution, which approaches the steady state
\((k^\infty, n^\infty)\) asymptotically.

Fourth step. In our final step, we derive the claimed properties of \(\tilde{v}\). First, since the
objective is strictly convex, \(\tilde{v}\) is strictly convex. It is also strictly increasing since the constraint
tightens with larger \(c_0\). \(\tilde{v}(0) > 0\) follows directly from \(g > 0\). For differentiability, pick any
\(\tilde{c}_0 \in \mathbb{R}_{++}\) and denote the associated optimal path for capital by \(\{\tilde{k}_t^\infty\}\). Following the logic in
Benveniste and Scheinkman (1979) we can define a strictly convex and differentiable function
\(w(c_0) = \int_0^\infty e^{-\rho t} \frac{1}{1+\zeta} N \left( \tilde{k}_t^\infty, c_0 e^{-\rho \rho t} + g + \tilde{k}_t^\infty + \delta \tilde{k}_t^\infty \right)^{1+\zeta} dt\) where \(N(k, y) \equiv f(k, \cdot)^{-1}(y)\) is
the level of labor needed to fund output \(y \geq 0\) given capital \(k > 0\). By construction,
\(w(\tilde{c}_0) = v(\tilde{c}_0)\) and \(w(c_0) \geq v(c_0)\) locally around \(\tilde{c}_0\).\(^{66}\) This implies that \(\tilde{v}\) is differentiable at
any \(c_0 \in \mathbb{R}_{++}\) with derivative\(^{67}\)

\[
\tilde{v}'(c_0) = \int_0^\infty e^{-(\rho + \rho/\sigma) t} \frac{v'(n_t^\infty)}{f_n(k_t^\infty, n_t^\infty)} dt.
\]

This formula for the derivative of \(\tilde{v}\) is equivalent to (3.60d) after expressing the former in
terms of \(\lambda_t\) using the first order condition for labor (3.62a). This concludes our proof of
Lemma 20.\(\square\)

The effective disutility \(\tilde{v}(c_0)\) is convenient since in the original planning problem (3.55a),
labor disutility appears in present value terms both in the objective as well as in the IC
constraint (3.55d). Moreover, due to the assumption of power disutility, both present values
are essentially \(\tilde{v}(c_0)\) up to a constant factor. The restricted version of the original planning
problem (3.55a) can now be simply written as restricted problem

\[
V_\infty(b_0) \equiv \max_{c_0 > 0} u(c_0)^{\sigma} - \tilde{v}(c_0)
\]

\[
c_0^{1-\sigma} \frac{\sigma}{\rho} (1 + \zeta) \tilde{v}(c_0) \geq c_0^{\sigma} - (k_0 + b_0).
\]

We obtained (3.64a) from the original problem (3.55a) by requiring that \(T = \infty\) and using
the definition of \(\tilde{v}\). We characterize the restricted problem (3.64a) in the following lemma.

**Lemma 21.** There exists a level of initial debt \(\bar{b} \in \mathbb{R}\) such that a solution to the restricted
planner's problem (3.64a) exists if and only if \(b_0 \leq \bar{b}\). For each \(b_0 \leq \bar{b}\), there is a unique
optimum \(c_0^\infty(b_0) \in \mathbb{R}_{++}\) and for each \(b_0 < \bar{b}\) there is a unique multiplier \(\mu^\infty(b_0) \in [0, \infty)\) on

\(^{66}\)The expression for \(w\) is obtained by substituting the resource constraint (3.59b) into the objective (3.59a).

\(^{67}\)Notice that the derivative must be finite since \(\tilde{v}\) is strictly convex and finite-valued for any \(c_0 \in \mathbb{R}_+\).
the IC constraint (3.64b) such that

\[ \Phi_u u'(c_0) \frac{\sigma}{\rho} - \Phi_v \tilde{v}'(c_0) = -\sigma \mu c_0^{-\sigma-1} (k_0 + b_0), \]

(3.65)

for \( c_0 = c_0^\infty(b_0), \) \( \mu = \mu^\infty(b_0) \). Finally, there exists some \( b^* < \bar{b}^* \) such that \( \mu^\infty : [b^*, \bar{b}^*) \to [0, \infty) \) is a continuous and strictly increasing bijection.

**Proof.** First, notice that the IC constraint of the restricted planning problem, (3.64b), can be rewritten as

\[ c_0^\sigma \rho - (1 + \zeta) c_0^\sigma \tilde{v}(c_0) \geq k_0 + b_0. \]

(3.66)

Observe that this is a convex constraint, as its left hand side is strictly concave. It is also strictly increasing at \( c_0 = 0 \) and diverges to \(-\infty\) for large \( c_0 \). Therefore, there exists an interior maximum at some \( \bar{c} > 0 \). By definition, \( c_0 = \bar{c} \) is the only value that is compatible with the IC constraint if \( b_0 = \bar{b}^* \), where we defined

\[ \bar{b}^* \equiv \max_{c_0 > 0} c_0^\sigma \rho - (1 + \zeta) c_0^\sigma \tilde{v}(c_0) - k_0. \]

(3.67)

The maximizer \( \bar{c} \) is then characterized by the first order conditions

\[ \frac{\sigma}{\rho} \bar{c} - (1 + \zeta) c_0^\sigma \tilde{v}(c_0) + (1 + \zeta) \tilde{v}'(c) = 0. \]

(3.68)

For any \( b_0 > \bar{b}^* \) the set of feasible \( c_0 \) compatible with the IC constraint (3.64b) is empty, so the restricted planning problem (3.64a) has a solution precisely when \( b_0 \leq \bar{b}^* \).

An advantage of writing the IC constraint as in (3.66) is that it allows us to see that the restricted problem (3.64a) has a strictly concave objective with a convex and bounded constraint set. The objective attains its unconstrained maximum at some \( c^* \in (0, \infty) \) satisfying

\[ u'(c^*) \frac{\sigma}{\rho} = \tilde{v}'(c^*). \]

We can show that \( c^* > \bar{c} \) since the objective is increasing at \( \bar{c} \),

\[ u'(\bar{c}) \frac{\sigma}{\rho} - \tilde{v}'(\bar{c}) = (1 + \zeta) \sigma \bar{c}^{-1} \tilde{v}(\bar{c}) + \zeta \tilde{v}'(\bar{c}) > 0, \]

where we used the first order condition for \( \bar{c} \), (3.68). Define \( b^* \equiv c^* \frac{\sigma}{\rho} - (1 + \zeta) c^* \tilde{v}(c^*) - k_0 \), so that \( c^* \) lies in the constraint set (3.64b) if and only if \( b_0 \leq b^* \)—or in other words, the constraint holds with equality for any \( b_0 \geq b^* \). We next show that there exists (a) a strictly decreasing

---

Note that for \( \sigma = 1 \), (3.66) reads \( c_0 (\frac{\sigma}{\rho} - (1 + \zeta) \tilde{v}(c_0)) \geq k_0 + b_0 \) and by positivity of \( k_0 + b_0 \) and monotonicity of \( \tilde{v} \), this means that \( \frac{\sigma}{\rho} - (1 + \zeta) \tilde{v}(0) > 0 \) (which is exactly equal to the derivative of the left hand side of (3.66) at \( c_0 = 0 \)).
(and hence continuous) bijection $c^\infty : [b^*, \overline{b}^*] \rightarrow (\overline{c}, c^*)$ and (b) a strictly increasing (and hence continuous) bijection $\mu^\infty : [b^*, \overline{b}^*] \rightarrow [0, \infty)$ such that $c^\infty(b_0)$ is the unique solution to the strictly concave problem (3.64a), and constraint (3.64b) has Lagrange multiplier $\mu^\infty(b_0)$, for any $b_0 \in [b^*, \overline{b}^*]$.

Take any $c_0 \in (\overline{c}, c^*)$. Clearly, $c_0$ is optimal with Lagrange multiplier $\mu$ when initial debt is $b_0$ if the three objects $c_0, \mu, b_0$ satisfy the first order condition of the problem—which can easily be seen to be given by (3.65)—and the constraint (3.64b). By substituting out $b_0$ from (3.65) using the constraint, the first order condition can be expressed as function of $\mu$,

$$\mu = \frac{\frac{\sigma}{\rho} - c_0^\sigma \partial'(c_0)}{(1 + \zeta) \sigma c_0^\sigma - 1 \partial + (1 + \zeta) c_0^\sigma \partial'(c_0) - \sigma / \rho} \equiv M(c_0).$$

For $c_0 \in (\overline{c}, c^*)$, the denominator is positive and strictly increasing in $c_0$, approaching 0 for $c_0 \searrow \overline{c}$; while the numerator is strictly decreasing and non-negative, with a zero at $c_0 = c^*$.

This defines a strictly decreasing bijection $M : (\overline{c}, c^*) \rightarrow [0, \infty)$. From the constraint (3.64b), we see that $b_0 = c_0 \frac{\sigma}{\rho} - (1 + \zeta) c_0^\sigma \partial(c_0) - k_0 \equiv B(c_0)$

which, by definition of $\overline{b}^*$ and $\overline{c}$, defines a strictly decreasing bijection $B : (\overline{c}, c^*) \rightarrow [\overline{b}, \overline{b}^*)$. It follows that for any $b_0 \in [\overline{b}, \overline{b}^*)$, the unique solution to (3.64a) is given by $c^\infty(b_0) = B^{-1}(b_0)$, with associated multiplier $\mu^\infty(b_0) = M(B^{-1}(b_0))$. This concludes the proof.

We finished our characterization of the restricted planning problem and are now ready for the second and main part of the proof of Proposition 18.

2nd step: Optimality of $T = \infty$ in the unrestricted problem. Before we proceed to prove the optimality of $T = \infty$ in the unrestricted problem, we establish that $\overline{b}^*$ is not just the upper bound of possible initial debt in the restricted planning problem, but equal to $\overline{b}$, the one in the unrestricted planning problem (3.55a).

Lemma 22. Let $b_0 \in \mathbb{R}$ and $\sigma \geq 1$. The constraints (3.55b), (3.55c), (3.55d) define a non-empty set for $\{c_t, n_t, k_t, r_t\}$ if and only if $b_0 \leq \overline{b}^*$. In particular, $\overline{b} = \overline{b}^*$. Moreover, if $b_0 = \overline{b}^*$ then capital is necessarily taxed at the maximum, $T = \infty$.

Proof. It suffices to show that the constraint set in the original problem is empty for $b_0 > \overline{b}^*$, and that $T = \infty$ is necessary for $b_0 = \overline{b}^*$. We show both by proving that any $b_0 \geq \overline{b}^*$ is infeasible with if capital is not taxed at its upper bound in all periods.

Hence fix some $b_0 \geq \overline{b}^*$ and assume it was achievable without $T = \infty$ by $\{c_t, n_t, k_t, r_t\}$. Then, it must be that $r_t > 0$ on some non-trivial interval, and the path of consumption
is described by the Euler equation (3.55b), as always. Let the initial consumption value be \( c_0 \) and denote by \( \hat{c}_t \) the path which starts at the same initial consumption \( \hat{c}_0 = c_0 \) but keeps falling at the fastest possible rate \(-\rho/\sigma\) forever, corresponding to \( T = \infty \). Similarly, define by \( \hat{n}_t \) the path for labor which keeps \( k_t \) fixed but satisfies the resource constraint with consumption equal to \( \hat{c}_t \). Clearly, \( \hat{n}_t \leq n_t \) for all \( t \) and \( \hat{n}_t < n_t \) on a positive-measure set of times \( t \). Because the left hand side of (3.55d) is weakly decreasing in \( c_t \) and strictly decreasing in \( n_t \), this strictly relaxes the IC constraint. Hence,

\[
\int_0^\infty e^{-\rho t} \hat{c}_t^{1-\sigma} dt - \int_0^\infty e^{-\rho t} \hat{v}(\hat{n}_t) dt > \hat{c}_0^{-\sigma} (k_0 + b_0).
\]

Notice, however, that for \( T = \infty \), we can do even better by optimizing over labor (not necessarily keeping capital constant, see (3.59a)), leading to

\[
\hat{c}_0^{1-\sigma} \frac{\sigma}{\rho} - (1 + \zeta) \hat{v}(\hat{c}_0) > \hat{c}_0^{-\sigma} (k_0 + b_0).
\]

By definition of \( \overline{b} \) in (3.67) this is a contradiction to \( b_0 \geq \overline{b} \). Therefore, \( \overline{b} \) is equal to the highest sustainable debt level in the original problem, \( \overline{b} \), and can only be achieved with \( T = \infty \).

Our next lemma establishes that the unrestricted problem (3.55a) is a strictly concave maximization problem with convex constraints. This will be helpful when proving uniqueness in Lemma 24 below.

**Lemma 23.** Suppose \( \sigma \geq 1 \). The unrestricted problem (3.55a) can be transformed into a strictly concave maximization problem with convex constraints, using variable substitution. Therefore, any optimum of (3.55a) is unique when \( \sigma \geq 1 \).

**Proof.** We rewrite (3.55a) in terms of the two variables \( u_t \equiv u(c_t) \in (-\infty, 0) \) and \( v_t \equiv v(n_t) \in [0, \infty) \) instead of \( c_t \) and \( n_t \). We only consider the case \( \sigma > 1 \); the case \( \sigma = 1 \) is analogous. For
\( \sigma > 1 \), the substitution yields

\[
V(b_0) \equiv \max_{\{u_t, v_t, k_t\}} \int_0^\infty e^{-\sigma t} (u_t - v_t) \, dt
\]

\[
\dot{u}_t \geq (\sigma - 1) \frac{\rho}{\sigma} u_t
\]

\[
((1 - \sigma)u_t)^{-1/(\sigma - 1)} + g + \delta k_t \leq f(k_t, ((1 + \zeta)u_t)^{1/(1 + \zeta)}) - \delta k_t
\]

\[
\int_0^\infty e^{-\sigma t} ((1 - \sigma)u_t - (1 + \zeta)v_t) \, dt \geq ((1 - \sigma)u_0)^{\sigma/(\sigma - 1)} (k_0 + b_0)
\]

\( u_t < 0, v_t \geq 0, k_t > 0 \).

We made two additional simplifications in (3.69): We incorporated the inequality for the control \( r_t \geq 0 \) in the Euler equation constraint (3.55b); and the (strictly convex) resource constraint was relaxed to be an inequality, which is without loss of generality since by (3.57a) we know that its Lagrange multiplier, the costate of capital \( \lambda_t \), is necessarily positive at any optimum. Since the resource constraint binds and is strictly convex, all other constraints in (3.69) are also convex and the objective is linear, this planning problem can at most have a single solution. And, (3.57a)-(3.57e), (3.57f), (3.57g), (3.55b)-(3.55d) are sufficient conditions to find this solution.

Our next lemma finally establishes the optimality of \( T = \infty \) in the unrestricted problem (3.55a).

Lemma 24. Suppose \( \sigma > 1 \) and define \( \bar{b} \equiv (\mu^\infty)^{-1} \left( \frac{1}{\sigma - 1} \right) \) with \( \mu^\infty \) as in Lemma 21. Indefinite capital taxation is optimal in the Chamley problem (3.55a) if and only if \( b_0 \in [\bar{b}, \bar{b}] \).

Proof. As a consequence of Lemma 23, the unrestricted planning problem (3.55a) can be transformed into a strictly concave maximization problem with convex constraints. This implies that the first order conditions (3.57a)-(3.57e), together with transversality conditions (3.57f), (3.57g), and constraints (3.55b)-(3.55d) are in fact sufficient to characterize the unique optimum of the unrestricted planning problem (3.55a). In this proof we guess a solution and verify the sufficient conditions in a first step. In a second step, we prove that any \( b_0 < \bar{b} \) does not imply positive long run capital taxation, where \( T < \infty \). Throughout the proof, we focus on \( b_0 < \bar{b} \) since we know from Lemma 22 that initial debt of \( \bar{b} \) requires indefinite capital taxation.

First step: Let \( b_0 \in [\bar{b}, \bar{b}] \). We now construct an allocation \( \{c_t, n_t, k_t, r_t\} \) and multipliers \( \{\lambda_t, \eta_t\}, \mu \) that satisfy all the sufficient conditions. We define \( c_0 \equiv c^\infty(b_0) \) as in Lemma 21; given \( c_0, \{c_t, n_t, k_t\} \equiv \{c_t^\infty, n_t^\infty, k_t^\infty\} \) and \( \lambda_t \equiv \Phi_v \cdot \lambda_t^\infty \) with notation as in Lemma 20; \( \mu \equiv \mu^\infty(b_0) \)
as in Lemma 21; \( \eta_0 = \Phi_u u'(c_0) \frac{\sigma}{\rho} - \Phi_v \ddot{v}'(c_0) \) (which is negative since \( \Phi_u \leq 0 \) by construction of \( \mu \)) and \( \eta_t \) as solution to the ODE (3.57c) with initial condition \( \eta_0 \). The first order conditions (3.57a)-(3.57d) are satisfied by construction and by the fact that the allocation \( \{n_t^\infty, k_t^\infty, \lambda_t^\infty\} \) satisfies (3.62a) and (3.62b). The first order condition for initial consumption (3.57e) is equivalent to (3.65) in Lemma 21. The Euler equation constraint (3.55b) is trivially satisfied by construction of \( \{c_t\} \). The resource constraint holds for \( \{c_t^\infty, n_t^\infty, k_t^\infty\} \) (see (3.59b) and Lemma 20) and therefore also for \( \{c_t, n_t, k_t\} \). Due to the fact that \( \{n_t^\infty, k_t^\infty\} \) solves (3.59a) and \( c_t = c_0 e^{-\rho/\sigma} t \), the IC constraint (3.55d) can be seen to be equivalent to (3.64b) and hence is satisfied since \( c_0 \) was chosen to be \( c^\infty(b_0) \). Finally, Lemma 20 implies that the transversality condition for capital, (3.57f), holds. And, concluding the second step, the transversality condition for consumption, (3.57g), holds since

\[
e^{-\rho t} \eta_t c_t = c_0 e^{-(\rho + \sigma) t} \eta_t = -c_0 \int_t^\infty e^{- (\rho + \sigma) s} \lambda_s \, ds + c_0 \Phi_u u'(c_0) \frac{\sigma}{\rho} e^{-\frac{\sigma}{\rho} t} \to 0. \tag{3.70}
\]

and by this expression it also follows that \( \eta_t < 0 \) at all times \( t \). The second equality in (3.70) builds on an integral version of the law of motion of \( \eta_t \), which we obtained by combining (3.57c) with our definition of \( \eta_0 \) as \( \Phi_u u'(c_0) \frac{\sigma}{\rho} - \Phi_v \ddot{v}'(c_0) \) and the expression for \( \ddot{v}'(c_0) \) in (3.60d) from Lemma 20. It will become important in the second step below that (3.70) also reveals the limiting behavior of \( \eta_t \) itself: \( \lim_{t \to \infty} \eta_t = -\infty \) but \( \lim_{t \to \infty} e^{-\rho t} \eta_t = \Phi_u u'(c_0) \frac{\sigma}{\rho} \).

**Second step:** We proceed by contradiction. Suppose \( b_0 < \bar{b} \) gave rise to indefinite capital taxation (at the maximum rate). Then, reversing the logic of the first step, it must be the case that the allocation \( \{c_t, n_t, k_t\} \) is also optimal in the labor disutility minimization problem (3.59a) with multipliers \( \lambda_t^\infty = \frac{1}{\Phi_u} \lambda_t \), given \( c_0 \), and \( c_0 \) and \( \mu \) must be optimal given \( b_0 \) in the restricted planning problem (3.64a), that is, \( c_0 = c^\infty(b_0) \) and \( \mu = \mu^\infty(b_0) \) \( < \frac{1}{\sigma - 1} \). Since the first order condition (3.57e) is necessary, it must then be the case that \( \eta_0 = \Phi_u u'(c_0) \frac{\sigma}{\rho} - \Phi_v \ddot{v}'(c_0) \) by comparing it to (3.65). Equation (3.70) thus holds as in the second step, implying \( \lim_{t \to \infty} e^{-\rho t} \eta_t = \Phi_u u'(c_0) \frac{\sigma}{\rho} \) which now is positive since \( \Phi_u > 0 \), a contradiction to the optimality of capital taxes.

**3rd step:** **Feasibility of finite capital taxation for all \( b_0 < \bar{b} \).** We now move to the third and last part of this section. Here, we establish:

**Lemma 25.** For any initial government debt level \( b_0 < \bar{b} \), there are implementable allocations with nonzero capital taxation for only a finite time, \( T < \infty \).

**Proof.** Fix \( b_0 \leq \bar{b} \) and fix the allocation \( \{c_t^\infty, n_t^\infty, k_t^\infty\} \) that is optimal among all allocations with indefinite capital tax. By construction, this allocation satisfies the restricted problem
We now explicitly construct an allocation \( \{ \tilde{c}_t, \tilde{n}_t, \tilde{k}_t \} \) for which there is no capital tax, \( \hat{c} = \frac{1}{\sigma}(r^*_t - \rho)\tilde{c}_t \), after time some time \( T < \infty \) but that is feasible—satisfying constraints (3.55b)-(3.55d)—with initial debt \( b_0 - \epsilon \), for \( \epsilon > 0 \) arbitrarily small. First, we describe the allocation for all times \( t \geq T \). Consider

\[
V_{\text{zero tax}}(\hat{k}) = \max_{\{c_t, n_t, k_t\}_{t \geq T}} \int_T^\infty e^{-\rho(t-T)}(u(c_t) - v(n_t))\, dt \\
\text{s.t. } c_t + g + \dot{k}_t = f(k_t, n_t) - \delta k_t \\
k_T = \hat{k} \\
c_t > 0, n_t \geq 0, k_t \geq 0
\]

which is the social planning problem of a standard neoclassical growth model with power utilities in consumption and labor, and a Cobb-Douglas technology (i.e. zero labor and zero capital taxes). It is known that such a model has optimal paths \( \{ c^*_t, n^*_t, k^*_t \} \) that monotonically converge to a unique positive steady state \( (c^*, n^*, k^*) \). This implies that \( \{ n^*_t \} \) is bounded from above by \( \overline{n}(\hat{k}) = \max\{n^*, n(\hat{k})\} \) where \( n(\cdot) \) denotes the (continuous) policy function for labor supply. Moreover, the undistorted Euler condition holds along the path for consumption \( \{ c^*_t \} \). Also, it is well known that the consumption policy function \( c(k) \) of this problem is continuous and strictly increasing, with \( c(k) > 0 \) for any \( k > 0 \). Fix \( k = k_T \). Since \( k^*_T \) converges to a positive limit \( k^* > 0 \) but \( c^*_T \to 0 \) (see Lemma 20), it is the case for sufficiently large \( T \) that \( c(\hat{k}) > c^*_T \). Focus on such \( T \). Also let \( \overline{n} \equiv \sup_T \overline{n}(k^*_T) < \infty \) be an upper bound for labor (which by construction is uniform in \( T \)). Notice that \( \overline{n} < \infty \) since \( k^*_T \) converges to some \( k^* > 0 \).

Now construct the paths \( \{ \tilde{c}_t, \tilde{n}_t, \tilde{k}_t \} \) by piecing together \( \{ c^*_t, n^*_t, k^*_t \} \) for \( t < T \) and a zero-tax path \( \{ c^*_t, n^*_t, k^*_t \} \), starting with \( k^*_T = k^*_T \), for \( t \geq T \). By design, the capital stock is continuous at \( t = T \) and consumption jumps upwards at \( t = T \).\(^{69}\) Using this construction, the allocation satisfies the resource constraint at all periods, and the Euler equation with equality for \( t > T \). Also,

\[
\int_0^\infty e^{-\rho t}(u'(c^*_t)c^*_t - v'(n^*_t)n^*_t)\, dt - \int_0^\infty e^{-\rho t}(u'(\tilde{c}_t)\tilde{c}_t - v'(\tilde{n}_t)\tilde{n}_t)\, dt = \]

\(^{69}\)We think of this as a very high capital subsidy for a very short amount of time (which would definitely not be violating any capital tax constraints). If one prefers to avoid this simple limit case, one could easily smooth out this jump over some very small interval. This makes no difference whatsoever for the argument that follows.
\[
\int_{T}^{\infty} e^{-\rho t} \left( u'(c_i^\infty)c_i^\infty - u'(c_t^*)c_t^* \right) dt + \int_{T}^{\infty} e^{-\rho t} \left( v'(n_t^*)n_t^* - v'(n_t^\infty)n_t^\infty \right) dt.
\]

As \( e^{-\rho T} u'(c_T^\infty)c_T^\infty \to 0 \) both terms in (3.71) approach zero. This is why for \( T \) sufficiently large, \( \int_{0}^{\infty} e^{-\rho t} \left( u'(\tilde{c}_t)\tilde{c}_t - v'(\tilde{n}_t)\tilde{n}_t \right) dt \) approaches \( u'(\tilde{c}_0)(k_0 + b_0) \). Thus, for any \( \epsilon > 0 \), there exists a \( T \) such that the allocation \( \{\tilde{c}_t, \tilde{n}_t, \tilde{k}_t\} \) is implementable without capital taxes after time \( T \), for initial debt \( b_0 - \epsilon \),

\[
\int_{0}^{\infty} e^{-\rho t} \left( u'(\tilde{c}_t)\tilde{c}_t - v'(\tilde{n}_t)\tilde{n}_t \right) dt \geq u'(\tilde{c}_0)(k_0 + b_0 - \epsilon)
\]

which is what we set out to show. This proves that for any \( b_0 < \tilde{b} \), there exists a feasible (but not necessarily optimal) path with only a finite period of positive capital taxation. \( \square \)

**Summary** This concludes the proof of parts A and B (i) of Proposition 18. For part A, we proved (i) in Lemma 22, (ii) in Lemma 24 and (iii) in Lemma 25. Part B (i) was shown in Lemma 22.

### 3.1.3 Proof of the bang-bang property and parts B (ii) and C

We proceed in three steps. We first establish a transversality condition that is necessary at any optimum (in general, transversality conditions are not necessary). Then, using this transversality condition, we derive the "bang-bang" property of capital taxes. Notice that previous proofs of this property relied on the assumption that indefinite capital taxation is not optimal, which we showed is not the case. The bang-bang property lets us summarize an optimal capital tax plan by the date \( T \in [0, \infty) \) at which capital taxes jump from the upper bound \( \bar{r} \) to zero. In the final step, we prove parts B (ii) and C, that is, \( T < \infty \) if either \( \sigma < 1 \) or \( \sigma = 1 \) and \( b_0 = \tilde{b} \).

**1st step: A necessary transversality condition.**

**Lemma 26.** Let \( \{c_t, n_t, k_t, r_t\} \) be a solution to problem (3.55a), with multipliers \( \{\lambda_t, \eta_t, \mu\} \). If \( \exists s \geq 0 \) such that \( c_t = c_se^{-\rho(t-s)/\sigma} \) for all \( t \geq s \), then the transversality condition for consumption (3.57g) holds.

**Proof.** We first establish that under the conditions of the lemma, \( \{k_t, n_t\} \) converges to a positive steady state. If \( c_t = c_se^{-\rho(t-s)/\sigma} \), then \( \{k_t, n_t\}_{t \geq s} \) must be minimizing the stream
of labor disutilities (3.61a) given initial capital $k_s$ and initial consumption $c_s$. Therefore, 
\$k_t, n_t \rightarrow \{k^\infty, n^\infty\}$, using the notation from Lemma 20.

Thus, there exists some large enough $\bar{n} > 0$ such that

$$f(k_t, \bar{n}) - \delta k_t - c_t - g > 0 \quad (3.72)$$

for all $t$. Since the time $t$ controls maximize the time $t$ Hamiltonian $H_t$ (see (3.56)), we then have for any $\bar{n}$

$$e^{-\rho t}(\Phi_u u(c_t) - \Phi_v v(\bar{n})) + e^{-\rho t} \eta_t \frac{1}{\sigma} (-\rho) + e^{-\rho t} \lambda(f(k_t, \bar{n}) - \delta k_t - c_t - g) \leq e^{-\rho t} H_t \rightarrow 0 \quad (3.73)$$

where the left hand side is the present value Hamiltonian with controls $r_t = 0$ and $n_t = \bar{n}$, and the right hand side is the present value Hamiltonian with optimal controls $r_t, n_t$ (both along the optimal path for $c_t, k_t$). The right hand side converges to zero following Michel (1982). Notice that in (3.73), $e^{-\rho t}(\Phi_u u(c_t) - \Phi_v v(\bar{n})) \rightarrow 0$. Suppose $\liminf_{t \rightarrow \infty} e^{-\rho t} \eta_t c_t$ were negative. Then, according to (3.73) it would have to be that

$$\limsup_{t \rightarrow \infty} e^{-\rho t} \lambda(f(k_t, \bar{n}) - \delta k_t - c_t - g) \leq \liminf_{t \rightarrow \infty} e^{-\rho t} \eta_t c_t \frac{1}{\sigma} \rho < 0$$

contradicting (3.72). This means the transversality condition for consumption (3.57g) holds.

\[ \square \]

2\textsuperscript{nd} step: The bang bang property. We move to the first main result of this subsection.

**Lemma 27.** A solution to problem (3.55a) is of the form that the capital tax $r_t$ binds at the upper bound for some time $T \in [0, \infty]$ and is equal to zero thereafter.

**Proof.** Let $\{c_t, n_t, k_t, r_t\}$ be an optimal allocation solving (3.55a), for some initial debt $b_0 \in \mathbb{R}$. Let $\{\lambda_t, \eta_t, \mu\}$ be a set of multipliers such that allocation and multipliers satisfy the necessary first order conditions for the case $b < \bar{b}$, (3.57a)-(3.57e) and constraints (3.55b)-(3.55d). Our proof is analogous if $b = \bar{b}$. We first show that if $r_t < \hat{r}$ on some non-trivial interval, then $r_t = 0$ on that interval. Second, we prove that $r_t = 0$ at all times after that interval as well. The proof utilizes the fact that once $r_t = 0$, it must not only be that $r_t^* \geq 0$ at that time (or else the $r_t \geq 0$ constraint would be binding); but also that $r_t^* > 0$ for all future times. We formally prove this fact in Lemma 28 below.

First, suppose $r_t < \hat{r}$ for some non-trivial interval $t \in [s_0, s_1]$. Then, by (3.57b), $r_t > 0$ and $\eta_t = 0$ on that interval. Hence, by (3.57c), $\lambda_t = \Phi_u u'(c_t)$. Taking logs and differentiating
implies an undistorted Euler equation of the agent. Therefore, \( \tau_t = 0 \) for \( t \in [s_0, s_1] \). Second, suppose there is a later time where capital taxes are positive, that is \( s' = \inf\{t > s_1 \mid \eta_t < 0\} < \infty \). Observe that, between \( t = s_1 \) and \( t = s' \), both \( u'(c_t) \) and \( \lambda_t \) grow at the common rate \( \rho - r_s^* \), so \( \lambda_{s'} = \Phi_s u'(c_{s'}) \). For any \( t > s' \), \( u'(c_t) \) still grows at least as fast as \( \lambda_t \), and, by definition of \( s' \), for a positive-measure set of times \( t \) after \( s' \), \( u'(c_t) \) grows at the faster rate \( \rho > \rho - r_s^* \) since \( \eta_t < 0 \) and \( \tau_t = \bar{\tau} \) for those \( t \). Therefore, for any \( t > s' \), \( \Phi_s u'(c_t) > \lambda_t \). By (3.57c), this means \( \bar{\eta}_t < \eta_t (\rho + \frac{\rho}{\sigma}) \), or in other words, \( \bar{\eta}_t < 0 \) and \( c_t = c_s e^{-\rho (t-s')/\sigma} \) for \( t > s' \). Moreover, \( \limsup_{t \to \infty} e^{-\rho \bar{\eta}_t} < 0 \), contradicting Lemma 26. This concludes our proof of Lemma 27.

**Lemma 28.** If \( \tau_s = 0 \) for \( s \geq 0 \), then \( r_s^* \geq 0 \) and \( r_s^* > 0 \) for all \( s' > s \).

**Proof.** For convenience we introduce \( R_t = f_k(k_t, n_t) \). \( R_t \) has the following law of motion,

\[
(\zeta + \alpha_t) \beta_t - 1 \frac{\dot{R}_t}{R_t} = \rho + (1 + \zeta)\delta + \zeta \frac{g + c_t}{k_t} - \zeta f(1, h(R_t)) - R_t,
\]

which was obtained by log-differentiating the first order condition of labor (3.57a) and combining it with the resource constraint (3.55c). Here, \( \alpha_t \equiv \frac{\partial \log f_k}{\partial (k_t/n_t)} \) as before, \( \beta_t \equiv \frac{\partial \log f_k}{\partial (n_t/k_t)} \), and \( h(x) \equiv f_k(1, \cdot)^{-1}(x) \). Observe that \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) is strictly increasing and bijective. Since \( \dot{R} \) depends implicitly (through \( \alpha \) and \( \beta \)) and explicitly on \( k_t, R_t \), and \( c_t \), we also write \( \dot{R}_t = \dot{R}(k, R, c) \).

Our proof of this lemma proceeds in two steps. First, we show an auxiliary result, namely that whenever

\[
(k_t, R_t, c_t) \in \mathcal{A} \equiv \{(k, R, c) \mid R \leq \delta, \dot{R}(k, R, c) \leq 0\},
\]

for some time \( t = t_0 \), then \( (k_t, R_t, c_t) \in \mathcal{A} \) for all later times \( t > t_0 \) too. Second, we establish the result stated in the lemma.

**First step.** To prove the auxiliary result, it suffices to consider points \( (k_t, R_t, c_t) \) at the boundary of \( \mathcal{A} \) and study whether the flows induced by the differential equation point to the inside of \( \mathcal{A} \). There are two kinds of boundary points. If \( R_t = \delta \), it trivially holds that \( \frac{d}{dt} R_t \leq \frac{d}{dt} \delta = 0 \). Suppose now that \( \dot{R}(k_t, R_t, c_t) = 0 \) and ask whether \( \frac{d}{dt} \dot{R}(k_t, R_t, c_t) \leq 0 \). Generally, whenever \( (k_t, R_t, c_t) \in \mathcal{A} \), it is straightforward to see that

\[
\frac{\dot{k}}{k} = \frac{f(k, n)}{k} - \delta - \frac{g + c_t}{k_t} \geq \frac{\rho}{\zeta} > 0.
\]

Moreover, \( \dot{c}_t = -\frac{\rho}{\sigma} c_t \) since \( r_s^* = R_t - \delta \leq 0 \). The fact that \( k_t \) is increasing and \( c_t \) is decreasing...

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mean that

\[
(\zeta + \alpha_t)\beta_t^{-1} \frac{d}{dt} \tilde{R}_t - R_t = \frac{d}{dt} \left( (\zeta + \alpha_t)\beta_t^{-1} \frac{\tilde{R}}{R} \right) = \frac{d}{dt} \left( \frac{g + c_t}{k_t} \right) < 0
\]

establishing the auxiliary result.

Second step. Suppose \( \tau_s = 0 \) for some \( s \geq 0 \). The fact that \( r^*_s \geq 0 \) follows directly from the constraint \( (1 - \tau_s)r^*_s = r_s \geq 0 \). Let \( s' \equiv \inf \{ t > s | r^*_t \leq 0 \} \) and suppose \( s' < \infty \). Since \( r^*_t \) is continuous and differentiable, this means that \( r^*_s = 0 \) and \( \frac{d}{dt} r^*_t |_{t=s'} \leq 0 \), or in terms of \( R_t, R_{s'} = \delta \) and \( \tilde{R}_{s'} \leq 0 \). Applying the auxiliary result, \( (k_t, R_t, c_t) \in \mathcal{A} \) for any \( t > s' \). Moreover, \( k_t \to \infty \) due to (3.74) at all times \( t > s' \). This is in sharp contradiction to Lemma 20 (which applies here using \( k_{s'} \) as initial capital stock since \( \dot{c}_t = -\frac{\rho}{\sigma} c_t \) for all \( t \geq s' \). Therefore \( r^*_t > 0 \) for all \( t > s \).

3rd step: Finite capital taxation \( T < \infty \) in parts B (ii) and C.

Lemma 29. If either \( \sigma < 1 \) or \( \sigma = 1 \) and \( b_0 < \overline{b} \), then \( T < \infty \).

Proof. If either \( \sigma < 1 \) or \( \sigma = 1 \) and \( b_0 < \overline{b} \), then \( \Phi_u > 0 \) for any \( \mu \geq 0 \).\(^{70}\) In the following, we prove that this is incompatible with \( T = \infty \). By contradiction, suppose it were the case that there exists an optimal allocation \( \{c_t, n_t, k_t, r_t\} \) with \( T = \infty \), i.e. \( c_t = c_0 e^{-\rho t/\sigma} \). Applying Lemma 26, \( (k_t, n_t) \to (k^\infty, n^\infty) \). In particular, \( r^*_t \to \rho > 0 \) following the definition of \( (k^\infty, n^\infty) \) in Lemma 20. Now, \( \Phi_u u'(c_t) \) grows at rate \( \rho \) while \( \lambda_t \) only grows at rate \( \rho - r^*_t < \rho \). Therefore, there exists some finite time \( s \) such that \( \lambda_t \leq \Phi_u u'(c_t) \) for all \( t > s \). Using law of motion of \( \eta_t \), (3.57c), this means \( \dot{r}_t \leq \eta_t (\rho + \frac{\rho}{\sigma}) \) for all \( t > s \) and so \( \limsup_{t \to \infty} e^{-\rho t} \eta_t c_t < e^{-\rho \sigma} \eta_s c_s < 0 \), contradicting Lemma 26.

Summary. This concludes our proofs of the bang bang property (Lemma 27) and parts B (ii) and C (Lemma 29).

3.8 Proof of Proposition 19

We proceed by providing an explicit solution to the first order and transversality conditions to problem (3.55a) with zero government spending and certain combinations of \( k_0, b_0 \). We do so in two steps. First, taking \( k_0 \) as given, we find paths \( \{c_t, n_t, k_t, r_t\}, \{\lambda_t, \eta_t\}, \mu \) and a level

\(^{70}\) If \( b_0 = \overline{b} \) and \( \sigma < 1 \), then as we explain in Section 3.1, \( \Phi_u \) can be taken to be \((1 - \sigma)\), and thus is positive here.
of initial debt $b_0$ which together satisfy all first order conditions, transversality conditions and constraints, with the one exception that $\eta_t$ need not necessarily be negative. In a second step, we choose $k_0$ such that $\mu \geq 1/(\sigma - 1)$ which will ensure that $\eta_t < 0$ at all times $t$.

The reason this construction is analytically tractable is that along the optimum, $c_t, n_t, k_t$ will all fall to zero at the exact same growth rate, which needs to equal $\frac{\rho}{\sigma}$ by the Euler equation (3.55b). At the same time, $r_t = 0$ (since $T = \infty$). Taken together, to find the solution for a given $k_0$, it is necessary to find $c_0, n_0, \{\lambda_t, \eta_t\}, \mu, b_0$. Again, we use the previous notation $\Phi_u = 1 + \mu(1 - \sigma)$ and $\Phi_v = 1 + \mu(1 + \zeta)$.

**First step.** We conjecture that $c_t = c_0e^{-\rho/\sigma t}$, $n_t = n_0e^{-\rho/\sigma t}$, $k_t = k_0e^{-\rho/\sigma t}$, $r_t = 0$, $\lambda_t = \lambda_0e^{-\zeta\rho/\sigma t}$. The Euler equation (3.55b) obviously holds. The resource constraint (3.55c) is satisfied iff

$$c_0 = f(k_0, n_0) - \delta k_0 + \frac{\rho}{\sigma} k_0. \tag{3.75}$$

The IC constraint (3.55d) is satisfied iff

$$b_0 = c_0 \frac{\sigma}{\rho} - \frac{1}{\rho + (1 + \zeta)\rho/\sigma} c_0^\sigma n_0^{1+\zeta} - k_0. \tag{3.76}$$

The first order condition for labor (3.57a) and the costate $\lambda_t$ (3.57d) hold iff

$$f_k(k_0, n_0) = \zeta \frac{\rho}{\sigma} + \rho + \delta \tag{3.77}$$

and

$$\Phi_v n_0^\zeta = \lambda_0 f_u(k_0, n_0). \tag{3.78}$$

Given $k_0$, (3.77) pins down $n_0$, (3.75) $c_0$, and (3.76) $b_0$. The law of motion of $\eta_t$ (3.57c) and the associated transversality condition (3.57g) are satisfied iff

$$\eta_t = -\frac{\lambda_0}{\rho + (1 + \zeta)\rho/\sigma} e^{-\zeta\rho/\sigma t} + \frac{\sigma}{\rho} \Phi_u c_0^{-\sigma} e^{\rho t}. \tag{3.79}$$

Notice that (3.57b) holds, i.e. $\eta_t < 0$, as long as $\Phi_u \leq 0$, requiring $\mu \geq \frac{1}{\sigma - 1}$. The transversality condition for capital (3.57f) obviously holds.

It remains to determine $\lambda_0$, $\eta_0$, and $\mu$ subject to (3.79) (at $t = 0$), $\mu \geq \frac{1}{\sigma - 1}$, (3.78), and the first order condition for $c_0$ (3.57e). For expositional reasons, define the initial labor tax as
\[ \tau_0^\ell = 1 - n_0^\ell \zeta_0^\ell / \omega^*. \] Then, we can solve for \( \mu \) as

\[
\mu = \frac{\tau_0^\ell + \sigma + \zeta}{\sigma \left( \left( 1 - \tau_0^\ell \frac{w^*}{c_0} \right) - 1 \right) - \tau_0^\ell (1 + \zeta)}. \tag{3.80}
\]

Notice that whenever \( \mu \in \left[ \frac{1}{\sigma - 1}, \infty \right) \), \( \lambda_0 > 0 \) is given by (3.78) and \( \eta_0 < 0 \) is given by (3.79) (at \( t = 0 \)). So the last step in our construction is to determine whether there are levels for \( k_0 \) for which \( \mu \in \left[ \frac{1}{\sigma - 1}, \infty \right) \).

**Second step.** The only object on the right hand side of (3.80) that depends on \( k_0 \) is \( \tau_0^\ell \), and \( \tau_0^\ell \) is a strictly decreasing function of \( k_0 \in (0, \infty) \), with \( \tau_0^\ell \to 1 \) as \( k_0 \to 0 \) and \( \tau_0^\ell \to -\infty \) as \( k_0 \to \infty \). Moreover, \( \mu \) is increasing in \( \tau_0^\ell \in (-\infty, 1] \) and has a pole at \( \tau_0^\ell_{\text{pole}} = \frac{\sigma w^* n_0 / \zeta_0 - \sigma}{\sigma w^* n_0 / \zeta_0 + 1 + \zeta} < 1 \), where it rises to \( +\infty \) from the left. For \( \tau_0^\ell = 1 \), \( \mu = -1 < 0 \). We define \( \bar{k} \) to be the value of \( k_0 \) corresponding to \( \tau_0^\ell_{\text{pole}} \). Putting the mapping from \( k_0 \) to \( \tau_0^\ell \) and the one from \( \tau_0^\ell \) to \( \mu \) together, we find a function \( \mu(k_0) \) with the properties that

\[
\begin{align*}
\mu(k_0) < 0 & \quad \text{for} \quad k_0 < \bar{k} \\
\mu(k_0) \geq 1/(\sigma - 1) & \quad \text{for} \quad k_0 \in (\bar{k}, \bar{k}] \\
\mu(k_0) < 1/(\sigma - 1) & \quad \text{for} \quad k_0 > \bar{k},
\end{align*}
\]

where \( \bar{k} \equiv \inf_{k_0 \geq \bar{k}} \{ k_0 \geq \bar{k} \mid \mu(k_0) < \frac{1}{\sigma - 1} \} \in (\bar{k}, \infty] \). This proves that for \( k_0 \in (\bar{k}, \bar{k}] \), there exists a debt level \( b_0(k_0) \) for which the quantities \( c_t, n_t, k_t \) all fall to zero at equal rate \(-\rho/\sigma\) and the sufficient optimality conditions of the problem are satisfied.

### 3.K Proof of Proposition 20

First, we show that the planner’s problem is equivalent to (3.13). Then we show that the functions \( \psi(T) \) and \( \tau(T) \) are increasing, have \( \psi(0) = \tau(0) = 0 \) and bounded derivatives.

The planner’s problem in this linear economy can be written using a present value resource
constraint, that is,
\[
\max \int_0^\infty e^{-\rho t} (u(c_t) - v(n_t)) \, dt \\
\text{s.t. } \dot{c} \geq c \frac{1}{\sigma} \left((1 - \bar{r})r^* - \rho\right) \\
\int_0^\infty e^{-r^* t} (c_t - w^* n_t) \, dt + G = k_0 \\
\int_0^\infty e^{-\rho t} [(1 - \sigma)u(c_t) - (1 + \zeta)v(n_t)] \, dt \geq u'(c_0) a_0.
\]

where \( G = \int_0^\infty e^{-r^* t} g(t) \, dt \) is the present value of government expenses, \( k_0 \) is the initial capital stock, \( a_0 \) is the representative agent’s initial asset position, and per-period utility from consumption and disutility from work are given by \( u(c_t) = c_t^{1-\sigma}/(1 - \sigma) \) and \( v(n_t) = n_t^{1+\zeta}/(1 + \zeta) \). Note that we assumed \( \sigma > 1 \). The FOCs for labor imply that given \( n_0 \),
\[
n_t = n_0 e^{-(r^* - \rho)t}/\zeta. \tag{3.82}
\]

An analogous argument to the bang-bang result in Appendix ?? implies the existence of \( T \in [0, \infty] \) such that \( \tau_t = \bar{\tau} \) for \( t \leq T \) and zero thereafter. In particular, the after-tax (net) interest rate will be \( r_t = (1 - \bar{\tau})r^* \equiv \bar{r} \) for \( t \leq T \) and \( r_t = r^* \) for \( t > T \). Then, by the representative agent’s Euler equation, the path for consumption is determined by
\[
c_t = c_0 e^{-r^* t + r^*/\sigma (t - T)^+}. \tag{3.83}
\]

Substituting equations (3.82) and (3.83) into (3.81), the planner’s problem simplifies to,
\[
\max_{T, c_0, n} \psi_1(T)u(c_0) - \psi_3 v(n_0) \\
\text{s.t. } \psi_2(T)(\chi^*)^{-1}c_0 + G = k_0 + \psi_3 w^* n_0 \\
\psi_1(T)u'(c_0)c_0 - \psi_3 v'(n_0)n_0 = \chi^* u'(c_0)a_0,
\]

where \( \psi_1(T) = \chi^* \left(1 - e^{-\chi T}\right) + e^{-\chi T}, \psi_2(T) = \chi^* \left(1 - e^{-\bar{\chi} T}\right) + e^{-\bar{\chi} T}, \psi_3 = \chi^* \left(r^* + \frac{r^* - \rho}{\zeta}\right)^{-1} \)
and \( \chi = \frac{\sigma - 1}{\sigma} \bar{r} + \frac{\rho}{\sigma}, \chi^* = \frac{\sigma - 1}{\sigma} r^* + \frac{\rho}{\sigma}, \bar{\chi} = r^* + \frac{r^* - \rho}{\zeta}. \) Notice that \( \bar{\chi} > \chi^* > \chi. \)

Now normalize consumption and labor
\[
c \equiv \psi_1(T)^{1/(1-\sigma)}c_0/\chi^* \quad \quad n \equiv \psi_3^{1/(1+\zeta)}n_0/\left(\chi^*/(1-\sigma)/(1+\zeta)\right)
\]
and define an efficiency cost \( \psi(T) \equiv \psi_2(T)\psi_1(T)^{1/(\sigma-1)} - 1 \), a capital levy \( \tau(T) \equiv 1 -
\( \psi_1(T)^{-\sigma/(\sigma-1)} \), and the present value of wage income \( \omega n = w^* \psi_3^{\psi/(1+\zeta)} n \). Here, we note that by definition, \( \psi \) is bounded away from infinity and \( \tau \) is bounded away from 1. Then, we can rewrite problem (3.84) as

\[
\begin{align*}
\max_{T,c,n} & \quad u(c) - v(n) \\
\text{s.t.} & \quad (1 + \psi(T))c + G = k_0 + \omega n \\
& \quad u'(c)c - v'(n)n = (1 - \tau(T))u'(c)\alpha_0,
\end{align*}
\]

which is what we set out to show. Notice that \( \psi(0) = \psi_1(0) = 1 \) and so \( \psi(0) = \tau(0) = 0 \). Further, given our assumption that \( \sigma > 1 \), \( \psi_1(T) \) and \( \tau(T) \) are increasing in \( T \). To show that \( \psi'(T) \geq 0 \), notice that, after some algebra,

\[
\frac{d}{dT} \left( \psi_2^{1/(\sigma-1)} \right) \geq 0 \iff \tilde{\chi} (e^{\chi T} - 1) \leq \chi (e^{\chi T} - 1),
\]

which is true for any \( T \geq 0 \) because \( \tilde{\chi} > \chi \). Therefore, \( \psi'(T) \geq 0 \), with strict inequality for \( T > 0 \), implying that \( \psi(T) \) is strictly increasing in \( T \).

Now consider the ratio of derivatives,

\[
\frac{\psi'(T)}{\tau'(T)} = \frac{1}{\sigma} \psi_2^{(1+\sigma)/(\sigma-1)} \left( (\sigma - 1) \frac{\psi''}{\psi'} + 1 \right).
\]

Notice that \( \psi_1(T) \in [1, \chi^*/\chi] \) and \( \psi_2(T) \in [\chi^*/\tilde{\chi}, 1] \), so both are bounded away from infinity and zero. Further, the ratio \( \psi_2'/\psi_1' \) is also bounded away from infinity, \( \psi_2'/\psi_1' = -\frac{1}{\sigma-1} e^{-(\tilde{\chi}-\chi)T} \in [-1/(\sigma-1), 0] \), implying that \( \psi'(T)/\tau'(T) \) is bounded away from \( \infty \).

### 3.L Proof of Proposition 21

The planning problem is given by

\[
\begin{align*}
\sup_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (3.85) \\
c_t + C_t + k_{t+1} = f(k_t) + (1 - \delta)k_t \quad (3.86) \\
\sum_{t=0}^{\infty} \beta^t C_t^{1-\sigma} = C_0^{-\sigma} a_0. \quad (3.87)
\end{align*}
\]
First, note that \( a_0 \) must be positive or else the IC constraint (3.87) cannot be satisfied (recall that \( \sigma > 1 \)). Second, note that there exists a unique solution \( C_0(\varphi) \) to the equation

\[
C_0^\sigma \varphi C_0^{1-\sigma} + C_0 = a_0
\]

for any \( \varphi_0, \varphi > 0 \), and that \( C_0(\varphi) \to 0 \) as \( \varphi \to 0 \). We now use this to construct a sequence of feasible paths \( \{C_t^{(n)}\}_{n=0}^\infty, \ n = 0, 1, \ldots \), with \( C_t^{(n)} \) uniformly converging to 0 as \( n \to \infty \). Take any sequence \( \{C_t^{(0)}\}_{n=0}^\infty \) that satisfies (3.87). Define

\[
C_t^{(n)} = \begin{cases} 
C_0(\varphi^n) & t = 0 \\
\varphi^n C_t^{(0)} & t > 0
\end{cases}
\]

for some \( \varphi \in (0, 1) \), \( \varphi_0 \equiv C_0^{-\sigma}(a_0 - C_0) \). By construction, \( C_t^{(n)} \to 0 \) uniformly and the supremum in (3.85) approaches the maximum of the planning problem of a standard neoclassical growth model,

\[
\max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^\infty \beta^t u(c_t)
\]

\[c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t.\]

The way \( \{C_t^{(n)}\} \) was constructed in this proof, it suggests an implementation via a wealth tax \( \tau_1 = R_1/R_1^* \to 100\% \). Analogous to this construction, a wealth tax approaching 100\% in any period would implement the same allocation. This also shows that only a single period of unconstrained taxation is necessary to implement the supremum.
3. M Proof of Proposition 22

As in Section 3.2, labor supply is inelastic at \( n_t = 1 \). Denote the capitalist’s initial wealth by \( a_0 \equiv R_0 k_0 + R^b_0 b_0 \). The planning problem is then

\[
\max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.88a}
\]

\[
C_{t+1} \geq C_t \beta^{1/\sigma} \tag{3.88b}
\]

\[
c_t + C_t + k_{t+1} = f(k_t) + (1 - \delta) k_t \tag{3.88c}
\]

\[
\sum_{t=0}^{\infty} \beta^t U''(C_t) C_t = U'(C_0) a_0. \tag{3.88d}
\]

The necessary first order conditions for \( C_t \) and \( c_t \) in problem (3.88a) are

\[
\beta^{1/\sigma} \eta_t - \beta^{-1} \eta_{t-1} = \lambda_t - \Phi_u U''(C_t) \tag{3.89}
\]

\[
u'(c_t) = \lambda_t \tag{3.90}
\]

\[
\beta^{1/\sigma} \eta_0 = \lambda_0 - \Phi_u U''(C_0) - \mu \sigma C_0^{-\sigma-1} a_0 \tag{3.91}
\]

where we defined \( \Phi_u \equiv \mu(1 - \sigma) \). Here, \( \mu \) is the multiplier on the IC constraint (3.88d), \( \lambda_t \) is the multiplier of the resource constraint (3.88c)—which is positive by (3.90)—and \( \eta_t \) denotes the costate of capitalists’ consumption \( C_t \). If \( \eta_t < 0 \), constraint (3.88b) is binding. Also, it follows from (3.88d) that

\[
\sigma C_0^{-\sigma-1} a_0 = \sigma C_0^{-1} U'(C_0) a_0 = \sigma C_0^{-1} \cdot \sum_{t=0}^{\infty} \beta^t U''(C_t) C_t > (\sigma - 1) U''(C_0),
\]

where the inequality is obtained by dropping all terms with \( t > 0 \) from the infinite sum and observing that \( \sigma > \sigma - 1 \). Using this inequality, (3.91) implies that \( \mu \) must be positive and \( \Phi_u < 0 \).

Suppose now there existed a period \( T \geq 0 \) where constraint (3.88b) is slack. In that case, \( \eta_T = 0 \) and (3.89) becomes for \( t = T + 1 \)

\[
\Phi_u U''(C_{T+1}) = \lambda_{T+1} - \beta^{1/\sigma} \eta_{T+1} > 0
\]

contradicting \( \Phi_u < 0 \). Therefore, (3.88b) binds in all periods, or equivalently, \( R_t = 1 \) for all \( t \).
3.N References


