THE CLASSIFYING RING OF GROUPS
WHOSE CLASSIFYING RING IS COMMUTATIVE

by

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ABSTRACT

We show that for groups whose classifying ring $U^K$
is commutative, the classifying ring is just the product
$U^K = U^G . U(k)^K$.

key words: lie group, classifying ring, lorentzide.

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To Alex Heller
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Preface. Why is $U^K$ a classifying ring?

Sophus Lie formalized the so-called "infinitesimal" method of studying groups by introducing the concept of their Lie algebras. This formalization was well suited to studying finite-dimensional representation theory, since the differential of such a group representation is a representation of its Lie algebra. However, the Lie algebra will not act on all the vectors in a Hilbert or Banach space representation of the group, for the same reason that not all $L^2$ or $L^1$ functions are differentiable.

Harish Chandra, noting that the irreducible representations of compact groups are all finite-dimensional, overcame this difficulty by restricting his attention to the subspace of $K$-finite vectors. More precisely, let $G$ be a semisimple Lie group with finite center, acting continuously on a Banach or Hilbert space $W$. If $K \subset G$ is a maximal compact subgroup, the subspace $V \subset W$ of $K$-finite vectors is defined as the union of all finite-dimensional subspaces of $W$ which are stable under $K$. Although the group $G$ does not preserve $V$, its Lie algebra does in the sense that for any $X \in \mathfrak{g}$, and any $v \in V$, the derivative $\frac{d}{dt}[\exp (tX)v] \bigg|_{t=0}$ exists and
V mirrors the structure of V in a number of important ways. It is irreducible (in the sense of having no subspaces invariant under g) if and only if W is irreducible. In this case, V is "locally K-finite"; that is, for every irreducible finite dimensional representation \( \gamma \) of K, the space \( \text{Hom}_K(\gamma, V) \) is finite dimensional. In fact every locally K-finite g module occurs as the space of K-finite vectors for some W. On the other hand, the set of K-finite vectors is always dense in V, and therefore determines V "up to questions of topology". In the unitary case the topology is determined by the restriction of the hermitian form to V, so that the correspondence between irreducible unitary representations and irreducible locally K-finite g modules which admit a g-invariant positive definite hermitian form, is bijective.

Harish-Chandra proved that these locally K-finite g modules are classified by the centralizer \( U^K \) of K in the enveloping algebra of g, in the following sense. Let \( \gamma \) be any irreducible finite dimensional K module; \( U^K \), since it preserves the K-primary decomposition of V, operates on \( \text{Hom}_K(\gamma, V) \). Then if V is irreducible, \( \text{Hom}_K(\gamma, V) \) is either 0 or is irreducible under the action of \( U^K \). Furthermore, for each \( \gamma \) there is a two sided ideal
I_γ \subset \mathcal{U}^K, \text{ such that the correspondence } V \leftrightarrow \text{Hom}_K(γ, V) \text{ induces a bijection between irreducible locally } K\text{-finite}\ g\ \text{modules "containing } γ\text{" (i.e. } \text{Hom}_K(γ, V) \neq 0\text{) and irreducible finite dimensional } \mathcal{U}^K\ \text{modules annihilated by } I_γ.
1. Introduction

In this paper, we prove that for real semisimple Lie groups whose classifying ring $U^K$ is commutative, the classifying ring is the product of the center of $U$ and the center of $U(k); U^K = U^G \cdot U(k)^K$. This leaves open the question of whether in general, $U^G \cdot U(k)^K$ is the center of $U^K$. That $U(k)^K$, and consequently the product $U^G \cdot U(k)^K$, is always contained in the center of $U^K$, follows from the fact that no nonzero element of the enveloping algebra can die in every finite dimensional representation of $g$. If $V$ is a finite dimensional representation of $g$, and $\gamma$ an irreducible representation of $K$, then since $U(k)^K$ acts via scalars in $\gamma$ it acts via the same scalars in the primary subspace $V[\gamma]$ of vectors which "transform according to $\gamma". Therefore $U(k)^K$ commutes in $V$ with any operators which preserve the $K$-decomposition, and in particular with those operators coming from $U^K$. Therefore any commutator $zx - xz$ with $x \in U^K$ and $z \in U(k)^K$ dies in all such $V$ and a fortiori was zero.

The Lie algebra of a semisimple group whose classifying ring is commutative is a sum of simple factors which are all either compact or "lorentzide" - $su(n,1)$ or $so(n,1)$. This follows from a theorem of Kostant which is a corollary to his (never published) multiplicity formula for restricting
representations of semisimple lie algebras to their reductive subalgebras. The corollary states that a semisimple proper subalgebra of a simple lie algebra, \( h \subset g \), has the "multiplicity 1 property" that every irreducible finite dimensional representation of \( g \) decomposes over \( h \) as a product of multiplicity 1 factors, if and only if the complexification of the pair \( h_C \subset g_C \) is isomorphic to one of the standard pairs \( \text{so}(n,\mathbb{C}) \subset \text{so}(n+1,\mathbb{C}) \), \( \text{su}(n,\mathbb{C}) \subset \text{su}(n+1,\mathbb{C}) \), or \( \text{u}(n,\mathbb{C}) \subset \text{su}(n+1,\mathbb{C}) \). Thus the real simple algebras for which \( k \subset g \) has this multiplicity 1 property are the compact and lorentzide algebras. On the other hand, the multiplicity 1 property implies that the image of \( U^K \) is commutative in any finite dimensional representation of \( g \), so that commutators die and \( U^K \) itself is commutative. Conversely, since the irreducible modules for commutative rings are all one dimensional, commutativity of the classifying ring implies the multiplicity 1 property.

For compact groups, of course, \( K = G \) and \( U(k)^K = U^K = U^G \). For the lorentzide groups, on the other hand, \( U^G \cap U(k)^K = \mathbb{C} \); moreover the multiplication map \( U^G \otimes U(k)^K \rightarrow U^K \) is injective. We shall prove, in fact, that this map is an isomorphism. Since a homomorphism of filtered rings is an isomorphism if and only if its
associated graded is, and since, by the Poincare-Birkoff-Witt theorem, the associated graded is just the corresponding multiplication map on symmetric algebras, this is equivalent to proving that $S_G \otimes S(k)^K + S^K$ is an isomorphism. It is equivalent as well to work over $\mathbb{C}$ and prove the corresponding statement for the complexification $K_C \subset G_C$, and this will allow us to adopt the point of view of algebraic geometry, that commutative rings are rings of functions on their associated maximal ideal spaces, or "spec"s.

With this in mind we adopt the convention that all groups, lie algebras and vector spaces are from here on taken to be complex. (e.g. $\text{So}(n,1)$ means $\text{So}(n,1;\mathbb{C}); K \subset \text{So}(n,1)$ is $\text{So}(n,\mathbb{C})$).

From this point of view the symmetric algebra $S = S(g)$ is the ring of polynomial functions on the linear dual $g'$ of $g$. Intuitively, $K$ invariant polynomials should be functions on an algebraic "quotient" space, and we take the liberty of defining $g'/K$ as $\text{spec } S(g)^K$. Since tensor product of rings corresponds to cartesian product on specs, our goal is to prove that $g'/K = g'/G \times k'/K$. Injectivity of the map on rings amounts to the fact that the image of $g'/K$ is dense in
$g'/G \times k'/K$, and this is demonstrated in §2. The next stage is to show that $S^K$ is actually a polynomial ring. The proof involves somewhat technical theorems of Kostant and Kostant-Rallis; these provide "local" results which are fit together according to a scenario using some mild theorems from algebraic geometry. This scenario is presented as §3, rather than at the end, in order to give the reader some geometric intuition for what is to follow. Basically the idea is to consider $g'/K$ as fibered over $g'/G$. Kostant has shown that the fibers of $g'$ over $g'/G$ are "essentially" orbits $G\cdot x$, and in §4 we show that the fibers of $g'/K$ over $g'/G$ are, as one might hope, "essentially" of the form $K G/G^X$. Although we cannot directly compute the fibers over every point, we are content by the discussion in §3, to compute them for the large open set defined in §5 - the principle being that for normal varieties, functions always extend over sets of codimension 2.

The idea involved in computing $K\backslash G/G^X$ is to interchange the order of taking quotients, and §6 is devoted to getting a good hold on $G/K$. In §7 we finally show that $G^X\backslash G/K$ can be identified with spec of a polynomial ring, and in §8 we show that this identification comes from a global map. At this point, according to the scenario in §3, we have proved that $S^K$ is a polynomial
ring. Since $S^G \otimes S(k)^K$ is also a polynomial ring, all that remains to be shown is that their generators occur in the same degrees, and this is done in §9.

At the conclusion, in §10, we use the results through §7 to show that similar statements about $S^K$ fail for $\text{Sp}(n,1)$. 
2. Injectivity

Let \( g'/G = \text{spec } (S(g)^G) \), \( g'/K = \text{spec } S(g)^K \).
Then multiplication \( S(g)^G \otimes S(k)^K \rightarrow S(g)^K \) induces a map \( g'/K \rightarrow g'/G \times k'/K \). We will prove that for \( G = \text{Su}(n,l) \) or \( \text{So}(n,l) \), this map has dense image; or what is the same, the map on rings is injective.

It clearly suffices to show that the map \( g' \rightarrow g'/G \times k'/K \) has dense image. If we use the killing form to identify \( k' \oplus p' = g' = g = k \oplus p \), then the projection onto the second factor is just the orthogonal projection of \( g \) onto \( k \) followed by the quotient map \( k \rightarrow k/K \), and is therefore surjective.

Our intent, then, is to show that for an open set in \( k/K \), each fiber has a dense image in \( g/G \): In fact, if \( x \in k \) is any regular semisimple element (i.e. \( k^X \) is a cartan subalgebra), then the image of the coset \( x + p \) is dense in \( g/G \).

We take for simplicity, \( G = \text{Su}(n,l) \) (the case of \( \text{So}(n,l) \) being left to the reader). Since \( x \) is semisimple we may take it to be a diagonal matrix, and since it is regular its eigenvalues \( \lambda_i \) must be distinct.
Thus the coset \( x + p \) consists in all matrices of the form
The invariant polynomials on $g$ are generated here by the coefficients of $t^{n-1}, t^{n-2} \ldots t^0$ in the characteristic polynomial of the matrix; of course the coefficient of $t^n$ is just the trace of the matrix and therefore is zero. We can compute the characteristic polynomial for our matrix in $x + p$, by inductively expanding the minors. Expanding by the first row shows that the characteristic polynomial, $\chi = (t-\lambda_1)\chi_1 + \prod b_{1}D$, where $\chi_1$ is the characteristic polynomial of the matrix

\[
\begin{pmatrix}
\lambda_1 & a_1 \\
\lambda_2 & a_2 \\
\vdots & \vdots \\
\lambda_n & a_n \\
b_1 b_2 \ldots b_n - \Sigma \lambda_i
\end{pmatrix}
\]
and $D$ is the determinant of the "semi-upper triangular" matrix

$$
\begin{pmatrix}
0 & -a_1 \\
(t-\lambda_2) 0 & -a_2 \\
\vdots & \ddots & \ddots \\
(t-\lambda_n) & \cdots & 0 & -a_n
\end{pmatrix}
$$

This in turn may be expanded by its first row to show that

$$D = \pm a_1 \prod_{j=2}^{n} (t-\lambda_j).$$

The sign occurring in this expansion is opposite from the one in the previous expansion, since the dimensionality has decreased by one.

Thus we have

$$\chi = (t-\lambda_1)\chi_1 - a_1 b_1 \prod_{j=2}^{n} (t-\lambda_j).$$

At each inductive step we will pick up a term $-a_i \prod_{j \neq i}^{n} (t-\lambda_j)$, so it is clear that

$$\chi = \tilde{\chi} - \sum_{j \neq i} a_i b_i \pi_i,$$

where

$$\pi_i = \prod_{j=1}^{n} (t-\lambda_j),$$

and $\tilde{\chi}$ is the characteristic polynomial of the diagonal matrix $x$.

Of course what we have actually done is to compute the map from $x + p$ to $g/G$; it is a map into a linear
space followed by an affine map. In coordinates, the map into the linear space, \( x + p + \mathbb{C}^n \) takes our matrix into the \( n \)-tuple \( a_i b_i \), and this map is clearly surjective. The affine map has a constant part which is given as the coefficients of \( t^{n-1}, t^{n-2}, \ldots \) in the polynomial \( \bar{\chi} \), and is, for our purposes, irrelevant. What we need to establish is that the linear part, which is given by the matrix \( c_{ij} \) of coefficients of the polynomials \( \pi_i \), is an epimorphism. Its domain and range are both \( n \)-dimensional spaces, so it will suffice that the linear part be injective. But the matrix of coefficients simply expresses the polynomials \( \pi_i \) in terms of the basis \( t^{n-1}, t^{n-2}, \ldots \); thus the matrix will be nonsingular precisely when the \( \pi_i \) are linearly independent. This will follow from the hypothesis that the eigenvalues \( \lambda_i \) are distinct. For any complex number, evaluation at that number is a linear functional on the space of polynomials; so, in particular, at the number \( \lambda_j \). But \( \pi_i(\lambda_j) = \pi_i(\lambda_j - \lambda_i) \) is zero for \( i \neq j \) and nonzero for \( i = j \), hence a nonsingular matrix. Thus the \( \pi_i \) are linearly independent, so we have proven that for regular semisimple points of \( k \), \( x + p \) maps onto \( g/G \), and therefore that the image of \( g \) is dense in \( g/G \times k/K \).
3. The scenario

We present here a scenario for proving that $S^K$ is a polynomial ring for the Lorentzide groups $SU(n,1)$ and $SO(n,1)$. The idea is to consider the projection $p : g'/K \rightarrow g'/G$, and to show that "enough" of the fibers can be identified with spec of a polynomial ring, in some global fashion.

If $L \subset S$ is any linear subspace of the symmetric algebra, the inclusion $L^K \subset S^K$ induces a ring homomorphism from the symmetric algebra on $L^K$, $i_* : S(L^K) \rightarrow S^K$ corresponding to a map of specs $i^* : g'/K \rightarrow (L')^K$. We propose to construct such an $L$, and an open set $U \subset g'/G$, such that

1) the complement of $U$ in $g'/G$ has codimension $\geq 2$

and

2) for each $u \in U$, the restriction of $i^*$ to that fiber is an isomorphism.

By "isomorphism" we mean in the sense of algebraic varieties, which in this case amounts to the fact that

$$i_* : S(L^K) \rightarrow S^K \otimes S^G/m_u$$

is an isomorphism of rings ($m_u$ is the maximal ideal corresponding to $u \in U$), which is the same as saying that $S^K/m_uS^K$ is a polynomial ring on $L^K$. Isomorphism in this sense implies that $i^*|_u$ is a bijection of point sets, and therefore $p^{-1}(u)$ is bijective with $U \times (L')^K$. Thus $i^* \times p$ induces a bijection between
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the two inverse images of $U$ in the following diagram:

$$
\begin{array}{c}
\text{g'}/K \\
\downarrow \pi \\
g'/G
\end{array}
\xrightarrow{i^* \times p}
\begin{array}{c}
(L')^K \times g'/G \\
\downarrow \text{p} \\
\text{g'}/G
\end{array}
$$

But now Zariski's Main Theorem tells us that $i^* \times p$ induces an algebraic isomorphism between the two inverse images; the theorem applies because $(L')^K \times g'/G$ is spec of a polynomial ring, hence normal, and $S^K$ is finitely generated, as was proved by Hilbert.

Let $\Gamma$ denote the "regular function" functor; thus the fact that $i^* \times p$ is an algebraic isomorphism on the inverse images implies that $\Gamma(p^{-1}U) \cong \Gamma(p^{-1}U)$. But $\Gamma(p^{-1}U) = S^K$. The point is that $g'/K$ is a normal variety, and rational functions on normal varieties always extend over sets of codimension 2. To see that the complement of $p^{-1}U$ has codimension $\geq 2$, we need to know that the fibers of $p$ are all equidimensional, and this follows because $S^K$ is flat over $S^G$. In fact Kostant has shown that there is a "harmonic" decomposition $S = S^G \otimes H$, where $H$ is not a subring of $S$ but a linear subspace stable under $G$, and this implies that $S^K = S^G \otimes H^K$, and is therefore free
over $S^G$.

A similar argument shows that $\Gamma(\pi^{-1}U) \approx S(L^K) \otimes S^G$. Thus $(i^* \otimes p)_*$ is an isomorphism of rings, and $i^* \times p$ is an isomorphism of their specs.
4. Fibers

Some of the charm in the notation $g'/G$ lies in the fact, due to Kostant $^{12}$, that for any complex reductive group $G$, the fibers of the projection $g' + g'/G$ can "essentially" be identified with the orbits of maximal dimension, called regular orbits. The fact is that although a fiber may be a union of many orbits, it will always contain one regular orbit $Gx$ as an open subset whose complement has codimension $\geq 2$; and since the fibers are normal, the rings of regular functions will be the same. If $m_x^{-} \subset S^G$ denotes the maximal ideal in $S^G$ corresponding to the image of $x$, this says that $S/m_x^{-}S = \Gamma(Gx)$.

What we are interested in, according to our scenario, are the rings $S^K/(m_x^{-})S^K$. Since $(m_x^{-}S)^K = (m_x^{-}S)S^K \subset S^K$, this is the same as $(S/m_x^{-}S)^K$, hence by Kostant, $\Gamma(Gx)^K$. On the other hand, the theory of algebraic groups says that functions on the quotients by a closed subgroup are just functions invariant under that subgroup. Of course, $G$ acts on $\Gamma(G)$ in two ways, and if we denote $\Gamma(Gx) = \Gamma(G/G^x)$ by $G^x(\Gamma(G))$ then our fiber over $\bar{x}$ is just $G^x\Gamma(G)^K$. 
Although the points of $g'/G$ can be identified with regular orbits, a more classical interpretation, which works more generally, is to identify them with closed orbits. If we use some invariant bilinear form to identify $g'$ with $g$ itself, the elements with closed orbits correspond exactly to the semisimple elements of $g$. In fact, if $x = x_s + x_n$ is the Jordan decomposition of any $x \in g$, then $\overline{x}$ and $(\overline{x_s})$ correspond to the same element of $g'/G$; this is a generalization of the fact that the characteristic polynomial of a matrix depends only on its semisimple part. It turns out, moreover, that $x$ is regular in $g$ if and only if $x_n$ is regular in $g_s$.

There is still another set of orbits with which we can identify $g'/G$; if $h \subset g$ is any cartan subalgebra, then $g/G$ is bijective with the set of Weyl group orbits in $h$. The point is that every semisimple orbit meets $h$. Thus it is easy to calculate the dimension of the stabilizer $x_s$; choosing an appropriate conjugate we may assume $x_s \in h$, and add to $\dim h$ the number of positive and negative roots which vanish on $x_s$.

We can now define $U_g$, or simply $U$, in two equivalent ways, corresponding either to Weyl group orbits in $h$ on
which at most one root vanishes, or regular orbits such that \( \dim g^x_s \leq \dim h + 2 \). It is clear from the second definition that \( U \) does not depend on the choice of \( h \), and from the first definition, that its complement has codimension > 2. In fact, if one root \( \alpha \) vanishes on \( x_s, x_n \in h \), \( g^s = h \oplus C_{\alpha} \oplus C_{-\alpha} = \ker \alpha \oplus t_{\alpha} \) where \( t_{\alpha} \) is the "root tds" through \( \alpha \). We have already seen that \( x = x_s + x_n \) is regular if and only if \( x_n \) is regular in \( g^s \); on the other hand, for \( SL(2) \) any two nonzero nilpotent elements are conjugate, so that up to conjugation we may take \( x_n = e_\alpha \) and \( g^x = \ker \alpha \oplus C_{e_\alpha} \). Since the Weyl group operates transitively on roots of a given length, \( g^x \) depends up to conjugacy only on the length of \( \alpha \). Kostant has shown that \( G^x \) is connected, so \( G^x \) also depends only on the length of \( \alpha \), and therefore \( G^x \Gamma(G) \) as well, up to equivariant ring isomorphism.

We conclude with a remark which will allow us later to reduce to small groups; namely, if \( g_1 \subset g \) is a reductive subalgebra containing \( h \) and \( t_{\alpha} \), then \( x \in g_1 \) and \( x \in U_{g_i} \).
We now pause for a discussion germane to a classical rank 1 group \( G = G_F(n, l) \); more precisely, if \( F \) is one of the three (skew) fields over \( R \), then \( G \) is the complexification of a maximal compact subgroup of \( \text{Aut}_F(F^{n+1}) \), and \( K \) consists in those automorphisms which preserve the complexification of the decomposition \( F^{n+1} = F^n \otimes F_e \). In particular \( G_e \subseteq K \). In fact it is a normal subgroup, and the quotient \( K/G_e = G_F(0, l) \); the identification arises from the fact that the action of \( K/G_e \) on \( K_e \) coincides with the action of \( G_F(0, l) \) on \( K_e \subseteq F \otimes_R C \).

But \( K/G_e \) acts on all of \( G_e \) by letting \( K \in K/G_e \) take \( g_e \) to \( g_k^{-1}e \); in fact this is just the action which induces the isomorphism

\[
G/K \simeq G/G_e/K/G_e.
\]

And again, if we denote by \( U_F \) the "unit" group of \( F \); that is, \( G_F(0, l) \) acting as scalars in all of \( F^{n+1} \otimes C \); we can identify \( K/G_e = U_F \) so that the actions coincide on \( G_e \subseteq F^{n+1} \otimes C \).

As fortune would have it, we know all about the action both of \( U_F \) and \( G \) as well on \( F^{n+1} \otimes C \). For if we let \( \tilde{G} = G_F(n+1, l) \), so that \( \tilde{G} = K \otimes \tilde{p} \), then we can identify \( \tilde{p} = F^{n+1} \otimes C \), and then \( \tilde{K}^e = G \). Furthermore, the action \( \rho : \tilde{K} \rightarrow \text{Aut} \tilde{p} \) identifies

\[
\rho(\tilde{K}) = G \cdot U_F \subseteq \text{Aut} (\tilde{p})
\]
Thus we are in a position to apply the results of Kostant-Rallis\(^5\), which generalize to \( \tilde{\mathfrak{p}} \) some of the theorems of Kostant which we have already quoted for the adjoint representation. In particular, \( \tilde{S} = S(\tilde{\mathfrak{p}}) = S^K \otimes \tilde{H} \), and since \( e \) is a regular semisimple element of \( \tilde{\mathfrak{p}} \), we have \( \tilde{H} = \Gamma(\tilde{K}_0 e) \) as \( \tilde{K}_0 \) modules, where \( \tilde{K}_0 \) is defined in Kostant-Rallis but, unless \( \tilde{g} = \mathfrak{so}(2,1) \) or smaller, \( \tilde{K}_0 e \) is connected, so \( \tilde{K}_0 e = \tilde{K} e \). But we have seen that \( \rho(\tilde{K}) = GU_F \), and \( U_F e = K e \). So \( \tilde{K}_0 e = G K e = G e \). Finally, since \( G/K \cong G e/U_F \), we have

\[
K \Gamma(G) \cong \tilde{U}_F^{\tilde{H}}
\]

as \( G \) modules.
7. $S^{U_F \cdot G^X}$

Let $x \in U_g$ be as in §5. Remember $G^X$ is connected, so that $S^{G^X} = S^g^X$. Claim:

(*) if $G = Su(n,1)$ or $So(n,1)$, then $S^{U_F \cdot G^X}$ is a polynomial ring with generators in degree 2 and, for $G = So(2m,1)$, one generator in degree 1.

Consider the case $G = Su(n,1)$. Then $U_F \cdot G$ acts as $U(n,1)$, and $U_F \cdot G^X = U(n,1)^X$. If no roots vanish on $a$ then we may take, up to conjugacy, $u(n,1)^X = d(n,1)$; that is, the set of matrices which are diagonal with respect to the decomposition $C^{n+1} = \oplus C_i$. If $z_i$ and $\bar{z}_i$ are eigenvectors in $C_i \oplus C$, then the desired generators for our ring are $z_i \bar{z}_i$.

Suppose then, that one root $\alpha$ vanishes on $x$. Since there is only one root length we may take, by our remarks concluding §5, $e_\alpha \in u(1,1)$; and $u(n,1)^X = d(n-1) \oplus u(1,1)^X$, compatible with the decomposition $C^{n+1} = C^{n-1} \oplus C^{1+1}$. Since each factor algebra operates trivially in the "other's" summand, we get an induced tensor product decomposition of the ring of invariants,

$$S^{u(n,1)^X} = S(C^{n-1} \oplus C)^{d(n-1)} \oplus S(C^{1+1} \oplus C)u(1,1)^X$$
We have already noted that the first factor is generated by elements of the form $z_i \overline{z}_i$. To compute the second factor, we may take $x \in U(1,1)$ or even $x \in U_{su(1,1)}$; thus we are reduced to the case $g = su(1,1)$. In fact, this is essentially the same as the case $g = so(3,1)$:

Observe that a hermitian form on $\mathbb{C}^2$ is an orthogonal form on the underlying $\mathbb{R}^4$, so that $u(1,1) \subset so(2,2)$, and since everything is complex there is no problem about signature; $U_F = U_R = 0$; and we may take $x_1 \in U_{so(3,1)}$ with $u(1,1)^x = so(3,1)^{x_1}$. So we are reduced to the following lemma: (*) is true for $G = So(3,1)$.

Proof: The proof when $g^x$ is a cartan is essentially contained in the discussion above. On the other hand, since $g = t_\alpha \oplus t_\beta$ is a sum of two roots $\alpha \beta$, we may take $g^x$ as being spanned by $e_\alpha$ and $h_\beta$. Let $V^\alpha_j$ be the irreducible $g$ module in which the highest eigenvalue of $h_\alpha$ is $j$, and in which $t_\beta$ acts trivially; similarly for $V^\beta_j$. Then a dimension argument shows that the harmonics in degree $j$, $H^\alpha_j = V^\alpha_j \otimes V^\beta_j$. In particular, all the harmonics killed by $h_\beta$ occur in even degrees, and $\dim(H^\beta_{2e}) e_\alpha = 1$. Similarly, since $S^G$ is a polynomial ring on one generator of degree 2, $\dim(S^G_{2e}) = 1$. Therefore,
as soon as we show that the two generators in degree 2 satisfy no relations, a dimension argument will show that they generate all of $\tilde{S}^G_X = \tilde{S}^G \circ \tilde{H}^G_X$.

But the relations span a homogeneous ideal in a polynomial ring on two generators of the same degree, and since $\tilde{S}$ is an integral domain the ideal must be prime. Such ideals correspond to single points in projective space, and are generated by linear subspaces of the generating subspace. Since our generators are linearly independent, there can be no relation of this form. QED

The proof for $\text{So}(n,1)$, when $n$ is odd, is entirely analogous to that for $\text{Su}(n,1)$. Again there is only one root length, and the reduction is to $\text{so}(3,1)$. When $n$ is even, the cartan is "too small" and has an invariant in degree 1, as does $G^X$ for $x$ corresponding to the long root, where again $t_\alpha \subset \text{so}(3,1)$. For the short root all the invariants of $\ker \alpha$ in $R^{n-2}$ have degree 2, but $t_\alpha^X = \text{so}(2,1)^e$ has one invariant of degree 2 and one of degree 1, on $R^3$. The proof of this last fact is similar to that of the lemma, except that the supposed relation is quadratic rather than linear.
We have essentially shown that $K_{\Gamma}(G)^{G^X}$ is a polynomial ring. Our intention is to prove this, and to give as well an "intrinsic" characterization of the generating subspace, a characterization which may be "transferred" to the fibers $\Gamma(G)^{K}$.

Assume, for the moment, $G \neq SO(2m,1)$. Let $L = \tilde{U}^F_p$. To restate the result of §7, $S^{\tilde{U}^F_p \cdot G^X}$ is the symmetric algebra on $(\tilde{S}_2)^{U^F_p \cdot G^X} = \tilde{S}_2 \otimes \tilde{G}^{G^X}$. Since $G$ has rank 1, by Chevalley's theorem $\tilde{S}_G$ is a polynomial ring on one generator in degree 2, so that $S(\tilde{S}_2) = \tilde{S}_G$ and $S^{\tilde{U}^F_p \cdot G^X} = \tilde{S}_G \otimes S(\tilde{L}^{G^X})$. By section 4, $K(\Gamma(G)) = \tilde{H}^U_p$, and if we call this isomorphism $j^*$, the inclusion of $j^*(\tilde{L}^{G^X})$ in $K_{\Gamma(G)^{G^X}}$ induces an isomorphism

$$j^* : S(\tilde{L}^{G^X}) \rightarrow K_{\Gamma(G)^{G^X}}$$

By a theorem of Helgason, a representation of $g$ of a rank one group has at most a one dimensional subspace of $K$-fixed vectors. Then by Frobenius reciprocity the multiplicity of a representation of $g$ in $K_{\Gamma(G)}$ is at most one. Thus $j^*\tilde{L}$ is a sum of primary subspaces of $K_{\Gamma(G)}$, over some index set $C$. This gives our alternate characterization.
Again using Frobenius reciprocity, we can decompose $\Gamma(G)$ as a sum of irreducible $G \times G$ modules,

$$\Gamma(G) = \bigoplus_{\gamma \in C} \nu_{\gamma} \oplus \nu_{\gamma}^*.$$ 

Now let $\gamma'$ be contragredient to $\gamma$, so we can "transfer"

$$(j*L)_{G^X} = \bigoplus_{\gamma' \in C} K_{\gamma'}(\gamma(G))_{G^X}$$

using awkward, but suggestive, notation.

We can now define $L$, as promised in §3, as

$$L = \bigoplus_{\gamma' \in C} (H_{\gamma'}).$$

By §4 we need to prove that

$$i^* : S(L^K) \rightarrow G^X_{\Gamma(G)^K}$$

is an isomorphism. But

$$i^*(L^K) = \bigoplus_{\gamma' \in C} (G^X_{\Gamma(G)})_{G^X}$$

is by design just the transpose of $(j*L^X)$; in fact $i^*$ is just the transpose of $j^*$, which we already know to be an isomorphism. So $i^*$ is an isomorphism as promised.

In case $G = S\mathfrak{o}(2m,1)$, we may view $S\mathfrak{u}_F$ as a polynomial ring on generators of degree 2, to which the square root of one of the generators is adjoined. In fact the subring to which we adjoin a square root has a geometric interpretation: here $K_0 = O(n)$, and only even degree polynomials are invariant under $K_0$. Moreover the entire
construction of the preceding paragraph goes through, with the conclusion if we care to make it that $S(g)^K$ is a polynomial ring. In any case the construction produces a subspace $L_2 \subseteq H^K$, and the obvious analogue applied to the generator in degree one produces a one dimensional subspace $L_1 \subseteq H^K$ which we will note as a corollary is related to the pfaffian in $S(k)^K$. $L$ may be taken to be the span of $L_1$ and any complement of the projection of $(L_1)^2 \subseteq S$ into $L_2 \subseteq H$. In the next section we will show that there is in fact a canonical complement with respect to the graded structure.
9. Degrees

We have seen now that $g'/K = g'/G \times (L')^K$; but $g'/K$ has a homogeneous structure as well, and in this section we would like to calculate the degrees in which $\mathfrak{h}$ occurs as a subspace of $S$.

Kostant has shown, for any irreducible representation $V$ of $g$, that the degrees in which $V$ occurs harmonically may be calculated directly from the structure of $V$ as a $g$ module, by using the principal tds of Dynkin. The principal tds is a subalgebra of $g$, isomorphic to $\mathfrak{sl}(2)$, conjugate to one defined relative to a cartan subalgebra and a system of positive roots; $e$ is a sum of positive simple root vectors and $h$ is in the cartan, characterized by the property that $\phi(h) = 2$ for every positive simple root $\phi$. The multiplicity with which $V$ occurs harmonically is given by $\dim V^\mathfrak{h}$, where $\mathfrak{h}$ is a cartan subalgebra of $g$; if this is not zero then $\dim V^\mathfrak{h} = \dim V^{(g^e)}$, and the degrees are given as the eigenvalues of $\frac{1}{2}h$ in $V^{(g^e)}$.

According to §7 and §8, we are interested in $S_2(\tilde{\mathfrak{p}}^U)$, and, in the case of $\text{So}(2m,1)$, in $\tilde{\mathfrak{p}}$ as well. For $\text{Su}(n,1)$, $\tilde{\mathfrak{p}} = \tilde{\mathfrak{p}}^+ \oplus \tilde{\mathfrak{p}}^-$, where the sign refers to the character of $\mathfrak{u}_F$. Thus $S_2(\tilde{\mathfrak{p}})^U = \tilde{\mathfrak{p}}^+ \oplus \tilde{\mathfrak{p}}^-$. It is easy to see, by writing explicitly the highest weight in
\( \tilde{\mathfrak{p}}^+ \) and \( \tilde{\mathfrak{p}}^- \), that they are irreducible under the principle tds; the point is that the eigenvalue of \( \hbar \) in the highest weight space is \( n \), corresponding to an \( n+1 \) dimensional irreducible representation, but \( \tilde{\mathfrak{p}} \) itself is only \( 2(n+1) \) dimensional. Using the Clebsch-Gordon formula, \( \tilde{\mathfrak{p}}^+ \oplus \tilde{\mathfrak{p}}^- \) is a sum of irreducible odd dimensional representations, with highest eigenvalue \( 2n, 2n-2, \ldots, 0 \). Thus \( (\tilde{\mathfrak{p}}^+ \oplus \tilde{\mathfrak{p}}^-)^e \) is \( n+1 \) dimensional. But we have already seen that \( S_2(\tilde{\mathfrak{p}}) \) is \( n+1 \) dimensional, so we must have \( (\tilde{\mathfrak{p}}^+ \oplus \tilde{\mathfrak{p}}^-)^e = (\tilde{\mathfrak{p}}^+ \oplus \tilde{\mathfrak{p}}^-)^G \). In defining \( \tilde{L} \) we used the complement of the invariant subspace \( S_2^G \), which must correspond to the eigenvalue 0; thus the eigenvalues of \( \frac{1}{2} h \) are \( n, n-1, \ldots, 1 \), and these are the degrees of \( L^K \).

We promised in §8 to give a more canonical construction of a complement for \( (L^K_1)^2 \), in the case of \( \text{So}(2m,1) \). Here again, \( \tilde{\mathfrak{p}} \) is irreducible over the principle tds, and of course the highest eigenvalue is \( 2m \), so that \( L^K_1 \) occurs in degree \( m \). \( \tilde{L}_2 \) is contained in \( S_2(\tilde{\mathfrak{p}}) \), which decomposes into irreducibles of highest weights \( 4m, 4m-4, \ldots, 0 \). All but the trivial weight are contained in \( \tilde{L}_2 \), so that \( L^K_2 \) has degrees \( 2m, 2m-2, \ldots, 2 \). Since \( S^K \) is a graded ring, \( (L^K_1)^2 \) is the component in degree \( 2m \), and the components of degree \( 2m-2, \ldots, 2 \) form a nice homogeneous complement.
For $\mathfrak{so}(2m-1,1)$ the situation is a bit more delicate. The highest weight vector of $\tilde{\mathfrak{p}}$ contributes a submodule of dimension $2m-1$, say $\tilde{\mathfrak{p}}_1$, which must then have a one dimensional complement $\tilde{\mathfrak{p}}_0$. The saving grace is that the principal tds may be taken to be in $k$, so that $S_2(\tilde{\mathfrak{p}}) = S_2(\tilde{\mathfrak{p}}_1) \otimes \tilde{\mathfrak{p}}_1 \otimes \tilde{\mathfrak{p}}_0 \otimes S_2(\tilde{\mathfrak{p}}_0)$ is a sum of $K$ modules. Thus although $(\tilde{\mathfrak{p}}_1 \otimes \tilde{\mathfrak{p}}_0)^e$ is one dimensional, we must have $(\tilde{\mathfrak{p}}_1 \otimes \tilde{\mathfrak{p}}_0)^g e = 0$. The point is that we may choose a sequence of regular semisimple elements of $k$, $x_1 + e$, so that $\tilde{S}_2(x_1) = S_2^e$. We know that $\tilde{S}_1^e = \tilde{p}_1^e = 0$, so also $(\tilde{p}_1 \otimes \tilde{p}_0)^g e = 0$ and hence $(\tilde{p}_1 \otimes \tilde{p}_0)^g e = 0$. Thus $\tilde{S}_2^e = S_2(\tilde{p}_1)^g e \otimes S_2(\tilde{p}_0)^g e$. By Clebsh-Gordon, this is contained in a space with eigenvalues for $h = 4m-4, 4m-8 \ldots 0$ and 0 again; the eigenvalue 0 may not occur, since it would correspond to harmonics of degree 0, and the dimension is $\dim \tilde{S}_2^e = m$. Thus $L^K$ has degrees $2m-2, 2m-4 \ldots 2$.

It is now a simple verification that $S^K$ and $S^G \otimes S(k)^K$ are both polynomial rings on generators of the same degree, hence equal.
10. Counterexamples

Everything up until §6 was written with sufficient generality to include the case of $\text{Sp}(n,1)$. These groups provide examples for which $S^K$ is not a polynomial ring, and for which $g'/K$ is not even a product of $g'/G$ and some complement.

It is quite easy from what we did in §6 to see what the general fibers are of the map $g'/K + g'/G$. $U_F$ in this case is $\text{Sp}(1)$. If we let $d = \oplus d_i$ be the sum of $n+1$ copies of $\text{Sp}(1)$ corresponding to the decomposition of $\mathfrak{p}$ as $n+1$ copies of $H \otimes C$, each $H \otimes C$ is isomorphic to $V_1 \otimes V_1$ as a $U_F \otimes d$ module. Any cartan subalgebra $h = \oplus h_i \subset d$ is also a cartan subalgebra of $\text{Sp}(m,1)$.

An argument similar to the one in section 8 will show that $S(H_i \otimes C) \cong S(V_2)[N]$ as a $U_F$ module, where $N$ (for "norm") is an invariant indeterminate whose square is equal to the invariant of degree 2 in $S(V_2)$. Thus

$$S(\mathfrak{p}) = (\oplus S(V_2')) \oplus [N_i].$$

The ring of invariants is well known (as in Weyl) to be a ring with certain generators in degree 2 ("dot products") and degree 3, ("triple products") satisfying certain relations coming from the evaluation of determinants in the quadratic generators. For $n = 2$, for instance, there are six
dot products and one triple product, which satisfy the relation

\[
\begin{vmatrix}
  x \cdot x & y \cdot x & z \cdot x \\
  x \cdot y & y \cdot y & z \cdot y \\
  x \cdot z & y \cdot z & z \cdot z
\end{vmatrix} = (x \times y \times z)^2.
\]

The point is that for \( n \geq 2 \), this ring is always singular, and therefore an open subset of \( g'/K \) is isomorphic to the product of an open subset of \( g'/G \) with a singular variety, hence \( g'/K \) is itself singular, and in particular is not spec of a polynomial ring.

To see that \( g'/K \) is not the product of \( g'/G \) and some complement, it is enough to show that the fibers are not all isomorphic. For \( \text{Sp}(n,1) \) there are two root lengths. The long roots correspond to the root tds's \( d_1 \), and the fibers over their zero sets in \( h/W \) are indeed isomorphic to the general fibers. The point is that \( g^X \subset d \), so that \( \tilde{\mathfrak{p}} \) decomposes into the same direct summands as for \( h \).

For the short roots the best we can do is to set \( s = \text{sp}(1,1) \), and write \( \tilde{\mathfrak{p}} = \tilde{\mathfrak{p}}' \oplus \tilde{\mathfrak{p}}_2 \) as a \( d' \oplus s \) module, so that \( g^X = h' \oplus s^X \). As an \( s \oplus \mathfrak{u}_F \) module, \( \tilde{\mathfrak{p}}_2 = V_{1,0} \oplus V_1 \); of course the subscript \( 1,0 \) refers to the weight lattice for \( s \) which is spanned by \( \varepsilon_1, \varepsilon_2 \):
Let $x$ be a regular element corresponding to a short root $\alpha$; that is, $x_n = e^\alpha$.

Lemma: Let $S^{ev}(\tilde{P}_2)$ be the subring generated by even degree elements. Then

$$S(\bar{p})^u_F \otimes g^x = [S(\bar{p}')^{h'} \otimes S^{ev}(\tilde{P}_2)^x]^{u_F}$$

Proof: Clearly $S(\bar{p})^u_F \otimes g^x = [S(\bar{p}')^{h'} \otimes S(\tilde{P}_2)^S]^{u_F}$. But $S(\bar{p}')^{h'}$, as we have seen, involves only odd dimensional representations of $u_F$; therefore only odd dimensional subrepresentations of $S(\tilde{P}_2)^S$ will make any contribution to the ring of invariants. And since $\tilde{P}_2 = V_{1,0} \otimes V_1$, the Clebsch-Gordon formula insures that all the odd dimensional subrepresentations of the tensor algebra are contained in even degree.

The significance of the lemma for us is that every $s$-submodule of $S^{ev}(\tilde{P}_2)$ contains a zero weight vector, and hence $\dim S_{2m}(\tilde{P}_2)^x = \dim S_{2m}(\tilde{P}_2)^{h^2}$, where $h^2$ is a
cartan subalgebra of $s$. That they all contain zero weight vectors follows from three facts:

1. $S^e_v$ is generated in degree 2
2. $V_{1,0} \otimes V_{1,0} = V_{2,0} \otimes V_{0,1} \otimes V_{0,0}$, all of which contain zero weight vectors, and
3. since an irreducible representation contains a zero weight vector if and only if any or all of its weights is a sum of roots, the property is stable under tensor product.

Now that we know what the dimension is in each degree of $S^e_v(\tilde{p}_2)^{s^x}$, it remains to demonstrate generators and relations. Choose a basis of weight vectors for $\tilde{p}_2$ such that $a,\bar{a}$ (etc.) generate two dimensional $u_p$ submodules and $e_\alpha a = b$, $e_\alpha c = d$, $e_\alpha \bar{a} = \bar{b}$ and $e_\alpha \bar{c} = \bar{d}$:

\begin{figure}
\begin{center}
\begin{tikzpicture}
    \node (0) at (0,0) {$0$};
    \node (a) at (1,0) {$a$};
    \node (b) at (2,0) {$b$};
    \node (c) at (1,-1) {$c$};
    \node (d) at (2,-1) {$d$};
    \node (a_bar) at (1,-1.5) {$\bar{a}$};
    \node (b_bar) at (2,-1.5) {$\bar{b}$};
    \node (c_bar) at (1,-2) {$\bar{c}$};
    \node (d_bar) at (2,-2) {$\bar{d}$};
    \draw[->] (a) -- (b);
    \draw[->] (c) -- (d);
    \draw[->] (a_bar) -- (b_bar);
    \draw[->] (c_bar) -- (d_bar);
\end{tikzpicture}
\end{center}
\end{figure}
Then $S_2(\tilde{\rho})^g$ consists of elements which are invariant under the root tds through $\alpha$, and highest weight vectors with respect to this tds: We may use the representation theory of the root tds through $\alpha$ to split $S_2(\tilde{\rho})^g$ into two four dimensional subspaces:

- fixed vectors $\tilde{ad} - \tilde{bc}$ and highest weight vectors $\tilde{bd}$
- $\tilde{ad} - \tilde{bc}$ and $\tilde{bd}$

These satisfy two relations, $(bd)(\tilde{bd}) = (b\tilde{d})(\tilde{bd})$ and

$$(bd)(\tilde{ad}-\tilde{bc}) + (\tilde{bd})(ad-bc) = (b\tilde{d})(\tilde{ad}-\tilde{bc}) - (\tilde{bd})(a\tilde{d}-b\tilde{c})$$

Each of these subspaces is stable under the action of $U_F$, and decomposes into a three plus one dimensional representation. If we call these pieces $X$ and $x$ for the fixed vectors, and $Y$ and $y$ for the highest weight vectors, our relations can be rewritten $Y \cdot Y = y^2$ and $X \cdot Y = xy$.

We leave it to the reader to show that the ring

$$\bigoplus_{i=1}^{n-1} [N_i] \bigoplus S(V_2) \bigoplus S(X \otimes Y)[x,y]$$

has a much larger singular set than the ring $\bigoplus_{i=1}^{n+1} [N_i]$, except for $n = 1$ (here $g = sp(1,1) = so(4,1)$), where $S(X \otimes Y)[x,y]$
is a polynomial ring on on $x$, $y$, and $X \cdot X$, and $S(V \oplus W)[N_1, N_2]$ is a polynomial ring on $N_1$, $N_2$ and $V \cdot W$. 
41.

BIBLIOGRAPHY


