Power hitting: finding the right implement

by

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Abstract

Striking a ball with an implement occurs often in sports. Athletes are given a large
variety of options to choose from when they select their implements. The motivation
for this study was the need for a simple method that athletes can use to choose
the implement that will allow them to perform their best. The specific focus of
this work is identifying the weight properties an implement should have in order to
have the most “powerful” shot. The “power” of a shot is measured by how fast the
athlete is able to make the ball move after hitting it (the outgoing ball speed). The
particular weight property of interest is the implement’s mass moment of inertia about
an axis through its handle. Five simple models for how the implement’s moment of
inertia affect the outgoing ball speed are developed and compared, primarily in a
field hockey case study. A new model based on the physiology of muscles proves to
be more successful in capturing the behavior observed in real striking of sports balls
and is the primary contribution of this study. Overall, the models predict that heavier
implements than are currently used would produce more powerful shots. This result
is reasonable, as implement’s are rarely selected with the sole purpose of hitting power
shots. Additional objectives should be incorporated into the model to more broadly
aid in an athlete’s implement selection process.

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Contents

1 Introduction ........................................... 15
  1.1 Motivation ........................................ 15
  1.2 Implement selection: state of the art .................. 16
  1.3 Implement selection: state of the science .............. 17
  1.4 Thesis roadmap .................................... 19

2 Collision model ........................................ 23
  2.1 One-dimensional model ................................ 23
  2.2 Coefficient of restitution ................................ 25
  2.3 Special case: stationary ball .......................... 26
  2.4 Analytically finding the optimum ....................... 27
    2.4.1 With an assumption for the form of the effective mass ... 28
    2.4.2 With an assumption for the form of the implement speed ... 30
    2.4.3 With an assumption for the incoming ball speed ........... 30

3 Effective mass .......................................... 33
  3.1 From geometry ...................................... 33
    3.1.1 Single-link model .................................. 34
    3.1.2 Double pendulum model .............................. 36
  3.2 From measured collisions ................................ 38
  3.3 Comparing the model to the measurements ................ 40

4 Implement speed ....................................... 43
4.1 Single-link model ........................................ 43
  4.1.1 Constant Torque ................................... 44
  4.1.2 Constant Power ................................... 45
  4.1.3 Hill Model ....................................... 46
4.2 Double pendulum model ............................. 48
4.3 Comparing the models ............................... 50
  4.3.1 Estimating parameters ......................... 51
  4.3.2 Results ......................................... 57

5 Optimum implements ................................. 59
  5.1 Combining the models ............................ 59
  5.2 As a function of incoming ball speed .......... 61
  5.3 As a function of the athlete .................... 62

6 Conclusion ............................................ 65
  6.1 Summary ......................................... 65
  6.2 Next steps ...................................... 67
List of Figures

1-1 The roadmap for the work in this thesis. The objective is to take limited information about the athlete and the sport and recommend the optimum implement for power hitting. There are two essential models developed in this work: a collision model and a biomechanical model of swinging. 

2-1 The collision model. The ball (circle) and implement (rectangle) are modeled as point particles interacting in one dimension with the depicted masses and speeds before and after the collision.

3-1 The single-link model of swinging. The implement is modeled as a rod attached to a pivot point (the wrist joint). The position of the rod is described by $\theta$.

3-2 The double pendulum model of swinging. The first link represents the athlete’s arm(s), and its position is described by $\theta_1$. The second link represents the implement, and its position is described by $\theta_2$.

3-3 An example of a video captured of a tennis collision. The blue points represent the LoggerPro® tracking of the impact point on the implement while the red represent the tracking of the ball. The tracking was done manually.
3-4 A comparison of the mass ratio measured from videos of collisions and that predicted by the single-link model. Each point represents the average of ten collisions. The data are correlated with $r = 0.762$, $p \leq 0.05$. Each color represents a different athlete (red is Subject A), and each symbol shape represents a different sport (labeled in the plot).

4-1 The famous muscle force-velocity curve observed by A.V. Hill[1]. How fast a muscle can contract ($v$) depends on the load it is carrying ($F$). This relationship was discovered to be hyperbolic and is depicted here. The curve depends on three biological parameters: the maximum isometric tension of the muscle ($F_{\text{max}}$), the maximum contraction speed under no load ($v_{\text{max}}$), and a parameter that determines the curvature of the relationship ($F^*$).

4-2 Implement speed vs. moment of inertia for the five swinging models. The plot is log-log and shows that the single-link models with constant torque and power applied both produce the power laws expected ($n = 1/2$ and $n = 1/3$ respectively). The other three models are only power laws locally, but the values of $n$ are closer to those observed in the literature (see Table 1.1).

5-1 Outgoing ball speed vs. implement moment of inertia for power hitting in field hockey. Each model for swinging displays a maximum performance (outgoing ball speed) for a different implement. However, only the constant torque model’s optimum implement is similar to the implements seen in the sport of field hockey.
5-2 Optimum implement moment of inertia vs. incoming ball speed. The best implement to use will depend on the speed of the ball typically seen in a given level of play; a heavier (higher moment of inertia) implement should be used for hitting balls that are moving quicker. The above plot uses the biological parameters discussed above for field hockey, despite it being a sport where power hitting is most important when the ball is not moving.

5-3 Optimum implement moment of inertia vs. athlete weight. The best implement to use will depend on the individual athlete. A “self-similarity” scaling is used to extrapolate the performance of a given athlete based on his or her weight. These results suggest that heavier athletes should generally use heavier (higher moment of inertia) implements.
List of Tables

1.1 A summary of literature on the power law relationship between the implement's speed and its weight properties (its mass moment of inertia) in the form of Eq. 1.1 .................................................. 18

4.1 A comparison of the parameters used in each of the five models. The models are labeled in shorthand where 1 = single-link, 2 = double pendulum, T = constant torque(s), P = constant power, and H = Hill model. Using a Hill-inspired model increases the number of parameters required, as does using a double pendulum model. ........................................ 51

4.2 Values and a typical range for each of the parameters used in the models. 52

4.3 Values of the fitted scaling parameter $\mu$ for each of the different models. 1 = single-link, 2 = double pendulum, T = constant torque(s), P = constant power, H = Hill model. All values are greater than one but are on the order one. .................................................. 57
Chapter 1

Introduction

1.1 Motivation

Many sports require players to hit a ball with an implement, like a bat or a racket. These implements come in different weights, weight distributions, dimensions, materials, and even colors. While each sport has some regulations on these properties, players are still left with a wide range of implements from which to choose. This study was motivated by the players' need for guidance in selecting the implements that will provide them with the best performance.

The goal of determining the best implement is quite broad, and thus a narrower question is explored here: what weight properties should the implement have in order to make the most "powerful" shot? Hitting "powerful" shots is a common objective in many sports because hitting the ball "harder" generally puts more pressure on the opponent to perform quickly. In this study, how "hard" an athlete hits the ball will be characterized by how fast the ball moves when it leaves its collision with the implement (the outgoing ball speed, $v_{out}$); the goal is to maximize this speed.

The effect of the implement weight properties on the outgoing ball speed is not straightforward. The speed will be determined by how much momentum transfer occurs in the collision and by extension, how much momentum the implement has entering the collision. While heavier implements have more mass (which would mean more momentum), they also are hard to accelerate, so they will be moving slower...
(which would mean less momentum). The weight properties of the implement therefore have an interesting, competing effect on the outgoing ball speed. Which of these two effects “wins out”? The answer depends on the nature of the collision and on how strongly the speed of the implement depends on its weight properties.

This study develops models of the collision and the biomechanics of swinging the implement to understand the competing effect of implement weight properties on the outgoing ball speed. These models are then used to examine which weight properties provide the optimum implement for “power hitting” in a given game. An underlying assumption of this work is that each athlete could have a different optimum implement; the models, then, must also be specific to the athlete in some way. The goal is to create a method for providing individual athletes with recommendations for implement selection to optimize their performance, improving the overall experience of participating in and viewing their sport.

1.2 Implement selection: state of the art

There are three sources upon which athletes can rely when deciding what sports implement to use. The first source is his or her personal experience. At a recreational level, athletes often buy implements simply because of their cost or appearance (as drives most purchasing decisions). Beyond that, athletes often try out the “feel” of different implements to determine what they prefer. The idea of “feel” is sometimes linked to the size of the “sweet spot”[2-4] or the swing-weight (a weight property)[5] of the implement; however, it is never discussed how the “feel” of the implement can be different for different athletes. Finally, the most common way personal experience plays a role in implement selection is that once athletes find implements they like, they will buy many of the same one so that when one (or two or three...) breaks, they are still able to use the implement they like best, even if it is no longer being manufactured. For this reason, athletes often miss out on new advances in implement technology.

The second source of information for athletes is the experience and recommenda-
tions of professionals. This advice can be found online\cite{6,9} or through one-on-one interactions with experts (like golf pros in a course’s pro shop). While these opinions may be based on more evidence than what a single athlete can experience, the advice can be conflicting. Particularly, there does not seem to be a consensus on the effect of implement weight properties on performance; some sources say that heavier, more “head-heavy” implements are more powerful\cite{6,7} while others say the opposite\cite{9} and some don’t draw any connection between the power of the shot and the implement’s weight properties\cite{8}. With only the first two sources of support, the athlete is still left to make a difficult and largely qualitative decision about what implement to use.

The third form of support available to athletes is the use of scientific models. Athletes are perhaps least aware and have less access to this type of guidance, but it exists nonetheless. This is the category of support in which the results of this study fall. Previous work on the topic of implement selection is discussed in the following section, and the role of this particular study in producing an accessible but accurate model is carved out.

### 1.3 Implement selection: state of the science

There are three common conclusions reached in the literature on implement-ball collisions in sports and the effects of implement weight properties. First, the collisions are quick and simple enough that they can be modeled as rigid body interactions\cite{5,10,11}. Such a model is used in this study and is discussed in detail in Chapter 2. Second, the speed of the implement (and also the outgoing ball speed) is a strong function of the implement’s weight distribution (its mass moment of inertia) but a weak function of its total weight\cite{3,12-14}. The third common conclusion is that this strong function of moment of inertia follows a power law that goes like

$$v_{imp} = \frac{A}{I_p^n} \quad (1.1)$$
where $A$ is a constant that is unique to the athlete, $v_{\text{imp}}$ is the speed of the implement at the time and point of impact, and $I_p$ is the second mass moment of inertia of the implement about an axis through the end of the handle. There is, however, some variation in the position from which the implement speed and moment are measured.

Though many agree that there is a power law relationship, there remains disagreement on the value of the power ($n$), and no underlying model has been able to explain this behavior. Additionally, no model has been created to determine the athlete-specific parameter ($A$) that reflects the uniqueness of the athlete. A summary of the studies proposing a power law dependence and evaluations of the parameter $n$ is laid out in Table 1.1. Overall the measured power of $n$ ranges between 0.2 and 0.3. As will be discussed in Sections 4.1.1 and 4.1.2, a power law arises in the case where a single link is swung with a constant torque or a constant power, but neither power law matches experimental observations ($n = 1/2$ and $n = 1/3$ respectively). Different models for accelerating the implement from rest to the point of impact are established in Chapter 4.

While many do agree on the power law relationship, others have proposed different models, both fitted models[16] and those based on physical (minimum work) principles[17]. Even Rod Cross, one of the most prominent researchers in the field of sports implements, has discussed fitting two separate power laws, one for light

Table 1.1: A summary of literature on the power law relationship between the implement's speed and its weight properties (its mass moment of inertia) in the form of Eq. 1.1

<table>
<thead>
<tr>
<th>Value of $n$</th>
<th>Sport(s)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.17 (forehand) - 0.27 (serve)</td>
<td>Tennis</td>
<td>[12]</td>
</tr>
<tr>
<td>0.3</td>
<td>Baseball</td>
<td>[2]</td>
</tr>
<tr>
<td>0.314</td>
<td>Tennis</td>
<td>[3]</td>
</tr>
<tr>
<td>0.28</td>
<td>Baseball, Tennis</td>
<td>[5]</td>
</tr>
<tr>
<td>0.25</td>
<td>Baseball, Tennis</td>
<td>[10]</td>
</tr>
<tr>
<td>0.25</td>
<td>Badminton</td>
<td>[13]</td>
</tr>
<tr>
<td>0.12 - 0.35</td>
<td>Different weighted rods</td>
<td>[14]</td>
</tr>
<tr>
<td>0.22</td>
<td>Meta-analysis</td>
<td>[15]</td>
</tr>
</tbody>
</table>
implements (low moment of inertia) and one for heavy implements (high moment of inertia)[12, 14]. Overall, the assumption of a power law is prevalent enough that it will be used as a meter-stick for evaluation of the models developed in Chapter 4.

The three conclusions discussed above still do not easily translate to a model that helps athletes choose their optimum implement; in fact, the uniqueness of the athlete is rarely considered. However, there has been at least one successful study on selecting implements for power hitting for individual athletes[18]. In this study, Watts and Bahill had athletes swing many differently weighted bats and recorded the implement speed each athlete was able to attain. From this data they fit a curve for each athlete’s implement speed as a function of the implement’s mass. From these fitted relationships and a simple collision model similar to the one discussed in Chapter 2, they were able to recommend bat weights to softball and baseball players to maximize their batted ball speed. In general, the optimum bat was lighter than athletes were using at the time, and their work started somewhat of a revolution towards lighter bats in collegiate softball[18].

While Watts and Bahill’s study showed its value in this application, it is actually too athlete-specific to be broadly applied. For a recreational athlete, it is not realistic to visit a testing facility and swing many implements to evaluate the individual’s intrinsic reaction to bats of different weights. They do try to extend their observations to a more generic model, but it is still largely based on fitted data, which makes it quite constrained, particularly in the application to other sports. The objective of this thesis is to create a model that does not require much interaction with the athlete but still is able to provide specific recommendations for each individual’s optimum implement in their sport of choice.

1.4 Thesis roadmap

The goal of this work is to take information about a particular athlete and sport and determine what the implement weight properties should be for he or she to perform best. This thesis follows the roadmap laid out in Fig. 1-1 to identify this optimum
implement. The success of an implement is measured as how fast the ball is moving after it is struck by the implement. One of the two essential models in the framework of this study is therefore a collision model. Here, the collision is modeled as a one-dimensional interaction between two point masses (the ball and the implement). This model is developed in Chapter 2.

The collision model depends both on parameters of the sport and on the model for swinging that is used. The three pieces of information about the sport that are used in this model are the coefficient of restitution, the mass of the ball, and most importantly, the typical speed at which the ball is traveling before the collision. The coefficient of restitution is discussed in depth in Section 2.2. Through some analysis of the collision model in Section 2.4, it is determined that the coefficient of restitution affects the magnitude of the outgoing ball speed, but it does not affect the value of the implement weight properties that give the maximum outgoing ball speed. This is an essential finding of the work presented here.

The driving hypothesis behind this study is that the outgoing ball speed is dependent on the implement's weight properties, so the collision model used must capture
this relationship. It does so through two parameters: the effective mass of the combined implement and athlete and the implement speed. Both of these depend on the model for implement swinging that is used. In Chapter 3, the effective mass is derived for two different swing set-ups: a single-link model and a double pendulum model. In Section 3.3, the effective mass predictions based on the single-link model are compared to measurements of the effective mass in actual collisions. The two are correlated strongly enough that there is not yet enough evidence to suggest the more complex double pendulum model should be used.

The effect of the implement's weight properties on its speed is explored in Chapter 4. This requires further elaboration of the swinging models; in addition to the two set-ups detailed in the discussion of effective mass, the torques that are applied to accelerate the links in these models must be determined. In Section 4.1, three different torque models are developed for the single-link set-up, and then in Section 4.2, two more are discussed for the double pendulum case. Each torque model requires some information about the particular athlete swinging the implement. Incorporating the individual athlete in the swinging model makes this study unique. In Section 4.3, all five of these swinging models are compared to each other and to the historical observations that were discussed in Section 1.3 for one specific case: an athlete hitting a field hockey ball with a field hockey stick (for which actual swinging data was collected). One of the primary innovations in this work is the use of a physiologically based torque model for swinging, and its success is highlighted in the comparison to more traditional models.

In Chapter 5, all of the pieces of the different models are combined to translate a given athlete and sport into a predicted outgoing ball speed, which is then used to determine the optimum implement weight properties for power hitting. The level of play of the sport is varied by varying the incoming ball speed, and the effects this has on the optimum implement parameters are explored in Section 5-2. The athlete is modeled simply by his or her weight, and the effects this has on the optimum implement are covered in Section 5-3. The results of this study are summarized in Chapter 6, and here, the shortcomings of the work are also discussed. Future work
to correct for these is then proposed in Section 6.2.
Chapter 2

Collision model

2.1 One-dimensional model

The momentum transferred to the ball is determined by its collision with the implement. For this study, the collision is modeled as a one-dimensional interaction between two point masses: the ball, \( m_{\text{ball}} \), and the combined implement and athlete, \( M_{\text{eff}} \). The one-dimensional model discussed here is commonly used in studies of sports collisions\([3, 5, 10, 11, 14, 18]\) and is depicted in Fig. 2-1. In this model, the two point masses are initially moving towards each other with speeds \( v_{\text{in}} \) (the ball) and \( v_{\text{imp}} \) (the implement). After the collision, the ball changes direction and moves at a speed \( v_{\text{out}} \), and the implement moves at \( v_{\text{imp.f}} \). The collision is assumed to be short, meaning the effect of any external forces (i.e. from the wrists) is negligible\([2, 5, 10]\).

(While negligible during the collision, these forces are not negligible and are in fact essential in accelerating the implement up to the moment of impact.) Conserving momentum in this situation then requires that

\[
-m_{\text{ball}} v_{\text{in}} + M_{\text{eff}} v_{\text{imp}} = m_{\text{ball}} v_{\text{out}} + M_{\text{eff}} v_{\text{imp.f}}. \tag{2.1}
\]

Collisions are often characterized by a coefficient of restitution, \( e \), whose value
indicates the elasticity of the interaction. It is defined as

\[ e = \frac{v_{\text{out}} - v_{\text{imp,f}}}{v_{\text{in}} + v_{\text{imp}}} \]  

(2.2)

and is discussed in further detail in Section 2.2. Using this definition, Eq. 2.1 simplifies to

\[ v_{\text{out}} = \frac{1 + e}{1 + r} v_{\text{imp}} + \frac{e - r}{1 + r} v_{\text{in}} \]  

(2.3)

where \( r \) is the mass ratio \( m_{\text{ball}}/M_{\text{eff}} \). This form of the momentum conservation equation helps illustrate what plays a role in determining the outgoing ball speed: the incoming ball speed \( (v_{\text{in}}) \), coefficient of restitution \( (e) \), mass ratio \( (r) \), and incoming implement speed \( (v_{\text{imp}}) \). This is one of the essential equations in this research and will be referred to as the collision equation.

The objective of this study is to understand how the weight properties of the implement affect the outgoing ball speed, and with a one-dimensional collision model, this can be done simply by exploring how the implement weight properties affect these
four parameters. The incoming ball speed, $v_{\text{in}}$, is something that is characteristic of the sport being played and the level at which it is being played. It will not be directly affected by the weight properties of the implement. The behavior of the other three parameters, $e$, $r$, and $v_{\text{imp}}$, are discussed in Section 2.2, Chapter 3, and Chapter 4 respectively. The result is that the outgoing ball speed is related to the sport through the ball mass, $m_{\text{ball}}$, the coefficient of restitution $e$, and the incoming ball speed $v_{\text{in}}$; it is related to the athlete and the implement through the effective mass $M_{\text{eff}}$ and the implement speed $v_{\text{imp}}$. The effect of the implement’s weight properties on the incoming implement speed is the least straightforward of these relationships, and the exploration of different models for this relationship is the primary contribution made by this study.

2.2 Coefficient of restitution

The coefficient of restitution, $e$, is a measure of the elasticity of the collision. It is most often discussed in terms of a collision between a ball and an infinite ($r = 0$), stationary ($v_{\text{imp}} = 0$) surface (like the ground). In that case, the collision equation reduces to

$$v_{\text{out}} = ev_{\text{in}}.$$  \hspace{1cm} (2.4)

From this perspective, the value of $e$ represents the percent of speed the rebounding ball is able to maintain from before the collision: from no speed lost (fully elastic with $e = 1$) to all speed lost (fully inelastic with $e = 0$). From this relationship between the incoming and outgoing speeds, it is simple to determine the percent of kinetic energy lost in the collision:

$$E_{\text{lost}} = 1 - e^2.$$  \hspace{1cm} (2.5)

This is a second way of interpreting the coefficient of restitution and is a relationship that holds even in the case where the implement is finite and moving. One small note, however, is that kinetic energy is frame-dependent, and the coefficient of restitution is defined with respect to the frame of the center of mass of the objects in the collision.
(and therefore may be moving). Nonetheless, this interpretation of the coefficient of restitution is helpful in developing a qualitative understanding of collision elasticity.

The amount of kinetic energy maintained and lost in a collision depends on many attributes of the interaction. The kinetic energy loss is dominated by the behavior of the ball. Its deformation and the amount of energy absorbed in hysteresis behavior are both governed by its material properties. Altering these properties has a significant effect on the coefficient of restitution[2, 10, 18, 19]. There are still kinetic energy losses associated with the implement rather than the ball. This energy is primarily transferred to the vibrational modes of the implement[3, 4]. Which vibrational modes are exited depends strongly on the location of the impact on the implement. (This is why athletes observe a “sweet spot” where vibrations are minimized, making the swing “feel” most comfortable.) The impact location therefore also plays an essential role in determining the coefficient of restitution.

This study makes one essential assumption about the coefficient of restitution: that it does not depend on the weight properties of the implement. This assumption is commonly made in the study of sports implement-ball collisions [2, 10, 11, 14, 18, 19]. The principle behind this is that the material properties of the ball (and implement) and the impact location of the collision on the implement play a much more important roll in determining the loss of kinetic energy in the interaction. The expectation is also that these properties can be held constant while altering the implement’s weight properties. This assumption plays an essential role in simplifying the collision equation, as now the outgoing ball speed is only affected by the implement’s weight properties through two parameters: the mass ratio \( r \) and the implement speed \( v_{\text{imp}} \). The models for the relationship between these two parameters and the implement weight properties are discussed in Chapters 3 and 4 respectively.

2.3 Special case: stationary ball

The collision equation (equation 2.3) can be reduced in a number of special cases. In Section 2.2, the case where the “implement” (the ground) is infinitely massive \((r = 0)\)
and stationary \((v_{\text{imp}} = 0)\) is presented. The result is Eq. 2.4. This section provides discussion of another important simplified case: hitting a stationary ball.

In some sports, like golf, a player will exclusively strike a stationary ball. However, in other sports, hitting stationary balls can also be important. In fact, the shots where “power” hitting is most advantageous are often those where the athlete is striking a stationary ball. In tennis, for instance, the serve is perhaps the shot that is most strategically made as a “power” shot, and the serving player is hitting a ball that is nearly stationary with respect to horizontal (the one direction in the one-dimensional model). In this case, the collision equation simplifies because the term involving the incoming ball speed disappears, leaving

\[
V_{\text{out}} = +V_{\text{imp}} (2.6)
\]

This version of the collision equation is not only applicable to a wide variety of sports situations, it is easier to work with than the full version in Eq. 2.3. It removes one parameter \(v_{\text{in}}\), representing the level of play, from the effect on the outgoing ball speed. In this special case, the outgoing speed will depend on the sport through the ball mass \(m_{\text{ball}}\) and the coefficient of restitution \(e\), and it will still depend on the implement and athlete through the effective mass \(M_{\text{eff}}\) and implement speed \(v_{\text{imp}}\).

### 2.4 Analytically finding the optimum

Theoretically, the weight properties of the implement that give the largest outgoing ball speed can be determined by a simple maximization. As is discussed in Section 1.3, the mass moment of inertia of an implement (not the mass) has the strongest effect on the outgoing ball speed. Assuming the other weight properties have negligible effects on the outgoing ball speed, the optimization problem can be solved by finding where

\[
\frac{\partial}{\partial I_p} (v_{\text{out}}) = 0
\]
where $I_p$ is the mass moment of inertia of the implement through an axis at its handle's end (the conventional way of describing the weight distribution of sporting implements)[2, 3, 5, 10–16].

For this particular case, the coefficient of restitution, which appears in the equation for the outgoing ball speed, is taken to be independent of the weight properties of the implement, as is discussed in Section 2.2. The resulting differentiation gives

$$\frac{\partial}{\partial I_p} (v_{out}) = \frac{(1 + e)}{(1 + r)^2} \left[ \frac{\partial v_{imp}}{\partial I_p} (1 + r) - \frac{\partial r}{\partial I_p} (v_{imp} + v_{in}) \right] = 0. \quad (2.8)$$

The multiplier $(1 + e)/(1 + r)^2$ is always a positive value and therefore does not play a role in determining the maximum; it “falls out” of the equation, leaving

$$\frac{\partial v_{imp}}{\partial I_p} (1 + r) - \frac{\partial r}{\partial I_p} (v_{imp} + v_{in}) = 0. \quad (2.9)$$

### 2.4.1 With an assumption for the form of the effective mass

The next assumption to make pertains to the mass ratio, $r = m_{ball}/M_{eff}$. The effective mass is discussed in depth in Chapter 3, but for one particular model of swing, it can be written as

$$M_{eff} = \frac{I_p}{l_c^2} \quad (2.10)$$

where $l_c$ is the distance from the end of the handle to the impact point on the implement. A justification for this form of the effective mass is presented in Section 3.1. In this case, the mass ratio and its derivative simplify to

$$r = \frac{m_{ball} l_c^2}{I_p} \quad \frac{\partial r}{\partial I_p} = -\frac{m_{ball} l_c^2}{I_p^2}. \quad (2.11)$$

To further clean up Eq. 2.9, we define the function $f$:

$$f(I_p) = v_{imp} \quad (2.12)$$
so that the optimization equation becomes

\[ I_p \left( \frac{I_p}{m_{\text{ball}}l_c^2} + 1 \right) f' + f = -v_{\text{in}}. \]  

(2.13)

This equation can be solved analytically for the function \( f \). The solution will represent the implement speed that should be chosen to optimize the outgoing ball speed. The particular solution to this differential equation is just \( f_p = -v_{\text{in}} \) while the homogeneous solution can found through separation of variables:

\[ f_h = C \frac{I_p + m_{\text{ball}}l_c^2}{I_p} \]  

(2.14)

where \( C \) is the constant of integration that will depend on the conditions of \( f \). To find the value of this constant, we will make the assumption that for infinitely massive implements (massive in terms of moment of inertia), the speed of the implement will be zero as it will be impossible to accelerate it. Applying this condition to the full form of \( f \) (meaning \( f = f_h + f_p \)) requires that \( C = v_{\text{in}} \). Finally the function \( f \) (the shorthand for the implement speed \( v_{\text{imp}} \)) becomes

\[ v_{\text{imp}} = v_{\text{in}} \frac{m_{\text{ball}}l_c^2}{I_p} = v_{\text{in}} r. \]  

(2.15)

What does this mean? If the athlete were able to choose the implement speed that optimizes the outgoing ball speed, this is the speed they should choose given a particular implement and sport. There is an optimum implement speed for each implement in each sport. There is no reason to believe athletes can or do choose the implement speed in this way. More importantly though, we do not yet know whether this optimum is a maximum, a minimum, or something else. The next step is to use this expression for the implement speed in the collision equation. The result is

\[ v_{\text{out}} = e v_{\text{in}}. \]  

(2.16)
If the athlete were to swing the implement at a speed $v_{in}$, the outgoing ball speed would always be $ev_{in}$. All of the derivatives with respect to the implement properties are zero, so this expression naturally satisfies Eq. 2.9. A natural next interpretation could be that this is the bound on the outgoing ball speed. However, this hypothesis is quickly negated by considering the case where the ball is stationary (certainly the maximum outgoing ball speed can be larger than zero). Solving for the function $f$ is therefore just a mathematical exercise; it does not reveal anything about the optimum implement properties.

### 2.4.2 With an assumption for the form of the implement speed

As solving for the implement speed did not provide insight into the problem at hand, we will instead assume it takes a particular form. As discussed in Section 1.3, the implement speed is often believed to be related to the implement’s moment of inertia via the power law of Eq. 1.1. If the implement speed is assumed to take this form and the effective mass is still assumed to take the form in Eq. 2.10, the maximization result of Eq. 2.9 becomes

$$
[Am_{\text{ball}}I_{c}^{2}(1 - n)] I_{p}^{-n} - An I_{p}^{-(n-1)} + m_{\text{ball}}I_{c}^{2}v_{in} = 0. \tag{2.17}
$$

This result is rather messy and cannot be solved analytically for the optimum value of the implement moment of inertia without further simplifications.

### 2.4.3 With an assumption for the incoming ball speed

The final simplification to make is that the ball is stationary (the incoming ball speed is zero). In this case, the equation for the maximization is reduced, and we can solve for the optimum implement moment of inertia:

$$
I_{p,\text{opt}} = m_{\text{ball}}I_{c}^{2}\frac{(1 - n)}{n}. \tag{2.18}
$$
This result suggests that the optimum implement is proportional to the “moment of inertia” of the ball about the handle of the implement. This suggests that implement moments of inertia should be selected to “match” that of the ball, a reasonable result. The matching factor (constant of proportionality) is related to the power in the presumed power law \((n)\). Another way to express this relationship is in terms of an optimum mass ratio:

\[
\rho_{\text{opt}} = \frac{n}{1 - n}.
\]  

(2.19)

As discussed in Section 1.3, this power is expected to be between 0.2 and 0.3, leading to optimum mass ratios of \(1/4\) and \(3/7\) respectively. This means that if the implement were a point mass colliding with the ball, it should be roughly two to four times as massive as the ball in order to produce the most powerful shot. Moving forwards, this result will serve as a useful guideline for the desired mass ratio. However, it is important to remember the restricting assumptions that were made to get to this result: a specific form of the effective mass, a power law relationship for the implement speed, and a stationary ball.
Chapter 3

Effective mass

3.1 From geometry

In sports collisions, the ball does not typically collide with the center of mass of the implement. Therefore in order for the one-dimensional collision model from Chapter 2 to hold, a way of determining the effective (point) mass of the implement acting in the collision must be established. Determining the effective mass of a complicated system (at a point not corresponding to the system’s center of mass) is a common requirement in the field of robotics. Understanding the inertial properties of a multi-link robot is essential for establishing appropriate control algorithms[20, 21]. In addition, understanding the inertial properties of a robot system can be important for making it safe enough to operate around humans[22]. For example, if a point on a robot arm were to collide with a human, the effective mass of the arm in that collision and speed at that point determine how much pain the interaction will cause the person. In this section, work on the effective mass of multi-link arms from the field of robotics will be adapted to two specific models of athletes swinging implements.

The results of the robotics studies on effective mass can be summarized by the following equation:

$$M_{\text{eff}} = \frac{1}{\hat{\mathbf{u}}^T \mathbf{J} \mathbf{I}^{-1} \mathbf{J}^T \hat{\mathbf{u}}}.$$ (3.1)
This equation, amazingly, applies to any system of \( N \) links, though here, the details will only be discussed for the one and two link cases. This fundamental equation contains three tensors. First, the unit vector \( \mathbf{u} \) represents the direction of the anticipated collision, the direction in which the effective mass is to be evaluated. Second, the matrix \( \mathbf{J} \) is the traditional Jacobian; it represents the mapping of the derivatives of the generalized coordinates to the linear velocity at the point of interest (where the collision will happen) on the system. This mapping often depends on the configuration of the system, meaning it is a function of the generalized coordinates themselves. In mathematical terms,

\[
\mathbf{v}_c = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}}
\]  

where \( \mathbf{v}_c \) is the velocity at the point on the arm where there will be a collision and \( \mathbf{q} \) is the vector of generalized coordinates.

The inertia tensor, \( \mathbf{I} \), is the final piece needed to compute the effective mass. The inertia tensor again is often a function of the configuration of the system, meaning it is also a function of the generalized coordinates. It is defined by the way it describes the kinetic energy \( (T) \) of the system:

\[
T = \frac{1}{2} \mathbf{q}^T \mathbf{I} \mathbf{q}
\]  

where, again, \( \mathbf{q} \) is the vector of generalized coordinates describing the system’s configuration. Each of these three tensors can be simplified for any number of cases where the effective mass is of interest, but two important ones that model swinging in sports are discussed in the following sections.

3.1.1 Single-link model

Perhaps the simplest model for swinging a sports implement is a single link rotating around a stationary pivot. This model is depicted in Fig. 3-1. The first question to answer is: what does the link correspond to in terms of actual swinging? Again, we
aim to find the simplest model that still captures the essential behavior we see in real collisions. The simplest interpretation of this link is that it represents just the implement. All motion of the body is ignored. However, the action of the athlete still plays a role in determining the outgoing ball speed: the athlete is the one applying a torque to the link, accelerating it from rest at $\theta = 0$ to the point and moment of contact at $\theta = \pi/2$. To determine the amount of torque the athlete is able to apply, we again ask: what do the pieces of the model correspond to in real swinging? If the link is assumed to be exclusively the implement, the torque applied is coming from the wrist (in one-handed sports) or wrists (in two-handed sports). The form of this torque and how it relates the properties of the athlete’s wrist(s) is discussed in detail in Chapter 4.

For now though, the effective mass of the implement is still the property of interest. As this model only has a single degree of freedom (a single generalized coordinate), Eq. 3.1 becomes much more simple. We first define two additional parameters. As discussed in Section 1.3, the inertial property of the implement most often studied in sports is its moment of inertia about an axis through the end of its handle $I_p$. In this model, $I_p$ is also the moment of inertia of the link about the pivot.

Figure 3-1: The single-link model of swinging. The implement is modeled as a rod attached to a pivot point (the wrist joint). The position of the rod is described by $\theta$. 
The second parameter to define is the distance from the end of the handle (the pivot point) to the point where the ball makes contact with the implement. This distance will be called $l_c$. In terms of the effective mass equation (Eq. 3.1), the inertia “tensor” is then simply $I_p$ with an inverse $1/I_p$, and the “Jacobian” is $l_c$ (because $v_c = l_c \dot{\theta}$). The effective mass for a collision in the direction perpendicular to the link is therefore

$$M_{eff} = \frac{I_p}{l_c^2}.$$  \hspace{1cm} (3.4)

This is the expression for the effective mass that was used to explore the collision model in Section 2.4. The expression is simple and depends directly on the property which, in the literature, is most often attributed to affecting implement performance ($I_p$). Again, it is important to note that the single link model only accounts for the individual athlete through the applied torque and that in real sports situations, the swing is a coordinated motion of many links with complex joints connecting them. Nonetheless, the behavior of this simple model can be instructive and is explored further in Chapters 4 and 5.

### 3.1.2 Double pendulum model

The next step up in complexity is the two-link system: the double pendulum model. Unlike the single-link model, this model captures the motion of the athlete as well as the implement. As depicted in Fig. 3-2, link 1 represents the athlete’s full arm (one-handed sports) or arms (two-handed sports). This link is connected on one end to a fixed pivot point representing shoulder joint(s). On the other end, it is connected to link 2, which represents the implement. The connection therefore is through the hand(s) and wrist(s) and is modeled as a frictionless pin joint. The configuration of the system is defined by the two absolute angles $\theta_1$ and $\theta_2$. Muscles at both the shoulder(s) (the pivot point) and the wrist(s) (the pin joint connection) accelerate the links from rest at $\theta_1 = 0$ and $\theta_2 = 0$ to the point of contact. In this case, we will still assume contact occurs at a point $l_c$ away from the handle of the implement (from the pin joint). The collision will occur when this point crosses the mid-plane.
of the body, which in the Fig. 3-2 corresponds to it moving from the right to the left of the pivot point.

Again, the quantity of interest is the effective mass of the implement in a collision for a given configuration of the links. To determine this value, some additional properties of the links must be known: the length of the first link ($l_{\text{arm}}$), the moment of inertia of the first link about the pivot point ($I_{\text{arm}}$), the mass of the second link ($m_{\text{imp}}$), and finally the distance from the pin joint to the center of mass of the second link ($h_{\text{imp}}$). More care must be taken this time in using Eq. 3.1. First, the ball is assumed to be traveling in the $x$-direction:

$$ \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  

(3.5)

The Jacobian, $\mathbf{J}$, will be

$$ \mathbf{J} = \begin{bmatrix} -l_{\text{arm}} \sin \theta_1 & -l_c \sin \theta_2 \\ l_{\text{arm}} \cos \theta_1 & l_c \cos \theta_2 \end{bmatrix}.$$  

(3.6)
so that the linear speed of the implement at the contact point is \( \mathbf{v}_c = J\dot{\theta} \). Finally, through some analysis of the kinetic energy, the inertia matrix is determined to be

\[
I = \begin{bmatrix}
I_{\text{imp}} + m_{\text{imp}}l_{\text{imp}}^2 & m_{\text{imp}}h_{\text{imp}}l_{\text{arm}}\cos(\theta_2 - \theta_1) \\
m_{\text{imp}}h_{\text{imp}}l_{\text{arm}}\cos(\theta_2 - \theta_1) & I_p
\end{bmatrix}.
\] (3.7)

where \( I_p \) is still the moment of inertia of the implement about an axis through its handle (through the pin joint). The three tensors of Eqs.3.5, 3.6, and 3.7 can then be combined via the relationship in Eq. 3.1 to determine the effective mass of this two link system.

This model is significantly more complicated than the single-link model and requires many more geometric and inertial properties. However, requiring additional properties of the athlete’s arms further allows for individualization of the model. Additionally, the double pendulum model allows for the possibility of coordinated motion by the muscles producing different torques at the two joints. The coordination of the muscles is not a focus of the work presented here, but one way to incorporate this aspect of swinging is discussed in Section 6.2.

### 3.2 From measured collisions

The effective mass of the combined body and implement can be calculated for any known collision without having to model the swinging dynamics. The conservation of momentum for the collision, described in Eq. 2.1, can be rearranged to determine the effective mass:

\[
M_{\text{eff}} = m_{\text{ball}} \left( \frac{\mathbf{v}_{\text{in}} + \mathbf{v}_{\text{out}}}{v_{\text{imp}} - v_{\text{imp, f}}} \right).
\] (3.8)

In this study, velocity data for a variety of different collisions was collected from videos in order to compute typical effective masses and to validate the models discussed in Section 3.1.

The recorded collisions were restricted to hits with stationary balls to reduce measurement errors. Seven different athletes participated. One, who will be called
Subject A, performed tests for golf, squash, and field hockey, while each other athlete performed tests for only one sport. Each athlete hit the ball ten separate times (per sport) with the same implement. For the field hockey trials, the same implement was used by each athlete. After the video collection, each implement was weighed on a scale and measured in length.

The videos were captured using the slow motion feature on an iPhone 6 to collect footage at 240 frames per second. Kinematic data was extracted from these videos using Vernier’s LoggerPro® software. Fig. 3-3 shows an example of the tracking of the ball and implement to determine their velocities during each of the collisions. The length scale of the motion in the image was set by reference to the implement’s length, while the frame rate determined the time scale. Notice, however, that in determining the effective mass and, rearranged, the mass ratio:

\[
    r = \frac{m_{\text{ball}}}{M_{\text{eff}}} = \frac{v_{\text{imp}} - v_{\text{imp,f}}}{v_{\text{in}} + v_{\text{out}}}
\]

that both the length and time scales will cancel out. This means error in setting the scales (particularly the length scale) do not propagate into the calculation of effective mass.

For each athlete and each sport, the measured mass ratio for each of the ten

Figure 3-3: An example of a video captured of a tennis collision. The blue points represent the LoggerPro® tracking of the impact point on the implement while the red represent the tracking of the ball. The tracking was done manually.
collisions was averaged, and the uncertainty in the measurements and its propagation were computed. These results are shown as values on the independent axis of Fig. 3-4. The results of these trials and the uncertainty measurements are discussed in the following section.

3.3 Comparing the model to the measurements

The measured mass ratios for each athlete and each sport are compared to the predicted mass ratios in Fig. 3-4. The predicted values shown are based on the single-link model. The double pendulum model predictions were not examined for two reasons. First, there is a large amount of uncertainty in evaluating the inertial properties of human body. Second, the double pendulum model assumes planar motion, but the actual swinging motion occurs in three-dimensions. It is therefore difficult to determine from videos what configuration the links are in at the moment of the collision. Both of these sources of uncertainty diminish the value of the predictions made for the effective mass using the double pendulum model.

In determining the predicted values of the mass ratio in terms of a single-link model, the field hockey stick, squash racket, and tennis racket were all assumed to be roughly uniform, and their moments of inertia about their handles were evaluated as

$$I_p = \frac{1}{3}m_{imp}l_{imp}^2. \quad (3.10)$$

The mass ($m_{imp}$) and length ($l_{imp}$) of these implements were measured using a scale and a measuring tape respectively. The vertical error bars presented in Fig. 3-4 represent uncertainty in these measurements as well as uncertainty in the assumption of uniform implements. The golf club is a special case because unlike the other implements, its mass is concentrated almost entirely at its head. For this reason, the effective mass of the golf club was consider simply to be the mass of the club. As this mass ratio depends only on the measurement of the implement's mass (and not also its length), the uncertainty in this prediction is smaller.
Figure 3-4: A comparison of the mass ratio measured from videos of collisions and that predicted by the single-link model. Each point represents the average of ten collisions. The data are correlated with $r = 0.762, p < 0.05$. Each color represents a different athlete (red is Subject A), and each symbol shape represents a different sport (labeled in the plot).

Overall, the model predicts the general trend of the measured observations; the correlation between the two is strong ($r = 0.762, p < 0.05$). Implements with smaller effective masses (larger mass ratios) are predicted to have smaller effective masses. This result holds for different athletes and different sports over many trials. The correlation, however, is not perfect. Notably, the single-link model predicts that each athlete swinging the same implement will have the same effective mass (like for the four different athletes performing field hockey tests). However, the measured values of the effective masses for these four athletes are different. Despite its simplicity, the single-link model produces predictions for the mass ratios in collisions that are correlated strongly enough with the measured values that it remains a viable model to be explored further in Chapter 4.
Chapter 4

Implement speed

In order to understand the effect of implement weight properties on the athlete's performance, the relationship between these weight properties and the swing speed of the implement must first be established. As is discussed in Section 1.3, the literature suggests that the swing speed depends primarily on the moment of inertia of the implement (about an axis through its handle), and it does so in a power law relationship. While there remains some disagreement on this functional form, the power law suggested in Eq. 1.1 (with observed powers discussed in Table 1.1, $n \approx 0.2 - 0.3$) will be used to evaluate the success of the different models of implement speed discussed in this chapter.

4.1 Single-link model

As in Chapter 3, the first model to explore is the simplest one: the single-link model shown in Fig. 3-1. This link represents the implement, and the athlete is represented by the torque, $\tau(t)$, applied at the pivot point causing the link to accelerate. The equation of motion for this single degree of freedom system is

$$\tau(t) = I_p \ddot{\theta}$$

(4.1)
where $I_p$ is the moment of inertia of the link (the implement) about the pivot point (an axis through its handle) and $\theta$ is defined as in the figure. The initial conditions for the swing are

$$\theta(0) = 0 \quad \dot{\theta}(0) = 0.$$  

(4.2)

The collision occurs at time $t_c$ when

$$\theta(t_c) = \pi/2 \quad \dot{\theta}(t_c) = \omega_c.$$  

(4.3)

Modeling the torque applied by the athlete is the remaining piece of this relationship to determine. Again, simple models (constant torque and constant power) are examined first. As these are known not to produce the desired power law, a new model is proposed whose roots lie in the fundamental force-velocity behavior of muscles.

### 4.1.1 Constant Torque

The first simple guess to make is to assume that athletes swing with a constant torque, $\tau_0$. The equation of motion then becomes

$$\tau_0 = I_p \ddot{\theta}.$$  

(4.4)

with the initial conditions presented in Eq. 4.2. By solving this equation, the angular position and speed are determined to be

$$\theta(t) = \frac{\tau_0 t^2}{2I_p} \quad \dot{\theta}(t) = \frac{\tau_0 t}{I_p}.$$  

(4.5)

The contact time $t_c$ is determined by using the condition in Eq. 4.3 and is

$$t_c = \sqrt{\frac{\tau_p}{\tau_0}}.$$  

(4.6)
Finally, the angular speed at contact is

$$\omega_c = \sqrt{\frac{\pi \tau_0}{I_p}}. \quad (4.7)$$

The linear contact speed (what is most often used in the power laws in the literature) is a simple factor of the “contact length” $l_c$.

Modeling the applied torque as constant therefore suggests that the implement speed is a function of the implement moment of inertia that fits the power law form of Eq. 1.1 with

$$C = l_c \sqrt{\pi \tau_0} \quad n = 0.5. \quad (4.8)$$

This formulation provides a description for the athlete specific parameter $C$ but evaluating it can be challenging: how much torque are athletes actually able to apply? Additionally, this approach produces a $1/2$ power law, which is not what has been observed experimentally. Further models need to be explored.

### 4.1.2 Constant Power

The second simple guess to make is to assume that athletes swing with a constant power, $P_0$. The equation of motion then becomes

$$\frac{P}{\dot{\theta}} = I_p \ddot{\theta} \quad (4.9)$$

with the initial conditions presented in equation 4.2. Again, the angular position and speed can be solved for:

$$\theta(t) = \sqrt{\frac{8P_0}{9I_p}} t^{3/2}, \quad \dot{\theta}(t) = \sqrt{\frac{2P_0 t}{I_p}}, \quad (4.10)$$

as well as the time of contact

$$t_c = \sqrt[3]{\frac{9\pi^2 I_p}{32P_0}}. \quad (4.11)$$
The angular speed at the time of contact in this case is

$$\omega_c = \sqrt[3]{\frac{3\pi P_0}{2I_p}}. \tag{4.12}$$

Again, the power law relationship arises with

$$C = l_c \sqrt[3]{\frac{3\pi P_0}{2}} \quad n = 1/3. \tag{4.13}$$

Determining what constant power athletes can produce is just as, if not more, challenging than determining their constant power. The underlying power law relationship, however, is much closer to the observed values than the constant torque case.

### 4.1.3 Hill Model

Both of the simple assumptions for the applied torque produce power law relationships between the implement speed and its moment of inertia, but neither relationship "matches" those observed experimentally. Additionally, there is no physical basis to assume that an athlete is likely to apply a constant torque (or power). In this section, the underlying behavior of muscles is discussed and is used to develop a new swinging model.

**Mechanics of muscles**

More than 80 years ago, A.V. Hill began exploring the mechanical behavior of muscle fibers. The simple relationships he discovered are still widely accepted in describing muscle function. One such canonical relationship is that for the force-velocity properties of muscle. Essentially, Hill discovered that how fast a muscle can contract depends on the load that it bears in the contraction; it will contract slower with a heavier load and faster with a lighter one. The particular form of this relationship is a hyperbola and is depicted in Fig. 4-1.

This hyperbolic relationship is characterized by three parameters. The first parameter is the muscle’s maximum isometric tension ($F_{\text{max}}$), the load at which the
Figure 4-1: The famous muscle force-velocity curve observed by A.V. Hill[1]. How fast a muscle can contract \((v)\) depends on the load it is carrying \((F)\). This relationship was discovered to be hyperbolic and is depicted here. The curve depends on three biological parameters: the maximum isometric tension of the muscle \((F_{\text{max}})\), the maximum contraction speed under no load \((v_{\text{max}})\), and a parameter that determines the curvature of the relationship \((F^*)\).

muscle stops being able to contract. The second is the maximum contraction speed \((v_{\text{max}})\), the speed the muscle contracts at when it is unloaded. Finally, the “curvature” of the relationship is characterized by the coefficient of heat shortening, \(a\), which is the amount of energy that is lost to heat per unit length of contraction. The “curvature” can equivalently be described by the ratio of the maximum isometric force and this coefficient of heat shortening and will here be referred to as \(F^* = F_{\text{max}}/a\). While Hill’s original experiments were performed by examining the contraction speed for a given load, this relationship can be interpreted in the opposite way. For a given contraction speed, there is a maximum load that can be supported. The equation for this relationship (the one normalized and shown in Fig. 4-1) is

\[
F = F_{\text{max}} \frac{1 - v/v_{\text{max}}}{1 + F^*v/v_{\text{max}}}.
\]
Application to swinging

While it is not certain whether or not the muscles driving the swinging motion pull with a constant torque, a constant power, or some other function of time, it is certain that they are what creates the motion. As Hill discovered, the amount of force these muscles can produce while contracting is limited by the speed of the contraction. Suppose that the muscles are pulling on a joint of radius $R$. The maximum amount of torque they are applying to the joint is related to the contraction speed (and thus how fast the joint is rotating). In the case of power hitting, we may assume that the muscles are working at their limits, trying to swing as “hard” as possible; the muscles are operating at this maximum torque. In the case of the single link, the muscles are acting around the wrist joint. Using the specific biological parameters that correspond to this joint and these muscles, a fairly clean function for the applied torque arises:

$$\tau(t) = F_{\text{max}, \text{wr}} R_{\text{wr}} \left( 1 - \frac{R_{\text{wr}} \dot{\theta}}{v_{\text{max}}} \right) = I_p \ddot{\theta}. \quad (4.15)$$

Subject to the initial conditions and collision criterion of Eq. 4.2 and 4.3 respectively, this relationship can be solved numerically to determine the effect of the implement moment of inertia on the speed of the implement at the time of the collision. The results of this numerical integration are discussed in Section 4.3.

4.2 Double pendulum model

As with the single-link model, the double pendulum model depends on the athlete to drive the swinging motion by applying torques at the joints. In this case however, there are two joints. A torque may be applied on link 1 (the arms) at the pivot point (the shoulder joints) but the arms may also apply a torque to link 2 (the implement) via the pin joint (the wrists) connecting them. If these two torques are $\tau_1(t)$ and $\tau_2(t)$
respectively, the equations of motion for this system can be written as

\[
\begin{bmatrix}
I_{\text{arm}} + m_{\text{imp}}l_{\text{arm}}^2 & m_{\text{imp}}h_{\text{imp}}l_{\text{arm}} \cos(\theta_2 - \theta_1) \\
m_{\text{imp}}h_{\text{imp}}l_{\text{arm}} \cos(\theta_2 - \theta_1) & I_{\text{p}} \\
+ m_{\text{imp}}h_{\text{imp}}l_{\text{arm}} \sin(\theta_2 - \theta_1)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
= \begin{bmatrix}
\tau_1(t) - \tau_2(t) \\
\tau_2(t)
\end{bmatrix}
\]  
(4.16)

Notice first that, in this model, a Coriolis term arises, and second that the torque applied to the second link by the first must be equal and opposite to the torque applied to the first by the second. All of the inertial and geometric properties are as defined in Section 3.1.2

To model the swinging motion, these equations are solved subject to the initial conditions

\[
\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = 0.
\]  
(4.17)

As discussed in Section 3.1.2, the collision in this case occurs at the time when the contact point (a distance \(l_c\) along the second link from the pin joint) crosses from the right to the left of the pivot point. If this happens at time \(t_c\), the speed of the implement in the \(x\)-direction at the time of the collision is

\[
v_c = l_{\text{arm}} \sin(\theta_1(t_c)) + l_c \sin(\theta_2(t_c)).
\]  
(4.18)

These equations can be solved numerically to determine the relationship between implement speed and its moment of inertia \(I_p\) about its handle. In this study, the double pendulum model was explored for two different cases of applied torques. The first is the case where constant torques are applied to both joints. The second is the equivalent Hill model case where the torque at each joint is governed by a speed-limited hyperbolic relationship (one for the shoulder joints and one for the wrists):

\[
\tau_1(t) = F_{\text{max,sh}} R_{sh} \frac{1 - (R_{sh} \dot{\theta}_1 / v_{\text{max,sh}})}{1 + F^* (R_{sh} \dot{\theta}_1 / v_{\text{max,sh}})}
\]

\[
\tau_2(t) = F_{\text{max,wr}} R_{wr} \frac{1 - (R_{wr} \dot{\theta}_2 / v_{\text{max,wr}})}{1 + F^* (R_{wr} \dot{\theta}_2 / v_{\text{max,wr}})}.
\]  
(4.19)
Here, the contraction speed of the muscle at the pin joint between the two links is actually related to the relative motion of the two by

\[ \dot{\theta}_{rel} = \dot{\theta}_2 - \dot{\theta}_1. \]  

(4.20)

There are two important things to notice. First, the curvature constant is assumed to be the same for each set of muscles. A justification for this assumption is discussed in Section 4.3.1. Second, this relative angular velocity could potentially be negative. While the hyperbolic relationship suggests that at a negative contraction the muscle can apply a torque larger than \( F_{\text{max,wr}} R_{\text{wr}} \), this is not possible physically. Therefore, in solving the equations of motion, anytime the relative speed is negative, the second torque is simply assumed to be its maximum, \( F_{\text{max,wr}} R_{\text{wr}} \).

Just as increasing the degrees of freedom to two greatly increased the number of parameters required to compute the effective mass, it also doubles the number of muscle parameters required to compute the implement speed. The parameters of the muscle, like the inertial properties of the athlete, can be quite difficult to measure. The single-link model is preferred, but both must be evaluated to determine if one link is enough to capture the essential behavior of swinging.

### 4.3 Comparing the models

For the sake of comparing the models, each one was solved (analytically or numerically) for the specific case of a field hockey stick hitting a field hockey ball. Values for all of the geometric, inertial, and biological parameters were estimated using a combination of literature review and measurement. The athlete specific parameters were based on the particular athlete (Subject A) who performed collision tests for the four different sports from Chapter 3. This decision was made for simplicity, but the work can easily be extended to the other sports or to the other field hockey athletes. This section discusses the estimation of the parameters for all of the models, and the result of solving the equations of motion for a variety of different implement moments.
Table 4.1: A comparison of the parameters used in each of the five models. The models are labeled in shorthand where 1 = single-link, 2 = double pendulum, T = constant torque(s), P = constant power, and H = Hill model. Using a Hill-inspired model increases the number of parameters required, as does using a double pendulum model.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>1T</th>
<th>1P</th>
<th>1H</th>
<th>2T</th>
<th>2H</th>
</tr>
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</table>

of inertia.

4.3.1 Estimating parameters

In this section, the methods for estimating the parameters for each model are discussed. Table 4.1 summarizes all of the parameters needed in each of the different models, and Table 4.2 presents the range of values each parameter is expected to take. Each parameter is related to one of the following: the sport, the implement, the muscles and the athlete.

First, there are five parameters that are specific to the sport, and each of the
Table 4.2: Values and a typical range for each of the parameters used in the models.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value used</th>
<th>Typical range</th>
<th>Units</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field hockey</td>
<td>( e )</td>
<td>0.20</td>
<td>0.11-0.39</td>
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</tr>
<tr>
<td></td>
<td>( m_{\text{ball}} )</td>
<td>159.5</td>
<td>156-163</td>
<td>g</td>
</tr>
<tr>
<td></td>
<td>( v_{\text{lin}} )</td>
<td>0</td>
<td>0-40</td>
<td>m/s</td>
</tr>
<tr>
<td></td>
<td>( N )</td>
<td>2</td>
<td>N/A</td>
<td>[]</td>
</tr>
<tr>
<td></td>
<td>( \alpha )</td>
<td>0.97</td>
<td>0.93-1.00</td>
<td>[]</td>
</tr>
<tr>
<td>Stick</td>
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<td>0.03-1.00</td>
<td>kgm(^2)</td>
</tr>
<tr>
<td></td>
<td>( l_{\text{imp}} )</td>
<td>36</td>
<td>( \leq 41.34 )</td>
<td>in</td>
</tr>
<tr>
<td></td>
<td>( h_{\text{imp}} )</td>
<td>18</td>
<td>12-24</td>
<td>in</td>
</tr>
<tr>
<td></td>
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<td>0.55</td>
<td>( \leq 0.737 )</td>
<td>kg</td>
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<td>44-70</td>
<td>kN/m(^2)</td>
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<td></td>
<td>( F^* )</td>
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<td>[]</td>
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<td></td>
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<td>3-10</td>
<td>cm</td>
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<tr>
<td></td>
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<td>( H )</td>
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<td></td>
<td>( W )</td>
<td>64.8</td>
<td>45-115</td>
<td>kg</td>
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</table>
models requires an estimate for all five of these parameters. The first three (the coefficient of restitution, the ball mass, and the incoming ball speed) are straightforward and were discussed in Chapter 2. In terms of estimating their values, the coefficient of restitution can be determined from videos of real collisions using the expression in Eq. 2.2; the mass of the ball is regulated by the governing body for a given sport; and the incoming ball speed is characteristic of the sport and level of play of the sport.

Table 4.1 presents two additional sport-specific parameters. First is the number of arms, \(N\), that are swung in the sport (is the sport one-handed or two-handed?). This is generally known, but may vary between athletes (particularly in the backhand in tennis). Second is the parameter \(\alpha\), which is simply another way to define the distance from the end of the implement handle to the point where the ball makes contact with it. In the work presented here, this quantity is labeled as \(l_c\). In order to estimate this value, it is more convenient to define

\[
\alpha = l_c/l_{\text{imp}}
\]  

(4.21)

where \(l_{\text{imp}}\) is the length of the implement and is easy to measure. The “length to contact \((l_c)\) can then be determined by estimating the parameter \(\alpha\) from background knowledge about different sports.

The next group of parameters to investigate contains those related exclusively to the implement. The implement moment of inertia, mass, and position of its center of mass were discussed in Section 3.1. Here, the (easily measured) implement length \(l_{\text{imp}}\) is included, again, in order to make the calculation of the “length to contact” \(l_{\text{imp}}\) more convenient to estimate. The implement mass and position of the center of mass can be measured with simple tools (a scale, a tape measure, and a point to balance the implement on). Notice that using the double pendulum model rather than the single-link model requires knowing two additional properties about the implement, increasing the complexity of the analysis.

The remaining parameter is the implement moment of inertia, which, rather than estimating, will serve as the independent variable in our analysis. The goal of this
work is to understand how the implement’s weight properties affect the outgoing ball speed. As discussed in Section 1.3, generally the implement’s moment of inertia (not its mass) has the largest affect on its speed (and the outgoing ball speed). The moment of inertia, therefore, is the focus of the work here and is used to quantify the implement’s “weight properties” from our original question. Also, for this reason, the implements mass, length, and position of center of mass are assumed to be fixed, while only the moment of inertia is allowed to vary. In actuality, the moment of inertia is likely to be changed by altering a combination of those three properties.

The next two categories (the muscles and the athlete) primarily contain parameters that govern the applied torques. The force-velocity relationship described by Hill and depicted in Fig. 4-1 can be used not only to understand the Hill swinging models but also to set bounds on the constant torques and power allowed in the simpler models. In the constant torque cases, we will assume that this applied torque cannot exceed the maximum value in the Hill relationship (where the muscles can no longer contract). This means the constant torque applied at the wrist should be at most $F_{\text{max,wr}}R_{\text{wr}}$ and $F_{\text{max,sh}}R_{\text{sh}}$ at the shoulder.

The power being output by a muscle in contracting is the product of the load it is carrying and its contraction speed. The load it can maintain, however, is also a function its contraction speed. The power is then

$$P = Fv = F_{\text{max}}v \frac{1 - v/v_{\text{max}}}{1 + F^*/v_{\text{max}}}. \quad (4.22)$$

By differentiating with respect to the contraction speed, the value of the contraction speed that operates at the highest power and further, the value of this maximum power, can be determined. The result is

$$P_{\text{max}} = \frac{F_{\text{max}}v_{\text{max}}}{F^*2} \left(\sqrt{1 + F^*} - 1\right)^2 \quad (4.23)$$

This is the value of power that will serve as a bound in the constant power model.

In order to evaluate the applied torque in all of the models, the next step is to estimate the values of the parameters that make up Hill’s force-velocity relationship.
For this, we turn to the literature. Beyond the work of Hill, others have continued to study the force-velocity properties of muscle tissue from different species and in different environments\[24-29\]. These studies suggest a slightly different set of three parameters to describe the muscle behavior of a given species than was laid out in Section 4.1.3. First, rather than the maximum isometric tension, what seems to characterize force behavior is this tension normalized by the cross-sectional area of the muscle. This means what is true of muscle tissue in a species is that it can sustain a roughly constant maximum internal stress, which will be called $\sigma$. The second parameter is related to the contraction speed. Again, what seems to be constant for a species’ muscle tissue is the contraction speed normalized by the length of the muscle fibers; this will be referred to as $v_{\text{norm}}$. The third parameter is unchanged as it is already normalized and represents the curvature of the relationship ($F^*$). As this value is expected to stay roughly the same across an individual’s different muscles, this will be used as the curvature parameter for both the muscles pulling at the shoulder and those pulling at the wrist\[24\].

One particular study by Botinelli et al performed tests on human skeletal muscle and published values for these three parameters; this is the work that the muscle parameter ranges and of Table 4.2 are based on. Additionally, to determine the non-normalized parameter values, dimensions of the muscle must be estimated. This estimation is challenging without slicing into the human body. Instead, measurements of the outside of the body were used to estimate the cross-section and length of the muscle as well as the radius of the joint. Because these measurements pose the greatest amount of uncertainty in the problem, one additional fitting parameter will be used for each model. The details of the fit are presented in Section 4.3.1.

Again, the double pendulum model requires the estimation of more parameters than the single-link model; the dimensions of the muscles pulling on the shoulder must be determined as well as those pulling on the wrist. Additionally, the use of Hill’s force-velocity relationship to determine the constants in the constant torque and power models means these models also require the estimation of muscle dimensions. In one way, this is good; it suggests an dependence of the swinging behavior on the
individual athlete, a goal for the models in this study. However, having more unknown parameters decreases the certainty in the result. Finally, requiring the estimation of muscle dimensions makes the model more difficult to implement with the everyday, recreational athlete. For this reason, Section 5.3 proposes a strategy for determining the athlete's muscle dimensions without having to explicitly measure them.

The final parameter to determine is the moment of inertia and length of the arm(s). For these, we again turn to the literature. In 1982, Zatsiorsky and Seluyanov collected X-ray data on the muscles of 100 subjects and fit regression curves based on human height and weight to all of the geometric and inertial properties of the different segments of the human body.[30] These regression curves were used in this study to determine the properties of the arm(s). The final parameters included in Tables 4.1 and 4.2 therefore are the athlete's height and weight, to be used in the regression models.

**Fitting parameters**

Given the number of parameters and the difficulty in measuring some of them (particularly the actual muscle dimensions), a fitted parameter was used for each model to ensure that each curve passed through the experimentally observed values. From the video footage of Subject A hitting a field hockey ball with a field hockey stick, the measured value of the implement speed was determined to be on average 21.8 [m/s]. The implement’s mass and length were measured, and if we assume its mass is uniformly distributed, its moment of inertia about the end of the handle is 0.153 [kgm²]. The fitted parameter $\mu$ was chosen with a specific physical meaning; it is a scaling factor on each length measurement made. It therefore represents the amount the muscle dimensions must be scaled up or down to produce the behavior observed in the collisions. Note that the area values then scale as $\mu^2$. This selection of $\mu$ means that the measurements of the body are not all for naught; they provide the relative muscle dimensions.

The value of the fitted parameter $\mu$ for each model is presented in Table 4.3. Unfortunately, they are all greater than one. This means we would expect only bigger
Table 4.3: Values of the fitted scaling parameter $\mu$ for each of the different models. 
1 = single-link, 2 = double pendulum, T = constant torque(s), P = constant power, H = Hill model. All values are greater than one but are on the order one.

<table>
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<th></th>
<th>1T</th>
<th>1P</th>
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<td>4.15</td>
<td>5.14</td>
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</table>

muscles to produce the observed swinging behavior, but our muscles certainly are no bigger than the outer dimensions of our body. This factor could however represent the rest of the muscles that are contributing to the acceleration (like those in the core) but are not directly pulling on the joints. Notice also that the model predictions are closest for the constant torque cases. This is because the constant power model and the Hill models both have an applied torque that is limited by the force-velocity relationship of muscles. While the applied torque in the constant torque case is inspired by this model, it does not share the same limitations. Nonetheless the scaling factor is still on the order one, meaning a physiological approach to determining the applied torque (and all of the athlete-specific parameters) is promising.

4.3.2 Results

The results of solving the equations of motion for each of the five models with the parameters listed in Table 4.2 are plotted on a log-log scale in Fig. 4-2. This scale allows for power laws to become clear. By design, the constant torque and power models for the single-link hold the power laws discussed in Sections 4.1.1 and 4.1.2 respectively. The other three models do not show consistent power laws. However, as was discussed in Section 1.3, while this proposed relationship is common, it is not exclusively used and is occasionally separated into one power law for high moments of inertia and one for low moments of inertia. Nonetheless, a power law can be established in the region immediately close to the measured value of the moment of inertia. The resulting powers ($n$) in this region are listed for each model in the figure.

Each of the three models that do not hold a power law relationship globally show a local power law that is “weaker” (smaller $n$) than that of the simple models.
Figure 4-2: Implement speed vs. moment of inertia for the five swinging models. The plot is log-log and shows that the single-link models with constant torque and power applied both produce the power laws expected ($n = 1/2$ and $n = 1/3$ respectively). The other three models are only power laws locally, but the values of $n$ are closer to those observed in the literature (see Table 1.1).
Chapter 5

Optimum implements

5.1 Combining the models

In Chapter 2, the outgoing ball speed $v_{\text{out}}$ was determined to be a function of four parameters: the mass ratio $r$, the implement speed $v_{\text{imp}}$, the incoming ball speed $v_{\text{in}}$, and the coefficient of restitution $e$. Models for the effect of the implement weight properties on the first of these two were developed in Chapters 3 and 4 respectively. These models can be combined with the collision model to produce a prediction for the outgoing ball speed as a function of the implement moment of inertia. Again, to start, the analysis will focus on one specific case: the athlete (Subject A) swinging a field hockey stick at a stationary ball.

The predicted outgoing ball speed is plotted as a function of the implement moment of inertia for each of the five models in Fig. 5-1. Each of the models predicts that a different moment of inertia will provide the maximum ball speed. The one that predicts the optimum to be closest to the estimated value for field hockey sticks is actually the single-link, constant torque model. All of the other models predict that a stick with a larger moment of inertia would perform better.

The constant torque model, however, is not necessarily the best. It is possible that athletes should actually be using heavier (larger moment of inertia) implements to improve their power hitting performance, but they are not making the correct implement selection (or manufacturers are not making the best implements). On that
Figure 5-1: Outgoing ball speed vs. implement moment of inertia for power hitting in field hockey. Each model for swinging displays a maximum performance (outgoing ball speed) for a different implement. However, only the constant torque model's optimum implement is similar to the implements seen in the sport of field hockey.

idea, some sports have regulations on the weight properties of the implements allowed during play. In field hockey, for instance, the stick may not be any larger than 737 g and may not be longer than 1.05 m \[23\]. Then, at absolute maximum (with all of the weight at the head of the implement), the moment of inertia is 0.813 kg m\(^2\). More realistically, if the stick is as heavy and large as possible but is still uniformly distributed, its moment of inertia would be 0.271 kg m\(^2\). This value is smaller than the optimum predicted by the single-link Hill model and both of the double pendulum models. Perhaps, these sticks would provide more powerful shots but are not allowed to be used.

The restrictions on implement weight suggest another important attribute of the results presented in Fig. 5-1: in general, models that predict the optimum moment of inertia is larger also predict that the maximum ball speed is larger in magnitude than for the optima in other models. Presumably, the restriction on the stick's properties is to put a cap on the maximum outgoing ball speed to ensure safety in the game play. In fact, another rule in field hockey is that the outgoing ball speed may be no more than 98% of the implement speed\[23\]. Heavier sticks may in fact produce higher
outgoing ball speeds, meaning the more complex models could still capture the true behavior seen in the sport, but such high performing sticks may not be within the regulations of the game.

One final remark on the difference between a typical stick moment of inertia and the predictions for optimal ones which are much heavier: implement selection in sports is not based solely on power hitting performance. In field hockey, as in most sports, the implement is required to do more than just hit the ball “hard.” Particularly, there is an energetic cost to carrying around a heavy stick, especially in a game like field hockey where the athlete must run with the stick at all times. While the models do suggest that heavier implements should be used to improve power hitting, perhaps there is another competing objective that influences implement selection.

5.2 As a function of incoming ball speed

The next parameter to explore is the incoming ball speed. The full collision model must be used in this case. There are some ball speeds that are more typically seen in different sports. For instance, in tennis, the ball speed is generally an indicator of the level of play. In others, like golf, the ball is always stationary. While in field hockey, a fast moving ball is rarely hit as a “power” shot, analyzing the effect of incoming ball speed on the optimum moment of inertia of the implement is still instructive. These results are shown in Fig. 5-2.

In general, the optimum implement predicted by each of the models is a monotonically increasing function of the incoming ball speed. This suggests two interesting conclusions. First, the optimum implement moment of inertia is lowest for the stationary ball case. Even though most of the models in the stationary case predict the optimum moment of inertia to be larger than that which is seen in the game, these are still the lowest predicted optima. Second, since the optimum value is not constant for different ball speed, there is some kind of momentum “matching” that is occurring. Finally, the slopes of each of the different curves indicate the models sensitivity to the incoming ball speed. The double pendulum model is significantly more robust to
Figure 5-2: Optimum implement moment of inertia vs. incoming ball speed. The best implement to use will depend on the speed of the ball typically seen in a given level of play; a heavier (higher moment of inertia) implement should be used for hitting balls that are moving quicker. The above plot uses the biological parameters discussed above for field hockey, despite it being a sport where power hitting is most important when the ball is not moving.

5.3 As a function of the athlete

So far, the analysis in this study has not explored how to extrapolate information about the optimum implement parameters from one athlete to another. In estimating the parameters to solve each of the different models, biometric data was required that could be a hassle to attain from a recreational athlete looking to select a new implement. To perform an extrapolation, the principles discussed in McMahon and Bonner's *On size and life* on animal self-similarity are used[31].

Self-similarity is the idea that all animals of the same species are geometrically the same, just at different length scales. Knowing the dimensions or properties of one person allows for the determination of that property for all other people through appropriate scaling laws. The athlete specific parameters in the models can then simply be scaled up based on athlete height. As discussed in Section 4.3.1, there are
three parameters that characterize the force-velocity relationship for all humans: the maximum internal stress $\sigma_{\text{max}}$, the normalized maximum contraction speed $v_{\text{max, norm}}$, and the curvature parameter $F^*$. The dimensions of the athletes' muscles were then required to determine the non-normalized parameters of the Hill model.

We will assume that each dimension scales like the athlete's height over the height of some reference athlete ($H/H_0$). The biological parameters for the new athlete will be assumed to be equal to the reference athlete's parameters (indicated by a subscript 0) times an appropriate scaling factor:

$$F_{\text{max}} = \left(\frac{H}{H_0}\right)^2 F_{\text{max,0}}, \quad R = \frac{H}{H_0} R_0, \quad v_{\text{max}} = \frac{H}{H_0} v_{\text{max,0}}, \quad F^* = F_0^*. \quad (5.1)$$

All together, this translates to the applied torque (in all models) to scale as

$$\tau(t) = \left(\frac{H}{H_0}\right)^3 \tau_0(t). \quad (5.2)$$

However, the cube of the dimensions of the person is really their volume, which is essentially their weight. The result is the assumption that any athlete's applied torque will be related to a reference athlete's torque by a factor of $W/W_0$ where $W$ is the weight of the new athlete and $W_0$ is the weight of the reference athlete.

These scales were used to evaluate the influence of the athlete (particularly their weight) on the optimum implement moment of inertia. The results are shown in Fig. 5-3. For the constant torque and power cases, the weight of the athlete does not affect the optimum. It will affect the overall magnitude of the outgoing ball speed but not which implement produces it. This is consistent with the analysis performed in Section 2.4 where the optimum implement for a power-law governed implement speed was explored. For the other models, a heavier implement is recommended for heavier athletes. These models capture some effect of the athlete on the optimum implement, but it is not clear whether the particular effect (heavier athletes should use heavier implements) is intuitive.
Figure 5-3: Optimum implement moment of inertia vs. athlete weight. The best implement to use will depend on the individual athlete. A “self-similarity” scaling is used to extrapolate the performance of a given athlete based on his or her weight. These results suggest that heavier athletes should generally use heavier (higher moment of inertia) implements.
Chapter 6

Conclusion

6.1 Summary

The goal of this study was to create a set of simple models that are able to predict what weight properties an implement should have for a given athlete playing a given sport to perform his or her best. The performance was focused on “power hitting” and was measured by the speed of the ball leaving its collision with the implement. The weight property of interest was the mass moment of inertia of the implement about an axis through its handle. This property was chosen as the focus of the study because historically, it was observed to have the largest effect on the behavior of swinging. A simple collision model (of the interaction of point masses in one dimension) was used to connect the properties of the sport, athlete, and implement to the outgoing ball speed (over which a maximum was determined for a fixed athlete and sport).

The collision model required the establishment of smaller models for the swinging of the implement. Five different models were evaluated in this study. Three were based around the implement being modeled as a single-link being swung about a fixed pivot. They differ in the torque that is applied to accelerate the link to the contact point. The three trial torques were based on the principles of applied constant torque, applied constant power, and a speed-varying torque based on a physiological muscle model coined by A.V. Hill. The other two models were based around a double pendulum set-up, where the athlete’s arm(s) were one link connected to a fixed pivot.
and the implement was a second link connected to the first by a pin joint. Again, the applied torque varied between the two models. In one case, constant torques were applied at each joint. In the other, speed-varying torques were applied, again, based on the physiological model by Hill.

These models were examined with respect to observations of real collisions (some collected exclusively for this study and some from the literature). The comparisons were made for the effective mass, the implement speed, outgoing ball speed, and optimum moment of inertia. The goal of the comparison was to determine the required level of model complexity (minimum level of simplicity) that can be used to capture the behavior of sports collisions in a sufficient way to provide recommendations for the implements that different athletes should use.

In Section 3.3, the simple single-link set-up was determined to be sufficient for capturing the effective mass observed in real collisions. Then, in Section 4.3, the single-link model with a Hill-like applied torque proved to be the model that locally, most resembled the observed power law relationship between implement speed and its moment of inertia that arises in much of the literature (see Table 1.1).

However, the comparison that is most important is that of the outgoing ball speed and for what implement properties it is a maximum. In this case, only the simplest model (a single-link with a constant applied torque) predicts that the optimum implement is close to those used in the game today. All of the other models recommend using heavier implements. This result, however, does not mean that the single-link with constant torque model is the best. The difference between the predicted optima and the implements used in real play are more likely to arise because the optimization here focuses solely on power hitting, but there are many other criteria that an implement must be able to meet beyond that to improve overall performance in a sport. Further work must be done to incorporate these objectives into the optimization.

There were two additional findings from the analysis that are important. First, heavier implements should be used in games where the ball is typically moving faster. This is most applicable in games like tennis, baseball, and softball, where it is important to hit powerful shots where the ball is incoming at a non-zero velocity. This
finding suggests that in higher levels of play (where the ball moves faster) athletes
should use heavier implements. Really what this means is the level of play should be
taken into account when selecting an implement to use.

The second important finding was that heavier athletes should use heavier im-
plements. This trend was observed for the single-link Hill model and the two double
pendulum cases. A heavier athlete has a larger capacity to wield a heavier implement,
and doing so will improve his or her overall power hitting performance. Though the
models tend to recommend athlete’s use heavier implements than they are currently
using, the work presented here still suggests valuable insights about the formulation
of an improved implement selection process.

6.2 Next steps

In order to truly help athletes find the best implement for their play, a broader set of
objectives need to be incorporated into the model. Rather than simply choosing an
implement that will perform best in power hitting, the model should take into account
other uses of the implement. As discussed for field hockey, another objective could
be to minimize energy required to run around carrying the stick during the course
of the game. The objectives will be different in different sports, so unfortunately the
models will become more and more specialized.

Other future work should include further validation of the models presented here.
More deliberate data should be collected to determine the relationship between im-
plement speed and weight properties. Is it truly a power law? Does the hill model
provide a better fit? Additionally, the parameter estimation would ideally be done
without a fitting parameter. There is presumably a missing level of complexity that
prevents the current models from achieving fast enough speeds. Perhaps this has to
do with the power provided to the shot by the athlete’s torso and the rest of his or her
body. Finally, the self-similarity principle should be validated in this specific case by
comparing the predicted swinging behavior using this scaling to the predicted swing-
ing behavior using measurements of different athletes’ muscles. These steps towards
deeper validation would justify the use of these simple mechanical models for sports collisions, but if they do not, the objective function must be expanded to incorporate other roles the implement must fulfill in the sport.
Bibliography


