Three Essays in Economic Theory

by

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Abstract

This dissertation is constituted by three independent essays. The first essay is an axiomatic model of knowledge, the second one is a model of predation and the third one addresses the issue of updating sets of probability measures.

In the first essay, An Axiomatic Model of Awareness and Knowledge, I argue that there are many circumstances where assuming that the economic agent knows the set of states of the world is inadequate. I then present a model where the agent’s representation of the world may be incomplete.

I develop an epistemic model, which uses a propositional framework. The model distinguishes between the knowledge of the existence of a proposition, which I call awareness, and the knowledge of the truth or the falsity of the proposition, which I call knowledge. This distinction, which is new in epistemic models, makes it possible to separate the assumption of introspection from the assumption of complete awareness which are implicit in traditional models of knowledge.

I do not assume that the agent knows the set of states of the world. Instead the agent is assumed to derive for herself a representation of the universe. Given her knowledge and her ability to reason about it the agent deduces a set of conceivable states of the world and a set of possible states of the world. Depending upon whether one assumes that the agent is aware or not of all the propositions she will or will not have a “complete model” of the world. When the agent is not aware of all the propositions, the states of the world and the possibility correspondence imaginable by her are coarser than the modeler’s. The agent has an incomplete knowledge of both the states of the world and the information structure.

In the last part of the essay I extend the model with "incompleteness" to a dynamic setting. Under the assumption that the agent’s knowledge is non-decreasing over time I show that the set of states of the world conceivable by the agent and her possibility correspondence get finer over time.

The second essay, Predation and Reputation Acquisition in the Debt Market, is a model of predation based on the existence of asymmetric information in the debt market. Since the cost of financing in the debt market is affected by the information investors have about a firm, I ask whether a competitor (the incumbent) of a new firm
(the entrant) might have incentives to engage in activities which change the information about the entrant received by the investors. In the model the incumbent uses predation for that purpose.

The model gives a formalization of the *deep pocket story*, based on the difference in the costs of financing for the entrant and the incumbent. The incumbent has a good reputation in the debt market, while the entrant is not yet known to the lenders. The difference in reputational assets, rather than financial assets, is what causes the asymmetry between incumbent and entrant.

The third essay, *Updating Ambiguous Beliefs*, explores using a decision theory framework, the issue of how to update sets of probability measures. I present an axiomatization of the rule which requires the updating of all the priors by Bayes rule.

In this work, I start by assuming that preferences over acts conditional on event $B$ happening, do not depend on the lotteries received on the complementary event $B^c$. In addition, they satisfy axioms which lead to the existence of a maxmin expected utility representation with non-unique subjective probability measure.

Conditional and unconditional preferences are related by another axiom, which implies that the set of probability measures derived in the utility representation of the conditional preferences given $B$ is the set of measures obtained by updating using Bayes rule the set of all prior probability measures derived in the utility representation of the unconditional preferences.

Each of the essays forms a chapter of the thesis. The chapters are independent of each other. They have their own introduction, their own conclusions and bibliography.

Thesis Supervisor: Professor Drew Fudenberg
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I am lucky to have the family I have. My parents did the best they could to give their children a better life than they had. They knew that education was essential, and so they gave us the opportunity to learn. As soon as they started working, my elder brothers and sisters helped the rest of us to achieve the best we could, to have opportunities they couldn’t have. How can I ever express in words my gratitude for them?

To my brother António I have to thank for his continuous support. Whenever I was happy or sad, excited or disappointed I knew I could call him and he would understand. I learned so much from him in our many discussions. Although he is not a economist I always admired his “intuition” and I cannot forget is very helpful comments to my papers.

My husband, Soumodip Sarkar, who I met at the end of my second year at MIT (and who distracted me for a while), helped me in so many ways. He was always the first one to hear about my ideas and to help with his insight. He would be the audience every time I was presenting a seminar. He had to polish the English in my writings. But above all he helped me through his understanding and love. With him it was so much easier to overcome any obstacle.

I cannot possibly identify the moment when I discovered I loved teaching (that must have been in my early years of high school when I would be a voluntary teacher of my colleagues). However the full discovery came when I taught at “my university” (Universidade Nova de Lisboa) for two years. That is why I have to say a word of thanks to my first students. They gave me two excellent years in my life, they gave me their friendship and they gave me the motivation I needed to do the Ph.D.

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With my advisor, Drew Fudenberg, I learned Game Theory and to give my first steps
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Chapter 1

An Axiomatic Model of Awareness and Knowledge

1.1 Introduction

In game and decision theory the set of conceivable states of the world is always assumed to be known. Given the information an agent has, there exists a subset of states that the agent considers possible. Learning involves being able to reduce the set of possible states. There is however nothing more learnt about how the world is, i.e., about the set of conceivable states.

However there are many circumstances where the assumption that the set of states of the world is known to the agent is inadequate. In fact, the role of unforeseen contingencies has been recognized in many areas of the economic literature (e.g., decision theory, contracting). Under these circumstances the agent has “an incomplete picture of the world”, which may get more complete over time.

In this paper I develop a model where the agent’s representation of the world may be incomplete. I do not assume that the agent knows $\Omega$. Instead the agent is assumed to derive for herself a representation of the universe. Given her knowledge and her ability to reason about it, the agent constructs her own view of the world.
I want to look at the world through the eyes of the economic agent. What does she know about the world? Which states of the world are imaginable by her? Which states of the world does she consider possible?

I analyze these questions using an axiomatic model of knowledge. There is an important distinction between the axiomatic model of knowledge in this paper and other models which have been used in economic theory to analyze issues like common knowledge\(^1\). Frequently a probability space \((\Omega, \Sigma, \mu)\) is considered, where \(\Omega\) is the space of states, \(\Sigma\) is a \(\sigma\)-field of events and \(\mu\) is a probability measure on \(\Sigma\). For each individual there is a given possibility correspondence \(P\) which describes the information available to the individual in each state of the world. Then it can be said that the individual knows event \(E\) at \(\omega\) if \(P(\omega) \subset E\). Hence the knowledge of events is a derived notion in these models. The properties of the knowledge operator can be derived from the characteristics of the possibility correspondence\(^2\).

The idea that I present is in a certain sense the opposite of the above. The primitives of the model are the properties of knowledge and the knowledge the agent has. The corresponding states of the world and properties of the possibility correspondence depend upon the assumptions about knowledge. The states of the world which are defined here are “epistemic states of the world”, they incorporate the knowledge the agent has. Two states of the world may differ only because the agent has different knowledge in each of them. One advantage of the explicit introduction of knowledge is that it enables us to reveal the underlying assumptions about knowledge which give rise to specific properties of the possibility correspondence. In this respect this work is closer to Bacharach [2] and Samet [10].

I assume, like Samet, a propositional world. The environment of interest is described by propositions, including propositions about the agent’s knowledge. The states of the world are assignments to the propositions of true and false values, which are consistent

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\(^1\)For example Aumann [1] and Brandenburger & Dekel [3].

\(^2\)Possibility correspondence is another name to information structure. The former expression is common in the literature of axiomatic models of knowledge.
with the axioms of the model.

The set of epistemic axioms which has been most frequently used includes three axioms. The first axiom which defines knowledge, says that if the agent knows a proposition, then that proposition is true. The second axiom (positive introspection) says that when the agent knows something, she knows that she knows it. The third axiom (negative introspection) states that when the agent doesn’t know something, she knows that she doesn’t. However a closer examination of the third axiom reveals that it entails two implicit assumptions: an assumption about the agent’s introspection capabilities and an assumption about the agent’s awareness of all the existing propositions.

Traditional epistemic models have been unable to separate the two assumptions because there is no reference to the agent’s knowledge of the existence of a proposition. In this paper I distinguish between knowledge of the existence of a proposition (which I call awareness) and knowledge of the logical value of a proposition (which I call knowledge). Consequently it becomes easy to separate the assumption of complete awareness from the assumption of introspection.

In my work the rationality assumption which is implicit in the axiom is maintained while the complete awareness assumption is given up. I assume that if the agent is aware of a proposition and she doesn’t know the proposition, then she knows that she doesn’t. In other words, the agent is capable of negative introspection, provided she is aware of the proposition.

Samet [10] proves that under the three epistemic axioms mentioned above the possibility correspondence is a partition and that this is not the case when only the first two axioms hold. I show that the partition result holds under the weaker assumption that negative introspection holds if and only if the agent is aware of the proposition.

Another important feature of my model is that the agent can reason. The agent is

---

3 The consequences for decision and game theory of non-partitional information structure has been discussed by Geanakoplos [5]. He argues that if the agent knows the information structure and the information structure is not a partition then the agent cannot be fully rational.

4 The idea of the agent being able to think is not original. In his model Gilboa [6] assumes the agent knows logic and the axioms of the model, hence the agent is able to deduce the properties of the model.
able to reason about her knowledge and the properties of her knowledge.

The analysis can proceed at two levels. One is at the modeler’s level, where he has a complete description of the world. The other is at the agent’s level. Her description of the world depends on the set of propositions she is aware of. Depending upon whether one assumes that the agent is aware or not of all the propositions, she will or will not have a “complete model” of the world.

In a world where the agent is not necessarily aware of all propositions the states of the world imaginable by the agent are not always a complete description of the world. The notion of a state of the world à la Savage, as resolving all uncertainty does not fit in this setup. In this case, the agent may be able to learn more about the set of conceivable states of the world, about how the world can be. I present a dynamic model of knowledge, where it is clear that the agent improves her description of the world over time.

The paper is organized as follows: the notation as well as some definitions are presented in section 2. This section reviews the axiomatic model of knowledge in Samet [10].

Section 3 describes the axiomatic model of awareness and knowledge which I developed. The following section explores some consequences of the axioms. In particular I show that under the axioms described in section 3, the possibility correspondence is a partition.

In section 5, I consider a particular state of the world and ask what is the agent’s view of the world in that state. The answer to this question is that the agent’s view of the world depends upon the set of propositions she is aware of in that state of the world. In the following section I study the relationship between the agent’s view of the world and the modeler’s universe. I show that when the agent is aware of all the propositions she has a “complete” model of the world. However if she is not aware of all propositions

---

5The modeler in this paper is an omniscient creature. He is aware of all propositions and knows the set of states of the world.
her view of the world is coarser than the modeler's one.

Section 7 extends the axiomatic model with incomplete awareness to a dynamic framework. Assuming that the agent doesn't forget, or in other words if she knows something at time \( t \) she also knows it at time \( t + 1 \), I show that the agent's view of the world might get finer and finer as she becomes aware of more propositions over time.

### 1.2 A Simple Epistemic Model

In this section I present a simple axiomatic model of knowledge. The objective is to introduce the notation and concepts used in these type of models and to provide a bridge to my model. I illustrate the concepts with examples which show the distinctive characteristics of epistemic models.

The concepts presented in this section are from Samet [10]. The notation I use is slightly different from the one used by Samet. The examples and interpretation of his model are my own.

#### 1.2.1 Concepts, Examples and a Result

Following Samet, I consider a propositional world. Let \( \Phi \) be a countable set of propositions describing a certain environment of interest. There exists a mapping \( \sim : \Phi \rightarrow \Phi \), where \( \sim \phi \) is interpreted as "not \( \phi \)". In addition, there is a mapping \( K : \Phi \rightarrow \Phi \), the meaning of \( K\phi \) is "the agent knows \( \phi \)".

\( \Phi \) includes propositions about the environment, as well as propositions about the agent's knowledge about the environment, about the agent’s knowledge about her knowledge, and so on. Let us now consider an example which illustrates the propositions included in \( \Phi \). Suppose that the universe of interest is only whether or not it is sunny. The propositions in \( \Phi \) are then: "it is sunny", its negation "it is not sunny", the propositions about the agent’s knowledge of the above propositions like "the agent knows it is sunny", the negation of this "the agent does not know it is sunny", the propositions about the agent’s knowledge about her knowledge: "the agent knows that she knows it
is sunny",... Even with a simple environment like this, the dimension of $\Phi$ is infinite because of the infinite hierarchy of knowledge.

We notice that propositions about knowledge have the same status than any other propositions. In Gilboa's words [6] there is no distinction between information and meta-information.

Each proposition can be true or false (1 or 0). Hence the set $\Sigma = \{0, 1\}^\Phi$ contains all the possible assignments of the truth and false values to elements of $\Phi$. An element $\omega$ of $\Sigma$ is called a state of the world if for each $\phi \in \Phi$, $\omega(\phi) + \omega(\neg \phi) = 1$, meaning that in a state of the world if a proposition is true its negation has to be false. Each state of the world can be described (and identified) by the set of propositions which are true in that state. It is equivalent to say "$\phi$ is true in $\omega$", "$\omega(\phi) = 1$", or "$\phi \in \omega$". Let us denote by $\Omega_0$ the set of all states of the world.

In the previous example, possible assignments of truth and false values for some of the propositions in $\Phi$, where $\phi_0$ is the proposition "it is sunny" and each row describes a state of the world, are:

\[
\begin{array}{cccccccc}
\phi_0 & \sim \phi_0 & K\phi_0 & \sim K\phi_0 & KK\phi_0 & \sim KK\phi_0 & K \sim K\phi_0 & \ldots \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & \ldots \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & \ldots \\
\end{array}
\]

As it is defined here, each state of the world includes propositions about the agents's knowledge as well as propositions about other (non-informational) aspects of the environment. Without additional restrictions the number of possible states of the world is infinite.

A remark about the proposition $\sim K\phi_0$. This proposition is simply the logical negation of $K\phi_0$, it does not necessarily imply ignorance in the common sense of the word. It says "the agent does not know that it is sunny". When this proposition holds
either "the agent knows that it is not sunny" or "the agent does not know it is not sunny" may be true. Only when the last combination happens can we say that the agent is ignorant of whether it is sunny or not.

At each state \( \omega \), one can define the set of propositions known by the agent in that state.

**Definition 1** The epistemic content of the state \( \omega \), \( \bar{K}(\omega) \) is:

\[
\bar{K}(\omega) = \{ \phi : K\phi \in \omega \} = \{ \phi : \omega(K\phi) = 1 \}
\]

We say that the state of the world \( \omega' \) is possible for the agent at \( \omega \) if and only if all that the agent knows at \( \omega \) is true in \( \omega' \) or, in other words, \( \omega' \) is compatible with the knowledge the agent has at \( \omega \). Let us define formally the possibility relation and possibility correspondence:

**Definition 2** The possibility relation \( p \) is a binary relation on \( \Omega \) such that:

\[\omega'p\omega \iff \bar{K}(\omega) \subseteq \omega' \text{ or equivalently } \omega(K\phi) = 1 \Rightarrow \omega'(\phi) = 1\]

The set of states which are possible at \( \omega \) is:

\[P(\omega) = \{ \omega' : \omega'p\omega \}\]

The possibility correspondence, \( P : \Omega \rightarrow 2^\Omega \) specifies the possibility sets of the agent for each state of the world.

The properties of the binary relation \( p \) are associated with the properties of the knowledge of the agent. Since I am interested in a model based on the knowledge the agent has, I shall consider the properties of knowledge as the basis.
There are three properties which have been frequently used in axiomatic models of knowledge:

\[ A1 \quad \text{If } \omega(K\phi) = 1 \text{ then } \omega(\phi) = 1, \forall \phi \in \Phi \]

\[ A2 \quad \text{If } \omega(K\phi) = 1 \text{ then } \omega(KK\phi) = 1, \forall \phi \in \Phi \]

\[ A3 \quad \text{If } \omega(\sim K\phi) = 1 \text{ then } \omega(K \sim K\phi) = 1, \forall \phi \in \Phi. \]

The first property states that if the agent knows a proposition, that proposition is true. This is the property that defines knowledge; if this property is not satisfied then we would be speaking of beliefs, not knowledge.

The second and third properties are introspective. The second axioms refers to positive introspection. Whenever the agent knows something she also knows that she knows it. If it holds the agent is able to tell what she knows.

The third property, often called the negative introspection axiom, is particularly objectionable. In my opinion this property makes sense in cases where the agent is aware of the existence of the proposition. In such a case, by introspection the agent should be able to recognize her ignorance. But I can imagine situations where the agent does not even know the existence of the proposition. In such cases, introspection would not be enough for the agent to recognize her ignorance\(^6\). In summary I believe that this axiom has implicit more than just introspection, it assumes that the agent is aware of all propositions.

Let us denote the set of states of the world which satisfies A1 by \( \Omega_1 \), the set of states which satisfies A1 and A2 by \( \Omega_2 \) and the set of states which satisfies the three properties by \( \Omega_3 \).

The number of states of the world when the three axioms are satisfied is much smaller.

\(^6\)A possible illustration which comes to my mind is the proposition: "nuclear bomb can potentially cause millions of deaths", which say to a Japanese farmer before Hiroshima, wasn't something he even knew existed.

Another example is when the proposition is about some speciality in which the agent is not an expert.
In previous example, the second, third, and fourth assignments in the table above are not permitted, because they do not obey axioms $A1$, $A2$ and $A3$, respectively. The states of the world in the example, under these axioms would be reduced to the four states:

<table>
<thead>
<tr>
<th>state</th>
<th>$\phi_0$</th>
<th>$\sim \phi_0$</th>
<th>$K\phi_0$</th>
<th>$K\sim \phi_0$</th>
<th>$\sim K\phi_0$</th>
<th>$\sim K\sim \phi_0$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
</tbody>
</table>

The assignments of truth and false values to all the other propositions in $\Phi$ is determined by using the axioms $A2$ and $A3$. Given the assignment rules for the negation and for the epistemic propositions it is enough to assign values to the propositions $\phi_0$, $K\phi_0$ and $K\sim \phi_0$, to generate the logical value of all the other propositions. In other words there are two elements of uncertainty - is the proposition $\phi_0$ true or false? Does the agent know whether the proposition is true or false or not?

It is easy to see that the possibility correspondence partitions the set of states of the world. In fact $P(1) = \{1\}$, $P(2) = \{2, 4\}$, $P(3) = \{3\}$ and $P(4) = \{2, 4\}$. We notice again that the knowledge the agent has is part of the definition of state of the world. This characteristic distinguishes this framework from the one normally used where the information structure is not an object of uncertainty, e.g. in Geanakoplos [5]. In the example above one could say that there are two possible information structures in the traditional sense, i.e. when the information structure is not subject to uncertainty. Under one information structure the agent knows if it is sunny or not (first and third rows), under the other the agent does not know whether it is sunny or not (second and fourth rows).

Samet proves several results about the relationship between the mentioned properties and the possibility relation and possibility correspondences. I summarize the results in the following proposition:
Proposition 1 (Samet) (i) What \( p \) implies in terms of the epistemic content:

For \( \Omega \subset \Omega_2, \omega' \models \omega \) iff \( \bar{K}(\omega') \supset \bar{K}(\omega) \)

For \( \Omega \subset \Omega_3, \omega' \models \omega \) iff \( \bar{K}(\omega') = \bar{K}(\omega) \)

(ii) Properties of the binary relation \( p \):

- If \( \Omega \subset \Omega_1, p \) is reflexive
- If \( \Omega \subset \Omega_2, p \) is reflexive and transitive
- If \( \Omega \subset \Omega_3, p \) is reflexive, transitive and symmetric.

(iii) Properties of the possibility correspondence

- If \( \Omega \subset \Omega_1, \omega \in P(\omega) \)
- If \( \Omega \subset \Omega_2, \omega \in P(\omega) \) and if \( \omega' \in P(\omega) \) then \( P(\omega) \supset P(\omega') \)
- If \( \Omega \subset \Omega_3, \omega \in P(\omega) \) and if \( \omega' \in P(\omega) \) then \( P(\omega) = P(\omega') \)

Proof: The complete proof can be found in Samet [10]. Here I prove the first part of the proposition. Consider the case where \( \omega \in \Omega_2 \). Suppose that \( \phi \in \bar{K}(\omega) \). That is equivalent to \( \omega(K\phi) = 1 \), which implies \( \omega(KK\phi) = 1 \). Hence \( K\phi \in \bar{K}(\omega) \). Since \( \omega' \) is possible at \( \omega \) iff \( \bar{K}(\omega) \subset \omega' \), the last result implies \( \omega'(K\phi) = 1 \) or in other words \( \phi \in \bar{K}(\omega') \). The implication in the other direction is a consequence of \( A1 \), in fact if \( \omega'(K\phi) = 1 \) it is also true that \( \omega'(\phi) = 1 \) hence \( \bar{K}(\omega') \subset \omega' \).

When \( A3 \) also holds one can prove that the inclusion cannot be strict. The proof is by contradiction. Suppose there is some proposition such that \( \omega'(K\phi) = 1 \) but \( \omega(K\phi) = 0 \). By \( A3 \) \( \omega(\sim K\phi) = 1 \) implies that \( \omega(K \sim K\phi) = 1 \) which implies, by definition of \( p \), that \( \omega'(\sim K\phi) = 1 \), a contradiction. Q.E.D.

So under the axioms \( A1-A3 \) the agent only considers possible states which are epistemologically equivalent and the possibility correspondence partitions the set of states of the world.
However when only $A1-A2$ hold the agent may consider possible states of the world such that if she was in those states she would know more propositions than the ones she actually knows. In terms of possibility correspondence this result says that the agent may consider possible states such that if she was in those states she would consider as possible only a subset of the states that she considers possible.

1.2.2 Some Comments

Some observations on Samet’s model would be appropriate here. The model is written from the view point of the modeler. He is describing a certain physical environment as well as the agent’s knowledge about it. The model does not say whether the agent knows or doesn’t know the information structure and the set of states of the world.

The concepts introduced are defined by the modeler. They would have a different meaning or no meaning at all from the agent’s perspective. For example, what would the agent answer if asked, at the state of the world $\omega$, which propositions does she know? The answer coincides with $\bar{K}(\omega)$ if $\omega \in \Omega_2$, in this case the agent knows exactly the propositions that she knows, she knows $\bar{K}(\omega)$. Otherwise it is only the modeler who knows that the agent knows the propositions in $\bar{K}(\omega)$.

More important, would it make sense (within the context of the model) to ask the agent, in the state of the world $\omega$, which states of the world does she consider possible? Generally it wouldn’t. The question may not be comprehensible to the agent (she may not recognize the expression “states of the world”). In order for the agent to be able to answer the question one has to assume that the set of propositions in $\Phi$ includes propositions referring to the possibility sets and that the agent knows these propositions.

The model I present in the next section can be analyzed from the agent’s perspective. The issues of whether the agent knows or doesn’t know the universe will be explicitly addressed within the model.

Let us assume for the moment that in the previous model the agent knows, in each state of the world, the set of conceivable states of the world (consider this as an informal
assumption of the model). Then the model without the assumption \( A3 \) implies that the agent is being partially irrational. Although the agent knows the complete description of the states of the world, and as such should be aware of all propositions which describe them, she does not always recognize her ignorance, which may lead the agent to consider possible states where she knows more propositions than the ones she actually knows!

In a different setup, which considers physical states of the world, Geanakoplos gives another example of how assuming that the agent knows the universe and knowledge doesn’t obey property \( A3 \), implies partial irrationality: Suppose there are only two states of the world \( \Omega = \{a, b\} \), where \( a \) is the layer of ozone is disintegrating and \( b \) is the layer of ozone is not disintegrating. Suppose \( P(a) = \{a\} \) and \( P(b) = \{a, b\} \), where the disintegration of the layer of ozone can be identified because if it happens there is emission of gamma rays, while otherwise nothing happens. Assume also that the agent knows the information structure (otherwise the definition of states of the world could not be as defined). If \( b \) occurs the agent doesn’t know whether state \( a \) or \( b \) occurred. But if the agent knows the information structure she should reason “I cannot possibly be in state \( a \) because if I was in state \( a \) I would detect gamma rays, since I do not detect them I have to be in state \( b \)”.

Hence if the agent uses all the information she has, including her knowledge of the possibility correspondence, the final information structure is a partition.

1.2.3 A Reinterpretation of the Model

Samet’s model does not distinguish between the agent’s knowledge of non-epistemic propositions from her knowledge of epistemic propositions. However under the axioms \( A1- A3 \) if the agent knows something, she can derive, by introspection, that she knows that she knows it,... So if the agent knows “it is sunny”, she knows “I know that it is sunny”, she knows “I know that I know that it is sunny”... This type of knowledge can be called inferred knowledge. Thus the knowledge of the proposition “it is sunny” is enough to generate all the sequence above.
A reinterpretation of the model can be made by assuming that at different states the agent has some non-inferred knowledge and the existence of certain deduction rules. Through reasoning and introspection the agent can derive other things, increasing the set of propositions she knows, the end result of this iterative process is what the axiomatic models of knowledge refer to. In fact, it is usual in axiomatic models of knowledge to include both inferred and non-inferred knowledge in the structure of the model\textsuperscript{7}.

Instead of defining the states of the world as I did before, using all the propositions in $\Phi$, one could consider a subset of propositions such that once the true and false values are assigned to these propositions all the assignments to the other propositions in $\Phi$ can be derived using the axioms.

The set of “basic propositions” one needs to consider to generate $\Omega$ depends on the axioms of the model. For example, under axioms $A1-A3$ by considering assignments to the propositions about the environment and the propositions about the agent’s knowledge of the environment one can “generate” all the other assignments.

I am interested in a model where the agent’s non-inferred knowledge includes, in any state of the world, knowing that a proposition can either be true or false, knowing the assignment rules of logic and knowing the notion of a state of the world. Through introspection and reasoning the agent can then derive a set of conceivable states of the world. Which means that in the end result of the agent’s reasoning process she will have a certain view about the set of states of the world.

### 1.3 The Description of My Model

#### 1.3.1 Features of The Model

I start this section by discussing the features of my model.

\textsuperscript{7}In Samet [10] $\Phi$ includes all the derived propositions. Bacharach [2] considers an atomic sigma-field of sets (the events), which includes events like knowing some event.
• The distinction between being aware of the existence of a proposition and knowing or not knowing its logical value is introduced in my model. Obviously an agent can only know the logical value of a proposition if she is aware of the proposition. One can imagine ourselves asking the agent to enumerate the propositions she can think about as the first step.

• I want the agent to be able to think logically. In order to achieve this goal I introduce logic in the model, in any state of the world the rules of propositional calculus hold. Thus the model considered here has high consistency requirements when compared with the previous one. Moreover the agent is assumed to be able to make deductions and to know the axioms of logic and therefore to know the theorems of logic, too.

• I continue to speak of knowledge, so axiom A1 continues to be valid.

• Positive introspection is maintained. I also preserve negative introspection whenever that makes sense, i.e. if the agent is aware of some proposition that she doesn’t know she is able to conclude that she doesn’t know the proposition.

• The agent is not assumed to know a priori the set of states of the world. Instead the agent knows the concepts of a state of the world and of the possibility correspondence. Since she can think she is able to deduce a set of conceivable states of the world and define the possibility correspondence on such universe.

Introducing logic  Besides the negation and knowledge operator all other operators of logic are defined in the model. The set of propositions \( \Phi \) is closed under the logical operators. For example if propositions \( \phi_1 \) and \( \phi_2 \) are elements of \( \Phi \) so is the proposition \( \phi_1 \land \phi_2 \). Hence the structure of \( \Phi \) is substantially more complex than the one used by Samet. I will return to the issue of the structure of \( \Phi \) later.

I assume that the assignment rules of propositional calculus hold in any state of the world. Thus once the logical value of the “basic propositions” is known one knows
the logical value the other propositions. In the example if both \( \phi_1 \) and \( \phi_2 \) are true the proposition \( \phi_1 \land \phi_2 \) is also true, otherwise it is false. While in the model of the previous section the only logical consistency requirement was that if a proposition is true its negation was false, here in any state of the world the assignments of true and false values have to satisfy all rules of logic.

The symbols \( \forall, \exists, \varepsilon, \ldots \), are part of the alphabets of the model, and can be used in propositions.

**The awareness operator** As referred above it seems to be important to distinguish between the agent’s knowledge of the existence of a given proposition from her knowledge about the logical value of the proposition. I will call the first kind of knowledge: awareness\(^8\).

For each proposition \( \phi \in \Phi \) there exists a proposition \( A\phi \). Which means “the agent is aware of proposition \( \phi \)”. Now each state of the world is also characterized by the agent being or not aware of a given proposition.

**Definition 3** The awareness set of the agent at state \( \omega \), \( \Phi^a(\omega) \), is the set of propositions the agent is aware of in the state \( \omega \):

\[
\Phi^a(\omega) = \{ \phi' \in \Phi : \omega(A\phi') = 1 \}
\]

I assume the agent always knows that if a proposition exists its negation also exists as well as propositions about the agent’s knowledge of the proposition. The agent also knows that if two propositions exist than one can define their conjunction, disjunction,...

\(^8\)The term awareness has been used in the computer science literature with a different meaning. Fagin and Halpern [4] use an awareness operator as a “filter” to what they call implicit beliefs. Implicit beliefs include all that could potentially be known by the agent, if she could derive all the consequences of her non-inferred knowledge. The use of the awareness operator is intended to capture the idea of bounded rationality. The agent beliefs explicitly only in a subset of all the consequences of her non-inferred knowledge.
As a consequence the set of propositions the agent is aware of has to be closed under the logical operators and the knowledge operator.

**The structure of Φ.** The set of propositions in my model is substantially more complex than the one in the model of the previous section. This is so partly because of the inclusion of the logical operators and the awareness operator in the model.

There is however a more fundamental difference. Since I am interested in a model where it makes sense to ask the agent her view about the set of states of the world and which states she considers possible,... a complete model includes propositions which refer to the conceivable states of the world, the epistemic content of these states, the set of propositions the agent is aware of. In other words the set of propositions Φ includes not only propositions about the environment and the agent’s awareness and knowledge. It includes propositions about the “model”, about the set of states of the world and the information structure.

To illustrate this idea consider the example of the previous section where the basic proposition about the environment was “it is sunny”. One can define the proposition “There exists a conceivable state of the world where it is sunny holds and the agent knows it...”. In order for us to say that the agent knows the universe of states of the world she has to know propositions like this one.

This leads to an element of circularity in the definition of states of the world, since I am defining states of the world based on the assignments of true and false values to propositions, where some propositions refer to states of the world. The element of circularity is not surprising. Circularity also exists in models where a probability space (Ω, P, μ) is considered and where it is assumed that the agent knows the information structure. This is so because in order to incorporate the idea that the agent knows the information structure one has to include the possibility correspondence in the full description of the state of the world ω⁹. Since my model is designed to answer questions

⁹Gilboa [6] gives an exposition of the circularity problem in that framework and presents a model
related to the agent’s knowledge of the set of states of the world one should expect the same type of circularity.

1.3.2 The Axioms

Let $\Phi$ be the set of all propositions. In order to simplify the list of axioms let $g(\phi)$ either be a proposition that combines $\phi$ with one or several of the operators $A, K, \sim$ (such as $K\phi, \sim \phi$) or the proposition “there exists a state of the world where $\phi$ holds”.

The states of the world defined in this model are “awareness-epistemic states of the world”.

**Definition 4** Consider $\Sigma = \{0, 1\}^\Phi$, the set of all the assignments of truth and false values to the propositions in $\Phi$. A state of the world $\omega$ is an element of $\Sigma$ such that the assignment of truth and false values satisfies the assignment rules of propositional calculus, the awareness and the epistemic rules stated below:

\begin{flalign*}
A0 & \quad \text{Axioms of propositional calculus.} \\
A1 & \quad \text{If } \omega(K\phi) = 1 \text{ then } \omega(\phi) = 1, \forall \phi \in \Phi. \\
A2 & \quad \text{If } \omega(K\phi) = 1 \text{ then } \omega(KK\phi) = 1, \forall \phi \in \Phi. \\
A3 & \quad \text{If } \omega(K(\phi_1 \Rightarrow \phi_2)) = 1, \text{ then } \omega(K\phi_1) = 1 \Rightarrow \omega(K\phi_2) = 1, \forall \phi_1, \phi_2 \in \Phi. \\
A4 & \quad \omega(A\phi) = \omega(Ag(\phi)), \forall \phi \in \Phi. \\
A5 & \quad \omega(A\phi_1) = 1 \text{ and } \omega(A\phi_2) = 1 \Rightarrow \omega(A(\phi_1 \cap \phi_2)) = 1, \forall \phi_1, \phi_2 \in \Phi. \\
A6 & \quad \text{If } \omega(A\phi) = 0 \text{ then } \omega(\sim K\phi) = 1 \text{ and } \omega(\sim K \sim \phi) = 1, \forall \phi \in \Phi. \\
A7 & \quad \text{If } \omega(A\phi) = 1 \text{ and } \omega(\sim K\phi) = 1 \text{ then } \omega(K \sim K\phi) = 1, \forall \phi \in \Phi. \\
A8 & \quad \text{The agent knows definitions 1, 2, 3, 4, and axioms A0 – A7.}
\end{flalign*}

where the agent’s knowledge of the information structure is specified within the model. His work is one of the few that doesn’t informally assume that the agent knows the information structure.
The first axiom says that the universe of states is logically consistent. It includes the requirement that in any state of the world either a proposition is true or its negation is true, but extends that to all other propositional calculus rules\(^10\).

The next three axioms refer to the assignment rules of epistemic propositions. \(\mathcal{A}1\) guarantees that one is speaking of knowledge. \(\mathcal{A}2\) says that the agent is able to do positive introspection and whenever she knows something she knows that she knows it. \(\mathcal{A}3\) says that the agent's knowledge has to be closed under implication. So if she knows "if it rains then there must be clouds in the sky" and additionally she knows "it is raining", then she knows that "there are clouds in the sky". This is obviously a very strong assumption. It imposes a very high level of rationality. For example, if the agent knows the axioms of logic, she knows all the theorems too.

Axioms \(\mathcal{A}4-\mathcal{A}5\) say that in any state of the world the set of propositions the agent is aware of is closed under the logical and the knowledge operator. The idea is that the agent knows that one can always define the negation of a proposition, propositions about the knowledge of the proposition, and so on. In addition if the agent knows that a proposition exists she knows that she can use the proposition in her description of the states of the world and vice-versa.

\(\mathcal{A}6\) says that if the agent is not aware of a proposition then she cannot know if the proposition is true or false. Knowledge of the existence of the proposition is a necessary

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\(^{10}\) Using the logic operators \(\land, \lor, \Rightarrow\) and \(\sim\) the axioms of propositional calculus are:

\[
\begin{align*}
(1) & \quad \phi_1 \Rightarrow (\phi_2 \Rightarrow \phi_1) \\
(2) & \quad (\phi_1 \Rightarrow (\phi_2 \Rightarrow \phi_3)) \Rightarrow ((\phi_1 \Rightarrow \phi_2) \Rightarrow (\phi_1 \Rightarrow \phi_3)) \\
(3) & \quad \phi_1 \land \phi_2 \Rightarrow \phi_1 \\
(4) & \quad \phi_1 \land \phi_2 \Rightarrow \phi_2 \\
(5) & \quad \phi_1 \Rightarrow (\phi_2 \Rightarrow (\phi_1 \land \phi_2)) \\
(6) & \quad \phi_1 \Rightarrow (\phi_1 \lor \phi_2) \\
(7) & \quad \phi_2 \Rightarrow (\phi_1 \lor \phi_2) \\
(8) & \quad (\phi_1 \Rightarrow \phi_3) \Rightarrow ((\phi_2 \Rightarrow \phi_3) \Rightarrow (\phi_1 \lor \phi_2 \Rightarrow \phi_3)) \\
(9) & \quad (\phi_1 \Rightarrow \phi_2) \Rightarrow ((\phi_1 \Rightarrow \sim \phi_2) \Rightarrow \sim \phi_1) \\
(10) & \quad \sim \sim \phi_1 \Rightarrow \phi_1
\end{align*}
\]

And the only rule of inference is *modus ponens*, i.e.; \(\phi_2\) is a direct consequence of \(\phi_1\) and \(\phi_1 \Rightarrow \phi_2\).
condition for the knowledge of its logical value.

$A7$ says that negative introspection holds if the agent is aware of the proposition. This axiom is weaker than the axiom of negative introspection in the model of section 2. In that model whenever the agent doesn’t know a proposition she knows that she doesn’t, here this property is true only if the agent is aware of the proposition.

If $A8$ holds the agent knows the properties of her knowledge. This in combination with $A0$ means that the agent knows logic, and with $A3$ that the agent can derive the logical consequences of the properties of her knowledge. A remark about the axiom is that the agent’s knowing the definition 4 doesn’t mean she knows $\Phi$. What it means is that the agent knows that a state of the world is an assignment of truth and false values to all the propositions which is consistent with the axioms.

Latter I will add one more axiom which is necessary for coherence between the modeler’s model and the agent’s view of model.

Axioms are by their own nature assumed to be always true. So, in a sense one can interpret them as propositions which are always valid. For these type of propositions, as well as any logical consequence of them, only the truth value can hold in any state of the world.

Let $\Omega$ be the set of all states of the world. I will also refer to $\Omega$ as the set of states of the world conceivable by the modeler.

An Alternative Approach Instead of considering the full set of propositions to define the states of the world one could consider “basic propositions” which are sufficient to generate the set of states of the world, given the set of axioms. Let us call the set of “basic propositions” $\Phi^b$. Let $\Sigma^b = \{0, 1\}^{\Phi^b}$ be the set of assignments of truth and false values to these propositions and let $\Omega^b$ be the subset of $\Sigma^b$ which satisfies the axioms of
the model. Let us define a correspondence from $\Omega^b$ to $\Omega$ as follows:

$$f(\omega^b) = \{ \omega \in \Omega : \omega^b(\phi) = \omega(\phi), \forall \phi \in \Phi^b \}$$

We can say that $\Phi^b$ generates the set of states of the world $\Omega$ if and only if $f$ is a one to one mapping. There are, of course, many sets of propositions which generate $\Omega$, the set of “basic propositions” is the minimal set of simple propositions which generates $\Omega$.

The set of “basic propositions” in my model includes propositions about the environment, propositions about the agent’s awareness of these propositions and propositions about the agent’s knowledge of the logical value of the propositions describing the environment. The other propositions, including propositions referring to the agent’s knowledge about her knowledge, what the agent knows about the true state of the world, what states does the agent consider possible, all have an assignment which can be derived by using the assignments to the basic propositions and the axioms.

Whenever I speak about a basic state of the world I use the superscript $b$. One can define the concepts of epistemic content and awareness set of a basic state of the world, just substituting $\Phi$ by $\Phi^b$ in the previous definitions.

In the paper I use often the basic states of the world, since they are easier to characterize and encode all the information we need. I call the states of the world $\omega \in \Omega$ “complete” states of the world.

**A reinterpretation of the notation** What does it mean to say that $\omega(K\phi) = 1$? It says literally that the proposition “the agent knows $\phi$” holds in the state of the world $\omega$. But what is the meaning of the agent knowing $\phi$? It means that the agent knows that $\phi = 1$, or to be more precise it means that the agent knows that the state of the world is such that $\phi = 1$ (such statement makes sense in this framework because the agent knows the meaning of a state of the world).

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11 For a given state of the world we may need a smaller set of propositions than $\Phi^b$, the set of basic propositions has to be large enough to generate all the states of the world.
Given $A1$, the agent knowing $\phi$, means she knows something about the actual state of the world. Let us denote by $\omega^*$ the true state of the world. One can express $\omega(K\phi) = 1$ in the following way: $\omega(K(\omega^*(\phi) = 1)) = 1$, the agent knows at $\omega$ that in the true state of the world $\phi = 1$ (the true state of the world is $\omega$, but the agent may not know). This gives a correspondence between the agent's knowledge of propositions and the agent's knowledge about the true state of the world. This interpretation is implicit in many steps which follow. However I will keep the previous notation for simplicity.

In addition one can interpret $\omega(A\phi) = 1$ as $\omega(K(\phi \in \Phi)) = 1$.

1.4 Consequences of the Axioms

The (complete) states of the world are full descriptions. They include all deductions made by the agent. In particular her knowledge about the information structure and about the conceivable states of the world is in its definition. In this section I use the axioms of the model to prove some properties which have to hold in the complete states of the world.

Since the agent knows the axioms of the model and knows logic, she can derive the consequences of the axioms. We notice that $\Phi$ includes the axioms of the model as well as their consequences, all these are objects that the agent can speak about.

**Lemma 1** (i) $\omega(K\phi) = 1 \Leftrightarrow \omega(KK\phi) = 1$

(ii) $\omega(K\phi) = 1 \Rightarrow \omega(A\phi) = 1$

(iii) $\omega(Kg(\phi)) = 1 \Rightarrow \omega(A\phi) = 1$ or equivalently $\omega(A\phi) = 0 \Rightarrow \omega(\sim Kg(\phi)) = 1$

(iv) If $\omega(A\phi) = 0$ then $\omega((\sim K)^n\phi) = 1$, for any $n$.

(v) $\omega(K\phi) = 1 \Rightarrow \omega(K \sim K \sim \phi) = 1$

**Proof:**
(i) By $A2$, $\omega(K\phi) = 1 \Rightarrow \omega(KK\phi) = 1$. Applying $A1$ to the proposition $K\phi$ one gets $\omega(KK\phi) = 1 \Rightarrow \omega(K\phi) = 1$. Hence the agent knows that a proposition holds iff she knows that she knows it.

(ii) This is an immediate consequence of $A6$, since $\omega(K\phi) = 1 \Leftrightarrow \omega(\neg K\phi) = 0$, thus $\omega(A\phi) = 1$.

(iii) If $\omega(Kg(\phi)) = 1$, by property (ii) $\omega(Ag(\phi)) = 1$ which is equivalent to $\omega(A\phi) = 1$, by axiom $A4$.

(iv) This property is a particular case of (iii) (for $g(\phi) = (\neg K)^{n-1}\phi$). I present a direct proof by induction. The property holds for $n = 1$, by axiom $A6$. Suppose that it holds for $n$, then I prove that it has to hold for $n + 1$. Suppose not, then we would have $\omega(K(\neg K)^n\phi) = 1$. Hence, by property (ii) $\omega(A(\neg K)^n\phi) = 1$. By $A4$ this implies that $\omega(A\phi) = 1$ which contradicts the initial assumption.

(v) If $\omega(K\phi) = 1$ then by $A1$ $\omega(\phi) = 1$. Hence $\omega(\neg \phi) = 0$. But $\omega(\neg \phi) = 0$ implies by $A1$ that $\omega(K \neg \phi) = 0$, or equivalently, $\omega(\neg K \neg \phi) = 1$. Thus one concludes that $\omega(K\phi) = 1 \Rightarrow \omega(\neg K \neg \phi) = 1$. Since the agent knows the axioms and knows logic she knows that this implication holds. Moreover, by $A2$, $\omega(KK\phi) = 1$. Hence by $A3$, $\omega(K \neg K \neg \phi) = 1^{12}$. Q.E.D.

These properties are immediate consequences of the axioms. One way of interpreting them is as propositions, which have to hold in any state of the world. Notice that they are propositions about properties of states of the world. However there is something else about these consequences which is important. Suppose that the state of the world is such that the agent knows the precedent of a given condition holds, then she knows the conclusion has to hold as well. In other words the knowledge of these properties

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12This proof uses implicitly the interpretation of the notation referred above. The property can be deduced by noticing that $\omega(A \neg \phi) = 1$ (by $A4$) and $\omega(\neg K \neg \phi) = 1$ and applying $A7$, but the proof above is more eliminating.
combined with the specific knowledge the agent has about the true state of the world leads the agent to derive further things about the true state of the world.

The next proposition refers to an extremely important consequence of the axioms.

**Proposition 2** \( \omega(A\phi) = 1 \) iff either \( \omega(K\phi) = 1 \) or \( \omega(K \sim K\phi) = 1 \).

**Proof:** Assume \( \omega(A\phi) = 1 \). In any state of the world either \( \omega(K\phi) = 1 \) or \( \omega(\sim K\phi) = 1 \) and when \( \omega(\sim K\phi) = 1 \) by axiom \( A7 \), \( \omega(K \sim K\phi) = 1 \). The implication in the other direction is a consequence of property (iii). Q.E.D.

Under the axioms of the model the agent being aware of a proposition can be interpreted as the agent being able to recognize whether she knows or doesn't know the logical value of that proposition.

A corollary to proposition 2 is that the agent being aware of a proposition is a "self-evident" event. If the agent is aware of a proposition she knows that she is aware of it:

**Corollary 1** \( \omega(A\phi) = 1 \Leftrightarrow \omega(K(A\phi)) = 1 \).

**Proof:** Suppose \( \omega(A\phi) = 1 \). Applying axiom \( A2 \) on the RHS of the equivalence in proposition 2 one gets \( \omega(KK\phi) = 1 \) or \( \omega(KK \sim K\phi) = 1 \). Since the agent knows that proposition 2 holds, axiom \( A3 \) implies \( \omega(K(A\phi)) = 1 \). The implication in the other direction is a consequence of axiom \( A1 \). Q.E.D.

The agent knows the consequences of the axioms, so she knows that this equivalence holds. Thus the agent knows that there is no state of the world and no \( \phi \in \Phi \) such that she is aware of \( \phi \) but she doesn't know so.

At any state of the world the agent knows exactly what is the set of propositions she is aware of. The set \( \{ \phi : \omega(A\phi) = 1 \} = \Phi^a(\omega) \) is the same than the set \( \{ \phi : \omega(KA\phi) = 1 \} \). Hence the proposition \( \{ \phi : \omega^*(A\phi) = 1 \} = \Phi^a(\omega) \) belongs to \( \hat{K}(\omega) \).

One may ask an interesting question- what are the properties of the possibility correspondence under these set of axioms? The rationality assumptions guarantee that the
possibility correspondence forms a partition:

**Proposition 3** Under the axioms $A0$-$A8$ the possibility relation is an equivalence relation.

*Proof:* Since $A1$ and $A2$ hold one knows, by proposition 1 that if $\omega' \sim \omega$ then $\bar{K}(\omega') \supset \bar{K}(\omega)$. So, at $\omega'$ the agent has to know at least as much as she knows at $\omega$.

Suppose there exists some proposition $\phi \in \Phi$ such that $\omega(\sim K \phi) = 1$ and $\omega'(K \phi) = 1$. Consider first the case where $\phi \in \Phi^a(\omega)$. In this case $\omega(\sim K \phi) = 1$ implies, by negative introspection, that $\omega(K \sim K \phi) = 1$. Since the agent has to know at $\omega'$ all that she knows at $\omega$ this implies $\omega'(K \sim K \phi) = 1$, and by $A1$, $\omega'(\sim K \phi) = 1$, which contradicts the initial assumption. Therefore $\phi$ cannot belong to $\Phi^a(\omega)$. But corollary 1 tells us that at $\omega$ the agent knows the set of propositions she is aware of and she knows that she is not aware of any other proposition. Hence the axioms of the model, which are known to the agent, imply that at state $\omega$ the agent doesn’t consider possible any epistemic state where she is aware of more propositions than the ones she is actually aware of. Q.E.D

The argument in the proof gives a characterization of the states that the agent considers possible at $\omega$. They have to be states where the set of propositions the agent is aware of is the same than at $\omega$ and where the set of propositions the agent knows is also the same. This characterization is true independently of the agent being or not being aware of all the existent propositions. An important conclusion is the result that the possibility correspondence partitions the set of states of the world depends upon the rationality assumption implicit on the axiom that if the agent doesn't know something she knows that she doesn't know. Hence by giving up complete awareness one does not loose the partitions result.

Given the one to one correspondence between complete states of the world and basic states of the world one can translate the possibility correspondence into the set of basic states of the world. The possibility correspondence partitions the set of basic states according to the awareness set and the epistemic content of the basic states of the world,
i.e:

\[ P(\omega^b) = \{ \omega^b \in \Omega^b : \Phi^a(\omega^b) = \Phi^a(\omega^b) \text{ and } \bar{K}(\omega^b) = \bar{K}(\omega^b) \} \]

1.5 The Agent's View of The Universe at \( \omega \)

In this section I consider a given basic state of the world, \( \omega^b \), and ask - what is the agent’s view of the world in the corresponding complete state of the world \( \omega \). Which states of the world can she conceive? Which states of the world does she consider possible?

1.5.1 Agent’s View About the Model

The properties of the model guarantee that the agent knows at \( \omega \) which propositions belong to the set \( \Phi^a(\omega^b) \). Moreover she knows that she is within a model where there is an abstract set of basic propositions \( \Phi^b \), which is the maximal set of basic propositions she can be aware of. If the agent doesn’t know anything more about \( \Phi^b \) she will be uncertain whether \( \Phi^b \supset \Phi^a(\omega^b) \) or \( \Phi^b = \Phi^a(\omega^b) \). But then a complete epistemic model should include this uncertainty within the model. In a sense what is happening is that the agent doesn’t know in which “model” she is in. She knows that the model has to satisfy the properties \( \mathcal{A}0 \) to \( \mathcal{A}7 \), but she doesn’t know which model in this class she is in.

One way to solve this internal consistency problem is to extend the model in order to include all the models in the class of models satisfying the axioms above. However there are three interesting cases where one doesn’t need to do so.

One case is when the agent is aware, in any state of the world, of all the existing propositions. Let us define the “completeness” axiom as follows:

\[
AC \quad \omega(A\phi) = 1, \forall \phi \in \Phi^b.
\]

\[
\omega(K(AC)) = 1.
\]
Adding this "completeness" axiom to the previous set of axioms, leads to a model where
the agent knows the set of states of the world.

The other interesting case is when the set $\Phi^b$ is infinite and in any state of the world
the set of propositions the agent is aware of is a strict subset of all the set of existing
propositions. This can be described by the following "incompleteness axiom":

$$AI \exists \phi \in \Phi^b : \omega(A\phi) = 0.$$  

$$\omega(K(AI)) = 1.$$  

A combination of these two cases which fits in this setup is when the completeness/incompleteness are self-evident, i.e. if the agent is not aware of all propositions she
knows it and if she is aware of all propositions she also knows it. Below I concentrate on
the case where the agent is never aware of all the basic propositions. I also refer briefly
to the opposing case where the agent is always aware of all propositions.

The next two subsections answer the question of what is the agent's description of
the set of states of the world and of the information structure. The concepts of states of
the world conceivable by the agent and agent's possible states of the world are introduced.

1.5.2 Agent's Description of The World

The agent is not assumed to know a priori $\Omega$, the set of states of the world. However
the agent knows the concept of state of the world, she knows that a state of the world is
an assignment of true and false values to the basic propositions which is consistent with
the axioms of the model.

The agent can construct her own description of the set of states of the world, using
her knowledge, at $\omega$, about $\Phi^b$ and the concept of state of the world.

**Definition 5** The set of basic states of the world conceivable by the agent at state $\omega$, is
the set of states of the world that the agent can construct using the concept of state of
the world and her knowledge about the set of basic propositions.
The question is how much does the agent know about $\Phi^b$? At $\omega$ the agent knows which are the elements of $\Phi^a(\omega^b)$. If the completeness axiom holds then the agent knows this is exactly the set $\Phi^b$.

If the incompleteness axiom holds then the agent knows that $\Phi^b \setminus \Phi^a(\omega^b) \neq \emptyset$. This implies that she knows that there exist states of the world conceivable by the modeler where she is aware of propositions not in $\Phi^a(\omega^b)$. She also knows that the set $\Phi^b \setminus \Phi^a(\omega^b)$ includes propositions about the environment, awareness and knowledge of the propositions about the environment. These are consequences of the agent’s knowledge about the structure of model.

Although the agent has a generic knowledge about the existence of states of the world conceivable by the modeler where she would be aware of propositions she is not aware of, she cannot know which specific propositions not in $\Phi^a(\omega^b)$ she would be aware in such states. Suppose the contrary, assume that the state $\omega^b$ is such that $\omega^b(A\phi^*) = 1$, while $\omega^b(A\phi^*) = 0$. The agent cannot be aware of the proposition “there exists a state such that $A\phi^* = 1$”, because if she was aware of such a proposition then by A4 she would be aware of proposition $\phi^*$ at the state $\omega$, a contradiction.

Obviously the agent is restricted in her description of the states of the world since she cannot use any proposition not in $\Phi^a(\omega^b)$. However the knowledge that $\Phi^b \setminus \Phi^a(\omega^b) \neq \emptyset$ and that this set includes propositions about the environment, awareness and knowledge of the propositions about the environment should be used in some manner by the agent in her description of the world.

In order to formalize this idea let us define the propositions: $\tilde{\phi}_\omega$ = “propositions about the environment not included in $\Phi^a(\omega^b)$ hold”, $\tilde{\phi}_\omega^a$ = “the agent is aware of some proposition about the environment not in $\Phi^a(\omega^b)$” and $\tilde{\phi}_\omega^k$ = “the agent knows some proposition about the environment not in $\Phi^a(\omega^b)$”.

The logical value of the propositions $\tilde{\phi}_\omega^a$ and $\tilde{\phi}_\omega^k$ can be determined for any basic state of the world $\omega^b$, as follows: $\omega^b(\tilde{\phi}_\omega^a) = 1$ if $\exists \phi \in \Phi^b \setminus \Phi^a(\omega^b)$ such that $\omega^b(A\phi) = 1$. And similarly for the proposition $\tilde{\phi}_\omega^k$. Once the basic states of the world are defined there is

35
no ambiguity on the logical value of these propositions.

Let \( \Phi^a(\omega^b) \) be the union of \( \Phi^a(\omega^b) \) with these three propositions. By adding these propositions to \( \Phi^a(\omega^b) \) all the agent's knowledge, at \( \omega \), about \( \Phi^b \) is captured - the specific knowledge about \( \Phi^a(\omega^b) \) and the generic knowledge that \( \Phi^b \setminus \Phi^a(\omega^b) \) is nonempty and includes propositions about the environment, awareness and knowledge of the propositions about the environment.

**Lemma 2** Let us consider the elements of \( \{0, 1\}^{\Phi^a(\omega^b)} \) which satisfy the axioms of the model. The agent can conceive this set of basic states of the world at state \( \omega \).

**Proof:** The agent is aware of the set of propositions \( \Phi^a(\omega^b) \) and knows the axioms of the model. Since the agent knows the incompleteness axiom, she knows that \( \Phi^b \setminus \Phi^a(\omega^b) \neq \emptyset \). In addition she knows the structure of the model and as such she knows there are states where she is aware/knows propositions not included in \( \Phi^a(\omega^b) \) and this is summarized in the propositions \( \Phi^b \omega, \Phi^a \omega \) and \( \Phi^k \omega \). Since she knows the axioms of the model she can imagine all the assignments of true and false values to the propositions in \( \Phi^a(\omega^b) \) which are consistent with the axioms. Q.E.D

Let us call a basic states of the world *conceivable by the agent at state \( \omega \)* by \( s^b \omega \). The subscript \( \omega \) is a reminder that this is a state conceived by the agent at state \( \omega \). Let \( \Omega^b(\Phi^a(\omega)) \) be the set of all basic states of the world conceivable by the agent, at \( \omega \). One can also define a conceivable complete state of the world, \( s_\omega \) as the closure under the axioms of a basic state of the world. Let \( \Omega(\Phi^a(\omega)) \) be the set of complete states imaginable by the agent.

The states of the world that the agent conceives are a function of the state of the world \( \omega \), because the set of propositions the agent is aware of and hence can use in her description of the states of the world changes with the state of the world.
1.5.3 Agent’s Possibility Correspondence

In the previous subsection we argued that the agent can construct her own description of the set of states of the world, using the concept of state of the world and her knowledge about the set of basic propositions. Similarly, the agent can use the concept of possibility correspondence to derive her own description of the information structure.

I start this subsection with a word about the interpretation of the possibility set $P(\omega)$, when one does not assume a priori that the agent knows $\Omega$. Then I define the agent’s possibility correspondence.

About the Possibility Set $P(\omega)$

As I said above the agent’s knowing a proposition in a certain state of the world can be understood as the agent knowing that the state of the world is such that the proposition is true. Hence the state of the world has to be such that the propositions that she knows are true. In addition, as I proved, the agent knows the set of propositions she is aware of and she knows that she is not aware of any other proposition.

The knowledge about the true state of the world need not be complete. If the agent doesn’t know the logical value of all the propositions then the agent cannot exactly determine the true state of the world. Moreover, the fact that the agent knows some of the properties of the true state of the world doesn’t mean that the agent can enumerate the elements of $\Omega$ which satisfy these properties. Since the agent is not assumed a priori to know $\Omega$, it may be impossible for her to do such an enumeration. Hence when one thinks about the possibility set $P(\omega)$, from the agent’s perspective, the set should be defined descriptively and not by enumeration. $P(\omega)$ can then be interpreted as the set of states of the world in $\Omega$, which are consistent with all that the agent knows at $\omega$.

In order for the agent to enumerate all the states which can possibly be the true state of the world she has to be aware of all the propositions she doesn’t know. However it is not always true that the agent is aware of all the propositions she doesn’t know. Hence the agent may fall short of a complete enumeration of all the possible states of the world.
The Agent's Possibility Correspondence

Although the agent can imagine all the states in $\Omega^b(\hat{\Phi}^a(\omega))$, she only considers possible a certain subset of these states. Given what she knows, some of the imaginable states cannot be the true state of the world. They are not possible states of the world.

Given the properties of the model the subset of $\Omega^b(\hat{\Phi}^a(\omega))$ that the agent considers possible at $\omega$ are the states where she is aware of the same propositions she is at $\omega^b$ and she knows the same basic propositions than in $\omega^b$. Let us call this set $\Gamma^*(\omega^b)$. Formally the agent's possible states of the world is:

$$\Gamma^*(\omega^b) = \{s^b_\omega \in \Omega^b(\hat{\Phi}^a(\omega)) : \Phi^a(s^b_\omega) = \hat{\Phi}^a(\omega^b), \hat{K}(s^b_\omega) = \hat{K}(\omega^b)\}$$

The difference between $\Gamma^*(\omega^b)$ and $P(\omega^b)$ is that the first is defined in the space of states of the world conceivable by the agent at $\omega$, while the last is defined in the set of states of the world conceivable by the modeler. The agent is able to enumerate the states which belong to $\Gamma^*(\omega^b)$ but she might not be able to enumerate the elements of $P(\omega^b)$.

Since the agent knows the concepts of possibility relation and possibility correspondence she can apply these notions to her set of conceivable states. Let $\gamma_\omega$ be the agent's possibility relation on $\Omega^b(\hat{\Phi}^a(\omega))$ and let $\Gamma_\omega : \Omega^b(\hat{\Phi}^a(\omega)) \rightarrow 2^{\Omega^b(\Phi^a(\omega))}$ be the possibility correspondence in $\Omega^b(\hat{\Phi}^a(\omega))$. $\gamma_\omega$ and $\Gamma_\omega$ depend on the state of the world $\omega$ since $\Omega^b(\hat{\Phi}^a(\omega))$ also depends on $\omega$.

Given the result in proposition (3) one can describe $\gamma_\omega$ and $\Gamma_\omega$ as follows:

$$s^b_\omega \gamma_\omega s^b_o \iff \hat{\Phi}^a(s^b_\omega) = \hat{\Phi}^a(s^b_o) \text{ and } \hat{K}(s^b_\omega) = \hat{K}(s^b_o)$$

$$\Gamma_\omega(s^b_\omega) = \{s^b_o \in \Omega^b(\hat{\Phi}^a(\omega)) : s^b_\omega \gamma_\omega s^b_o \iff \hat{\Phi}^a(s^b_\omega) = \hat{\Phi}^a(s^b_o) \text{ and } \hat{K}(s^b_\omega) = \hat{K}(s^b_o)\}$$

The possibility correspondence $\Gamma_\omega$ tells us the information structure in the set of states.
of the world imaginable by the agent.

1.5.4 Conceivable and Possible States - An Example

The concepts of *states of the world conceivable by the agent* and the *agent's possible states of the world* are relevant both under the completeness/incompleteness axioms as well as under the case where the completeness/incompleteness is "self-evident". In order to consider a tractable example, keeping the spirit of incomplete awareness I present an example where completeness/incompleteness are "self-evident".

Assume that there are only three unrelated propositions about the environment \( \phi_1 = \text{"it is sunny in Boston"}, \phi_2 = \text{"it is sunny in Lisbon"} \) and \( \phi_3 = \text{"it is sunny in Paris"} \).

Let us assume that in the actual state of the world the agent is only aware of the proposition "it is sunny in Boston", and she knows that she is not aware of all the propositions. Assume in addition, that \( \phi_1, \phi_2 \) and \( \phi_3 \) hold and the agent doesn't know whether the proposition \( \phi_1 \) is true or not.

Given her knowledge, the states of the world that the agent can conceive in the actual state of the world are:

<table>
<thead>
<tr>
<th>state</th>
<th>( \phi_1 )</th>
<th>( A\phi_1 )</th>
<th>( K\phi_1 )</th>
<th>( K \sim \phi_1 )</th>
<th>( \bar{\phi}_w )</th>
<th>( \bar{\phi}_o )</th>
<th>( \bar{\phi}_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( s_6 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( s_7 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>
We notice that the set of states of the world conceivable by the agent would be as above in any other state of the world where the agent is only aware of the proposition \( \phi_1 \). The agent is missing in her description of the states of the world the propositions \( \phi_2, \phi_3, A\phi_2, A\phi_3, K\phi_2, K\phi_3 \).

Given her knowledge, the states of the world the agent considers possible are \( s_1 \) and \( s_2 \). Notice that for the purpose of distinguishing between the possible states of the world what matters are the set of propositions that the agent doesn’t know (but knows that she doesn’t).

It may seem that there is no reason for the agent to care about those conceivable states which are not possible. However there are many circumstances when the ability to think about what would happen if other conceivable states of the world were the true state of the world is crucial\(^{13}\).

### 1.6 The Relationship Between the Agent’s and the Modeler’s Universe

In this section I ask what is the relationship between the agent’s and the modeler’s universe under the completeness and incompleteness axioms.

#### 1.6.1 The Model With The Completeness Axiom

Adding the “completeness axiom” has several implications: Axioms \( \mathcal{A}4-\mathcal{A}5 \) are redundant since it is assumed that \( \Phi \) is closed under the logical and knowledge operators. In addition \( \mathcal{A}6 \) is irrelevant because it is never the case that \( \omega(\mathcal{A}\phi) = 0 \). Finally, \( \mathcal{A}7 \) is equivalent to having negative introspection over all the propositions the agent does not know.

\(^{13}\)One example: suppose a duopolist has a low cost technology, but her competitor doesn’t know this. The competitor only knows that the rival may have either low or high cost technology. Although the agent knows exactly the true state of the world, her ability to reason about how she would act had her technology been high cost is important because that influences the competitor’s choice of strategy and as consequence it affects her prediction about how the competitor will act.
It is trivial to show the following:

**Proposition 4** When the completeness axiom is satisfied $\Omega^b(\Phi^a(\omega)) = \Omega^b, \forall \omega$. In addition $\Gamma_\omega = P$.

*Proof:* By the “completeness axiom” $\Phi^a(\omega^b) = \Phi^b$ in any state of the world, and the agent knows that she is aware of all propositions. In addition she knows the definition of state of the world and all the axioms of the model. Therefore the set of conceivable basic states of the world coincides with $\Omega^b$. The fact that $\Gamma_\omega$ coincides with $P$ is obvious from their definition. Q.E.D.

The most important difference between the model in section 1.2 with axioms $A_{i-.43}$ and this model is that here one can interpret the model from the agent's perspective, which is a result of the agent being able to reason and know the properties of her knowledge. Here it makes sense to ask the agent which set of states does she conceive, which set of states does she consider possible given what she knows, even which set of states would she consider possible if she was in some other state of the world. In any complete state of the world the agent knows what is the set of states of the world and what is the information structure.

The model with the completeness axiom has the same consequences as assuming informally that the agent knows the information structure, a common assumption in many economic models. However in this model the agent's knowledge about the information structure is not assumed. Instead it is “inferred knowledge”, in the sense that the agent derives herself what is the set of states of the world and the information structure.

1.6.2 The Model With The Incompleteness Axiom

My interpretation of this case is that $\Phi$ is not finitely generated. It doesn't matter how many (basic) propositions the agent is aware of, there are always more propositions that she could be aware of.
Before analyzing the relationship between the agent’s model and the global model let me introduce here the concept of refinement.

Formally, a set Ω' is a refinement of another Ω if there is a mapping, \(\hat{\Pi}\), from the set of subsets of Ω to the set of subsets of Ω' (\(\hat{\Pi} : 2^\Omega \to 2^{\Omega'}\)), such that:

(i) \(\hat{\Pi}(\{\omega\}) \neq \emptyset\), for all \(\omega \in \Omega\)
(ii) \(\hat{\Pi}(\{\omega\}) \cap \hat{\Pi}(\{\omega'\}) = \emptyset\), if \(\omega \neq \omega'\)
(iii) \(\bigcup \{\hat{\Pi}(\{\omega\}) : \omega \in \Omega\} = \Omega'\)
(iv) \(\hat{\Pi}(A) = \bigcup \{\hat{\Pi}(\{\omega\}) : \omega \in A\}\)

Since (iv) implies that the image set of a subset of Ω is the union of the image sets of each element of that subset, one can define refinement in a slightly different and more familiar way. We can say that Ω' is a refinement of another set Ω if there is a correspondence from Ω onto Ω', such that the image sets corresponding to any two different elements of Ω do not intersect.

The mapping \(\Pi : 2^{\Omega'} \to 2^\Omega\) defined as:

\[
\Pi(A') = \{\omega \in \Omega : \hat{\Pi}(\omega) \cap A' \neq \emptyset\}
\]

is called the outer reduction induced by \(\hat{\Pi}\).

The relationship between the agent's model and the true model is the following:

**Proposition 5** \(\Omega^b\) is a refinement of \(\hat{\Omega}^b(\hat{\Phi}^a(\omega))\).

**Proof**: Define the refinement map\(^{14}\):

\[
\hat{\Pi}(\{s^b_\omega\}) = \{\omega^b \in \Omega^b : \omega^b(\phi) = s^b_\omega(\phi), \forall \phi \in \hat{\Phi}^a(\omega)\}
\]

\[
\hat{\Pi}(A) = \bigcup \{\hat{\Pi}(\{s^b_\omega\}) : s^b_\omega \in A\}
\]

---

\(^{14}\)The refinement mapping defined depends on ω, however I will not include the index ω in the notation for simplicity.
One needs to prove that $\hat{\Pi}$ satisfies the requirements (i) to (iv).

i The only way the property could fail is if a consistent assignment over $\hat{\Phi}^a(\omega^b)$ implies something which is incompatible with all the assignment over $\Phi^b \setminus \Phi^a(\omega^b)$. However, for any given assignment over $\hat{\Phi}^a(\omega^b)$, the only implications over the set of propositions that the agent is not aware of are the ones implied by the propositions $\bar{\phi}_\omega^a, \bar{\phi}_\omega^b$.

ii If $s^a_\omega \neq s^b_\omega$ then there exists some $\phi \in \hat{\Phi}^a(\omega^b)$ such that $s^b_\omega(\phi) \neq s^b_\omega(\phi)$, call it $\phi^*$. By definition of the refinement map for any $\omega^b \in \hat{\Pi}(\{s^a_\omega\})$ and any $\omega^b \in \hat{\Pi}(\{s^a_\omega\})$ we have $\omega^b(\phi^*) = s^b_\omega(\phi^*) \neq s^b_\omega(\phi^*) = \omega^b(\phi^*)$. Hence the requirement holds.

iii One needs to prove that there is no $\omega^b \in \Omega^b$ such that there is no $s^b_\omega \in \Omega^b(\hat{\Phi}^a(\omega))$ such that $\omega^b \in \hat{\Pi}(s^b_\omega)$. This can be proven by contradiction. Suppose there exists such state of the world, call it $\bar{\omega}^b$. By construction $\bar{\omega}^b$ is an element of $\{0,1\}^{\phi^a}$ which is consistent with the axioms of the model. This can also be written as $\omega^b \in \{0,1\}^{\phi^a(\omega)}\{0,1\}^{\phi^a(\omega)}$. The non-existence of a state of the world conceivable by the agent which image is $\bar{\omega}^b$ is equivalent to say that if one projects $\bar{\omega}^b$ on the space $\hat{\Phi}^a(\omega)$, the assignment obtained in that subset of propositions is not consistent with the axioms. Which means that either the assignment over $\Phi^a(\omega^b)$ is not consistent with the axioms or the assignments to the propositions $\bar{\phi}_\omega^a$ and $\bar{\phi}_\omega^b$ is not consistent with the axioms. But then $\bar{\omega}^b$ cannot be a consistent assignment, which is a contradiction.

iv Holds by definition. Q.E.D.

What this proposition means is that there is a one to many mapping from the set of states of the world conceivable by the agent to the set of states of the world of the modeler. The agent can only make an incomplete description of each state of the world. Hence if we consider a certain state of the world as described by the agent, say $s^b_\omega$, and ask what is the set of states of the world (in the modeler’s model) which satisfy the
agent's description there will be several states which will do (precisely $\hat{\Pi}(s^k_\omega)$). In other words, the states of the world imaginable by the agent in the state $\omega$ are coarser than the modeler’s states of the world.

The example in section 1.5.4 illustrates this relationship. Consider, for example, the agent’s state $s_1$. There are four states of the world in the modeler’s model which correspond to $s_1$. These states have in common the propositions that “it is sunny in Boston” and “the agent doesn’t know that it is sunny in Boston”, but in one of them “it is sunny in Paris and it is sunny in Lisbon”, in a second one “it is sunny in Paris and it is not sunny in Lisbon”, in the third “it is not sunny in Paris and it is sunny in Lisbon” and in the fourth “it is not sunny in Paris and it is not sunny in Lisbon”.

In the case of $s_1$, since the agent is not aware of propositions $\phi_2$ and $\phi_3$ she cannot know their logical value. Hence the four states above are the only states which correspond to $s_1$. The same reasoning holds in the case of $s_2$, $s_3$, $s_4$. However not all the states of the world conceivable by the agent, correspond only to four states in the modeler’s universe. For example, in the case of $s_5$, $s_6$ and $s_7$ there are twelve states corresponding to each of them in the modeler’s universe ($\phi_2$ and $\phi_3$ can have true or false values as above, and the agent can be aware of $\phi_2$, be aware of $\phi_3$ or be aware of both).\textsuperscript{15}

The definition of states of the world conceivable by the agent only involves the set of propositions the agent is aware of and the knowledge the agent has of the axioms of the model. Since in any state of the world the agent knows the axioms, the fact that the agent may have different views of the world in alternative states of the world is driven by the differences in the set of propositions the agent is aware of in these alternative states. Two states of the world belonging to the same “awareness equivalence class” will have

\textsuperscript{15}In Kreps [7] the agent lists a set of physical states of the world, where the list is not completely fine. In each of these states there are subcontingencies that can occur but the agent did not foresee. In my framework, which differs from Kreps because the states of the world are epistemic, the set of subcontingencies that the agent doesn’t foresee, in state $\omega$, is the same for the states of the world which belong to the same possibility set of the agent. Implicitly Kreps only considers the set of possible states of the world (conceivable states of the world which are not possible are not relevant in a context where there is only one decision maker). Thus if one embedded Kreps’s framework in my model we would get the result that the set of subcontingencies in his framework is the same for all states of the world that he considers.
the same set of states of the world conceivable by the agent.

Furthermore one can prove, using the same type of argument as in the proof of proposition 5, that if two states of the world are such that the set of propositions the agent is aware of in the first state is a superset of the set of propositions the agent is aware in the second state, then the set of the states conceivable by the agent in the first state, is a refinement of the set of states conceivable by her in the second state.

The next question one may ask is - how does \( \Gamma_\omega \) relate with \( P \)? Proposition 6 gives an answer to this question.

The proposition compares the possibility set in a given state, say \( \omega^b \), with the agent's view at \( \omega \) of that possibility set. What do I mean by this? Since at \( \omega \) there exists some state imaginable by the agent which corresponds to \( \omega^b \), \( s_\omega^b = \Pi(\omega^b) \), one can ask which are the states of the world, conceivable by the agent at \( \omega \), which she thinks at \( \omega \) that she would consider possible if she was in that state, \( \Gamma_\omega(s_\omega^b) \). This is the agent's view at \( \omega \) of the possibility set in state \( \omega^b \).

However the agent's and the modeler's possibility correspondences are defined on different spaces. As a consequence one needs to express both correspondences in the same space in order to make such a comparison. I do this by using the refinement mapping on the agent's possibility sets, hence expressing both correspondences in \( \Omega^b \). By applying the refinement mapping on the agent's possibility sets one identifies the states of the world (in the modeler's model) which are consistent with the agent's description of each possibility set. The comparison of these sets with the modeler's possibility sets is basically a comparison of the agent's description at \( \omega \) of the information structure with the true information structure.

The result of this comparison is the following:

**Proposition 6**

i) If \( \Phi^a(\omega^b) \subset \Phi^a(\omega^b) \) then \( \hat{\Pi}(\Gamma_\omega(\Pi(\omega^b))) = P(\omega^b) \).

ii) If \( \exists \phi \in \Phi^a(\omega^b) : \phi \notin \Phi^a(\omega^b) \) then \( \hat{\Pi}(\Gamma_\omega(\Pi(\omega^b))) \supset P(\omega^b) \).

**Proof:**
i Let us consider a state \( \omega^b \) such that \( \Phi^a(\omega^b) \subset \Phi^a(\omega^b) \). I construct \( \hat{\text{II}}(\Gamma_\omega(\Pi(\omega^b))) \) and show that it is equal to \( P(\omega^b) \).

Let us find the projection of \( \omega^b \) on \( \Omega^b(\Phi^a(\omega^b)), \Pi(\omega^b) \), say it is the state \( s^b_\omega \). By construction \( s^b_\omega \) has the same assignments than \( \omega^b \) over the propositions in \( \Phi^a(\omega^b) \). Thus \( \hat{\text{K}}(s^b_\omega) = \hat{\text{K}}(\omega^b) \) and \( \Phi^a(s^b_\omega) = \Phi^a(\omega^b) \subset \Phi^a(\omega^b) \). Moreover \( s^b_\omega(\Phi^b_\omega) = 0 \) and \( s^b_\omega(\Phi^b_\omega) = 0 \).

Any state in \( \Gamma_\omega(s^b_\omega) \) has the same awareness and epistemic states than \( s^b_\omega \) and \( \Gamma_\omega(s^b_\omega) \) includes all the states conceivable by the agent which satisfy this property. Hence for any \( s^b_\omega \in \Gamma_\omega(s^b_\omega) \), \( \hat{\text{K}}(s^b_\omega) = \hat{\text{K}}(\omega^b) \), \( \Phi^a(s^b_\omega) = \Phi^a(\omega^b) \), \( s^{\phi^b}(S^b_\omega) = 0 \), and \( s^{\phi^b}(S^b_\omega) = 0 \).

Applying the refinement mapping to \( \Gamma_\omega(s^b_\omega) \) one gets:

\[
\hat{\text{II}}(\Gamma_\omega(s^b_\omega)) = \{ \omega^{\phi^b} \in \Omega^b : \omega^{\phi^b}(\phi) = s^{\phi^b}(\phi) \ \forall \phi \in \Phi^a(\omega), \text{ where } s^{\phi^b} \in \Gamma_\omega(s^b_\omega) \}
\]

Obviously any \( \omega^{\phi^b} \) which belongs to this set has to satisfy \( \hat{\text{K}}(\omega^{\phi^b}) = \hat{\text{K}}(\omega^b) \) and \( \Phi^a(\omega^{\phi^b}) = \Phi^a(\omega^b) \) and hence it belongs to \( P(\omega^b) \). In addition this set has to include all the elements of \( \Omega^b \) which satisfy these properties because otherwise \( \hat{\text{II}} \) wouldn’t be a refinement mapping.

ii Let us consider a state \( \omega^b \) where the condition holds, this implies that \( \omega^b(\Phi^b_\omega) = 1 \).

One can divide the set of propositions the agent is aware of at \( \omega^b \) in two disjoint sets: \( A = \Phi^a(\omega^b) \cap \Phi^a(\omega^b) \), and \( B = \Phi^a(\omega^b) \setminus A \), where by assumption \( B \) is nonempty. Similarly one can partition the set \( \hat{\text{K}}(\omega^b) \) into the sets \( C \) and \( D \), \( C = \hat{\text{K}}(\omega^b) \cap \Phi^a(\omega^b) \), and \( D = \hat{\text{K}}(\omega^b) \setminus C \), \( D \) may or may not be empty. Let us assume it is not empty, the proof can be adjusted easily if otherwise.

Applying the outer reduction mapping to \( \omega^b \) one gets a state, \( s^b_\omega \), such that \( s^b_\omega(\Phi^b_\omega) = 1 \) and all the assignments of truth and false values to the propositions in \( \Phi^a(\omega^b) \) coincide with the ones of \( \omega^b \). Hence \( \Phi^a(s^b_\omega) = A \cup \Phi^b_\omega \) and \( \hat{\text{K}}(s^b_\omega) = C \cup \Phi^b_\omega \).
The definition of $\Gamma_\omega(s'^b_\omega)$ implies that for any state in the agent's possibility set $\Phi^a(s'^n_{\omega}) = A \cup \phi^a_\omega$ and $\bar{K}(s'^n_{\omega}) = C \cup \bar{\phi}^k_\omega$.

I only need to prove that applying the outer reduction mapping to any state $\omega'^{nb} \in P(\omega'^b)$ one gets one element of $\Gamma_\omega(s'^{nb}_\omega)$. Let us assume that $\Pi(\omega'^{nb}) = s'^{nb}_\omega$. Since $\Phi^a(\omega'^{nb}) = A \cup B$, and $\bar{K}(\omega'^{nb}) = C \cup D$, it is clear that $\Phi^a(s'^{nb}_\omega) = A \cup \phi^a_\omega$ and $\bar{K}(s'^{nb}_\omega) = C \cup \bar{\phi}^k_\omega$. Hence $s'^{nb}_\omega \in \Gamma_\omega(s'^{nb}_\omega)$.

It is obvious that in general $P(\omega'^b)$ is strictly included in $\hat{\Pi}(\Gamma_\omega(s'^b_\omega))$ because any state, $\omega^*$ such that $\Phi^a(\omega^*b) \cap \Phi^a(\omega^b) = A$ and $\bar{K}(\omega^*b) \cap \Phi^a(\omega^b) = C$ and such that $\Phi^a(\omega^*b) \setminus A$ and $\bar{K}(\omega^*b) \setminus C$ are nonempty is included in $\hat{\Pi}(\Gamma_\omega(s'^b_\omega))$ while $P(\omega'^b)$ only includes the states $\omega'^{nb}$ for which $\Phi^a(\omega'^{nb}) = A \cup B$ and $\bar{K}(\omega'^{nb}) = C \cup D$.

Q.E.D.

The first part of the proposition says literally that the set of states in the modeler's model which are consistent with the agent's description at $\omega$, of her possibility set in the conceivable state of the world corresponding to $\omega'$ is exactly $P(\omega')$. The comparison is between the set of states of the world which are consistent with the agent's description of the possibility set, not a comparison of the description of the possibility sets per se (that would differ if the awareness set at $\omega$ is different from the awareness set at $s'_\omega$).

Roughly speaking, one can say that the agent's description at $\omega$ of the possibility set in the state of the world $\omega'^b$, where she is not aware of any proposition which she is not aware of in $\omega$, is as good as the description she would make if that state of the world had been the true state of the world. This result is not surprising. Since the agent is aware at $\omega$ of all the propositions she would be aware of in $\omega'^b$ she can replicate what would have been her view of the world if she was in that state. In particular, she can make the counterfactual reasoning of describing the set of states that she would consider possible if she were in that state.

However in imagining a state, $\omega'^b$, where she would be aware of propositions she is not aware of in the true state of the world the agent cannot describe the possibility set
as well as she would do if she was in that state. The reason is that if the agent was at $\omega^b$ she would know the specific propositions she would be aware of and the states she would consider possible would be the ones which have exactly the same awareness and epistemic sets. On the other hand at $\omega$ the agent doesn’t know the specific propositions she would be aware of if she was in state $s^b_\omega$. So she cannot perform the referred contrafactual reasoning.

To summarize, in the model with the incompleteness axiom, the states of the world and the possibility correspondence imaginable by the agent are coarser than the modeler’s. The agent has an incomplete knowledge of both the set of states of the world and the information structure. I would like to stress that the agent knows this property and that saying that the agent’s possibility correspondence is coarser than the modeler’s one does not contradict the result that $P$ is a partition.

The agent’s description is the best one she can do, given her awareness and her knowledge at $\omega$. The relationship of the agent’s possibility correspondence and the modeler’s possibility correspondence is fully consistent with the fact that the agent has an incomplete view of the states of the world. If the agent had a complete knowledge of the possibility correspondence how could one argue that she can be rational and have an incomplete description of the states of the world?

1.7 Dynamic Model of Knowledge

1.7.1 Description

In this section I extend the model where the agent is never aware of all propositions to more than one period. As usual a state of the world is to be understood as a complete path, a complete history of the world. Awareness and knowledge are now time dependent. The proposition $A_t\phi$ means that the agent is aware of proposition $\phi$ at time $t$, and the epistemic proposition $K_t\phi$ means that the agent knows $\phi$ at period $t$. The type of propositions to which the knowledge operator is applied is quite general, in particular it
can apply to propositions about past or future knowledge.

Let us rewrite the axioms using the time dependent awareness and knowledge operators and add one more axiom, saying that the agent doesn’t forget, hence knowledge is non-decreasing:

\[ A9 \quad \text{If } \omega(K_t \phi) = 1 \text{ then } \omega(K_{t+1} \phi) = 1. \]

A basic state of the world \( \omega^b \) now includes propositions about the environment, which can refer to different point in time, it says which propositions about the environment the agent is aware of at each point of time, which propositions about the environment she knows at each point in time.

The axioms of the model imply that the knowledge of the agent about her past knowledge is well defined: if the agent knows at time \( t \) a proposition the agent knows a time \( t + 1 \) that she knew that proposition at time \( t \) (by \( A2 \) and \( A9 \)), if the agent was aware of a proposition at time \( t \) but she didn’t know its logical value then she knows at \( t + 1 \) that she did not know the proposition at time \( t \) (by \( A7 \) and \( A9 \)), if the agent is not aware of a proposition at time \( t \) and becomes aware of the proposition at time \( t + 1 \) the agent knows at \( t + 1 \) that she did not know the proposition in time \( t \).

However the knowledge that the agent has about her future knowledge is not completely specified once one knows which propositions about the environment the agent is aware of and which propositions she knows at each point of time. Therefore a basic state of the world has to say what the agent knows about the structure of revelation of information. One can have cases where the agent knows at which point in time she will learn the logical values of a certain proposition and cases where she doesn’t know when will she learn the logical value of certain propositions.

The states of the world conceivable by the agent are also descriptions, eventually incomplete, of a full history of the world. As the agent becomes aware of more and more propositions, her description of the world will change. Let us denote by \( s_{\omega, t} \) a state of
the world as described by the agent at period $t$, when the state of the world is $\omega$. The index $\omega, t$ is only a reminder that this is a state of the world which can be conceived by the agent at period $t$, in state $\omega$. One can have statements like $s_{\omega,t}(K_t \phi) = 1$, which means that at state $s_{\omega,t}$, imaginable by the agent at period $t$, the agent knows at period $t$ that $\phi$ is true. Or like $s_{\omega,t}[K_t(K_{t+1} \phi \lor K_{t+1} \sim \phi)] = 1$, which means that the agent knows at period $t$ that in the next period she will know whether $\phi$ is true or false.

### 1.7.2 Law of Iterated Knowledge

One can prove a law of iterated knowledge, saying that the agent cannot have at time $t$ knowledge about future knowledge that exceeds what she knows at period $t$.

**Proposition 7 (LIK)** If $\omega(\sim K_t \phi) = 1$ then it cannot be the case that $\omega(K_t K_{t+1} \phi) = 1$.

**Proof**: The proof is by contradiction. If $\omega(K_t K_{t+1} \phi) = 1$, then by $A1$, $\omega(K_{t+1} \phi) = 1$ and once again by $A1$, $\omega(\phi) = 1$. Since the agent knows the properties of her knowledge she knows that $\omega(K_{t+1} \phi) = 1 \Rightarrow \omega(\phi) = 1$, but then by $A3$, $\omega(K_t \phi) = 1$, which contradicts the assumption $\omega(\sim K_t \phi) = 1$. Q.E.D.

It is easy to prove that, the set of propositions the agent is aware of is non decreasing over time:

**Lemma 3** If $\phi \in \Phi^a_t(\omega)$ then $\phi \in \Phi^a_{t+1}(\omega)$

**Proof**: If $\phi \in \Phi^a_t(\omega)$ that implies that either $\omega(K_t \phi) = 1$ or $\omega(K_t \sim K_t \phi) = 1$. In the first case by $A9$ one knows that $\omega(K_{t+1} \phi) = 1$, and thus $\phi \in \Phi^a_{t+1}(\omega)$. In the second case, again by $A9$ one gets $\omega(K_{t+1} \sim K_t \phi) = 1$. Hence $\omega(A_{t+1} \sim K_t \phi) = 1$ but the closure axiom ($A4$) guarantees that $\omega(A_{t+1} K_t \phi) = 1$. Q.E.D.

It is interesting that one doesn't need any axiom relating the awareness operator at different points of time to get the non-decreasing awareness result. The reason goes back to the equivalence of the agent's awareness of a proposition and her knowledge about
knowing or not the logical value of a proposition. Since knowledge is non-decreasing the mentioned equivalence is all we need to guarantee that awareness is non-decreasing.

A corollary of the LIK is that \( \Phi^a_t(\omega) \) is the best forecast the agent has of \( \Phi^a_{t+1}(\omega) \).

**Corollary 2** If \( K_t(\phi \in \Phi^a_{t+1}(\omega)) \) then \( \phi \in \Phi^a_t(\omega) \).

**Proof:** Suppose not, suppose \( \exists \phi \) not belonging to \( \Phi^a_t(\omega) \) and such that \( K_t(\phi \in \Phi^a_{t+1}(\omega)) \). By property [v], \( \omega(\sim K_t\phi) = 1 \) and \( \omega(\sim K_{t+1} \sim K_t\phi) = 1 \). By LIK that implies that it cannot be the case that:

\[
\omega(K_tK_{t+1}\phi) = 1 \text{ or } \omega(K_tK_{t+1} \sim K_t\phi) = 1
\]

Or equivalently:

\[
\omega(\sim K_t[\omega(K_{t+1}\phi) = 1 \text{ or } \omega(K_{t+1} \sim K_t\phi) = 1]) = 1 \Leftrightarrow \omega(\sim K_tA_{t+1}\phi) = 1
\]

Hence the agent cannot know at period \( t \) that some proposition belongs to \( \Phi^a_{t+1} \), unless she is aware of that proposition in period \( t \). Q.E.D

### 1.7.3 Relationship Between \( \Omega(\Phi^a_t(\omega)) \) and \( \Omega(\Phi^a_{t+1}(\omega)) \)

In this section I analyze how the agent’s view about the world changes over time. I show that the set of states of the world conceivable by the agent and her possibility correspondence get finer as time passes.

The set of conceivable basic states of the world by the agent at state \( \omega \) at time \( t \) depends on \( \omega \) and \( t \). Like before the knowledge of the agent about the structure of the model should be used to get a comprehensive picture of \( \Phi^b_t \). Let us extend the set \( \Phi^a_{\omega, t} \) by propositions like \( \bar{\phi}_{\omega,t} \), \( \bar{\sigma}^a_{\omega,t} \), and \( \bar{\delta}^b_{\omega,t} \). These propositions are defined as before except that now because the awareness and knowledge operators are time dependent there will be propositions referring to the awareness or knowledge of propositions which do not belong to \( \Phi^a_{\omega,t} \), at each point of time, let \( \hat{\Phi}^a_{\omega,t} \) be union of these propositions with \( \Phi^a_{\omega,t} \)
Consider a state of the world $\omega$. I showed before that under the axioms the set of propositions the agent is aware of, in state $\omega$, is non-decreasing over time. Hence we should expect the agent's description of the set of states of the world to improve over time. One can prove the following:

**Proposition 8** $\Omega^b(\Phi_{t+1}^\omega(\omega))$ is a refinement of $\Omega^b(\Phi_t^\omega(\omega))$

**Proof:** Define the refinement mapping:

$$\tilde{\Pi}(s_{\omega,t}^b) = \{ s_{\omega,t+1}^b \in \Omega^b(\Phi_{t+1}^\omega(\omega)) : s_{\omega,t+1}^b(\phi) = s_{\omega,t}^b(\phi), \forall \phi \in \Phi_t^\omega(\omega^b) \}$$

$$\tilde{\Pi}(A) = \bigcup\{ \tilde{\Pi}(s_{\omega,t}^b) : s_{\omega,t}^b \in A \}$$

The rest of the proof follows the same reasoning than the proof of proposition 5. **Q.E.D.**

In section 1.6.2 I mentioned that if one compares the set of states of the world conceivable by the agent in two states of the world, such that the set of propositions the agent is aware of in the first set is a superset of the set of propositions she is aware of in the second state, one concludes that the set of conceivable states of the world in the first state is a refinement of the set of conceivable states of the world in the second state. The result in the previous proposition has implicitly exactly the same idea. The difference being that we is comparing the set of conceivable states of the world at two different points in time, instead of two different states of the world.

Since $\Phi_t^\omega(\omega) \subset \Phi_{t+1}^\omega(\omega)$, it is obvious that the set of conceivable states in $t + 1$ is finer than the set of conceivable states in $t$. In other words the agent learns more about the set of conceivable states of the world over time.

Before I describe how the possibility correspondence evolves over time let me mention some important aspects about the possibility correspondence in the dynamic model.

I mentioned before that the knowledge and awareness operators are time dependent. As such the possibility correspondence is also time dependent. $P_t(\omega)$ is the set of states of the world which are compatible with what the agent knows at $\omega$ at time $t$. Using
the definition of possibility set and axiom \(A9\) one can prove that \(P_{t+1}(\omega) \subseteq P_t(\omega), \forall \omega \in \Omega\). In other words, the possibility correspondence gets finer and finer over time. This phenomena is common in traditional models of knowledge. The only difference is that in our model the structure of revelation of information may not be known. In other words, although the agent learns more over time about the set of possible states of the world she may not know the sequence of learning.

By the same reasoning as before the agent’s possibility correspondence is also time dependent. Let us call \(\Gamma_{\omega, t, \tau}\) the agent’s possibility correspondence at state \(\omega\) and time \(t\), corresponding to time \(\tau\). \(\Gamma_{\omega, t, \tau}\) specifies for each state of the world conceivable by the agent at time \(t\) in state \(\omega\), which conceivable states of the world does the agent consider possible when she is in a given conceivable state, at time \(\tau\). In summary \(\Gamma_{\omega, t, \tau}\) is a correspondence defined from the set of states conceivable in state \(\omega\) at time \(t\) to the set of subsets of such states.

Since the states of the world conceivable by the agent satisfy the axioms of the model \(\Gamma_{\omega, t, \tau}\) is coarser than \(\Gamma_{\omega, t, \tau+1}\). We notice that in this statement we are fixing the point in time in which the agent is conceiving the set of states of the world (time \(t\)) and what we are comparing is the possibility sets, in some conceivable state, at time \(\tau\) and at time \(\tau+1\). The reason why the relationship is as mentioned is the same than the reason why \(P_{t+1}(\omega) \subseteq P_t(\omega)\).

The question of how the possibility correspondence of the agent changes over time can be phrased as follows: What is the relationship between \(\Gamma_{\omega, t, \tau}\) and \(\Gamma_{\omega, t+1, \tau}\)? In words how does the possibility correspondence corresponding to time \(\tau\) differ if the agent conceives of it at time \(t\) or at time \(t+1\)?

In order to make the comparison one needs to express both correspondences in the same space. Since, \(\Omega^b(\Phi_{t+1}(\omega))\) is a refinement of \(\Omega^b(\Phi_t(\omega))\), if we project \(\Gamma_{\omega, t+1, \tau}\) in \(\Omega^b(\Phi_t(\omega))\) we would loose information. Therefore the right thing approach is to apply the refinement mapping defined in the proof of proposition 8 to \(\Gamma_{\omega, t, \tau}\) and compare its image with \(\Gamma_{\omega, t+1, \tau}\). Let us call \(\tilde{\Gamma}_{\omega, t, \tau}\) the image of the possibility correspondence as
conceivable in period \( t \) in \( \Omega^b(\Phi^a_{t+1}(\omega)) \). One gets the following:

**Proposition 9** i) \( \hat{\Gamma}_{\omega,t,\tau} \) is coarser than \( \Gamma_{\omega,t+1,\tau} \). I.e., for any \( s^b_{\omega,t+1} \in \Omega^b(\Phi^a_{t+1}(\omega)) \) one has \( \Gamma_{\omega,t+1,\tau}(s^b_{\omega,t+1}) \subset \hat{\Gamma}_{\omega,t,\tau}[\Pi(\{s^b_{\omega,t+1}\})] \).

ii) If \( \Phi^a_{t}(s^b_{\omega,t+1}) \subset \Phi^a_{\tau}(\omega) \) then \( \Gamma_{\omega,t+1,\tau}(s^b_{\omega,t+1}) = \hat{\Gamma}_{\omega,t,\tau}[\Pi(\{s^b_{\omega,t+1}\})] \).

**Proof**: The argument is the same than in proposition 6. Q.E.D.

One should stress that this result is not about the agent learning more about the possible states of the world over time. This result is about how the agent's view of the information structure changes over time. What the proposition says is that the agent learns more about the information structure as time passes. Hence this is a result about learning about how the world is.

The second part of the proposition states that if the conceivable state is such that its awareness set at time \( \tau \) is contained in \( \Phi^a_{\tau}(\omega) \) (which means that the agent can replicate at time \( t \) what would be her reasoning if that conceivable state had occurred and she was at time \( \tau \)) then the agent's description at time \( t \) of the possibility set in that conceivable state at time \( \tau \), is consistent with the description she would make at time \( t+1 \).

In summary, the set of states of the world conceivable by the agent and her possibility correspondence get finer over time. The agent learns about how the world is as time passes.

### 1.8 Final Comments

The new element in the epistemic model presented in this paper is the distinction between knowing the existence of a proposition and knowing the logical value of a proposition.

A conclusion from this paper is that the states of the world as viewed by the agent, if she is not aware of all the existing propositions, are not a complete description of all that can happen. In a sense they are "events", instead of states of nature.
Although my model says nothing about probabilities I would like to note an implication of the idea that the agent’s model is incomplete - the agent cannot have a prior probability on the states of nature as viewed by the modeler. Even if the agent is Bayesian and has a probability distribution on \( \Omega(\Phi_t) \), when translated in the modeler’s universe what one has is a probability measure only on the “events” the agent can express about. The two kinds of learning: learning more about which state of the world can possibly be true and learning more about how the world can be, will imply that conditionalization will be important but besides that one also has to consider the specialization of the probability measure.

This work doesn’t address the issue of decision making when the agent has an incomplete view of the world. The study of choice in the presence of unforeseen contingencies is a natural extension of the current paper and I hope to pursue it in future work. An obvious implication of the agent having an incomplete view of the world is that she cannot have a complete contingent plan of action, her plan cannot be contingent on anything she is not aware of. Furthermore one expects the agent’s plan of action to be adapted when she becomes aware of more things. Her plan of action will get more and more complete over time.
Bibliography


Chapter 2

Predation and Reputation
Acquisition in the Debt Market

2.1 Introduction

The literature on industrial organization contains a host of arguments on the practice of predation. The most common models are either based on asymmetric or/and incomplete information in the product market (signaling and reputation models) or based on some sort of capital market imperfection.

A common feature of the models based on capital market imperfections is that they stress some difference in the "financial vulnerability" of the incumbent and the entrant.

A first subset of these models simply assumes that the entrant is financially constrained while the incumbent is not. This idea was first proposed by Telser [13], who argues that predation would not be observed as a consequence of existing financial constraints. The point is that if the entrant is financially constrained and the incumbent knows it, it will pay to the incumbent to prey till the entrant reaches her financial upper-bound and exits. However the entrant anticipates this and consequently does not enter. A formal model of this argument was developed by Benoit [2]. Benoit [3] extends his
previous model to the case where the incumbent has incomplete information about the entrant. In this case he shows that entry can occur and predation can be observed in the equilibrium path, contrarily to the result of the complete information model.

Another subset of the models based on capital market imperfections assumes that the incumbent and the entrant have different financial structures: the incumbent finances totally with equity while the entrant finances (at least partially) with debt. The use of debt makes the entrant vulnerable to predation. Examples of this type of idea are Fudenberg and Tirole [7], Bolton and Scharfstein [4] and Poitevin [11].

In this essay I explore an additional reason why the incumbent and the entrant may have different financial vulnerability. Let us assume that both the incumbent and the entrant finance with debt. However while the incumbent has an established reputation in the debt market the entrant’s quality is not known in the debt market. As a consequence while the incumbent is able to get financing at a fair interest rate a good quality entrant is pooled with bad quality ones thus facing an higher cost of financing than if her quality was known. It is even possible that the entrant does not get financing as a consequence of the asymmetry of information in the debt market.

Diamond [5] developed an adverse selection/moral hazard model which explains the reputation acquisition in the debt market. Ignoring the moral hazard component of his model one could describe his argument as follows: When a new firm enters in the debt market her riskiness/quality is not known with full accuracy by the investors, there is some asymmetry of information in the debt market. The pooling of good and bad risk/quality firms increases the cost of financing for the good ones. However the asymmetry of information decreases over time, whenever more information about the firm is obtained. Hence the credit history of a firm is an important variable in determining whether the firm is able or not to get financing and, her cost of financing. A firm with a long record of no defaults in repayments will be able to get financing at a lower interest rate than when she first entered the market. On the other hand if a firm has a history of defaults, her cost of financing will increase, eventually reaching a point where she is
not able to get further financing. A firm who has a good reputation in the debt market will be able to get financing at a lower interest rate than a firm unknown to the market.

I will use a transformed version of Diamond's model, in combination with a product market, to explore the question of whether a competitor (the incumbent) of a new firm (the entrant) might benefit from engaging in activities which somehow change the information about the entrant received by the lenders, and as a consequence interfere with "reputation acquisition" of the entrant. The incumbent uses predation, which has a certain cost, to change the information about the entrant received by the lenders.

There are two sorts of questions which can be explored in this framework. One relates to the influence of predation on the reputation acquisition in the debt market. For example: How does predation influence the process of learning the quality of the entrant? Is it easier or more difficult to get financing when there exists the possibility of predation? The other set of questions refers to the influence of the reputation acquisition in the debt market on the incentive of the incumbent to prey. In particular, how does the incentive to prey change over time as more information exists about the entrant. If one believes in the "reputation acquisition" story one expects predation to occur while the entrant is still building her reputation.

In this model predation changes the quality of the signals received by the lenders. We analyze a case where predation makes default and no default more informative signals about the quality of the entrant, a case where predation makes these signals less informative of the quality of the entrant, and a mixed case.

We show that the incumbent may gain by engaging in predation and that the entrant is less likely to get financing when predation is profitable. The optimal pattern of predation may involve periods of no predation followed by periods of predation. In other words, it is not generally true that if predation is observed in a particular period than it is observed in all previous periods. Predation seems to be likely when the entrant is in bad shape, in the sense that if the entrant has another round of bad luck (defaults) she will not get further lending.
Besides the predation literature this work is related with the issue of manipulation of information. Consider any model where there is some uncertainty about some characteristic of a firm (e.g., quality of the product, cost structure), and where a competitor can influence the signals received by a third agent (e.g.; lenders, consumers), then the question arises whether the competitor may benefit from manipulating the agent’s information\(^1\).

The chapter begins with a description of the models and its assumptions. In the following section I explain the reputation acquisition story by analyzing the debt market game. I proceed with the study of the interaction of predation and reputation acquisition and explore through examples the question of the optimal pattern of predation.

### 2.2 The Model

#### 2.2.1 Description

The model has two blocks: One describes the way firms interact with the debt market, the other refers to the product market.

The first block is a transformed version of the model used by Diamond (1989) [5] in his paper “Reputation Acquisition in the Debt Market”. While Diamond uses an *adverse selection/moral hazard* model to explain the reputation acquisition in the debt market, this version ignores the moral hazard story of Diamond’s model. The basic idea of this model is that there is *adverse selection* in the debt market. The lenders do not know the “riskiness” of the firms. The pooling of the good and the bad firms increases the cost of

\(^1\)The literature on *pioneering brands* (Schmalensee [12], Bagwell [1]) is a good example. There the incumbent’s quality is already known, while the entrant’s is not. The incumbent can affect the learning process of the consumers by changing his price. In the existing literature this happens because the consumer might not try the entrant’s product. In these papers it is assumed that the product’s quality is known with certainty once the product is tried one period. The analogy with the model presented here is more obvious if one assumed that learning the exact quality of the product takes some time.

One important difference between this model and the literature on pioneering brands is that in this model the incumbent changes the quality of the information received by the lenders, but the incumbent does not change the information received by consumers in models on pioneering brands.
financing for the good firms.

In the product market block there are two firms. One is an entrant, a young firm which has no reputation in the debt market. The other one is an incumbent, well known by the investors, who can get financing at a fair interest rate. The entrant quality is not known either by the debt market or by the incumbent. She might be good $G$, or bad $B$. The incumbent can prey at a cost $c$. The effect of predation on the entrant's profit depends on the quality of the entrant.

### 2.2.2 Assumptions of The Model

The model has a long list of assumptions, the most part of which are inherited from Diamond's model.

1. There are $T$ time periods; $T$ is finite.

2. All the agents (lenders, entrant, incumbent) are risk neutral.

3. The entrant may be good, $G$, with prior probability $\theta_0$, or bad, $B$, with probability $(1 - \theta_0)$. The entrant's type is private information. The entrant profit is stochastic. It can either be high, $\pi^H$, or low, $\pi^L$, with $\pi^H > 0$ and $\pi^L = 0$. The probability of high profit is $\mu_G$ for the good firm and $\mu_B$ for the bad one, with $\mu_G > \mu_B$.

In summary, the probability distribution of the profit for each type of entrant is:

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<td>$B$</td>
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4. The entrant needs to get financing. She has not enough of her own funds to finance the project.

5. The incumbent might prey or not prey. Predation has a cost, $c$. Predation is observed by the lenders, but cannot be verified. Hence, no debt contracts contingent on predation can be made.
Predation works by decreasing the probability of high profits for the entrant. The effect of predation is not necessarily the same for good and bad entrants. Let \( \lambda_G \) and \( \lambda_B \), \( 0 \leq \lambda_i \leq 1 \), \( i = G, B \), be the predation vulnerability parameters for the good and bad firm, respectively. The probability of high profits if the incumbent preys is \( \lambda_i \mu_i \), \( i = G, B \). Hence the closer \( \lambda_i \) is to 1 the lower is the effect of predation. In summary, the distribution of profit, when the incumbent preys is:

<table>
<thead>
<tr>
<th></th>
<th>( \pi^H )</th>
<th>( \pi^L )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>G</strong></td>
<td>( \lambda_G \mu_G )</td>
<td>( 1 - \lambda_G \mu_G )</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>( \lambda_B \mu_B )</td>
<td>( 1 - \lambda_B \mu_B )</td>
</tr>
</tbody>
</table>

6. Inputs are endowed to each lender at the beginning of each period. These can be used to lend or be stored for consumption at the end of the period. Storage has a sure gross return of \( \bar{r} \) (\( \bar{r} \) is one plus the interest rate). Projects are in short supply, as such the storage technology is in use, and borrowers can offer just the rate of return of the storage. Each lender lives only a period \(^2\). Each loan is \( \$ \) 1, that is also the scale of borrowers project.

7. There is no commitment technology for the lenders. Borrowers can commit to use a liquidation technology conditional on some payments to the lenders and not to use it given other payments.

The liquidation technology allows the output to be destroyed before the borrower consumes it. This assumption is crucial - liquidation works as an enforcement mechanism.

8. Entrant’s profits are not observable by the lenders and the incumbent. Hence no profit contingent contracts can be made\(^3\).

---

\(^2\)This assumption is intended to model lenders as an anonymous capital market rather than a financial institution.

\(^3\)The assumption that profits are not observable is a strong one. I am now studying a model where profits are verifiable: profits can either be low or high, the probability of getting low or high profits
9. It is assumed that $\mu_B \pi^H < \bar{\pi}$, and $\mu_G \pi^H > \bar{\pi}$.

10. Each borrower's track record of repayment or default is observed by current lenders. Moreover it is also known whether the entrant was or not preyed upon.

11. Profits are distributed in each period. There are no retained earnings.

Assumption 5 implies that under predation the relative likelihood of being a good or a bad entrant, given that default/ non-default is observed, is changed. In addition the assumption implies that the expected profit of the entrant decreases when the incumbent preys. In other words, predation not only affects the quality of information about the entrant as received by the lenders, it also affects negatively the expected profit of the entrant.

Although this seems to be a realistic scenario, it can be argued that in order to focus on the informational role of predation one should consider an alternative assumption: predation changes the spread of the profit distributions but not their mean. A simple way to formalize this idea would be to assume that $\pi^H$ increases if the incumbent preys, in such a way that expected profits remain the same as without predation. The level of high profit under predation would be: $\pi^H_i = \pi^H_i \frac{1}{\lambda_i}$, $i = G, B$. For future reference let me call this assumption $5'$.

Throughout this essay I use assumption 5 because it seems to be a more natural assumption and it does not change the most of the results presented here. However, I will mention the cases where considering the alternative hypothesis could change the results.

Since the value of profit observed under liquidation is always zero, assumption 7 implies that liquidation cannot be used to get information about the profit of the entrant.
Liquidation is useful only to provide incentives for repayment. This assumption is crucial when the support of the profit distribution for the *good* and *bad* entrants is not the same, as under assumption 5'. In this case if the liquidation technology made the realized profits observable, liquidation could be useful in determining the type of the entrant.

If the support of the profit distributions is the same for *good* and *bad* entrants, as under assumption 5, an alternative assumption could be used. We could allow for non-pecuniary penalties which are contingent on the observed repayment as in Diamond (1984) [6].

### 2.3 The Debt Market Game

I will first revise the implications of having *adverse selection* in the debt market, ignoring predation.

The sequence of moves in each time period is the following:

<table>
<thead>
<tr>
<th>profits are realized and loans repaid</th>
</tr>
</thead>
<tbody>
<tr>
<td>borrowers offer contract</td>
</tr>
<tr>
<td>lenders accept or not</td>
</tr>
</tbody>
</table>

Since the profit is unobservable, no financing contract can be written contingent on profits (like equity contracts).

In this model, the idea is to use the liquidation technology to give incentives for repayment: The debt contract in period \( t \) specifies a repayment of \( r_t \) for each loan of one unit. If repayment is made no liquidation occurs, otherwise there is liquidation\(^4\).

\(^4\)The optimality of this contract can be derived formally. Let \( z \) be the repayment made at the end of the period by the entrant. Let \( L(z) \) be the "liquidation indicator function" which can take the values zero or one. Let the profit be the random variable \( \hat{x} \). Then the optimal contract solves the following problem: \( \text{MAX}_{L(z)} E[\hat{x} - L(z)\hat{x} - z] \) subject to \( z \in \text{argmax}_{z \in [0, \pi]} \{\pi - L(z)\pi - z\} \) and \( E\{\text{argmax}_{z \in [0, \pi]} \{\pi - L(z)\pi - z\} \geq \hat{r} \). The solution to this problem is: \( L(z) = 0 \) if \( z \geq r \), \( L(z) = 1 \) if \( z < r \), where \( r \) is the lowest number satisfying \( P(\hat{x} \geq r) r \geq \hat{r} \).
With this debt contract it is never in the interest of the borrower to default when her profit is high, as long as, the repayment amount is less than $\pi^H$. This is so because by repaying the borrower gets a positive net profit, $\pi^H - r_t$, while by defaulting she gets zero, given the assumptions about the liquidation technology\(^5\).

If the profit is low, the borrower has to default, since profits are not enough to repay the debt.

The lenders know that the borrower will default if the profit is low or if the promised repayment is above $\pi^H$.

### 2.3.1 One Period Model / The Last Period

In this section I discuss the results of a one period model. The formulae presented are written as if $T = 1$. However, the analysis is fully valid for the last period of a model with a longer horizon - one just needs to substitute $\theta_0$ by $\theta_{T-1}$, in other words one needs to use the prior beliefs when the last period decisions are made.

Let $r_1$ be the repayment offered in the contract by the entrant. The expected gross rate of return of lending to an entrant in period 1 is:

$$
\begin{cases}
    r_1 \left( P(G/r_1) \mu_G + (1 - P(G/r_1)) \mu_B \right) & \text{if } r_1 \leq \pi^H \\
    0 & \text{otherwise}
\end{cases}
$$

where $P(G/r_1)$ is the updated belief that the entrant is good, given that she offers $r_1$. The decision to lend is based on the comparison of the expected gross rate of return of lending to an entrant with the storage rate of return. They will lend if:

$$r_1 \left( P(G/r_1) \mu_G + (1 - P(G/r_1)) \mu_B \right) \geq \bar{r}, \text{ and } r_1 \leq \pi^H.$$

The contract choice cannot separate good and bad entrants. Any payment which is

\(^5\)I am ignoring the possibility of renegotiation. If renegotiation was possible lending would not occur in this model.
feasible for the good entrant is also feasible for the bad entrant. Hence the bad entrant can always mimic the strategy of the good entrant. Since a separating equilibrium would imply no financing for the bad entrant, she is always better off by imitating the good entrant.

On the other hand there are many “pooling” equilibria of the game. Under pooling the equilibrium path beliefs are $P(G/r_1) = \theta_0$. Thus if the offered interest rate, $r_1$, is such that the lending condition stated above is satisfied when $P(G/r_1)$ equals the prior beliefs, the equilibrium strategy of the lenders is to lend. Let $r_i^*$ be the lowest interest rate which is acceptable by the lenders under “pooling”:

$$r_i^* = \frac{\bar{r}}{\theta_0 \mu_G + (1 - \theta_0) \mu_B} \quad (2.1)$$

Any interest rate in the interval $[r_i^*, \pi^H]$ can be supported as a “pooling” equilibria$^6$.

I argue that the most reasonable of these equilibria involves all the entrants offering $r_i^*$, and the lenders policy is to lend if the offered interest rate is in $[r_i^*, \pi^H]$ and not to lend otherwise. This equilibrium is supported by the lender’s belief that the interest rate offered by the borrower tells nothing about the type of the borrower. Given this belief no entrant can gain by deviating from the mentioned equilibrium$^7$.

The entrants “pooling” at the lowest rate of return acceptable by the lenders is the only equilibrium that satisfies the definition of perfect sequential equilibria proposed by Grossman and Perry [10]. Consider any equilibrium interest rate above $r_i^*$. Both types would prefer to offer a lower interest rate, if the beliefs of the lender were such that lending is the best response. Hence the only “consistent” belief is that a lower interest rate tells nothing about the type of the borrower. However under these belief both types would deviate from the proposed equilibrium. The Grossman and Perry criteria puts no

---

$^6$Consider any interest rate in that interval. Assume that the lender believes that if any other interest rate is offered, then the borrower is bad. Under this belief no borrower would gain by deviating.

$^7$Offering an interest rate lower than $r_i^*$ will lead to no lending and offering an interest rate higher than $r_i^*$ has a higher cost. The lender’s belief also satisfies the Cho-Kreps intuitive criteria. However the intuitive criteria also holds in the other equilibria.
restrictions on the beliefs of the lender if the offered interest rate is above the equilibrium one, since in that case no type would wish to deviate.

Let me summarize the results of the one period transformed Diamond’s model in the following lemma:

**Lemma 4** In a one period model, the following is an equilibrium outcome: Both types of entrants offer a debt contract with the lowest interest rate which provides an expected gross return of $\bar{\pi}$, i.e.:

$$r_1 = \frac{\bar{\pi}}{\theta_0 \mu_G + (1 - \theta_0) \mu_B}$$  \hspace{1cm} (2.2)

The lenders will accept the contract provided the offered gross rate of return is less than $\pi^H$. In other words, the lenders will accept if the prior probability of the entrant being good is sufficiently high:

$$\theta_0 \geq \frac{\bar{\pi} - \mu_B \pi^H}{(\mu_G - \mu_B) \pi^H}$$  \hspace{1cm} (2.3)

For future reference, let us call $\theta^*$ the value of the prior beliefs for which the inequality (2.3) holds with equality.

The main observation to make about the lending condition is that for a good entrant the cost of financing is higher than if there was perfect information. On the other hand, a bad entrant might be able to get financing, something that would not happen under perfect information, by assumption 9.

### 2.3.2 Finite Number of Periods

The analysis of a longer horizon does not differ very much from the one verified in the one period model. The reason is that, the repayment policy followed by the borrower is exactly the same.

The lenders anticipate that behavior and, as a consequence, their lending decision follows the same criteria (given the prior beliefs, in each period, about the quality of the entrant). Finally the borrower, correctly anticipates the lending rule and offers the
minimum rate accepted by the lenders\(^8\).

The evolution of the interest rates depends on the history. At each point of time the beliefs about the entrant's type are updated according to Bayes rule. Let \( h_t \) be the history up to period \( t \), which is constituted by the series of default/non-default. Let \( \theta_t \) be the probability of the entrant being a good firm, after \( t \) periods of history, i.e. \( \theta_t \) is the posterior belief at \( t \) and the prior belief at \( (t+1) \). Obviously \( \theta_t \) depends on the credit history of the borrower, the series of defaults and non-defaults\(^9\).

**Lemma 5** Let \( n \) be the number of defaults observed before period \( t \). The gross rate of return offered by both types of entrants in period \( t \) is:

\[
\tau_t = \frac{\theta_0(\mu_G)^{t-1-n}(1 - \mu_G)^n + (1 - \theta_0)(\mu_B)^{t-1-n}(1 - \mu_B)^n}{\theta_0(\mu_G)^{t-1-n}(1 - \mu_G)^n + (1 - \theta_0)(\mu_B)^{t-1-n}(1 - \mu_B)^n} \bar{\tau} \tag{2.4}
\]

**Proof:** If \( n \) defaults (and \( t - 1 - n \) non-defaults) were observed before period \( t \), the prior beliefs at period \( t \), \( \theta_{t-1} \), are:

\[
P[G|h_{t-1}] = \frac{\theta_0(\mu_G)^{t-1-n}(1 - \mu_G)^n}{\theta_0(\mu_G)^{t-1-n}(1 - \mu_G)^n + (1 - \theta_0)(\mu_B)^{t-1-n}(1 - \mu_B)^n} \tag{2.5}
\]

The lending condition in period \( t \) is:

\[
\tau_t = \frac{\bar{\tau}}{\theta_{t-1}\mu_G + (1 - \theta_{t-1})\mu_B} \tag{2.6}
\]

Substituting the expression (2.5) on (2.6) one gets the result. **Q.E.D.**

When the number of periods is very large, the lenders may get to know the quality of the entrant.

\(^8\)The supporting beliefs are the same as above. Lenders believe that the interest rate offered at the beginning of each period reveals nothing about the entrant's type.

\(^9\)The lenders observe the credit history of the borrower. The only fulfilling beliefs of the lenders are the following: If there was a default in a given period, that was because the entrant was constrained to do so.

Given this belief, if the entrant's profit is high in a given period she will avoid default, since by defaulting she loses both in the current and in the future periods.
Lemma 6 If the entrant is good, provided that at no point in time the posterior beliefs drop below \( \theta^* \), when \( t \to \infty \), the proportion of non-defaults converges to \( \mu_G \), the posterior beliefs converge to 1 and, the gross rate of return converges to \( r_t = \frac{\bar{r}}{\mu_G} \).

Proof: The lemma is just a consequence of the central limit theorem. As long as lending never stops (posterior beliefs do not drop below \( \theta^* \)), the history about the entrant will get larger and larger. But, by the CLT, when the sample gets very large, the observed proportion of non-defaults will tend to the population proportion. As such, the lenders can identify the quality of the entrant with great accuracy. The interest rate converges to the fair interest rate of a good entrant under perfect information. Q.E.D.

If it happens that a good entrant never has a negative profit (\( \mu_G = 1 \))\(^{10}\), the rate of return tends to the storage rate of return, \( \bar{r} \).

2.4 The Complete Game

The timing of the model in each period is the following: (1) The entrant offers a debt contract to the lenders, (2) the lenders accept or reject, (3) The incumbent decides whether to prey or not and, (4) Profits are realized and the entrant decides how much to repay to the lenders.

<table>
<thead>
<tr>
<th>entrant offers contract</th>
<th>lenders accept or not</th>
<th>Incumbent preys or not</th>
<th>profits are realized and loans repaid</th>
</tr>
</thead>
</table>

The assumptions that \( \mu_B \pi^H < \bar{r} \) and that \( \mu_G \pi^H > \bar{r} \) are maintained. Hence with

\(^{10}\)This is what Diamond assumes in his model.
perfect information only the good entrant would get financing. Furthermore it is assumed that $\lambda_G \mu_G \pi^H > \bar{r}$; i.e., the good entrant, even if preyed upon continues to be profitable.

As usual the game has to be solved backwards:

1. The entrant decides how much to repay to the lender, depending on the realized profits.
2. The incumbent decides whether to prey or not to prey contingent on his beliefs about the quality of the entrant and lending decision and, anticipating the entrant repayment behavior.
3. The lending decision is based on the comparison of the expected gross rate of return resulting from the contract offered by the entrant and the storage technology. The lenders have rational beliefs about the behavior of the incumbent and the entrant.
4. Finally, the borrowers offer the lowest interest rate acceptable by the lenders.

The optimal repayment policy in this model is exactly the same as the one before: the entrant pays $r_t$ as long as the realized profit exceeds it, and defaults otherwise.

The lending rule follows the same logic than in the previous section model. The beliefs of the lenders and the incumbent about the quality of the entrant as a function of the offered interest rate are as before. The only difference now is that lenders have to think if the incumbent will prey or not. Hence, the critical value of $\theta_t$ above which they lend depends on whether they anticipate predation or not.

Before I analyze the model closely for diverse time periods, I am going to make some remarks about the informational role of predation and about how beliefs change with history.

### 2.4.1 The Informational Role of Predation

There are three issues to take into account when deciding whether to prey or not: The first one is the cost of preying, the second is that predation implies an increase in the
probability of default, the third (and most interesting) is the alteration of the inference process about the quality of the borrowers.

With predation, the relative likelihood of being a good or a bad entrant, given that default/non-default was observed, is changed. Consequently predation has an informational role. The question is - how does predation change the relative likelihoods? The answer to this question depends on the parameters $\mu_G, \mu_B, \lambda_G, \lambda_B$. There are three possible cases:

- Predation makes default/non-default stronger signals; i.e., non-default is stronger evidence that the firm is good and default a stronger evidence the firm is bad. I will call this the informative case.

The informative case holds if:

\[
\begin{align*}
(i) & \quad \lambda_G > \lambda_B \\
(ii) & \quad \frac{1-\lambda_G}{1-\mu_G} < \frac{1-\lambda_B}{1-\mu_B}
\end{align*}
\]

Condition (i) is the condition for non-default under predation to be a stronger signal that the firm is good, condition (ii) is the condition for default under predation to be a stronger evidence that the entrant is bad than if predation had not occurred.

Notice that (ii) implies (i), hence it is enough to refer to condition (ii)\(^{11}\).

- Predation makes default/non-default worse signals - The disinformative case. This happens if $\lambda_G \leq \lambda_B$. In other words, if the good firm is more vulnerable to predation than the bad firm, predation plays a disinformative role.

\(^{11}\)The fact the (ii) $\rightarrow$ (i) means that we cannot have a case where predation makes, simultaneously, non-default a weaker signal of being good and default a stronger signal of being bad. That is why one has only three cases, not the four combinations of weaker/stronger.
- Predation makes non-default a stronger signal that the firm is good, and default a weaker signal that the firm is bad. This will be the case if:

\[
\begin{align*}
(i) & \quad \lambda_G > \lambda_B \\
(ii) & \quad \frac{1-\lambda_G\mu_G}{1-\mu_G} > \frac{1-\lambda_B\mu_B}{1-\mu_B}
\end{align*}
\]

This case is, in my opinion, quite intuitive. On one hand no default in the presence of predation is a very good evidence of being good (in spite of predation the entrant did not default), on the other side defaulting is not a so strong indication that the firm is bad (maybe it was just because of predation).

### 2.4.2 History

If lending doesn’t occur in a certain period, no inference about the entrant can be made. Hence the prior beliefs don’t change. If lending occurs in a certain period, the history in that period is the observation whether default and predation occurred or not.

At any point of time the beliefs about the quality of the entrant depend on the number of times each pair: predation(yes/no)-default(yes/no) was observed. Let us call \( p_i \) the indicator function which takes the value one if predation was observed in period \( i \) and zero otherwise, and let us define \( d_i \) similarly for default. Then the value of \( \theta_{t-1} \) is:

\[
P[G|h_{t-1}] = \frac{\theta_0 \prod_{i=1}^{t-1} A_i}{\theta_0 \prod_{i=1}^{t-1} A_i + (1-\theta_0) \prod_{i=1}^{t-1} B_i}
\]

where \( A_i \) and \( B_i \) are:

\[
A_i = (\lambda_G\mu_G)^{(1-d_i)p_i}(1 - \lambda_G\mu_G)^{p_id_i}(\mu_G)^{(1-p_i)(1-d_i)}(1 - \mu_G)^{(1-p_i)d_i}
\]

\[
B_i = (\lambda_B\mu_B)^{(1-d_i)p_i}(1 - \lambda_B\mu_B)^{p_id_i}(\mu_B)^{(1-p_i)(1-d_i)}(1 - \mu_B)^{(1-p_i)d_i}
\]
if lending occurred at time \( i \), and are equal to 1 otherwise. As in the debt market model, these beliefs are the only ones which are self-fulfilling.

### 2.4.3 One period Model

Let \( \pi^I \) be the profit of the incumbent when the entrant is in the market and \( \pi^M \) be the profit when the incumbent is a monopolist. In a one period model the incumbent has certainly no interest in preying since he gets \( \pi^I \) by not preying and \( (\pi^I - c) \) by preying. In other words preying is a dominated strategy in a one period model, the idea is that there is no future hence the incumbent has no benefits, only the cost of preying.

The lenders know that the incumbent will not prey. Their lending decision is based on the probability of non-default with no predation. *Adverse selection* in the debt market is the only phenomena at work, lending occurs if:

\[
r_1 \geq \frac{\bar{r}}{\theta_0 \mu_G + (1- \theta_0) \mu_B} \quad \text{and} \quad r_1 \leq \pi^H
\]

### 2.4.4 Two period model

One needs to start reasoning from the second period. In the second period, only one period is left, the incumbent has no incentive to prey and the lenders anticipate that correctly. Given the beliefs at the beginning of period two, \( \theta_1 \), the gross rate of return is:

\[
r_2 = \frac{\bar{r}}{\theta_1 \mu_G + (1- \theta_1) \mu_B}, \quad \text{provided} \quad r_2 < \pi^H.
\]

The value of \( \theta_1 \) is contingent on the history in the first period. Using the formula for
the updating of beliefs, it is easy to derive the values of the gross rate of return in the second period if lending occurred in the first period:

\[ d_1 = 0 \quad d_1 = 1 \]

\[ p_1 = 0 \quad \frac{\theta_0 \mu_G + (1 - \theta_0) \mu_B}{\theta_0 \mu_G^2 + (1 - \theta_0) \mu_B^2} \tilde{r} \quad \frac{\theta_0 (1 - \mu_G) + (1 - \theta_0) (1 - \mu_B)}{\theta_0 \mu_G (1 - \mu_G) + (1 - \theta_0) \mu_B (1 - \mu_B)} \tilde{r} \]

\[ p_1 = 1 \quad \frac{\theta_0 \lambda_G \mu_G + (1 - \theta_0) \lambda_B \mu_B}{\theta_0 \lambda_G \mu_G^2 + (1 - \theta_0) \lambda_B \mu_B^2} \tilde{r} \quad \frac{\theta_0 (1 - \lambda_G \mu_G) + (1 - \theta_0) (1 - \lambda_B \mu_B)}{\theta_0 \lambda_G \mu_G (1 - \lambda_G \mu_G) + (1 - \theta_0) \lambda_B \mu_B (1 - \lambda_B \mu_B)} \tilde{r} \]

Let us call \( r_{2,(i,j)} \) the gross rate of return in period two, when history in period one takes the values \((i, j)\) for predation and default, respectively. Similarly define the posterior beliefs in period one \( \theta_{1,(i,j)} \).

It is instructive to analyze the relationship among the four possible values of \( r_2 \) under each of the informational cases.

**Lemma 7** In the informative case one has:

\[ r_{2,(1,0)} < r_{2,(0,0)} < r_1 < r_{2,(0,1)} < r_{2,(1,1)} \]

In the disinformative case:

\[ r_{2,(0,0)} < r_{2,(1,0)} < r_1 < r_{2,(1,1)} < r_{2,(0,1)} \]

Finally, in the mixed case:

\[ r_{2,(1,0)} < r_{2,(0,0)} < r_1 < r_{2,(1,1)} < r_{2,(0,1)} \]
Proof: I will prove the relation between \( r_{2,(1,0)} \) and \( r_{2,(0,0)} \). The other proofs are similar.

First divide both numerator and denominator in \( r_{2,(1,0)} \) by \( \mu_G \lambda_G \), and in \( r_{2,(0,0)} \) by \( \mu_G \). One gets the following:

\[
\begin{align*}
    r_{2,(1,0)} &= \frac{\theta_0 + (1 - \theta_0) \frac{\lambda_B}{\lambda_G} \frac{\mu_B}{\mu_G}}{\theta_0 \mu_G + (1 - \theta_0) \frac{\lambda_B}{\lambda_G} \frac{\mu_B}{\mu_G}} \tilde{r} \\
    r_{2,(0,0)} &= \frac{\theta_0 + (1 - \theta_0) \frac{\mu_B}{\mu_G}}{\theta_0 \mu_G + (1 - \theta_0) \frac{\mu_B}{\mu_G}} \tilde{r}
\end{align*}
\]

The two expressions look similar, except for the second terms in both the numerator and denominator. These terms have a product by \( \frac{\lambda_B}{\lambda_G} \) in \( r_{2,(1,0)} \), but not in \( r_{2,(0,0)} \). In particular, if that ratio is 1 the two interest rates are exactly the same.

Taking all the parameters in \( r_{2,(1,0)} \) as fixed but the ratio \( \frac{\lambda_B}{\lambda_G} \) and computing the derivative of \( r_{2,(1,0)} \) with respect to the ratio \( \frac{\lambda_B}{\lambda_G} \) one finds that the derivative is positive as long as \( \mu_G > \mu_B \), which is always true in this model.

Consequently, if \( \frac{\lambda_B}{\lambda_G} < 1 \), which happens in the informative and mixed cases, \( r_{2,(1,0)} < r_{2,(0,0)} \). If \( \frac{\lambda_B}{\lambda_G} > 1 \), which happens in the disinformative case, the opposite holds. Q.E.D.

The decision of the incumbent in the first period

How does the incumbent decide whether or not to prey in the first period? The incumbent compares the expected discounted stream of profits (given his beliefs about the future behavior of lenders and entrant) when he preys and when he does not prey in the first period and, he chooses the action with the highest expected value.

The description of the optimal strategy of the incumbent should be, by definition of strategy, a complete plan of action contingent on the prior beliefs and the lenders decision.
When no lending occurs the incumbent does not have the move, because he will be a monopolist, so he cannot prey. When lending occurs the optimal action depends on the prior probability of the entrant being good and, on the values of the other parameters of the model.

Now suppose the prior beliefs are such that the lenders should not lend, even in the absence of predation, but yet the lenders do lend (by mistake). In this case the incumbent would need to be careful, since the posterior beliefs if he preys and the entrant doesn’t default could increase enough to make next period lending attractive (in the informative and mixed cases, no-default increases more $\theta$ when the incumbent preys).

This case, although interesting for completeness, doesn’t need to be evaluated to describe the equilibrium strategy. The reason is that for the lenders, if they ask the question - "what happens if we deviate from the equilibrium strategy and lend when prior beliefs are very low"? The answer is always - "we are worse off, because if we lend when $\theta_0 < \theta^*$ we have an expected loss this period and, next period we are dead anyway!”. In other words, to lend is a dominated strategy when $\theta_0 < \theta^*$.

In conclusion, when the a priori probability of the entrant being good is below $\theta^*$, the equilibrium path story is very short: no lending occurs, the incumbent remains a monopolist, the beliefs about the entrant do not change (no credit history) and, the history repeats...

In what follows, I will only describe the optimal strategy of the incumbent, given that $\theta_0$ is above $\theta^*$.

Predation always changes the probability of default (which increases). Moreover it might change the lending decision, depending on how predation modifies the beliefs.

The lending decision and, as a consequence, the incentive to prey, depend on the comparison of $r_{2,(i,j)}$ with $\pi^H$.

Lemma 8 Suppose that both $r_{2,(0,1)}$ and $r_{2,(1,1)}$ are smaller than $\pi^H$, meaning that even if default is observed (with or without predation) lending still occurs in the last period. Then the incumbent will not prey in the first period.
Proof: It is obvious. No matter what the incumbent does (prey or not prey) the entrant will get financing and be in the market in the next period, hence the future benefit of preying is zero. Since preying has a cost, preying is a dominated strategy under the referred circumstances. Q.E.D.

However, in this two period model, preying may be the best strategy for certain values of the parameter. Consider the case contrary to the one just mentioned:

Lemma 9 Suppose that \( r_{2,0,1}, r_{2,1,1} > \pi^H \); i.e., the parameter are such that when default is observed in the first period (with or without predation) no lending occurs in the next period, the incumbent will prey in the first period as long as:

\[
c \leq \delta[\theta_0 \mu_G (1 - \lambda_G) + (1 - \theta_0)\mu_B(1 - \lambda_B)](\pi^M - \pi^I) \tag{2.7}
\]

Proof: If the incumbent preys in first period his expected profit is:

\[
V(p_1 = 1) = (\pi^I - c) + \delta[\theta_0 \lambda_G \mu_G + (1 - \theta_0)\lambda_B \mu_B]\pi^I + \\
\delta[\theta_0(1 - \lambda_G \mu_G) + (1 - \theta_0)(1 - \lambda_B \mu_B)]\pi^M \tag{2.8}
\]

While if he does not prey it is:

\[
V(p_1 = 0) = \pi^I + \delta[\theta_0 \mu_G + (1 - \theta_0)\mu_B]\pi^I + \\
\delta[\theta_0(1 - \mu_G) + (1 - \theta_0)(1 - \mu_B)]\pi^M \tag{2.9}
\]

In order for preying to be optimal \([V(p_1 = 1) - V(p_1 = 0)]\) has to be positive. Subtracting (2.9) from (2.8):

\[
V(1) - V(0) = -c + \delta[\theta_0 \mu_G (\lambda_G - 1) + (1 - \theta_0)\mu_B(\lambda_B - 1)]\pi^I + \\
\delta[\theta_0 \mu_G (1 - \lambda_G) + (1 - \theta_0)\mu_B(1 - \lambda_B)]\pi^M \tag{2.10}
\]

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Noticing that the first expression within parentheses is the symmetric of the second expression within parentheses, one concludes that (2.10) is positive if and only if (2.7) holds. **Q.E.D.**

The interpretation of the above condition is straightforward. Since default implies no further lending, the incumbent will be a monopolist if default occurs. Hence the expected marginal benefit from predation is given by the increase in the probability of the entrant getting no further lending (which is equal to the increase in the probability of default in this case) times the additional profit the incumbent gets if he becomes a monopolist ($\pi^M - \pi^I$). In other words, equation (2.7) is no more than a comparison of marginal costs with expected marginal benefits.

Interestingly, there is a situation where the incentive to prey is even larger than under the previous case. That happens when $r_{2,1,1} > \pi^H > r_{2,0,1}$, which can happen only in the **informative case**. Under this conditions if default is observed and no predation occurred the entrant gets financing in the second period, however the entrant does not get financing if predation occurred.

**Lemma 10** Suppose $r_{2,1,1} > \pi^H > r_{2,0,1}$; i.e., if default is observed under predation, no lending occurs in the last period. However if the incumbent doesn’t prey and default is observed, lending occurs in the last period. Then predation is the optimal action in the first period if:

$$c \leq \delta[\theta_0(1 - \lambda_G \mu_G) + (1 - \theta_0)(1 - \lambda_B \mu_B)](\pi^M - \pi^I)$$

(2.11)

**Proof:** The expected profits are, with predation:

$$V(p_1 = 1) = (\pi^I - c) + \delta[\theta_0 \lambda_G \mu_G + (1 - \theta_0) \lambda_B \mu_B] \pi^I +$$

$$\delta[\theta_0(1 - \lambda_G \mu_G) + (1 - \theta_0)(1 - \lambda_B \mu_B)] \pi^M$$

(2.12)
Without predation:

\[ V(p_1 = 0) = \pi^I + \delta \pi^I \]  \hspace{1cm} (2.13)

Subtracting (2.13) from (2.12) one concludes that the difference is positive if and only if inequality (2.11) holds. **Q.E.D.**

The interpretation of condition (2.11) is the same than above. The difference is that in this case the probability of the entrant not getting financing in the second period if she defaults in the first period is zero if the incumbent does not prey in the first period. Hence the increase in the probability of the entrant getting no further lending when the incumbent preys is equal to the probability of default under predation.

In the **disinformative** and **mixed cases**, there is a circumstance under which predation might be detrimental for future profits.

**Lemma 11** When \( r_{2,(1,1)} < \pi^H < r_{2,(0,1)} \), the incumbent will never prey. The total loss if he preys is:

\[ c + \delta[\theta_0(1 - \mu_G) + (1 - \theta_0)(1 - \mu_B)](\pi^M - \pi^I) \]  \hspace{1cm} (2.14)

**Proof**: If the incumbent preys he gets:

\[ V(p_1 = 1) = (\pi^I - c) + \delta \pi^I \]  \hspace{1cm} (2.15)

If he does not prey he gets:

\[ V(p_1 = 0) = \pi^I + \delta[\theta_0 \mu_G + (1 - \theta_0)\mu_B] \pi^I + \\
\delta[\theta_0(1 - \mu_G) + (1 - \theta_0)(1 - \mu_B)] \pi^M \]  \hspace{1cm} (2.16)
The difference between (2.16) and (2.15) gives the loss that the incumbent would incur if he preys. The difference is equal to (2.14). Q.E.D.

In this case, predation may change the decision from no lending to lending and hence have a negative impact in future profit.

I summarize the results of this section in the following proposition:

**Proposition 10** If \( \theta_0 > \theta^* \), and lending occurs the optimal action of the incumbent in the first period is:

(i) If \( r_{2, (1,1)} < \pi^H \) do not prey.

(ii) If \( r_{2, (1,1)} \) and \( r_{2, (0,1)} \) are both larger than \( \pi^H \) prey if:

\[
c \leq \delta[\theta_0 \mu_G (1 - \lambda_G) + (1 - \theta_0) \mu_B (1 - \lambda_B)](\pi^M - \pi^I)
\]

(iii) If \( r_{2, (1,1)} > \pi^H > r_{2, (0,1)} \) prey if:

\[
c \leq \delta[\theta_0 (1 - \lambda_G \mu_G) + (1 - \theta_0)(1 - \lambda_B \mu_B)](\pi^M - \pi^I)
\]

**Proof**: Use lemmas 8-11. Q.E.D.

This proposition can be rewritten by defining the several cases as a function of the prior beliefs.

Let us call \( \theta' \) and \( \theta'' \) the critical values of the prior beliefs, \( \theta_0 \), for which \( r_{2, (1,1)} \) and \( r_{2, (0,1)} \) are equal to \( \pi^H \), respectively. Or equivalently, the values of the prior beliefs such that the posterior beliefs, \( \theta_{1, (1,1)} \) and \( \theta_{1, (0,1)} \), are equal to \( \theta^* \).

\( \theta' \) and \( \theta'' \) are the maximum value that \( \theta_0 \) can have so that default in period one (with and without predation, respectively) implies the updating of beliefs to be such that no lending occurs in the next period. Or, in other words, the maximum value of \( \theta_0 \) so that the posterior beliefs when default occurs are below \( \theta^* \).
Intuitively, for default to imply no more lending, the value of $\theta_0$ cannot be too much above $\theta^*$, unless the default signal is very informative and, as a consequence implies a big *jump* in the beliefs.

To find $\theta'$ and $\theta''$ one just needs to use the expressions of $\theta_{1,(1,1)}$ and $\theta_{1,(0,1)}$ and relate them with $\theta^*$.

For example, suppose that the informative case holds and the condition $r_{2,(1,1)} \succ \pi^H \succ r_{2,(0,1)}$ is verified. This condition is equivalent to $\theta_{1,(1,1)} < \theta^* < \theta_{1,(0,1)}$, which can be expressed in terms of the prior beliefs as $\theta'' < \theta_0 < \theta'$.

The values of $\theta'$ and $\theta''$, are:

$$\theta' = \frac{(1 - \lambda_B \mu_B)(\bar{r} - \mu_B \pi^H)}{(1 - \lambda_B \mu_B)(\bar{r} - \mu_B \pi^H) + (1 - \lambda_G \mu_G)(\mu_G \pi^H - \bar{r})}$$

$$\theta'' = \frac{(1 - \mu_B)(\bar{r} - \mu_B \pi^H)}{(1 - \mu_B)(\bar{r} - \mu_B \pi^H) + (1 - \mu_G)(\mu_G \pi^H - \bar{r})}$$

In the *informative case*: $\theta'' < \theta'$. In the *disinformative* and *mixed cases*: $\theta'' > \theta'$.

In summary, in order for predation to be worthwhile, there are two kinds of conditions which need to be satisfied. One is that $\theta_0$ cannot be so high that even with default the posterior beliefs of period 1 are still above $\theta^*$. The other one is the expected profit comparison condition, (2.17) or (2.18). Obviously, this last condition can also be written with respect to $\theta_0$. These two conditions are necessary for predation to happen.

Instead of rewriting the proposition with respect to the prior beliefs. I will do the representation of each case. In the *informative case*:

<table>
<thead>
<tr>
<th>$\theta^*$</th>
<th>$\theta''$</th>
<th>$\theta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>prey if (2.17) holds</td>
<td>prey if (2.18) holds</td>
<td>do not prey</td>
</tr>
</tbody>
</table>

What happens if $\theta^* < \theta_0 < \theta''$ is that if default occurs, either with or without
predation, beliefs are revised to values below $\theta^*$. If $\theta'' < \theta_0 < \theta'$ when default occurs under predation the posterior beliefs are below $\theta^*$, but if default occurs without predation posterior beliefs are still above $\theta^*$.

In the disinformative and mixed cases, $\theta' < \theta''$:

<table>
<thead>
<tr>
<th>$\theta^*$</th>
<th>$\theta'$</th>
<th>$\theta''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>prey if (2.17) holds</td>
<td>do not prey</td>
<td></td>
</tr>
</tbody>
</table>

For $\theta' < \theta_0 < \theta''$ the incumbent decreases future profits if he preys by (2.14).

The decision to lend in the first period

When the investors decide whether to lend or not they have to think about what the incumbent will do.

If the lenders do not anticipate predation then they accept the entrant’s offer to repay $r_1$ as long as:

$$r_1[\theta_0\mu_G + (1 - \theta_0)\mu_B] \geq \bar{r} \quad \text{and} \quad r_1 \leq \pi^H$$

On the other hand, if the lenders expect predation they only lend if:

$$r_1[\theta_0\lambda_G\mu_G + (1 - \theta_0)\lambda_B\mu_B] \geq \bar{r} \quad \text{and} \quad r_1 \leq \pi^H$$

Since the entrant anticipates the lenders behavior she will offer the minimum return acceptable to them. One can compute the value of $\theta$ above which the lenders lend in each circumstance. When no predation his expected that value is $\theta^*$ defined in lemma.
1. If predation is expected then:

\[ \theta^{**} = \frac{\bar{r} - \lambda_B \mu_B \pi^H}{[\lambda_G \mu_G - \lambda_B \mu_B] \pi^H} \] (2.19)

It can be shown that \( \theta^{**} > \theta^* \). So one concludes that the threat of predation makes financing more difficult to obtain for the entrant.

I would like to point out however, that this result would be different if one considers assumption 5'. In that case, if the lenders expect predation they only lend if:

\[ r_1[\theta_0 \lambda_G \mu_G + (1 - \theta_0) \lambda_B \mu_B] \geq \bar{r} \]

\[ \text{and } r_1 \leq \pi^H_G \]

Since \( \pi^H_G \geq \pi^H \) the value of \( \theta_0 \) above which the lenders lend when they expect predation, is smaller under assumption 5' than using assumption 5. However \( \theta^{**} > \theta^* \) continues to hold under assumption 5' as long as \( \lambda_G > \lambda_B \). If \( \lambda_G \leq \lambda_B \), this inequality is reversed. Hence under assumption 5', if the disinformative case holds, and lenders expect the incumbent to prey, financing is easier to obtain than if no predation was expected.

Let me summarize the optimal strategy of the lenders in the first period, under assumption 5: If \( \theta_0 < \theta^* \) not to lend is the dominant strategy. If \( \theta_0 > \theta^{**} \) to lend is the optimal strategy conditional on \( r_1 < \pi^H \), no matter if the incumbent is expected to prey or not. This implies that if predation is profitable then it will be observed in the equilibrium path. Finally if \( \theta^* < \theta_0 < \theta^{**} \), the optimal strategy of the lenders (conditional on \( r_1 < \pi^H \)), depends on whether the lenders expect the incumbent to prey or not.

### 2.5 When is Predation More Likely to Occur?

In this section I wish to answer the following question: is the optimal pattern of predation such that whenever predation is optimal in period \( t \) than it is also optimal in all periods
before $t$?

Instead of extending the previous model to a longer horizon, I will answer the question above by considering two examples. Each one gives a different insight about the optimal pattern of predation over time. In the first example we find that if predation is observed in a given period, then it also occurred in all periods before that. In the second example this pattern doesn’t hold.

I end this section by analyzing a related question: suppose in period $t+1$, for prior beliefs $\hat{\theta}$ the optimal strategy involves preying in period $t+1$. Is it the case that in period $t$ if beliefs are $\hat{\theta}$, the optimal strategy involves preying in period $t$?

2.5.1 First Example - Default Is a Perfect Signal

Consider the case where $\mu_G = \lambda_G = 1$; i.e., the good firm never defaults and it is not vulnerable to predation.

Notice that in such a model default is a perfect signal - if it is observed one knows the firm is bad. In addition, provided $\lambda_B < 1$, predation improves the quality of the non-default signal, hence the informative case holds$^{12}$.

One Period

In a one period model preying never occurs, hence lending occurs if:

$$\frac{\bar{r}}{\theta_0 + (1 - \theta_0)\mu_B} = r_1 < \pi^H$$

For given values of $\mu_B, \pi^H, \bar{r}$ it is possible to calculate the value of $\theta_0$ above which lending occurs. This value is:

$$\theta^* = \frac{\bar{r} - \mu_B \pi^H}{(1 - \mu_B) \pi^H} \quad (2.20)$$

In summary, if the proportion of good entrants is high enough, lending occurs. The

$^{12}$Actually the mixed case also holds, because default is neither a better nor a worse signal that the firm is bad.
necessary proportion depends on how bad are the bad entrants. Note that the critical value of $\theta$ for lending/not lending when no predation is expected is the same for all periods. What changes are the beliefs about the quality of the entrant. Depending on the history, either $\theta_t$ increases (if no default is observed), or $\theta_t = 0$ if default is observed.

Two Periods

In a two period model, preying will be optimal in period 1 (conditional on lending occurring and $\theta_0 > \theta^*$) if the following condition holds:

$$c \leq \delta (1 - \theta_0) \mu_B (1 - \lambda_B) (\pi^M - \pi^I)$$  \hspace{1cm} (2.21)

To prove the condition (2.21) just use lemma 9 with $\mu_G = 1, \lambda_G = 1$.

The condition above is the only one which needs to be satisfied for predation to be worth in this model. The reason is that both $\theta'$ and $\theta''$ defined on section 2.4.4 are equal to 1 in this model. No matter what are the prior beliefs, if default is observed the posterior beliefs drop to zero. This makes the model much easier to analyze.

Solving the inequality (2.21) for $\theta_0$:

$$\theta_0 \leq \frac{\delta \mu_B (1 - \lambda_B) (\pi^M - \pi^I) - c}{\delta \mu_B (1 - \lambda_B) (\pi^M - \pi^I)}$$  \hspace{1cm} (2.22)

Let us call $\theta^*$ the value for which (2.22) holds with equality. In principle, one can have $\theta^*$ larger or smaller than $\theta^*$, depending on the values of parameters. When $\theta^*$ is smaller than $\theta^*$ predation never occurs in the equilibrium path. What happens if $\theta^*$ is larger than $\theta^*$?

If $\theta_0$ is between the two critical values, $\theta^*$ and $\theta^*$, then predation will occur, if the entrant gets financing. However the lenders anticipate the incumbent's behavior, implying that the lending decision needs to be done taking in account that predation
will happen. Using expression (2.19), with \( \lambda_G, \mu_G = 1 \), one gets:

\[
\theta^{**} = \frac{\bar{r} - \lambda_B \mu_B \pi^H}{1 - \lambda_B \mu_B \pi^H}
\]

(2.23)

Where \( \theta^{**} \) is the critical value for the lending decision when predation is expected\(^{13}\).

In summary:

For values of \( \theta_0 \) below \( \theta^* \) there is no lending, the incumbent remains a monopolist forever.

If \( \theta_0 \) is between \( \theta^* \) and \( \theta^{**} \) lending will not occur in the first period, because the lenders know that if they lend the incumbent will prey, which implies that their expected return would be less than \( \bar{r} \). So, in that case, the threat of predation worsens the entrants position in the debt market, they do not get financing in the first period for a larger set of values of \( \theta_0 \).

If \( \theta_0 \) is above \( \theta^{**} \) but below \( \theta^* \) lending occurs and the incumbent preys.

Finally, if \( \theta_0 \) is above \( \theta^* \) lending occurs but the incumbent does not prey.

\[\begin{array}{c|c|c}
\text{predation} & \theta^{**} & \theta^* \\
\hline
\text{no lending} & \theta^* & \theta^{**} \\
\text{no predation} & \text{lending} & \text{lending}
\end{array}\]

It is interesting to observe that predation occurs for medium-range values of \( \theta_0 \). For a very low prior of the entrant being good lending does not occur, hence no predation is needed, for very high values of \( \theta_0 \) predation might not be profitable.

It might happen that the entrant does not get financing in the first period but she gets in the second (for values of \( \theta_0 \) between \( \theta^* \) and \( \theta^{**} \)).

In conclusion, if the prior beliefs are such that \( \theta^{**} < \theta_0 < \theta^* \) predation occurs in the

\(^{13}\)Since \( \lambda_G = 1 \), this would also be the critical value if we had considered assumption 5'.

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equilibrium path. If $\theta^* < \theta_0 < \theta^{**} < \theta^*$ predation is not observed in the equilibrium path, however the threat of predation influences the financing decision.

In what follows I assume that $\theta_0 > \theta^{**}$, which means that the lenders lend even if they expect predation. If no default is observed the posterior beliefs will always be above $\theta^{**}$. Hence lending is the optimal continuation strategy unless default is observed. And if a default is observed, lending will stop forever. Thus the equilibrium lending strategy depends on whether default occurred anytime in the past or not. It is not contingent on whether predation is expected or not. Without this assumption the pattern of predation could differ from the one presented.

**Three Period Model**

Since predation is never observed in the last period, one needs at least three periods to analyze how the incentive to prey evolves over time. For this reason, I will be detailed in the explanation of the three period case.

Let us go back one period, knowing that the continuation equilibrium strategies, if no default occurs in the first period, are the ones discussed in the previous section. And that, when default is observed in the first period, beliefs drop to $\theta_1 = 0$, lending is a dominated strategy from then on, and the incumbent becomes a monopolist.

Recall that $\theta_{1,(1,0)} > \theta_{1,(0,0)}$. The important point is how do these two values relate to $\theta^*$; i.e., the value below which is optimal to prey in the second period. The optimal continuation strategy in the second period if the entrant doesn’t default in the first period can be summarized as follows:

(i) $\theta_{1,(1,0)} > \theta_{1,(0,0)} > \theta^*$ - the optimal policy in the second period, if no default was observed in the first period, is not to prey, no matter what action (prey/not prey) was done in the first period.

(ii) $\theta_{1,(0,0)} < \theta_{1,(1,0)} < \theta^*$ - the optimal policy is to prey in the second period, no matter what action the incumbent has taken in the first period.
(iii) $\theta_{1,(1,0)} > \theta^* > \theta_{1,(0,0)}$ - the optimal policy is not to prey in the second period if predation was done in the first period, and to prey otherwise.

What should the incumbent do in the first period?

In case (i) the incumbent doesn’t prey in the last two periods. One may wonder if he wants to prey in the first period. A sub case of (i) is when $\theta_0 > \theta^*$, and hence with a two period horizon the incumbent wouldn’t prey. Is it possible that the incumbent preys in the three period horizon? Intuitively, one expects the condition to prey to be worthwhile, to be less stringent in a longer horizon model, because the benefits of being a monopolist extend for a larger number of periods.

In case (ii) predation is optimal in the second period when the entrant doesn’t default in the first period, no matter what the incumbent did in the first period. The question here is, can it ever be optimal not to prey in the first period in this circumstance?

In the last case it is optimal to prey once, if the entrant doesn’t default in the first period. The question is, does the incumbent prefer to prey in the first or in the second period?

The optimal strategy under each of these cases is:

**Lemma 12** (i) *If $\theta_{1,(1,0)} > \theta_{1,(0,0)} > \theta^*$, the incumbent never preys in the second and third periods. He preys in the first period if:*

$$c \leq \delta \left[ (1 - \delta_0) \mu_B (1 - \lambda_B) [\mu_B (1 - \pi^M_1 - \pi^I_1) + \delta \mu_B (\pi^M - \cdot \pi^I)] \right]$$

(2.24)

(ii) *If $\theta_{1,(0,0)} < \theta_{1,(1,0)} < \theta^*$, predation is optimal in the first period. The incumbent also preys in the second period if no default occurs in the first period.*

(iii) *If $\theta_{1,(1,0)} > \theta^* > \theta_{1,(0,0)}$ it is optimal to prey in the first period and not to prey in the second and third periods.*

**Proof:** The proof consists of comparing the expected stream of profits with and without predation in the first period, taking in account the optimal continuation strategies
described above. For more details on the computations see the appendix 2.7.1. Q.E.D.

For which parameter values will the incumbent never prey, prey only in the first period, prey in the first period and prey in the second period if the entrant doesn’t default in the first period?

In order to answer one just needs to write the conditions for in each of the three cases in terms of the prior beliefs, $\theta_0$. Doing that, one arrives to the following conclusion: For high values of $\theta_0$ predation never occurs, for slight less high values predation occurs only in the first period and, for low values of $\theta_0$ it occurs in both first and second period (if no default is observed in the first period). Let us call $\theta^1$ and $\theta^2$ the values below which predation occurs the first period and the first and second period, respectively. In, summary:

<table>
<thead>
<tr>
<th>$\theta^{**}$</th>
<th>$\theta^2$</th>
<th>$\theta^*$</th>
<th>$\theta^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>prey in</td>
<td>prey in</td>
<td>never</td>
<td></td>
</tr>
<tr>
<td>periods 1,2</td>
<td>period 1</td>
<td>prey</td>
<td></td>
</tr>
</tbody>
</table>

There are some characteristics which deserve to be mentioned. One is the dependence of the optimal predation policy on the values of $\theta_0$, with cut-off points $\theta^1$ and $\theta^2$. The other one is the fact that a longer horizon implies that predation is more likely to occur (the value of $\theta_0$ below which predation occurs in a three period horizon is larger than in a two period horizon, $\theta^1 > \theta^*$). Finally, when predation occurs it is better to do it at the beginning of the horizon, as it is clear from lemma 12.

$T$ period Model

Given the results of the previous section, it is natural to wonder if the properties of the model with three periods extend to a longer horizon.
Proposition 11 Suppose $\theta_0 > \theta^*$. In a $T$ period model the equilibrium strategy of the incumbent can be described as follows:

There are cut-off values of the prior beliefs $\theta^{T-1} < \theta^{T-2} < \ldots < \theta^3 < \theta^2 < \theta^1$, such that:

- If $\theta_0 > \theta^1$ the incumbent never preys.
- If $\theta^2 < \theta_0 < \theta^1$ he preys only the first period.
- If $\theta^3 < \theta_0 < \theta^2$, he preys in the first period. If no default is observed in the first period he also preys in the second period, otherwise he doesn't prey. The incumbent never preys in the remaining periods.
- ...
- If $\theta^{T-1} < \theta_0 < \theta^{T-2}$ he preys in all the periods until the entrant defaults or the first $T - 2$ periods end up, whichever happens first. He does not prey in any period after a default is observed. If no default is ever observed up to period $T - 2$ he does not prey the last two periods.
- If $\theta_0 < \theta^{T-1}$ the incumbent preys in all periods until the entrant defaults or the first $T - 1$ periods end up, whichever happens first. He does not prey in any period after a default is observed. He does not prey in the last period under any circumstance.

Remark: Two important features of the incumbent's equilibrium strategy are: if the entrant defaults in a given period, the incumbent does not prey for all the remaining periods; and if the incumbent ever preys, he will prey at the beginning of the horizon.

Proof: First we notice that if default is observed posterior beliefs drop immediately to $\theta = 0$, and do not change from then on. Hence lending is a dominated strategy after default is observed. In the equilibrium path no lending is observed after a default.

Since predation has cost $c$, to prey any time after a default is observed cannot be optimal, given the optimal strategy of the lenders.
The rest of the proof is by induction. It was proved already that in a three period model one cannot have a sequence where no predation is the optimal action in the first period and predation is optimal in the second period if no default was observed in the first period. In terms of my notation the only possible predation sequences, \((p_1, p_2, p_3)\), are \((0,0,0),(1,0,0)\), and \((1,1,0)\).

Now, let us assume that this type of sequence, where the 1’s always come first, holds in a \(n\) periods model. I want to prove it also holds in a \(n + 1\) periods model. Notice that, if the property holds for \(n\), it holds for all cases before \(n\) and hence for \(n - 1\).

In the \(n + 1\) model one needs to evaluate the following types of sequences: \((1,1,\ldots,0)\), \((1,0,\ldots,0)\), \((0,1,\ldots,0)\), \((0,0,\ldots,0)\). Recall that it is enough to consider the strategies which have optimal continuation strategies\(^{14}\). Notice that, by the induction hypothesis the second and forth type of sequences have only zeros, after the second period.

Only the third sequence does not satisfy the property that the ones come first (the others satisfy it, by the induction assumption). However, one can prove that the third sequence can never be optimal because it is dominated by the second one.

Let us start with certain prior beliefs. If no default is observed either in the first or in the second period, the posterior beliefs when predation is done in the first period but not in the second, are the same than if the order of the predation action was reversed. In other words, the posterior beliefs at the beginning of the third period are equal in the second and third sequences. But that means that the optimal continuation strategy, if default did not occur in the two first periods, is the same under the two sequences. On the other hand if default is observed in period one or two, the incumbent becomes a monopolist from then on. As such it is enough to compare the expected stream of profits for the first three periods. The sequence \((1,0,\ldots,0)\) implies:

\[
V(p_1 = 1) = (\pi^I - c) + \delta(1 - \theta_0)(1 - \lambda_B \mu_B)(\pi^M + \delta \pi^M + \delta^2 \pi^M + \ldots) + \\
\delta(1 - \theta_0)\lambda_B \mu_B (1 - \mu_B)(\pi^I + \delta \pi^M + \delta^2 \pi^M + \ldots) +
\]

\(^{14}\)A strategy with non-optimal continuation strategies can never be optimal.
\( \delta(\theta_0 + (1 - \theta_0)\lambda_B\mu_B^2)(\pi^I + \delta V(n - 1) ) \) \hspace{1cm} (2.25)

The sequence \((0,1,\ldots,0)\) implies:

\[
V(p_1 = 0) = \pi^I + \delta(1 - \theta_0)(1 - \mu_B)(\pi^M + \delta\pi^M + \delta^2\pi^M + \ldots) + \\
\delta(1 - \theta_0)\mu_B(1 - \lambda_B\mu_B)(\pi^I - c + \delta\pi^M + \delta^2\pi^M + \ldots) + \\
\delta(\theta_0 + (1 - \theta_0)\lambda_B\mu_B^2)(\pi^I - c + \delta V(n - 1) )
\] \hspace{1cm} (2.26)

Where \(V(n - 1)\) is the value function starting on the third period when no default was observed in the first two periods.

After some manipulations one concludes that in order for \(V(1) - V(0)\) to be positive the following condition has to hold:

\[
c \leq \frac{\delta(1 - \theta_0)(1 - \lambda_B)\mu_B(\pi^M - \pi^I)}{1 - \delta[(1 - \theta_0)\mu_B + \theta_0]}
\] \hspace{1cm} (2.27)

The condition can also be written as a geometric series with ratio \((\delta(1 - \theta_0)\mu_B + \theta_0)\).

Let us imagine the incumbent chose the third sequence and the first period is over, with no default. In order for preying to be the optimal action in this period, when the continuation strategy is not to prey, it has to be true that:

\[
c \leq \delta(1 - \theta_0)(1 - \lambda_B)\mu_B[(\pi^M - \pi^I) + (\delta\mu_B)(\pi^M - \pi^I) + \ldots + (\delta\mu_B)^{n-2}(\pi^M - \pi^I)]
\] \hspace{1cm} (2.28)

It is easily verified that the previous condition implies (2.27). In conclusion, the first property holds for a \((n+1)\) horizon model.

To derive the cut-off levels is enough to order the different optimal sequences by the number of ones and, to compare each strategy with each successor. \textbf{Q.E.D.}

In conclusion, the features of the three period model hold for any \(T\): a longer horizon makes predation more likely to occur and if the incumbent ever preys he will prey at the beginning of the horizon. In this example the optimal pattern of predation is such that
if predation is optimal in a certain period than it is also optimal in all previous periods.

2.5.2 Second Example - Same Vulnerability to Predation

Should we expect always the result of the previous section? My second example shows that the answer is no.

I will analyze only a three period model, that is enough to show that we can have a period of no predation followed by a period of predation. The reason for this is quite intuitive - if the initial beliefs are such that there is a high probability of the entrant being good, then preying in the first period may be a waste (especially if predation makes default a weaker signal of the entrant being bad). Instead the incumbent may prefer to wait and see what happens in the first period, if the entrant does well forget it, if he does badly, and posterior beliefs get to a point where one more default implies stop lending, then may be predation is a good action.

In other words, predation is more likely to happen when the entrant is in a bad shape.

Let us consider the special case where $\lambda_G = \lambda_B = \lambda$. This corresponds to a disinformative case\(^{15}\). Let us recall the results of the general model with $T = 2$:

\[
| \begin{array}{c|c|c} 
\theta^* & \theta' & \theta'' \\
\hline
\text{prey if} & \text{do not} & \text{prey} \\
(2.17) \text{ holds} & \text{prey} & \\
\end{array}
\]

Where (2.17) can be simplified to:

\[
c \leq \delta(1 - \lambda)[\theta_0\mu_G + (1 - \theta_0)\mu_B](\pi^M - \pi^I)
\]

I will assume, to simplify my analysis, that this condition holds.

\(^{15}\)Actually it is also a mixed case because under predation no default is neither a better nor worse signal of the entrant being good.
If $\theta^{**} < \theta_0 < \theta'$ predation is observed in the equilibrium path\textsuperscript{16}.

Let us go back one period and ask ourselves what is the optimal strategy in the first period, given that the optimal continuation strategy are as defined above. The idea is the same than in the first example. One needs to evaluate the prior beliefs and first period histories which make the posterior beliefs be in each of the regions above.

Let us define $\theta^{a'}$ as the value of the prior beliefs, $\theta_0$, such that if predation and no default are observed in the first period the posterior beliefs are equal to $\theta'$; i.e., $\theta^{a'} = \{\theta_0 : \theta_{1,(1,0)} = \theta'\}$. Since $\theta_{1,(1,0)} = \theta_{1,(0,0)}$ and by assumption condition (2.17) holds, one can interpret $\theta^{a'}$ as the maximum value of the prior beliefs such that if no default is observed in the first period, the posterior beliefs are such that it is still optimal to prey in the second period. Let us define $\theta^{b'} = \{\theta_0 : \theta_{1,(1,1)} = \theta'\}$ and $\theta^{c'} = \{\theta_0 : \theta_{1,(0,1)} = \theta'\}$. The interpretation of $\theta^{b'}$ and $\theta^{c'}$ is like the one of $\theta^{a'}$, the only difference is the history in period one which leads to the posterior beliefs $\theta'$.

Let $\theta^{a''}, \theta^{b''}, \theta^{c''}$ be defined similarly, where we substitute $\theta'$ by $\theta''$.

It can be shown that $\theta^{a'} < \theta^{a''}, \theta^{b'} < \theta^{b''}, \theta^{c'} < \theta^{c''}$ and that $\theta^{b''} = \theta^{c'}$. Obviously $\theta^{a'} < \theta'$, and $\theta^{b'}$ and $\theta^{c'}$ are larger than $\theta'$. Similar relationships hold for $\theta''$. Let us represent the different points:

\begin{center}
\begin{tikzpicture}
\draw [->] (-5,0) -- (5,0);
\draw (-5,0) node [below] {$\theta^*$} -- (-4,0) node [below] {$\theta^{a'}$} -- (-3,0) node [below] {$\theta^{a''}$} -- (-2,0) node [below] {$\theta'$} -- (-1,0) node [below] {$\theta''$} -- (0,0) node [below] {$\theta^{b'}$} -- (1,0) node [below] {$\theta^{c'}$} -- (2,0) node [below] {$\theta^{c''}$} -- (3,0) node [below] {$\theta^{b''}$};
\end{tikzpicture}
\end{center}

The relative position of $\theta^{a''}$ and $\theta'$ and of $\theta''$ and $\theta^{b'}$ may be the reverse of the one above. In what follows I consider the case where the relative position of the parameter values is as represented.

To show that we might observe no predation in the first period followed by predation in the second let us analyze the case where $\theta^{b'} < \theta_0 < \theta^{c'}$.

\textsuperscript{16}If we had considered assumption 5', $\theta^{**}$ would be equal to $\theta^*$, the lending decision would not depend on whether the lenders expect predation to occur or not.

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What happens if the incumbent preys in the first period? If the incumbent preys in the first period the posterior beliefs are such that lending will occur in the second period (even if the entrant defaults in the first period). Moreover if default is observed, posterior beliefs will lie between $\theta'$ and $\theta''$, which implies that the optimal continuation strategy is not to prey in the second period.

However, if the incumbent does not prey and default is observed in the first period, then the posterior probability is below $\theta'$. As such the optimal continuation strategy is to prey in the second period if default is observed in the first period and not to prey otherwise. It can be proved that the optimal strategy of the incumbent is the last one:

**Lemma 13** If $\theta' < \theta_0 < \theta''$ the optimal strategy of the incumbent is not to prey in the first period, prey in the second period only if default is observed in the first period, and do not prey in the third period. In addition if $\theta_{1,\{0,1\}} > \theta^{**}$ this pattern of predation is observed with positive probability.

**Proof**: See appendix 2.7.2. **Q.E.D.**

If $\theta_{1,\{0,1\}} > \theta^{**}$, the intuitive explanation of the result is the following: In the two strategies above, no lending can only happen in the third period and the probability of this happening is the same in the two strategies. However under the second strategy, predation will occur only if the entrant defaults in the first period while in the first strategy predation happens for sure. In addition the cost $c$ is delayed till period two under the second strategy. Hence the second strategy implies a lower probability of spending $c$ as well as a delay of the cost $c$ to the second period if predation occurs.

Hence the optimal pattern of predation may include periods of no predation followed by periods with predation.

### 2.5.3 A Last Question About the Pattern of Predation

A final question of interest is the following: Suppose in period $t + 1$, for prior beliefs $\hat{\theta}$ the optimal strategy involves preying in period $t + 1$. Would it be possible to show that
in period $t$ if beliefs are $\hat{\theta}$, then the optimal strategy involves preying in period $t$?

In the examples I presented this property holds. However it cannot be generalized. Consider a three period model under a strictly mixed case. Assume that $\theta_1 < \theta'$. This implies that if the entrant defaults, with or without predation, she will not get financing in the second period and the incumbent will become a monopolist. Predation increases the probability of default and thus it increases the probability of the incumbent becoming a monopolist in the second period. Provided this benefit is greater than $c$ (condition (2.17) holds) the incumbent would prey in a two period model. In a three period model the benefit of being a monopolist extends to an additional period, hence the effect above is stronger in a three period model.

However one also needs to consider a possible contrary effect. If the entrant doesn't default in the second period we know that $\theta_{1,(1,0)} > \theta_{1,(0,0)}$. Consider the case where $\theta_{1,(1,0)} > \theta'' > \theta_{1,(0,0)}$. What happens in this case is that if the incumbent preys and the entrant does not default in the first period then the entrant will get financing for the two remaining periods. On the other hand if the incumbent does not prey and the entrant does not default in the first period there exists some probability of the entrant not getting financing in the third period (e.g., if she is not preyed upon and defaults in the second period). It is possible to find examples where the second effect is larger than the first. In conclusion there exists cases where for certain prior beliefs predation would occur in a two period model but not in a three period model.

2.6 Conclusion

The main idea underlying this essay is that a firm who is entering a market and needs to get financing, has a disadvantage relatively to the established firms, because she has not established reputation in the debt market.

This clearly distinguishes this work from the existing literature on predation which focuses on capital market imperfections. In previous models the difference between the
incumbent and the entrant was that the incumbent has retained earnings while the entrant did not. In this model the difference between the incumbent and the entrant is that the former has acquired a reputation, while the latter has not. Hence the incumbent has *deep pockets* of reputational assets, rather than financial assets.

Predation is used by the incumbent to change information received by lenders about the entrant, therefore interfering with the reputation acquisition process of the entrant. Predation can be observed in equilibrium and the threat of predation might imply that financing is harder to obtain for the entrant.

This essay further addresses the question of what is the optimal pattern of predation. In the model predation is not effective after a very long history. This is because with a very long history the entrant's quality will be known, and the incumbent's gains in preying will be lost by then. Under certain circumstances the optimal pattern of predation is such that if predation occurs in a particular period, it also occurs in all previous periods. However this pattern doesn't hold in general. Predation seems more likely to occur when the entrant is in *bad shape*, in the sense that her credit history is bad.

The model could be extended such that the entrant chooses in a first stage of the game the degree of differentiation of her product. Both the profit and the vulnerability to the predation of the competitor may depend on that choice. The entrant can choose a highly differentiated product, maybe with a small expected net profit, but with very little sensitivity to predation. This may be a good strategy till reputation is acquired in the debt market.
2.7 Appendix

2.7.1 First Example - Proof of Lemma 9

In this appendix I prove lemma 12, which describes the incumbent's optimal strategy in a three period model:

Lemma 12

(i) If $\theta_{1,(1,0)} > \theta_{1,(0,0)} > \theta^*$, the incumbent never preys in the second and third period. He preys in the first period if:

$$ c \leq \delta \left[ (1 - \theta_0)\mu_B(1 - \lambda_B)(\pi^M - \pi^I) + \delta\mu_B(\pi^M - \pi^I) \right] $$

(2.29)

(ii) If $\theta_{1,(0,0)} < \theta_{1,(1,0)} < \theta^*$, predation is optimal in the first period. The incumbent also preys in the second period if no default occurs in the first period.

(iii) If $\theta_{1,(1,0)} > \theta^* > \theta_{1,(0,0)}$ it is optimal to prey in the first period and not to prey in the second and third period.

Proof: The proof consists of comparing the expected stream of profits with and without predation in the first period, taking into account the optimal continuation strategies. I will not present the proof for all the cases in complete detail, it would be too long and not too instructive.

In case (i), the optimal continuation strategy is not to prey in the second and third periods, no matter what happened in the first period. If the incumbent preys in the first period he gets:

$$ V(p_1 = 1) = (\pi^I - c) + \delta[(1 - \theta_0)(1 - \lambda_B\mu_B)(\pi^M + \delta\pi^M)] + $$

$$ \delta[(1 - \theta_0)\mu_B\lambda_B(1 - \mu_B)(\pi^I + \delta\pi^M)] + $$

$$ \delta[((1 - \theta_0)\lambda_B\mu_B^2 + \theta_0)(\pi^I + \delta\pi^I)] $$

(2.30)
While if he does not prey in the first period he gets:

\[ V(p_1 = 0) = \pi^I + \delta[(1 - \theta_0)(1 - \mu_B)(\pi^M + \delta\pi^M)] + \delta[(1 - \theta_0)\mu_B(1 - \mu_B)(\pi^I + \delta\pi^M)] + \delta[((1 - \theta_0)\mu_B^2 + \theta_0)(\pi^I + \delta\pi^I)] \]  

(2.31)

Computing \( V(1) - V(0) \) one concludes that this difference is positive as long as (2.29) holds.

In case (ii), the optimal continuation strategy if no default is observed in the first period, is to prey in the second period. Otherwise the incumbent doesn’t prey in the second period. He doesn’t prey in the third period (no matter what happened before). If one writes the value functions with and without predation and evaluate when is \( V(1) - V(0) \) positive, after some algebraic manipulations one gets the condition:

\[ c \leq \delta [(1 - \theta_0)(1 - \lambda_B)\mu_B(\pi^M - \pi^I) + c + \delta\lambda_B\mu_B(\pi^M - \pi^I)] \]  

(2.32)

Since preying is optimal in the second period if no default is observed in the first period, one knows that the following condition has to be true:

\[ c \leq \delta(1 - \theta_1,(*,0))(1 - \lambda_B)\mu_B(\pi^M - \pi^I) \]  

(2.33)

Where \( \theta_1,(*,0) \) are the posterior beliefs if no default is observed. But since \( \theta_1,(*,0) > \theta_0 \) condition (2.33) implies:

\[ c \leq \delta(1 - \theta_0)(1 - \lambda_B)\mu_B(\pi^M - \pi^I) \]  

(2.34)

Now, the RHS of (2.32) can be decomposed into two terms: the first one is \( \delta(1 - \theta_0)(1 - \lambda_B)\mu_B(\pi^M - \pi^I) \), which is greater or equal to \( c \) and, the second one \( \delta c + \delta^2\lambda_B\mu_B(\pi^M - \pi^I) \) is positive. Consequently, if predation is optimal in the second period when no default is observed in the first period, it has to be optimal in the first period too.
In case (iii), the optimal continuation strategy is to prey in the second period if no default and no predation were observed in the first period, and not to prey in the second period if predation was observed in the first period. The difference of the value function when preying is done in the first period relatively to the case when predation is done in the second period is positive if and only if the following condition holds:

\[ c \leq \frac{\delta(1 - \theta_0)(1 - \lambda_B)\mu_B(\pi^M - \pi^I)}{1 - \delta[(1 - \theta_0)\mu_B + \theta_0]} \]  

(2.35)

Since predation is optimal in the second period when no predation occurred in the first period, condition (2.33) holds for \( \theta_{1,0,0} \). But \( \theta_{1,0,0} \) is larger then \( \theta_0 \). Hence, it has to be the case that condition (2.34) holds. This implies that the condition (2.35) has to be true, because the denominator of the RHS is less than one. In other words, if the incumbent preys in only one period it is better to prey in the first period than in the second. Q.E.D.

2.7.2 Second Example - No Predation can be Followed by Predation

I show that a period of no predation can be followed by a period of predation, as stated in lemma 13:

**Lemma 13** If \( \theta^b < \theta_0 < \theta^c \) the optimal strategy of the incumbent is not to prey in the first period, prey in the second period only if default is observed in the first period and do not prey in the third period. In addition if \( \theta_{1,0,1} > \theta^{**} \) this pattern of predation is observed with positive probability.

**Proof**: As I mentioned in the text, if \( \theta^b < \theta_0 < \theta^c \), the optimal continuation strategy if the incumbent preys in the first period, is not to prey in the second and third periods, no matter what the entrant does in the first period. The entrant always gets financing in the second period. Thus the total profit by preying in the first period is:

\[ V(p_1 = 1) = \pi^I - c + \delta\pi^I + \delta^2\pi^I + \delta^2[\theta_0(1 - \mu_G)(1 - \lambda\mu_G) + \]
\[ (1 - \theta_0)(1 - \mu_B)(1 - \lambda \mu_B)(\pi^M - \pi^I) \] (2.36)

On the other hand if the incumbent does not prey in the first period, then the optimal continuation strategy is not to prey in the second period if the entrant does not default, and prey if the entrant defaults in the first period and the lenders lend in the second period, and do not prey in the third period. The expected profit of this strategy depends on whether the lenders lend or not in the second period. If they lend, which happens when \( \theta_{1,0,1} > \theta^{**} \), the expected profit is:

\[
V(p_1 = 0) = \pi^I + \delta[\theta_0 \mu_G + (1 - \theta_0)\mu_B](\pi^I + \delta\pi^I) + \\
\delta \lambda[\theta_0(1 - \mu_G)\mu_G + (1 - \theta_0)(1 - \mu_B)\mu_B](\pi^I - c + \delta\pi^I) + \\
\delta[\theta_0(1 - \mu_G)(1 - \lambda \mu_G) + (1 - \theta_0)(1 - \lambda \mu_B)(1 - \mu_B)] \\
(\pi^I - c + \delta\pi^M) \tag{2.37}
\]

The gain by not preying in the first period \((2.37)-(2.36)\) is:

\[
c[1 - \delta[\theta_0(1 - \mu_G) + (1 - \theta_0)(1 - \mu_B)] \tag{2.38}
\]

This value is clearly positive. Hence the optimal strategy is not to prey in the first period, prey in the second period only if default is observed in the first period and do not prey in the third period.

On the other hand if \( \theta_{1,0,1} < \theta^{**} \), it is optimal for the lenders not to lend in the second period, and to lend in the third period. Hence the expected profit for the incumbent is:

\[
V(p_1 = 0) = \pi^I + \delta[\theta_0 \mu_G + (1 - \theta_0)\mu_B](\pi^I + \delta\pi^I) + \\
\delta[\theta_0(1 - \mu_G) + (1 - \theta_0)(1 - \mu_B)](\pi^M + \delta\pi^I) \tag{2.39}
\]

It is easy to verify that not to prey is optimal in the first period.
If $\theta_{1,(0,1)} > \theta^{**}$, predation is observed in the second period with probability $(\theta_0(1 - \mu_C) + (1 - \theta_0)(1 - \mu_B))$. Q.E.D.
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Chapter 3

Updating Ambiguous Beliefs

3.1 Introduction

In Savage’s framework there is a natural way of defining conditional preferences from the overall/unconditional preferences of the agent. Suppose we want to derive the conditional preferences given \( B \) between the acts \( f \) and \( g \). Then we only need to consider acts which coincide on \( B \) with \( f \) and \( g \) respectively, and offer the same lotteries \( h \) over \( B^c \), call them \( f|_B^h \) and \( g|_B^h \). The sure-thing principle says that the preferences of the agent between these two acts, does not depend on \( h \). Hence if \( f|_B^h \succeq g|_B^h \) for some \( h \), one knows that \( f|_B^h \succeq g|_B^h \) for all \( h \). Then if \( f|_B^h \succeq g|_B^h \) for some \( h \) we say that \( f \succeq g \), given \( B \)

Such a natural way of defining conditional preferences does not exist in the non-additive expected utility framework because the sure-thing principle does not hold. Here the ordering of \( f \) and \( g \) derived from the unconditional preferences as above depends on \( h \). The “induced” conditional preferences depend on \( h \).

Gilboa and Schmeidler [4], extend the idea of deducing conditional preferences from unconditional preferences. In the setup of non-additive expected utility, they define “updating rule” of a set of preference relations. An updating rule specifies for any preference relation and for any event of a measurable partition of the set of states of the world, what is the preference relation once that event is known to have occurred. The
only two conditions the "updating rule" has to satisfy are that if the only event known to be true is the set of all states of the world, then the updating function is the identity and that the preference relation given that event $B$ is known to have happened depends only on the outcomes on states of the world in $B$.

They then consider a particular set of "updating rules", where the preference relation given $B$ is "induced" from the unconditional preferences by choosing a common $h$ on $B^c$; i.e., $f \succeq g$ given $B$ if and only if $f|_{B^c}^h \succeq g|_{B^c}^h$. They call this rule $h$-Bayesian update rule. Clearly, when preferences don't obey the sure-thing principle, the updated preferences depend on $h$.

Gilboa and Schmeidler argue that choosing $h = x^*$, where $x^*$ is the best prize, is a pessimist update rule which fits well with the pessimism of an uncertainty averse agent. The decision maker implicitly assumes that if $B$ had not occurred the best possible outcome would have happened. Hence the rule reflects disappointment that $B$ occurred. This rule for updating preference relations corresponds to the maximum likelihood principle. The agent would choose from the initial set of priors only the probability measures which give the maximum probability to event $B$, and update these probability measures using Bayes rule.

In this work I do not follow the approach of deriving the conditional preferences from the unconditional preferences, by specifying an update rule. Instead I characterize the conditional preferences and how they are related to unconditional preferences.

I assume that preferences over acts conditional on event $B$ happening do not depend on the lotteries received on $B^c$ and obey axioms which are equivalent, under that assumption, to the ones postulated by Gilboa and Schmeidler [3]. As a consequence one can derive a representation of the conditional preferences as in Gilboa and Schmeidler [3] (from now on, G/S).

There are two axioms which link the different preferences orderings. One assumes that the conditional preferences have common induced preferences over lotteries. The other one relates unconditional preferences to conditional preferences. It states that if
the agent is indifferent given $B$, between a certain act $f$ and a constant act $\bar{g}$, then the unconditional preferences should be such that the agent is also indifferent between an act which gives $f$ if $B$ happens and the constant act $\bar{g}$ if $B^c$ happens and the constant act $\bar{g}$. I show that, under this axiom, the set of probability measures which appears in $G/S$ representation of the conditional preferences, is the set of measures obtained by updating using Bayes rule the set of all prior probability measures derived in the $G/S$ representation of the unconditional preferences.

The axiomatization presented in this essay is consistent with the existence of an "updating rule" of preferences as defined by Gilboa and Schmeidler, as it is clear from the assumption that preferences given $B$ only depend on the outcomes in states of the world in $B$. However the updating rule implicit in this work is quite more complex than the "h-Bayesian update rule". Such rule is imbedded in the axiom which relates conditional and unconditional preferences.

In terms of updating a set of probability measures, the rule implicit in this work is that the agent updates all the priors using Bayes rule. Although it can be argued that this rule is extreme because it assumes that the agent continues to use probability measures which give very small probability to the events which occurred, I believe it is the most appropriate rule to use. The reason is that the agent who has $G/S$ type of preferences is likely to have a "conservative" attitude. He would not reject some probability measure unless he is sure that it cannot be the true probability measure.

In the remaining of the chapter I describe the notation and the axioms satisfied by the conditional preferences and present the results.

### 3.2 Framework

I will adopt the "lottery-acts" framework of Anscombe and Aumann. Let $X$ be the set of consequences. Let $Y$ be the set of distributions over $X$, with finite support. The elements of $Y$ are called lotteries (roulette lotteries). Let $S$ be the set of states of the
world, and $\Sigma$ be an algebra on $S$. The elements of $\Sigma$ are events. $L_0$ is defined as the set of $\Sigma$-measurable finite step functions from $S$ to $Y$. Let $L_c$ be the set of constant functions in $L_0$. Let $L$ be a convex subset of $Y^S$ which includes $L_c$.

Acts are functions from the set of states of the world to the set of lotteries ($S \rightarrow Y$). A constant act $\bar{y}$ is an act which gives the lottery $y$ in any state of the world. We denote by $\bar{f(s)}$ the constant act which gives in every state of the world $s'$, the same lottery that $f$ gives in the state $s$, $f(s)$.

The decision maker has conditional preferences over acts. I will denote by $\succeq_B$ the preferences given $B$. In other words if the agent knew that $B$ happened, then $\succeq_B$ is his preference ordering. Let $\succeq$ be the unconditional preference ordering ($\succeq = \succeq^S$).

Let $g|_B^T$ be the act which coincides with $g$ on $B^c$ and with $f$ on $B$. In particular $g|_B^{f(s)}$ is the act which gives the lottery $f(s)$ in any state belonging to $B$ and coincides with $g$ otherwise. Let $L^c_B$ be the set of acts which are constant on $B$.

Given a preference ordering $\succeq_B$, an event $E \in \Sigma$ is $\succeq_B$-null, or in other words $E$ is null with respect to $\succeq_B$ iff $\forall f, g \in L$, s.t. $\forall s \in E^c, f(s) \sim_B g(s)$, it is true that $f \sim_B g$. Otherwise we say that $E$ is $\succeq_B$-non-null.

The conditional preferences are assumed to obey the following axioms:

$\mathcal{A}_1$ (Weak Order) For all $f$ and $g$ in $L$, $f \succeq_B g$, or $g \succeq_B f$ or both. If $f \succeq_B g$ and $g \succeq_B h$ then $f \succeq_B h$.

$\mathcal{A}_2$ $B^c$ is a null-event with respect to $\succeq_B$.

$\mathcal{A}_3$ (State-Independence) $f \succeq^{(s)} (s) g$ if and only if $h|_B^{f(s)} \succeq_B h|_B^{g(s)}$, $\forall h$ for any $B$ in $\Sigma$.

$\mathcal{A}_4$ (Certainty Independence) For all $f$ and $g$ in $L$ and $h$ in $L^c_B$: $f \succeq_B g$ if and only if $\alpha f + (1 - \alpha) h \succeq_B \alpha g + (1 - \alpha) h$ for $\alpha \in (0, 1)$.

$\mathcal{A}_5$ (Continuity) For all $f, g, h$ such that $f \succeq_B g$ and $g \succeq_B h$, there exist $\alpha_B$ and $\beta_B \in (0, 1)$ such that $\alpha_B f + (1 - \alpha_B) h \succeq_B g$ and $g \succeq_B \beta_B f + (1 - \beta_B) h$.

$\mathcal{A}_6$ (Monotonicity) For all $f, g \in L$ such that $f(s) \succeq^{(s)} (s) g(s)$ for all $s \in B$, $f \succeq_B g$. 

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A7 (Uncertainty Aversion) For all $f, g \in L$ and $\alpha \in (0, 1)$, if $f \sim^B g$ then $\alpha f + (1 - \alpha)g \succeq^B f$.

A8 (Non-degeneracy) For all $B$, not for all $f, g \in L$, $f \succeq^B g$.

A9 For all $B \in \Sigma$ such that $B$ is $\succeq$-non-null, if $f \sim^B \bar{y}$ then $f|_{B^c} \sim \bar{y}$.

Axiom A2 captures the idea that if the agent knew that $B$ happened his (conditional) preferences over the acts on $L$ should not depend on the lotteries received on $B^c$. This property is important in defining conditional preferences. It is the same condition that is present in the definition of an “updating rule”.

Axiom A3 guarantees that the preferences over lotteries are well defined and common to all conditional preferences. A constant act on $B$ means that the agent knows which lottery he will receive if $B$ happens. If we restrict $\succeq^B$ to constant acts on $B$, uncertainty will be irrelevant. Hence the ordering given $B$ of the constant acts on $B$, should coincide with the ordering of the lotteries. A3 states that this happens for any $B$. We notice that A3 is implicit in Gilboa-Schmeidler [3]. In fact they derive preferences over lotteries by considering the preference ordering over constant acts.

The axioms A1, A4, A5, A6, A7, A8 are Gilboa-Schmeidler axioms, adapted to the case of conditional preferences. Since by A2, $B^c$ is a null event with respect to $\succeq^B$, the monotonicity and certainty independence axioms impose conditions only on the lotteries on $B$.

The axiom of certainty independence is weaker than the standard independence axiom. While the certainty independence axiom requires $h$ to be a constant act, the standard independence axiom holds for any act $h$. The certainty independence axiom allows for the phenomenon of hedging, whereas the standard independence axiom does not.

Axiom A9 relates conditional and unconditional preferences. It says that if the agent is indifferent given $B$, between a given act $f$ and a constant act $\bar{y}$, then the unconditional preferences should be such that the agent is also indifferent between an act which gives $f$
if $B$ happens and the constant act $\bar{y}$ if $B^c$ happens and the constant act $\bar{y}$. With the act $f|_{B^c}$ the agent receives something indifferent to the constant act $\bar{y}$ both on $B$ and $B^c$. However this is not why the agent is indifferent between $\bar{y}$ and $f|_{B^c}$. As it will become clear latter, the reason why the agent whose preferences obey the previous axioms is indifferent between these two acts is because the relative weight given to states in $B$ in the evaluation of $f|_{B^c}$ coincides with the relative weights used in evaluating the act $f$, given $B$.

Axiom $A9$ imposes a certain coherence in the pessimism implicit in the conditional preferences. A decision maker with ambiguous subjective beliefs and who obeys to G/S axioms, evaluates each act in such a way that it is as if he chooses among the set of possible subjective probability measures the one which gives less weight to the states with the best utility outcomes and more weight to the states with the worst utility outcomes and take... the expectation of his utility with respect to this probability measure.

Let us assume that the decision maker uses this pessimist evaluation criteria, given $B$ and concludes $f \sim_B \bar{y}$. Using also the pessimist criteria in the evaluation of the act $f|_{B^c}$, the agent gives the least possible weight to states in $B$ where the utility is higher than the utility of $\bar{y}$, and the most possible weight to states in $B$ where the utility is lower than the utility of $\bar{y}$. It is clear that no matter what are the weights given to $B$ and $B^c$, provided the relative weights of the states in $B$ are the same than the one used in the evaluation of $f$ given $B$, the agent will be indifferent between $\bar{y}$ and $f|_{B^c}$.

### 3.3 Results

In this section I present the main result of this essay. I start by stating two lemmas which are useful to the proof of the main result.

**Lemma 14** Under $A1$ and $A3$ the preference ordering over lotteries is the same for all the states of the world. I.e., if $f(s) = f(t)$, $g(s) = g(t)$ and $f \succ^s g$ then $f \succ^t g$. 
Proof: Suppose not, suppose \(\sim (f \succ^{(t)} g)\). Then by A1, \(g \succeq^{(t)} f\). Applying A3 for \(B = S, g(t) \succeq f(t)\), which by assumption is equivalent to \(g(s) \succeq f(s)\). However by A3 this is equivalent to \(g \succ^{(s)} f\), contradicting the assumption that \(f \succ^{(s)} g\). Q.E.D.

Lemma 15 If the conditional preferences \(\succeq^B\) satisfy axioms A1, A2 and A4-A7 there exists a G/S representation of the conditional preferences \(\succeq^B\). I.e. there exist an affine function \(u : Y \to R\) and a non-empty, closed, convex set \(C^B\) of finitely additive probability measures on \(\Sigma\), s.t. \(\forall f, g \in L_0, f \succeq^B g\) iff \(\min_{p \in C^B} \int u \circ f\, dp \geq \min_{p \in C^B} \int u \circ g\, dp\). In addition \(C^B\) is unique if A8 holds.

Proof: Since, by A2, preferences given \(B\) do not depend on the lotteries received in \(B^c\), it is enough to verify that restricting the acts to \(B\) all the axioms which are necessary and sufficient for G/S representation hold. Axioms A1, A4-A8 guarantee that is true. Q.E.D.

The previous lemma guarantees that the preference ordering given \(B\) has a G/S representation. Axiom A3 implies that the function \(u : Y \to R\) is the same, up to a positive linear transformation, to all preference orderings \(\succeq^B\). The next proposition summarizes these results and establishes the link between the sets of additive probability measures \(C^B\) and \(C^S = C\):

Proposition 12 Let \(\succeq^B\) be a set of binary relations on \(L_0\). Then the following conditions are equivalent:

(i) The binary relations \(\succeq^B\), for all \(B \in \Sigma\) satisfies axioms A1 - A7 for \(L = L_0\).

(ii) There exists an affine function \(u : Y \to R\), and non-empty, closed and convex sets, \(C^B\), of finitely additive measures on \(\Sigma\) such that:

\[
f \succeq^B g \iff \min_{p \in C^B} \int u \circ f\, dp \geq \min_{p \in C^B} \int u \circ g\, dp
\]
In addition:

(a) \( u \) is unique up to a positive linear transformation and the sets \( C^B \) are unique iff \( A8 \) holds.

(b) \( C^B \) is equal to the set of posterior probability measures using Bayes rule of \( C \) given \( B \), \( C^B = C/B \), iff \( A9 \) is satisfied.

Proof: By lemma 15 one knows that:

\[
I^B(u \circ f) = \min_{p \in C^B} \int u \circ f \, dp^B \quad \text{and} \quad I(u \circ f|_{B^c}) = \min_{p \in C} \left[ \int_B u \circ f \, dp + \int_{B^c} u \circ \tilde{y} \, dp \right]
\]

where \( u \) is common in both functionals by lemma 14. Now if \( f \sim^B \tilde{y} \) the functionals \( I^B(u \circ f) = u(\tilde{y}) \) and \( I(u \circ f|_{B^c}) \) should have the same value by \( A9 \). I claim that this happens iff \( C^B \) is the set of posterior measures of \( C \), given \( B \).

Let us first suppose \( C^B \) is the set of posterior measures of \( C \), given \( B \). I want to show that if \( f \sim^B \tilde{y} \) then \( I^B(u \circ f) = I(u \circ f|_{B^c}) = u(\tilde{y}) \); i.e., the solution to the problems (3.1) and (3.2) is such that the functionals have the same value.

Assume \( p^* \) is the minimand in problem (3.1). Since \( f \sim^B \tilde{y} \), \( u(\tilde{y}) \) is the value of the functional in (3.1). Define the set \( C^* \) as the set of probability measures in \( C \) with posterior probability \( p^* \); i.e. \( C^* = \{ \mu \in C : p/B = p^* \} \). Problem (3.2) can be written as:

\[
\min_{p \in C} \left[ p(B) \int_B \frac{u \circ f}{p(B)} \, dp + (1 - p(B))u(\tilde{y}) \right]
\]

Since \( C^* \subset C \) the solution to this problem will be no greater than:

\[
\min_{p \in C^*} \left[ p(B) \int_B \frac{u \circ f}{p(B)} \, dp + (1 - p(B))u(\tilde{y}) \right]
\]

Which is equal to \( u(\tilde{y}) \). We need to prove that the solution to (3.2) cannot be smaller than \( u(\tilde{y}) \). Suppose not, suppose \( \tilde{p} \in C \) such that \( \tilde{p} \) is the solution to problem (3.2) such
that the value of \( I(u \circ f|_{B^c}) < u(\bar{y}) \). The only way this can happen is if \( \int_{B^c} \frac{u \circ f}{p(B)} \, d\bar{p} < u(\bar{y}) \).

But since the posterior of \( \bar{p} \) belongs to \( C^B \) this would mean that \( p^*B \) could not be the solution to (3.1), a contradiction.

To prove the converse fix \( B \) and assume that \( I^B(u \circ f) = u(\bar{y}) = I(u \circ f|_{B^c}) \), for all \( f \in L_0 \) and \( \bar{y} \in L_c \) such that \( f \sim^B \bar{y} \). We need to show that \( C^B = C/B \).

The equality \( I(u \circ f|_{B^c}) = u(\bar{y}) \) is equivalent to:

\[
\min_{p \in C} \left[ \int_B u \circ f \, dp + (1 - p(B))u(\bar{y}) \right] = u(\bar{y}) \tag{3.5}
\]

Let \( p^* \) be the minimand in the LHS problem. Then:

\[
\int_B u \circ f \, dp^* = p^*(B)u(\bar{y}) \tag{3.6}
\]

However this means that \( u(\bar{y}) \) is the conditional expectation, with respect to \( p^* \), of \( u \circ f \). Hence \( p^*B \) is equal to the posterior probability of \( p^* \).

By varying \( f \) one gets the set of all possible subjective probability measures on \( B \). \( C^B \) is the closure of the convex hull of all these possible \( p^*B \). Since each \( p^*B \) is a posterior probability of some \( p^* \in C \) one concludes that \( C^B \subset C/B \). I claim that \( C^B = C/B \). Suppose not, then without loss of generality one may assume that there exists \( p' \in C/B \) but such that \( p' \notin C^B \). By a separation theorem (Dunford and Schwartz), there exists an \( f \in L_0 \), such that:

\[
\int_B u \circ f \, dp' < \min_{p \in C \cup B} \int_B u \circ f \, dp \tag{3.7}
\]

Let \( \bar{y} \in L_c \) be such that \( f \sim^B \bar{y} \). The previous inequality implies that \( I(u \circ f|_{B^c}) < I^B(u \circ f) \), a contradiction. Q.E.D.
Bibliography


