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Ryan Hynd[1] · Erik Lindgren[2]

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Abstract
For a bounded domain $\Omega \subset \mathbb{R}^n$ and $p > n$, Morrey's inequality implies that there is $c > 0$ such that

$$c \, \|u\|_\infty^p \leq \int_\Omega |Du|^p \, dx$$

for each u belonging to the Sobolev space $W_0^{1,p}(\Omega)$. We show that the ratio of any two extremal functions is constant provided that Ω is convex. We also show with concrete examples why this property fails to hold in general and verify that convexity is not a necessary condition for a domain to have this feature. As a by product, we obtain the uniqueness of an optimization problem involving the Green's function for the p-Laplacian.

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1 Introduction

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $p > n$. Morrey's inequality for $W_0^{1,p}(\Omega)$ functions u may be expressed as

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✉ Erik Lindgren
erik.lindgren@math.uu.se

[1] Department of Mathematics, MIT, Cambridge, USA

[2] Department of Mathematics, Uppsala University, Uppsala, Sweden

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$$c \|u\|^p_{C^{1-\frac{n}{p}}(\overline{\Omega})} \leq \int_\Omega |Du|^p \, dx,$$

where $c > 0$ is a constant that is independent of u. In particular,

$$c \|u\|^p_\infty \leq \int_\Omega |Du|^p \, dx. \tag{1.1}$$

Let us define

$$\lambda_p := \inf \left\{ \frac{\int_\Omega |Du|^p dx}{\|u\|^p_\infty} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}.$$

Observe that

$$\lambda_p \|u\|^p_\infty \leq \int_\Omega |Du|^p \, dx \tag{1.2}$$

and that $c = \lambda_p$ is the largest constant such that (1.1) is valid. Furthermore, if there is a function $u \in W^{1,p}_0(\Omega)\setminus\{0\}$ such that equality holds in (1.1), then $c = \lambda_p$.

Definition A function $u \in W^{1,p}_0(\Omega)\setminus\{0\}$ is an *extremal* if equality holds in (1.2).

It is plain to see that any multiple of an extremal is also an extremal. Using routine compactness arguments, it is not difficult to verify that extremal functions exist. We will argue below that any extremal u satisfies the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda_p \, |u(x_0)|^{p-2} \, u(x_0) \delta_{x_0} & x \in \Omega \\ \qquad u = 0 & x \in \partial\Omega, \end{cases} \tag{1.3}$$

which was derived by Ercole and Pereira in [9]. Here $\Delta_p \psi := \operatorname{div}(|D\psi|^{p-2} D\psi)$ is the p-Laplacian, and x_0 is the *unique* point for which $|u|$ is maximized in Ω. Moreover, using (1.3) we will be able to conclude that any extremal has a definite sign in Ω. And as the PDE in (1.3) is homogeneous, the optimal constant λ_p can be interpreted as being an eigenvalue.

The primary goal of this work is to address the extent to which extremal functions can be different. In particular, we would like to know if any two extremal functions are necessarily multiples of one another. If they are, we consider the set of extremals to be uniquely determined. For once one extremal is found, all others can be obtained by scaling. We will argue that annuli never have this uniqueness property. We will also exhibit star-shaped domains for which this uniqueness property fails. However, we will see that if a planar domain has certain symmetry, then its extremals are one dimensional.

Our main result is that convex domains always have the aforementioned uniqueness property.

Theorem 1.1 *Assume that $\Omega \subset \mathbb{R}^n$ is open, convex and bounded. If u and v are extremal, then u/v is constant throughout Ω.*

We will also explain how Theorem 1.1 implies the following corollary involving the Green's function of the p-Laplacian in Ω.

Corollary 1.2 *Assume that $\Omega \subset \mathbb{R}^n$ is open, convex and bounded. Suppose that $G(\cdot, y)$ is the Green's function of the p-Laplacian in Ω with pole $y \in \Omega$; that is, $G(\cdot, y)$ satisfies*

$$\begin{cases} -\Delta_p w = \delta_y & x \in \Omega \\ w = 0 & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Then x_0 given in Eq. (1.3) is the unique point in Ω for which

$$G(x_0, x_0) = \max\{G(y, y) : y \in \Omega\}.$$

Part of our motivation was to extend a previous result of Talenti. He considered extremal functions for the following inequality, which is also due to Morrey. For each weakly differentiable function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\|u\|^p_{L^\infty(\mathbb{R}^n)} \le c(n, p)\, |\mathrm{supp}(u)|^{\frac{p}{n}-1} \int_{\mathbb{R}^n} |Du|^p \, dx. \tag{1.5}$$

Here $c(n, p)$ is an explicit constant depending only on p and n, and $|\mathrm{supp}(u)|$ is the Lebesgue measure of the support of u. Employing Schwarz symmetrization, Talenti showed in [22] that if equality holds in (1.5) there are $a \in \mathbb{R}$, $r > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$u(x) = \begin{cases} a\left(r^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}}\right), & |x - x_0| < r \\ 0, & |x - x_0| \ge r. \end{cases}$$

We remark that a quantitative version of this result has been established by Cianchi [5], and we refer the reader to [4,8,21] for work on sharp constants of related inequalities.

Unfortunately, $\mathbb{R}^n$ and balls are the only known domains for which the extremals have such convenient characterizations. Nevertheless, in this paper, we believe that we have taken significant steps in understanding precisely which domains have a one dimensional collection of extremals. In Sect. 2, we will derive basic properties of solutions of (1.3), and in Sect. 3, we consider the support function of an extremal. In Sect. 5, we will provide examples of domains for which uniqueness fails; these include annuli, bow tie and dumbbell shaped planar domains. In Sect. 4, we verify Theorem 1.1, and in Sect. 6, we use Steiner symmetrization to exhibit some nonconvex planar domains that have unique extremals.

2 Properties of extremals

We now proceed to deriving some properties of extremal functions. These properties will be crucial to our uniqueness study. First, we verify that extremal functions satisfy

the boundary value problem (1.3). Then we will study the behavior of solutions of (1.3) near their global maximum or minimum points. We also refer the reader to the recent paper [9] by Ercole and Pereira, where they studied properties of extremal functions in a wide class of inequalities that include (1.1). In particular, they obtained analogous results to Corollaries 2.2 and 2.4 below.

Lemma 2.1 *A function $u \in W_0^{1,p}(\Omega)$ is extremal if and only if*

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi dx = \lambda_p \max\left\{|u(x)|^{p-2} u(x)\phi(x) : x \in \overline{\Omega}, |u(x)| = \|u\|_\infty\right\} \tag{2.1}$$

for all $\phi \in W_0^{1,p}(\Omega)$.

Proof 1. First let us establish the following identity

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \frac{\|u+\epsilon\phi\|_\infty^p - \|u\|_\infty^p}{\epsilon}\\ &\quad = p \max\left\{|u(x)|^{p-2} u(x)\phi(x) : x \in \overline{\Omega}, |u(x)| = \|u\|_\infty\right\},\end{aligned} \tag{2.2}$$

for $u, \phi \in C(\overline{\Omega})$. For any x_0 such that $|u(x_0)| = \|u\|_\infty$,

$$\begin{aligned}\frac{1}{p}\|u+\epsilon\phi\|_\infty^p &\geq \frac{1}{p}|u(x_0)+\epsilon\phi(x_0)|^p\\ &\geq \frac{1}{p}|u(x_0)|^p + \epsilon\,|u(x_0)|^{p-2}u(x_0)\phi(x_0)\\ &= \frac{1}{p}\|u\|_\infty^p + \epsilon\,|u(x_0)|^{p-2}u(x_0)\phi(x_0).\end{aligned}$$

Therefore,

$$\liminf_{\epsilon\to 0^+} \frac{\|u+\epsilon\phi\|_\infty^p - \|u\|_\infty^p}{\epsilon} \geq p\,|u(x_0)|^{p-2}u(x_0)\phi(x_0),$$

and so "$\geq$" holds in (2.2).

Now choose a sequence of positive numbers $(\epsilon_j)_{j\in\mathbb{N}}$ tending to 0 such that

$$\limsup_{\epsilon\to 0^+} \frac{\|u+\epsilon\phi\|_\infty^p - \|u\|_\infty^p}{\epsilon} = \limsup_{j\to\infty} \frac{\|u+\epsilon_j\phi\|_\infty^p - \|u\|_\infty^p}{\epsilon_j},$$

and select a sequence $(x_j)_{j\in\mathbb{N}}$ maximizing $|u+\epsilon_j\phi|$ that converges to a maximizer x_0 of $|u|$. Such sequences exist by the continuity of u and ϕ, the compactness of $\overline{\Omega}$, and the inequalities $|u(x)+\epsilon_j\phi(x)| \leq \|u+\epsilon_j\phi\|_\infty = |u(x_j)+\epsilon_j\phi(x_j)|$. As $\mathbb{R} \ni z \mapsto \frac{1}{p}|z|^p$ is continuously differentiable,

$$\limsup_{j\to\infty} \frac{\|u+\epsilon_j\phi\|_\infty^p - \|u\|_\infty^p}{\epsilon_j} \leq \limsup_{j\to\infty} \frac{|u(x_j)+\epsilon_j\phi(x_j)|^p - |u(x_j)|^p}{\epsilon_j}$$

$$= p\,|u(x_0)|^{p-2}\,u(x_0)\phi(x_0)$$
$$\leq p\max\{|u(x)|^{p-2}\,u(x)\phi(x) : |u(x)| = \|u\|_\infty\}.$$

We conclude "$\leq$" in (2.2).

2. Any extremal $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ satisfies

$$\lambda_p = \frac{\displaystyle\int_\Omega |Du|^p\,dx}{\|u\|_\infty^p} \leq \frac{\displaystyle\int_\Omega |Du + \epsilon D\phi|^p\,dx}{\|u + \epsilon\phi\|_\infty^p},$$

for each $\phi \in W_0^{1,p}(\Omega)$ and $\epsilon > 0$ sufficiently small. Exploiting (2.2)

$$0 \leq \lim_{\epsilon\to 0^+} \frac{1}{p\epsilon}\left(\frac{\displaystyle\int_\Omega |Du + \epsilon D\phi|^p\,dx}{\|u + \epsilon\phi\|_\infty^p} - \frac{\displaystyle\int_\Omega |Du|^p\,dx}{\|u\|_\infty^p}\right)$$
$$= \frac{\displaystyle\int_\Omega |Du|^{p-2}\,Du\cdot D\phi dx}{\|u\|_\infty^p}$$
$$- \frac{\displaystyle\int_\Omega |Du|^p\,dx}{\|u\|_\infty^{2p}}\max\{|u(x)|^{p-2}u(x)\phi(x) : x\in\overline{\Omega}, |u(x)| = \|u\|_\infty\}$$
$$= \frac{1}{\|u\|_\infty^p}\left(\int_\Omega |Du|^{p-2}\,Du\cdot D\phi dx\right.$$
$$\left.-\lambda_p\max\{|u(x)|^{p-2}u(x)\phi(x) : x\in\overline{\Omega}, |u(x)| = \|u\|_\infty\}\right).$$

Canceling the factor $1/\|u\|_\infty^p$ and replacing ϕ with $-\phi$ gives (2.1).

3. Of course if (2.1) holds, we can choose $\phi = u$ to verify that u is extremal. □

Corollary 2.2 *Each extremal function is everywhere positive or everywhere negative in* Ω.

Proof Assume that $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ is extremal. Then $w := |u|$ is extremal, as well. Moreover, (2.1) implies

$$\int_\Omega |Dw|^{p-2}\,Dw\cdot D\phi dx \geq 0$$

for all $\phi \geq 0$. Therefore, w is p-superharmonic, $w \geq 0$ and $w|_{\partial\Omega} = 0$. Since w doesn't vanish identically, $w = |u| > 0$ (Theorem 11.1 in [18]). Hence, u doesn't vanish in Ω and so u has a definite sign in Ω. □

Observe that the left hand side of (2.1) is linear in ϕ, while the right hand side appears to be nonlinear in ϕ. We will argue that this forces the set $\{x \in \overline{\Omega} : |u(x)| = \|u\|_\infty\}$ to be a singleton for any extremal function.

Proposition 2.3 *Assume u is an extremal function. Then $\{x \in \Omega : |u(x)| = \|u\|_\infty\}$ is a singleton.*

Proof Without any loss of generality, we may assume $u > 0$ and $\|u\|_\infty = 1$. In view of (2.1),

$$\max_{\{u=1\}} \{\phi_1 + \phi_2\} = \max_{\{u=1\}} \phi_1 + \max_{\{u=1\}} \phi_2, \tag{2.3}$$

for any two $\phi_1, \phi_2 \in C_c^\infty(\Omega)$. Suppose that there are distinct points x_1 and x_2 for which $u(x_1) = u(x_2) = 1$. In this case, there are balls $B_\delta(x_1)$, $B_\delta(x_2) \subset \Omega$ that are disjoint for some $\delta > 0$ small enough. We choose functions $\phi_1 \in C_c^\infty(\Omega)$, $\phi_2 \in C_c^\infty(\Omega)$ that are nonnegative, have maximum value 1, and are supported in $B_\delta(x_1)$ and $B_\delta(x_2)$, respectively. It follows that

$$\max_{\{u=1\}} \{\phi_1 + \phi_2\} = 1,$$

while $\max_{\{u=1\}} \phi_1 = 1$ and $\max_{\{u=1\}} \phi_2 = 1$. This contradicts (2.3). □

Corollary 2.4 *Assume u is an extremal function. Then $|u|$ attains its maximum value uniquely at some $x_0 \in \Omega$. Moreover,*

$$\int_\Omega |Du|^{p-2}\, Du \cdot D\phi dx = \lambda_p |u(x_0)|^{p-2} u(x_0)\phi(x_0)$$

for each $\phi \in W_0^{1,p}(\Omega)$. In particular, u is a weak solution of (1.3).

We note that any solution u of (1.3) is differentiable with a locally Hölder continuous gradient in $\Omega \backslash \{x_0\}$, see [10,15,23]. However, we show below that u is not differentiable at x_0.

Example 2.5 As we noted above, when $\Omega = B_r(x_0)$, we have an explicit extremal function

$$u(x) = a\left(r^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}}\right), \quad x \in B_r(x_0) \tag{2.4}$$

for each $a \in \mathbb{R}$ (See Fig. 1). Moreover, any extremal is of the form (2.4) for some $a \in \mathbb{R}$; in particular, any ball has a one dimensional collection of extremal functions. The corresponding optimal constant in (1.2) is

$$\lambda_p = \frac{\displaystyle\int_{B_r(x_0)} |Du|^p\, dx}{\|u\|_\infty^p} = \left(\frac{p-n}{p-1}\right)^{p-1} r^{n-p} n\omega_n.$$

We can use the extremals for balls (2.4) to study the behavior of general extremals near the points which maximize their absolute values. Note in particular, that the family of extremals (2.4) are Hölder continuous with exponent $\frac{p-n}{p-1} \in (0, 1]$, which is a slight improvement of the exponent $\frac{p-n}{p}$ one has from the Sobolev embedding $W_0^{1,p}(\Omega) \subset C^{1-\frac{n}{p}}(\overline{\Omega})$. We will first argue that solutions of (1.3), and in particular

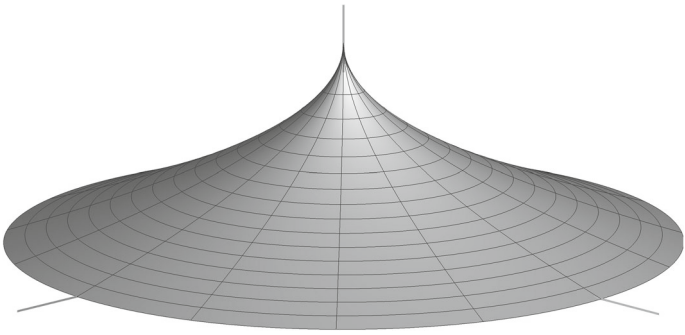

Fig. 1 Example 2.5 with $a = 1$, $p = 4$, $n = 2$, $r = 1$ and $x_0 = 0$

extremals, have exactly this type of continuity at their maximizing or minimizing points.

Proposition 2.6 *Assume u is a solution of (1.3) and that $B_r(x_0) \subset \Omega \subset B_R(x_0)$. Then*

$$\frac{\|u\|_\infty}{R^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}} \leq |u(x) - u(x_0)| \leq \frac{\|u\|_\infty}{r^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}}$$

for each $x \in \Omega$.

Proof Without loss of generality we may assume that $u > 0$ and $u(x_0) = 1$. Define

$$\begin{aligned} v(x) &:= \frac{1}{r^{\frac{p-n}{p-1}}} \left(r^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}} \right) \\ &= 1 - \frac{1}{r^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}} \end{aligned}$$

for $x \in B_r(x_0)$. Observe that u and v are p-harmonic in $B_r(x_0)\backslash\{x_0\}$, $u(x_0) = v(x_0)$ and $u \geq v$ on $\partial B_r(x_0)$. By weak comparison, $u \geq v$ in $B_r(x_0)$. That is

$$u(x) \geq 1 - \frac{1}{r^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}}, \quad x \in B_r(x_0).$$

Since $v(x) \leq 0$ for $x \notin B_r(x_0)$, the above inequality trivially holds for $x \in \Omega\backslash B_r(x_0)$.

Now set

$$\begin{aligned} w(x) &:= \frac{1}{R^{\frac{p-n}{p-1}}} \left(R^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}} \right) \\ &= 1 - \frac{1}{R^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}}. \end{aligned}$$

Observe that u and w are p-harmonic in $\Omega\backslash\{x_0\}$, $u(x_0) = w(x_0)$ and $u \leq w$ on $\partial\Omega$ as $\Omega \subset B_R(x_0)$. By weak comparison, $u \leq w$ in Ω. That is

$$u(x) \leq 1 - \frac{1}{R^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}}, \quad x \in \Omega.$$

□

Corollary 2.7 *Suppose that u is a non-zero solution of (1.3). Then u is not differentiable at x_0.*

Proof First assume $\frac{p-n}{p-1} \in (0, 1)$. By hypothesis, $u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + o(|x - x_0|)$, as $x \to x_0$. Choosing R so large that $\Omega \subset B_R(x_0)$, we have by the previous proposition that

$$\frac{\|u\|_\infty}{R^{\frac{p-n}{p-1}}} |x - x_0|^{\frac{p-n}{p-1}} \leq |u(x) - u(x_0)| \leq |Du(x_0)|\,|x - x_0| + o\left(|x - x_0|\right),$$

as $x \to x_0$. That is,

$$\frac{\|u\|_\infty}{R^{\frac{p-n}{p-1}}} \frac{1}{|x - x_0|^{1-\frac{p-n}{p-1}}} \leq |Du(x_0)| + o(1).$$

This inequality can not be true since $\frac{p-n}{p-1} \in (0, 1)$. If $\frac{p-n}{p-1} = 1$, then $n = 1$ and the claim trivially holds since u is then of the form (2.4). □

We will now refine the above estimates to deduce the exact behavior of a solution u of (1.3) near x_0. The following proposition relies on the results of Kichenassamy and Veron in [12].

Proposition 2.8 *Assume that u is a solution of (1.3). Then*

$$\lim_{x\to x_0} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\frac{p-n}{p-1}}} = \|u\|_\infty \left(\frac{p-1}{p-n}\right) \left(\frac{\lambda_p}{n\omega_n}\right)^{\frac{1}{p-1}},$$

and

$$\lim_{x\to x_0} \frac{|Du(x)|}{|x - x_0|^{\frac{p-n}{p-1}-1}} = \|u\|_\infty \left(\frac{\lambda_p}{n\omega_n}\right)^{\frac{1}{p-1}}.$$

Here ω_n is the Lebesgue measure of $B_1(0) \subset \mathbb{R}^n$.

Proof Without any loss of generality, we may assume that u is positive in Ω and that $u(x_0) = 1$. Recall that u is p-harmonic in $\Omega\backslash\{x_0\}$; and in view of Proposition 2.6, u

satisfies $0 \leq 1 - u(x) \leq C|x - x_0|^{\frac{p-n}{p-1}}$ in Ω for some constant C. This permits us to use Theorem 1.1 and Remark 1.6 in [12] to conclude that there is $\gamma > 0$ such that

$$\lim_{x \to x_0} \frac{1 - u(x)}{|x - x_0|^{\frac{p-n}{p-1}}} = \gamma, \tag{2.5}$$

$$\lim_{x \to x_0} \frac{|Du(x)|}{|x - x_0|^{\frac{p-n}{p-1} - 1}} = \left(\frac{p-n}{p-1}\right)\gamma, \tag{2.6}$$

and

$$\lim_{x \to x_0} \left(\frac{Du(x)}{|Du(x)|} + \frac{x - x_0}{|x - x_0|}\right) = 0. \tag{2.7}$$

We may integrate by parts and exploit (2.7) to get

$$\begin{aligned}
\lambda_p &= \int_\Omega |Du|^p \, dx \\
&= \lim_{r \to 0^+} \int_{\Omega \setminus B_r(x_0)} |Du|^p \, dx \\
&= \lim_{r \to 0^+} \int_{\Omega \setminus B_r(x_0)} \operatorname{div}(u \, |Du|^{p-2} \, Du) dx \\
&= \lim_{r \to 0^+} \int_{\partial B_r(x_0)} u \, |Du|^{p-2} \, Du \cdot \left(-\frac{x - x_0}{|x - x_0|}\right) d\sigma \\
&= \lim_{r \to 0^+} \int_{\partial B_r(x_0)} u \, |Du|^{p-1} \, d\sigma.
\end{aligned}$$

Here σ is $n-1$ dimensional Hausdorff measure. In view of (2.5), we actually have

$$\lambda_p = \lim_{r \to 0^+} \int_{\partial B_r(x_0)} |Du|^{p-1} \, d\sigma. \tag{2.8}$$

Now we can apply (2.6). This limit gives

$$|Du(x)|^{p-1} = \left(\frac{p-n}{p-1}\gamma\right)^{p-1} |x - x_0|^{-(n-1)} + o(1),$$

as $x \to x_0$. By (2.8),

$$\lambda_p = \lim_{r \to 0^+} \int_{\partial B_r(x_0)} \left[\left(\frac{p-n}{p-1}\gamma\right)^{p-1} r^{-(n-1)} + o(1)\right] d\sigma = \left(\frac{p-n}{p-1}\gamma\right)^{p-1} n\omega_n,$$

which concludes the proof. □

Remark 2.9 A function $w \in W_0^{1,p}(\Omega)$ that satisfies

$$\begin{cases} -\Delta_p w = 0, & x \in \Omega \backslash \{x_0\} \\ w = 1, & x = x_0 \\ w = 0, & x \in \partial\Omega \end{cases} \tag{2.9}$$

weakly is called a *potential* function. Observe that every extremal is a multiple of a potential function but not vice versa. For instance, if $\Omega = B_1(0)$, then w is an extremal if and only if $x_0 = 0$.

The strong maximum principle for p-harmonic functions implies that $0 < w < 1$ in $\Omega \backslash \{x_0\}$. In particular, w is uniquely maximized at x_0. Using similar arguments as in the proof of Proposition 2.8, one can easily show that

$$-\Delta_p w = \lambda \delta_{x_0}$$

in Ω where

$$\lambda := \int_\Omega |Dw|^p \, dx.$$

Therefore, the conclusion of Proposition 2.8 holds for w with λ replacing λ_p.

3 Support function of an extremal

Suppose now that Ω is convex and that $u \in W_0^{1,p}(\Omega)$ is a positive extremal which achieves its maximum at $x_0 \in \Omega$. By Corollary 2.4, the results of Lewis [14] imply that

$$\begin{cases} -(x - x_0) \cdot Du(x) > 0 \text{ for } x \in \Omega \backslash \{x_0\}, \\ u \text{ is locally real analytic in } \Omega \backslash \{x_0\}, \\ \{u > t\} \text{ is convex for each } t \in \mathbb{R}, \text{ and} \\ \{u = t\} \text{ has positive Gaussian curvature for each } t \in (0, \|u\|_\infty). \end{cases} \tag{3.1}$$

By the implicit function theorem, it also follows that the level sets of u are smooth.

We define the *support function* of u as

$$h(\xi, t) := \sup \left\{ x \cdot \xi : x \in \overline{\Omega}, u(x) \geq t \right\}. \tag{3.2}$$

$\xi \in \mathbb{R}^n$, $t \in [0, \|u\|_\infty]$. For $t \in [0, \|u\|_\infty]$, $h(\cdot, t)$ is the usual support function of the convex set $\{u \geq t\}$; if $u(0) \geq t$ and $|\xi| = 1$, $h(\xi, t)$ represents the distance from the origin to the hyperplane that supports $\{u \geq t\}$ with outward normal ξ. It follows from (3.1) and Theorem 4 of [7], $h \in C^\infty(\mathbb{R}^n \backslash \{0\} \times (0, \|u\|_\infty))$ and $h_t < 0$.

Suppose $u(x_0) = \|u\|_\infty$. Then for

$$x \in \Omega \backslash \{x_0\}, \quad \xi = -\frac{Du(x)}{|Du(x)|}, \quad \text{and} \quad t = u(x),$$

we have

$$h(\xi,t)=x\cdot\xi,\quad h_t(\xi,t)=-\frac{1}{|Du(x)|},\quad\text{and}\quad D_\xi h(\xi,t)=x.$$

See [16]. In particular, since ξ is the outward unit normal to the hypersurface $\{u=t\}$ at the point x, $D_\xi h(\xi,t)$ is the inverse image of the Gauss map at x. Moreover, as $D^2_\xi h(\xi,t)\xi=0$, the restriction of the linear transformation $D^2_\xi h(\xi,t):\mathbb{R}^n\to\mathbb{R}^n$ to $\xi^\perp:=\{z\in\mathbb{R}^n: z\cdot\xi=0\}$ is the inverse of the second fundamental form of $\{u=t\}$ at the point x (see Section 2.5 of [20] for more on this point). In particular, $D^2_\xi h(\xi,t)|_{\xi^\perp}:\xi^\perp\to\xi^\perp$ is positive definite and its eigenvalues are the reciprocals of the principle curvatures of $\{u=t\}$ at x.

Recall that $-\Delta_p u=0$ in $\Omega\backslash\{x_0\}$. Using this equation, Colesanti and Salani proved (in Proposition 1 of [7]) that h satisfies

$$h_t^2\,\mathrm{tr}\left[\left(D^2_\xi h|_{\xi^\perp}\right)^{-1}\right]+(p-1)\left(\left(D^2_\xi h|_{\xi^\perp}\right)^{-1}\nabla_\xi h_t\cdot\nabla_\xi h_t-h_{tt}\right)=0 \tag{3.3}$$

for each $|\xi|=1$ and $t\in(0,\|u\|_\infty)$. Here $\nabla_\xi h_t:=(I_n-\xi\otimes\xi)D_\xi h_t$ is the projection of the gradient of h_t onto $\xi^\perp$. Equation (3.3) will have an important role in our Proof of Theorem 1.1.

4 Convex domains

Throughout this section, we will assume that $\Omega\subset\mathbb{R}^n$ is a bounded convex domain. We will also suppose that $u_0,u_1\in W^{1,p}_0(\Omega)$ are positive extremal functions which satisfy

$$\begin{cases}u_0(x_0)=\|u_0\|_\infty=1\\ u_1(x_1)=\|u_1\|_\infty=1,\end{cases}$$

for some $x_0,x_1\in\Omega$. We aim to show that

$$u_0\equiv u_1. \tag{4.1}$$

It is easy to see that Theorem 1.1 follows from (4.1).

For each $\rho\in(0,1)$, we define the *Minkowski combination* of u_0 and u_1

$$u_\rho(z):=\sup\left\{\min\{u_0(x),u_1(y)\}: z=(1-\rho)\,x+\rho y,\ x,y\in\overline{\Omega}\right\}$$

$z\in\overline{\Omega}$. We recall that u_0 and u_1 are quasiconcave. Using the definition above, it is straightforward to verify

$$\left\{u_\rho>t\right\}=(1-\rho)\,\{u_0>t\}+\rho\,\{u_1>t\}, \tag{4.2}$$

for each $t \in \mathbb{R}$. Here the addition is the usual Minkowski addition of convex sets. In particular, u_ρ itself is quasiconcave.

The Minkowski combination was introduced in work of Borell in [1] when he studied capacitary functions; although his work was motivated by the previous papers of Lewis [14] and Gabriel [11]. We also were particularly inspired to utilize the Minkowski combination after we became aware of the work of Colesanti and Salani in [7], who verified a Brunn-Minkowski inequality for p-capacitary functions ($1 < p < n$), and the work of Cardaliaguet and Tahraoui in [3] on the strict concavity of the harmonic radius.

Along the way to proving (4.1), we will need some other useful properties of u_ρ.

Proposition 4.1 *Define*

$$x_\rho := (1-\rho)x_0 + \rho x_1.$$

Then the following hold:

(i) $u_\rho(x_\rho) = \|u_\rho\|_\infty = 1$.
(ii) $u_\rho|_{\partial\Omega} = 0$.
(iii) $u_\rho \in C^\infty(\Omega\setminus\{x_\rho\}) \cap C(\overline{\Omega})$.
(iv) *For each* $z \in \Omega\setminus\{x_\rho\}$, *there are* $x \in \Omega\setminus\{x_0\}$ *and* $y \in \Omega\setminus\{x_1\}$ *such that*

$$\begin{aligned}
z &= (1-\rho)x + \rho y,\\
u_\rho(z) &= u_0(x) = u_1(y),\\
\frac{Du_\rho(z)}{|Du_\rho(z)|} &= \frac{Du_0(x)}{|Du_0(x)|} = \frac{Du_1(y)}{|Du_1(y)|},\\
\frac{1}{|Du_\rho(z)|} &= (1-\rho)\frac{1}{|Du_0(x)|} + \rho\frac{1}{|Du_1(y)|},\\
\frac{D^2u_\rho(z)}{|Du_\rho(z)|^3} &\geq (1-\rho)\frac{D^2u_0(x)}{|Du_0(x)|^3} + \rho\frac{D^2u_1(y)}{|Du_1(y)|^3}.
\end{aligned}$$

We omit the proof of the above proposition. However, we remark that (i) and (ii) are elementary; Theorem 4 of [7] and Theorem 1 of [14] together imply (iii); and (iv) follows from Sect. 2, [3] or Sect. 7 of [16]. Using these properties we will verify that u_ρ itself is an extremal for each $\rho \in (0,1)$.

Lemma 4.2 u_ρ *is extremal.*

Proof We first show that u_ρ is p-subharmonic and integrate by parts to derive an upper bound on the integral $\int_\Omega |Du_\rho|^p dz$. Then we show that u_ρ satisfies the limits in Proposition 2.8 (that are also satisfied by every extremal function). Finally, we combine the upper bound and limits to arrive at the desired conclusion.

1. Let $z \in \Omega\setminus\{x_\rho\}$, and select $x \in \Omega\setminus\{x_0\}$ and $y \in \Omega\setminus\{x_1\}$ such that

$$e := \frac{Du_\rho(z)}{|Du_\rho(z)|} = \frac{Du_0(x)}{|Du_0(x)|} = \frac{Du_1(y)}{|Du_1(y)|}$$

and

$$\frac{D^2u_\rho(z)}{\left|Du_\rho(z)\right|^3} \geq (1-\rho)\frac{D^2u_0(x)}{|Du_0(x)|^3} + \rho\frac{D^2u_1(y)}{|Du_1(y)|^3}.$$

Recall that such x, y exist by Proposition 4.1. We have

$$\begin{aligned}\left|Du_\rho(z)\right|^{-(p+1)}\Delta_p u_\rho(z) &= \frac{\Delta u_\rho(z)}{\left|Du_\rho(z)\right|^3} + (p-2)\frac{D^2u_\rho(z)e\cdot e}{\left|Du_\rho(z)\right|^3}\\ &= (I_n + (p-2)e\otimes e)\cdot\frac{D^2u_\rho(z)}{|Du_\rho(z)|^3}.\end{aligned}$$

Note that $\min\{1, p-1\} > 0$ is a lower bound on the eigenvalues of the matrix $I_n + (p-2)e\otimes e$. Therefore,

$$\begin{aligned}&\left|Du_\rho(z)\right|^{-(p+1)}\Delta_p u_\rho(z)\\ &\geq (I_n + (p-2)e\otimes e)\cdot\left((1-\rho)\frac{D^2u_0(x)}{|Du_0(x)|^3} + \rho\frac{D^2u_1(y)}{|Du_1(y)|^3}\right)\\ &= (1-\rho)\,|Du_0(x)|^{-(p+1)}\Delta_p u_0(x) + \rho\,|Du_1(y)|^{-(p+1)}\Delta_p u_1(y)\\ &= (1-\rho)\cdot 0 + \rho\cdot 0\\ &= 0.\end{aligned}$$

Consequently, $-\Delta_p u_\rho \leq 0$ in $\Omega\backslash\{x_\rho\}$.

2. The divergence theorem gives

$$\begin{aligned}&\int_{\Omega\backslash B_r(x_\rho)}\operatorname{div}(u_\rho\left|Du_\rho\right|^{p-2}Du_\rho)dz\\ &= \int_{\partial B_r(x_\rho)}u_\rho\left|Du_\rho\right|^{p-2}Du_\rho\cdot\left(-\frac{z-x_\rho}{|z-x_\rho|}\right)d\sigma.\end{aligned}$$

On the other hand, since u_ρ is a positive p-subharmonic function in $\Omega\backslash\{x_\rho\}$

$$\begin{aligned}\int_{\Omega\backslash B_r(x_\rho)}\operatorname{div}(u_\rho\left|Du_\rho\right|^{p-2}Du_\rho)dz &= \int_{\Omega\backslash B_r(x_\rho)}\left(u_\rho\Delta_p u_\rho + \left|Du_\rho\right|^p\right)dz\\ &\geq \int_{\Omega\backslash B_r(x_\rho)}\left|Du_\rho\right|^p dz.\end{aligned}$$

As $u_\rho \leq 1$,

$$\int_\Omega\left|Du_\rho\right|^p dz \leq \liminf_{r\to 0^+}\int_{\partial B_r(x_\rho)}\left|Du_\rho\right|^{p-1}d\sigma. \tag{4.3}$$

3. Let w_ρ be a solution of the PDE (2.9) with x_ρ replacing x_0. As u_ρ is p-subharmonic, $u_\rho(x_\rho) = 1$ and $u_\rho|_{\partial\Omega} = 0$, weak comparison implies $u_\rho \le w_\rho$. This is a version of Borell's inequality; see [1,3]. In particular,

$$w_\rho(z) \ge u_\rho(z) \ge \min\{u_0(x), u_1(y)\}, \tag{4.4}$$

whenever $z = (1-\rho)x + \rho y$.
Now let $z^k \to x_\rho$ with $z^k \neq x_\rho$ for all $k \in \mathbb{N}$ sufficiently large. Set

$$\begin{cases} x^k := z^k + (x_0 - x_\rho) \\ y^k := z^k + (x_1 - x_\rho) \end{cases}$$

for each $k \in \mathbb{N}$. Observe $z^k = (1-\rho)x^k + \rho y^k$ and

$$\left|z^k - x_\rho\right| = \left|x^k - x_0\right| = \left|y^k - x_1\right|.$$

Setting $\lambda := \int_\Omega |Dw_\rho|^p dz$, we have from Proposition 2.8, Remark 2.9 and (4.4) that

$$\begin{aligned} \left(\frac{p-1}{p-n}\right)\left(\frac{\lambda}{n\omega_n}\right)^{\frac{1}{p-1}} &= \lim_{k\to\infty} \frac{1 - w_\rho(z^k)}{|z^k - x_\rho|^{\frac{p-n}{p-1}}} \\ &\le \lim_{k\to\infty} \max\left\{ \frac{1-u_0(x^k)}{|x^k - x_0|^{\frac{p-n}{p-1}}}, \frac{1-u_1(y^k)}{|y^k - x_1|^{\frac{p-n}{p-1}}} \right\} \\ &= \left(\frac{p-1}{p-n}\right)\left(\frac{\lambda_p}{n\omega_n}\right)^{\frac{1}{p-1}}. \end{aligned}$$

It follows that $\lambda = \lambda_p$. In view of (4.4), and since the sequence z^k was arbitrary,

$$\lim_{z\to x_\rho} \frac{1 - u_\rho(z)}{|z - x_\rho|^{\frac{p-n}{p-1}}} = \left(\frac{p-1}{p-n}\right)\left(\frac{\lambda_p}{n\omega_n}\right)^{\frac{1}{p-1}}. \tag{4.5}$$

4. Again let $z^k \to x_\rho$ with $z^k \neq x_\rho$ for all $k \in \mathbb{N}$ sufficiently large. By Proposition 4.1, there are $x^k \in \Omega\backslash\{x_0\}$ and $y^k \in \Omega\backslash\{x_1\}$ such that $z^k = (1-\rho)x^k + \rho y^k$,

$$u_\rho\left(z^k\right) = u_0\left(x^k\right) = u_1\left(y^k\right), \tag{4.6}$$

and

$$\frac{1}{\left|Du_\rho(z^k)\right|} = (1-\rho)\frac{1}{\left|Du_0(x^k)\right|} + \rho\frac{1}{\left|Du_1(y^k)\right|}. \tag{4.7}$$

Since $u_\rho(z^k) \to 1$, (4.6) implies that $x^k \to x_0$ and $y^k \to x_1$ as u_0 and u_1 are uniquely maximized as these points, respectively. Combining this fact with Proposition 2.8, (4.5) and again with (4.6) also gives

$$\lim_{k\to\infty} \frac{|y^k - x_1|}{|z^k - x_\rho|} = \lim_{k\to\infty} \frac{|x^k - x_0|}{|z^k - x_\rho|} = 1. \tag{4.8}$$

By (4.7),

$$\begin{aligned}\frac{|z^k - x_\rho|^{\frac{p-n}{p-1}-1}}{|Du_\rho(z^k)|} &= (1-\rho)\frac{|z^k - x_\rho|^{\frac{p-n}{p-1}-1}}{|Du_0(x^k)|} + \rho\frac{|z^k - x_\rho|^{\frac{p-n}{p-1}-1}}{|Du_1(y^k)|} \\ &= \left(\frac{|z^k - x_\rho|}{|x^k - x_0|}\right)^{\frac{p-n}{p-1}-1}(1-\rho)\frac{|x^k - x_0|^{\frac{p-n}{p-1}-1}}{|Du_0(x^k)|} \\ &\quad + \left(\frac{|z^k - x_\rho|}{|y^k - x_1|}\right)^{\frac{p-n}{p-1}-1}\rho\frac{|y^k - x_1|^{\frac{p-n}{p-1}-1}}{|Du_1(y^k)|}.\end{aligned}$$

We can now employ the second limit in Proposition 2.8 and (4.8) to obtain

$$\lim_{k\to\infty} \frac{|Du_\rho(z^k)|}{|z^k - x_\rho|^{\frac{p-n}{p-1}-1}} = \left(\frac{\lambda_p}{n\omega_n}\right)^{\frac{1}{p-1}}.$$

And since z^k was arbitrary,

$$\lim_{z\to x_\rho} \frac{|Du_\rho(z)|}{|z - x_\rho|^{\frac{p-n}{p-1}-1}} = \left(\frac{\lambda_p}{n\omega_n}\right)^{\frac{1}{p-1}}. \tag{4.9}$$

5. Using the upper bound (4.3) and the limits (4.5) and (4.9), we can proceed with the same arguments as in the proof of Proposition 2.8 to conclude

$$\int_\Omega |Du_\rho|^p \, dz \le \liminf_{r\to 0^+} \int_{\partial B_r(x_\rho)} |Du_\rho|^{p-1} \, d\sigma = \lambda_p.$$

□

In order to verify (4.1), we will employ the respective support functions h_0, h_1, and h_ρ of u_0, u_1 and u_ρ; recall the support function of an extremal was defined in (3.2). In particular, we note that the identity (4.2) implies

$$h_\rho = (1-\rho)h_0 + \rho h_1.$$

Using this identity with the fact that h_0, h_1 and h_ρ all satisfy Eq. (3.3), Colesanti and Salani showed for each $t \in (0, 1)$ there is $C(t) > 0$ such that

$$D_\xi^2 h_0(\xi, t)\Big|_{\xi^\perp} = C(t) D_\xi^2 h_1(\xi, t)\Big|_{\xi^\perp} \tag{4.10}$$

and

$$\partial_t h_0(\xi, t) = C(t)\partial_t h_1(\xi, t), \tag{4.11}$$

for all $|\xi| = 1$ (see the proof of Theorem 1 in [7]).

As $D_\xi^2 h_0(\xi, t)\xi = D_\xi^2 h_0(\xi, t)\xi = 0 \in \mathbb{R}^n$, it follows from (4.10) and the homogeneity of h_0 and h_1 that

$$D_\xi^2 h_0(\xi, t) = C(t) D_\xi^2 h_1(\xi, t),$$

for $(\xi, t) \in (\mathbb{R}^n \backslash \{0\}) \times (0, 1)$. Upon integration we find

$$h_0(\xi, t) = C(t)h_1(\xi, t) + a(t) \cdot \xi + b(t)$$

for some $a(t) \in \mathbb{R}^n$ and $b(t) \in \mathbb{R}$. Since h_0 and h_1 are homogeneous of degree one, it must be that $b(t) = 0$ for each $t \in (0, 1)$. Thus,

$$h_0(\xi, t) = C(t)h_1(\xi, t) + a(t) \cdot \xi.$$

Taking the time derivative of both sides of this equation gives

$$\partial_t h_0(\xi, t) = C(t)\partial_t h_1(\xi, t) + C'(t)h_1(\xi, t) + a'(t) \cdot \xi.$$

Comparing with (4.11) leads us to

$$C'(t)h_1(\xi, t) + a'(t) \cdot \xi = 0.$$

Suppose that $C'(t_0) \neq 0$ for some $t_0 \in (0, 1)$. Then

$$h_1(\xi, t_0) = \left[\frac{-a'(t_0)}{C'(t_0)}\right] \cdot \xi.$$

This would imply the level set $\{u_1 = t_0\}$ is the singleton $\{-a'(t_0)/C'(t_0)\}$, which is not possible. Therefore, $C'(t) = 0$ for $t \in (0, 1)$ and thus $a'(t) \cdot \xi = 0$ for all ξ. Consequently, $a'(t) = 0$ for $t \in (0, 1)$. Since h_0 and h_1 coincide at $t = 0$, $C(t) = 1$ and $a(t) = 0$ for all $t \in [0, 1]$. As a result, $h_0 \equiv h_1$ and so $\{u_0 \geq t\} = \{u_1 \geq t\}$ for each $t \in [0, 1]$ (Theorem 8.24 in [19]). This verifies (4.1).

Proof of Corollary 1.2 By Eq. (1.4) and inequality (1.2),

$$G(y, y) = \int_\Omega |DG(x, y)|^p \, dx$$

$$\geq \lambda_p \|G(\cdot, y)\|_\infty^p$$
$$\geq \lambda_p G(y, y)^p.$$

Therefore, $G(y, y) \leq \lambda_p^{-1/(p-1)}$ and equality holds if and only if $G(\cdot, y)$ is extremal. Theorem 1.1 in turn implies that equality occurs if and only if $y = x_0$. □

5 Nonuniqueness

We will now explain that uniqueness does not hold for general domains by providing a few explicit examples. These instances include planar annuli, bow tie and dumbbell shaped domains. The perceptive reader will also see how to construct other examples from our remarks below.

Example 5.1 Define

$$\Omega_{r_1,r_2} := \left\{x \in \mathbb{R}^n : r_1 < |x| < r_2\right\},$$

for $r_1, r_2 > 0$ with $r_1 < r_2$. As mentioned above, there is a positive extremal u that achieves is maximum at a single point $x_0 \in \Omega_{r_1,r_2}$. Notice that for any $n \times n$ orthogonal matrix O, $v := u \circ O$ is a positive extremal and $\|v\|_\infty = \|u\|_\infty$. Consequently, for each $y_0 \in \Omega_{r_1,r_2}$ with $|y_0| = |x_0|$, there is a distinct positive extremal with supremum norm equal to $\|u\|_\infty$. Thus, uniqueness of extremals does not hold for annuli as showed in Fig. 2.

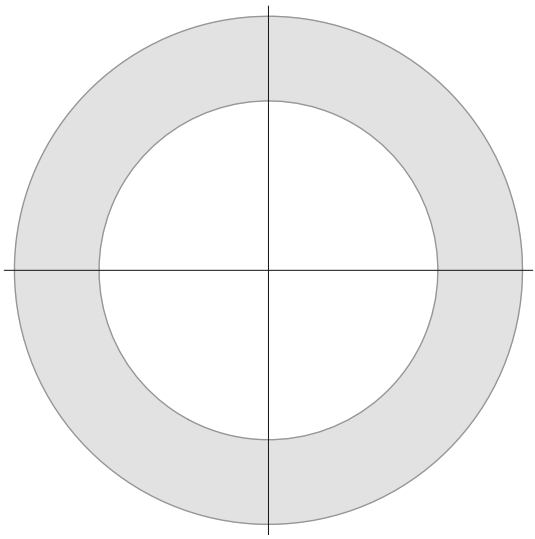

Fig. 2 $\Omega_{4,6}$ when $n = 2$

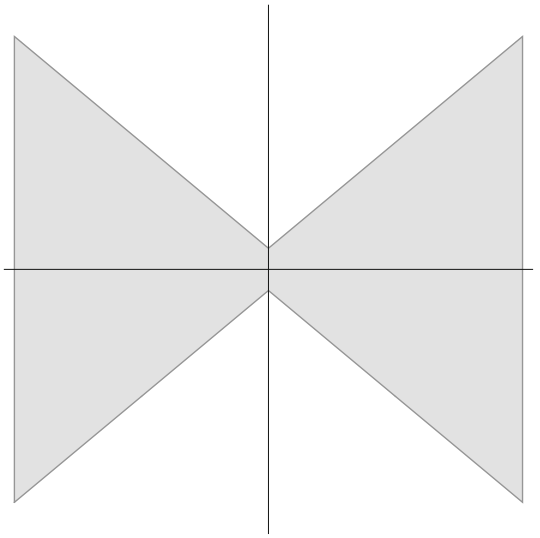

Fig. 3 $\Omega_{1/10}$

Example 5.2 Consider the "bow tie" domain in the plane

$$\Omega_\epsilon := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < |x_1| + \epsilon, |x_1| < 1\right\},$$

for $\epsilon > 0$. Note, in particular, that Ω_ϵ is star-shaped with respect to the origin; see Fig. 3. Let u_ϵ be a positive extremal for Ω_ϵ with $\|u_\epsilon\|_\infty = 1$. If u_ϵ is unique, then it must be that

$$u_\epsilon(0, 0) = 1. \tag{5.1}$$

This is due to the fact that the Ω_ϵ and the p-Laplacian are invariant with respect to reflection about the x_1 and x_2 axes.

Let us assume (5.1) holds for each $\epsilon > 0$ and extend u_ϵ to be 0 outside of Ω_ϵ. Notice that the resulting function, which we also denote as u_ϵ, belongs to $W^{1,p}(\mathbb{R}^2)$. Also note that since $\Omega_0 \subset \Omega_\epsilon$

$$\int_{\mathbb{R}^2} |Du_\epsilon|^p \, dx = \int_{\Omega_\epsilon} |Du_\epsilon|^p \, dx = \lambda_p(\Omega_\epsilon) \leq \lambda_p(\Omega_0).$$

Consequently, there is a decreasing sequence of positive numbers $(\epsilon_j)_{j\in\mathbb{N}}$ tending to 0 and a continuous function $u_0 : \mathbb{R}^2 \to [0, 1]$ for which $u_{\epsilon_j} \to u_0$ locally uniformly on $\mathbb{R}^2$. In view of (5.1),

$$u_0(0, 0) = 1.$$

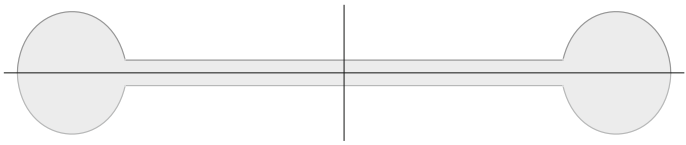

Fig. 4 Dumbbell-shaped domain for $\delta = 1/5$

On the other hand, $u_\epsilon(0, 2\epsilon) = 0$ for all $\epsilon > 0$. Thus

$$u_0(0, 0) = \lim_{j\to\infty} u_{\epsilon_j}(0, 2\epsilon_j) = 0,$$

which is a contradiction.

As a result, we conclude that there is some $\epsilon > 0$ such that u_ϵ does not achieve its maximum value at $(0, 0)$. For this value of ϵ, Ω_ϵ will have a least two positive extremals with supremum norm equal to 1.

Example 5.3 The same ideas used in Example 5.2, can be used to show the dumbbell-shaped domain

$$B_1(-5, 0) \cup ([-5, 5] \times [-\delta, \delta]) \cup B_1(5, 0) \quad (0 < \delta < 1),$$

does not have unique extremals for some $\delta > 0$ chosen small enough. See Fig. 4.

6 Steiner symmetric domains

Theorem 1.1 implies that if a convex domain has some reflectional symmetry, then we have additional information on the location of the maximum points of positive extremals. More precisely, we can make the following observation.

Corollary 6.1 *Assume $\Omega \subset \mathbb{R}^n$ is a convex domain that is invariant with respect to reflection across the hyperplanes $\{x \in \mathbb{R}^n : x_j = 0\}$ for $j = 1, \ldots, n$. Then any positive (negative) extremal achieves its maximum (minimum) value at $0 \in \mathbb{R}^n$.*

Proof Assume $u \in W_0^{1,p}(\Omega)$ is a positive extremal that achieves it maximum value at z. As Ω is invariant with respect to $\{x \in \mathbb{R}^n : x_1 = 0\}$, the function

$$u_1(x_1, x_2, \ldots, x_n) := u(-x_1, x_2, \ldots, x_n), \quad (x_1, x_2, \ldots, x_n) \in \Omega,$$

belongs to $W_0^{1,p}(\Omega)$ and $\|u\|_\infty = \|u_1\|_\infty$. Moreover, it is routine to verify that u_1 is also a positive extremal that achieves it maximum at the reflection of z about the plane $\{x \in \mathbb{R}^n : x_1 = 0\}$. By Theorem 1.1, $u_1 = u$ which forces $z \in \{x \in \mathbb{R}^n : x_1 = 0\}$. Repeating this argument for $j = 2, \ldots, n$, we find $z \in \{x \in \mathbb{R}^n : x_j = 0\}$ for $j = 1, \ldots, n$. As a result, $z = 0$. □

We now seek to extend this observation. We will show below that certain symmetric two dimensional domains have unique extremals without assuming the domains were

convex to begin with. To this end, we employ Steiner symmetrization. In particular, we will make use of the results by Cianchi and Fusco in [6] on the equality condition in the Pólya-Szegö inequality associated with Steiner symmetrization. We also use special properties of the critical points of p-harmonic functions in two dimensions due to Manfredi in [17].

Let us first briefly recall the notion of the Steiner symmetrization of a subset of $\mathbb{R}^2$. For a given $A \subset \mathbb{R}^2$ and $a \in \mathbb{R}$, we will denote $A \cap \{x_1 = a\}$ as the intersection of A with the vertical line $x_1 = a$. We also will write $\mathcal{L}^m$ for the outer Lebesgue measure defined on all subsets of $\mathbb{R}^m$ $(m = 1, 2)$.

Definition 6.2 Assume $A \subset \mathbb{R}^2$. The *Steiner symmetrization* of A with respect to the x_1 axis is

$$A_1^* = \left\{ (a, b) \in \mathbb{R}^2 : |b| < \frac{1}{2}\mathcal{L}^1 \left(A \cap \{x_1 = a\} \right) \right\}.$$

A is said to be *Steiner symmetric* with respect to the x_1 axis if $A_1^* = A$.

Now suppose $u : \mathbb{R}^2 \to [0, \infty)$ is Lebesgue measurable. We can use the above definition to provide the following rearrangement of u

$$u_1^*(x) := \int_0^\infty \chi_{\{u>t\}_1^*}(x) dt, \quad x \in \mathbb{R}^2.$$

This function is called the *Steiner rearrangement* of u with respect to the x_1 axis. Observe that

$$\{u_1^* > t\} = \{u > t\}_1^* \tag{6.1}$$

for each $t \geq 0$. Note also that $u_1^*(x_1, \cdot)$ and $u(x_1, \cdot)$ have the same distribution for $\mathcal{L}^1$ almost every $x_1 \in \mathbb{R}$.

It is known that if $p \in [1, \infty)$, $\Omega \subset \mathbb{R}^2$ is a bounded domain and $u \in W_0^{1,p}(\Omega)$, then $u_1^* \in W_0^{1,p}(\Omega)$. Moreover, if Ω_1^* is Lebesgue measurable, the *Pólya-Szegö inequality*

$$\int_{\Omega_1^*} |Du_1^*|^p dx \leq \int_\Omega |Du|^p dx \tag{6.2}$$

holds, see [2,6]. Cianchi and Fusco showed that if $\Omega_1^* = \Omega$ and equality holds in (6.2), then $u_1^* = u$ provided

$$\mathcal{L}^2 \left(\{u_{x_2} = 0\} \right) = 0 \tag{6.3}$$

(Theorem 2.2 in [6]). All of the above definitions and facts regarding Steiner symmetrization and rearrangements with respect to the x_1 axis have obvious counterparts with respect to the x_2 axis.

Our main assertion regarding the uniqueness of extremals on Steiner symmetric domains is as follows.

Proposition 6.3 *Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain that is equal to its Steiner symmetrization with respect to the x_1 and x_2 axes. Then any positive (negative) extremal achieves its maximum (minimum) value at $0 \in \mathbb{R}^2$.*

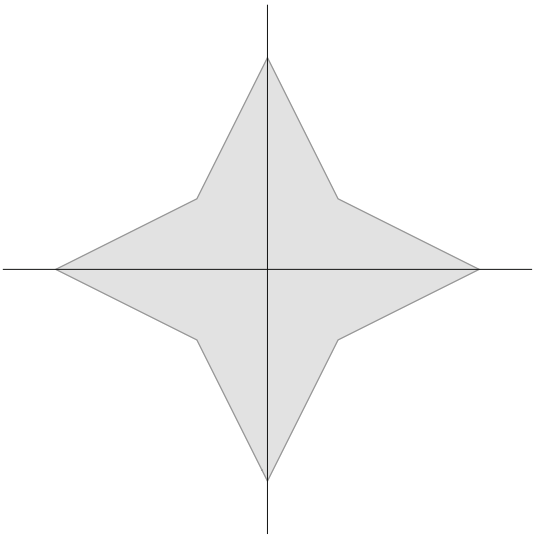

Fig. 5 The domain $\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + 2|x_2| < 1, 2|x_1| + |x_2| < 1\}$

Proof Assume $u \in W_0^{1,p}(\Omega)$ is a positive extremal with $\|u\|_\infty = u(z) = 1$. In view of (6.1), $\|u_1^*\|_\infty = 1$, as well. By the Pólya-Szegö inequality (6.2), we easily conclude u_1^* is extremal and $\int_\Omega |Du_1^*|^p dx = \int_\Omega |Du|^p dx$. We now claim that u satisfies (6.3). Once we verify this assertion, we would have $u = u_1^*$ which implies $u(x_1, x_2) = u(x_1, -x_2)$ for all $(x_1, x_2) \in \Omega$. As a result z belongs to the x_1 axis, and very similarly we would have that z also belongs to the x_2 axis. Therefore, $z = 0 \in \mathbb{R}^2$.

Let us now show that any positive extremal $u \in W_0^{1,p}(\Omega)$ satisfies (6.3). Recall that u is p-harmonic in $\Omega\backslash\{z\}$ and therefore, $u \in C^1_{\text{loc}}(\Omega\backslash\{z\})$. By the results of Manfredi in [17], we know the zeros of Du are isolated in $\Omega\backslash\{z\}$. Consequently, u is locally real analytic in $S := \Omega\backslash(\{z\} \cup \{|Du| = 0\})$, which is an open set of full measure. In particular, u_{x_2} is also locally real analytic in S. Therefore, if

$$\mathcal{L}^2\left(\{x \in S : u_{x_2}(x) = 0\}\right) > 0,$$

then it must be that $u_{x_2} \equiv 0$ in S; see section 3.1 of [13]. Since u_{x_2} is continuous in $\Omega\backslash\{z\}$, it would then follow that $u_{x_2} \equiv 0$ in $\Omega\backslash\{z\}$, as well. However, this is clearly not possible as the function

$$[0, \infty) \ni t \mapsto u(z + te_2)$$

is positive at $t = 0$ and vanishes for all $t > 0$ sufficiently large. As a result, (6.3) holds and the assertion follows. □

See Figs. 5, 6 and 7 for Steiner symmetric, nonconvex domains Ω for which Proposition 6.3 applies to. Figure 5 displays

$$\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + 2\,|x_2| < 1, 2\,|x_1| + |x_2| < 1\}.$$

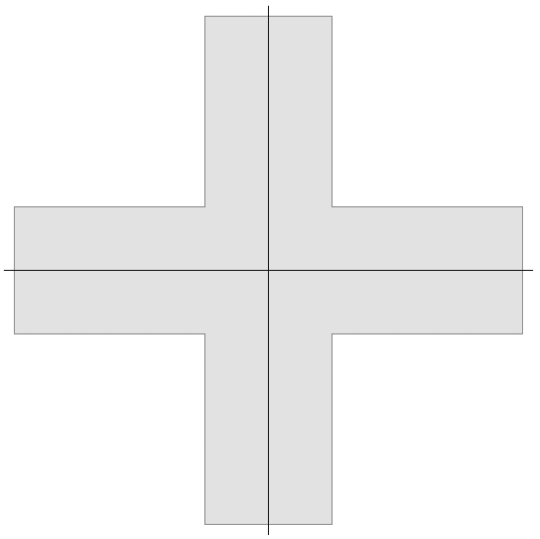

Fig. 6 The domain $([-3, 3] \times [-1, 1]) \cup ([-1, 1] \times [-3, 3])$

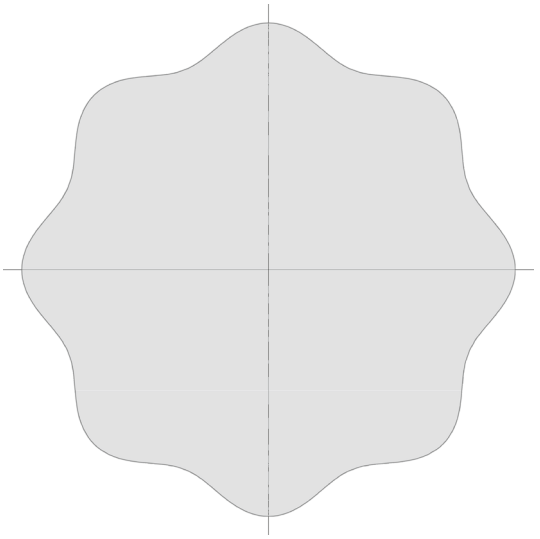

Fig. 7 The region bounded by the curve $r = 10 + \frac{13}{20}\cos(8\theta)$

Figure 6 shows

$$([-3, 3] \times [-1, 1]) \cup ([-1, 1] \times [-3, 3])$$

and Fig. 7 exhibits the region bounded by the curve

$$r = 10 + \frac{13}{20}\cos(8\theta)$$

given in polar coordinates.

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