THE EFFECT OF LEARNING ON THE ECONOMIC LOT SIZE

by

David Byron Lawrence

B.E., Yale University (1957)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

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Signature of Author

School of Industrial Management

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THE EFFECT OF INTERVENTION ON THE ECONOMIC LOT SIZE

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David Byron Lawrence

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REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science

Massachusetts Institute of Technology 1959

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School of Industrial Management

Certifying the oral examination of the thesis

JUL 21 1959

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Dear Professor Sloane:

In accordance with the requirements for graduation, I here-with submit a thesis entitled "The Effect of Learning on the Economic Lot Size."

I would like to express my appreciation for the aid received of the faculty in preparing this work, especially that of Professor T. M. Whitin who was Faculty Advisor. I am also grateful to Professors R. B. Maffei and M. E. Shaw for their critical comment and advice.

Two individuals in industry have also given much in the development of some of the ideas presented here, and for that assistance I am indebted to Mr. R. P. Jones, Jr. and Mr. E. S. Bottum.

Sincerely,

David B. Lawrence
ABSTRACT

TITLE: The Effect of Learning on the Economic Lot Size

AUTHOR: David Byron Lawrence

Submitted to the School of Industrial Management on May 20, 1959 in partial fulfillment of the requirements for the degree of Master of Science.

A mathematical expression was found to describe the phenomenon of worker learning by specifying the rate of production as a function of time. This expression differed in important ways from the learning curves traditionally used in the airframe and other industries.

A model describing the least costly size of production lot was derived to fit the industrial situation where worker learning is an important source of cost.

An analysis of the total variable cost was used to derive the models given. This was accomplished by specifying the relation of the cost components to size of lot and taking the first derivative with respect to lot size of the resulting function of total cost. This derivative was set equal to zero and was solved for lot size. This lot size is the one with the lowest associated total cost.

Thesis Advisor: Thomson M. Whitin

Title: Associate Professor of Industrial Management
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The Problem

There are many manufacturing firms in existence whose product mix contains a number of items with low but stable rates of demand. The sales rate of these items is so low that a full schedule of production of any one item cannot be supported, but the cost of preparing for production is high enough that it is uneconomical to produce for each order as it arrives. The usual policy in such a situation is to manufacture in lots for storage in inventory. The demand for the product is then fulfilled by issuing stock from inventory until the supply is depleted. At that time a new lot is manufactured and placed in stock.

Businessmen have recognized that the size of the manufacturing lot is an important consideration. Large lots raise the average level of inventory while small lots require a more frequent replenishment of the inventory with an associated cost of preparing for production. The managers have attempted, in some fashion, to balance between the disadvantages of large and small stock levels.

In recent years some businessmen have come to rely upon a formula to help them decide the question of lot size. This formula, the economic lot size equation, minimizes the total cost of carrying inventory and preparing for production.
In assembly operations and others requiring manual skill, some of the cost of making ready for a production run can be traced to the process of training workers to perform the required tasks. Strictly speaking these costs are usually incurred during the early part of the production run rather than during preparation for the run. Nonetheless, they are a direct consequence of the decision to run a separate lot, and they must be treated as the preparation costs are.

Until the present time there has been no formalized way of handling the training costs where the economic lot size models have been used. Where it has been possible to estimate the cost of worker learning this has, in general, been of no consequence. However, estimating the cost of learning can be difficult under certain circumstances, and a formalized method of doing so would be useful, especially in the situation where it would be uneconomical (because of the high inventory costs) to allow the workers to learn fully their task.

**Methods**

The work presented here was an attempt to provide a useful variant of the economic lot size model which would deal explicitly with the problem of industrial learning. The approach used was one of specifying logical relationships among the variables, followed by a mathematical determination of the minimum cost.
The first part of the work was directed at finding a mathematical expression which adequately describes the industrial learning phenomenon. The measure of learning was taken to be the rate of production.

The expression found to satisfy that requirement was then used as a basis for estimating the cost of learning and the cost of inventory. These costs, as well as the set-up cost, were expressed as average costs for units of a given lot size, and the total of these average costs was differentiated with respect to lot size. The derivative was set equal to zero and was solved for the least costly size of lot.

The Results

It was concluded that learning, as measured by the rate of production, could be expressed by the formula

\[ P = L(1 - e^{-Ax}) \]

where \( P \) = production rate,
\( L \) = ultimate production rate at completion of learning,
\( A \) = a constant,
\( x \) = the length of time in production, and
\( e \) = base of the natural logarithm.

Evidence supporting this choice of form was demonstrated.

The most economical size of lot in the presence of learning was found to be \( Q^* \) where \( Q^* \), \( P^* \), and \( X^* \) satisfy the equations
\[ P^* = L(1 - e^{-AX^*}), \]
\[ Q^* = LX^* \frac{L}{A}(1 - e^{-AX^*}), \]
\[ Q^* = \frac{2R[S + W(X^* - Q^*/P^*) + ILX^*(X^*/2 - Q^*/P^*)]}{I(1 - 2R/P^*)} \]

where \( L, A, \) and \( e \) are as defined above and
\[ I = \text{cost of carrying inventory}, \]
\[ R = \text{rate of demand}, \]
\[ S = \text{set-up cost per lot}, \]
\[ W = \text{wage rate}. \]

Because of the difficulty of determining this solution for \( Q^* \), an alternate solution was derived for use where learning could be expected to be completed. This solution is
\[ Q^* = \sqrt{\frac{2R(S + 0.317WT + 0.039ILT^2)}{I(1 - R/L)}} \]

where \( T \) is the length of time required for completion of the learning process.
CHAPTER II - WHAT ARE THE CHARACTERISTICS OF LEARNING?

Worker learning as a factor in industrial output has been recognized for a long while. A worker must be trained to do his task. In modern times industry is compelled to maintain extensive training programs for workers in order to decrease somewhat the time required for learning.

The cost of learning has also been understood for several centuries. The economic fact that a worker in the process of learning his job does not always pay for himself was implicit in the apprentice system of the Medieval Period, when it was possible that a young boy might have to pay a fee so that he could work under a journeyman and learn the trade.

In the age of industrialism the need for workers has grown to such an extent that an unskilled laborer can demand and get pay during the period in which he is being trained. In such a setting it is apparent that training can be the cause of considerable expense, and where it does it is a legitimate object of study. An investigation of learning should be made with the objective of gaining sufficient knowledge of the subject to predict its behavior. Once the ability to predict is achieved, further effort can be made to control learning or at least to adapt a line of action that will minimize the undesirable aspect of it which is cost.
The 80 Per Cent Learning Curve

One model of the industrial learning phenomenon has gained fairly widespread use in the air-frame industry. It was first introduced in 1936 by T. P. Wright in an article entitled "Factors Affecting the Cost of Airplanes." Since that time the concept has been used by the aircraft industry for virtually every kind of decision possible, including those of manpower requirements, cost estimates, and scheduling. It has also been used by the federal government in setting contract prices. In the more recent literature this model has come to be called the linear progress curve, although it is linear only when it is plotted on log-log coordinates.

This model postulates that the average direct labor hours required to manufacture a given quantity of product is related to that quantity in the following way:

\[ \frac{X}{Q} = KQ^{-a} \]

where \( X \) = total number of direct labor hours,
\( Q \) = total quantity produced,

---


3 The notation has been changed to make it comparable with other models given below.
K = direct labor hours for first unit,
a = a constant which must satisfy condition 0 < a < 1.

One can perform certain mathematical operations on this equation to find some of its implications.

For instance, if \( \frac{X}{Q} \) is the average time required to build Q items, then \( Q(\frac{X}{Q}) \) is the total amount of time required.

\[
Q\left(\frac{X}{Q}\right) = Q(KQ^{-a})
\]

\[
X = KQ(l - a)
\]

The derivative \( \frac{dX}{dQ} \) will give the approximate amount of time needed to produce one unit.

\[
\frac{dX}{dQ} = K(l - a)Q^{-a},
\]

and the reciprocal of this, \( \frac{dQ}{dX} \), will give the production rate, a figure that is closely watched in most operations.

\[
\frac{dQ}{dX} = \frac{Q^a}{K(l - a)} = \text{production rate } P.
\]

The production rate can be given as a function of the length of time in production by first solving the time-quantity relation for quantity Q and substituting this value in the production rate formula.

---

1 This progress function becomes the 80% learning curve when the constant a is equal to 0.322. Doubling cumulative output decreases the average labor cost by 20 per cent where this progress function is followed. The value of 0.322 can be derived as follows:

\[
\frac{X}{Q} = KQ^{-a}
\]

\[
0.8\left(\frac{X}{Q}\right) = K(2Q)^{-a}
\]

\[
0.8 = 2^{-a}
\]

\[
\log 0.8 = -a \log 2
\]

\[
a = 0.322
\]
\[ X = KQ (1 - a) \]
\[ \frac{X}{K} = Q (1 - a) \]
\[ \left( \frac{X}{K} \right)^{1/(1 - a)} = Q \]

Substituting: \[ P = \frac{Q^a}{K(1 - a)} \]
\[ P = \frac{(X/K)^a/(1 - a)}{K(1 - a)} \]
\[ = \frac{X^a/(1 - a)}{(1 - a)K^{1/(1 - a)}} \]

The implications of this expression for production rates cause some doubt about the correctness of the "eighty per cent" learning curve. Plotting a curve of production rate against time in production demonstrates the reason for this doubt. (See Figure 2.1.)

FIGURE 2.1
THE LINEAR PROGRESS FUNCTION
For all values of \(a\) which satisfy the condition that \(0 < a < 1\), the exponent of \(X\) takes a positive value. The dashed line shows the form taken when \(a\) is greater than 0.5. This kind of learning phenomena is clearly in disagreement with what is known by intuition—that practice does not enable one to increase his output at an ever-increasing rate of improvement. The solid line shows a typical curve for values of \(a\) that are less than 0.5 but greater than zero. This curve is also subject to some suspicion for it does not approach any asymptotic level of production. If true, this curve would indicate that the rate of production keeps increasing but at a decreasing rate of increase. Even so the eventual outcome can be seen by taking the limit of \(P\) as \(X\) approaches infinity.

\[
\lim_{X \to \infty} P = \infty
\]

This, then, is a real difficulty with the linear progress function that has been used over the last twenty years.

The original author, T. P. Wright, apparently was aware of this problem for he formulated a segmented curve for use with wide changes in the total output. Thus the curve showed a rate of improvement of eighty-three per cent for the first 100 units (a seventeen per cent decrease in average cost for each doubling of output). The production coming in the interval from 100 units to 1,000 units followed a progress function with a slope of eighty-five per cent,
and further production to the level of 10,000 units was described by a function with an eighty-seven per cent slope. Any further production would proceed with a ninety per cent slope.¹

A good analysis of the propriety of the traditional learning curve was done by Harold Asher of the RAND Corporation.² He confined his study to the production of fighter aircraft in the post-World War II period and was very careful to adjust his data to eliminate, where possible, distortions introduced by such factors as the cost accounting system, sub-contracts, and strikes.

After careful study he concluded as follows:

"It is safe to conclude on the basis of the admittedly limited sample examined in this study, that the conventional linear progress curve is not an accurate description of the relationship between cost and cumulative output. Beyond certain values of cumulative output, both the labor and the production-cost curves develop convexities."³

He estimated that the curves are reliable up to about 300 units.

Because the cumulative output of a given model of airplane by any one of the manufacturers has always been fairly

¹ Wright, T. P., Ibid., p. 127-128.
³ Asher, Ibid., p. 129.
low, this problem has not made itself apparent.¹ The cumulative average cost has demonstrated a straight-line relationship when plotted against cumulative output on log-log coordinates. Because of this performance, the airframe manufacturers have had little incentive to improve the form of their learning curves to agree with more reasonable assumptions about the ultimate rate of production.

One might conclude, then, that the relationship postulated by the airframe learning curve was a valid one and that improvement in the theory is necessary only if the curves are to be used in totally different situations. One criticism of this conclusion is important however. It is best expressed by the following quotation:

"On the other hand Industrial Engineers (sic) have long known that once a control or quantitative objective is imposed on an organization, there are strong forces created to make the performance fit the data. Hence the argument that it has 'worked' in practice is at best a weak defense of it. It is the authors' opinion that belief in a constant percentage model such as the 80% learning curve is dangerous and is logically indefensible on an

¹ Most analytical studies of progress functions have been based on Source Book of World War II Basic Data: Airframe Industry, Vol. 1, published by the Air Material Command, Dayton, Ohio, in 1947. The largest production run (called model-facility combination in the literature) studies was one of 13,686 P-40 fighters built by Curtiss-Wright Corporation at their Buffalo, New York Plant, as reported by Armen Alchian in "Reliability of Progress Curves in Airframe Production," a research memorandum published by the RAND Corporation, (p.20). The total labor hours for the first 8,494 planes could predict within six per cent the total labor hours for the last 4,192 using a linear relation on log-log grids. The error, while slight, predicted more improvement than was experienced (p.22). The average error of predictions made in this fashion was found to be 22 per cent with no particular bias in direction (p.22).
empirical basis if during the empirical period it was used as a control. ¹

This criticism is probably well-founded. It is generally agreed that foremen and workers have succeeded in evading scheduling, budgeting, and cost controls in the past, and their ability for evasion should not be underestimated where the learning curve is being used to judge their performance.

If the imposition of the progress curve could command its being met without economic effects elsewhere in the firm, the situation would be a happy one indeed, but the side-effects almost invariably make their appearance in management's "blind spots". Since this may introduce more problems than have been solved, it is felt that the learning curve should be as nearly perfect in predicting what can be done as possible.

An Alternative Model of Learning Behavior

A more reasonable assumption about production rates is that a saturation effect occurs as learning progresses. This implies that there is some maximum rate at which a worker can do his job for a sustained period of time. Once this level is reached, further improvement cannot be expected.

One model which will describe this kind of behavior is one in which the difference between the actual level of production and the ultimate level of production decreases exponentially with time.

FIGURE 2.2

THE EXPONENTIAL LEARNING CURVE

Thus, if the ultimate level of production is to be one unit per unit time, the production rate at time \( x \) will fall short of the ultimate level by \( e^{-Ax} \). Therefore

\[ P = 1 - e^{-Ax} \]

where \( P \) is the rate of production.

Many factors might be present to cause different rates of ultimate production, such as the dexterity required or the weight of the parts being handled. Therefore, this expression needs a parameter to allow it to be applied to a general class of learning situations. Thus the production
at a given time $X$ would be a percentage of the ultimate level which can be called $L$. The resulting expression would be

$$P = L(1 - e^{-AX}).$$

The coefficient $A$ has the effect of timing the learning process. A large value for $A$ describes a rapid learning process, a small value indicates a slower one.

Psychologists have shown little interest in the mathematical expressions that will define a learning curve. The functions which were suggested early in this century were readily accepted and have been subject only to re-statement since then.

Two forms have been proposed\(^1\), one of them the simple exponential decay of the potential improvement that is shown in Figure 2.2. The other form is Gompertz or logistic curve which is S-shaped and is asymptotic at both the beginning and termination of learning. This second form suggests


that some cumulative effects are necessary for learning to take place. ¹

There is one important difference between these learning curves and the one presented here. That difference is in the choice of abscissa for the learning curve. Usually the choice has been "trials" which would correspond closely to cumulative production.

The learning curve on a cumulative basis that is implied by the learning curve on a time basis can be shown by plotting theoretical values of production rate against cumulative output. This has been done in Figure 2.3, where the result is compared to an exponential form on a cumulative output basis. As can be seen, the two forms are very similar. ²

¹ Fredrick Mosteller & Robert R. Bush in their book *Stochastic Models for Learning,* (New York, John Wiley & Sons, 1955) develop an attractive model which describes the probability of a response as a function of training using a stochastic matrix operator. They show that the repeated application of a single matrix operator drives the probability toward an asymptote in an exponential fashion for repeated trials. Although the resulting mathematical form is the same as that proposed here, the Bush and Mosteller model is difficult to relate to industrial production because of the difficulty in finding a causal relation between response probability and elapsed time in assembly.

² The exact form of the learning curve on a production basis that is implied by the learning curve suggested in this chapter is given by the following expression:

\[ P = L(1 - e^{-\frac{A}{L}(q + \frac{F}{A})}). \]

The appearance of the production rate \( P \) in the exponent causes this curve to accelerate towards the asymptotic level more rapidly in the final stage of learning.
FIGURE 2.3
PRODUCTION RATE VS. PRODUCTION TIME AND CUMULATIVE OUTPUT

The learning curve with a time abscissa is not without precedent, however. Bryan and Harter used a time basis in their studies of the effect of practice on the ability to transmit Morse code, and they found what appears to be an exponential form for their learning curve.¹

It is of at least passing interest to see how closely T. P. Wright's linear progress function approximates the behavior of average direct labor input under the conditions suggested by the exponential learning curve form. The relationship between average direct labor hours and cumulative


output cannot be solved explicitly, although it is possible to choose values of \( x \) and calculate the cumulative output \( q \) for that value.\(^1\) From these values it is a simple matter to find the average time per unit \( x/q \) and to plot these on a log-log scale against \( q \). This has been done in Figure 2.4.

Immediately apparent is the curvature in the line of average cost. This model definitely does not imply an ultimate unit labor cost of zero. This curvature has a familiar appearance for it can be seen in some of the graphs that have been presented in the literature.\(^2\) This is at least some indication that the model has some usefulness, but Harold Asher's warning about the "linear" curve is applicable to this form as well.

"It is clearly a matter of judgment whether or not in a specific instance the linear curve is appropriate."\(^3\)

This is sage advice. Careful study of the production pattern is necessary for the correct application of any learning curve, and as pointed out by Conway and Schultz, the study should be made before the curve is applied as a method of control.

\(^1\) The cumulative production which has been designated as \( q \) is expressed by the definite integral \( \int_0^x Pdx \). As will be shown in Chapter III the value of this integral is

\[
q = Lx - \frac{L}{A}(1 - e^{-Ax})
\]


\(^3\) Asher, H., op. cit., p. 129.
FIGURE 2.4
PROGRESS FUNCTION FROM
HYPOTHESIZED LEARNING CURVE PERFORMANCE

\[ P = 50(1 - e^{-x/3}) \]
The mathematical expression using exponential decay of the unlearned potential was chosen for pragmatic reasons. First, as will be shown below, it gives a fairly good fit to the data. It does not give a better fit than some other models, but no other expression has been found that is superior to it. Practicability was the factor that dictated the choice of this form from among the others, for in a relative sense, it is easier to use in the operations of integration and differentiation that will follow.

In some respects this decision is a fairly sensitive one because the slopes along the learning curve will be shown later to be the curve's important characteristic where the economic lot size decision is concerned. The choice among mathematical expressions can change the position of a given value of slope without causing an apparent difference in fit to the data, since the fit is judged entirely on the basis of distance from the axes.

By other criteria, however, the problem is not so serious since small mistakes in lot size cause only a small percentage increase in total cost. In the conventional lot size model overestimating any of the costs by a factor of two (or underestimating by the same factor) will cause an increase from the minimum cost of less than ten per cent.¹

The model that is to include the costs of learning has a similar tendency to minimize the effect of errors. However, where large sums of money are involved, a small percentage error can still be costly, therefore an effort should be made to get the best possible curve to describe the learning process. In support of this form of learning curve one can first turn to the empirical data presented in Figures 2.5 and 2.6. These data are from an electronics firm, and represent daily production figures of a team of assembly workers who were performing an operation requiring a high degree of finger dexterity. (The scales of the coordinates are omitted to protect the source.) In each case, tens of thousands of units were assembled, giving each worker ample opportunity to become familiar with his particular step in the assembly operation. The thin solid line shows the variations in production from day to day. A seven-day moving average is represented by the heavier solid line. A broken line shows a learning curve of the form \( P = L(1 - e^{-Ax}) \) fitted to the data. In each case the fit appears to be good or at least adequate.

There is some evidence that this form of learning curve can be generalized to almost any repetitive industrial task. J. G. Taylor and P. C. Smith attempted to determine whether a typical learning curve exists for all kinds of tasks regardless of the complexity of the job or the length of time required to learn how to do it. To do this they studied a clothing factory and developed seventy learning curves. After adjusting
FIGURE 2.5
PRODUCTION DATA WITH APPARENT LEARNING

DAILY PRODUCTION (P)

CUMULATIVE DAYS IN PRODUCTION (X)

Moving Average

Daily Production

\[ P = L(1 - e^{-AX}) \]
FIGURE 2.6
PRODUCTION DATA WITH APPARENT LEARNING

\[ P = L(1 - e^{-Ax}) \]

- Daily Production
- Moving Average
the scale on each curve to compensate for differences in ultimate production level and length of learning time, they combined the curves into two learning curves by plotting the median percent of learning at given per cents of learning time. In comparing the two curves they found no significant differences. Because they had picked learning curves with widely varying degrees of complexity and with learning periods of length from seven weeks to twenty-seven weeks, they concluded that "a single negatively accelerated curve would serve as 'typical'"\(^1\) for all tasks studied by them.

The composite learning curve as determined by Taylor and Smith is plotted in Figure 2.7. The exponential learning function is also presented for the purpose of comparison. The workers studied by Taylor and Smith appear to exceed the expected performance during the early stages of learning. Of curious interest is the nearly linear portion of the observed learning pattern which does not appear to have any negative acceleration.\(^2\)

---


\(^2\) The exponential learning curve is negatively accelerated throughout its entire range.

\[
P = L(1 - e^{-Ax})
\]

\[
\frac{dP}{dx} = ALe^{-Ax}
\]

\[
\frac{d^2P}{dx^2} = - A^2Le^{-Ax}
\]
FIGURE 2.7
LEARNING CURVE AFTER TAYLOR & SMITH

PER CENT ATTAINMENT

Composite Curve
Taylor & Smith

Exponential Learning Curve

PER CENT LEARNING TIME
To explain why the learning curves for industrial tasks take the particular shape that they do is an exceedingly difficult task. First it is difficult to know what constitutes the learning. It is possible that all the improvement in rate of production is due to increased skill in performing the motions and manipulations required by the task. There is some indication that individuals do undergo pure motor improvement with practice. E. C. Tolman included "the acquisition of motor patterns purely as such"\(^1\) among his six distinct kinds of learning.

However, motor learning is not the solitary cause of improvement. Workers are able to devise subtle changes in the way that they perform their task, with the result that they are able to speed up significantly the pace of their job. That they hide these tricks of the trade when time-study men are in view is legend. Clearly this kind of improvement in proficiency is the result of a much higher mental activity than pure motor learning.

To claim for the production workers all of the improvement in output would be incorrect. Otherwise the industrial engineers' improved methods and jigs would be valueless. Also there are likely to be improvements in the raw materials and in the way in which they are handled. These improvements could not be explained in terms of psychological theory, but

their existence is difficult to deny. Although they can be expected to introduce discontinuities, they are automatically included in any empirical learning curve. Furthermore they are considered to be "irreversible" by one writer on progress functions, Werner Hirsch.\(^1\) The effect of these improvements will eventually show up in later production runs even where a complete turnover of labor has taken place.

It appears that a fairly complex blending of kinds of learning takes place in the work situation and that there is insufficient knowledge of what transpires to cause improvement for any type of learning that might be identified. In the face of this kind of difficulty two assumptions are made — first, that all the kinds of learning react alike to experience, and second, that the improvement in the measured variable is a linear combination of the increase in proficiency in each of the areas of learning. The purpose of the first assumption is apparent; it allows the consideration of a single variable in measuring learning rather than a number of variables to indicate the progress of each of the types of learning suspected to be present. The second assumption is necessary if the measured variable is to present a smooth function with no intervening "plateaus" before the final levelling.

---

A remaining problem is one of translating the performance of an individual tested under laboratory conditions so that it will logically explain the performance of another individual who is in a totally different environment. Some effects are likely to be introduced by the presence of supervisors and fellow workers and by different methods of reward.

By extending this problem one step further, one is faced with the task of relating the performance of an individual to the performance of a team. Possibly the output of the group is limited at each point in time by their weakest member. The group may cope with this problem itself by redividing the tasks. It is not unlikely that conditions would arise to cause the group to consistently mask the evidence of learning by deliberate slowdowns.

For all of these reasons an empirical approach to industrial learning has been chosen. Relating the multitudinous variables is an immense task, and it will not be attempted.

Explaining the shape of the learning curve is reduced then to the level of making a few simple statements about the nature of the human organism. First, the human body has finite limitations to its performance. Second, as the factors which allow improvement are increased, the organism moves only partially through the remaining distance to the finite bound.
The exponential decay of unlearned potential makes this statement in mathematical terms. It proposes that the added application of a factor favorable to output (time) causes an amount of improvement which is a fraction of the maximum possible increase in proficiency. This is a naive theory of learning, but so long as it describes the phenomenon of learning consistently it will be perfectly serviceable to the industrial decision-maker, who has always been a believer in pragmatism.¹

The reliance on an empirical approach to industrial learning limits to a large extent the conclusions that may be drawn in evaluating the learning curve that has been selected here, for before it is applied to any specific work situation, it should have to be tested and cleared for use in that situation. Otherwise there is no guarantee that it will describe the behavior of production rate.

On the other hand examples have been drawn from a variety of situations. Individual performance in learning Morse code, individual performance in the presence of a group

¹ In the background, of course, is the psychologist, disappointed in the retreat from progress taken by this treatment of learning. Refuge by the writer is taken in a statement by Herbert Woodrow before the American Psychological Association in 1941. "Quantitative psychological laws so derived are, if you will, superficial laws - superficial in the sense that they state the relationship between events observable from a position outside the behaving organisms. They are, nevertheless, as explanatory, and as truly psychological as any laws which psychologist may hope to formulate." From "The Problem of General Quantitative Laws in Psychology," Psychological Bulletin, 39:1-6, January, 1942.
in a clothing factory, and team performance in an electronics assembly line have been demonstrated. In each case there was evidence of the type of learning curve suggested here, and there is some hope that this form will be applicable to a general class of industrial learning situations.

Under the assumption that the propriety of the exponential form of learning has been established, an attempt will be made in the next chapter to utilize it in determining the size of production run with the lowest associated costs. Primarily the learning curve will be used to predict average labor costs for various lengths of production time, and against these costs will be balanced the other potential costs that would result from each run length. Using the methods of elementary differential calculus a run length with the minimum cost will be found.
CHAPTER III  THE QUESTION OF SIZE OF LOT

A worker learning a new assembly operation nearly always takes more time to complete the first unit he builds than he takes to build his hundredth unit. The acquired skill which causes this improvement can be gained only through the experience of building the intervening one hundred units. This skill can be lost (or at least dimmed) through the passage of time if he does not continue to practice the job by performing it over and over. Because of the expectation that existing skill will be lost, a manager in charge of an operation in which learning is important will be inclined to allow his workman to "make a few extra--just in case." Certainly the extra labor cost of the added "few" appears to be less than the cost that would arise from having to begin anew with a "green" workman.

To demonstrate this effect one might use the exponential learning form discussed in Chapter II to obtain typical ratios of cost performance under the effect of learning. Suppose, for example, that a worker earns $20 for a day's work and that he is expected to eventually produce 50 units per day for these wages. If his performance follows the pattern of the exponential form of learning in such a manner that the worker will be able to produce 47.5 units on the tenth day, then the first five units that he will produce can be shown to have a total labor cost of $14.60 while the last five units of a lot of one hundred will require the expenditure of $2.80
The reduction in unit labor cost has a ratio of five to one. The average labor cost of a lot of one hundred units amounts to $0.87 under these conditions of learning while the average cost of a lot of five units would be $2.92. This three to one improvement in cost figures in a strong incentive for larger lot sizes.

An identical line of reasoning can be applied to the fixed cost of going into production. The added fixed cost of producing the last five units is zero by definition. If the average fixed cost for the first five units is $5, then the average fixed cost for a hundred units would be $0.25, an even larger improvement than found with the average labor cost during learning.

In the case of fixed cost though, the well-known economic lot size model shows mathematically what has been known intuitively - that the attractiveness of the much lower average fixed cost does not necessarily justify very large production or purchase quantities, since other considerations, particularly inventory costs, need to be taken into account, and the same point can be made about the decreasing cost of labor where learning is taking place. In this chapter, an attempt will be made to relate the important factors that bear on the decision of size of manufacturing lot in the presence of learning.
Two expressions for the economic lot size will be derived. The first one will be a generalized expression good for any economic lot size question where learning is a factor. This expression does not provide an explicit solution for lot size, although an iterative solution is possible. Because of the difficulty in making this iterative solution, a restricted expression for lot size will be derived for use where learning is to be substantially completed before the production run is terminated. In using the lot size expressions the restricted form should be used first and the answer tested for validity. Where the suggested lot size falls outside the permissable ranges, the generalized form will have to be used.

An analysis of the total variable cost of satisfying customer demand will be used to find the optimum lot size.

To perform the analysis five steps are necessary. They are, 1) identify the components of cost which vary, 2.) specify how each varies with relation to the variable being controlled (lot size), 3.) construct a function which relates the total variable cost to lot size using the relations established in 2.) above, 4.) differentiate the total cost function with respect to lot size, and 5.) set the resulting derivative equal to zero and solve for the lot size. This procedure will yield the lot size with the minimum associated costs, provided that the second derivative is positive and provided especially that the cost relations as postulated
in the third step are correct and inclusive of all costs which would be affected by a change in lot size.

Determining the Economic Size of Order

The now well-known economic order size formula will be derived here because it demonstrates the procedure outlined above and leads directly to the problem at hand. First, it is assumed that the only costs of importance are the fixed cost of placing an order (or the set-up cost in a manufacturing lot size problem) and the cost of carrying inventory. These will be denoted by:

\[
\begin{align*}
S &= \text{Fixed cost of placing an order} \\
I &= \text{Inventory carrying cost per unit per year} \\
Q &= \text{Order quantity, in units} \\
R &= \text{Demand for product in units per year.}
\end{align*}
\]

The relation between lot size and the average fixed cost is equal to the fixed cost divided by the number of units in the order and is expressed mathematically as \( S/Q. \)

Similarly a relation must be found that specifies the average cost of the carrying charges that are incurred by the lot. The nature of these costs makes it likely that they

\[1\] Many writers have chosen to derive the economic order quantity rule by considering the total annual expenditure as a function of the size of order. This form can be obtained by multiplying each average cost shown here by the constant \( R. \) This does not affect the result. The method of minimizing average costs was used here because it will be found by some readers to be a fresh approach. A derivation using the annual expenditure approach may be found in Whitin, T. M., The Theory of Inventory Management, Second Edition, Princeton: The Princeton University Press, 1957, p. 32.
vary directly with both the level of inventory and with the length of time the stock is held, and the total cost should be a function of a product between time and inventory level.

Certain assumptions about the flow of goods need to be made: first, that the entire lot is placed in the inventory simultaneously and second, that the withdrawals from inventory take place at a steady rate. This causes the inventory to have a "sawtooth" characteristic with respect to time, as shown in Figure 3.1.

FIGURE 3.1

INVENTORY LEVEL VS. TIME

The total inventory cost incurred by one of the lots will be equal to the cost rate \( I \) times the instantaneous inventory level multiplied by the length of time that that level is held. Integral calculus demonstrates that this product is equivalent to the area contained by the sawtooths.
This area can easily be evaluated without resorting to an integral since it is contained by a 90-degree triangle.

The height of the triangle is equal to Q, the order size. The triangle's base is the length of time that the order will last considering the rate of usage. Expressed in years, this time will be equal to Q/R. (Thus if the order size is 20 and the annual demand is 100, some of the order will remain in inventory for 1/5 year.) The area of the triangle is one-half the product of its base times its height. Substituting, the area is Q^2/2R.

When multiplied by the cost coefficient, this area gives the total carrying cost incurred by the order - IQ^2/2R. However, it is the average cost that is needed, and this is IQ/2R.¹

The total average cost of each item in the lot is a sum of the cost components due to all factors.

Total Average Cost = Avg. Fixed Cost + Avg. Inventory Cost

\[ TAC = \frac{IQ}{2R} + \frac{S}{Q} \]

It is this total cost function that must be differentiated with respect to the order size.

¹ The absence of safety allowances is assumed here. These are necessary for the case of uncertain demand, a case that is treated by Whitin, op. cit., p. 56.
\[
\frac{d(TAC)}{dQ} = \frac{I}{2R} - \frac{S}{Q^2}
\]

Setting this derivative equal to zero allows one to solve for the order size with the lowest associated costs.

\[
0 = \frac{I}{2R} - \frac{S}{(Q^*)^2}
\]

\[
I(Q^*)^2 = 2RS
\]

\[
Q^* = \sqrt{\frac{2RS}{I}}
\]

This model specifies that lot size which will have the lowest total of inventory and set-up costs through time for any configuration of costs and demand, provided that the real situation can satisfy all the assumptions made.

In a number of ways this model is not applicable, however, to the manufacturing situation with concomitant learning. First the additions to finished goods inventory are spaced through time instead of entering the stock room simultaneously. Second, the labor component of cost changes with lot size and its effect needs to be considered.

With this in mind one can proceed with the analysis of total variable cost as outlined earlier in this chapter. It should be recalled that two models are to be built, one for small lot sizes and one for large lot sizes. The simpler model for large lot sizes will be erected first.

The General Model

The components of cost which are relevant in considering the question of lot size where learning is a factor are the
same for both models to be derived. The costs of set-up and inventory remain important as they were in the simple order quantity models, and they are joined by the variable cost of direct labor.

The cost of set-up is handled exactly as it was in the order quantity model - the average cost of set-up for the lot is equal to the set-up cost divided by the quantity.

The average inventory cost may be found by determining the shaded area in Figure 3.2, multiplying it by the cost rate to get total carrying cost for the lot, and dividing the result by the lot size.

**FIGURE 3.2**

INVENTORY LEVEL AFFECTED BY LEARNING

Figure 3.2 above shows the behavior of inventory through time. The heavy solid line shows the cumulative production
from the lot at each point in time.\footnote{The cumulative production at any time $x$ is equal to the definite integral of the function describing the production rate. Thus}

\[
q = \int_0^x P \, dx \\
= \int_0^x L(1 - e^{-Ax}) \, dx \\
= \left[ Lx + \frac{Le^{-Ax}}{A} \right]_0^x \\
= Lx - \frac{L}{A}(1 - e^{-Ax})
\]
The right-hand portion of the cross-hatched area is a right triangle with a base \(Y\) and a height \(M\). Since

\[ Y = \frac{Q}{R} - X, \text{ and} \]
\[ M = Q - RX, \]

the area of this portion is equal to

\[ \frac{1}{2R} (Q - RX)^2. \]

The left-hand portion of the cross-hatched area is more difficult to evaluate. The line forming its upper boundary must be defined by a function which must then be integrated with respect to time.

At any point of time during the period of production the level of inventory will be equal to the cumulative production at that time less the cumulative demand at that time (assuming that there was no initial level of inventory). Since the cumulative demand at any point \(x\) is equal to the rate \(R\) times the time \(x\), the line defining level of inventory can be defined as

\[ m = q - Rx \]
\[ = Lx - \frac{L}{A}(1 - e^{-Ax}) - Rx \]

This can be integrated to obtain the area under consideration.

\[ \int_{0}^{X} mx \, dx = \int_{0}^{X} Lx - Rx - \frac{L}{A}(1 - e^{-Ax}) \, dx \]
\[ \int_0^x \left( \frac{L - R}{2} x^2 - \frac{L}{A} \left( x + \frac{e^{At}}{A} \right) \right) dx = \left( \frac{L - R}{2} \right) x^2 - \frac{L}{A} \left( x - \frac{1}{A} + \frac{e^{-At}}{A} \right) = \left( \frac{L - R}{2} \right) x^2 - \frac{Q}{A} \]

At the end of the production run the variable \( x \) takes the value \( X \) and the variable \( q \) takes the value \( Q \) so that the left-hand portion of the cross-hatched area is expressed mathematically as

\[ \left( \frac{L - R}{2} \right) x^2 - \frac{Q}{A} \]

After summing the two portions of area, the result must be multiplied by the cost rate \( I \) and divided by lot size \( Q \) to obtain the average carrying cost for each item in the lot. The resulting expression is as follows:

Average Carrying Cost = \( I \left[ \frac{(Q - RX)^2}{2R} + \frac{(L - R)x^2}{2} - \frac{Q}{A} \right] \)

= \( I \left( \frac{Q}{2R} - X + \frac{Lx^2}{2Q} - \frac{1}{A} \right) \)

The relationship between average labor cost and size of lot is easily formulated. The amount of time required for the manufacture of \( Q \) units is \( X \) by definition, and accordingly the labor cost is equal to the wage rate \( W \) multiplied by the time \( X \). The average labor cost is expressed below.

Average Labor Cost = \( \frac{WX}{Q} \).

Summing up the various components gives the average cost due to all factors.
Total Average Cost = Average Set-up Cost + Average Labor Cost + Average Carrying Cost

\[ TAC = \frac{S}{Q} + \frac{WX}{Q} + \frac{I(Q - X + \frac{LX^2}{2Q} - \frac{1}{A})}{2R} \]

The total average cost function may be rewritten

\[ TAC = \frac{S}{Q} + \frac{WX}{Q} + \frac{QI}{2R} - IX + \frac{ILX^2}{2Q} - \frac{I}{A}. \]

Then taking the derivative with respect to the lot size Q, one gets

\[ \frac{d(TAC)}{dQ} = -\frac{S}{Q^2} + \frac{WQ}{PQ} - \frac{WX}{Q^2} + \frac{I}{2R} - \frac{I}{P} + \frac{ILX}{PQ} - \frac{ILX^2}{2Q^2} \]

Making the substitution \( \frac{dX}{dQ} = \frac{1}{P} \) reduces this derivative to the following form:

\[ \frac{d(TAC)}{dQ} = -\frac{S}{Q^2} + \frac{W}{PQ} - \frac{WX}{Q^2} + \frac{I}{2R} - \frac{I}{P} + \frac{ILX}{PQ} - \frac{ILX^2}{2Q^2} \]

When this derivative is set equal to zero, the solution for Q will yield the optimum value Q*. However when Q is equal to Q*, X and P must have special values X* and P*.

X* is defined by the relation \( Q^* = L - \frac{1}{A}(1 - e^{-AX^*}) \), and P* by the relation \( P^* = L(1 - e^{-AX^*}) \).

\[ \frac{d(TAC)}{dQ} = 0 = \frac{S}{Q^2} + \frac{WX^* + \frac{1}{2} IL(X^*)^2}{(Q^*)^2} - \frac{W + ILX^*}{P^*Q^*} - I\left(\frac{1}{2R} - \frac{1}{P^*}\right) \]

\( (Q^*)^2 \cdot \frac{I}{2R} \left(1 - \frac{2R}{P^*}\right) = S + WX^* + \frac{1}{2} IL(X^*)^2 - (Q^*/P^*)(W + ILX^*) \)

\[ 1 \text{ Since } Q \text{ is, by definition, } \int_0^X PdX \text{ then} \]

\[ \frac{dQ}{dX} = P \text{ and } \frac{dX}{dQ} = \frac{1}{P}. \]
Thus the optimal lot size may be found if one can solve the three equations for the three unknowns $Q^*$, $X^*$, and $P^*$. The task of reducing these equations and unknowns to an explicit solution of $Q^*$ is very difficult, if it can be done at all. This suggests that an iterative solution is necessary. This is a tedious job and one not suggested for the production floor personnel. For those firms using electronic computers, the solution of this problem will be easily performed. For those without such powerful help, a short cut method will be presented to provide a solution a large portion of the time.

The Restricted Form for Large Lot Sizes

The simplified form of the model is designed for use where the economic lot size is sufficiently large that the process of learning can be substantially completed. For lots of this size the daily production proceeds at the average rate of $L$ units per day, and any change in the average cost of direct labor can be attributed to the allocation of the initial cost of learning over lots of different quantities. Thus the learning cost is treated as simply another set-up cost.
The cost of learning can be estimated through the device of subtracting the recoverable cost from total outlay for labor during the learning period. The recoverable cost is the minimum labor cost for which the production achieved during the learning period could be produced. This analysis requires some method of determining when learning has been completed. The exponential form of learning curve implies that learning can never be "completed". That is the production rate $P$ never equals $L$ in a finite length of time, although it does attain $0.9L$, $0.99L$, $0.999L$, or any other value less than $1.0$ in a measurable length of time.

In view of this difficulty a rule of thumb is offered and will be used in deriving these models. That convention is to declare learning to be "completed" when the production rate $P$ is equal to $0.95L$. The length of time required to accomplish this level of production will be called $T$.\(^1\)

At time $T$ the behavior of production level that is implied by this convention is an instantaneous increase in production level from $0.95L$ to $L$. This has some minor effects

\[ P = 0.95L = L(1 - e^{-AT}) \]
\[ e^{-AT} = 0.05 \]
\[ AT = 3 \]
\[ A = 3/T \]

This provides a useful way of estimating the constant $A$ since an experienced manager is often able to predict the approximate length of time necessary for learning.

\[^1\] This definition of $T$ allows the coefficient $A$ to be estimated in terms of $T$. 

\[ P = 0.95L = L(1 - e^{-AT}) \]
\[ e^{-AT} = 0.05 \]
\[ AT = 3 \]
\[ A = 3/T \]
in estimating the cost of learning, but they will be ignored.¹
The shift in level is insignificant statistically because of
the inherent variability of the measure of learning. (See
Figures 2.5 and 2.6 for an example of the variability of
production level after learning has been "completed".)

To find the cost of learning, one first estimates the
total labor cost during the learning period which is equal
to the wage rate multiplied by the learning time. The
recoverable portion of this cost is calculated on a unit
basis as being equal to the wage rate divided by the maximum
possible production rate, this rate being $L$ theoretically.
Multiplying this cost rate by the number of units gives the
total recoverable cost which should be subtracted from the
total wages paid during the learning period. Expressed
mathematically this cost is $0.317WT$. (The derivation is
shown in Appendix A.)

The result of following the remainder of the procedure
for the analysis of the average costs is an expression for
the optimum lot size as shown below. The derivation of this
equation is a procedure identical to that followed for the
generalized lot size model given above. (This derivation is
also shown in Appendix A.)

¹ The error involved here is a 5% underestimation of
the cost of learning, provided, of course, that the facts
of the situation fit this model of learning.
\[ Q^* = \sqrt{\frac{2R(S + 0.317WT + 0.039ILT^2)}{I(1 - R/L)}} \]

An inspection of this formula shows it to be similar to the simplest lot size formula first discussed. One exception is the cost of learning which is added to the normal set-up cost. Another change is the adjustment \((1 - R/L)\) which is a result of the distribution through time of the input to finished goods inventory. The remaining adjustment compensates for a fixed component of the cost of carrying inventory during the learning period.

To use these models, one needs first to estimate the constants involved, \(W, L, T, R, I,\) and \(S\) and to substitute them first in the simplified model. This procedure will produce a quantity \(Q^*\) which may be the optimal size of lot. To qualify it must satisfy the condition that learning would be "complete" for a lot of that size. This will be so, provided that

\[ Q^* > 0.683LT. \]

If this test cannot be satisfied, then the more complex solution for optimum lot size must be used. The procedure requires the substitution of the values of \(W, L,\) etc., and the use of \(3/T\) for \(A\) where better estimates of \(A\) are not available.

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1 For an extended discussion of the problem of determining these values, see Goetz, B. E., Management Planning and Control, New York: McGraw-Hill, 1949.
Finally a guess at the lot size is made, based on the estimate provided by the "simple" lot size formula. When this size of lot is not satisfactory, another estimate is made and tested. This procedure is continued until successive differences do not improve the degree to which the "complex" lot size expression is satisfied.

It should be pointed out that there is some overlap in the range of answers provided by the two models. Particularly this means that the initial solution for lot size will not meet the test for completion of learning (by a small margin), and the subsequent solution by the iterative method may show a lot size slightly larger, even to the extent of satisfying the test for completion of learning that was failed in the first step. This overlap is not great, and due to the generally insensitive nature of the average cost function either solution will prove satisfactory. A lot size that is too large will cause more inventory cost than necessary, but some compensation for this extra cost will be in the decreased labor cost and set-up cost. The reverse is also true, so that a small error in lot size in either direction tends to be unimportant.¹

CHAPTER IV THE IMPORTANCE OF THE MODEL; ITS LIMITATIONS

The importance of proper inventory control has been correctly emphasized by many writers. That inventory control can achieve substantial reductions in expenses has been demonstrated in a number of instances.¹ In the case of the marginal firm of an industry this savings could mean the difference between survival and bankruptcy.

The potential cost reduction offered by inventory control is not surprising when one considers the cost of carrying an item for a year. Expressed as a percentage of the inventory's value, estimates of annual inventory costs have been higher than twenty per cent.² A carrying charge of that magnitude can easily be the source of a very large expense in the profit and loss statement where the inventory has a high value.

Also significant is the cost of low inventory levels. To maintain a small inventory while handling a high volume of business requires that closer attention be paid to the inventory. Particularly the inventory has to be replenished more often. Where there is a cost to accomplish this


² Trundle, G. T., Jr., "Your Inventory a Graveyard?" Factory Management and Maintenance, 94:45, December, 1936.
replenishment, frequent reorder or manufacture of small amounts may well cost more than a higher inventory level.

Mathematical inventory control is an attempt to arbitrate these conflicting demands on inventory level. Its advantage over the intuitive methods of setting inventory level lies in its systematic consideration of the cost of different policies. It brings assumptions into a prominent position for study and consideration, and it can cross departmental lines to show the production management why shorter production runs have merit and to show the comptroller why higher inventory levels can be less costly.

These virtues of mathematical inventory control have in recent years made them attractive to some of the largest businesses in the United States including the largest of all - the national military establishment. However, many complaints have been directed at the mathematical devices, a great number of them to the effect that the models do not describe the physical realities of business.

One of these realities is the incidence of industrial learning, and the work presented here is an attempt to improve the basic model so that it will fit the manufacturing operations where learning is apparent. Actually, only two of the assumptions underlying the basic model have been relaxed. One of them is the assumption that the unit labor cost remains constant; the other - that all additions to
finished goods inventory for a given lot take place simultane- ously.

There are several comments that might be made about the applicability of the learning curve modification of the economic lot size equation, the most obvious of them that it will prove useful only where learning has a significant cost. Where a task can be learned in an hour's time, there will be little need to make the effort of applying this model, but a six man team that learns for ten weeks should be studied carefully to determine what savings would be possible through the use of the formulation given here.

The individual manager will know whether he is faced with the learning problem or not, but certain attributes of a job are more likely than others to give rise to a predictable cost of learning.\(^1\) For one thing, the task should have a certain amount of inherent difficulty in comprehension and performance. If it does not, then it can be learned almost instantaneously. Second, a sizeable proportion of the time required to accomplish a task must be taken in some manual operation. The task requiring fifteen seconds to feed new stock into an automatic turret lathe that operates for fifteen minutes without reloading will not

\(^1\) A discussion of the characteristics of product and facilities that are conducive to learning may be found in Brenner, J. R., "Techniques and Methods of the Learning Curve for Manufacturing Operations," a Masters Thesis presented at Massachusetts Institute of Technology, 1957, p. 46-65.
offer a chance for significant improvement unless the lathe can be speeded up. Even where the task is almost one hundred per cent manual labor there is another possibility of machine control of the worker's production rate. A case in point would be the conveyor belt on an assembly line. Even though the worker may repeat his task many times, he may never be able to improve his production rate since the conveyor does not bring him his raw materials quickly enough.¹

Another requirement for the fruitful application of this variation on the economic lot size model is that a production item with reasonable expectation of continuing demand be under consideration. A "style" good is not suitable for economic lot size analysis because it is uncertain whether or not another lot of the product will ever be built. The economic lot size equation balances the marginal cost of carrying inventory against the marginal cost of setting-up for production in a future period.² Unless one cares to

¹ One manufacturer of rubber footwear deliberately controls the rate of learning on the production lines by setting the speed of the conveyor belts. It is interesting to report that the belt's speed is increased to the maximum rate for a job according to an exponential form of learning curve using a time basis.

² The line of reasoning used in developing the lot size models in Chapter III is more persuasive than it is valid. Once a decision to produce a lot has been made, the cost of setting up for that lot is a "sunk" cost, and manipulations on the size of lot will not change the amount of set-up cost. However, a future set-up cost can be forestalled by increasing the size of lot. A similar line of reasoning can be applied to the cost of learning.
think about the probability of set-up in the future period, the marginal cost of set-up cannot be defined.

Closely related to this difficulty is a series of problems arising from the costs and parameters of the economic lot size equations. These problems arise from the implicit assumption that there will be a chain of production lots (or at least two of them), each with identical values for all of the costs and quantities. The economic lot size equation gives the least costly size of lot subject to the conditions that demand and the cost of carrying inventory during the period to the subsequent lot are as predicted and that the cost of set-up, the wage rate, the ultimate production level, and the rate of learning of the subsequent lot are the same as in the lot under question and are as estimated. In a fast-moving economy such as has been seen in recent years, the time between lots must be short, or in all likelihood one or more of these conditions will not be met.

Hirsch mentions that the increase in technical knowledge from lot to lot tends to decrease the costs of production. Because of this trend he includes the cumulative number of lots as a variable in his firm progress functions, and both lot size and lot number are used to predict average labor costs. Due to this effect, the economic lot size formulas

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can lead to uneconomic performance by a firm, as can a shift in demand or wages.

One further difficulty in using the model is the task of estimating the quantities and cost figures that are required. The procedural problem of this task is large and can be difficult. More important is the conceptual problem of what kinds of cost should be used. The mathematical operation of setting the derivative of the total cost function equal to zero is simply another way of making the economists' statement, "Marginal cost equals marginal revenue." Most often the practitioner will want to rely on the average cost of set-up, for example, while he should be interested in the marginal cost of set-up.

Marginal cost is not easy to estimate from the standard cost accounting figures which usually contain allocated overhead. Marginal cost can differ according to the size of change in operation considered because of discontinuities in the total cost function. For instance small changes in the annual number of set-ups might be absorbed at low cost, but large changes could require the hiring or firing of personnel at a disproportionately higher cost. In short, the task of obtaining appropriate figures requires skill and effort.

As has been mentioned, small errors in estimating the data for the conventional economic lot size model produce a very slight amount of unnecessary cost. This characteristic
of the model presented here has not been analyzed thoroughly, but it appears that the new model will be even less sensitive than the basic model to errors due to the additive nature of most of the costs.¹

A final limitation in the model is the possibility that the learning curve chosen will not accurately describe the learning process in some particular situation. As was pointed out in Chapter II, this problem can be handled only by an analysis of the particular situation. Where the exponential learning curve is appropriate, the model of economic lot size given here can be used.

In spite of these many limitations, it is hoped that this new model can be used as one more profitable tool in the kit of the mathematical inventory analyst. Because industrial learning seems to be almost universally present, there should be an ample supply of situations where an attempt to use this model might prove profitable.

¹ A fifty per cent error in the cost of learning will not produce a fifty per cent error in the total of all the fixed costs of going into production for one lot. The exact amount of this change would depend on the relative magnitudes of set-up cost and learning costs.
Conclusions

The single major conclusion that may be drawn from the work presented here is that, subject to a number of restrictions, the most economical size of manufacturing lot may be determined even in the presence of worker learning which causes changes in rate of production and in the unit cost of the product.

The major restriction on this conclusion is the possibility that the pattern of learning used in deriving the inventory model will not match the pattern to be found in a particular industrial situation. However, evidence of the pattern selected has been drawn from widely dissimilar activities, and it is likely that the performance of many particular tasks will demonstrate the same form of learning.

The conclusion is also dependent on the ability to estimate the parameters of learning such as the ultimate rate of production and the time required to reach this rate. The stability of these factors from lot to lot is also vital, and, as is true with the basic economic lot size model, the ability to estimate the costs and rates of demand is necessary.

In analyzing the proper form of learning curves, it was concluded that the linear progress function that has
been in use by the airframe industry has some logical implications that are incompatible with common sense. In particular, the progress function suggests that the time required to perform a task approaches zero after a large number of repetitions. This conclusion has been drawn before by many others, but because of the low number of units manufactured and the possibility of biased behavior on the part of production management, the linear progress function has described the cost and production figures to the satisfaction of the industry. Therefore, they have not seen it necessary to use an alternative function.

The linear progress function does not, however, adequately describe assembly line work with a high number of repetitions of the task to be learned. For this situation especially, there was derived a learning equation with an exponential decay of the unlearned potential, and upon this equation was built the economic lot size model.

Suggestions for Further Study

Several questions of interest appeared during the course of preparing this work. For various reasons they were not pursued although they are most likely worth of further study.

An important question that needs to be answered in full is one of determining the consequence of using incorrect estimates of each of the costs and quantities in the model.
If the model is extremely sensitive to these changes, then it will be difficult and perhaps dangerous to use. If it has almost no reaction at all to wide shifts in the parameters, the worth of the model would be in doubt since there would be no point in controlling the lot size.

Closely associated with this question is one of deciding if the model that includes learning is significantly better than the basic economic lot size model. Also, there is a question of whether the complex form of the learning model of economic lot size would give a significantly less costly policy than would the simplified form, regardless of whether learning was "complete" or not.

Two questions directed at the process of industrial learning should be studied. One of them is a question of reward. Some of the theories of learning view the reinforcement of behavior to be the vital element of learning. If this is so, then different patterns of industrial learning might result from the different payment systems in use today. One method of testing this would be a study of learning of industrial tasks with incentive pay and with straight-time pay but otherwise identical in every respect.

The effect of previous experience on the rate of learning might also prove a worthwhile subject of study. Such factors as the similarity of the experience to the new task and the intervening length of time might prove to have an important influence on the pattern of learning.
Another potentially valuable area of study would determine how the quality of output varies during the learning period. Where there is a definite level of quality required, it is possible that a higher percentage of rejects would be experienced during the early stages of learning. This hypothesis needs to be tested, and if found to be true, the varying cost of materials would need to be introduced into the learning model of economic lot size.

Finally one might consider a manufacturing situation which required the product to be built in two or more distinct operations, each with different rates of learning. The object of this would be to find if the average total learning cost followed the form of the average learning cost of a one step operation, and if so, to determine how the parameters of the learning cost function could be estimated. If the average cost of all learning followed a new pattern, the equation describing it would need to be determined.

Perhaps the answer to the questions raised here will show the learning variations on the economic lot size model to be valueless in all but a few isolated instances, or perhaps they will lead to a new and better model. Because of the widespread incidence of learning in industry, there is considerable potential value in understanding the nature of learning. It is hoped that the analysis here will lead to interest in the subject and subsequent study and understanding of it.
Let $x$ = measure of time in production,

$X = \text{total time to produce a lot},$

$L = \text{ultimate production rate of completion of learning},$

$P = \text{production rate at time } x = L(1 - e^{-Ax}),$

$q = \text{cumulative production at time } x = Lx - \frac{L}{A}(1 - e^{-Ax}),$

$Q = \text{total production for one lot} = LX - \frac{L}{A}(1 - e^{-AX}),$

$A = \text{a constant which describes the speed of learning},$

$W = \text{wages per unit time},$

$I = \text{inventory cost per unit per unit time},$

$R = \text{rate of demand for product}.$

Assume that learning is completed at time $T$ when

$P = 0.95L,$

and that the production rate then jumps to rate of $L$ units.

This allows an estimate of the constant $A.$

$$P = 0.95L = L(1 - e^{-AT})$$

$$e^{-AT} = 0.5$$

$$AT = 3.00$$

$$A = \frac{3}{T}$$

The Average Cost of Learning

Total Learning Cost = Outlay for Labor - Recoverable Labor Cost

$$= WT - W\left[\frac{LT}{L} - \frac{L}{A}(1 - e^{-AT})\right]$$

$$= WT - W\left[ T - \frac{T}{3}(0.95)\right]$$

$$= 0.317WT$$
Average Learning Cost = \frac{0.317WT}{Q}

\frac{d}{dQ}(Average Learning Cost) = -\frac{0.317WT}{Q^2}

The Average Cost of Set-ups

Total Set-up Cost = S

Average Set-up Cost = \frac{S}{Q}

\frac{d}{dQ}(Average Set-up Cost) = -\frac{S}{Q^2}

The Average Cost of Carrying Inventory

(See Figure A.1.)

Total Carrying Cost = I \times \text{Avg. Inventory Level} \times \text{Cycle Time}

= I \times (\text{Cross-hatched Area of Figure A.1.})

= I \times (\text{Area A} + \text{Area B} + \text{Area C})

Average Carrying Cost = \frac{I}{Q}(\text{Area A}) + \frac{I}{Q}(\text{Area B}) + \frac{I}{Q}(\text{Area C})

\text{Area A}

\text{Area A} = \left[ \frac{Lx^2}{2} - \frac{1}{A}(x - \frac{R}{A}) \right]_0^T - \left[ \frac{Rx^2}{2} \right]_0^T

= \frac{0.272LT^2}{2} - \frac{0.5RT^2}{2}

Avg. Carrying Cost_A = \frac{I}{Q}(0.272LT^2 - 0.5RT^2)

(Avg. Carrying Cost_A) = -\frac{I}{Q^2}(0.272LT^2 - 0.5RT^2)

\text{Area B}

\text{Area B} = \frac{1}{2}(X - T) \left[ (0.683LT - RT) + (Q - RX) \right]

\text{Substitute } X = Q/L + 0.317T

\text{Area B} = \frac{1}{2}(Q/L - 0.683T)(0.683LT + Q - \frac{RQ}{L} - 1.317RT)
Avg. Carrying Cost

\[ \text{Avg. Carrying Cost}_B = \frac{1}{2}(Q/L - \frac{RQ}{L^2} - \frac{0.466LT^2}{Q} - 0.634\frac{RT}{L} + 0.900\frac{RT^2}{Q}) \]

\[ \frac{d}{dQ}(\text{Avg. Carrying Cost}_B) = \frac{1}{2}\left(\frac{1}{L^2} - \frac{R}{L^2} + \frac{0.466LT^2}{Q^2} - 0.900\frac{RT^2}{Q^2}\right) \]

Area C

\[ \text{Area C} = \frac{1}{2}(Q - RX)(\frac{Q}{R} - X) \]

Substitute \( X = \frac{Q}{L} + 0.317T \)

\[ \text{Area C} = \frac{1}{2}(Q - \frac{RQ}{L} - 0.317RT)(\frac{Q}{R} - \frac{Q}{L} - 0.317T) \]

\[ = \frac{1}{2}\left(\frac{Q^2}{R} - 2\frac{Q^2}{L} - 0.634QT + \frac{RQ^2}{L^2} + 0.634\frac{RT}{L} + 0.100RT^2\right) \]

Avg. Carrying Cost

\[ \text{Avg. Carrying Cost}_C = \frac{1}{2}\left(\frac{Q}{R} - \frac{2Q}{L} - 0.634T + \frac{RQ^2}{L^2} + 0.634\frac{RT}{L} + 0.100\frac{RT^2}{Q}\right) \]

\[ \frac{d}{dQ}(\text{Avg. Carrying Cost}_C) = \frac{1}{2}\left(\frac{1}{R^2} - \frac{2}{L^2} + \frac{R}{L^2} - 0.100\frac{RT^2}{Q^2}\right) \]

\[ \frac{d}{dQ}(\text{Avg. Carrying Cost}) = \frac{d}{dQ}(\text{A.C.C.}_A) + \frac{d}{dQ}(\text{A.C.C.}_B) + \frac{d}{dQ}(\text{A.C.C.}_C) \]

\[ = \frac{1}{2}\left(\frac{1}{R} - \frac{1}{L} - 0.078\frac{LT^2}{Q^2}\right) \]

Total Average Cost

\[ \frac{d}{dQ}(\text{Total Average Cost}) = \frac{d}{dQ}(\text{Avg. Set-up Cost} + \text{Avg. Learning Cost} + \text{Avg. Carrying Cost}) \]

\[ = -\frac{S}{Q^2} - \frac{0.317WT}{Q^2} + \frac{1}{2}\left(\frac{1}{R} - \frac{1}{L} - 0.078\frac{LT^2}{Q^2}\right) \]

\[ \frac{d}{dQ}(\text{Total Average Cost}) = 0 \]

\[ \frac{S + 0.317WT}{(Q^*)^2} = \frac{1}{2}\left(\frac{1}{R} - \frac{1}{L}\right) - 0.039\frac{LT^2}{(Q^*)^2} \]

\[ S + \frac{0.317WT + 0.039LT^2}{(Q^*)^2} = \frac{1}{2R}\left(1 - \frac{R}{L}\right) \]

\[ Q^* = \sqrt{\frac{2R(S + 0.317WT + 0.039LT^2)}{I(1 - \frac{R}{L})}} \]
FIGURE A.1

INVENTORY LEVEL VS. TIME
APPENDIX B - THE MODEL WITH INSTANTANEOUS LEARNING

By hypothesizing instantaneous learning one can test
the validity of the learning variation on the economic lot
size equation. Assigning an infinitely large value to the
constant A creates a condition of instantaneous learning
which implies that the production rate immediately reaches
the maximum level and holds there.

\[ P = L(1 - e^{-Ax}) \]

Under this condition the cumulative production is simply the
maximum rate multiplied by the time in production.

\[ Q = Lx - \frac{L}{A}(1 - e^{-Ax}) \]

Thus \( P^* = L \). Since \( Q^* = LX^* \), it then follows that \( X^* = Q^*/L \).

These values can be substituted into the economic lot
size formula and manipulated as follows.

\[
(Q^*)^2 = \frac{2R[S + W(X^* - Q^*/P^*) + IQLX^*/L(Q^*/2L - Q^*/L)]}{I(1 - 2R/L)}
\]

\[
\frac{I(Q^*)^2 - (1 - \frac{2R}{L})}{2R} = 2R[S + IQ^*(- Q^*/2L)]
\]

\[
\frac{I(Q^*)^2 - (1 - \frac{2R}{L})}{2R} = 2RS - 2RQ^2/2L
\]

\[
\frac{I(Q^*)^2 - (1 - \frac{2R}{L})}{2R} = 2RS
\]

\[
\frac{I(Q^*)^2(1 - \frac{R}{L}) = 2RS}{I(1-R/L)}
\]

\[
(Q^*)^2 = \frac{2RS}{I(1-R/L)}
\]
This is identical to the expression for economic lot size under the special condition of a distributed flow of finished goods into inventory.¹

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