STABILITY AND TRANSIENT ANALYSIS
OF CONTROLLED LONGITUDINAL MOTION OF AIRCRAFT
WITH NONIDEAL AUTOMATIC CONTROLS

by

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Submitted in partial fulfillment of the requirement
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STABILITY AND TRANSIENT ANALYSIS
OF CONTINUOUS POSITIVE NOCTURNAL MOTION OF AIRPLANE
WITH NONIDEAL AUTOMATIC CONTROL

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Dear Sir:

I hereby submit a thesis entitled "Stability and Transient Analysis of the Longitudinal Motion of Aircraft with Nonideal Automatic Controls" in partial fulfillment of the requirement for the degree of Doctor of Science.

Very respectfully yours,

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TABLE OF CONTENTS

LETTER OF SUBMITTAL
ACKNOWLEDGMENT
ABSTRACT
INTRODUCTION

PART I
GENERAL THEORY OF AUTOMATIC CONTROL ON
LONGITUDINAL MOTION OF AIRCRAFT

CHAPTER ONE Controlled Longitudinal Motion of Aircraft with Conventional $\theta$ Control
CHAPTER TWO Theory of Deparasitization

PART II
PROPERTIES OF THE QUARTIC EQUATION

CHAPTER THREE The Importance of the Quartic Equation to the Generalized Automatic Control Problem
CHAPTER FOUR Stability Transition Curve of the Quartic Equation with Standardized Nondimensional Coefficients
CHAPTER FIVE The Development of the Quartic Chart and Stability Criteria
CHAPTER SIX Detailed Analysis of Stability Criteria $M$ and $N$

PART III
STABILITY IMPROVEMENT WITH DIFFERENT CONTROLS

CHAPTER SEVEN Stability Transition Curve with Different Coupling Coefficients
CHAPTER EIGHT Controls with High Natural Frequency
TABLE OF CONTENTS CONT'D

CHAPTER NINE  Compounding Controls  128
CHAPTER TEN  Tuning Controls  148

PART IV
ANALYSIS OF TRANSIENT

CHAPTER ELEVEN  Stability Function, Quality Function, Disturbance Function and Response Function  166
CHAPTER TWELVE  Characteristic Decomposition  173
CHAPTER THIRTEEN  Surging Error and Surging Disturbance  186

PART V
PERFORMANCE OF TYPICAL AIRPLANE WITH NONIDEAL CONTROL AND SOME REFINEMENT CONSIDERATIONS

CHAPTER FOURTEEN  Airplane Controlled by Pitching-Velocity-Elevator Control Coupling  198
CHAPTER FIFTEEN  Some Refinement Considerations in the Deparasitized Nonideal Longitudinal Control  213

CONCLUSION AND SUGGESTIONS TO FURTHER DEVELOPMENT  216
BIBLIOGRAPHY  218
BIOGRAPHY  221
APPENDICES:

A  Nondimensional Cubic Equation and the Cubic Chart  Al-A5
B  Directions for the Quartic Chart  B1-B13
C  Response to Unit Surging Disturbance Developed by Heaviside Expansion Theorem  C1-C3
D  Semigraphical Application of De Moivre's Theorem in Evaluating Polynomial Functions with a Complex Number  D1-D3
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Longitudinal Disturbance Detectors Available for Automatic Controls</td>
<td>19</td>
</tr>
<tr>
<td>II</td>
<td>Nondimensional Coefficients for Longitudinal Stability of the Airplane</td>
<td>27</td>
</tr>
<tr>
<td>III</td>
<td>Stability Behavior of the Quartic Equation in Nondimensional Form</td>
<td>92</td>
</tr>
<tr>
<td>IV</td>
<td>Notations and Definitions of &quot;Advantages&quot; Due to Control in Action</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>Approximate Stability Improvement Due to Various Controls of High Natural Frequency at Certain</td>
<td>129</td>
</tr>
<tr>
<td>VI</td>
<td>Comparison Between $+\Gamma$, and $-\Gamma$, Tuning Controls</td>
<td>164</td>
</tr>
<tr>
<td>Figure No.</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Stability Transition Curve of the Standardized Nondimensional Quartic Equation</td>
<td>53</td>
</tr>
<tr>
<td>2 A,B,C</td>
<td>Stability Criteria M and N of the Quartic Equation</td>
<td>65,66</td>
</tr>
<tr>
<td>3</td>
<td>Effect of Error-Sensitive Control on Stability Transition Curve</td>
<td>98</td>
</tr>
<tr>
<td>4</td>
<td>Effect of Error-Velocity Control on Transition Curve</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>Effect of Error-Acceleration Control on Transition Curve</td>
<td>103</td>
</tr>
<tr>
<td>6</td>
<td>Characteristics of Error-Sensitive Control at large $\rho$'s</td>
<td>110</td>
</tr>
<tr>
<td>7 A,B,C,D</td>
<td>Characteristics of Error-Acceleration Control at Large $\rho$'s</td>
<td>114-117</td>
</tr>
<tr>
<td>8 A,B,C</td>
<td>Characteristics of Error-Velocity Controls at Large $\rho$'s $(\pm)\gamma$</td>
<td>122-124</td>
</tr>
<tr>
<td>9</td>
<td>Characteristics of Error-Acceleration Control at Large $\rho$'s</td>
<td>126</td>
</tr>
<tr>
<td>10</td>
<td>Characteristics of $\gamma_1, \gamma_2$ Compound Controls, Damping and Damping Ratio Improving Only (For finding $\rho_0$)</td>
<td>135</td>
</tr>
<tr>
<td>11</td>
<td>Characteristics of $\gamma_1, \gamma_2$ Compound Controls, Damping and Damping Ratio Improving Only (For finding $\gamma_1$)</td>
<td>136</td>
</tr>
<tr>
<td>12 A,B,C,D,E</td>
<td>Characteristics of $\gamma_1, \gamma_2$ Compound Controls, Damping and Damping Ratio Improving Only (For finding $\gamma_2$)</td>
<td>137-141</td>
</tr>
<tr>
<td>13</td>
<td>Characteristics of $\gamma_1, \gamma_2$ Compound Controls, Damping and Damping Ratio Improving Only (For finding $\rho_0$)</td>
<td>144</td>
</tr>
<tr>
<td>14</td>
<td>Characteristics of $\gamma_1, \gamma_3$ Compound Controls, Damping and Damping Ratio Improving Only (For finding $\gamma_3$)</td>
<td>145</td>
</tr>
<tr>
<td>Figure No.</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>------------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>15</td>
<td>Characteristics of ( \lambda - \lambda_s ) Compound Controls, Damping and Damping Ratio Improving Only (For finding ( \lambda ))</td>
<td>146</td>
</tr>
<tr>
<td>16</td>
<td>Design Data of Tuning Controls with (+\Gamma) Coupling, (\rho_{\omega_0} vs. \xi_0) at Various (\xi_t)</td>
<td>154</td>
</tr>
<tr>
<td>17</td>
<td>Design Data of Tuning Controls with (+\Gamma) Coupling, (\xi_c vs. \xi_0) at Various (\xi_t)</td>
<td>155</td>
</tr>
<tr>
<td>18</td>
<td>Design Data of Tuning Controls with (+\Gamma) Coupling, (\xi_{\omega_0} vs. \xi_0) at Various (\xi_t)</td>
<td>156</td>
</tr>
<tr>
<td>19</td>
<td>Design Data of Tuning Controls with (-\Gamma) Coupling, (\rho_{\omega_0} vs. \xi_0) at Various (\xi_t)</td>
<td>160</td>
</tr>
<tr>
<td>20</td>
<td>Design Data of Tuning Controls with (-\Gamma) Coupling, (\xi_c vs. \xi_0) at Various (\xi_t)</td>
<td>161</td>
</tr>
<tr>
<td>21</td>
<td>Design Data of Tuning Controls with (-\Gamma) Coupling, (\xi_{\omega_0} vs. \xi_0) at Various (\xi_t)</td>
<td>162</td>
</tr>
<tr>
<td>22</td>
<td>Unit Disturbance Function with Various Apparent Surging Factors (\xi)</td>
<td>189</td>
</tr>
<tr>
<td>23</td>
<td>Controlled and Uncontrolled Pitching Motion Due To Vertical Surging Gust ((\xi = 1.0)) with respect To Slow Oscillatory Component</td>
<td>203</td>
</tr>
<tr>
<td>24</td>
<td>Controlled and Uncontrolled Vertical Speed and Acceleration Response due to Vertical Surging Disturbance ((\xi = 1.0)) with Respect to Slow Oscillatory Component</td>
<td>204</td>
</tr>
<tr>
<td>25</td>
<td>Uncontrolled Pitching Motion Due to Vertical Surging Gust ((\xi = 1.0)) with Respect to Fast Oscillatory Component</td>
<td>203</td>
</tr>
</tbody>
</table>
ABSTRACT

Automatic control systems have long been in practice with considerable success based upon years of cut and try experiences. The present type of automatic pilot used in aircraft has followed the same track of development. Problems dealing with automatic control of aircraft are too complicated for solution by ordinary mathematical methods because of the number of freedom involved. The equations of motion, when reduced to one dependent variable, give a linear differential equation of the sixth or higher order. However, for longitudinal stability, the pitching response can be represented by a fourth order differential equation if the control is properly designed. The theory involved is presented in Part I and its justification is carried out in Part V.

Many problems in the field of automatic control, both in aircraft and other systems, lead to the differential equation of the fourth order. For this reason it is important to develop a means for solving these quartic equations which will be useful in engineering practice. In the present thesis systematic and convenient methods for solving quartic equations are developed and presented in the form of curves and charts based on nondimensional variables. Simplified methods of determining stability criteria are also discussed.
The physical significance of the nondimensional resolvent cubic derived from the nondimensionalized quartic equation is demonstrated. It is shown that at least one of the three roots of the resolvent cubic equation is the sum of the ratio of the natural frequencies of the two components and its reciprocal. A quartic chart is developed from the modified resolvent cubic equation. This quartic chart makes possible the practical solution of the quartic equation in terms of nondimensional physical constants. The scheme used gives results which are in error by less than 2%.

Possibilities for the improvement of stability with controls of high frequency and different coupling coefficients are investigated and the results are presented in charts. These results are expressed in terms of figures of merit which are called advantages. A table of such advantages is given which is very very useful for finding the compounding effect of controls with exciting forces proportional to different order of time derivatives of error. The theory of compounding effects is investigated and a generalized rule for the expression of the control advantage is presented. Special compounding controls are investigated with the purpose of improving the damping ratio of the system to be controlled, and the results are presented in a series of charts.

Particular relationships between the damping ratio and natural frequency of the control and the damping ratio and natural frequency of the member to be controlled should be
maintained so that the controlled results may have unique frequency and unique damping ratio. The use of a properly selected frequency and damping ratio for the control is called tuning. The requirements for proper tuning are presented in the form of charts.

A knowledge of the stability of a system is not sufficient to give the whole picture of the response to a forcing function. A method suitable for a generalized transient analysis of automatic control problems is given in detail as a result of the application of the Heaviside Expansion Theorem. For aircraft applications particular attention is paid to the response of pitching and vertical motion of the airplane when a vertical surging gust is encountered. Both controlled and uncontrolled results are given in the form of plots. The control system considered in the analysis is based upon an assumed specification of keeping the frequency of the oscillations of the control and the aircraft unchanged. Any other suitable specification can be used and the control and coupling factor can be easily evaluated with the aid of the previously developed study of stability.

Four appendices are presented in this thesis.

Appendix A is the development of the cubic chart upon the established nondimensional form used by Weiss. Its improvement for use in evaluation is that only one chart is sufficient for the evaluation of physical constants (nondimensional) of the cubic equation which is often met in constant speed control systems.
Appendix B is a concise set of directions for using the quartic chart.

Appendix C is the derivation of the response when a system (with no repeating roots in its stability equation) is encountered with a surging disturbance similar to that often met in bumpy air.

Appendix D is a semi-graphical application of De Moivre's Theorem to the problem of evaluating polynomial functions involving complex numbers.

The material presented in this thesis should be useful as a common tool for the automatic control designer when the control system involves linear differential equations of the fourth order. The longitudinal control of an airplane is used as a particular illustration of the methods developed in this thesis.
I N T R O D U C T I O N

For the modern aircraft, whether commercial or military, a dependable and efficient piloting system is of growing importance. A dependable and efficient piloting system may be defined as one which consists of a combination of groups of dependable pilot mechanisms operated by a certain efficient agency. The proper coordination must be determined according to the condition of flight as well as the condition of the environment. However, during long range flight the aircraft must be kept constantly in course whether the course is straight, follows the path of a great circle, or even follows a number of connected broken lines determined by the convenience of available radio beacons. Keeping in course is the only way for the airplane, which is already available in service, to save the total fuel consumption, to reduce wearing of the engines, to save time for the passengers of the commercial aircraft or increase the swiftness of military operation. In rough weather, not only may the course be subject to drift, but annoying oscillations are always associated with it if the aircraft is not properly piloted or controlled. It is advisable to eliminate as far as possible any annoying oscillations due to such disturbances because it assures the comfort of passengers and diminishes the fatigue of the crew. Briefly, we obtain greater efficiency in our flying activity with a proper control system.
In the early stages of aviation, the pilot was the sole factor carrying the responsibility of control. Due to the lack of sensitivity to certain modes of motion and the sluggishness of reaction of the human pilot, the aircraft could never be expected to achieve its best performance. Moreover, the constant vigilance of the human pilot led him to a state of extreme fatigue, and erroneous controlling of the plane was frequently the result.

Since World War I the automatic pilot has been introduced to aircraft for normal flight. The result has been very promising. As one of its leading industries, the Sperry Gyroscope Company has developed an automatic pilot system to perfection entirely through a series of elaborate experiments. The success of the present Sperry system depends upon its logical procedure of development, but strict mathematical investigation has never been bothered with until recently when Weiss and Lin made a mathematical study of the controlled motion of airplanes. Haus has tabulated every possible means of detection of deviation from normal quantities which is available as control source. However, the Sperry Company has so far only made an exhaustive development of simple deviation control (simple displacement control, using \( \theta \), the absolute inclination of the airplane as the control source). This is true also of the British Smith automatic pilot system. Incidentally, the fact that Sperry and Smith have followed the same track might be due to a lack of theoretical light for other possible or even better control.
such as the compounding control used aboard the S. S. New
Mexico reported by Minorsky. Of course, due to the wide
difference of mediums of support, and henceforth the differ-
ence in construction of steamships and aircraft, entirely
different types of control may be needed, respectively, to
produce successfully controlled motion. However, the prin-
ciple of continuity and the advantage of higher derivative
control has been explained very clearly by Minorsky in his
paper based upon the expansion theory of Taylor's series
for a continuous function. In a simple word, "the higher
time derivatives are capable of giving us the necessary warn-
ing as to what is going to happen a few instants ahead of the
present time". It is logical to predict that with proper
coordination of the compounding of derivative and displace-
ment controls, the controlled result may be better than that
obtainable from a simple control. It is the purpose of this
thesis to throw considerable light on the exploration of sim-
ple controls other than the displacement type, as well as
compounding controls which are hitherto hidden from aeronaut-
ical engineers.

From the general law of motion of a system composed of
many degrees of freedom which are intercoupled by the nature
of its construction, as many simultaneous linear differential
equations can be written if the coefficients of each variable
and its time derivatives are constant; that is, if each of
the component forces varies linearly with its corresponding
variable or time derivative of the variable. Due to physical properties such strict linear variation could not exist for certain variables, especially when their variations are of large range. However, if we are dealing with small variations, the assumption of linearity is acceptable especially when the motion is reduced to small magnitudes by well designed controls. The solution of the general simultaneous linear differential equations can be written using the principle of determinants and the principle of operational calculus. However, the task is not a pleasant one when the number of degree of freedom gets higher. Even a two-degrees-of-freedom system, which usually involves a solution of fourth order linear differential equation, becomes unmanageable as far as the general characteristic of the system is concerned.

Uncontrolled longitudinal motion of aircraft already yields a linear differential equation of the fourth order. The addition of nonideal control raises the equation from fourth to sixth order in which it is unmanageable to investigate systematically the possibility of better control than the conventionally successful simple displacement type. Probably, being handicapped by the unmanageability of the higher order differential equation, both Weiss and Lin confined their investigation to the simple displacement control. With utmost effort, a better displacement control may be discovered, but the improvement
shall be predictably limited as the manufacturers have definitely fought their way with long experience to the present degree of success on their simple displacement control.

It is well known that the uncontrolled longitudinal motion after disturbance consists of two oscillatory motions of different frequency with different damping. The so-called short oscillation is usually associated with considerable damping which damps the motion very quickly. The long oscillation contains only a little damping so that it dies away in a few oscillations. Sometimes the damping is slightly negative, so the long oscillation will accordingly increase in magnitude after a disturbance. The application of control aims to redistribute the original poor distribution of damping among the two oscillations. In fact, as we shall see from the following parts of this thesis, a sole change of coefficient or coefficients of the fourth order differential equation, which represents the stability of the uncontrolled longitudinal motion, will attain the end of redistribution of damping. The nature really leaves plenty of flexibility for us to play these coefficients of the fourth order differential equation. The raising of the order of the differential equation by the addition of control is inevitable. But could we manage the additional degree of freedom in such a way that it remains its characteristic (damping and frequency) no matter what the coupling may be, and thus let it only affect the coefficient or coefficients of the original uncontrolled fourth order differential equa-
tion? The answer to this question is "yes, we can." From the analysis of the equation for controlled longitudinal motion which appears in Part I, Chapter II of this thesis, we shall see the mathematical proof of this answer together with a logical argument of good reasons for the introduction of such kind of control for the airplane.

For lateral disturbed motion the lack of course stability of an uncontrolled airplane can be overcome by the addition of control of rudder movement. The order of differential equation shall be raised from fifth to ninth if aileron control is provided together. The solution for a best control is far more difficult than in the case of longitudinal motion.

However, the thesis is carried on to insure full understanding of fourth order differential equations with which the design of a most desired longitudinal control is made possible. As the quartic equation has been a stumbling block to the understanding of higher degree equations, its full understanding shall improve the ladder for us to attack the still higher degree equation as that which we shall face on the lateral controls.

Maxwell has done a good deal of work on servos involving the two-degree-of-freedom equation, yet due to lack of systematization his results are not readily applicable.

Besides the automatic control of airplanes, the growing importance of other control problems encourages the writer...
to systematize the mathematical presentation in a much wider range than it should be if only applicable to aircraft control engineers, in the hope that it will minimize the effort for every control engineer in his particular design work.
PART I

GENERAL THEORY OF AUTOMATIC CONTROL ON
LONGITUDINAL MOTION OF AIRCRAFT
1. Separability of Longitudinal and Lateral Motions of the Airplane

A rigid body moving in free space possesses six degrees of freedom, three translational ones along three mutually perpendicular axes and three rotational ones about the same three axes. If the motion away from its steady state is small in magnitude, or all the physical coefficients are of "linearity", the unsteady components of motion of such a body would comprise six simultaneous oscillations having different frequencies and different dampings (or their equivalent in subsidences) which can be solved from the six simultaneous equations of motion -- each for one degree of freedom -- or their resultant represented by a linear differential equation of the twelfth order. The stability equation of such motion is, therefore, of the twelfth order.

However, the forces and moments on the airplane are indifferent to displacements along each of its three axes, so three roots of the resultant equation are zero. In addition to this, the uncontrolled airplane lacks the sense in azimuth (about vertical axis) so another root is also zero. The stability equation is therefore reduced to the eighth degree.
Due to the plane-symmetrical construction of the airplane, the six degrees of freedom can be separated into two groups of three each. One group involves only motion in the plane of symmetry and the solution of this group of equations gives the longitudinal motion of the airplane. The second group, if considered for small displacement alone, involves only asymmetric or lateral motion of the airplane. The separation of longitudinal motion from the lateral one greatly simplifies the mathematical mess otherwise involved.

2. Uncontrolled Longitudinal Motion

For the static and dynamic stabilities of longitudinal motion the aircraft engineer has designed a horizontal tail for his airplane. The tail is again split into a stabilizer and an elevator immediately behind the stabilizer for the convenience of controlling the airplane when it is disturbed by gusts of wind. However, when the elevator is locked, the airplane is said to be uncontrolled and the longitudinal motion is then defined as uncontrolled longitudinal motion.

For convenience of study, equations of uncontrolled longitudinal motion may be written, with reference to wind axes, as follows:

\[ X = m(\dot{u} + Wq) \]

*The notation adopted in this thesis will be the same as that used by Metcalf in his Resume on Airplane Longitudinal Stability. A table of symbols and definitions is prepared in the beginning of this part of the thesis.
where the left sides of the above equations represent aero-
dynamic forces and moments while the right sides represent
the inertial forces and moments necessary for the balance of
the state of motion (small oscillation).

On expanding,

\[ X = \frac{\partial X}{\partial u} u + \frac{\partial X}{\partial w} w + \frac{\partial X}{\partial q} q + \frac{\partial X}{\partial \theta} \theta \quad \ldots \quad (1.04) \]

\[ Z = \frac{\partial Z}{\partial u} u + \frac{\partial Z}{\partial w} w + \frac{\partial Z}{\partial q} q + \frac{\partial Z}{\partial \theta} \theta \quad \ldots \quad (1.05) \]

\[ M = \frac{\partial M}{\partial u} u + \frac{\partial M}{\partial w} w + \frac{\partial M}{\partial q} q + \frac{\partial M}{\partial \theta} \theta \quad \ldots \quad (1.06) \]

Let \( X_u = \frac{1}{m} \frac{\partial X}{\partial u} \) \( Z_w = \frac{1}{m} \frac{\partial Z}{\partial w} \) \( M_q = \frac{1}{B} \frac{\partial M}{\partial q} \)

Then Eqs. (1.01), (1.02) and (1.03) can be written in the
following form:

\[ \dot{u} + W_q = X_u u + X_w w + X_q q + g \cos \theta \quad (1.10) \]

\[ \dot{w} - U_q = Z_u u + Z_w w + Z_q q + g \sin \theta \quad (1.11) \]

\[ \ddot{\theta} = M_u u + M_w w + M_q q \quad (1.12) \]

Rearrange the terms and replace the time operator \( \frac{d}{dt} \) by the
symbol D. The above equations take the following form on
neglecting terms with negligible coefficients:

\[ (D - X_u) u - X_w w - g \cos \theta = 0 \quad (1.13) \]

\[ - Z_u u + (D - Z_w) w - (D U_q + g \sin \theta) \theta = 0 \quad (1.14) \]

\[ - M_u u + M_w w + (D^2 - D M_q) \theta = 0 \quad (1.15) \]
Equations (1.13), (1.14) and (1.15) represent the motion when the airplane is disturbed, but the disturbing force has already ceased. However, when the airplane encounters a gust such as the vertical one \(w_0\) of magnitude \(w_0\) beginning from a state of equilibrium, equations (1.13), (1.14) and (1.15) become:

\[
(D - X_u)u + X_ww - g\cos\Theta = -X_ww_0l \\
- Z_uu + (D - Z_w)w - (DU_o + g)\Theta = -Z_ww_0l \\
- M_uu - Mww + (D^2 - DM_q)\Theta = -M_ww_0l
\]

By the principle of determinants, the solution can be expressed in operational form:

\[
u = \frac{\Delta'_u}{\Delta'_o} w_0l \quad (1.19)
\]

\[
\Theta = \frac{\Delta'_\Theta}{\Delta'_o} w_0l \quad (1.20)
\]

\[
w = \frac{\Delta'_w}{\Delta'_o} w_0l \quad (1.21)
\]

where \(\Delta'_o\) is the stability determinant of the system

\[
\Delta'_o = \begin{vmatrix}
D - X_u & -X_w & -g\cos \Theta \\
- Z_u & D - Z_w & -DU_o - g\Theta \\
- M_u & -M_w & D^2 - DM_q
\end{vmatrix} \quad (1.22)
\]

and \(\Delta'_u\), \(\Delta'_\Theta\) and \(\Delta'_w\) are the quality determinants of forward velocity, inclination angle and vertical velocity.

\[
\Delta'_u = \begin{vmatrix}
-X_w & -X_w & -g\cos H \\
-Z_w & D - Z_w & -DU_o - g H \\
-M_w & -M_w & D^2 - DM_q
\end{vmatrix} = D
\]

*The symbol 1 means unit step function; that is, when \(t < 0\) the function is zero, when \(t \geq 0\), the function is unity.*
\[
\Delta^T = \begin{vmatrix}
-D-Xu & -Xw & -Xw \\
-Zu & D-Zw & -Zw \\
-Mu & -Mw & -Mw
\end{vmatrix} = D
\]

\[
\Delta^T_w = \begin{vmatrix}
-D-Xu & -Xw & -g \cos \Theta \\
-Zu & -Zw & -DU_0 -g \cos \Theta \\
-Mu & -Mw & D^2 - DM_q
\end{vmatrix} = \Delta^T_0 - D
\]

Eq. (1.25) can also be written as

\[
\Delta^T_w = \Delta^T_0 - \left( \Delta^T_u + \Delta^T_\Theta + \Delta^T' \right)
\]

where \[
\Delta' = \begin{vmatrix}
1 & 1 & 1 \\
-Mu & -Mw & D^2 - DM_q
\end{vmatrix}
\] (1.26)

The numerical solution of Equations (1.19), (1.20), and (1.21) is deferred until all the determinants are nondimensionalized and the control theory is well established.

When Eq. (1.22) is developed and equated to zero, an equation of fourth degree in algebraic form is obtained. It is this equation from which stability criteria of the disturbed motion can be evaluated. The extensive study of stability criteria of such equations is deferred to Part II of this thesis.

From the previous work of other investigators, it is understood that the longitudinal motion of the airplane after being disturbed consists of two components:

(a) A heavily damped oscillation of short period (of the order of a few seconds). This component disappears almost at once and in most airplanes is not noticeable.
(b) A lightly damped slow oscillation, during which the airplane produces noticeable changes of forward speed, altitude, and attitude.

It is customary to neglect the quick oscillation because of its almost immediate disappearance and no report is available from the pilots as they are never bothered by such rapid oscillation. Some writers even imply denunciation of the actual presence of such rapid yet fast dying oscillation. However, Jones points out that although the heavy damping of this mode of motion insures its rapid subsidence in calm air, it imposes an effective restraint against movements of the airplane relative to the air, which results in violent movements of the airplane in gusts. This conclusion has been paid attention by Weiss when he started his theory of automatic control in an attempt to reduce the sharp response of the airplane to the gusts by introducing the automatic control.

3. The Conventional θ Control and the Controlled Longitudinal Motion

The most convenient way to control the disturbed longitudinal motion is to operate the elevator manually or automatically. The elevator, when operated by automatic control, follows definite law in accordance with a certain disturbance detector. Haus gives the following table of disturbance detectors and the quantity to which each is sensitive:
TABLE I

LONGITUDINAL DISTURBANCE DETECTORS AVAILABLE FOR AUTOMATIC CONTROLS

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Recording Quantity</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Airspeed Indicator</td>
<td>Relative Speed</td>
<td>U</td>
</tr>
<tr>
<td>b. Wind Vane</td>
<td>Incidence</td>
<td>α = - \frac{W}{U}</td>
</tr>
<tr>
<td>c. Free Gyro suspended at its center of gravity</td>
<td>Absolute Inclination</td>
<td>θ</td>
</tr>
<tr>
<td>d. Motor-driven Gyro with Precessional Moment</td>
<td>Angular Velocity</td>
<td>q = \dot{θ}</td>
</tr>
<tr>
<td>e. Accelerometer along X axis</td>
<td>Direction of Apparent Gravity</td>
<td>\frac{du}{dt} - g\sinθ</td>
</tr>
<tr>
<td>f. Accelerometer along Z axis</td>
<td>Magnitude of Apparent Gravity</td>
<td>\frac{dw}{dt} + g\cosθ</td>
</tr>
<tr>
<td>g. Lift Indicator</td>
<td>Magnitude of Lift</td>
<td>wU</td>
</tr>
<tr>
<td>h. Rate of Climb Meter</td>
<td>Vertical Airspeed</td>
<td>w or Usinθ</td>
</tr>
<tr>
<td>i. Torsional About Y Axis</td>
<td>Angular Acceleration</td>
<td>\ddot{θ}</td>
</tr>
</tbody>
</table>

Lin\textsuperscript{17} gives the generalized equation of motion due to several controls combined together, but no conclusions are drawn. Both Weiss and Lin eventually resigned their work within the scope of conventional θ control (type c. according to the above table) as manufactured by Sperry\textsuperscript{1} and Smith\textsuperscript{5}.

In order to discuss briefly the advantages and disadvantages of this type of control and henceforth lead to new control theory, it is necessary to start with the equation of motion of the system including the control.
The fundamental equation takes the following form:

\[ X = m(\dot{u} + W_q) \]  
\[ Z = m(w - U_q) \]  
\[ M' + M = B \dot{q} \]  
\[ -K_0 \dot{\theta} = m_c \ddot{\sigma} + F_c \dot{\sigma} + K_c \sigma \]

where \( M' \) = moment exerted by the elevator which is controlled by the \( \theta \) control

\( \sigma \) = control movement with respect to position when control is locked

\( m_c \) = equivalent mass of the control system

\( F_c \) = equivalent damping coefficient of the control system

\( K_c \) = equivalent spring constant of the control system

\( K_0 \) = equivalent exciting force coefficient for \( \theta \) control

If Eq. (1.30) is divided throughout by \( m_c \), we have

\[ -F_0 \theta = \ddot{\sigma} + 2 \beta \omega_{nc} \dot{\sigma} + \omega^2_{nc} \sigma \]

where \( -F_0 = -\frac{K_0}{m_c} \) = equivalent exciting coefficient per unit equivalent mass of the \( \theta \) control.

\[ \beta = \frac{F_c}{2 \sqrt{m_c K_c}} \] = damping ratio of the equivalent control system

\[ \omega_{nc} = \frac{K_c}{m_c} \] = undamped angular natural frequency of the equivalent control system.

The value of natural frequency and damping ratio of the entire control system can be obtained from free vibration experiment when the exciting force is kept zero. The detail of such technique is referred to General Principles of Instrumentation.

*The subscript \( c \) here is used to indicate the belonging of control. It does not mean critical as those used in Draper's paper.*
ment Analysis by Draper and Schliestett.

If Eqs. (1.27) and (1.28) are divided throughout by \( m \) and Eq. (1.29) divided by \( B \) and developed as we did Eqs. (1.01), (1.02) and (1.03), the equations will appear in the following form:

\[
\begin{align*}
\dot{u} + W_Q &= X_{uu} + X_{ww} + X_{qq} + g \cos \Theta \\
\dot{w} - U_Q &= Z_{uu} + Z_{ww} + Z_{qq} + g \sin \Theta \\
\dot{\theta} &= M_{uu} + M_{ww} + M_{qq} + M_\sigma \\
\dot{\sigma} &= -F_0 \theta - 2f_\omega \omega_{nc} - \omega_{nc} \sigma
\end{align*}
\]  

(1.35) \hspace{1cm} (1.36) \hspace{1cm} (1.37) \hspace{1cm} (1.38)

where \( M_\sigma = \frac{1}{B} \frac{dM}{d\sigma} \)  

(1.39)

Use \( D \) for \( \frac{d}{dt} \), rearrange the terms and neglect the insignificantly small terms, the above group of equations appear as follows:

\[
\begin{align*}
(D-X_u)u - X_{ww} - g \cos \Theta \theta + 0 &= 0 \\
-Z_uu + (D-Z_w)w - (D_0 + g \sin \Theta) \theta &= 0 \\
-M_{uu} - M_{ww} + (D^2 - D\theta) \theta - M_\sigma \sigma &= 0 \\
0 + 0 &= F_0 \theta + (D + 2f_\omega \omega_{nc} D + \omega_{nc}) \sigma = 0
\end{align*}
\]  

(1.40) \hspace{1cm} (1.41) \hspace{1cm} (1.42) \hspace{1cm} (1.43)

When the controlled airplane encounters a vertical gust of \( w_0 \) from equilibrium, the above equations are modified on their right sides.

\[
\begin{align*}
(D-X_u)u - X_{ww} - g \cos \Theta \theta + 0 &= -X_{ww} w_0 \\
-Z_uu + (D-Z_w)w - (D_0 + g \sin \Theta) \theta &= -Z_{ww} w_0 \\
-M_{uu} - M_{ww} + (D^2 - D\theta) \theta - M_\sigma \sigma &= -M_{ww} w_0 \\
0 + 0 + F_0 \theta + (D^2 + 2f_\omega \omega_{nc} D + \omega_{nc}) \sigma &= 0
\end{align*}
\]  

(1.44) \hspace{1cm} (1.45) \hspace{1cm} (1.46) \hspace{1cm} (1.47)

The solution of each of the variables \( u, w, \theta, \) and \( \sigma \) can be expressed in the following form:
\( u = \frac{\Delta u_c}{\Delta c} w_{ol} \)  \hspace{1cm} (1.48)

\( w = \frac{\Delta w_c}{\Delta c} w_{ol} \)  \hspace{1cm} (1.49)

\( \theta = \frac{\Delta \theta_c}{\Delta c} w_{ol} \)  \hspace{1cm} (1.50)

\( \sigma = \frac{\Delta \sigma_c}{\Delta c} w_{ol} \)  \hspace{1cm} (1.51)

where

\[
\Delta_c = \begin{bmatrix}
D-X_u & -X_w & -\cos \theta & 0 \\
-Z_u & D-Z_w & -DU_0 \sin \theta & 0 \\
-M_u & -M_w & D^2 - DM_q & -M_\sigma \\
0 & 0 & F_\theta & D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c
\end{bmatrix}
\]

or

\[
\Delta_c = (D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c) \Delta_c + F_\theta M_\sigma \begin{bmatrix}
D-X_u - X_w & -Z_u - D-Z_w \\
0 & 0 & F_\theta & D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c
\end{bmatrix}
\]

\( \Delta_u = \begin{bmatrix}
-D-X_u - X_w & -\cos \theta & 0 \\
-Z_u & D-Z_w & -DU_0 \sin \theta & 0 \\
-M_u & -M_w & D^2 - DM_q & -M_\sigma \\
0 & 0 & F_\theta & D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c
\end{bmatrix}
\]

or

\[
\Delta_u = (D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c) \Delta_u + F_\theta M_\sigma \begin{bmatrix}
D-X_u - X_w & -Z_u & -D(Z_u) \omega n D^2 + \omega^2 n c \\
0 & 0 & F_\theta & D^2 - DM_q
\end{bmatrix}
\]

\( \Delta_w = \Delta_c - D(D+ 2 \gamma c \omega n D^2 + \omega^2 n c) \begin{bmatrix}
D-X_u & -\cos \theta & 0 \\
-M_u & D^2 - DM_q & -M_\sigma \\
0 & 0 & 0 & D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c
\end{bmatrix}
\]

or

\[
\Delta_w = \Delta_c - D(D+ 2 \gamma c \omega n D^2 + \omega^2 n c) \begin{bmatrix}
D-X_u & -X_w & -X_w & 0 \\
-Z_u & D-Z_w & -Z_w & 0 \\
-M_u & -M_w & -M_w & -M_\sigma \\
0 & 0 & 0 & D^2 + 2 \gamma c \omega n D^2 + \omega^2 n c
\end{bmatrix}
\]
or \( \Delta'_{\theta c} = (D^2 + 2f_0 \omega_{nc} D + \omega_{nc}^2) \Delta'_{\theta} \) \( \text{(1.55)} \)

\[
\Delta'_{\sigma c} = \begin{vmatrix}
D - X_u & -X_w & -g \cos H & -X_w \\
-Z_u & D - Z_w & -D U_0 \cdot g \sin H & -Z_w \\
-M_u & -M_w & D^2 - D M_q & -M_w \\
0 & 0 & F_\theta & 0
\end{vmatrix}
\]

\( \Delta'_{\sigma c} = -F_\theta D \begin{vmatrix}
D - X_u & -X_w \\
-M_u & -M_w
\end{vmatrix} = -F_\theta \Delta'_{\theta} \) \( \text{(1.56)} \)

Equation (1.52) or (1.52)a gives sixth degree algebraic equation when developed and equated to zero.

Or \( \Delta'_{\theta c} = 0 \) will specify the stability of the disturbed motion.

Eqs. (1.48) to (1.51) can be solved, but the result will be more advantageous if it is converted into nondimensional form. \( ^9, ^{11} \)

In terms of its dimensions, \( \Delta'_{\sigma c} \) becomes

\[
\Delta'_{\sigma c} = \begin{vmatrix}
T^{-1} & T^{-1} & L^{-1} & T^{-1} & X \\
T^{-1} & T^{-1} & L^{-1} & T^{-1} & X \\
L^{-1} & L^{-1} & T^{-1} & L^{-1} & T^{-1} \\
X & X & L^{-1} & T^{-1} & T^{-1}
\end{vmatrix}
\]

The procedure of nondimensionalization can be performed as follows:

First step -- multiply the coefficients in the third column by unit length \( L \). \( \text{(1.57)} \)

Second step -- multiply the coefficients in the first column by unit time \( T \).

Third step -- multiply the coefficient in the second column by unit time \( T \).
Fourth step -- multiply the coefficient in the third column by \( \frac{T}{V} = L^{-1}T^{-2} \). (1.57)

Fifth step -- multiply the coefficient in the last column by \( T^2 \).

\[
\Delta_c = (L)(T)(T^2L^{-1})(T^2) \Delta_c'
\]

\[
\Delta_c = T^6 \Delta_c'
\]

(1.58)

where \( \Delta_c \) is defined as nondimensional stability determinant of the controlled longitudinal motion.

Likewise,

\[
\Delta_{uc} = T^6 \Delta_{uc}'
\]

(1.59)

\[
\Delta_{wc} = T^6 \Delta_{wc}'
\]

(1.60)

\[
\Delta_{\theta c} = LT^5 \Delta_{\theta c}'
\]

(1.61)

\[
\Delta_{\sigma c} = T^5 \Delta_{\sigma c}'
\]

(1.62)

where \( \Delta_{uc}, \Delta_{wc}, \Delta_{\theta c} \) and \( \Delta_{\sigma c} \) are defined as nondimensional quality determinants of forward speed, vertical speed, inclination and control movement of the controlled longitudinal motion.

Substitute Eqs. (1.58) to (1.62) into equations (1.48) to (1.51). The expressions of \( u, w, \theta \) and \( \sigma \) will be in terms of nondimensional determinants:

\[
u = \frac{\Delta_{uc}}{\Delta_c} w_{0l}
\]

(1.63)

\[
w = \frac{\Delta_{wc}}{\Delta_c} w_{0l}
\]

(1.64)

\[
\theta = \frac{\Delta_{\theta c}}{\Delta_c} \left[ \frac{T}{L} \right] w_{0l}
\]

(1.65)

\[
\sigma = \frac{\Delta_{\sigma c}}{\Delta_c} \left[ T \right] w_{0l}
\]

(1.66)
or \[
\frac{u}{w_0} = \frac{\Delta u c}{\Delta c} \quad (1.63)a
\]
\[
\frac{\dot{u}}{w_0} = \frac{\Delta w c}{\Delta c} \quad (1.64)a
\]
\[
\frac{\theta}{w_0} = \left[ \frac{L}{T} \right] \frac{\Delta \theta c}{\Delta c} \quad (1.65)a
\]
\[
\frac{\sigma}{w_0} = \left[ T \right] \frac{\Delta \sigma c}{\Delta c} \quad (1.66)a
\]

Eqs. (1.63)a, (1.64)a, (1.65)a and (1.66)a are defined as
unit response of forward speed, vertical speed, inclination
and control movement of the airplane to vertical gusts of the
shape of the step function.

**Unit Time T, Unit Length L and Compact Ratio \( \mu \)**

In order to change the dimensional coefficients such as
\( X_w, Z_u, \) etc., into nondimensional coefficients \( x_w, z_u, \) etc.,
expressible in terms of those established fundamental aero-
dynamic coefficients, the length of the tail moment arm (from
tail post to center of gravity of the airplane) is taken as
the unit length (or characteristic length as defined else-
where\(^{20} \)). The unit time of the nondimensional system is
defined by the following equation:
\[
T = \frac{m}{\rho SU} \quad (1.67)
\]

On this basis, the unit velocity of the nondimensional
system should be
\[
V = \frac{L}{T} = \frac{L}{m} \quad \frac{\rho}{\rho SU} \quad \text{or} \quad \frac{U}{m} \quad \frac{\rho}{\rho SL} \quad (1.68)
\]
where \( \frac{m}{\rho \frac{x}{2} SL} \) receives the symbol \( \mu \)

or \( \mu = \frac{m}{\rho \frac{x}{2} SL} \) \( (1.69) \)

which is defined as compact ratio (or relative density as defined elsewhere\(^{20}\)).

The general procedure to reduce the dimensional coefficients to nondimensional one is omitted here (as it can be found elsewhere\(^{11}\)) with the exception of those which have relations with the control. However a complete table is given on the next page for all the nondimensional coefficients. These coefficients correspond to those dimensional ones in Eq. (1.52) for the convenience of application.
# Table II

Nondimensional Coefficients for Longitudinal Stability of the Airplane

<table>
<thead>
<tr>
<th>$x_u$</th>
<th>$2C_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_u$</td>
<td>$2C_L$</td>
</tr>
<tr>
<td>$m_u$</td>
<td>0</td>
</tr>
</tbody>
</table>

$$2C_D$$

$$2C_L$$

$$\left[ -C_{m_w} \frac{c}{l^*} \frac{Q}{1+Q(r-1)} \frac{dQ}{du} \right] \frac{1}{b}$$

$$+ 0.62C_{Dl} \left[ \frac{C_l}{C_{L*}} \frac{x-h}{l^*} \right] \frac{1}{b}$$

<table>
<thead>
<tr>
<th>$x_w$</th>
<th>$\frac{dC_x}{d\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_w$</td>
<td>$\frac{dC_L}{d\alpha}$</td>
</tr>
<tr>
<td>$m_w$</td>
<td>$\frac{1}{b} \frac{dC_m}{d\alpha}$</td>
</tr>
</tbody>
</table>

$$\frac{dC_X}{d\alpha}$$

$$\frac{dC_L}{d\alpha}$$

$$\frac{1}{b} \frac{dC_m}{d\alpha}$$

| $x_\theta$ | $-\frac{\mu}{\theta_C}$ |
| $z_\theta$ | $-\frac{\mu}{\theta_C \theta_0}$ |
| $m_q$ | $\frac{5}{4} e \left( \frac{dC_L}{d\alpha} \right) S' \frac{1}{S} \frac{1}{b}$ |

$$\frac{5}{4} e \left( \frac{dC_L}{d\alpha} \right) S' \left[ 1+Q(r-1) \right] \frac{1}{b}$$

| $f_\theta$ | $\frac{1}{\theta_{C\alpha}} \frac{\mu}{\alpha}$ |
| $f_\sigma$ | $\frac{1}{\sigma_{C\alpha}} \frac{\mu}{\alpha}$ |
| $f_{\omega_{nnc}}$ | $\omega_{nnc} = \frac{1}{\sigma_{C\alpha}} \frac{C_L}{L}$ |

| $f_{\sigma_{\omega_{nnc}}}$ | $2f_c \omega_{nnc} = 2f_c \frac{1}{\sigma_{C\alpha}} \frac{\mu C_L}{L}$ |
| $f_{\sigma_{\omega_{nnc}}}$ | $2f_c \omega_{nnc} = 2f_c \frac{1}{\sigma_{C\alpha}} \frac{\mu C_L}{L}$ |

| $m_{\sigma}$ | $\frac{\mu}{3} e \left( \frac{3C_L}{\alpha} \right) \frac{S'}{S} \frac{\alpha}{S} \frac{\beta}{b} \frac{L}{b} \left[ 1+Q(r-1) \right] e$ |

* Lower case L. Throughout this thesis lower case L will be so designated by the addition of an asterisk (L*) to differentiate from the figure one (l).
Control Coefficients $F_\theta, \omega_{nc}, M_\sigma$ and Their Corresponding

Nondimensional Coefficients $f_\theta, \omega_{nnc}$ and $m_\sigma$

(a) \[ F_\theta = \frac{1}{m_c} \frac{\partial F}{\partial \theta} = \frac{K_\theta}{m_c} \quad (1.70) \]

\[ f_\theta = \frac{K_\theta}{m_c} X L^{-1} T^2 = \frac{K_\theta}{W_c} g X L^{-1} T^2 \quad (1.71) \]

where $W_c$ is the weight equivalent of the control system.

Substitute Eq. (1.67) into Eq. (1.71) and assume level flight so that

\[ W = C_L \rho U^2 S \quad (1.80) \]

then

\[ f_\theta = \frac{1}{\theta_{W_c}} \mu C_L \quad (1.81) \]

and \[ \theta_{W_c} = \frac{W_c}{K_\theta} \quad (1.82) \]

where $\theta_{W_c}$ is the magnitude of disturbed inclination of the airplane with which the force exerted on the control shall equal the equivalent weight of the control system.

(b) \[ \omega_{nc} = \frac{K_c}{m_c} = \frac{K_c}{W_c} g = \frac{g}{\sigma_W} \quad (1.83) \]

\[ \sigma_W = \frac{W_c}{K_c} \quad (1.84) \]

where $\sigma_W$ is defined as the static deflection of the control system due to its equivalent weight. Since

\[ \omega_{nnc} = \omega_{nc} T^2 = \frac{1}{\sigma_{W_c}} g T^2 \quad (1.85) \]

substitute Eq. (1.67) into Eq. (1.85) and assume the level flight condition. We then have

\[ \omega_{nnc} = \frac{1}{\sigma_W} \frac{\mu C_L}{L} = f_\sigma \quad (1.86) \]

Likewise

\[ 2\frac{1}{5} c \omega_{nnc} = 2\frac{1}{5} \sqrt{\frac{1}{\sigma_{W_c}} \frac{\mu C_L}{L}} = f_\sigma \quad (1.87) \]
(c) \( \frac{\Delta M'}{\Delta \sigma} = \frac{1}{B} \frac{\partial M'}{\partial \sigma} \) \hspace{1cm} (1.88)

\[ M' = C_L \frac{S'}{2} U'^2 L \] \hspace{1cm} (1.89)

where \( M' \) = tail moment

\( C_L' \) = tail lift coefficient

\( S' \) = tail area

\( U' \) = air stream velocity at tail plane which equals \( U \) if slip stream effect is neglected.

\( e \) = tail plane efficiency

\( \frac{\partial M'}{\partial \sigma} = e \left( \frac{\partial C_L}{\partial \alpha} \right) \left( \frac{\partial \alpha}{\partial \beta} \right) \left( \frac{\partial \beta}{\partial \sigma} \right) \left( \frac{S'}{S} \right) U'^2 L \) \hspace{1cm} (1.90)

where \( \left( \frac{\partial C_L}{\partial \alpha} \right)' \) = static lift coefficient slope

\( \beta \) = elevator deflection from balanced position

\( \sigma \) = control movement

\[ M_\sigma = \frac{1}{b m L^2} e \left( \frac{\partial C_L}{\partial \alpha} \right) \left( \frac{\partial \alpha}{\partial \beta} \right) \left( \frac{\partial \beta}{\partial \sigma} \right) \left( \frac{S'}{S} \right) U'^2 L \] \hspace{1cm} (1.91)

where \( b \) = distribution factor of longitudinal moment of inertia.

Now, \( m_\sigma = \frac{1}{B} \frac{\partial M'}{\partial \sigma} L T^2 \) \hspace{1cm} (1.92)

Substitute equations (1.90) and (1.67) into equation (1.92) and simplify the expression,

\[ m_\sigma = \mu e \left( \frac{\partial C_L}{\partial \alpha} \right) \left( \frac{S'}{S} \right) \left( \frac{\partial \alpha}{\partial \beta} \right) \left( \frac{\partial \beta}{\partial \sigma} \right) \left( \frac{L}{b} \right) \] \hspace{1cm} (1.93)

where \( \frac{\partial \alpha}{\partial \beta} \) depends upon the design of tail plane and \( \frac{\partial \beta}{\partial \sigma} \) depends upon the design of coupling of the control movement to the elevator deflection.
Stability Determinant in Nondimensional Form

Now Eq. (1.52) can be changed into nondimensional form:

\[
\Delta c = \begin{vmatrix}
-d-x_u & -x_w & -\mu C_L & 0 \\
-z_u & d-x_w & -d & -\mu C_L d \\
-m_u & -m_w & d^2 & -d \delta q \\
0 & 0 & f_\Theta & d^2 + df_\Theta + f_\Theta
\end{vmatrix}
\]

(1.94)

where \( f_\Theta \) may be written as \( \omega_{nnc} \) and \( f_\Theta^* \) as \( 2Tc \omega_{nnc} \).

Likewise all the nondimensional quality determinants shall retain the same form as the dimensional ones \((1.53)a, (1.54)a, (1.55)a\) and \((1.56)a\) with the capital letters replaced by the small letters; also \( \omega_{nc} \) replaced by \( \omega_{nnc} \).

Equation (1.94) can be partially developed into the following form:

\[
\Delta c = (d^2 + 2Tc \omega_{nnc} d + \omega_{nnc}^2) \Delta_0 + f_{\Theta \Theta} \Delta_0 \begin{vmatrix}
-d-x_u & -x_w \\
-z_u & d-z_w
\end{vmatrix}
\]

(1.94)a

It is very advantageous in one way to have the minor

\[
\begin{vmatrix}
-d-x_u & -x_w \\
-z_u & d-z_w
\end{vmatrix}
\]

especially when the control approaches ideal condition, because it raises the coefficient of the first degree operator of the developed stability equation with great predominance.

However, it is disadvantageous in the other way to have the same minor even when the control is ideal because it limits our freedom to adjust the coefficients of the stability equation in the most desirable way. (A detailed explanation will be given in the next chapter.)
For a nonideal control, the stability equation becomes a sixth degree equation from which three quadratic factors can be abstracted. The disturbed motion therefore comprises three oscillatory components (or the equivalent subsidence) among which one is due to the additional degree of freedom of the control, but it may be entirely different from the isolated control response. There is no literature available to draw conclusions as to the exact effect of the control characteristic upon the controlled motion and the reaction of the controlled motion to the control characteristic. The result of introducing a control is considered successful as to easing the motion in pitch, but not at all as regards the vertical motion. Motion of pitch is eased because the control seeks to equalize the damping of the slow and fast oscillations of the uncontrolled motion so that both of them may disappear much sooner after being disturbed. For a detailed discussion the reader is referred to Lin's work.17

Weiss16 carefully examines the vertical motion (due to vertical gust) assuming a full restraint in pitch (possible if the control is very fast and powerful). He points out that the sharp response (quick following-up characteristic) is almost entirely contributed by the coefficient $z_w$ which is the slope of the lift coefficient curve and depends upon the aspect ratio of the wing. For the aerodynamic efficiency, larger aspect ratio is required, but such an airplane will give sharp response to vertical gust; i.e., the airplane will experience a large vertical acceleration dur-
ing the gust-picking-up period. The vertical acceleration is painful to unaccustomed occupants. Any means of reducing the sharp response to vertical gust is therefore worth while investigating. However, the airplane's efficiency must not be violated.

Suggestions in the aim of reducing the sharpness of vertical response will be given and discussed in Part V of this thesis.
4. The Parasite Minor of the Conventional e Control

Going back to equation (1.94)a, let the control be very fast as approaching ideal condition; then:

$$\Delta_{ci} \equiv \omega_{nnc}^2 \left\{ \Delta_0 + \frac{f_0 m_e}{\omega_{nnc}^2} \begin{vmatrix} d-x_u & -x_w \\ -3_u & d-3_w \end{vmatrix} \right\}$$  \hspace{1cm} (1.95)

where $\Delta_{ci}$ is the non-dimensional stability determinant of the controlled motion with an ideal control.

Develop $\Delta_0$, the uncontrolled non-dimensional stability determinant, into the quartic form as:

$$\Delta_0 = b_4 d^4 + b_3 d^3 + b_2 d^2 + b_1 d + b_0$$ \hspace{1cm} (1.96)

where

- $b_4 = 1$
- $b_3 = -(x_u + 3w + m_e)$
- $b_2 = (x_u 3w - x_w 3_u) + m_e (x_u - 3w) - \mu m_w$
- $b_1 = -m_e (x_u 3w - x_w 3_u) + \mu m_w (x_u - c_L)$
- $b_0 = -c_L \mu m_w (3_u - x_u) - c_L \mu m_w (x_w - 3_w)$ \hspace{1cm} (1.96)a

Let

$$\begin{vmatrix} d-x_u & -x_w \\ -3_u & d-3_w \end{vmatrix} = c_2 d^2 + c_1 d + c_0$$ \hspace{1cm} (1.97)

where

- $c_2 = 1$
- $c_1 = -(x_u + 3w)$
- $c_0 = x_u 3w - x_w 3_u$ \hspace{1cm} (1.98)
It is seen that the coefficients $b_0$, $b_1$ and $b_2$ are affected by $c_1$, $c_2$ and $c_0$ respectively with definite relative magnitudes. The common multiplier or coupling factor $\frac{f_0 m_c}{\omega n c^2}$ cannot alter these relative magnitudes at all. From the standpoint of controlling, we have surrended our liberty of adjusting the uncontrolled coefficients $b_4$, $b_3$, $b_2$, $b_1$ and $b_0$ by the application of control to this determinant:

$$
\begin{vmatrix}
-d-x_u & -x_w \\
-\gamma_u & d-\gamma_w
\end{vmatrix}
$$

which is entirely fixed by the design of the airplane and by the flying attitude. It is therefore justifiably defined as the parasite determinant of the conventional $e$ control, and shall bear the notation $\Delta p_0$.

$$
\Delta p_0 = \begin{vmatrix}
-d-x_u & -x_w \\
-\gamma_u & d-\gamma_w
\end{vmatrix}
$$

If the control is non-ideal, the parasite determinant still holds its characteristic as to affect the coefficients of the stability equation with a definite relative magnitude.

5. Theory of De parasitization

From the discussion of the previous chapter, it is seen that:

(a) The uncontrolled disturbed motion is comprised of a heavily damped fast oscillation and a lightly damped slow oscillation.

(b) The purpose of control (so far as $e$ control is concerned) is to equalize the damping of these two components and to ease the pitching oscillation in magnitude.

With ideal control, it is possible to achieve the purpose to
a certain extent without introducing the complication of additional oscillatory component. But when the control is non-ideal, the presence of additional component of motion in pitch is inevitable.

Equation (1.65)a can be developed completely into operational form:

\[
\frac{\theta}{W_0} = \frac{1}{\omega_c^4} \left( \frac{(d^2 + 2y_c \omega_{nnc} d + \omega_{nnc}^2) d (-d m_w + x_w m_w - m_w x_w)}{(d^2 + 2y_c \omega_{nnc} d + \omega_{nnc}^2) \Delta_0 + f_0 m_\sigma \Delta p_\sigma} \right) (1.100)
\]

As the denominator is a sixth degree equation, three oscillatory components of motion should be expected.

There is no special advantage to have this complication. In fact, we have plenty to do with those two original components. By proper adjustment of coefficient, optimum distribution of damping between the two components is obtainable. The addition of third component is really unnecessary.

Fortunately, the parasite determinant itself represents a one-degree-of-freedom system. Its undamped natural frequency lies in between those of the slow and the fast oscillation and is nearer to the slow one. The damping ratio of the parasite determinant is in the vicinity of two for the average airplane. Now, if we allow the following condition:

\[
\Delta p_\sigma = (d^2 + 2y_c \omega_{nnc} d + \omega_{nnc}^2) p = d^2 + 2y_p \omega_{nnp} d + \omega_{nnp}^2 (1.101)
\]

that is, let the equivalent control system be designed according to the characteristic of parasite determinant, such non-ideal control will not introduce third oscillatory component as others. In a
strict sense, the third component (pitch) has a magnitude of zero, as can be seen from the following equation:

\[
\frac{\Theta}{W_0} = \frac{1}{[\frac{1}{T}]} \left( \frac{\left( d^2 + 2 \gamma_c \omega_{n_m} d + \omega_{n_m}^2 \right) d(-d w + x w m_w - m_x x_w)}{(d^2 + 2 \gamma_c \omega_{n_m} d + \omega_{n_m}^2) (\Delta_0 + f_\Theta m_\sigma)} \right)
\]

or

\[
\frac{\Theta}{W_0} = \frac{1}{[\frac{1}{T}]} \left\{ \frac{\left( d^2 + 2 \gamma_c \omega_{n_m} d + \omega_{n_m}^2 \right) 1 + \frac{d(-d w + x w m_w - m_x x_w)}{\Delta_0 + f_\Theta m_\sigma}}{1} \right\} \quad \text{(1.102)}
\]

Therefore, when the control is designed with the identical dynamic characteristic of the parasite determinant, it is defined as **deparasitized non-ideal control**.

It is easy to get confused by equation (1.102) where the coupling factor \( f_\Theta m_\sigma \) can only affect the constant term of \( \Delta_0 \). There is no particular advantage gained by such control. However, equation (1.102) is only responsible to a \( \varepsilon \) control; that is, the control movement is only excited by force which is proportional \( \varepsilon \) away from the equilibrium value. If mechanical complication is allowed so that the control system is simultaneously excited by forces which are proportional to pitching velocity \( \dot{\Theta} \), pitching acceleration \( \ddot{\Theta} \), etc., the denominator of the second fraction of equation (1.102) will have the following form:

\[
\Delta_0 + \left( d^2 f_\Theta + \ldots + d^2 f_\Theta + d f_\Theta + f_\Theta \right) m_\sigma \quad \text{(1.103)}
\]

where \( f_\Theta, f_\Theta, f_\Theta \) and \( f_\Theta \) are entirely independent constants, the choice of which is entirely up to the control designer.

In practice, one or two exciting forces are needed to obtain desirable results, so the mechanical complication is not as bad as one would imagine from expression (1.103).
It should be noted, although the magnitude of the third component or the control component is zero in pitch, it is not so for forward speed nor for vertical speed. Equation (1.64)a can be fully developed as follows:

\[
\frac{w}{w_0} = \frac{(d^2 + 2\bar{f}_p \omega_{\text{mp}}d + \omega_{\text{mp}}^2)(\Delta_0 + f_0 \omega_0) - f_0 \omega_0 (d - x_0)}{(d^2 + 2\bar{f}_p \omega_{\text{mp}}d + \omega_{\text{mp}}^2)(\Delta_0 + f_0 \omega_0)}
\]

or

\[
\frac{w}{w_0} = \frac{h_5 d^5 + h_4 d^4 + h_3 d^3 + h_2 d^2 + h_1 d + b_0 \omega_{\text{mp}}^2 + \omega_{\text{mp}}^2 f_0 \omega_0 - f_0 \omega_0 (d - x_0)}{(d^2 + 2\bar{f}_p \omega_{\text{mp}}d + \omega_{\text{mp}}^2)(\Delta_0 + f_0 \omega_0)}
\]

The above equation gives the evidence of the presence of the control component in vertical speed. It may be seen more clearly if the reader is referred to Part V of this thesis.

Objection might be raised from the standpoint of fast control. Control lag is indeed troublesome when slow control is used as the deparasitized control, but it can be overcome by using higher derivative force or moment to excite the control. The physical significance of this overcoming property is evidently due to that higher derivative excited control controls earlier than the deviation or error; it controls the tendency of being disturbed.

The much overdamping characteristic of the deparasitized control is doubtful in its advantage. But due to the slow natural frequency of the control, the absolute damping force is not as tremendous as one might think it would be. However, even if it needs more energy to operate this type of control, it pays to do so if the controlled motion is in the most desired mode.
As far as pitching motion is concerned, the deparasitized non-ideal control may be considered as an ideal control. For one-degree-of-freedom system, the application of ideal controls 6, 12, 21 of the first class does not increase the degree of freedom. The (e) deparasitized non-ideal control holds the same principle as far as the pitching motion is concerned. The e control is primarily designed for pitching motion. It is for this reason that the writer feels the promise of this type of control.

It should be noted that the theory of deparasitization can be applied to controls other than the e-elevator coupled type.

6. The Stability Determinant of the e Deparasitized Non-Ideal Control

From equations (1.102) and (1.104) and expression (1.103) the stability determinant of the deparasitized control can be factored into one quadratic factor, which is actually the parasite minor, and one quartic factor. Let $\Delta_{cpe}$ represent the stability determinant of the deparasitized controlled motion, then:

$$
\Delta_{cpe} = \Delta_{pe} \left[ \Delta_{o} + m_{o} (d^{n}f_{e}^{\prime} + \cdots + d^{2}f_{e}^{\prime} + df_{e} + f_{e}) \right] \tag{1.105}
$$

Assume the highest derivative exciting force is $f_{e}',$ then we have:

$$
\Delta_{cpe} = (d^{2} + z_{p} \omega_{n_{p}} d + \omega_{n_{p}}^{2}) \left[ d^{2} + b_{1}d^{2} + (b_{2} + m_{o}f_{e})d + (b_{1} + m_{o}f_{e}) \right] + b_{0} + m_{o}f_{e} \tag{1.105a}
$$

The quadratic factor depends upon the design of the airplane and flying attitude. In general, it gives two (real) negative roots for $\Delta$ when equated to zero, which indicates the presence of one subsiding pair in motion; in other words, an overdamped stable component. The quartic factor will give additional criteria of the stability of the motion.
In order to handle the distribution of damping of the slow and fast oscillation in an optimum way, thorough knowledge about the quartic equation is necessary. Part II of this thesis will be devoted to this purpose.

It should be noted when the control is isolated or the elevator is locked, $m_0$ is zero, the quartic factor returns to the uncontrolled form. The presence of the quadratic factor might lead to some misunderstanding. In fact, it should not be present. With close examination on equation (1.104) we may see that the expression can be reduced to the following uncontrolled form:

$$\frac{W}{W_0} = \frac{\Delta_0 - d [d^2 (x_u + m_0) - x_u m_0 - \mu m u c_L]}{\Delta_0}$$  (1.106)

when $m_0 = 0$; thus the stability of the disturbed motion is only determined by $\Delta_0$, the uncontrolled stability determinant.
PART II

PROPERTIES OF THE QUARTIC EQUATION
CHAPTER THREE

THE IMPORTANCE OF THE QUARTIC EQUATION TO THE GENERALIZED AUTOMATIC CONTROL PROBLEM

7. Self-excited Vibrations

Den Hartog, in treating the "hunting of steam-engine governors", points out that when the engine is rigidly coupled to an electric generator feeding a large network, the presence of "engine spring" causing the stability equation of the system goes up to a quartic one. Many other problems such as "Axial Oscillation of Turbines Caused by Steam Leakage", "Airplane Wing Flutter", etc., involve quartic equations.

To realize the importance of the quartic equation in the so-called "self-excited" problem, the steam engine governor system is quoted here with a few changes of notation adapted to the text of this thesis.

Let $I$ = moment of inertia of the rotor

$\varphi$ = angular displacement of the rotor from the equilibrium position

$c_e$ = coefficient of damping torque resulting from the damper winding

$k_e$ = coefficient of restoring torque or the magnetic spring constant in the air gap of the generator.

$m$ = equivalent mass of the governor

$x$ = displacement of the governor

$\varsigma_g$ = damping coefficient of the governor system
\( k_g = \) spring constant of the governor system

\( C_1 = \) coefficient of velocity exciting force on the governor

\( C_2 = \) coefficient of displacement controlling torque

Then the two simultaneous differential equations of the problem are:

\[
(mD^2 + C_g D + k_g)x = C_1 \frac{d\phi}{dt}
\]

\[
(ID^2 + C_e D + k_e)\phi = -C_2 x
\]

where \( D = \frac{d}{dt} \)

With some algebraic manipulation, the stability equation is established as follows:

\[
D^4 + D^3 \left( \frac{Ce}{I} + \frac{C_e}{m} \right) + D^2 \left( \frac{ke}{I} + \frac{k_g}{I} + \frac{Ce \cdot C_g}{Im} \right) + D \left( \frac{C_g \cdot k_e}{m} + \frac{C_e \cdot k_g}{m} + \frac{C_1 \cdot C_2}{Im} \right) + \frac{ke \cdot k_g}{Im} = 0
\]

in which all coefficients are seen to be positive. The criterion of stability of Eq. (2.04) by the application of the Routh's discriminant becomes

\[
(\frac{Ce}{I} + \frac{C_g}{m}) (\frac{ke}{I} + \frac{k_g}{m} + \frac{Ce \cdot C_g}{Im}) (\frac{C_g \cdot k_e}{m} + \frac{C_e \cdot k_g}{m} + \frac{C_1 \cdot C_2}{Im})
\]

\[
> (\frac{C_g}{m} \frac{ke}{I} + \frac{C_e}{m} \frac{k_g}{m} + \frac{C_1 \cdot C_2}{Im})^2 + \frac{ke \cdot k_g (\frac{Ce}{I} + \frac{C_g}{m})^2}{Im}
\]

A generalized conclusion cannot be drawn from Eq. (2.05). Den Hartog emphasizes its physical meaning only by assuming some special cases. However, in general, quantitative, not qualitative, criteria will be more desirable.

8. Constant Azimuth or Displacement Follow-up Control System

Ships as well as airplanes usually do not possess sensitivity of direction. Constant steering by means of a rudder is necessary to keep them in course or in constant azimuth. Minorsky treats automatically steered bodies (using a ship
as the primary subject) with ideal controls which are classified into three groups and considers the control lag due to mass and inertia of the transmission mechanism by introducing constant time lag.

In fact, control lag due to mass and damping cannot be exactly replaced by a constant time lag, as Lin\textsuperscript{24} has verified the invalidity of this replacement when the coupling factor becomes large.

Displacement follow-up control has been in practice for some years. Hazen\textsuperscript{21} defines this type of control as servomechanism, and treats them only with ideal controls. The automatic direction finder, manufactured by different companies, used in airplanes and the acoustic detector used in anti-aircraft artilleries belong to this type of control.

Now, let us take the automatic direction finder as an example leading to the important quartic equation. In its usual construction\textsuperscript{25} the enclosed antenna loop is geared to a motor which supplies the following-up torque controlled by the error and error derivative signals from the antenna through a nonideal control system. The equation of motion for the rotor can be written as:

\[ I\ddot{\phi}_d + c_e\dot{\phi}_d + k_e\phi_d = -c\sigma \quad (2.06) \]

and the equation of motion for the control can be written as

\[ m_c\ddot{\sigma} + c_c\dot{\sigma} + k_c\sigma = C_2\dot{E} + C_1\dot{E} + C_0E \quad (2.07) \]

where \( I = \) equivalent moment of inertia of the rotating part.

\( \phi_d = \) displacement of the rotor from zero position.

\( c_e = \) equivalent damping coefficient of the rotating part.
\[ k_e = \text{spring constant of the rotating part} \]
\[ m_c = \text{equivalent mass of the control} \]
\[ c_e = \text{equivalent damping coefficient of the control} \]
\[ C = \text{coefficient of displacement controlling torque} \]
\[ E = \text{error between the driving and driven angular displacement}. \]

where \( E = \varphi_d - \varphi_1 \) \( (2.08) \)

\( \varphi_1 = \text{the angle to be followed} \)

\( C_0 = \text{coefficient of displacement error exciting force} \)

\( C_1 = \text{coefficient of first error derivative exciting force} \)

\( C_2 = \text{coefficient of second error derivative exciting force} \)

Substitute \( D = \frac{d}{dt} \) for the overhead dot; also, substitute equation (2.08) into (2.07); then,

\[ (ID^2 + C_0 D + k_e) \varphi_d = -C_0 \sigma \quad (2.09) \]

\[ (m_c D^2 + c_c D + k_c) \sigma = (C_2 D^2 + C_1 D + C_0) \varphi_d - (C_2 D^2 + C_1 D + C_0) \varphi_1 \quad (2.10) \]

By canceling \( \sigma \) between equations (2.09) and (2.10), \( \varphi_d \) can be solved in terms of \( \varphi_1 \).

\[ \varphi_d = \frac{C(C_2 D^2 + C_1 D + C_0)}{(ID^2 + C_0 D + k_e)(m_c D^2 + c_c D + k_c) + C(C_2 D^2 + C_1 D + C_0)} \varphi_1 \quad (2.11) \]

The solution can be expressed in terms of \( E \) and \( \varphi_1 \) where

\[ E = \frac{(ID^2 + c_c D + k_c)(m_c D^2 + c_c D + k_c)}{(ID^2 + c_c D + k_c)(m_c D^2 + c_c D + k_c) + C(C_2 D^2 + C_1 D + C_0)} \varphi_1 \quad (2.12) \]

In practice \( \varphi_1 \) may be any function of time, but it is fair enough to assume a step function for the automatic direction finder when the loop is suddenly called into operation.

Eq. (2.11) can also be written in the developed form:

\[ \varphi_d = \frac{B_2 D^2 + B_1 D + B_0}{A_4 D^4 + A_3 D^3 + A_2 D^2 + A_1 D + A_0} \varphi_1 \quad (2.13) \]
Where $A_{14} = \text{Im}_c$

\[ A_3 = Ic_m + \text{Im}_c c_c \]

\[ A_2 = Ik_c + m_k c_e + c_e c_c + B_2 \]

\[ A_1 = k_e c_c + k_c c_e + B_1 \]

\[ A_0 = k_e k_c + B_0 \]

\[ B_2 = cC_2 \]

\[ B_1 = cC_1 \]

\[ B_0 = cC_0 \]

(2.13)a

It can be seen from Eqs. (2.13) and (2.13)a that the steady state value of $\phi_d$ cannot be equal to $\phi_1$ unless either $k_e$ or $k_c$ is zero or at least the product of $k_e k_c$ is very very small compared to $B_0$. In most follow-up systems, the driven part usually possesses no stiffness; that is, with zero $k_c$ so that the steady state reading is accurate or the following-up characteristic is perfect as far as the steady state is concerned.

For mathematical analysis, we may allow a very weak stiffness in the member to be controlled, which will simplify the analysis considerably.

Again the denominator of Eq. (2.13) is defined as the stability function of the controlled system, the numerator the quality function of $\phi_d$. Divide both denominator and numerator of Eq. (2.13) by $\text{Im}_c$

\[ \phi_d = \frac{B_2 D^2 + B_1 D + B_0}{D^4 + A_3 D^3 + A_2 D^2 + A_1 D + A_0} \phi_1 \]  

(2.13)b

where $A_3 = \frac{c_c}{m_c} + \frac{c_e}{I}$
\[ A'_2 = \frac{k_c}{m_c} + \frac{k_e}{I} + \frac{c_c}{m_c} + \frac{c_e}{I} + B'_2 \]
\[ A'_1 = \frac{c_c}{m_c} \frac{k_e}{I} + \frac{c_e}{m_c} \frac{k_c}{I} + B'_1 \]
\[ A'_0 = \frac{k_e}{I} \frac{k_c}{m} + B'_0 \]
\[ B'_1 = \frac{c_c}{m} \frac{k_e}{I} \]
\[ B'_2 = \frac{c_c}{m} \frac{k_e}{I} \]
\[ B'_0 = \frac{c_c}{m} \frac{k_e}{I} \]

where \( B'_0 \) is defined as coupling factor of error-sensitive coupling,

\( B'_1 \) is defined as coupling factor or error-velocity coupling,

\( B'_2 \) is defined as coupling factor of error-acceleration coupling.

It can be seen that the stability function of Eq. (2.13)b (with the substitution of Eq. (2.13)c) takes the same form as Eq. (2.04), the stability equation of heavily laden generator controlled by ordinary flyball governor.

Now let \[ \frac{c_c}{2k_c m_c} = \frac{c_e}{2k_e I} = \frac{26}{c_c} = \text{damping ratio of control} \]
\[ \frac{c_c}{k_c m_c} = \omega_{nc} = \text{undamped angular natural frequency of the control} \]
\[ \frac{c_e}{k_e I} = \omega_{no} = \text{undamped angular natural frequency of the member to be controlled} \]

Then, \[ A'_2 = 2\frac{c_c}{c_c} \omega_{nc} + 2\frac{c_e}{c_e} \omega_{no} \]
\[ A'_2 = \omega_{nc} + \omega_{no} + \frac{4}{c_c} \frac{c_e}{c_e} \omega_{no} \omega_{nc} + B'_2 \]
\[ A'_1 = 2\xi c\omega_{nc}\omega_{no}^2 + 2\xi c\omega_{no}\omega_{nc} + B'_1 \]

\[ A'_0 = \omega_{nc}^2 + B'_0 \]  

From the above examples, the importance of quartic equations has been established. It is believed that the better controlled result can only be obtained with a thorough knowledge of the quartic equation.
CHAPTER FOUR

STABILITY TRANSITION CURVE OF QUARTIC EQUATION WITH STANDARDIZED NONDIMENSIONAL COEFFICIENTS

9. Nondimensionalization of Stability Function

The stability function of nonideally controlled motion has been shown to be in quartic form in many cases. The general form of such function is

\[ S(D) = D^4 + A_1 D^3 + A_2 D^2 + A_3 D + A_4 = 0 \]  \hspace{1cm} (2.14)

where \( A_1, A_2, A_3 \) and \( A_4 \) are physical constants with dimensions of \( T^0, T^1, T^2 \) and \( T^3 \). (For longitudinal stability function of the airplane, these coefficients are nondimensional as in Eq. (1.105).

The first requirement for stability is that all coefficients must have the same sign.

Eq. (2.14) can be nondimensionalized by introducing a nondimensional operator \( \lambda \) which is equal to \( \frac{D}{A_0^{\frac{1}{4}}} \)
or \( D = A_0^{\frac{1}{4}} \lambda \) \hspace{1cm} (2.15)

Substitute Eq. (2.15) into Eq. (2.14)

\[ S(D) = A_0^{\frac{1}{4}} (\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1) = 0 \]  \hspace{1cm} (2.16)

where 
\[ \alpha_3 = \frac{A_3^{\frac{1}{4}}}{A_0^{\frac{1}{4}}} , \quad \alpha_2 = \frac{A_2^{\frac{1}{2}}}{A_0^{\frac{1}{4}}} , \quad \alpha_1 = \frac{A_1^{\frac{3}{4}}}{A_0^{\frac{1}{4}}} \] \hspace{1cm} (2.17)

or 
\[ \psi(\lambda) = \frac{S(D)}{A_0^{\frac{1}{4}}} = (\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1) = 0 \] \hspace{1cm} (2.18)

By the process of nondimensionalization the stability of the equation is not affected, which will be proven a little
later. The sole effect is that every root of the old equation is equivalent to $A_i^{1/4}$ times the corresponding one of the new equation which is nondimensional.

10. Factorization of the Nondimensional Quartic Equation

Physically, a differential equation of fourth order with constant coefficients represents a motion or its equivalent, such as current, which consists of two vibratory components. The components may be really vibratory, divergent or convergent, or subsiding, but they are in general of different natural frequency and of different damping.

When the operator is used for the differential, the root of the equation indicates the frequencies and dampings. The same is true for nondimensionalized equations.

The factorization of the fourth degree equation or the root-finding process of the same is not an easy task, especially when it should give two pair of complex roots. Several outstanding methods have been found by different investigators.

Graeffe uses the root-squaring process to insure the wide separation of roots. Lyon considers the complex root as a two-dimensional vector and solves its magnitude and direction from which the roots are finally computed. Woodruff extends Lyon's principle to sixth and higher order equations in which the calculating machine must be used. Ku first gives the evidence of two quadratic factors for a quartic equation and finds the coefficient of the
quadratics analytically from the general resolvant cubic equation with the coefficients of the original quartic and solves the roots thereon from the quadratics. Recently Lin discovered a method of successive approximation to factor quartic and higher degree equations. The method is simpler than any mentioned above, but analytic proof is still being sought.

None of the methods mentioned above is started from the physical significance of the fourth degree linear differential equation. The present method is based on the physical significance of the two components. Let us take the nondimensional quartic equation as our starting point and factor it into the following form:

$$\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = (\lambda^2 + \omega_1 \lambda + \omega_0^2)(\lambda^2 + \omega_2 \lambda + \omega_0^2).$$

where

- $\omega_1 =$ dimensionless undamped natural angular frequency of component 1,
- $\omega_2 =$ dimensionless undamped natural angular frequency of component 2,
- $\zeta_1 =$ damping ratio of component 1,
- $\zeta_2 =$ damping ratio of component 2.

As the components are of different frequencies and different dampings, it is convenient to take one component as reference component, and express other quantities in ratios with the reference quantity. Suppose that component 1 is of lower frequency (from now on we shall call it the low frequency component, and call component 2 the high frequency component) and is taken as the reference component. This is purely arbitrary.
Now let $\omega_0 = \omega_r$, (arbitrarily), dimensionless undamped angular frequency of reference component,

$\zeta_r = \zeta_1, \quad (\text{arbitrarily}), \quad \text{damping ratio of reference component,}$

$\rho_w = \frac{\omega_{n2}}{\omega_{n1}} \quad \text{ratio of undamped natural frequency}$

where $\omega_{n1} = \text{undamped natural angular frequency of component 1}$

$\omega_{n2} = \text{undamped natural angular frequency of component 2}$

$\rho_s = \frac{\omega_{s2}}{\omega_{s1}}, \quad \text{ratio of damping ratio}$

Eq. (2.19) becomes

$$\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = \left(\lambda^2 + 2 \zeta_r \omega_r \lambda + \omega_r^2\right)$$

(2.20)

On developing,

$$\alpha_3 = 2 \zeta_r \omega_r + 2 \zeta_r \rho_s \omega_r \rho_w \quad (2.21)$$

$$\alpha_2 = \omega_r^2 + \omega_r \rho_s^2 \rho_w^2 + \rho_s^2 \omega_r \rho_w \quad (2.22)$$

$$\alpha_1 = 2 \zeta_r \rho_b \omega_r \rho_w + 2 \zeta_r \omega_r \rho_w^2 \quad (2.23)$$

$$1 = \omega_r^4 \rho_w \quad (2.24)$$

Eq. (2.24) gives us a real advantage, that since

$$\omega_r \rho_w^2 = 1 \quad (2.24a)$$

so

$$\omega_r \rho_w^2 = 1 \quad (2.24b)$$

and

$$\omega_r \rho_w^2 = 1 \quad (2.24c)$$

Substitute Eqs. (2.24)a, (2.24)b and (2.24)c into Eqs. (2.21), (2.22) and (2.23). The following equations are established.

$$\alpha_3 = 2 \zeta_r \left(\frac{1}{\rho_w^2} + \rho_s \sqrt{\rho_w}\right) \quad (2.25)$$

$$\alpha_2 = \frac{1}{\rho_w} + \rho_w + 4 \zeta_r^2 \rho_s \quad (2.26)$$

$$\alpha_1 = 2 \zeta_r \left(\frac{\rho_s}{\sqrt{\rho_w}} + \sqrt{\rho_w}\right) \quad (2.27)$$

The above three simultaneous equations will give solutions to the three important physical unknown quantities.
(1) $\rho_\omega$, the ratio of undamped natural frequencies with respect to reference component,

(2) $\xi_r$, the damping ratio of the reference component,

(3) $\rho_\xi$, the ratio of damping ratios referring to the reference component.

In addition to the above three quantities, the fourth one, $\omega_r$, can be solved from Eq. (2.24) or

$$\omega_r = \rho_\omega^{1/2} \tag{2.28}$$

The actual process of solving these equations is deferred to the next chapter in which the Quartic Chart is designed to render the practical convenience.

11. Stability Transition Curve (Fig. 1)

It is understood that when the damping ratio of either component is zero, the system must be at a state of unending oscillation. If either damping is slightly negative, the oscillation is unstable. The loci of such transition plotted in terms of the nondimensional coefficients will be defined as stability transition curve.

Obtain the ratio $\frac{d_2}{d_1}$ from Eqs. (2.25) and (2.27)

$$\frac{d_2}{d_1} = \frac{1}{\rho_\xi + \rho_\omega} \tag{2.29}$$

when $\xi_r = 0$ and the other component does possess some damping, $\rho_\xi$ must be infinity, so at that condition

$$\frac{d_2}{d_1} = \rho_\omega \tag{2.30}$$

when $\xi_r = 0$, or $\rho_\xi = 0$, Eq. (2.26) is reduced to

$$\alpha_2 = \frac{1}{\rho_\omega} + \rho_\omega \tag{2.31}$$
STABILITY TRANSITION CURVE

\[ \alpha_2 = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \]

FOR EQUATION

\[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0 \]

Y J. Liu
Feb 40

FIG. 1
It is evident when $\xi = 0$ that the stability transition curve may be represented by the following equation

$$\alpha_x = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \quad (2.32)$$

Moreover, if $\xi \neq 0$, but $\rho^2 = 0$, Eq. (2.29) is reduced to

$$\frac{\alpha_3}{\alpha_1} = \frac{1}{\rho^2} \quad (2.32a)$$

So the stability transition curve can still be represented by Eq. (2.32) for the case.

We shall proceed to find stability criterion from the transition curve.

Take the unrestricted value of $\alpha_1$, $\alpha_2$, and $\alpha_3$ from Eqs. (2.25), (2.26) and (2.27) and find the value of $\alpha_x - \left( \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \right) \times \alpha_x$.

It can be shown

$$\alpha_x - \left( \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \right) = \frac{\alpha^3}{\alpha^2} \frac{(\rho^2 - 1)^2}{\rho^2 (1 + \rho^2) + \rho^2 (1 + \rho^2)} + 4 \xi^2 \rho^2 \quad (2.33)$$

When both $\xi$ and $\rho^2$ are positive, the system is stable and the right side of Eq. (2.33) is positive or greater than zero. That means

$$\alpha_x > \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \quad (2.34)$$

is necessary for stability. The inequality (2.34) is therefore defined as the stability criterion of the standardized nondimensional equation (2.18).

The reverse condition for instability needs no further proof, but to show the rigorousness of Eq. (2.33), it is advisable to do so. Eq. (2.33) can be changed into the following form

$$\alpha_x - \left( \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \right) = \frac{1}{\rho^2} \frac{(\rho^2 - 1)^2}{(1 + \rho^2) + \rho^2 (\frac{1}{\rho^2} + \rho^2)} + 4 \xi^2 \rho^2 \quad (2.33a)$$
If the reference component possesses positive damping, the other component just possesses an infinitesimal negative damping; then \( \rho \) is negative, but is of infinitesimal magnitude. That is, the system is just off the transition condition and shows instability.

Let \( \rho = -\epsilon \) (2.35)

where \( \epsilon \) is an infinitesimal quantity. Substitute the above equation into Eq. (2.33a)

\[
d_2 = \left( \frac{\sigma_2}{a_1} + \frac{d_1}{a_3} \right) \leq - \left[ (\rho \omega - \frac{1}{\rho \omega})^2 + \zeta_2^2 \right] \epsilon \quad (2.36)
\]

Eq. (2.36) shows that as soon as the motion departs from the transition condition, \( d_2 \) becomes negative, or

\[
d_2 < \frac{\sigma_2}{a_1} + \frac{d_1}{a_3} \quad (2.37)
\]

If the reference component possesses an infinitesimal negative damping while the other component possesses certain positive damping \( \zeta_2 \) so that \( \rho = -\frac{\zeta_2}{\epsilon} \) which approaches infinity and has a negative sign

\[
\zeta_{2} = -\epsilon \quad , \quad \rho = -\frac{\zeta_{2}}{\epsilon} \leq -\infty \quad (2.35a)
\]

Then Eq. (2.33)a becomes

\[
d_2 = \left( \frac{\sigma_2}{a_1} + \frac{d_1}{a_3} \right) \leq - \left[ \frac{1}{\epsilon^2} (\rho \omega - \frac{1}{\rho \omega})^2 + \zeta_2^2 \right] \epsilon \quad (2.36a)
\]

The right side of Eq. (2.36)a is negative no matter how small \( \epsilon \) is. Therefore the inequality (2.37) holds true for instability in any case.
Since the low frequency component has arbitrarily assigned as reference component, \( \omega_0 \) has to be greater than one. Let

\[
\omega_0 = 1 + \rho'
\]

where \( \rho' \) is any positive number.

Substitute this value of \( \omega_0 \) into Eq. (2.29). The expression of \( \frac{\alpha_3}{\alpha_1} \) becomes:

\[
\frac{\alpha_3}{\alpha_1} = 1 + \frac{\rho'(\rho_3^2 - 1)}{1 + \rho + \rho_3^2}
\]  

(2.39)

For a stable system (the condition is \( \frac{\alpha_3}{\alpha_1} \times (\frac{\alpha_3^2}{\alpha_1^2} + \frac{\alpha_3}{\alpha_1}) \)

\( \beta \) and \( \rho_3^2 \) are positive. Remember \( \rho_3^2 \) is the ratio of damping ratios referred to the low frequency component. If \( \rho_3^2 \) is greater than one, it means the high frequency component has a greater damping ratio. If \( \rho_3^2 \) is less than one, it means that the high frequency component has a smaller damping ratio.

Now examine Eq. (2.39). If

\[
\rho_3^2 > 1 \quad (2.40)
\]

\[
\frac{\alpha_3}{\alpha_1} = 1 + \rho'' \quad \text{where } \rho'' \text{ is a positive number}
\]

or

\[
\frac{\alpha_3}{\alpha_1} > 1 \quad (2.40)a
\]

If

\[
\rho_3^2 < 1 \quad (2.41)
\]

\[
\frac{\alpha_3^2}{\alpha_1^2} = 1 - \rho''' \quad \text{where } \rho''' \text{ is a positive number and less than one}
\]

or

\[
\frac{\alpha_3^2}{\alpha_1^2} < 1 \quad (2.41)a
\]

The above condition can be extended even if \( \rho_3^2 \) is negative; that is, the system is dynamically unstable.
It can therefore be safely concluded that:

(a) When \( \frac{\alpha^2}{\alpha^1} > 1 \), \( \beta^2 > 1 \); that is, the high frequency component possesses the greater damping ratio.

(b) When \( \frac{\alpha^2}{\alpha^1} < 1 \), \( \beta^2 < 1 \); that is, the high frequency component possesses the smaller damping ratio.

Apparently we have forgotten the condition when \( \frac{\alpha^2}{\alpha^1} = 1 \)

By reexamining Eq. (2.29)

\[
\frac{\alpha^2}{\alpha^1} = \frac{1 + \beta^2 \rho_w}{\beta^2 + \rho_w}
\]

two parallel conclusions can be made immediately:

(c-1) When \( \frac{\alpha^2}{\alpha^1} = 1 \), \( \beta^2 = 1 \) at any value of \( \rho_w \) (2.42)

That means that the two components possess the same dynamic behavior except that they are of different undamped natural frequencies. Or

(c-2) When \( \frac{\alpha^2}{\alpha^1} = 1 \), \( \rho_w = 1 \) at any value of \( \beta^2 \) (2.42a)

That means that the two components are only of the same frequency, but their dynamic behaviors are different.

(c-3) There is also a possible case that both \( \rho_w \) and \( \beta^2 \) are unity. Mathematically it means repetition of the quadratic factor. Physically it means the system is critically damped* quadratically.

The case of \( \rho^2 = \rho_w = 1 \) for \( \frac{\alpha^2}{\alpha^1} = 1 \) is merely a special case which is common to both (c-1) and (c-2). It is the coefficient \( \alpha^2 \) which will decide the fate of \( \frac{\alpha^2}{\alpha^1} = 1 \). Further detailed discussion is deferred until the development of the Quartic Chart and again when the theory of tuning is presented.

*Critical damping in simple degree of freedom means repetition of the binomial factor mathematically, so for two degrees of freedom the repetition of the quadratic factor is also a kind of critical damping physically.
13. The Dimensional Resolvent Cubic Equation

The resolvent cubic equation may be defined as one derived from an ordinary quartic equation and serves the latter as a tool to evaluate its roots (roots of the quartic). The mathematical approach of the resolvent cubic equation varies as various mathematical attacks. With purely algebraic manipulation Ferrari\(^{32}\) reaches the form:

\[ y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0 \]  \hspace{1cm} (2.43)

from the quartic equation:

\[ x^4 + bx^3 + cx^2 + dx + e = 0 \]  \hspace{1cm} (2.44)

Lyon\(^{29}\) starting with vector conception reaches the same form only in different notations. Ku\(^{31}\) with biquadratic manipulation also obtains the same equation as Lyon does. The physical meaning of such resolvent cubic equation is so far hidden from the engineers' retina. The only mathematical interpretation is the relation between the roots of the quartic and of its resolvent. Such relation is expressed by the following equations:

\[ y_1 = x_1x_2 + x_3x_4 \]  \hspace{1cm} (2.45)

\[ y_2 = x_1x_3 + x_2x_4 \]

\[ y_3 = x_1x_4 + x_2x_3 \]

However, equation (2.45) does not furnish any light for engineers' understanding.
14. The Non-dimensional Resolvent Cubic Equation and Its Physical Significance

In paragraph 10 of the last chapter we have established equations (2.25), (2.26) and (2.27), which we shall rewrite here for the starting point of the non-dimensional resolvent cubic equation:

\[
\begin{align*}
\sigma_3 &= 2 \eta \left( \frac{1}{\epsilon_w} + \frac{\epsilon_z}{\epsilon_w} \right) \\
\sigma_2 &= \frac{1}{\epsilon_w} + \epsilon_z + 4 \eta^2 \epsilon_z \\
\sigma_1 &= 2 \eta \left( \frac{\epsilon_z}{\epsilon_w} + \sqrt{\epsilon_z} \right)
\end{align*}
\]

From equations (2.25) and (2.26) we have the product:

\[
\sigma_1 \sigma_3 = 4 \eta^2 \epsilon_z \left[ \epsilon_z + \frac{1}{\epsilon_w^2} + \frac{1}{\epsilon_z} \right]
\]

or

\[
4 \eta^2 \epsilon_z = \frac{\sigma_1 \sigma_3}{\epsilon_z + \frac{1}{\epsilon_w^2} + \frac{1}{\epsilon_z}}
\]

The ratio of \( \sigma_3 \) to \( \sigma_1 \) has been obtained as equation (2.29).

\[
\frac{\sigma_3}{\sigma_1} = \frac{1 + \epsilon_z}{\epsilon_z + \frac{1}{\epsilon_w^2} + \frac{1}{\epsilon_z}}
\]

or solve for \( \epsilon_z \):

\[
\epsilon_z = \frac{\epsilon_z \left( \frac{\sigma_3}{\sigma_1} \right) - 1}{\epsilon_w - \frac{\sigma_3}{\sigma_1}}
\]

With purely algebraic manipulation on the sum of equation (2.47) and its reciprocal expression, we get:

\[
\epsilon_z + \frac{1}{\epsilon_z} = \frac{\left( \epsilon_z + \frac{1}{\epsilon_w^2} \right)\left( \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3} \right) - 4}{\left( \epsilon_z + \frac{1}{\epsilon_w^2} \right) - \left( \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3} \right)}
\]

Now, define: \( \epsilon_w + \frac{1}{\epsilon_w} = \epsilon_m \)

where \( \epsilon_m \) is called mutual frequency ratio.
Substitute the notation \( e \) into equations (2.26), (2.46) and (2.48); and substitute equation (2.48) into equation (2.46), and finally substitute the expression thus obtained for \( \frac{\sqrt{e^2} \ell_y}{\ell_y} \) into equation (2.26), and we will get the following expression:

\[
\alpha_2 = \ell_m + \frac{\sigma_3 \sigma_1}{\ell_m \left( \frac{\sigma_3^2}{\sigma_1} + \frac{\sigma_1^2}{\sigma_3} \right) - 4} \tag{2.50a}
\]

Clear up the fraction of the above equation and the non-dimensional resolvent cubic equation is obtained:

\[
\ell_m^3 - \sigma_2 \ell_m^2 + \left( \sigma_3 \sigma_1 - 4 \right) \ell_m - \left[ \left( \sigma_3^2 + \sigma_1^2 \right) - 4 \alpha_2 \right] = 0 \tag{2.50}
\]

One can immediately observe the similarity between the dimensional and non-dimensional resolvent cubic equation. The equivalent \( e \) term in the non-dimensional equation is unity, so the non-dimensional resolvent cubic equation is much simplified. Yet the physical significance is evident in the non-dimensional resolvent cubic equation whose root (or roots) is the mutual frequency ratio, or:

\[
\ell_\omega + \frac{1}{\ell_\omega}
\]

Three roots are obtainable from the non-dimensional resolvent cubic equation. At least one of them is real and greater than the other two to furnish real value of \( \ell_\omega \); the other two may be real or complex. What do these mean? We shall discuss them when the quartic chart is constructed.

15. The Modified Non-dimensional Resolvent Cubic and Stability Criteria

Let \( l_m = \sigma_2 l_\alpha \) and substitute into equation (2.50); the non-
dimensional resolvent is modified to the following form:

\[ l_\sigma^3 - l_\sigma^2 + \left( \frac{\sigma_3 \sigma_1 - 4}{\sigma_2^2} \right) l_\sigma - \left[ \frac{(\sigma_3^2 + \sigma_1^2) - 4 \sigma_2}{\sigma_2^3} \right] = 0 \]  

(2.51)

where \( \sigma_2 \) is the middle coefficient of the original non-dimensional quartic equation and \( l_\sigma \) is defined as modified mutual frequency ratio, or:

\[ l_\sigma = \frac{1}{\sigma_2} \theta = \frac{1}{\sigma_2} \left( \theta + \frac{1}{\theta} \right) \]  

(2.52)

Equation (2.51) will look much simpler if we define the lumped coefficients:

\[ \frac{\sigma_3 \sigma_1 - 4}{\sigma_2^2} = M \]  

(2.53)

and

\[ \frac{\sigma_3^2 + \sigma_1^2 - 4 \sigma_2}{\sigma_2^3} = N \]  

(2.54)

So equation (2.51) assumes the following form:

\[ l_\sigma^3 - l_\sigma^2 + M l_\sigma - N = 0 \]  

(2.55)

It is seen from equation (2.55) that \( M \) and \( N \) are the constants which will give solution to \( l_\sigma \), and then the process can be traced back until all the non-dimensional constants with physical significance are determined. It is therefore believed that certain special combination of \( M \) and \( N \) will mark:

(a) the boundary line between stably and unstably oscillatory regions

(b) the boundary line between stably oscillatory and stably non-oscillatory regions

Therefore, \( M \) and \( N \) are defined as stability criteria of the non-dimensional quartic equation.
(a) The Boundary Line Between Stably and Unstably Oscillatory Region

Substitute the equation of stability transition curve:

$$\sigma_2 = \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}$$

into equations (2.53) and (2.54):

$$\frac{M}{N} = \frac{(\sigma_3 \sigma_1 - 4 \sigma_2)}{(\sigma_3^2 + \sigma_1^2) - 4 \sigma_2^2} = \frac{\sigma_3 \sigma_1 - 4 \sigma_2}{\sigma_3^2 + \sigma_1^2 - 4 \sigma_2^2} = 1 \quad (2.56)$$

or when $M=N$, the system is in the state of unending oscillation. When $g_r$ or $e_0$, $\sigma_3$, $\sigma_1$ are positive, it is possible $M = N$ = positive value or negative value depending upon their relative magnitude. But when $g_r = e_0$ = 0, $\sigma_3 = \sigma_1 = 0$; therefore, it is only possible that $M=N$ = negative value; it can never be positive.

$\therefore M=N = \text{positive}$ One component is at unending oscillation, and the ratio $\frac{\sigma_3}{\sigma_1}$ will tell whether this component is a high frequency one or low frequency one.

$M=N = \text{negative}$ One or both components are at unending oscillation.

When $\sigma_3 = \sigma_2 = 0$, and $\sigma_2 = 2$ (or $e_0 = 1$), then $M = N = -1$. This is a special condition; when the equation has two identical quadratic factors, both of them miss their damping term.

The inequality $\sigma_2 > \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}$ must be maintained if both components are to be stable, or:

$$\sigma_2 \sigma_1 \sigma_3 > \sigma_3^2 + \sigma_1^2 \quad (2.57)$$

When this is substituted into equation (2.56), it becomes the
following inequality:
\[
\frac{M}{N} > 1
\]  \hspace{1cm} (2.58)
or \( M > N \) is required to have stable operation.

If the inequality \( \alpha_2 < \frac{\sigma_3}{\sigma_1} + \frac{\alpha_1}{\alpha_3} \) is substituted for the unstable region, equation (2.56) becomes the following inequality:
\[
\frac{M}{N} < 1
\]  \hspace{1cm} (2.59)
or \( N > M \) is the region of unstable operation.

Therefore, when \( N \) is plotted as ordinate against \( M \) as abscissa, this boundary condition \( M = N \) is a straight line passing through origin. The region above this line is unstable and the region below it stable.

(b) The Boundary Line Between Stably Oscillatory and Stably Non-oscillatory Regions, Or the Locus of Critical Damping

When the two components are both critically damped,
\[
\mathcal{J}_r = \rho_{\xi \nu} \mathcal{J}_r = 1
\]
that makes:
\[
\sigma_3 = \sigma_1 = 2 \left( \frac{1}{\sqrt{\rho_1 \rho_2}} + \sqrt{\rho_1 \rho_2} \right)
\]  \hspace{1cm} (2.60)
and
\[
\sigma_2 = \rho_2 + \frac{1}{\rho_2} + 4
\]  \hspace{1cm} (2.61)

This condition will mark the oscillatory and non-oscillatory region, because if both \( \mathcal{J}_r \) and \( \rho_{\xi \nu} \mathcal{J}_r \) are greater than unity, both components are overdamped, and four distinct real roots shall be observed from the quartic equation.

Substitute equations (2.60) and (2.61) into the general expressions of \( M \) and \( N \) (equations (2.53) and (2.54)); \( M \) and \( N \) can be evaluated as:
\[
M = \frac{4 \left( \rho_2 \frac{1}{\rho_1 \rho_2} + 1 \right)}{\left( \rho_2 + \frac{1}{\rho_1 \rho_2} + 4 \right)^2} = \frac{4 \left( \rho_2 + 1 \right)}{\left( \rho_2 + 4 \right)^2}
\]  \hspace{1cm} (2.62)
The useful range of $\rho_{w}$ can be extended from $+\infty$ to $-2$. When $-2 < e_{w} < 2$, $\rho_{w}$ becomes a complex number. Mathematically it is correct, but it is hard to be interpreted physically. (The confusion will be cleared up when the Quartic Chart is completed.)

Another form of expression of $M$ and $N$ for this oscillatory and non-oscillatory boundary can be obtained by considering starting with both components critically damped so that their factorized expression of the quartic equation can be written as:

$$
\lambda^4 + \sigma_3 \lambda^3 + \sigma_2 \lambda^2 + \lambda = (\lambda + \frac{1}{\rho_{w}})(\lambda + e_{w})(\lambda + \rho_{w})(\lambda + \rho_{w})
$$

(2.64)

$$
= \left[ (\lambda + \frac{1}{\rho_{w}})(\lambda + e_{w}) \right]^2
$$

$$
= \left[ \lambda^2 + (\frac{1}{\rho_{w}} + e_{w}) \lambda + 1 \right]^2
$$

(2.64a)

The physical meaning of equation (2.64a) is that the two vibratory components are of the same frequency and same damping ratio; in fact, the system is critically damped quadratically.

In symbol, they are:

$$
\rho_{w} = 1, \quad \rho_{y} = 1
$$

(2.65)

That makes:

$$
\sigma_3 = \sigma_1 = 4 \gamma_r
$$

(2.66)

and

$$
\sigma_2 = 2 \left( 1 + 2 \gamma_r^2 \right)
$$

(2.67)
STABILITY CRITERION \( N \) vs. \( M \) for
\[
\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0
\]

**STABLE REGION, OSCILLATORY**

- \( N > M \) UNSTABLE REGION
- \( N = M \) UNENDING OSCILLATION 1 COMPONENT ONLY
- 3 or 4 Equal Roots
- 2 Equal roots or 2 pairs of equal roots
- STABLE REGION; NON-OSCILLATORY OR DOUBLE OSC.
- \( J_{2,1} < \sqrt{1 - \frac{\alpha_1^2}{\alpha_2^2}} \)
- 2 Equal quadratic factors or 2 Critically damped quadratics with different frequencies

\[
M = \frac{\alpha_3\alpha_1 - 4}{\alpha_2^2}
\]

Y. J. Liu
Feb. 1940

FIG. II C
When equations (2.66) and (2.67) are substituted into the general form of $\mathcal{M}$ and $\mathcal{N}$, they appear as:

$$\mathcal{M} = \frac{4g_r^2 - 1}{(1 + 2g_r^2)^2}$$  \hspace{1cm} (2.68)

and

$$\mathcal{N} = \frac{2g_r^2 - 1}{(1 + 2g_r^2)^3}$$  \hspace{1cm} (2.69)

Here the usable range of $g_r$ extends from $+\infty$ to 0.

Figures IIA, B and C show the plot of equations (2.68), (2.69) and (2.58). Figure IIA covers a wide range of $\mathcal{M}$ and $\mathcal{N}$. Stability can be verified with practically every possible combination of $\mathcal{M}$ and $\mathcal{N}$. The non-oscillatory region is bounded by the curve BA, AC and BC, and is shaded. Figure IIB is an enlargement of the non-oscillatory region where four unequal real roots are present. Figure IIC is plotted in logarithmic scales to render better the visualization of very small quantities of $\mathcal{M}$ and $\mathcal{N}$ forming part of the boundary between the oscillatory and non-oscillatory regions. Figure I can be referred to as an indication of relative damping between the two components. Such indication is not available in Figure II.

The shaded area, being the non-oscillatory region, is not so evident; the pure mathematical proof which is tedious is excluded here. However, a simple logic proof will be given when the Quartic Chart is completed.

Equations (2.62) and (2.68), and equations (2.63) and (2.69) are mutually transferable with the following relation:

$$\ell_m + 4 = 2(1 + 2g_r^2)$$  \hspace{1cm} (2.70)
When \( g_f = 1 \) for equations (2.68) and (2.69), it means that the two quadratic factors are identically the same, and both are critically damped, and

\[
\begin{align*}
\therefore (M, N) &= \left( \frac{1}{3}, \frac{1}{37} \right) \\
\end{align*}
\]

is the point of cusp where four equal real roots are possibly obtainable from the quartic equation.

16. The Development of the Quartic Chart for the Non-dimensional Quartic Equation

\[
\lambda^4 + \sigma_3 \lambda^3 + \sigma_2 \lambda^2 + \sigma_1 \lambda + 1 = 0
\]

When the stability criteria \( M \) and \( N \) is obtained from the coefficients \( \sigma_3, \sigma_2 \) and \( \sigma_1 \), the modified resolvent cubic equation is fixed:

\[
\ell_\alpha^3 - \ell_\alpha^2 + M \ell_\alpha - N = 0
\]

It looks like a simple matter to solve \( \ell_\alpha \) from the above equation, but solving a cubic equation analytically is usually tedious. Had the equation been transferred to Weiss' form, much effort could be saved. However, to preserve a simpler form of stability criterion and to simplify the further direct graphical solution, the writer decided to take the form as obtained above. Nevertheless, Weiss' chart does not extend to regions of three unequal real roots, while at the present study of quartic equations, the region of four unequal real roots cannot be logically proven without the help of the region of three unequal real roots of its resolvent cubic equation.

* The writer has revised Weiss' cubic charts to a single chart for the cubic equation in Weiss' form. It is presented in Appendix A for the interest of readers.
THE QUARTIC CHART

FOR EQUATION

\[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0 \]

AND EQUATION

\[ a_4 \frac{d^4 x}{dt^4} + a_3 \frac{d^3 x}{dt^3} + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \]

VALUES OF \( \sqrt{3} (\alpha_3 + \alpha_1) \)
Graphical solution of equation (2.55) is constructed as Chart I of the Quartic Chart. The curves are plotted for constant values of $N$ with $M$ as ordinate and $P_\alpha$ as abscissa. For the convenience of further application of the graphical result, the scale of $M$ is linear, and that of $P_\alpha$ logarithmic, and they are both rotated 45 degrees from ordinary horizontal and vertical axes. It is noticed, when $N=0$, equation (2.55) becomes:

$$P_\alpha (P_\alpha^2 - P_\alpha + M) = 0 \quad (2.72)$$

one of the root becomes zero, and the other roots are:

$$P_\alpha = \frac{1 \pm \sqrt{1-4M}}{2} \quad (2.73)$$

When $M$ is negative, one of the roots is negative, which is of no interest in our problem; therefore it is excluded. The root zero implies pure imaginary $P_\alpha$ which is also of no interest. Therefore, only the root $P_\alpha = \frac{1 + \sqrt{1-4M}}{2}$ is shown on the chart. (That is, the only intersection of constant $N(=0)$ and constant $M$ appears on the chart.)

When $M$ is greater than zero, $P_\alpha$ has two values until $M = \frac{1}{4}$ where the curve $N=0$ reaches its maximum point. Beyond $M = \frac{1}{4}$, there will be no intersection of $N=0$ with any $M$; that is, there is no real value for $P_\alpha$.

For $N<0$, the curves have simpler shape with one maximum, because negative roots are excluded. When $\frac{1}{34} > N > 0$ the curves give one maximum at greater $P_\alpha$ and a minimum at smaller $P_\alpha$. Again, when $N$ exceeds $\frac{1}{34}$, neither maximum nor minimum can be seen for positive values of $P_\alpha$. 
It is very interesting to notice the relation between the locus of maxima and minima of the constant $N$ curves of the Quartic Chart with the boundary line that separates the oscillatory and non-oscillatory regions on the stability criteria plot (Figure II).

Write equation (2.55) in the following form:

$$M = \frac{N - e_\alpha^3 + e_\alpha^2}{e_\alpha}$$

Let $\frac{dM}{d e_\alpha} = 0$, $N = -2 e_\alpha^3 + e_\alpha^2$ for $M_{\text{max}}$ or $M_{\text{min}}$

$$M_{\text{min}} \text{ or } M_{\text{max}} = 3 e_\alpha^2 + 2 e_\alpha$$

Assume the locus of $M_{\text{max}}$ and $M_{\text{min}}$ represents the repetition of quadratic factor; that is, at $e_\omega = 1$ and $e_y = 1$. Then,

$$e_\alpha = \frac{e_\omega + \frac{1}{2} e_\omega}{\alpha^2} = \frac{e_\omega + \frac{1}{2} e_\omega}{e_\omega + \frac{1}{2} e_\omega + 4 y_r^2} = \frac{1}{1 + 2 y_r^2}$$

(2.76)

Substitute equation (2.76) into equations (2.75) and (2.75) and simplify the expressions; we have:

$$M_{\text{min}} \text{ or } M_{\text{max}} = \frac{4 y_r^2 - 1}{(1 + 2 y_r^2)^2}$$

(2.77)

$$N = \frac{2 y_r^2 - 1}{(1 + 2 y_r^2)^3} \text{ for } M_{\text{min}} \text{ or } M_{\text{max}}$$

(2.78)

Equations (2.77) and (2.78) are exactly the same as equations (2.68) and (2.69) which are derived directly from the condition of repetition of quadratic factor. Therefore, the assumption just made is correct.

This locus $M_{\text{max}}$ and $M_{\text{min}}$ (or $\frac{dM}{d e_\alpha} = 0$) is plotted on Chart I.
as the dotted curve. The part of the dotted curve to the left of its vertex (or passing through $M_{\text{min}}$) corresponds to the boundary line BA of Figure II. (Both of them start from $M,N=0,0$ and end at $M,N=\frac{1}{3},\frac{1}{3}$).

The part of the dotted curve to the right of its vertex (or passing through $M_{\text{max}}$) corresponds to the boundary line ACD. (Both start from $M,N=\frac{1}{3},\frac{1}{3}$ and end at $M,N=-1,-1$).

The scale of $\ell_\alpha$ is marked along the line $M=0$ whenever the intersection is fixed by a particular pair of $M$ and $N$; its projection onto the line $K=0$ will give the value of $\ell_\alpha$. (Precaution: Take the rightmost intersection if more than one intersection are observed for a particular pair of $M$ and $N$ to avoid the trouble in getting complex quantity in ratio of undamped natural frequencies.)

The Scale of $\alpha_2$, The Concealed Scale of $\ell_m$,* and Scale of $\ell_\omega$, Ratio of Undamped Natural Frequencies

Lines (which are perpendicular to $M=\text{constant}$) of 135 degrees to horizontal lines represent constant $\ell_\alpha$. The vertical scale on the left is provided for the middle constant $\alpha_2$ in logarithmic scale. The intersection of the horizontal line of constant $\alpha_2$ and the 145 degree line of constant $\ell_\alpha$ gives the product $\alpha_2 \ell_\alpha$ or $\ell_m$ when it is projected vertically down to the horizontal base of which the scale should be provided for $\ell_m$, but it is concealed. In other words, the vertical line represents constant $\ell_m$.

The Curve $P$

The curve $P$ is the plotted result of $\ell_\omega + \frac{1}{\ell_\omega} = \ell_m$. When constant

* The detailed explanation of charting is omitted. If the reader is interested in it, please read "Graphical and Mechanical Computation", Chapter II, by J. Lipka, John Wiley and Sons, Inc., 1918.
Line (vertical) falls on P curve and deflects horizontally to the right until it reaches the scale on the right ordinate of Chart I, the reading thus obtained is $P_w$.

There are two curves of $P$, one appearing at the center part of the curve, another in the crowded zone. The former should be matched with the left scale of $\alpha_z$ (range $10 - 1000$); the latter with the right scale of $\alpha_z$ (range $1 - 10$). (The matching of scales is exactly the same for $Q$ curves).

The Scale of $\omega_r$, the Dimensionless Angular Natural Frequency

Of Reference Component

The relation between $P_w$ and $\omega_r$ ($\omega_r = P_w^{-1/2}$, equation (2.28)) offers a simple nomogramic solution. Prolong the horizontal line just obtained from the deflection on P curve until it meets the left ordinate scale on Chart II. This intersection gives the value of $\omega_r$.

The Curve $Q$ and Chart II

With much juggling for further practical convenience of charting, a new variable $Q$ is introduced with the following relation:

$$Q = \left( P_m + 2 \right)$$  \hspace{1cm} (2.79) \\

or

$$Q = \left( \frac{1}{\sqrt{P_w}} + \sqrt{P_w} \right)^2$$  \hspace{1cm} (2.79a)

Equation (2.79) is plotted on Chart I as the Q curve with right ordinate scale of Chart I as Q's ordinate, and the horizontal concealed scale for $P_m$. 
As the left ordinate scale of Chart II bears a nomogramic relation \( \omega_r = e^{-\frac{\alpha}{2}} \) to the right ordinate scale of Chart I, therefore when vertical \( e_\omega \) is deflected on \( Q \) horizontally toward the right and continued until it hits the left ordinate scale of Chart II, the reading on this scale will be \( e^{-\frac{\alpha}{2}} \).

or

\[
Q = e^{-\frac{\alpha}{2}} = \frac{1}{\sqrt{\omega} + \sqrt{\omega}}
\]

(2.80)

We have solved \( e_\omega \) from equations (2.25) and (2.27) that:

\[
e_\omega = \frac{\omega_\omega (\frac{\alpha_3}{\alpha}) - 1}{\omega_\omega - (\frac{\alpha_3}{\alpha})}
\]

(2.47)

are plotted on Chart II with \( e_\omega \) as abscissa and \( e \) as ordinate (left). So when certain deflected horizontal line from \( Q \), curve on Chart I cuts the constant \( \frac{\alpha_3}{\alpha} \), curve, the intersection projected onto the bottom scale of Chart II gives \( e_\omega \), the ratio of damping ratios with respect to the reference frequency.

It is seen from Chart II that \( \frac{\alpha_3}{\alpha} \) actually does the function of damping distribution as has been discussed in Paragraph 12, Chapter Four.

The Effective Damping Parameter \( 1/2 (\alpha_3 + \alpha) \)

The addition of equations (2.25) and (2.27) gives:

\[
\alpha_3 + \alpha_1 = 2j \omega_\omega (1 + e_\omega) (\frac{1}{\sqrt{\omega}} + \sqrt{\omega}) = 2j (1 + e_\omega) (\frac{1}{\sqrt{\omega}} + \sqrt{\omega})
\]

(2.81a)
or

\[
\frac{\alpha_3 + \alpha_1}{2} = j \omega_\omega (1 + e_\omega) (\frac{1}{\sqrt{\omega}} + \sqrt{\omega}) = j \omega_\omega (1 + e_\omega) / \omega
\]

(2.81)

The definition of effective damping parameter is evident from equation (2.81a), because it is proportional to the sum of damping ratios of the two components of the system.
The Scales of $q$, $e_y$, and $y_r$, and The Constant $\frac{q}{1+e_y}$ Lines

The vertical scale of $q (= \frac{1}{\sqrt{\varepsilon_w}+\frac{1}{\varepsilon_w}})$ is drawn logarithmically upward on the left side of Chart II. It is exactly the same as that for $a_t$ but the denomination of the scale is omitted.

The scale of $e_y$ is drawn on the bottom of Chart II with values of $1+e_y$ logarithmically rightward, but the scale is numbered according to values of $e_y$ itself.

With the above arrangement of scales of $q$ and $e_y$, each line 45 degrees inclined to horizontal one represents a constant value of $\frac{1+e_y}{q}$.

Transfer equation (2.81) into the following form:

$$\frac{1}{\sqrt{a_t}} (\alpha_3+\alpha_1) = \frac{1+e_y}{q}$$

which offers a simple way to obtain $y_r$ from the chart by scaling $1/2 (\alpha_3+\alpha_1)$ horizontally rightward and $y_r$ vertically upward, both logarithmically identical with those for $1+e_y$ and $q$. To avoid confusion, the scale of $1/2 (\alpha_3+\alpha_1)$ is laid on top of Chart II and scale of $y_r$ along the right ordinate.

For any horizontal line deflected from $Q$ curve on Chart I which meets a certain $\frac{\alpha_3}{\alpha_1}$ curve, the projection of the intersection onto the bottom of Chart II gives $e_y$. Start from the same intersection on the particular $\frac{\alpha_3}{\alpha_1}$; draw a 45 degree line until it hits a particular $1/2 (\alpha_3+\alpha_1)$ vertical line. The intersection on such vertical line will give the value of $y_r$ on the rightmost ordinate scale.
17. Cyclic Shifting of Logarithmic Scales

In case no intersection on the particular 135 degree inclined line (for constant \( \ell_\alpha \)) and the horizontal line of particular value of \( a_z \) can be found within Chart I, we may shift the proper 135 degree inclined line one logarithmic cycle left (or right), but the scale of \( a_z \) should never be changed. By this process the matched P and Q curves are automatically shifted one logarithmic cycle left (or right) with the shifted 135 degree inclined line. So the local P and Q curves are available.

Shifting of the 45 degree lines on Chart II one logarithmic cycle up (or down) is also permissible. However, the decimal points of the ordinates scale for \( \gamma_r \) must be shifted one figure left (or right). Moreover, the scales of \( 1/2 \left( \frac{a_3}{a_2} \right) \) and of \( \gamma_r \) can be multiplied by a common factor, for instance, 10 simultaneously. In the latter case the constancy of the 45 degree lines are not affected.

18. Option in Reference Component

Sometimes the horizontal line from Q curve does not intersect the particular \( \frac{a_3}{a_2} \) curve within the range of the chart (Chart II). In that case the ratio \( \frac{a_1}{a_3} \) may be used instead of \( \frac{a_3}{a_2} \) (or considering that all the numbers marked on \( \frac{a_3}{a_2} \) curve are now for \( \frac{a_r}{a_3} \)). The rest of the procedure is exactly the same as before, but the data obtained is referred to high frequency as reference component; that is, \( \gamma_r \) is the damping ratio of high frequency component, and \( \ell_y \) the ratio of damping ratio of low frequency
component to that of the high frequency component.

Mathematical proof can be given for such transformation.

Let:

\[ f_r = f_1, \quad f_r' = f_2, \]
\[ e_y = \frac{f_2}{f_1}, \quad e_y' = \frac{f_1}{f_2} = \frac{1}{e_y} \]

Starting from equation (2.47):

\[ e_y = \frac{e_\omega (\frac{\alpha_3}{\alpha_3}) - 1}{e_\omega - \frac{\alpha_3}{\alpha_3}} \tag{2.47} \]

take the reciprocal expression of the above equation and multiply both numerator and denominator by \( \frac{\alpha_1}{\alpha_3} \). We have:

\[ e_y' = \frac{1}{e_y} = \frac{e_\omega \left( \frac{\alpha_1}{\alpha_3} \right) - 1}{e_\omega - \frac{\alpha_1}{\alpha_3}} \tag{2.83} \]

Equation (2.83) is of the same form as equation (2.47), so the transformation of \( \frac{\alpha_3}{\alpha_1} \) to \( \frac{\alpha_1}{\alpha_3} \) is to change \( e_y \) to \( e_y' \).

Next, take equation (2.82) and substitute \( e_y = \frac{1}{e_y'} \). There we have:

\[ \frac{1}{2} (\alpha_3 + \alpha_1) = \frac{e_y' + 1}{e_y} \tag{2.84} \]

What is \( \frac{f_r}{e_y'} \)? As \( f_r = f_1, \quad e_y' = \frac{f_1}{f_2} \)

Therefore,

\[ \frac{f_r}{e_y'} = f_2 = f_r' \]

or

\[ \frac{1}{2} (\alpha_3 + \alpha_1) = \frac{e_y' + 1}{e_y} \tag{2.84'a} \]

which is of the same form as equation (2.82); therefore, the graphical procedure is entirely the same if \( \frac{\alpha_1}{\alpha_3} \) is used instead of \( \frac{\alpha_3}{\alpha_1} \).
Factorization of Quartic Equation by Means of the Quartic Chart

With the understanding of the development of the Quartic Chart, one should be able to find the four non-dimensional physical constants $c, \omega_r, c_y$ and $\gamma_r$. As soon as they are obtained, the factorized quartic equation can be written in the form of equation (2.19) or (2.20).

Returning to the dimensional quartic equation, with

$$D = A_o \frac{1}{\lambda^4} \lambda$$

or

$$\lambda = A_o \frac{1}{\lambda^4}$$

(2.85a)

$$\omega_{n1} = \omega_r A_o \frac{1}{\lambda^4}$$

(undamped angular natural frequency of component 1)

(2.85b)

$$\omega_{n2} = \omega_r A_o \frac{1}{\lambda^4}$$

(undamped angular natural frequency of component 2)

(2.85c)

Substitution of equations (2.85)a, b and c into equation (2.14) will give:

$$D^4 + A_o D^3 + A_2 D^2 + A_1 D + A_o = (D^2 + a \gamma D + \omega^2_n)(D^2 + a \gamma D + \omega^2_n)$$

(2.86)

Graphical and Analytic Solutions

$\alpha$ is a principal datum to the four dimensionless physical quantities $c, \omega_r, c_y$ and $\gamma_r$. Chart I serves the most practical and convenient way to find the value of $\alpha$, which could be obtained analytically only after elaborate formulation and substitution. However, when $\alpha$ is obtained from Chart I, the following formulae can be used for the evaluation of $c, \omega_r, c_y$ and $\gamma_r$:

* Complete directions for the Quartic Chart are presented as Appendix E, which has been issued to the Class of Servo-mechanism at the Institute, and which proved to be practical. A limited number of mimeographed prints is available at the Institute Instrumentation Laboratory.
\[ \frac{1}{\xi} \left( \rho_\omega + \frac{1}{\rho_0} \right) = \rho_{\omega} \quad \text{or} \quad \rho_{\omega} = \frac{1}{2} \left[ \xi \rho_\omega + \sqrt{\xi^2 \rho_\omega^2 - 4} \right] \]  

(2.87)

\[ \omega_r^2 \rho_{\omega} = 1 \quad \text{or} \quad \omega_r = \frac{1}{\sqrt[2]{\rho_{\omega}}} \]  

(2.88)

\[ \frac{\xi}{\xi_i} = \frac{1 + \xi \rho_\omega}{\rho_i + \rho_\omega} \quad \text{or} \quad \rho_i = \frac{\rho_\omega \left( \frac{\xi}{\xi_i} \right) - 1}{\rho_\omega - \frac{\xi}{\xi_i}} \]  

(2.89)

\[ \frac{\xi + \xi_i}{2} = \xi_r \left( 1 + \frac{\rho_i}{\rho_\omega} + \frac{1}{\rho_\omega} \right) \quad \text{or} \quad \xi_r = \frac{\frac{1}{2} \left( \xi + \xi_i \right)}{\left( 1 + \frac{\rho_i}{\rho_\omega} + \frac{1}{\rho_\omega} \right)} \]  

(2.90)

\[ \xi_2 = \frac{1}{\rho_\omega} + \rho_\omega + 4 \xi_r \rho_i \quad \text{or} \quad \xi_r = \frac{1}{2} \sqrt{\frac{\xi_2 \left( 1 - \rho_\omega \right)}{\rho_i}} \]  

(2.90a)

\[ \xi_3 = 2 \xi_r \left( \frac{\rho_i}{\rho_\omega} + \sqrt{\rho_\omega} \right) \quad \text{or} \quad \xi_r = \frac{\xi_3}{2 \left( \frac{\rho_i}{\rho_\omega} + \sqrt{\rho_\omega} \right)} \]  

(2.90b)

\[ \xi_4 = 2 \xi_r \left( \frac{\rho_i^2}{\rho_\omega} + \sqrt{\rho_\omega} \right) \quad \text{or} \quad \xi_r = \frac{\xi_4}{2 \left( \frac{\rho_i^2}{\rho_\omega} + \sqrt{\rho_\omega} \right)} \]  

(2.90c)
CHAPTER SIX

DETAILED ANALYSIS OF STABILITY CRITERIA M AND N

21. The Rigorous Proof of The Nonoscillatory Region

Bounded By BACB

As we have proven the correspondence of the boundary line ABC to the dotted curve on the quartic chart, the bottom line CB on Fig. 2 is \( N = 0 \), so it corresponds to the curve \( N = 0 \) on the quartic chart. Therefore, the shaded area BCAB on Fig. 2 may be considered as the area bounded by the curves \( \frac{\partial M}{\partial \rho_2} = 0 \), and \( N = 0 \) on the quartic chart.

It is easily seen that any curve for \( 0 < N < \frac{1}{27} \) enters the area (bounded by \( \frac{\partial M}{\partial \rho_2} = 0 \) and \( N = 0 \)) at \( M = \text{max} \), and leaves the same at \( M = \text{min} \). A particular \( M \) (which must be greater than zero, but less than \( 1/3 \)) which intersects one particular \( N \) curve inside this area gives two additional intersections outside the area -- one to its left and another to its right. That is, for the particular \( M \) and \( N \) which do intersect inside the shaded area BACB, three intersections are obtainable to give three distinct \( \rho_a \). That means that this particular pair, \( M \) and \( N \), will give three values of \( \rho_m \) because \( \alpha_2 \) is constant for the particular problem. This in turn gives three values of \( \rho_\omega \).

By common sense, if a quartic equation can be factored into four real and distinct binomial factors as

\[
(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4) = 0
\]

(2.91)
there are three ways to combine them into two distinct quadratics which are both critically damped from physical point of view, and accordingly three real distinct $\omega$'s can be observed.

It can also be observed that:

(a) The constant $M$ line which is minimum to a particular $N$ curve has another intersection with the same constant $M$ to the right of the dotted curve. This corresponds to $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$ so that two different ways can be at our liberty to combine the four factors into two distinct quadratics. Thus we have two real distinct $\omega$. The intersection at the minimum point evidently will give $\omega = \beta = 1$ and $\xi > 1$, while the right intersection will give $\omega = \beta = 1$ and $\omega > 1$.

Hold the particular $N$ curve. The slightest increase of $M$ immediately gives three intersections with this $N$ curve. However, increasing $M$ from its minimum value actually corresponds to the rightward increment of $M$ from the boundary line $BA$ of Fig. 2. This means entering the shaded area on Fig. 2 which corresponds to giving 4 distinct real roots for the quartic equation.

With the slightest decrease of $M$ only one intersection with the same $N$ is observed which means that two of the roots are a conjugate pair. The natural way to factor such a quartic into two simple quadratics (with real coefficients) is restricted to one. Physically it means to the left of the boundary line $BA$ (Fig. 2) the system is oscillatory.

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This conclusion is only valid for $\frac{\omega}{\omega_1} = 1$. Further discussion for $\frac{\omega}{\omega_1} \neq 1$ appears in section 22.
(b) The constant M line which is maximum to a particular N curve has another intersection with the same N curve to the left of the dotted curve. Along $\frac{dM}{d\alpha} = 0$ and $M = \text{max.}$ the quartic equation has been shown to have two identical quadratic factors with $\xi_Y$ less than one (at the vertex of $\frac{dM}{d\alpha} = 0$, or at the cusp of M vs. N curve, $\xi_Y = 1$). Mathematically,

For $\xi_Y < 1$ \[ (\lambda^2 + 2\xi_Y \lambda + 1)^2 = [\lambda + (\xi_Y + i\sqrt{1-\xi_Y^2})][\lambda + (\xi_Y - i\sqrt{1-\xi_Y^2})]^2 \] (2.92)

From the right side of Eq. (2.90) it is easily seen that

\[ \rho' = \frac{(\xi_Y + i\sqrt{1-\xi_Y^2})^2}{(\xi_Y - i\sqrt{1-\xi_Y^2})^2} \quad \text{or} \quad \rho'' = \frac{\xi_Y + i\sqrt{1-\xi_Y^2}}{\xi_Y - i\sqrt{1-\xi_Y^2}} \tag{2.93} \]

Rationalizing equation (2.93) we get

\[ \rho'' = \frac{2\xi_Y^4 - 1 + 2i\xi_Y\sqrt{1-\xi_Y^2}}{\xi_Y - i\sqrt{1-\xi_Y^2}} \tag{2.94a} \]

and

\[ \rho_m = \rho'' + \frac{1}{\rho''} = 2(2\xi_Y^2 - 1) \tag{2.94} \]

which is the same thing as $\rho + 4 = 2(2\xi_Y^2 + 1)$ (2.70)

That makes the expressions of M and also of N mutually transferable either in terms of $\rho_m$ or $\xi_Y$.

From the above mathematical analysis we can understand that the left intersection of M and N (the same M is tangent to the same N at the right) is mathematically correct giving real $\rho$, thence real $\rho_m$, but complex $\rho''$ which is not interested physically.

It is clear that the right intersection gives larger $\rho_m$ than the left one for the same pair of M and N. In other words, right $\rho_m >$ left $\rho_m$ (if $\alpha_z$ is kept constant, and it is so because we are dealing with one and the same quartic equation).
And because

\[ \rho_m = \frac{1}{\rho_\omega} + \rho_\omega \quad \text{or} \quad \rho_\omega = \frac{\rho_m \pm \sqrt{\rho_m^2 - 4}}{2} \]  

(2.95)

Therefore, when \( \rho_m = 2 \), \( \rho_\omega = 1 \)

\[-2 < \rho_m < 2, \quad \rho_\omega = \text{complex}\]

\( \rho_m \) for \( \rho_\omega = 1 \) is larger than \( \rho_m \) for \( \rho_\omega = \text{complex} \). There is no further confusion to take the tangential point instead of the left intersection whose physical significance is hard to interpret, although it is mathematically correct.

Hold the particular \( N \) curve. The slightest decrease of \( M \) immediately gives three intersections with this \( N \) curve. However, decreasing in \( M \) from its maximum value actually corresponds to leftward increment of \( M \) from the boundary line \( AC \) of Fig. 2. This means that the entering of the shaded area on Fig. 2, which corresponds to giving four distinct real roots for the quartic equation. With the slightest increase in \( M \), only one intersection with the same \( N \) is observed which means that two of the roots are a conjugate pair. The natural way to factor such a quartic equation into two simple quadratics with real coefficients is restricted to one. Physically it means that to the right of the boundary line \( AC \) (Fig. 2) the system is oscillatory.

(c) The boundary curve \( N = 0 \) shows that when \( N \) is less than zero, there is no chance to have three intersections with any \( M \). But slightly above the boundary of \( N > 0 \) (but less than \( \frac{1}{27} \)) three intersections are obtainable with suitable \( M \)'s. This boundary curve \( N = 0 \) corresponds to the base line \( N = 0 \) of the shaded area \( BACB \) on Fig. 2. It is therefore safe to say that
the shaded area of Fig. 2 bounded by 

\[ N = \frac{2 \beta^2 - 1}{(1 - 2 \beta^2)^2} \]

and \( N = 0 \) is the nonoscillatory region

where four distinct real roots are present.

It is possible to show that for \( N = 0 \) the two components fall on either one of the following conditions:

1. \( \rho_0 > 1, \quad \beta = 0.707 = \frac{\beta_z}{(2.96)} \)
2. \( \rho_0 = 1, \quad \beta = \sqrt{1 - \frac{\beta^2}{(2.97)}} \)

22. The Most Lenient Behavior Affixing to
The Boundary Lines That Separate The Oscillatory and Nonoscillatory Region

All the above conditions for the boundary lines between the oscillatory and nonoscillatory are apparently derived from the least condition that \( \frac{\alpha_3}{\alpha_1} = 1^* \) which forms the following possible combinations:

(a) \( \rho_0 = 1, \quad \beta \neq 1 \) Boundary lines BA and CB
(b) \( \beta = 1, \quad \rho_0 > 1 \) Boundary lines BA and CB
(c) \( \rho_0 = 1, \quad \beta = 1 \) Boundary lines BA and AC

However, with simple and logical reasoning, if one component is being critically damped, but another is being overdamped, the factored form of the quartic equation can be arranged in two and only two combinations.

\[ [(\lambda + \lambda_{12})(\lambda + \lambda_{12})(\lambda + \lambda_3)(\lambda + \lambda_4)] = 0 \quad (2.98) \]

or

\[ [(\lambda + \lambda_{12})(\lambda + \lambda_3)(\lambda + \lambda_{12})(\lambda + \lambda_4)] = 0 \quad (2.98a) \]

In this case two and only two different real frequency ratios are obtainable. On the quartic chart two and only two \( \rho_0^* \)'s should be present. Therefore, one of the two \( M - N \) inter-

*See Section 12, Chapter II of this thesis
sections must be at \( \frac{dM}{d\rho} = 0 \). With this example, it is evident that \( \lambda_3 \) may not be equal to \( \lambda_4 \), yet their stability criteria \( M \) and \( N \) fall on the boundary line between oscillatory and nonoscillatory regions as \( \lambda_3 = \lambda_4 \) does.

Apparently the lenient condition is \( \bar{\rho}_r = 1, \rho_\omega > 1 \) and \( \bar{\rho}_5 \neq 1 \) (\( \bar{\rho}_5 \) may be \( \geq 1 \), the condition \( \bar{\rho}_5 = 1 \) has been treated). From that condition

\[
\alpha_3 = 2 \left( \frac{1}{\sqrt{\rho_\omega}} + \frac{\bar{\rho}_5}{\sqrt{\rho_\omega}} \right) 
\frac{\rho_\omega}{\bar{\rho}_5}
\]

\( \alpha_2 = \frac{1}{\rho_\omega} + \rho_\omega + 4 \bar{\rho}_5 \)

\( \alpha_1 = 2 \left( \frac{\bar{\rho}_5}{\rho_\omega} + \sqrt{\rho_\omega} \right) \)

The first evidence of the above condition is

\[
\frac{\alpha_3}{\alpha_1} = \frac{1 + \frac{\bar{\rho}_5}{\rho_\omega}}{\bar{\rho}_5 + \rho_\omega} \neq 1 
\]

With the above condition substitute Eqs. (2.99) to (2.101) into Eqs. (2.53) and (2.54) and simplify with \( \rho_m = \rho_\omega + \frac{1}{\rho_\omega} \)

\[
M = \frac{4 \bar{\rho}_5 (\rho_m + \bar{\rho}_5)}{(\rho_m + 4 \bar{\rho}_5)^2} \quad (2.103)
\]

and

\[
N = \frac{4 \bar{\rho}_5^2 \rho_m}{(\rho_m + 4 \bar{\rho}_5)^3} \quad (2.104)
\]

Eqs. (2.103) and (2.104) cannot be directly identified as anything along the boundary line, but if both denominator and numerator of Eq. (2.103) are divided by \( \bar{\rho}_5^2 \) and those of Eq. (2.104) by \( \bar{\rho}_5^3 \), and let \( \rho'_m = \frac{\rho_m}{\bar{\rho}_5} \) for both equations.

Then

\[
M = \frac{4(\rho'_m + 1)}{(\rho'_m + 4)^2} \quad (2.103)\text{a}
\]

and

\[
N = \frac{4 \rho'_m}{(\rho'_m + 4)^3} \quad (2.104)\text{a}
\]
Eqs. (2.103)a and (2.104)a take the same form as Eqs. (2.62) and (2.63), only with $P_m$ changed to $P'_m$. It is true that with one critically damped component (or two equal roots), the stability criteria $M$ and $N$ fall also on the boundary line that separates the oscillatory and nonoscillatory regions. However, the point is shifted from $M, N(P_m)$ to $M, N(P'_m)$. Such shifting sometimes means shifting from branch BA to branch ACD. An example will show this statement clearly.

Suppose that the original conditions are:

$$\rho = 5.0, \quad \frac{\rho}{\rho} = 5.2$$

Then $\rho_m = 5.2$ or $M, N(P_m) = 0.293, 0.0266$ (On BA)

but $\rho'_m = \frac{5.2}{1.0} = 0.32, 0.032$ (On AC)

For the bottom line, or $N = 0$, the lenient condition will reduce to $M = 0$; that is, one component is at unending oscillation. Apparently this is only one point of $N = 0$ (at $M = 0$) so it is not applicable to the whole range of $N = 0$. In other words, such lenient condition is not applicable to the boundary line $N = 0$.

23. The Lenient Behavior Affixing To The Cusp A of The Shaded Area

Logically speaking, at the cusp A (Fig. 2) or the corresponding vertex of $\frac{\partial M}{\partial \rho} = 0$, only one way of factoring the quartic equation is possible to give real $\rho$. The condition of four equal real roots is too strict. However, only three equal real roots are sufficient to reach the cusp of the shaded area.
This can be proved mathematically by the following considerations. Let the quartic equation be factored as:

$$\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = (\lambda + \lambda_1) (\lambda + \lambda_1^*) \left( \lambda + \frac{1}{\lambda_1^*} \right)$$

where \( \lambda_1 \neq 1 \)

(2.105)

\( \alpha_3 = 3 \lambda_1 + \frac{1}{\lambda_1^*} \)

(2.105a)

\( \alpha_2 = 3 (\lambda_1^2 + \frac{1}{\lambda_1^*}) \)

(2.105b)

and \( \alpha_1 = \frac{3}{\lambda_1^*} + \lambda_1^3 \)

(2.105c)

from which \( \frac{\alpha_3}{\alpha_1} = \frac{3 \lambda_1^4 + 1}{(3 + \lambda_1^*) \lambda_1^*} \neq 1 \) (This is an essential indication)

\[ M = \frac{\alpha_3 \alpha_2 - 4}{\alpha_2^2} = \frac{1}{3}, \quad N = \frac{\alpha_1^2 + 4 \alpha_1 - 4 \alpha_2}{\alpha_2^2} = \frac{1}{27} \]

Therefore the logical prediction is correct.

### 24. Double and Single Oscillatory Regions

(A) Double Oscillatory Regions

From the quartic chart (Chart I) it is seen that all curves of \( N < 0 \) give two intersections with a constant \( M \) line which is less than \( M_{\text{max}} \) for that \( N \) curve. By logical reasoning for such condition of \( M \) and \( N \), two ways of factoring may yield two different \( \rho_m \)'s. We know that when \( N < 0 \), the system is oscillatory. At least one of the two components is oscillatory; the other component may or may not be oscillatory. However, the two-intersection behavior will help us to clear up any such uncertainty.

Assume both components to be oscillatory --

$$\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = (\lambda^2 + 2 \tilde{\omega}_m \omega_{m1} \lambda + \omega_{m1}^2) (\lambda^2 + 2 \tilde{\omega}_m \omega_{m2} \lambda + \omega_{m2}^2)$$

(2.106)
where \[ \omega_{nn_1} \omega_{nn_2} = 1 \] (2.106a)
\[ s_1 < 1, \ s_2 < 1 \quad \text{and} \quad s_1 \neq s_2 \] (2.106b)

The right hand side of Eq. (2.106) can also be written as
\[ (\lambda + (l^*_1 + i\sqrt{1-l^2_1})\omega_{nn_1}) \left[ \lambda + (l^*_2 + i\sqrt{1-l^2_2})\omega_{nn_2} \right] \left[ \lambda + (l^*_2 - i\sqrt{1-l^2_2})\omega_{nn_2} \right] \] (2.107)

From expression (2.107) we have three ways to express \( \rho_\omega \):

\[ \rho_{\omega_1} = \frac{\omega_{nn_2}}{\omega_{nn_1}} \] (2.107a)
\[ \rho_{\omega_2} = \left[ \frac{(l^*_1 + i\sqrt{1-l^2_1})(l^*_2 + i\sqrt{1-l^2_2})}{(l^*_1 - i\sqrt{1-l^2_1})(l^*_2 - i\sqrt{1-l^2_2})} \right]^{\frac{1}{2}} \] (2.107b)
\[ \rho_{\omega_3} = \left[ \frac{(l^*_1 + i\sqrt{1-l^2_1})(l^*_2 - i\sqrt{1-l^2_2})}{(l^*_1 - i\sqrt{1-l^2_1})(l^*_2 + i\sqrt{1-l^2_2})} \right]^{\frac{1}{2}} \] (2.107c)

From the above expressions of \( \rho_\omega \), expressions of \( \rho_m \) can be deduced:
\[ \rho_{m_1} = \frac{\omega_{nn_2}}{\omega_{nn_1}} + \frac{\omega_{nn_1}}{\omega_{nn_2}} \quad \text{real,} \quad > 2 \] (2.108a)
\[ \rho_{m_2} = 2 \left[ l, l_2 - \sqrt{(1-l^2_1)(1-l^2_2)} \right] \quad \text{real,} \quad < 2 \quad \forall \lambda \geq 0 \] (2.108b)
\[ \rho_{m_3} = 2 \left[ l, l_2 + \sqrt{(1-l^2_1)(1-l^2_2)} \right] \quad \text{real,} \quad < 2 \quad \forall \lambda > 0 \] (2.108c)

It is necessary to prove \( \frac{\rho_{m_2}}{2} \) or \( \frac{\rho_{m_3}}{2} < 1 \).

Let us assume
\[ l, l_2 \neq \sqrt{(1-l^2_1)(1-l^2_2)} \quad < 1 \] (2.109)

add \(-\frac{l, l_2}{2}\) to both sides of the inequality and square both sides
\[ 1 - \frac{l^2_1 - l^2_2 + 1}{2} l, l_2 \quad < 1 - 2 \frac{l, l_2}{2} + \frac{l^2_1 - l^2_2}{2} \]

Canceling and transferring the terms we have
Equation (2.109)a is a true fact except \( \xi_1 = \xi_2 \) which has been excluded.

\( \rho_{m_3} \) is always positive because \( \xi_1 \) and \( \xi_2 \) are both less than unity. \( \rho_{m_2} \) may be positive, negative, or zero. The condition for the transition of \( \rho_{m_2} \) from negative to positive can be found in the following treatment:

\[
\begin{align*}
\xi_1, \xi_2 - \sqrt{(1-\xi_1^2)(1-\xi_2^2)} &> 0 \\
(1-\xi_1^2)(1-\xi_2^2) &> \xi_1^2 \xi_2^2 \\
1 - \xi_1^2 - \xi_2^2 &> 0 \\
\xi_2 &< \sqrt{1-\xi_1^2} \\
\text{or} & \\
\xi_1 &< \sqrt{1-\xi_2^2}
\end{align*}
\]  

(2.110)

With the help of Chart I and the above analysis, it can be seen that

(a) if \( \xi_1 \neq \xi_2 \), \( \xi_1 < 1 \), \( \xi_2 < 1 \) and \( \xi_{2,1} > \sqrt{1-\xi_1^2} \), only two intersections at positive \( \rho \) can be obtained from the stability criteria \( M \) and \( N \) of such a system. All the curves for \( N < 0 \) satisfy this condition, therefore the region below \( N = 0 \) is a double oscillatory region restricted to the condition \( \xi_{2,1} > \sqrt{1-\xi_{1,2}^2} \)

(b) If \( \xi_1 \neq \xi_2 \), \( \xi_1 < 1 \), \( \xi_2 < 1 \) and \( \xi_{2,1} < \sqrt{1-\xi_{1,2}^2} \), three intersections at positive \( \rho \) should be observed from the stability criteria \( M \) and \( N \) of such a system. The shaded region of Fig. 2, which so far has been claimed to be nonoscillatory, satisfies this condition. Therefore this shaded region serves
twofold to indicate (1) nonoscillatory, and (2) double oscillatory, restricted to the condition \( s_{2,1} < \sqrt{1 - s_{1,2}^2} \). The decision between the two states of motion merely depends upon the magnitude of the middle constant \( \alpha_2 \) of the quartic equation.

From the above analysis it is understood that if two or three intersections are observed for certain pairs of \( M \) and \( N \) of a quartic equation, the rightmost intersection will always give results according to physical ways of decomponentization (that is, give real \( \rho_\omega \) and real \( s_r \) etc.). The other intersection or intersections may or may not lead to complex \( \rho_\omega \), \( s_r \) etc. according to whether this system is doubly oscillatory or nonoscillatory.

(B) Single Oscillatory Region

Going back to Eq. (2.146) and letting one of the \( \xi \)'s be greater than one, the other less than one, (for convenience let \( s_2 > 1, \quad s_1 < 1 \)), the four factors of equation (2.106) become:

\[
[\lambda + (s_1 + i \sqrt{1 - s_1^2}) \omega_{nn1}] [\lambda + (s_1 - i \sqrt{1 - s_1^2}) \omega_{nn1}] [\lambda + (s_2 + i \sqrt{1 - s_2^2}) \omega_{nn2}] [\lambda + (s_2 - i \sqrt{1 - s_2^2}) \omega_{nn2}] \tag{2.111}
\]

For expression (2.111) we also have three ways in which to express \( \rho_\omega \):

\[
\rho_\omega = \frac{\omega_{nn2}}{\omega_{nn1}} \tag{2.111a}
\]

\[
\rho_{\omega_2} = \frac{(s_2 + i \sqrt{1 - s_2^2}) (s_2 + i \sqrt{1 - s_2^2})}{(s_1 - i \sqrt{1 - s_1^2}) (s_2 - i \sqrt{1 - s_2^2})} \frac{1}{2} \tag{2.111b}
\]

\[
\rho_{\omega_3} = \frac{(s_2 + i \sqrt{1 - s_2^2}) (s_2 - i \sqrt{1 - s_2^2})}{(s_1 - i \sqrt{1 - s_1^2}) (s_2 + i \sqrt{1 - s_2^2})} \frac{1}{2} \tag{2.111c}
\]
From the above expressions for $p_w$, expressions for $p_m$ can be deduced:

\[ p_m = \frac{\omega_{n_2}}{\omega_{n_1}} + \frac{\omega_{n_1}}{\omega_{n_2}} \]  

(2.112)a

\[ p_{m_2} = 2 \left( \xi_1 \xi_2 - i \sqrt{(1 - \xi_1^2)(\xi_2^2 - i)} \right) \]  

(2.112)b

\[ p_{m_3} = 2 \left( \xi_1 \xi_2 + i \sqrt{(1 - \xi_1^2)(\xi_2^2 - 1)} \right) \]  

(2.112)c

The presence of $i$ in the expression of $p_m$ can be interpreted as meaning that no more real intersections can be observed for the $M-N$ pair obtained from a single oscillatory system other than the one which possesses physical significance; that is, giving solution to real $p_w$, real $\xi$, etc. Accordingly the region above $N = 0$ and outside the nonoscillatory (or double oscillatory) region satisfies the above condition. Therefore, the region above $N = 0$ and outside the nonoscillatory region is defined as the single oscillatory region.

25. Summary of Stability Analysis

The following table (Table III) may serve as a good summary of stability analysis for the quartic equation.
| \( N > M \) | Unstable |
| N = M | Unending oscillation |
| N = M = -1 | Unending oscillation of both components |
| N < M | Outside region BABC |
| Stable | Along boundary BABC |
| \( N = \frac{2 \tilde{f}^2 - 1}{(1 + 2 \tilde{f}^2)^3} \) | At least one component is oscillatory |
| \( M = \frac{4 \tilde{f}^2 - 1}{(1 + 2 \tilde{f}^2)^2} \) | Two equal quadratic factors or one critically damped quadratic factor accompanying another of any damping ratio |
| \( \tilde{f} \) may be any real number |
| \( N > 0 \) | One component only |
| \( N < 0 \) | One or both components |
| \( N = 0 \) | Nonphysical existence |
| \( N < 0 \) | Both components oscillatory |

| Stable | Along boundary BABC |
| Stable | Along AB |
| \( \alpha_3 = \alpha_1 \) | Two pair of equal roots |
| Two equally overdamped quadratics |
| \( \alpha_3 \neq \alpha_1 \) | Two equal roots. Another component may be over-damped. |
| Along AC |
| \( \alpha_3 = \alpha_1 \) | Two equal conjugate-paired roots |
| Also |
| \( \alpha_3 \neq \alpha_1 \) | Two equal roots. Another component overdamped |
| At Vertex |
| \( \alpha_3 = \alpha_1 \) | Four equal roots |
| \( \alpha_3 \neq \alpha_1 \) | Three equal roots |
| Along CB |
| \( \tilde{f}_1 = \tilde{f}_2 = 0.707 \) at \( \tilde{f}_2 > 1 \) or \( \tilde{f}_3 = 1 \) |
| \( N = 27 \) |
| \( M = \frac{1}{3} \) |
| Inside region ABC |
| Four distinct real roots or both components oscillatory |
| limited by \( \tilde{f}_2 < \sqrt{1 - \tilde{f}^2} \) |
| \( \alpha_3 > \alpha_1 \) | High frequency component has greater damping ratio |
| \( \alpha_3 < \alpha_1 \) | High frequency component has lesser damping ratio |
| \( \alpha_3 = \alpha_1 \) | Two components of same frequency, but with different damping |
| Two components of different frequency, but with same damping |
| or two components of same frequency and same damping |
PART III

STABILITY IMPROVEMENT WITH DIFFERENT CONTROLS
CHAPTER SEVEN

STABILITY TRANSITION CURVE WITH DIFFERENT COUPLING COEFFICIENTS

26. Coupling Factor and Coupling Coefficient

In reviewing Section 8, Chapter Three, the coupling factors of a control system are recollected. The coefficients of the quartic equation of the controlled motion can be broken into two parts; one is due to the idly dynamic combination of the control and controlled member, and the other due to the coupling effect.

The idly dynamic combination may be defined as one in which the coupling factors of the system are all zero (physically the control is locked) and the coefficients of the quartic behave in such a way that the quartic equation may be resolved into two factors, one identified as the identical characteristic of the member to be controlled and the other the identical characteristic of the control.

In symbols, Eq. (2.13)c -- or (2.13)d -- can be written as follows:

\[ A_3^1 = A_{30}^1 \]
\[ A_2^1 = A_{20}^1 + B_2^1 \]
\[ A_1^1 = A_{10}^1 + B_1^1 \]
\[ A_0^1 = A_{00}^1 + B_0^1 \] (3.01)
where \[ A'_{30} = \frac{c_c}{m_c} + \frac{c_e}{I} = 2 \frac{c_c}{m_c} \omega_{nc} + 2 \frac{c_e}{I} \omega_{no} \]

\[ A'_{20} = \frac{k_c}{m_c} + \frac{k_e}{I} + \frac{c_c}{m_c} \omega_{nc}^2 + \omega_{no}^2 + 4 \frac{c_e}{I} \omega_{nc} \omega_{no} \omega_{nc} \]

\[ A'_{10} = \frac{c_c}{m_c} \omega_{nc} + \frac{c_e}{I} \omega_{no}^2 + 2 \frac{c_c}{m_c} \omega_{nc} \omega_{no} \omega_{nc} \]

and \[ A'_{oo} = \frac{k_e}{I} \omega_{no}^2 = \omega_{no}^2 \omega_{nc}^2 \] (3.01a)

are defined as idle coefficients.

Eq. (3.01) can be written in the following form:

\[ A'_{1} = A'_{30} \]

\[ A'_{2} = A'_{20} (1 + \gamma_2) \]

\[ A'_{1} = A'_{10} (1 + \gamma_1) \]

\[ A'_{0} = A'_{oo} (1 + \gamma_o) \] (3.02)

where \[ \gamma_2 = \frac{B'_{1}}{A'_{20}} \]

\[ \gamma_1 = \frac{B'_{1}}{A'_{10}} \]

\[ \gamma_o = \frac{B'_{1}}{A'_{oo}} \]

defined as second derivative coupling coefficient

defined as first derivative coupling coefficient

defined as error sensitive coupling coefficient

The stability function \( S(D) \) in the form of Eq. (2.14) can be expressed in the following way:

\[ S(D) = D^4 + A'_{30} D^3 + A'_{20} (1 + \gamma_2) D^2 + A'_{10} (1 + \gamma_1) D + A'_{oo} (1 + \gamma_o) \] (3.03)

By introducing

\[ D = A'_{oo} \lambda \] (3.04)

We may write the stability function in nondimensional form:

\[ \psi_o (\lambda) = \frac{S(D)}{A'_{oo}} = \lambda^4 + \alpha_{30} \lambda^3 + \alpha_{20} (1 + \gamma_2) \lambda^2 + \alpha_{10} (1 + \gamma_1) \lambda + \alpha_{oo} (1 + \gamma_o) \] (3.05)

where \[ \alpha_{30} = \frac{A'_{20}}{A'_{oo}^{1/3}} = 2 \frac{c_c}{m_c} \left( \frac{1}{\sqrt{\rho_{wo}} + p \sqrt{\rho_{wo}}} \right) \] (3.06)

\[ \alpha_{20} = \frac{A'_{20}}{A'_{oo}^{1/2}} = \rho_{wo} + \frac{1}{\rho_{wo}} + 4 \frac{c_e}{I} \rho_{wo} \] (3.07)
\[
\alpha_{1o} = \frac{A_{1o}}{A_{1o}^{0.64}} = 2 \left( \frac{\rho_{o}}{\sqrt{\rho_{o} \omega_{o}}} + \sqrt{\rho_{o} \omega_{o}} \right) \tag{3.08}
\]

\[
\alpha_{00} = \frac{A_{3o}^{0}}{A_{3o}^{0 \circ}} = 1 \tag{3.09}
\]

with \( \rho_{o} = \frac{\omega_{nc}}{\omega_{no}} \) Undamped natural frequency of control \( \tag{3.10} \)

\( \omega_{nc} \) Undamped natural frequency of member to be controlled

\[
\rho_{c} = \frac{f_{c}}{f_{o}} = \frac{\text{Damping ratio of the control}}{\text{Damping ratio of the member to be controlled}} \tag{3.11}
\]

and \( \alpha_{3o}, \alpha_{2o}, \alpha_{1o}, \text{ and } \alpha_{00} \) are defined as nondimensional idle coefficients.

27. Effect of Coupling Coefficients on Stability

Transition Curve

It is understood that a mechanical system of one degree of freedom cannot be unstable. When such member is controlled by a mechanical non-ideal control at idle condition, the resultant quartic equation must show the stable behavior. Graphically, the point \( \alpha_{2o} \) vs. \( \frac{\alpha_{3o}}{\alpha_{1o}} \) must lie above the stability transition curve.

\[
\alpha_{2o} = \frac{\alpha_{3o}}{\alpha_{1o}} + \frac{\alpha_{1o}}{\alpha_{3o}} \tag{3.12}
\]

However, in a two-degree-of-freedom system such as the uncontrolled longitudinal stability of an airplane, it may be stable or unstable; that is \( \alpha_{2o} \) vs. \( \frac{\alpha_{3o}}{\alpha_{1o}} \) of such a system may lie above or below the transition curve.

When the control is put into action, the same point \( \alpha_{2o} \) vs. \( \frac{\alpha_{3o}}{\alpha_{1o}} \) will change its relative position with the new
transition curve which is affected by the coupling coefficient. The effect of different coupling coefficients can be analyzed separately and their resultant effect can be easily interpreted.

(A) Effect of Error Sensitive Coupling Coefficient

With only error sensitive coupling coefficient, Eq. (3.05) appears in the following form:

\[\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + (1 + \gamma_0) = 0\]  \hspace{1cm} (3.13)

which can be transformed into:

\[\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0\]  \hspace{1cm} (3.14)

where

\[\alpha_3 = \frac{\alpha_3 \alpha}{(1 + \gamma_0)^{1/4}}, \quad \alpha_2 = \frac{\alpha_2 \alpha}{(1 + \gamma_0)^{1/2}},\]

\[\alpha_1 = \frac{\alpha_1 \alpha}{(1 + \gamma_0)^{3/4}} \quad \text{and} \quad \lambda = (1 + \gamma_0)^{1/4} \lambda'\]  \hspace{1cm} (3.14)\text{a}

The new stability transition curve evidently is

\[\alpha_2 = \frac{\alpha_3 \alpha}{\alpha_1} + \frac{\alpha_1 \alpha}{\alpha_3} \quad \text{or} \quad \alpha_2 = \frac{\alpha_3 \alpha}{\alpha_1 (1 + \gamma_0)} + \frac{\alpha_1 \alpha}{\alpha_3}\]  \hspace{1cm} (3.15)

or in symmetric form

\[\frac{\alpha_2}{(1 + \gamma_0)^{1/2}} = \frac{\alpha_3 \alpha (1 + \gamma_0)^{1/2}}{\alpha_1 \alpha} + \frac{\alpha_1 \alpha (1 + \gamma_0)^{1/2}}{\alpha_3}\]  \hspace{1cm} (3.15)\text{a}

Eq. (3.15) is plotted as Fig. 3 with \(\alpha_2\) as ordinate against \(\alpha_3\) as abscissa with \(\gamma_0\) as varying parameter.

In the high performance control (which will be discussed later in Chapter Eight) the control frequency is usually higher than that of the controlled member, and the damping ratio of control is usually higher than that of the controlled member. Symbolically the high performance controlled system possesses
FIG. III

EFFECT OF ERROR SENSITIVE CONTROL ON STABILITY TRANSITION CURVE:

\[ \frac{\alpha_2}{\alpha_{10}} = \frac{\alpha_3}{\alpha_{10}}(1 + \gamma) + \frac{\alpha_{10}}{\alpha_{30}} \]

FOR EQUATION

\[ \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + (1 + \gamma) = 0 \]
the two conditions \( p_{o} \gg 1 \) and \( f_{o} > 1 \). With control at idle condition, the point \( \alpha_{20} \) \( \times \frac{\alpha_{20}}{\alpha_{10}} \) is slightly above the transition curve of \( \gamma_{o} = 0 \) at some values of \( \frac{\alpha_{20}}{\alpha_{10}} > 1 \) (because \( f_{o} > 1 \)). It is therefore seen that with a positive coupling coefficient, the system will soon reach unending oscillation when the transition curve, raised by the coupling coefficient, passes through the point \( \alpha_{20} \times \frac{\alpha_{20}}{\alpha_{10}} \). This sets a limit to \( \gamma_{o} \) beyond which the system will be unstable. However, if negative coupling coefficient is used, there is no such limit, and the system is always stable until \( \gamma_{o} = -1 \) when the quartic equation is reduced to cubic one with one root equal to zero. At \( \gamma_{o} = -1 \) the transition curve becomes a straight line* \( \alpha_{20} = \frac{\alpha_{10}}{\alpha_{20}} \) in a log-log plot and there the meaning of high frequency or low frequency becomes obscure.

It is also interesting to notice the shifting of the vertices of the stability transition curves along a straight line \( \alpha_{20} = 2 \times \frac{\alpha_{10}}{\alpha_{30}} \) in the log-log plot. It is therefore possible to make the high frequency component possess the smaller damping ratio by introducing negative \( \gamma_{o} \) (to make the specified point \( \alpha_{20} \times \frac{\alpha_{20}}{\alpha_{10}} \) appear above the left branch of the transition curve). For a good follow-up control, the least requirement is to have steady state reading equal to quantity which is to be followed. Going back to Eq. (2.13)b the above requirement cannot be fulfilled unless \( B_{o} = A_{o} \).

*Compare this with the stability criteria chart of the cubic equation in Appendix A.
EFFECT OF FIRST-TIME-DERIVATIVE OF-ERROR CONTROL ON STABILITY TRANSITION CURVE:

\[ \alpha_2 = \frac{\alpha_3}{\alpha_0 (1+\gamma)} + \frac{\alpha_4 (1+\gamma)}{\alpha_3} \]

FOR EQUATION

\[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 (1+\gamma) \lambda + 1 = 0 \]

Region above each particular curve is stable for that particular value of \( \gamma \), that below the curve unstable. Along the curve the resultant motion is in a state of unending oscillation. Transition of stability along the curve and to the left of its vertex is referred to the component of higher frequency & that to its right the low one.
But from Eq. (2.13)c it is seen that \( A'_o = \frac{kek_c}{Im} + B'_o \), so \( A'_o \) cannot be equal to \( B'_o \) unless \( k_e \) (or \( k_o \)) is zero. Approximate truth can be reached if \( k_e \) is very small, so that \( \frac{kek_c}{Im} \) is negligible to \( B'_o \). In such type control it is necessary to use large positive \( \gamma \) to satisfy the requirement of true steady state reading. However, as positive \( \gamma \) easily leads the system into an unstable condition, (such as is shown in Fig. 3), means of improving damping is therefore of equal importance after the introduction of overcontrolled positive \( \gamma \).

If the system is not aimed at the following-up characteristic like the controlled longitudinal motion of the airplane, only moderate positive \( \gamma \) may be needed.

(B) Effect of First Derivative Coupling Coefficient

With only error velocity coupling, Eq. (3.05) appears in the following form:

\[
\lambda^4 + \alpha_{30} \lambda^3 + \alpha_{20} \lambda^2 + \alpha_{10} (1 + \gamma_i) \lambda + 1 = 0
\]  

of which the stability transition curve becomes

\[
\alpha_{20} = \frac{\alpha_{30}}{\alpha_{10} (1 + \gamma_i)} + \frac{\alpha_{10} (1 + \gamma_i)}{\alpha_{30}}
\]  

Eq. (3.17) is plotted as Fig. 4 with \( \alpha_{20} \) as ordinate against \( \frac{\alpha_{30}}{\alpha_{10}} \) as abscissa with \( \gamma_i \) as varying parameter. Positive \( \gamma_i \) shifts the curve to the right of the original one while negative \( \gamma_i \) moves it to the left.

With particular control and controlled member the point \( \alpha_{20} \) vs. \( \frac{\alpha_{30}}{\alpha_{10}} \) may be located slightly above the right branch of \( \gamma_i = 0 \). Too big a positive \( \gamma_i \) will lead the system to instability of which the high frequency component will first
undergo unending oscillation. A slight negative \( \gamma \), will also lead the system to instability of which the low frequency component will first pass through a stable oscillation to an unstable one.

Introducing positive \( \gamma \), decreases the ratio \( \frac{\alpha_{30}}{\alpha_{10}(1+\gamma)} \) which adjusts the distribution of damping ratio between the fast and the slow components. In fact, the damping ratio of the controlled member (low frequency component) is noticeably improved with a slight positive \( \gamma \), while the damping ratio of the control (high frequency component) is only slightly decreased. It is possible to adjust the coupling coefficient so that \( \frac{\alpha_{30}}{\alpha_{10}(1+\gamma)} = 1 \). In most cases of high performance control, such a condition gives two components of motion with the same frequency, but with different damping ratio. When \( \gamma \) exceeds such a limit, the high frequency component will have less damping ratio than the low frequency component.

Physically, the change in damping ratio of either component should be a continuous variation. However, Maxwell^{36} finds that there abrupt change of damping distribution when coupling coefficient (defined in a different way with the symbols used in this thesis) reaches a certain value. The abrupt change is only due to the exchange of title of the high and the low frequency components. Detailed application on the condition \( \frac{\alpha_{30}}{\alpha_{10}(1+\gamma)} = 1 \) will be developed into another chapter.*

*Chapter Ten: Tuning Control System.
FIG. V

EFFECT OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL ON STABILITY TRANSITION CURVE:

\[ \alpha_{20}(1+\gamma_2) = \frac{\alpha_3}{\alpha_{10}} + \frac{\alpha_2}{\alpha_{30}} \]

FOR EQUATION

\[ \lambda^4 + \alpha_3\lambda^3 + \alpha_2(1+\gamma_2)\lambda^2 + \alpha_1\lambda + 1 = 0 \]

Region above curve of a particular value of \( \gamma_2 \) is stable, that below unstable.
Along the curve the resultant motion is in a state of unending oscillation.

Transition of stability along the curve and to the left of its vertex is referred to the component of higher frequency and to its right the low one.
(C) Effect of Second Derivative Coupling

With only error-acceleration coupling, Eq. (3.05) appears in the following form:

$$\lambda^4 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_1 \lambda + 1 = 0 \quad (3.18)$$

of which the stability transition curve becomes

$$\alpha_2 \lambda + \gamma_2 = \frac{\alpha_3}{\alpha_{10}} + \frac{\alpha_1}{\alpha_{30}} \quad (3.19)$$

Eq. (3.19) is plotted as Fig. 5 with $\alpha_2$ as ordinate against $\frac{\alpha_3}{\alpha_{10}}$ as abscissa with $\gamma_2$ as varying parameter. The transition curve is shifted upward with negative coupling coefficient; that means that if overcontrolled with negative coupling, the system might be led into instability. However, no such instability would occur if positive coupling is used.

(D) Effect of Higher Derivative Coupling

As detective instruments are limited to error sensitive, velocity sensitive, and acceleration sensitive types, higher derivative instruments are not yet available. Therefore, no complication is needed to explore their effect upon the transition curve. Moreover, the combination of $\gamma_0$, $\gamma_1$, and $\gamma_2$ is widely open to yield desirable results. Therefore the analysis of transition of stability is confined to the three types of coupling.
28. High Performance Controls and Definition of Advantages

It is desirable to have least control lag in a follow-up control. Controls with high natural frequency can achieve this object with ideal control. A simple figure of merit can be derived.

With nonideal control, it is impossible to use a single expression to sum up all the relative merits. However, to simplify the effort of design the following definitions and notations are introduced in a tabular form.

(See Table IV next page)
### TABLE IV

<table>
<thead>
<tr>
<th>Results with Control in idle</th>
<th>Results with control in action</th>
<th>Ratio and Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{s0}$</td>
<td>$d_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{s0}$</td>
<td>$d_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{s0}$</td>
<td>$d_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_0^*$</td>
<td>$w_1 = (w_0)$</td>
<td>$w_1/w_0 = \eta_w$</td>
<td>Advantage of undamped natural frequency</td>
</tr>
<tr>
<td>$w_c$</td>
<td>$w_2$</td>
<td>$w_2/w_c = \eta_w$</td>
<td>Advantage of undamped natural frequency (control component)</td>
</tr>
<tr>
<td>$\rho_w = \frac{w_c}{w_0}$</td>
<td>$\rho_w = \frac{w_2}{w_1}$</td>
<td>$\rho_w/\rho_w = \eta_{\rho_w}$</td>
<td>Advantage of ratio of undamped natural frequencies</td>
</tr>
<tr>
<td>$\delta_0$</td>
<td>$\delta_1 = (\delta_0)$</td>
<td>$\delta_1/\delta_0 = \eta_{\delta}$</td>
<td>Advantage of damping ratio</td>
</tr>
<tr>
<td>$\delta_c$</td>
<td>$\delta_2$</td>
<td>$\delta_2/\delta_c = \eta_{\delta}$</td>
<td>Advantage of damping ratio (control component)</td>
</tr>
<tr>
<td>$\rho_\delta = \frac{\delta_c}{\delta_0}$</td>
<td>$\rho_\delta = \frac{\delta_2}{\delta_0}$</td>
<td>$\rho_\delta/\rho_\delta = \eta_{\rho_\delta}$</td>
<td>Advantage of ratio of damping ratios</td>
</tr>
<tr>
<td>$w_0\delta_0$</td>
<td>$w_1\delta_1$</td>
<td>$w_1\delta_1/\delta_0 = \eta_{w_0\delta_0}$</td>
<td>Advantage of damping</td>
</tr>
<tr>
<td>$w_c\delta_c$</td>
<td>$w_2\delta_2$</td>
<td>$w_2\delta_2/\delta_c = \eta_{w_c\delta_c}$</td>
<td>Advantage of damping (control component)</td>
</tr>
</tbody>
</table>

* All $w$'s in this table and henceforth in Chapters 8, 9, and 10 are referred to undamped natural angular frequency.
It should be noted here that although every ratio referred to control at idle condition is defined as advantage, it is merely a mathematical symbol because a control cannot be expected to influence every quantity in the advantageous sense. Some of the "advantages" may actually work on the disadvantageous side. The designer has to use his own judgment to make a satisfactory compromise.

29. Error Sensitive Control with High Natural Frequency

Going back to equation (3.14) and (3.14)a and with substitution of Eqs. (2.25) to (2.27) and (3.06) to (3.08) we have

\[ 25. \left( \frac{1}{\rho_w} + \frac{b}{\sqrt{\rho_w}} \right) = \frac{25.0 \left( \frac{1}{\sqrt{\rho_{wo}} + \frac{b}{\sqrt{\rho_{wo}}} \right)}{(1 + \gamma_o)^{3/4}} \]  

\[ \frac{\rho_w + \frac{1}{\rho_w} + 4 \frac{b}{\rho_w} + \frac{b^2}{\rho_w}}{(1 + \gamma_o)^{3/2}} = \frac{\rho_{wo} + \frac{1}{\rho_{wo}} + 4 \frac{b}{\rho_{wo}} + \frac{b^2}{\rho_{wo}}}{(1 + \gamma_o)^{3/2}} \]  

\[ 25. \left( \frac{\rho_w}{\sqrt{\rho_w}} \right) = \frac{25.0 \left( \frac{\rho_{wo}}{\sqrt{\rho_{wo}}} \right)}{(1 + \gamma_o)^{3/4}} \]  

\[ \rho_w^2 \omega_{1}^{4} = \rho_{wo}^2 \omega_{o}^{4} (1 + \gamma_o) \]  

If \( \rho_w >> 1 \), approximation can be safely made to obtain those advantages as defined in Table IV in terms of control specification and coupling coefficient explicitly.

From Eq. (3.21) it is seen that \( \frac{1}{\rho_w} \) and \( \frac{1}{\rho_{wo}} \) are definitely negligible when \( \rho_w >> 1 \). Since \( \frac{b^2}{\rho_{wo}} = \frac{b^2}{\rho_{c}} \) and in general \( \rho_{c} \) is small, (may be in the order of .1), \( \rho_{c} \) is in the order of 1.0, so \( 4 \frac{b^2}{\rho_{wo}} \) is also negligible to \( \rho_{wo} \). It
will be seen soon that the product \( \frac{5}{5} \cdot \frac{5}{5} \) or \( \frac{5}{5} \cdot \frac{5}{5} \) is of the
same magnitude as \( \frac{5}{5} \cdot \frac{5}{5} \). So \( \frac{4}{4} \cdot \frac{5}{5} \) is also negligible to
\( \rho_w \). Therefore Eq. (3.21) is simplified to

\[
\eta \rho_w = \frac{\rho_w}{\rho_w} = \frac{1}{(1 + \gamma_0)^{1/2}} \tag{3.24}
\]

From Eq. (3.23) expression of \( \eta_w \) can be obtained as

\[
\eta_w = \frac{\omega_1}{\omega_0} = \left( \frac{\rho_w}{\rho_w} \right)^{1/2} \left( 1 + \gamma_0 \right)^{1/4} = \left( 1 + \gamma_0 \right)^{1/2} \tag{3.25}
\]

In Eq. (3.20) it is safe to neglect \( \frac{1}{\sqrt{\rho_w}} \) against \( \rho_w \) and \( \sqrt{\rho_w} \) against
\( \rho_w \sqrt{\rho_w} \) because \( \rho_w, \rho_w \gg 1 \) and \( \rho_w, \rho_w \) also \( \gg 1 \) in
general. Therefore we get

\[
\frac{5}{5} \left( 1 + \gamma_0 \right)^{1/4} = \frac{\rho_w \sqrt{\rho_w}}{\rho_w} \tag{3.26a}
\]

Substituting Eq. (3.24) into Eq. (3.26a) we get

\[
\frac{5}{5} = \frac{\rho_w}{\rho_w} \tag{3.26b}
\]

or

\[
\eta \frac{5}{5} = \frac{5}{5} \frac{\rho_w}{\rho_w} = 1 \tag{3.26}
\]

which means the damping ratio of the control component is not
(essentially) changed.

Equation (3.26)b can also be written in the following
form

\[
\rho_w = \frac{\rho_w}{\rho_w} \tag{3.26c}
\]

Substitute the value of \( \rho_w \) into Eq. (3.22) and leave every
term in because \( \rho_w \) is not negligible to \( \sqrt{\rho_w} \) and \( \rho_w \) is
not negligible to \( \sqrt{\rho_w} \). We have

\[
\eta \frac{5}{5} = \frac{\sqrt{\rho_w} + \rho_w}{(\gamma_0)^{1/4} \left( \sqrt{\rho_w} + \frac{\rho_w}{\rho_w} \right)} \tag{3.27a}
\]
Multiply both the numerator and denominator by \( \frac{1}{\sqrt{\rho_{wo}}} \) and substitute \( \rho_{wa} = \rho_{wo} \frac{\rho}{\rho_{wo}} \), Eq. (3.27)a appears in the following form:

\[
\eta_{5} = \frac{1}{(1 + r_{o})^{1/2}} \left[ 1 + \frac{\rho_{60}}{\rho_{wo}} (1 + r_{o})^{1/2} \right]^{1/2} \]

(3.27)b

From Eq. (3.27)b can be solved thus

\[
\eta_{5} = \frac{1}{(1 + r_{o})^{1/2}} \left[ 1 - \frac{r_{0}}{\rho_{wo}/\rho_{60}} \right] \]

(3.27)

Eq. (3.27) is rather interesting in which the advantage of damping ratio is a function of coupling coefficient \( r_{o} \) and the control specification \( \rho_{wo} \) which relates the ratio of natural frequencies with the ratio of damping ratios between the controlled member and the control in a form of simple ratio. Multiply Eq. (3.27) by Eq. (3.25). \( \eta_{5} \omega \) is obtained.

\[
\eta_{5} \omega = 1 - \frac{r_{0}}{\rho_{wo}/\rho_{60}} \]

(3.28)

From the expressions for \( \eta_{5} \) and \( \eta_{5} \omega \) it is seen that when \( r_{o} = \frac{\rho_{wo}}{\rho_{60}} \) (positive). Both \( \eta_{5} \) and \( \eta_{5} \omega \) become zero which means that at that value of \( r_{o} \) the system becomes unendingly oscillatory and the unendingly oscillatory component is the low frequency component. But when \( r_{o} \) is negative, but less than 1.0, the system can never be unstable. When \( r_{o} \) slightly exceeds -1.0, \( \eta_{5} \) becomes negative while \( \eta_{5} \omega \) is still positive. This contradiction should be considered as a result of negligence of certain terms during the course of derivation unjustifiable at the region where \( r_{o} \) exceeds -1.0. Fortunately too great a coupling coefficient is not used in practice.

By definition, the following advantages can be found:
FIG VI  CHARACTERISTICS OF ERROR SENSITIVE CONTROL AT LARGE $\rho_\omega$ S
\[ \eta'_{\omega} = \frac{\omega^2}{\omega_{c}} = \frac{P_{o}}{P_{s}} \omega = \frac{1}{(1 + \gamma_o)^{\nu}} (1 + \gamma_o)^{\nu} = 1 \]  
(3.29)

\[ \eta'_{s\omega} = \eta'_{s} \eta'_{\omega} = 1 \]  
(3.30)

\[ \eta_{p} = \frac{s_{c}}{s_{o}} = \frac{\eta_{s}}{\eta_{o}} = \frac{1}{1 - \frac{\gamma_o}{\gamma_{p}}} \]  
(3.40)

Eqs. (3.24), (3.25), (3.27), (3.40) and (3.28) are plotted as Fig. 6 with \( \eta_{s}, \eta'_{s\omega}, \eta_{\omega}, \eta_{p}, \eta_{p_{s}} \) as ordinates against the coupling coefficient \( \gamma_o \) as abscissa with the control specification \( \frac{P_{o}}{P_{s}} \) as varying parameter.

All other quantities -- \( \eta'_{s}, \eta_{\omega}, \eta'_{s\omega} \) -- which are approximately equal to unity are not plotted.

With Fig. 6 the stability improvement of a controlled system by a particular control with particular coupling coefficient can be picked up without any effort. When the member to be controlled possesses excess damping ratio, a control with high frequency of the error sensitive type will be satisfactory with positive coupling coefficient so long as the damping is concerned.

In case the member to be controlled possesses sufficient stiffness, but not sufficient damping, the error sensitive control will be satisfactory with slightly negative coupling coefficient.

However, the error sensitive control is primarily designed to supply the azimuthal or following-up characteristic (positively coupled). The real advantage is to increase the natural frequency of the controlled member. However, the damping is inherently spoiled. Therefore, damping improving coupling
is of great necessity to compensate the spoiled damping.

30. Error-Acceleration Control with High Natural Frequency

Because of simplicity in analysis of this type of control, it is taken up ahead of the error-velocity type.

In reviewing Eqs. (3.18), (3.06) to (3.08) and (2.25) to (2.27), the following relations are obtained for the error-acceleration control.

\[ \alpha_3 = \alpha_{30} \]
\[ \alpha_2 = \alpha_{20} (1 + \gamma_2) \]
\[ \alpha_1 = \alpha_{10} \]

or

\[ 2 \phi_1 \left( \frac{1}{\rho_{\omega}} + \frac{1}{\rho_{\omega}^0} \right) = 2 \phi_0 \left( \frac{1}{\rho_{\omega}^0} + \frac{\rho_{\omega}}{\sqrt{\rho_{\omega}^0 \rho_{\omega}}} \right) \quad (3.41) \]

\[ \rho_{\omega} + \frac{1}{\rho_{\omega}} + 4 \phi_1^2 \rho_{\omega}^1 = (\rho_{\omega}^0 + \frac{1}{\rho_{\omega}^0} + 4 \phi_0^2 \rho_{\omega}^0) (1 + \gamma_2) \quad (3.42) \]

\[ 2 \phi_1 \left( \sqrt{\rho_{\omega}} + \frac{\rho_{\omega}}{\sqrt{\rho_{\omega}^0}} \right) = 2 \phi_0 \left( \sqrt{\rho_{\omega}^0} + \frac{\rho_{\omega}^0}{\sqrt{\rho_{\omega}^0}} \right) \quad (3.43) \]

Eq. (3.42) can be simplified, if \( \rho_{\omega} \gg 1 \), with neglect of \( \frac{1}{\rho_{\omega}} \), \( 4 \phi_1^2 \rho_{\omega}^1 \), \( \frac{1}{\rho_{\omega}^0} \) and \( 4 \phi_0^2 \rho_{\omega}^0 \) so that

\[ \eta_{\rho\omega} = \frac{\rho_{\omega}}{\rho_{\omega}^0} = 1 + \gamma_2 \quad (3.44) \]

\[ \frac{\omega_{\rho\omega}^2}{\omega_0^2} \frac{\rho_{\omega}^0}{\rho_{\omega}} = 1 \]

\[ \eta_\omega = \frac{\omega_{\rho\omega}^2}{\omega_0^2} \frac{\rho_{\omega}^0}{\rho_{\omega}} = \frac{1}{(1 + \gamma_2)^{1/2}} \quad (3.45) \]

By neglecting \( \frac{1}{\sqrt{\rho_{\omega}^0}} \) against \( \rho_{\omega}^0 \sqrt{\rho_{\omega}} \) and \( \frac{1}{\sqrt{\rho_{\omega}^0}} \) against \( \rho_{\omega}^0 \sqrt{\rho_{\omega}} \), Eq. (3.41) can be put into the following form:
\[ \eta_s = \frac{\xi}{\xi_0} = \eta_\omega \frac{\rho_\omega}{\rho_f} \quad (3.47) a \]

or
\[ \rho_s = \frac{\eta_\omega \rho_\omega}{\eta_s} \quad (3.47) \]

Nothing can be allowed to be neglected in equation (3.43).

It can be written as
\[ \eta_s = \frac{1 + \frac{\rho_s}{\rho_\omega}}{\sqrt{\frac{\rho_\omega}{\rho_f}} (1 + \frac{\rho_s}{\rho_\omega})} \quad (3.48) \]

With the substitution of \( \rho_\omega = \rho_\omega \frac{\rho_\omega}{\rho_\omega} = \frac{\rho_\omega}{\eta_\omega^2} \) and Eqs. (3.47) and (3.44), Eq. (3.48) gives the following solution
\[ \eta_s = \eta_\omega \left[ 1 + (1 - \eta_\omega^2) \frac{\rho_\omega}{\rho_\omega} \right] \quad (3.49) a \]

or
\[ \eta_s = \frac{1}{(1 + \gamma_s)^{1/2}} \left[ 1 + \frac{\gamma_s}{1 + \gamma_2} \cdot \frac{1}{\frac{\rho_\omega}{\rho_f}} \right] \quad (3.49) \]

Hence
\[ \eta_{s\omega} = \eta_s \eta_\omega = \frac{1}{1 + \gamma_2} \left[ 1 + \frac{\gamma_s}{1 + \gamma_2} \cdot \frac{1}{\frac{\rho_\omega}{\rho_f}} \right] \quad (3.50) \]

\[ \eta_{\rho_s} = \frac{\eta_\omega}{\eta_s} = \frac{1}{1 + \frac{\gamma_s}{1 + \gamma_2} \cdot \frac{1}{\frac{\rho_\omega}{\rho_f}}} \quad (3.51) \]

\[ \eta_\omega' = \frac{\omega_\omega}{\omega_c} = \frac{\rho_\omega \omega_c}{\rho_\omega \omega_\omega} = \eta_{s\omega} \eta_\omega = (1 + \gamma_2)^{1/2} \quad (3.52) \]

\[ \eta_s' = \frac{\xi_s}{\xi_c} = \frac{\xi_f \xi_c}{\xi_f \xi_c} = \gamma_s \eta_{s\omega} = (1 + \gamma_2)^{1/2} \quad (3.53) \]

\[ \eta_{s\omega}' = \eta_s' \eta_{\omega}' = \frac{1}{(1 + \gamma_2)^{1/2}} (1 + \gamma_2)^{1/2} = 1 \quad (3.54) \]

Eqs. (3.49) and (3.50) are separately plotted as Fig. 7A and Fig. 7B with \( \eta_s \) and \( \eta_{s\omega} \) as ordinates against the coupling coefficient as abscissa with \( \frac{\rho_\omega}{\rho_f} \) as varying parameter.
FIG. VII A

CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega$S

(For $\eta_3$)

$\eta_3$, ADVANTAGE OF DAMPING RATIO

$\tau_2$, SECOND DERIVATIVE COUPLING COEFFICIENT
CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega S$

(for $\eta_{\omega_2}$)
FIG. VII C

CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $p_\omega S$
(For max. $\eta _1$, $\eta _{3w}$ & zero $\eta _3$)

SECOND DERIVATIVE COUPLING COEFFICIENT

$\gamma _2$

$\gamma$ for max. $\eta _1$

$\gamma$ for max. $\eta _{3w}$

Unstable region

For Unending oscillation

$\eta _1 = \eta _{3w} = 0$

$\frac{p_\omega}{p_{\omega_0}}$

0 0.1 0.3 0.6 1.0 3 6 10 30 60 100

MAXIMUM ADVANTAGE

0 1.0 2.0 3.0 4.0

0 0.3 0.6 1.0 3 6 10 30 60 100

-10 -3 -1 0 10
ADVANTAGE OF DAMPING RATIO (CONTROL COMPONENT)

ADVANTAGE OF RATIO OF UNDAMPED NATURAL FREQUENCIES

ADVANTAGE OF UNDAMPED NATURAL FREQUENCY

FIG VII D

CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\delta$S

$\eta_{p\omega}$

$\eta_{\omega}$, $\eta'_{\omega}$

$\eta_{5\omega}$

$\eta_{5}$

SECOND DERIVATIVE COUPLING COEFFICIENT

$\gamma_2$
positive coupling the system is always stable as \( \eta_s \) and \( \eta_{s\omega} \) approach a certain asymptotic value no matter what \( \frac{\rho_{\infty}}{\rho_{f_0}} \) is used. But with negative coupling, unending oscillation occurs when it exceeds a certain amount (because at that time, \( \eta_s \) and \( \eta_{s\omega} \) become zero).

For a constant value of \( \frac{\rho_{\infty}}{\rho_{f_0}} \), there is one coupling coefficient that will give maximum \( \eta_s \) and another \( \gamma_2 \) that will give maximum \( \eta_{s\omega} \). Such loci are plotted as dotted curves in Figs. 7A and 7B.

In terms of this control specification \( \frac{\rho_{\infty}}{\rho_{f_0}} \), maximum \( \eta_s \), \( \eta_{s\omega} \) and their corresponding \( \gamma_2 \) can be expressed:

\[
\text{max. } \eta_s = \frac{2 \left(1 + \frac{\rho_{\infty}}{\rho_{f_0}}\right)^{3/2}}{\frac{3^{3/2} \rho_{\infty}}{\rho_{f_0}}} = \frac{0.384 \left(1 + \frac{\rho_{\infty}}{\rho_{f_0}}\right)^{3/2}}{\frac{\rho_{\infty}}{\rho_{f_0}}}
\]

(3.55)

for max. \( \eta_s \), \( \gamma_2 = \frac{2 - \rho_{\infty}}{1 + \frac{\rho_{\infty}}{\rho_{f_0}}} \) \hspace{1cm} (3.55a)

\[
\text{max. } \eta_{s\omega} = \frac{(1 + \frac{\rho_{\infty}}{\rho_{f_0}})^2}{2 \frac{\rho_{\infty}}{\rho_{f_0}}}
\]

(3.56)

for max. \( \eta_{s\omega} \), \( \gamma_2 = \frac{1 - \frac{\rho_{\infty}}{\rho_{f_0}}}{1 + \frac{\rho_{\infty}}{\rho_{f_0}}} \) \hspace{1cm} (3.56a)

The limiting value of \( \gamma_2 \) at which the system becomes unendingly oscillatory can also be expressed in terms of \( \frac{\rho_{\infty}}{\rho_{f_0}} \), for zero \( \eta_s \), \( \gamma_2 = -\frac{\rho_{\infty}}{1 + \frac{\rho_{\infty}}{\rho_{f_0}}} \) \hspace{1cm} (3.57)
The above results -- Eqs. (3.55) to (3.57) -- are plotted as Fig. 7C with $\gamma_2$'s and $\eta$'s as ordinates against $\frac{f_2}{f_0}$ as abscissae. It can be seen that for the same $\frac{f_2}{f_0}$, max. $\eta$ obtainable is greater than max. $\eta$ obtainable when $\frac{f_2}{f_0} > \sqrt{2}$, and the condition is reversed when $\frac{f_2}{f_0} < \sqrt{2}$. To obtain max. $\eta$, $\gamma_2$ cannot be greater than 2. To obtain max. $\eta$, $\gamma_2$ cannot be greater than 1.0.

Fig. 7D are plotted for $\omega$, $\omega'$, $\gamma_2$, $\gamma_2'$, and $\omega$ vs. $\gamma_2$. Fig. 7D supplies the information that the damping of the control (high frequency) component is not essentially changed ($\eta_2' = 1$) because the reduction of damping ratio ($\eta_2' < 1$ for $\gamma_2 > 1$) is compensated by increase in its natural frequency ($\eta_2' > 1$ for $\gamma_2 > 1$). The compensation in the region for $\gamma_2 < 1$ is in the opposite way.

For the principal (or low frequency) component, natural frequency is decreased with positive coupling and increased with negative coupling. Because of the complication of $\gamma_2$, which is not only a function of $\gamma_2$ but also a function of $\frac{f_2}{f_0}$, the effect of the control upon the principal natural frequency therefore cannot compensate the effect upon $\gamma$. Real advantage is then taken from negatively coupled control of the error-acceleration type with proper magnitude of the coupling coefficient so that both $\eta_2$ and $\omega_2$ are greater than 1.0 ($\frac{f_2}{f_0} > 2$) which greatly improves the stability of the principal component, and yet does not substantially affect the stability of the control component as far as damping ($\gamma_2^2$) is concerned. Such advantage is available only at the sacrifice of decreasing the
inertia relating to the principal component. When the system encounters a prolonged disturbance, the diminished inertia due to negative $\gamma_2$ coupling will throw the system immediately into the disturbance. However, if such negative coupling is only called into action when the disturbance has ceased, it will definitely "quench" the disturbed motion.

Error-acceleration type control can be considered as inertia improving control. Therefore the improvement in damping, if there is any, is only a secondary effect. Hence, a system which only possesses a negligible damping cannot be improved to a satisfactory degree by the error-acceleration type control.

\section*{31. Error-Velocity Control with High Natural Frequency}

In reviewing equations (3.17), (3.06) to (3.08) and (2.25) to (2.27) the following relations are obtained for the error-velocity control.

\begin{align*}
\alpha_3 &= \alpha_{30} \\
\alpha_2 &= \alpha_{20} \\
\alpha_1 &= \alpha_{10}(1 + \gamma_1)
\end{align*}

or

\begin{align*}
2 \, \jmath \left( \frac{1}{\sqrt{\rho_0}} + \rho_0 \sqrt{\rho_0} \right) &= 2 \, \jmath_{00} \left( \frac{1}{\sqrt{\rho_{00}}} + \rho_{00} \sqrt{\rho_{00}} \right) \\
\rho_{00} + \frac{1}{\rho_{00}} + 4 \, \jmath_{00}^2 \rho_0 &= \rho_{000} + \frac{1}{\rho_{000}} + 4 \, \jmath_{000}^2 \rho_{000} \\
2 \, \jmath (\sqrt{\rho_{00}} + \rho_0) &= 2 \, \jmath_{00} \left( \sqrt{\rho_{000}} + \rho_{000} \right) (1 + \gamma_1)
\end{align*}

Write Eq. (3.58) as

\begin{align*}
\eta &= \frac{1}{\sqrt{\rho_{00}} + \rho_{00} \sqrt{\rho_{00}}} = \frac{1 + \rho_{00} \rho_{000}}{\sqrt{\rho_{000}} (1 + \rho_{000} \rho_{00})}
\end{align*}
Since such error-velocity control is usually called for improving damping, \( \rho_f > 1 \), and \( \lambda_f > 1 \). With high natural frequency control, \( \rho_f \rho_\omega > 1 \), and \( \lambda_f \lambda_\omega > 1 \); therefore Eq. (3.61) may be simplified as

\[
\eta_f = \eta_\omega \frac{\rho_f}{\lambda_f} \tag{3.62}
\]

where \( \eta_\omega = \frac{\omega_f}{\omega_\omega} = \frac{\sqrt{\rho_\omega}}{\sqrt{\rho_\omega}} \) because \( \rho_\omega^2 \omega_\omega^4 = \rho_\omega^2 \omega_\omega^4 \).

To allow frequency change, Eq. (3.59) can be simplified to the following form by neglecting \( \frac{1}{\rho_\omega} \) and \( \frac{1}{\lambda_\omega} \) terms only.

\[
\rho_\omega + 4 \lambda_\omega \rho_\omega = \rho_\omega + 4 \lambda_\omega \rho_f \tag{3.63}
\]

Substitute Eq. (3.62) into the above one, and the following relation can be obtained:

\[
\eta_f = \frac{\rho_\omega}{4 \lambda_\omega \rho_f} \left( 1 - \frac{1}{\eta_\omega} \right) + \frac{1}{\eta_\omega} \tag{3.64a}
\]

or

\[
\eta_f = \frac{\rho_\omega}{4 \lambda_\omega \rho_f} \left( 1 - \frac{1}{\eta_\omega} \right) + \frac{1}{\eta_\omega} \tag{3.64}
\]

Write Eq. (3.60) as:

\[
\eta_f = \frac{(1 + \gamma_f)(1 + \frac{\rho_f}{\rho_\omega})}{\sqrt{\rho_\omega} \left( 1 + \frac{\rho_f}{\rho_\omega} \right)}
\]

and equate to Eq. (3.62). The following relation is obtained:

\[
\frac{\rho_f}{\rho_f} = \frac{\eta_f}{\eta_\omega} = (1 + \gamma_f) \left( 1 + \frac{1}{\rho_\omega \rho_f} \right) - \frac{\eta_\omega^2}{\rho_\omega \rho_f} \tag{3.65}
\]

Eq. (3.64) is plotted as Fig. 8A with \( \eta_f \) as ordinate against \( \frac{\rho_\omega}{\lambda_\omega \rho_f} \) as abscissa with \( \eta_\omega \) as varying parameter. It is seen, although the variation of \( \omega_\omega \) is small, that it is very sensitive to \( \gamma_f \). If \( \eta_\omega \) is allowed to be unity, \( \eta_f \) would be unity.
CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_0$ S

FIG VIII A

ADiVANTAGE OF DAMPING RATIO

$\eta_z$

$\frac{\rho_0}{\zeta}$
CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega$ S
FIG. VIII C

CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega$'S
at any $\gamma$, which is apparently not true. When $\eta_0$ reaches 1.75, $\eta_f$ becomes maximum, so the curves retreat as $\eta_0$ further increases. However, with ordinary high frequency control, such optimum operation cannot be attained unless the natural frequency of the control is only moderately high (say $\omega$ around 10) and the damping ratio of control is well above 1.0.

To use Fig. 8A, the designer is supposed to know how many times the damping ratio of the controlled member is to be raised; that is, he must know what $\eta_f$ he wants. He has also to know what frequency variation to allow for; that is, $\eta_0$. Then the abscissa value or $f_0$ is fixed from which $f_c$ can be easily found because $f_c$ is known from the problem. $f_c$ is somewhere between 1 and 2 for conservative design.

Eq. (3.65) is plotted as Fig. 8B with $\gamma_f$ as ordinate against $\frac{f_0}{f_c}$ as abscissa with $\frac{\eta_f}{\eta_0}$ (or $\frac{f_0}{f_f}$) as principal varying parameter and $\eta_0$ as the secondary one.

When $\frac{f_0}{f_c}$ is obtained from Fig. 8A, $\frac{f_0}{f_c}$ can be figured out. Therefore, $\gamma_f$ can be found with the known values $\frac{\eta_f}{\eta_0}$, $\eta_0$, and $\frac{f_0}{f_c}$.

Any attempt to apply the Figs. (8A and 8B) in reverse order is easily confusing because the conditions are not well defined when $\gamma_f$ is chosen as starting datum.

For academic interest or rather, prospective design, optimum operation curves are plotted as Fig. 8C, with $\gamma_f$ as ordinate against $\eta_0$ as abscissa, with $\frac{f_0}{f_c}$ as varying parameter. The curves show a flat top at maximum $\gamma_f$. With values of $\eta_0$ lying between 1.6 and 1.8 the purpose of maximum $\gamma_f$ may be
FIG. IX
CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_o S$

[Graph showing characteristics of first-time-derivative-of-error control at large $\rho_o S$.]

- Axis labels and values are indicated on the graph.
- The graph illustrates the relationship between damping ratio $\eta_3$ and $\rho_o S$.
- Various curves represent different values of the coupling coefficient $\eta_3 / \eta_o$.
satisfactorily attained. However, the lower the value of is, the less the coupling coefficient will be required.

When $\eta_\omega$ is slightly less than one, $\eta_1$ is less than one and negative coupling coefficient is required. Eqs. (3.64) and (3.65) are plotted side by side as Fig. 9 for $\eta_\omega < 1$. The left part of Fig. 9 should be applied first, leaving the coupling coefficient $\gamma_i$ to be found last from the right part of the figure.

When $\rho_\omega \gg 1$, ought to be only slightly away from 1.0. Eq. (3.65) can be simplified further as

$$
\eta_1 \approx 1 + \gamma_i \left(1 + \frac{1}{\rho_\omega \eta_1}ight)
$$

$$
\eta_\omega \approx 1.0
$$

This simplified equation serves as a quick estimate of the coupling coefficient when $\rho_\omega \gg 1$ and required $\eta_1$ is known.
32. Single Coupling and Compound Coupling

In Chapter Eight three types of control have been discussed. They are only excited by a single force derived from the error or error derivative. Such controls may be defined as single coupling control. When a control is excited by two or more forces simultaneously, it is defined as compound control. From the analysis in Chapter Eight it is understood that neither one of the three types of single coupling will give a result which may be considered as "all round". Therefore proper "compounding" should be studied.

When a control is excited by two or more forces, the result obtained is neither the sum nor the product of the separately excited controlled system. Mechanically the system adjusts itself to give a compounding result without complication. But in analysis, step-by-step consideration has to be followed.

A control started with specification \( \frac{\nu_o}{\tau_o} \) will not keep on as such when the system is improved by the application of a certain coupling. Therefore the control specification \( \frac{\nu_o}{\tau_o} \) has to be modified in order to consider the improvement of stability of the system with the other coupling (which actually acts simultaneously with the first one).

It is advisable to have the results obtained in the last chapter in the tabulated form in order to minimize the effort
<table>
<thead>
<tr>
<th>COUPLING COEFF.</th>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
<th>$\gamma_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_w$</td>
<td>$(1+\gamma_0)^{1/2}$</td>
<td>1</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\gamma_t$</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}\left[1-\gamma_0 - \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$(1+\gamma_1)^{-1}$</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}\left[1+\gamma_1 + \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>1</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>$\eta_{w_0}$</td>
<td>$1-\gamma_0 \frac{\rho_{w_o}}{P_{t_o}}$</td>
<td>$1+\gamma_1 \frac{\rho_{w_o}}{P_{t_o}}$</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}\left[1+\gamma_2 + \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$1-\gamma_3 \frac{\rho_{w_o}}{P_{t_o}}$</td>
<td>1</td>
</tr>
<tr>
<td>$\eta'_t$</td>
<td>$1$</td>
<td>1</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}$</td>
<td>1</td>
<td>$\gamma_3$</td>
</tr>
<tr>
<td>$\eta'_w$</td>
<td>$1$</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}$</td>
<td>1</td>
<td>$\gamma_3$</td>
<td></td>
</tr>
<tr>
<td>$\eta'_{w_0}$</td>
<td>$1$</td>
<td>1</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}$</td>
<td>$\gamma_3$</td>
<td></td>
</tr>
<tr>
<td>$\eta_p$</td>
<td>$\frac{1}{(1+\gamma_0)^{1/2}}$</td>
<td>1</td>
<td>$\gamma_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta_{p_0}$</td>
<td>$\frac{1+\gamma_0}{(1+\gamma_0)^{1/2}}\left[1-\gamma_0 - \frac{\rho_{w_o}}{P_{t_o}}\right]^{-1}$</td>
<td>$\frac{1+\gamma_1}{(1+\gamma_0)^{1/2}}\left[1+\gamma_1 + \frac{\rho_{w_o}}{P_{t_o}}\right]^{-1}$</td>
<td>$\frac{1+\gamma_2}{(1+\gamma_0)^{1/2}}\left[1+\gamma_2 + \frac{\rho_{w_o}}{P_{t_o}}\right]^{-1}$</td>
<td>$\frac{1+\gamma_3}{(1+\gamma_0)^{1/2}}\left[1+\gamma_3 + \frac{\rho_{w_o}}{P_{t_o}}\right]^{-1}$</td>
<td>$\frac{1+\gamma_4}{(1+\gamma_0)^{1/2}}\left[1+\gamma_4 + \frac{\rho_{w_o}}{P_{t_o}}\right]^{-1}$</td>
</tr>
<tr>
<td>$\eta_{p_0}^t$</td>
<td>$\frac{1}{1+\gamma_0}\left[1+\gamma_0 - \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$(1+\gamma_1)^{-1}$</td>
<td>$\frac{1}{1+\gamma_0}\left[1+\gamma_1 + \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$\frac{1}{1+\gamma_2}\left[1+\gamma_2 + \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$\frac{1}{1+\gamma_3}\left[1+\gamma_3 + \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
</tr>
<tr>
<td>$\eta_{p_0}^t$</td>
<td>$\frac{1}{1+\gamma_0}\left[\gamma_0 + (1+\gamma_0) - \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$\gamma_2 + (1+\gamma_0) - \frac{\rho_{w_o}}{P_{t_o}}$</td>
<td>$\frac{1}{1+\gamma_0}\left[\gamma_2 + (1+\gamma_0) - \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$\frac{1}{1+\gamma_3}\left[\gamma_3 + (1+\gamma_0) - \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
<td>$\frac{1}{1+\gamma_4}\left[\gamma_4 + (1+\gamma_0) - \frac{\rho_{w_o}}{P_{t_o}}\right]$</td>
</tr>
</tbody>
</table>

* Expression is true for very small $\xi_0$ and very high $\rho_{w_o}$ with conservative $\xi_0$ (around one). For more accurate results, consult Fig. VIII A-B. If much overdamping is allowed in the control, $\eta'_t$ can be derived by Bairstow's approximation as: $1+\gamma_1\left[1+\frac{\rho_{w_o}}{P_{t_o}} + 4\xi_0^2\left(1+\frac{1}{P_{t_o}}\right)\right]^{-1} - \gamma_2\left[1+\frac{\rho_{w_o}}{P_{t_o}} + 4\xi_0^2\left(1+\frac{1}{P_{t_o}}\right)\right]^{-1} - \gamma_3\left[1+\frac{\rho_{w_o}}{P_{t_o}} + 4\xi_0^2\left(1+\frac{1}{P_{t_o}}\right)\right]^{-1} - \gamma_4\left[1+\frac{\rho_{w_o}}{P_{t_o}} + 4\xi_0^2\left(1+\frac{1}{P_{t_o}}\right)\right]^{-1}$. 

1-29
in an attempt to consider the compounding effect of two and more couplings. Table V not only serves as a summary of the results in the last chapter, but also extends to higher derivative coupling. (The derivations are omitted as they can be done by carefully neglecting terms which are negligible). The inclusion of the higher terms would be found useful should the time lag of the detecting instrument be considered.

33. Theory of Compounding for Controls with High Natural Frequency

It is seen from Table V that some expressions of \( \eta \) only contain the coupling coefficient itself. The specification of control has no influence upon it. For such a \( \eta \) simple multiplication of the individual expression is sufficient to give the resultant \( \eta \). For instance,

\[
\eta_{o1z} = \eta_{wo} \eta_{w}, \quad \eta_{o2} = \left( \frac{1 + \gamma_{o}}{1 + \gamma_{z}} \right)^{1/2}
\]

(3.67)

where the subscripts after the \( \eta \)'s and \( \gamma \)'s are referred to degrees of derivative. When they appear together, it means that the system is being compoundly coupled.

But some \( \eta \)'s are functions of \( \gamma \) as well as \( \frac{\omega_{o}}{\gamma_{o}} \). In such a case the expression can be, in general, expressed by

\[
\eta_{q} = f_{q}(\gamma_{q})^{*} \left[ 1 + h_{q}(\gamma_{q})^{*} \frac{1}{\frac{\omega_{o}}{\gamma_{o}}} \right]
\]

(3.68)

* For convenience in typing, \( f_{q}(\gamma_{q}) \), \( h_{q}(\gamma_{q}) \) and \( k_{q}(\gamma_{q}) \) shall be abbreviated as \( f_{q} \), \( h_{q} \) and \( k_{q} \) respectively throughout this section, and \( \frac{\omega_{o}}{\gamma_{o}} \) abbreviated as \( s_{c} \) signifying specification of control.
where the subscript \( q \) is referred to the order of derivative and \( f \) and \( h \) represent two different functions. The control specification should be multiplied by the factor

\[
\eta_{\frac{\omega}{\gamma}} = k_q(\gamma_q) \left[ 1 + h_q(\gamma_q) \frac{1}{\frac{\omega}{\gamma_q}} \right]
\]  

(3.69)

When the compounding is made between the \( q \)th and \( s \)th derivative, the expression can be written

\[
q_s = f_q(1 + \frac{h_q}{s_c}) \left\{ f_s* \left[ 1 + \frac{h_s}{s_c} \left( 1 - \frac{1}{k_q} + \frac{h_q}{s_c} \right) \right] \right\}
\]

\[
= f_q(1 + \frac{h_q}{s_c}) \left\{ f_s \left[ \frac{h_s}{s_c}(l - \frac{1}{k_q} + \frac{h_q}{s_c}) \right] \right\}
\]

\[
= f_q(1 + \frac{h_q}{s_c})f_s(1 + \frac{h_s}{s_c}) - f_qf_s h_q \frac{1}{s_c} \left( l - \frac{1}{k_q} + \frac{h_q}{s_c} \right)/s_c
\]

\[
= \eta_q \eta_s - f_qf_s h_q \frac{1}{h_q} \frac{1}{s_c} \left( l - \frac{1}{k_q} + \frac{1}{s_c} \right)/s_c
\]  

(3.70)

It can be shown that

\[
(1/h_q)(1-1/k_q) = 1
\]

or

\[
1 - 1/k_q = h_q
\]

(3.71)

For instance, take \( q = 2 \)

\[
k_2 = k_2(\gamma_2) = 1 + \gamma_2
\]

and

\[
1 - \frac{1}{1+\gamma_2} = \frac{\gamma_2}{1+\gamma_2} = h_2(\gamma_2) = h_2
\]

or take \( q = 0 \)

\[
1 - \frac{1}{k_0(\gamma_o)} = 1 - (1 + \gamma_0) = -\gamma_o = h_o(\gamma_o) = h_o
\]

*Where \( h_s, f_s, \) etc. stand for \( h_s(\gamma_s), f_s(\gamma_s), \) etc.*
Therefore

\[ \eta_{qs} = \eta_q \eta_s - f_q f_s h_q h_s (1 + 1/s_c)/s_c \]  

(3.72)

On developing, and canceling the terms containing \(1/s_c\) Eq. (3.72) becomes

\[ \eta_{qs} = f_q f_s \left[ 1 + (h_q + h_s - h_q h_s) / s_c \right] \]

\[ = f_{q,s} (1 + h_{q,s}/s_c) \]  

(3.73)

where

\[ f_{q,s} = f_{q,s}(\gamma_q, \gamma_s) = f_q(\gamma_q) f_s(\gamma_s) = f_q f_s \]  

(3.73)a

\[ h_{q,s} = h_{q,s}(\gamma_q, \gamma_s) = h_q + h_s - h_q h_s \]  

(3.73)b

It should be noted that the order of substitution is indifferent to the result. In the bracket of Eq. (3.70) it would look like \((1/h_s - 1/h_s k_s + 1/s_c)\) if subscript \(s\) is substituted for subscript \(q\), but the simplified form takes the same expression \(1 + 1/s_c\).

By the same reasoning and by following the same procedure as that given above, it can be shown that

\[ \eta_{qs} = k_{q,s} (1 + h_{q,s}/s_c) \]  

(3.74)

again, \( k_{q,s} = k_{q,k_s} \)  

(3.74)a

and \( h_{q,s} = h_q + h_s - h_q h_s \)  

(3.73)b

It can also be proven that

\[ 1 - 1/k_{q,s} = h_{q,s} \] by the same method in proving Eq. (3.71).

Therefore

\[ \eta_{qsp} = \eta_{qs} \eta_p - f_{q,s,p} h_{q,s} h_p (1 + 1/s_c)/s_c \]  

(3.76)

On developing

\[ \eta_{qsp} = f_{q,s,p} (1 + h_{q,s,p}/s_c) \]  

(3.77)
where
\[ f_{q,s,p} = f_q f_s f_p = f_q(\gamma_q) f_s(\gamma_s) f_p(\gamma_p) \] (3.77a)
and
\[ h_{q,s,p} = h_q + h_s + h_p + h_q h_s h_p - h_q h_s - h_s h_p - h_p h_q \] (3.77b)

From the above analysis several general rules can be deduced.

(1) When the advantage of a singly coupled control is a function of coupling coefficient as well as control specification, it can be expressed by the product of two factors, of which one is a simple function of the coupling coefficient; that is, \( f_q(\gamma_q) \), and the other is the sum of unity and the quotient of another simple function of the coupling coefficient; that is, \( h_q(\gamma_q) \), divided by the control specification.

(2) When the control is compoundly coupled, such "advantage" takes the same form although the two functions are modified according to the following rules:

(A) The function that has nothing to do with the control specification becomes the product of the individual function, or in symbol,
\[ f_{q,s,\ldots,p} = f_q f_s \ldots f_p \] (3.78a)

(B) The function that divided by the control specification takes the form
\[ h_{q,s,\ldots,p} = h_q + h_s \ldots + h_p \]
\[-h_q h_s - h_s h_r - \ldots - h_q h_p \]
\[+h_q h_s h_{s+1} \ldots + h_q h_s h_p \] (3.78b)
\[-h_q h_s h_{s+1} h_p - \ldots + \ldots \]

where \( h_q \) stands for \( h_q(\gamma_q) \), etc.

It is beneficial to master the compoundly coupled control, as the "advantages" can be actually attained to a better degree.
by compromising the coupling coefficients. The practical example will be left to the practical designers.

34. Special Compounding Controls

In some cases where only the damping ratio is required to be improved, the natural frequency of the member to be controlled is satisfactory and required to be kept the same no matter how much improvement has been made on the damping ratio. The problem may therefore be specified as follows:

(a) What \( \rho \) is convenient to design for a specific \( \omega \)?

(b) What is the initial damping present in the member to be controlled and how big is the advantage of the damping ratio expected to be obtained from the control?

(c) What is the relative damping ratio \( \rho \) between the control component and the principal component?

(d) Based upon the validity of the detecting instrument, what compounding couplings are to be used?

From the above specification, the following data can be determined:

(e) The relative damping ratio \( \rho_0 \) between the control and the member to be controlled. This datum virtually fixes the damping ratio of the control.

(f) The coupling coefficients.

35. Constant \( \omega \), Velocity-Acceleration Compounding Controls

For such a controlled system the nondimensional coefficients of the standardized quartic equation are

\[
\begin{align*}
\alpha_3 &= \alpha_{30} \\
\alpha_2 &= \alpha_{20} (1 + \gamma_2) \\
\alpha_1 &= \alpha_{10} (1 + \gamma_1)
\end{align*}
\]

(3.79)
Because nothing has been added to the constant term
(or \( \alpha_{00} \)) ,
\[ \rho_0^2 \omega_i^4 = \rho_0^2 \omega_i^4 \]
, but \( \omega_0 = \omega_i \); therefore
\[ \rho_0 = \rho_i . \]

Therefore,
\[ 2 \sqrt{\frac{1}{\rho_i} + \frac{\rho_0}{\rho_i} } = \alpha_3 \sqrt{\frac{1}{\rho_i} + \frac{\rho_0}{\rho_i} } = \alpha_{20} = 2 \int_0^1 \left( \frac{1}{\sqrt{\rho_i}} + \frac{\rho_0}{\sqrt{\rho_i}} \right) \]

or
\[ \frac{\int}{\int_0} = \frac{1 + \rho_0 \rho_i}{1 + \rho_i \rho_i} \]

(3.80)

or
\[ \rho_0 = \frac{1}{\rho_i} \left( \eta - 1 \right) + \rho_i \eta \]

(3.80)

or
\[ \gamma_i = \frac{\alpha_i - \alpha_{10}}{\alpha_{10}} = \frac{\alpha_i}{\alpha_{10}} - 1 \]

\[ = \frac{\int}{\int_0} \left( \frac{\rho_0 + \rho_i}{\sqrt{\rho_i}} \right) \]

\[ \frac{\int}{\int_0} \left( \frac{\rho_0 + \rho_i}{\sqrt{\rho_i}} \right) \]

(3.81)

or
\[ \gamma_i = \eta \left( \frac{\rho_0 + \rho_i}{\rho_0 + \rho_i} \right) - 1 \]

(3.82)

or
\[ \gamma_2 = \frac{\alpha_2 - \alpha_{20}}{\alpha_{20}} = \frac{4 \left( \int \frac{\rho_i^2}{\rho_i^2} - \int \rho_0^2 \right) \rho_0}{\rho_i + \frac{1}{\rho_i} + 4 \int \rho_0 \rho_i} \]

(3.82a)

or
\[ \gamma_2 = \frac{\eta \rho_i^2 \rho_i - \rho_0^2}{4 \int \rho_i^2 \left( \rho_i + \frac{1}{\rho_i} \right) + \rho_0} \]

(3.82)

or
\[ \gamma_2 = \frac{\rho_i - \frac{1}{\rho_i^2} \rho_i}{4 \int \rho_i^2 \left( \rho_i + \frac{1}{\rho_i} \right) + \frac{1}{\eta} \rho_i^2 \rho_i} \]

(3.82)

Eqs. (3.80), (3.81) and (3.82) are plotted as series of charts keeping \( \rho_j \) as the leading parameter which only varies from one sheet to another.
FIG. X

\( x - x_e \) COMPOUND CONTROLS
DAMPING, DAMPING RATIO
IMPROVING ONLY
FIG. XI

$\chi_1 - \chi_2$ COMPOUND CONTROLS
DAMPING & DAMPING RATIO
IMPROVING ONLY
FIG. XII A

$\gamma_1 - \gamma_2$, COMPOUND CONTROLS
DAMPING & DAMPING RATIO
IMPROVING ONLY
FIG. XII B  \( x - \xi \) COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY

\[
\frac{1}{\zeta} = 1.25, \quad \frac{1}{\zeta} = 0.8, \quad \frac{1}{\zeta} = 0.707, \quad \frac{1}{\zeta} = 0.6, \quad \frac{1}{\zeta} = 0.4, \quad \frac{1}{\zeta} = 0.2
\]
FIG. XII C  \( \gamma - \gamma_s \) COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY
FIG. XII D $\gamma - \gamma_z$ COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY

- $R_e = 0.75 \quad \zeta = 0.1$
- $R_e = 0.75 \quad \zeta = 0.6$
- $R_e = 0.75 \quad \zeta = 0.4$
- $R_e = 0.75 \quad \zeta = 0.2$

- $R_e = 0.75 \quad \zeta = 0.707$

- $R_e = 0.75 \quad \zeta = 0.4$
FIG. XII E \( \gamma - \gamma \) COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY
With known specification, \(\rho_0\), \(r_s\), and \(\rho_s\), \(\rho_{s0}\) can be found from Fig. 10, (Eq. 3.80), \(\gamma\) from Fig. 11 (Eq. 3.81), and \(\gamma_2\) from Figs. 12A to 12E.

From these figures, it is seen that the positive acceleration coupling is required together with a positive velocity coupling. With high frequency control, analysis has been made for singly coupled control and the results are plotted in Fig. 7A-D. Slight positive acceleration-coupling means slight reduction of undamped natural frequency of the controlled member and slight variation of the damping ratio. If the system is not strictly restricted to constant \(\omega\), neglect of \(\gamma\), determined by Fig. 12, in most cases would act in favor of improving damping. Of course, if \(\gamma\) is neglected, and \(\gamma_2\) determined by Fig. 11 is applied along, the quadratic factors of the quartic equation are to be determined according to the method described in Part II, in order to get the true improvement of damping ratio, frequency ratio, etc.

36. Constant \(\omega\), First- and-Third Derivative Coupling

**Compound Controls**

It is well known that \(\frac{\alpha_3}{\alpha_1}\) affects the distribution of damping ratio between the high and low frequency components. If \(\frac{\alpha_{30}}{\alpha_{10}}\), when the control is idle, deviates from unity, the first derivative and third derivative couplings may be applied-compoundly in such a way that tends to make \(\frac{\alpha_3}{\alpha_1}\) more or less equal to unity. The effect of third derivative coupling should be much more prominent than that due to acceleration
coupling so far as improvement of the distribution of the damping ratio distribution is concerned. The coefficients of the standard quartic equation of such compound controls are

\[ \alpha_3 = \alpha_3 \gamma_3 \]
\[ \alpha_2 = \alpha_2 \gamma_2 \]
\[ \alpha_1 = \alpha_1 \gamma_1 \]  

in which

\[ \frac{p_0 + \frac{1}{p_0} + 4 \gamma_j p_j}{\gamma} = \alpha_j = \alpha_j \gamma_j = \frac{p_0 + \frac{1}{p_0} + 4 \gamma_j p_j}{\gamma} \]  

\[ \gamma_j = \left( \frac{1}{p_0 + p_j \gamma_j} \right) \gamma_j - 1 \]

\[ \gamma_j = \left( \frac{1}{p_0 + p_j \gamma_j} \right) \gamma_j - 1 \]

Eqs. (3.84), (3.85) and (3.86) are plotted as Fig. 13, Fig. 14, and Fig. 15, respectively. The leading parameter \( \gamma_j \) is taken below unity, otherwise the damping ratio in the control would be too large. It is interesting to note that normally \( \gamma_3 \) needs negative sign, while \( \gamma_4 \) needs a positive one. But when \( \gamma_j \) is too small, positive \( \gamma_3 \) is required for controls with moderate frequency.

Even at normal \( \gamma_j \), negative \( \gamma_4 \) is required for controls with \( p_0 < 10 \). It is hard to accept this result from the ordinary point of view that a negative velocity coupling could improve damping, but if the idle damping ratio of the control
FIG. XIII

$r_1 - r_3$ COMPOUND CONTROLS

Damping & Damping ratio

Improving only
FIG. XIV
$\gamma - \gamma_s$ COMPOUND CONTROLS
DAMPING, DAMPING RATIO
IMPROVING ONLY
FIG. XV

$\gamma - \gamma_s$ COMPOUND CONTROLS
DAMPING, DAMPING RATIO
IMPROVING ONLY
is considered, which should be high for such compound control, it can be understood that the improvement of damping of the controlled member is at the expense of decreasing the damping ratio of the control component.
CHAPTER TEN

TUNING CONTROLS

37. Tuned System and Tuning Control

A system, when controlled by a nonideal control, singly or compoundly coupled, gives two identical quadratic factors to the quartic equation of the stability function of the system which is then defined as a tuned system. Such control is defined as the tuning control of the system. The damping ratio of the quadratic factor shall be called tuning ratio, $\zeta_t$. If $\zeta_t$ is equal to one, the system is said to be critically tuned. If $\zeta_t > 1$, it is overtuned, and if $\zeta_t < 1$, it is undertuned.

38. Tuning Control with Single Coupling and Theory of Tuning

It is more practical to design a singly coupled control than a compound one. If that control can be made a tuning one at desirable tuning ratio, it would receive more application than any other type of control. The following analysis verifies the possibility of such a tuning control.

It is believed that error-velocity coupling gives best results in improving the damping of the system. It is our purpose, therefore, to develop the theory for such a type of control in different degrees of tuning.
Let us assume:

\[ \xi_0 = \text{damping ratio of the member to be controlled.} \]
\[ \xi_c = \text{damping ratio of the control.} \]
\[ \rho = \frac{\omega_{nc}}{\omega_{n0}} = \text{ratio of damping ratios between the control and the member to be controlled.} \]
\[ \rho_{\omega} = \frac{\omega_{nc}}{\omega_{n0}} = \text{ratio of undamped natural frequencies between the control and the controlled member.} \]
\[ \gamma = \text{error-velocity coupling coefficient.} \]
\[ \alpha_i = \text{dimensionless error-velocity coupling factor} \]
\[ (\alpha_i = \alpha_{10} \gamma_i \rho_{\omega}^{3/2}) = \frac{B_i}{\omega_0^2} \]

Also assume:

(1) The two quadratic factors are only of the same frequency so that \( \rho_{\omega} = 1.0 \).

(2) The damping ratios of the two components are \( \xi_1 \) and \( \xi_2 \), which may or may not be equal to each other.

(3) If \( \xi_1 = \xi_2 \), their designation is \( \xi_t \), and sufficient conditions will be developed thereon.

It is understood that:

\[ \alpha_3 = 2(\xi_1 + \xi_2) ; \quad \alpha_{30} = 2 \xi_0 \left( \frac{1}{\sqrt{\rho_{\omega}}} + \frac{\xi_0}{\sqrt{\rho_{n0}}} \right) \quad (3.87) \]
\[ \alpha_2 = 2 + 4 \xi_1 \xi_2 ; \quad \alpha_{20} = \frac{1}{\rho_{\omega}} + \frac{\rho_{n0}}{\rho_{\omega}} + 4 \xi_0^2 \rho_{n0} \quad (3.87) \]
\[ \alpha_1 = \alpha_{10} (1 + \gamma) ; \quad \alpha_{10} = 2 \xi_0 \left( \frac{1}{\sqrt{\rho_{n0}}} + \sqrt{\rho_{\omega}} \right) \quad (3.87) \]

By equating \( \alpha_3 = \alpha_{30} \), and \( \alpha_2 = \alpha_{20} \), we have

\[ \xi_1 + \xi_2 = \xi_0 \left( \frac{1}{\sqrt{\rho_{\omega}}} + \frac{\xi_0}{\sqrt{\rho_{n0}}} \right) \quad (3.88) \]
\[ 2 + 4 \xi_1 \xi_2 = \frac{1}{\rho_{\omega}} + \rho_{n0} + 4 \xi_0^2 \rho_{n0} \quad (3.88) \]
By solving Eqs. (3.88)a and (3.88)b for \( \xi_1 \) and \( \xi_2 \), we have

\[
\frac{\xi_1}{s_0}, \frac{\xi_2}{s_0} = \frac{1}{2} \left\{ \frac{1}{s_0^2} + s_0 \sqrt{s_0} \pm \left( \frac{1}{s_0^2} - s_0 \sqrt{s_0} \right)^{1/2} - \frac{1}{s_0^2} \left( \frac{p_{wo} - 1}{s_0^2 p_{wo} - 1} \right)^2 \right\}
\]

(3.89)

When \( |s_c p_{wo} - s_0| > |(p_{wo} - 1)| \) the radical is real. With proper adjustment of \( \gamma \) the controlled results would be two components of motion of the same frequency, but with different damping ratios. When \( |s_c p_{wo} - s_0| < |(p_{wo} - 1)| \) the radical is imaginary, with proper adjustment of \( \gamma \) the controlled results would be two components of the same frequency, but with complex damping ratio conjugate to one another. The statement is mathematically correct, but what is the physical meaning of such a pair of components of motion?

With some algebraic juggling the above statement can be changed to: the controlled results would be two components of different frequency, but with the same damping ratio. Let

\[
S(\lambda) = \left\{ \lambda^2 + 2(a + ib)\lambda + 1 \right\} \left[ \lambda^2 + 2(a - ib)\lambda + 1 \right]
\]

(3.90)

\[
\lambda_{1,2} = -(a + ib) \pm \sqrt{a^2 - b^2 + 21ab - 1} = -(a + ib) \pm (c + id)
\]

(3.91)a

\[
\lambda_{3,4} = -(a - ib) \pm \sqrt{a^2 - b^2 - 21ab - 1} = -(a - ib) \pm (c - id)
\]

(3.91)b

where \( a, b, c, d \), are real positive quantities and

\[
(c + id)^2 = a^2 + b^2 - l + 21ab
\]

or \( (c - id)^2 = a^2 + b^2 - l - 21ab \)

or \( c^2 - d^2 = a^2 + b^2 - l \), and \( cd = ab \)

(3.92)

On developing,

\[
\lambda_1 = -a + c - i(b - d)
\]

\[
\lambda_2 = -(a + c) - i(b + d)
\]

\[
\lambda_3 = -(a + c) + i(b - d)
\]

\[
\lambda_4 = -(a + c) + i(b + d)
\]
Rearrange the sequence so that

\[ S(\lambda) = \left[ \lambda - \lambda_1 \right] \left[ \lambda - \lambda_2 \right] \left[ \lambda - \lambda_3 \right] \]

\[ = \left[ \lambda^2 - 2(c-a)\lambda + (c-a)^2 + (b-d)^2 \right] \lambda \]

\[ = \left[ \lambda^2 + 2(a+c)\lambda + (a+c)^2 + (b+d)^2 \right] \]

\[
\therefore \quad \rho_\omega = \frac{(a+c)^2 + (b+d)^2}{(c-a)^2 + (b-d)^2} \neq 1
\] (3.93)

It is necessary to prove

\[ \zeta' = \frac{c-a}{\sqrt{(c-a)^2 + (b-d)^2}} = \frac{a+c}{\sqrt{(a+c)^2 + (b+d)^2}} = \zeta_2
\] (3.95)

or the same thing

\[ \sqrt{\zeta'_1 - 1} = \frac{b-d}{c-a} = \frac{b+d}{c+a} = \sqrt{\zeta'_2 - 1}
\] (3.95a)

Finally, the above equation can be reduced to \( cd = ab \) which is fundamentally true as indicated by Eq. (3.92). Therefore both Eqs. (3.94) and (3.95) are true, so physically the controlled results would consist of two components of different frequency, but with the same damping ratio.

When

\[ \zeta c \rho_\omega - \zeta_0 = \pm (\rho_\omega - 1)
\] (3.96)

the radical is zero, therefore, at that condition, \( \zeta = \zeta_2 = \zeta_t \)

if the coupling coefficient \( \gamma \) is properly adjusted, and

\[ \zeta_t = \frac{1}{2} \zeta_0 \left( \frac{1}{\sqrt{\rho_\omega}} + \rho_\omega \sqrt{\rho_\omega} \right) = \frac{1}{2} \left( \frac{\zeta_0}{\sqrt{\rho_\omega}} + \zeta_0 \sqrt{\rho_\omega} \right)
\] (3.97)

39. Tuning Control With Positive Error Velocity Coupling

Take the positive sign of Eq. (3.96) and solve for \( \zeta_c \)

\[ \zeta_c = 1 - \frac{1 - \zeta_0}{\rho_\omega}
\] (3.98)
Squaring both sides of Eq. (3.97), we have

\[ 4\delta_t^2 = \delta_c^2 - \delta_c^2 \rho_{\omega} + 2\delta_c \delta_{\omega} \quad (3.99) \]

Substitute \( \delta_c \) of Eq. (3.98) into Eq. (3.99) and we have

\[ \rho_{\omega} - 2(1 + \delta_t^2 - 2\delta_c)\rho_{\omega} + (1 - 2\delta_c)^2 = 0 \]

\[ \therefore \quad \rho_{\omega} = 1 + 2\delta_t^2 - 2\delta_c + 2\delta_t \sqrt{1 + \delta_t^2 - 2\delta_c} \quad (3.100) \]

It is seen from Eq. (3.100) that the condition

\[ 1 + \delta_t^2 - 2\delta_c > 0 \quad (3.101) \]

must be fulfilled in order to get a real value of \( \rho_{\omega} \). The negative sign before the radical of Eq. (3.100) has been discarded, which not only gives negative value to \( \rho_{\omega} \) in most cases, but also yields negative value to \( \delta_c \) which is impossible to get from ordinary mechanical nonideal controls.

In Section 12, Chapter IV, it has been mentioned that when \( \delta_t / \delta_c \), = 1, it is possible that:

(C1) \( \rho_t = 1 \) at any value of \( \rho_{\omega} \)

(C2) \( \rho_{\omega} = 1 \) at any value of \( \rho_t \)

or (C3) \( \rho_t = \rho_{\omega} = 1 \)

Apparenty the first case is only possible when

\[ |\delta_c \rho_{\omega} - \delta_c| < |\rho_{\omega} - 1| \]

and the second case is only possible when

\[ |\delta_c \rho_{\omega} - \delta_c| > |\rho_{\omega} - 1| \]

It is the third case that is held true by the condition

\[ \delta_c \rho_{\omega} - \delta_c = \pm (\rho_{\omega} - 1) \] (for the time being + sign is used)

Therefore, \( \delta_c / \delta_c \) must be unity, or
Therefore \( \frac{\alpha_{30}}{\alpha_{10}(1+\gamma)} = 1 \) or \( \alpha_{10} \gamma = \alpha_{30} - \alpha_{10} \) (3.102)

Therefore \( \Gamma_i = \frac{B_i}{\omega_0^3} = \alpha_{10} \gamma \rho_{\omega_0}^{3/2} \) (3.102)

Substitute the value of \( \alpha_{30} \) and \( \alpha_{10} \) of Eqs. (3.87)a and (3.87)b into equation (3.102) and simplify the result by substituting \( 1 - \frac{1}{\rho_{\omega_0}} \) for \( \gamma_c \). The result will be

\[
\Gamma_i = 2 \left( \rho_{\omega_0} - 1 \right)^2 (1 - \gamma_c)^2
\]

(3.103)

where \( \rho_{\omega_0} \) is the value obtained from Eq. (3.100). This value of \( \Gamma_i \), when multiplied by \( \omega_0^3 \), gives the value of \( B_1 \) defined as error-velocity coupling factor in Eq. (2.13)c.

Eq. (3.100) is plotted as Fig. 16 with \( \rho_{\omega_0} \) as ordinate against \( \gamma_c \) as abscissa with \( \gamma_t \) as varying parameter (range plotted \( \gamma_t = 0.4 -- 1.1 \)). It is clearly seen that a constant \( \gamma_t \) curve turns back when it reaches its furthermost point to the right. Beyond that furthermost \( \gamma_c \), it is impossible to get a ratio of tuning below that particular \( \gamma_t \). If the negative sign before the radical of Eq. (3.100) is retained, the curve will have its turned back position as shown by the two dotted branches. The locus of these horizontal vertices can be represented by a straight line with Eq. \( \rho_{\omega_0} = 2 \gamma_c - 1 \) as shown by the dotted one.

From the relation

\[
A_{10} = \rho_{\omega_0}^2 \omega_0^4 = \rho_{\omega}^2 \omega^4 = A_0 \]

for the constant term of the quartic equation
FIG. XVI  DESIGN DATA OF TUNING CONTROLS
(+\sum_i COUPLING)  \rho_{wo} VS. \zeta_0 AT VARIOUS \zeta_t, \zeta_t = TUNING RATIO
FIG. XVII DESIGN DATA OF TUNING CONTROLS (+Γ COUPLING)

ζ_0 V.S. ζ_0 AT VARIOUS ζ_r, ζ_r = TUNING RATIO
FIG. XVIII  DESIGN DATA OF TUNING CONTROLS (+Γ COUPLING)

\( \Pi^{\frac{1}{2}} \) VS. \( \zeta_0 \) AT VARIOUS \( \zeta_t \), \( \zeta_t = \) TUNING RATIO
\[ \frac{\omega}{\omega_0} = \sqrt{\frac{\omega}{\omega_0}} = 1.0 \]

As all \( \frac{\omega}{\omega_0} \)'s are seen to be greater than 1.0, so \( \frac{\omega}{\omega_0} > 1 \), which means that such a tuned system possesses a favorable \( \frac{\omega}{\omega_0} \) in improving damping. The conclusion may also be reached that if the member to be tuned is originally overdamped (that is, \( \zeta_0 > 1.0 \)) it is impossible to tune this system back to \( \zeta_t < 1.0 \) by using a control having \( \frac{\omega}{\omega_0} > 1.0 \).

A mechanical member of 1 degree of freedom cannot possess negative damping. However, the exploration of Eq. (3.100) has been extended to negative \( \zeta_0 \). The adequacy of such an extension will find its utility in a system of two-degree-of-freedom which is unstable without being controlled. Therefore, even an airplane which shows longitudinal instability on free flight can be tuned to have \( \zeta_t \) if originally the frequency ratio between the two uncontrolled components is favorable for doing so.

Eq. (3.98) is plotted as Fig. 17 with \( \zeta_c \) as ordinate against \( \zeta_0 \) as abscissa with \( \zeta_t \) as varying parameter. From Fig. 17 it is interesting to notice when \( \frac{\omega}{\omega_0} \) reaches its horizontal vertex on Fig. 16 where \( \zeta_c \) becomes tangential to the vertical line on Fig. 17. In approaching this region the damping ratio of the control is too sensitive to the damping ratio of the controlled member so that the calculated \( \zeta_c \) may not be at tuned condition if the determination of \( \zeta_0 \) is slightly in error. When \( \frac{\omega}{\omega_0} = 1 \), it corresponds to \( \zeta_c = \zeta_0 \) in Fig. 17 shown by the dotted 45° line. To the left of this line it corresponds to \( \frac{\omega}{\omega_0} > 1 \); to the right, \( \frac{\omega}{\omega_0} < 1 \). It is
therefore seen that when \( \omega > 1 \), \( J_c \) is very stable with respect to \( J_0 \), which means that with small error in determination of \( J_0 \), the calculated value of \( J_c \) for the tuned condition does not vary considerably.

In general, within the operating range the damping ratio of the control necessary for the tuned condition never exceeds 1.0; yet in some cases the tuned result is amazing in that the tuning ratio \( J_t \) is greater than either \( J_c \) or \( J_0 \). In common language, such an amazing case may be stated thus: that an oscillatory member can be tuned to give less or nonoscillatory motion by an oscillatory control.

Eq. (3.103) is plotted as Fig. 18 with \( \Gamma_{\frac{1}{2}} \) as ordinate against \( J_0 \) as abscissa. When \( \Gamma_{\frac{1}{2}} = 0 \) naturally \( \omega_0 \) for tuning must be 1.0 and \( J_c = J_0 \); and the cusps of the solid curves form the locus \( \Gamma_{\frac{1}{2}} = 0 \). If \( \Gamma_{\frac{1}{2}} \) is plotted instead of \( \Gamma_{\frac{1}{2}} \), there are no cusps, but continuous curves with minima at the cusps as shown by the dotted curves for part of \( J_t = 0.4 \) and \( J_t = 0.5 \). The dot-dash straight line on Fig. 18 is equivalent locus through the vertices of \( J_t \) = constant on Fig. 16. Such a dot-dash line can be represented by Eq. \( \Gamma_{\frac{1}{2}} = 2 - 2 \cdot J_0 \) (or \( \Gamma_{\frac{1}{2}} = 2(1 - J_0)^{\frac{1}{2}} \), if \( \Gamma_{\frac{1}{2}} \) is plotted instead of \( \Gamma_{\frac{1}{2}} \).

It is evident that keeping the condition \( J_c \omega_0 - J_0 = + \omega_0^{-1} \) for the tuning coupled (factor, dimensional or nondimensional) is required within the working range \( \omega_0 \) is greater than one and the control must be underdamped.
40. Tuning Controls with Negative Error-Velocity Coupling

By doing the same algebraic work as has been done in Section 39, only with

\[ \xi_c \rho_0 - \xi_o = -(\rho_0 - 1) \]  \hspace{1cm} (3.104)

instead of

\[ \xi_c \rho_0 - \xi_o = + (\rho_0 - 1) \]  \hspace{1cm} (3.98)

the following results are obtained

\[ \xi_c = -1 + \frac{1 + \xi_o}{\rho_0} \] \hspace{1cm} (3.105)

\[ \rho_0 = 1 + 2 \xi_c^2 + 2 \xi_o - 2 \xi_c \sqrt{1 + \xi_c^2 + 2 \xi_o} \] \hspace{1cm} (3.106)

\[ \Gamma = - (1 + \xi_o)(1 - \rho_0)^2 \]  \hspace{1cm} (3.107)

Apparently the + sign before the radical in Eq. (3.106) has been discarded. It is because otherwise the damping ratio of the control needs to be negative which is impossible from ordinary mechanical nonideal controls.

Eq. (3.106) is plotted as Fig. 19 with \( \rho_0 \) as ordinate against \( \xi_o \) as abscissa. To tune a member with \( \xi_o = -0.5 \) a control with no stiffness should be used; that is, \( \rho_0 = 0 \). Beyond \( \xi_o = -0.5 \) ( \( \xi_o < -0.5 \)) negative \( \rho_0 \) is needed which has no physical significance. From Fig. 19 it is evident that no matter how much (+) damping is possessed by the controlled member, it is always possible to be tuned back to \( \xi_t \) of any + magnitude. The dot-dash locus represents the bound-
FIG. XIX DESIGN DATA OF TUNING CONTROLS (-F COUPLING)

$P_{\omega_0}^{1/2}$ VS. $\zeta_0$ AT VARIOUS $\zeta_t$. $\zeta_t$ = TUNING RATIO
FIG. XX  DESIGN DATA OF TUNING CONTROLS (–Ω COUPLING)

\( \zeta_c^{1/3} \) VS. \( \zeta_0 \) AT VARIOUS \( \zeta_t \). \( \zeta_t = \) TUNING RATIO

\( \rho_{\omega_0} = 1 \)
FIG. XXI DESIGN DATA OF TUNING CONTROLS (-\Gamma COUPLING)

\( \Gamma \) vs. \( \zeta_0 \) at various \( \zeta_t \). \( \zeta_t \) = TUNING RATIO
ary of $\xi_c = 0$ beyond which negative $\xi_c$ is required which is impossible from ordinary mechanical control, but possible with a vacuum tube circuit as the control element. The dotted curves indicate negative damping controls are required for tuning.

Eq. (3.105) is plotted as Fig. 20 with $\xi_c^{1/3}$ as ordinate against $\xi_0$ as abscissa with $\xi_t$ as varying parameter. When $\xi_0 > 1.0$, tuning back to $\xi_t < 1.0$ needs a control of $\rho_{\omega o} > 1.0$ and $\xi_c < 1.0$. Such requirement is quite normal. To the left of $\rho_{\omega o} = 1$, $\rho_{\omega o}$ is less than 1.0; high damping ratio in control is therefore needed for tuned results. However, the damping coefficient is not excessive because of the small value of $\rho_{\omega o}$. When $\rho_{\omega o}$ is small or negative, $\xi_c$ is too sensitive to be tuned with $\xi_0$ or the tuned condition is not very stable in such a region. The abrupt inflection along constant $\xi_t$ is caused by the scale of $\xi_c^{1/3}$. If $\xi_c$ is plotted as ordinate instead of $\xi_c^{1/3}$, such inflection is missed and the curve is continuous at $\xi_c = 0$ as shown by the dotted curve for $\xi_t = 0.4$. Below $\xi_c = 0$, only electric tuning control may possibly be practical.

Eq. (3.107) is plotted as Fig. 21 with $F'_c$ as ordinate against $\xi_0$ as abscissa with $\xi_t$ as varying parameter. $F'_c$ is plotted instead of $F_c$ when $-F_c > 1.0$ and $\rho_{\omega o} > 1.0$. It is seen that no matter how large the value of the $\rho_{\omega o}$ used, a negative coupling factor (nondimensional or dimensional) is needed for the error-velocity coupled control in tuned condition. The dot-dash locus separates the mechanical nonideal
control below and electric nonideal control above (shown by dotted curves.)

41. Comparison Between $+\Gamma$ and $-\Gamma$ Tuning Controls

It is much simpler to make a table (as follows) for the comparison of the $+\Gamma$ and $-\Gamma$ tuning controls.

**TABLE VI**

**Comparison Between $+\Gamma$ and $-\Gamma$ Tuning Controls**

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>$+\Gamma$ Tuning Control</th>
<th>$-\Gamma$ Tuning Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. What makes the consistency of the sign</td>
<td>$\Delta_c = \pi(\rho_\infty - \pi_0)$; $\pi_0 &gt; \pi_1$ When $\rho_\infty &gt; 1.0$</td>
<td>$\Delta_c = \pi(\rho_\infty - \pi_0)$; $\pi_0 &lt; \pi_1$ When $\rho_\infty &lt; 1.0$</td>
</tr>
<tr>
<td></td>
<td>At idle condition, high freq. comp. has higher damping ratio</td>
<td>At idle condition, low freq. comp. has higher damping ratio</td>
</tr>
<tr>
<td>2. Best working range</td>
<td>$\rho_\infty &gt; 1.0$, $\pi_0 &lt; 1.0$, $\Delta_c &lt; 1.0$</td>
<td>$\rho_\infty &gt; 1.0$, $\pi_0 &gt; 1.0$, $\Delta_c &lt; 1.0$</td>
</tr>
<tr>
<td>3. Tuning result not very stable</td>
<td>$\rho_\infty &lt; 1.0$, $\pi_0 &lt; 1.0$, $\Delta_c &lt; 1.0$</td>
<td>$\rho_\infty &lt; 1.0$, $-0.5 &lt; \pi_0 &lt; 1.0$, $\Delta_c &gt; 1.0$</td>
</tr>
<tr>
<td>4. Range of physical nonexistence</td>
<td>$\rho_\infty &lt; 1.0$, $\Delta_c^2 &gt; 2\pi_0 - 1$</td>
<td>$\rho_\infty &lt; 1.0$, $\pi_0 &lt; -0.5$</td>
</tr>
</tbody>
</table>

It is understood that control of higher frequency is preferable to control of lower frequency simply because at tuned condition the frequency of the controlled member is increased by $\sqrt{\rho_\infty}$. However, if high frequency control is not available by some physical restriction, the tuning condition is obtainable by controls with $\rho_\infty < 1.0$ only at the expense of (1) less assurance in tuned condition, and (2) the undamped natural frequency of the controlled member is lowered, which means that such control spoils the rapidity of response of the controlled member.
PART IV

ANALYSIS OF TRANSIENT
42. Stability and Transient Analysis of Controlled System

Sufficient knowledge of stability analysis of automatic controlled problem may assure (a) stable operation in general, and (b) better distribution of damping between the components by providing a well compromised control. However, such assurances do not inform us how the controlled system responds to a disturbance of any characteristic. Analysis of transient response is therefore important in order to obtain better performance when the system is subject to sudden disturbance or to one which does not repeat periodically.

43. Stability Function, Quality Function, Disturbance Function and Response Function

From the analysis in Section 5, Chapter II, and in Section 8, Chapter III, Eqs. (1.102), (1.104) and (2.136) are noticed in the general form.

\[ R_h(t) = \frac{Q_h(D)}{S(D)} I(t) I(t) \quad \text{(4.01a)} \]

or \[ R_h(\tau) = \frac{Q_h(d)}{S(d)} I(\tau) I(\tau) \quad \text{(4.01)} \]

*Such a form of equation is originated by Heaviside in treating network responses. For the convenience of this thesis, the symbols are entirely different from the original form. The reader is referred to Chapter V in Bush's "Operational Circuit Analysis".
where \( D = \frac{d}{dt} \) = time differential operator

\[
d = \frac{d}{d\tau} = \text{dimensionless time differential operator}
\]

\[
S(D) = \sum_{k=0}^{k=n} A_k D^k \text{ and is defined as stability function of the controlled system (dimensional).}
\]

\[
S(d) = \sum_{k=0}^{k=n} a_k d^k \text{ also defined as stability function of the controlled system (non-dimensional).}
\]

\[
Q_h(D) = \sum_{k=0}^{k=m} B_{nk} D^k \text{ and is defined as quality function of h-wise motion with } m = n.
\]

\[
Q_h(d) = \sum_{k=0}^{k=m} b_{nk} d^k \text{ also defined as quality function of h-wise motion (non-dimensional) with } m = n.
\]

\[
I(t) = \text{function of } (t), \text{ defined as disturbance function or input function applied to the controlled system.}
\]

The symbol \( \mathbf{1} \) is defined as unit function which specifies a discontinuous function of time which is zero until \( t \) equals zero and unity thereafter. Any function, such as \( I(t) \), followed by the symbol \( \mathbf{1} \), indicates its discontinuity at \( t = 0 \); and that the function is zero until \( t \) equals zero and equal to \( I(t) \) thereafter.

\( R_h(t) \) is function of \( (t) \) for h-wise response to the disturbance function \( I(t) \mathbf{1} \). It is defined as response function of the h-wise motion.

\( I(\tau) \) and \( R_h(\tau) \) are defined in the same way but they are referred to non-dimensional quantities.
Eq. (4.01) has the same form as Eq. (4.01)a. Therefore full development of either equation will enable one to handle the other. Eq. (4.01) shall be fully developed into a form which is more familiar to engineers.

44. **Expanded Form of Response Function when Unit Step Disturbance is Applied**

When \( I(\tau)1 = 1 \), the response function is defined as unit response.

\[
R_h(\tau) = \frac{Q_h(d)}{S(d)} 1 \quad (4.02)\text{a}
\]

The subscript \( h \) is omitted hereafter temporarily for the simplicity in appearance.

\[
R(\tau) = \frac{Q(d)}{S(d)} 1 \quad (4.02)
\]

On expanding,

\[
R(\tau) = \frac{Q(0)}{S(0)} + \sum_{k=1}^{k=n} \frac{Q(d_k)}{d_k S'(d_k)} e^{d_k \tau} \quad (4.03)
\]

where \( \frac{Q(0)}{S(0)} \) is the steady state response and \( \sum_{k=1}^{k=n} C_k e^{d_k \tau} \) is the transient response. \( d_k \)'s are the roots of \( S(d) = 0 \).

\( S'(d_k) \) is the first derivative of the stability function, in purely algebraic form,

\[
S'(d_k) = a_0(d_k-d_1)(d_k-d_2)...(d_k-d_{k-1})(d_k-d_{k+1})...(d_k-d_n) \quad (4.04)
\]

which should be substituted in Eq. (4.03).

When all \( d_k \)'s are real, solution (4.03) is really handy. But when some of them are complex quantities, Eq. (4.03)
is too tedious to be rationalized. The following procedure will reduce the effort in doing so.

Physically, when one of the roots of $S(d)$ is complex there must be another complex root which is conjugate to the first. Let the root be $d_r$ and its conjugate be $\overline{d_r}$, where

$$d_r, \overline{d_r} = (-\frac{1}{t_f} + j \sqrt{1 - \frac{1}{t_f^2}}) \omega_{mnf} \quad (4.05)$$

where $\omega_{mnf}$ = non-dimensional natural frequency of component $f$. $t_f$ = damping ratio of component $f$.

Now,

$$\sum_{k=1}^{n} \frac{Q(d_k)}{d_k S'(d_k)} d_k \tau = \sum_{f=1}^{n} \frac{Q(d_f)}{d_f S'(d_f)} d_f \tau + \sum_{f=1}^{n} \frac{Q(\overline{d_f})}{d_f S'(d_f)} \overline{d_f} \tau \quad (4.06)$$

$$= \sum_{f=1}^{n} \frac{1}{d_f d_f S'(d_f) S'(d_f)} \left[ Q(d_f) d_f S'(d_f) d_f \tau + Q(\overline{d_f}) d_f S'(d_f) \overline{d_f} \tau \right] \quad (4.07)$$

Inside the bracket of $(4.06)$, $Q(d_f) d_f S'(d_f)$ and $Q(\overline{d_f}) d_f S'(d_f)$ are conjugate functions, so let

$$\frac{1}{2} (R_f + j I_f) = Q(d_f) d_f S'(d_f), \quad Q(\overline{d_f}) d_f S'(d_f) \quad (4.08)$$

where $\frac{1}{2} (R_f) = \text{real part of } Q(d_f) d_f S'(d_f)$

$\frac{1}{2} (I_f) = \text{imaginary part of } Q(d_f) d_f S'(d_f)$.

Substitute $(4.08)$ and $(4.05)$ into $(4.07)$. The result will be
\[ \sum_{k=1}^{n} \frac{Q(d_k)}{d_k s'(d_k)} e^{d_k \tau} = \sum_{f=1}^{n} \left[ \frac{1}{2} \frac{1}{d_f d_f s'(d_f) s'(d_f)} \left\{ (R_f + iI_f) \epsilon (-i_f + i1)^2 \omega\right. \right. \\
+ (R_f - iI_f) \epsilon \left. \left(-i_f - i1 \right)^2 \omega\right. \right] \]

\[ = \sum_{f=1}^{n} \frac{\epsilon^{-i \omega \tau}}{d_f d_f s'(d_f) s'(d_f)} \left[ R_f \cos(1 - i_f) \omega\right. \right. \\
- i_f \sin(1 - i_f) \omega\right. \right. \]

(4.09)

or

\[ \sum_{k=1}^{n} \frac{Q(d_k)}{d_k s'(d_k)} e^{d_k \tau} = \sum_{f=1}^{n} \frac{\mu_f e^{-i \omega \tau}}{d_f d_f s'(d_f) s'(d_f)} \cos(\sqrt{1 - i_f^2} \omega\right. \right. \\
+ \phi_f) \right. \right. \]

(4.10)

where \( \mu_f = \sqrt{R_f^2 + I_f^2} = 2Q(d_f)Q(d_f) d_f d_f s'(d_f) s'(d_f) \)

(4.11a)

\[ \phi_f = \tan^{-1} \frac{I_f}{R_f} = \tan^{-1} \frac{I_f}{Q_f} + \tan^{-1} \frac{I_f}{S' f} + \tan^{-1} \frac{1 - i_f^2}{i_f} \]

(4.11b)

with \( R_{Qf} = \text{real part of } Q(d_f) \)

\( I_{Qf} = \text{imaginary part of } Q(d_f) \)

\( R_{S'f} = \text{real part of } S'(d_f) \)

\( I_{S'f} = \text{imaginary part of } S'(d_f) \).

Substitute (4.11a) in (4.11) and the transient response will appear as
\[
2 \sum_{f=1}^{n} \frac{\sqrt{Q(d_f)Q(\overline{d_f})}}{d_f d_f' S'(d_f) S'(\overline{d_f})} \ e^{-\frac{f_w n_f}{\omega_{nnf}}} \tau \cos(\sqrt{1-\frac{\xi_f^2}{\omega_{nnf}^2}} \omega_{nnf} \tau + \phi_f) \quad (4.12)
\]

Eq. (4.12) indicates the physical conception of the transient response when all components are oscillatory so that each undergoes a damped sinusoidal motion with its own damping rate, \(-\xi_f \omega_{nnf}\), with a certain natural frequency and a certain phase shift. The phase shift apparently consists of three parts: (1) due to the quality function \(Q_f\), (2) due to the stability derivative function \(S'_f\), and (3) due to the damping ratio \(\xi_f\).

Since \(\sqrt{d_f d_f'} = \left(\xi_f - j \sqrt{1-\xi_f^2}\right) \left(-\xi_f - j \sqrt{1-\xi_f^2}\right) \frac{1}{\sqrt{2}} \omega_{nnf} \)

\[= \frac{\xi_f^2 - j \sqrt{1-\xi_f^2} \frac{1}{\sqrt{2}} \omega_{nnf}}{\sqrt{-1}} = -\omega_{nnf} \quad (4.13)_a \]

Therefore expression (4.12) can be simplified as

\[-2 \sum_{f=1}^{n} \left[ \frac{Q(d_f)Q(\overline{d_f})}{S'(d_f) S'(\overline{d_f})} \right] \frac{1}{\sqrt{2}} e^{-\frac{f_w n_f}{\omega_{nnf}}} \tau \cos(\sqrt{1-\frac{\xi_f^2}{\omega_{nnf}^2}} \omega_{nnf} \tau + \phi_f) \quad (4.13)\]

It should be noted that when the component \(f\) is non-oscillatory, that is, it is overdamped or \(\xi_f > 1\),

(1) both \(d_f\) and \(d_f'\) become real and their magnitude ratios are

\[
\frac{Q(d_f)}{S'(d_f)} \text{ and } \frac{Q(d_f')}{S'(d_f')}.
\]
\[
(2) \sqrt{1-\xi_f^2} = j\sqrt{\xi_f^2-1}
\]

and \(\phi_f\) becomes \(j\psi_f^*\) where \(\phi_f = \psi_f\).

Therefore expression (4.13) becomes

Transient response of component \(f = \)

\[
-2 \left[ \frac{Q(d_f)Q'(d'_f)}{S'(d_f)S'(d'_f)} \right] \frac{1}{2} \epsilon -j \omega_{nnf} \tau \cosh(\sqrt{\xi_f^2-1} \omega_{nnf} \tau + \psi_f) \quad (4.14)
\]

with \(\xi_f > 1\).

The above derivation is valid only for non-repeating roots. If some real root or complex root occurs more than once in \(S(d) = 0\), the evaluation is much more complicated. It will be analyzed in the next chapter.

*because

\[
j \tanh^{-1}\sqrt{\xi^2-1} \xi = \tan^{-1} j \sqrt{\xi^2-1} \xi
\]

and all the imaginary quantities for \(\xi < 1\) become real when \(\xi > 1\) and vice versa; therefore (4.11)b may be used to find \(\psi_f\).
CHAPTER TWELVE

Characteristic Decomposition

45. **Characteristic Decomposition or Quadratically Partial Fraction**

For engineers' better understanding of transient response of controlled system it is better to fractionalize the expression \( \frac{Q(d)}{S(d)} \) before the unit function or any disturbance function is attached to it. The partial fractions will be done according to physical significance, so that each one of them will represent a mode of vibration when disturbance is applied.

For the purpose of this thesis, the highest power of \( d \) in \( S(d) \) will be limited to six. The highest power of \( d \) in \( Q(d) \) is limited either to six or less. In case the highest powers of \( d \) in \( Q(d) \) and in \( S(d) \) are the same, their coefficient is usually the same (and equal to unity) as shown in Eq. (1.104), and the fraction \( \frac{Q(d)}{S(d)} \) may be changed to the form \( 1 + \frac{Q(d)}{S(d)} \) in which the highest power of \( d \) in \( Q(d) \) is at least one less than that in \( S(d) \). It is this \( \frac{Q(d)}{S(d)} \) which will be partially fractionalized.

Assume

\[
\frac{Q(d)}{S(d)} = \frac{q_1(d)}{s_1(d)} + \frac{q_2(d)}{s_2(d)} + \frac{q_3(d)}{s_3(d)}
\]  

(4.15)

where

\[
s_1(d) = d^2 + 2 \sum_1 \omega_{\text{nnl}} d + \omega_{\text{nnl}}^2, \text{ etc.} \]  

(4.15a)

\( \omega_{\text{nnl}} \) represents the non-dimensional natural frequency of component 1.)
\[ q_1(d) = \int_1^\infty \omega_{nnl} k_{11} d + \kappa_{01} \omega_{nnl}^2, \text{ etc.} \quad (4.15)b \]

(First subscript of \( \kappa \) refers to component number; second subscript of \( \kappa \) indicates the associated exponential power of \( d \).)

On summing up the right sides of (4.15), the following form is obtained.

\[
\frac{Q(d)}{S(d)} = \frac{q_1(d)s_2(d)s_3(d) + q_2(d)s_1(d)s_2(d) + q_3(d)s_1(d)s_2(d)}{S(d)}
\]

Let \( d_1, \bar{d}_1 = \text{roots of } S(d) = 0 \), where \( d_1 \) and \( \bar{d}_1 \) are conjugate pair, because \( s_1(d) = 0 \).

Substitute \( d_1 \) into (4.16). It appears that

\[
Q(d_1) = q_1(d_1)s_2(d_1)s_3(d_1)
\]

or

\[
q_1(d_1) = \frac{Q(d_1)}{s_2(d_1)s_3(d_1)} = F_1
\]

(4.17)

From Eq. (4.15)a, we obtain

\[
d_1 = (-f_1 + j\sqrt{1-f_1^2}) \omega_{nnl}
\]

(4.18)

Substitute (4.18) into (4.15)b, and then (4.17).

\[
\omega_{nnl}^2 (\kappa_{01} - f_1^2 \kappa_{11} + i \kappa_{11} f_1 \sqrt{1-f_1^2}) = F_1
\]

or

\[
\kappa_{01} - f_1^2 \kappa_{11} + i \kappa_{11} f_1 \sqrt{1-f_1^2} = \frac{1}{\omega_{nnl}^2} F_1
\]

(4.19)

Therefore

\[
\kappa_{11} = \frac{1}{\omega_{nnl}^2 f_1 \sqrt{1-f_1^2}} \frac{F_1}{F_1} \quad (4.20)
\]
\[ k_{01} = \frac{1}{\omega_{nnl}} R_{F_1} + r_1^2 \kappa_{11} \]  \hspace{1cm} (4.21)

or

\[ \kappa_{01} = \frac{1}{\omega_{nnl}} \left[ R_{F_1} + \frac{r_1}{\sqrt{1-r_1^2}} I_{F_1} \right] \]  \hspace{1cm} (4.21a)

where \( L_{F_1} \) and \( R_{F_1} \) represent imaginary and real parts of \( F_1 \) respectively.

Now we have to break \( F_1 \) into its real and imaginary parts.

\[ F_1 = \frac{Q(d_1)}{s_2(d_1)s_3(d_1)} = \frac{Q(d_1)s_2(\overline{d_1})s_3(\overline{d_1})}{s_2(d_1)s_2(\overline{d_1})s_3(d_1)s_3(\overline{d_1})} \]  \hspace{1cm} (4.22)

where the denominator is rationalized and equal to

\[ s_2(d_1)s_2(\overline{d_1})s_3(d_1)s_3(\overline{d_1}) = \left[ (s_2 \omega_2^* - s_1 \omega_1)^2 + (\sqrt{1-s_1^2} \omega_1 + \sqrt{1-s_2^2} \omega_2)^2 \right] \]

\[ \times \left[ (s_3 \omega_3^* - s_1 \omega_1)^2 + (\sqrt{1-s_1^2} \omega_1 - \sqrt{1-s_3^2} \omega_3)^2 \right] \]

\[ \times \left[ (s_3 \omega_3^* - s_1 \omega_1)^2 + (\sqrt{1-s_1^2} \omega_1 + \sqrt{1-s_3^2} \omega_3)^2 \right] \]

\[ \times \left[ (s_3 \omega_3^* - s_1 \omega_1)^2 + (\sqrt{1-s_1^2} \omega_1 - \sqrt{1-s_3^2} \omega_3)^2 \right] \]  \hspace{1cm} (4.23)

and the denominator can be expanded by the binomial theorem.

*For convenience in typing, \( \omega_1, \omega_2 \) and \( \omega_3 \) are used for \( \omega_{nnl}, \omega_{nn2} \) and \( \omega_{nn3} \) respectively.*
\[ Q(d_1) = Q(-\frac{j}{2} \omega_1 + j \omega_1 \sqrt{1 - \frac{1}{2}}) \]

\[ = Q(-\frac{j}{2} \omega_1) - \omega_1^2 (1 - \frac{1}{2}) Q''(-\frac{j}{2} \omega_1) \frac{Q''(-\frac{j}{2} \omega_1)}{2!} + \omega_1^4 (1 - \frac{1}{2}) Q''''(-\frac{j}{2} \omega_1) \frac{Q''''(-\frac{j}{2} \omega_1)}{4!} \]

\[ + j \left[ \omega_1 (1 - \frac{1}{2}) \frac{1}{2} Q'(-\frac{j}{2} \omega_1) - \omega_1^3 (1 - \frac{1}{2}) \frac{3}{4} Q''''(-\frac{j}{2} \omega_1) + \omega_1 \mathcal{S}(-\frac{j}{2} \omega_1) \right] \frac{Q''(-\frac{j}{2} \omega_1)}{2} \left[ \frac{Q''''(-\omega_1)}{3} \right] \]

(4.24)

\[ s_2(\overline{a_1}) = s_2(-\frac{j}{2} \omega_1 - \frac{j}{2} \omega_1 \sqrt{1 - \frac{1}{2}}) = 2(\frac{j}{2} \omega_2 - \frac{j}{2} \omega_1)(-\frac{j}{2} \omega_1 - \frac{j}{2} \omega_1 \sqrt{1 - \frac{1}{2}}) \]

\[ = 2(\frac{j}{2} \omega_2 - \frac{j}{2} \omega_1)(\frac{j}{2} \omega_1 + \frac{j}{2} \omega_1 \sqrt{1 - \frac{1}{2}}) \]

(4.25)

\[ s_3(\overline{a_1}) = s_3(-\frac{j}{2} \omega_1 - \frac{j}{2} \omega_1 \sqrt{1 - \frac{1}{2}}) = 2(\frac{j}{2} \omega_3 - \frac{j}{2} \omega_1)(-\frac{j}{2} \omega_1 - \frac{j}{2} \omega_1 \sqrt{1 - \frac{1}{2}}) \]

\[ = 2(\frac{j}{2} \omega_3 - \frac{j}{2} \omega_1)(\frac{j}{2} \omega_1 + \frac{j}{2} \omega_1 \sqrt{1 - \frac{1}{2}}) \]

(4.26)

With the help of Eqs. (4.23) to (4.26), the real and imaginary parts of (4.22) can be evaluated. It looks very messy but when numerical values are substituted in, the work is much simplified.

Likewise, we can evaluate \( \kappa_{12} \), \( \kappa_{22} \), \( \kappa_{13} \) and \( \kappa_{23} \).

\[ \kappa_{12} = \frac{1}{2 \omega_{mn}^2 \sqrt{1 - \frac{1}{2}} \mathcal{I}_F} \frac{\mathcal{I}_p}{2} \]

(4.27)

\[ \kappa_{22} = \frac{1}{\omega_{mn}^2} \left( \frac{R_{F_2}}{\mathcal{I}_F} + \frac{\mathcal{I}_2}{\sqrt{1 - \frac{1}{2}}} \right) \]

(4.28)

*Practical evaluation of polynomial function of complex variables is much simplified by applying DeMoivre's theorem graphically. See Appendix D.*
\[ \kappa_{18} = \frac{1}{2} \frac{\omega_{nn3}^2}{\omega_{nn3}} \frac{1}{\sqrt{1 - \kappa_{20}^2}} \frac{I_{F_2}}{F_2} \]  \hspace{1cm} (4.29) \\
\[ \kappa_{08} = \frac{1}{\omega_{nn3}^3} \left( \frac{R_{F_3}}{F_3} + \frac{\kappa_{30}}{\sqrt{1 - \kappa_{20}^2}} \frac{I_{F_3}}{F_3} \right) \]  \hspace{1cm} (4.30) \\
where \\
\[ F_2 = \frac{Q(d_2)}{s_1(d_2) s_3(d_2)} = \frac{Q(d_2) s_1(d_2) s_3(d_2)}{s_1(d_2) s_1(d_2) s_2(d_2) s_3(d_2)} \]  \hspace{1cm} (4.31) \\
\[ F_3 = \frac{Q(d_3)}{s_1(d_3) s_3(d_3)} = \frac{Q(d_3) s_1(d_3) s_3(d_3)}{s_1(d_3) s_1(d_3) s_2(d_3) s_3(d_3)} \]  \hspace{1cm} (4.32) \\

The expansion of \( F_2 \) and \( F_3 \) can be made with the same procedure as has been used for \( F_1 \).

When the numerical evaluation of all the constants \( \kappa_{01}, \kappa_{11}, \kappa_{02}, \kappa_{12}, \kappa_{03} \) and \( \kappa_{18} \) is completed, Eq. (4.15) can be written in the following form.

\[ \frac{Q(d)}{S(d)} = \frac{\int_{\omega_{nn1}} \kappa_{11} d + \kappa_{01} \omega_{nn1}^2}{d^2 + 2 \int_{\omega_{nn1}} d + \omega_{nn1}^2} + \frac{\int_{\omega_{nn2}} \kappa_{12} d + \kappa_{22} \omega_{nn2}^2}{d^2 + 2 \int_{\omega_{nn2}} d + \omega_{nn2}^2} \]

\[ + \frac{\int_{\omega_{nn3}} \kappa_{13} d + \kappa_{33} \omega_{nn3}^2}{d^2 + 2 \int_{\omega_{nn3}} d + \omega_{nn3}^2} \]  \hspace{1cm} (4.33)  \\

Hence \\
\[ \frac{Q(d)}{S(d)} = \frac{\int_{\omega_{nn1}} \kappa_{11} d + \kappa_{01} \omega_{nn1}^2}{d^2 + 2 \int_{\omega_{nn1}} d + \omega_{nn1}^2} + \frac{\int_{\omega_{nn2}} \kappa_{12} d + \kappa_{22} \omega_{nn2}^2}{d^2 + 2 \int_{\omega_{nn2}} d + \omega_{nn2}^2} \]

\[ + \frac{\int_{\omega_{nn3}} \kappa_{13} d + \kappa_{33} \omega_{nn3}^2}{d^2 + 2 \int_{\omega_{nn3}} d + \omega_{nn3}^2} \]  \hspace{1cm} (4.34)  \\

Each of the three terms on the right side of Eq. (4.34) represents one of the three components of vibratory motion (steady and transient response). Each one has its
own steady state response and the respective transient response. The transient response is referred to this component steady state response. In symbols,

\[
\text{Steady state } = \frac{Q(0)}{S(0)} = \frac{q_1(0)}{s_1(0)} + \frac{q_2(0)}{s_2(0)} + \frac{q_3(0)}{s_3(0)}
\]

or \[
\frac{Q(0)}{S(0)} = q_1 + q_2 + q_3 \quad \text{(4.35)}
\]

Such fractionalization (as into several quadratic fractions) which is made before operating on the disturbance function is defined as characteristic decomposition of the controlled system.

After being characteristically decomposed, each component then operates on the unit function and the following formula is common to all:

\[
\frac{j \kappa_1 \omega_{nn} d + \kappa_0 \omega_{nn}^2}{d^2 + 2j \omega_{nn} d + \omega_{nn}^2} 1 = \kappa_0 \left[ 1 - \frac{e^{-j \omega_{nn} \tau}}{\sqrt{1 - \frac{\kappa_0}{\kappa_0} - 1}^2} \sin\left(\sqrt{1 - \frac{\kappa_0}{\kappa_0} - 1} \omega_{nn} \tau + \phi\right) \right]
\]

or \[
\frac{j \kappa_1 \omega_{nn} d + \kappa_0 \omega_{nn}^2}{d^2 + 2j \omega_{nn} d + \omega_{nn}^2} 1 = \kappa_0 \left[ 1 - \frac{e^{-j \omega_{nn} \tau}}{\sqrt{1 - \frac{\kappa_0}{\kappa_0} - 1}^2} \sin\left(\sqrt{1 - \frac{\kappa_0}{\kappa_0} - 1} \omega_{nn} \tau \right) \right]
\]

or \[
\frac{j \kappa_1 \omega_{nn} d + \kappa_0 \omega_{nn}^2}{d^2 + 2j \omega_{nn} d + \omega_{nn}^2} 1 = \kappa_0 \left[ 1 - \frac{1}{\sqrt{1 - \frac{\kappa_0}{\kappa_0} - 1}^2} \left( \frac{\kappa_0}{\kappa_0} - 1 \right)^2 + 1 \right] e^{-j \omega_{nn} \tau} \cos\left(\sqrt{1 - \frac{\kappa_0}{\kappa_0} - 1} \omega_{nn} \tau + \phi\right) \quad \text{**(4.37)}
\]

where \( \phi = \tan^{-1} \sqrt{\frac{1 - \frac{\kappa_0}{\kappa_0} - 1}{\frac{\kappa_0}{\kappa_0} - 1}} \)

\text{(4.36a)}

*Derivation based upon basic operational formula.

**Same result can be obtained by carefully manipulating Eqs. (4.13) and (4.11)b and counting the steady state response in them.
and $\phi' = \tan^{-1} \left( \frac{\frac{\kappa}{\kappa_0} - 1}{\sqrt{1 - \frac{\kappa}{\kappa_0}^2}} \right)$ (4.37a)

When the forcing function is of sinusoidal shape continuous with respect to time: that is,

$$I(t) = \sin(\mathcal{N}_F n \tau)$$ (4.38)

where the magnitude of the forcing sinusoidal function is assumed to be unity, $\mathcal{N}_F n$ means the angular frequency of the forcing function in non-dimensional units matched to non-dimensional time unit ($T$).

The steady state response from each characteristic component to such sinusoidal forcing function will be

$$R(t)_{ss} = \mu_F \sin(\mathcal{N}_F n \tau + \phi_F)$$ (4.39)

where

$$\mu_F = \frac{\kappa_0 \sqrt{1 + (\frac{\kappa}{\kappa_0})^2}}{\sqrt{(1-\beta^2)^2 + (2\beta \beta')^2}}$$ is defined as sinusoidal magnification factor.

$$\phi_F = \tan^{-1} \frac{\kappa}{\kappa_0} \beta - \tan^{-1} \frac{2\beta \beta'}{1-\beta^2}$$ is defined as sinusoidal phase shift.

$$\beta = \frac{\mathcal{N}_F n}{\omega_{nn}} = \frac{\omega_F}{\omega_n}$$ is defined as forcing frequency ratio.

(4.40)

(4.41)

(4.42)

Both $\mu_F$ and $\phi_F$ are referred to steady state response of the particular characteristic component.

When all the (three) components are evaluated, they can be put together with proper attention to magnitude and phase shift and the response curve, whether steady state or transient, can be plotted.
46. Characteristic Decomposition with One or More Components

Overdamped

Oftentimes some of the components are overdamped. The process of decomposing is somewhat simplified. Now assuming component 1 of Eq. (4.15) being an overdamped one,

\[ \frac{q_1(d)}{S_1(d)} = \frac{q_a_1(d)}{S_a_1(d)} + \frac{q_b_1(d)}{S_b_1(d)} \quad (4.42) \]

where

\[ S_a_1(d) = d + (\gamma + \sqrt{\gamma^2 - 1}) \omega_{nn}, \quad c_a_1 = -(\gamma + \sqrt{\gamma^2 - 1}) \quad (4.43a) \]
\[ S_b_1(d) = d + (\gamma - \sqrt{\gamma^2 - 1}) \omega_{nn}, \quad c_b_1 = -(\gamma - \sqrt{\gamma^2 - 1}) \quad (4.43b) \]
\[ q_a(d) = k_a_1 (\gamma + \sqrt{\gamma^2 - 1}) \omega_{nn}, \quad (4.43c) \]
\[ q_b(d) = k_b_1 (\gamma - \sqrt{\gamma^2 - 1}) \omega_{nn} \quad (4.43d) \]

Rewrite Eq. (4.16) in the following form

\[ \frac{Q(d)}{S(d)} = \frac{q_{a_1}(d)S_{b_1}(d)S_3(d)S_2(d) + q_{b_1}(d)S_{a_1}(d)S_3(d)S_2(d) + q_{2}(d)S(d)S_3(d) + q_3(d)S_3(d)S_2(d)}{S(d)} \quad (4.45) \]

Substituting \( d_a \) into (4.45) we have

\[ Q(d_a) = q_{a_1}(d_a)S_{b_1}(d_a)S_3(d_a)S_2(d_a) \]

or

\[ q_{a_1}(d_a) = \frac{Q(d_a)}{S_{b_1}(d_a)S_3(d_a)S_2(d_a)} \quad (4.46) \]

\[ k_a_1 (\gamma + \sqrt{\gamma^2 - 1}) \omega_{nn} = \frac{Q(-\gamma - \sqrt{\gamma^2 - 1})}{-2 \sqrt{\gamma^2 - 1} S_2(-\gamma - \sqrt{\gamma^2 - 1})S_3(-\gamma - \sqrt{\gamma^2 - 1}) \omega_{nn}} \]

\[ k_a_1 = \frac{2 \sqrt{\gamma^2 - 1} d_a S_2(d_a) S_3(d_a) \omega_{nn}^2}{Q(d_a)} \quad (4.47) \]

and likewise

\[ k_{b_1} = \frac{Q(d_{b_1})}{-2 \sqrt{\gamma^2 - 1} d_{b_1} S_2(d_{b_1}) S_3(d_{b_1}) \omega_{nn}^2} \quad (4.48) \]
When this is done, the fraction \( \frac{Q(d)}{S(d)} \) can be reduced to a simpler one with two degrees reduced in \( d \) in the denominator.

Let us represent this reduced fraction by

\[
\frac{Q(d)}{S(d)} = \frac{Q(d)}{S(d)} \left[ \frac{q_{a_1}(d)}{s_{a_1}(d)} + \frac{q_{b_1}(d)}{s_{b_1}(d)} \right] \tag{4.49}
\]

The reduced fraction contains one less degree of freedom than the original one. It can be decomposed with much less effort through the same procedure, but in a simpler way, should be followed as described in Section 45.

Evaluation of \( \frac{q_a(d)}{S_a(d)} \) and \( \frac{q_b(d)}{S_b(d)} \) is very simple:

\[
\frac{q_a(d)}{S_a(d)} = \frac{\kappa_{a_1}(f_1 + \sqrt{f_2^2 + i}) \omega_{nn}}{d + (f_1 + \sqrt{f_2^2 + i}) \omega_{nn}} \tag{4.50}
\]

Likewise,

\[
\frac{q_{b_1}(d)}{S_{a_1}(d)} = k_{b_1} \left[ 1 - \epsilon^{-\left(f_1 - \sqrt{f_2^2 + i}\right) \omega_{nn} \tau} \right] \tag{4.50a}
\]

### 47. Characteristic Decomposition when \( S(d) \) Has Repeating Quadratic Factors

Let us again limit our scope to \( \frac{Q(d)}{S(d)} \) in which the highest power in \( S(d) \) is six and that in \( Q(d) \) five or less. And assume \( S_2(d) = S_3(d) \) and \( S_1(d) \) is oscillatory.

\[
\frac{Q(d)}{S(d)} = \frac{q_1(d)}{S_1(d)} + \frac{q_2(d)}{S_2(d)} + \frac{q_2r(d)}{S_2^2(d)} \tag{4.51}
\]

*The reduction of \( \frac{Q(d)}{S(d)} \) into \( \frac{Q(d)}{S(d)} \) is also applicable to the case where all components are oscillatory. In such a case,

\[
\frac{Q(d)}{S(d)} = \frac{Q(d)}{S(d)} - \frac{q_1(d)}{S_1(d)}
\]
where $S_2^2(d) = \left[ S_2(d) \right]^2 = (d^2 + 2 \sum \omega_{nn} a + \omega_{nn}^2)^2$ \hspace{1cm} (4.51a)

$q_{2r}(d) = \int \omega_{nn}^2 \kappa_{12r} d + \kappa_{02r} \omega_{nn}^2$ \hspace{1cm} (4.51b)

with subscript $r$ indicating the belonging of repeating factor.

\[ \therefore \quad Q(d) = q_1(d)S_2^2(d) + q_2(d)S_1(d)S_2(d) + q_{2r}(d)S_1(d) \] \hspace{1cm} (4.52)

Substitute $d_2$ in equation (4.52) so that $S_2(d_2) = 0$

\[ \therefore \quad q_{2r}(d_2) = \frac{Q(d_2)}{S(d_2)} = F_{2r} \] \hspace{1cm} (4.53)

By the same procedure used in Section 45 we obtain

\[ k_{12r} = \frac{1}{\omega_{nn}^2 \xi_2 \sqrt{1-\frac{\xi_2^2}{\eta_2^2}}} F_{2r} \] \hspace{1cm} (4.54)

\[ k_{02r} = \frac{1}{\omega_{nn}^2 \left[ \frac{R F_{2r}}{\sqrt{1-\frac{\xi_2^2}{\eta_2^2}}} F_{2r} \right] } \] \hspace{1cm} (4.55)

Now the reduced fraction can be obtained as

\[ \frac{Q(d)}{S(d)} = \frac{Q(d) - q_{2r}(d)}{S(d) - S_2(d)} = \frac{Q(d) - q_{2r}(d)}{S(d) - S_2(d)} \] \hspace{1cm} (4.55a)

From this reduced fraction the constants $\kappa_n$, $\kappa_{01}$, $\kappa_{12}$, and $\kappa_{02}$ can be evaluated with less effort.

48. Transient Response with Repeating Binomial Factors in $S(d) = 0$

Let the highest power of the binomial factor be $r$, then

\[ \frac{k_r \omega_{nn}^r}{(d + d_o \omega_{nn})^r} \] \hspace{1cm} (4.56)
49. Transient Response with Repeating Quadratic Factors in $S(d) = 0$

Suppose $\frac{q(d)}{S(d)}$ takes the following form:

$$
\frac{\omega_{nn}^4}{(d^2 + 2\omega_{nn}d + \omega_{nn}^2)^2} = \frac{\omega_{nn}^4}{(d + d_a \omega_{nn})^2(d + d_b \omega_{nn})^2}
$$

where $d_a = \ell + \sqrt{\ell^2 - 1}$, $d_b = \ell - \sqrt{\ell^2 - 1}$ when $\ell > 1$ (4.57)

or $d_a = \ell + j\sqrt{\ell^2 - 1}$, $d_b = \ell - j\sqrt{\ell^2 - 1}$ when $\ell < 1$ (4.57)

$$
\frac{\omega_{nn}^4}{(d + d_a \omega_{nn})^2(d + d_b \omega_{nn})^2} \frac{1}{1 - \frac{1}{(d_a - d_b)^2} \left[ \frac{\omega_{nn}^2}{(d + d_a \omega_{nn})^2} + \frac{\omega_{nn}^2}{(d + d_b \omega_{nn})^2} - \frac{2}{d_a - d_b} \left[ \frac{\omega_{nn}}{d + d_a \omega_{nn}} - \frac{\omega_{nn}}{d + d_b \omega_{nn}} \right] \right]} 
$$

$$
= \frac{1}{(d_a - d_b)^2} - \frac{1}{(d_a - d_b)^2} \left[ \frac{d_a - 3d_b}{d_a^2} \epsilon - \frac{d_b}{d_b^2} \epsilon \right] 
$$

$$
+ (d_a - d_b) \omega_{nn} \tau \left[ \frac{\epsilon}{d_b} - \frac{\epsilon}{d_a} \right]
$$

$$
\frac{\omega_{nn}^4}{(d + d_a \omega_{nn})^2(d + d_b \omega_{nn})^2} \frac{1}{1 - \frac{1}{(d_a - d_b)^3} \left[ \frac{\omega_{nn}^3}{(d + d_a \omega_{nn})^2} + \frac{\omega_{nn}^3}{(d + d_b \omega_{nn})^2} - \frac{2}{(d_a - d_b)^3} \left( \epsilon - \frac{d_b}{d_a} \omega_{nn} \tau \right) \right]} 
$$

$$
= \frac{1}{(d_a - d_b)^3} \left( \epsilon - \frac{d_b}{d_a} \omega_{nn} \tau \right)
$$

and

$$
\frac{\omega_{nn}^3 d}{(d + d_a \omega_{nn})^2(d + d_b \omega_{nn})^2} \frac{1}{1 - \frac{1}{(d_a - d_b)^3} \left[ \frac{\omega_{nn}^3}{(d + d_a \omega_{nn})^2} + \frac{\omega_{nn}^3}{(d + d_b \omega_{nn})^2} - \frac{2}{(d_a - d_b)^3} \left( \epsilon - \frac{d_b}{d_a} \omega_{nn} \tau \right) \right]} 
$$

$$
+ \frac{\omega_{nn} \tau}{(d_a - d_b)^2} \left( \epsilon - \frac{d_b}{d_a} \omega_{nn} \tau \right)
$$

Hence

$$
\frac{1}{(d + d_a \omega_{nn})^2(d + d_b \omega_{nn})^2} \frac{\epsilon - \frac{d_b}{d_a} \omega_{nn} \tau}{\epsilon - \frac{d_b}{d_a} \omega_{nn} \tau}
$$

$$
= k_{or} \left\{ 1 - \frac{1}{(d_a - d_b)^3} \left[ \left( \frac{d_a - 3d_b}{d_a^2} + \frac{2k_{ir}}{k_{or}} \right) \epsilon - \frac{d_b}{d_b^2} \epsilon \right] - \left( \frac{d_a - 3d_b}{d_a^2} + \frac{2k_{ir}}{k_{or}} \right) \epsilon - \frac{d_a}{d_a} \omega_{nn} \tau \right\}
$$

$$
- \frac{\omega_{nn} \tau}{(d_a - d_b)^3} \left[ \left( \frac{1}{d_a} - \frac{k_{ir}}{k_{or}} \right) \epsilon - \frac{d_b}{d_a} \omega_{nn} \tau \right]
$$

$$
+ \left( \frac{1}{d_a} - \frac{k_{ir}}{k_{or}} \right) \epsilon - \frac{d_a}{d_a} \omega_{nn} \tau \right\}
$$

(4.60)
Eqs. (4.58), (4.59) and (4.60) are convenient for application when \( \mathcal{J} > 1 \).

When \( \mathcal{J} < 1 \), substitute the complex values of \( \delta_a \) and \( \delta_b \) into Eq. (4.58). After long manipulation, we get

\[
\frac{\omega_{nn}}{\sqrt{d^2 + 2\omega_{nn} d + \omega_{nn}^2}} \frac{1}{1 - \frac{e^{-i\omega_{nn} \tau}}{1 - \frac{1}{\mathcal{J}}}} \left[ \frac{1}{\sqrt{1 - \frac{1}{\mathcal{J}^2}}} \sin \left( \sqrt{1 - \frac{1}{\mathcal{J}^2}} \omega_{nn} \tau + \phi_{or} \right) + \frac{\omega_{nn}^2}{2} \sin \left( \sqrt{1 - \frac{1}{\mathcal{J}^2}} \omega_{nn} \tau + \phi_{or}' \right) \right] (4.61)
\]

where \( \phi_{or} = \tan^{-1} \left( \frac{2(1 - \mathcal{J}^2)^{1/2}}{\mathcal{J}(3 - 2\mathcal{J}^2)} \right) \); \( \phi_{or}' = -\tan^{-1} \left( \frac{5}{\sqrt{1 - \mathcal{J}^2}} \right) \) (4.61a)

Eq. (4.59) can be transformed into the following form for \( \mathcal{J} < 1 \)

\[
\frac{\omega_{nn}^2 \delta}{\sqrt{d^2 + 2\omega_{nn} d + \omega_{nn}^2}} \frac{1}{1 - \frac{e^{-i\omega_{nn} \tau}}{1 - \frac{1}{\mathcal{J}}}} \left[ \frac{1}{\sqrt{1 - \frac{1}{\mathcal{J}^2}}} \cos \left( \sqrt{1 - \frac{1}{\mathcal{J}^2}} \omega_{nn} \tau \right) - \frac{i}{\sqrt{1 - \frac{1}{\mathcal{J}^2}}} \sin \left( \sqrt{1 - \frac{1}{\mathcal{J}^2}} \omega_{nn} \tau \right) \right] (4.62)
\]

Careful manipulation of Eq. (4.60) gives the following result for \( \mathcal{J} < 1 \)

\[
\frac{\kappa_{ir} \omega_{nn} \delta + \kappa_{or} \omega_{nn}^2}{\sqrt{d^2 + 2\omega_{nn} d + \omega_{nn}^2}} \frac{1}{1 - \frac{e^{-i\omega_{nn} \tau}}{1 - \frac{1}{\mathcal{J}}}} \left[ 1 - \frac{e^{-i\omega_{nn} \tau}}{2(1 - \frac{1}{\mathcal{J}})} \left[ -\sigma \omega_{nn} \tau \cos \left( \sqrt{1 - \frac{1}{\mathcal{J}^2}} \omega_{nn} \tau + \phi_{or}' \right) \right] \right] (4.63)
\]

where \( \sigma = \left[ \left( \frac{\kappa_{ir}}{\kappa_{or}} + \mathcal{J} \right)^2 + 1 - \frac{1}{\mathcal{J}^2} \right]^{1/2} \) (4.63a)

\[
\sigma = \left[ \left( \frac{\kappa_{ir}}{\kappa_{or}} + \frac{(2 - \mathcal{J}^2)(3 - 2\mathcal{J}^2)}{2\sqrt{1 - \mathcal{J}^2}} \right)^2 + (2 - \mathcal{J}^2)^2(1 - \mathcal{J}^2)^2 \right]^{1/2} \frac{1}{\sqrt{1 - \mathcal{J}^2}} (4.63b)
\]

\[
\phi_{or} = \tan^{-1} \left( \frac{(2 - \mathcal{J}^2)(1 - \mathcal{J}^2)^2}{\frac{\kappa_{ir}}{\kappa_{or}} + \frac{(2 - \mathcal{J}^2)(3 - 2\mathcal{J}^2)}{2\sqrt{1 - \mathcal{J}^2}}} \right) (4.63c)
\]

\[
\phi_{or}' = \tan^{-1} \left( \frac{\sqrt{1 - \mathcal{J}^2}}{\frac{\kappa_{ir}}{\kappa_{or}} + \mathcal{J}} \right) (4.63d)
\]
Summary

From the above analysis, the response of an automatically controlled system on unit function can be calculated first by decomposing the function $Q(d)$ into elementary characteristic component of one degree of freedom and then each of the components will respond to the step function simultaneously. For one-degree-of-freedom component, the unit response can be easily found by the nondimensional operational formulae developed in this chapter. Nondimensional operation formulae of unit response with repeating roots in $S(d) = 0$ are also developed for both $f \geq 1$ and $f = 1$. 

50. **Surging Error or Surge**

When a system is disturbed, its motion goes off from equilibrium state and gradually comes back to the original state of equilibrium, or continues on until it reaches another equilibrium state. During this course of change, the deviation from the equilibrium state (original or the new one) may reach a largest magnitude which shall never be exceeded until another disturbance comes to affect it. Such maximum deviation is defined as surge error, or often it is named the first surging error. Mathematically it can be expressed as:

\[
R(t) = \frac{Q(o)}{S(o)} + \sum_{k} \frac{Q(d_k)}{d_k S'(d_k)} e^{-d_k t}
\]

where \( t \) satisfies \( R'(t) = 0 \) for the first occurrence.

However, when there are a number of components, it is difficult to solve \( t \) from the condition \( R'(t) = 0 \) analytically. When the response \( R(t) \) is plotted, the surging error can be measured then. Actually, only the predominant component, usually the low frequency one, plays an important role on this surging error. Differentiation may therefore be applied to that component only and solve \( t' \) from \( r_f'(t) = 0 \) and find \( r_f(t') \) instead of \( R(t') \). Such approximate value usually gives a good check to that measured from the response curve.
51. Surging Disturbance and Unit Surging Disturbance Function

Step function disturbance is justifiable in many cases. Sudden application of D.C. voltage to a network falls in this type of disturbance. When an automatic direction finder is called to action by suddenly switching in, this also belongs to this type of function. However, "rough air" does not possess step-function characteristics unless the airplane is flown into a storm where air current rises steadily. A gust, whether horizontal, vertical or rotary, does not keep its magnitude. In fact, it rises to a certain magnitude and then dies away. Later on a second gust follows. Physically, the rise of a gust from zero to a certain magnitude takes time no matter how fast it is. That is, \( \frac{dw}{dt} \neq \infty \) as what step function is. Such rise and fall of a gust is evident, however, due to lack of experimental data of such rise and fall; true representation of a gust train is impossible.

However, a single gust probably can be represented by \( Kte^{-bt} \), for such function has its surging phenomenon; that is, it rises to maximum at \( t = \frac{1}{b} \) and then dies away gradually to zero. When \( b \) is larger, it reaches its maximum sooner and then dies away faster and vice versa.

It is more convenient to study the effect of such surging disturbance by keeping its maximum magnitude at unity.

Let \( I_s(t) = e^{bt}te^{-bt} \) be such function.

Then \( I_s'(t) = e^{bt}b(e^{-bt}(1-bt)) \) for \( t > 0 \).

Put \( I_s''(t) = 0 \); we have \( t = \frac{1}{b} \).
Substitute $t = \frac{1}{b}$ into equation (4.64), and we have:

$$I_g(t)_{\text{max}} = 1$$

(4.65)

where subscript $s$ refers to "surging", and the function

$I_g(t) = e^{bt}t$ is defined as unit surging disturbance function or unit surging input function. The larger the constant $b$ is, the faster the surging phenomenon is. When $b$ approaches $\infty$, the surging phenomenon approaches a quick surging impulse with peak value equal to unity.

The unit surging disturbance can be put in non-dimensional form:

Let $\frac{t}{T} = \tau$  

(4.66)

where $T = \frac{m}{c}$, equation (1.67), for the longitudinal motion of the airplane. Then $bt = bT \tau$ where $bT$ is non-dimensional damping.

Let $bT = \xi, \omega_n, \omega_{nf} = \xi, \omega_{nf}$  

(4.67)

where $\xi$ is damping ratio of the surging disturbance by assuming its natural frequency to be the same as that of the characteristic component $1$, etc.

$$I_g(\tau) = \epsilon \xi, \omega_{nf} \tau e^{-\xi, \omega_{nf} \tau} 1$$

(4.68)

For simplicity, just drop the subscript $f$:

$$I_g(\tau) = \epsilon \xi, \omega_n \tau e^{-\xi, \omega_n \tau} 1$$

(4.68a)

A plot showing $I_g(\tau) = \epsilon \xi, \omega_n \tau e^{-\xi, \omega_n \tau} 1$ is shown in Figure XXIII with $\xi$ as varying parameter to show its effect upon the rapidity of surging.
FIG. XXII

UNIT SURGING DISTURBANCE FUNCTION WITH

VARIOUS APPARENT SURGING FACTORS $\xi$

\[ I_s(t) = \text{INSTANTANEOUS MAGNITUDE OF UNIT SURGING DISTURBANCE} \]

\[ t = \text{NON-DIMENSIONAL TIME} \]
As the rapidity of surging is determined by $\xi$ when the disturbance is applied to the characteristic component, and because $\xi \omega_m$ = constant for a certain disturbance, therefore it will be a fast surging disturbance when the dealt component is a slow oscillatory one, and a slow surging disturbance when the dealt component is a fast one.

Differentiate equation (4.68)b:

$$I_{sl}'(\tau) = \varepsilon \xi \left[ e^{-\xi \tau} (1 - \xi \tau) \right] \quad \text{for } \tau > 0 \quad (4.69)$$

When $\tau$ approaches zero from the positive side, we have:

$$I_{sl}'(\tau) \bigg|_{\tau \to 0} = \varepsilon \xi \quad (4.69)a$$

Equation (4.69)a indicates the fact that the apparent (apparent to the characteristic component we are dealing) initial surging speed is proportional to $\xi$. Therefore, $\xi$ is defined as apparent surging factor and $\varepsilon \xi$ is defined as apparent initial surging speed.

52. Response Due to Surging Disturbance

Repeat equation (4.01) with addition of subscripts; it will be the operational form of unit surging response:

$$R_{hs}(\tau) = \frac{Q_h(d)}{S(d)} I_s(\tau) 1 \quad (4.70)$$

To evaluate equation (4.70), several directions of approach can be equally applied. But for the interest of engineers, we shall first follow the method of characteristic decomposition to have:

$$\frac{Q(d)}{S(d)} = \sum_{f=1}^{f=\infty} \frac{q_f(d)}{S_f(d)} + \sum_{n=1}^{n=\infty} \sum_{r=1}^{r=\infty} \frac{q_{nr}(d)}{S_r(d)} \quad (4.71)$$
When the above procedure has been done, the effect of surging disturbance can be studied component after component.

Now assuming one of the components may be represented by the expression:

\[
\frac{q(d)}{S(d)} = \frac{\kappa_0 \omega_n^2}{d^2 + 2\xi \omega_n d + \omega_n^2} \tag{4.71a}
\]

Then applying the surging disturbance function, we have:

\[
\tilde{S} (\tau) = \frac{\kappa_0 \omega_n^2}{d^2 + 2\xi \omega_n d + \omega_n^2} \epsilon \xi \omega_n \tau \epsilon^{-\xi \omega_n \tau} \tag{4.72}
\]

By application of transformation formulae of operational calculus we may derive the expression \( r_s (\tau) \) according to the following steps:

* 

\[
r_s (\tau) = (\epsilon^{-\xi \omega_n \tau}) \frac{\kappa_0 \omega_n^2}{d + (d_1 - \xi) \omega_n} \frac{\xi \omega_n \tau}{d + (d_1 - \xi) \omega_n} \tag{4.72a}
\]

where 

\[
d_1, \quad \bar{d}_1 = \xi \pm i \sqrt{1 - \xi^2} \]

\[
r_s (\tau) = (\epsilon^{-\xi \omega_n \tau}) \frac{\kappa_0 \omega_n^2}{d + (d_1 - \xi) \omega_n} \frac{\xi \omega_n \tau}{d + (d_1 - \xi) \omega_n} \tag{4.72c}
\]

\[
r_s (\tau) = \epsilon \kappa_0 \xi \epsilon^{-\xi \omega_n \tau} \left\{ \frac{\omega_n}{(d_1 - \xi)(d_1 - \xi)} d - \frac{\omega_n}{(d_1 - d_1)(d_1 - \xi)} [d + (d_1 - \xi) \omega_n] \right. \]

\[
- \left. \frac{\omega_n}{(d_1 - \bar{d}_1)(d_1 - \xi)} [d + (d_1 - \xi) \omega_n] \right\} \tag{4.72d}
\]

\[
\therefore \quad r_s (\tau) = \kappa_0 \epsilon \xi \left[ \frac{\epsilon^{-\xi \omega_n \tau}}{\sqrt{1 - \xi^2}} \cos \left( \frac{\epsilon^{-\xi \omega_n \tau}}{\sqrt{1 - \xi^2}} \omega_n \tau + \phi \right) + \epsilon^{-\xi \omega_n \tau} \left( \mu_1 \omega_n \tau + \mu_2 \omega_n \tau \right) \right] \tag{4.73}
\]

* by formula 293, p. 134, Bush's "Operational Circuit Analysis"

** by formula 205, p. 117, Bush's "Operational Circuit Analysis"

*** obtained by characteristic decomposition
where

$$\kappa_1 = \frac{1}{(\xi - \xi)^2 + 1 - \xi^2} = \kappa_1^\prime$$  \hspace{1cm} (4.73a)

$$\kappa_2^\prime = \frac{2(\xi - \xi)}{[(\xi - \xi)^2 + 1 - \xi^2]^2} = 2\kappa_2^\prime(\xi - \xi)$$  \hspace{1cm} (4.73b)

$$\kappa_3^\prime = \frac{1}{(\xi - \xi)^2 + 1 - \xi^2} = \kappa_3^\prime$$  \hspace{1cm} (4.73c)

$$\phi = \tan^{-1} \left( \frac{(\xi - \xi)^2 - (1 - \xi^2)}{2(\xi - \xi) \sqrt{1 - \xi^2}} \right)$$  \hspace{1cm} (4.73d)

It is interesting to notice that:

(1) The oscillatory component keeps its essential characteristic; that is, its damping, and angular velocity.

(2) The surging disturbance is magnified by a factor equal to $$\kappa_0$$.

(3) An additional subsiding component takes place of the steady state (when unit step disturbance is applied).

(4) However, the guiding magnitude factor* of the oscillatory component is multiplied by a factor equal to $$\xi \kappa_5^\prime$$, and the phase shift is modified too.

(5) When $$\xi$$ is very large or it is an apparent fast surging disturbance, all these factors $$\xi \kappa_5^\prime$$, $$\xi \kappa_2^\prime$$, and $$\xi \kappa_3^\prime$$ become small and approach $$\frac{1}{\xi}$$, $$\frac{2}{\xi}$$ and $$\frac{1}{\xi}$$ as limits. Physically it means that

* Compare to equation (4.36)
this component is so inert to the surging disturbance which is very fast apparent to the system (or rather to this component we are dealing). On the other hand, when $\xi$ is small, their magnitudes are small too because of this small factor $\xi$. Physically it means a very sluggish surging disturbance would be so gentle that it can only affect or disturb the system unnoticeably.

It is therefore believed that the system will be disturbed most violently by a surging disturbance of certain particular apparent surging factor.

At first glance, one would suggest determining such maximum disturbance by maximizing $\xi f$, $\xi \mu_{\xi}$ and $\mu_{\xi}$. Unfortunately, this cannot be done for simultaneous occurrence. Besides, when a system is comprised of several degrees of freedom, analytical maximization is impossible. The conclusion can be safely obtained after the response is plotted.

(6) If the numerator of the right side of equation (4.71) contains a $d$ term such as $\kappa_i w_\eta d$, the complete solution of such component when disturbed by surging disturbance shall be:

$$r_s(\tau) + \frac{\kappa_i}{\kappa_0} r_s'(\tau)$$

where $r_s(\tau)$ is the equation of (4.73) and $r_s'(\tau) = \frac{d}{d\tau} r_s(\tau)$

(7) The simultaneous effect of the same surging disturbance upon all the characteristic components will yield as many surging components and as many subsiding components as the number of characteristic components. Each of them has the
same factor \( e^{-\frac{3}{2}\omega_m t} \), but the factors \( k_0 \frac{3}{2} k_0 \frac{3}{2} \) and \( k_0 \frac{5}{2} k_0 \frac{5}{2} \) are all different. The total surging and subsiding components can be written as:

\[
e^{-\frac{3}{2}\omega_m t} \left\{ \sum \left( k_0 \frac{3}{2} k_0 \frac{3}{2} \right)_f + \left[ \sum \left( k_0 \frac{5}{2} k_0 \frac{5}{2} \right)_f \right] \xi \omega_{mn} t \right\}
\]

Additional terms that contain \( (K_1)_f \) should be included.

53. Substitution of Operational Expression for the Unit Surging Disturbance

It is quite tedious to follow the analysis given in Paragraph 52. An alternative method is briefly formulated here.

The unit surging disturbance \( e^{-\frac{3}{2}\omega_m t} e^{-\frac{3}{2}\omega_m t} \) can be substituted by its operational form:

\[
I_d(t) = e^{-\frac{3}{2}\omega_m t} e^{-\frac{3}{2}\omega_m t} I = \frac{e^{-\frac{3}{2}\omega_m d}}{(d + \frac{3}{2}\omega_m)^2} I
\]  \( (4.74) \)

It should be noted that \( \frac{3}{2}\omega_m \) is a constant for a particular unit surging disturbance.

Equation (4.74) is then substituted into equation (4.70) and we have:

\[
R_{hs}(\tau) = \frac{Q_h(d)}{S(d)} \cdot \frac{e^{\frac{3}{2}\omega_m d}}{(d + \frac{3}{2}\omega_m)^2} I
\]  \( (4.75) \)

The procedure of characteristic decomposition can be applied with two additional terms, as:

\[
\frac{Q_h(d)e^{\frac{3}{2}\omega_m d}}{S(d)(d + \frac{3}{2}\omega_m)^2} = e^{\frac{3}{2}\omega_m} \left[ \frac{k_s \frac{5}{2} \omega_m}{d + \frac{3}{2}\omega_m} + \left( \sum \frac{S_f(d)}{S_f(d)} + \sum \frac{S_{mn}(d)}{S_{mn}(d)} \right) \right]
\]  \( (4.76) \)

Equation (4.76) gives the separation of the characteristic components, each of which can be solved upon the unit step function.
PART V

PERFORMANCE OF TYPICAL AIRPLANE WITH NONIDEAL CONTROL AND SOME REFINEMENT CONSIDERATIONS
INTRODUCTION

In the foregoing chapters, the general procedure in analyzing an automatically controlled problem has been well established. Stability improvement is usually analyzed before the transient response. In fact, with the aid of the foregoing chapters, we may specify the stability improvement for the problem in which we are interested, and determine the control constant as well as the coupling coefficients thereof. With such control and coupling coefficients, the transient response can be analyzed. If the transient is considered satisfactory, the problem is solved; otherwise the specification of stability improvement should be revised until a satisfactory transient is also obtained. Therefore, actual designing of control is a matter of compromise.

In order to gain the freedom to control the uncontrolled quartic stability function of the longitudinal motion of an airplane, the control specification is fixed by the parasite minor of the stability determinant of the uncontrolled longitudinal motion. The compromise between the stability improvement and the transient improvement will be left entirely to the variation of coupling coefficients.

A numerical example becomes necessary to show the validity and facility of the theory and analysis which have been established in the foregoing chapters. We shall take the Fair-

*See Chapter Two, Part I.
child 22 as a studying subject of which the aerodynamic characteristics have been thoroughly investigated and are believed in the average region of a good many modern airplanes so far as nondimensional derivatives are concerned.

We shall assume the disturbance to be a surging vertical gust of unit magnitude.
CHAPTER FOURTEEN

AIRPLANE CONTROLLED BY PITCHING VELOCITY-
ELEVATOR CONTROL COUPLING

51. Dimensional Data, Nondimensional Aerodynamic Derivatives, and Flight Conditions of the Fairchild 22

(a) Dimensional Data:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wing area (S) (high wing, monoplane)</td>
<td>171 sq. ft.</td>
</tr>
<tr>
<td>Span (b)</td>
<td>32.83 ft.</td>
</tr>
<tr>
<td>Stabilizer area</td>
<td>15.8 sq. ft.</td>
</tr>
<tr>
<td>Elevator area</td>
<td>10.4 sq. ft.</td>
</tr>
<tr>
<td>Tail length (L)</td>
<td>14.69 ft.</td>
</tr>
<tr>
<td>Weight (W)</td>
<td>1600 pounds</td>
</tr>
<tr>
<td>Radius of gyration in pitch</td>
<td>4.41 ft.</td>
</tr>
<tr>
<td>Wing setting</td>
<td>1 degree</td>
</tr>
</tbody>
</table>

(b) Horizontal Flight Condition

Power-off, horizontal flight

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airspeed (U₀)</td>
<td>133 f.p.s.</td>
</tr>
<tr>
<td>Altitude (H) (above sea level)</td>
<td>3000 ft.</td>
</tr>
<tr>
<td>Air density at 3000 ft. above sea level</td>
<td>0.00218 slugs ft³</td>
</tr>
<tr>
<td>C_L</td>
<td>-0.45</td>
</tr>
</tbody>
</table>

(c) Fundamental Units in Nondimensional System

| Unit of mass (m)           | 50 slugs               |
| Unit of time (T)           | 2 seconds              |
| Unit of length (L)         | 15 feet                |
| Compact parameter μ        | 20                      |
(d) The Nondimensional Aerodynamic Derivatives* 

\[
\begin{align*}
\text{x}_u & \quad -0.15** \\
\text{x}_w & \quad 0.40 \\
\text{z}_u & \quad -1.00 \\
\text{z}_w & \quad -4.50 \\
\text{m}_w & \quad -3.00 \\
\text{m}_q & \quad -6.00 \\
\end{align*}
\]

\(\Theta_0, \alpha_0, \text{x}_q, \text{z}_q\) and \(\text{m}_u\) are all assumed negligible compared with the other terms in the stability equations.

52. Uncontrolled Pitching Motion with Vertical Surging Gust

We shall examine the response in pitching and in vertical motion when a vertical surging gust of unit magnitude acts on the uncontrolled airplane.

\[
\frac{Q}{W_0} = \frac{1}{L/T} \left[ \frac{d}{\Delta_0} \left| \begin{array}{cc} -x_u & -x_w \\ -m_u & -m_w \end{array} \right| \right] \frac{\xi \omega_{nn} \text{d}}{(d + \xi \omega_{nn})^2} \]

\[
= \frac{1}{L/T} \frac{d(-d m_w + x_u m_w - m_u x_w)}{\Delta_0} \frac{\xi \omega_{nn} \text{d}}{(d + \xi \omega_{nn})^2} 
\]

\[
= \frac{1}{L/T} \left[ \frac{3 \text{d}(d + .15)}{d^4 + 10.65d^3 + 89.0d^2 + 15.5d + 27.0} \right] \frac{\xi \omega_{nn} \text{d}}{(d + \xi \omega_{nn})^2} \]  

Here, \(Q(d) = 3d(d + .15) = 3d^2 + .45d\)  

\(S(d) = d^4 + 10.65d^3 + 89.0d^2 + 15.5d + 27.0\)

* See Table II, Chapter Two, Part I.

**Figures are transformed from the measured results (Klemin's T.N. 666) and rounded off to the nearest significant figures.
Convert $S(d)$ into $S_\lambda(\lambda)$

$$S_\lambda(\lambda) = \lambda^4 + 4.67\lambda^3 + 17.09\lambda^2 + 1.31\lambda + 1$$

(5.02)

Here $\alpha_3 = 4.67$, $\alpha_2 = 17.09$, $\alpha_1 = 1.31$

Immediately we know that the high frequency component has a larger damping ratio, (because $\alpha_3 > \alpha_1$) and the system is stable (because $\alpha_2 > \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}$).

Then obtain

$$M = \frac{-\frac{4}{\alpha_3^2} \alpha_1^2 \alpha_2^2}{\alpha_3^2} = 0.0072$$

$$N = \frac{-\frac{4}{\alpha_3^2} \alpha_1^2 \alpha_2^2}{\alpha_3^2} = -0.0094$$

Because $M$ is positive and $N$ is negative the system is doubly oscillatory; that is, both high and low components are oscillatory.

Follow the direction of the quartic chart and we shall get

$$\rho_\omega = 16.72$$

$$\omega_r = 00.244, \quad \omega_{nn1} = \omega_{127.2} = 0.556, \quad \omega_{nn2} = 16.72 \quad \omega_{nn1} = 9.31$$

$$\rho_s = 4.52$$

$$s_1 = 0.1245$$

$$s_2 = \rho_s = 0.561$$

Therefore, the characteristic roots in $d$ are:

$$d_1 = (-0.1245 + 1.992)$$

$$\bar{d}_1 = (-0.1245 - 1.992)$$

$$d_2 = (-0.561 + 1.827)$$

$$\bar{d}_2 = (-0.561 - 1.827)$$

(5.03)

Since there is no repeating root in the uncontrolled quartic stability equation, it can be shown* that

*See Appendix C
\[
\frac{Q(d)}{S(d)} = \frac{\epsilon \xi \omega_{nn}}{(d + \xi \omega_{nn})^2} = 1
\]

\[
= \sum_{k} \epsilon \xi \omega_{nn} A_k e^{d_k \omega_{nnk} \tau} + \sum_{f} \epsilon \xi \omega_{nn} \sqrt{A_f A_r} e^{\xi \omega_{nn} \tau} \cos(\omega_{nnf} \sqrt{1 - \xi^2 \tau} + \phi_f) \\
+ (B_k + B_f) \xi \omega_{nn} e^{-\xi \omega_{nn} \tau} \\
+ \left[ C_k + C_f + \frac{Q(0)}{S(0)} \right] \epsilon \xi \omega_{nn} \tau e^{-\xi \omega_{nn} \tau}
\]

where \( A_k = \frac{Q(d_k \omega_{nnk})}{S'(d_k \omega_{nnk})(\xi_k + d_k) \omega_{nnk}} \)

\( A_f = \frac{Q(d_f \omega_{nnf})}{S'(d_f \omega_{nnf})(\xi_f + d_f)^2 \omega_{nnf}} \)

\( \bar{A}_f = \frac{Q(\bar{d}_f \omega_{nnf})}{S'(\bar{d}_f \omega_{nnf})(\xi_f + \bar{d}_f)^2 \omega_{nnf}} \)

\( B_k = -\sum A_k \)

\( B_f = -\sum (A_f + \bar{A}_f) \)

\( C_k = \sum_{k} \frac{Q(d_k \omega_{nnk})}{S'(d_k \omega_{nnk})(\xi_k + d_k)} \)

\( C_f = \sum_{f} \left[ \frac{Q(d_f \omega_{nnf})}{S'(d_f \omega_{nnf})(\xi_f + d_f)^2 \omega_{nnf}} \right] \frac{\xi_f}{S'(d_f \omega_{nnf})(\bar{d}_r \omega_{nnf})(\bar{d}_r + \xi_f)} \)

\( \phi_f = \tan^{-1} \frac{I_{Q_f}}{R_{Q_f}} - \tan^{-1} \frac{I_{S'} f}{R_{S'} f} - \tan^{-1} \frac{2(\xi_f - \xi_f)^2}{(\xi_f - \xi_f)^2 - (1 - \xi_f^2)} \)

(5.04a)

where \( k \) goes with nonoscillatory components and \( f \) oscillatory one.

---

* Practical evaluation of polynomial function of complex variable is greatly simplified by applying the De Moivre theorem graphically. See Appendix D.
In case the stability function only shows oscillatory characteristic, $k = 0$, and $A_k = B_k = C_k = 0$.

Assuming the surging disturbance has an apparent surging factor equal to unity with respect to the predominant component or component 1 (low frequency component), it follows that

$$
\xi_1 \omega_{nn1} = \omega_{nn1}, \quad \text{or} \quad \xi_1 = 1; \quad \text{therefore} \quad \xi_2 \omega_{nn2} = \omega_{nn1}, \quad \text{or} \quad \xi_2 = \frac{\omega_{nn1}}{\omega_{nn2}} = \frac{1}{f_0} = 0.0592 \quad (5.05)
$$

that is, with the same surging disturbance, it appears to be surging slowly with respect to the high frequency component.

Substitute the values of $\xi_1$, $\xi_2$, $\xi_2$, $\xi_2$, $\xi_1$, and $\xi_2$ of Eq. (5.05) into Eq. (5.04)a, and we have

$$
2\sqrt{A_1A_1} = 0.0366, \quad \phi_1 = -3.0^\circ \quad A_k = B_k = C_k = 0
$$

$$
2\sqrt{A_2A_2} = 0.00532, \quad \phi_2 = 28.5^\circ \quad B_1 = -0.0405, \quad c_1 = 0.0145
$$

When these values are substituted into Eq. (5.04) and multiplied by $\frac{1}{L_T} \times 57.3$, we have

$$
\frac{\theta}{W_0} = 0.421 \epsilon - 0.0693 \epsilon \cos(0.533 \tau - 3.0^\circ) \quad \text{In degrees per unit surging disturbance (max. 1 ft./sec. with apparent surging factor equal to one with respect to low frequency component.)}
$$

$$
+0.0612 \epsilon - 5.22 \epsilon \cos(7.7 \tau + 28.5^\circ)
$$

$$
-0.466 \epsilon - 0.556 \epsilon
$$

$$
+0.167 \epsilon - 0.556 \epsilon
$$

The second component, (quick oscillation) owing to its small initial magnitude and fast damping characteristic, dies away in less than one (or two seconds for our particular airplane at the assumed flight condition.) However it is an important component to adjust the initial condition of the response.
Damping of contr.
slow osc. comp. slow osc. comp.

Uncontrolled
response

Controlled
response

Slow oscillatory comp.

Coupling coeff. \( r = 3.6 \)
\( \xi = 0.084 \)

Uncontrolled
Pitching motion due to vertical surging gust, apparent surging factor \( \xi = 1.0 \) with respect to fast oscillatory component
FIG. XXIV
CONTROLLED AND UNCONTROLLED VERTICAL SPEED AND ACCELERATION RESPONSES DUE TO VERTICAL SURGING DISTURBANCE, $\xi = 1.0$ WITH RESPECT TO SLOW OSCILLATORY COMPONENT

![Graph showing controlled and uncontrolled vertical speed and acceleration responses due to surging disturbance.](image-url)
All four components are plotted in Fig. 23 with the same scale separately in dotted curves to show their relative magnitudes and phases and their resultant is plotted in a light solid curve. It is interesting to notice, due to the surging characteristic of disturbance, the resultant curve shows a slightly apparent divergence at the first two peak values. But after the second peak the resultant is essentially the same as the component of slow oscillation with slow convergence indicated by guiding curve of damping $e^{-0.0693\tau}$.

53. Uncontrolled Vertical Motion with Vertical Surging Disturbance

The uncontrolled vertical motion when an airplane is encountered with a vertical surging gust can be expressed by

$$\frac{W}{W_0} = \frac{\Delta_0 - d \left| \frac{d-x_u}{-mu} - \frac{C_L}{d^2 - dq} \right|}{\Delta_0} \frac{\xi \omega_n d}{(d + \xi \omega_n)^2} 1$$

(5.08)

With substitution of the particular constants of the airplane and the assumed flight condition and surging disturbance, Eq. (5.08) can be written in the following form

$$\frac{W}{W_0} = \frac{4.5d^3 + 88.1d^2 + 15.5d + 27}{d^4 + 10.65d^3 + 89d^2 + 15.5d + 27} \times \frac{0.556d}{(d + 0.556)}$$

(5.08a)

which gives the following solution:

$$\frac{W}{W_0} = 0.01283 \ e^{-0.0693\tau} \ \cos(0.533\tau - 9.08^\circ)
+ 0.192 \ e^{-5.22\tau} \ \cos(7.7\tau + 56.4^\circ)
- 0.1055 \ e^{-0.556\tau}
+ 1.546 \ e^{-0.556\tau}$$

(5.09)
It is very interesting to notice that the slowly oscillatory component is of negligible importance in the vertical motion because of its minute magnitude. The response is highly predominated by the last surging term of the same shape with the disturbance. The second and third terms of (5.09) play as important a role in adjusting the initial condition of the response with a slight lagging in the rising part of the gust. This solution verifies the good following-up characteristic of the airplane in longitudinal disturbance which is contributed by the component of quick oscillation. As the apparent surging of the disturbance with respect to quick oscillation is very slow, such slow disturbance should be easily followed.

It is more important to know the vertical acceleration when the airplane is subjected to vertical surging gust. By differentiating Eq. (5.09) and dividing the result by \( T \) we have:

\[
\frac{\ddot{W}}{W_0} = \frac{1}{T} \left[ -0.01283 \ e^{-0.0693\tau} \ \omega_{nn1} \sin(0.533\tau - 90.8^0 + \phi'_1) \\
-0.192 \ e^{-5.22\tau} \ \omega_{nn2} \sin(7.7\tau + 56.4^0 + \phi'_2) \\
+ (1.546 + 1.055 \times 556) \ e^{-0.556\tau} + (1.546 + 4.105 \times 0.556) \ e^{-0.556\tau}
\right]
\]

where \( \phi'_1 = \tan^{-1} \frac{\dot{f}_1}{\sqrt{1 - f_1^2}} \) and \( \phi'_2 = \tan^{-1} \frac{\dot{f}_2}{\sqrt{1 - f_2^2}} \)

\[
\frac{\ddot{W}}{W_0} = -0.00358 \ e^{-0.0693\tau} \ \sin(0.533\tau - 83.6^0) \\
-0.902 \ e^{-5.22\tau} \ \sin(7.7\tau + 90.7^0) \\
+ 0.815 \ e^{-0.556\tau} \\
-0.43 \ \tau \ e^{-0.556\tau}
\]  

(5.10)
Again, the slow oscillation is negligible, while the quick oscillatory term is very important in adjusting the zero time condition, although after \( T = 3 \) its effect is negligible.

Both the vertical speed and vertical acceleration are plotted as Fig. 24. The oscillation of the vertical motion is unnoticeable. The acceleration shows a peak value of \(.65\) ft. per second per second with unit surging gust of \(1\) ft. per second of unit apparent surging factor with respect to the slow component. During the decaying part of the gust, the vertical acceleration of the aircraft is negative and then gradually dies away with the gust.

With a vertical surging gust of apparent surging factor equal to unity with respect to the quick oscillation, the slow oscillation in pitch becomes unnoticeable while the quick oscillation gives maximum peak of \(.2\) degree per unit surging disturbance, but it dies away before \( T = 1 \) (2 seconds) as shown in Fig. 25.

54. Disturbed Pitching Motion of the Airplane with Nonideal Control of Deparasitized Type When Encountered By Vertical Surging Gust

By the theory developed in Chapter Two we shall adopt the deparasitized control to allow better controlability. Such control should have the nondimensional undamped natural frequency \( \omega_{\text{nnp}} \) and damping ratio \( \zeta_p \) according to the following equation.

\[
\frac{d^2}{dt^2} + 2 \zeta_p \omega_{\text{nnp}} \frac{d}{dt} + \omega_{\text{nnp}}^2 = \begin{bmatrix}
\frac{d}{dt} - x_u & -x_w \\
-z_u & \frac{d}{dt} - z_w
\end{bmatrix}
\]
With the substitution of particular values of $x_u$, $x_w$, $z_u$ and $z_w$, it is found:

$$\omega_{nnp} = 1.035$$

$$f_p = 2.25$$

With such control, we can apply different $\theta$ derivative exciting forces and couple the control movement to the elevator. Now let the two components of uncontrolled motion be designated by the subscripts 0 and c instead of 1 and 2. The subscript zero is for the low frequency component, while c is for the high frequency component which is acting as a control component especially to the pitching motion, because the component introduced by the parasitized control is of zero magnitude. Through the action of control, the component 0 becomes component 1 and the component c becomes component 2 as usual. For the vertical motion there shall be an additional component which will be designated by subscript p.

The purpose of applying control is to regulate the distribution of damping ratio between the two components. For the uncontrolled motion $\rho_0 = \frac{\zeta}{\zeta_0} = 4.52$. It is reasonable to specify the controlled $\rho_\zeta$ less than unity because in such a case the decreased damping ratio in high frequency component is not serious for there is a time factor to affect the rate of decay. Now let us assume the coupling shall not affect the frequency (natural undamped) of either component (so that $\rho_\omega$ is kept at 16.72) while $\rho_\zeta$ is expected to be 0.75. From Fig. 10 it is found that the damping advantage $\zeta_\zeta$ arising from such specification is 5.7, so that $f_1 = 0.707$ and $f_2 = 0.531$. 
Because the frequencies are not to be changed, it is necessary not to use the error in \( \theta \) itself as a quantity to excite the control; error derivative coupling is therefore needed. From Fig. 11 with \( \omega_\theta = .75, \quad \omega_\phi = 5.7, \quad \omega = 16.72 \) it is found that \( \gamma_\theta \) is the first derivative coupling coefficient, must be 3.6 and from Fig. 12D \( \gamma_\phi \) is the second derivative coupling coefficient, must be 0.084.

Therefore the stability equation of the controlled longitudinal stability becomes

\[
\Delta_c = (d^2 + 2\omega_{nnp}^2 + \omega_{nnp}^2)(\ddot{d} + 10.65\dot{d} + 89(1 + .084)\dot{d}^2 + 15.5(1 + 3.6)d + 27.0) \quad \text{Eq. (5.11)}
\]

\[
= (d^2 + 2\omega_{nnp}^2 + \omega_{nnp}^2)(\ddot{d}^2 + 10.65\dot{d}^2 + 96.5d^2 + 71.3d + 27.0)
\]

\[
= (d^2 + 2\omega_{nnp}^2 + \omega_{nnp}^2)S_c(d)
\]

with \( f_{\omega m} = 15.5 \times 3.6 = 55.8 \) \( \text{(5.12)} \)

and \( f_{gm} = 89 \times 0.084 = 7.5 \) \( \text{(5.13)} \)

From Eqs. (5.12) and (5.13) the coefficient of exciting forces \( f_\theta \) and \( f_\phi \) (nondimensional) can be evaluated because \( m_\sigma \) is fixed by the design of the elevator.

For the pitching motion, the response to a vertical surging disturbance can be expressed as

\[
\frac{\theta}{w_0} = \frac{1}{\frac{L}{T}} \left( \frac{d^2 + 2\omega_{nnp}^2 + \omega_{nnp}^2}{(d^2 + 2\omega_{nnp}^2 + \omega_{nnp}^2)S_c(d)} \right) \begin{vmatrix} d-x_u & -x_w \\ -m_u & -m_w \end{vmatrix} X \frac{\xi \omega_{nn} d}{(d + \xi \omega_{nn})^2}
\]

\[
= \frac{1}{\frac{L}{T}} \frac{d-x_u & -x_w}{S_c(d)} X \frac{\xi \omega_{nn} d}{(d + \xi \omega_{nn})^2}
\]

\( \text{(5.14)} \)
It is noticed that the numerator of the operational expression does not change; the only difference is the denominator, which changes from $S(d)$ to $S_0(d)$.

Here the characteristic of $S_0(d)$ has been known from the specification of the problem, so it is not necessary to use the quartic chart to find those physical nondimensional quantities.

The solution to (5.14) can be written as

$$\frac{\Theta}{W_0} = 1.242 e^{-0.393\tau} \cos(0.393\tau + 28.3^\circ)$$

$$+ 0.057 e^{-4.93\tau} \cos(7.38\tau + 34.1^\circ)$$

$$- 1.385 e^{-0.556\tau}$$

$$+ 0.52 \tau e^{-0.556\tau} \text{ in degrees}$$

in unit surging disturbance with $\xi = 1$

Compare Eq. (5.15) with Eq. (5.07). It may be concluded that

(a) the quick oscillation is essentially unchanged,

(b) the surging component is increased because the approaching of $\xi$, with $\xi$, (refer to Eq. (4.73)c).

(c) the slow oscillation is also magnified due to the same reason and with approximately the same ratio (should be the same as (4.73)a and (4.73)c indicate). However, the magnification does not do any harm because it converges rapidly for having a large damping ratio,

(d) the simple exponential term is also magnified to suit the zero condition.

Eq. (5.15) is plotted superimposed on Fig. 23 as a heavy solid curve. It is interesting to notice that the beginning part of the curve coincides with the uncontrolled disturbed pitching motion. This is no doubt due to the unavoidable control lag that the control cannot produce noticeable effect
when the disturbance just begins. But after two seconds or
one τ the control turns down the motion very rapidly; at the
end of one cycle the disturbed pitching motion almost disap-
pears along with the disappearance of the disturbance itself.

55. Disturbed Vertical Motion of an Airplane with Nonideal
Control of Deparasitized Type when Encountered with Vertical
Surging Gust

The qualitative function of the disturbed motion can be
written as

\[ \Delta_w = \Delta_{p0} \left( \Delta_{wo} + f_{\theta m_0} \frac{d^2}{d^2 + f_{\theta m_0} d} \right) \]
\[ - d(d - x_u) \left( f_{\theta m_0} \frac{d^2}{d^2 + f_{\theta m_0} d} \right) \]  \hspace{1cm} (5.16)

Substitute the value of \( f_{\theta m_0} \) and \( f_{\theta m_0} \) into Eq. (5.16) and
expand the equation numerically so that we have

\[ \Delta_w = 4.5d^4 + 116.5d^3 + 467.0d^2 + 223.3d + 29 - 7.5d^2 - 57.0d - 8.4d \]
\[ (5.16)a \]

The negative terms are derived from \(-d(d-x_u)\left(f_{\theta m_0} \frac{d^2}{d^2 + f_{\theta m_0} d}\right)\)
the presence of which causes the addition of overdamped compon-
ent. However, by the examination on relative magnitude of the
numerical coefficients, it can be roughly stated that the addi-
tional control component is of small magnitude. With the
neglect of these negative terms, the response of vertical motion
can be simplified as

\[ \frac{W}{W_0} = \frac{\Delta_{p0} \left( \Delta_{wo} + f_{\theta m_0} \frac{d^2}{d^2 + f_{\theta m_0} d} \right)}{\Delta_{p0} - \Delta_c} \times \frac{\xi \omega_{nn}}{(d + \xi \omega_{nn})^2} \]
\[ = \frac{\Delta_{wo} + f_{\theta m_0} \frac{d^2}{d^2 + f_{\theta m_0} d}}{\Delta_0 + f_{\theta m_0} \frac{d^2}{d^2 + f_{\theta m_0} d}} \times \frac{\xi \omega_{nn}}{(d + \xi \omega_{nn})^2} \]  \hspace{1cm} (5.17)
The solution to Eq. (5.17) should be a few per cent off the true value (greater than the true value). When it is plotted superimposedly on Fig. 23 with the allowance of being over-estimated, the difference between the controlled and uncontrolled motion is almost undistinguishable. And the variation in the vertical acceleration is approximately the same in both cases.

It is therefore believed that with such (9) deparasitized control the pitching motion is greatly eased on surging disturbance while the vertical motion cannot be appreciably improved.
CHAPTER FIFTEEN

SOME REFINEMENT CONSIDERATIONS IN THE $\theta$ DEPARASITIZED NONIDEAL LONGITUDINAL CONTROL

56. Possibility of Reducing Vertical Acceleration by Introducing Flap Control

The $\theta$ deparasitized control is effective in reducing the pitching oscillation discussed in the last chapter. Due to the high value of $z_w$, the vertical acceleration cannot be appreciably reduced. Weiss\(^4\) has pointed out that even with a fully restrained pitching motion by a powerful fast $\theta$ control, the vertical motion cannot be eased unless $z_w$ has been reduced. But due to the requirement of airplane efficiency, $z_w$ has to be high by using large aspect ratio. When the airplane is disturbed, or is operating under disturbing air conditions, the only way to reduce $z_w$ is to use flap control with the flap up when the airplane is struck by an upward gust. The moment variation of wing due to the flap movement is somewhat neutralized by the variation of the downward angle which affects the tail moment. There is some variation in $x_w$ due to the same flap movement. Such flap control should be excited by the relative vertical velocity between the airplane and the vertical gust; in other words, the detecting instrument must be an airspeed meter with pitot tube heading along the vertical axis. The detailed analysis can be done by the aid of mathematics.
developed in this thesis, but due to lack of time the conclusion is not yet reached.

57. Effect of Time Lag on the Detecting Instrument

In the θ deparasitized longitudinal control, the control lag is entirely offset by the parasite minor so far as the pitching motion is concerned; however, the detecting instrument itself usually possesses certain lagging effects. If such lagging is counted in the operational form of the response, it would become more complicated. However, if such time lag is very short compared to the natural frequencies of the system to be controlled, approximation can be made from the expanded form of Taylor’s theorem.

\[ \theta(\tau - \zeta_1) = \theta(\tau) - \zeta_1 \theta'(\tau) + \frac{\zeta_1^2}{2!} \theta''(\tau) \]

\[ \theta'(\tau - \zeta_2) = \theta'(\tau) - \zeta_2 \theta''(\tau) + \frac{\zeta_2^2}{2!} \theta'''(\tau), \text{ etc.} \]

where \( \zeta_1 \) and \( \zeta_2 \) are respectively time lag of error and error derivative of the detecting instrument. In operational form, they appear as

\[ \theta(\tau - \zeta_1) = (1 - \zeta_1 \frac{\zeta_1^2}{2!} \frac{d^2}{d\tau^2})\theta(\tau) \]

\[ \theta'(\tau - \zeta_2) = (\frac{d - \zeta_2 d^2 + \frac{\zeta_2^2}{2!} d^3}{2!})\theta(\tau) \]

Therefore, with an error-sensitive control and coupling factor \( f_{\theta m \sigma} \), a slight negative damping coupling factor \(-\zeta_1 f_{\theta m \sigma}\) and a slight positive accelerating coupling of coupling factor \( \frac{\zeta_1^2}{2!} f_{\theta m \sigma} \) are naturally involved due to the time lag of the detecting instrument. It therefore acts like a compound control and the advantages of the lag-compounding can be eval-
uated by the theory of compounding developed in Chapter Nine. With a second derivative coupling control, the lagging effect may reach the fourth derivative of the stability function. It is for this reason that Table V is made up to the fourth derivative coupling coefficient $\gamma_4$. 
CONCLUSION
AND SUGGESTION TO FURTHER DEVELOPMENT

In the present thesis, attention has been centered on the stability of a controlled system involving the fourth order linear differential equation. Ordinary nonideal control for the longitudinal motion of aircraft involves a sixth order linear differential equation, but when the control is properly designed, the stability function in pitching remains as a fourth order differential equation.

The damping ratios and the undamped natural frequencies of the two components (between which there may be wide difference when in their original uncontrolled state) are free to be adjusted. The stability function alone does not reveal the whole story of the response. Transient response must also be analyzed. In reviewing the present thesis, the writer feels the following points are worth while developing or investigating further.

(1) Simple pitching velocity elevator control coupling of the θ deparasitized type should be investigated thoroughly to compare with ordinary θ-elevator control.

(2) Experimental method of determining the resultant lagging from the variable, which is needed to work on the control, up to the valve movement which produces the exciting force on the control. By knowing this lagging time the compounding theory can be applied to determine the effect of lagging.
(3) To investigate the stability and transient response of vertical velocity flap control. If possible, actual tests should be conducted in order to know the effectiveness of reducing vertical acceleration when the airplane encounters a vertical surging gust.

(4) To develop an instrument for recording gusts in bumpy air from which the spectrum of the apparent surging factor can be determined thereby enabling the designer to attain a compromise in selecting the most suitable coupling coefficient.

(5) To investigate a simple course for following-up an equation for disturbed lateral motion of aircraft parallel to what has been done by Minorsky for steamship course stabilization by assuming full restraint in roll and pitch.

(6) For constant azimuth control investigate the relative merits in using a control of high natural frequency and a positive first derivative tuning control including the transient analysis of the response.

(7) Theoretically an overdamped slow control of first derivative coupling is advantageous to distributing more evenly the damping ratio between the two components. It is not easily seen. For this reason, actual tests should be conducted in seeking convincible evidence.

(8) To investigate, following the method of attack on the quartic equation, the property of the sextic equation. If possible, summarize the stability criteria in the form of a plot and develop a sextic chart for evaluating the nondimensional physical constants of the system involving the sextic equation.
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BIOGRAPHY

The author, Thomas Yee-Ying Liu, of the present thesis, was born in Jukao, Kiangsu, China on January 28, 1909. He received his elementary education in his home town until 1924 when he went to Nanking to have his senior middle school education. There his English was promoted one year ahead of the rest of the courses. He felt proud of such a promotion then, but later on he found it was a terrible missing that his English could never be advanced beyond the basic language.

On 1926 he entered Chiao Tung University at Shanghai to study Electrical Engineering and was graduated in July, 1930. From then he spent five years in teaching Physics and Chemistry at Kiangsu Provincial Taitsang and Hawaiian middle schools. He was honored twice in being appointed by the Provincial Education Bureau to take the charge of joint examination in Physics and Chemistry for the middle schools. With the five year patience in the provincial education circle he was qualified to take the competitive examination for abroad study. In winning such scholarship on 1935 he was sent to United States of America on January 1936.

In attempting to improve the safety and comfort of flying, he was permitted to study Aeronautical Engineering at Massachusetts Institute of Technology with special attention to Aeronautical Instrumentation. He received his Master of Science degree in 1938 at the Institute.
Due to the invasion of Japanese, Kiangsu has been occupied by the invader since December 1937. The author could not get support from the Provincial Government. In seeking financial aid for his further study, he applied the scholarship of the Institute and the scholarship of the China Foundation for the Promotion of Education and Culture. Fortunately he was granted by both for two years up to 1940, when the present thesis has been partially worked out. For the present academic year Tsieng Hua University grants him a partial scholarship for his completion of writing up of the present thesis titled "Stability and Transient Analysis of Longitudinal Controlled Motion of Aircraft with Nonideal Automatic Controls".

During the course in working the thesis, frequent stalls were encountered. But being inspired by L. Kronecker's famous statement, "God made the integers -- the rest is the work of man", he was much encouraged by the Catholicism. The fruitfulness of the present thesis in fact is made after he was baptized on December 8, 1940, when he received his Christian name, Thomas.

The year after he entered Chiao Tung University he suffered the loss of his father. By the struggle of his mother in earning money he was able to finish the college education. Recently, when hoping to be back soon to have a reunion with his dear mother, he is again struck by the news of her passing away. The only dream he is seeking now is hoping to see his wife and their two children soon after his completion of the thesis.
APPENDIX A

Nondimensional Cubic Equation and the Cubic Chart

by

Y. J. Liu

In the analysis of the performance of nonideal constant speed control, a cubic equation* representing the stability function is often met. In terms of natural frequency $\omega_n$, the damping ratio $\zeta_c$ of the control and the control constant $S_v$, the stability function can be written as

$$\frac{d^3U}{dt^3} + 2\zeta_c \omega_n \frac{d^2U}{dt^2} + \omega_n^2 \frac{dU}{dt} + S_v \omega_n^3 U = 0$$

Equation A1

Let $\frac{d}{dt} = D$, then

$$D^3 + 2\zeta_c \omega_n D^2 + \omega_n^2 D + S_v \omega_n^3 = 0$$

Equation A2

Eq. A2 can be nondimensionalized by introducing nondimensional operator $d$ for $\frac{D}{\omega_n}$ giving the form

$$d^3 + 2\zeta_c d^2 + d + S_v = 0$$

Equation A3

The roots of Eq. A3 are then evaluated.

Before the process of evaluating the roots of Eq. A3, stability can be verified from the form

$$\lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0$$

Equation A4

where $\lambda = \frac{d}{S_v^{1/3}}$, $\alpha_2 = \frac{2\zeta_c}{S_v^{1/3}}$, and $\alpha_1 = \frac{1}{S_v^{2/3}}$

Equation A5

The condition necessary for stable performance is

$$\alpha_2 > \alpha_1$$

For nonoscillatory performance, the reader is referred to Fig. A1. Assume that Eq. A3 can be factored into the following form.

$$d^3 + 2\zeta_o d^2 + d + S_v = (d^2 + 2\gamma \omega_r d + \omega_r^2)(d + \zeta \omega_r)$$

Equation A6

where $\omega_r$ is the reference angular frequency of the oscillatory component of the controlled motion, $\zeta$ the damping ratio of this

component, and $\xi$ the apparent subsiding coefficient of the subsiding component. On developing,

$$2\zeta_c = (\xi + 2\xi)\omega_r$$  \hspace{1cm} (A7)

$$1 = (2\xi + 1)\omega^2$$  \hspace{1cm} (A8)

$$S_v = \xi\omega_r^3$$  \hspace{1cm} (A9)

From Eq. A8, $\omega_r = \frac{1}{\sqrt{2\xi + 1}}$ \hspace{1cm} (A8')

Substitute Eq. A8' into A7 and A9. We then have

$$2\zeta_c = \frac{\xi + 2\xi}{\sqrt{2\xi + 1}}$$  \hspace{1cm} (A7')

and

$$S_v = \frac{\xi}{(2\xi + 1)^{3/2}}$$  \hspace{1cm} (A9')

From Eq. A9' we may solve for

$$\xi = \frac{1}{2\xi} \left( \frac{\xi}{S_v} \right)^{2/3} - 1$$  \hspace{1cm} (A10)

From Eq. A7' we may also solve for

$$\xi = \xi \left( \frac{\xi^2}{2} - \frac{1}{2} \pm \xi \sqrt{\frac{\xi^2}{2} - 1 + \frac{1}{\xi^2}} \right)$$  \hspace{1cm} (A11)

Eq. A10 is plotted with $\xi$ as ordinate against $\xi$ as abscissa and $(S_v)^{1/3}$ as varying parameter. Eq. A11 is plotted also with $\xi$ as ordinate against $\xi$ as abscissa, but with $\zeta_c$ as varying parameter. These two sets of curves mutually intersect one another and form the cubic chart. Therefore, when a controlled system is known by its damping ratio in control ($\zeta_c$) and control constant ($S_v$), the nondimensional characteristic of the controlled system can be found from the cubic chart by locating the intersection of $\zeta_c$ and $S_v^{1/3}$ which determines the damping ratio $\xi$ of the oscillatory component of the controlled result and the apparent subsiding coefficient of the subsiding component, $\xi$.

When $\xi$ is known,

$$\omega_r = \left( \frac{S_v}{\xi} \right)^{1/3}$$  \hspace{1cm} (A12)
The dimensional natural angular frequency is then

\[ \omega_n \omega_r = \omega_n \left( \frac{S_v}{\xi} \right)^{\frac{1}{3}} \]  \hspace{1cm} A13

With \( \xi \), \( \xi \), and \( \omega_r \) known, Eq. A6 can be written immediately and Eq. A2 can be factored as

\[ (D^2 + 2\xi \omega_n \omega_r D + \omega_n^2 \omega_r^2) (D + \xi \omega_n \omega_r) = 0 \] \hspace{1cm} A14

It can be seen from the cubic chart that in the region \( \xi > 1 \), three intersections can be found from a pair of \( \xi_c \) and \( S_v^{\frac{1}{3}} \). This is naturally true because the three binomial factors can be arranged in three different ways to form the factorized Eq. A14.
STABILITY CRITERIA $\alpha_2$ VS $\alpha_1$ OF THE CUBIC EQUATION

$$\lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0$$

The graph shows regions for different types of roots:

- **Stable Region, Non-Oscillatory**
- **Stable Region, Oscillatory**
- **Unstable Region**
- **Unending Oscillation**

Three equal real roots
Two equal real roots, third one is larger
Two equal real roots, third one is smaller
Three unequal real roots
THE CUBIC CHART

For equation \( d^3 + 25c d^2 + d + S_v = 0 \)
SUMMARY: In the analysis of servo-mechanism or automatic follow-up systems linear differential equations of fourth order are often met when a non-ideal controller is used (that is, a controller in which the effects of inertia damping and spring constants are not negligible). Physically a linear differential equation of fourth order as
\[ a_4 \frac{d^4x}{dt^4} + a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0x = 0 \tag{1} \]
represents two vibratory (divergent, critically or over damped) components of motion or other quantities provided at least the first and the last coefficients are of the same sign. Equation (1) can be transformed into an algebraic one by introducing \( p \) as the time operator symbol for \( \frac{d}{dt} \), giving the form:
\[ p^4 + a_3p^3 + a_2p^2 + a_1p + a_0 = 0 \tag{2} \]
Where, \( a_3 = \frac{a_3}{a_4} \), \( a_2 = \frac{a_2}{a_4} \), \( a_1 = \frac{a_1}{a_4} \), and \( a_0 = \frac{a_0}{a_4} \).
Again, equation (2) can be non-dimensionalized by introducing a non-dimensional operation \( \lambda \) for \( \frac{D}{a_0} \) giving the form:
\[ \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0 \tag{3} \]
Where, \( \alpha_3 = a_3/a_0^{1/4} \); \( \alpha_2 = a_2/a_0^{1/2} \); \( \alpha_1 = a_1/a_0^{3/4} \).

The roots of equation (3) are then to be evaluated.

However, instead of finding the roots of equation (3) directly, we may according to the physical state of the original differential equation factorize equation (3) into two quadratics indicating the dynamic characteristics of the two components respectively. The factorized equation can be written as:
\[ (\lambda^2 + 2\zeta_1\omega_{1r}\lambda + \omega_{1r}^2)(\lambda^2 + 2\zeta_2\omega_{2r}\lambda + \omega_{2r}^2) = 0 \tag{4} \]
Where, \( \omega_{1r} \) = dimensionless natural frequency(1) of component 1 (angular)
\( \omega_{2r} \) = dimensionless natural frequency of component 2 (angular)
\( \zeta_1 \) = damping ratio(1) of component 1
\( \zeta_2 \) = damping ratio of component 2
and, \( \omega_{1o}^{1/4} = \omega_{n1} \) = undamped natural frequency of component 1 of equation (1)
\( \omega_{2o}^{1/4} = \omega_{n2} \) = undamped natural frequency of component 2 of equation (1)
It is advisable to have one component as reference. Arbitrarily component 1 is considered as the reference. Then the following symbols are defined:
\( \omega_r = \omega_{1r} \) = dimensionless natural frequency of reference component

Superscripts are referred to notes in appendices.

APPENDIX B
DIRECTIONS FOR THE QUARTIC CHART
By Y. J. Liu
\[ \ddot{z}_r = \dot{z}_r = \text{damping ratio of reference component} \]

\[ \rho_\omega = \frac{\omega_{\text{ref}}}{\omega_{\text{ref}}} = \text{ratio of undamped natural frequency between components} \]

\[ \rho_\zeta = \frac{\zeta_2}{\zeta_1} = \text{ratio of damping ratios between components} \]

Hence equation (4) can be written as

\[ (\lambda^2 + 2\zeta_r \omega_r \lambda + \omega_r^2) [\lambda^2 + 2(\zeta_r \rho_\Omega) \lambda + (\omega_r \rho_\Omega)^2] = 0 \quad \text{--------(5)} \]

Evidently it is of primary interest to know these four important physical or rather characteristic constants, namely \( \omega_r, \zeta_r, \rho_\omega, \) and \( \rho_\zeta, \) because the dynamic characteristics of the system are well defined if these four quantities are known.

To facilitate the evaluation of these four quantities a quartic chart has been designed by the writer based on the non-dimensionalized equation (3). Three auxiliary figures are presented together with the quartic chart for the verification of stability of the original system.

The quartic chart itself is composed of two parts; the left part (or Chart I) is designed for the ratio of undamped natural frequencies and the dimensionless natural frequency of the reference component, and the right part (or Chart II) for the ratio of damping ratios and the damping ratio of the reference component. To minimize the effort in applying the chart, steps are listed below and should be followed orderly. Symbols and definitions of other interesting quantities that have not been introduced in the summary will be given as they appear along with the steps. A list of symbols and definitions is given as Table I.

1. Non-dimensionalization Of Equation

   (Ia) Given equation in the general form

   \[ a_4 \frac{d^4x}{dt^4} + a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \quad \text{---------(1)} \]

   (Ib) Introduce time operator \( p \) for \( \frac{dx}{dt}, \) then

   \[ p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0 \quad \text{---------(2)} \]

   Where, \( a_i = \frac{a_i}{a_4}, \) and \( a_0 = \frac{a_0}{a_4}. \)

   (Ic) Introduce non-dimensional operator \( \lambda, \)

   with \( \lambda = \frac{p}{a_0^{1/4}}, \) then

   \[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0 \quad \text{---------(3)} \]

   Where, \( \alpha_3 = \frac{a_3}{a_0^{1/4}}, \) \( \alpha_2 = \frac{a_2}{a_0^{1/4}}, \) \( \alpha_1 = \frac{a_1}{a_0^{1/4}}. \)

2. Verification Of The Stability Of The Original Equation [Equation (1)] with the non-dimensional coefficients \( \alpha_3, \alpha_2, \) and \( \alpha_1. \)

   (2a) Obtain stability criteria \( M \) (Greek capital Mu) and \( N \) (Greek-capital Nu), and damping parameters
\[ \frac{\alpha_2}{\alpha_1} \text{ and } \frac{1}{2}(\alpha_3 + \alpha_4); \]

where, \( M = \frac{\alpha_3 \alpha_1 - 4}{\alpha_2^2} \), \( N = \frac{\alpha_3 \alpha_2^2 - 4 \alpha_2}{\alpha_2^3} \).

(2b) Verify the stability with the stability criteria curves \( M \) vs. \( N \) (Fig. II A, II B, II C)\(^{(2)}\).

3. Application Of Chart I To Find The Ratio Of Undamped Natural Frequencies Between The Two Components, \( \rho \omega \), And The Dimensionless Frequency Of The Reference Component, \( \omega_r \).

(3a) Locate intersection (3a)\(^{(3)}\) of the particular pair of \( M \) and \( N \) on Chart I (see sketch on next page). Draw a line through (3a) and parallel to the 135\(^{\circ}\) inclined lines until it intersects at the 45\(^{\circ}\) inclined scale. The intersection on this scale gives the value of \( \rho_\alpha \); where

\[ \rho_\alpha = \frac{1}{\alpha_2} (\rho_\omega + \frac{1}{\rho_\omega}). \]

The value of \( \rho_\alpha \), however, is not required for further application of the chart, but it serves as a principal datum from which other data can be evaluated\(^{(4)}\) thereon by calculation in a more elaborate way.

(3b) Pick up the particular value of \( \alpha_2 \) from the left hand scale\(^{(5)}\) and draw a horizontal line until it meets the particular 135\(^{\circ}\) inclined line. Call this intersection (3b).

(3c) From the intersection (3b) draw a vertical line which will intersect the curve P\(^{(5)}\) at (3P) and curve Q at (3Q).

(3d) A horizontal line then drawn through the intersection (3P) intersects on the immediate right scale of ordinates at (3d) showing the value of \( \rho_\omega \) and on the next right scale (that is, the left hand scale of ordinates of Chart II) at (3d\(^{\prime}\)) showing the value of \( \omega_r \). Record the values of \( \rho_\omega \) and \( \omega_r \).

4. Application Of Chart II To Find The Ratio Of Damping Ratios Between Components, \( \rho_\zeta \), And The Damping Ratio Of The Reference Component, \( \zeta_r \).

(4a). Starting from the intersection (3Q) on Chart I run a horizontal line until it meets a curve of the particular value of \( \rho_\omega \)\(^{(6)}\) [which has been found in (2a)] on Chart II. This intersection (4a) projected onto the abscissa scale gives the value of \( \rho_\zeta \). Record this value of \( \rho_\zeta \).

(4b) Through the intersection (4a) run a line parallel to those 45\(^{\circ}\) inclined lines until it meets a vertical line of the corresponding particular value of \( 1/2(\alpha_3 + \alpha_4) \) [which has been found in (2a)]. This intersection (4b)\(^{(7)}\) projected onto the extreme right hand ordinates of Chart II gives the value of \( \zeta_r \). Record this value of \( \zeta_r \).

5. Factorization of Equation (3) By Utilization Of The Values Of \( \rho_\omega \), \( \omega_r \), \( \rho_\zeta \), And \( \zeta_r \).
SKETCH OF THE QUARTIC CHART

CHART I

- Scale of $\alpha_2$
- Scale of $\rho_x$
- Scale of $\omega_r$
- Scale of $\xi$ 

Const. $N$
Const. $M$

Point $P$
Point $Q$
Point $3a$
Point $3b$
Point $3p$
Point $3d$
Point $3d'$

CHART II

- Scale of $\frac{1}{2}(\alpha_3 + \alpha_1)$
- Scale of $\alpha_3$
- Scale of $\alpha_1$

Point $4a$
Point $4b$

Const. $\frac{\alpha_3}{\alpha_1}$
As pointed in the summary equation (3) can be written in terms of the four characteristic constants in the form of equation (5)

\[(\lambda^2 + 2\zeta_1 \omega_n^2 \lambda + \omega_n^2) [\lambda^2 + 2(\zeta_1 \rho_0^4 \lambda + (\omega_r \rho_0^2)^2] = 0 \quad \text{---------(5)}\]

6. Factorization Of Equation (2) By Utilization Of The Values of \(\rho_w, \omega_r, \rho_\zeta, \zeta_r,\) And \(a_0^{1/2}.\)

By writing:

\[\omega_{n1} = \omega_r a_0^{1/2}, \quad \omega_{n2} = \omega_n \rho_w, \quad \zeta_1 = \zeta_r, \quad \text{and} \quad \zeta_2 = \zeta_1 \rho_\zeta,\]

equation (2) can be written as:

\[(p^2 + 2\zeta_1 \omega_{n1} p + \omega_{n1}^2) (p^2 + 2\zeta_2 \omega_{n2} p + \omega_{n2}^2) = 0 \quad \text{---------(6)}\]

or one step further as:

\[[p + (\zeta_1 + \sqrt{\zeta_1^2 - 1}) \omega_{n1}] [p + (\zeta_2 - \sqrt{\zeta_2^2 - 1}) \omega_{n1}] [p + (\zeta_2 + \sqrt{\zeta_2^2 - 1}) \omega_{n2}] [p + (\zeta_2 - \sqrt{\zeta_2^2 - 1}) \omega_{n2}] = 0 \quad \text{---------(7)}\]

The values of \(\sqrt{\zeta_2^2 - 1}\) can be obtained easily by means of Fig. III. It is evident that in case \(\zeta\) is less than unity, \(\sqrt{\zeta_2^2 - 1}\) comes out naturally with imaginary value which indicates the presence of oscillatory component.
(1) Natural frequency $\omega_n$ and damping ratio $\zeta$ are two very significant quantities of the dynamic characteristic of vibratory system with one degree of freedom. In symbol, $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m \omega_n}$; where $k$ is the spring coefficient of the simple system, $m$ the vibratory mass of the system, $c$ the actual damping present in the system and $c_c$ is the amount of damping which would cause critical damping to the system. As our problem is one which may be resolved into two components of motion, so each has its own natural frequency and damping ratio. Further reference may be found in various papers published by C. S. Draper and his co-authors such as "General Principles of Instrument Analysis" by C. S. Draper and G. V. Schliestett and "An Instrument for Measuring Low Frequency Accelerations in Flight" by C. S. Draper and W. Wrigley.

(2) Fig. IIA covers wide range of $M$ and $N$. Stability can be verified practically with every possible combination of $M$ and $N$. Fig. IIB is an enlargement of the non-oscillatory region where four unequal real roots are present. Fig. IIb is plotted in logarithmic scales to render better the visualization of very small quantities of $M$ and $N$ forming part of the boundary between the oscillatory and the non-oscillatory regions. Fig. I can be referred to as an indication of relative degree of damping between the two components; such indication is not available in Fig. II. Table II is a summary of Figs. I, IIA, IIB, & IIb. With the table and these figures one can find out whether the system is oscillatory, stable or not immediately after he gets the values of $M$ and $N$.

(3) It is preferable to take the right-and-uppermast intersection of a given pair of $M$ and $N$; because in case oscillatory component or components are present the left intersection will lead to a complex frequency ratio which, though reasonable from the mathematical point of view, is not usable. In case two non-oscillatory components are present, three intersections may be observed with one pair of $M$ and $N$. They are equally usable, but the rightmost one gives the largest value of frequency ratio which makes the further application of the Chart easier. The dotted curve appearing on Chart I is nothing but equivalent to the boundary lines $AB$ and $ACD$ on Fig. IIA. To the left of the vertex, the dotted curve is equivalent to the boundary line $AB$ and to the right, equivalent to the part $ACD$. Discard any intersection of $M$ and $N$ which appears below and to the left of the dotted curve. The significance of the boundary lines (and so of the dotted curve) is summarized in Table I.

(4) When $p_\alpha$ is obtained, the following formulae can be used for the evaluation of $p_\omega$, $p_\zeta$, $\omega_r$, and $\zeta_r$:

$$p_\omega = \frac{1}{2} \left[ \alpha_2 p_\alpha + \sqrt{(\alpha_2 p_\alpha)^2 - 4} \right]$$  \hspace{1cm} (8)

$$\omega_r = \frac{1}{\sqrt{p_\omega}}$$  \hspace{1cm} (9)
\[
\rho_\zeta = \frac{\rho_\omega (\alpha_3)}{\rho_\omega - (\alpha_3)} - 1 \quad \text{--------------------- (10)}
\]

\[
\zeta_r = \frac{\frac{1}{2}(\alpha_3 + \alpha_4)}{(1+\rho_\zeta)(\sqrt{1+\frac{1}{1+\rho_\omega}})} \quad \text{--------------------- (11a)}
\]

\[
= \frac{\frac{1}{2} \sqrt{\alpha_3 (1-\rho_\zeta)}}{\rho_\zeta} \quad \text{--------------------- (11b)}
\]

\[
= \frac{\alpha_3}{2(\sqrt{1+\rho_\omega} + \rho_\zeta \sqrt{\rho_\omega})} \quad \text{--------------------- (11c)}
\]

\[
= \frac{\alpha_4}{2(\sqrt{1+\rho_\omega} + \rho_\zeta \sqrt{\rho_\omega})} \quad \text{--------------------- (11d)}
\]

(5) Curves P and Q appearing to the right upper corner (in the crowded zone) of Chart I are matched with the right scale of \( \alpha_3 \) (range: 1-20). The centered P and Q are matched with left scale of \( \alpha_3 \) (range: 10-1000). In case no intersection on the particular 135° inclined line and the horizontal line of particular value of \( \alpha_3 \) can be found within Chart I, one is at liberty to shift the proper 135° inclined line one logarithmic cycle left or right, but the scale of \( \alpha_3 \) should never be changed. By this process the matched P and Q curves are automatically shifted one logarithmic cycle left or right with the shifted 135° inclined line. So the local P and Q curves are available.

(6) As curve Q on Chart I has its top value corresponding to \( \omega_r = 0.5 \), any linear differential equation of 4th order with positive real coefficients does not go beyond the limit. Therefore any horizontal line run from Q can always intersect with constant \( \alpha_3 \) curves. In case \( \alpha_3 \) is greater than 4, \( \alpha_4 \) may be used instead of \( \alpha_3 \). Then the ratio of damping ratios \( \rho_\zeta \) read from the abscissa scale of Chart II will be referred to the damping ratio of component 2 and the value of \( \zeta_r \) on the rightmost ordinate scale is still the damping ratio of reference component, but it is the damping ratio of component 2. As \( \rho_\omega \) is always given greater than unity, component 1 is always referred as the low frequency one, and component 2 as the high frequency one.

(7) Shifting of the 45° lines on Chart II one logarithmic cycle up or down on Chart II is permissible. However the decimal points of the ordinates scale for \( \xi_r \) must be shifted one figure left when the 45° line is shifted up one logarithmic cycle or one figure right when the latter shifted one cycle down. Moreover, the scales of \( \frac{1}{2}(\alpha_3 + \alpha_4) \) and of \( \zeta_r \) can be multiplied by a common factor, for instance, 10, simultaneously.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>time, independent variable</td>
</tr>
<tr>
<td>$x$</td>
<td>dependent variable</td>
</tr>
<tr>
<td>$a_k$</td>
<td>physical coefficient associated with $\frac{d^k x}{dt^k}$, kth time derivative of the dependent variable</td>
</tr>
<tr>
<td>$p$</td>
<td>time operator symbol, $p = \frac{d}{dt}$</td>
</tr>
<tr>
<td>$p^k$</td>
<td>$p^k = \frac{d^k}{dt^k}$</td>
</tr>
<tr>
<td>$a_k'$</td>
<td>time coefficient associated with $p^k$ ($a_4' = 1$)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>dimensionless operator, $\lambda = \frac{p}{a_4'}$</td>
</tr>
<tr>
<td>$\lambda^k$</td>
<td>$\lambda^k = \left(\frac{p}{a_4'}\right)^k$</td>
</tr>
<tr>
<td>$\alpha_k$</td>
<td>dimensionless coefficient associated with $\lambda^k$ ($\alpha_4 = 1$, and $\alpha_4 = 1$)</td>
</tr>
<tr>
<td>$\frac{a_3}{a_1}$</td>
<td>damping parameter 1</td>
</tr>
<tr>
<td>$\frac{1}{2}(a_3 + a_1)$</td>
<td>damping parameter 2</td>
</tr>
<tr>
<td>$M$ (Greek Capital Mu)</td>
<td>$M = \frac{a_3 a_1 - 4}{\alpha_3^2} = \frac{a_3 a_1 - 4 a_4 a_0}{\alpha_3^2}$</td>
</tr>
<tr>
<td>$N$ (Greek Capital Nu)</td>
<td>$N = \frac{\alpha_3^2 + \alpha_2^2 - 4 \alpha_2}{\alpha_3^2} = \frac{\alpha_3^2 a_0 + a_4 a_1^2 - 4 a_4 a_2 a_0}{\alpha_3^2}$</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>dimensionless angular frequency of component 1 (Low freq.)</td>
</tr>
<tr>
<td>$\omega_{r2}$</td>
<td>dimensionless angular frequency of component 2 (High freq.)</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>dimensionless angular frequency of reference component (component 1, arbitrarily)</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>undamped natural angular frequency in general</td>
</tr>
<tr>
<td>$\omega_{n1}$</td>
<td>undamped natural angular frequency of component 1</td>
</tr>
<tr>
<td>$\omega_{n2}$</td>
<td>undamped natural angular frequency of component 2</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>damping ratio in general</td>
</tr>
<tr>
<td>$\zeta_1$</td>
<td>damping ratio of component 1</td>
</tr>
<tr>
<td>$\zeta_2$</td>
<td>damping ratio of component 2</td>
</tr>
<tr>
<td>$\zeta_r$</td>
<td>damping ratio of reference component (component 1, arbitrarily)</td>
</tr>
<tr>
<td>$\rho_\omega$</td>
<td>$\rho_\omega = \omega_{r2}/\omega_r = \omega_{n2}/\omega_{n1}$, ratio of undamped natural frequency</td>
</tr>
<tr>
<td>$\rho_\zeta$</td>
<td>$\rho_\zeta = \zeta_2/\zeta_1$, ratio of damping ratios</td>
</tr>
<tr>
<td>$\rho_\alpha$</td>
<td>$\rho_\alpha = (1/\alpha_2)\left(\rho_\omega + 1/\rho_\omega\right)$</td>
</tr>
</tbody>
</table>
## TABLE II

### STABILITY BEHAVIOR OF THE QUARTIC EQUATION IN NONDIMENSIONAL FORM

\[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0 \]

\[ N = \frac{\alpha_3 \alpha_2 - \alpha_1^2}{\alpha_2^2} \quad M = \frac{\alpha_3^2 + \alpha_2^2 - 4\alpha_1}{\alpha_2^2} \]

<table>
<thead>
<tr>
<th>( N &gt; M )</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = M )</td>
<td>Unending oscillation</td>
</tr>
<tr>
<td>( N = M = -1 )</td>
<td>Unending oscillation of both components</td>
</tr>
<tr>
<td>( N &lt; M )</td>
<td>Outside region BAAC</td>
</tr>
</tbody>
</table>

#### Stable

| \( N > 0 \) | One component only |
| \( N < 0 \) | One or both components |

#### Inside region ABC

- Four distinct real roots or both components oscillatory
- \( \alpha_3 > \alpha_1 \) High frequency component has greater damping ratio
- \( \alpha_3 < \alpha_1 \) High frequency component has lesser damping ratio
- \( \alpha_3 = \alpha_1 \) Two components of same frequency, but with different damping
- Two components of different frequency, but with same damping or two components of same frequency and same damping

\[ f \] may be any real number
STABILITY TRANSITION CURVE

\[ \alpha_2 = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \]

FOR EQUATION

\[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0 \]

Y J. LIU
FEB 40

FIG. 1
\[ M = \frac{\alpha_3 \alpha_1 - \alpha}{\alpha^2} \]

**Stability Criterion:**

\[ \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + 1 = 0 \]

**Figure II A & II B:**

- Unending oscillation both components
- Single oscillatory region (S.O.R.)
- Stable region, oscillatory
- Double oscillatory region
- Unstable region
- \( N > M \)

- \( N = M \), unending oscillation
- 2 equal quadratic factors
- Critically damped quadratic factors
- 4 unequal real roots or double oscillatory
- 3 or 4 equal real roots

**Critical Points:**

- Point A
- Point B
- Points 1 through 16

**Lines & Regions:**

- \( Y > 1 \)
- \( y = 0 \)
- \( y^* \)
- \( y > 0 \)
- \( 0 < y < 1 \)
- \( y < 0 \)
- \( y^* < y < 0 \)
- \( y^* > y > 0 \)
- \( y^* > y^* > 0 \)

**Graphical Representation:**

- Graph showing the relationship between \( N \) and \( M \)
- Different regions indicated by shading and lines
- Critical points and regions defined by algebraic expressions

**Y.J. Liu Feb 40**
STABILITY CRITERION  \( N \) vs. \( M \) for

\[
\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0
\]

- **N > M**
  - UNSTABLE REGION

- **N = M**
  - UNENDING OSCILLATION
  - 1 COMPONENT ONLY

- **3 or 4 Equal Roots**

- **SINGLE OSCILLATORY REGION**
  - 2 equal roots or 2 pairs of equal roots

- **STABLE REGION, NON-OSCILLATORY**
  - \( \frac{\alpha_3^2 + \alpha_1^2 - \alpha_2^2}{\alpha_2^2} \)

- **STABLE REGION, DOUBLE OSC.**
  - \( J_{2,1} < \sqrt{1 - \frac{\alpha_2^2}{\alpha_2^2}} \)
  - 2 Equal quadratic factors
  - or 2 Critically damped quadratics with different frequencies

\[
M = \frac{\alpha_3\alpha_1 - 4}{\alpha_2^2}
\]
APPENDIX C

Response to Unit Surging Disturbance

Developed by Heaviside Expansion Theorem

by

Y. J. Liu

The unit surging disturbance is represented by

\[ e^{\xi \omega_n t} e^{-\xi \omega_n t} \]

of which the maximum is unity at \( t = \frac{1}{\xi \omega_n} \) where \( \xi \omega_n \) is defined as the surging factor of the surging disturbance which is constant and equals \( \xi \omega_{nk} \) in referring to the \( k^{th} \) component of motion of a system. In this system \( \omega_{nk} \) is the undamped natural frequency (angular) of the \( k^{th} \) component and \( \xi \omega_{nk} \) is defined as the apparent surging factor of the surging disturbance with respect to the \( k^{th} \) component.

When the surging disturbance is applied to a system whose response to the unit step function is \( \frac{Q_x(d)}{S(d)}I \), the response can be written as

\[ I_s(t) = \frac{Q_x(d)}{S(d)} e^{\xi \omega_n t} e^{-\xi \omega_n t} \]

where \( S(d) \) is the stability function of the system

\( Q_x(d) \) is the quality function of the \( x \)-wise motion

\( I_s(t) \) is the unit surging response

Eq. C2 can be changed into the following form

\[ I_s(t) = \frac{Q_x(d)}{S(d)} \frac{e^{\xi \omega_n d}}{(d + \xi \omega_n)^2} \]

\[ = \frac{e^{\xi \omega_n d}}{(d + \xi \omega_n)} \left( \frac{Q_x(d)}{S(d)} \right)^2 \]

If both \( Q_x(d) \) and \( S(d) \) are polynomials in \( d \) with constant coefficients and with equal or less degree in \( d \) in the numerator

\[ -C1- \]
than that in the denominator, $C_3$ can be expanded by the Heaviside Expansion Theorem as follows provided the $S(d) = 0$ has no repeating roots

$$I_s(t) = \frac{\varepsilon \xi \omega_n d}{(d + \xi \omega_n)^2} \left[ \frac{Q_x(0)}{S(0)} + \sum_{k=1}^{k} \frac{d}{(d + \xi_k \omega_n)^2} C_{x_k} \frac{d_k \omega_{nk}}{d_k \omega_{nk} S'(d_k \omega_{nk})} \right]$$

$$= \varepsilon \xi \omega_n \left[ \frac{d}{(d + \xi \omega_n)^2} \frac{Q_x(0)}{S(0)} \left( \sum_{k=1}^{k} \frac{d}{(d + \xi_k \omega_n)^2} C_{x_k} \frac{d_k \omega_{nk}}{d_k \omega_{nk} S'(d_k \omega_{nk})} \right) \right]$$

where $d_k \omega_{nk} = \text{the } k\text{th root of } S(d) = 0$ is real or complex $S'$ is the first derivative of the stability function. The first term in the bracket of $C_4$ can be written as:

$$\frac{Q_x(0)}{S(0)} \varepsilon t e^{-\xi \omega_n t}$$

and the second term, by applying the shifting formula** can be written as:

$$\sum_{k=1}^{k} \frac{Q_x(d_k \omega_{nk})}{d_k \omega_{nk} S'(d_k \omega_{nk})} \left[ \frac{d}{(d + \xi_k \omega_n + d_k \omega_{nk})^2} \right]$$

On developing, 06 gives

$$\sum_{k=1}^{k} \frac{Q_x(d_k \omega_{nk})}{d_k \omega_{nk} S'(d_k \omega_{nk})} \left[ \frac{\frac{d}{(\xi_k + d_k)^2 \omega_{nk}}} e^{-\xi \omega_n t} - \frac{d_k}{(\xi_k + d_k)^2 \omega_{nk}} e^{-\xi \omega_n t} \right]$$

$$\left[ \frac{d_k}{(\xi_k + d_k)^2 \omega_{nk}} e^{-\xi \omega_n t} \right]$$

$$I_s(t) = \varepsilon \xi \omega_n \left[ \sum_{k=1}^{k} \frac{Q_x(d_k \omega_{nk})}{S'(d_k \omega_{nk})(\xi_k + d_k)^2 \omega_{nk}} \right]$$

$$= \varepsilon \xi \omega_n \left[ \sum_{k=1}^{k} \frac{Q_x(d_k \omega_{nk})}{S'(d_k \omega_{nk})(\xi_k + d_k)^2 \omega_{nk}} \right] e^{-\xi \omega_n t}$$

$$+ \left[ \frac{Q_x(0)}{S(0)} + \sum_{k=1}^{k} \frac{Q_x(d_k \omega_{nk})}{d_k \omega_{nk} S'(d_k \omega_{nk})} \right] e^{-\xi \omega_n t} \varepsilon t e^{-\xi \omega_n t}$$

* V. Bush, "Operational Circuit Analysis", Chapter VII
** V. Bush, "Operational Circuit Analysis", Chapter VIII, p. 130.
when the \( f \)th pair of roots is a conjugate pair, \( d_f \) and \( d_f \) can
be used to represent the conjugate pair;

\[
-\xi_f = i \sqrt{1 - \xi_f^2}
\]

Then \( I_s(t) = \sum_k e^{\xi_k \omega_n t} A_k e^{d_k \omega_n t} + \sum_f e^{\xi_f \omega_n t} 2 \sqrt{A_f \xi_f} \cos(\xi_f \omega_n t + \phi_f) \]

\[
+ (B_k + B_f) e^{\xi_k \omega_n t} e^{-\xi_k \omega_n t} + [C_k + C_f + \frac{Q_x(0)}{S(0)}] e^{\xi_k \omega_n t} e^{-\xi_k \omega_n t}
\]

where

\[
A_k = \frac{Q_x(d_k \omega_{nf})}{S(1)(d_k \omega_{nk})(\xi_k + d_k)^2 \omega_{nk}^2}
\]

\[
A_f = \frac{Q_x(d_f \omega_{nf})}{S(1)(d_f \omega_{nf})(\xi_f + d_f)^2 \omega_{nf}^2}, \quad \bar{A}_f = \frac{Q_x(\bar{d}_f \omega_{nf})}{S(1)(\bar{d}_f \omega_{nf})(\bar{d}_f \omega_{nf})^2 \omega_{nf}^2}
\]

\[
B_k = - \sum_k A_k
\]

\[
B_f = - \sum_f (A_f + \bar{A}_f)
\]

\[
C_k = \sum_k \frac{Q_x(d_k \omega_{nk})}{S(1)(d_k \omega_{nk})(d_k \omega_{nk})(\xi_k + d_k)} \cdot \xi_k
\]

\[
C_f = \sum_f \left[ \frac{Q_x(d_f \omega_{nf})}{S(1)(d_f \omega_{nf})(d_f \omega_{nf})} \cdot \xi_f + \frac{Q_x(\bar{d}_f \omega_{nf})}{S(1)(\bar{d}_f \omega_{nf})(\bar{d}_f \omega_{nf})} \cdot \bar{\xi}_f \right]
\]

\[
\phi_f = \tan^{-1} \frac{I_{Q_f}}{R_{Q_f}} - \tan^{-1} \frac{I_{S'} f}{R_{S'} f} - \tan^{-1} \frac{2(\xi_f - \bar{f}) \sqrt{1 - \xi_f^2}}{(\xi_f - \bar{f})^2 - (1 - \xi_f^2)}
\]

and \( I_{Q_f} \) means the imaginary part of the quality function
\( Q_x(d_f \omega_{nf}) \) and \( R \) means the real part, etc.

From the above solution, it is seen that the surging disturbance does not affect the stability of the system. However, it produces a surging component with a magnitude factor

\[
[C_k + C_f + \frac{Q_x(0)}{S(0)}] \]

comparable to the disturbance function and another subsiding component which adjusts the zero condition of the response.

- C3 -
APPENDIX D

Semigraphical Application of De Moivre's Theorem
in Evaluating Polynomial Functions with a Complex Number

by

Y. J. Liu

In evaluating the polynomial \( Q(d) \) or \( S'(d) \) with the substitution of a complex number \((-\xi + i\sqrt{1 - \xi^2})\omega_n\), the work is tedious. However, much time may be saved by using De Moivre's Theorem graphically. Assume:

\[ S'(d) = a_m d^m + a_{m-1} d^{m-1} + \ldots + a_4 d^4 + a_3 d^3 + a_2 d^2 + a_1 d + a_0 \]  

and \( d_k \omega_{nk} = (-\xi_k + i\sqrt{1 - \xi_k^2})\omega_{nk} \)

Then \( S'(d_k \omega_{nk}) = a_m \omega_{nk}^m d_k^m + \ldots + a_3 \omega_{nk}^3 d_k^3 + a_2 \omega_{nk}^2 d_k^2 + a_1 \omega_{nk} d_k + a_0 \)

\[ d_k = -\xi_k + i\sqrt{1 - \xi_k^2} \]

where \( \cos \theta_k = -\xi_k \) and \( \sin \theta_k = \sqrt{1 - \xi_k^2} \)

\( \theta_k \) is in the second quadrant or

\[ \theta_k = \frac{\pi}{2} + \tan^{-1} \frac{\xi_k}{\sqrt{1 - \xi_k^2}} \]

where \( \theta_{kp} \) is defined as proper angle.

By De Moivre's Theorem

\[ d_k^m = \cos m\theta_k + i \sin m\theta_k \]

\[ = \cos m\left(\frac{\pi}{2} + \theta_{kp}\right) + i \sin m\left(\frac{\pi}{2} + \theta_{kp}\right) \]

\[ = \cos \left(\frac{m\pi}{2} + m\theta_{kp}\right) + i \sin \left(\frac{m\pi}{2} + m\theta_{kp}\right) \]

\[ = \cos \frac{m\pi}{2} \cos m\theta_{kp} - \sin \frac{m\pi}{2} \sin m\theta_{kp} \]

\[ + i \sin \frac{m\pi}{2} \cos m\theta_{kp} + i \cos \frac{m\pi}{2} \sin m\theta_{kp} \]

\[ = (\cos \frac{m\pi}{2} + i \sin \frac{m\pi}{2})(\cos m\theta_{kp} + i \sin m\theta_{kp}) \]

\[ = e^{im\frac{\pi}{2}} (\cos m\theta_{kp} + i \sin m\theta_{kp}) \]

\[ = i^m (\cos m\theta_{kp} + i \sin m\theta_{kp}) \]

From Eq. D7 a simple graphical method can be developed for finding \( d_k^m \) by a circle superimposed on rectangular coordinate paper.
At the upper left hand part of Fig. D1 is the scale of $\zeta$. Pick up $\zeta_k$ and drop it down to the circle. The arc between this point and zenith of the circle represents the value of $\theta_{kp}$. The coordinates of this point are $-\zeta_k + 1\sqrt{1 - \zeta_k^2}$. Use a pair of dividers to pick up the chord length of the arc and divide the circle with this chord length counterclockwisely until the $m$th point or $m$ times the $\theta_{kp}$ is obtained, where

$$d_k^m = i^m (\cos m\theta_{kp} + i \sin m\theta_{kp})$$

$$= i^m (y_m + ix_m)$$

$$= i^{m-1} (-x_m + iy_m)$$

where $-x_m$ and $y_m$ are coordinates of $m\theta_{kp}$

For instance, let $\zeta_k = .372$ (See Fig. D1)

$$d_k^2 = i(-0.69 + 10.72) = - .72 + 10.69$$

$$d_k^3 = i^2(-0.908 + 10.42) = + .908 - 10.42$$

$$d_k^4 = i^3(-1.0 + 10.03) = + 0.03 + 11.0$$

$$d_k^5 = i^4(-0.93 - 10.36) = - 0.93 - 10.36$$

Such a graphical application to evaluate $d_k^m$ does not need a known angle, but a dividers and circle diagram should be provided. The result is accurate enough for engineering purposes, yet the method is so simple that not even a trigonometric table is needed.

With all $d_k^m$ known, $S^i(d_k \omega_{nk})$ and $Q(d_k \omega_{nk})$ can be easily evaluated in the form of Eq. D3.
FIG D1

Semigraphical Application of DeMoivre's Theorem in Evaluating Polynomial Functions with complex number