FINITE ELEMENT METHODS BASED ON MODIFIED VERSIONS OF HU-WASHIZU PRINCIPLE WITH APPLICATIONS TO PLATE AND SHELL ANALYSES

by

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SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY June 1983 © 1983 Hely Ricardo Costa Savio

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APR 7 1983 Archives
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Submitted to the Department of Aeronautics and Astronautics on March 30, 1983 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Abstract

Finite element methods based on modified versions of the generalized or Hu-Washizu principle are presented for linear elastic static analysis of structures. A family of cubic displacement field plate bending elements is studied in the light of the new formulation. Two 30 degrees-of-freedom shell elements with cubic displacement fields are developed. Numerical applications are made and comparisons with existing exact and numerical solutions are carried out.
ACKNOWLEDGEMENTS

I would like to thank the members of my thesis committee, Prof. Theodore H. H. Pian, Prof. Emmett A. Witmer and Prof. John Dugundji for their guidance throughout this work.

I am eternally indebted to my wife Martha and my son Alexandre, without whom this work would never had been accomplished.
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7.1 Summary
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LIST OF SYMBOLS

\( A, A_m \)  
area of plate or shallow shell finite element

\( A_{\alpha\beta\delta\gamma} \)  
elastic coefficients

\( A_{\alpha\beta\delta\gamma}^* \)  
elastic compliance coefficients

\( B_{\alpha\beta\delta\gamma} \)  
elastic coefficients

\( B_{\alpha\beta\delta\gamma}^* \)  
elastic compliance coefficients

\( B \)  
interpolation matrix defined in eqn.(3.6)

\( C_{ijkm} \)  
elastic coefficients

\( C_{u}, C_{u_m} \)  
boundary of plate or shell where displacements are prescribed

\( C_{\sigma}, C_{\sigma_m} \)  
boundary of plate or shell where stresses are prescribed

\( C_m \)  
plate or shell interelement boundary

\( D_u \)  
matrix of differential operators for defining the strain-displacement relations

\( D_{\alpha\beta\delta\gamma} \)  
elastic coefficients

\( D_{\alpha\beta\delta\gamma}^* \)  
elastic compliance coefficients

\( D_{\sigma} \)  
matrix of differential operators for defining the stress equilibrium equations

\( E \)  
Young's modulus

\( E \)  
interpolation matrix defined in eqn.(3.11)

\( e_{\alpha\beta} \)  
permutation symbols

\( C_{e_i} \)  
base vectors of the common coordinate system

\( L_{e_i} \)  
base vectors of the local coordinate system

\( G_{e_i} \)  
base vectors of the global coordinate system

\( F_i \)  
body force components
-7b-

G matrix defined in eqns. (3.18) and (3.19)
H matrix defined in eqn. (3.14)
h plate or shell thickness
J matrix defined in eqn. (3.13)
K global (or structure) stiffness matrix
k_n element stiffness matrix
L interpolation matrix for boundary displacement field, defined in eqn. (3.4)
l_i length of side i of plate element
\( M_{\alpha\beta}, M_x, M_y, M_{xy} \) moment resultants
\( M_n, M_{ns} \) boundary tractions related to moment resultants
\( N_{\alpha\beta}, N_x, N_y, N_{xy} \) in-plane stress resultants
\( N_n, N_{ns} \) boundary tractions related to in-plane stress resultants
\( N_{1i}, N_{2i}, N_{3i} \) interpolation functions defined in eqn. (5.2)
\( N_1, N_2, N_3, N_4 \) interpolation functions defined in eqn. (5.19)
N interpolation matrix that relates the displacement field to the generalized nodal displacements, defined in eqn. (3.3)
P, P_i interpolation matrices that relate stresses to stress parameters and also strains to strain parameters
P_z lateral distributed load applied to plates
P_{\alpha}, P_3 body forces applied to shells
P_v interpolation matrix that relates boundary tractions to stress parameters, defined in eqn. (3.10)
q vector of generalized nodal displacements
Q global (or structure) applied load vector
Q_n element applied load vector
out-of-plane shear force in Kirchhoff plate bending theory

vector defined in eqn. (3.17)

vector defined in eqn. (3.16)

matrix defined in eqn. (3.15)

coordinate along plate (or shell) side

elastic compliance coefficients

boundary where displacements are prescribed

interelement boundary

boundary where stresses are prescribed

boundary traction components

components of transformation matrix from a local to a common coordinate system

displacement components

components of boundary displacement field

displacement component normal to the boundary

displacement component tangent to the boundary

displacement field interpolated in terms of generalized nodal displacements

additional internal displacement field

in-plane displacements in plate or shell theory

displacement components in the local coordinate system

displacement components in the common coordinate system

volume or problem domain

element domain
$V_n$ out-of-plane shear force in Marguerre shallow shell theory
$v$ in-plane displacement component in plate or shell theory
$x_{\alpha',x}$ Cartesian coordinates
$Lx_i$ local system Cartesian coordinates
$Cx_i$ common system Cartesian coordinates
$Gx_i$ global system Cartesian coordinates
$y$ Cartesian coordinate
$w$ out-of-plane displacement component in plate or shallow shell theory
$w^q$ out-of-plane displacement field interpolated in terms of generalized nodal displacements
$w^\lambda$ additional internal out-of-plane displacement field
$\omega_{w,n}$ boundary rotation field
$W$ matrix defined in eqn.(3.26)
$z$ coordinate normal to plate surface
$z$ height of shell surface above the base plane
$\alpha$ strain parameters
$\beta$ stress parameters
$\Gamma$ interpolation matrix for additional displacement fields, defined in eqn.(3.5)
$\varepsilon_{ij},\varepsilon_{\alpha\beta}$ strain components
$\Phi_i$ stress functions
$K_{\alpha\beta}$ curvature-strain components
$\lambda$ generalized Lagrange multipliers
$\pi_{\text{HW}}$ Hu-Washizu principle
$\pi_{\text{mHW}}$ modified Hu-Washizu principle
\( \pi_{mR} \)  
modified Hellinger-Reissner principle

\( \sigma_{ij} \)  
stress components

\( \Theta_{x_i} \)  
x-axis component of nodal rotation

\( \Theta_{y_i} \)  
y-axis component of nodal rotation

\( \Theta_n \)  
rotation component oriented along boundary

\( \nu_i, \nu_{\alpha}, \nu_x, \nu_y \)  
direction cosines of surface normal vector

\( \xi_i \)  
area (or triangular) coordinates

\( \partial V_n \)  
entire boundary of element

\( \partial A_n \)  
entire boundary of plate or shell element

(\( \bar{\cdot} \))  
prescribed quantities
Chapter 1
Introduction

1.1 The Finite Element Method

The Finite Element Method is a general discretization procedure for approximate solutions to field problems. In this method a domain is divided into a finite number of regions or subdomains called elements. The first step in the process is to assume, within each element, a distribution of the field variables in terms of a finite number of unknown parameters. These parameters usually are the value of the field variables at a finite number of points called nodes, which may be located either on the boundary of the elements or inside them. Employing a variational principle or variational statement, we combine the relations for the individual elements yielding a finite number of simultaneous equations to be solved for the unknown parameters.

The method received a forwarding impulse in the mid 1950's when, in applications to aircraft structural analysis, Turner et al\(^1\) and Argyris\(^2\) extended the so-called matrix methods in structural analysis to plane stress problems. The idea being introduced\(^*\) was that in the conventional matrix structural analysis, the relation between forces and displacements for each structural

\[^*\] In 1943, Courant\(^3\) presented an approximate solution of the St.Venant torsion problem by assuming a linear distribution of the stress function within triangular subdomains, but it seems that the engineering community only recognized the possibilities of the new method after the two works mentioned above.
component is derived exactly, while in the solution of the plane stress problem, only approximate displacement distributions were assumed inside each element. This now can be interpreted as an application of the Rayleigh-Ritz method used in conjunction with the principle of virtual work or principle of minimum potential energy using piecewise assumed functions in terms of nodal displacements as generalized coordinates.

The term "Finite Element Method" was first introduced in 1960 by Clough. A new and powerful method was then emerging, its potentiality being recognized and its popularity growing very rapidly.

Since the early 1960's much progress has been made. Different Finite Element models based on the various variational principles were derived. The models based on conventional variational principles require various continuity conditions to be satisfied throughout the entire domain. Modified variational principles have been derived to allow for the discontinuity of the field variables across interelement boundaries, and several different models were studied in the past.

The displacement model is derived from the principle of minimum potential energy, and for monotonic convergence of the finite element solutions, the assumed displacement distributions should satisfy the interelement boundary displacement compatibility, then called the "compatible displacement model" as discussed by Melosh. Numerous examples of the applications of the assumed displacement model may be found in texts by Zienkiewicz, Desai and Abel, Cook and many others.

The equilibrium model is derived from the principle of minimum complementary energy, and an equilibrating stress field should be assumed
throughout the domain. This model was first discussed by Fraeijs de Veubeke\(^9\).

A mixed model based on Hellinger-Reissner principle was first introduced by Herrmann\(^10\). Distributions with nodal values of both displacements and stresses as unknowns were assumed, yielding a balanced accuracy in displacement and stress solutions. But, for a given mesh size the number of unknowns by a mixed formulation is always larger than that by the conventional matrix displacement method. Thus the advantage of better predictions for stress distributions is offset by the larger computational effort required. For a review of mixed methods based on Hellinger-Reissner principle and its modified forms see Pian and Tong\(^11\).

A hybrid displacement model based on a modified potential energy principle was first discussed by Jones\(^12\). Here displacement and traction parameters are left as final unknowns in the system of equations producing a mixed matrix method.

Another hybrid displacement model based on a different modified potential energy principle was discussed by Tong\(^13, 14\) and Atluri\(^15\). This model is based on three independently assumed fields: a noncompatible displacement distribution is assumed within the element, a nonequilibrating traction and compatible displacement distributions are assumed along element boundaries. The final set of equations are in the form of a matrix displacement method.

The assumed stress hybrid model developed by Pian\(^16, 17\) is based on a modified complementary energy principle. This is a two-field principle in which equilibrating stress fields are assumed within the element and a compatible displacement field must be chosen only along element boundaries. The stress field is assumed in terms of stress parameters which are independent from one
element to another, and which are then eliminated at the level of element formulation. This leaves us with only nodal displacements as unknowns in the final system of equations. Features of this method and applications in the past will be discussed in more detail in the next section.

General discussion of variational principles with relaxed continuity requirements have been made by Prager\textsuperscript{18, 19}, Pian and Tong\textsuperscript{20}, Pian\textsuperscript{21} and by Nemat-Nasser\textsuperscript{22, 23}.

Systematic classification of the various finite element models and the variational principles upon which they are based are given by Pian and Tong\textsuperscript{20, 24}, Pian\textsuperscript{25, 21, 26} and Atluri and Pian\textsuperscript{27}. A very comprehensive survey of modified variational principles is given in a text by Washizu\textsuperscript{28}.

1.2 The Assumed Stress Hybrid Model

The assumed stress hybrid model was first introduced by Pian\textsuperscript{16} using as an example a rectangular plane stress element. Applications to the plate bending problem were immediately done by Pian\textsuperscript{17}, Severn and Taylor\textsuperscript{29}, Allwood and Cornes\textsuperscript{30}, Allman\textsuperscript{31} and others. From the elements derived, one of the most effective nine degrees-of-freedom triangular plate bending element available today was obtained\textsuperscript{32}. Several different ways of obtaining the stiffness matrix of this element have been devised depending on the variational principle being used. Allman\textsuperscript{33} and Yoshida\textsuperscript{34} derived its stiffness matrix using a hybrid displacement approach. In the present work, as we shall see later, this element will be derived by using a modified form of Hu-Washizu principle. So, as we can observe, there are alternative ways of formulating stiffness matrices of hybrid stress elements through the various variational principles.
In the late 1960's Pian and Tong[^35, 20, 36, 37] established more rigorously the basis of the method. They proved mathematically that the assumed stress hybrid model yields a structure which is more flexible than the compatible model of the same boundary displacement approximation and more rigid than the equilibrium model of the same interior stress distribution, thus they suggested that hybrid stress models most likely can provide more accurate results than both the compatible and equilibrium models. Besides, by that time, it was quite clear from numerical experimentation that assumed stress methods (equilibrium and hybrid models) give better accuracy in stress predictions than assumed displacement models.

Applications to St. Venant torsion problem were done by Yamada et al.[^38, 39]. Their first work was for linear elastic analysis while the second one, for elastic-plastic torsional deformation studies of structural members with complicated cross sections. Strain-hardening and anisotropic plastic effects were included in an incremental tangent stiffness approach. This is one example in which stress-free conditions exist on the external boundaries and assumed displacement elements often fail to satisfy these conditions, giving rise to somewhat inaccurate stress solution near the boundary. But in the elastic-plastic analysis, an accurate prediction of the stress distribution is required for determination of yielding as well as unloading of the elements. The hybrid stress model provided remarkable improvement in the stress solution due not only to its more accurate stress predictions than the assumed displacement models but also due to its flexibility for the creation of special elements with stress distributions intrinsically satisfying the stress-free boundary conditions.

Boland[^40] applied the method to the bending of laminated plates by formulating a general quadrilateral multilayer plate element based on Kirchhoff
or classical plate bending theory. Spilker\textsuperscript{41} and Mau \textit{et al}\textsuperscript{42} extended the study by including the transverse shear effect. Spilker used Reissner-Mindlin\textsuperscript{43, 44} plate bending theory while Mau \textit{et al} derived elements based on the assumption that shear strains may be different in different layers. Obviously, the last assumption is essential for very thick plates, while for moderately thick plates with many layers, Spilker’s approach would be more efficient. Tong \textit{et al}\textsuperscript{45} described consistent ways of deriving geometric stiffness and mass matrices for finite element hybrid models. Mau and Witmer\textsuperscript{46} applied the previous conceptions to static, vibration and thermal stress analyses of laminated plates and shells using flat elements. The free vibration problem had been studied earlier by Tabarrok\textsuperscript{47} who employed the Toupin principle. His approach, not being applicable to transient dynamic problems, produced neither a mass nor a stiffness matrix, but rather a frequency matrix.

Mau and Pian\textsuperscript{48} solved the linear transient response problem of laminated plates and shells. Atluri\textsuperscript{49}, utilizing convolution integrals in time, also studied the linear dynamic problem.

The linear static thin shell problem was solved by Tanaka\textsuperscript{50} using deep, doubly curved triangular elements. He assumed a stress field in equilibrium through the use of stress-stress function relations obtained by the so-called static-geometric analogy\textsuperscript{51, 27}.

The assumed stress hybrid model also proved to be very versatile in the analysis of stress states around cracks\textsuperscript{52}. For a review of the subject, see Pian\textsuperscript{52, 26}; and for some applications using assumed stress hybrid elements, we could mention: Pian \textit{et al}\textsuperscript{53}, Luk\textsuperscript{54}, Tong \textit{et al}\textsuperscript{55}, Wang \textit{et al}\textsuperscript{56, 57}, Lin and Mar\textsuperscript{58}, Rhee \textit{et al}\textsuperscript{59}, Tong\textsuperscript{60}, Pian and Moriya\textsuperscript{61} and Nishioka and Atluri\textsuperscript{62}. 
One interesting development is the work by Tong et al.\textsuperscript{55} for two-dimensional crack problems. An element with an embedded crack was constructed. A compatible boundary displacement field is assumed, and the stress field is based on a complete series expansion near the tip of the crack obtained from exact elasticity solution, thus including singular terms, satisfying compatibility conditions, equilibrium equations and stress-free conditions along the crack surface. This element gives a quite superior performance in plane crack problems when compared to other schemes\textsuperscript{52}. Based on the same idea, Mei and co-workers\textsuperscript{63, 64, 65} applied similarly developed elements in studies of diffraction and radiation of water waves. Their elements were used as a transition between a region discretized by conventional finite elements and a region where analytic representation was possible. They concluded that accurate solutions can be obtained with substantial economy. Other applications to fluid flow have been reported in the area of biomechanics by Tong and Fung\textsuperscript{66} and Tong and Vawter\textsuperscript{67}.

Solution to the linear prebuckling problem of flat sandwich plates using the assumed stress hybrid model was first attempted by Lundgren\textsuperscript{68}. He derived a linear elastic stiffness matrix by standard assumed stress hybrid methods and simply used the geometric stiffness matrix obtained from the displacement model. Later, however, Pian\textsuperscript{21} showed that the above method could be justified as being based on a modified Hellinger-Reissner principle.

The study of the geometric nonlinear problem based on the hybrid stress model was first approached by Pirotin\textsuperscript{69}. He derived a basic functional for incremental solution without any consideration of correction terms that may result from stress equilibrium imbalance and compatibility mismatch at the beginning of each increment\textsuperscript{70}.
Atluri\textsuperscript{71}, Horrigmoe and Bergan\textsuperscript{72} and Boland\textsuperscript{73} reported on incremental forms of variational principles with applications to the geometrically-nonlinearity problem. Boland's work represents the most complete survey and consistent derivation of the incremental forms of variational principles for large deflection applications, regarding the inclusion of corrective terms due to stress equilibrium imbalance and compatibility mismatch.

Horrigmoe and Bergan\textsuperscript{74} applied the hybrid stress model to the problem of large deflections of shells using flat elements. Very simple triangular and quadrilateral elements were used. They were constructed from the superposition of linear moment distribution to describe bending behavior and constant stress distribution to represent stretching behavior. They found the approach to be computationally efficient and to give accurate solutions.

Boland and Pian\textsuperscript{73, 75} studied the problem of large deflection of shells using triangular flat and shallow shell elements. An important conclusion from their work is that by including stress equilibrium imbalance correction terms in the incremental solution, there is no need to assume a stress field satisfying the complete equilibrium equations which, in the present case, also involve unknown displacements. We may satisfy only a portion of those, like the linear homogeneous equilibrium equations as suggested in their work. The incorporation of the correction terms was found to be essential in improving the efficiency of the iterative-incremental finite element solution.

Efficient elastic-plastic analyses require accurate estimation of the stress field. The hybrid stress model then emerges as a more attractive tool than the conventional assumed displacement method. Studies based on the hybrid stress model have been reported by: Yamada et al\textsuperscript{79}, Luk\textsuperscript{54}, Spilker\textsuperscript{76}, Pian\textsuperscript{77},
Spilker and Pian\textsuperscript{78, 79}, Horrigmoe and Eidsheim\textsuperscript{80}, Barnard and Sharman\textsuperscript{81} and Spilker and Munir\textsuperscript{82}. These include applications of all three commonly used elastic-plastic procedures: the tangent stiffness, initial-strain and initial-stress methods. It must be noted that the initial-strain approach corresponding to the assumed-stress hybrid model is not feasible for problems involving elastic, perfectly-plastic or nearly perfectly-plastic material behaviour\textsuperscript{54, 76}. A systematic development of incremental variational principles for small deflection materially nonlinear problems has been presented by Pian\textsuperscript{70}. His work gives special emphasis on the inclusion of correction terms due to both equilibrium imbalance and compatibility mismatch in the various principles.

An assumed stress hybrid model for small deformation creep analysis has been formulated by Pian\textsuperscript{70}. Creep and elastoplastic problems can also be solved by a unified approach, as discussed by Zienkiewicz and Cormeau\textsuperscript{83, 84}, utilizing a viscoplastic constitutive law proposed by Perzina\textsuperscript{85}. Pian and Lee\textsuperscript{86} developed a hybrid stress model based on this viscoplastic method, and very accurate and efficient finite element solutions were achieved. Wang\textsuperscript{87} extended the above model to the study of large strain viscoplastic behavior and made applications to simple two-dimensional continuum problems.

The assumed stress hybrid model is an effective scheme for reducing the severe constraints for the strain components that appear in the conventional assumed displacement finite element method in some limiting cases\textsuperscript{26}, like the solution of problems of nearly incompressible materials\textsuperscript{88}, thin plates with transverse shear\textsuperscript{48, 89, 90, 91}, and membrane strain constraints on curved beams and shells\textsuperscript{92, 93}.

The assumed stress hybrid model is particularly useful in the derivation of
special purpose elements with stress distributions inherently satisfying traction-free boundary conditions. Spilker and Chou\textsuperscript{94} reported on such an element for analysis of edge effects in multilayer plates and concluded that reliable stress predictions near the free edge can only be obtained with pointwise exact satisfaction of traction-free edge conditions. Atluri and Rhee\textsuperscript{95} presented a work on the subject of enforcing traction boundary conditions, within any desired accuracy, along arbitrarily curved boundaries using the hybrid stress model. Clearly an important application of this study is in enforcing traction-free conditions along crack surfaces.

Inspired by the work of Tang \textit{et al}\textsuperscript{96}, Pian \textit{et al}\textsuperscript{97, 98} recently suggested alternative ways of obtaining hybrid stress elements with improved computational efficiency. In this new model the complete stress equilibrium equations are introduced as conditions of constraint into a modified Hellinger-Reissner or modified Hu-Washizu principle. The Lagrange multipliers can be easily identified as additional internal displacement parameters. These will be statically condensed, enforcing the stress equilibrium equations to be satisfied exactly or only approximately at element level.

1.3 Synopsis of the Research

The assumed stress hybrid model has been applied to the most diverse areas in continuum mechanics, all of which attests to the model being a very effective and accurate tool. It has been proven to give improved accuracy for displacements and stress solutions and to be more accurate and efficient for elastic-plastic and creep analyses than the conventional assumed displacement method. Its flexibility in tayloring element stress distributions for special
applications (fracture mechanics, satisfaction of stress-free conditions, etc.); the absence of "locking" difficulty for problems with strain constraints (nearly incompressible materials, thin plates with transverse shear effects, membrane strain constraints on curved beams and shells) are some of its important features. Now, the chance of improving its computational efficiency makes it an even more attractive model.

One important application of the new formulation is in the development of shell elements. As we know, in shell theory, equilibrium equations can be satisfied by the use of the static-geometric analogy. But this may be quite complicated and will involve parameters of the element geometry. So, if we are formulating a general shell element which, in general, represents the actual shell geometry only approximately, there will not be much meaning in trying to satisfy the equilibrium equations exactly using the above analogy. Thus, this new approach, not requiring equilibrium to be satisfied \textit{a priori}, but still enforcing it at element level - exactly or with a certain degree of approximation - most likely will yield more flexible and efficient methods with simpler computing procedures.

Regarding applications of the finite element method to the analysis of thin shells, very comprehensive surveys, reviewing the various approaches to element formulations, are given by Gallagher.\textsuperscript{99, 100}

It is our intent, in the present work, to make a contribution in the exploration of the possibilities of the newly proposed method by developing elements for general shell analysis. As a first step, a family of cubic displacement field plate bending elements is studied. Then, two 30 degrees-of-freedom shell elements are constructed based on shallow shell theory applied at element level.
As an introductory section, Chapter 2 presents, within the framework of three-dimensional elasticity theory, Hu-Washizu principle and some modified forms.

The finite element implementation of two of the modified forms of Hu-Washizu principle presented in Chapter 2 is described in detail in Chapter 3, Section 3.1. Section 3.2 discusses some of the features of this model.

Hu-Washizu principle and its modified forms are written in particular form for plate bending applications in Chapter 4, Section 4.1. The classical or Kirchhoff plate bending theory is used. In Section 4.2 those principles are written for shell analysis based on shallow shell theory due to Marguerre101.

Chapter 5 describes the several elements developed, details of their construction and assumptions used. Section 5.1 presents a family of cubic displacement triangular plate bending elements: two elements with linear and three with quadratic moment distributions. One of them, the nine degrees-of-freedom linear moment distribution element, is identical to the HSM element32 discussed previously in Section 1.2 and is the one to which all comparisons in performance are made. Section 5.2 describes two 30 degrees-of-freedom shell elements with cubic displacement fields.

Numerical applications are discussed in Chapter 6. Three problems are studied to analyze the performance of the plate bending elements: a simply-supported square plate under uniform load and under a central concentrated load are two of them; a clamped square plate under central concentrated load is the third one. For the shell elements, to test their in-plane behavior, a problem of stretching of a square plate under parabolic edge loading is analyzed and comparisons with the assumed displacement quadratic strain triangle are made.
A shell problem widely used by researchers to test finite elements is studied using the new ones: a shallow spherical cap under uniform pressure.

Finally, summary and conclusions are presented as a final chapter, along with some suggestions for future study.
Chapter 2
Hu-Washizu Principle
and Some Modified Forms


In its usual form, the functional for the Hu-Washizu or generalized principle may be written as:

\[
\pi_{HW}(u_i, \epsilon_{ij}, \sigma_{ij}) = \\
= \int_V \left[ (1/2)C_{ijkm}\epsilon_{ij}\epsilon_{km} - \sigma_{ij}\epsilon_{ij} + \sigma_{ij}(1/2)(u_{ij} + u_{ji}) - \bar{F}_i u_i \right] dV \\
- \int_{S_u} T_i(u_i - \bar{u}_i) ds - \int_{S_\sigma} \bar{T}_i u_i ds = \text{stationary} \tag{2.1}
\]

with no subsidiary conditions

where*

\[
\sigma_{ij} = \text{stresses} \\
\epsilon_{ij} = \text{strains} \\
C_{ijkm} = \text{elastic coefficients} \\
\bar{F}_i = \text{body forces}
\]

*Repetition of latin indices means summation from 1 to 3
\( T_i = \text{boundary tractions} = \sigma_{ij} \nu_j \)

\( \nu_j = \text{direction cosines of surface normal vector} \)

\( V = \text{entire domain} \)

\( S_\sigma = \text{boundary where tractions are prescribed} \)

\( S_u = \text{boundary where displacements are prescribed} \)

\( ( ) = \text{prescribed quantities} \)

Its stationary condition \( \delta \pi_{HW} = 0 \) gives us as Euler equations, all equations of elasticity:

in the domain \( V \):

**equilibrium equations**

\[ \sigma_{ij} + \bar{F}_i = 0 \]  \( (2.2) \)

**strain-displacement relations**

\[ \epsilon_{ij} - (1/2)(u_{ij} + u_{ji}) = 0 \]  \( (2.3) \)

**stress-strain relations**

\[ \sigma_{ij} - C_{ijkm}\epsilon_{km} = 0 \]  \( (2.4) \)

on the boundary:

**mechanical boundary condition on \( S_\sigma \)**

\[ T_i - \bar{T}_i = 0 \]  \( (2.5) \)

**geometrical boundary condition on \( S_u \)**

\[ u_i - \bar{u}_i = 0 \]  \( (2.6) \)
### 2.2 Modified Forms of Hu-Washizu Principle for the Small Displacement Theory of Elastostatics.

Dividing our domain \( V \) into \( n \) finite elements and relaxing the continuity requirement for \( u_i \) we may write\(^{20} \):

\[
\int_V \sigma_{ij} (1/2)(u_{ij} + u_{ji}) \, dV = \sum_n \left\{ \int_{V_n} \sigma_{ij} (1/2)(u_{ij} + u_{ji}) \, dV - \int_{S_n} T_i (u_i - \overline{u_i}) \, ds \right\} (2.7)
\]

where

- \( \overline{u_i} \) is a new field variable
- \( S_n \) is the interelement boundary

and \( \sigma_{ij} \) and \( u_i \) are allowed to be discontinuous while \( \overline{u_i} \) must be continuous across interelement boundaries.

Then a modified version of Hu-Washizu principle can be written as\(^* \):

\[
\pi_{mHW}^{(1)}( u_i, \epsilon_{ij}, \sigma_{ij}, \overline{u_i} ) =
\]

\[
= \sum_n \left\{ \int_{V_n} \left[ (1/2)C_{ijkl} \epsilon_{ij} \epsilon_{kl} - \sigma_{ij} \epsilon_{ij} + \sigma_{ij} (1/2)(u_{ij} + u_{ji}) - F_i u_i \right] \, dV
\]

\[
- \int_{S_n} T_i (u_i - \overline{u_i}) \, ds - \int_{S_{n_{u_n}}} \overline{T_i} \overline{u_i} \, ds - \int_{S_{n_{\partial V_n}}} T_i (u_i - \overline{u_i}) \, ds \right\} (2.8)
\]

where \( \partial V_n = S_n + S_{n_u} + S_{\sigma_n} \) = entire boundary of element

The Euler equations obtained from the stationary condition of \( \pi_{mHW}^{(1)} \) are eqns. (2.2) to (2.6) plus

\*

The several modified forms of the variational principle will be identified by a superscripted number inserted between parenthesis
eqns. (2.2) to (2.6) plus

on the boundary:

compatibility of displacement on \( \partial V_n \)

\[
u_i - \tilde{u}_i = 0
\]  \( (2.9) \)

equilibrium of interelement tractions on \( S_n \) (see Fig. 1)

\[
T_i^{(a)} + T_i^{(b)} = 0
\]  \( (2.10) \)

Another modified form was recently proposed by Pian et al.\(^{97, 98}\) in which the element displacements \( u_i \) are divided into two parts,

\[
u_i = u_i^q + u_i^\lambda
\]  \( (2.11) \)

where

\( u_i^q \) are incompatible displacements which can be expressed in terms of the nodal unknowns \( q_i \).

\( u_i^\lambda \) are additional displacements which are expressed in terms of internal displacement parameters \( \lambda_i \) that can be eliminated at element level. They may be incompatible along the boundary or they may be bubble functions which are zero along the boundary.

Inserting eqn. (2.11) into (2.8) and using the fact that:

\[
\int_V \sigma_{ij}(1/2)(u_{i,j} + u_{j,i}) \, dV = \int_{\partial V} T_{i,j} \, du_i \, ds - \int_V \sigma_{ij,j} u_i \, dV
\]  \( (2.12) \)

one obtains

\[
\pi_{\text{mHW}}^{(2)}(\ u_i^q, u_i^\lambda, \epsilon_{ij}, \sigma_{ij}, \tilde{u}_i) = \sum_n \left\{ \int_{V_n} \left[ (1/2)C_{ijkm}\epsilon_{ij}\epsilon_{km} - \sigma_{ij}\epsilon_{ij} + \sigma_{ij}(1/2)(u_{i,j}^q + u_{j,i}^q) - \tilde{F}_{i}u_i^q \right] \, dV \right\}
\]
\[-\int_{V_n} (\sigma_{ij,j} + \overline{F}_i) u_i^\lambda \ dV \]
\[-\int_{S_u_n} T_i(\overline{u}_i - \overline{u}_i) \ ds - \int_{S_{\sigma_n}} \overline{T_i u_i} \ ds - \int_{\partial V_n} T_i(u_i^q - \overline{u}_i) \ ds \} \quad (2.13)\]

It is seen that the second volume integral in \(\pi^{(2)}_{mHW}\) contains the stress equilibrium equation with \(u_i^\lambda\) as Lagrange multipliers. Then the additional displacement field will play the role of constraining the stresses to satisfy equilibrium.

Another form of the principle, completely equivalent to the preceding one is obtained using eqn. (2.12) once more:

\[\pi^{(3)}_{mHW}(u_i^q, u_i^\lambda, \epsilon_{ij}, \sigma_{ij}, \overline{u}_i) = \]

\[= \sum_n \left\{ \int_{V_n} \left[ (1/2)C_{ijkm}\epsilon_{ij}\epsilon_{km} - \sigma_{ij}\epsilon_{ij} - (\sigma_{ij,j} + \overline{F}_i) u_i^q \right] \ dV \right. \]
\[-\int_{V_n} (\sigma_{ij,j} + \overline{F}_i) u_i^\lambda \ dV \]
\[-\int_{S_u_n} T_i(\overline{u}_i - \overline{u}_i) \ ds - \int_{S_{\sigma_n}} \overline{T_i u_i} \ ds + \int_{\partial V_n} T_i(u_i^q - \overline{u}_i) \ ds \} \quad (2.14)\]

If by choosing an adequate distribution for \(u_i^\lambda\) the equilibrium equations can be identically satisfied we may write the above equations as:

\[\pi^{(4)}_{mHW}(u_i^\lambda, \epsilon_{ij}, \sigma_{ij}, \overline{u}_i) = \]

\[= \sum_n \left\{ \int_{V_n} \left[ (1/2)C_{ijkm}\epsilon_{ij}\epsilon_{km} - \sigma_{ij}\epsilon_{ij} \right] \ dV \right. \]
\[-\int_{V_n} (\sigma_{ij,j} + \overline{F}_i) u_i^\lambda \ dV \]
At this point it must be observed that the introduction of strain-stress relation

\[ \epsilon_{ij} = S_{ijkl} \sigma_{kl} \]

where \( S_{ijkl} = \) elastic compliance coefficients

into the above modified forms of Hu-Washizu principle generates formulations for modified versions of Hellinger-Reissner principle, and if the stress equilibrium equations are satisfied pointwise it reduces to the formulation of a modified complementary energy principle. Then we have here a formulation completely equivalent to an assumed stress hybrid model. This new formulation, however, is a more general and more flexible method, as will be seen in the following chapters and as is discussed by Pian et al.\(^{97,98}\).

Another form that can be briefly mentioned, and will be discussed later on in the context of shallow shell formulations, is the one that comes from \( \pi_{mHW}^{(1)} \) in which only part of the equilibrium equations are satisfied exactly. Let us say, as an example, if we satisfy only homogeneous equilibrium equations

\[ \sigma_{ij,j} = 0 \quad \text{in} \quad V_n \]

using eqn. (2.8) we obtain:

\[ \begin{align*}
\pi_{mHW}^{(5)}(u_i, \epsilon_{ij}, \sigma_{ij}, \tilde{\epsilon}_i) &= \\
&= \sum_n \left\{ \int_{V_n} \left[ (1/2)C_{ijkl} \epsilon_{ij} \epsilon_{km} - \sigma_{ij} \epsilon_{ij} - \tilde{F}_i u_i \right] dV \\
&\quad - \int_{S_{u_n}} T_i (\tilde{\epsilon}_i - \tilde{u}_i) ds - \int_{S_{\sigma_n}} \tilde{T}_i \tilde{u}_i ds + \int_{\partial V_n} T_i \tilde{u}_i ds \right\} \\
&\quad - \int_{S_{u_n}} T_i (\tilde{\epsilon}_i - \tilde{u}_i) ds - \int_{S_{\sigma_n}} \tilde{T}_i \tilde{u}_i ds + \int_{\partial V_n} T_i \tilde{u}_i ds \right\} \\
&\quad - \int_{S_{u_n}} T_i (\tilde{\epsilon}_i - \tilde{u}_i) ds - \int_{S_{\sigma_n}} \tilde{T}_i \tilde{u}_i ds + \int_{\partial V_n} T_i \tilde{u}_i ds \right\} \tag{2.16}
\end{align*} \]
where, as will be seen later, in the case of shallow shell formulation, still can be used in such a way as to preserve some of the computational advantages of the new formulation.
Chapter 3
Finite Element Implementation of Modified Forms of the Hu-Washizu Principle

For convenience, matrix notation will be used instead of the indicial notation used in the preceding chapter.

Implementation of two of the modified forms, \( \pi_{mHW}^{(2)} \) and \( \pi_{mHW}^{(3)} \), will be shown and its features discussed.

To introduce the new notation let us rewrite these modified forms as:

\[
\pi_{mHW}^{(2)}(\mathbf{u}^q, \mathbf{u}^\lambda, \epsilon, \sigma, \mathbf{\widetilde{u}}) = 
\]

\[
\sum_n \left\{ \int_{V_n} \left[ \frac{1}{2} \epsilon^T C \epsilon - \sigma^T \epsilon + \sigma^T (D_u \mathbf{u}^q) - F^T \mathbf{u}^q \right] \, dV 
- \int_{V_n} (D_o \sigma + \mathbf{F})^T \mathbf{u}^\lambda \, dV 
- \int_{S_{u_n}} T^T (\mathbf{u}^q - \mathbf{\widetilde{u}}) \, ds 
- \int_{S_{\sigma_n}} \mathbf{\widetilde{T}}^T \mathbf{\widetilde{u}}' \, ds 
- \int_{\partial V_n} T^T (\mathbf{u}^q - \mathbf{\widetilde{u}}) \, ds \right\} \tag{3.1}
\]

and

*boldface and greek letters will represent matrices and vectors*
\[ \pi_{mHW}^{(3)}(u^q, u^\lambda, \epsilon, \sigma, \tilde{u}) = \]
\[ = \sum \left\{ \int_{V_n} \left[ (1/2)\epsilon^T C \epsilon - \sigma^T \epsilon - (D_0 \sigma + \bar{F})^T u^q \right] dV \right. \]
\[ \left. - \int_{V_n} (D_0 \sigma + \bar{F})^T u^\lambda dV \right. \]
\[ \left. - \int_{S_{u_n}} T^T (\tilde{u} - \bar{u}) ds - \int_{S_{\sigma_n}} \overline{T^T \tilde{u}} ds + \int_{\partial V_n} T^T \tilde{u} ds \right\} \]  
(3.2)

where

\( D_0 \) represents a matrix of differential operators for defining the stress equilibrium equations,

\( D_u \) is a matrix of differential operators for defining the strain-displacement relations,

and all the other symbols are easily identified as those shown in eqns. (2.13) and (2.14).

3.1 Implementation of \( \Pi_{mHW}^{(2)} \) and \( \Pi_{mHW}^{(3)} \)

In the finite element formulation the displacement field \( u^q \) is expressed in terms of nodal unknowns \( q \) as:

\[ u^q = N q \]  
(3.3)

The boundary displacements \( \tilde{u} \) are interpolated with respect to the same set of nodal unknowns:

\[ \tilde{u} = L q \]  
(3.4)
and the additional displacement field $\mathbf{u}^\lambda$ is expressed in terms of internal parameters $\lambda$:

$$\mathbf{u}^\lambda = \Gamma \lambda \quad (3.5)$$

From eqn. (3.3) we obtain a matrix $\mathbf{B}$ such that

$$\mathbf{D}_u \mathbf{u}^q = (\mathbf{D}_u \mathbf{N}) \mathbf{q} = \mathbf{B} \mathbf{q} \quad (3.6)$$

The important assumption in this new formulation is that stresses $\sigma$ and strains $\epsilon$ are approximated by using the same interpolation functions, in an uncoupled form, i.e.

$$\sigma = \mathbf{P} \beta \quad (3.7)$$

and

$$\epsilon = \mathbf{P} \alpha \quad (3.8)$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_3 & \ddots \\ \mathbf{0} & \mathbf{0} & \ddots \\ \end{bmatrix}$$

and $\mathbf{P}_i$ are row matrices.

$\beta$'s and $\alpha$'s are called stress and strain parameters, respectively.

Boundary tractions are expressed as:

$$\mathbf{T} = \nu^T \sigma = \mathbf{P}_\nu \beta \quad (3.9)$$

where

$$\mathbf{P}_\nu = \nu^T \mathbf{P} \quad (3.10)$$
From eqn. (3.7) we define a matrix $E$ as:

$$D_o \sigma = (D_o P) \beta = E \beta \quad (3.11)$$

Substituting the above assumptions into the variational principles, eqns. (3.1) and (3.2), we obtain:

$$\pi_{\text{mHW}} = \sum_n \left\{ \frac{1}{2} \alpha^T J \alpha - \beta^T H \alpha + \beta^T G q - \beta^T R \lambda - q^T Q^q - \lambda^T Q^\lambda \right\} \quad (3.12)$$

where

$$J = \int_{V_n} P^T C P \, dV \quad (3.13)$$

$$H = \int_{V_n} P^T P \, dV \quad (3.14)$$

$$R = \int_{V_n} E^T \Gamma \, dV \quad (3.15)$$

$$Q^\lambda = \int_{V_n} \Gamma^T F \, dV \quad (3.16)$$

$$Q^q = \int_{V_n} N^T F \, dV + \int_{S} L^T T \, ds \quad (3.17)$$

$\pi_{\text{mHW}}$ stands for both functionals, $\pi_{\text{mHW}}^{(2)}$ and $\pi_{\text{mHW}}^{(3)}$, the only difference being in the definition of matrix $G$. 
for $\pi_{mHW}^{(2)}$:

$$G = \int_{V_n} P^T B \, dV - \int_{\partial V_n} P^T (N - L) \, ds$$

(3.18)

and for $\pi_{mHW}^{(3)}$:

$$G = \int_{\partial V_n} P^T L \, ds - \int_{V_n} E^T N \, dV$$

(3.19)

The parameters $\alpha$, $\beta$ and $\lambda$ are independent from one element to another, then they can be eliminated at element level. From the first variation of $\pi_{mHW}$ with respect to stress parameters $\beta$ we obtain:

$$\alpha = H^1(G \, q - R \, \lambda)$$

(3.20)

with respect to strain parameters $\alpha$:

$$\beta = H^1 J \, \alpha$$

(3.21)

and with respect to $\lambda$:

$$R^T \beta + Q^\lambda = 0$$

(3.22)

We observe that the above expression, eqn. (3.22), is a constraint equation for the stress parameters, enforcing satisfaction of the complete equilibrium equation.

From eqns. (3.20) to (3.22) we can obtain parameters $\alpha$, $\beta$ and $\lambda$ in terms of nodal variables $q$:

$$\lambda = (R^T WR)^{-1}(R^T WG q + Q^\lambda)$$

(3.23)

$$\beta = [WG - WR(R^T WR)^{-1}R^T WG] q - WR(R^T WR)^{-1}Q^\lambda$$

(3.24)
\[ \alpha = [H^{-1}G - H^{-1}R(R^TWR)^{-1}R^TWG]q - H^{-1}R(R^TWR)^{-1}Q^\lambda \]  \hspace{1cm} (3.25)

where

\[ W = H^{-1}JH^{-1} \]  \hspace{1cm} (3.26)

Then, \( \pi_{mHW} \) becomes:

\[ \pi_{mHW} = \sum_n \left\{ (1/2)q^T k_n q - q^T Q_n + \text{constant} \right\} \]  \hspace{1cm} (3.27)

where \( k_n \) is the element stiffness matrix and \( Q_n \) is the element external load vector. These are given by:

\[ k_n = G^T WG - G^T WR (R^T WR)^{-1} R^T WG \]  \hspace{1cm} (3.28)

and

\[ Q_n = Q^q + G^T WR (R^T WR)^{-1} Q^\lambda \]  \hspace{1cm} (3.29)

By assembling the element matrices, the global stiffness matrix and load vector can be obtained and \( \pi_{mHW} \) becomes,

\[ \pi_{mHW} = (1/2)q^T K q - q^T Q \]  \hspace{1cm} (3.30)

The first variation with respect to \( q \) gives the system of equations:

\[ K q = Q \]  \hspace{1cm} (3.31)

After solving the above equations for the nodal displacements \( q \), stresses are obtained using eqns. (3.24) and (3.7) while strains are obtained using equations (3.25) and (3.8).

At this point we must observe that the element stiffness matrix, given by eqn. (3.28), can also be obtained by first writing the functional of \( \pi_{mHW} \) as:
where

\[
\begin{bmatrix}
G^T WG & -G^T WR \\
-R^T WG & R^T WR
\end{bmatrix}
\]

and then eliminating \( \lambda \) by static condensation.

### 3.2 Features of the Formulation

One of the main advantages of using the modified Hu-Washizu principle \( \pi_{mHW} \) with uncoupled stress assumption, over the conventional assumed stress hybrid model\textsuperscript{16, 20}, is that the inversion of the \( H \) matrix (see eqn. (3.14)) can be done more efficiently. Matrix \( H \), which is symmetric positive definite, in the present formulation has the form

\[
H = \begin{bmatrix}
H_1 & H_2 & 0 \\
H_2 & H_3 & \sim \\
0 & \sim & H_3
\end{bmatrix}
\]

where \( H_i = \int_{V_n} P_i^T P_i \, dV \quad i=1,2,3,... \) (3.35)

Thus its inversion is reduced to that of the individual submatrices \( H_i \).

It should be observed that in this method it is also required to invert matrix \( R^T WR \), but this is, in general, of much smaller order than that of \( H \).

Expanding the assumed stresses in terms of natural coordinates, such as
the isoparametric coordinates, which are inherent in an element itself, the submatrices $H_i$ of $H$ can be made totally independent of element geometry, as proposed by Pian and Sumihara.\(^{102}\) Hence, submatrices $H_i$ need to be inverted only once for all elements.

Another important aspect of the new formulation is in the development of shell elements. As we know, equilibrium equations can be satisfied \textit{a priori} by the use of the static-geometric analogy. But these may be quite complex, will, in general, couple membrane and moment stresses in the $H$ matrix, and will involve parameters of the element geometry. So, if we are formulating a general shell element, which, in general, only approximates the actual shell geometry, there will not be much meaning in trying to satisfy the equilibrium equations exactly using the above analogy. Thus, this new approach, most likely, will yield more flexible and efficient methods, with simpler computing procedures.

For further discussions about the present formulation, see Pian \textit{et al.}\(^{97, 98}\).
Chapter 4
Forms of the Hu-Washizu Principle for Analysis of Plates and Shells

The intent of this chapter is to rewrite the functionals presented in chapter 2, but in particular forms such as for Kirchhoff theory for plate bending analysis and for Marguerre shallow shell theory for shell analysis.

4.1 Hu-Washizu Functionals for Kirchhoff Plate Bending Theory

4.1.1 Conventional Form of the Hu-Washizu Principle

In its usual form, the Hu-Washizu functional for plate bending analysis under the Kirchhoff theory may be written as:

\[ \pi_{HW}(w, \kappa_{\alpha\beta}, M_{\alpha\beta}) = \]
\[ = \int_A \left[ (1/2)D_{\alpha\beta\gamma\delta}^{\kappa\kappa\gamma\delta} - M_{\alpha\beta}^{\kappa\kappa_{\alpha\beta}} - M_{\alpha\beta}w_{,\alpha\beta} - \bar{p}_z w \right] \, dx \, dy \]
\[ - \int_{C_u} \left[ Q_n(w-n\bar{w}) - M_n(w_n-n\bar{w}_n) - M_{ns}(w_{,s}-\bar{w}_{,s}) \right] \, ds \]
\[ - \int_{C_\sigma} \left( Q_n w - M_n w_{,n} - M_{ns} w_{,s} \right) \, ds \]

where

* repetition of Greek indices means summation from 1 to 2
Boundary tractions can be written as:

\[ Q_n = \nu_\alpha M_{\alpha\beta,\beta} \]
\[ M_n = \nu_\alpha \nu_\beta M_{\alpha\beta} \quad (4.2) \]
\[ M_{ns} = e_{\alpha\gamma} \nu_\alpha \nu_\beta M_{\beta\gamma} \]

where \( \nu_\alpha \) = outward normals
\[ e_{\alpha\beta} \] = permutation symbol

\[ \nu_\alpha \] = moment resultants
\[ \kappa_{\alpha\beta} \] = curvature strains
\[ D_{\alpha\beta\gamma\delta} \] = elastic coefficients
\( w \) = lateral displacement
\( \bar{p}_z \) = lateral load
\( h \) = plate thickness
\( z \) = coordinate normal to plane of plate
\( x, y \) or \( x_\alpha \) = Cartesian coordinates in the plane of plate
\( A \) = area of plate
\( C_u \) = boundary where displacements \( w \) are prescribed
\( C_\sigma \) = boundary where tractions are prescribed
\( (\_\_\_\_\_\_\_\_) \) = prescribed quantities

*subscripts \( n \) and \( s \) will be used to denote boundary tractions and summation convention will not be applicable in this case
\[ ( e_{12} = 1 ; e_{21} = -1 ; e_{11} = e_{22} = 0 ) \]

\((\cdot)_n\) and \((\cdot)_s\) represent differentiations, where:

\[ (\cdot)_n = \frac{\partial}{\partial n} = \nu_\alpha \frac{\partial}{\partial x_\alpha} \]

\[ (\cdot)_s = \frac{\partial}{\partial s} = e_{\alpha\beta} \nu_\alpha \frac{\partial}{\partial x_\beta} \]

The Euler equations obtained by setting the first variation of \(\pi_{HW}\) equal to zero are:

in the domain A:

- equilibrium equation
  \[ M_{\alpha\beta,\alpha\beta} + \overline{p}_z = 0 \quad (4.3) \]

- curvature-displacement relations
  \[ \kappa_{\alpha\beta} + w_{,\alpha\beta} = 0 \quad (4.4) \]

- moment-curvature relations
  \[ M_{\alpha\beta} - D_{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} = 0 \quad (4.5) \]

on the boundary:

- mechanical boundary conditions on \(C_\sigma\)
  \[ Q - \overline{Q}_n = 0 \]
  \[ M_n - \overline{M}_n = 0 \quad (4.6) \]
  \[ M_{ns} - \overline{M}_{ns} = 0 \]

- geometrical boundary conditions on \(C_u\)
  \[ w - \overline{w} = 0 \]
  \[ w_{,n} - \overline{w}_{,n} = 0 \quad (4.7) \]
4.1.2 Modified Forms of the Hu-Washizu Principle

When we relax the continuity requirement for the rotation $w, _n$ we may write for a domain divided into $m$ finite elements:

$$\int_{A} M_{\alpha \beta} w,_{\alpha \beta} \, dx \, dy = \sum_{m} \left\{ \int_{A_{m}} M_{\alpha \beta} w,_{\alpha \beta} \, dx \, dy - \int_{C_{m}} M_{n}(w,_{n} - \tilde{w},_{n}) \, ds \right\} \quad (4.8)$$

where $\tilde{w},_{n}$ is a new field variable

$C_{m}$ is the interelement boundary

and $M_{n}$ and $w,_{n}$ are allowed to be discontinuous while $w$ and $\tilde{w},_{n}$ must be continuous across interelement boundaries.

Then a modified version of the Hu-Washizu principle for plate bending can be written as:

$$\pi_{mHW}^{(1)}(\, w, \kappa_{\alpha \beta}, M_{\alpha \beta}, \tilde{w},_{n} \,) =$$

$$= \sum \left\{ \int_{A_{m}} \left[ (1/2)D_{\alpha \beta \gamma \delta} \kappa_{\alpha \beta} \kappa_{\gamma \delta} - M_{\alpha \beta} \kappa_{\alpha \beta} - M_{\alpha \beta} w,_{\alpha \beta} - \bar{p}_{z} w \right] \, dx \, dy$$

$$+ \int_{\partial A_{m}} M_{n}(w,_{n} - \tilde{w},_{n}) \, ds$$

$$- \int_{C_{u_{m}}} \left[ \bar{Q}_{n}(w - \tilde{w}) - M_{n}(\tilde{w},_{n} - \bar{w},_{n}) - M_{n_{s}}(w,_{s} - \bar{w},_{s}) \right] \, ds$$

$$- \int_{C_{\sigma_{m}}} (\bar{Q}_{n} w - \bar{M}_{n} \tilde{w},_{n} - \bar{M}_{n_{s}} w,_{s}) \, ds \right\} \quad (4.9)$$

where $\partial A_{m} = C_{m} + C_{u_{m}} + C_{\sigma_{m}} = $ entire boundary of the plate element
The Euler equations obtained from $\delta \pi_{mH}^{(1)} = 0$ are eqns. (4.3) to (4.7) plus the following ones:

on the boundary:

- compatibility of normal rotation on $\partial V_m$
  \[ w_n \mathbf{\tilde{w}}_n = 0 \quad (4.10) \]

- equilibrium of interelement normal moment on $C_m$
  \[ M_n^{(a)} - M_n^{(b)} = 0 \quad (4.11) \]

Following the same ideas presented before, we divide the displacement $w$ into two parts,

\[ w = w^q + w^\lambda \quad (4.12) \]

where

- $w^q$ is the part of lateral displacement which will be expressed in terms of nodal unknowns $q_i$. In this case $w^q$ must be continuous while $w_n^q$ is allowed to be discontinuous across interelement boundaries.

- $w^\lambda$ is the additional displacement which will be written in terms of internal displacement parameters $\lambda_i$ that can be statically condensed at the element level. Here they must be bubble functions (i.e., $w^\lambda = 0$ on $\partial A_m$) and $w_n^\lambda$ is allowed to have any value along element boundaries.

Realizing that:

\[ \int_A M_{\alpha \beta} w_{,\alpha \beta} \, dx\, dy = \int_A w \, M_{\alpha \beta,\alpha \beta} \, dx\, dy - \int_{\partial A} (Q_n w - M_n w_{,n} - M_{ns} w_{,s}) \, ds \quad (4.13) \]

and using the fact that

\[ w^\lambda = w_{,s}^\lambda = 0 \quad \text{on } \partial A_m \]
we obtain:

\[
\int_{A_m} M_{\alpha \beta} w_{\alpha \beta}^\lambda \, dx dy = \int_{A_m} w^\lambda M_{\alpha \beta,\alpha \beta} \, dx dy + \int_{\partial A_m} M_n w_n^\lambda \, ds \quad (4.14)
\]

Substituting the above equation and eqn. (4.12) into eqn. (4.9) we get:

\[
\pi_{mI}^{(2)}(w^q, w^\lambda, \kappa_{\alpha \beta}, M_{\alpha \beta}, \bar{w}_n) =
\]

\[
= \sum \left\{ \int_{A_m} \left[ (1/2) D_{\alpha \beta \gamma \delta} \kappa_{\alpha \beta} \kappa_{\gamma \delta} - M_{\alpha \beta} \kappa_{\alpha \beta} - M_{\alpha \beta,\alpha \beta} w^q - \bar{p}_z w^q \right] \, dx dy
\]

\[
- \int_{A_m} (M_{\alpha \beta,\alpha \beta} + \bar{p}_z) w^\lambda \, dx dy
\]

\[
+ \int_{\partial A_m} M_n (w_n^q - \bar{w}_n) \, ds
\]

\[
- \int_{C_u_m} \left[ Q_n (w_n^q - \bar{w}_n) - M_n (\bar{w}_n - \bar{\bar{w}}_n) - M_n (w_n^q - \bar{w}_n) \right] \, ds
\]

\[
- \int_{C_{\sigma_m}} (Q_n w_n^q - M_n \bar{w}_n - M_{ns} w_{ns}^q) \, ds \right\}
\]

Another form which is identical to the preceding one is obtained by using eqn. (4.13) once more:

\[
\pi_{mI}^{(3)}(w^q, w^\lambda, \kappa_{\alpha \beta}, M_{\alpha \beta}, \bar{w}_n) =
\]

\[
= \sum \left\{ \int_{A_m} \left[ (1/2) D_{\alpha \beta \gamma \delta} \kappa_{\alpha \beta} \kappa_{\gamma \delta} - M_{\alpha \beta} \kappa_{\alpha \beta} - (M_{\alpha \beta,\alpha \beta} + \bar{p}_z) w^q \right] \, dx dy
\]

\[
- \int_{A_m} (M_{\alpha \beta,\alpha \beta} + \bar{p}_z) w^\lambda \, dx dy
\]

\[
+ \int_{\partial A_m} (Q_n w_n^q - M_n \bar{w}_n - M_{ns} w_{ns}^q) \, ds
\]

\[
- \int_{C_u_m} \left[ Q_n (w_n^q - \bar{w}_n) - M_n (\bar{w}_n - \bar{\bar{w}}_n) - M_n (w_n^q - \bar{w}_n) \right] \, ds
\]

\[
- \int_{C_{\sigma_m}} (Q_n w_n^q - M_n \bar{w}_n - M_{ns} w_{ns}^q) \, ds \right\}
\]
A form equivalent to the one in eqn. (2.16) in which the homogeneous equilibrium equation is satisfied exactly:

\[ M_{\alpha\beta,\alpha\beta} = 0 \text{ in } A_m \]  

(4.17)

is written as

\[
\pi_{mHW}^{(4)}(w, \kappa_{\alpha\beta}, M_{\alpha\beta}, \tilde{w}_{,n}) =
\]

\[
= \sum_{m} \left\{ \int_{A_m} \left[ \frac{1}{2} D_{\alpha\beta\gamma\delta} \kappa_{\alpha\beta} \kappa_{\gamma\delta} - M_{\alpha\beta} \kappa_{\alpha\beta} - \bar{p}_w \right] \, dx \, dy
+ \int_{\partial A_m} (Q_n w - M_{n,n} \tilde{w}_{,n} - M_{n,s} w_s) \, ds
- \int_{C_{um}} (Q_n (w - \tilde{w}) - M_{n,n} (\tilde{w}_{,n} - \tilde{w}_{,n}) - M_{n,s} (w_s - \tilde{w}_s)) \, ds
- \int_{C_{\sigma m}} (\bar{Q}_n w - \bar{M}_{n,n} \tilde{w}_{,n} - \bar{M}_{n,s} w_s) \, ds \right\}
\]

(4.18)

Another set of modified forms is obtained when the continuity requirements for \( w \) and \( w_{,n} \) are relaxed. For a domain divided into \( m \) elements we may write:\[20\]

\[
\int_{A} M_{\alpha\beta} w_{,\alpha\beta} \, dx \, dy =
\]

\[
= \sum_{m} \left\{ \int_{A_m} M_{\alpha\beta} w_{,\alpha\beta} + \int_{C_m} \left[ Q_n (w - \tilde{w}) - M_{n,n} (w_{,n} - \tilde{w}_{,n}) - M_{n,s} (w_s - \tilde{w}_s) \right] \, ds \right\}
\]

(4.19)

now, \( M_n, w \) and \( w_{,n} \) are allowed to be discontinuous while \( \tilde{w} \) and \( \tilde{w}_{,n} \) must be continuous across interelement boundaries. Then a form corresponding to \( \pi_{mHW}^{(1)} \), eqn. (4.9), is given by:
\[
\pi^{(5)}_{mHW}(w, \kappa_{\alpha \beta}, M_{\alpha \beta}, \tilde{w}) =
\]
\[
= \sum \left\{ \int_{A_m} \left[ (1/2)D_{\alpha \beta \gamma \delta} \kappa_{\alpha \beta} \kappa_{\gamma \delta} - M_{\alpha \beta} \kappa_{\alpha \beta} - M_{\alpha \beta} w_{,\alpha \beta} - \bar{p}_z w \right] \, dx \, dy
\]
\[
- \int_{\partial A_m} \left[ Q_n (w - \tilde{w}) - M_n (w_{,n} - \tilde{w}_{,n}) - M_{ns} (w_{,s} - \tilde{w}_{,s}) \right] \, ds
\]
\[
- \int_{C_{nm}} \left[ Q_n (\tilde{w} - \bar{w}) - M_n (\tilde{w}_{,n} - \bar{w}_{,n}) - M_{ns} (\tilde{w}_{,s} - \bar{w}_{,s}) \right] \, ds
\]
\[
- \int_{C_{om}} (\tilde{Q}_n \tilde{w} - \tilde{M}_n \tilde{w}_{,n} - \tilde{M}_{ns} \tilde{w}_{,s}) \, ds \right\}
\]
\[
(4.20)
\]

Dividing displacement \( w \) into two parts:
\[
w = w^q + w^\lambda
\]
\[
(4.21)
\]

where

\( w^q \)

is an incompatible displacement which will be expressed in terms of nodal unknowns \( q_i \). Now, \( w^q \) and \( w^q \) are allowed to be discontinuous across interelement boundaries.

\( w^\lambda \)

is the additional displacement field expressed in terms of internal parameters \( \lambda_i \). Here they may be incompatible along the element boundary or they may be bubble functions.

Using eqns. (4.14), (4.20) and (4.21) we may write:
\[
\pi^{(6)}_{mHW}(w^q, w^\lambda, \kappa_{\alpha \beta}, M_{\alpha \beta}, \tilde{w}) =
\]
\[
= \sum \left\{ \int_{A_m} \left[ (1/2)D_{\alpha \beta \gamma \delta} \kappa_{\alpha \beta} \kappa_{\gamma \delta} - M_{\alpha \beta} \kappa_{\alpha \beta} - M_{\alpha \beta} w_{,\alpha \beta} - \bar{p}_z w^q \right] \, dx \, dy
\]
\[
- \int_{A_m} (M_{\alpha \beta, \alpha \beta} + \bar{p}_z) w^\lambda \, dx \, dy
\]
\[ \int_{\partial A_m} \left[ Q_n (w_q - \bar{w}) - M_n (w_q - \bar{w})_n - M_{ns} (w_q - \bar{w})_s \right] \, ds \\
\int_{C_{um}} \left[ Q_n (\bar{w} - \bar{w}) - M_n (\bar{w}_n - \bar{w})_n - M_{ns} (\bar{w}_s - \bar{w})_s \right] \, ds \\
\int_{C_{om}} \left( Q_n \bar{w} - M_n \bar{w}_n - M_{ns} \bar{w}_s \right) \, ds \right) \]

(4.22)

With one more integration by parts we get a form corresponding to \( \pi_{mHW}^{(3)} \):

\[
\pi_{mHW}^{(7)}(w^q, w^\gamma, \kappa_{\alpha\beta}, M_{\alpha\beta}, \bar{w}) = \\
= \sum_m \left\{ \int_{A_m} \left[ \frac{1}{2} D \alpha\beta\gamma \delta \kappa_{\alpha\beta} \kappa_{\gamma \delta} - M_{\alpha\beta} \kappa_{\alpha\beta} \right] \, dx \, dy \\
- \int_{A_m} (M_{\alpha\beta, \alpha\beta} + \bar{p}_z) \, w^\gamma \, dx \, dy \\
+ \int_{\partial A_m} (Q_n \bar{w} - M_n \bar{w}_n - M_{ns} \bar{w}_s) \, ds \\
- \int_{C_{um}} \left[ Q_n (\bar{w} - \bar{w}) - M_n (\bar{w}_n - \bar{w})_n - M_{ns} (\bar{w}_s - \bar{w})_s \right] \, ds \\
- \int_{C_{om}} \left( Q_n \bar{w} - M_n \bar{w}_n - M_{ns} \bar{w}_s \right) \, ds \right\} 
\]

(4.23)

The form corresponding to \( \pi_{mHW}^{(4)} \), eqn. (4.18), which is obtained from \( \pi_{mHW}^{(5)} \) and satisfying the homogeneous equilibrium equation, eqn.(4.17), is:

\[
\pi_{mHW}^{(8)}(w, \kappa_{\alpha\beta}, M_{\alpha\beta}, \bar{w}) = \\
= \sum_m \left\{ \int_{A_m} \left[ \frac{1}{2} D \alpha\beta\gamma \delta \kappa_{\alpha\beta} \kappa_{\gamma \delta} - M_{\alpha\beta} \kappa_{\alpha\beta} - \bar{p}_z w \right] \, dx \, dy \\
+ \int_{\partial A_m} (Q_n \bar{w} - M_n \bar{w}_n - M_{ns} \bar{w}_s) \, ds \right\} 
\]
4.2 Hu-Washizu Functionals for Marguerre Shallow Shell Theory

4.2.1 Conventional Form of the Hu-Washizu Principle

The conventional form of the generalized principle for shell analysis under Marguerre shallow shell theory which includes bending and stretching behaviour may be written as:

\[
\pi_{HW}( u_\alpha , w, \epsilon_{\alpha \beta} , \kappa_{\alpha \beta} , N_{\alpha \beta} , M_{\alpha \beta} ) = \]

\[
= \int_A \left[ (1/2)A_{\alpha \beta \gamma} \epsilon_{\alpha \beta} \epsilon_{\beta \gamma} + B_{\alpha \beta} \epsilon_{\alpha \beta} \kappa_{\gamma \delta} + (1/2)D_{\alpha \beta \gamma \delta} \epsilon_{\alpha \beta} \kappa_{\gamma \delta} \right] dxdy
\]

\[- \int_A \left[ N_{\alpha \beta} \epsilon_{\alpha \beta} + M_{\alpha \beta} \kappa_{\alpha \beta} \right] dxdy
\]

\[+ \int_A \left[ N_{\alpha \beta}(1/2)(u_{\alpha, \beta} + u_{\beta, \alpha} + z_{\alpha} w_{\beta} + z_{\beta} w_{\alpha}) + M_{\alpha \beta}( -w_{, \alpha \beta} ) \right] dxdy
\]

\[- \int_A \left( \bar{p}_{\alpha} u_{\alpha} + \bar{p}_{\beta} w \right) dxdy
\]

\[- \int_{C_u} \left[ N_n(u_{n} - \bar{u}_{n}) + N_{ns}(u_s - \bar{u}_s) - M_n(w_{, n} - \bar{w}_{, n}) - M_{ns}(w_s - \bar{w}_s) + V_n(w - \bar{w}) \right] ds
\]

\[- \int_{C_o} \left[ \bar{N}_n u_{n} + \bar{N}_{ns} u_s - \bar{M}_n w_{, n} - \bar{M}_{ns} w_s + \bar{V}_n w \right] ds \]

where

\[ N_{\alpha \beta} = \text{membrane stress resultants} \]
\[ M_{\alpha\beta} = \text{moment resultants} \]
\[ \epsilon_{\alpha\beta} = \text{in-plane strains} \]
\[ \kappa_{\alpha\beta} = \text{curvature strains} \]
\[ A_{\alpha\beta\gamma\delta}, B_{\alpha\beta\gamma\delta}, D_{\alpha\beta\gamma\delta} = \text{elastic constants} \]
\[ u_\alpha = \text{in-plane displacements} \]
\[ w = \text{out-of-plane displacement} \]
\[ \zeta = \text{coordinate normal to shell surface} \]
\[ z = \text{height of shell mid-surface with respect to base plane} \]
\[ p_\alpha, p_3 = \text{body forces} \]

The displacements normal and tangent to the boundary are given by:

\[ u_n = \nu_\alpha u_\alpha \quad (4.26) \]
\[ u_s = e_{\alpha\beta} \nu_\alpha u_\beta \quad (4.27) \]

Boundary tractions are:

\[ N_n = \nu_\alpha \nu_\beta N_{\alpha\beta} \]
\[ N_{ns} = e_{\alpha\gamma} \nu_\alpha \nu_\beta N_{\beta\gamma} \]
\[ M_n = \nu_\alpha \nu_\beta M_{\alpha\beta} \quad (4.28) \]
\[ M_{ns} = e_{\alpha\gamma} \nu_\alpha \nu_\beta M_{\beta\gamma} \]
\[ V_n = Q_n + N_n z_{,n} + N_{ns} z_{,s} \]

and
\[ Q_n = \nu_\alpha M_{\alpha\beta\beta} \]

displacements are measured in the base plane while stresses and strains are
measured in the shell surface.

The Euler equations obtained from the stationary condition of the principle 
\(\delta \pi_{HW} = 0\) are:

in the domain \(A\):

- **equilibrium equations**
  \[
  N_{\alpha\beta,\alpha} + p_\alpha = 0 \quad (4.29)
  \]
  \[
  M_{\alpha\beta,\alpha} + (N_{\alpha\beta}z_\beta)_\alpha + p_3 = 0 \quad (4.30)
  \]

- **strain-displacement relations**
  \[
  \epsilon_{\alpha\beta} = 1/2(u_{\alpha,\beta} + u_\beta,\alpha + z_\alpha w_\beta + z_\beta w_\alpha) = 0 \quad (4.31)
  \]
  \[
  \kappa_{\alpha\beta} + w_{,\alpha\beta} = 0 \quad (4.32)
  \]

- **stress-strain relations**
  \[
  N_{\alpha \beta} - A_{\alpha \beta \gamma \sigma} \epsilon_{\gamma \sigma} - B_{\alpha \beta \gamma \sigma} \kappa_{\gamma \sigma} = 0 \quad (4.33)
  \]
  \[
  M_{\alpha \beta} - B_{\alpha \beta \gamma \sigma} \epsilon_{\gamma \sigma} - D_{\alpha \beta \gamma \sigma} \kappa_{\gamma \sigma} = 0 \quad (4.34)
  \]

on the boundary:

- **mechanical boundary conditions on** \(C_\sigma\)
  \[
  N_n - \bar{N}_n = 0
  \]
  \[
  N_{ns} - \bar{N}_{ns} = 0
  \]
  \[
  M_n - \bar{M}_n = 0
  \]
  \[
  M_{ns} - \bar{M}_{ns} = 0 \quad (4.35)
  \]
  \[
  V_n - \bar{V}_n = 0
  \]

- **geometrical boundary conditions on** \(C_u\)
  \[
  u_n - \bar{u}_n = 0
  \]
\begin{equation}
\begin{align*}
    u_s - \bar{u}_s &= 0 \\
    w - \bar{w} &= 0 \\
    w_{,n} - \bar{w}_{,n} &= 0 \\
    w_{,s} - \bar{w}_{,s} &= 0
\end{align*}
\end{equation}

4.2.2 Modified Forms of the Hu-Washizu Principle

In the same way as done for plates, we could form and describe several possible modified forms of $\pi_{\text{HW}}$ for shells, but we will limit ourselves to the description of only three functionals that are of some interest, two of which will be used later for the development of new elements.

The first modified form presented will be the one obtained by relaxing the continuity requirement for $w_{,n}$. Dividing the domain $A$ into $m$ elements we may write:

\begin{equation}
\int_A M_{\alpha\beta} w_{,\alpha\beta} \, dx \, dy = \sum \left\{ \int_{A_m} M_{\alpha\beta} w_{,\alpha\beta} \, dx \, dy - \int_{C_m} M_n (w_{,n} - \bar{w}_{,n}) \, ds \right\}
\end{equation}

Then we obtain:

$$
\pi_{\text{mHW}}^{(1)}(u_\alpha, w, \epsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N_{\alpha\beta}, M_{\alpha\beta}, \bar{w}_{,n}) =
$$

\begin{align*}
    &= \sum \left\{ \int_{A_m} \left[ (1/2)A_{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} + B_{\alpha\beta\gamma} \epsilon_{\alpha\beta} \kappa_{\gamma} + (1/2)D_{\alpha\beta\gamma\delta} \kappa_{\alpha\beta} \kappa_{\gamma\delta} \right] \, dx \, dy \\
    &\quad - \int_{A_m} \left[ N_{\alpha\beta} \epsilon_{\alpha\beta} + M_{\alpha\beta} \kappa_{\alpha\beta} \right] \, dx \, dy \\
    &\quad + \int_{A_m} \left[ N_{\alpha\beta}(1/2)(u_{,\alpha\beta} + u_{,\beta\alpha} + z_{,\alpha} w_{,\beta} + z_{,\beta} w_{,\alpha}) + M_{\alpha\beta}( -w_{,\alpha\beta} ) \right] \, dx \, dy \\
    &\quad - \int_{A_m} ( -p_{,\alpha} u_{,\alpha} + p_{,3} w ) \, dx \, dy
\end{align*}
Let us separate displacements into two components:

\[ u_\alpha = u^q_\alpha + u^\lambda_\alpha \]
\[ w = w^q + w^\lambda \] (4.39)

where

\( u^q_\alpha \) and \( w^q \) are the part of the displacements which will be expressed in terms of nodal unknowns \( q_i \). In this case \( u^q_\alpha \) and \( w^q \) must be continuous while \( w^q_n \) is allowed to be discontinuous across interelement boundaries.

\( u^\lambda_\alpha \) and \( w^\lambda \) are the additional displacement fields which will be written in terms of internal displacement parameters \( \lambda_i \) that can be statically condensed at the element level. Here they must be bubble functions (i.e., \( u^\lambda_\alpha = w^\lambda = 0 \) on \( \partial A_m \)) and \( w^\lambda \) is allowed to have any value along element boundaries.

Performing integration by parts, the \( \pi_{mHW}^{(1)} \) functional becomes modified to give:

\[ \pi_{mHW}^{(2)}( u^q_\alpha, u^\lambda_\alpha, w^q, w^\lambda, \epsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N_{\alpha\beta}, M_{\alpha\beta}, \tilde{w}_n ) = \]

\[ = \sum_m \left\{ \int_{A_m} \left[ (1/2)A_{\alpha\beta\gamma}\epsilon_{\alpha\beta}\epsilon_{\gamma} + B_{\alpha\beta\gamma}\epsilon_{\alpha\beta}\kappa_{\gamma} + (1/2)D_{\alpha\beta\gamma\delta}\kappa_{\alpha\beta}\kappa_{\gamma\delta} \right] dx dy 
- \int_{A_m} \left[ N_{\alpha\beta}\epsilon_{\alpha\beta} + M_{\alpha\beta}\kappa_{\alpha\beta} \right] dx dy \right\} \]
The satisfaction of the complete homogeneous stress equilibrium equations in shell theory can be done by the use of stress-stress function relations which are obtained through the so-called static-geometric analogy for shells.  

For a shallow shell under Marguerre's theory, the relation between stresses and stress functions is relatively simple:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & -\frac{\partial^2}{\partial y^2} \\
0 & 0 & -\frac{\partial^2}{\partial x^2} \\
0 & 0 & \frac{\partial^2}{\partial x \partial y} \\
0 & -\frac{\partial}{\partial y} & z_{,yy} \\
-\frac{\partial}{\partial x} & 0 & z_{,xx} \\
\frac{1}{2} \frac{\partial}{\partial y} & \frac{1}{2} \frac{\partial}{\partial x} & -z_{,yy}
\end{bmatrix} 
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{bmatrix}
\]
where $\Phi_i$ are stress functions.

The use of the above relations would introduce couplings between membrane stress resultants and moment resultants in the $P$ matrix and the efficiency we are trying to achieve in the inversion of $H$ matrix (see eqns.(3.14),(3.34) and (3.35)) could not be obtained in this manner.

Another idea is to satisfy only part of the homogeneous equilibrium equations (or the flat shell equilibrium equations) specialized from eqns.(4.29) and (4.30):

$$N_{\alpha\beta,\gamma} = 0 \quad (4.42)$$
$$M_{\alpha\beta,\gamma} = 0 \quad (4.43)$$

Integrating $\pi^{(1)}_{mHw}$ by parts and substituting the above equations, we obtain

$$\pi^{(3)}_{mHw}(u, w, \epsilon, \kappa, N_{\alpha\beta}, M_{\alpha\beta}, w_n) = \sum \left\{ \int_{A_m} \left[ (1/2)A_{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} + B_{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta} \kappa_{\gamma\delta} + (1/2)D_{\alpha\beta\gamma\delta} \kappa_{\alpha\beta} \kappa_{\gamma\delta} \right] dx dy 
- \int_{A_m} \left[ N_{\alpha\beta} \epsilon_{\alpha\beta} + M_{\alpha\beta} \kappa_{\alpha\beta} \right] dx dy 
+ \int_{A_m} N_{\alpha\beta} 1/2(z_\alpha w_\beta + z_\beta w_\alpha) dx dy 
- \int_{A_m} (\vec{p}_\alpha u_\alpha + \vec{p}_3 w) dx dy 
+ \int_{\partial A_m} \left[ N_n u_n + N_{ns} w_n - M_n \vec{w}_n + M_{ns} w_s + Q_n w \right] ds 
- \int_{C_m} \left[ N_n (u_n w_n) + N_{ns} (u_s w_s) - M_n (\vec{w}_n w_n) - M_{ns} (w_s \vec{w}_s) + V_n (w \vec{w}) \right] ds \right\}$$
The use of this functional would permit assuming a \( P \) matrix with membrane stresses uncoupled from moments and vice-versa. The third integral in the principle above would enforce this coupling in a variational sense.
Chapter 5
Formulation of Plate and Shell Finite Elements

It is the purpose of this chapter to describe in detail all assumptions used in the development of the plate and shell elements studied in the present work. Section 5.1 presents a family of five triangular plate bending elements with a cubic \( w \) displacement distribution. Two of them have linear assumed moment distributions and three have quadratic assumed moment distributions. Section 5.2 describes two shell elements of cubic displacement fields having 30 degrees-of-freedom each. They are formulated based on Marguerre shallow shell theory.

5.1 Family of Cubic Displacement Triangular Plate Bending Elements

A family of triangular plate bending elements will be introduced at this point. Construction of \( C^{(0)} \) continuous shape functions for the lateral displacement \( w \) is an easy task, therefore the variational principles where only \( w \),\( n \) continuity requirement is relaxed, will be employed. Then, the additional displacement field \( w^\lambda \) must be represented in terms of bubble functions (i.e., \( w^\lambda = 0 \) along element boundary) as discussed in the explanations following eqn.(4.12).

The interior transverse displacement \( w^q \) is interpolated as an incomplete (9-term) cubic function\(^6\):

\[ w^q = \sum_{i=1}^{9} N_i \phi_i \]
\[ w^q = \sum_{i=1}^{3} \left[ \begin{array}{ccc} N_{1i} & N_{2i} & N_{3i} \end{array} \right] \left\{ \begin{array}{c} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{array} \right\} \] (5.1)

where

\[ N_{1i} = \xi_i + \xi_i^2 \xi_j + \xi_i^2 \xi_k - \xi_i \xi_j^2 - \xi_i \xi_k^2 \]
\[ N_{2i} = b_j [\xi_i^2 \xi_k + (1/2) \xi_i \xi_j \xi_k] - b_k [\xi_i^2 \xi_j + (1/2) \xi_i \xi_j \xi_k] \] (5.2)
\[ N_{3i} = a_j [\xi_i^2 \xi_k + (1/2) \xi_i \xi_j \xi_k] - a_k [\xi_i^2 \xi_j + (1/2) \xi_i \xi_j \xi_k] \]

\[ (i,j,k) = 1,2,3 \text{ in cyclic order} \]

\[ a_i = x_k - x_j \]
\[ b_i = y_j - y_k \]

\[ x_i, y_i = \text{nodal values of Cartesian coordinates } x, y \text{ respectively} \]
\[ \xi_i = \text{area (triangular) coordinates} \]
\[ \theta_x = w_y \]
\[ \theta_y = -w_x \]

with this information we are able to construct the \( N \) matrix defined in eqn.(3.3).

5.1.1 Elements with Linear Moment Distribution

A linear moment distribution satisfies the homogeneous equilibrium equation \( M_{\alpha\beta,\alpha\beta} = 0 \) automatically. Then the functional of \( \pi_{mHW}^{(4)} \) presented in eqn. (4.18) will be used.

A nine degree-of-freedom element (see fig.2) formulated in this way, will be
identical to the previously known HSM element\textsuperscript{20, 32}. Here it will be called T9L, T because it is a triangular element, the number 9 is used to indicate the number of degrees-of-freedom and L means linear assumed moment distribution. A twelve degree-of-freedom element (see fig.2), called T12L, will also be introduced at this point. It includes one rotational degree-of-freedom at each of its mid-sides which is being only used to interpolate the boundary rotation field.

The moment distribution of these elements will be written as:

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
P_1 & 0 & 0 \\
0 & P_1 & 0 \\
0 & 0 & P_1
\end{bmatrix} \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_9
\end{bmatrix} = P \beta
\] (5.3)

where

\[P_1 = \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}
\] (5.4)

Curvature strains will be interpolated in the same way:

\[\kappa = P \alpha
\] (5.5)

With the P matrix defined above, the H matrix defined in eqn.(3.14) can be obtained and its inverse \(H^{-1}\) will be:

\[
H^{-1} = \begin{bmatrix}
H_1^{-1} & 0 & 0 \\
0 & H_1^{-1} & 0 \\
0 & 0 & H_1^{-1}
\end{bmatrix}
\] (5.6)
where

\[ H_1^{-1} = \frac{3}{A} \begin{bmatrix} 3 & -1 & -1 \\ 3 & -1 \\ \text{sym} & 3 \end{bmatrix} \]  (5.7)

\( A = \text{area of triangular element} \)

The \( P_v \) matrix which interpolates the boundary tractions, can easily be obtained using eqns.(4.2) and matrix \( P \) above.

The rotation field \( \tilde{\omega}_n \) on the boundary, is interpolated as a linear function in the T9L element while it is quadratic in the T12L element. For element side 1 (connecting nodes 2 and 3, see fig.2) we may write:

for T9L:

\[ \tilde{\omega}_n = -(1/2)(1-\xi)\tilde{\theta}_{n_2} - (1/2)(1+\xi)\tilde{\theta}_{n_3} \]  (5.8)

for T12L:

\[ \tilde{\omega}_n = - (1/2)\xi(\xi-1)\tilde{\theta}_{n_2} - (1/2)\xi(\xi+1)\tilde{\theta}_{n_3} - (1-\xi)(1+\xi)\tilde{\theta}_{n_5} \]  (5.9)

where

\[ \tilde{\theta}_{n_i} = - \nu_y \theta_{x_i} + \nu_x \theta_{y_i} \quad i=2,3 \]

\( \xi = 2s/l_1 \quad -1 \leq \xi \leq 1 \)

\( l_1 = \text{length of side 1} \)

\( s = \text{coordinate along side, with origin at the mid-point} \)

Similar expressions can be obtained for sides 2 and 3. The above information enables us to construct the \( L \) matrix defined in eqn.(3.4).

Having \( N, L, P \) and \( P_v \) matrices as defined above, the stiffness matrix of
these plate bending elements with assumed linear moment distribution can be obtained by:

\[ k_n = G^T W G \]  \hspace{1cm} (5.10)

where \( G \) is defined in eqn.(3.18) and \( W \) in eqn.(3.26).

A consistent element load vector due to a distributed loading \( \overrightarrow{p_z} \) may be obtained by:

\[ Q_n = \int_{A_m} N^{T_{\overrightarrow{p_z}}} \ dx\,dy \]  \hspace{1cm} (5.11)

Element T12L contains 2 kinematic modes\(^{33,103}\). Then, this element has to be used carefully since it may not produce correct results for certain problems. These kinematic modes can be eliminated by adding more terms to the stress distribution. Therefore the elements with quadratic moment assumptions are introduced in the next section. These assumptions will eliminate those 2 kinematic modes. A nine degree-of-freedom element with assumed quadratic moment distribution (T9Q1) is included only for the sake of completeness, but one should realize that it will certainly produce an element of much stiffer behaviour than the T9L element.

5.1.2 Elements with Quadratic Moment Distribution

For these elements the functional \( \pi^{(2)}_{mHW} \), presented as eqn. (4.15), is used.

Three elements are studied (see Figs.2 and 3) : T9Q1, T12Q1 and T15Q1, a nine, twelve and fifteen degrees-of-freedom elements respectively. Again, according to the previous convention, \( T \) stands for triangular element, \( Q \) for quadratic moment distribution and the number 1 at the end indicates that only
one mode for \( w^\lambda \) is assumed, which is enough to satisfy the homogeneous equilibrium equation exactly. It should be noted that more than one mode for \( w^\lambda \) may be required if one seeks to satisfy various forms of the complete (inhomogeneous) equilibrium equation exactly or approximately as presented in eqn.(3.22).

The moment distribution can be written as:

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
P_2 & 0 & 0 \\
0 & P_2 & 0 \\
0 & 0 & P_2
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\ddots \\
\beta_{18}
\end{bmatrix}
= P \beta
\]  

(5.12)

where

\[
P_2 = \begin{bmatrix}
(2\xi_1-1)\xi_1 & (2\xi_2-1)\xi_2 & (2\xi_3-1)\xi_3 \\
4\xi_1\xi_2 & 4\xi_2\xi_3 & 4\xi_3\xi_1
\end{bmatrix}
\]  

(5.13)

Curvature strains are interpolated using the same \( P \) matrix:

\[
k = P \alpha
\]  

(5.14)

With the \( P \) matrix defined above the \( H \) matrix defined in eqn.(3.14) can be obtained and its inverse \( H^{-1} \) will be:

\[
H^{-1} = 
\begin{bmatrix}
H_2^{-1} & 0 & 0 \\
0 & H_2^{-1} & 0 \\
0 & 0 & H_2^{-1}
\end{bmatrix}
\]  

(5.15)
where

\[
\begin{bmatrix}
96 & 16 & 16 & -4 & 16 & -4 \\
96 & 16 & -4 & -4 & 16 \\
96 & 16 & -4 & -4 \\
26 & -9 & -9 \\
26 & -9 \\
\text{sym} & 26
\end{bmatrix}
\]

(5.16)

The \( P_v \) matrix is obtained by

\[
P_v = \begin{bmatrix} \nu_x^2 & \nu_y^2 & 2\nu_x\nu_y \end{bmatrix} P
\]

(5.17)

The rotation field \( \tilde{w}_n \) on the boundary is interpolated as a linear function for the T9Q1 (see eq.(5.8)), quadratic for the T12Q1 (see eq.(5.9)), and cubic for the T15Q1 element. For element T15Q1 we have along side 1:

\[
\tilde{w}_n = - \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} \Theta_{n_2} \\ \Theta_{n_3} \\ \Theta_{n_6} \\ \Theta_{n_7} \end{bmatrix}
\]

(5.18)

where

\[
N_1 = -(1/16) \ (1+3\xi)(1-3\xi)(1-\xi)
\]

\[
N_2 = -(1/16) \ (1+\xi)(1+3\xi)(1-3\xi)
\]

\[
N_3 = \ (9/16) \ (1+\xi)(1-3\xi)(1-\xi)
\]

\[
N_4 = \ (9/16) \ (1+\xi)(1+3\xi)(1-\xi)
\]

(5.19)

where all symbols were defined previously. Similar expressions can be obtained
for sides 2 and 3. This information is sufficient for the construction of the L matrix defined in eqn.(3.4).

The additional displacement field \( w^\lambda \) is expressed as:

\[
  w^\lambda = [ \xi_1 \xi_2 \xi_3 ] [ \lambda_1 ]
\]  

(5.20)

Having \( N, L, P, P_\nu \) and \( \Gamma \) matrices the stiffness matrix of these plate bending elements with quadratic assumed moment distribution can be obtained by using eqn.(3.28) and consistent element load vectors \( Q_n \) can be obtained by using eqn.(3.29).

5.2 Shell Finite Elements

The details for the construction of two shell finite elements are presented in this section. The shell elements will be called TS30, which is based on \( \Pi^{(2)}_{mHW} \) presented in eq.(4.40); and TS30E which is based on \( \Pi^{(4)}_{mHW} \) presented in eq.(4.44). The letter T stands for triangular element, S for shallow shell formulation, the number 30 is used to indicate 30 degrees-of-freedom and the letter E is used to differentiate between them. It is noted that for the TS30E element, parts of the equilibrium equations are satisfied directly on the assumptions for the stress distribution (i.e., \( N_{\alpha\beta,\beta} = 0 \), \( M_{\alpha\beta,\alpha} = 0 \)).

It has also been employed in the present study, a shell element developed previously by Boland\textsuperscript{73} based on a modified Hellinger-Reissner principle as follows:

\[
  \pi_{mR}( u_\alpha, w, N_{\alpha\beta}, M_{\alpha\beta}, \overset{\wedge}{w}_n ) =
\]
This element has been modified by including one rotational degree-of-freedom at each of its mid-sides (this being only used to interpolate the boundary rotation field). This element, following our convention, is called TS18.

As a first part, a discussion of coordinates systems and transformations relating these systems, is presented in section 5.2.1. Then, matrix expressions describing all assumptions used in the construction of the shell elements are presented in section 5.2.2.

5.2.1 Coordinate Systems and Transformations

The coordinate systems to be discussed here, are those required for the generation of element matrices and assembly procedures.

Three basic and, in general, distinct, rectangular Cartesian coordinate systems will be used (see fig.4):
1. A global system \((G_X_i = 1, 2, 3)\). Used for the description of the geometry of the structure. Usually chosen such that this description can be easily done. We will call its orthonormal base vectors, \(G_i e\).

2. A local system \((L_X_i = 1, 2, 3)\). This is a system where individual element properties and processes can be easily described. It will be defined at each element base plane (see fig.4). All element matrices will be generated initially referred to this system and then, for assemblage purposes, a transformation to a common coordinate system will be required. Its orthonormal base vectors will be called \(L_i e\).

3. A common system \((C_X_i = 1, 2, 3)\). This is the system where all the element matrices will be transformed to for assembling and solution purposes. At each nodal point in the structure a vector \(V_n\) normal to the actual shell surface at the point, should be given. Then, at any point \(P\), we will make \((C_X_3)_P\) direction coincide with \((V_n)_P\), or

\[
C_e_3 = V_n / |V_n| \quad \text{at } P \tag{5.22}
\]

Having \(C_e_3\), we define two vectors that are perpendicular to it and to each other. These vectors will be:

\[
C_e_1 = (G_e_2 \times C_e_3) / |G_e_2 \times C_e_3| \tag{5.23}
\]

\[
C_e_2 = C_e_3 \times C_e_1 \tag{5.24}
\]

Then \(C_e_i \ i = 1, 2, 3\) are the three orthonormal base vectors defining the common coordinate system.

For the special case when global \(G_X_2\)-axis and \(C_e_3\) are parallel \((G_e_2 \times C_e_3\)
we will define:

\[ C_e_1 = G_e_3 \text{ and } C_e_2 = G_e_1 \] (5.25)

Note that when \( C_e_3 \) is along the global \( G_X_3 \)-axis (\( C_e_3 = G_e_3 \)), \( C_e_1 \) and \( C_e_2 \) are along the global \( G_X_1 \) and \( G_X_2 \)-axes respectively.

Any two sets of such systems may be related to each other via the direction cosines between their axes. Thus,

\[ LX_i = LCT_{ij} C_j \] (5.26)

where \( LCT_{ij} = \cos(LX_i, C_j) = L_e_i \cdot C_j \) (5.27)

Displacements are transformed as follows:

\[ L_u_i = LCT_{ij} C_j \] (5.28)

and derivatives of displacements, assuming that all derivatives with respect to \( C_X_3 \) will vanish, are transformed as follows:

\[ \partial L_u_i / \partial L X_\alpha = LCT_{\alpha\beta} LCT_{im} \partial C_m / \partial C X_\beta + LCT_{\alpha\beta} (\partial LCT_{im} / \partial C X_3) C_u_m \]

Taking into consideration the shallow shell approximations:

\[ LCT_{33} \approx 1 \]
\[ LCT_{\alpha 3} \approx 0 \] (5.29)
\[ \partial LCT_{\alpha\beta} / \partial L X_\gamma \approx 0 \]

we may write the following relations:

\[ \partial L u_\sigma / \partial L X_\alpha \approx LCT_{\alpha\beta} LCT_{\sigma\gamma} \partial C_m / \partial C X_3 \]
5.2.2 Matrix Formulation of Shell Elements

All element matrices are initially generated referred to the local coordinate system and then transformed by using eqns.(5.28) and (5.30) to a common system for assemblage and solution purposes.

In Marguerre shallow shell theory the displacements that appear in the equations are all referred to a base plane (or local) coordinate system, while the stresses are referred to the shell surface (common) coordinates. Keeping this in mind, we will simply write the displacements as \( u, v, w \) instead of \( L_u, i=1,2,3 \) as done in the previous section. Also the stresses will be written without the superscripts denoting the common system.

The displacements are interpolated as cubic functions:

\[
\begin{bmatrix}
  u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{3} \begin{bmatrix}
  N_{1i} & 0 & 0 & -N_{3i} & N_{2i} & 0 & 0 & 0 & 0 \\
  0 & N_{1i} & 0 & 0 & -N_{3i} & N_{2i} & 0 & 0 & 0 \\
  0 & 0 & N_{1i} & 0 & 0 & 0 & -N_{3i} & N_{2i} & 0
\end{bmatrix} \begin{bmatrix}
  q_i 
\end{bmatrix}
\]

(5.31)

where

\[
\begin{bmatrix}
  q_i 
\end{bmatrix} = \begin{bmatrix}
  u_i \\
v_i \\
w_i \\
u_{x_i} \\
v_{x_i} \\
w_{x_i} \\
v_{y_i} \\
w_{y_i}
\end{bmatrix}^T
\]

(\( x = \partial/\partial X_1 \))

(\( y = \partial/\partial X_2 \))

and \( N_{ij} \) were presented in eqn.(5.2).

This information enables us to construct the \( N \) matrix defined in eqn.(3.3).

The height above the base plane, \( z \), is interpolated in the same way as
displacements:

\[ z = \sum_{i=1}^{3} \begin{bmatrix} -N_{3i} & N_{2i} \end{bmatrix} \begin{bmatrix} Z, Y_i \\ Z, Y_i \end{bmatrix} \]

(Note that \( z_i = 0 \) \( i = 1, 2, 3 \))

Stresses are interpolated as quadratic functions in TS30 element:

\[
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} P_2 & & & & \\ & P_2 & & & \\ & & P_2 & & \\ & & & P_2 & \\ & & & & P_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \vdots \\ \vdots \\ \beta_{36} \end{bmatrix}
\]

(5.33)

where \( P_2 \) is given in eqn.(5.13).

For the TS30E element, the stresses are given by:

\[
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = P_N \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{18} \end{bmatrix} = P_N \beta_N
\]

(5.34)

where \( P_N \) is given by
\[
\begin{bmatrix}
0 & 0 & 1 & \cdots & 0 & x^2 & 2xy & y^2 & \cdots & x^3 & 3x^2y & 3xy^2 & y^3 \\
1 & 0 & 0 & \cdots & x^2 & 2xy & y^2 & \cdots & x^3 & 3x^2y & 3y^2x & y^3 & \cdots \\
0 & 1 & 0 & \cdots & -x & -y & \cdots & -x^2 & 2xy & y^2 & \cdots & -x^3 & -3x^2y & -3xy^2 & y^3 & \cdots \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
P_1 & 0 & 0 \\
0 & P_1 & 0 \\
0 & 0 & P_1
\end{bmatrix}\begin{bmatrix}
\beta_{19} \\
\vdots \\
\beta_{27}
\end{bmatrix} = P_M \beta_M
\]

where \( P_1 \) is given in eqn.(5.4).

The \( P \) matrix is then obtained by:

\[
P = \begin{bmatrix}
P_N & 0 \\
0 & P_M
\end{bmatrix}
\]

(5.36)

For the TS30 element, the \( P_v \) matrix which relates boundary traction \( M_n \) and stress parameters can be obtained by:

\[
P_v = \begin{bmatrix}
0 & 0 & 0 & \nu_x^2 & \nu_y^2 & 2\nu_x\nu_y
\end{bmatrix}
\]

(5.37)

For the TS30E element, the \( P_v \) matrix relates all boundary tractions \( N_n, N_{ns}, M_n, M_{ns} \) and \( Q_n \) to the stress parameters. Therefore to obtain this matrix use eqns.(4.28) and matrix \( P \) presented above.

The rotation field \( \tilde{w}_{\alpha} \) on the boundary, is interpolated as a quadratic function for the TS30 and the TS30E element, in a similar way as done for
For element TS30 the additional displacement fields are written as:

\[
\begin{bmatrix}
u^\lambda \\
v^\lambda \\
w^\lambda \\
\end{bmatrix} = \begin{bmatrix}
\Gamma_1 & 0 & 0 \\
0 & \Gamma_1 & 0 \\
0 & 0 & \Gamma_1 \\
\end{bmatrix}\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\end{bmatrix}
\]  
(5.38)

where

\[\Gamma_1 = \xi_1 \xi_2 \xi_3\]

With the matrices presented above we are able to obtain matrices J, H, R and G as defined in eqns.(3.13), (3.14), (3.15) and (3.18), respectively. All those integrations are performed over the element base plane. Subsequently the element stiffness matrix can be obtained by using eqn.(3.28). Consistent element load vectors \(Q_n\) can be obtained from distributed loading by first calculating vectors \(Q^\lambda\) and \(Q^q\) defined in eqns.(3.16) and (3.17), respectively, and then by using eq.(3.29). Again, all integrations in eqns. (3.16) and (3.17) are performed over the element base plane.
Chapter 6
Applications, Evaluation and Discussion

In order to assess the performance of the present elements which are based upon modified versions of the Hu-Washizu principle, these elements have been applied to analyze several types of problems. Those problems include:

1. bending of a thin square flat plate with all four edges simply-supported, with the plate subjected to either a uniform lateral load or a concentrated lateral load applied at the center of the plate,

2. bending of a thin square flat plate with all four edges clamped, with the plate subjected to a concentrated lateral load applied at the center of the plate,

3. the "stretching" behaviour of a square flat plate with two of its opposite edges stress-free and the other two edges subjected to an equal and opposite in-plane parabolic tension loading, and

4. a thin shallow spherical cap segment of a square planform subjected to uniform pressure loading, while all four edges are freely-supported (both membrane and bending behaviour are present).

For each of these problems, exact solutions as well as other finite element solutions are available for comparison with the predictions produced from the present new elements.

To assist the reader in his appraisal of these comparisons, the principal features of each of the finite elements used in these comparisons are summarized in tables 1 and 2. Table 1 pertains to the present flat-plate bending elements which are known as T9L, T9Q1, T12L, T12Q1 and T15Q1, identifying the shape, number of nodes, number of degrees of freedom per each node, the degree-
of-freedom quantities used per each side, the total number of generalized
displacement \( q \) degrees of freedom for the element, the number of assumed
stress parameters, the number of kinematic modes possessed by the element, and
the specific variational principle upon which each element is based. Table 2
contains similar information for the triangular shallow shell elements used in
these comparisons: (a) the present TS18, TS30 and TS30E elements and (b) the
assumed displacement element developed by Cowper et al\(^{104}\).

Predictions and comparisons for each of these problems are shown and
discussed.

6.1 Simply-Supported Square Plate under Uniform Loading

This problem consists of a square plate of side length \( a \), simply-supported
on all four edges. The plate is loaded by an uniformly distributed pressure
acting over its entire surface. Because of symmetry, only one quarter of the
plate is modelled with uniform meshes of 2x2 and 4x4 with orientations as
shown in fig. 5. Only two element stiffness matrices are generated per mesh for
the solution. Consistent element load vectors are calculated based on the
assumed cubic displacement field. The numerical results are presented in tables
3 to 7 and figures 6 and 7.

Table 3 shows strain energy predictions, from where it can be concluded
that, on a mesh size basis, element T12L presents the fastest convergence to
the exact value of strain energy. Shown in parenthesis in Table 3 is the final
number of unknown degrees of freedom for each case.

Regarding convergence of the displacement at the center, as shown in
table 4, elements T12L and T15Q1 show the best results (again on a mesh size
basis), yielding comparable predictions. Note in table 4 that the element T12L prediction with a 2x2 mesh (24 equations) gives the same order of error (absolute value) as a 4x4 mesh (48 equations) of T9L elements. Also in this table, two columns are shown for T9L results; one is obtained from the work by Batoz et al.\textsuperscript{32} where lumped loading was used, while the other pertains to the use of consistent nodal loads. Clearly the use of consistent load vectors improves the results substantially.

Observing tables 5 and 6 for predictions of bending moments at the center and corner reaction, respectively, we might initially surmise that element T9L performance is superior to the other elements, but observing the complete stress distribution as shown in figures 6 and 7 we can clearly see that element T12L gives better results for stress-free boundary conditions, and stress predictions at the mid-side nodes are always very close to exact values, showing then, in general, a better overall predicted stress distribution. This is due mainly to the fact that the sole function of the mid-side node degree-of-freedom, is that of enforcing traction equilibrium across interelement boundaries; in fact, this condition is satisfied exactly at those points. Also, on stress-free boundaries the normal stress will be zero at those mid-side nodes.

Table 7 presents the CPU time required for the solution of the entire problem (data input, generation of stiffness matrix and load vectors, solution of equations, and data output).

For the elements studied, the use of the quadratic moment distribution increases computational time without improving the stress predictions. In element T15Q1, two nodes at each of its sides are only able to enforce a linear moment solution by tying the element's moment distribution to neighbouring elements. This is clearly confirmed in the results presented in fig.7. In element
T12Q1, as fig.7 also shows, the rotational degree-of-freedom at the mid-side nodes plus the ones at the corner nodes are also able to predict a linear moment solution. In addition, note that a linear moment solution is also enforced by the cubic displacement field through the variational principle (strain-displacement and stress-strain relations); however, this seems to be a very weak enforcement, as we observe in fig.7 the results of element T9Q1 which has no mid-side nodes.

As a global performance we could say that elements T9L and T12L are the preferred elements in the family. The CPU time required to run this problem, with these elements, is essentially the same despite the fact that for a given mesh size, the T12L element will yield, basically, a number of equations to be solved which is the double of that produced by the T9L element. This is due to the fact that for the present problem, with the mesh sizes used, the computer time required is governed by the generation of elements stiffness matrices and not by the solution of the equations. For very refined meshes or problems that require a large number of elements, element T9L may be the better choice. We should also note that, for certain problems, element T12L may not produce correct results because of its kinematic modes.

6.2 Simply-Supported Square Plate under Central Concentrated Load

This problem, as the preceding one, consists of a square plate of side length $a$, simply-supported on all four edges. The plate is now loaded by a concentrated normal load applied at the center point. Because of symmetry, only one quarter of the plate is modelled with uniform meshes with orientations as shown in fig.5. Only two element stiffness matrices are generated per mesh
for each solution. The numerical results are presented in tables 8 and 9 and figures 8 and 9.

Table 8 shows results for the displacement at the center where comparisons on a mesh size basis can be made. Good predictions are obtained from elements T12L, T12Q1 and T15Q1, with poorer results from element T9L.

Table 9 presents corner reaction results where element T9L and T9Q1 give the best answers for each mesh size.

Observing the stress distributions in figures 8 and 9, we clearly see better overall predictions by element T12L; the T12Q1 and T15Q1 predictions are also reasonably good. This is mainly due to the quite good stress-free predictions and enforcement of traction equilibrium at the mid-side nodes.

For this problem, as a global performance, it seems that element T12L gives slightly superior results compared with those of element T9L. The same comments regarding CPU time as made in section 6.1 also apply here.

6.3 Clamped Square Plate under Central Concentrated Load

This problem consists of a square plate of side length a, clamped on all four edges. The plate is loaded by a concentrated normal load applied at the center point. Because of symmetry, only one quarter of the plate is modelled with uniform meshes of 2x2 and 4x4 with orientations as shown in fig.5. Only two stiffness matrices are generated per mesh for each solution. Only convergence of the displacement at the center is presented as results (in table 10).

Slightly superior results are provided by element T12L compared with
those of element T9L. Elements T9L and T12Q1, for the same mesh size, give comparable predictions for the center displacement.

6.4 Stretching of a Square Plate under an In-Plane Parabolic Edge Loading

This plane stress problem, illustrated in fig.10, is used to check the in-plane behavior of the shell element. The problem is of a square plate loaded on two opposite sides by a parabolically distributed in-plane normal stress. The other two sides are stress-free. Using symmetry, only one quarter of the plate is modelled as indicated in fig.10. A consistent load vector is used to represent the edge load. Some of the numerical results for displacements and stresses at points A, B, C and D as well as strain energy predictions are presented in tables 11 to 15 and are compared with the exact solution presented by Cowper et al 104.

The convergence characteristics of the shell element developed by Cowper are also shown in the tables, where it is called QST, because its in-plane behavior, when flat, should be identical to the assumed displacement quadratic strain triangle.

As shown in table 13(a), the strain energy convergence properties of the QST and TS30 elements are identical, while TS30E exhibits a slightly stiffer behaviour.

Convergence to the largest displacement, $v_D$, is presented in table 12(b). Element QST converges from above while elements TS30 and TS30E exhibit some oscillating behavior but results do converge with grid refinement.
From observation of tables 11(a) to 12(b) we may conclude, regarding convergence to displacement values, that TS30 and QST yield essentially the same results, converging quite fast to the exact solutions. TS30E exhibits, in general, not as fast convergence. These results are expected since QST and TS30 both have cubic displacement distributions for the in-plane displacements; TS30 has two additional displacement fields being eliminated at the element level and QST also has two internal in-plane displacement degrees-of-freedom that are statically condensed at the element level. Then, both will be enforcing the satisfaction of the equilibrium equations at the element level of formulation in a similar way. The only difference between the two is in the shape functions used to interpolate the in-plane displacement fields. TS30 and TS30E have the same displacement distributions but the in-plane stresses in TS30E are interpolated by cubic functions, while they are quadratic in TS30. Obviously, then, TS30E will exhibit stiffer behaviour.

Regarding the prediction of stresses, QST and TS30 display nearly the same behavior giving, in general, slightly better results than those of TS30E (see tables 13(b) to 15(b)).

6.5 Shallow Spherical Cap under Uniform Pressure

The geometry of this problem is described in fig.11. The shell is loaded by a uniform normal pressure acting over its whole surface. The shell edges are freely-supported (i.e. \( w \) and \( v \) are zero along edge AD, \( w \) and \( u \) are zero along edge AB) on a boundary of square planform. The problem has an exact solution in the form of a double sine series presented by Ambartsumyam based on Vlasov shallow shell theory. This analysis is done for the value of
the shell parameter $Rh/a^2$ equal to 0.02. Both membrane and bending behaviour are of importance in this problem. The shell functions primarily as a membrane with zones of bending near the supporting edges. For the above shell parameter of 0.02, the effective shell boundary layer width is about $a/6$.

Using symmetry, only one quarter of the shell is modelled, and the calculations are carried out for uniform meshes of 2x2, 3x3 and 4x4 with triangular elements oriented as shown in fig.11. The use of a nonuniform mesh of elements would be a more efficient arrangement in this problem, as done in the works by Cowper et al\[^{104}\] and Dawe\[^{107}\], but this is not done in the present work.

For the present problem the element load vectors are calculated consistently with the shallow shell formulation (applied at the element level), with integrations performed over the element base planes.

Numerical results are presented in tables 16 to 20 and figures 12 to 16 where comparisons with both exact solutions and the results by Cowper et al\[^{104}\] for an assumed-displacement shallow shell triangular element are made. It should be emphasized that the exact solution and Cowper's element are both based on the shallow shell theory due to Vlasov\[^{106}\], while our elements are all based on shallow shell theory due to Marguerre\[^{101}\]. Slightly different exact solutions may be expected from these two shell theories.

Table 16 and fig.12 show strain energy results. Note that the TS30 element prediction convergence to the exact strain energy solution is not as fast as that exhibited by Cowper's element, which has an incomplete quintic, but complete quartic polynomial assumption for the lateral displacement and complete cubic distributions for the in-plane displacements. Element TS18
predictions shows faster convergence than does element TS30E on a degree-of-freedom basis. Table 16 also shows the computation time required for the solution on a per element basis. Elements TS30 and TS30E require, respectively, about 42 and 15 times the time needed by element TS18 to solve the problem.

Table 17 shows predictions for the lateral displacement at the center. Element TS18 gives results comparable to those of TS30 and TS30E. The element TS18 results converge from above while those of elements TS30 and TS30E converge from below. Figure 13 shows predictions of the lateral displacement along BC for the 4x4 mesh. Note that the best distribution is given by element TS18, which is slightly stiff near point B and then becomes more flexible as we approach point C. Element TS30E behaves in exactly the opposite manner to that of element TS18, while TS30 exhibits stiffer behaviour than the exact solution provides all along BC.

Convergence of the in-plane displacement at point B is shown in table 18 where reasonable results are given by elements TS18 and TS30E. Element TS30 converges from below while TS30E converges from above.

At the center point, the stress resultant $N_x$ and the moment $M_x$ results are presented in tables 19 and 20, respectively. Elements TS30 and TS30E apparently yield better results than TS18.

Stress distributions predicted by the elements on a 4x4 mesh are plotted in figures 14 to 16. Figure 14 shows results for stress resultants $N_x$ and $N_y$ along BC. TS18 with its constant stress in-plane behaviour follows the general trend of the exact solution in a reasonable way. Some oscillatory behaviour is noticed in the predictions by the TS30 and TS30E elements. This may be due
to the fact that the equilibrium equations are not being satisfied exactly by using only one mode for each of the three internal displacement fields. This fact can be confirmed through the work by Pian and Sumihara\textsuperscript{102}, where in the analysis of a cylindrical shell roof problem a similar oscillatory behaviour was observed in the stress distributions. By increasing the number of $\lambda$'s, or by better satisfying the equilibrium equations, the amplitudes of these oscillations tend to decrease, approaching the exact solution.

Figures 15 and 16, respectively, present $M_y$ and $M_x$ moment distributions along BC. Element TS30E with its linear moment distribution assumption predicts the general behaviour of the exact solution reasonably. Better results are given by element TS30, especially for $M_x$. The element TS30 predictions exhibit some oscillatory behaviour (the probable reasons have been discussed before). Element TS18 predicts the general trend of the exact curves reasonably if we consider only the results at the mid-side nodes, but the results at the corner nodes are somewhat awkward. These bad results may be due to the presence of mid-side nodes or to the assumed moment distribution which includes the height above the base plane, $z$. A comparison of predictions obtained by using Boland's original element\textsuperscript{73} to those obtained by using TS18 element would be necessary to assess the nature and reasons for its bad behaviour and for determining ways of improvement.
Chapter 7
Summary, Conclusions, and Recommendations

7.1 Summary

Alternative ways of obtaining hybrid stress elements based on modified versions of the Hu-Washizu principle have been presented. In these models the complete stress equilibrium equations are introduced as conditions of constraint through additional internal displacement parameters acting as Lagrange multipliers. This procedure permits satisfying the equilibrium equations either approximately or exactly in the variational principle; the associated (Lagrange multiplier) internal displacement parameters are subsequently condensed out at the element level.

The main features of these new models have been discussed and their versatility in the development of shell elements has been emphasized.

A family of triangular plate elements has been developed and numerical results obtained. Two new shallow shell elements have been constructed based on two different versions of the Hu-Washizu (generalized) principle. A third shell element, developed previously by Boland\textsuperscript{73}, based on a modified Hellinger-Reissner principle, has also been employed in the present study. This element has been modified by including one rotational degree-of-freedom at each of its mid-sides (this degree-of-freedom being used only to interpolate the boundary rotation field). Numerical studies have been made and comparisons with known
exact and numerical solutions presented. The present flat plate elements have been applied to plate bending problems to assess the behaviour of these elements in comparison with exact solutions. The present shallow shell elements have been tested in a pure plate stretching problem to assess their behaviour in comparison with both exact and other finite element solutions. The shallow shell elements also have been applied to analyze the bending and stretching behaviour of a pressure loaded spherical-shell segment; comparisons with exact and other finite element solutions permitted assessing the performance of these new elements.

7.2 Conclusions

Despite the fact of this work being a rather limited experience in the early stages of this new method based on modified versions of Hu-Washizu principle, the following set of conclusions can be drawn:

1. The convenience and versatility of formulating shell finite elements based on this new formulation is very clear. Here we are assuming stress distributions in terms of natural coordinates, without worrying about the a priori satisfaction of the equilibrium equations, but still enforcing it at the element level. In this respect, this new technique is very simple and versatile since several levels of approximately satisfying the equilibrium equations may be achieved by varying the number of additional internal displacement parameters. We may also think in terms of satisfying only a portion of the equilibrium equations by introducing this portion into the variational principle properly constrained by the Lagrange multipliers (here they may not be identified to displacement parameters anymore).

2. Now we should compare the efficiency of obtaining hybrid stress elements through this new approach with the conventional formulative way using Minimum Complementary Energy or Hellinger-
Reissner principles. For a direct efficiency comparison, the elements presently studied should also have been implemented using the conventional formulation. But, since this has not been done, we can only express personal feelings about the matter. In the conventional technique, the computing time required to generate an element stiffness matrix is governed by the inversion of a flexibility matrix. The order of this matrix is equal to the number of stress parameters. In the present technique we are also required to invert a symmetric positive definite matrix (see eqn.(3.14)) of order equal to the number of stress parameters, but this can be done only once and for all, as shown previously. In this new method we also have to invert a matrix of order equal to the number of additional displacement parameters (or Lagrange multipliers), and perform a series of matrix multiplications, as shown in eqn.(3.28), that are not present in the conventional method. Therefore, computational advantage of using one method instead of another is governed by the number of stress parameters and by a balance between this number and the number of additional displacement parameters. Then, regarding computational efficiency, it seems that for elements with a large number of stress parameters like TS30 and TS30E (36 and 27 betas, respectively), the use of Hu-Washizu principle is preferred, while this may not be true for elements like TS18 which has only 12 stress parameters. Obviously, one has to establish, for each individual case under study, the number of stress parameters that determines this "boundary of efficiency" and one has to study how the number of additional displacement parameters affects this "boundary".

3. A clear advantage of this new approach over the conventional assumed-stress hybrid model in the formulation of shell finite elements is the simplicity of its formulation since we are not required to satisfy equilibrium equations \textit{a priori}.

4. A drawback of the present approach, detected in our limited experience, is in the formulation of consistent element load vectors from applied body forces. The calculation of these vectors is dependent on several matrices that are generated at the time of stiffness matrix calculation (see eqns.(3.16),(3.17) and (3.29)). To avoid the recalculation of these matrices, which may require significant effort, element load vectors have to be generated at the same time stiffness matrices are being calculated. This is one feature that may not be convenient for general purpose finite element programs where a higher degree of independence between
the generation of load vectors and generation of element stiffnesses may be required. But, this problem can be circumvented by satisfying only the homogeneous part of the equilibrium equations by proper use of Lagrange multipliers instead of the complete inhomogeneous equations as studied in the present work.

7.3 Recommendations for Future Study

Despite the fact that element TS18 is quite inexpensive to use, its results for moments are not very good (at least in the present configuration). One should carefully compare its results to predictions by Boland's original element 73 (rather than the present modification of Boland's element) as to assess the nature of and reasons for its bad behaviour and for determining ways of improvement.

Element TS30 can be improved in several ways. First, the effect of increasing the number of \( \lambda \)'s should be verified. Another way of improving stress distributions, but at a risk of excessively increasing the total number of equations to be solved for a given mesh, is by adding one more rotational degree-of-freedom at each of its sides, as in the T15Q1 bending element. The use of deep shell theory may also be worth trying, since it will not add much complexity in this new method.

From the study of plate bending elements we have clearly determined the role of the rotational degrees-of-freedom at the side nodes. Since they are being used only to represent the boundary rotation field, their sole function is to enforce traction equilibrium along element boundaries and that is exactly satisfied at those nodes. The rotational degrees-of-freedom at corner nodes will have a double role, since they are representing not only the displacement field
w but also the boundary rotation field. Then, these will be responsible for the
description of element straining modes and also for interelement traction
equilibrium enforcement, but in a weaker sense. Since these side nodes are
being used to connect the stress distributions assumed inside each element, we
may think in terms of reducing the order of the polynomials assumed for the
displacement fields. Then, the line of development presented by Pian and
Sumihara\textsuperscript{102} on hybrid semiloof shell elements, is very promising. There,
rotational degrees-of-freedom are located at the sides and only translational
degrees-of-freedom are present at the corner nodes leading to simple formulation
and making those elements very easy to use.

As a final note, we could say that we are, right now, facing the early
stages of a very promising method, with many options still available, several
paths still open and many questions not answered.
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<th>Element*</th>
<th>Shape</th>
<th>Nodes</th>
<th>DOF/Node</th>
<th>Nodal DOF's</th>
<th>Total q-DOF</th>
<th>No. of Stress Param. β</th>
<th>No. of Kinem. Modes</th>
<th>Element Basis</th>
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<td>w, w, x, w, y</td>
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<td>3</td>
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<td>Modified Hu-Washizu, Eq. 4.15</td>
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*Nomenclature:
T  Triangle Element  
N  = 9,12,15: No. of q DOF's  
L  Linear Assumed Moment Distribution  
Q  Quadratic assumed Moment of Distribution
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<th>Element*</th>
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<th>Nodes</th>
<th>DOF/Node</th>
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<td>( \tilde{w}, n )</td>
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<td>9</td>
<td>( u, v, w, x, y, v, x, y )</td>
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<td>( \tilde{w}, n )</td>
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<td>12</td>
<td>( u, v, w, x, y, v, x, y, w, x, y, w, \tilde{x}, xy )</td>
<td>36</td>
<td>--</td>
<td>--</td>
<td>Principle of Stationary Total Potential Energy Assumed Displacement Model</td>
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*Nomenclature:  
T: Triangular  
S: Shallow Shell  
N = 18, 30, etc. = No. of q-DOF's
Table 3
Simply-Supported Square Plate under Uniform Loading

Convergence of Strain Energy \[ \frac{10^3 UD}{a^6 P_z^2} \]

exact solution = 0.851255

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<th>Δ%</th>
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\[ Δ\% = \left( \frac{\text{sol} - \text{exact sol}}{\text{exact sol}} \right) \times 100 \]

( ) : final number of unknown degrees of freedom
Table 4
Simply-Supported Square Plate under Uniform Loading
Convergence of displacement at center $10^3 \frac{w_c D}{a^4 p_z}$

<table>
<thead>
<tr>
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<th>T9L+</th>
<th>T9L</th>
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</table>

* Lumped loading (Batoz et al.)

All other results are for consistent nodal loading.

The exact solution is $4.062$. 

[Image - Table 4]
Table 5
Simply-Supported Square Plate under Uniform Loading
Convergence of bending moment at center \( \frac{10^2 (M_X)_C}{a^2 p_z} \) exact solution = 4.789

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⁺ lumped loading (Batoz et al)
Table 6
Simply-Supported Square Plate under Uniform Loading

Convergence of corner reaction \( \frac{2M_{xy}}{a^2p_z} \)

<table>
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<th>MESH</th>
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<th>T9QI</th>
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<td>(192)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^+\) lumped loading (Batoz et al)

exact solution = -0.0650
Table 7

Simply-Supported Square Plate under Uniform Loading

CPU time [sec] on Vax 11/782 computer for solution of entire problem

dof = final number of degrees-of-freedom

<table>
<thead>
<tr>
<th>MESH</th>
<th>T9L</th>
<th>T9QI</th>
<th>T12L</th>
<th>T12QI</th>
<th>T15QI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>time</td>
<td>dof</td>
<td>time</td>
<td>dof</td>
<td>time</td>
</tr>
<tr>
<td>2x2</td>
<td>6.00</td>
<td>12</td>
<td>13.0</td>
<td>12</td>
<td>6.43</td>
</tr>
<tr>
<td>4x4</td>
<td>9.58</td>
<td>48</td>
<td>16.9</td>
<td>48</td>
<td>10.7</td>
</tr>
</tbody>
</table>
Table 8  
Simply-Supported Square Plate under Central Concentrated Load  
Convergence of Displacement at Center $\frac{10^2 w_c D}{a^2 P}$  

<table>
<thead>
<tr>
<th>MESH</th>
<th>T9L</th>
<th>T9QI</th>
<th>T12L</th>
<th>T12QI</th>
<th>T15QI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>sol</td>
<td>sol</td>
<td>sol</td>
<td>sol</td>
</tr>
<tr>
<td></td>
<td>$\Delta%$</td>
<td>$\Delta%$</td>
<td>$\Delta%$</td>
<td>$\Delta%$</td>
<td>$\Delta%$</td>
</tr>
<tr>
<td>1x1</td>
<td>0.9838</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>1.0876</td>
<td>1.0620</td>
<td>1.1320</td>
<td>1.1295</td>
<td>1.1311</td>
</tr>
<tr>
<td></td>
<td>(12)</td>
<td>(12)</td>
<td>(24)</td>
<td>(24)</td>
<td>(36)</td>
</tr>
<tr>
<td>4x4</td>
<td>1.1399</td>
<td>1.1328</td>
<td>1.1534</td>
<td>1.1522</td>
<td>1.1526</td>
</tr>
<tr>
<td></td>
<td>(48)</td>
<td>(48)</td>
<td>(96)</td>
<td>(96)</td>
<td>(144)</td>
</tr>
<tr>
<td>8x8</td>
<td>1.1546</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>(192)</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

exact solution = 1.1600
Table 9

Simply-Supported Square Plate under Central Concentrated Load

Convergence of Corner Reaction \( \frac{2 M_{xy}}{P} \)

exact solution = -0.1219

<table>
<thead>
<tr>
<th>MESH</th>
<th>T9L</th>
<th>T9QI</th>
<th>T12L</th>
<th>T12QI</th>
<th>T15QI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>Δ%</td>
<td>sol</td>
<td>Δ%</td>
<td>sol</td>
</tr>
<tr>
<td>1x1</td>
<td>-0.1653 ((3))</td>
<td>35.6</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2x2</td>
<td>-0.1317 ((12))</td>
<td>8.02</td>
<td>-0.1305 ((12))</td>
<td>7.00</td>
<td>-0.1397 ((24))</td>
</tr>
<tr>
<td>4x4</td>
<td>-0.1231 ((48))</td>
<td>0.97</td>
<td>-0.1231 ((48))</td>
<td>0.97</td>
<td>-0.1275 ((96))</td>
</tr>
<tr>
<td>8x8</td>
<td>-0.1221 ((192))</td>
<td>0.14</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 10
Clamped Square Plate under Central Concentrated Load
Convergence of Displacement at Center \( \frac{10^3 w_c D}{a^2 P} \)

exact solution = 5.600

<table>
<thead>
<tr>
<th>MESH</th>
<th>T9L</th>
<th>Δ%</th>
<th>T9Q1</th>
<th>Δ%</th>
<th>T12L</th>
<th>Δ%</th>
<th>T12Q1</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td></td>
<td>sol</td>
<td></td>
<td>sol</td>
<td></td>
<td>sol</td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>5.022 (8)</td>
<td>-10.3</td>
<td>4.808 (8)</td>
<td>-14.1</td>
<td>5.153 (16)</td>
<td>-7.98</td>
<td>5.021 (16)</td>
<td>-10.3</td>
</tr>
<tr>
<td>4x4</td>
<td>5.441 (40)</td>
<td>-2.84</td>
<td>5.373 (40)</td>
<td>-4.05</td>
<td>5.530 (80)</td>
<td>-1.25</td>
<td>5.508 (80)</td>
<td>-1.64</td>
</tr>
</tbody>
</table>
Table II
Stretching of a Square Plate under an In-Plane Parabolic Edge Loading

(a) Convergence of \( \frac{10Eh_{u_B}}{(1-\nu^2)N_0L} \) exact sol. = -1.519928

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>sol</td>
<td>sol</td>
</tr>
<tr>
<td></td>
<td>( \Delta%)</td>
<td>( \Delta%)</td>
<td>( \Delta%)</td>
</tr>
<tr>
<td>1x1</td>
<td>-1.507941</td>
<td>-1.535426</td>
<td>-1.476470</td>
</tr>
<tr>
<td></td>
<td>-0.79</td>
<td>1.02</td>
<td>-2.90</td>
</tr>
<tr>
<td>(14)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>-1.519812</td>
<td>-1.519396</td>
<td>-1.523228</td>
</tr>
<tr>
<td></td>
<td>-0.008</td>
<td>-0.035</td>
<td>0.22</td>
</tr>
<tr>
<td>(38)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4x4</td>
<td>-1.519862</td>
<td>-1.519753</td>
<td>-1.521648</td>
</tr>
<tr>
<td></td>
<td>-0.004</td>
<td>-0.012</td>
<td>0.11</td>
</tr>
<tr>
<td>(122)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

( ) = Total number of unknown degrees-of-freedom

(b) Convergence of \( \frac{10^2Eh_{u_C}}{(1-\nu^2)N_0L} \) exact sol. = 1.7837

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>sol</td>
<td>sol</td>
</tr>
<tr>
<td></td>
<td>( \Delta%)</td>
<td>( \Delta%)</td>
<td>( \Delta%)</td>
</tr>
<tr>
<td>1x1</td>
<td>2.1934</td>
<td>2.1500</td>
<td>0.7207</td>
</tr>
<tr>
<td></td>
<td>23.0</td>
<td>20.5</td>
<td>-59.6</td>
</tr>
<tr>
<td>(14)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>1.8684</td>
<td>1.8679</td>
<td>1.8178</td>
</tr>
<tr>
<td></td>
<td>4.75</td>
<td>4.72</td>
<td>1.91</td>
</tr>
<tr>
<td>(38)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4x4</td>
<td>1.7896</td>
<td>1.7896</td>
<td>1.7833</td>
</tr>
<tr>
<td></td>
<td>0.33</td>
<td>0.33</td>
<td>-0.025</td>
</tr>
<tr>
<td>(122)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 12
Stretching of a Square Plate under an In-Plane Parabolic Edge Loading

(a) Convergence of \( \frac{10Eh v_c}{(1-v^2)N_0L} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>( \Delta % )</td>
<td>sol</td>
</tr>
<tr>
<td>1 x 1 (14)</td>
<td>1.31824</td>
<td>3.21</td>
<td>1.31390</td>
</tr>
<tr>
<td>2 x 2 (38)</td>
<td>1.28574</td>
<td>0.66</td>
<td>1.28569</td>
</tr>
<tr>
<td>4 x 4 (122)</td>
<td>1.27787</td>
<td>0.047</td>
<td>1.27786</td>
</tr>
</tbody>
</table>

(b) Convergence of \( \frac{10 E h v_D}{(1-v^2)N_0L} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>( \Delta % )</td>
<td>sol</td>
</tr>
<tr>
<td>1 x 1 (14)</td>
<td>5.085466</td>
<td>0.24</td>
<td>5.057995</td>
</tr>
<tr>
<td>2 x 2 (38)</td>
<td>5.073595</td>
<td>0.0023</td>
<td>5.074011</td>
</tr>
<tr>
<td>4 x 4 (122)</td>
<td>5.073544</td>
<td>0.0013</td>
<td>5.073654</td>
</tr>
</tbody>
</table>
Table 13
Stretching of a Square Plate under an In-Plane Parabolic Edge Loading

(a) Convergence of \( \frac{10EhU}{(1-\nu^2)N_0^2L^2} \)  
exact sol. = 2.7935695

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>Δ%</td>
<td>sol</td>
</tr>
<tr>
<td>1 x 1</td>
<td>2.7879813</td>
<td>-0.20</td>
<td>2.788181</td>
</tr>
<tr>
<td>(14)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 x 2</td>
<td>2.7933662</td>
<td>-0.70</td>
<td>2.793379</td>
</tr>
<tr>
<td>(38)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 x 4</td>
<td>2.7935617</td>
<td>-0.57</td>
<td>2.793562</td>
</tr>
<tr>
<td>(122)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Convergence of \( \frac{10N_0X_A}{N_0} \)  
exact sol. = -1.40954

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>Δ%</td>
<td>sol</td>
</tr>
<tr>
<td>1 x 1</td>
<td>-1.44190</td>
<td>2.30</td>
<td>-1.42831</td>
</tr>
<tr>
<td>(14)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 x 2</td>
<td>-1.40137</td>
<td>-0.58</td>
<td>-1.38768</td>
</tr>
<tr>
<td>(38)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 x 4</td>
<td>-1.40789</td>
<td>-0.12</td>
<td>-1.40689</td>
</tr>
<tr>
<td>(122)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 14
Stretching of a Square Plate under an In-Plane Parabolic Edge Loading

(a) Convergence of \( \frac{10 N_y}{N_0} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th></th>
<th>TS30</th>
<th></th>
<th>TS30E</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>( \Delta % )</td>
<td>sol</td>
<td>( \Delta % )</td>
<td>sol</td>
<td>( \Delta % )</td>
</tr>
<tr>
<td>1 x 1</td>
<td>8.55810</td>
<td>-0.38</td>
<td>9.57169</td>
<td>-0.20</td>
<td>8.65182</td>
<td>0.71</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 x 2</td>
<td>8.59863</td>
<td>0.095</td>
<td>8.61232</td>
<td>0.25</td>
<td>8.65323</td>
<td>0.73</td>
</tr>
<tr>
<td>(38)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 x 4</td>
<td>8.59211</td>
<td>0.019</td>
<td>8.59311</td>
<td>0.030</td>
<td>8.59828</td>
<td>0.090</td>
</tr>
<tr>
<td>(122)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Convergence of \( \frac{10 N_y B}{N_0} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th></th>
<th>TS30</th>
<th></th>
<th>TS30E</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol</td>
<td>( \Delta % )</td>
<td>sol</td>
<td>( \Delta % )</td>
<td>sol</td>
<td>( \Delta % )</td>
</tr>
<tr>
<td>1 x 1</td>
<td>4.70735</td>
<td>14.6</td>
<td>4.66466</td>
<td>13.6</td>
<td>4.57532</td>
<td>11.4</td>
</tr>
<tr>
<td>(14)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 x 2</td>
<td>4.17500</td>
<td>1.66</td>
<td>4.17106</td>
<td>1.57</td>
<td>4.12623</td>
<td>0.48</td>
</tr>
<tr>
<td>(38)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 x 4</td>
<td>4.10971</td>
<td>0.073</td>
<td>4.10939</td>
<td>0.065</td>
<td>4.10107</td>
<td>-0.14</td>
</tr>
<tr>
<td>(122)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 15

Stretching of a Square Plate under an In-Plane Parabolic Edge Loading

(a) Convergence of \( \frac{N_{xB}}{N_0} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x 1 (14)</td>
<td>0.040</td>
<td>0.023</td>
<td>-0.071</td>
</tr>
<tr>
<td>2 x 2 (38)</td>
<td>0.0042</td>
<td>0.0021</td>
<td>0.0096</td>
</tr>
<tr>
<td>4 x 4 (122)</td>
<td>0.00033</td>
<td>0.00011</td>
<td>0.00075</td>
</tr>
</tbody>
</table>

(b) Convergence of \( \frac{N_{xC}}{N_0} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>QST</th>
<th>TS30</th>
<th>TS30E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x 1 (14)</td>
<td>0.032</td>
<td>0.024</td>
<td>0.011</td>
</tr>
<tr>
<td>2 x 2 (38)</td>
<td>-0.000050</td>
<td>-0.00029</td>
<td>0.015</td>
</tr>
<tr>
<td>4 x 4 (122)</td>
<td>-0.0029</td>
<td>-0.0028</td>
<td>0.014</td>
</tr>
</tbody>
</table>
Table 16

Shallow Spherical Cap under Uniform Pressure

Convergence of Strain Energy

\[
\frac{10 E h U}{p_0^2 a^2 R^2}
\]

exact solution = 3.89958

<table>
<thead>
<tr>
<th>MESH</th>
<th>EL</th>
<th>TS18</th>
<th>TS30</th>
<th>TS30E</th>
<th>Cowper et al</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>DOF</td>
<td>T/EL</td>
<td>sol.</td>
<td>Δ%</td>
</tr>
<tr>
<td>2x2</td>
<td>8</td>
<td>32</td>
<td>1.47</td>
<td>5.45305</td>
<td>39.8</td>
</tr>
<tr>
<td>3x3</td>
<td>18</td>
<td>72</td>
<td>1.22</td>
<td>4.17694</td>
<td>7.11</td>
</tr>
<tr>
<td>4x4</td>
<td>32</td>
<td>128</td>
<td>1.11</td>
<td>3.96544</td>
<td>1.69</td>
</tr>
</tbody>
</table>

EL = number of elements

DOF = final number of degrees-of-freedom

T = CPU time (VAX 11/782) for entire solution [sec]
Table 17
Shallow Spherical Cap under Uniform Pressure
Convergence of Lateral Displacement at Center \( \frac{E h w_c}{p_o R^2} \)

<table>
<thead>
<tr>
<th>MESH</th>
<th>TS18</th>
<th>Δ%</th>
<th>TS30</th>
<th>Δ%</th>
<th>TS30E</th>
<th>Δ%</th>
<th>Cowper et al</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x2</td>
<td>-1.340456</td>
<td>32.7</td>
<td>-0.968592</td>
<td>-4.08</td>
<td>-0.894302</td>
<td>-11.4</td>
<td>-1.012863</td>
<td>0.30</td>
</tr>
<tr>
<td>3x3</td>
<td>-1.071786</td>
<td>6.14</td>
<td>-0.968042</td>
<td>-4.13</td>
<td>-0.951276</td>
<td>-5.79</td>
<td>-1.009819</td>
<td>0.0031</td>
</tr>
<tr>
<td>4x4</td>
<td>-1.028092</td>
<td>1.81</td>
<td>-0.984632</td>
<td>-2.49</td>
<td>-0.975472</td>
<td>-3.40</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

exact solution = -1.009788
Table 18

Shallow Spherical Cap under Uniform Pressure

Convergence of In-Plane Displacement at Point B

\[ \frac{10 E h v_B}{p_0 a R} \]

exact solution = 3.67936

<table>
<thead>
<tr>
<th>MESH</th>
<th>TS18 sol.</th>
<th>Δ%</th>
<th>TS30 sol.</th>
<th>Δ%</th>
<th>TS30E sol.</th>
<th>Δ%</th>
<th>Cowper et al sol.</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x2</td>
<td>4.32879</td>
<td>17.7</td>
<td>3.37735</td>
<td>-8.21</td>
<td>3.85330</td>
<td>4.73</td>
<td>3.66585</td>
<td>-0.37</td>
</tr>
<tr>
<td>3x3</td>
<td>3.71487</td>
<td>0.97</td>
<td>3.54433</td>
<td>-3.67</td>
<td>3.70049</td>
<td>0.57</td>
<td>3.67742</td>
<td>-0.053</td>
</tr>
<tr>
<td>4x4</td>
<td>3.64947</td>
<td>-0.81</td>
<td>3.59961</td>
<td>-2.17</td>
<td>3.68172</td>
<td>0.064</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
Table 19
Shallow Spherical Cap under Uniform Pressure
Convergence of Stress Resultant $N_x$ at Center

$$\frac{10(N_x)_c}{R_0 R}$$

**exact solution** = -5.049

<table>
<thead>
<tr>
<th>MESH</th>
<th>TS18</th>
<th></th>
<th>TS30</th>
<th></th>
<th>TS30E</th>
<th></th>
<th>Cowper et al</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sol.</td>
<td>Δ%</td>
<td>sol.</td>
<td>Δ%</td>
<td>sol.</td>
<td>Δ%</td>
<td>sol.</td>
</tr>
<tr>
<td>2x2</td>
<td>-5.384</td>
<td>6.63</td>
<td>-6.491</td>
<td>28.6</td>
<td>-4.301</td>
<td>-14.8</td>
<td>-5.248</td>
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<tr>
<td>3x3</td>
<td>-5.426</td>
<td>7.47</td>
<td>-5.115</td>
<td>1.31</td>
<td>-5.061</td>
<td>0.24</td>
<td>-5.111</td>
</tr>
<tr>
<td>4x4</td>
<td>-5.320</td>
<td>5.37</td>
<td>-4.999</td>
<td>-0.99</td>
<td>-4.756</td>
<td>-5.80</td>
<td>—</td>
</tr>
</tbody>
</table>
Table 20

Shallow Spherical Cap under Uniform Pressure

Convergence of Moment $M_x$ at Center

\[
\frac{10^3 (M_x)_C}{p_0 R}
\]

exact solution = -8.488

<table>
<thead>
<tr>
<th>MESH</th>
<th>TS18</th>
<th>TS30</th>
<th>TS30E</th>
<th>Cowper et al</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x2</td>
<td>392.1</td>
<td>-2.366</td>
<td>-19.43</td>
<td>-1.635</td>
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<tr>
<td>3x3</td>
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<td>-7.857</td>
<td>-5.077</td>
<td>-8.873</td>
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<tr>
<td>4x4</td>
<td>82.78</td>
<td>-10.28</td>
<td>-6.873</td>
<td>——</td>
</tr>
</tbody>
</table>
Fig. 1  Equilibrium of boundary tractions along interelement boundary.

\[ T_\alpha^{(a)}(s) + T_\alpha^{(b)}(s) = 0 \quad \alpha = 1, 2 \]
Fig. 2 Nine and Twelve Degrees-of-Freedom Triangular Plate Bending Elements
Fig. 3 Fifteen Degrees-of-Freedom
Triangular Plate Bending Element
Fig. 4 Global and Local Cartesian Coordinate Systems for Shallow Shell Elements
D = flexural rigidity
a = side length
h = thickness
E = Young's modulus
ν = Poisson's ratio = 0.3
P = concentrated load at center
p_z = uniform distributed load

Fig. 5 Nomenclature and geometry description for plate bending problems. (mesh 2x2 shown)
Fig. 6 Simply-supported square plate under uniform loading. $M_x$ distribution along center line. Elements with linear moment distribution.
Fig. 7 Simply-supported square plate under uniform loading. 
$M_x$ distribution along center line. 
Elements with quadratic moment distribution.
Fig. 8 Simply-supported square plate under concentrated load. 
$M_x$ distribution along center line. 
Elements with linear moment distribution.
Fig. 9 Simply-supported square plate under concentrated load. 
$M_x$ distribution along center line. 
Elements with quadratic moment distribution.
Fig. 10 Stretching of a square plate under parabolic edge loading (mesh 2x2 shown).

\[ v = \text{Poisson's ratio} = 0.3 \]
\[ E = \text{Young's modulus} \]
\[ h = \text{plate thickness} \]
E = Young's modulus
\( \nu = \) Poisson's ratio = 0.3

Fig. 11 Shallow Spherical Cap under Uniform Pressure
(mesh 2 x 2 shown)
Fig. 12 Shallow Spherical Cap under Uniform Pressure. Convergence of Strain Energy Results.
Fig. 13 Shallow Spherical Cap under Uniform Pressure
Displacement $w$ predictions along BC (4x4 mesh)
Fig. 14 Shallow Spherical Cap under Uniform Pressure
Stress resultants predicted along BC (4x4 mesh)
Fig. 15 Shallow Spherical Cap under Uniform Pressure
My distribution predicted along BC (4x4 mesh)
Fig. 16 Shallow Spherical Cap under Uniform Pressure $M_x$ distribution predicted along BC (4x4 mesh)