Multidisciplinary Design of Thermally Radiating Structures Using A Level Set Based Topology Optimization Approach

by

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Abstract

The need for efficient thermally radiating structures for aerospace applications is apparent in many system designs including satellites, launch vehicles and hypersonic aircraft. While multidisciplinary structural optimization methods have been employed to design these systems, few have incorporated thermal radiation as part of their multiphysics analysis capability. Moreover, with the rapid advancement of manufacturing technologies, the ability to fabricate flight hardware with unprecedented geometric complexity has challenged the limits of human design intuition. As a result, there is a critical need for free-form design methods to harness the full potential of these new manufacturing techniques. This thesis presents a level set based topology optimization approach for designing thermally efficient radiating structures considering multiple objectives, constraints and disciplines. Level set based methods offer a key advantage of defining crisp structural boundaries while seamlessly handling complex geometric transformations.

The first contribution of this thesis is deriving a shape sensitivity of the thermal heat power radiated objective function using the adjoint method. This sensitivity is a necessary ingredient for our gradient-based algorithm. The second contribution is developing a topology optimization framework capable of handling multiple objectives and constraints via the augmented Lagrangian method. Both von Mises stress and first mode frequency constraints are implemented since they are commonly found in aerospace design problems where severe structural loads can lead to catastrophic failure. The final contribution is testing several data mining techniques to identify prominent topological features of a large set of designs found using our topology optimization approach. The CUR matrix decomposition is best suited for identifying the principal designs and also aids in categorizing the solution topologies.

These contributions are demonstrated on several 2D example problems where the goal is to identify thermally efficient radiating structures able to reject heat using limited material. Both von Mises stress and first mode frequency constraints are enforced to ensure that solutions are sufficiently rigid. The results indicate that many locally optimal solutions exist in the design tradespace with topologically diverse features.
Additionally, we show that the thermal compliance objective function commonly used to design conductive structures is not sufficient when a radiating surface exists.

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Thanks Dad.
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Nomenclature

Roman Symbols

\( a \) \quad \text{SIMP penalty exponent for element density}

\( b \) \quad \text{Global stress function aggregation parameter}

\( C \) \quad \text{Courant number for CFL condition}

\( c \) \quad \text{Column index}

\( d \) \quad \text{Dimension}

\( d(\cdot, \cdot) \) \quad \text{Shortest distance function}

\( D_s(\cdot) \) \quad \text{Signed distance function}

\( dA \) \quad \text{Differential area}

\( ds \) \quad \text{Differential edge length}

\( dV \) \quad \text{Differential volume}

\( E \) \quad \text{Young's modulus}

\( e \) \quad \text{Element index}

\( F \) \quad \text{Domain integrand for shape functional}

\( f \) \quad \text{Volumetric force}

\( G \) \quad \text{Domain boundary integrand for shape functional}
\( g \) Inequality constraint function
\( H \) Boundary curvature
\( h \) Equality constraint function
\( I \) Identity mapping
\( i \) Node index
\( J \) Objective function
\( K \) Number of clusters for k-means clustering analysis
\( k \) Iteration counter for optimization routine
\( k_{\text{max}} \) Maximum number of iterations allowed
\( k_{\text{min}} \) Minimum number of iterations required
\( L \) Lagrange function used for unconstrained optimization formulation
\( M \) Mapping from level set function to structural boundary \( \partial \Omega \)
\( m \) Total number of elements in finite element mesh
\( n \) Boundary normal
\( N_i \) \( i^{th} \) element interpolation function
\( p \) Total number of nodes in finite element mesh
\( P_{\text{CUR}}(\cdot) \) Probability function for CUR matrix approximation algorithm
\( P_{\text{k-means}}(\cdot) \) Probability function for k-means algorithm
\( q \) Lagrange multiplier for shape sensitivity derivation
\( q^* \) Adjoint solution
\( Q_{\text{gen}} \) Volumetric heat generated
\( Q_{\text{rad}} \) Surface heat flux radiated

\( Q_{\text{rad}}^{\text{max}} \) Maximum surface heat flux radiated

\( r \) Matrix rank

\( RR \) Rejection ratio for evolutionary structural optimization algorithm

\( S \) Set of all cluster sets \( S_1, \ldots, S_K \) using k-means clustering

\( S_j \) Set of data points within \( j^{th} \) cluster using k-means clustering

\( T \) Number of Hamilton Jacobi time steps

\( t \) Time

\( T_t \) Shape transformation

\( T_{\text{max}} \) Maximum number of Hamilton Jacobi time steps for a single optimization iteration

\( T_{\text{max}}^0 \) Initial maximum number of Hamilton Jacobi time steps for the first optimization iteration

\( U \) Generic integrand for shape derivative

\( u \) Displacement

\( U_j \) \( j^{th} \) left singular vector

\( u_x \) X-displacement

\( u_y \) Y-displacement

\( v \) Hamilton Jacobi equation normal velocity magnitude scalar field

\( V_j \) \( j^{th} \) right singular vector

\( v_n \) Hamilton Jacobi equation normal velocity vector field

\( v_{\text{max}} \) Hamilton Jacobi equation maximum normal velocity magnitude
\( V_{\text{req}} \)  Volume constraint

\( w \)  Number of first natural frequencies to calculate

\( w_j \)  \( j^{\text{th}} \) objective function weighting factor

\( x \)  Position

\( y \)  Number of solutions in data set

\( z \)  Number of top right singular vectors used in CUR decomposition

**Accented Symbols**

\( \bar{U}_r \)  \( U \) matrix for rank-\( r \) matrix approximation using CUR decomposition

\( \bar{\omega} \)  Lower bound on frequency constraint

\( \bar{\psi}_{e,1} \)  Mode shape displacement at element \( e \) centroid

\( \bar{\sigma} \)  Upper bound on global stress constraint

\( \bar{C} \)  Set of all cluster centroids for k-means clustering

\( \bar{c} \)  Desired number of columns to sample for CUR decomposition algorithm

\( \bar{c}^* \)  Actual number of columns sampled for CUR decomposition algorithm

\( \bar{u}_e \)  Displacement at the barycenter of element \( e \)

\( \hat{r}, \hat{s}, \hat{t} \)  Local coordinates

\( \hat{x}, \hat{y}, \hat{z} \)  Global coordinates

\( \bar{\phi} \)  Updated level set function

\( \bar{\epsilon} \)  Small positive scalar value

\( \bar{\epsilon} \)  Thermal conductivity of Ersatz material

\( \bar{v} \)  Velocity field projected onto the Hilbert inner product space
$\tilde{T}_{\text{max}}$  Maximum number of Hamilton Jacobi time steps for any iteration in optimization routine

$\tilde{T}_{\text{max}}$  Updated maximum number of Hamilton Jacobi time steps after single optimization iteration

**Greek Symbols**

$\alpha_j$  $j^{th}$ weight for Gauss quadrature

$\alpha_{H1}$  Scaling constant for Hilbertian smoothing technique

$\alpha_{KS}$  Scaling constant for global KS stress functional

$\alpha_{p\text{-norm}}$  Scaling constant for global p-norm stress functional

$\beta_{\mu}$  Tolerance on updating the penalty parameter using the equality/inequality constraint

$\beta_{g}$  Tolerance on the inequality constraint function

$\beta_{h}$  Tolerance on the equality constraint function

$\beta_{f}$  Tolerance on the objective function or weighted objective functions

$\beta_{L}$  Tolerance on accepting the updated augmented Lagrangian

$\chi$  Domain flag for radiating surface

$\Delta t$  Step size for Hamilton Jacobi solver

$\Delta x$  Minimum element size

$\delta$  Lower bound for ersatz material density

$\epsilon$  Strain tensor ($2^{nd}$ order)

$\eta$  Constant to reduce the number of steps for backtracking method

$\eta$  Update tolerance for equality constraint
\( \Gamma_N \)  Structure Neumann boundary

\( \Gamma_{0,\psi} \)  Structure tractionless boundary for modal model

\( \Gamma_{0,\theta} \)  Structure tractionless boundary for thermal model

\( \Gamma_{0,u} \)  Structure tractionless boundary for statics model

\( \gamma_\mu \)  Penalty parameter update factor

\( \gamma_{T\text{-max}} \)  Update factor for increasing/decreasing \( T_{\text{max}} \) in backtracking routine

\( \gamma_T \)  Update factor for decreasing \( T \) in backtracking routine

\( \Gamma_{D,\psi} \)  Structure Dirichlet boundary for modal model

\( \Gamma_{D,\theta} \)  Structure Dirichlet boundary for thermal model

\( \Gamma_{D,u} \)  Structure Dirichlet boundary for statics model

\( \iota_{\text{max}} \)  Maximum number of augmented Lagrangian re-evaluations in backtracking algorithm

\( \iota_{\text{retry}} \)  Counter for number of augmented Lagrangian re-evaluations in backtracking algorithm

\( \kappa \)  Conductivity tensor (2\textsuperscript{nd} order)

\( \lambda \)  Lagrange multiplier estimate for augmented Lagrangian

\( \lambda^* \)  Lagrange multiplier

\( \mu \)  Penalty parameter for augmented Lagrangian

\( \nu \)  Poisson’s ratio

\( \Omega \)  Structure domain

\( \omega \)  Frequency

\( \Omega^* \)  Optimal structure domain
\( \Omega_e \) Element material sub-domain

\( \Omega_{\text{load}} \) Static pressure loaded sub-domain

\( \Omega_{\text{rad}} \) Radiating sub-domain

\( \partial \Omega \) Structure boundary

\( \phi \) Level set function

\( \phi^e_i \) \( i^{th} \) nodal value of level set function in element \( e \)

\( \pi_c \) Normalized statistical leverage score of column \( c \) for CUR decomposition

\( \psi \) Eigenvector from modal PDE

\( \psi^e_i \) \( i^{th} \) nodal value of eigenvector in element \( e \)

\( \rho \) Material density

\( \rho_e \) Element density

\( \sigma \) Stress tensor (2nd order)

\( \sigma_{\text{VM}} \) Von Mises stress

\( \sigma_{\text{VM}}^{\text{max}} \) Maximum von Mises stress

\( \sigma_Y \) Yield strength

\( \sigma_j \) \( j^{th} \) singular value for SVD

\( \tau \) Heat flux

\( \theta \) Temperature

\( \theta_0 \) Fixed temperature

\( \theta^e_i \) \( i^{th} \) nodal value of temperature solution in element \( e \)

\( \theta_{\text{surf}} \) Surface temperature
\( \gamma \) Stefan-Boltzmann Constant
\( \varepsilon \) Emissivity
\( \varrho \) Thermal conductivity for isotropic materials
\( \xi \) Shape perturbation
\( \zeta \) Direction to take derivatives of the Lagrangian for shape sensitivity analysis

**Matrix Symbols**

\( \Phi \) Level set function array at varying time steps
\( \phi \) Level set function vector
\( \phi_e \) Element level set function vector
\( \phi_{\text{reinit}} \) Reinitialized level set function vector
\( \psi \) Mode shape vector
\( \psi_{e,1} \) Element first mode shape vector
\( \rho \) Element density vector
\( \sigma \) Element stress vector
\( \Sigma_r \) Diagonal matrix of largest \( r \) singular values for SVD
\( \tau_e \) Element heat flux vector
\( \theta \) Temperature vector
\( \theta_e \) Element temperature vector
\( B \) Strain-displacement matrix
\( C \) Stress-strain matrix
\( c^* \) Closest centroid previously chosen in k-means clustering algorithm
\(c^j\) \(j^{th}\) cluster centroid for k-means clustering

\(C_r\) Column matrix for rank-\(r\) matrix approximation using CUR decomposition

\(D\) Constitutive matrix for thermal finite element method

\(f\) Force vector

\(f^*\) Force vector for adjoint system of equations

\(g\) Heat vector

\(H\) Shape function matrix

\(J_e\) Element Jacobian matrix

\(K\) Stiffness matrix

\(K_e\) Element stiffness matrix

\(L\) Conductivity matrix

\(L_e\) Element conductivity matrix

\(M\) Mass matrix

\(M_e\) Element mass matrix

\(P\) Solution data set

\(q^*\) Adjoint state vector

\(R_r\) Row matrix for rank-\(r\) matrix approximation using CUR decomposition

\(S\) Coefficient matrix to calculate the von Mises stress

\(u\) Displacement vector

\(u_e\) Element displacement vector

\(U_r\) Matrix of first \(r\) left singular vectors for SVD
\( V_r \)  Matrix of first \( r \) right singular vectors for SVD
\( x \)  Position vector

**Other Symbols**

\( \ell \)  Number of nodes/shape functions per element
\( \mathcal{L} \)  Lagrangian used for adjoint analysis to derive shape sensitivity
\( C \)  Elasticity tensor (4th order)
\( \mathbb{R} \)  Set of all real numbers
\( \mathbb{R}_{>0} \)  Set of positive real numbers
\( \mathbb{R}_{\geq0} \)  Set of non-negative real numbers
\( \mathbb{Z}_{>0} \)  Set of all positive integers
\( 1 \)  Indicator function
\( D \)  Design domain
\( \mathcal{E} \)  Set of all indices for equality constraints
\( \mathcal{I} \)  Set of all indices for inequality constraints
\( J \)  Shape functional
\( T_e \)  Domain for element \( e \)
\( \partial D \)  Design domain boundary

30
Abbreviations

AM  additive manufacturing.
AOS  additive operator splitting.
BESO  bi-directional evolutionary structural optimization.
CAD  computer aided design.
CCD  charge-coupled device.
CFL  Courant-Friedrichs-Lewy.
CONLIN  convex linearization method.
CPU  central processing unit.
DOF  degree of freedom.
ESO  evolutionary structural optimization.
GPU  graphics processing unit.
HJE  Hamilton Jacobi equation.
KKT  Karush-Kuhn-Tucker.
KS  Kreisselmeier-Steinhauser.
LSF  level set function.
**LSM** level set method.

**MMA** method of moving asymptotes.

**NSGA** Non-dominated Sorting Genetic Algorithm.

**PDE** partial differential equation.

**RAMP** rational approximation of material properties.

**REXIS** Regolith X-ray Imaging Spectrometer.

**SIMP** solid isotropic material with penalization.

**SLP** sequential linear programming.

**SNOPT** Sparse Nonlinear Optimizer.

**SVD** singular value decomposition.
Chapter 1

Introduction

Thermal radiation is a physical phenomenon affecting the design of many aerospace systems including hypersonic vehicles and spacecraft. The need to design lightweight, efficient and multi-functional structural elements requires the implementation of computational tools to perform multidisciplinary analysis and optimization. These tools are used to gain valuable insight on performance and limitations on complex multidisciplinary systems early in the design cycle, reducing cost and schedule overruns caused by iterative redesigns. Topology optimization is a specific method which has become increasingly popular for free-form structural design. Engineers typically apply these methods to reduce weight, improve structural rigidity, and optimize designs subject to a wide variety of objectives and constraints. However, limited research exists on developing multi-objective, multi-constraint topology optimization methods for designing thermally radiating structures.

This chapter introduces the need for developing a topology optimization method for solving problems incorporating thermal radiation. Section 1.1 provides the motivation by introducing several real-world applications where thermal radiation plays a significant role. Section 1.2 introduces the field of topology optimization, its development in both academic and industrial settings, and several examples within aerospace engineering where components have been designed using this method. Section 1.3 presents our goals of advancing topology optimization methods to solve real-world thermal radiation problems and enumerates the thesis objectives. And finally Sec-
Thermal radiation plays a significant role in many aerospace systems. In spacecraft design, thermally sensitive equipment such as optics devices and power systems must remain within narrow ranges of operating temperatures. Because spacecraft operate in a vacuum environment, the primary method for rejecting waste heat and maintaining thermodynamic equilibrium is via thermal radiation. The New Horizons spacecraft, an interplanetary space probe which performed a flyby of the Pluto system in 2015, required a radioisotope thermoelectric generator (RTG) for generating electrical power. The spacecraft is depicted in Figure 1-1(a) along with the mounted RTG in Figure 1-1(b). Waste heat produced by the RTG is radiated from the cylindrical housing via an array of radially protruding aluminum fins.

Figure 1-1: New Horizons spacecraft reproduced from [120]
Another example spacecraft is the James Webb Space Telescope, a large space-based telescope equipped to observe the universe in the infrared spectrum and scheduled to launch in 2019. This telescope houses a suite of optics instrumentation (the Integrated Science Instrument Module) which requires massive cryogenic radiators to maintain operating temperatures as low as 7 K [66]. Figure 1-2 illustrates the telescope along with the radiator panels fixed to the Integrated Science Instrument Module on the backside of the primary mirror.

![Figure 1-2: James Webb Space Telescope](a) Artist rendering of James Webb Space Telescope. *NASA*

(b) Artist rendering of James Webb Space Telescope’s radiator panels *NASA*

Efficient radiator designs are also needed for smaller payloads, such as the Regolith X-ray Imaging Spectrometer (REXIS) instrument, mounted on board the OSIRIS-REx spacecraft launched in 2016 to measure the abundance of elements on the asteroid Bennu’s surface. The computer aided design (CAD) model for the REXIS instrument is shown in Figure 1-3. An aluminum panel radiator was needed to cool the charge-coupled device (CCD) to temperatures below -55°C.

Yet even more harsh environments are encountered by planetary reentry vehicles which experience extremely high heat loads while traveling in the upper atmosphere at hypersonic speeds. These vehicles require thermal protection systems made from
composites and other refractory metal alloys to protect against the heat of reentry where a primary mode of heat transport is thermal radiation. Figure 1-4(a) depicts the Orion Crew Module during reentry into earth's atmosphere on its maiden test flight in late 2014. The module's charred heat shield post-reentry is also shown in Figure 1-4(b).

The handful of above examples serve to motivate the need for computational tools to perform design optimization on thermally radiating structures where mass and volume are constrained. While the implementation of formal optimization methods to the structural design of aerospace systems is abundant in the literature, only limited research exists on design optimization methods for radiating structures. We find that several factors have contributed to the need and growing interest in structural design optimization methods: (1) growth in system complexity requiring the need for multifunctional and integrated structures, (2) limits of human intuition and commonly followed design practices, (3) availability of high fidelity computational modeling and analysis tools and (4) improvements in manufacturing technologies. These factors have created a need and opportunity to augment the art of engineering design with accurate, multidisciplinary structural analysis and optimization tools.
1.2 Topology Optimization

Topology optimization is a popular method used for free-form structural design. This method identifies an optimal material layout within a fixed design domain subject to a set of boundary conditions and other constraints to minimize a cost functional. Deaton and Grandhi define topology optimization as “the process of determining the connectivity, shape, and location of voids inside a given design domain” [49]. While
other structural optimization methods handle a fixed set of design variables such as the lengths, thicknesses and other relevant dimensions of predetermined features, topology optimization allows for variation in the overall shape and connectivity of the structure. This procedure is typically more advantageous in earlier design stages where a variety of geometries can be explored. Several papers provide overviews of the various topology optimization methods and current areas of research [63, 138, 150, 49]. Bendsøe’s landmark textbook “Topology Optimization: Theory, Methods and Applications” [21] is also widely referenced by many researchers in the field.

Since its inception in the late 1970’s, topology optimization has been adopted for both academic and industrial applications. After Sigmund’s “99 Line Topology Optimization Code Written in MATLAB” was published in 2001 [149], several other researchers have also developed codes for interested readers. Andreassen et al. further simplified the 99-line topology code developed by Sigmund [14], Challis implemented a discrete level set method on rectangular plane stress elements under static load conditions [33], Talischi et al. devised a framework to handle unstructured 2D polygonal finite element meshes [165] and Liu and Tovar extended Sigmund’s work to 3D meshes [103]. These examples encompass only a partial list of contributors to open-source topology optimization programs. Likewise, engineering software firms such as Dassault Systèmes, COMSOL and MSC Software Corporation have to date augmented their finite element analysis tools with a topology optimization capability [154, 166, 41, 117]. Other companies, most notably Altair Engineering, have dedicated whole standalone packages such as OptiStruct and HyperStudy to perform topology optimization, design of experiments and multi-disciplinary design optimization [11]. In particular, OptiStruct has been used in a number of aircraft and automotive applications [186, 187, 73].

Since the mid-2000’s, many researchers have applied topology optimization methods to hypersonic vehicle and spacecraft design problems. Kim et al. used an evolutionary structural optimization algorithm to perform a 3D design of a thermal protection system for atmospheric reentry vehicles [94]. Kang et al. optimized the primary load-bearing structure of a spacecraft while constraining its moment of in-
ertia to minimize fuel consumption for attitude maneuvers [88]. Liu et al. used OptiStruct to lightweight a spacecraft antenna pedestal subject to random excitation loads [102]. Cismilianu et al. designed an aluminum truss structure to support an array of thrusters subject to modal frequency and stress constraints [38]. And Orme et al. demonstrated an end-to-end design of an additively manufactured engine mount for a lunar landing craft [123].

Topology optimization methods typically generate organic-looking shapes which cannot be fabricated using traditional manufacturing techniques. To successfully apply these methods to real-world design problems, a significant amount of research has been invested in devising and implementing manufacturing constraints [194, 77, 156, 114]. However, additive manufacturing (AM) technologies which can fabricate geometrically complex structures have largely closed the gap. Additive manufacturing, a process which deposits successive layers of material to print a 3D structure, has improved tremendously during the recent years in many key ways: (1) larger varieties of materials (e.g., plastics, metal alloys, and composites) can be printed, (2) higher resolution and smaller tolerances can be achieved, and (3) greater bed capacities allow for building larger monolithic structures. These advancements have changed the role of additive manufacturing from building small prototype and toy plastic parts to making flight hardware for aerospace systems. A few recent examples include rocket chambers made by SpaceX [157], propulsion fuel tanks made by Lockheed Martin [104], and an antenna support structure fabricated by RUAG for the Sentinel satellite [67]. Figure 1-5 illustrates these examples. An emerging synergy between topology optimization and additive manufacturing is now clear. The free-form design methodology provided by topology optimization can leverage the full potential of manufacturing flexibility afforded by AM. Several researchers have recognized this synergy and demonstrated end-to-end designs of a variety of aerospace system components using topology optimization in conjunction with additive manufacturing [25, 142, 123, 38].
Figure 1-5: Real-world examples of additively manufactured flight hardware for aerospace structures
1.3 Thesis Objectives

The goal of this thesis is to develop and test a level set based topology optimization approach to identify optimal geometries for radiating structures. The case studies considered range in complexity from single objective, single constraint to multi-objective, multi-constraint optimization problems on 2D finite element models. We also explore the design landscape by identifying Pareto optimal solutions which trade mass and thermal heat power radiated for each of the case studies. By visually inspecting the Pareto optimal shapes, we gain better intuition on dominant topological features and their contributions to making the solution both feasible and optimal. For larger data sets where it would become arduous for a designer to inspect the entire Pareto front, we apply data mining techniques to extract this information and help categorize the solutions into design families.

The specific thesis objectives are enumerated below:

1. Develop a framework for performing multi-objective, multi-constraint level set based topology optimization on thermally radiating structures.

2. Derive an analytic shape sensitivity for maximizing thermal heat power radiated and apply a gradient-based optimizer to implement this sensitivity.

3. Test our methodology on a 2D design problem to demonstrate that locally optimal solutions can be identified.

4. Develop a methodology to extract prominent topological features of the non-dominated set of solutions and help categorize them into design families.

In this thesis we lay the groundwork for applying topology optimization methods to design problems where the goal is to maximize radiated heat power. To date, there was no literature or research found on applying formal topology optimization methods to this class of problems.
1.4 Thesis Roadmap

The remainder of this thesis is organized into seven chapters. In Chapter 2 we provide some background on topology optimization methods and focus on level set methods specifically, highlighting their advantages and disadvantages. In Chapter 3 we introduce a single-objective, single-constraint optimization problem to design a thermally efficient radiating structure. The level set based topology optimization approach is defined using an augmented Lagrangian method to handle constraints. We also provide an overview of the computational framework developed using MATLAB®’s Partial Differential Equation Toolbox. In Chapter 4 we demonstrate the proposed methodology on a 2D radiating aluminum plate problem using a novel shape sensitivity for maximizing thermal heat power radiated. Several initial material configurations are tested and we demonstrate that our method is identifying locally optimal solutions. In Chapter 5 we introduce a multi-objective, multi-constraint problem to maximize thermal heat radiated and minimize volume fraction. Both a von Mises stress and first mode frequency constraint are enforced. We again demonstrate our methodology using the previously introduced 2D radiating aluminum plate case study and illustrate the resulting Pareto plots. In Chapter 6 we test four data mining techniques including a frequency distribution analysis, SVD and CUR matrix approximation methods, and a k-means clustering analysis to extract the dominant features of a large dataset of shapes. This dataset is generated by solving the 2D multi-objective, multi-constraint problem previously presented in Chapter 5 using varied initialization parameters. We find that the CUR matrix decomposition is best suited for visualizing and identifying the various topological features in the dataset. Finally in Chapter 7 we provide a summary of the contributions presented in the thesis and discuss future work.
Chapter 2

Background

Modern structural optimization paralleled the development of computational methods to analyze structures such as the finite element method. Since Schmidt’s seminal work on “systematic synthesis” [141] a wealth of literature has been published on applying formal optimization methods to structural design. While much of the literature dealt with problems where the structural features were controlled via a handful of design variables, the emergence of topology optimization methods enabled free-form design where the solution space encompasses a nearly limitless number of shapes and geometries.

This chapter presents a brief background on topology optimization methods and introduces several key terminologies. In Section 2.1 we present the origins of structural design optimization and discuss the limits of parametric approaches. In Section 2.2 we introduce the fundamental building blocks of topology optimization and provide an overview of three popular methods, including their strengths and weaknesses. Finally, in Section 2.3 we argue that the level set method is the best approach for thermally radiating problems introduced in this thesis as well as more advanced problems where irradiation effects are encountered.
2.1 Origins of Structural Design Optimization

Structural design optimization is the process of identifying or altering an existing structure to carry a set of loads in an efficient manner. In this context the term "efficient" is meant to describe a design which uses the least amount of material. In many fields such as aerospace and civil engineering, reducing structural weight plays a huge role in performance and cost-reduction. Additionally the term "loads" includes not only mechanical forces, but also thermal heat, and electromagnetic fluxes.

The classical structural design optimization problem is written as follows:

\[
\text{(SDO): } \min_{x} \quad J(x) \\
\text{subject to } \quad g_j(x) \leq 0 \quad j \in \mathcal{I} \\
\quad h_j(x) = 0 \quad j \in \mathcal{E} \\
\quad x \subseteq \mathbb{R}^d
\]

where \( x \) is the design vector which lies in the \( d \)-dimensional real number space \( \mathbb{R} \), \( J \) is the objective function, \( g_j(x) \) and \( h_j(x) \) are the \( j^{th} \) inequality and equality constraint respectively, and \( \mathcal{I} \) and \( \mathcal{E} \) are the set of indices for the inequality and equality constraints respectively. Typically the equality constraints will include the governing equations which model the structural response to set of load conditions. Other constraints on displacements, stresses, buckling criteria and dynamic responses (e.g., modal frequency) can also be enforced.

Since the 1960’s much effort has been placed on developing efficient implementations of gradient-based optimizers and approximation techniques to circumvent limited processing power for solving (SDO) [170, 169, 108]. By the 1970’s, these methods had fallen into two different approaches: numerical optimization using (1) nonlinear programming [80] and (2) optimality criteria [171]. The former method suffered from its inability to handle a large number of design variables, whereas the latter could not handle large numbers of constraints. While several researchers have also more recently applied heuristic methods, they are quickly out-performed by gradient-based algorithms as the design space grows. Coroneos and Pai illustrate the effectiveness of
a variety of solvers in the openMDAO framework [42]. They only report performance metrics of a genetic algorithm in a three-bar truss problem where six design variables are specified. Other larger problems are solved using only gradient-based methods.

By the late 1970’s, one major drawback of structural optimization methods was their inability to create new topologies. Design variables consisted mainly of sizing parameters which maintained the overall configuration and connectivity of the structure throughout the optimization procedure. Consequently designers needed to initialize their optimization methods with a structural design based on some a priori knowledge or intuition. Limitations in manufacturing and the abundance of structurally robust examples aided engineers in selecting these initial designs for civil engineering applications. Such examples included truss assemblies, frames and tensioned rods [169, 39].

2.2 Overview of Topology Optimization Methods

The field of topology optimization began to blossom in the late 1970’s. Rather than maintaining a predefined shape or structural configuration, topology optimization methods can generate a nearly limitless number of material layouts affording much greater design freedom. However, different methods have evolved based on their strategies for discretizing and updating the shape and connectivity of the structure. The choice in method is largely problem-dependent, although tradeoffs also exist in robustness, computational efficiency and complexity. Section 2.2.1 provides a brief history of topology optimization, while Sections 2.2.2 to 2.2.4 outline three popular methods: homogenization, evolutionary and boundary variation methods.
2.2.1 A Brief History

Michell provided the first written account of topology or generalized shape optimization in 1904 where he derived the optimality criteria for a minimal weight truss design [115]. It was not until the 1970's that topology optimization gained traction with the work of Prager and Rozvany [138] and later in 1988 with Bendsøe and Kikuchi [20]. The major catalyst was the development of computational methods for analyzing structural models.

Topology optimization methods typically couple a finite element model of the structure under consideration with a numerical optimization algorithm. While other analysis techniques have been implemented such as finite volume [27] and boundary element methods [183], finite element analysis is the most prevalent. The first account of pairing finite element methods with numerical optimization techniques can be found in Rossow and Taylor’s 1973 paper [136]. The design variables (discrete or continuous) are usually assigned to the elements or nodes of the mesh to signify whether a region is filled with material (i.e. solid) or empty. Consequently, due to the large dimensionality of these problems a major effort on implementing efficient gradient-based algorithms had been undertaken. Example algorithms include Newton’s method, sequential linear programming (SLP), convex linearization method (CONLIN), and the method of moving asymptotes (MMA). Several of these algorithms, including CONLIN [65] and MMA [160] have been developed solely for structural optimization applications. Other more advanced algorithms have been implemented to solve larger problems such as Sparse Nonlinear Optimizer (SNOPT) [69] and a globally convergent variant of MMA developed by Svanberg [161].

To illustrate how topology optimization methods couple with a finite element model, we present the classic statically loaded 2D cantilever topology optimization problem. The goal is to design an efficient load carrying cantilever constrained on the left edge and subjected to a tip load on the bottom right corner. The rectangular design domain illustrated in Figure 2-1(a) is meshed using square plane-stress elements with two translational degree of freedoms (DOFs) on each node. An isotropic linear
elastic material is modeled with a Young’s modulus $E = 1.0$ and a Poisson’s ratio $\nu = 0.3$.

![Meshed design domain](image1.png)  ![Optimal topology solution](image2.png)

(a) Meshed design domain  (b) Optimal topology solution

Figure 2-1: Classic cantilever topology optimization problem for minimizing structural compliance

The optimization problem (TO) is formally stated as follows:

$$(TO): \quad \min_{\rho_e} \quad u^T K(\rho_e) u$$

subject to $\sum_{e=1}^{m} \rho_e |T_e| \leq V_{\text{req}}$

$K(\rho_e) u = f$

$0 \leq \rho_e \leq 1 \quad e = 1, \ldots, m$

where $K$ is the global finite element stiffness matrix, $u$ is the displacement vector, $f$ is the load vector, $|T_e|$ is the area of element $e$ (of which there are $m$ total elements), $V_{\text{req}}$ is the total volume constraint, and $\rho_e \in [0, 1]$ is the element density where a value of 0 and 1 indicates void (no material) and solid (fully material) respectively. The constraint $Ku = f$ is the system of equations derived using the finite element method for an elastostatic problem, and $u^T K u$ represents the compliance of the structure. Structural compliance is twice the strain energy, the energy stored in the material as it is deformed by the load condition. Figure 2-1(b) illustrates the optimal shape where black elements are solid and white elements are void. The gray elements have intermediary densities $\rho_e \in (0, 1)$ which, depending on the problem and method used, has varying implications. If we consider the integer problem where elements are only
allowed an element density of 0 or 1, a total of $2^m$ configurations are possible.

What differentiate topology optimization methods from one another are how the element densities are defined, their effect on the finite element matrices, and the procedure for updating the density values. The three most common methods are homogenization, evolutionary, and boundary variation. These methods are presented next in Section 2.2.2.

2.2.2 Homogenization Methods

Homogenization techniques work by averaging or “smearing” the properties of a heterogeneous material to approximate them as homogeneous. This method was primarily intended for composite materials, but has found large acceptance in the topology optimization community for isotropic materials as well. Perhaps the most popular homogenization method is the solid isotropic material with penalization (SIMP) method [19]. SIMP utilizes a fixed Eulerian finite element mesh, where the density of each element is prescribed as a design variable $\rho_e$. The optimization routine then adjusts the densities of these elements until a given objective function is minimized. A mapping between the element density and element stiffness matrix must be established in order to iteratively solve the finite element model and evaluate the objective function. Equation (2.1) defines a commonly used relationship known as the proportional stiffness model for linear elastic problems such as (TO) defined in the previous section.

$$C^* = \rho_e^a C$$  \hspace{1cm} (2.1)

In Equation (2.1), $C$ and $C^*$ are the nominal and penalized constitutive matrices respectively for the element $e$ in the mesh, $\rho_e$ is the element design variable (also referred to as the element density) which interpolates the stiffness across the mesh, and $a$ is an exponential penalty parameter. The purpose of $a$ is to help drive the optimal solution to a “0/1” design where the majority of elements demonstrate clear void/solid representations. Due to the ambiguous nature of intermediary element densities (e.g. representing a boundary, porous material, etc.), they are undesirable
in the final material configuration. The exponential penalty strategy is useful for synthesizing designs of a single isotropic material under various load conditions. Figure 2-2 illustrates the difference between optimized solutions to a cantilever beam problem with and without an exponential penalization.

![Figure 2-2: Homogenization method for composite (left) and penalized (right) applications reproduced from [2]](image)

Other relationships have been implemented including the rational approximation of material properties (RAMP) [132] and SINH methods which relate the density and stiffness through rational and hyperbolic sine functions respectively. These other models have been shown to be more effective in a variety of circumstances such as when low volume fraction solutions are desired under design dependent loading [49]. While SIMP is the most commonly used method today to perform topology optimization, it suffers from a number of drawbacks such as “checkerboarding”. As the name implies, locally optimal solutions can be found which exhibit periodic solid and void elements as the mesh is refined (resulting in infeasible designs which are porous in nature). However, this dilemma can be managed to some degree through the appropriate use of filtering techniques and geometric constraints [21].
2.2.3 Evolutionary Methods

Evolutionary methods work by adding or removing material based on a heuristic criteria related to the objective function. While several methods have emerged since the 1990's [111, 134, 90], the most common of them is known as evolutionary structural optimization (ESO) [82]. One common example criteria used with ESO for linear elastic problems is based on the principle that all elements within the mesh should be equally stressed. Thus, elements that have low stress values should be removed, and elements with high stress values should be reinforced. Whereas ESO is based on the removal of low-stressed elements only, bi-directional evolutionary structural optimization (BESO) [133] is able to both remove and add elements based on the programmed criteria. An example criteria is presented below:

\[
\frac{\sigma_{\text{VM}}}{\sigma_{\text{VM}}^{\text{max}}} \leq RR^k
\]  

(2.2)

where \(\sigma_{\text{VM}}\) is the elemental von Mises stress, \(\sigma_{\text{VM}}^{\text{max}}\) is the maximum von Mises stress, and \(RR^k\) is the current rejection ratio for optimization iteration \(k\). Thus all elements meeting the criteria from Equation (2.2) should be removed. The rejection ratio is then increased at each iteration \(k\) and asymptotically approaches 1.0. A similar criteria can also be written for strain energy to minimize displacement. Evolutionary methods are commonly referred to as “hard-kill” methods since they do not allow for intermediary element densities. The design variables are discrete with values of either 0 or 1. Conversely, homogenization techniques such as SIMP are sometimes considered “soft-kill” methods since intermediary element densities are permitted.

ESO was developed by Xie and Steve in 1992 [180]. The method can be implemented quickly and integrates well with commercial finite element analysis software. Additionally ESO is a gradient-free method, meaning each iteration requires fewer computations than a gradient-based algorithm. However, ESO methods suffer from several drawbacks, including slow convergence rates, no guarantee of finding an optimal solution, no direct control over the desired material volume fraction and difficulty implementing multiple constraints and loads [138].
2.2.4 Boundary Variation Methods

Boundary variation methods are the most recently developed within the topology optimization community. They are based upon an implicit representation of the structure's boundaries rather than an explicit parameterization of the design domain [49]. Two popular methods which fall under this category are level set methods (LSMs) and phase-field methods. Both employ an implicit function $\phi$ to define the structural boundary as well as an evolution equation used to update $\phi$ between optimization iterations. While phase-field methods offer some numerical advantages over level set methods, they have not been widely employed for topology optimization problems and the reader is referred to [163, 36, 86] for more information.

In level set methods, $\phi$ is defined as the level set function (LSF) and implicitly defines the boundary of the structure through the following conditions:

\[
\begin{align*}
\phi(x) &< 0 \quad x \in \Omega \\
\phi(x) &= 0 \quad x \in \partial\Omega \\
\phi(x) &> 0 \quad x \in D \setminus \Omega
\end{align*}
\]  \tag{2.3}

where $\Omega$ is the structure domain, $\partial\Omega$ is the structure boundary and $D$ is the design domain. Note that $D$ defines the physical limitations of how large the structure can become, and in all cases $\Omega \subset D$. Figure 2-3 illustrates an example level set function and the zero level set which identifies $\Omega$ and $\partial\Omega$ defined in Equation (2.3).
The level set method was developed by Osher and Sethian [125] for modeling moving boundaries encountered in many applications including combustion, computer vision and path planning [144]. Sethian and Wiegmann [143] introduced a topology optimization routine where the structural boundary is defined by a level set function and updated using a stress-based heuristic criteria. A complete review of level set methods can be found in [124, 168]. The traditional algorithm requires solving the Hamilton Jacobi equation (HJE) to update $\phi$ using a numerical procedure such as an upwind finite difference scheme. The Hamilton Jacobi equation is written as follows:

$$\frac{\partial \phi(x,t)}{\partial t} + v_n \cdot \nabla \phi(x,t) = 0$$ (2.4)

where $t$ is a fictitious time and $v_n$ represents a velocity field normal to the structural boundary defined on $x \in \partial \Omega$. Many techniques have been implemented to extend
the normal velocity field such that is redefined on the entire design domain \( x \in D \) [29]. The velocity field is constructed using a shape sensitivity analysis to drive the design towards a local optimum [5]. As the solution converges, the velocity field on the structural boundary will converge to zero, and the boundaries will stagnate.

In the simplest and most common cases a fixed Eulerian mesh, as was used in the SIMP method, is implemented while the level set function is continuously changing. Mesh elements intersected by the zero level set \( \phi = 0 \) are given an intermediary density based on the ratio of solid to total area or volume. However, some researchers have developed Lagrangian schemes which re-mesh the solid domain \( \Omega \) at every iteration in the evolution of \( \phi \) and eliminate elements which lie in the void domains \( D \setminus \Omega \) [3]. This approach removes intermediary elements and improves overall accuracy of the method, but suffers from a higher computational cost to iteratively re-mesh the structure domain.

To date, LSMs have been applied to a number of topology optimization problems in 2D and 3D, both with structured and unstructured meshes [85]. These include both single and multi-load static compliance problems [6, 5, 4, 47], minimal stress designs [9], dynamic analyses including the eigenvalue problem for computing natural frequencies [8, 148], aerodynamic loading problems considering buckling constraints [61], and both linear [75, 193, 72, 70] and non-linear [92] heat conduction problems.

Due to the growing popularity of LSMs, several major improvements have been made in the past decade. One of the drawbacks of early level set topology optimization methods were their inability to nucleate holes in the interior of a solid domain. This limitation was due to implementing explicit numerical solvers constrained by the Courant-Friedrichs-Lewy (CFL) condition for solving the Hamilton-Jacobi equation as well as periodically regularizing the level set function to avoid ill-conditioning. To circumvent this problem topological derivatives, defined as the sensitivity of a functional subject to an infinitesimally small hole generated within a material, were introduced. Prior strategies required initializing the design domains with arrays of holes to find locally optimal solutions containing void cutouts. Topological derivatives are typically employed in 2D problems only, as the solutions of 3D problems
tend to be less sensitive to the initial material configuration [4]. A full derivation of this derivative can be found in [12]. Several techniques have been implemented to incorporate topological derivatives into an optimization routine, including augmenting the Hamilton-Jacobi equation [28, 33], performing separate topological gradient steps [4], and ranking mesh elements according to their individual topological sensitivities and removing those with the highest value [71]. Dunning and Kim also developed a novel approach where a secondary level set function is defined solely to nucleate holes in the interior of a 2D material domain using the original shape derivative [58].

Other researchers have worked on implementing implicit schemes to reduce the computational cost of solving the Hamilton-Jacobi equation and avoid time stepping constraints such as the CFL condition. Luo et al. developed a semi-implicit additive operator splitting (AOS) scheme to approximate the solution of the HJE using the average of several one-dimensional solutions obtained through the mathematically efficient “Thomas Algorithm” [105]. This technique has been applied to image processing and computer vision problems where nonlinear diffusion equations are frequently found. Shojaee [147] published a level set topology optimization method also using an AOS scheme where the gradient of $\phi$ was approximated using a series of Delta functions. Bui et al. [26] implemented a numerical scheme based on the method of characteristics to evolve the HJE on an unstructured mesh. This method can be parallelized to further reduce central processing unit (CPU) runtime.

Another area of research in LSMs includes finding strategies to parameterize the level set function in more advantageous ways. For example, Chen et al. [87] devised a novel technique to define a smooth and differentiable level set function by combining both B-splines and parameterized primitives using R-functions. This approach allows a designer the flexibility of updating the structural shape in a parametric, topological or hybrid manner which in turn can improve convergence and retain some design intent. Luo et al. [106] developed a parameterization using radial basis functions which can significantly reduce the number of design variables and allows for mathematical programming algorithms (e.g., MMA) to be employed. Another technique was introduced by Estellers et al. [64] to retain the level set function as a signed distance
function, thereby avoiding ill-conditioning when $\phi$ becomes too steep or flat near the structural boundaries. The authors then solve the problem using a split-Bregman method for efficient computation.

Other attempts to decrease CPU runtime of level set topology optimization algorithms include the development of parallelized architectures, such as those built on graphics processing units (GPUs) [79, 34, 178]. Sivapuram et al. [152] also devised a parallelized decomposition approach to enable multiscale design in a level set topology optimization framework. The macroscale features of the structure are designed concurrently with the material’s microstructure in a select number of user-defined regions.

### 2.3 Advantages of Level Set Methods

Level set based methods offer a number of advantages over SIMP based and evolutionary methods. The main advantage is that level set methods produce a crisp structural boundary which is implicitly defined via the level set function. Because other methods vary the densities of the elements directly, the resulting optimal shapes will oftentimes have gray zones where elements have an intermediate density, and “staircasing” features where material and void elements form a pattern similar to a staircase. These awkward patterns lead to numerical errors and require some form of filtering to produce a smooth boundary.

A well-defined clear structural boundary is advantageous for design-dependent problems where the boundary conditions change during the optimization routine. As an example, Allaire et al. [10] considered a 3D structure subjected to a constant pressure load acting on its surface. As the optimizer updated the boundary of this structure, the pressure load acting normal to the boundary needed to be recomputed. The optimal topology closely resembled a starfish. Thermal radiation is also a surface phenomenon where the shape and orientation of the boundary greatly impacts the amount of heat transferred between the structure and surrounding environment. We argue that a level set based method would be advantageous for future work considering
design-dependent thermal radiation loads.

Moreover, problems where the objective or constraint functions are formulated on the structural boundary are easily handled using the shape gradient derivation. The perimeter [113, 54] and curvature [147] of the boundary can also be incorporated as penalty functions or augmented to the velocity field in order to help regularize the solutions. And finally, level set methods allow for complex topological changes to occur seamlessly. Because the level set function implicitly defines the structural boundary, combining holes, splitting material and merging boundaries are some of the topological transformations that can be easily handled.

2.4 Chapter Summary

In this chapter we have provided some background on the origins of modern structural design optimization which began in the early 1960’s. By the late 1970’s topology optimization became the most versatile method for free-form structural design. Topology optimization typically uses a finite element model to evaluate the objective and constraint functions along with the sensitivities needed to guide the optimizer towards an optimal solution. Three popular topology optimization methods include homogenization, evolutionary and boundary variation methods. SIMP, the most popular homogenization method, employs element densities as the design variables to interpolate the material properties within the design domain. Evolutionary methods are heuristic and while easy to implement and interface with existing commercial finite element software, suffer from slow convergence rates and limited control over the desired material volume fraction. Boundary variation methods implicitly define the structural boundaries to seamlessly handle complex topological changes and avoid grayscale effects such as “checkboarding” which is common in homogenization methods. The level set method defines the structural boundary via a level set function which is updated by the Hamilton Jacobi equation during the optimization routine. Gradient-based schemes typically define an analytic shape sensitivity which is translated into a velocity field used to update the Hamilton Jacobi equation and drive the
design towards a local optimum. A decade’s worth of research has been devoted to advancing level set methods in order to improve accuracy and stability, and reduce computational runtime. We argue that a level set method is the most appropriate for designing efficient thermally radiating structures since thermal radiation in a boundary phenomenon. As such, level set methods offer the advantages of generating crisp structural boundaries and implementing shape sensitivities which can be derived for functionals defined explicitly on the structural boundary.
Chapter 3

Topology Optimization of Thermally Radiating Structures

While a substantial amount of research has been devoted to designing thermally conductive structures, relatively little attention has been given to design problems where the boundary conditions are nonlinear. Thermal radiation is a common example of such a nonlinearity where the heat power radiated is directly proportional to the fourth power of the surface temperature. The Stefan-Boltzmann equation quantifies the heat power radiated from a surface across all wavelengths in the electromagnetic spectrum.

\[ Q_{\text{rad}} = \varepsilon \Upsilon \theta_{\text{surf}}^4 \]  

(3.1)

In Equation (3.1), \( Q_{\text{rad}} \) is the heat power flux radiated in units of W m\(^{-2}\), \( \varepsilon \) is the emissivity, \( \Upsilon \) is the Stefan-Boltzmann constant \( 5.670 \times 10^{-8} \text{W m}^{-2} \text{K}^{-4} \), and \( \theta_{\text{surf}} \) is the temperature field of the radiating surface. The emissivity \( \varepsilon \) varies between 0 and 1 and accounts for the fact that real surfaces do not absorb or emit as much energy as a blackbody (\( \varepsilon = 1 \)). While in reality the emissivity is a function of wavelength, designers oftentimes assume it to be constant as an approximation.

This chapter reviews previous work on optimizing the designs of thermally radiating structures in Section 3.1. These studies generally rely on parametric models which employ a handful of design variables to describe the structural configuration and
thereby limit the design space. An optimization problem is posed in Section 3.2 with a single objective to maximize heat power radiated and a single volume constraint. The problem is reformulated as a series of unconstrained subproblems defined using the augmented Lagrangian method in Section 3.3. Section 3.4 introduces a level set method which defines the structural boundaries implicitly via the level set function and allows for free-form shape design. Several key numerical ingredients such as updating and reinitializing the level set function are presented. Section 3.5 provides an overview of the computational implementation and software architecture used to run the analysis.

### 3.1 Literature Review

Various aerospace engineering applications exist which require lightweight efficient thermally radiating structures. One common application presented in Section 1.1 is the design of spacecraft structural components, such as radiators and thermal insulation blankets. Because spacecraft operate in high vacuum environments, they rely on thermal radiation to bleed waste heat from warm electrical components and maintain thermodynamic equilibrium with the surrounding environment. This section presents an overview of existing research on optimizing the design of radiating structures.

#### 3.1.1 Case Studies Using Parametric Design

Parametric design studies involve exploring a design tradespace or optimizing a structure which can be described by a handful of geometric variables. These variables may include for example, dimensions (e.g., the length, height or width of an element), or number of repeating features (e.g., number of ribs on a reinforced plate). Designers choose this strategy for a number of reasons including (1) the ease of implementation using computational methods, (2) the desire to maintain a specific design intent, and (3) the need to enforce geometric constraints which substantially limit the design space. For these reasons parametric studies are common in structural design applications and provide valuable insight into the trade-offs between weight and performance.
Contributions to spacecraft radiator design date back as far back as the 1960’s. Cockfield [40] performed a parametric analysis of tube radiators using two geometric variables to minimize weight. Cuco et al. [43] applied a range of heuristic algorithms to perform a multi-objective optimization of a variable emittance space radiator using an algebraic thermal model with five variables. Kim et al. [93] also employed a similar one-dimensional thermal model of a honeycomb radiator and analytically derived optimal solutions minimizing mass per unit heat rejected by linearizing the governing energy equation. Sam and Deng [140] employed a lumped parameter thermal model to minimize the mass of a heat pipe network for transferring heat from a communications satellite payload to a radiating panel. The model was solved using Thermal Desktop, an industry standard software tool for spacecraft thermal analysis, and two gradient-based optimizers embedded in the SINDA/Fluint software package. Kim et al. [91] also employed the same analysis tools on a low-fidelity model to minimize radiator surface area for a cube-shaped satellite. The problem was formulated as an integer program and solved using the Non-dominated Sorting Genetic Algorithm (NSGA)-II multi-objective optimizer. Krishnaprakas [98] solved a two-dimensional radiator problem consisting of a cylindrical core with radiating fins. The radiative heat transfer between different surfaces on the same structure, known as irradiation, was also included and required that view factors were updated as the geometry changed during the optimization routine. Krikkis and Razelos [96, 97] have also performed optimizations on similar tubular structures with radiating fins while incorporating irradiation. Both Krishnaprakas, and Krikkis and Razelos developed analytic thermal models and used a handful of design variables to define the geometry and number of fins. Bialecki et al. [22] also considered an array of radiating fins where thermal conduction, radiation, and thermoelasticity were considered. The approach employed a boundary element method to solve the governing partial differential equations (PDEs) and a genetic algorithm to maximize the dissipated flux and minimize surface area considering a total of 14 design variables.

Researchers have also performed parametric design studies on multilayer insula-
tion for spacecraft, which comes in many forms from flexible blankets to rigid honeycombed foam-filled structures. Daryabeigi [46] used a design-of-experiments approach to minimize the weight of a multilayer thermal protection system for reentry vehicles with three design variables. A finite volume numerical model was developed and validated with published test data. Xie et al. [179] investigated a similar problem but with a higher fidelity finite element model and included four design variables. A globally convergent method of moving asymptotes was used to find the minimal mass solution subject to maximum temperature and stress constraints.

### 3.1.2 Case Studies Using Topology Optimization

Topology optimization offers the designer much greater flexibility in the geometric shape of the structure under consideration. Typically a finite element model is employed to evaluate the objective and constraint functions, in addition to any sensitivities needed for implementing a gradient-based optimizer. A density value is assigned to each element of the mesh which interpolates the stiffness, conductivity or other material property throughout the design domain. Because the number of design variables in topology optimization problems scales directly with the number of nodes or elements in the finite element model, they tend to be much larger in dimensionality than parametric design problems. However, topology optimization offers much greater design freedom since a nearly limitless number of shapes can be obtained. The reader is referred to Section 2.2 for a review of topology optimization methods. Only a handful of papers can be found which implement a topology optimization method to design a thermally radiating structure.

Castro et al. [31] investigated the design of radiative enclosures by redistributing reflective material using a SIMP-based approach. The 2D element densities variables were used to interpolate surface reflectivity within the enclosure. The authors then calculated the view factors for each element and used a finite element method coupled with an iterative solver to determine the temperature and heat flux fields within the enclosures.

Munk et al. [118] used an evolutionary structural topology optimization approach
on a three-dimensional panel model of a hypersonic aircraft wing incorporating aerothermoelastic effects. The evolutionary optimization method was a hard-kill bi-directional method, meaning that element densities were integer variables and that void elements were excluded from the finite element and radiation analysis altogether. In addition only radiation cooling from the skin of the wing to the ambient surroundings was considered and internal irradiation effects were ignored. The objective was to minimize weight and stress.

In both the above papers the radiating surface itself never changed topologically. The evolutionary algorithm used by Munk et al. is a heuristic approach which removes material based on optimality criteria that are tuned by the designer. Finally only Castro et al. considers the heat radiated or reflected by regions of the interior enclosure as an objective function. Munk et al. does not consider the radiative heat transfer as an objective or constraint function explicitly.

### 3.2 Generalized 2D Single Objective Thermal Radiation Problems

We consider 2D problems where a surface is radiating solely to an ambient environment and a fixed temperature boundary condition is enforced. We apply the Dirichlet boundary condition to an edge segment on the structural boundary and the radiating surface is defined as a fixed sub-domain. Since a level set method is applied to solve the thermally radiating problems presented in this thesis, the notation defined in Section 2.2.4 is adopted to describe the geometry. Figure 3-1 illustrates the design domain $\mathcal{D}$, structure domain $\Omega$, radiating sub-domain $\Omega_{\text{rad}}$ and structure boundary $\partial\Omega$. Note that the structural boundary consists of two disjoint parts: the Dirichlet $\Gamma_{D,\theta}$, and insulating $\Gamma_{0,\theta}$ boundaries.

$$\partial\Omega = \Gamma_{D,\theta} \cup \Gamma_{0,\theta}$$ (3.2)
The Dirichlet and insulating boundaries do not intersect. Also note that the structure domain always remains a subset of the design domain \((\Omega \subset D)\) as does the radiating sub-domain always remain a subset of the structure domain \((\Omega_{\text{rad}} \subset \Omega)\). These relations are illustrated in Figure 3-1.

![Figure 3-1: Structural domains and boundaries for thermal radiation problem](image)

In this thesis we define a thermally efficient radiating structure as one that can exchange as much heat power between itself and its surroundings as possible provided some restriction on material or weight. Recall that the heat power radiated by a surface is governed by the Stefan-Boltzmann law stated in Equation (3.1). The total heat power radiated (in units of watts W) is calculated by integrating Equation (3.1) over the radiating surface \(\Omega_{\text{rad}}\). The heat power radiated functional \(J(\Omega_{\text{rad}}, \theta)\) is written as follows:

\[
J(\Omega_{\text{rad}}, \theta) = \int_{\Omega_{\text{rad}}} \epsilon \tau \theta^4 \, dA
\]  

(3.3)

where \(\theta = \theta(x)\) is the temperature field within the design domain \(x \in \mathcal{D}\). Since \(\Omega_{\text{rad}} \subset \mathcal{D}\)
\( \Omega \), the objective function \( J(\Omega_{\text{rad}}, \theta) \) can also be defined without loss of generality as follows:

\[
J(\Omega, \theta) = \int_{\Omega} \chi \varepsilon \gamma \theta^4 \, dA 
\]

(3.4)

where \( \chi \) is a domain flag defined in Equation (3.5).

\[
\chi = \begin{cases} 
1 & \text{for } x \in \Omega_{\text{rad}} \\
0 & \text{for } x \in \mathcal{D} \setminus \Omega_{\text{rad}} 
\end{cases}
\]

(3.5)

The reason why the objective is transformed in this way will be apparent when the notion of shape derivatives are introduced in Section 3.4.4.

To limit the amount of material that can be used to design the structure, a volume fraction equality constraint is enforced. The volume fraction is defined as the volume of the structure domain \( \Omega \) normalized by the volume of the design domain \( \mathcal{D} \). Thus the equality constraint function is written as follows:

\[
h(\Omega) = \frac{\int_{\Omega} dV - V_{\text{req}}}{\int_{\mathcal{D}} dV} 
\]

(3.6)

where \( V_{\text{req}} \) is the required material volume. The integrals in the numerator and denominator represent the material and total design domain volumes respectively.

Finally, the governing partial differential equation defined by Fourier's law of conduction coupled with the Dirichlet boundary and radiation conditions is written in Equation (3.7).

\[
\begin{align*}
\text{div} \left( \kappa \nabla \theta \right) &= \varepsilon \gamma \theta^4 & \text{in } \Omega_{\text{rad}} \\
\text{div} \left( \kappa \nabla \theta \right) &= 0 & \text{in } \Omega \setminus \Omega_{\text{rad}} \\
(\kappa \nabla \theta) \cdot n &= 0 & \text{on } \Gamma_{0,\theta} \\
\theta &= \theta_0 & \text{on } \Gamma_{D,\theta}
\end{align*}
\]

(3.7)

In Equation (3.7), \( \kappa \) is the second order thermal conductivity tensor, and \( \theta_0 \) is the prescribed temperature on the Dirichlet boundary \( \Gamma_{D,\theta} \). We assume that the 2D elements have a uniform thickness of 1 and thus the radiating term \( \varepsilon \gamma \theta^4 \) on the first line of Equation (3.7) is in units of W m\(^{-2}\). For isotropic materials the thermal
conductivity tensor reduces to a single term $\rho$ defined in units of $\text{W m}^{-1} \text{K}^{-1}$ and also assumed to be invariant with temperature.

The above objective and constraint functions along with the governing PDE can now be formally written as the following optimization problem (SP1):

$$
\begin{align*}
\text{(SP1):} \quad \max_{\Omega} & \int_{\Omega} \chi \varepsilon \theta \partial^{4} dA \\
\text{subject to} & \int_{\Omega} \frac{dV - V_{\text{req}}}{dV} = 0 \\
& \text{div} (\kappa \nabla \theta) = \varepsilon \theta \partial^{4} \quad \text{in} \quad \Omega_{\text{rad}} \\
& \text{div} (\kappa \nabla \theta) = 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{rad}} \\
& (\kappa \nabla \theta) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta} \\
& \theta = \theta_{0} \quad \text{on} \quad \Gamma_{D,\theta}
\end{align*}
$$

The volume fraction equality constraint is added to formulate a well-posed topology optimization problem. Without this constraint, the optimizer would select a fully filled material design domain to maximize the radiated heat power. While the volume fraction could also be posed as an inequality constraint, any solution with an inactive volume fraction inequality constraint could be improved by the addition of material to further increase the thermal conductivity of the structure. The augmented Lagrangian method used to solve (SP1) is introduced next.

### 3.3 Augmented Lagrangian Method

A common way to solve constrained optimization problems is to replace the original problem by a series of unconstrained subproblems which are iteratively solved until a converged solution is achieved. Each subproblem seeks to minimize a single equation constructed by adding terms derived from the constraint functions to the original objective function [121]. Example methods which uniquely define these subproblems include penalty methods, interior point methods, method of Lagrange multipliers and augmented Lagrangian methods.

We select an augmented Lagrangian method to solve optimization problem (SP1). The augmented Lagrangian combines both the Lagrangian function and a penalty
function derived from the equality constraint. Equation (3.8) defines the augmented Lagrangian:

\[
L(\Omega, \theta, \lambda, \mu) = J(\Omega, \theta) + \lambda h(\Omega) + \frac{\mu}{2} [h(\Omega)]^2
\]  

(3.8)

where \( J(\Omega, \theta) \) is the original objective function found in Equation (3.4), \( h(\Omega) \) is the equality constraint function found in Equation (3.6), \( \lambda \) is the Lagrange multiplier estimate, and \( \mu \) is the penalty parameter. The unconstrained subproblem to be solved at every iteration \( k \) during the optimization routine is then:

\[
\arg \min_{\Omega} L(\Omega, \theta(\Omega), \lambda^k, \mu^k)
\]  

(3.9)

where the superscript \( k \) denotes the current iteration. Recall that the optimizer is changing the structural boundaries via updates to the level set function. However, for simplicity we indicate \( \Omega \) as the design variable describing the structural shape. Also, we have defined the temperature field as a function of the structural domain \( \theta = \theta(\Omega) \) since at every iteration at least one finite element solve is performed using the current shape \( \Omega \) for the temperature field. More details on this step are provided in Section 3.4.2. The Lagrange multiplier estimate and penalty parameter are also iteratively updated to drive the design towards a feasible solution where \( \lambda \) approaches the Lagrange multiplier \( \lambda^* \) that satisfies the Karush-Kuhn-Tucker (KKT) conditions. The Lagrange multiplier estimate is updated at each optimization iteration \( k \) according to Equation (3.10) derived from the optimality conditions for unconstrained minimization of \( L \) [121].

\[
\lambda^{k+1} = \lambda^k + \mu^k h(\Omega^k)
\]  

(3.10)

The penalty parameter is also iteratively updated according to Equation (3.11):

\[
\mu^{k+1} = \begin{cases} 
\mu^k & \text{if } \mu^k = \mu_{\text{max}} \text{ or } \|h(\Omega^{k+1})\| < \beta_{\mu}\|h(\Omega^k)\| \\
\gamma_\mu \mu^k & \text{otherwise}
\end{cases}
\]  

(3.11)

where \( \mu_{\text{max}} > 0 \) is the upper bound on the penalty parameter, \( \beta_{\mu} \) is the update tolerance on the equality constraint such that \( 0 < \beta_{\mu} \leq 1 \), and \( \gamma_\mu > 1 \) is the penalty
parameter update factor. Throughout the optimization routine, the penalty parameter is monotonically increasing. Keeping $\mu$ bounded, however, avoids ill-conditioning and numerical instabilities. From Equation (3.10), it is evident that if $\mu$ becomes too large, the Lagrange multiplier updates can become too aggressive and drive the optimizer unstable. Appropriate choices for $\mu_{\text{max}}, \beta_\mu$, and $\gamma_\mu$ depend on several factors including the convexity of the problem and the choice of optimizer [121].

Algorithm 1 provides a general overview of how the augmented Lagrangian method in our framework is implemented. Superscripts indicate the optimizer iteration and $\Omega^*$ is the optimal structural domain. Recall that the level set method defines the structural boundary $\Omega$ using the level set function $\phi$ according to Equation (2.3). We employ the finite element method to numerically solve the governing equations (3.7), and evaluate the objective and constraint functions in lines 1–3. The level set function is also computed at the node points of the finite element mesh using the same basis functions defining the temperature solution $\theta$. More details regarding our implementation of the level set function is provided in Section 3.4.

The approximate minimizer to $L(\Omega, \theta(\Omega), \lambda^k, \mu^k)$ found on line 6 can be determined using a variety of methods, several which will be discussed in Section 3.5. Iterative updates to the augmented Lagrangian requires solving the finite element equations, and evaluating the objective and constraint functions. This sequence of instructions is identical to that from lines 1–4, and for the sake of brevity is not explicitly rewritten on line 6. An initial starting condition $\Omega_{s}^k$ is also needed to solve for the approximate minimizer. In order for the updated shape $\Omega^k$ to be accepted, the augmented Lagrangian must have sufficiently decreased according to the condition on line 7. The tolerance $\beta_L^k$ is specified by the user for each iteration up to $k_{\text{max}}$ and can thus be changed as the optimizer progresses. If a new augmented Lagrangian $L(\Omega^k, \theta^k, \lambda^k, \mu^k)$ is found, then the algorithm continues by updating the estimated Lagrange multipliers $\lambda^{k+1}$ and penalty parameters $\mu^{k+1}$, and evaluating the convergence criteria. If an approximate minimizer cannot be found which satisfies the condition on line 7, then the algorithm terminates. For more details on augmented Lagrangian methods the reader is referred to [23] and [121].
Algorithm 1: Augmented Lagrangian method for solving topology optimization problem (SP1)

Input: Initial structural domain \( \Omega_0 \) and design domain \( \mathcal{D} \)
- Maximum number of iterations \( k_{\text{max}} \)
- Initial Lagrange multiplier estimate \( \lambda^0 \) and penalty parameter \( \mu^0 \)
- Augmented Lagrangian tolerances \( \beta_L^k \) for \( k = 1, \ldots, k_{\text{max}} \)
- Update parameters \( \mu_{\text{max}}, 0 < \beta_\mu \leq 1 \) and \( \gamma_\mu > 1 \) for penalty \( \mu \)

Output: Optimal structural domain \( \Omega^* \)

1. Calculate temperature solution \( \theta_0 \) to the governing equations (3.7) using \( \Omega_0 \)
2. Evaluate objective function \( J^0 = J(\Omega_0, \theta_0) \) using Equation (3.4)
3. Evaluate constraint function \( h^0 = h(\Omega_0) \) using Equation (3.6)
4. Evaluate augmented Lagrangian \( L(\Omega_0, \theta_0, \lambda^0, \mu^0) \) using Equation (3.8)

for \( k \leftarrow 0 \) to \( k_{\text{max}} \) do

8. Find approximate minimizer \( \Omega^k \approx \arg\min_{\Omega} L(\Omega, \theta(\Omega), \lambda^k, \mu^k) \) starting at \( \Omega_0 \)

9. if \( L(\Omega^k, \theta^k, \lambda^k, \mu^k) \leq \beta_L^k L(\Omega^k, \theta^k, \lambda^k, \mu^k) \) then

12. Calculate new Lagrange multiplier estimate \( \lambda^{k+1} \) using Equation (3.10)
13. Calculate new penalty parameter \( \mu^{k+1} \) using Equation (3.11)
14. \( \Omega^{k+1} \leftarrow \Omega^k \) and \( \theta^{k+1} \leftarrow \theta^k \)
15. Evaluate new augmented Lagrangian \( L^k(\Omega^{k+1}, \theta^{k+1}, \lambda^{k+1}, \mu^{k+1}) \)
16. Evaluate convergence criteria
17. if convergence criteria are satisfied then

19. Return to top of the for loop
20. else
21. Return to top of the for loop
22. end
23. end
24. \( \Omega^* \leftarrow \Omega^k \)
25. return \( \Omega^* \)
3.4 Level Set Method

A level set method is selected to define the structural boundaries implicitly via the level set function $\phi$. Several key advantages of using level set methods for solving topology optimization problems were introduced in Section 2.3. This section outlines the methodology of defining, updating and regularizing the level set function, and translating the shape derivative to the normal velocity field so that it drives the design towards a local optimum.

3.4.1 Defining the Level Set Function

The level set function $\phi = \phi(x)$ is a scalar valued function defined on the design domain $x \in \mathcal{D}$ illustrated in Figure 3-1. Equation (2.3), reproduced below, defines the structure domain $\Omega$ where an isotropic material exists and the void domain $\mathcal{D} \setminus \Omega$ where no material exists.

$$\begin{align*}
\phi(x) < 0 & \quad x \in \Omega \\
\phi(x) = 0 & \quad x \in \partial \Omega \\
\phi(x) > 0 & \quad x \in \mathcal{D} \setminus \Omega
\end{align*}$$

(3.12)

The boundary $\partial \Omega$ between the material and void domains can be calculated using a variety of interpolation techniques employing an approximate Heaviside function [168]. In our approach, however, the level set function is interpolated within each element $e$ via the shape functions $N_i^e$ from the finite element mesh used to analyze the structure:

$$\phi(x) = \sum_{i=1}^{\ell} N_i^e(x) \phi_i^e \quad \text{for } x \in \mathcal{T}_e$$

(3.13)

In Equation (3.13) $i = \{1, 2, \ldots, \ell\}$ where $\ell$ is the total number of shape functions (and also the number of nodes) in element $e$, $\phi_i^e$ is the $i^{th}$ nodal value of the level set function in element $e$, and $\mathcal{T}_e$ is the domain of element $e$. Recall from Section 3.1.2 that topology optimization methods generally rely on a finite element model to evaluate the objective and constraint functions. For our approach, we use a finite element model to perform these evaluations as well as to define the level set function. Thus
the structure boundary $\partial \Omega$ is defined within each element cut by $\phi = 0$ by solving the following equation for $x$:

$$\sum_{i=1}^{\ell} N_i^e(x) \phi_i^e = 0$$

(3.14)

For quadratic and higher order elements, a Heaviside function is a useful approximation for determining the structural boundary $x \in \partial \Omega$ within each element. However, this approximation will introduce numerical errors and care must be taken to devise a suitable function which balances numerical error and smoothness (since the Heaviside function itself is non-differentiable). In this thesis, we use only linear finite elements and thus the structural boundary is piecewise linear and hence straightforward to calculate. Next we discuss how elements cut by the structural boundary are handled, including their contributions to the global finite element equations.

### 3.4.2 Ersatz Material Technique

Two popular strategies for handling elements cut by the zero level set $\phi = 0$ are (1) the Eulerian approach and (2) the Lagrangian approach. In this thesis we adopt an Eulerian approach where the mesh for the design domain $\mathcal{D}$ is kept fixed for the entirety of the optimization procedure. The Ersatz material technique then dictates that elements cut by $\phi = 0$ are assigned an interpolated value for the material stiffness, conductivity or other relevant physical property. In the Lagrangian approach the structural domain $\Omega$ is re-meshed at each iteration in the optimizer such that all elements are either solid material or completely void. Void elements are removed entirely from the finite element model. Recent work by Dapogny et al. [44] and Allaire et al. [7] has shown that Lagrangian methods improve accuracy but suffer from the added computational effort needed to perform a meshing operation at every iteration. Using the Ersatz material approach, a density value $\rho_e$ is assigned to each element $e$ similar to the design variables used in a SIMP based approach. These values are used to interpolate the material properties within cut elements and are calculated based on the volume fraction within element $e$ lying inside the structure.
domain according to Equation (3.15):

\[
\rho_e = \frac{|\Omega_e|}{|\mathcal{T}_e|}
\]  

(3.15)

where \(\Omega_e\) is the material sub-domain of element \(e\) defined as \(\Omega_e := \{x \mid x \in \Omega \cap \mathcal{T}_e\}\), \(|\Omega_e|\) is the area of \(\Omega_e\), and \(|\mathcal{T}_e|\) is the total element area. According to Equation (3.15), \(\rho_e\) is the volume fraction of solid material within element \(e\).

To define the sub-domain \(\Omega_e\) for element \(e\), a mapping \(M\) is needed which transforms the nodal values of the level set function \(\phi_i^e\) to the material boundary \(\partial\Omega_e\):

\[
M : \phi_i^e \mapsto \partial\Omega_e
\]  

(3.16)

Recall from Section 3.4.1 that Equation (3.14) provides an analytic expression for defining the zero level set using the element shape functions. Using this relation along with the known edges for each element, we have fully defined the material boundary \(\partial\Omega_e\). Subsequently it is straightforward to calculate the element density using Equation (3.15). Clearly for elements where the level set nodal values are all positive (void), \(\rho_e = 0\) and conversely where the nodal value are all non-positive (solid), \(\rho_e = 1\).

\[
\begin{align*}
\rho_e &= 0, \quad \text{if } \forall i = 1, \ldots, \ell \quad \phi_i^e > 0 \\
\rho_e &= 1, \quad \text{if } \forall i = 1, \ldots, \ell \quad \phi_i^e \leq 0 \\
\rho_e \in (0, 1), & \quad \text{otherwise}
\end{align*}
\]  

(3.17)

The element densities \(\rho_e\) linearly interpolate the element stiffness, mass and/or conductivity matrices (depending on the PDEs to be solved). The global stiffness, mass and/or conductivity matrices are assembled according to the following equation:

\[
\begin{align*}
K &= \sum_{e=1}^{m} \max \left( \delta, \frac{|\Omega_e|}{|\mathcal{T}_e|} \right) K_e & \text{(stiffness)} \\
M &= \sum_{e=1}^{m} \max \left( \delta, \frac{|\Omega_e|}{|\mathcal{T}_e|} \right) M_e & \text{(mass)} \\
L &= \sum_{e=1}^{m} \max \left( \delta, \frac{|\Omega_e|}{|\mathcal{T}_e|} \right) L_e & \text{(conductivity)}
\end{align*}
\]  

(3.18)
where $\delta$ is a small positive number, $K_e$, $M_e$ and $L_e$ are the element stiffness, mass and conductivity matrices respectively and $m$ is the total number of elements in the mesh. If an element density lies in the range $0 \leq \rho_e \leq \delta$, then the element matrix is multiplied by $\delta$ to avoid ill-conditioning or generating a singular global matrix. The smaller the value of $\delta$ the more ill-conditioned the global matrix becomes. However, the larger the value of $\delta$, the more a void element maintains its capacity to support a load or heat flux. Hence the analyst must make a judicious choice which balances numerical stability and model accuracy.

### 3.4.3 Solving the Hamilton Jacobi Equation

The level set function defined in Equation (3.12) is updated at each optimization iteration by solving the Hamilton-Jacobi equation using a fictitious time $t$ and advection velocity field $v_n$ normal to the structural boundary. The Hamilton Jacobi equation is restated below:

$$ \frac{\partial \phi(x,t)}{\partial t} + v_n \cdot \nabla \phi(x,t) = 0 $$ (3.19)

Several metrics can be computed using the level set function $\phi$, such as the boundary normal $n = \nabla \phi / |\nabla \phi|$ and curvature $H = \text{div} \ (\nabla \phi / |\nabla \phi|)$. The normal advection velocity $v_n$ can therefore be rewritten as a vector with magnitude $v$ and direction $n$:

$$ v_n = v \frac{\nabla \phi}{|\nabla \phi|} $$ (3.20)

Equation (3.20) is substituted into Equation (3.19) to yield the following form of the Hamilton Jacobi equation.

$$ \frac{\partial \phi}{\partial t} + v |\nabla \phi| = 0 $$ (3.21)

Equation (3.21) can be solved using a variety of upwind finite difference schemes for structured meshes [144, 10, 30] as well as for unstructured meshes [144, 192] and more recently using the method of characteristics [7]. We choose a second order upwind scheme to solve Equation (3.21) on a structured Cartesian grid. Details regarding this scheme can be found on page 66 of [144]. Neumann type boundary conditions
are implemented and written as follows:

\[
\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \partial D
\]  

(3.22)

where \( \partial D \) is the boundary of the design domain. While there is no clear correct choice for the level set function boundary conditions, the Neumann type seem to be the most natural and most common \([8, 168]\). This boundary condition alleviates any artificial effect that the design domain boundary would have on the structural boundary. If for example, a Dirichlet boundary condition were used instead, then it would be possible for spurious voids to be created near the design domain boundary due to numerical error.

### 3.4.4 Constructing a Velocity Field for the Hamilton Jacobi Equation

The final ingredient in solving the Hamilton Jacobi equation defined in (3.21) is to construct an appropriate velocity field \( v \). Recall in optimization problem \((SPI)\) that the design variable is presented as the structural shape \( \Omega \). Since a level set function implicitly defines the structural shape according to Equation (3.12), the variable \( \Omega \) is in fact a function of \( \phi \).

\[
\Omega = \Omega(\phi)
\]  

(3.23)

As such, the velocity field should alter the structural shape to minimize the augmented Lagrangian stated in Equation (3.8). In order to employ a gradient-based optimizer, we require sensitivity information of the augmented Lagrangian to perturbations in the structural boundary. This sensitivity information is commonly referred to as a shape derivative.

The notion of shape derivatives goes as far back as Hadamard \([76]\) and was extensively studied in the mid-1970s to 1980s by many researchers including Murat \([119]\) and Simon \([151]\), Sokolowski and Zolesio \([153]\), Céa \([32]\), Rousselet \([137]\), and Haug \([78]\). The definition of a shape derivative is summarized here but the interested reader
The shape derivative of a functional $J$ is defined by considering a perturbation in the current shape $\Omega^0$ to a deformed shape $\Omega^t$. The perturbation is carried out by a transformation $T_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined in Equation (3.24) where $\mathbb{R}^d$ represents the dimensionality of the shape (i.e., $d = 2$ for 2D or $d = 3$ for 3D).

$$T_t = I + \xi \quad \text{(3.24)}$$

In Equation (3.24), $I$ is the identity mapping in $\mathbb{R}^d$ and $\xi$ is a smooth vector field defined over $\mathbb{R}^d$ which meets the regularity conditions $\xi \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ where $W^{1,\infty}$ is a Sobolev space. This transformation $\Omega^t = T_t(\Omega^0)$ is illustrated in Figure 3-2. The shape derivative is defined as the Fréchet derivative $D\mathcal{J}$ of the shape functional $\mathcal{J}$ according to the following equation:

$$\mathcal{J}(\Omega^t) = \mathcal{J}(\Omega^0) + D\mathcal{J}(\Omega^0)(\xi) + o(\|\xi\|) \quad \text{such that} \quad \lim_{\xi \rightarrow 0} \frac{|o(\xi)|}{\|\xi\|} = 0 \quad \text{(3.25)}$$

is referred to [153] for more details on shape optimization and shape derivatives.

Figure 3-2: Shape perturbation of a domain $\Omega^0$ by transformation $T_t$.
Two common types of functionals encountered in shape optimization problems are domain and boundary integrals. These generalized functionals and their shape derivatives are summarized in [10] and restated here.

Consider the function $F(x) \in W^{1,1}(\mathbb{R}^d)$ for $x \in \Omega$, used to define the following domain integral functional $\mathcal{J}(\Omega)$:

$$\mathcal{J}(\Omega) = \int_{\Omega} F(x) \, dV$$  \hspace{1cm} (3.26)

The shape derivative of Equation (3.26) is defined as follows:

$$DJ(\Omega)(\xi) = \int_{\partial \Omega} \xi \cdot n(F(x)) \, ds$$  \hspace{1cm} (3.27)

where $n$ is the normal to the boundary $\partial \Omega$ and $\xi \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Similarly consider the functional $G(x) \in W^{2,1}(\mathbb{R}^d)$ for $x \in \partial \Omega$, used to define the following boundary integral functional $\mathcal{J}(\Omega)$:

$$\mathcal{J}(\Omega) = \int_{\partial \Omega} G(x) \, ds$$  \hspace{1cm} (3.28)

The shape derivative for Equation (3.26) is defined as follows:

$$DJ(\Omega)(\xi) = \int_{\partial \Omega} \xi \cdot n \left( \frac{\partial G(x)}{\partial n} + HG(x) \right) ds$$  \hspace{1cm} (3.29)

where $H$ is the curvature of the boundary. Note that both Equations (3.27) and (3.29) can be written in the following generic form:

$$DJ(\Omega)(\xi) = \int_{\partial \Omega} \xi \cdot n(U) \, ds$$  \hspace{1cm} (3.30)

where $U$ is scalar vector field defined on $x \in \mathcal{D}$. The goal then is to define a perturbation $\xi$ that will drive the unconstrained subproblem in Equation (3.9) towards a local minimum. A common strategy is to define the perturbation as a steepest descent direction of the augmented Lagrangian function:

$$\xi = -Un$$  \hspace{1cm} (3.31)
Substituting Equation (3.31) into Equation (3.30), the shape functional is guaranteed to decrease:

\[ D\mathcal{J}(\Omega)(-Un) = \int_{\partial\Omega} -Un \cdot n(U) \, ds = \int_{\partial\Omega} -U^2 \, ds \]  

(3.32)

Since the level set function which defines the shape \( \Omega \) is updated via the Hamilton-Jacobi equation found in (3.21), the normal component of the perturbation field is set as the velocity field \( v \).

\[ v = -Un \cdot n = -U \]  

(3.33)

Using Equation (3.33), we can construct the velocity field by evaluating the integrand of the augmented Lagrangian shape derivative at all points \( x \in \mathcal{D} \). By calculating the velocity field at all points \( x \in \mathcal{D} \) and not just on the structural boundary \( x \in \partial\Omega \), we implement a natural velocity extension. This allows us to use the Hamilton-Jacobi equation to update the level set function at all points in the design domain.

Using the chain rule and the fact that the Fréchet derivative is a linear operator, the shape derivative of the augmented Lagrangian defined in Equation (3.8) is written as follows:

\[ DL(\Omega, \theta, \lambda, \mu)(\xi) = DJ(\Omega, \theta)(\xi) + [\lambda + \mu h(\Omega)] Dh(\Omega)(\xi) \]  

(3.34)

where \( Dh(\Omega)(\xi) \) is the shape derivative of the volume fraction functional \( h(\Omega) \) defined in Equation (3.6). The shape derivative of this functional is well known [10, 33] and restated below:

\[ Dh(\Omega)(\xi) = \frac{\int_{\Omega} \xi \cdot n \, ds}{\int_{\mathcal{D}} dV} \]  

(3.35)

Analytic expressions for shape sensitivities of other common functions such as structural compliance, target displacement, fundamental natural frequency and stress exist in numerous papers including [10, 8, 9]. However, the shape derivative for the thermal heat power radiated objective function defined in Equation (3.4) has not yet been derived. This derivation is presented in Chapter 4.
3.4.5 Level Set Function Reinitialization

To construct the velocity field and update the level set function using the Hamilton-Jacobi equation, it is necessary to accurately calculate the boundary normal \( n(x) \) and/or the boundary curvature \( H(x) \) for \( x \in \mathcal{D} \) using the level set function. If the level set function becomes too steep or flat, this can lead to numerical errors which yield poor approximations of \( n(x) \) and \( H(x) \). To regularize the level set function, it is necessary to periodically reinitialize it as a signed distance function [10, 173, 168]. This function is reproduced from [45] and presented in Equation (3.36) below:

\[
D_\varepsilon(x) = \begin{cases} 
-d(x, \partial \Omega) & \text{if } x \in \Omega \\
0 & \text{if } x \in \partial \Omega \\
d(x, \partial \Omega) & \text{if } x \in \mathcal{D} \setminus \Omega 
\end{cases} \tag{3.36}
\]

where \( d(x, \partial \Omega) \) is the shortest distance between the point \( x \) and the boundary \( \partial \Omega \).

Several methods exist to reinitialize the level set function and generally implement one of two approaches: (1) solving a boundary value problem where the distance is known in a small region around the structural boundary \( \partial \Omega \) and (2) iteratively solving an unsteady problem in fictitious time \( t \) via a propagation technique to extend the signed distance field from the boundary \( \partial \Omega \). While there are benefits and disadvantages to both methods [45], we select the former approach in our topology optimization framework.

The boundary value problem that needs to be solved in order to reinitialize \( \phi \) to the signed distance function is stated below:

\[
|\nabla \phi(x)| = 1 \\
\phi(x) = 0 \quad \forall x \in \partial \Omega \tag{3.37}
\]

Equation (3.37) is a special case of the Eikonal equation where the speed function is unity for all points in the domain. A popular method for solving this equation was introduced by Sethian in [144] known as the fast marching method. The details
of this method can be found on pages 86 – 100 of [144]. A summary is provided in
Algorithm 2.

\begin{algorithm}
\caption{Fast marching method for solving the Eikonal equation $|\nabla \phi| = 1$
on a structured 2D grid}
\begin{algorithmic}[1]
\STATE \textbf{Input} : Level set function $\phi$
\hspace{1em} Node position vector $x_{\text{nodes}}$ of size $p \times 2$
\STATE \textbf{Output}: Reinitialized level set function vector $\phi_{\text{reinit}}$
\STATE 1 Calculate the set of points $x_{\phi=0}$ along the element edges where $\phi = 0$
\STATE 2 Identify the set of nodes $p_{\text{known}}$ nearest to the points $x_{\phi=0}$
\STATE 3 Calculate $\phi_{\text{reinit}}[p_{\text{known}}]$, the distance between $p_{\text{known}}$ and $\phi = 0$
\STATE 4 Identify the set of trial nodes $p_{\text{trial}}$ that are neighboring $p_{\text{known}}$
\FOR {$i \leftarrow 1$ \TO number of nodes in the set $p_{\text{trial}}$}
\STATE 5 Identify the set of points $x_{\phi=0}$ along the element edges where $\phi = 0$
\STATE 6 Identify the neighboring nodes to $p_{\text{trial}}[i]$, the $i$th trial node
\STATE 7 Calculate $\phi_{\text{reinit}}[p_{\text{trial}}[i]]$, the distance between $p_{\text{trial}}[i]$ and $\phi = 0$ by using
\hspace{1em} the second order finite difference scheme to solve $|\nabla \phi| = 1$
\ENDFOR
\WHILE {the number of nodes in the set $p_{\text{trial}} \neq 0$}
\STATE 8 Identify the smallest element in $\phi_{\text{reinit}}[p_{\text{trial}}]$, $i_{\text{min}} \leftarrow \arg \min_i \phi_{\text{reinit}}[p_{\text{trial}}[i]]$
\STATE 9 Move the element $p_{\text{trial}}[i_{\text{min}}]$ to the set $p_{\text{known}}$
\STATE 10 Identify the neighboring nodes $p_{\text{neighbors}} \notin p_{\text{known}}$ to $p_{\text{trial}}[i_{\text{min}}]$
\STATE 11 Add the set $p_{\text{neighbors}}$ to the set $p_{\text{trial}}$
\STATE 12 Delete $p_{\text{trial}}[i_{\text{min}}]$
\FOR {$i \leftarrow 1$ \TO number of nodes in $p_{\text{neighbors}}$}
\STATE 13 Identify the neighboring nodes to $p_{\text{neighbors}}[i]$
\STATE 14 Calculate $\phi_{\text{reinit}}[p_{\text{neighbors}}[i]]$, the distance between $p_{\text{neighbors}}[i]$ and $\phi = 0$
\hspace{1em} by using the second order finite difference scheme to solve $|\nabla \phi| = 1$
\ENDFOR
\ENDWHILE
\STATE 16 \textbf{return} $\phi_{\text{reinit}}$
\end{algorithmic}
\end{algorithm}
Symbols in boldface represent vectors and arrays. The level set function is represented as $\phi$, the $p \times 1$ vector of nodal values of $\phi$, where $p$ is the number of nodes in the mesh. The arrays $x_{\text{nodes}}$ and $x_{\phi=0}$ include coordinates $(\hat{x}, \hat{y})$ for the node locations and points where the level set function crosses zero in the design domain $D$, respectively. The vectors $p$ include node indices belonging to the set $[1, ..., p]$. Brackets $[\cdot]$ are used as an indexing notation (e.g., $p[1]$ is the first element of the vector $p$). The neighboring nodes identified in lines 4, 6, 12, and 16 are predetermined using a stencil for Cartesian structured meshes. We implement a second order finite difference upwinding scheme to compute the signed distance function between neighboring nodes. However when points are not available to calculate a second order derivative (e.g., near the design domain boundary), a first order scheme is used instead. In addition, the same Neumann type boundary conditions in Equation (3.22) used for solving the Hamilton Jacobi equation are also used for solving the Eikonal equation. The fast marching method is significantly faster than iterative algorithms which solve the unsteady formulation. Because the level set function needs to be reinitialized often, this advantage will play an important role in speeding up our level set based approach.

3.5 Computational Implementation

Modeling, analysis and optimization are all performed within the MATLAB$^\text{®}$ environment [110]. All numerical analyses are performed and timed on a Dell laptop equipped with an Intel$^\text{®}$ Core$^\text{TM}$ i7-4700MQ processor running at 2.40 GHz, and 16.0 GB of DDR3 SDRAM. All code was compiled and run using MATLAB$^\text{®}$ version 8.6 (R2015b). The Partial Differential Equation Toolbox is employed to handle finite element modeling and analysis using simplicial meshes. Text files are used to import modeling and optimization parameters as well as store results and output data. The architecture is depicted in Figure 3-3. The white boxes indicate text files, the blue box labeled Build Model is an initialization routine and the green box labeled Topology Optimization is the main program. The Build Model routine also allows the user to run a finite element analysis to verify that the model has been properly initialized.
Figure 3-3: Computational architecture for level set based topology optimization framework

The **Topology Optimization** routine is defined in greater detail in Figure 3-4. The gray trapezoids indicate input/output data, the blue boxes indicate functions, and the green diamonds indicate decision junctions. In the remainder of this thesis, the term “steps” will be applied to those taken by the Hamilton-Jacobi equation solver to update the level set function, while the term “iterations” is used for the number of times the optimization routine updated the level set function by achieving an acceptable decrease of the augmented Lagrangian. In Figure 3-4, the term “steps” refers the loop enclosed within the red dotted line where the Hamilton-Jacobi solver and reinitialization scheme are updating the level set function, whereas “iterations” refers to the larger loop between the input and output data blocks.

Several key features of this implementation are the level set function reinitialization, the decision on whether the augmented Lagrangian has sufficiently decreased after the Hamilton Jacobi solver updates the level set function, the number of steps $T$ and step size $\Delta t$ provided to the Hamilton Jacobi solver, the smoothing/normalization
Figure 3-4: Level set based topology optimization routine

of the velocity field, and the convergence criteria. Each of these features requires the user to pick a set of parameters which will ultimately need to be tuned in order to achieve convergence in a suitable number of iterations. Tuning these parameters can be challenging depending on the nature of the problem, and few researchers discuss the inherent difficulties involved [168, 150].

Periodic level set function reinitializations are performed at regular intervals (number of Hamilton Jacobi solver steps). The size of this interval is dependent on how fast the gradient of the level set function deviates from 1. Another consideration is that reinitializing the level set function is roughly as computationally expensive as solving the finite element model. To reduce CPU runtime, reinitializations are often not performed after every time step taken by the Hamilton Jacobi solver. A suitable balance between numerical accuracy and computational speed needs to be found.

During each iteration the optimizer must decide whether a sufficient decrease of the augmented Lagrangian was achieved. The augmented Lagrangian method introduced in Algorithm 1 requires that an approximate minimizer of the augmented Lagrangian
is found. However, this condition also implemented in [121, 23] does not carry any specific criteria for selecting the new design point. Consequently, many strategies have been adopted. A common approach is to allow the augmented Lagrangian some margin to increase for the first set of iterations, and subsequently require a decrease during the remaining iterations. This strategy allows the initial structure shape to make a substantial transformation before converging to a locally optimal design [10]. In the framework presented here, we allow the user to specify a margin on the augmented Lagrangian, such that the following inequality must be satisfied in order for the optimizer to accept the new design point:

\[ L(\Omega^k, \theta^k, \lambda_k^+, \mu_k^+) \leq \beta_L^k L(\Omega^k_s, \theta^k_s, \lambda^k_s, \mu^k_s) \]  

(3.38)

The condition in Equation (3.38) is implemented on line 7 of Algorithm 1. If \( \beta_L^k \) is greater than one, a margin of growth is allowed for the augmented Lagrangian to increase between iterations. In our framework, the value of \( \beta_L^k \) can be selected for each iteration \( k \) up to the maximum number of iterations \( k_{\text{max}} \).

At iteration \( k \), the level set function \( \phi^k \) is updated by solving the Hamilton Jacobi equation using the velocity field \( v^k \). The solver requires two arguments: (1) the number of steps \( T \) where \( T \in \mathbb{Z}_{>0} \) and \( \mathbb{Z}_{>0} \) is the set of positive integers and (2) the step size \( \Delta t \) where \( \Delta t \in \mathbb{R}_{>0} \) and \( \mathbb{R}_{>0} \) is the set of positive real numbers. These arguments are chosen at each iteration of the optimizer such that a suitable decrease in the augmented Lagrangian is found, as was previously discussed. Several strategies have already been implemented including marching a constant number of steps with a fixed step size for every iteration [173, 33, 85, 188], performing a backtracking line search [10, 13], and several other methods discussed in [24]. Certain methods are effective for specific problems. The choice of method also relies on the balance between numerical accuracy and computational effort. Performing an exhaustive search in the steepest descent direction, however, can be computationally prohibitive as each evaluation of the augmented Lagrangian requires updating the level set function, solving the finite element model, and calculating the objective.
and constraint functions. In our framework, we allow for the user to select from a variety of line search strategies, along with the parameters which are specific to those strategies (e.g., marching forward by a constant number of steps for each iteration requires defining the number of steps $T$ taken at each iteration and the step size $\Delta t$). These line search strategies include:

- **Constant number of steps** $T$ at a fixed step size $\Delta t$ for $k = 1, \ldots, k_{\text{max}}$.

- **Variable number of steps** at each iteration according to a predefined sequence $T = T(k)$ as well as variable step size $\Delta t = \Delta t(k)$.

- **Backtracking method** where $T \in [1, T_{\text{max}}]$ for $T_{\text{max}} \in \mathbb{Z}_{>0}$. The step size $\Delta t$ can also be reduced if only a single step $T = 1$ is taken. A trust region method is implemented to allow $T_{\text{max}}$ to change between optimization iterations. The backtracking method is discussed in greater detail in Section 4.4.

- **Golden section search technique** where $T \in [1, T_{\text{max}}]$ and the step size $\Delta t$ is fixed for all iterations $k = 1, \ldots, k_{\text{max}}$.

- **Simulated annealing technique** where $T \in [1, T_{\text{max}}]$ and the step size $\Delta t$ is fixed for all iterations $k = 1, \ldots, k_{\text{max}}$.

The other consideration in selecting $T$ and $\Delta t$ is the degree to which the optimizer will remove small topological features. With a larger number of steps (and larger time steps), the structural boundary updates will be more aggressive. As a result, locally optimal solutions with many small features will be avoided in favor of those with fewer and larger features. In some cases, designs with smaller features may be better performing than those with larger and fewer features, as in the cases of minimally compliant Michell type structures [21].

The purpose of regularizing the velocity field $v$ is to improve convergence and stability by removing sharp spikes and discontinuities. While some authors have used the terms filtering [33, 189] and smoothing [21], the goal is still to improve convergence and stability. Regularization, however, typically implies that the velocity...
field is redefined by projecting it onto an inner product space such as the Hilbert space [29, 48]. On the other hand, smoothing or filtering imply techniques which average the velocity field using spatially local information (e.g., convolutions) [33, 189]. In our framework, we allow the user to select a Hilbertian regularization technique which was presented by Gournay [48]. This technique projects the velocity field onto the Hilbert inner product space \((\cdot, \cdot)_{H^1}\) defined as follows:

\[
(\vartheta, \varphi)_{H^1} = \int_D [\alpha_{H^1} \nabla \vartheta \cdot \nabla \varphi + \vartheta \varphi] \, dV
\]  

(3.39)

where \(\vartheta\) and \(\varphi\) are scalar fields defined on \(D\) and \(\alpha_{H^1} \in \mathbb{R}_{>0}\). The projected velocity field \(\hat{v}\) is found by solving the following equation for \(\vartheta\):

\[
(\vartheta, \varphi)_{H^1} = \int_{\partial\Omega} \varphi v \, ds \quad \forall \varphi \in H^1(\Omega)
\]  

(3.40)

where \(v\) is original velocity field, \(\varphi\) is a test function, \(H^1(\Omega)\) is the Hilbert space (also a Sobolev space) whose functions and first order weak derivatives are both Lebesgue square-integrable, and the projected velocity field \(\hat{v}\) is defined as follows:

\[
\hat{v} = -\frac{\vartheta}{\|\vartheta\|_{H^1}}
\]  

(3.41)

The unique solution to \(\vartheta\) in Equation (3.40) is found using the finite element method. For more details on projecting the velocity field into the Hilbert space \(H^1(\Omega)\) the interested reader is referred to [48]. The projected velocity field \(\hat{v}\) provided by this regularization technique has been shown to speed and improve convergence. However, the user must select a scalar \(\alpha_{H^1}\) which controls the amount of smoothing to apply. If \(\alpha_{H^1}\) is too large, many of the smaller features are lost in the optimal design as was the case in choosing a large number of time steps for the Hamilton Jacobi solver. Conversely, choosing small values for \(\alpha_{H^1}\) result in poor regularization and do not provide enough smoothing to speed convergence. Our numerical testing indicates that \(\alpha_{H^1}\) should be roughly equal to the mesh size \(\Delta x\), or minimum side length of the mesh elements.
Lastly we present three convergence criteria utilized in our framework. These criteria are provided in the following equations:

\[ k > k_{\text{min}} \]  \hspace{1cm} (3.42a)
\[ \frac{|J^k - J^{k-l}|}{|J^k|} < \beta_J \quad \text{for } l = 1, ..., k_{\text{min}} \]  \hspace{1cm} (3.42b)
\[ |h^k| < \beta_h \]  \hspace{1cm} (3.42c)

where \( \beta_J \) and \( \beta_h \) are the objective function and equality constraint tolerances respectively and \( k_{\text{min}} \) is the number of previous iterations to consider when evaluating criteria 3.42b. According to criteria 3.42b, the change \( (J^k - J^{k-l}) / J^k \) must lie in the range \([-\beta_J, \beta_J]\) for a minimum of \( k_{\text{min}} \) steps. This criteria is meant to ensure that the objective function has stabilized. According to the equality constraint criteria 3.42c, \( h^k \) must lie in the range \([-\beta_h, \beta_h]\). If the optimization never reaches a converged solution, the program terminates when the maximum number of iterations \( k_{\text{max}} \) has been reached.

3.6 Chapter Summary

In this chapter, a brief literature review on designing thermally efficient radiating structures revealed that analysis and optimization techniques are limited to parametric approaches incorporating only a handful of design variables. Therefore a need exists for developing a method which allows for free-form design optimization of thermally radiating structures. To meet this need a level set based topology optimization method has been introduced which defines the structural shape implicitly via the level set function. The 2D optimization problem (SP1) maximizes the total heat radiated subject to an equality constraint on the volume fraction. The purpose of the volume fraction constraint is to formulate a well-posed optimization problem. We employ a steepest descent method where the velocity field used in the Hamilton Jacobi equation to update the level set function is chosen as the vector field which guarantees a decrease in the shape functional. Finally, a computational framework built in the
MATLAB® environment was introduced to perform level set based topology optimization. We discussed several subroutines responsible for assembling and solving the finite element equations, reinitializing the level set function, solving the Hamilton Jacobi equation, and constructing a velocity field from the shape derivative of the augmented Lagrangian. This framework will be applied in subsequent chapters to solve a series of 2D problems.
Chapter 4

Single Objective 2D Thermally Radiating Plate Case Study

This chapter presents a 2D case study to verify that the computational framework developed in Chapter 3 identifies locally optimal solutions for efficient radiating structures. In Section 4.1, we introduce the example problem where a 2D aluminum plate is radiating on a sub-domain with a fixed temperature boundary condition. In Section 4.2 we demonstrate that minimizing the thermal compliance objective function commonly found in many topology optimization problems designing thermally efficient structures does not correlate with maximizing the heat power radiated, and hence cannot be used for solving (SP1). In Section 4.3 we derive a novel shape sensitivity for the thermal heat power radiated function which is used to define a steepest descent direction for updating the level set function. In Section 4.4 we solve optimization problem (SP1) using our computational framework described in Chapter 3 with a series of perforated initial material configurations. The results illustrate that the methodology converges to different topologies, suggesting that the design space is populated by many locally optimal solutions of varying geometric complexity.
4.1 Example 2D Single Objective Thermal Radiation Problem

In this section a 2D problem is presented to demonstrate that the optimizer is converging to locally optimal solutions and validate that the shape sensitivity is correct. Section 4.1.1 defines the problem geometry, boundary conditions and material properties. Section 4.1.2 provides an overview of the finite element model used to evaluate the structural performance and sensitivities. A structured mesh is employed such that both the Hamilton Jacobi and Eikonal equations can be solved using second order finite difference schemes. These numerical methods are more computationally efficient than those for unstructured meshes.

4.1.1 Problem Definition

The problem analyzes a thin square plate with dimensions 100 cm × 100 cm and thickness 1 cm. A Dirichlet boundary condition is enforced on a 20 cm segment along the center of the left edge where the temperature is fixed at 300 K. The plate rejects heat via radiation on both sides of a thin strip along the right edge with size 4 cm × 100 cm. This sub-domain $\Omega_{\text{rad}}$ is the only region where heat is being radiated. For simplicity the radiating surface is modeled as a perfect blackbody, meaning the emissivity $\varepsilon = 1$ and the plate does not absorb any energy from the surrounding environment (assumes that the mean ambient temperature is 0 K). Figure 4-1 illustrates the problem geometry.

In Figure 4-1 the gray region indicates the radiating sub-domain $\Omega_{\text{rad}}$ as well as the non-design sub-domain. The term “non-design” is used to indicate that the topology optimizer cannot remove material from this region and it remains filled with material. Essentially, the radiating material is fixed during the optimization and cannot change shape. This is a common tactic used in the topology optimization community to enforce a design-independent load condition. Some authors have also considered design-dependent load conditions where the boundary conditions are affected by the
Figure 4-1: Geometry, boundary conditions and sub-domain definitions of 2D thermally radiating plate

structural shape [24, 10, 83, 182], although these load conditions are computationally challenging to implement and for the purposes of verifying the thermal heat radiated sensitivity analysis are not necessary.

The plate is made from an aluminum alloy (Al-6061-T6) whose physical properties are listed in Table 4.1. This material is isotropic meaning that the physical properties are uniform and identical in all directions.

Table 4.1: Summary of material properties for Al-6061-T6.

<table>
<thead>
<tr>
<th>Property</th>
<th>Units</th>
<th>Design Material*</th>
<th>Non-Design Material*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus</td>
<td>GPa</td>
<td>68.9</td>
<td>68.9</td>
</tr>
<tr>
<td>Tensile Yield Strength</td>
<td>MPa</td>
<td>276</td>
<td>276</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>n/a</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>Thermal Conductivity</td>
<td>W m(^{-1}) K(^{-1})</td>
<td>167</td>
<td>167</td>
</tr>
<tr>
<td>Density</td>
<td>g cm(^{-3})</td>
<td>2.7</td>
<td>2.7</td>
</tr>
</tbody>
</table>

* Material properties for Al-6061-T6.

The material properties are identical for both the design and non-design domains. They are listed separately in Table 4.1 since the optimization framework can account for different materials in each domain.
4.1.2 Finite Element Model

A finite element model is used to solve the governing PDE from Equation (3.7) for the 2D radiating plate problem illustrated in Figure 4-1. For an in-depth review of finite element methods, the interested reader is referred to [17]. A structured mesh is generated using 5000 linear triangle elements including 2601 nodes. This mesh is illustrated in Figure 4-2.

![Mesh Illustration]

Figure 4-2: Mesh of 2D thermally radiating plate

Figure 4-3 illustrates the local and global coordinate systems of the linear triangle elements used.

![Coordinate Systems Illustration]

Figure 4-3: Local and global coordinate systems for linear triangle element
The local and global element coordinates are denoted by \((\hat{r}, \hat{s})\) and \((\hat{x}, \hat{y})\) respectively. Linear shape functions \(N_i^e(\hat{r}, \hat{s})\) are used to interpolate the solution \(\theta(\hat{r}, \hat{s})\) within each element \(e\) according to the following equation:

\[
\theta(\hat{r}, \hat{s}) = \sum_{i=1}^{3} N_i^e(\hat{r}, \hat{s}) \theta_i^e
\]  

(4.1)

where \(\theta_i^e\) are the nodal temperature values for element \(e\). The shape functions are listed in Equation (4.2):

\[
\begin{align*}
N_1^e &= 1 - \hat{r} - \hat{s} \\
N_2^e &= \hat{r} \\
N_3^e &= \hat{s}
\end{align*}
\]

(4.2)

Since we implement linear triangle elements, the spatial derivatives of the solution are constant within each element:

\[
\begin{align*}
\frac{\partial \theta^e}{\partial \hat{r}} &= \sum_{i=1}^{3} \frac{\partial N_i^e}{\partial \hat{r}} \theta_i^e = \theta_2^e - \theta_1^e \\
\frac{\partial \theta^e}{\partial \hat{s}} &= \sum_{i=1}^{3} \frac{\partial N_i^e}{\partial \hat{s}} \theta_i^e = \theta_3^e - \theta_1^e
\end{align*}
\]

(4.3)

Equation (4.3) can be written in matrix format using the nodal temperature vector \(\theta_e = [\theta_1^e, \theta_2^e, \theta_3^e]^T\) and strain-displacement matrix \(B\) as follows:

\[
\begin{bmatrix}
\frac{\partial \theta^e}{\partial \hat{r}} \\
\frac{\partial \theta^e}{\partial \hat{s}}
\end{bmatrix} = B \theta_e
\]

(4.4)

where \(B\) is defined as the 2 \times 3 matrix:

\[
B = \begin{bmatrix}
\frac{\partial N_1}{\partial \hat{r}} & \frac{\partial N_2}{\partial \hat{r}} & \frac{\partial N_3}{\partial \hat{r}} \\
\frac{\partial N_1}{\partial \hat{s}} & \frac{\partial N_2}{\partial \hat{s}} & \frac{\partial N_3}{\partial \hat{s}}
\end{bmatrix}
\]

(4.5)
To translate the spatial derivatives from the local to global coordinate system, each element $e$ is assigned a Jacobian matrix $J$ defined as follows:

$$J_e = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{bmatrix}$$ (4.6)

The temperature gradient in the global coordinate system $(\hat{x}, \hat{y})$ for each element $e$ is then calculated using the following:

$$\nabla \theta_e = \begin{bmatrix} \frac{\partial \theta}{\partial \hat{x}} \\ \frac{\partial \theta}{\partial \hat{y}} \end{bmatrix} = J_e^{-1} B \theta_e$$ (4.7)

where $\nabla \theta_e$ is the element temperature gradient in the global coordinate system. Likewise the heat fluxes, which are proportional to the temperature gradient according to Fourier's law, are also constant within each element.

$$\tau_e = \begin{bmatrix} \tau_{11} \\ \tau_{22} \end{bmatrix} = D \nabla \theta_e$$ (4.8)

where $\tau_e$ is the element heat flux vector, the terms $\tau_{11}$ and $\tau_{22}$ are the heat fluxes in the $\hat{x}$ and $\hat{y}$ directions respectively, and $D$ is the constitutive matrix for thermal conductivity defined in Equation (4.9):

$$D = \begin{bmatrix} \varrho_{11} & 0 \\ 0 & \varrho_{22} \end{bmatrix}$$ (4.9)

where $\varrho$ is the material thermal conductivity and the subscripts 11 and 22 again indicate the $\hat{x}$ and $\hat{y}$ directions respectively in the material. Because the example problem here uses an isotropic material (although anisotropic materials can also be handled), the diagonal entries of the above matrix are simply $\varrho$, which is the uniform thermal conductivity in all directions.
The finite element program solves the following global linear system of equations:

\[ \mathbf{L}\mathbf{\theta} = \mathbf{g} \quad (4.10) \]

where \( \mathbf{L} \) is the \( p \times p \) global conductivity matrix, \( \mathbf{\theta} \) is the \( p \times 1 \) temperature vector, and \( \mathbf{g} \) is the \( p \times 1 \) heat load vector. The dimension \( p \) is the number of nodes (and degrees of freedom) in the finite element model. MATLAB®’s PDE Toolbox implements a Gauss-Seidel solver since the heat load vector \( \mathbf{g} \) is a nonlinear function of the temperature according to the Stefan-Boltzmann law. The global conductivity matrix is assembled from the elemental conductivity matrices according to Equation (3.18). The mapping \( M \) from Equation (3.16) translates the level set function to the structural boundary, and subsequently the element densities \( \rho_e \) are calculated using Equations (3.14) and (3.15). Therefore the global conductivity matrix is dependent on the level set function:

\[ \mathbf{L} = \mathbf{L}(\rho(\phi)) \quad (4.11) \]

where \( \rho \) is the vector of element densities \( \rho_e \) for \( e = 1, \ldots, m \) and \( m \) is the total number of elements in the mesh. Likewise \( \phi \) is the vector of nodal values of the level set function. The element conductivity matrix \( \mathbf{L}_e \) is defined as follows:

\[ \mathbf{L}_e = \int_{\Gamma_e} \mathbf{B}^T \mathbf{D} \mathbf{B} \det(\mathbf{J}_e) \, dA \quad (4.12) \]

Because the element thickness is unity (1 cm) it is not explicitly shown in Equation (4.12).

The shape functions in Equation (4.2) are also used to interpolate the level set function according to Equation (3.13). This interpolation defines a piecewise linear structural boundary \( \partial\Omega \) where \( \phi = 0 \). In other words, \( \phi = 0 \) is defined as a line segment within each element intersected by the zero level set. See Figure 4-4 for a 2D and 3D illustration of this interpolation.
According to Equation (3.15) the element densities $\rho_e$ are defined as the ratio between the area of the element where $\phi \leq 0$, $|\Omega_e|$, and the total area $|\mathcal{T}_e|$. This computation becomes straightforward when the intersecting zero level set is a line segment.

To calculate the heat power radiated, a Gauss quadrature scheme is implemented to integrate the fourth power of the temperature in each radiating element. This scheme is written in Equation (4.13) for a 2D linear triangle element in the local coordinate system $(\hat{r}, \hat{s})$.

$$ \int_{\mathcal{T}_e} \theta^4(\hat{r}, \hat{s}) \, dA = \sum_{j=1}^{3} \alpha_j \theta^4(\hat{r}_j, \hat{s}_j) $$  \hspace{1cm} (4.13)

where $\alpha_j$ is the $j^{th}$ Gauss weight and $(\hat{r}_j, \hat{s}_j)$ is the corresponding $j^{th}$ Gauss point. To transform the integration into the physical domain, the Jacobian matrix is once again employed:

$$ \int_{\mathcal{T}_e} \theta^4(\hat{x}, \hat{y}) \, dA = \frac{1}{2} \det(J_e) \sum_{j=1}^{3} \alpha_j \theta^4(\hat{r}_j, \hat{s}_j) $$  \hspace{1cm} (4.14)

where the temperature at point $(\hat{r}, \hat{s})$ is calculated using the element shape functions $N_i(\hat{r}, \hat{s})$ according to Equation (4.1). Since the temperature varies linearly within each element, a 3-point integration scheme is used to exactly calculate the integration in Equation (4.14). The Gauss quadrature points and weights in this scheme are found
in Table 4.2.

Table 4.2: Gauss points and weights for 3-point integration scheme.

<table>
<thead>
<tr>
<th>Gauss Point</th>
<th>Local-Coordinate ((\hat{r}, \hat{s}))</th>
<th>Gauss Weight (\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point 1</td>
<td>(1/6, 1/6)</td>
<td>1/3</td>
</tr>
<tr>
<td>Point 2</td>
<td>(2/3, 1/6)</td>
<td>1/3</td>
</tr>
<tr>
<td>Point 3</td>
<td>(1/6, 2/3)</td>
<td>1/3</td>
</tr>
</tbody>
</table>

4.2 Efficient Thermal Design Using a Compliance Objective Function

Various other researchers, while not tackling problems that incorporate thermally radiating boundary conditions in particular, have investigated the design of efficient thermally conductive structures [68, 83, 27, 103, 191]. The common objective function used in these problems is a thermal compliance function written as follows:

\[
J = \nabla \theta \cdot \kappa \nabla \theta
\]  

(4.15)

where \(\nabla \theta\) is the temperature gradient vector field, and \(\kappa\) is the second order conductivity tensor. Equation (4.15) is analogous to the structural compliance objective where \(\nabla \theta\) and \(\kappa\) are replaced respectively by the second order strain tensor \(\epsilon(u)\) and fourth order elasticity tensor \(\mathbb{C}\), and the dot product is replaced by the tensor inner product. The thermal compliance functional is a measure of how well the structure conducts heat. By minimizing the thermal compliance the optimizer is minimizing the temperature gradients and hence keeping the structure as isothermal as possible. Thus the goal of these problems is to design for thermal efficiency since a limited amount of material is allocated to minimize thermal resistance and hence transport heat as effectively as possible.

A classic example is the 2D design of a thermal heat sink in a square design domain. This problem has been solved using both SIMP [21, 112] and level set methods [182].
In this problem a uniformly distributed heat load is imparted on the design domain, with a fixed temperature on a segment of one edge. The solution is a symmetric branched tree structure which attempts to capture as much of the distributed heat load and transfer it to the Dirichlet boundary. This classic problem has also been solved in 3D by Burger et al. [27].

We find that minimizing the thermal compliance of a thermally radiating structure does not necessarily maximize the heat power radiated. If these two objective functions were negatively correlated, we could pose the optimization problem using a thermal compliance objective function. However, we demonstrate a counterexample where the two objective functions are instead positively correlated. Consider the three test designs illustrated in Figure 4-5, each with a volume fraction $V_{\text{vol}}/V_{\text{tot}} = 0.5$, along with their corresponding finite element solutions. The boundary conditions and material properties are the same as those presented in Section 4.1.1. Each design is evaluated for thermal compliance and heat power radiated.

\[
J = \theta^T L \theta \quad \text{and} \quad J = \int \epsilon Y \theta^T \, dA
\]

<table>
<thead>
<tr>
<th>Shape</th>
<th>Temperature</th>
<th>Heat Flux</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Shape1" /></td>
<td><img src="image2.png" alt="Temperature1" /></td>
<td><img src="image3.png" alt="Heat Flux1" /></td>
</tr>
<tr>
<td>$2.64 \times 10^{-5}$</td>
<td>28.3</td>
<td></td>
</tr>
<tr>
<td><img src="image4.png" alt="Shape2" /></td>
<td><img src="image5.png" alt="Temperature2" /></td>
<td><img src="image6.png" alt="Heat Flux2" /></td>
</tr>
<tr>
<td>$2.39 \times 10^{-5}$</td>
<td>23.8</td>
<td></td>
</tr>
<tr>
<td><img src="image7.png" alt="Shape3" /></td>
<td><img src="image8.png" alt="Temperature3" /></td>
<td><img src="image9.png" alt="Heat Flux3" /></td>
</tr>
<tr>
<td>$0.43 \times 10^{-5}$</td>
<td>0.76</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4-5: Thermal compliance vs. radiated heat power objective function comparison for three structural designs.
The material and void domains in the illustrated shapes are designated by the cyan and white colors, respectively. The radiating strip is shown in gray on the right side of the design domain as was depicted in Figure 4-1. Note that although the third design (bottom row in Figure 4-5) completely disconnects the material domain from the radiating domain, the structure is still able to radiate some heat. Using the Er-satz material approach described in Section 3.4.2, the void domain retains a small nonzero thermal conductivity. The thermal conductivity of the void region is $10^3$ times smaller than the nominal material conductivity.

It is clear from Figure 4-5 that there is a positive rather than negative correlation between the thermal compliance and heat power radiated functions of the three test designs. From top to bottom in Figure 4-5 the designs have progressively decreasing thermal compliances, but also decrease in the amount of heat power radiated. This finding debunks our hypothesis that these two objective functions are strictly negatively correlated, such that a lower thermal compliance design would always be radiating a larger amount of heat. While there are possible cases in which these two objectives do negatively correlate, it is clear from the above counterexamples that this is not always guaranteed. As such, it is crucial to develop a topology optimization framework which specifically defines a radiated heat power objective function and its corresponding shape derivative.
4.3 Shape Sensitivity for Radiated Heat Power

A shape sensitivity analysis is performed on the radiated heat power objective functional defined in Equation (3.4) and restated below:

\[
J(\Omega, \theta) = -\int_{\Omega} \chi \varepsilon \mathcal{U} \theta^4 \, dA
\]  

(4.16)

We negate the function found in Equation (3.4) since our goal is to maximize the heat power radiated. We also restate the governing PDE here for reference:

\[
\text{div} (\kappa \nabla \theta) = \chi \varepsilon \mathcal{U} \theta^4 \quad \text{in } \Omega
\]
\[
(\kappa \nabla \theta) \cdot n = 0 \quad \text{on } \Gamma_{0,\theta}
\]
\[
\theta = \theta_0 \quad \text{on } \Gamma_{D,\theta}
\]  

(4.17)

where the domain flag \( \chi \) multiplies the radiating heat power term \( \varepsilon \mathcal{U} \theta^4 \) to remove the second line in Equation (3.7), \( \text{div} (\kappa \nabla \theta) = 0 \) on \( \Omega \setminus \Omega_{\text{rad}} \). The domain flag \( \chi \) allows us to write a single PDE on the entire domain \( \Omega \) rather than specify the radiating and non-radiating sub-domains separately. The derivation closely follows those found in [10] which are credited to Murat [119] and Simon [151]. While only the highlights are presented in this section, the interested reader should refer to Appendix A where a full derivation is provided. The definition of a shape derivative was introduced in Section 3.4.4 but again the interested reader should refer to [131] and [153] for a more detailed review.

We use the adjoint method to perform the necessary shape sensitivity analysis. The adjoint method allows for efficient calculation of sensitivities for large dimensional problems oftentimes encountered in PDE constrained optimization. This process is enumerated in the following steps:
1. Define the Lagrangian $L(\Omega, \theta, q)$ from the objective function in (4.16), governing PDE in (4.17) and corresponding Lagrange multiplier $q$.

2. Find the stationary point $(\theta^*, q^*)$ of the Lagrangian by setting the derivatives with respect to the Lagrange multiplier $q$ and solution $\theta$ to zero. These equations define the optimality conditions for minimizing the Lagrangian.

3. Calculate the shape derivative of the Lagrangian at the stationary point $L(\Omega, \theta^*, q^*)$, where $q^*$ is the adjoint state vector.

Beginning with step (1), the Lagrangian is defined as follows:

$$L(\Omega, \theta, q) = -\int_{\Omega} \chi \varepsilon \nabla \theta^4 \, dV + \int_{\Omega} q \cdot \left( \text{div} \left( \kappa \nabla \theta \right) - \chi \varepsilon \nabla \theta^4 \right) \, dV$$

$$+ \int_{\Gamma_{D, \theta}} \left( (\kappa \nabla q) \cdot n \right) (\theta - \theta_0) \, ds$$

$$+ \int_{\Gamma_{0, \theta}} q \cdot \left( - (\kappa \nabla \theta) \cdot n \right) \, ds$$

In Equation (4.18), an insulating boundary condition on $\Gamma_{0, \theta}$ has been imposed. Recall that the boundary $\partial \Omega$ is composed of both the Dirichlet and insulating boundaries according to Figure 3-1:

$$\partial \Omega = \Gamma_{D, \theta} \cup \Gamma_{0, \theta}$$

After applying Green's first identity and grouping terms, the Lagrangian is transformed into Equation (4.20).

$$L(\Omega, \theta, q) = -\int_{\Omega} \chi \varepsilon \nabla \theta^4 \, dV - \int_{\Omega} (\nabla q \cdot \kappa \nabla \theta) \, dV - \int_{\Omega} q \cdot \chi \varepsilon \nabla \theta^4 \, dV$$

$$+ \int_{\Gamma_{D, \theta}} [q \cdot ((\kappa \nabla q) \cdot n) + (\theta - \theta_0) ((\kappa \nabla \theta) \cdot n)] \, ds$$

In step (2), we find the stationary point $(\theta^*, q^*)$ where the partial derivatives of the Lagrangian with respect to $q$ and $\theta$ are equal to zero. Both $\theta \in H^1(\Omega)$ and $q \in H^1(\Omega)$ where $H^1(\Omega)$ is the Sobolev space whose functions and first order weak derivatives are both Lebesgue square-integrable. Because $q$ and $\theta$ belong to a functional space
which does not depend on $\Omega$, we can differentiate the Lagrangian first with respect to $q$ in the direction $\zeta \in H^1(\Omega)$

\[
\left< \frac{\partial L}{\partial q} (\Omega, \theta^*, q^*), \zeta \right> = -\int_{\Omega} (\nabla \zeta \cdot \kappa \nabla \theta^*) \, dV - \int_{\Omega} \zeta \cdot \chi \varepsilon \mathcal{Y} \theta^{*4} \, dV
\]
\[+ \int_{\Gamma_{D,\theta}} \zeta \cdot ((\kappa \nabla \theta^*) \cdot n) \, ds + \int_{\Gamma_{D,\theta}} (\theta^* - \theta_0) \, ((\kappa \nabla \zeta) \cdot n) \, ds \tag{4.21}
\]

Using Green’s first identity again to rewrite the first integral and collecting terms on the Dirichlet and insulating boundaries, Equation (4.21) is transformed as follows:

\[
\left< \frac{\partial L}{\partial q} (\Omega, \theta^*, q^*), \zeta \right> = \int_{\Omega} \zeta \cdot \left( \text{div}(\kappa \nabla \theta^*) - \chi \varepsilon \mathcal{Y} \theta^{*4} \right) \, dV
\]
\[+ \int_{\Gamma_{0,\theta}} \zeta \cdot (- (\kappa \nabla \theta^*) \cdot n) \, ds
\]
\[+ \int_{\Gamma_{D,\theta}} (\theta^* - \theta_0) \, ((\kappa \nabla \zeta) \cdot n) \, ds \tag{4.22}
\]

Setting Equation (4.22) equal to zero yields the original state equation and boundary conditions found in Equation (4.17). The first integral gives the state equation for any $\zeta$ with compact support in $\Omega$. Varying the trace function $\zeta$ on $\Gamma_{0,\theta}$ gives the insulating boundary condition and by varying $(\kappa \nabla \zeta) \cdot n$ on $\Gamma_{D,\theta}$ gives the Dirichlet boundary condition.

Next the derivative of the Lagrangian with respect to the state variable $\theta$ is used to define the adjoint equations.

\[
\left< \frac{\partial L}{\partial \theta} (\Omega, \theta^*, q^*), \zeta \right> = -\int_{\Omega} \zeta \cdot 4\chi \varepsilon \mathcal{Y} \theta^{*3} \, dV
\]
\[+ \int_{\Omega} \nabla q^* \cdot \kappa \nabla \zeta \, dV - \int_{\Omega} \zeta \cdot \left( q^* \cdot 4\chi \varepsilon \mathcal{Y} \theta^{*3} \right) \, dV
\]
\[+ \int_{\Gamma_{D,\theta}} q^* \cdot ((\kappa \nabla \zeta) \cdot n) \, ds + \int_{\Gamma_{D,\theta}} \zeta \cdot ((\kappa \nabla q^*) \cdot n) \, ds \tag{4.23}
\]

Again by using Green’s first identity and grouping terms on the Dirichlet and insu-
lating boundary conditions, Equation (4.23) is transformed as follows:

\[
\left\langle \frac{\partial L}{\partial \theta} (\Omega, \theta^*, q^*), \zeta \right\rangle = \int_{\Omega} \zeta \cdot \left( \text{div}(\kappa \nabla q^*) - 4\chi\varepsilon \Upsilon \theta^{*3} - q^* \cdot 4\chi\varepsilon \Upsilon \theta^{*3} \right) dV \\
+ \int_{\Gamma_{0,\theta}} \zeta \cdot (-(\kappa \nabla q^*) \cdot n) \, ds \\
+ \int_{\Gamma_{D,\theta}} q^* \cdot ((\kappa \nabla \zeta) \cdot n) \, ds
\]  

(4.24)

Setting Equation (4.24) equal to zero yields the adjoint state equations. The first integral gives the adjoint state equation for any \( \zeta \) with compact support in \( \Omega \):

\[
\text{div}(\kappa \nabla q^*) - 4\chi\varepsilon \Upsilon \theta^{*3} - 4q^* \cdot \chi\varepsilon \Upsilon \theta^{*3} = 0 \quad \text{in} \quad \Omega
\]  

(4.25)

Similarly the insulating boundary condition is given by varying the trace function \( \zeta \) on the boundary \( \Gamma_{0,\theta} \):

\[
-(\kappa \nabla q^*) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta}
\]  

(4.26)

Finally, the Dirichlet boundary conditions is given by varying \( \kappa \nabla \zeta \) on the boundary \( \Gamma_{D,\theta} \):

\[
q^* = 0 \quad \text{on} \quad \Gamma_{D,\theta}
\]  

(4.27)

Note that \( q^* \) is the adjoint state vector. In summary, the adjoint state is defined by the following set of equations:

\[
\text{div}(\kappa \nabla q^*) - 4\chi\varepsilon \Upsilon \theta^{*3} - 4q^* \cdot \chi\varepsilon \Upsilon \theta^{*3} = 0 \quad \text{in} \quad \Omega
\]

\[
(\kappa \nabla q^*) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta}
\]  

(4.28)

\[
q^* = 0 \quad \text{on} \quad \Gamma_{D,\theta}
\]

While the thermal radiation problem is shown to be well-posed, it is not self-adjoint (i.e., \( q^* \neq \theta \)). In order to calculate the shape derivative we need to first solve the original governing PDE in Equation (4.17). Next the solution \( \theta \) is substituted into Equation (4.28) as \( \theta^* \) and the equation is solved for \( q^* \). We use the finite element method to solve both PDEs numerically.
Finally the shape derivative of the Lagrangian defined in Equation (4.20) is calculated using Equations (3.27) and (3.29) which provide the general forms of the shape derivatives for domain and boundary integral functionals respectively. The shape derivative is taken at the stationary point \((\theta^*, q^*)\):

\[
\mathcal{L}(\Omega, \theta^*, q^*) = -\int_{\Omega} \chi \varepsilon \Upsilon \theta^{*4} dV - \int_{\Omega} (\nabla q^* \cdot \kappa \nabla \theta^*) dV - \int_{\Omega} q^* \cdot \chi \varepsilon \Upsilon \theta^{*4} dV \\
+ \int_{\Gamma_{D,\theta}} [q^* \cdot ((\kappa \nabla \theta^*) \cdot n) + (\theta^* - \theta_0) ((\kappa \nabla q^*) \cdot n)] ds
\]

(4.29)

Applying Equations (3.27) and (3.29) and grouping all of the surface integrals we have the following:

\[
D\mathcal{L}(\Omega)(\xi) = \int_{\partial \Omega} \xi \cdot n \left( -\chi \varepsilon \Upsilon \theta^{*4} - \nabla q^* \cdot \kappa \nabla \theta^* - q^* \cdot \chi \varepsilon \Upsilon \theta^{*4} \right) ds \\
+ \int_{\Gamma_{D,\theta}} \xi \cdot n \left( \frac{\partial}{\partial n} \left( q^* \cdot ((\kappa \nabla \theta^*) \cdot n) + (\theta^* - \theta_0) ((\kappa \nabla q^*) \cdot n) \right) \right) ds \\
+ \int_{\Gamma_{D,\theta}} \xi \cdot n \left( H [q^* \cdot ((\kappa \nabla \theta^*) \cdot n) + (\theta^* - \theta_0) ((\kappa \nabla q^*) \cdot n)] \right) ds
\]

(4.30)

Next the first integral is split into the Dirichlet and insulating boundaries:

\[
D\mathcal{L}(\Omega)(\xi) = \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon \Upsilon \theta^{*4} - \nabla q^* \cdot \kappa \nabla \theta^* - q^* \cdot \chi \varepsilon \Upsilon \theta^{*4} \right) ds \\
+ \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon \Upsilon \theta^{*4} - \nabla q^* \cdot \kappa \nabla \theta^* - q^* \cdot \chi \varepsilon \Upsilon \theta^{*4} \right) ds \\
+ \int_{\Gamma_{D,\theta}} \xi \cdot n \left( \frac{\partial}{\partial n} \left( q^* \cdot ((\kappa \nabla \theta^*) \cdot n) + (\theta^* - \theta_0) ((\kappa \nabla q^*) \cdot n) \right) \right) ds \\
+ \int_{\Gamma_{D,\theta}} \xi \cdot n \left( H [q^* \cdot ((\kappa \nabla \theta^*) \cdot n) + (\theta^* - \theta_0) ((\kappa \nabla q^*) \cdot n)] \right) ds
\]

(4.31)

Note from Equations (4.17) and (4.28) that \(\hat{\theta}^* - \theta_0 = 0\) and \(q^* = 0\) on the boundary.
\( \Gamma_{D,\theta} \) and thus the above equation simplifies greatly:

\[
D\mathcal{L}(\Omega)(\xi) = \int_{\Gamma_{0,\theta}} \xi \cdot n \left( -\chi \varepsilon \mathcal{Y} \theta^{*4} - \nabla q^* \cdot \kappa \nabla \theta^* - q^* \cdot \chi \varepsilon \mathcal{Y} \theta^{*4} \right) ds \\
+ \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon \mathcal{Y} \theta^{*4} - \nabla q^* \cdot \kappa \nabla \theta^* \right) ds
\]  

(4.32)

In practice additional simplifications are made to shorten Equation (4.32). First, the optimizer should not be removing material from the Dirichlet boundaries since this area is the source of the heat power being radiated by the structure. Isolating or removing material in this region would cause the temperature on the radiating sub-domain to drop, causing the radiated heat power to decrease. Therefore, on \( \Gamma_{D,\theta} \) the optimizer is forced to retain the structural boundary and the dot product \( \xi \cdot n \) goes to zero, eliminating the second integral in Equation (4.32).

Using a similar reasoning, the radiation flag \( \chi \) is only active (\( \chi = 1 \)) on the sub-domain \( \Omega_{\text{rad}} \). This sub-domain should also remain fixed since removing radiating material will reduce the radiated heat power. As such, the optimizer is only affecting regions with \( \chi(x) = 0 \) where there is no thermal radiation. Note that regions and boundaries that are not able to be modified by the optimizer are termed “non-design”, meaning they are not subject to design (i.e., removing or adding material). By enforcing these practices, Equation (4.32) is reduced to the following single term:

\[
D\mathcal{L}(\Omega)(\xi) = \int_{\Gamma_{0,\theta}} \xi \cdot n \left( -\nabla q^* \cdot \kappa \nabla \theta^* \right) ds
\]

(4.33)

As previously mentioned, solutions for both \( \theta^* \) and \( q^* \) are required to calculate the shape derivative. These solutions are found numerically using the finite element method.

The sensitivity field for the radiating heat power functional defined in Equation (4.33) is also calculated using the finite element method according to Equation (4.34):

\[
D\mathcal{L}(\Omega)(\xi)_{e} = -\frac{1}{|T_e|} q_e^T L_e \theta_e
\]

(4.34)

where the subscript \( e \) identifies the local element, \( q_e \) is the element adjoint vector.
\( \mathbf{L}_e \) is the element conductivity matrix and \( \mathbf{\theta}_e \) is the element nodal temperatures. Equation (4.34) is normalized by the element volume \( |T_e| \) (i.e., area in 2D) due to the fact that the finite element equations are written in the weak form, and hence the term \( \mathbf{q}_e^T \mathbf{L}_e \mathbf{\theta}_e \) is an integral over the element volume.

\[
\mathbf{q}_e^T \mathbf{L}_e \mathbf{\theta}_e = \mathbf{q}_e^T \left[ \int_{T_e} \mathbf{B}^T \mathbf{B} \det (\mathbf{J}_e) \, dV \right] \mathbf{\theta}_e \tag{4.35}
\]

Equation (4.35) must be normalized by \( |T_e| \) to arrive at the shape sensitivity at the element centroid.

### 4.4 Numerical Results

A finite element analysis is first performed on the plate with all of the design domain filled by material. The results of the finite element analysis are shown in Figure 4-6.

The temperature on the Dirichlet boundary is fixed at 300 K and heat flows from this boundary to the radiating domain, where it is then radiated out to the ambient surroundings. The average temperature of the radiating boundary is approximately 278 K and the total heat power radiated from this surface is 27.25 W. Furthermore, because the Dirichlet boundary is a small segment of the left edge of the plate and
the radiating domain occupies the entire right side, the heat flow pattern is curved.

Before running the topology optimization method, two designs are tested to evaluate their performance (i.e., total radiated thermal heat power). Both designs have a volume fraction of 0.5 which will be the equality constraint for the remainder of the analyses in this chapter. These two designs are baselines to compare with those generated by the topology optimization method. They are shown in Figure 4-7.

Test Design 1 and Test Design 2 have total radiated heat powers of 22.58 W and 22.90 W respectively. Because the boundary conditions are symmetric about the mid-line of the design domain parallel to the x-axis, both design shapes also exhibit this symmetry. The intuition leading to these design shapes is that thermal conduc-
tion is a linear phenomenon, and the goal is to maximize the radiated heat power by designing a structure with minimal thermal gradients between the Dirichlet boundary and radiating sub-domain. As such keeping the thermal pathways as direct (i.e., as short) as possible to the radiating sub-domain should yield a well-performing structure. Again these two designs will be revisited later to compare to the results from the topology optimization method.

The topology optimization problem (SP1) is solved with a volume fraction equality constraint of $\frac{V_{\text{vol}}}{f_0} dV = 0.5$. The first case is initiated with a fully-filled design domain, and a Lagrange multiplier estimate and penalty parameter of $\lambda^0 = 1$ and $\mu^0 = 10$ respectively. The initial and final shapes are illustrated in Figure 4-8.

![Initial shape (fully filled)](image1)

![Optimal shape at iteration $k = 55$](image2)

Figure 4-8: Initial and optimal shapes for optimization problem (SP1)

The analysis was completed in 51 minutes, 48 seconds. The optimal shape radiates 24.1 W of heat power. Thus the structural weight is reduced by 50% while the heat rejected drops by 11.5%. The optimal design is analyzed to show the temperature and heat flow fields in Figure 4-9. It is clear from Figure 4-9(b) that heat flows primarily through the material domain $\Omega$. Conversely, little heat is flowing in the void domain where a weak Ersatz material is defined with thermal conductivity $\bar{\sigma} = \sigma \times 10^{-3}$. The factor $10^{-3}$ is the lower bound $\delta$ used in Equation (3.18) for assembling the global finite element equations, and the material density $\sigma$ is given in Table 4.1.
Figure 4-9: Finite element solution of optimal design for optimization problem (SP1)

The objective, constraint and augmented Lagrangian histories are given next in Figure 4-10. The constraint function presented in Equation (3.6) is calculated by summing the volume fractions of each element $e$ using the following equation:

$$h = \left( \sum_{e=1}^{m} \frac{|\Omega_e|}{|T_e|} \right) - 0.5$$

where $m$ is the total number of elements. Therefore the volume fraction constraint is satisfied at $h(\Omega) = 0$. From Figure 4-10(b), the shape converges after approximately 45 iterations. The prior oscillations are due to several factors including that (1) the initial design is infeasible ($h(\Omega) \neq 0$), (2) the Lagrange multiplier estimate is updated at each iteration using a monotonically increasing penalty parameter, and (3) the iterative updates to the structural boundary allow for a small increase in the augmented Lagrangian up until iteration 50. Allowing the augmented Lagrangian to increase in the first few iterations allows the optimizer to find a locally optimal solution that is sufficiently far, or topologically different, from the initial design. The number of iterations in which the augmented Lagrangian is allowed to increase is a heuristic parameter. Allaire et al. [10] mentions that different choices of this parameter can yield different locally optimal solutions.
In the above analysis, we implement a backtracking line search method introduced in Section 3.5 to speed convergence. Our backtracking routine iteratively reduces the number of time steps $T$ and step size $\Delta t$ to advance the level set function until the updated augmented Lagrangian satisfies Equation (3.38). If the optimizer selects an update corresponding to the maximum number of time steps $T = T_{\text{max}}$, the next optimization iteration will attempt to advance the level set function by a larger number of time steps. Conversely, if the optimizer selects a single time step, $T = 1$, the next iteration will use a fewer number of time steps. This technique is similar to the trust region methods discussed in [121]. Algorithm 3 details our backtracking routine.
Algorithm 3: Backtracking algorithm for updating the level set function \( \phi \)

**Input:** Level set function \( \phi \)
- Lagrange multiplier estimate \( \lambda \) and penalty parameter \( \mu \)
- Maximum number of time steps \( T_{\text{max}} \) and time step size \( \Delta t \)
- Tolerance on accepting updated augmented Lagrangian \( \beta_L \)
- Maximum retries \( t_{\text{max}} \) to reduce number of time steps and step size
- Factor \( \gamma_T \leq 1 \) and constant \( \eta \geq 0 \) to decrease number of steps
- Factor to increase (decrease) maximum number of steps \( \gamma_{T_{\text{max}}} \geq 1 \)

**Output:** Updated level set function \( \hat{\phi} \)
- Updated maximum number of time steps \( \hat{T}_{\text{max}} \)

1. Define \( \Omega \) using \( \phi \) and solve finite element equations for \( \theta \)
2. Calculate the current augmented Lagrangian \( L(\Omega, \theta, \lambda, \mu) \)
3. Assemble \( \Phi \), the \( p \times T_{\text{max}} \) array of level set functions found by solving the HJE using \( 1, ..., T_{\text{max}} \) steps of size \( \Delta t \)
4. Choose \( T \leftarrow T_{\text{max}} \)
5. Choose the level set function \( \phi_{\text{try}} \) at time step \( T \) such that \( \phi_{\text{try}} \leftarrow \Phi[:,T] \)
6. Initialize counter for the number of retries \( t_{\text{retry}} \leftarrow 0 \)

while \( t_{\text{retry}} < t_{\text{max}} \) do

7. Define \( \Omega_{\text{try}} \) using \( \phi_{\text{try}} \) and solve finite element equations for \( \theta_{\text{try}} \)
8. Evaluate the augmented Lagrangian \( L(\Omega_{\text{try}}, \theta_{\text{try}}, \lambda, \mu) \)
9. if \( L(\Omega_{\text{try}}, \theta_{\text{try}}, \lambda, \mu) \leq \beta_L L(\Omega, \theta, \lambda, \mu) \) then
10. Exit while loop
11. else
12. Increment the counter for number of retries \( t_{\text{retry}} \leftarrow t_{\text{retry}} + 1 \)
13. if only a single time step was used, \( T = 1 \) then
14. Reduce the step size by half, \( \Delta t \leftarrow \frac{\Delta t}{2} \)
15. Calculate \( \phi_{\text{try}} \) by solving the HJE using \( T = 1 \) and \( \Delta t \)
16. else
17. Reduce the number of time steps \( T \leftarrow \lfloor \gamma_T T - \eta \rfloor \)
18. Choose the level set function \( \phi_{\text{try}} \) at time step \( T \), \( \phi_{\text{try}} \leftarrow \Phi[:,T] \)
19. end
20. end
21. end
22. if maximum number of retries have been attempted, \( t_{\text{retry}} = t_{\text{max}} \) then
23. No updates accepted, \( \hat{\phi} \leftarrow \phi \) and \( T_{\text{max}} \leftarrow T_{\text{max}} \)
24. else if no retries have been attempted, \( t_{\text{retry}} = 0 \) then
25. \( \hat{\phi} \leftarrow \phi_{\text{try}} \) and increase \( T_{\text{max}} \leftarrow T_{\text{max}} \)
26. else if retries have been attempted \( t_{\text{retry}} \neq 0 \) and \( T = 1 \) then
27. \( \hat{\phi} \leftarrow \phi_{\text{try}} \) and decrease \( T_{\text{max}} \leftarrow \frac{1}{\gamma_{T_{\text{max}}}} T_{\text{max}} \)
28. else
29. \( \hat{\phi} \leftarrow \phi_{\text{try}} \) and \( T_{\text{max}} \leftarrow T_{\text{max}} \)
30. end
31. end
32. return \( \hat{\phi}, \hat{T}_{\text{max}} \)
In line 1 the structural domain $\Omega$ is defined by the level set function $\phi$ and the finite element equations are solved once the element densities are computed. This procedure is outlined in Section 3.3. In line 2 the augmented Lagrangian is calculated by evaluating the objective and constraint functions enumerated in lines 2–3 of Algorithm 1. For the sake of brevity they are implied in line 2 of Algorithm 3. In line 3, the array $\Phi$ is created by solving the Hamilton Jacobi equation with the time steps $1, \ldots, T_{\text{max}}$. Thus $\Phi$ contains the evolution history of $\phi$ at each time step up to $T_{\text{max}}$. Rather than recomputing the updated level set function when the number of time steps $T$ is reduced in line 18, we found it was more computationally efficient to precompute the entire level set function history for $T_{\text{max}}$ steps before the while loop in line 7. The array $\Phi$ can then be indexed as is shown in lines 5 and 19. However, when the step size $\Delta t$ is reduced by half in line 15, then the HJE must be resolved using a single time step and the reduced step size. Lines 23–31 allow the maximum number of steps $T_{\text{max}}$ to change between optimization iterations. If the maximum number of steps is chosen $T = T_{\text{max}}$, then the optimizer can advance the level set function by a larger number of steps in the subsequent iteration $\bar{T}_{\text{max}} \geq T_{\text{max}}$. Conversely the optimizer can attempt a fewer number of steps $\bar{T}_{\text{max}} \leq T_{\text{max}}$ if only a single time step $T = 1$ was previously chosen.

The initial step size is dictated by the Courant-Friedrichs-Lewy (CFL) condition since we are using an explicit time integration scheme to solve the Hamilton Jacobi equation. The CFL condition is stated as follows:

$$C = \frac{v_{\text{max}} \Delta t}{\Delta x}$$  \hspace{1cm} (4.37)

where $C$ is the dimensionless Courant number, $v_{\text{max}}$ is the maximum normal velocity magnitude defined on $D$, and $\Delta x$ is the minimum element size. For explicit time integration schemes, such as the upwind finite difference scheme used here, the Courant number must be less than or equal to one. We choose $C = \frac{1}{2}$, and normalize the
velocity field such that \( v_{\text{max}} = 1 \) to yield the following:

\[
\Delta t = \frac{\Delta x}{2} \tag{4.38}
\]

The minimum element size of the mesh described in Section 4.1.2 is \( \Delta x = 2 \), and hence the time step is set to \( \Delta t = 1 \). Recall from Algorithm 3 that the step size shrinks only if the optimizer rejects the level set function update after a single time step was attempted. At this point the optimizer will reduce the step size further until an acceptable update is found or a maximum number of augmented Lagrangian re-evaluations have been made.

For a converged solution, the gradient of the augmented Lagrangian should be approximately zero. This criterion is used in augmented Lagrangian methods to determine that a KKT point has been reached. Because the velocity field is constructed from the derivative of the augmented Lagrangian with respect to a perturbation in the normal direction of the boundary, we expect the velocity at the boundary of the shape to approach zero. Figure 4-11 illustrates several metrics of the velocity field on the structural boundary over the optimization history.

![Figure 4-11: Convergence history using boundary velocity metrics for optimization problem (SP1)](image)

In Figure 4-11(a), the metrics labeled as “Min\( (v_{\partial \Omega}) \)” and “Max\( (v_{\partial \Omega}) \)” refer to the
minimum and maximum value of the velocity on the boundary respectively. The normed metrics are defined using Equation (4.39) for the $L_1$ norm and (4.40) for the $L_2$ norm:

\[
\|v_{\partial \Omega}\|_1 = \frac{\int_{\partial \Omega} |v| \, ds}{|\partial \Omega|}
\] (4.39)

\[
\|v_{\partial \Omega}\|_2 = \left[\frac{\int_{\partial \Omega} |v|^2 \, ds}{|\partial \Omega|}\right]^{1/2}
\] (4.40)

where $|\partial \Omega|$ is the perimeter of the structural boundary $\partial \Omega$. Intuitively, as the velocity field approaches zero on the boundary $\partial \Omega$, updates to the level set function from solving the Hamilton-Jacobi equations will cause smaller changes in the shape. Therefore the structure domain $\Omega$ will ultimately converge.

From the plots in Figure 4-11(b), it is clear that a converged solution has been achieved since the velocity field norms are asymptotically becoming smaller. Note that the velocity fields will never achieve zero due to numerical errors inherent in the level set method. One of the primary errors arises from interpolating the shape sensitivity field from the center of each mesh element to the node points [54]. The interpolated sensitivity field is then used to construct the velocity field. Additionally, the reinitialization of the level set function using the fast marching method is not a volume-preserving procedure, meaning that the structural boundary will shift by some small amount. In the analyses considered here, the level set function is reinitialized every other time step taken by the Hamilton Jacobi solver.

Since a gradient-based optimization method is employed, the solution shown in Figure 4-8 is only guaranteed to be a local optimum. While much of the published research in topology optimization presents a single local optimum solution for a particular design problem of interest, the goal of this thesis is to explore the design landscape. As such, two additional analyses are also run with different initial conditions. Specifically, $2 \times 2$ and $4 \times 4$ arrays of circular holes are introduced in the interior of the material domain to allow for voids in the final design solution. Because the step sizes for our explicit Hamilton Jacobi equation solver must be limited by the CFL condition coupled with the frequent reinitializations of the level set function,
it is impossible to nucleate new holes in the interior of a material domain [10]. By initiating the level set function with voids in the interior, locally optimal solutions not achievable with a fully-filled initial design domain can now be reached. We choose symmetric hole patterns for the initial material configurations since the boundary conditions and previous optimal design exhibit a line of symmetry parallel to the x-axis through the center of the design domain.

The optimal shape for the initial design containing a $2 \times 2$ array of holes is illustrated in Figure 4-12. This analysis was completed in 36 minutes, 54 seconds. The corresponding finite element solution for the temperature field and heat flux of the optimal design are also presented in Figure 4-13. The optimal solution retains several voids from the initial design. These voids align with the direction of heat flow and place material farther from the horizontal axis of symmetry. Although the complexity of this design has increased, the heat power radiated has also increased by 0.18 W from the optimal solution found using an initial fully-filled design domain.

Figure 4-12: Initial and optimal shapes for optimization problem (SP1) with initial material configuration of $2 \times 2$ array of equally spaced circular holes
Figure 4-13: Finite element solution of optimal design for optimization problem (SP1) with initial material configuration of $2 \times 2$ array of equally spaced circular holes.

The objective, constraint and augmented Lagrangian histories are plotted in Figure 4-14. These histories show that after approximately 15 iterations the design converges. The same velocity field metrics previously introduced are also tracked to illustrate that convergence has been reached. Figure 4-15 plots the histories of these metrics and demonstrates that the velocity approaches zero on the structural boundaries.
Figure 4-14: Objective, constraint, augmented Lagrangian and Lagrange multiplier estimate histories for optimization problem (SP1) with initial material configuration of $2 \times 2$ array of equally spaced circular holes.
Figure 4-15: Convergence history using boundary velocity metrics for optimization problem (SP1) with initial material configuration of $2 \times 2$ array of equally spaced circular holes.

Next the results for the initial material configuration containing a $4 \times 4$ array of holes are displayed in Figures 4-16, 4-17, 4-18, and 4-19. The analysis was completed in 5 minutes, 33 seconds. Our method converges to a final design which again retains some of the voids from the initial shape. Having increased the number of holes in the initial shape has resulted in more voids in the final design, and more intricate structural features near the radiating surface.
Figure 4-16: Initial and optimal shapes for optimization problem (SP1) with initial material configuration of $4 \times 4$ array of holes

Figure 4-17: Finite element solution of optimal design for optimization problem (SP1) with initial material configuration of $4 \times 4$ array of holes
Figure 4-18: Objective, constraint, augmented Lagrangian and Lagrange multiplier estimate histories for optimization problem (SP1) with initial material configuration of $4 \times 4$ array of equally spaced circular holes.
A summary of the two test design shapes along with the three topologically optimized solutions, designated as “Design A”, “Design B” and “Design C” is provided in Table 4.3.

Table 4.3: Comparison of test designs with locally optimal solutions for optimization problem (SP1).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Design 1</td>
<td>Filled Rectangle</td>
<td>22.58</td>
<td>-4.6004</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>Test Design 2</td>
<td>Filled Trapezoid</td>
<td>22.90</td>
<td>-2.9995</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>Design A</td>
<td>Fully Filled</td>
<td>23.41</td>
<td>2.8106</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Design B</td>
<td>2 x 2 Array of Holes</td>
<td>23.59</td>
<td>1.7066</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Design C</td>
<td>4 x 4 Array of Holes</td>
<td>23.59</td>
<td>4.1403</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

All of the topologically optimized designs outperformed the two test designs presented earlier. Design A achieved a 3.5% and 2.2% improvement in radiated thermal heat power from Test Designs 1 and 2 respectively. Likewise, both Designs B and C achieved a 4.3% and 2.9% improvement in performance over Test Designs 1 and 2 respectively. Despite the simplicity of the problem (e.g., square design domain, sin-
ingle radiating surface and fixed temperature boundary, single objective and constraint, etc.), our design intuition yielded a suboptimal geometry compared to those generated by the topology optimizer. This finding further supports the need originally stated in Chapter 1 for computational tools to augment the designer’s intuition regarding structural design problems. For even more complex problems involving multiple objectives, constraints and physics, an even greater performance gap would likely be found between a solution that is topologically optimized and one that is generated based on intuition alone. Additionally, as the problem complexity grows the number of design iterations needed to arrive at an optimal or even feasible solution using intuition alone would also increase.

Another important finding here is that the added topological complexity in Design B and C offer only a 0.8% improvement in performance (in terms of radiated heat power) over Design A. Additionally both Designs B and C are iso-performing, meaning they both radiate the same heat power. Therefore, the additional geometric complexity found in Design C compared to Design B in no way improves performance. This finding suggests that design problems exist where too much geometric complexity could stunt or even degrade performance. Whether or not added geometric complexity affords better performing designs is problem-dependent. This thesis will later explore the design tradespace of similar problems to better understand the performance bounds considering a wide variety of topologically complex solutions.

### 4.5 Chapter Summary

In this chapter, we introduced an example 2D problem of an aluminum plate radiating heat to the surrounding environment. A Dirichlet boundary condition was imposed on a line segment opposite the side of a fixed radiating sub-domain. We applied our level set based topology optimization method introduced in Chapter 3 to solve the problem for three different initial material configurations, each yielding a distinct locally optimal design. Our method was shown to converge smoothly without the need for velocity regularization, filtering or added penalty functions on the structural
perimeter and/or boundary curvature. The results from our analysis on this 2D example problem indicate that adding geometric complexity, in the form of voids and branching features, can in fact improve performance. However, the performance benefit gained by increasing topological complexity can also stagnate. Because of these findings, it is important that we explore the design tradespace in order to not only identify new unintuitive topological features, but also better understand their cost-benefit between geometric complexity and performance.
Chapter 5

Multi-Objective 2D Thermally Radiating Plate Case Study

Many topology optimization problems found in the literature include a single objective and single constraint function. In order to formulate a well-posed problem, one of these functions is usually the total or fractional material volume used within the design domain. However, real design applications oftentimes require a multidisciplinary approach where multiple requirements and goals are considered in addition to weight reduction. In order to capture this design intent, a clear need has been established for topology optimization methods which are able to handle multiple objectives and/or multiple constraints. Sigmund and Maute [150], and Deaton and Grandhi [49] present this need within the topology optimization community. To date some research has been published on design problems formulated with multiple objectives and/or multiple constraints.

This chapter reviews some previous work on multi-objective, multi-constraint and multi-physics topology optimization approaches in Section 5.1. Many of these studies involve solid isotropic material with penalization (SIMP) based approaches although some research in recent years has emerged involving boundary variation methods. Section 5.2 introduces optimization problem (SP2) which optimizes a weighted sum of thermal heat power radiated and volume fraction subject to a maximum von Mises stress and minimum first mode frequency constraint. A detailed discussion is also
included in this section on stress-based and natural frequency design problems. Section 5.3 discusses how the augmented Lagrangian method introduced in Section 3.3 for single objective, single equality constraint problems is extended to allow for multiple objectives and constraints. Several challenges with the proposed method are also discussed. Finally in Section 5.4, our level set based topology optimization approach previously introduced in Chapter 3 is applied to solve a 2D problem similar to the one solved in Chapter 4. In addition to shrinking the feasible design space, the additional inequality constraints promote more intricate geometries than those encountered in Chapter 4.

5.1 Literature Review

Current areas of research within the topology optimization community involve solving multi-objective, multi-constraint and/or multi-physics problems. Because most real-world systems require a multidisciplinary design approach, developing topology optimization routines to handle a wide variety of objective and constraint functions is an important next step for these methods. Unfortunately, a number of challenges still need to be overcome in order to make these routines versatile and robust enough for real-world applications. In this section we also provide a brief overview of the different classes of problems which have been solved to motivate the need for robust multidisciplinary topology optimization methods.

5.1.1 Multi-Objective and Multi-Constraint Topology Optimization

Most of the existing research implements a SIMP based approach since it is the most popular method used to solve topology optimization problems. As far back as 1997, Yang [185] employed a sequential linear programming method to solve a beam problem where the objective was to minimize weight subject to frequency, displacement and compliance constraints. Shao et al. [146] implemented a neural network algorithm to
solve a series of multi-load static problems where the load conditions were aggregated using a weighted sum. Kruijf et al. [99] also used a weighted sum approach to solve a multi-physics problem subjected to both structural and thermal loads and identified Pareto fronts for thermal and structural compliance objectives. Takezawa et al. [164] considered a set of multi-physics problems whose objectives were to minimize weight subject to both stress and thermal compliance constraints incorporating the effects of thermal strain due to temperature changes. Several commercial software tools such as Altair OptiStruct [11] and NASTRAN [167] have also implemented multi-constraint capabilities using the SIMP method. While less popular than SIMP methods, Kim et al. [94] applied an evolutionary structural optimization method to design a 3D thermal protection system to maximize the fundamental natural frequency while minimizing the maximum thermal stress.

Few papers have been published on multi-objective, multi-constraint problems solved using boundary variation methods. Suresh [159] developed a novel method to efficiently trace the Pareto front of a 2D cantilever subjected to a single static load using a weighted sum of structural compliance and volume fraction. By restarting each optimization run using a perturbation of the previously found optimized solution, his approach identifies “nearby” Pareto optimal designs. Deng and Suresh [51] extended this approach to handle multiple stress and displacement constraints using an augmented Lagrangian method. They also extended their method to solve 3D problems [52]. The approach uses a topological sensitivity field only so that holes can be nucleated in 2D design domains. Zhu et al. [190] and Xia and Shi [175] optimized 2D compliant mechanisms by formulating a weighted sum of several objective functions including target displacement, structural compliance, perimeter and volume fraction. Guirguis et al. [74] explored a hybrid method which optimized both material and component layouts for sheet metal structures considering weight, assembly and manufacturing costs. The level set function was updated using an NSGA-II algorithm since the objectives did not have analytic shape derivatives. Ryu et al. [139] implemented an adaptive weighted sums approach to maximize the magnetic force of an actuator while minimizing the volume. This approach more evenly distributed the
design solutions along the Pareto front compared to a weighted sums method.

5.1.2 Multi-Physics Topology Optimization

While structural compliance is the most commonly found objective function within the topology optimization literature, a number of other problems involving thermal heat flow, structural dynamics, and aerodynamics have also been solved. Moreover, since level set methods are oftentimes reliant on gradient-based optimizers, the derivation of analytic shape sensitivities is crucial to research in multi-objective, multi-constraint problems. Many of the papers considered in this section provide analytic shape derivatives.

Structural compliance is a measure of how much work is being done on a structure by an external load. Many authors have used level set based topology optimization methods to minimize structural compliance to improve rigidity subject to a volume fraction constraint [172, 10, 4, 105, 33, 168]. Researchers have also solved problems designing compliant mechanisms (e.g., grippers and clamps) where a target displacement is desired at specific locations. Allaire et al. [10] derived a shape sensitivity for the least square error on a desired displacement. Luo et al. [107] also solved a compliant mechanism problem, but accounted for large deformations where the design material exhibits a nonlinear response. Dunning et al. [59] also solved a structural compliance problem subject to uncertainty in the load conditions such that the direction and magnitude of the static force vector was described by Gaussian distributions.

Stress-based design has also received substantial focus where the goal is to minimize stress or ensure that it does not exceed a critical level (e.g., rising above the von Mises failure criterion). Sethian and Wiegmann [143] first considered the minimization of stress within elastic structures subjected to a single static load. Since then Allaire et al. [9] have developed a general framework to minimize a class of smooth global stress functions to approximate the maximum stress value within the structure. James et al. [85] then applied Allaire’s method to minimize the von Mises stress of an L-bracket using isoparametric elements. More details on stress-based design will be addressed in Section 5.2.1.
Authors have further extended the capabilities of level set based optimization techniques to include dynamics analyses, which involve structural responses occurring in the frequency or time domain. Several dynamics analyses include normal modes, acoustics, and random vibrations. Allaire and Jouve [8] derived a shape sensitivity for maximizing the fundamental natural frequency of a structure. The approach developed by Allaire and Jouve for maximizing first mode frequency including the shape sensitivity will be discussed in Section 5.2.2. Ansari et al. [15] took one step further and maximized the modal loss factor of the fundamental natural frequency by optimizing the shapes and locations of damping patches to dissipate the vibration energy. Shu et al. [148] solved a series of acoustics problems for minimizing sound pressure levels within a predefined frequency range at specified locations in the design domain. Noguchi et al. [122] applied a similar method to the design of acoustic cavities by formulating a two-phase material problem where an acoustic medium and elastic structure are modeled. The objective in this problem was also to minimize the average sound pressure level at a fixed boundary.

More recently, application of level set based methods to problems involving heat and fluid flow has emerged in the literature. For heat conduction problems, recall from Section 4.2 that many authors have used thermal compliance to design efficient conducting structures. Giusti et al. [72] also derived a topological derivative of the thermal compliance function to facilitate nucleation of holes in the interior of a 2D material domain. Kim et al. [92] rederived this topological derivative for materials with nonlinear heat conduction properties (i.e., materials whose conductivity is temperature dependent). Similarly, Giusti and Novotny [70] rederived the topological derivative for anisotropic and heterogeneous materials. Finally, Deng et al. [50] optimized the designs of lightweight thermoelastic structures accounting for thermal strain. For fluid flow problems, much of the initial work on shape analysis was done by Mohammadi and Pironneau [116]. Since the publication of [116], several researchers have tackled similar problems. Deng et al. [53] solved the Navier-Stokes equations with body forces to minimize the dissipated power and thus identify a fluid system with minimal pressure drop. Yaji et al. [181] optimized heat exchangers by solving a
coupled thermal-fluid problem governed by the steady-state Navier-Stokes equations to maximize the heat exchanged between a solid-fluid interface. Dunning et al. [62] optimized a 3D model of the NASA Common Research Model wing box subjected to coupled aerodynamic loading. Duan and Li [57] developed a framework to solve the Stokes equations for incompressible fluids to also minimize the dissipated power.

Yet another field where level set based topology optimization methods have been employed is electromagnetics. Yamasaki et al. [184], Hirayama et al. [81] and Otomori et al. [127] all developed frameworks to design ferromagnetic waveguides by solving the Helmholtz equations to maximize the transmission power at specified frequencies. Otomori et al. [126] then later developed a framework to design an electromagnetic cloak made of a ferrite material to minimize the scattering of an electric field.

Clearly, the versatility of level set based methods demonstrated by the above research supports the notion that a wide variety of functions and their corresponding shape and/or topological derivatives can be employed to tackle multi-physics problems. Moreover, as was discussed in Section 2.3, the implicit definition of a crisp structural boundary is advantageous for problems where loads are dependent on the orientation and smoothness of the boundary (e.g., acoustic problems). The obvious next step is to develop multidisciplinary approaches that can handle a variety of these functions for real-world multidisciplinary applications.

5.2 Multi-Objective, Multi-Constraint Thermally Radiating Problems

In this section we consider a multi-objective, multi-constraint thermally radiating 2D design problem. The design intent is to produce a thermally efficient radiating structure, where we have defined thermal efficiency in Section 3.2 as the ability to exchange as much heat power between itself and its surroundings as possible provided some restriction on material or weight. In addition, we now levy von Mises stress and first mode frequency constraints to ensure that the structure is sufficiently rigid.
5.2.1 Stress-Based Design

We consider 2D problems loaded by a volumetric or body force and constrained by a fixed displacement boundary condition along a subset of the boundary $\partial \Omega$. Figure 5-1 illustrates the design domain $\mathcal{D}$, structural domain $\Omega$, structural boundary $\partial \Omega$. In addition we have defined a pressure loaded sub-domain $\Omega_{\text{load}}$ where the volumetric force is applied. The structural boundary consists of two disjoint parts: the Dirichlet $\Gamma_{D,u}$, and traction-free $\Gamma_{0,u}$ boundaries.

$$\partial \Omega = \Gamma_{D,u} \cup \Gamma_{0,u}$$  \hspace{1cm} (5.1)

The Dirichlet and traction-free boundaries do not intersect. Also note that the pressure loaded sub-domain always remains a subset of the structure domain ($\Omega_{\text{load}} \subset \Omega$). These relations are illustrated in Figure 5-1.

Figure 5-1: Structural domains and boundaries for static loads problem
In this thesis we consider an isotropic, linearly elastic material subjected to a static load condition. The governing PDE for this class of problems is provided in Equation (5.2):

\[
\begin{align*}
\text{div} \ (\sigma) + f &= 0 \quad \text{in} \quad \Omega_{\text{load}} \\
\text{div} \ (\sigma) &= 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{load}} \\
\sigma n &= 0 \quad \text{on} \quad \Gamma_{0,u} \\
u &= 0 \quad \text{on} \quad \Gamma_{D,u}
\end{align*}
\] (5.2)

where \( \sigma \) is the 2\textsuperscript{nd} order stress tensor, \( f \) is the volumetric body force, \( u \) is the displacement field, and \( n \) is the unit normal to the boundary. The stress-strain relationship is governed by Hooke’s law which is given in Equation (5.3):

\[
\sigma = C \varepsilon (u)
\] (5.3)

where \( C \) is the 4\textsuperscript{th} order constitutive tensor derived from the material properties, and \( \varepsilon (\cdot) \) is the strain function which takes a vector field as an argument and outputs a 2\textsuperscript{nd} order tensor.

\[
\varepsilon (u) = \frac{1}{2} \left( \nabla u^T + \nabla u \right)
\] (5.4)

Equation (5.4) is derived with the assumption of infinitesimal strain within a continuum body. The governing PDE in Equation (5.2) is solved using a finite element model where the degrees of freedom are the displacements in the \( \hat{x} \) and \( \hat{y} \) directions, \( u_x \) and \( u_y \) respectively.

Stress-based design problems seek to minimize or constrain the maximum stress level within a structure. This class of problems can be particularly challenging because stress is a vector field defined at each point in the structure. Thus if the designer wanted to ensure that a specific stress level was not exceeded within the finite element model, a constraint would need to be imposed on each element. While some authors have implemented local stress constraints in a topology optimization framework [128], this approach would quickly render larger design problems intractable. Others have used global functions to aggregate the stress constraints into a single differentiable
A global constraint function approximates the maximum value of stress within a domain. Two of the most common functions include the Kreisselmeier-Steinhauser (KS) function, originally used in control systems design [95], and the p-norm function which has been employed in several topology optimization applications for stress-based design [89]. The KS function is defined in Equation (5.5).

\[ g(\sigma, b) = \frac{1}{b} \ln \left[ \frac{1}{\alpha_{KS}} \int_{\Omega} e^{b\sigma} dV \right] \]  \hspace{1cm} (5.5)

where \( b \) is the aggregation parameter, and \( \alpha_{KS} \) is an arbitrary constant used to scale the KS function. In the limit as \( b \rightarrow \infty \) the KS function approaches \( \max(\sigma) \). The p-norm functional is written in Equation (5.6).

\[ g(\sigma, b) = \left[ \frac{1}{\alpha_{p-norm}} \int_{\Omega} |\sigma|^b dV \right]^{\frac{1}{b}} \]  \hspace{1cm} (5.6)

where again \( b \) is the aggregation parameter and \( \alpha_{p-norm} \) is an arbitrary scaling constant. An important distinction between the KS and p-norm functionals is that the p-norm functional approximates \( \max(|\sigma|) \) rather than \( \max(\sigma) \).

We choose a p-norm functional to enforce a von Mises stress inequality constraint with \( \alpha_{p-norm} = 1 \). Our approach closely follows that laid out by James et al. [85], although many others have chosen to use a p-norm functional for their stress-based topology optimization methods [9, 129, 176, 86]. The global stress constraint function is provided below:

\[ g(\sigma_{VM}, b) = \left[ \int_{\Omega} \sigma_{VM}^b dV \right]^{\frac{1}{b}} \]  \hspace{1cm} (5.7)

where \( \sigma_{VM} \) is the von Mises stress defined from the Cauchy stress tensor \( \sigma \) in the following equations:

\[ \sigma_{VM} = \sqrt{\frac{1}{2} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2) \right]} \]  \hspace{1cm} (5.8)

The subscripts in Equation (5.8) represent the components of the 2\textsuperscript{nd} order Cauchy
stress tensor $\sigma$. Because the von Mises stress is strictly non-negative we can drop the absolute value from Equation (5.6). Designers use the von Mises stress in structural analyses to evaluate the following yield failure criteria:

$$\sigma_{VM} = \sigma_Y$$ (5.9)

where $\sigma_Y$ is the yield strength of a homogeneous material. If the von Mises stress exceeds the yield strength of the material, then permanent deformation and failure is expected.

Several challenges exist in stress-based topology optimization design. Because the aggregation functionals only approximate the maximum stress, there is no guarantee that the yield strength is not exceeded at all points within the structure. While some authors have developed techniques to adaptively scale the stress constraint during the optimization routine [100], this approach requires some tuning of an update parameter and renders the stress constraint function non-differentiable between optimization iterations. If the aggregation parameter $b$ in Equation (5.7) is increased, than the p-norm functional will more closely estimate the maximum stress value in the structure. However, increasing this parameter also reduces the smoothness of the stress functional and adversely affects convergence [100, 164]. Stresses at corners and other sharp edges in the structure can reach very high values which also reduces the smoothness of the shape sensitivity and further hinders convergence.

5.2.2 Frequency-Based Design

Modal analysis is used to find the natural frequencies and mode shapes of a structure. These results can be used in further dynamic analyses and to predict when the structure will encounter resonance with external excitation forces. Figure 5-2 illustrates the geometry of a 2D modal problem where a single Dirichlet boundary $\Gamma_{D,u}$ is enforced. Note that this boundary is identical to that from Figure 5-1 since the degrees of freedom are the same between the statics and modal model (i.e., the displacements $u_x$ and $u_y$).
Figure 5-2: Structural domains and boundaries for eigenfrequency problem

The governing PDE for the eigenvalue problem illustrated in Figure 5-2 is given in Equation (5.10)

\[
\begin{align*}
\text{div} \left( C\epsilon(\psi) \right) + \omega^2 \rho \psi &= 0 & \text{in } \Omega \\
(C\epsilon(\psi)) n &= 0 & \text{on } \Gamma_{0,u} \\
\psi &= 0 & \text{on } \Gamma_{D,u}
\end{align*}
\]

where \( \omega \) is the natural frequency corresponding to the mode shape \( \psi \), and \( \rho \) is the material density. The above equation admits an infinite number of solutions \((\omega_j, \psi_j)\) for \( j \in \mathbb{Z}_{>0} \) where \( \mathbb{Z}_{>0} \) is the set of all positive integers. Designers are oftentimes concerned with the first mode frequency \( \omega_1 \) (also known as the fundamental frequency) and corresponding first mode shape \( \psi_1 \) since low frequency excitations are abundant in many aerospace design applications and can cause catastrophic failure. As an example, the Space Mission Engineering, or more commonly known “SMAD” reference book [174] discusses how understanding and designing for the launch environment is
critical to every satellite program. Specifically on pages 847–850, a discussion regarding coupled loads analysis explains how dynamic mechanical loads imparted by the launch vehicle can resonate with the natural frequencies of the satellite payload. Such excitations are caused by the rocket motor, thrust ramping and aerodynamic forces as the launch vehicle travels through the atmosphere. On page 848, Table 26-4 lists the minimum recommended spacecraft structural frequencies for a variety of launch vehicles. While these numbers are rough estimates and depend on the spacecraft mass, size and configuration, they motivate the need to design rigid structural elements.

Because of such design concerns, a number of researchers have implemented topology optimization techniques to maximize the natural frequencies of a structure [130, 84, 8, 177, 101]. To do so a modal analysis is performed to calculate a finite set of modes, typically corresponding to the first \( w \) frequencies starting with the fundamental frequency \( \{\omega_1, ..., \omega_w\} \). In this thesis, constraints are levied on the first mode frequency alone.

Several challenges exist when trying to maximize the first mode frequency using topology optimization. First, both the mass and stiffness of the structure need to be accounted for when performing a modal analysis. Since many methods use the Ersatz material approach where voids are considered to have a small nonzero stiffness and mass, \( \delta_{\text{stiff}} \) and \( \delta_{\text{mass}} \) respectively, it is important that these parameters are chosen such that spurious modes do not originate in void domains. These modes can take the place of the fundamental mode and are entirely nonsensical. Another challenge arises when the first mode frequency has repeated eigenvalues since the differentiability of the objective function is lost. However, several authors have developed methods implementing the directional derivative to overcome this phenomenon and identify converged solutions [145, 48, 177]. It is also possible for the natural frequencies to cross during the optimization routine as the structural shape evolves. Similar to the aforementioned repeated eigenvalue problem, these crossings can cause the optimizer to become unstable and prevent convergence [8].
5.2.3 Optimization Problem

The multi-objective, multi-constraint problem considered here is an extension of optimization problem (SP1) developed in Section 3.2. As before, the thermal heat power radiated is defined as the first objective. The second objective is the volume fraction which was previously defined as an equality constraint in Equation (3.6). Because a weighted sums approach is used to formulate a single objective, both functions are normalized. While the volume fraction is already defined such that it lies in the range $[0, 1]$, the heat power radiated is not normalized. Therefore a constant $Q_{\text{rad}}^{\text{max}}$ is used to normalize the heat power radiated objective function. This constant is the largest possible amount of heat power that can be radiated by a structural shape $\Omega \in \mathcal{D}$. We compute $Q_{\text{rad}}^{\text{max}}$ by solving the finite element model with all element densities $\rho_e = 1$, where $e = 1, \ldots, m$ and $m$ is the total number of elements in the finite element model. The entire design domain is filled with material since the inclusion of any voids will only decrease the conductivity of the structure. The normalized heat radiated objective functions $J_1$ is defined below:

$$J_1(\Omega, \theta) = \frac{\int_{\Omega} -\chi \epsilon \gamma \theta^4 dA}{Q_{\text{rad}}^{\text{max}}}$$  \hspace{1cm} (5.11)

The volume fraction objective $J_2$ is unchanged from Equation (3.6):

$$J_2(\Omega) = \frac{\int_{\Omega} dV - V_{\text{req}}}{\int_{\partial\Omega} dV}$$  \hspace{1cm} (5.12)

Next two inequality constraints are considered: (1) an upper limit on the maximum von Mises stress and (2) a lower limit on the first mode frequency. As was discussed in Sections 5.2.1 and 5.2.2, these constraints are used to ensure that the structure is sufficiently rigid. Rearranging the global function for the von Mises stress introduced in Equation (5.7), the first constraint $g_1$ is defined in the following equation:

$$g_1(\Omega, \sigma_{VM}, b) = \frac{\int_{\Omega} \sigma_{VM}^b dV}{\sigma_b^b} - 1 \leq 0$$  \hspace{1cm} (5.13)
where $\bar{\sigma}$ is the upper bound on the global von Mises stress function. Recall from Section 5.2.1 that the global stress function only approximates the maximum, and by setting $\bar{\sigma} = \sigma^*$ we are making a conservative estimate. Since the p-norm function integrates the stress field over the structural domain $\Omega$, it is also included as an argument to the function $g_1$. The first mode frequency constraint is defined next:

$$g_2(\Omega, \omega_1) = 1 - \frac{\omega_1^2}{\omega^2} \leq 0 \quad (5.14)$$

where $\bar{\omega}$ is the lower bound for the first mode frequency.

Optimization problem (SP2) is posed using the objective and constraint functions defined above along with the governing PDEs in Equations (3.7), (5.2), and (5.10):

$$\text{(SP2)}: \min_{\Omega} \quad w_1 \left( \frac{\int_{\Omega} - \chi T^4 dA}{Q_{\max}} \right) + w_2 \left( \frac{\int_{\Omega} \sigma_b V_{\text{rad}} dV}{\bar{\sigma}^b} \right) - 1 \leq 0$$

subject to

\[
\begin{align*}
\text{div} (\kappa \nabla \theta) &= \varepsilon T^4 \quad \text{in} \ \Omega_{\text{rad}} \\
\text{div} (\kappa \nabla \theta) &= 0 \quad \text{in} \ \Omega \setminus \Omega_{\text{rad}} \\
(\kappa \nabla \theta) \cdot n &= 0 \quad \text{on} \ \Gamma_{0,\theta} \\
\theta &= \theta_0 \quad \text{on} \ \Gamma_{D,\theta} \\
\text{div} (\sigma) + f &= 0 \quad \text{in} \ \Omega_{\text{load}} \\
\text{div} (\sigma) &= 0 \quad \text{in} \ \Omega \setminus \Omega_{\text{load}} \\
\sigma n &= 0 \quad \text{on} \ \Gamma_{0,u} \\
u &= 0 \quad \text{on} \ \Gamma_{D,u} \\
\text{div} (\mathcal{C} \varepsilon (\psi)) + \omega^2 \rho \psi &= 0 \quad \text{in} \ \Omega \\
(\mathcal{C} \varepsilon (\psi)) n &= 0 \quad \text{on} \ \Gamma_{0,u} \\
\psi &= 0 \quad \text{on} \ \Gamma_{D,u}
\end{align*}
\]

where $w_1$ and $w_2$ are the weighting parameters for the thermal heat radiated and volume fraction objective functions respectively. The equality constraints are satisfied by solving the governing PDEs using the finite element method, which is discussed in greater detail in Section 5.4.2.
5.3 Augmented Lagrangian Method

As was done previously in Section 3.4, we again select an augmented Lagrangian method to reformulate the objectives and inequality constraints in (SP2) into a series of unconstrained subproblems. For additional information on current algorithms, extensions and practical implementation issues of augmented Lagrangian methods, the interested reader is referred to [121] and [23].

The augmented Lagrangian used here implements a weighted sums technique to combine the objective functions $J_1$ and $J_2$. The two inequality constraints $g_1$ and $g_2$ each have an associated Lagrange multiplier estimate and penalty parameter which is iteratively updated in the optimization routine. The augmented Lagrangian is provided in Equation (5.15):

$$L(Q, \theta, \lambda_1, \mu_1, \lambda_2, \mu_2) = w_1 J_1(\Omega, \theta) + w_2 J_2(\Omega)$$

$$+ \begin{cases} 
\lambda_1 g_1(\Omega) + \frac{1}{2} \mu_1 g_1(\Omega)^2 & \text{if } \lambda_1 + g_1(\Omega)\mu_1 > 0 \\
-\frac{1}{2} \frac{\lambda_1^2}{\mu_1} & \text{otherwise} 
\end{cases}$$

$$+ \begin{cases} 
\lambda_2 g_2(\Omega) + \frac{1}{2} \mu_2 g_2(\Omega)^2 & \text{if } \lambda_2 + g_2(\Omega)\mu_2 > 0 \\
-\frac{1}{2} \frac{\lambda_2^2}{\mu_2} & \text{otherwise} 
\end{cases}$$

(5.15)

where the constraint function dependencies on $\sigma_{VM}$, $b$, and $\omega_1$ have been removed for clarity but still implied. In Equation (5.15), $\lambda_1$ and $\mu_1$ are the Lagrange multiplier estimate and penalty parameter respectively for the first constraint, and $\lambda_2$ and $\mu_2$ correspond to the second constraint. Because we are working with inequality constraints, both $\lambda_{1,2} \in \mathbb{R}_{\geq 0}$ and $\mu_{1,2} \in \mathbb{R}_{\geq 0}$ where $\mathbb{R}_{\geq 0}$ is the set of strictly non-negative real numbers.

According to Equation (5.15) the inequality constraints are violated if the condition $g_{1,2} > -\lambda_{1,2}/\mu_{1,2}$ is met rather than $g_{1,2} > 0$. The term $\lambda_{1,2}/\mu_{1,2}$ can be interpreted as a shift and is commonly used to improve the coincidence of the approximate minimizer of the augmented Lagrangian in Equation (5.15) with that of the local solution to the optimization problem in (SP2) [23]. The penalty terms $\frac{1}{2} \mu_{1,2} g_{1,2}(\Omega)^2$ affect the
augmented Lagrangian only if the corresponding inequality constraint becomes active (i.e. if $g_{1,2} > -\frac{\lambda_{1,2}}{\mu_{1,2}}$).

The updates to the Lagrange multiplier estimates for the inequality constraints closely follow those for the equality constraints found in Equation (3.10).

$$\lambda^{k+1} = \max \left[ 0, \lambda^k + \mu^k g(\Omega^k) \right] \quad (5.16)$$

In Equation (5.16), the superscript $k$ indicates the optimizer iteration. The $\max[\cdot]$ function forces the Lagrange multiplier estimate to 0 if the condition $g_{1,2} \leq -\frac{\lambda_{1,2}}{\mu_{1,2}}$ is met, meaning that the constraint becomes inactive. The penalty parameters monotonically increase as before in Section 3.3 according to Equation (3.11).

## 5.4 Example 2D Multi-Objective Thermal Radiation Problem

The example problem considered here is adopted from Section 4.1 with the addition of a static load acting on the radiating sub-domain $\Omega_{\text{rad}}$ to generate a stress field within the plate. We present the geometry of the problem as well as the uniform mesh used for thermal, static load and modal finite element analyses. We also introduce the shape derivatives of the von Mises global stress and first mode frequency functions.

### 5.4.1 Problem Definition

The geometry and material properties for the problem presented here are identical to that in the example problem from Section 4.1. The same thermal boundary conditions, a fixed temperature constraint on a line segment of the left edge and a radiating sub-domain on the right side, are also applied to this problem. However, we now consider the static load condition where a vertical body force of 300 N cm$^{-3}$ is also placed on the radiating sub-domain $\Omega_{\text{rad}}$ in the negative $\hat{y}$-direction. Therefore the loaded and radiating sub-domains are equivalent $\Omega_{\text{rad}} = \Omega_{\text{load}}$. In order to properly constrain the plate, the left edge is fixed with zero displacement. These boundary conditions
are illustrated in Figure 5-3.

![Figure 5-3](image)

**Figure 5-3**: Geometry, boundary conditions and sub-domain definition with additional static load condition of 2D thermally radiating plate

For the modal analysis, we also use the same Dirichlet boundary condition on the left edge of the plate. External loads are not considered when calculating natural frequencies.

While the left edge of the plate has a fixed displacement condition, the optimizer may move the structural boundary $\partial \Omega$ away from or onto this edge. However, only the structural boundary that is coincident with the left edge will enforce a fixed displacement condition. This rule also applies to the Dirichlet boundary condition for the thermal model illustrated in Figure 4-1. The Dirichlet boundary conditions are applied to the segments of $\partial \Omega$ coincident with the blue edges illustrated in Figures 4-1 and 5-3. This condition removes any design-dependent load considerations as was discussed in Section 4.1.1

We require that the stresses in the plate do not exceed the yield strength $\sigma_Y = 276 \text{ MPa}$ of the aluminum alloy material according to Table 4.1. We set $\bar{\sigma} = \sigma_Y$ knowing that the global constraint function will be a conservative estimate of the maximum von Mises stress over all elements. Unfortunately we do not have a method for estimating the margin a priori. Additionally, we also require that the first mode
frequency lies at or above $\bar{\omega} = 50$ Hz. The frequency limit of $50$ Hz and vertical body force $300 \text{ N cm}^{-3}$ were chosen such that for an equal weighting of the objective functions at least one of the constraints would be active when solving optimization problem (SP2). Additionally we also choose the constraints and load values such that a feasible design space existed.

### 5.4.2 Finite Element Models

In this multidisciplinary framework, finite element analyses are employed to solve each of the three PDEs defined in Equations (3.7), (5.2), and (5.10) for the thermal, static loading and modal analyses respectively on the same mesh. For each finite element model the same mesh is used to alleviate any need for interpolating sensitivities between models. The finite element mesh is also the same as that used in Section 4.1.2. Recall that a total of 5000 linear triangular elements are assembled in a structured array to allow the Hamilton Jacobi equation solver to use an upwind finite difference scheme. The linear elements are constant strain, meaning that both strain and stress are uniform within each element, and discontinuous between elements.

For the static loading and frequency analysis, motion is restricted to the $\hat{x} - \hat{y}$ plane, and the plane stress assumption is made since the element thickness is much smaller than the dimensions of the plate. This assumption sets the normal $\sigma_{33}$ and shear stresses $\sigma_{13}$ and $\sigma_{23}$ in the $\hat{z}$-direction (normal to the $\hat{x} - \hat{y}$ plane) to be zero. The material deformation is described at each node by two degrees of freedom $u_x$ and $u_y$ representing the $\hat{x}$ and $\hat{y}$ displacement in the global coordinate system respectively. The displacement fields are interpolated using the same linear shape functions found in Equation (4.2)

The strain is calculated in each element $e$ in the global coordinate system $(\hat{x}, \hat{y})$ by using the following matrix calculation:

$$
\varepsilon_e = \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12}
\end{bmatrix} = J^{-1}Bu_e \quad (5.17)
$$

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where $\epsilon_e$ is the element strain vector, $\epsilon_{11}$ and $\epsilon_{22}$ are the normal strains in $\hat{x}$ and $\hat{y}$ directions respectively, $\epsilon_{12}$ is the shear strain in the $\hat{x} - \hat{y}$ plane, $J_e$ is the Jacobian matrix defined in Equation (4.6), $B$ is the element strain-displacement matrix defined in Equation (4.5), and $u_e$ is the element displacement vector. The element stresses and strains are related through Hooke's law which was stated in Equation (5.3) and is rewritten in matrix form below:

$$
\sigma_e = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = C \epsilon_e 
$$

(5.18)

where $\sigma_e$ is the element stress vector, $C$ is the stress-strain matrix defined in Equation (5.19), and the subscripts 11 and 22 again indicate the normal stress components whereas 12 indicates the shear stress component.

$$
C = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}
$$

(5.19)

In Equation (5.19), $\nu$ is the Poisson’s ratio and $E$ is the Young’s modulus of the material. This stress-strain matrix is specific to the plane stress assumption. To calculate the scalar von Mises stress for any single element, the following matrix calculation is adopted from [162]:

$$
\sigma_{VM} = \sqrt{\sigma_e^T S \sigma_e}
$$

(5.20)

where $S$ is a $3 \times 3$ symmetric coefficient matrix:

$$
S = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}
$$

(5.21)
The finite element program solves the following linear system of equations:

\[ \mathbf{Ku} = \mathbf{f} \quad (5.22) \]

where \( \mathbf{K} \) is the \( 2p \times 2p \) global stiffness matrix, \( \mathbf{u} \) is the \( 2p \times 1 \) displacement vector and \( \mathbf{f} \) is the \( 2p \times 1 \) force vector. The dimension \( 2p \) corresponds to the total degree of freedoms (DOFs) of the finite element model where \( p \) is the total number of nodes (i.e., there are two DOFs per node). The global stiffness matrix is assembled according to Equation (3.18) where the element densities \( \rho_e \) are mapped from the level set function. The element stiffness matrix \( \mathbf{K}_e \) is defined as follows:

\[ \mathbf{K}_e = \int_{\tau_e} \mathbf{B}^T \mathbf{C} \mathbf{B} \det(\mathbf{J}_e) dA \quad (5.23) \]

Because the element thickness is unity (1 cm) it is not explicitly shown in Equation (5.23).

A modal finite element analysis involves solving the following eigenvalue problem:

\[ \mathbf{K}\psi_i = \omega_i^2 \mathbf{M}\psi_i \quad (5.24) \]

where \( \omega_i \) and \( \psi_i \) are the \( i^{th} \) mode frequency and shape respectively and \( \mathbf{M} \) is the global mass matrix. The global mass matrix is assembled according to Equation (3.18) where again the element densities \( \rho_e \) are mapped from the level set function. The element mass matrix \( \mathbf{M}_e \) is defined as follows:

\[ \mathbf{M}_e = \int_{\tau_e} \rho \mathbf{H}^T \mathbf{H} \det(\mathbf{J}_e) dA \quad (5.25) \]

where \( \rho \) is the material density from Table 4.1 and \( \mathbf{H} \) is the displacement interpolation matrix defined in the following equation:

\[ \mathbf{H} = \begin{bmatrix} N_1^e & N_2^e & N_3^e & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^e & N_2^e & N_3^e \end{bmatrix} \quad (5.26) \]
Because the element thickness is unity (1 cm) it is not explicitly shown in Equation (5.25).

The first $w$ modes are calculated and the corresponding mode shapes are mass-orthonormalized:

$$
\psi_i^T M \psi_j = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{for } i \neq j 
\end{cases}
$$

(5.27)

where $i = 1, ..., w$ and $j = 1, ..., w$. Recall that repeated mode frequencies may be encountered depending on the geometry and boundary conditions of the problem.

### 5.4.3 Shape Sensitivities

The von Mises stress constraint found in optimization problem (SP2) is defined in Equation (5.13). The denominator in this function is a constant (the yield strength raised to the power $b$) and thus we require an analytic sensitivity for the numerator. The shape derivative for this functional is provided in [85] and restated below:

$$
Dg_1(\Omega)(\xi) = \int_{\partial \Omega} \xi \cdot n \left( \sigma_{VM}^b + \epsilon(q^*) : C\epsilon(u) \right) ds
$$

(5.28)

where $q^*$ is the adjoint vector, the colon operator $:$ is the tensor inner product, and $u$ is the displacement solution to the governing PDE defined in Equation (5.2). The adjoint solution $q^*$ is computed by solving the adjoint equation which is rewritten from [9]:

$$
\text{div} \left( C\epsilon(q^*) \right) = -\text{div} \left( C \frac{\partial (\sigma_{VM}^p)}{\partial \sigma} \right) \quad \text{in} \quad \Omega \\
(C\epsilon(q^*))n = -\left( C \frac{\partial (\sigma_{VM}^p)}{\partial \sigma} \right) n \quad \text{on} \quad \Gamma_{0,u}
$$

(5.29)

$$
q^* = 0 \quad \text{on} \quad \Gamma_{D,u}
$$

A numerical solution to Equation (5.29) is also obtained by solving a finite element model with an identical mesh to the static loads finite element model. According to Equation (5.29), the same global stiffness matrix can be used as in Equation (5.22) but a new body force vector $f^*$ needs to be calculated using the right-hand-sides of
Equation (5.29). Thus we solve the new linear system of equations:

\[ K q^* = f^* \]  

(5.30)

where \( q^* \) is the adjoint vector. Once the solution \( q^* \) is found, it is then used to compute the shape derivative in Equation (5.28). The term \( (\sigma_{\text{YM}}^b + \epsilon(q^*) : C\epsilon(u)) \) in Equation (5.28) is used to calculate the velocity field (using a steepest descent direction) for the Hamilton Jacobi solver and is written in matrix form as follows:

\[ v_e = - \left( \left( \sigma^T_e S \sigma_e \right)^{b/2} + \frac{1}{|T_e|} q_e^T K_e u_e \right) \]  

(5.31)

where the subscript \( e \) identifies the local element, and \( v_e \) is the velocity at the element centroid. The first term is equivalent to the von Mises stress as was given by Equation (5.20) raised to the power \( b \). The second term resembles the element strain but is computed using both the adjoint and displacement vectors \( q_e \) and \( u_e \) respectively, rather than just the displacement vector. The second term is again normalized by the element volume \( |T_e| \) (i.e., area in 2D) in Equation (5.31) due to the fact that the finite element equations are written in the weak form, as was previously discussed in Section 4.3.

The shape derivative for maximizing the first mode frequency is restated from [8] in the following equation:

\[ Dg_1(\Omega)(\xi) = \frac{\int_{\partial_1} \xi \cdot n \left( \omega_1^2 \rho |\psi_1|^2 - \epsilon(\psi_1) : C\epsilon(\psi_1) \right) ds}{\int_{\Omega} \rho |\psi_1|^2 dV} \]  

(5.32)

where \( \psi_1 \) is the first mode shape associated with the fundamental frequency \( \omega_1 \), and \( \rho \) is the material density (not to be confused with the element densities \( \rho_e \) used to interpolate the material properties). A second term found in the original derivation [8] has been removed since it was defined on the Dirichlet boundary which is considered fixed (i.e., the optimizer cannot move or alter the Dirichlet boundary).

We again use the finite element method to evaluate the shape sensitivities at the element centroids defined in Equation (5.32). The term \( \frac{\omega_1^2 \rho |\psi_1|^2 - \epsilon(\psi_1) : C\epsilon(\psi_1)}{\int_{\Omega} \rho |\psi_1|^2 dV} \) in Equa-
tion (5.32) is used to calculate the velocity field (using a steepest descent direction) for the Hamilton Jacobi solver and is written in matrix form as follows

\[
v_e = \frac{\omega_i^2 \rho \left| \tilde{\psi}_{e,1} \right|^2}{\psi_{e,1}^T M_e \psi_{e,1}} - \frac{1}{\left| \tilde{\tau}_e \right|} \psi_{e,1}^T K_e \psi_{e,1}
\]

(5.33)

where \( \psi_{e,1} \) is the first mode shape displacement vector for element \( e \), and \( \tilde{\psi}_{e,1} \) is the displacement at the element centroid. Recall that for 2D linear triangle elements, the displacement at the centroid \( \tilde{\psi}_e \) is computed as follows:

\[
\tilde{\psi}_e = \frac{1}{3} \sum_{i=1}^{3} \psi_i^e
\]

(5.34)

where \( \psi_i^e \) is \( i^{th} \) nodal value of the eigenvector in element \( e \). Again the velocity field is evaluated at the element centroids and interpolated to the nodes before being passed to the Hamilton Jacobi equation solver.

### 5.5 Numerical Results

The objective function is a weighted sum of both the thermal heat radiated and the volume fraction, which without any constraints is still a well-posed problem. Before solving optimization problem (SP2), we solve a set of precursor problems (SP2a), (SP2b) and (SP2c) which share the same objective functions as (SP2) but include (a) neither constraints, (b) only the von Mises stress constraint, and (c) only the first mode frequency constraint, respectively. These precursor problems help to gain some intuition of the optimization problem (SP2) and understand how each constraint is affecting the final optimal shape. Finally, both constraints are considered simultaneously and we solve optimization problem (SP2). For all cases Pareto plots are used to illustrate the trade-offs between the competing objectives.
5.5.1 Unconstrained Problem

Initially we consider a problem where no constraints are levied, and only maintain the weighted sum of the heat power radiated and volume fraction objectives. This problem, designated (SP2a), is provided below:

\[
\text{(SP2a):} \quad \begin{align*}
\min_{\Omega} \quad & w_1 \left( \frac{\int_{\Omega} - \chi_\epsilon \gamma \theta^4 \, dA}{Q_{\text{rad}}^{\text{max}}} \right) + w_2 \left( \frac{\int_D dV - V_{\text{req}}}{f_D dV} \right) \\
\text{subject to} \quad & \text{div} (\kappa \nabla \theta) = \varepsilon \gamma \theta^4 \quad \text{in} \quad \Omega_{\text{rad}} \\
& \text{div} (\kappa \nabla \theta) = 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{rad}} \\
& (\kappa \nabla \theta) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta} \\
& \theta = \theta_0 \quad \text{on} \quad \Gamma_{D,\theta}
\end{align*}
\]

Recall that the maximum power radiated $Q_{\text{rad}}^{\text{max}}$ is found when the entire design domain is fully filled with material and previously determined to be 27.25 W. This constant is used in Equation (5.11) to normalize the heat power radiated objective function. We perform 71 separate optimization runs with the weights $w_1$ and $w_2$ varying between $[0.25, 0.95]$ and $[0.75, 0.05]$ respectively such that $w_1 + w_2 = 1$. The total runtime to complete all 71 analyses was 38 hours, 14 minutes, 15 seconds. We use the same backtracking line search method described in Section 4.4 and allow the Hamilton Jacobi solver to take up to 100 time steps between optimization iterations. The level set function is reinitialized every other time step.

The optimal objective function values from each optimization run are plotted in Figure 5-4. The point designs trace out a Pareto curve which illustrates the trade between both objective functions. Clearly, a design with less material will have a reduced capacity for radiating heat power and vice versa. These plots are useful to designers because they indicate the cost of increasing one objective at the expense of the competing objective. Note that the volume fraction is always greater than zero because it includes the radiating domain which cannot be removed by the optimizer. Also, the radiated heat power will never drop to zero since a completely void design domain is filled with a weakly conducting material according to the Ersatz approach from Section 3.4.2. Thus a very small amount of heat is still conducted from the
Figure 5-4: Pareto optimal point designs for optimization problem (SP2a)

Dirichlet boundary to the radiating domain.

In Figure 5-5 we plot the Pareto curve along with four highlighted point designs designated A through D. For low volume fraction designs (e.g., point Design A), the structural shape essentially becomes a thin rectangle connecting the Dirichlet boundary and radiating sub-domain. For high volume fraction designs (e.g., point Design D), material fills most of the design domain $\mathcal{D}$ except for the upper and lower left corners where less heat is being conducted through the structure.
Figure 5-5: Point design topologies along Pareto front for optimization problem (SP2a)

As a comparison, we also solve optimization problem (SP1) for 17 different volume fraction constraints between $V_{\text{req}} = 0.1$ and $V_{\text{req}} = 0.9$. We expect that the design points identified by solving (SP1) and (SP2a) should both lie on the same Pareto optimal curve. We illustrate this comparison in Figure 5-6 where point designs from both optimization problems are plotted. Figure 5-6 clearly shows that these points align on the same Pareto optimal curve, and thus optimization problems (SP1) and (SP2a) yield the same solution depending on the volume fraction requirement and objective weights chosen respectively. The next step is to investigate the effect of adding the von Mises stress constraint to see how it changes the Pareto curve.
5.5.2 Stress-Constrained Problem

The PDE from Equation (5.2) is solved using the finite element method, and we first look at the results of a fully filled design domain as we had done in Section 4.4. This analysis will help verify that the finite element model has been correctly formulated, and that the solver is generating sensible results. The $\hat{x}$ and $\hat{y}$ displacement, and von Mises stress fields are illustrated in Figure 5-7.
A stress concentration clearly exists at the upper and lower left corners of the plate. Moreover, the maximum von Mises stress is approximately 120 MPa which is 43% of the yield stress of the aluminum alloy material.
We now define optimization problem (SP2b) which includes only the von Mises stress constraint defined in Equation (5.13) below:

\[
\begin{align*}
\text{(SP2b):} & \quad \min_{\Omega} w_1 \left( \frac{\int_{\Omega} - \chi \theta^4 dA}{Q_{\text{rad}}^{\max}} \right) + w_2 \left( \frac{\int_{\Omega} dV - V_{\text{req}}}{\int_{\Omega} dV} \right) \\
\text{subject to} & \quad \int_{\Omega} \sigma_{\text{VM}} dV - 1 \leq 0 \\
& \quad \text{div} (\kappa \nabla \theta) = \varepsilon \theta^4 \quad \text{in} \quad \Omega_{\text{rad}} \\
& \quad \text{div} (\kappa \nabla \theta) = 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{rad}} \\
& \quad (\kappa \nabla \theta) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta} \\
& \quad \theta = \theta_0 \quad \text{on} \quad \Gamma_{D,\theta} \\
& \quad \text{div} (\sigma) + f = 0 \quad \text{in} \quad \Omega_{\text{load}} \\
& \quad \text{div} (\sigma) = 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{load}} \\
& \quad \sigma n = 0 \quad \text{on} \quad \Gamma_{0,u} \\
& \quad u = 0 \quad \text{on} \quad \Gamma_{D,u}
\end{align*}
\]

First a single optimization is performed with the weights \( w_1 = 0.5 \) and \( w_2 = 0.5 \) to ensure that the optimizer can converge to a solution. We initialize the Lagrange multiplier estimate and penalty parameter corresponding to the stress inequality constraint to \( \lambda^0 = 0.01 \) and \( \mu^0 = 0.1 \) respectively. The aggregation parameter for the global stress function in Equation (5.7) is chosen to be \( b = 6 \). Recall from Section 5.2.1 that the global stress function can become non-smooth if the aggregation parameter is too large. We find through numerical experiments that \( b = 6 \) is the largest value we can use before the optimizer becomes unstable. However, we also find that the same backtracking line search method used in Section 5.5.1 easily causes instabilities and hinders convergence. Alternatively we apply a constant time step scheme where for each optimization iteration we solve the Hamilton Jacobi equation using only a single time step. We update the level set function a total of 500 steps during each optimization run. The step size is the same as was used in Equation (4.38) since the minimum element size is the same and the velocity field is normalized \( (v_{\text{max}} = 1) \). Again we reinitialize the level set function every other time step which now also occurs every other iteration in the optimization routine.
We consider an initial design perforated by a $2 \times 2$ array of equally spaced circular holes. The initial and optimal shapes are illustrated in Figure 5-8.

(a) Initial shape ($3 \times 3$ array of holes)  
(b) Optimal shape at iteration $k = 500$

Figure 5-8: Initial and optimal shapes for optimization problem (SP2b)

Figure 5-9: Stress field of optimal design for optimization problem (SP2b)
The analysis was completed in 13 minutes, 12 seconds. The von Mises stress field of the optimal shape is also plotted in Figure 5-9. Note that the material on the center left side and upper and lower right corners is hardly stressed and hence these features are not load-bearing. The reason the optimizer retained material in these areas is to transport heat between the Dirichlet boundary and radiating sub-domain.

The objective functions, constraint function, augmented Lagrangian and Lagrange multiplier estimate histories are all plotted in Figure 5-10. The histories indicate that the optimizer has converged and the augmented Lagrangian has monotonically decreased. The oscillations are due to numerical errors previously discussed as well as the non-smoothness of the velocity field influenced by the von Mises stress constraint sensitivity. After approximately 100 iterations, the von Mises stress constraint is shown to be active indicating that the maximum von Mises stress should be close to 276 MPa. To see how closely the global stress function approximates the maximum value, we plot the maximum von Mises stress history in Figure 5-11. Initially the maximum stress is reduced and then rebounds until it converges near 140 MPa. Hence the actual maximum von Mises stress is roughly half the yield strength of the material.
While added safety margin is typically desirable from a designer’s viewpoint, the magnitude of this safety margin is not controllable. Thus the designer cannot know how large the margin is until the optimization has finished. This problem was discussed in Section 5.2.1 and several publications were cited in which researchers attempted to resolve the issue. For now, we accept the discrepancy between the global stress function and the maximum von Mises stress.
Figure 5-11: Maximum von Mises stress history for optimization problem (SP2b)

Finally we plot the $L_1$ and $L_2$ norms of the boundary velocity in Figure 5-12. The boundary velocity is clearly decreasing and the normed metrics asymptotically approach a small non-zero value. Recall that the velocity will never be exactly zero on the structural boundary $\partial \Omega$ due to numerical errors arising from interpolating the velocity field, updating and reinitializing the level set function, and the non-smoothness of the global stress function sensitivity.
Figure 5-12: Convergence history using boundary velocity norms for optimization problem (SP2b)
Having demonstrated that our method converged to a feasible solution whose objective function has decreased, we now consider the influence of changing the initial material configuration. We run the same analysis using the same parameters with a varying number of holes in the initial design domain as well as a fully filled design domain. The results of all 5 optimization runs are presented in Table 5.1. Each of these designs perform similarly and, with the exception of the design initialized by a $2 \times 2$ array of holes, is non-dominated. The fully filled initial design retains the most material but has the greatest radiating power. Conversely the design initialized by the $4 \times 4$ array of holes has the lowest volume fraction but also radiates the least amount of heat power. Figure 5-13 illustrates the optimal shapes for each of these optimization runs.

Table 5.1: Comparison of locally optimal solutions using varied initial material configurations for optimization problem (SP2b).

<table>
<thead>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>Fully Filled</td>
<td>23.79</td>
<td>0.6671</td>
<td>$-3.143 \times 10^{-3}$</td>
<td>100.2</td>
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<td>Holes (1 x 1 array)</td>
<td>21.73</td>
<td>0.5933</td>
<td>$5.698 \times 10^{-4}$</td>
<td>136.7</td>
</tr>
<tr>
<td>Holes (2 x 2 array)</td>
<td>21.87</td>
<td>0.5965</td>
<td>$8.010 \times 10^{-4}$</td>
<td>134.1</td>
</tr>
<tr>
<td>Holes (3 x 3 array)</td>
<td>21.91</td>
<td>0.5947</td>
<td>$1.428 \times 10^{-3}$</td>
<td>136.2</td>
</tr>
<tr>
<td>Holes (4 x 4 array)</td>
<td>21.58</td>
<td>0.5829</td>
<td>$8.779 \times 10^{-4}$</td>
<td>135.9</td>
</tr>
<tr>
<td></td>
<td>Initial Shape</td>
<td>Optimal Shape</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>---------------</td>
<td>---------------</td>
<td></td>
<td></td>
</tr>
<tr>
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<td><img src="image2" alt="Optimal Shape" /></td>
<td></td>
<td></td>
</tr>
<tr>
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<td><img src="image3" alt="Initial Shape" /></td>
<td><img src="image4" alt="Optimal Shape" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 × 2 Holes</td>
<td><img src="image5" alt="Initial Shape" /></td>
<td><img src="image6" alt="Optimal Shape" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 × 3 Holes</td>
<td><img src="image7" alt="Initial Shape" /></td>
<td><img src="image8" alt="Optimal Shape" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 × 4 Holes</td>
<td><img src="image9" alt="Initial Shape" /></td>
<td><img src="image10" alt="Optimal Shape" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 5-13: Optimal shapes using varied initial material configurations for optimization problem (SP2b)
Finally, we generate a Pareto plot by performing 15 optimization runs varying the first objective function weight between $[0.25, 0.95]$ and the second objective function weight between $[0.05, 0.75]$ such that $w_1 + w_2 = 1$. The total runtime to complete all 15 analyses was 3 hours, 15 minutes, 2 seconds. The Pareto plot is illustrated in Figure 5-14.

![Pareto plot](image)

**Figure 5-14:** Pareto optimal point designs for optimization problem (SP2b)

The feasible design space has been significantly curtailed by the von Mises stress constraint. From the Pareto plot we see that designs with volume fractions under 0.59 are not feasible. However, because such a large amount of material is required to keep the design rigid enough, the cost of improving the heat power radiated in terms of volume fraction is relatively high. Aluminum is a good heat conductor and hence between volume fractions of 0.6 and 1.0, the difference in heat power radiated is only approximately 6 W or 22% of the heat power radiated with a fully filled design domain. Four point designs along the Pareto front are illustrated next in Figure 5-15. The high volume fraction designs (e.g., Design D) are less interesting since they...
are mostly filled in with material. However, at lower volume fractions near 0.6, the Pareto designs have more elaborate truss structures which effectively carry the vertical static pressure load and transport heat to the radiating sub-domain. In this sense the structure is multi-functional since it is able to both transport heat and support a load.

5.5.3 Frequency-Constrained Problem

We now consider the same weighted sum of the objective functions with a single first mode frequency constraint. The constraint is evaluated by solving the governing PDE
found in Equation (5.10) using the finite element method. As before the finite element model is initially solved with a fully filled design domain to verify that it is correctly formulated and that the eigenvalue solver is giving sensible results. The first three mode shapes are illustrated in Figure 5-16, along with their corresponding frequencies. From Figure 5-16 we see that the first and third mode shapes are sinusoidal and resemble those found in a classic cantilever problem. The second mode shape of the plate is a widening along the y-axis and shortening along the x-axis. Since the second mode frequency is substantially higher than the first mode frequency (by approximately a factor of 2.4), we don’t expect any problems arising from repeating eigenvalues or mode frequency crossings as was discussed in Section 5.2.2.

This optimization problem considered here, designated (SP2c), includes only the first mode frequency constraint from Equation (5.14). The optimization problem (SP2c) is presented below:

\[
(SP2c): \quad \min \Omega \quad w_1 \left( \frac{\int_{\Omega} -\chi \varepsilon \theta^3 dA}{\sigma_{\text{rad}}^{\text{max}}} \right) + w_2 \left( \frac{\int_{\Omega} dV - V_{\text{req}}}{f_{\text{D}} dV} \right)
\]

subject to

\[
1 - \frac{\omega^2}{\omega_{\text{lim}}^2} \leq 0
\]

\[
\text{div} (\kappa \nabla \theta) = \varepsilon \theta A \quad \text{in} \quad \Omega_{\text{rad}}
\]

\[
\text{div} (\kappa \nabla \theta) = 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{rad}}
\]

\[
(\kappa \nabla \theta) \cdot n = 0 \quad \text{on} \quad \Gamma_{\theta,0}
\]

\[
\theta = \theta_0 \quad \text{on} \quad \Gamma_{\theta,0}
\]

\[
\text{div} \left( \mathbb{C} \varepsilon (\psi) \right) + \omega^2 \rho \psi = 0 \quad \text{in} \quad \Omega
\]

\[
(\mathbb{C} \varepsilon (\psi)) n = 0 \quad \text{on} \quad \Gamma_{\theta,0}
\]

\[
\psi = 0 \quad \text{on} \quad \Gamma_{\theta,0}
\]

A single optimization run is performed with weights \(w_1 = 0.5\) and \(w_2 = 0.5\) to ensure that the optimizer converges to a solution. The Lagrange multiplier estimate and penalty parameter are initialized to \(\lambda^0 = 0.1\) and \(\mu^0 = 0.5\) respectively. We continue to use the constant step line search method that was discussed in the previous section and again allow for a maximum of 500 optimization iterations. All the parameters regarding the step size, and level set function reinitialization are also kept the same.
(a) Mode shape 1 @ 52.8 Hz

(b) Mode shape 2 @ 127.2 Hz

(c) Mode shape 3 @ 141.9 Hz

Figure 5-16: Modal finite element solution of 2D thermally radiating plate
from the previous section on solving (SP2b). A summary of the main parameters used for solving (SP2b) and (SP2c) is presented in Table 5.2.

Table 5.2: Initialization parameters for solving optimization problems (SP2b) and (SP2c).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(SP2b) Values</th>
<th>(SP2c) Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max no. of optimization iterations</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>Lagrange multiplier $\lambda^0$</td>
<td>0.01</td>
<td>0.1</td>
</tr>
<tr>
<td>Penalty parameter $\mu^0$</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>Max penalty parameter $\mu_{\text{max}}$</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>No. of steps between $\phi$ re-initializations</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Line search method</td>
<td>Constant step</td>
<td>Constant step</td>
</tr>
<tr>
<td>Lower bound $\delta_{\text{cond}}$ for void material ($L_e$)</td>
<td>$10^{-3}$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Lower bound $\delta_{\text{stiff}}$ for void material ($K_e$)</td>
<td>$10^{-3}$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Lower bound $\delta_{\text{mass}}$ for void material ($M_e$)</td>
<td>n/a</td>
<td>$10^{-4}$</td>
</tr>
</tbody>
</table>

The finite element equations for solving the eigenvalue problem require both a mass and stiffness matrix. Recall that void elements are represented by a small nonzero factor $\delta$ that multiplies the element stiffness, mass, and conductivity matrices to prevent the global matrices from becoming ill-conditioned or singular. As was discussed in Section 5.2.2, we choose separate values of $\delta$ to multiply each element matrix. We define these values as $\delta_{\text{stiff}} = 10^{-2}$ and $\delta_{\text{mass}} = 10^{-4}$ for the element stiffness $K_e$ and mass $M_e$ matrices respectively. These lower bounds ensure that regions of void material do not cause spurious modes of oscillation since the element mass matrices are more heavily penalized than their stiffness matrices. A lower bound $\delta_{\text{cond}} = 10^{-2}$ is given for the element conductivity matrices $L_e$. 
The initial material configuration is again perforated by a $2 \times 2$ array of holes and both initial and optimal shapes are illustrated in Figure 5-17. The analysis was completed in 13 minutes, 54 seconds. When solving the eigenvalue equations using MATLAB®'s built-in `sptarn.m` function, the user specifies an upper limit on the natural frequencies rather than a desired number of modes to calculate. Thus the number of mode shapes that are calculated is variable between optimization iterations and highly dependent on the evolution of the structural shape. The more modes that are identified, the longer the CPU runtime required.

The optimal shape uses much less material than the stress-constrained design illustrated in Figure 5-8. The stress constraint is evidently more restricting, meaning that more material is needed to keep the von Mises stress within acceptable limits. The first mode occurs at a frequency of 50.01 Hz and is illustrated in Figure 5-18. This mode shape exhibits a single vertical beat similar to that shown in Figure 5-16 for the fully filled design domain.
Figure 5-18: First mode shape at 50.01 Hz of optimal design for optimization problem (SP2c)

The objective functions, constraint function, augmented Lagrangian and Lagrange multiplier estimate histories are all plotted in Figure 5-19. The histories indicate that the optimizer has converged to a feasible solution with a reduced weighted objective function value. Other than iterations 23-25, the inequality constraint remains active and thus the first mode frequency lies close to the limit $\bar{\omega} = 50$ Hz.
Figure 5-19: Objective, constraint, augmented Lagrangian and Lagrange multiplier estimate histories for optimization problem (SP2c)
We plot the first three mode frequencies to ensure that they are all greater than or equal to 50 Hz and that no mode crossing occurred with the initial first mode. The mode frequency histories are illustrated in Figure 5-20.

![Figure 5-20: First three mode frequency histories for optimization problem (SP2c)](image)

While the second and third mode frequencies come close to crossing, it is clear that the first mode frequency quickly converges to 50 Hz and is not crossed. Moreover, no repeated eigenvalues were found in this problem.
Finally we plot the $L_1$ and $L_2$ norms of the boundary velocity in Figure 5-21. The boundary velocity is clearly decreasing and the normed metrics asymptotically approach a small non-zero value.

![Figure 5-21: Convergence history using boundary velocity norms for optimization problem (SP2c)](image)

Having demonstrated that our method converged to a feasible and improved solution, we again consider the influence of changing the initial material configuration. We run the same analysis using several initial level set functions where different arrays of equally spaced circular holes are present in the design domain. We also run the analysis for a fully filled design domain as was done in the previous section solving (SP2b). The results of all 5 optimization runs are presented in Table 5.3.
Table 5.3: Comparison of locally optimal solutions using varied initial material configurations for optimization problem (SP2c).

<table>
<thead>
<tr>
<th>Initial Material Configuration</th>
<th>$J_1$ [W]</th>
<th>$J_2$ Vol. Frac. [n/a]</th>
<th>$g_2$ [n/a]</th>
<th>1st Mode Frequency [Hz]</th>
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<tr>
<td>Fully Filled</td>
<td>14.88</td>
<td>0.3472</td>
<td>$-2.368 \times 10^{-4}$</td>
<td>50.01</td>
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<td>Holes (1 x 1 array)</td>
<td>11.81</td>
<td>0.2278</td>
<td>$-3.310 \times 10^{-4}$</td>
<td>50.01</td>
</tr>
<tr>
<td>Holes (2 x 2 array)</td>
<td>11.58</td>
<td>0.2262</td>
<td>$1.421 \times 10^{-4}$</td>
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<td>Holes (3 x 3 array)</td>
<td>11.82</td>
<td>0.2226</td>
<td>$-2.861 \times 10^{-4}$</td>
<td>50.01</td>
</tr>
<tr>
<td>Holes (4 x 4 array)</td>
<td>11.98</td>
<td>0.2282</td>
<td>$-3.026 \times 10^{-4}$</td>
<td>50.01</td>
</tr>
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</table>

Each of these solutions perform similarly and, with the exception of the designs initialized by a 1 x 1 and 2 x 2 array of holes, is non-dominated. The fully filled initial design retains more material but has the greatest radiating power. Conversely the design initialized by the 3 x 3 array of holes has the lowest volume fraction but also radiates the least amount of heat power. Figure 5-22 illustrates the optimal shapes for each of these optimization runs.
Figure 5-22: Optimal shapes using varied initial material configurations for optimization problem (SP2c)
Each of the optimal shapes in Figure 5-22 has a triangular form with varying truss features in the interior. Some of these designs are symmetric while others are not. Since the mode shape of the structure need not be symmetric, it is evident that the optimizer does not always generate symmetric designs. While our intuition may be that any optimal solution should have a symmetric material configuration, many locally optimal asymmetric solutions exist.

Finally, we generate a Pareto plot by performing 15 optimization runs varying the first objective function weight between [0.25, 0.95] and the second objective function weight between [0.05, 0.75] such that \( w_1 + w_2 = 1 \). The total runtime to complete all 15 analyses was 3 hours, 7 minutes, 21 seconds. The Pareto plot is illustrated in Figure 5-23.

![Pareto plot](image)

**Figure 5-23: Pareto optimal point designs for optimization problem (SP2c)**

The feasible design space is much larger than the one found solving problem (SP2b). Designs with volume fractions greater than or equal to 0.22 are now feasible. Four point designs along the Pareto front are illustrated next in Figure 5-24.
The high volume fraction designs (e.g., Design D) are less interesting since they mostly fill the design domain with material. However, at lower volume fractions near 0.3, the Pareto designs have more elaborate truss structures which provide needed rigidity to maintain a first mode frequency greater than or equal to 50 Hz and transport heat to the radiating sub-domain. Another interesting finding in Figure 5-24 is that Design C closely resembles the Pareto optimal topologies encountered in the von Mises stress constrained optimization problem (SP2b). We would then expect in a multi-constraint problem where both stress and frequency constraints are enforced,
that the designs would primarily resemble the stress-constrained topologies from the previous section due to (1) the fact that low volume fraction designs would be infeasible and (2) high volume fraction designs for both the stress and frequency constraint optimization problems (SP2b) and (SP2c), respectively, resemble a similar trussed geometry. The multi-constraint problem is examined next.

5.5.4 Stress and Frequency Constrained Problem

The precursor problems (SP2a) - (SP2c) provided valuable insight as to how the constraints affect the optimal topologies and limit the design space. Again we perform a single optimization run with the weights $w_1 = 0.5$ and $w_2 = 0.5$ to ensure that the optimizer can converge to a solution. We choose the same initial Lagrange multipliers and penalty parameters that were used to solve (SP2b) and (SP2c). The constant step line search method that was used in the previous sections is again selected with a maximum of 500 iterations. The lower bound factors for the element densities $\rho_e$ are chosen to be $\delta_{\text{cond}} = 10^{-3}$ for the conductivity matrices, $\delta_{\text{stiff}} = 10^{-3}$ for the stiffness matrices and $\delta_{\text{mass}} = 10^{-6}$ for the mass matrices. A summary of the optimization parameters is provided in Table 5.4.

<table>
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<tr>
<th>Parameter</th>
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</tr>
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<td>Max no. of optimization iterations</td>
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<td>Lagrange multiplier for stress constraint $\lambda_0^0$</td>
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<tr>
<td>Lower bound $\delta_{\text{cond}}$ for void material ($L_e$)</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Lower bound $\delta_{\text{stiff}}$ for void material ($K_e$)</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Lower bound $\delta_{\text{mass}}$ for void material ($M_e$)</td>
<td>$10^{-6}$</td>
</tr>
</tbody>
</table>
We again consider an initial material configuration perforated by a $2 \times 2$ array of equally spaced circular holes. The initial and optimal shapes are illustrated in Figure 5-25.

![Initial shape (2 x 2 array of holes) vs. Optimal shape at iteration k = 500](image)

Figure 5-25: Initial and optimal shapes for optimization problem (SP2)

The analysis was completed in 15 minutes, 32 seconds. Note that the CPU runtime for this analysis was only approximately 2 minutes longer than those solving (SP2b) and (SP2c) in Sections 5.5.2 and 5.5.3 respectively. The additional finite element solves for this multi-constraint problem require only a marginal increase in CPU resources, indicating that the routines for updating and reinitializing the level set function are more costly. Moreover, because the optimal shape is nearly identical to that found in Figure 5-8 from Section 5.5.2, we expect the stress constraint is active. The von Mises stress field and first mode shape of the optimal design are both plotted in Figure 5-26.
Figure 5-26: Stress field and first mode shape of optimal design for optimization problem (SP2)

The objective, constraint, augmented Lagrangian and Lagrange multiplier estimate histories are plotted in Figure 5-27. From Figure 5-27(b) we can now confirm that the stress constraint is indeed active during the final iterations. Additionally, we find that both inequality constraints are active at different iterations during the optimization routine. Initially both constraints are violated, but between iterations 3 and 40 only the first mode frequency constraint is active or violated. The Lagrange multiplier estimate histories also support this finding since $\lambda_1 = 0$ for much the first 50 iterations, indicating the von Mises stress constraint is inactive. Afterwards the first mode frequency constraint becomes inactive and the von Mises stress constraint becomes active. The decreasing augmented Lagrangian illustrated in Figure 5-27(c) also indicates that optimizer is consistently choosing a descent direction. The histories collectively indicate that a converged solution has been reached.
We also plot the maximum von Mises stress and modal frequency histories in Figure 5-28. The maximum stress converges to approximately 120 MPa whereas the first mode frequency converges near 52 Hz. The second and third mode frequencies remain well above the first mode and no mode crossings exist. We also verify in the optimization history that multiple eigenvalues did not exist.
Finally the $L_1$ and $L_2$ norms are plotted for the boundary velocity in Figure 5-29 to further illustrate that the solution has converged. Both norms are clearly more noisy than those found in Figures 5-12 and 5-21 for optimization problems (SP2b) and (SP2c) respectively. Because the velocity field is a compilation of the objective and constraint function shape sensitivities, the addition of more objectives and constraints will contribute to its non-smoothness. Moreover the velocity field is not differentiable between optimization iterations where the constraint functions change from active to inactive (or vice versa). Section 3.4.4 presented a brief overview of some of the research that has been devoted to smoothing and regularizing the velocity fields and sensitivity information. Unfortunately the task becomes much more arduous as the optimization problems grow in the number of constraint and objective functions.
Having demonstrated that our method converged to a feasible and improved solution, we again consider the influence of changing the initial material configuration. We run the same analysis using the parameters listed in Table 5.4 with a varying number of equally spaced circular holes in the initial design domain as well as a fully filled design domain. The results of all 5 optimization runs are presented in Table 5.5. Each of these solutions perform similarly and are non-dominated, with the exception of the design initialized by a single void at the center of the design domain \((1 \times 1\) array). As before the fully filled initial design retains more material but has the greatest radiating power. Conversely the design initialized by the \(4 \times 4\) array of holes has the lowest volume fraction but also radiates the least amount of heat power.
Table 5.5: Comparison of locally optimal solutions using varied initial material configurations for optimization problem (SP2).

<table>
<thead>
<tr>
<th>Initial Material Configuration</th>
<th>$J_1$ (W)</th>
<th>$J_2$ (n/a)</th>
<th>Maximum von Mises Stress (MPa)</th>
<th>1st Mode Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fully Filled</td>
<td>24.80</td>
<td>0.6969</td>
<td>108.9</td>
<td>51.08</td>
</tr>
<tr>
<td>Holes (1 × 1 array)</td>
<td>22.27</td>
<td>0.6019</td>
<td>125.6</td>
<td>52.35</td>
</tr>
<tr>
<td>Holes (2 × 2 array)</td>
<td>22.92</td>
<td>0.6284</td>
<td>119.3</td>
<td>52.06</td>
</tr>
<tr>
<td>Holes (3 × 3 array)</td>
<td>22.61</td>
<td>0.6055</td>
<td>127.7</td>
<td>53.13</td>
</tr>
<tr>
<td>Holes (4 × 4 array)</td>
<td>22.47</td>
<td>0.5997</td>
<td>125.6</td>
<td>53.60</td>
</tr>
</tbody>
</table>

The optimal shapes are illustrated in Figure 5-30 for each of these optimization runs. The topologies closely resemble those found in Figure 5-13 from solving optimization problem (SP2b). We conclude that the stress constraint is the more limiting of the two constraints and principally affecting the final topologies. However, closer scrutiny of the optimal designs in both Figures 5-13 and 5-30 reveals several differences in their geometric features (e.g., the optimal topologies found when initializing the material configuration with a 4 × 4 array of holes). Because the frequency constraint was active during parts of the optimization history when solving (SP2), the first mode frequency shape sensitivity influenced the final design. On the other hand, the optimal topologies found when solving (SP2b) were only influenced by the stress constraint. Again we further assert that many local optimum solutions exist in the design space. In Chapter 6 we discuss several methodologies to help identify the prominent features commonly found across many of these local optimums.
Figure 5-30: Optimal shapes using varied initial material configurations for optimization problem (SP2)
Finally, a Pareto plot is generated by performing 15 optimization runs varying the first objective function weight between [0.25, 0.95] and the second objective function weight between [0.05, 0.75] such that \( w_1 + w_2 = 1 \). The total runtime to complete all 15 analyses was 3 hours, 15 minutes, 11 seconds. The Pareto plot is illustrated in Figure 5-31.

![Pareto plot](image)

Figure 5-31: Pareto optimal point designs for optimization problem (SP2)

The feasible design space is comparable to the previous problem (SP2b) using a von Mises stress constraint. Designs with volume fractions less than 0.59 are not feasible. The optimal shapes of four point designs along the Pareto front are illustrated next in Figure 5-32.
Figure 5-32: Point design topologies along Pareto front for optimization problem (SP2)
While the stress constraint is more limiting to the design space than the frequency constraint, according to Figure 5-28(a) the maximum von Mises stress only reaches approximately 127 MPa which is far below the yield strength of 276 MPa. Recall that the global stress function underestimates the maximum von Mises stress by roughly a factor of 2. As such we try modifying the stress constraint in Equation (5.13) by increasing $\bar{\sigma}$ to twice the yield strength. Hence the new upper bound for the global stress constraint is $\bar{\sigma} = 552$ MPa. The goal of making this adjustment is to narrow the margin between the maximum von Mises stress and the yield strength, effectively growing the feasible design space.

We perform a total of 15 optimization runs using the same objective weights and parameters used previously for solving (SP2). The total runtime to complete all 15 analyses was 4 hours, 6 minutes, 59 seconds. The objective values for each optimal topology are plotted in Figure 5-33.

![Figure 5-33: Pareto optimal point designs for optimization problem (SP2) using $\bar{\sigma} = 552$ MPa](image)
The gray dots indicate dominated designs whose solutions are outperformed by the non-dominated designs (red dots). A dominated design is one in which either objective function can be improved without trading the competing objective. The feasible design space has now been expanded such that designs with volume fractions equal to or above 0.3875 are now feasible. Four of the non-dominated point designs are illustrated next in Figure 5-34.

![Graph showing radiated heat vs volume fraction with highlighted design points.](image)

(a) Pareto front with highlighted design points

(b) Design A  (c) Design B  (d) Design C  (e) Design D

Figure 5-34: Point design topologies along Pareto front for optimization problem (SP2) using $\sigma = 552$ MPa

The majority of the designs now have active first mode frequency constraints, and the maximum von Mises stresses vary between 160 MPa and 276 MPa. In four of the 15 cases the optimizer converged to a solution where the maximum von Mises...
stress constraint exceeded the yield strength (now that \( \bar{\sigma} \) is set to twice the yield strength). In these cases an optimal and feasible solution within the optimizer history was selected and plotted in Figure 5-34. As before we see that both the stress and frequency constraints become active during various phases of the optimization history. Figure 5-35 illustrates this phenomenon by plotting the inequality constraint histories for Design A from Figure 5-34.

![Constraint History](image)

Figure 5-35: Inequality constraint history for Design A indicated in Figure 5-34

Both the active stress and frequency constraints affect the velocity fields such that the final designs inherit features found in the solutions of both optimization problems (SP2b) and (SP2c). Design A for example is largely triangular in shape similar to the designs found when solving (SP2c), but also exhibits thicker internal truss elements found in designs attained from solving (SP2b). Hence the optimal topologies found solving (SP2) now resemble combinations of the designs found solving (SP2b) and (SP2c). While this finding is not unexpected, it does motivate a further need to classify the non-dominated design set into families and to extract prominent features. In doing so an analyst can more easily understand which geometric features promote
efficient structural designs under specific load conditions. In Chapter 6 we present a methodology to accomplish this task.

5.6 Chapter Summary

In this chapter, we perform a brief literature review to motivate the need for a multidisciplinary framework for level set based topology optimization methods. We discussed the various objective and constraint functions along with their respective analytic shape derivatives that have been developed in the topology optimization community to demonstrate the versatility of level set based methods. We extended the optimization problem (SP1) introduced in Chapter 3 to include a weighted sum of the radiated heat power and volume fraction objective functions along with two inequality constraints on the maximum von Mises stress and first mode frequency. This new optimization problem is designated (SP2). We tested our level set based framework on a design problem where the 2D aluminum radiating plate in Section 4.1.1 now included a static vertical pressure load applied on the radiating sub-domain. We solve the new governing PDE for a statically loaded linear elastic material to yield the displacement and stress field within the plate. We also perform a modal analysis by solving the governing eigenvalue PDE to determine the first few mode shapes and corresponding frequencies. Initially we solved a series of precursor problems to (SP2) where we consider (a) no inequality constraints, (b) a single von Mises stress constraint, and (c) a single first mode frequency constraint. These problems provided valuable insight into how the constraints affected the optimal shapes and altered the Pareto curves.
Chapter 6

Feature Analysis Using
Decomposition and Clustering Methods

When designing or improving a structure, topology optimization methods usually do not capture the full design intent since many factors cannot be easily transcribed to a formal optimization problem statement. Some examples include enforcing regularity or patterns (i.e., identical, equally spaced cutouts), interface configurations (e.g., fastened or welded joints), and integration considerations (e.g., positioning and orientation of supported subcomponents). Moreover, the topologies identified by the optimizer must undergo a number of post-processing steps such as surface or edge smoothing and tolerance checks before fabrication. Converting the solution to a manufacture-ready blueprint or computer aided design (CAD) model is not yet a well-defined process. As a consequence, most designers use topology optimization as a tool to guide their design intuition rather than an end-to-end process for generating a manufacturable structure. As such, an effective method for identifying prominent topological features in a multidisciplinary framework can provide several benefits.

This chapter demonstrates a variety of numerical techniques for visualizing and categorizing the topological features found in large datasets of solutions acquired using our topology optimization method. In Section 6.1 we discuss the utility of
identifying these prominent topological features in a design study and the type of information which would be most valuable to a designer. In Section 6.2 we introduce four candidate numerical techniques including a frequency distribution, singular value decomposition (SVD), CUR matrix approximation, and k-means clustering analysis. Section 6.3 then presents results from applying these numerical techniques on a large solution dataset acquired by solving optimization problem (SP2) using the 2D radiating plate example from Chapter 5. Concluding remarks on the effectiveness of each of the numerical techniques are also given.

6.1 Multi-Objective Design Space Exploration

We illustrated in Chapter 5 several Pareto fronts which quantified the trade-offs between volume fraction and thermal heat power radiated. After solving (SP2), we found that the non-dominated design solutions exhibited varying topological features at different locations in the design space. We would like to categorize these topological features and determine which ones are the most prominent at different regions along the Pareto front. Since only a single constraint was active for most of the optimal designs, we expect that the non-dominated design set can be decomposed into shapes similar to those from solving (SP2b) and (SP2c) where only a single constraint was enforced. Understanding the dominant topological features and how they correlate to the active constraint helps a designer to understand not only the optimal material configuration for a structural design, but also how that design should be altered if a load condition or structural requirement were to change.

Understanding the dominant topological features in iso-performing design sets can also be useful to a designer. Iso-performing designs exhibit similar performance such that their objective function values lie within predetermined tolerances set by the designer. In Section 5.5 recall that multiple solutions were obtained keeping the objective weights constant but varying the initial material configuration. Each of the design solutions radiated a similar amount of heat but exhibited different geometries. These iso-performing designs can then be evaluated by the designer using metrics and
criteria that are not explicitly stated in the optimization problem.

So far we have shown that the resultant topology is affected by the objective weights and initial material configuration. However, since we are applying a gradient-based augmented Lagrangian algorithm, different topologies can also be found choosing varied initialization parameters. For example, the initial Lagrange multiplier estimates and level set update parameters can impact the final converged topology. To illustrate this effect, we return to the 2D radiating aluminum plate example presented in Chapter 5. We solve optimization problem (SP2) where both the frequency and stress constraints are enforced. The upper bound on the global von Mises stress constraint is kept at \( \bar{\sigma} = 552 \text{ MPa} \). An equal weighting \( w_1 = w_2 = 0.5 \) is assigned to each objective function and the initial material configuration is a \( 2 \times 2 \) array of equally spaced circular holes. The results of two optimization analyses, designated Analysis A and B, are compared with initialization parameters listed in Table 6.1.

Table 6.1: Initialization parameters for solving optimization problem (SP2) resulting in varied solution topologies.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Analysis A</th>
<th>Analysis B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max no. of optimization iterations</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Initial Lagrange multiplier (stress constraint)</td>
<td>0.01</td>
<td>0.1</td>
</tr>
<tr>
<td>Initial penalty parameter (stress constraint)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Max penalty parameter (stress constraint)</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Initial Lagrange multiplier (freq. constraint)</td>
<td>0.1</td>
<td>0.02</td>
</tr>
<tr>
<td>Initial penalty parameter (freq. constraint)</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Max penalty parameter (freq. constraint)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Line search method</td>
<td>Backtrack</td>
<td>Backtrack</td>
</tr>
<tr>
<td>Initial trust region size ( T_{\text{max}}^0 )</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Largest trust region size ( T_{\text{max}} )</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>Factor to change trust region size ( \gamma_{T_{\text{max}}} )</td>
<td>1.2</td>
<td>1.2</td>
</tr>
<tr>
<td>Factor to backtrack no. of time steps ( \gamma_T )</td>
<td>1</td>
<td>0.91</td>
</tr>
<tr>
<td>Constant to backtrack no. of time steps ( \eta )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Augmented Lagrangian update tolerance ( \beta_L^{[k1,k2]} )</td>
<td>1.01[0,10])</td>
<td>1.2[0,50])</td>
</tr>
</tbody>
</table>

\* \( \beta_L^{[k1,k2]} \) specifies the augmented Lagrangian update tolerance \( \beta_L \) for optimization iterations \( k1 \) through \( k2 \).
The backtracking line search method presented in Algorithm 3 from Section 4.4 is selected to update the level set function. As shown in Table 6.1, Analyses A and B choose different arguments for Algorithm 3 to influence the level set function updates. Specifically, $\gamma_T$ and $\eta$ both affect how quickly the line search method backtracks to a single time step via the following update on line 18: $T \leftarrow [\gamma_T T - \eta]$. In addition, the trust region size, denoted by the maximum number of steps the Hamilton Jacobi equation solver can take, is updated by $\gamma_{T_{\text{max}}}$ and bounded by $T_{\text{max}}$. For Analysis B the trust region is allowed to expand much more than Analysis A, and the augmented Lagrangian update tolerances are more relaxed. This combination of parameters allows the optimizer to make more aggressive changes to the topology between iterations by taking greater numbers of steps with the HJE solver.

The final topologies for Analyses A and B are illustrated in Figure 6-1.

Figure 6-1: Optimal shapes for Analyses A and B using parameters from Table 6.1

The optimal shapes clearly exhibit different topological features, yet both designs were identified using the same initial material configurations and objective weights. While the heat power radiated and volume fractions are similar for both designs, there
may be attributes in each design that make them more desirable from a designer's perspective. For example, the more slender features found in the optimal shape from Analysis A may be more susceptible to buckling or more difficult to manufacture.

The convergence histories are provided in Figures 6-2 and 6-3 for Analysis A and B respectively.

![Figure 6-2: Objective, constraint, augmented Lagrangian, and Lagrange multiplier estimate histories for Analysis A using initial parameters from Table 6.1](image)

Because Analysis A restricts the level set function updates more than Analysis B, the histories are much less noisy. Additionally the smaller updated augmented Lagrangian tolerances help to speed convergence to a nearby locally optimal solution. In Analysis B, the topology is undergoing a more drastic change from the initial material
configuration where a larger number of HJE solver steps are taken at each iteration in the optimizer. However, the optimal solution identified by Analysis B outperforms that from Analysis A in both thermal heat power radiated and volume fraction. The notion that many locally optimal solutions exist in the design space is once again confirmed.
We also show that the velocity smoothing technique introduced in Section 3.5 can alter the features of a topology optimization analysis. Recall that this smoothing technique projects the velocity field onto a Hilbert inner product space \((\cdot, \cdot)_{H^1}\) restated below:

\[
(\vartheta, \varphi)_{H^1} = \int_D \left[ \alpha_{H^1} \nabla \vartheta \cdot \nabla \varphi + \vartheta \varphi \right] dV
\]

(6.1)

where \(\vartheta\) and \(\varphi\) are scalar fields defined on \(D\) and \(\alpha_{H^1} \in \mathbb{R}_{>0}\) where \(\mathbb{R}_{>0}\) is the set of positive real numbers. The parameter \(\alpha_{H^1}\) is user-defined and has a significant impact on the smoothing operation. Larger values of \(\alpha_{H^1}\) will smooth the velocity field to a greater extent. The benefit of smoothing the velocity field is to remove disconnected material and jagged boundaries from the final solution. The presence of smaller features will also diminish as the parameter \(\alpha_{H^1}\) is increased, eliminating the possibility of porous features resembling the "checkerboarding" effect encountered in many SIMP based approaches.

We again demonstrate the effect of velocity smoothing by solving optimization problem (SP2) where both the frequency and stress constraints are enforced on the 2D aluminum plate example. The upper bound on the global von Mises stress constraint is lowered to \(\bar{\sigma} = 276\) MPa to ensure that the stress constraint is active. The sensitivities from this constraint are inherently more noisy than those from the frequency constraint due to the \(p\)-norm aggregation functional. Hence any velocity smoothing should have a more significant effect on the topology. An equal weighting \(w_1 = w_2 = 0.5\) is assigned to each objective function and the initial material configuration is a \(4 \times 4\) array of equally spaced circular holes. Two optimization analyses are performed: Analysis C without velocity smoothing and (2) Analysis D with velocity smoothing using a fixed scaling \(\alpha_{H^1} = 12\). A constant time step method is used in both analyses where a single time step is taken by HJE solver each iteration in the optimizer.

To demonstrate the effects of velocity smoothing, Figure 6-4 illustrates the sign of the velocity magnitude at the first iteration for Analyses C and D. Areas with positive velocity magnitudes (indicated by the magenta color) will drive the struc-
tural boundaries outwards, closing void spaces, whereas areas with negative velocity magnitudes (indicated by the cyan color) will do the opposite. In Figure 6-4(a), the unsmoothed velocity field is more irregularly shaped and influences many of the voids to elongate near the left edge. Conversely the smoothed velocity field in Figure 6-4(b) is more regularly shaped and seeks to completely close many of the voids near the left edge. This illustration exemplifies how smaller features are more likely to be removed during the optimization routine when velocity smoothing is applied. Some benefits of velocity smoothing include removing small, potentially non-manufacturable features, and improving convergence.

(a) Analysis C
(b) Analysis D

Figure 6-4: Velocity field magnitude sign (magenta—positive and cyan—negative) at iteration $k = 1$ for Analyses C and D
The final topologies for Analyses C and D are illustrated in Figure 6-5 at iteration \( k = 320 \). Again the optimal shapes clearly exhibit different topological features as before, despite having identical objective function weightings and initial material configurations. The design produced by Analysis C retains many more of the initial voids than Analysis D, which has eliminated some of the smaller features. Note too that the optimal shape from Analysis D is not symmetric which can be attributed to a wide range of numerical artifacts introduced by the mesh configuration, sensitivity interpolation and level set function reinitialization.

(a) Analysis C
\[ J_1 = 21.38 \text{ W}, \ J_2 = 0.5791 \]

(b) Analysis D
\[ J_1 = 20.42 \text{ W}, \ J_2 = 0.5495 \]

Figure 6-5: Optimal shapes for Analyses C and D at optimization iteration \( k = 320 \)
The convergence histories are provided in Figures 6-6 and 6-7 for Analysis C and D respectively. Neither optimal solutions dominate in both objectives (i.e. heat power radiated and volume fraction) in this example comparison. Again, we have shown that multiple locally optimal solutions exist in the design space.

![Graphs showing convergence histories](image)

Figure 6-6: Objective, constraint, augmented Lagrangian, and Lagrange multiplier estimate histories for Analysis C

We have shown that the objective function weights, initial material configuration and choice of initialization parameters can all lead to topologically diverse local optimal solutions. Exploring these options can be valuable to a designer since the design intent cannot be fully captured by the optimizer. Therefore we can perform a large set of topology optimization analyses with varying objective function weights.
Figure 6-7: Objective, constraint, augmented Lagrangian, and Lagrange multiplier estimate histories for Analysis D
initial material configurations and initialization parameters to identify many topologically diverse solutions in a tradespace study. The designer must then visually inspect the solutions and evaluate their performance in order to gain insight into the design problem. This process can become cumbersome, if not intractable, for large datasets. Thus we explore several candidate data mining techniques for extracting the dominant features identified by the optimizer.

6.2 Numerical Techniques for Data Mining Large Topology Optimization Solution Datasets

Data mining is the process of extracting patterns from large datasets using a variety of techniques arising from machine learning, statistics, and database management. In Section 6.2.1, we first define a topology optimization solution dataset using the element density vectors. In Sections 6.2.2–6.2.5 we introduce four candidate data mining techniques that will be later tested on a sample dataset in Section 6.3.

6.2.1 Defining a Topology Optimization Solution Dataset

Recall that a solution using a level set based approach is a structural shape defined implicitly by the level set function $\phi$. Instead of using the design vector consisting of the nodal values of the level set function $\phi$, we choose to use the element density vector $p$. The level set function maps to an element density field according to Equation (3.17) and interpolates the material conductivity, stiffness, and density across the design domain according to Equation (3.18). The reason for choosing the element density vector is that the level set function is not unique for any given shape unless it is regularized to the signed distance function. However, the level set function is not necessarily reinitialized after the final iteration in the topology optimizer, and moreover the fast marching method only approximates the signed distance function. Consequently, two different level set functions may map to the same topology. Using the level set function vector $\phi$ to define a single solution can create an ambiguity in
the dataset which is avoided by choosing the element density vector instead. Hence a single solution is defined as follows:

\[
\mathbf{\rho} = \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\vdots \\
\rho_m
\end{bmatrix}
\]  

(6.2)

where \(\rho_e\) is the density for element \(e\) and \(m\) is the total number of elements in the mesh. If there exists a total of \(y\) solutions, the solution dataset \(\mathbf{P}\) is written in the following matrix format:

\[
\mathbf{P} = \begin{bmatrix}
\rho^1 & \rho^2 & \cdots & \rho^y \\
\rho^1 & \rho^2 & \cdots & \rho^y \\
\vdots & \vdots & \ddots & \vdots \\
\rho^1 & \rho^2 & \cdots & \rho^y
\end{bmatrix} = \begin{bmatrix}
\rho^1_1 & \rho^2_1 & \cdots & \rho^y_1 \\
\rho^1_2 & \rho^2_2 & \cdots & \rho^y_2 \\
\vdots & \vdots & \ddots & \vdots \\
\rho^1_m & \rho^2_m & \cdots & \rho^y_m
\end{bmatrix}
\]  

(6.3)

where the superscripts on \(\mathbf{\rho}\) and \(\mathbf{\rho}\) are solution indices for the dataset. Using Equation (6.3) to define a solution dataset, we next seek a numerical technique which can analyze \(\mathbf{P}\) in a way to identify the prominent topological features. The following subsections describe four candidate techniques to accomplish this task.

### 6.2.2 Frequency Distribution

Frequency distribution analysis counts the number of full, void, or intermediate element densities across all designs in \(\mathbf{P}\) for each element of the mesh. The elements with intermediary densities are cut by the zero level set and therefore lie on the structural boundary. This procedure generates three histograms of size \(m \times 1\) from the solution dataset, one for each category (full, boundary, void). Each histogram can be plotted to visualize the geometric features most common in the dataset.
\[ \bar{\rho}_{\text{full}} = \begin{bmatrix} \sum_{j=1}^{y} 1_{\text{full}}(\rho_{1}^{j}) \\ \sum_{j=1}^{y} 1_{\text{full}}(\rho_{2}^{j}) \\ \vdots \\ \sum_{j=1}^{y} 1_{\text{full}}(\rho_{m}^{j}) \end{bmatrix} \]  

(6.4a)

\[ \bar{\rho}_{\text{boundary}} = \begin{bmatrix} \sum_{j=1}^{y} 1_{\text{boundary}}(\rho_{1}^{j}) \\ \sum_{j=1}^{y} 1_{\text{boundary}}(\rho_{2}^{j}) \\ \vdots \\ \sum_{j=1}^{y} 1_{\text{boundary}}(\rho_{m}^{j}) \end{bmatrix} \]  

(6.4b)

\[ \bar{\rho}_{\text{void}} = \begin{bmatrix} \sum_{j=1}^{y} 1_{\text{void}}(\rho_{1}^{j}) \\ \sum_{j=1}^{y} 1_{\text{void}}(\rho_{2}^{j}) \\ \vdots \\ \sum_{j=1}^{y} 1_{\text{void}}(\rho_{m}^{j}) \end{bmatrix} \]  

(6.4c)

where \(1_{\text{full}}\), \(1_{\text{boundary}}\) and \(1_{\text{void}}\) are the indicator functions defined below:

\[ 1_{\text{full}}(\rho_{e}) = \begin{cases} 1 & \text{if } \rho_{e} = 1 \\ 0 & \text{if } \rho_{e} \neq 1 \end{cases} \]  

(6.5a)

\[ 1_{\text{boundary}}(\rho_{e}) = \begin{cases} 1 & \text{if } \rho_{e} \in (0,1) \\ 0 & \text{if } \rho_{e} \notin (0,1) \end{cases} \]  

(6.5b)

\[ 1_{\text{void}}(\rho_{e}) = \begin{cases} 1 & \text{if } \rho_{e} = 0 \\ 0 & \text{if } \rho_{e} \neq 0 \end{cases} \]  

(6.5c)

(6.5d)
The singular value decomposition plays an important role in data compression, image processing, regression analysis, spectral analysis and other areas. We consider the SVD since it is a commonly used tool for principal component analysis and low rank matrix approximations where the goal is to characterize the structure of a large multivariate dataset.

We use the singular value decomposition to compute a rank-$r$ matrix approximation $P_r$ to our original dataset $P$ where $r < y$. The matrix approximation is written as follows:

$$P_r = U_r \Sigma_r V_r^T$$  \hspace{1cm} (6.6)

where $U_r$ is an $m \times r$ matrix whose columns comprise of the first $r$ left singular vectors $U_j$ for $j = 1, ..., r$, $\Sigma_r$ is an $r \times r$ matrix with the largest $r$ singular values $\sigma_j$, $j = 1, ..., r$ located on the diagonal, and $V_r^T$ is a $r \times y$ matrix whose rows comprise of the first $r$ right singular vectors $V_j$ for $j = 1, ..., r$. Both the columns of $U_r$ and rows of $V_r^T$ are orthonormal. The columns of matrix $U_r$ represent a orthogonal basis set where the principal axis is defined by the first column. This axis is the direction which is most representative of (or most closely correlates to) the dataset. The subsequent columns correlate less to the dataset, as indicated by their singular values which are monotonically non-increasing (i.e., $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r$).

We can visualize the columns of $U_r$ since they are the same length as the element density vector $\rho$. We expect however that unlike $\rho$, some element values of $U_j$ may be negative. However, by plotting the first few left singular vectors which best approximate the dataset, we are still able to discern geometric features. Moreover we hypothesize that the first few basis vectors will represent solutions which capture the most prominent topological features in the design set. To perform the SVD we employ MATLAB®'s built-in function `svds.m`. 

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6.2.4 CUR Matrix Approximation

The CUR decomposition, like the SVD, is also used to perform low rank matrix approximations. The CUR decomposition however, makes the approximation using a subset of the columns and rows from the original dataset. Thus a key advantage of using CUR decomposition analysis is that the basis vectors are a subset of the data points (in our case design solutions represented by \( \rho \)) from the dataset. Hence those solutions which best represent the whole data set are found with a CUR decomposition analysis, as opposed to SVD where an orthogonal basis is found with the principal component lying in the direction which most correlates to the dataset. This important difference can make SVD analysis difficult to interpret since it does not necessarily resemble any single data point in the set. On the other hand, CUR matrix approximations can provide more intuitive results to an analyst because the basis vectors are now composed of actual data points (i.e., design solutions).

Using the dataset \( P \), the CUR matrix approximation provides a rank-\( r \) matrix \( P_r \), written as follows:

\[
P_r = C_r \bar{U}_r R_r
\]

where \( C_r \) is an \( m \times r \) matrix whose \( r \) columns come from the dataset \( P \), \( \bar{U}_r \) is an \( r \times r \) matrix, and \( R_r \) is an \( r \times y \) matrix whose \( r \) rows also come from the dataset \( P \). These matrices are not unique and several techniques have been devised to perform the decomposition [35, 55, 56, 135, 155]. We adopt the algorithm presented by Mahoney and Drineas [109]. This algorithm computes a probability distribution for all columns in the dataset matrix \( P \). Using this distribution, each column is then randomly selected to be included in the matrix \( C_r \). The column-wise probabilities, also termed the normalized statistical leverage scores, are computed from the top \( z \) right singular vectors \( V_j \), \( j = 1, ... , z \) of \( P \) according to the following equation given in [109]:

\[
\pi_c = \frac{1}{z} \sum_{j=1}^{z} (V_{cj})^2 \quad \forall c = 1, ... , y
\]

where \( V_{cj} \) is the \( c^{th} \) element of the \( j^{th} \) right singular vector, and \( \pi_c \) is the normalized
statistical leverage score of column \( c \) in \( P \) such that \( \pi_c \geq 0 \) and \( \sum_{c=1}^{y} \pi_c = 1 \). The algorithm implemented for choosing the columns of \( P \) to construct the matrix \( C_r \) is given in Algorithm 4.

**Algorithm 4: Column selection algorithm for constructing \( C \) matrix**

**Input:** Dataset matrix \( P \in \mathbb{R}^{m \times y} \)
- Desired number of columns to sample \( \bar{c} \)
- Number of right singular values to compute \( z \)

**Output:** Column matrix \( C_r \in \mathbb{R}^{m \times r} \)

1. Compute the top \( z \) singular vectors \( V^1, \ldots, V^z \)
2. for \( c \leftarrow 1 \) to \( y \) do
   3. Compute the normalized statistical leverage scores \( \pi_c \) using Equation (6.8)
4. end
5. for \( c \leftarrow 1 \) to \( y \) do
   6. Compute the probability \( P(c) = \min[1, \bar{c}\pi_c] \)
   7. Keep the \( c^{th} \) column of \( P \) with probability \( P_{\text{CUR}}(c) \) and append to matrix \( C_r \)
8. end
9. return \( C_r \)

Because each column of \( P \) is randomly selected using a probability score, the final size of \( C_r \) is not predetermined. While the argument \( \bar{c} \) defines the desired number of columns to sample, the actual number of columns in \( C_r \) is \( r \). Using the probability \( P_{\text{CUR}}(c) \) defined in Algorithm 4, \( r \leq \bar{c} \) in expectation [109].

The matrix \( R_r \) is also computed using Algorithm 4 where the argument \( P \) is replaced by its transpose \( P^T \). The matrix \( U_r \) is computed using the following equation:

\[
\tilde{U}_r = C_r^+ P R^+_r \tag{6.9}
\]

where the superscript \( + \) denotes the Moore-Penrose generalized inverse.

Mahoney and Drineas prove that the rank-\( r \) approximation error of \( P \) is bounded with a probability of at least 98% by the following inequality [109]:

\[
\| P - C_r \tilde{U}_r R_r \|_F \leq (2 + \bar{\epsilon}) \| P - P_z \|_F \tag{6.10}
\]

where \( P_z \) is the rank-\( z \) approximation computed using the SVD, \( \bar{\epsilon} \) is a small positive
number \( \tilde{c} \in (0, 1] \) and \( \| \cdot \|_F \) indicates the Frobenius norm. In order for Equation (6.10) to hold, the desired number of columns \( \tilde{c} \) in Algorithm 4 must be chosen such that \( \tilde{c} = \mathcal{O}\left( \frac{z \log z}{\varepsilon^2} \right) \) where \( z \) was defined as the number of right singular vectors to use in calculating the normalized statistical leverage scores. The same holds for the number of rows to select from \( P^T \) when computing \( R_r \). Equation (6.10) is valid for any random matrix \( P \). In our analysis we choose \( \varepsilon = 1 \) and \( \tilde{c} = z \log z \) rounded to the nearest integer. According to Equation (6.10), the rank-\( r \) approximation to \( P \) using Mahoney and Drineas’ CUR algorithm is at most three times less accurate than using a rank-\( z \) SVD approximation. For more details on the CUR decomposition algorithm and computing the matrix \( \bar{U}_r \) the interested reader is referred to [109].

The columns of the matrix \( C_r \) best approximate the dataset \( P \). Each column represents a single design solution in the dataset, and hence we can plot the density field to visualize the solution and its topological features. In addition, these solutions can be evaluated based on their objective and constraint function values indicating their overall performance.

6.2.5 K-Means Clustering

K-means clustering is an iterative process which partitions a dataset into \( K \) clusters where each cluster is defined by a centroid \( c^j \) for \( j = 1, \ldots, K \). We choose a squared Euclidean metric to describe the distance between each design solution \( \rho \) in the dataset \( P \) and the cluster centroid, although the designer can choose from a variety of distance metrics. We use MATLAB®’s built-in function kmeans.m to perform the clustering analysis which implements the \( k\)-means++ algorithm for seeding the initial centroids developed by Arthur and Vassilvitskii [16].

K-means clustering seeks to find a solution to the following problem:

\[
\arg\min_{S, \bar{C}} \sum_{j=1}^{K} \sum_{\rho \in S_j} \| \rho - c^j \|^2
\]

where \( S \) is the set of all clusters \( S = \{S_1, \ldots, S_K\} \) with corresponding centroids \( \bar{C} = \{c^1, \ldots, c^K\} \), and \( \| \cdot \| \) is the Euclidean distance. Therefore all data points \( \rho \in S_j \) are
part of the $j^{th}$ cluster with a centroid located at $c^j$. The algorithm initializes the centroids by randomly selecting a data point $\rho^c$ in the dataset $\mathbf{P}$ using the following probability metric:

$$P_{k\text{-means}}(c) = \frac{\|\rho^c - c^*\|^2}{\sum_{\rho \in \mathbf{P}}\|\rho - c^*\|^2} \quad (6.12)$$

where $c^*$ is closest center that has already been chosen. In other words, if the algorithm were choosing an initial second centroid $c_2$, then $c_1$ would be used as $c^*$ in Equation (6.12) to determine the probability distribution over all columns of $\mathbf{P}$, $c = 1, ..., y$. Once the initial centroids have been chosen, the algorithm iteratively recomputes the sets $S_j$ for $j = 1, ..., K$ by choosing the closest points to each centroid $c_j$, and subsequently updates the centroids $c_j$ for $j = 1, ..., K$ by calculating the center of mass for each set $S_j$. The center of mass of a set $S_j$ is calculated using the following equation:

$$c^j = \frac{1}{|S_j|} \sum_{\rho \in S_j} \rho \quad (6.13)$$

where $|S_j|$ is the number of data points in the cluster $S_j$. Note that the center of mass is not restricted to coincide with a data point $\rho$ in the dataset $\mathbf{P}$. Instead the center of mass is purely an average of all data points within a cluster. Thus cluster centroids, which we later visualize to represent the dataset in our analysis, can have more intermediate density elements than any of the design solutions $\rho$ in the dataset.

Next, we detail the k-means algorithm below in Algorithm 5.
Algorithm 5: K-means algorithm for partitioning dataset into $K$ clusters

**Input**: Dataset matrix $\mathbf{P} \in \mathbb{R}^{m \times y}$
- Desired number of clusters $K$

**Output**: Cluster sets $S = \{S_1, ..., S_K\}$
- Cluster centroids $\mathbf{C} = \{\mathbf{c}^1, ..., \mathbf{c}^K\}$

1. Choose the first centroid $\mathbf{c}^1$ at random from $\mathbf{P}$
2. for $j \leftarrow 2$ to $K$
   3. Choose the next centroid $\mathbf{c}^j$ from $\mathbf{P}$ using the probability distribution defined in Equation (6.12)
4. end
5. while The set $S = \{S_1, ..., S_K\}$ is changing do
   6. for $j \leftarrow 1$ to $K$
      7. Set the cluster $S_j$ to be set of points closest to $\mathbf{c}^j$ using the Euclidean norm
   8. end
   9. for $j \leftarrow 1$ to $K$
      10. Set the centroid $\mathbf{c}^j$ to be the center of mass of all points in $S_j$ using Equation (6.13)
11. end
12. end
13. return $S, \mathbf{C}$

Algorithm 5 provides the essential steps of the k-means algorithm. MATLAB's built-in `kmeans.m` function does offer additional features which can help ensure that a locally optimal solution to the problem posed in Equation 6.11 is found. The user also has the ability to repeat the clustering for various initial cluster centroids, and use different distance metrics other than the Euclidean norm. For more information, the interested reader should refer to the MATLAB help documentation on `kmeans.m` [110].
6.3 Multi-Objective 2D Thermally Radiating Plate Dataset Analysis

We revisit the 2D radiating aluminum plate example presented in Chapter 5 where optimization problem (SP2) was solved using our level set based topology optimization method. The minimum frequency $\bar{\omega} = 50\text{Hz}$ and upper bound on the von Mises stress constraint $\bar{\sigma} = 552\text{MPa}$ have the same values used in the final analysis of Section 5.5.4. Now we are interested in more thoroughly exploring the design tradespace. In Section 6.3.1 we present the varied initialization parameters used to perform the tradespace exploration. In Section 6.3.2 we present and categorize the dataset by visualizing each of the design solutions in the objective space. And finally in Sections 6.3.3–6.3.6 we present the results using each of the numerical techniques discussed in Section 6.2.

6.3.1 Design Exploration Setup

We design a full-factorial experiment where the initialization parameters are varied. These parameters include the objective function weights, initial material configuration, initial Lagrange multiplier estimates, and a boolean variable for applying velocity smoothing. Table 6.2 provides a complete listing of the all the initialization parameters and their discrete values used in the tradespace exploration.
Table 6.2: Initialization parameters for solving optimization problem (SP2) in a full-factorial experiment.

<table>
<thead>
<tr>
<th>Initialization Parameter</th>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal heat power objective function weight $w_1$</td>
<td>0.25, 0.275, 0.30, …, 0.95</td>
</tr>
<tr>
<td>Volume fraction objective function weight $w_2$</td>
<td>0.75, 0.725, 0.70, …, 0.05</td>
</tr>
<tr>
<td>Initial Material Configuration</td>
<td>Fully Filled</td>
</tr>
<tr>
<td></td>
<td>Holes (1 x 1 array)</td>
</tr>
<tr>
<td></td>
<td>Holes (3 x 3 array)</td>
</tr>
<tr>
<td></td>
<td>Holes (5 x 5 array)</td>
</tr>
<tr>
<td></td>
<td>Holes (7 x 7 array)</td>
</tr>
<tr>
<td>Lagrange multiplier for stress constraint $\lambda_1^0$</td>
<td>0.01, 0.10</td>
</tr>
<tr>
<td>Lagrange multiplier for frequency constraint $\lambda_2^0$</td>
<td>0.10, 0.02</td>
</tr>
<tr>
<td>Use Hilbert inner product space smoothing</td>
<td>yes ($\alpha_{HI} = \Delta x$), no</td>
</tr>
</tbody>
</table>

† The objective functions weights always sum to one, $w_1 + w_2 = 1$

According to Table 6.2, 29 different objective weights, 5 initial material configurations, 2 values for each of the Lagrange multiplier estimates and a boolean variable to enable a velocity smoothing operation yields a total of 1160 possible combinations.

Recall that if the velocity smoothing $\alpha_{HI}$ is too large, the velocity field becomes over-smoothed and many of the smaller topological features will disappear. On the other hand if $\alpha_{HI}$ is too small than the velocity is under-smoothed and there will be little to no effect on the final design solution. From numerical experiments, we determined that $\alpha_{HI} = \Delta x$, where $\Delta x$ is the minimum element size in the mesh, provides a suitable balance between under and over-smoothing the velocity field.

We perform a topology optimization run for each combination of parameters. The total CPU runtime to complete all 1160 analyses was 13 days, 7 hours, 54 minutes, 35 seconds. For each optimization run, the histories of the objective and constraint functions, along with the level set function and element density vectors are stored. We use a constant step approach for updating the Hamilton Jacobi equation since it is possible that the optimizer may traverse several local optimums during the optimization routine before converging. This occurrence can be due to a number of reasons including (1) large tolerance $\beta_L$ allowing the augmented Lagrangian to increase for most of the optimization history, (2) numerical errors arising from interpolating the
shape sensitivity, calculating the boundary normals and reinitializing the level set function, and (3) non-smoothness of the shape sensitivities (e.g., from the global von Mises stress function). After each optimization run the objective and constraint histories are exhaustively searched for the optimal, feasible design. If such a design is found, it is added as a data point when plotting the objective function values from the dataset. While this approach is more computationally expensive than using a backtracking method, it is more likely to find better performing solutions and does not require enforcing an arbitrary update tolerance on the augmented Lagrangian.

6.3.2 Categorizing the Design Set

We categorize the design solutions on the objective space by four criteria: (1) non-dominated and dominated, (2) initial material configuration, (3) the active constraint and (4) whether velocity smoothing was used or not. A total of 2242 design points are plotted in the design space. The non-dominated and dominated designs are illustrated in Figure 6-8. The designs categorized by their initial material configuration are illustrated in Figure 6-9. The designs categorized by their active constraint are illustrated in Figure 6-10. And finally the designs categorized by whether or not a smoothing operation was performed on the velocity field are illustrated in Figure 6-11.
Figure 6-8: Solution dataset categorized into non-dominated and dominated design sets

Figure 6-9: Solution dataset categorized by the initial material configurations
Figure 6-10: Solution dataset categorized by the constraint(s) active

Figure 6-11: Solution dataset categorized into designs identified using smoothed and non-smoothed velocity fields
6.3.3 Frequency Distribution Analysis

Having categorized the dataset, we now apply a frequency distribution analysis to extract the prominent topological features. Each element $e$ in the mesh is tagged as either "full", "void" or "boundary" according to Equations (6.5) for each solution in the dataset. Then using Equations (6.4), we sum the number of occurrences over all design solutions that each element is categorized as either "full", "void" or "boundary". These categories correspond to element densities of $\rho_e = 1$, $\rho_e = 0$ and $\rho_e \in (0, 1)$, respectively. This calculation is performed on the entire dataset, in addition to each of the subgroups outlined in the previous section.

For each category, a multivariate histogram is illustrated by plotting the finite element mesh where each element is colored according to its frequency of being tagged in each of the three categories. Thus the more red an element appears, the more times it was tagged in the respective category. For example, if an element $e$ was tagged as "full" over all the solutions $y$ in the dataset, it would appear red in the multivariate histogram for "full" elements. This same element would also then never be tagged as either "void" or "boundary", and therefore appear blue in histograms for those categories. The reason for plotting each of the categories, "full", "void" and "boundary", is to more easily depict structural boundaries and therefore better visualize the topological features.

First we consider the entire dataset. The histograms are illustrated in Figure 6-12 where we can distinguish a few design features: (1) a nozzle-like shape where the structure converges from the left edge and then diverges again before interfacing with the radiating sub-domain, (2) a large cross feature which nearly intersects the corners of the design domain, (3) a truss structure in the interior and (4) a semi-circular protrusion abutting the Dirichlet edge where temperature is held constant (also the source of heat power flowing through the structure). Because the entire dataset of 2242 design solutions is included, the multivariate histogram is not particularly easy to interpret and all of the prominent design solutions are overlapped, meaning we can not associate any particular features to a single design. Moreover, the colors
Figure 6-12: Histograms of complete design dataset (2242 designs)

are graduated and no clear structural boundary is defined. Instead the three images resemble the grayscale results found using a SIMP based approach. However, the frequency distribution analysis does provide a general sense of common features in the dataset. Those features which are more clearly depicted are more frequently occurring, meaning the optimizer tends to favor these designs.

The first subgroup we consider within the dataset is the non-dominated design solutions. Figure 6-13 illustrates the histogram plots for this subgroup. The general features illustrated are very similar to those in Figure 6-12.

Figure 6-13: Histograms of non-dominated design set (372 designs)

The major differences are that the converging and diverging outer structure is less curved, and the internal truss structure is more intricate (i.e., truss elements are shorter, thicker and more abundant). Again because this dataset consists of a larger
number of design solutions, many of the more intricate features are difficult to discern. In addition, most of the non-dominated design solutions have high volume fractions. As a result, low volume fraction non-dominated designs are blurred out by the abundance of higher volume fraction designs. Because we have chosen a weighted sums approach for combining the objective functions, we cannot guarantee that the non-dominated design solutions will be equally distributed along the Pareto front.

Next we consider the subgroups containing all designs with either active stress or frequency constraints only. The histograms generated from analyzing the stress constrained and frequency constrained designs are illustrated in Figures 6-14 and 6-15 respectively. Recall from Figure 6-10 that relatively few designs with active stress constraints exist in our dataset. Only 13 of the 2242 topologies have only an active stress constraint. Conversely 1992 topologies have only an active frequency constraint. The remainder of the designs in the dataset have either both constraints active or inactive. Since the active stress constraint subgroup is small, the histograms are much sharper. On the other hand for the active frequency constraint subgroup, the topological features are much more blurry. While having a smaller dataset produces more crisp topological features, less geometric diversity exists in the dataset and any numerical issues such as disconnected material regions and non-smooth edges will manifest themselves to a greater degree. Figure 6-14(a) exhibits some of these nonsensical features where the interior truss members are not all connected and excess material is protruding from the lower right outer truss. For this reason we also plot the void and boundary element histograms which help to clarify the feature shapes.
The next two datasets we consider are the design solutions achieved by smoothing the velocity field and those which were generated using an unsmoothed velocity field. The histograms for the subgroup of unsmoothed velocity solutions are illustrated in Figure 6-16 whereas those for the smoothed velocity solutions are illustrated in Figure 6-17. The only conclusion we can draw when comparing these two sets of histograms is that the velocity smoothing did not seem to change the smoothness or size of the topological features appreciably. Both sets of histograms are very similar and hence we cannot draw any further conclusions.
Finally the last two datasets we consider are the sets of design solutions whose volume fractions lie in the range $[0.35, 0.45]$ and $[0.55, 0.65]$, designated the low and high volume fraction subgroups respectively. These two datasets are illustrated in Figures 6-18 and 6-19. The low volume fraction designs exhibit the nozzle-like converging and diverging shape seen previously with a single pair of inner crossed trusses. The high volume fraction designs also exhibit a converging-diverging shape, but the structural boundaries are straighter and the interior is more filled with solid material. In the high volume fraction designs, we also note the same cross depicted in Figure 6-12 and a pattern of small voids dispersed throughout the material domain.
Figure 6-18: Histograms of design subgroup with volume fractions between 0.35 and 0.45 (587 designs)

Figure 6-19: Histograms of design subgroup with volume fractions between 0.55 and 0.65 (282 designs)

### 6.3.4 SVD Analysis

Next we apply the SVD to the same dataset and subgroups from the previous section. The first three left singular vectors $U^1$, $U^2$ and $U^3$ are plotted in each of the figures from this section along with their corresponding singular values, $\sigma_1$, $\sigma_2$ and $\sigma_3$ respectively. Recall that the left singular vectors are orthogonal to each other and are not coincident with any of the particular design solutions in the dataset. Therefore the SVD will also yield mesh plots where topological features are blurred and more difficult to discern. However, we are now choosing multiple left singular vectors corresponding to the largest singular values which best approximate all topologies in
the dataset. As opposed to the frequency distribution analysis where all the features were superimposed on a single image, the prominent topological features will now be separated amongst the different vectors $U^1$, $U^2$, and $U^3$ which we hypothesize will allow the designer to better interpret the dominant features. We calculate only the first three left singular vectors to balance the cost of CPU runtime with the goal of finding distinct discernible features.

First we consider the entire dataset, followed by the non-dominated design dataset. The two sets of left singular vectors are illustrated in Figure 6-20 and 6-21, respectively.

![Figure 6-20: First three left singular vectors of complete design dataset](image1)

- (a) $U^1, \sigma_1 = 2190$
- (b) $U^2, \sigma_2 = 586.0$
- (c) $U^3, \sigma_3 = 474.3$

![Figure 6-21: First three left singular vectors of non-dominated design set](image2)

- (a) $U^1, \sigma_1 = 1051$
- (b) $U^2, \sigma_2 = 222.1$
- (c) $U^3, \sigma_3 = 170.5$

The first singular vector $U^1$ for both sets is blurred as we had seen before in Section 6.3.3. However the second and third singular vectors display much more clear
discernible trussed features. Unlike the frequency distribution analysis which provided images of the full and void domains separately, the singular vectors are defined as an orthogonal basis and therefore the colored regions are not specifically correlated to either material or void. In fact some of the elements in the singular vectors have negative values which is not possible for the element density vector. By inspection, it is straightforward to determine which regions indicate material or voids.

The subgroups corresponding to a single active stress or frequency constraint are considered next. Figures 6-22 and 6-23 illustrate the left singular vectors for these subgroups respectively.

Figure 6-22: First three left singular vectors of active stress constraint design subgroup

Figure 6-23: First three left singular vectors of active frequency constraint design subgroup
As before, the active stress constraint subgroup has many fewer design solutions than the active frequency constraint subgroup. Consequently, plots from the former subgroup are clearer but the features can appear to be broken and in some instances nonsensical (e.g., in Figure 6-22(b)). The images derived from the active frequency constraint subgroup are very similar to those generated using the entire dataset since most of those designs have only an active frequency constraint. The solutions with an active stress constraint exhibit a more intricate truss structure on the interior of the design as well as less of a diverging shape close to the radiating sub-domain.

The subgroup of design solutions whose velocity fields have and have not been smoothed are illustrated in Figures 6-24 and 6-25 respectively.

![Figure 6-24: First three left singular vectors of design subgroup using smoothed velocity field](image)

(a) $U^1, \sigma_1 = 1512$  
(b) $U^2, \sigma_2 = 404.3$  
(c) $U^3, \sigma_3 = 329.5$

---

![Figure 6-25: First three left singular vectors of design subgroup using non-smoothed velocity field](image)

(a) $U^1, \sigma_1 = 1585$  
(b) $U^2, \sigma_2 = 424.8$  
(c) $U^3, \sigma_3 = 341.2$
Again it appears that the smoothing operation did not alter the final topologies appreciably. This result suggests that it might be possible to increase the smoothing parameter $\alpha_{\text{H}}$ in order to remove some of the finer topological features and improve convergence by considering a more aggressive strategy for advancing the level set function such as the backtracking method.

Finally we analyze the low and high volume fraction subgroups. The left singular vectors for these subgroups are illustrated in Figures 6-26 and 6-27 respectively.

![Figure 6-26](image)

(a) $U^1, \sigma_1 = 836.9$  
(b) $U^2, \sigma_2 = 382.2$  
(c) $U^3, \sigma_3 = 172.5$

Figure 6-26: First three left singular vectors of design subgroup with volume fractions between 0.35 and 0.45

![Figure 6-27](image)

(a) $U^1, \sigma_1 = 783.6$  
(b) $U^2, \sigma_2 = 240.1$  
(c) $U^3, \sigma_3 = 160.7$

Figure 6-27: First three left singular vectors of design subgroup with volume fractions between 0.55 and 0.65

The low volume fraction designs exhibit a converging and diverging structure from the left side of the design domain to the right. In addition a single internal cross feature is depicted. While the first left singular vector $U^1$ for the high volume fraction
designs exhibits a similar converging/diverging structure, the interior is more filled with material and the outer edges of the converging region are straighter. The second and third singular vectors depict a porous structure with a larger cross feature that nearly reaches the corners of the design domain. This feature provides needed rigidity while also dispersing heat to the outer edges of the radiating sub-domain.

6.3.5 CUR Analysis

The CUR factorization is applied by implementing Algorithm 4 to select the columns of \( \mathbf{P} \) used to assemble \( \mathbf{C}_r \). Because the columns of \( \mathbf{C}_r \) are specific solutions from the dataset, we can directly plot grayscaled images of the element density vectors on the mesh. In addition, these solutions have already been evaluated for their performance and the objective function values \( J_1 \) and \( J_2 \) will also be provided with each element density plot. The ability to extract actual design solutions from the dataset exemplifies a major advantage over the results produced using the frequency distribution and SVD analyses. In addition to highlighting the common topological features, the selected designs from the \( \mathbf{C}_r \) matrix have already been evaluated for their objective and constraints values. Moreover, the entire set of initialization parameters can be recalled for each of these designs. In summary, using the CUR factorization the analyst is provided a series of design solutions which best represent the dataset, illustrate the dominant topological features in a black and white image, and associate with specific performance metrics and initialization parameters.

For each dataset, we calculated \( z = 10 \) left singular vectors to select \( \tilde{c} = 23 \) desired number of columns in \( \mathbf{P} \). Through numerical experiments we found these values for \( z \) and \( \tilde{c} \) to yield a sufficient number of topologically diverse designs. In general the analyst would need to make a judgment on what constitutes topologically diverse, since this metric is purely qualitative and largely dependent on the specific problem at hand. Choosing \( \tilde{c} \) to be large will result in many similar design solutions whereas a value too low will not fully represent the entire range of topologies. In addition, the larger \( \tilde{c} \) becomes, the more designs are used to represent the dataset and hence the more cumbersome it will be for the analyst to visually inspect each of the designs.
First, we consider the complete and non-dominated design sets. Figures 6-28 and 6-29 provide six of the selected designs in matrix $C_r$. The illustrated topologies represent actual solutions generated by the topology optimizer. As such they depict clear geometric features with crisp structural boundaries, and have also been evaluated for their objective and constraint function values. The CUR factorizations of both the entire dataset and non-dominated design set generate topologies with converging and diverging shapes well as porous structures. The analyst could easily inspect these handful of designs and make comparisons between performance, structural rigidity, ease of manufacturing and other relevant factors to the design intent.

Figure 6-28: Sample columns from $C_r$ matrix for complete design dataset
(a) $J_1 = 12.60$, $J_2 = 0.2868$  
(b) $J_1 = 17.91$, $J_2 = 0.3697$  
(c) $J_1 = 19.84$, $J_2 = 0.4342$  
(d) $J_1 = 22.39$, $J_2 = 0.5442$  
(e) $J_1 = 22.59$, $J_2 = 0.5549$  
(f) $J_1 = 23.69$, $J_2 = 0.6339$

Figure 6-29: Sample columns from $C_r$ matrix for non-dominated design set

In Figure 6-30 we plot the six representative designs from Figure 6-29 on the objective space to illustrate their position along the Pareto front. Clearly the CUR matrix approximation technique has selected designs that vary both geometrically (i.e., dissimilar topological features) as well as volumetrically (i.e., dissimilar material volume fraction). Hence, these designs provide an excellent basis for the dataset. Note however, that geometrically diverse designs can have similar heat power radiated and volume fraction performances, as indicated by those in Figures 6-29(d) and 6-29(e). The CUR matrix approximation searches for designs whose element density vectors best span the $m$-dimensional space. The total volume fraction, however, is a normed metric of the density vector which correlates with its length. Hence even a perfectly orthogonal basis may have designs that are close to one another from a performance standpoint. We draw a similar conclusion later in Section 6.3.6 where clustered design solutions can lie in varying proximities from one another on the objective space.
Next we consider the active stress and active frequency constraint subgroups. Because the active stress constraint subgroup is substantially smaller than any other subgroups from the dataset, we only select two designs from the $C_p$ matrix. Similarly we select four designs to represent the active frequency constraint subgroup. Figures 6-31 and 6-32 illustrated the selected designs from the matrix $C_p$. Note that the two solutions for the active stress constraint subgroup are very similar to the two lowest volume fraction designs found in the non-dominated design dataset from Figures 6-29(a-b), whereas the four active frequency constraint topologies are more similar to the higher volume fraction designs from the same non-dominated set in Figures 6-29(c-f). This distinction provides some clear intuition on associating topological features desirable for satisfying either of the two constraints.
Finally the lower and higher volume fraction design subgroups are considered. We again select six sample columns from the $C_r$ matrix for each subgroup and plot them in Figures 6-33 and 6-34, respectively. The low volume fraction designs exhibit...
more intricate truss structures whereas the high volume fraction designs exhibit either fewer and larger (i.e., thicker) features or more porous designs with a large array of voids. The porous designs tend to dominate those with few larger features in terms of performance. However, there may be several concerns with porous structures including manufacturability, cost, and failure mechanisms not considered by the optimizer such as buckling. To summarize, the benefit of this analysis is to provide the designer with real solutions and intuition on the topological features that are beneficial for optimizing the objectives and satisfying the constraints.

Figure 6-33: Sample columns from $C_r$ matrix for design dataset with volume fractions between 0.35 and 0.45
Figure 6-34: Sample columns from $C_p$ matrix for design dataset with volume fractions between 0.55 and 0.65
6.3.6 K-Means Clustering Analysis

Finally we perform a k-means clustering analysis on the dataset \( P \). We select a second “online update phase” when calling the \textit{kmeans.m} function which guarantees that the solution to Equation (6.11) is a local minimum. This second phase, which occurs after the \textbf{while} loop on line 5 of Algorithm 5, exhaustively checks the reassignment of each data point to an alternate set \( S_j \) by recomputing the sum of distances squared function found in Equation (6.11). This search algorithm can be computationally expensive, but executes in approximately 10 seconds for our largest dataset of 2242 design points. In addition we also specify that the k-means algorithm repeat five consecutive analyses, each with a varying set of initial cluster centroids. The best of the five solutions is then selected as the clustering for our dataset. Again while this option adds computational cost, the algorithm is able to execute in seconds for our largest dataset.

First we consider the entire dataset. We choose to calculate six clusters and illustrate them in Figure 6-35 on the objective space. The centroids of each cluster are next illustrated in Figure 6-36 where we again see a larger number of intermediate density elements since the centroids do not lie on any particular design solution. Note however that the centroid of \( S_1 \) depicts a clear black and white solution with crisp structural boundaries. As expected, the \( S_1 \) cluster was the smallest with only 173 design points included. Overall, the cluster centroids are much clearer than the results generated using the frequency distribution and SVD analyses. However, in order to remove the intermediate density domains within the centroid designs, an additional post-processing step is needed. One possible strategy is to identify the closest actual design point to each centroid. Depending on the size of the given dataset and the proximity of the design solutions to one another, the nearest design to the centroid could be very geometrically dissimilar. In other words, the topological features found in the centroid may not match those of the nearest design point. This would be more of a concern if considering, for example, the centroid of \( S_6 \) where a significant number of internal trusses with intermediate element densities exist.
Figure 6-35: K-means clustering for complete design dataset

(a) $S_1$ Centroid
(b) $S_2$ Centroid
(c) $S_3$ Centroid
(d) $S_4$ Centroid
(e) $S_5$ Centroid
(f) $S_6$ Centroid

Figure 6-36: Cluster centroids for complete design dataset
Next we consider the non-dominated design set. Figure 6-37 plots the six clusters on the objective space, whose centroids are then illustrated in Figure 6-38. Again we see that the centroid images have some gray features, but overall give a much clearer depiction of the structural shapes as compared to the frequency distribution and SVD analyses. Another interesting result is that the lower volume fraction designs belonging to a single cluster tend to be more dispersed on the objective space in Figure 6-37 than the higher volume fraction designs. Clustered designs exhibiting similar geometric features can have a wide variety of performance ranges. Similar geometric shapes can be varied by thinning or thickening structural elements (e.g., narrowing or widening a truss) or adding smaller features to the general overall design (e.g., smaller internal gray truss elements of the $S_3$ centroid). This result is similar to that found in Figure 6-30 using the CUR decomposition where we saw that geometrically diverse solutions can be both close and far in proximity from one another on the objective space.

![Figure 6-37: K-means clustering for non-dominated design set](image)
Next we consider the active stress and active frequency constraint subgroups. Because the active stress constraint subgroup is substantially smaller than any of the other subgroups, we only partition the data points into two clusters. However for the frequency constraint subgroup, we retain six clusters as was done previously. The centroids are illustrated in Figures 6-39 and 6-40 for the stress and frequency constraint subgroups, respectively. Note that because the active stress constraint subgroup is small, the features depicted in the centroid for cluster $S_1$ are disconnected and the edges are rougher. The fewer the number of design solutions in the dataset, the less any data mining technique is able to smooth the affects from the various numerical artifacts previously mentioned (e.g., sensitivity interpolation, mesh configuration, etc.). In addition, some of the features noted earlier using the frequency distribution and SVD analysis are once again depicted. For example, the converging and diverging shape in Figure 6-40(b) and large cross element in Figure 6-40(e) were both identified in the earlier analyses.
Figure 6-39: Cluster centroids for the active stress constraint design subgroup

Figure 6-40: Cluster centroids for the active frequency constraint design subgroup
Finally we plot the results from clustering the low and high volume fraction designs. For the low volume fraction design dataset, we plot both the clustering in the objective space in Figure 6-41, and the corresponding centroids in Figure 6-42.

![Figure 6-41: K-means clustering for design dataset with volume fractions between 0.35 and 0.45](image)

From these plots we see that the centroids from clusters $S_1$ and $S_3$ are more crisp in terms of clearly depicting a structural boundary (i.e., less gray). These clusters are also more closely spaced in Figure 6-41. We conclude that the designs within each of these clusters are essentially the same geometric shape with varying feature thicknesses. As an example, consider the bifurcated feature originating from the center left edge in Figure 6-42(c). The gray regions surrounding the edges of this feature indicate that the designs within this cluster have the same feature, but with varying width. Conversely, the gray internal trusses in Figure 6-42(b) indicate that some designs incorporate these smaller features within the cluster, and some do not.
Therefore, these internal trusses can be considered less significant in terms of optimizing the performance or maintaining feasibility of the design compared to those structural elements illustrated in black. The darker a region, the more designs within the cluster place material at this location. A possible strategy for removing intermediate density elements from the centroid images might be to increase the number of clusters, since less common features can then be segmented. Similar to choosing the number of left singular vectors using the SVD matrix approximation, or desired number of columns from the $\mathbf{C}_r$ matrix using the CUR matrix approximation, the number of clusters to choose is based solely on the number of topological solutions the analyst wishes to visualize.

Next, the cluster centroids from the high volume fraction design dataset are illustrated in Figure 6-43. The designs in Figures 6-43(a) and 6-43(d) depict the porous structures that were identified previously, whereas the design in Figure 6-43(b) resem-
bles a more solid shape with no internal voids. Also, from inspecting Figure 6-43(a), we see that a ghosted image of a less porous, more solid structural shape is present. By re-running the k-means analysis using a larger number of clusters, we can segment out this alternate topology. Figure 6-44 illustrates the centroids of six clusters, now clearly depicting the ghosted image in Figure 6-44(e).

Figure 6-43: Cluster centroids for design dataset with volume fractions between 0.55 and 0.65
Figure 6-44: Additional cluster centroids for design dataset with volume fractions between 0.55 and 0.65

6.4 Chapter Summary

In this chapter, we introduced four candidate numerical techniques for identifying the prominent topological features in a large dataset of design solutions. The utility of this analysis is to aid the designer by providing intuition on features which are beneficiary to maximizing performance while maintaining feasibility. For large datasets it can be cumbersome for an analyst to review all the solutions and assimilate or categorize the geometric features desired by the optimizer. In addition, level set based optimization methods can be difficult to tune since there are many initialization parameters, which if not chosen carefully can lead to instability, slow convergence and nonsensical design features. Therefore a strategy to identify the prominent topological features will allow a designer to perform a large tradespace exploration using varied initialization parameters, rather than tune each topology optimization run to arrive at the non-dominated or Pareto optimal set. The benefit is then to offset some
of the burden from the designer by utilizing added computational effort. The candidate techniques included a frequency distribution (i.e., histogram) analysis, the SVD and CUR decompositions, and a k-means analysis. These analyses were then tested on a large dataset of 2242 designs produced from solving the optimization problem (SP2) for the 2D radiating aluminum plate example introduced in Chapter 5. The results from the frequency distribution and SVD analyses provided images which depict some of the prominent topological features. However, using the frequency distribution analysis, it can be very difficult to depict these features since many topologies are overlapped onto a single image. Likewise when using the SVD there is no correlation between the left singular vectors and any actual design solution in the objective space. The images of the left singular vectors represent principal components of the dataset and not the element density maps used to define crisp structural domains. The k-means clustering analysis provided much clearer design solutions, but again is not able to directly correlate any of the performance metrics with the structural topology. The CUR decomposition alleviates these disadvantages by providing clear design solutions with their respective objective and constraint function evaluations. As a result, the designer can now more easily compare candidate topologies with one another, in addition to associating geometric features with a design family.
Chapter 7

Conclusions and Future Work

The goal of this thesis was to extend the capability of topology optimization methods to designing thermally efficient radiating structures. These structures are found in many aerospace systems ranging from hypersonic reentry vehicles to interplanetary probes. We define a thermally efficient structure in the context of this thesis as a Pareto optimal design which trades thermal radiated heat power for material mass or volume. Section 7.1 provides a summary of the tasks accomplished and key points in each chapter. Section 7.2 enumerates the thesis contributions and provides conclusions on the research presented in this thesis. Finally, Section 7.3 provides several key areas for future work to further broaden the capabilities of our level set based topology optimization methodology so that it can be applied to real-world design applications.

7.1 Thesis Summary

This thesis expands the current capabilities of level set based topology optimization to identify efficient thermally radiating structures and demonstrates an approach for exploring the design space of multi-objective, multi-constraint problems as well as extracting meaningful information regarding dominant topological features. The thesis was divided into five body chapters.

Chapter 2 provided some background on structural design optimization and the
common implementation of parametric models to optimize structures with only a handful of design variables. We then introduced topology optimization as a class of free-form design methods. Three major approaches were discussed including homogenization methods, evolutionary methods and boundary variation methods. The advantages and disadvantages of each approach were highlighted. Ultimately we found that the level set based method was the most suitable for thermally radiating problems since the nonlinear boundary effect could be more easily modeled and the heat power radiated objective function optimized.

Chapter 3 introduced some previous research on optimizing thermally radiating structures using a variety of techniques including parametric or sizing optimization. A well-posed optimization problem (SP1) was presented to maximize thermal heat power radiated using a single volume fraction constraint. To solve this problem we implemented an augmented Lagrangian method to handle the equality constraint and used a level set method to implicitly define the structural boundary. The level set function is updated by solving a Hamilton Jacobi equation in fictitious time where the velocity field is constructed from the shape derivatives of the objective and constraint functions. By using a steepest descent method, the optimizer is able to drive the shape towards a local optimum. Finally an overview was provided on the computational framework developed in the MATLAB® environment using the Partial Differential Equation Toolbox to generate and solve the finite element models.

Chapter 4 introduced a 2D example problem where an aluminum square plate is radiating in a small sub-domain and a Dirichlet boundary condition is enforced. The radiating sub-domain is considered fixed meaning the optimizer cannot alter its shape. This restraint is needed to avoid a design-dependent loading scenario. Next we demonstrated that using the thermal compliance objective function, commonly found in problems for designing efficient heat sinks and other structures considering thermal conduction only, is insufficient for problems where a radiating boundary condition exists. To overcome this drawback, a radiated heat power functional is formulated and a novel shape sensitivity analysis is performed using the adjoint method. The shape derivative is used in a steepest descent method and tested on the 2D example
problem. The results show that the solution converged to a local optimum and that varying the initial material configuration yields different locally optimal solutions.

Chapter 5 presented some previous research on solving multi-objective, multi-constraint topology optimization problems as well as an overview of the different physics problems solved using level set based topology optimization methods. Next we formulated a 2D optimization problem (SP2) where the objective function was a weighted sum of the heat power radiated and volume fraction. Two constraints were also enforced, one on the maximum von Mises stress and one on the minimum first mode frequency. A single global p-norm functional constraint was used to replace the many stress constraints needed to ensure that the structure was not exceeding the von Mises stress criterion. The augmented Lagrangian method was again implemented to handle the multiple inequality constraints. The previous 2D aluminum plate problem was expanded to include two additional PDEs for static loading and modal frequency analysis. These PDEs were also solved using the finite element method. The approach was to solve the problem in stages, considering only a single constraint at a time before solving the multi-constraint formulation. The results indicated that our methodology converged to locally optimal solutions which were plotted in the objective space to illustrate the Pareto fronts. We found that for our 2D radiating aluminum plate problem, the majority of Pareto optimal designs had only one active constraint.

Chapter 6 revisited the multi-objective, multi-constraint 2D thermally radiating plate problem considered in Chapter 5 where we demonstrated that altering the initial material configuration affects the final design solution. Another common practice used in multi-objective problems to explore the design space is to vary the objective function weights. We demonstrated that changing the initialization parameters relating to the augmented Lagrangian method and level set update algorithm can also yield different geometric features in our topology optimization framework. Given the presence of many locally optimal solutions and the desire to explore these diverse topologies, a methodology is proposed where the designer runs a full-factorial experiment sampling several objective function weights, initial material configurations and initialization parameters. A data mining technique is then used to extract
the dominant topological features from the solution dataset. The utility of our proposed method is to both (1) gain valuable design insight for problems with multiple competing objectives and multiple constraints and (2) help categorize the solution topologies into design families, associating geometric features to specific performance metrics or feasibility criteria. Four numerical techniques were used to extract the dominant topological features from the solution data set, including a frequency distribution, SVD and CUR matrix approximation, and k-means clustering analysis. These techniques were then tested on a large dataset of solutions generated by solving optimization problem (SP2) for the 2D radiating aluminum plate. The frequency distribution and SVD analyses provided graduated color images where the topologies were overlapping, making geometric features difficult to discern and impossible to associate. The k-means clustering analysis provided much more discernible geometric features, but was still unable to correlate the structure's objective and constraint function evaluations to a specific topology. The CUR decomposition, however, identified actual design solutions which best represented the dataset and provided clear black and white images of the optimal topologies. Subsequently we were able to compare the performance metrics (i.e., objective function values) of these representative design solutions with one another and associate them to specific active constraints. We concluded that the CUR decomposition provides a significant benefit by allowing the designer to categorize the solutions into several design families and make both qualitative and quantitative comparisons.
7.2 Contributions and Conclusions

This thesis makes several contributions which are enumerated below:

1. Created a framework implementing a level set based topology optimization method to optimize 2D thermally radiating structures in a free-form manner using a gradient-based optimizer.

2. Derived an analytic shape sensitivity for the thermal heat power radiated functional and implemented it numerically using the finite element method.

3. Tested the level set based topology optimization framework on a 2D example problem with the goal of minimizing mass and maximizing thermal heat radiated subject to von Mises stress and first mode frequency constraints.

4. Applied frequency distribution, SVD and CUR matrix approximation, and k-means clustering methods as data mining tools to extract dominant topological features from a large dataset of topology solutions generated by varying the objective function weights and initialization parameters.

This thesis has demonstrated that a free-form design methodology can be implemented to optimize thermally radiating structures. The advantages of free-form design are clear: an almost limitless number of topologies can be identified where feature sizes, shapes and connectivity are all variable during the optimization. The growing capabilities of advanced manufacturing technologies (e.g. 3D printing) and the growing computational power to perform high fidelity analyses of structures have enabled the use of topology optimization methods in real-world design problems. To date, thermally radiating structures have not been considered in level set based topology optimization. Thus our thesis presents the first approach to solving this class of problems using a level set based method. One of the major limitations is that we only consider the radiation exchange between the structure and the ambient surroundings. Without accounting for radiation heat exchange between disparate surfaces, many real-world design problems cannot yet be considered. However, the challenges of including radiation exchange between structural surfaces, as well as design-dependent
loading where the radiating surface can also change shape are formidable. These future work tasks are discussed in the following section.

While topology optimization has proved to be a powerful tool for augmenting the designer’s intuition and identifying higher performance designs, it is only recently that larger design exploration studies are being considered. The benefit of these tradespace studies is apparent, providing the designer multiple real options and assessing the tradeoffs between competing goals such as weight and load capacity. In addition, designers can evaluate the multiple topology solutions for criteria not easily embedded in an optimization routine (e.g. manufacturing costs, clearances for part assembly, interface compatibility, etc.). For such tradespace studies where many topologies are identified, the analyst’s ability to inspect each design and draw insight into which features are beneficial for performance is limited. Mechanical design requires a strong cognitive ability to visualize concepts, interfaces and other complex geometric entities. Dealing with 3D designs further exacerbates this problem since the entire geometry cannot be viewed in a single still image.

The numerical techniques tested in Chapter 6 were an attempt to mitigate this problem by extracting a handful of dominant geometric features which benefit performance and help maintain feasibility. The results from such a data mining technique could be used in two ways: (1) to initialize a topology optimization analysis with an initial material configuration exhibiting these geometric features and hence speeding convergence to a local optimum and (2) to guide the designer’s intuition and insight into the problem while generating his or her own designs. We found that the CUR decomposition was the most effective way at visualizing the dominant topological features represented in a large dataset. The other benefit afforded by the CUR decomposition technique was the ability to associate a selected design exhibiting specific geometric features to a quantitative measure of performance. Thus the variety of topological features identified by the optimizer are grouped rather than overlapped as in the frequency distribution and SVD analyses. Additionally we were able to analyze subgroups of the dataset (e.g. designs with only one active constraint) to further categorize topologies which were beneficial to that particular subgroup.
Unfortunately the process of tuning gradient-based topology optimization routines is not well discussed in the literature, arguably because the analyst must use heuristics and numerical experiments to arrive at a suitable set of initialization parameters. Many authors offer their own methods to optimize the problem but rarely discuss the parameters chosen to solve the Hamilton Jacobi equation or other update strategy for the level set function. While we cannot avoid the need for tuning these algorithms to arrive at improved topologies to our design problems, we can use the techniques presented in Chapter 6 to provide some relief to the analyst. Rather than discarding solutions which exhibit some undesirable convergence characteristics (e.g. oscillating objective and constraint functions, unstable boundary velocity norms, regions of disconnected material, etc.) or spending additional time tuning the algorithms to remove these characteristics, we can use these designs in larger tradespace studies. A suitable data mining technique, such as the CUR decomposition, can complement the analysis by extracting the major topological features the optimizer is driving towards across many design solutions. Moreover the analyst is able to visualize these features in a handful of images rather than inspect the full set of solutions generated in the tradespace study.

7.3 Future Work

In this thesis we have laid the groundwork for applying level set based topology optimization methods to the design of thermally radiating structures. While there are many directions to continue this research, we highlight several of the most important ones for enabling the application of topology optimization methods to real-world design problems involving thermal radiation. In particular we present the need for incorporating design-dependent radiative heat transfer, developing strategies for tackling more complex problems with larger finite element models and added constraints, and interfacing with commercial finite element analysis software.

A growing area of interest in the topology optimization community is design-dependent loads analysis. Problems where design-dependent loads exist require that
the boundary conditions be modified as the optimizer updates the shape of the structure. These boundary conditions could include, for example, a pressure force. In Section 2.3 we presented a problem solved by Allaire et al. [10] where the authors optimized a structure subjected to a uniform pressure load. As the optimizer altered the structural shape, the load vectors were also updated such that they remained normal to the boundary. The final design closely resembled that of a starfish.

The same design-dependent load scenario can also be applied to thermally radiating structures. If the optimizer was allowed to change the topology of the radiating sub-domain or surfaces in the example problems considered in this thesis, then the design spaces would be substantially larger and we could possibly find better performing solutions. Consider the 2D aluminum radiating plate presented in Chapter 4. We expect that the optimizer would have identified a variety of other topologies if the radiating sub-domain were allowed to change shape. Without a constraint limiting which elements or the number of elements able to radiate heat, the entire structural domain would be made to radiate \( \Omega_{\text{rad}} = \Omega \). Furthermore, the distance between the Dirichlet boundary and radiating sub-domain would be shortened as much as possible so that the structure can radiate at the highest temperature possible (rejecting the most heat power). With these postulations in mind, we hypothesize that the optimal structure would resemble a circular segment where the chord is abutting the Dirichlet boundary. The optimal shape considering the design-independent load scenario presented in Section 4.4 is illustrated in Figure 7-1(a) along with our hypothesized optimal shape considering a design-dependent load scenario in Figure 7-1(b). Assuming that no additional constraints would prohibit these new topologies, the added design freedom of allowing the radiating surface to change shape could be very advantageous.
In order to solve topology optimization problems considering design-dependent loading, the analyst must devise a strategy for recalculating the load conditions as the structural boundaries move. In thermally radiating problems the radiative heat exchange between surfaces, also known as irradiation, must be calculated. This exchange of heat power occurs when two surfaces have a direct line of sight between each other, such that a fraction of the energy radiating out from each surface will impinge on the other. Figure 7-2 illustrates the radiative exchange in a convex, flat and concave surface.

Figure 7-1: Optimal shapes for optimization problem (SP2) considering design-independent (actual) and design-dependent (hypothesized) load conditions.
The term $F_{11}$ is the view factor between the surface and itself, defined as the fraction of radiated energy that is being reabsorbed by the surface. For a convex or flat surface (as was considered in the example problems in this thesis), all of the energy leaving the radiating surface is transferred to the ambient environment which is kept at a fixed temperature $\theta_0$. However for a concave surface, a radiative exchange is made between all points on the surface as well as the ambient environment. Typically an analyst will use a computational ray-tracing tool to calculate the view factors and subsequently evaluate the amount of heat transferred between each surface element.

Thermal irradiation is an important phenomenon in the design of radiatively cooled spacecraft components which use fins or other protruding structures to increase the radiating surface area. An example is the outer aluminum shell of the radioisotope thermoelectric generator on board the New Horizons spacecraft introduced in Chapter 1. The outer shell consisting of a cylindrical tube structure with radial fins cools the device by radiating heat to ambient space. The radial fins, however, will also radiate to each other and to the core structure. A cross-section of the radioisotope thermoelectric generator used on board the Mars Science Laboratory is illustrated in Figure 7-3 for reference. A strategy for handling irradiation in a topology optimization framework is necessary to consider these types of design problems.
Two major challenges must be overcome in order to implement a methodology for handling irradiation as a design-dependent load condition. First, the shape sensitivity must be re-derived to incorporate the view factors for all elements lying on the radiating surface. This derivation is not trivial and it is possible that an analytic solution may not exist. An alternate approach would be to calculate the shape sensitivity numerically via, for example, a finite difference scheme. While numerically evaluating this sensitivity could be computationally expensive, several strategies for reducing the CPU runtime of our methodology exist and are later discussed. Second, a ray-tracing tool would need to be integrated in order to calculate the view factors for each element on the radiating surface. The ray-tracing tool would require a well-defined crisp structural boundary where the surface normal is accurately calculated. This requirement makes level set methods particularly well suited for solving problems where irradiation is considered.

Another future work task is testing our topology optimization framework on problems with additional constraints. In real-world applications a multitude of systems requirements are defined to ensure that the structure can survive the external load conditions experienced during it’s lifetime. Aerospace systems are oftentimes ana-
lyzed for g-force loading, buckling, acoustics, sinusoidal and random vibration, and shock. Many of these structural requirements can be translated to a set of constraints in a topology optimization problem. Several authors have already applied level set based topology optimization methods to solving buckling [60] and acoustics [122] problems. However, as the number of constraints increases, the number of initialization parameters will also grow proportionally making it more difficult to tune our algorithm. The geometric features may also become more complex and intricate as additional constraints are enforced.

Another potential approach to consider is decomposing the problem into a series of subproblems, each enforcing only a single constraint from the original set. A feasible solution could then be reconstructed by assembling the resulting topologies of these subproblems in a novel way. The feasible design could be evaluated for its performance and accepted, or used as an initial configuration for a topology optimization analysis including all constraints simultaneously. Recall in Section 5.5.4 that along the Pareto fronts generated by solving (SP2) of our 2D radiating plate case study, typically only a single constraint was active while the other was inactive. Consequently, the topologies identified along the Pareto front either resembled those from solving (SP2b) or (SP2c) where a single constraint was enforced, or a hybrid topology exhibiting geometric features from both solution sets. This trend may continue for larger optimization problems with additional constraints. On the other hand, some constraints may be correlated and favor similar geometric features, such as a first mode frequency and buckling constraint [8]. Both these constraints would for example, prevent the final topologies from developing long and slender features. In summary, the alternate methodology would decompose the optimization problem into a series of single constraint subproblems and reconstruct a feasible solution.

In order to incorporate design-dependent load scenarios, calculate view factors for radiative heat exchange, solve problems with added constraints, and evaluate sensitivities numerically, it would be necessary to speed up our numerical routines. Perhaps the most obvious way this could be accomplished is through the careful use of parallelization. Many subroutines such as the fast marching method used to regularize the
level set function can be parallelized. Other routines that are not currently implemented could also be parallelized and exchanged for the routines used in this thesis. For example, the Hamilton-Jacobi solver we implemented is an upwind scheme which cannot be parallelized. However, a parallelized version of the method of characteristics has been successfully implemented to solve the Hamilton-Jacobi equation [7] and can be swapped for our current method. In addition, many commercial finite element analysis software packages exist and typically offer application programming interfaces for code developers. These commercial software packages, which oftentimes have been optimized for speed and memory usage, could be called to solve the finite element models in order to evaluate the objective and constraint functions, as well as the shape sensitivity fields. A major challenge, however, would be assembling the global stiffness, mass and conductivity matrices after scaling each of the element matrices with their respective density value $\rho_e$. One potential solution would be to implement a Lagrangian approach as discussed in Section 3.4.2 where the structure is re-meshed at every optimization iteration, removing the void and intermediate density elements.
Appendix A

Shape Sensitivity Analysis for Radiated Heat Power

Consider the linear thermal problem introduced in Section 3.2 with a nonlinear radiative boundary condition. This problem is illustrated in Figure 3-1. Thermal radiation can be modeled using the Stefan-Boltzmann Law which is written as follows:

$$Q_{\text{rad}} = \varepsilon \Upsilon \theta_{\text{surf}}^4$$  \hspace{1cm} (A.1)

where $Q_{\text{rad}}$ is the thermal heat power flux radiated in units of W m$^{-2}$, $\varepsilon$ is the surface emissivity ($0 < \varepsilon \leq 1$), $\Upsilon$ is the Stefan-Boltzmann constant $5.670 \times 10^{-8}$ Wm$^{-2}$K$^{-4}$ and $\theta_{\text{surf}}$ is the temperature field if the radiating surface in K. For $\varepsilon = 1$, the structure is assumed to be radiating as a perfect black body (otherwise for $\varepsilon < 1$ the structure is considered a grey body). The Stefan-Boltzmann Law is derived by integrating the irradiated power over all wavelengths within a $2\pi$ steradian half sphere.
The governing PDE, reproduced below in Equation (A.2) from Section 3.2 includes a sub-domain \( \Omega_{\text{rad}} \subset \Omega \) where the structure is radiating heat power.

\[
\begin{align*}
\text{div} (\kappa \nabla \theta) &= \varepsilon \nabla \theta^4 \quad \text{in} \quad \Omega_{\text{rad}} \\
\text{div} (\kappa \nabla \theta) &= 0 \quad \text{in} \quad \Omega \setminus \Omega_{\text{rad}} \\
(\kappa \nabla \theta) \cdot n &= 0 \quad \text{on} \quad \Gamma_{0,\theta} \\
\theta &= \theta_0 \quad \text{on} \quad \Gamma_{D,\theta}
\end{align*}
\]  

(A.2)

A domain flag \( \chi = \chi(x) \) is introduced to evaluate to 1 where the structure is radiating \( (x \in \Omega_{\text{rad}}) \) and 0 where the structure is not radiating \( (x \in D \setminus \Omega_{\text{rad}}) \). Recall that \( \Gamma_{0,\theta} \) is defined in Figure 3-1 as an insulated boundary which does not exchange any heat with the surroundings. The above governing PDE is transformed into the following equivalent PDE:

\[
\begin{align*}
\text{div} (\kappa \nabla \theta) - \chi \varepsilon \nabla \theta^4 &= 0 \quad \text{in} \quad \Omega \\
\theta - \theta_0 &= 0 \quad \text{on} \quad \Gamma_{D,\theta} \\
(\kappa \nabla \theta) \cdot n &= 0 \quad \text{on} \quad \Gamma_{0,\theta}
\end{align*}
\]  

(A.3)

A.1 Objective Function

The objective function is the total heat radiated from the domain \( \Omega_{\text{rad}} \). Using the domain flag \( \chi \), this function is written as follows:

\[
J(\Omega, \theta) = \int_{\Omega} -\chi \varepsilon \nabla \theta^4 dA
\]  

(A.4)

Although our objective is to maximize the radiated heat power, we can define an equivalent objective which is to minimize the negative of the radiated heat power. This is a common strategy to reformulate a maximization problem into a minimization problem.
A.2 Lagrangian and Adjoint Approach

The Lagrangian is a summation of the objective function and the constraints weighted by a Lagrange multiplier $q$. The objective function is stated in Equation (A.4), whereas the constraints are given by the state equation and boundary conditions listed in Equation (A.3) for the nonlinear thermal problem. The solution to this set of governing equations $\theta$ belongs to the Sobolev space whose functions and first order weak derivatives are both Lebesgue square-integrable: $\theta \in H^1(\Omega)$. The Lagrange multiplier $q$ also belongs to the same space $q \in H^1(\Omega)$. The Lagrangian is defined in Equation (A.5):

$$\mathcal{L}(\Omega, \theta, q) = -\int_{\Omega} \chi \varepsilon \theta^4 \, dV + \int_{\Omega} q \cdot \left( \text{div} (\kappa \nabla \theta) - \chi \varepsilon \theta^4 \right) \, dV + \int_{\Gamma_{D,\theta}} ((\kappa \nabla q) \cdot n) (\theta - \theta_0) \, ds + \int_{\Gamma_{o,\theta}} q \cdot (- (\kappa \nabla \theta) \cdot n) \, ds$$

(A.5)

The second integral containing the volume integrand $q \cdot \text{div} (\kappa \nabla \theta)$ is split using Green’s first identity to yield the following expansion:

$$\int_{\Omega} [\nabla q \cdot \kappa \nabla \theta + q \cdot (\nabla \cdot \kappa \nabla \theta)] \, dV = \int_{\partial \Omega} q \cdot ((\kappa \nabla \theta) \cdot n) \, ds$$

(A.6)

Substituting Equation (A.6) into Equation (A.5) yields the following:

$$\mathcal{L}(\Omega, \theta, q) = -\int_{\Omega} \chi \varepsilon \theta^4 \, dV - \int_{\Omega} (\nabla q \cdot \kappa \nabla \theta) \, dV + \int_{\partial \Omega} q \cdot ((\kappa \nabla \theta) \cdot n) \, ds - \int_{\Omega} q \cdot \chi \varepsilon \theta^4 \, dV + \int_{\Gamma_{D,\theta}} ((\kappa \nabla q) \cdot n) (\theta - \theta_0) \, ds + \int_{\Gamma_{o,\theta}} q \cdot (- (\kappa \nabla \theta) \cdot n) \, ds$$

(A.7)
Dividing the third integral into the Dirichlet and insulated boundaries:

\[
\mathcal{L}(\Omega, \theta, q) = - \int_{\Omega} \chi \varepsilon \nabla \theta^4 \, dV - \int_{\Omega} (\nabla q \cdot \kappa \nabla \theta) \, dV - \int_{\Omega} q \cdot \chi \varepsilon \nabla \theta^4 \, dV \\
+ \int_{\Gamma_{D,\theta}} q \cdot ((\kappa \nabla \theta) \cdot n) \, ds + \int_{\Gamma_{D,\theta}} ((\kappa \nabla q) \cdot n) \cdot (\theta - \theta_0) \, ds \\
+ \int_{\Gamma_{0,\theta}} q \cdot ((\kappa \nabla \theta) \cdot n) \, ds + \int_{\Gamma_{0,\theta}} q \cdot (-(\kappa \nabla \theta) \cdot n) \, ds
\]  

(A.8)

In the above equation, the final two terms on the insulated boundary are canceled out and the equation simplifies to the following:

\[
\mathcal{L}(\Omega, \theta, q) = - \int_{\Omega} \chi \varepsilon \nabla \theta^4 \, dV - \int_{\Omega} (\nabla q \cdot \kappa \nabla \theta) \, dV - \int_{\Omega} q \cdot \chi \varepsilon \nabla \theta^4 \, dV \\
+ \int_{\Gamma_{D,\theta}} q \cdot ((\kappa \nabla \theta) \cdot n) \, ds + \int_{\Gamma_{D,\theta}} ((\kappa \nabla q) \cdot n) \cdot (\theta - \theta_0) \, ds
\]  

(A.9)

Finally, the two Dirichlet boundary integrals are combined:

\[
\mathcal{L}(\Omega, \theta, q) = - \int_{\Omega} \chi \varepsilon \nabla \theta^4 \, dV - \int_{\Omega} (\nabla q \cdot \kappa \nabla \theta) \, dV - \int_{\Omega} q \cdot \chi \varepsilon \nabla \theta^4 \, dV \\
+ \int_{\Gamma_{D,\theta}} q \cdot ((\kappa \nabla \theta) \cdot n) \, ds + \int_{\Gamma_{D,\theta}} ((\kappa \nabla q) \cdot n) \cdot (\theta - \theta_0) \, ds
\]  

(A.10)

**A.3 Derivatives of the Lagrange Equation**

The optimality conditions for minimizing the Lagrangian in Equation (A.10) are defined using the derivatives of \( \mathcal{L}(\Omega, \theta, q) \) with respect to \( \theta \) and \( q \) in the direction \( \zeta \in H^1(\mathbb{R}) \). The saddle point where these derivatives evaluate to zero is defined by \( (\theta^*, q^*) \).

Because (A.10) is linear, or affine in \( q \), the directional derivative \( \left\langle \frac{\partial \mathcal{L}}{\partial q} (\Omega, \theta^*, q^*) \ , \zeta \right\rangle \) is evaluated by replacing \( q \) with \( \zeta \), and eliminating the first integral which does not depend on \( q \).

\[
\left\langle \frac{\partial \mathcal{L}}{\partial q} (\Omega, \theta^*, q^*) \ , \zeta \right\rangle = - \int_{\Omega} (\nabla \zeta \cdot \kappa \nabla \theta^*) \, dV - \int_{\Omega} \zeta \cdot \chi \varepsilon \nabla \theta^* \, dV \\
+ \int_{\Gamma_{D,\theta}} \zeta \cdot ((\kappa \nabla \theta^*) \cdot n) \, ds + \int_{\Gamma_{D,\theta}} (\theta^* - \theta_0) \cdot ((\kappa \nabla \zeta) \cdot n) \, ds
\]  

(A.11)

The first term in Equation (A.11) was derived using the Fréchet derivative \( DF(q) \) of
$\nabla q \cdot \kappa \nabla \theta$ with respect to $q$. If the Fréchet derivative exists, the following limit must hold:

$$\lim_{s \to 0} \frac{\| \nabla (q + s \zeta) \cdot \kappa \nabla \theta - \nabla q \cdot \kappa \nabla \theta - DF(q)s \|}{\| s \|} = 0$$ (A.12)

where $s$ is a scalar and $DF(q)$ is the Fréchet derivative of $\nabla q \cdot \kappa \nabla \theta$. Because the gradient operator $\nabla$ is linear the first term is split as follows:

$$\lim_{s \to 0} \frac{\| \nabla q \cdot \kappa \nabla \theta + \nabla (s \zeta) \cdot \kappa \nabla \theta - \nabla q \cdot \kappa \nabla \theta - DF(q)s \|}{\| s \|} = 0$$ (A.13)

After canceling the first and third terms and bringing the scalar $s$ out of the gradient operator in the second term, the above equation is simplified:

$$\lim_{s \to 0} \frac{\| s \nabla \zeta \cdot \kappa \nabla \theta - DF(q)s \|}{\| s \|} = 0$$ (A.14)

Canceling the scalar $s$ form both the numerator and the denominator, the Fréchet derivative in the direction $\zeta$ is given in the following relation:

$$DF(q) = \nabla \zeta \cdot \kappa \nabla \theta$$ (A.15)

Thus we have derived the first integral term in Equation (A.11). Applying Green's first identity, we can rewrite this term as follows:

$$\int_{\Omega} (\nabla \zeta \cdot \kappa \nabla \theta^*) dV = \int_{\partial \Omega} \zeta \cdot ((\kappa \nabla \theta^*) \cdot n) ds - \int_{\Omega} \zeta \cdot (\nabla \cdot \kappa \nabla \theta^*) dV$$ (A.16)

Substituting this result back into Equation (A.11):

$$\left< \frac{\partial \mathcal{L}}{\partial q} (\Omega, \theta^*, q^*), \zeta \right> = \int_{\Omega} \zeta \cdot (\nabla \cdot \kappa \nabla \theta^*) dV - \int_{\partial \Omega} \zeta \cdot ((\kappa \nabla \theta^*) \cdot n) ds$$

$$- \int_{\Omega} \zeta \cdot \chi \varepsilon \gamma \theta^4 dV$$ (A.17)

$$+ \int_{\Gamma_{D,\theta}} \zeta \cdot ((\kappa \nabla \theta^*) \cdot n) ds + \int_{\Gamma_{D,\theta}} (\theta^* - \theta_0) ((\kappa \nabla \zeta) \cdot n) ds$$

Next the second integral is split into the Dirichlet and insulating boundaries, and
after combining terms yield the following:

\[
\left\langle \frac{\partial L}{\partial q} (\Omega, \theta^*, q^*), \zeta \right\rangle = \int_{\Omega} \zeta \cdot \left( \text{div}(\kappa \nabla \theta^*) - \chi \varepsilon \Upsilon \theta^{*4} \right) dV \\
+ \int_{\Gamma_{0,\theta}} \zeta \cdot (-\kappa \nabla \theta^*) \cdot n) ds \\
+ \int_{\Gamma_{D,\theta}} (\theta^* - \theta_0) (\kappa \nabla \zeta) \cdot n) ds 
\] (A.18)

Setting the above result to zero yields the original governing PDE and boundary conditions. For any \( \zeta \) with compact support in \( \Omega \), the first integral term yields the original state equation:

\[
\text{div}(\kappa \nabla \theta^*) - \chi \varepsilon \Upsilon \theta^{*4} = 0 \quad \text{in} \quad \Omega 
\] (A.19)

Similarly the insulated boundary condition is given for any trace function \( \zeta \) on \( \Gamma_{0,\theta} \):

\[-(\kappa \nabla \theta^*) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta} \] (A.20)

The Dirichlet boundary condition is given by varying \((\kappa \nabla \zeta) \cdot n \) on \( \Gamma_{D,\theta} \):

\[
\theta^* - \theta_0 = 0 \quad \text{on} \quad \Gamma_{D,\theta} 
\] (A.21)

Next we calculate the derivative of the Lagrangian with respect to the state variable \( \theta \) to arrive at the adjoint equation.

\[
\left\langle \frac{\partial L}{\partial \theta} (\Omega, \theta^*, q^*), \zeta \right\rangle = -\int_{\Omega} \zeta \cdot 4\chi \varepsilon \Upsilon \theta^{*3} dV \\
- \int_{\Omega} (\nabla q^* \cdot \kappa \nabla \zeta) dV - \int_{\Omega} \zeta \cdot (q^* \cdot 4\chi \varepsilon \Upsilon \theta^{*3}) dV \\
+ \int_{\Gamma_{D,\theta}} q^* \cdot ((\kappa \nabla \zeta) \cdot n) ds + \int_{\Gamma_{D,\theta}} \zeta \cdot ((\kappa \nabla q^*) \cdot n) ds 
\] (A.22)

Again we have used the definition of the Fréchet derivative to calculate the second and fourth integrals. Next, we apply Green's first identity to split the second integral.
into the following two terms:

\[
\int_{\Omega} (\nabla q^* \cdot \kappa \nabla \zeta) \, dV = \int_{\partial \Omega} \zeta \cdot ((\kappa \nabla q^*) \cdot n) \, ds - \int_{\Omega} \zeta \cdot (\nabla \cdot \kappa \nabla q^*) \, dV \quad (A.23)
\]

The above result is substituted back into Equation (A.22).

\[
\left\langle \frac{\partial L}{\partial \theta} (\Omega, \theta^*, q^*), \zeta \right\rangle = -\int_{\Omega} \zeta \cdot 4\chi\varepsilon\gamma\theta^{*3} \, dV - \int_{\Omega} \zeta \cdot ((\kappa \nabla q^*) \cdot n) \, ds + \int_{\Omega} \zeta \cdot (\nabla \cdot \kappa \nabla q^*) \, dV - \int_{\Omega} \zeta \cdot (q^* \cdot 4\chi\varepsilon\gamma\theta^{*3}) \, dV \quad (A.24)
\]

\[+ \int_{\Gamma_{D,\theta}} \left[ q^* \cdot ((\kappa \nabla \zeta) \cdot n) + \zeta \cdot ((\kappa \nabla q^*) \cdot n) \right] \, ds \]

The second integral is split into the Dirichlet and insulating boundaries. Collecting terms yields the following:

\[
\left\langle \frac{\partial L}{\partial \theta} (\Omega, \theta^*, q^*), \zeta \right\rangle = \int_{\Omega} \zeta \cdot \left( \text{div}(\kappa \nabla q^*) - 4\chi\varepsilon\gamma\theta^{*3} - q^* \cdot 4\chi\varepsilon\gamma\theta^{*3} \right) \, dV + \int_{\Gamma_{D,\theta}} \zeta \cdot (-\kappa \nabla q^*) \cdot n \, ds \quad (A.25)
\]

\[+ \int_{\Gamma_{D,\theta}} q^* \cdot ((\kappa \nabla \zeta) \cdot n) \, ds \]

For any \( \zeta \) with compact support in \( \Omega \) yields the adjoint state equation:

\[
\text{div}(\kappa \nabla q^*) - 4\chi\varepsilon\gamma\theta^{*3} - 4q^* \cdot \chi\varepsilon\gamma\theta^{*3} = 0 \quad \text{in} \quad \Omega \quad (A.26)
\]

Similarly the insulating boundary condition is given by varying the trace function \( \zeta \) on the boundary \( \Gamma_{0,\theta} \):

\[-(\kappa \nabla q^*) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta} \quad (A.27)\]

Finally, the Dirichlet boundary condition is given by varying \( \kappa \nabla \zeta \) on the boundary \( \Gamma_{D,\theta} \):

\[q^* = 0 \quad \text{on} \quad \Gamma_{D,\theta} \quad (A.28)\]

Note that \( q^* \) is the adjoint state vector. In summary, the adjoint state is defined by
the following set of equations:

\[
\text{div} \left( \kappa \nabla q^* \right) - 4 \chi \varepsilon \Gamma \theta^{*3} - 4 q^* \cdot \chi \varepsilon \Gamma \theta^{*3} = 0 \quad \text{in} \quad \Omega \\
q^* = 0 \quad \text{on} \quad \Gamma_{D,\theta} \\
(k \nabla q^*) \cdot n = 0 \quad \text{on} \quad \Gamma_{0,\theta}
\]

While the thermal radiation problem is shown to be well-posed, it is not self-adjoint (i.e., \( q^* \neq \theta \)). In order to calculate the shape derivative we need to first solve the original governing PDE in Equation (A.2). Next the solution \( \theta \) is substituted into Equation (A.29) as \( \theta^* \) and solved for \( q^* \). Typically we use the finite element method to solve these PDEs numerically.

### A.4 Shape Derivative

The shape derivative is a measure of the change in a functional \( J \) subject to a perturbation of the structure domain \( \Omega \), such that the perturbation is a small enough to be considered a diffeomorphism. The perturbation is defined by a smooth vector field \( \xi = \xi(x) \) such that the perturbed domain is defined by:

\[
\Omega^t = (I + \xi)(\Omega^0)
\]

where \( I \) is the identity mapping in \( \mathbb{R}^d \) and \( \Omega^0 \) is the initial shape. Section 3.4.4 provides a more detailed description of the shape derivative.

To calculate the shape derivative of Equation (A.10) we refer to Equations (3.27) and (3.29) which provide the general forms of the shape derivatives for domain and boundary integral functionals respectively. The shape derivative is taken at the stationary point \( (\theta^*, q^*) \):

\[
\mathcal{L}(\Omega, \theta^*, q^*) = -\int_{\Omega} \chi \varepsilon \Gamma \theta^{*4} \, dV - \int_{\Omega} (\nabla q^* \cdot \kappa \nabla \theta^*) \, dV - \int_{\Omega} q^* \cdot \chi \varepsilon \Gamma \theta^{*4} \, dV \\
+ \int_{\Gamma_{D,\theta}} [q^* \cdot ((k \nabla \theta^*) \cdot n) + \theta^* - \theta_0] (\kappa \nabla q^*) \cdot n \, ds
\]

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Applying Equations (3.27) and (3.29) and grouping all of the surface integrals we arrive at the following shape derivative $DL$:

\[
DL(\Omega)(\xi) = \int_{\partial \Omega} \xi \cdot n \left( -\chi \varepsilon_\Omega \theta^* + \nabla q^* \cdot \kappa \nabla \theta^* + q_0 \varepsilon_\Omega \theta^* \right) ds + \int_{\Gamma_{D,\theta}} \xi \cdot n \left( \frac{\partial \left( q^* \cdot \left( (\kappa \nabla \theta^*) \cdot n \right) + (\theta^* - \theta_0) \left( (\kappa \nabla q) \cdot n \right) \right)}{\partial n} \right) ds + \int_{\Gamma_{D,\theta}} \xi \cdot n \left( H \left( q^* \cdot \left( (\kappa \nabla \theta^*) \cdot n \right) + (\theta^* - \theta_0) \left( (\kappa \nabla q) \cdot n \right) \right) \right) ds
\]  

(A.32)

Next the first integral is split into the Dirichlet and insulating boundaries:

\[
DL(\Omega)(\xi) = \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon_\Omega \theta^* + \nabla q^* \cdot \kappa \nabla \theta^* + q^*_0 \varepsilon_\Omega \theta^* \right) ds + \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon_\Omega \theta^* + \nabla q^* \cdot \kappa \nabla \theta^* + q^*_0 \varepsilon_\Omega \theta^* \right) ds + \int_{\Gamma_{D,\theta}} \xi \cdot n \left( \frac{\partial \left( q^* \cdot \left( (\kappa \nabla \theta^*) \cdot n \right) + (\theta^* - \theta_0) \left( (\kappa \nabla q) \cdot n \right) \right)}{\partial n} \right) ds + \int_{\Gamma_{D,\theta}} \xi \cdot n \left( H \left( q^* \cdot \left( (\kappa \nabla \theta^*) \cdot n \right) + (\theta^* - \theta_0) \left( (\kappa \nabla q) \cdot n \right) \right) \right) ds
\]  

(A.33)

Note from Equations (A.3) and (A.29) that $\theta^* - \theta_0 = 0$ and $q^* = 0$ on the boundary $\Gamma_{D,\theta}$ and thus the above equation simplifies greatly:

\[
DL(\Omega)(\xi) = \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon_\Omega \theta^* + \nabla q^* \cdot \kappa \nabla \theta^* + q^*_0 \varepsilon_\Omega \theta^* \right) ds + \int_{\Gamma_{D,\theta}} \xi \cdot n \left( -\chi \varepsilon_\Omega \theta^* + \nabla q^* \cdot \kappa \nabla \theta^* \right) ds
\]  

(A.34)
Bibliography


