From a thin membrane to an unbounded solid:
dynamics and instabilities in radial motion of
nonlinearly viscoelastic spheres

by
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Submitted to the Department of Mechanical Engineering
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Abstract

In this thesis, a theoretical investigation of the dynamic motion of spherically symmetric bodies is presented, considering nonlinear viscoelastic material responses. To explore the stability thresholds of the dynamic motion and to compare them with available formula for the quasi-static limit, the present formulation employs a generalized constitutive relation and accounts for different loading scenarios. Specifically for instantaneously applied load, by studying the entire spectrum of radii ratios of the spherical body, ranging from a thin membrane to an unbounded medium, we show that geometric effects can significantly reduce the dynamic stability limit while viscoelasticity has a stabilizing effect. Additionally, we show that in finite spheres, rate-dependence can induce a bifurcation of the long-time response. The stability thresholds derived in this thesis, together with their geometric and constitutive sensitivities, can inform the design of more resilient material systems that employ soft materials in dynamic settings, with examples including seismic bearings that are designed to absorb shocks but often fail due to rupture of internal cavities, and thin inflatable membrane structures like rubber balloons, which may exhibit snap-through instabilities and consequent ruptures. By accounting for rate-dependence, the results of this thesis also shed light on the response of biological materials to dynamic load and the possible instabilities that can lead to injury in vulnerable organs, such as the brain and the lungs. Moreover, while modern therapeutic ultrasound techniques intentionally generate cavities within the tissue, the present investigation of the material response to harmonic excitations across various frequencies can lead to safer practice.

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Chapter 1

Introduction

In 1959, Gent and Lindley reported an “unusual rupture process” that occurs in rubbers [18]. In their experiments, they subjected cylindrical specimens to tensile load, and observed that the damage is initiated by the nucleation of voids (Fig. 1-1a). Since their pioneering work, the rupture phenomenon that they observed has been studied extensively and is now known as cavitation. Additionally, several examples of cavitation have been reported in the literature. In ductile metals, cavitation has been indicated as a precursor to fracture [2], and thus defines the limit of the material to sustain load. Rubber bearings that are used to absorb seismic vibrations in critical infrastructure have also been reported to experience a remarkable decrease in tensile stiffness due to cavitation damage [27], as shown in Fig. 1-1b. In addition to engineering materials, cavitation can also appear in biological tissue and can cause significant injury. It has been indicated as a primary mechanism of damage in traumatic brain injury that can be induced by impact or blast [10, 16], and has been associated with aneurysms rupture [25, 3]. On the other hand, cavitation can be used to benefit medical processes, as reviewed in [9, 37]. For example, medical ultrasound techniques have been shown to intentionally generate cavities to enhance drug delivery, to fracture kidney stones, and to treat cancer. Moreover, cavitation can be used to measure local material properties [43, 11]. Overall, cavitation occurs in various materials and can be either beneficial or harmful. Hence, to fully take advantage of or avoid cavitation, we must obtain a comprehensive understanding of its sensitivity
to different loading conditions, material properties, and geometrical settings.

(a) Internal rupture of cylindrical rubber specimens [18].

(b) Cavitation in seismic rubber bearings [27].

Figure 1-1: Cavitation induced damage in rubbers.

Theoretically, spontaneous expansion of a spherical void due to the application of a constant load is defined as cavitation. The load can be applied remotely or at the wall of a pre-existing cavity. Typically, in cavitation, the undeformed body (Fig. 1-2a) is assumed to be spherically symmetric and unbounded such that the ratio between the undeformed outer radius $B$ and the inner radius $A$ is infinite ($B/A \to \infty$). Upon application of load, the spherical body will deform to a new equilibrium configuration.
with $a$, $b$ representing the deformed inner and outer radii, respectively, as shown in Fig. 1-2b for the case of remote tension ($P_{\text{out}}$) or inner pressure ($P_{\text{in}}$). However, as shown in Fig. 1-2c, the loading-expansion curve approaches an asymptotic maximum $P_{\text{cr}}$ as the deformed inner radius $a$ increases to infinity, above which no equilibrium solution exists. This critical pressure ($P_{\text{cr}}$) is referred as the cavitation pressure.

A rigorous mathematical analysis of the onset of cavitation in both compressible and incompressible hyperelastic solids was carried out by Ball [4]. A review of additional theoretical studies can be found in [19]. Since cavitation can be triggered by dynamic loading conditions and can also generate dynamic response, accounting for inertial effects is important. For hyperelastic materials such as biological tissue and rubbers, incompressibility can be reasonably assumed, which leads to significant simplification in determining the kinematics. Knowles and Jakub [23] as well as Chou-Wang and Horgan [7] investigated the onset of cavitation in unbounded incompressible hyperelastic bodies subjected to step pressure increase. It was shown that the critical pressure in the dynamic scenario is identical to the quasi-static value.
Figure 1-3: Representative loading-expansion curve for the snap-through instability.

$(P_{cr})$, and that below the critical threshold, a periodic solution appears, which is dynamically stable.

Up to now, we have focused our attention to the instability that occurs in an unbounded medium $(B/A \to \infty)$. Alternatively at the other limit $B/A \to 1$, membrane or membrane-like structures can also exhibit instability. The most common example is the party balloon, which upon inflation, may suddenly continue to expand without increase of pressure. This is due to a well-known snap-through instability, which is explained by the non-monotonic loading-expansion curve that permits bistable states. As schematically plotted in Fig. 1-3, the equilibrium radius jumps from $a_1$ to $a_2$ when the applied pressure is perturbed from $P_{cr}^-$ to $P_{cr}^+$. A similar instability could also appear for the local minimum upon deflation. The existence of bistable states in rubber membranes has been shown experimentally in Fig. 1-4, where two identically inflated balloons are connected by opening the valve in the middle, resulting in different stable deformations after the pressure is rebalanced [34]. Inflated membranes appear in several additional settings such as catheters, weather monitoring systems, and biological organs [25, 3], hence understanding their behavior and onset of the snap-through instability is essential. In particular, it can over-stretch the material to an undesirable extent and lead to damage. Or, on the other hand, it could be carefully designed to achieve large deformations and actuation [22, 35]. In contrast to cavitation with $B/A \to \infty$, it was shown that for membranes dynamic load can lead to a lower critical pressure as investigated in [1, 39, 36].

While the instabilities that occur in spherical membranes and at the cavitation
Figure 1-4: Bistable states in rubber membranes [34].

limit are well understood, the only difference between them is the geometric ratio $B/A$. Nonetheless, they exhibit significantly different behaviors both quasi-statically and dynamically. Hence in this thesis, we are interested in examining the complete geometric spectrum between these two limits while accounting for the complex constitutive response associated with both material nonlinearity and viscoelasticity. To examine the dynamic stability thresholds, in this work we consider the response to instantaneous application of load that are then held constant over time. This loading scenario may be triggered in situations for which the pre-existing cavity is filled by a fluid. Thence the nucleation of a void in the fluid would generate a sudden release of the internal pressure applied to the solid, followed by damped oscillation of the cavity. Remarkably this behavior has been recently observed in [31]. Although our work focuses on the dynamic response of an isolated cavity, the stability thresholds derived herein can be applied to explain stability of fluid-saturated porous media under hydrostatic stress, as schematically plotted in Fig. 1-5. Instantaneous application of load can also appear due to an internal explosion. Such behavior has recently been captured experimentally in [30], where a micro-explosion was generated in a polymer material by thermal activation of an embedded explosive crystal.

In this thesis, the elastodynamics of radial motion under step pressure increase are discussed in Chapter 3, where an emphasis is put on the comparison between stability
limits for quasi-static and dynamic load with an arbitrary geometric parameter $1 < B/A < \infty$.

Most of the elastomeric and biological materials considered in the previous examples exhibit significant viscoelastic effects during deformation, which indicates the insufficiency of a purely elastic description. Moreover, high-performance actuators and medical devices typically require high precision, hence prediction of viscoelastic behavior is imperative. As discussed in [38], several questions remain open with regards to the nonlinear oscillation of viscoelastic spherical shells, especially on its application in biological tissue, for example understanding the heart beating.

Following an investigation of purely elastic responses of spherical shells [5], an early discussion on effects of viscoelasticity on stability was presented by Calderer [6]. Later, Fosdick et al. [14, 15] proposed two constitutive models for viscoelasticity and studied free oscillations of spherical shells as examples. Then dynamic motion of the spherical shell induced by periodic forcing was thoroughly explored, where periodic and chaotic responses were observed [13]. As for viscoelastic membranes, theoretical and numerical studies on dynamic inflation can be found in the work by Verron et al. [40]. Cohen and Molinari [8] derived analytical solutions for cavity relaxation and steady-state expansion in incompressible and nonlinear viscoelastic solids. Specifically in the time variation of the applied pressure during expansion, overshoot and two bounds of the pressure were observed. Subsequently, Kumar et al. [26] made some insightful remarks on consequences of viscosity and inertia on the onset of cavitation. By examining contributions of each component in the cavity
expansion induced by a constant pressure rate, it is demonstrated that the viscosity can significantly stabilize the solution while the inertia has negligible effects when resultant stretch rates are relatively small. Furthermore, Faye et al. [12] incorporated failure criteria and inspected these effects using finite element analysis.

In this thesis, effects of viscosity are discussed in Chapter 4 for the entire geometric spectrum, where levels of load that result in stable or unstable solutions are fully determined. Bifurcation in final solutions induced by the variation of viscoelastic time scales are also observed.

Finally, in Chapter 5 we combine results from previous chapters to study the dynamic response of spherical shells subjected to harmonic excitations. Harmonic excitations are particularly important in relation to therapeutic ultrasound [9, 37]. However, recent research suggests that it could induce cavitation and corresponding damage in tissue that is outside of the target region and is exposed to ultrasound bursts [33, 32]. Some researchers have considered bubble dynamics under step pressure changes and harmonic excitations assuming an unbounded medium with various viscoelastic constitutive relations [20, 17, 41]. Given the increasing interest in more broad applications of ultrasound, we investigate the natural frequency of the system, especially with considerations of different material and geometric parameters. Then harmonic excitations are applied with different amplitudes and frequencies either close or away from the natural frequency.
Chapter 2

Problem Formulation

In this chapter, we formulate the governing equations that describe the dynamic motion of hollow spheres. Specifically, we focus on spherically symmetric deformations and consider incompressible, isotropic, and homogeneous bodies. The constitutive relations that account for both purely elastic and viscoelastic response are introduced and we derive expressions for elastic and kinetic energies that will be shown useful in determining the stability thresholds in the next chapter.

2.1 General formulation

2.1.1 Kinematic relations

Consider a hollow spherical body with undeformed outer radius $B$ and inner radius $A$ that, by application of time-dependent radial traction on its boundaries ($P_a(t)$, $P_b(t)$), undergoes spherically symmetric deformation, as illustrated in Fig. 2-1. The reference location of a material particle is denoted by $X = X(R, \Theta, \Psi)$, where the spherical coordinates are defined such that

$$ R \in [A, B], \quad \Theta \in [0, 2\pi), \quad \Psi \in [0, \pi]. \quad (2.1) $$

Considering spherically symmetric deformations, in the current configuration, the spacial location of a material particle is denoted by $x = x(r, \theta, \psi)$, where the deformed
spherical coordinates are defined as

\[ r(R, t) \in [a(t), b(t)], \quad \theta = \Theta \in [0, 2\pi), \quad \psi = \Psi \in [0, \pi]. \quad (2.2) \]

The principal stretch components are

\[ \lambda_r = \frac{\partial r}{\partial R}, \quad \lambda = \lambda_\theta = \lambda_\psi = \frac{r}{R}. \quad (2.3) \]

Restricting our attention to incompressible materials, we have \( J = \det \mathbf{F} = 1 \), where \( \mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X} \) is the deformation gradient, and thus \( \lambda_r = \lambda^{-2} \). Accordingly, radial expansion is described by \( \lambda \in (1, +\infty) \) and contraction by \( \lambda \in (0, 1) \). In addition, the identity \( \partial r / \partial R = (R/r)^2 \) can be integrated with the boundary condition on the deformed inner radius \( r(A, t) = a(t) \) to obtain the volume constraint

\[ r(R, t)^3 - a(t)^3 = R^3 - A^3. \quad (2.4) \]

By taking the first and second time derivatives of the above equation, we obtain the radial velocity \( \dot{r}(R, t) \) and acceleration \( \ddot{r}(R, t) \) of a particle labeled by \( R \) at moment \( t \) as functions of the inner radial motion \( a(t) \) and its derivatives, with their arguments
dropped for clarity, as follows
\[
\dot{r} = \frac{a^2 \dot{a}}{r^2}, \quad \ddot{r} = \frac{2a\ddot{a}^2 + a^2 \dddot{a} - 2r \ddot{r}^2}{r^2}.
\] (2.5)

### 2.1.2 Governing equations

For the spherically symmetric deformation field the only non-trivial equation of motion is along the radial direction

\[
\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} s = \rho \ddot{r},
\] (2.6)

where the shorthand \( s = \sigma_r - \sigma_\theta \) is the difference between \( \sigma_r \) and \( \sigma_\theta = \sigma_\psi \), namely the radial and circumferential principal stress components, respectively, and \( \rho \) is the constant mass density. Substitution of the acceleration \( \ddot{r}(R, t) \) in (2.5) into the radial equation of motion (2.6) generates

\[
\frac{\partial \sigma_r}{\partial r} + \frac{2s}{r} = \rho \left( \frac{2a\ddot{a}^2 + a^2 \dddot{a}}{r^2} - \frac{2a^4 \dddot{a}^2}{r^5} \right).
\] (2.7)

Integrating the above equation along radius over the whole body from \( r = a(t) \) to \( b(t) \), we have
\[
\sigma_r(b, t) - \sigma_r(a, t) = \rho \left[ (2\ddot{a} + \dot{a}) \left(1 - \frac{a}{b}\right) - \frac{a^2}{2} \left(1 - \frac{a^4}{b^4}\right) \right] - S(t),
\] (2.8)

where the notation
\[
S(t) = \int_a^b \frac{2s(\bar{r})}{\bar{r}} d\bar{r}.
\] (2.9)

Notice that in the above equation of motion (2.8), the first term on the right-hand-side represents the inertial contribution and the second term depends on material properties, and will be defined by specific choice of constitutive relations as discussed in the next section. Using the undeformed inner radius \( A \) as a characteristic length scale for this problem, the equation (2.8) emerges a natural time and a velocity scale
of this system read
\[ \ddot{t} = A \sqrt{\frac{\mu}{\rho}}, \quad \ddot{v} = \sqrt{\frac{\mu}{\rho}}. \quad (2.10) \]

Given the incompressibility, we can uniquely determine the applied tractions from (2.8) with (2.4) calculated at \((r, R) = (b, B)\), as long as we know the inner radius \(a(t)\) as a function of time. On the other hand, if the applied tractions are known, the motion of the body is determined by solving the initial-boundary value problem:

1. initial conditions, including cavity radius and velocity
   \[ a(t)|_{t=0} = a_0, \quad \dot{a}(t)|_{t=0} = \dot{a_0}. \quad (2.11) \]

2. boundary conditions at the inner and outer walls require that \(\sigma_r(a, t) = P_a(t)\) and \(\sigma_r(b, t) = P_b(t)\), which can be contracted into a single variable
   \[ P(t) = P_b(t) - P_a(t). \quad (2.12) \]

Governing equation (2.8) along with initial and boundary conditions (2.11) and (2.12) fully define the dynamic problem for an arbitrary material with material response entering through (2.9). In next sections, we will evaluate (2.9) for a family of elastic and viscoelastic materials.

### 2.2 Rate-independent constitutive relations

For arbitrary incompressible isotropic materials, the strain energy density function can be written in terms of the principal stretch components \(\lambda_i\), with \(i = r, \theta, \psi\) in this problem, such that \(W = W(\lambda_r, \lambda_\theta, \lambda_\psi)\). For the purpose of the present investigation, we adopt the Mooney–Rivlin energy density function characterized by two material parameters \(C_1, C_2 > 0\) such that

\[ W(I_1, I_2) = C_1 (I_1 - 3) + C_2 (I_2 - 3), \quad (2.13) \]
where $I_1$ and $I_2$ are invariants of the left Cauchy-Green tensor $B = FF^\top$ and read

$$
I_1 = \lambda_r^2 + \lambda_\theta^2 + \lambda_\psi^2,
$$

$$
I_2 = \lambda_r^2 \lambda_\theta^2 + \lambda_\psi^2 \lambda_\psi^2 + \lambda_\theta^2 \lambda_\psi^2.
$$

(2.14)

Due to spherical symmetry, the Cauchy stress tensor is diagonal with its principal components expressed in the form

$$
\sigma_i = -p + \lambda_i \frac{\partial W}{\partial \lambda_i},
$$

(2.15)

where $p$ is the hydrostatic pressure that arises due to the incompressibility constraint. Recalling that $\lambda_r = \lambda^{-2}$ and $\lambda_\theta = \lambda_\psi = \lambda$ and inserting (2.15) we may write the stress components as follows

$$
\sigma_r = -p + 2C_1 \frac{1}{\lambda^4} + 4C_2 \frac{1}{\lambda^2},
$$

$$
\sigma_\theta = \sigma_\psi = -p + 2C_1 \lambda^2 + 2C_2 \frac{1}{\lambda^2} + 2C_2 \lambda^4,
$$

(2.16)

Therefore, the principal stress difference $s = \sigma_r - \sigma_\theta$ can be written in terms of material constants and the hoop stretch as

$$
s = \mu \left( \frac{1}{\lambda^4} - \lambda^2 \right) + n \mu \left( \frac{1}{\lambda^2} - \lambda^4 \right),
$$

(2.17)

where we have replaced $C_1 = \mu/2$ and $C_2 = n\mu/2$, with $n$ being dimensionless. Note that when $n = 0$ we reduce the formulation to account for the neo-Hookean material. Substituting the above result into the elastic term (2.9) and performing integration along with the transformation of integration variable

$$
\frac{d\lambda}{\lambda(1-\lambda^3)} = \frac{dr}{r},
$$

(2.18)

we obtain the explicit equation

$$
S = 2\mu \left[ \left( n\lambda_b - \frac{1}{\lambda_b} - \frac{n}{2\lambda_b^2} - \frac{1}{4\lambda_b^4} \right) - \left( n\lambda_a - \frac{1}{\lambda_a} - \frac{n}{2\lambda_a^2} - \frac{1}{4\lambda_a^4} \right) \right],
$$

(2.19)
where \( \lambda_a = \lambda(A, t) \) and \( \lambda_b = \lambda(B, t) \) are the hoop stretches at boundaries of the body. Finally by substituting the above elastic term into (2.8) we rewrite the equation of motion in the form

\[
P = \rho \left[ (2\dot{a}^2 + a\ddot{a}) \left( 1 - \frac{a}{b} \right) - \frac{\dot{a}^2}{2} \left( 1 - \frac{a^4}{b^4} \right) \right] \\
- \mu \left[ \left( \frac{A^4}{a^4} + 4\frac{A}{a} \right) - \left( \frac{B^4}{b^4} + 4\frac{B}{b} \right) \right] - \frac{n\mu}{2} \left[ \left( \frac{4b}{b} - \frac{2B^2}{B^2} \right) - \left( \frac{4a}{A} - \frac{2A^2}{a^2} \right) \right].
\]  

(2.20)

Notice that in the above equation, \( A \) and \( B \) are constant undeformed radii and \( b(t) \) is related to \( a(t) \) through the volume constraint (2.4). The right-hand-side is thus a function of \( a(t) \) and its derivatives, and can be numerically integrated to obtain the dynamic response under arbitrary load \( P(t) \). Meanwhile, (2.20) allows us to consider the response for ratio \( B/A \) in the range \((1, +\infty)\), where the limits represent the thin membrane and a cavity embedded in an unbounded solid medium as we will discuss next, respectively.

1. **Thin membrane** \( B/A \rightarrow 1 \)

At the membrane limit, we assume the undeformed body has a finite inner radius \( A \) and infinitesimal thickness \( H \rightarrow 0 \), therefore the undeformed outer radius is \( B = A + H \). The conservation of volume (2.4) yields the deformed outer radius

\[
b(t) = \left( a(t)^3 - A^3 + (A + H)^3 \right)^{1/3}.
\]  

(2.21)

Accordingly, from (2.20) we can write the pressure in the form \( P(a(t), H) \), which by Taylor expansion with respect to \( H \) to the first order terms and noting the fact that \( P(a(t), 0) = 0 \), we are left with the following approximation

\[
\frac{P}{H} \simeq \left. \frac{\partial P}{\partial H} \right|_{H=0}.
\]  

(2.22)

Substituting the equation of motion (2.20) along with (2.21) into the above
result we have

\[
\frac{P}{\mu} \simeq \frac{H}{a} \left[ \frac{\rho A^2 \ddot{a}}{\mu a} + 2 \left( \frac{a^6 - A^6}{a^6} + n \frac{a^6 - A^6}{a^4 A^2} \right) \right],
\]  

(2.23)

where \(P/\mu\) scales with \(H/a\). At the quasi-static limit (\(\ddot{a} = 0\)), this reduces to the result in [29].

2. **Unbounded solid** \(B/A \to \infty\)

Now we consider an unbounded solid, or equivalently an infinitesimal cavity, with \(B/A \to \infty\). From the incompressibility condition (2.4) we find that at this limit the outer radius \(b \simeq B\). Hence, we can reduce the equation of motion to the form

\[
P \simeq \rho \left( \frac{3}{2} a^2 + a \ddot{a} \right) - \frac{\mu}{2} \left( \frac{A^4}{a^4} + \frac{4A}{a} - 5 \right) + n \frac{\mu}{2} \left( \frac{4a - 2A^2}{A} - 2 \right).
\]  

(2.24)

Note that quasi-static response of this case may be easily obtained by removing the inertial term. In particular, by imposing \(n = 0\) we obtain corresponding relation for neo-Hookean material, which coincides the result in [18].

2.3 **Rate-dependent constitutive relations**

To capture realistic dynamic response of soft materials that may be highly dissipative, the rate-dependent material properties should be accounted for. An example of this rate-dependent behavior is observed in the experiment by Milner et al. [31], where the free oscillation is damped by viscoelastic effects. Here we extend the Mooney-Rivlin constitutive relation to account for an additional relaxable term in the strain energy function to account for finite viscoelastic deformations by adapting the theoretical framework in [42]. This model can be schematically represented as in Fig. 2-2, showing an elastic spring connected in parallel to a viscoelastic Maxwell element composed of an elastic spring with a viscous dashpot.

The first spring behaves as a Mooney-Rivlin material defined in the previous
section with shear modulus $\mu$, while the second one has shear modulus $\alpha\mu$, and
the dashpot characterized by a time scale $\tau$. The deformation of the viscoelastic
component can be decomposed into two parts as in the Kröner–Lee decomposition
[24, 28] such that $F = F^e F^v$, which in terms of hoop stretches can be written as

$$\lambda_i(t) = \xi_i(t)\zeta_i(t), \quad (2.25)$$

where $\lambda_i$ denotes the total principal stretch components and equivalently the principal
stretches of spring $\mu$, while $\zeta_i$ denotes the principal stretches of the spring $\alpha\mu$, and $\xi_i$
denotes the principal stretches of the dashpot.

Thus, the strain energy density function can be written in the form

$$W = W^e + W^v = \frac{\mu}{2}(I^e_1 - 3) + \frac{n}{2}\mu(I^e_2 - 3) + \frac{\alpha\mu}{2}(I^v_1 - 3) + \frac{n}{2}\alpha\mu(I^v_2 - 3), \quad (2.26)$$

where $I^e_1$ and $I^e_2$ follow the form in (2.14), while $I^v_1$ and $I^v_2$ read

$$I^e_1 = \zeta_r^2 + \zeta_\theta^2 + \zeta_\psi^2,$$
$$I^e_2 = \zeta_r^2\zeta_\theta^2 + \zeta_r^2\zeta_\psi^2 + \zeta_\theta^2\zeta_\psi^2. \quad (2.27)$$

Next to comply with the incompressibility condition while preserving spherical sym-
metry, we assume the elastic deformation in the Maxwell element $F^e$ is incompressible
and therefore so is $F^v$ such that

$$\zeta_\theta = \xi_\psi = \zeta, \quad \zeta_r = \zeta^{-2},$$
$$\xi_\theta = \xi_\psi = \xi, \quad \xi_r = \xi^{-2}. \quad (2.28)$$

Figure 2-2: Schematic representation of the viscoelastic model.
The principal components of the Cauchy stress tensor are given by the sum of elastic and viscoelastic contributions

\[ \sigma_i = \sigma_i^e + \sigma_i^v, \]  

(2.29)

where the viscoelastic part, calculated by (2.15), reads

\[ \sigma_i^v = -p^v + \alpha \mu \frac{1}{\zeta^4} + 2n\alpha\mu \frac{1}{\zeta^2}, \]

\[ \sigma_{\psi}^v = -p^v + \alpha \mu \zeta^2 + n\alpha\mu \left( \frac{1}{\zeta^2} + \zeta^4 \right). \]  

(2.30)

Therefore the stress difference \( s(t) = s^e(t) + s^v(t) \) is

\[ s = \mu \left( \frac{1}{\lambda^4} - \lambda^2 \right) + n\mu \left( \frac{1}{\lambda^2} - \lambda^4 \right) + \alpha \mu \left( \frac{1}{\zeta^4} - \zeta^2 \right) + n\alpha\mu \left( \frac{1}{\zeta^2} - \zeta^4 \right), \]  

(2.31)

which upon integration gives the following expression

\[ S = 2\mu \left[ \left( n\lambda_b - \frac{1}{\lambda_b} - \frac{n}{2\lambda_b^2} - \frac{1}{4\lambda_b^4} \right) - \left( n\lambda_a - \frac{1}{\lambda_a} - \frac{n}{2\lambda_a^2} - \frac{1}{4\lambda_a^4} \right) \right] + 2\alpha \mu \left[ \left( n\zeta_b - \frac{1}{\zeta_b} - \frac{n}{2\zeta_b^2} - \frac{1}{4\zeta_b^4} \right) - \left( n\zeta_a - \frac{1}{\zeta_a} - \frac{n}{2\zeta_a^2} - \frac{1}{4\zeta_a^4} \right) \right]. \]  

(2.32)

Recalling that \( \zeta = \lambda/\xi \) represents the hoop stretch of spring in the Maxwell element, this viscoelastic term can be substituted into the general equation of motion (2.8), together with stretch rate of dashpot as we will discuss next, to complete the description of dynamical behavior in viscoelastic materials.

Since the radial and hoop stretches are coupled due to incompressibility, we can write our evolution equation as a function of the dashpot hoop stretch \( \xi \) in the general form,

\[ \dot{\xi} = \frac{\partial \xi}{\partial t} = f(\xi, \lambda), \]  

(2.33)

which, as explained in [42], must obey the following requirement to comply with the thermodynamic inequality (see Appendix A)

\[ \frac{\partial W}{\partial \xi} \xi \leq 0, \quad \text{for} \quad \xi \in (0, \infty), \]  

(2.34)
and
\[ f(\xi = \lambda, \lambda) = 0. \]  
(2.35)

For the purpose of the present study, we choose a minimal form of the evolution equation that satisfies (2.34), (2.35), and the incompressible kinetic constraint of the Maxwell spring (2.39), reads

\[ \dot{\xi} = \frac{\partial \xi}{\partial t} = \frac{1}{\tau} \frac{\lambda^3 - \xi^3}{\xi^2}, \]  
(2.36)

where \( \tau \) is the viscoelastic time scale as mentioned above. Specifically the incompressibility requires

\[ (\xi^3_a - 1) A^3 = (\xi^3 - 1) R^3, \]
\[ (\lambda^3_a - 1) A^3 = (\lambda^3 - 1) R^3, \]  
(2.37)

which generate constraints

\[ A^3 \xi_a^2 \dot{\xi}_a = R^3 \xi^2 \dot{\xi}, \quad (\lambda^3_a - \xi^3) A^3 = (\lambda^3 - \xi^3) R^3. \]  
(2.38)

By rearranging we obtain

\[ \frac{\dot{\xi}_a}{\xi} = \frac{\xi^2 R^3}{\xi_a^2 A^3} = \frac{\xi^2 (\lambda^3_a - \xi^3)}{\xi_a^2 (\lambda^3 - \xi^3)}, \]  
(2.39)

which is satisfied by the specific form (2.36) we adopted above. It is worth mentioning again that there are numerous evolution equations available to describe viscoelastic behavior of materials as long as (2.34), (2.35) and (2.39) are satisfied.

In summary, the kinetic relation (2.25) and evolution equation (2.36) gives description of real-time response for \( \zeta(t) \), which thence determines viscoelastic term by using (2.32) to obtain \( S(t) \) in the equation of motion (2.8).
2.4 Elastic and kinetic energies

In evaluation of the dynamic response in our system, it is useful to consider the elastic and kinetic energies. First we can rewrite the elastic energy density function in terms of \( r \) and \( R \) in the form

\[
W = \frac{\mu}{2} \left( \frac{R^4}{r^4} + \frac{2r^2}{R^2} - 3 \right) + \frac{n\mu}{2} \left( \frac{2R^2}{r^2} + \frac{r^4}{R^4} - 3 \right).
\]

The total elastic energy is obtained by integrating over the deformed body as follows

\[
U(a) = 4\pi \int_a^b \left[ \frac{\mu}{2} \left( \frac{R^4}{r^4} + \frac{2r^2}{R^2} - 3 \right) + \frac{n\mu}{2} \left( \frac{2R^2}{r^2} + \frac{r^4}{R^4} - 3 \right) \right] r^2 dr
\]

\[
= 2\pi\mu \int_a^b \left[ \left( \frac{R^4}{r^4} + \frac{2r^2}{R^2} \right) + n \left( \frac{2R^2}{r^2} + \frac{r^4}{R^4} \right) \right] r^2 dr - 2\pi\mu (1 + n) (B^3 - A^3).
\]

The first integral can be directly integrated as

\[
\int_a^b \left[ \left( \frac{R^4}{r^4} + \frac{2r^2}{R^2} \right) + n \left( \frac{2R^2}{r^2} + \frac{r^4}{R^4} \right) \right] r^2 dr = \left[ \frac{R}{r} (2r^3 - R^3) + n \frac{r}{R} (2R^3 - r^3) \right]_a^b,
\]

substitution of which provides an explicit expression for the elastic energy

\[
U(a) = 2\pi\mu \left[ b^3 \left( \frac{2}{b} - \frac{B^4}{b^4} - \frac{B^3}{b^3} \right) - a^3 \left( \frac{2A}{a} - \frac{A^4}{a^4} - \frac{A^3}{a^3} \right) \right] + 2n\pi\mu \left[ B^3 \left( \frac{2b}{B} - \frac{b^4}{B^4} - 1 \right) - A^3 \left( \frac{2a}{A} - \frac{a^4}{A^4} - 1 \right) \right].
\]

The rate of change of the total elastic energy is

\[
\frac{dU}{dt} = 2\pi a^2 \dot{a} \left\{ \mu \left[ \left( \frac{B^4}{b^4} + \frac{4B}{b} \right) - \left( \frac{A^4}{a^4} + \frac{4A}{a} \right) \right] + n\mu \left[ \left( \frac{2B^2}{b^2} - \frac{4b}{B} \right) - \left( \frac{2A^2}{a^2} - \frac{4a}{A} \right) \right] \right\}.
\]

Similarly, the kinetic energy \( T \) is obtained by integration over the volume of the
body

\[ T(a, \dot{a}) = \int_\Omega \frac{1}{2} \rho \left( \frac{a^2}{r^2} \dot{a} \right)^2 \, dV = 2\pi \rho a^4 \dot{a}^2 \left( \frac{1}{a} - \frac{1}{b} \right), \tag{2.45} \]

which has the time derivative

\[ \frac{dT}{dt} = 2\pi a^2 \dot{a} \rho \left[ (4\dot{a}^2 + 2a\ddot{a}) \left( 1 - \frac{a}{b} \right) - \dot{a}^2 \left( 1 - \frac{a^4}{b^4} \right) \right]. \tag{2.46} \]

We can now consider the elastic and kinetic energies at the limits of the spherical membrane and the unbounded solid.

1. **Thin membrane** \( B/A \to 1 \)

For spherical membrane, we may reduce the elastic and kinetic energy following similar step as we did in Section 2.2. By writing the Taylor expansion of \( U(a(t), H) \). Keeping up to the first order terms and noting \( U(a(t), 0) = 0 \), we obtain the following expression

\[ U = \frac{U}{H} \approx \frac{\partial U}{\partial H} \bigg|_{H=0}, \tag{2.47} \]

which can be written explicitly as

\[ U = 2\pi \mu H \left[ \left( 2\dot{a}^2 - 3A^2 + \frac{A^4}{a^4} \right) + n \left( \frac{a^4}{A^2} - 3A^2 + 2\frac{A^4}{a^2} \right) \right]. \tag{2.48} \]

For kinetic energy we can carry out the same process and get

\[ T = 2\pi \rho HA^2 \dot{a}^2. \tag{2.49} \]

2. **Unbounded solid** \( B/A \to \infty \)

For a cavity in an unbounded solid with \( b \approx B \), the elastic energy can be obtained by taking limit

\[ U = \frac{2\pi \mu}{3} (a - A)^2 \left[ \left( 3\frac{A^2}{a} + 4A + 5a \right) + n \left( 3\frac{a^2}{A} + 4a + 5A \right) \right], \tag{2.50} \]
and for the kinetic energy we have

\[ T = 2\pi \rho {\dot{a}}^2. \]  

(2.51)
Chapter 3

Analysis of Constant Cauchy Traction Problem

In this chapter, we will discuss the dynamic behavior of hollow spheres with arbitrary radii ratio $B/A$ subjected to instantaneous application of a constant pressure difference between its inner and outer boundaries. We are particularly interested in and will focus on the dynamic expansion process induced by $P(t) > 0$. However the general approach used in this chapter is also valid for negative load unless buckling occurs [21].

First, we introduce the quasi-static response and investigate its stability. Next, the dynamic motion are studied with its stability threshold being compared to the quasi-static case. The general formulation for hollow spheres of arbitrary dimensions reveals the geometric sensitivity of both quasi-static and dynamic responses. Using the elastic and kinetic energies derived in the previous chapter and accounting for the energy invested by external load, we obtain an energetic representation for this problem, which provides an insightful phase plane description.

3.1 Quasi-static responses

With dynamic motion being suppressed such that $\dot{a}(t) \to 0$ and $\ddot{a}(t) \to 0$, we eliminate the inertial term multiplied by $\rho$ in the equation of motion (2.20) and denote the elastic
term as \( f(a) \), which now has the form

\[
f(a) = -\frac{\mu}{2} \left[ \frac{A(A^3 + 4a^3)}{a^4} - \frac{B(B^3 + 4b^3)}{b^4} \right] - \frac{n\mu}{2} \left[ \left( \frac{4b}{B} - \frac{2B^2}{b^2} \right) - \left( \frac{4a}{A} - \frac{2A^2}{a^2} \right) \right].
\]  

(3.1)

Based on this equation we can study the quasi-static response at various limits.

For a neo-Hookean sphere with \( n = 0 \), the relation for two finite ratios \( B/A = 10, 30 \) are plotted in Fig. 3-1, where \( f(a) \) is shown to have a global maximum denoted by \( P_* \) that represents the stability threshold. However, for an unbounded medium with \( B/A \to \infty \), \( f(a) \) increases monotonically along \( a \) to an asymptotic value of \( 2.5\mu \) such that \( P_* \to 2.5\mu \), which retrieves the famous critical traction that causes internal rupture of a void reported in [18]. It is known that an equilibrium solution exists only if the applied load \( P \) is not larger than \( P_* \).

![Figure 3-1](image)

Figure 3-1: Quasi-static relation between dimensionless pressure \( P/\mu \) and inner radius \( a/A \) for various dimensions in neo-Hookean material.

For Mooney-Rivlin material where \( n > 0 \), the additional term on the right-hand-side of (3.1) accounts for the nonlinear elastic contribution. It is shown that due to this nonlinearity, the global maximum disappears but may be replaced by a local
maximum. We plot the quasi-static curves for a ratio $B/A = 10$ in Fig. 3-2 with
different values of the parameter $n$. In this case, an arbitrary load $P$ admits at least
one equilibrium solution. Specifically, when $n$ is relatively small, as represented by the
dash line, a bistable response appears and permit a snap-through instability between
two metastable states. The critical value $n_s$ determines the transition from a bistable
response to a monostable response, which can be derived analytically. First, we take
the derivative of $f(a)$ and find when it is non-negative such that

$$
\frac{df}{da} = -\frac{\mu}{2} \left[ \left( \frac{12A}{a^2} - \frac{4A(A^3 + 4a^3)}{a^5} \right) - \frac{a^2}{b^2} \left( \frac{12B}{b^2} - \frac{4B(B^3 + 4b^3)}{b^5} \right) \right]
$$

which should be valid for all $a > A$. By rearranging the above expression, we obtain

$$
n_s = \max_{a > A} \left\{ \left[ \left( \frac{12A}{a^2} - \frac{4A(A^3 + 4a^3)}{a^5} \right) - \frac{a^2}{b^2} \left( \frac{12B}{b^2} - \frac{4B(B^3 + 4b^3)}{b^5} \right) \right] \right\}
$$

which is a function of $B/A$. Taking the radii ratio $B/A = 10$ for example, we find
the critical value $n_s \approx 0.055$ as the solid line in Fig. 3-2.

As for the membrane limit ($B/A \to 1$), given the derivation in the previous
chapter, we have the following quasi-static relation

$$
\frac{P}{\mu} \approx 2H \left( \frac{a^6 - A^6}{a^6} + n \frac{a^6 - A^6}{a^4 A^2} \right).
$$

Similar to the analysis above, the critical value $n_s$ to maintain monotonicity is in the
form

$$
n_s = \max_{a > A} \left\{ \frac{A^2 (a^6 - 7A^6)}{a^2 (a^6 + 5A^6)} \right\} \approx 0.2145.
$$

On the other hand, for an unbounded medium ($B/A \to \infty$), the relation between $P$
and $a$ is always monotonic, therefore in this case $n_s = 0$.  

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3.2 Dynamic responses under instantaneous load

The quasi-static assumption is only valid when the load rate is sufficiently small such that inertial effects can be neglected. This assumption will no longer be reasonable if the load is applied rapidly or at an onset of instability. The most extreme case is the application of instantaneous load such that

$$P(t) = \begin{cases} 
0, & t < 0, \\
 P_b - P_a, & t \geq 0.
\end{cases} \quad (3.6)$$

As a first step, we solve the equation of motion (2.20) directly by numerical integration with boundary condition (3.6) and assuming the body is initially undeformed and at rest

$$a_0 = A, \quad \dot{a}_0 = 0. \quad (3.7)$$

For the neo-Hookean material with $n = 0$, the quasi-static response has been
shown in Section 3.1 to have a global maximum when the radii ratio $B/A$ is finite, where $P_s$ serves as the static critical load such that unbounded expansion will happen once that value is exceeded. In Fig. 3-3 we plot time histories for hollow spheres with radii ratios $B/A = 10$ and $100$ under different levels of instantaneous load, where the corresponding static critical load is approximately $P_s \approx 1.95\mu$ and $2.40\mu$, respectively. Here we have normalized the time with respect to time scale $\bar{t}$ as define in (2.10). It is shown that for relatively small values of load, a steady oscillatory solution appears and the undeformed inner radius serves as the lower limit. Quite remarkably all of the curves in Fig. 3-3a represent the dynamic response due to load below the quasi-static stability threshold $P < P_s$, nonetheless for $P = 1.90\mu$ dynamic instability is clearly observed. Similar oscillatory and unstable solutions appear for a sphere of $B/A = 100$ as shown in Fig. 3-3b. Therefore it suggests that unstable solutions may appear earlier in dynamic settings. We denote the critical dynamic threshold pressure by $P_d$. In following sections, we will study the sensitivity of this dynamic threshold to both geometric and constitutive model parameters and we will show that for all cases $P_d \leq P_s$ while the equality is obtained only at the limit of $B/A \to \infty$.

For Mooney-Rivlin materials with $n > 0$, the quasi-static response no longer has a global maximum (as shown in the previous section) and thus unbounded expansion
will not appear. This is observed in Fig. 3-4, where we show responses under instantaneous load for hollow spheres of radii ratio $B/A = 10$ with two different values of $n$. It is observed that the dynamic motion of Mooney-Rivlin spheres can be highly sensitive to both the applied load and the value of $n$. This phenomenon is associated with non-monotonic behavior of Mooney-Rivlin material response (Fig. 3-2), which will be discussed in detail later.

3.3 Energy conservation in dynamic motion

The total energy in the considered dynamic system consists of three energy contributions, namely the work done by the traction $W$, the elastic energy stored by the deformation $U$, and the kinetic energy $T$. The work done by the constant tractions $P_b$ and $P_a$ is obtained by integration

$$W(a, b) = \int_B^b 4\pi b^2 P_b db - \int_A^a 4\pi a^2 P_a da.$$  \hspace{1cm} (3.8)

Using the incompressibility condition $b^3 - B^3 = a^3 - A^3$ and recalling that $P = P_b - P_a$, we have

$$W(a) = \frac{4}{3} P\pi (a^3 - A^3),$$  \hspace{1cm} (3.9)
which is determined as long as the cavity radius \( a(t) \) is known for each moment. Now we take the time derivative to write the power invested as

\[
\frac{dW}{dt} = 4\pi a^2 \dot{a} P. \tag{3.10}
\]

Since \( W(a) \) only depends on \( a(t) \) such that the load \( P \) can be recognized as a conservative force. Hence the sum of kinetic and potential energy \( (V = U - W) \) of the system must remain constant and thus

\[
\frac{d}{dt} (T + V) = 0. \tag{3.11}
\]

Recalling that we have readily derived the rate of change of elastic and kinetic energies in (2.44) and (2.46), and inserting (3.10), we obtain the following expression

\[
\frac{d}{dt} (T + V) = 4\pi a^2 \dot{a} \left\{ \rho \left[ (2\ddot{a} + \dot{a}^2) \left( 1 - \frac{a}{b} \right) + \frac{a^2}{2} \dot{a}^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \dot{b}^2 \right) \right] \right. \\
+ \frac{\mu}{2} \left[ \left( \frac{B^4}{b^4} + \frac{4B}{b^2} \right) - \left( \frac{A^4}{a^4} + \frac{4A}{a} \right) \right] \\
+ \frac{n\mu}{2} \left[ \left( \frac{2B^2}{b^2} - \frac{4b}{B} \right) - \left( \frac{2A^2}{a^2} - \frac{4a}{A} \right) \right] \\
- P \right\} = 0, \tag{3.12}
\]

notice that we have recovered the equation of motion in the curly brackets, which confirms our previous results for the case of constant boundary traction. Therefore, for arbitrary initial conditions \( a_0 \) and \( \dot{a}_0 \), we can write

\[
T + V = T_0 + V_0, \tag{3.13}
\]

with constants on the right-hand-side calculated from initial states at \( t = 0 \) by substituting \( a_0 \) and \( \dot{a}_0 \). This basic energy argument will be useful in describing phase portraits of dynamic motion in the next sections.
3.4 Dynamic instabilities and critical load

If an oscillatory solution exists, the inner radius \( a(t) \) will change periodically between two oscillation limits, namely \( a_{\text{min}} \) and \( a_{\text{max}} \) (as shown in Figs. 3-3 and 3-4). Since these limits correspond to zero velocity and therefore to zero kinetic energy, we impose \( T(a) = 0 \) in (3.13), together with \( T_0 = V_0 = 0 \) from initial conditions \( a_0 = A \) and \( \dot{a}_0 = 0 \), and obtain an implicit relation that \( V(a_{\text{min}}) = V(a_{\text{max}}) = 0 \). By studying these limits we can investigate the stability of this system.

3.4.1 Elastic neo-Hookean materials

At the neo-Hookean limit \((n = 0)\), the potential energy \( V = U - W \) can be expressed in the form

\[
V(a) = 2\pi\mu \left[ b^3 \left( \frac{2B}{b} - \frac{B^4}{b^4} - \frac{B^3}{b^3} \right) - a^3 \left( \frac{2A}{a} - \frac{A^4}{a^4} - \frac{A^3}{a^3} \right) \right] - \frac{4}{3} P\pi(a^3 - A^3). \tag{3.14}
\]

We are generally interested in points of zero potential which appear at the oscillation limits. Note that \( a = A \), and consequently \( b = B \), is necessarily a trivial solution of \( V(a) = 0 \) for an arbitrary \( P \), and thus \( A = a_{\text{min}} \) is the lower oscillation limit given that \( P > 0 \) and the upper limit for \( P < 0 \). However we will restrict our attention to the first case. In this motion the value of cavity radius belongs to the range \( a \in [A, a_{\text{max}}] \) with a bounded upper limit \( a_{\text{max}} > A \) to be determined.

We plot \( V(a) \) in Fig. 3-5 for three different values of \( P \). It is shown that for \( P < P_d \), three roots appear and are denoted in an increasing order by \( \{ z_0, z_1, z_2 \} \) where \( z_0 \equiv A \). The dashed portion of the curve between \( z_1 \) and \( z_2 \) indicates an unphysical regime where kinetic energy becomes negative. Thus our admissible solutions for positive \( P \) will be in ranges

\[
a \in \{a | z_0 \leq a \leq z_1 \cup z_2 \leq a < \infty} \}.
\tag{3.15}
\]

For the first range, the oscillation about the quasi-static solution \( a_s \) (which corresponds to a local minimum of \( V(a) \)) is confined by the two limits \( z_0 \) and \( z_1 = a_{\text{max}} \). If \( a_0 \geq z_2 \), non-oscillatory solution appears and the system becomes dynamically un-
Figure 3-5: Potential energy for different applied load $P$ compared with critical value of dynamic instability $P_d$. For radii ratio $B/A = 10$ we have $P_d \approx 1.89\mu$. Dash line represents unphysical solution.

stable, which will be put aside from our discussion given initial conditions in (3.7). The nature of this solution can be alternatively understood using phase portraits in Section 3.6. Note that since the two branches that emerge for $P < P_d$ are separated by an unphysical regime, the system cannot jump between them. As the load is increased, the non-physical regime vanishes. At the critical state that we denote by $P = P_d$ only two roots appear. For higher values of $P > P_d$, no roots appear in the range $a \in (A, \infty)$. Accordingly in this regime a non-trivial state of zero kinetic energy does not exist, and the inner radius will expand indefinitely. Thus $P = P_d$ represents the critical value of dynamic stability threshold.

According to the above argument, the system is stable if $V(a)$ has non-trivial roots. Hence by setting (3.14) to zero, we rearrange and write $P = g(a)$ where

$$g(a) = \frac{3\mu}{2(a^3 - A^3)} \left[ b^3 \left( \frac{2B}{b} - \frac{B^4}{b^4} - \frac{B^3}{b^3} \right) - a^3 \left( \frac{2A}{a} - \frac{A^4}{a^4} - \frac{A^3}{a^3} \right) \right], \quad (3.16)$$

which indicates the relation between applied load $P$ and upper limit $z_1 = a_{\text{max}}$ when
\( a \in (A, \infty) \). Since dynamic instabilities will occur if the kinetic energy \( T(a) \geq 0 \), and thus \( V(a) \leq 0 \), for all \( a \in (A, \infty) \), by using the expressions of \( V(a) \), we obtain the critical pressure

\[
P_d = \max_{a > A} \{ g(a) \} .
\]  
(3.17)

Eventually oscillatory and unstable solutions correspond to two ranges of positive load

oscillatory: \( P \in [0, g_{\text{max}}] \), unstable: \( P \in (g_{\text{max}}, \infty) \).

We can identify the extrema of \( g(a) \) by imposing \( \frac{dg}{da} = 0 \). First noticing that the quasi-static solution can be written in the general form

\[
f(a) = \frac{1}{4\pi a^2} \frac{dU}{da} ,
\]  
(3.19)

as well as for oscillation limits

\[
g(a) = \frac{3}{4\pi} \frac{U(a)}{(a^3 - A^3)} ,
\]  
(3.20)

taking derivative of the above equation we have

\[
\frac{dg(a)}{da} = \frac{3}{4\pi} \frac{-3a^2}{(a^3 - A^3)^2} U(a) + \frac{3}{4\pi} \frac{1}{(a^3 - A^3)} \frac{dU(a)}{da} \]
\[
= -\frac{3a^2}{a^3 - A^3} [g(a) - f(a)] ,
\]  
(3.21)

which indicates that the intersections between \( g(a) \) and \( f(a) \) are extrema of \( g(a) \). We can visualize regimes corresponds to the stable and unstable solutions by plotting the \( g(a) \) together with \( f(a) \) as well as the trivial lower limit \( a_{\text{min}} = A \) in Fig. 3-6. It is observed that in this representative case, the dynamic stability threshold \( P_d \) is lower than the quasi-static stability threshold \( P_s \). Specifically, the solution under instantaneous load \( P < P_d \) has the inner radius \( a \) oscillating around the quasi-static solution \( a_s \) between two limits indicated by solid blue curves. For \( P > P_d \), no upper limit exists and thus indefinite expansion appears.

In summary, by analyzing the elastic energy at the limit states in which the
kinetic energy vanishes, we have identified the upper and lower bounds of the dynamic oscillation generated by a constant and instantaneously applied load. In contrast to the quasi-static process, instantaneous application of pressure requires additional kinetic energy that at the oscillation limits translates to larger expansion ($a_{\text{max}} > a_s$) and could push the system beyond the stability threshold. Based on this argument we can now consider the stability threshold at the geometric limits of a thin membrane with $B/A \to 1$ and a cavity inside an unbounded solid with $B/A \to \infty$.

1. **Thin membrane** $B/A \to 1$

Recall that for the thin-walled membrane we have the following quasi-static relation by eliminating the inertial term in (2.23) as follows

$$\frac{f(a)}{\mu} \approx \frac{2H}{a} \left( \frac{a^6 - A^6}{a^6} \right), \quad (3.22)$$

which is a non-monotonic function of $a$. The maximum appears at $a \approx 1.38A$ with the quasi-static critical pressure $P_s/\mu \approx 1.24H/A$. 

Figure 3-6: Oscillation limits and quasi-static solution for neo-Hookean spheres with radii ratio $B/A = 10$. 

- Quasi-static solution
- Oscillation limits
- Unstable branches
Similar to the process of deriving $f(a)$ for membranes in (2.23), by Taylor expansion on $H$ we can evaluate the first order terms of (3.16) and obtain $g(a)$ at the limit of $B/A \to 1$ in the form

$$
\frac{g(a)}{\mu} \simeq \frac{3H}{2a} \frac{2a^3 + a^{-3}A^6 - 3aA^2}{(a^3 - A^3)}.
$$

(3.23)

According to (3.21), the dynamic stability limit can be found by equating (3.22) and (3.23)

$$
\frac{a^6 - A^6}{a^6} = \frac{3}{4} \frac{2a^3 + a^{-3}A^6 - 3aA^2}{a^3 - A^3}, \quad \text{for} \quad a > A,
$$

(3.24)

which yields a critical dynamic threshold $P_d/\mu \simeq 1.11H/A$ at $a \simeq 1.73A$. Therefore in the membrane limit, it is shown that $P_d < P_s$.

2. **Unbounded solid $B/A \to \infty$**

As derived in (2.24), we keep only the elastic term on the right-hand-side and obtain the quasi-static solution for infinite medium as

$$
f(a) \simeq -\frac{\mu}{2} \left( \frac{A^4}{a^4} + \frac{4A}{a} - 5 \right).
$$

(3.25)

It monotonically increases along $a$ to the asymptotic value $2.5\mu$ and thus does not have a maximum (as shown previously in Fig. 3-1).

For an infinite ratio of $B/A$, the relation between the instantaneously applied load $P$ and the upper limit of cavity radius $a = a_{\text{max}}$ in oscillation is obtained by the limit of $g(a)$ at $b \to B$ such that

$$
g(a) = \frac{5\mu}{2} - \frac{3\mu a^3}{2(a^3 - A^3)} \left( \frac{2A}{a} - \frac{A^3}{a^3} - \frac{A^4}{a^4} \right),
$$

(3.26)

which has the same asymptotic value as the quasi-static solution when $\lambda_a =$
By subtracting \( f(a) \) from \( g(a) \), it is shown that

\[
\frac{g(a) - f(a)}{\mu/2} = \frac{A^6 + aA^5 + a^2A^4 + a^3A^3 - 2a^4A^2 - 2a^5A}{a^4A^2 + a^5A + a^6} < 0, \quad \text{for} \quad a > A.
\] (3.27)

Accordingly for an infinite medium there is no intersection between \( f(a) \) and \( g(a) \) as plotted in Fig. 3-7, therefore the dynamic and static stability thresholds are identical, which was previously concluded in [23, 7].

### 3.4.2 Elastic Mooney-Rivlin materials

As in the previous section, we impose kinetic energy equal to zero and write out the relation between applied load and the oscillation upper limit

\[
g(a) = \frac{3\mu}{2(a^3 - A^3)} \left[ b^3 \left( \frac{2B}{b} - \frac{B^4}{b^4} - B^3 \right) - a^3 \left( \frac{2A}{a} - \frac{A^4}{a^4} - A^3 \right) \right] + \frac{3\mu}{2(a^3 - A^3)} \left[ B^3 \left( \frac{2b}{B} - \frac{b^4}{B^4} - 1 \right) - A^3 \left( \frac{2a}{A} - \frac{a^4}{A^4} - 1 \right) \right].
\] (3.28)
It will be shown later that \( g(a) \) could possess a local maximum as does \( f(a) \) when \( n \) is relatively small. This maximum is responsible for the dynamic snap-through instability. It is important to identify the critical value \( n_d \) that distinguishes the dynamically bistable state and the monostable state. First we recall from (3.21) that the derivative of \( g(a) \) has the general form

\[
\frac{dg(a)}{da} = -\frac{3a^2}{a^3 - A^3} [g(a) - f(a)].
\] (3.29)

In order to avoid local maximum we require the derivative to be non-negative when \( a > A \). After rearrangement, we obtain

\[
n_d = \max_{a > A} \left\{ -\frac{f_1 - g_1}{f_2 - g_2} \right\},
\] (3.30)

where we used shorthands \( f(a) = f_1(a) + nf_2(a) \) and \( g(a) = g_1(a) + ng_2(a) \) such that

\[
f_1 = -\frac{\mu}{2} \left[ \left( \frac{A^4}{a^4} + \frac{4A}{a} \right) - \left( \frac{B^4}{b^4} + \frac{4B}{b} \right) \right],
\]

\[
f_2 = -\frac{\mu}{2} \left[ \left( \frac{4b}{B} - \frac{2B^2}{b^2} \right) - \left( \frac{4a}{A} - \frac{2A^2}{a^2} \right) \right],
\] (3.31)

as well as

\[
g_1 = \frac{3\mu}{2(a^3 - A^3)} \left[ b^3 \left( \frac{2B}{b} - \frac{B^4}{b^4} - \frac{B^3}{b^3} \right) - a^3 \left( \frac{2A}{a} - \frac{A^4}{a^4} - \frac{A^3}{a^3} \right) \right],
\]

\[
g_2 = \frac{3\mu}{2(a^3 - A^3)} \left[ a^3 \left( \frac{2b}{B} - \frac{b^4}{B^4} - 1 \right) - A^3 \left( \frac{2a}{A} - \frac{a^4}{A^4} - 1 \right) \right].
\] (3.32)

**Dynamically monostable and bistable states**

In order to examine the effect of \( n \), we will take the radii ratio \( B/A = 10 \) for example, which exhibits a transition from bistable to monostable behavior at \( n_d \approx 0.047 \), as calculated numerically. For Mooney-Rivlin spheres with \( n > n_d \), the potential energy profile is plotted in Fig. 3-8, and the curves for \( f(a) \) and \( g(a) \) are plotted in Fig. 3-9. It is straightforward to observe that for arbitrary applied load \( P > 0 \), there is a unique root and thus only one oscillatory solution exists. In this case, neither
Figure 3-8: Potential energy for different applied load $P$ on Mooney-Rivlin spheres with $B/A = 10$ and $n = 0.06 > n_d$. Dash lines represent unphysical solutions.

sudden increment of upper limit nor indefinite expansion will occur. For Mooney-Rivlin spheres with $n < n_d$, the curve of $g(a)$ has a local maximum. It allows a sudden increment of the oscillation upper limit as previously shown in Fig. 3-4 when applied load crosses the local maximum of $g(a)$.

Taking $n = 0.04 < n_d$ for example, the curves for $f(a)$ and $g(a)$ are plotted in Fig. 3-10, where two intersections are denoted as $g_{\text{max}}$ and $g_{\text{min}}$, respectively, and are shown also to be the local maximum and minimum for $g(a)$. We plot the potential energy $V(a)$ with a series of applied load $P$ in Fig. 3-11. For relatively small or large values of the applied load such that $0 < P < g_{\text{min}}$ or $P > g_{\text{max}}$, $V(a)$ has single root at $a > A$, which serves as the unique upper limit. For intermediate values $g_{\text{min}} < P < g_{\text{max}}$, there exist three roots and thus we obtain two distinct oscillatory solutions that will be determined by initial states.

Restricting our attention to the initial conditions of an undeformed and static sphere given by (3.7), we find that infinitesimal perturbations about $P = g_{\text{max}}$ can generate finite changes in the upper limit of oscillation, which is analogous to the snap-through instability in the quasi-static process. Thus we define the dynamic stability
Figure 3-9: Oscillation limits and quasi-static solution for Mooney-Rivlin spheres with radii ratio $B/A = 10$ and $n = 0.06 > n_d$.

Figure 3-10: Oscillation limits and quasi-static solution for Mooney-Rivlin spheres with radii ratio $B/A = 10$ and $n = 0.04 < n_d$. The two intersections corresponds to values of $g_{\text{max}} \simeq 2.31\mu$ and $g_{\text{min}} \simeq 2.27\mu$. 
threshold \( P_d := g_{\text{max}} \). In contrast to the unbounded expansion in neo-Hookean spheres when crossing this dynamic stability threshold, since the upper limit solution does not have global maximum, the motion always has a bounded \( a_{\text{max}} \) even for \( P > P_d \) and thus indefinite expansion will not appear.

**Membrane and unbounded body approximations**

To understand the motion when \( B/A \to 1 \) and \( B/A \to \infty \), we alternatively investigate \( n_d \) at these limits, which is previously defined in (3.30). Using the \( f(a) \) derived in (2.23) and repeating this process for \( g(a) \) we obtain the following expressions

\[
\begin{align*}
    f_1 &= 2\mu H \frac{a^6 - A^6}{a^6}, \\
    f_2 &= 2\mu H \frac{a^6 - A^6}{a^5 A^2}, \\
    g_1 &= \frac{3\mu H}{2} \frac{2a^2 + a^{-4} A^6 - 3A^2}{a^3 - A^3}, \\
    g_2 &= \frac{3\mu H}{2} \frac{a^4 A^{-2} + 2a^{-2} A^4 - 3A^2}{a^3 - A^3}.
\end{align*}
\] (3.33)
By solving (3.30) we find that \( n_d \approx 0.1449 \) for the thin membrane limit. On the other hand, we may evaluate \( n_d \) for \( B/A \to \infty \) with following components

\[
\begin{align*}
    f_1 &= \frac{5\mu}{2} - \frac{\mu}{2} \left( \frac{4A}{a} + \frac{A^4}{a^4} \right), \\
    f_2 &= -\mu + \mu \left( \frac{2a}{A} - \frac{A^2}{a^2} \right), \\
    g_1 &= \frac{\mu a - A}{2} \frac{5a^2 + 4aA + 3A^2}{a^2 + aA + A^2}, \\
    g_2 &= \frac{\mu a - A}{2} \frac{3a^2 + 4aA + 5A^2}{a^2 + aA + A^2},
\end{align*}
\]

(3.34)

which yields \( n_d = 0 \). Thus for a cavity embedded in an unbounded solid, there is no dynamically bistable state.

### 3.5 Geometric sensitivity

In this section, we will investigate the geometric effects with the geometry characterized by \( B/A \). For the neo-Hookean material, the dynamic and quasi-static stability thresholds against radii ratio \( B/A \) are plotted in Fig. 3-12. Specifically, it is observed that \( P_d/P_s \) approaches 1 at \( B/A \to \infty \) as we have already shown that \( P_d = P_s \), while for the thin membrane with \( B/A \to 1 \) we obtain the analytical limit \( P_d \approx 0.897P_s \) by using the solution from (3.24). The actual values of \( P_d \) and \( P_s \) decrease from \( 2.5\mu \) in infinite bodies to nearly zero at the membrane limit. It is worth mentioning that both the ratio and the actual values become remarkably sensitive to the geometry when approaching the membrane limit.

For the Mooney-Rivlin material, we plot the critical values \( n_s \) and \( n_d \) in Fig. 3-13 that represent the transition from non-monotonic to monotonic behaviors of \( f(a) \) and \( g(a) \), respectively. Both curves reach their maxima at the membrane limit and approach zero at the unbounded solid limit. Notice that \( n_s \geq n_d \) for any \( B/A \), thus for the range of \( n \in (n_d, n_s) \), \( g(a) \) is monotonic while \( f(a) \) is not.

It is not straightforward to define quasi-static and dynamic stability thresholds for Mooney-Rivlin spheres since they may not always exist. For different values of \( n \), there is an upper bound of \( B/A \) where the \( f(a) \) or \( g(a) \) becomes monotonic. Thus for \( B/A \) that is below the upper bound, we can define the local maximum as the corresponding quasi-static, or dynamic, stability thresholds in Mooney-Rivlin spheres.
(a) Sensitivity of the ratio $P_d/P_s$

(b) Sensitivity of actually values $P_d$ and $P_s$

Figure 3-12: Geometric sensitivity of quasi-static and dynamic stability thresholds for neo-Hookean spheres.
By plotting these thresholds against admissible $B/A$ in Fig. 3-14, we find that the neo-Hookean thresholds serve as the lower bound. On the other hand, there exists an upper bound of thresholds obtained by connecting inflection points as indicated by the dash-dot lines, which approach $P_s/\mu \simeq 3.02$ and $P_d/\mu \simeq 2.93$ for quasi-static and dynamic responses, respectively. Notice that even for large $B/A$ where $n$ becomes small but still non-zero, the neo-Hookean limit is not retrieved. This is explained by the fact that inflection points appear at the expansion of $O(a/A) \sim O(B/A)$, which is allowable in this elastic problem where no material failure is considered.

### 3.6 Phase portraits of dynamic responses

Apart from conducting integration of the equation of motion directly, we may alternatively derive the relation between $a(t)$ and $\dot{a}(t)$ by substituting energy components into the equation (3.13), which yields a phase portrait description of the dynamic motion for arbitrary initial conditions. It provides a different way to examine the dynamic motion, and is particularly helpful when viscoelastic effects are included as
Figure 3-14: Geometric sensitivity of quasi-static and dynamic stability limits for Mooney-Rivlin spheres.
will be discussed in the next chapter.

For spheres that are at rest and undeformed before being loaded, the initial conditions \( a_0 = A \) and \( \dot{a}_0 = 0 \) yield \( V_0 = T_0 = 0 \). Hence the relation between \( a(t) \) and \( \dot{a}(t) \) reduces to the following form

\[
\frac{2P}{3\rho} (a^3 - A^3) = a^4 \dot{a}^2 \left( \frac{1}{a} - \frac{1}{b} \right) + \frac{\mu}{\rho} \left[ b^3 \left( \frac{2B}{b} - \frac{B^4}{b^4} - \frac{B^3}{b^3} \right) - a^3 \left( \frac{2A}{a} - \frac{A^4}{a^4} - \frac{A^3}{a^3} \right) \right] \\
+ n \frac{\mu}{\rho} \left[ B^3 \left( \frac{2b}{B} - \frac{b^4}{B^4} - 1 \right) - A^3 \left( \frac{2a}{A} - \frac{A^4}{A^4} - 1 \right) \right],
\]

which can be further rearranged to write

\[
P(a, \dot{a}) = \frac{3\rho}{2} \frac{a^4 \dot{a}^2}{a^3 - A^3} \left( \frac{1}{a} - \frac{1}{b} \right) + g(a).
\]

with \( g(a) \) as derived in (3.28). Therefore by calculating the field \( P \) for different \( a \) and \( \dot{a} \), we obtain the phase portrait by finding constant contours of \( P \).

For the neo-Hookean spheres with \( n = 0 \), the phase portraits for three different values of \( P \) are plotted in Fig. 3-15a. Specifically for \( P < P_d \), the closed loop on the left corresponds to the oscillatory solution, while the open path indicates an expansion solution as mentioned in Section 3.4. When \( P = P_d \), the closed loop starts to connect with the open path. Thence the system is no longer stable if higher load is applied as indicated by the dash-dot path. For other radii ratios \( B/A \) in the neo-Hookean material, the phase portraits possess similar motion but with different dynamic stability threshold \( P_d \).

Phase portraits for Mooney-Rivlin spheres with \( n < n_d \) are plotted in Fig. 3-15b, where \( g_{\text{min}} \simeq 0.98g_{\text{max}} \). Recall that we have defined \( P_d := g_{\text{max}} \). Note that for \( P < g_{\text{min}} \) or \( P > P_d \), there is only one closed loop and therefore a single oscillatory solution. For intermediate load \( g_{\text{min}} \leq P \leq P_d \), two closed loops and thus two distinct oscillatory solutions may appear depending on initial conditions. We plot phase portraits with \( n > n_d \) in Fig. 3-15c. In this case, there is only one closed loop for arbitrary applied load, which confirms our conclusion in Section 3.4 that unique oscillatory motion exists and instabilities disappear.
Figure 3-15: Phase portraits for different applied load $P$ on spheres with radii ratio $B/A = 10$ and different parameters $n$, where $n_d \approx 0.047$. 

(a) $n = 0.0$, $P_d \approx 1.89\mu$

(b) $n = 0.04 < n_d$, $P_d \approx 2.31\mu$

(c) $n = 0.06 > n_d$
Chapter 4

Effects of Viscous Dissipation

In the previous analysis we assumed the material is purely elastic. However, it is not always appropriate to make this assumption since many materials that can undergo large elastic deformations are accompanied with rate-dependent behaviors. In this section we are going to investigate the effects of viscous dissipation on dynamic motion induced by instantaneous load.

We may study viscoelastic responses by first looking at quasi-static response and oscillation limits for instantaneous and long-time shear moduli as we did for the purely elastic material, which is determined by $\alpha$ and $\mu$ as well as the radii ratio $B/A$. There will be a transition from the instantaneous to long-time behavior characterized by the viscous time scale $\tau$. In this study, we will restrict our attention to dynamic motion with stable instantaneous behaviors. Specifically, we will take a large enough $\alpha = 0.5$ such that the instantaneous dynamic stability threshold $P_d^0$ is larger than long-time quasi-static $P_s^\infty$, meanwhile the applied load will not cause instantaneous instabilities with $P < P_d^0$.

4.1 Neo-Hookean spheres

For neo-Hookean spheres with $\eta = 0$, the quasi-static response and oscillation limits are plotted in Fig. 4-1, where the global maximums of $f^\infty(a)$ and $g^\infty(a)$ corresponding to the long-time shear modulus are labeled as $f_{\text{max}}^\infty$ and $g_{\text{max}}^\infty$, respectively. We define
The two maxima give \( P_d^\infty \approx 1.89\mu \) and \( P_s^\infty \approx 1.95\mu \). Instantaneous responses are indicated in dash-dot lines.

4.1.1 Behavior of \( 0 < P \leq P_d^\infty \)

Given the load \( 0 < P \leq P_d^\infty \), which corresponds to monostable portions of both functions \( f^\infty(a) \) and \( g^\infty(a) \), we have already shown that there exists an equilibrium solution for quasi-static response and a bounded upper limit for the oscillation. In the presence of viscosity, the dynamic motion is accompanied with dissipation of kinetic energy and eventually the system will be at rest in the equilibrium solution.

For example, we plot time histories and phase portraits of \( a(t) \) from numerical integration in Fig. 4-2 for a neo-Hookean sphere with radii ratio \( B/A = 10 \) under instantaneous load \( P = 1.8\mu < P_d^\infty \). It is shown that the viscoelastic response starts...
with oscillation and the amplitude decreases along time, which finally converges exactly to the quasi-static solution \( a \simeq 3.04A \). Different from the purely elastic behavior, the viscoelastic trajectory is not closed and turns into a stable spiral, which starts from the initial condition and ends at the quasi-static solution with zero velocity.

### 4.1.2 Behavior of \( P_d^\infty < P \leq P_s^\infty \)

Since amplitudes of oscillation in viscoelastic materials are reduced when there is dissipation, viscosity could suppress the onset of dynamic instability such that it will not necessarily appear.

We plot time histories and phase portraits in Fig. 4-3 for a neo-Hookean sphere with radii ratio \( B/A = 10 \) under sudden load of \( P_d^\infty < P = 1.9\mu < P_s^\infty \). The elastic dynamic response is unbounded in this regime, as is the viscoelastic solution with \( \tau = 0.3\bar{\ell} \). However, for larger \( \tau = 0.5\bar{\ell} \) the system equilibrates at \( a \simeq 3.77A \). Hence the viscoelasticity has a stabilizing effect.

### 4.1.3 Behavior of \( P_s^\infty < P < \infty \)

In this regime there is no long-time quasi-static solution. We plot time histories and phase portraits in Fig. 4-4 for a instantaneous application of load \( P = 2.0\mu > P_s^\infty \) on
Figure 4-3: Time histories and phase portraits for neo-Hookean spheres with radii ratio $B/A = 10$ under instantaneous application of $P = 1.9\mu$.

For Mooney-Rivlin materials with $n > 0$, the quasi-static response and oscillation limits are plotted in Fig. 4-5, where the local extrema of $f(a)$ and $g(a)$ are labeled if they exist. As in the previous section, we define stability thresholds $P_s^\infty := f_{\max}^\infty$ and $P_d^\infty := g_{\max}^\infty$, and we will separate our discussion into two parts according to the applied load in comparison with these extrema.

### 4.2.1 Monostable responses

For Mooney-Rivlin spheres, the long-time solution of the viscoelastic response arrives at the corresponding quasi-static solution with same applied load, if one of the
Figure 4-4: Unbounded expansion instability under $P = 2.0\mu$ in viscoelastic neo-Hookean spheres with radii ratio $B/A = 10$ and viscous parameter $\alpha = 0.5$.

Figure 4-5: Quasi-static response and oscillation limits for instantaneous and long-time behaviors in viscoelastic Mooney-Rivlin spheres with radii ratio $B/A = 10$ and $\alpha = 0.5$. Instantaneous responses are indicated in dash-dot lines.
Figure 4-6: Dynamic responses under $P = 2.40\mu$ in viscoelastic Mooney-Rivlin spheres $n = 0.04 < n_c$ with radii ratio $B/A = 10$ and viscous parameter $\alpha = 0.5$.

following conditions is satisfied:

(M1) $n \in \{n|0 < n < n_d\} \cap P \in \{P|0 < P \leq P_d^\infty \cup P_s^\infty < P < \infty\}$.

(M2) $n \in \{n|n_d \leq n < n_s\} \cap P \in \{P|0 < P < f_{\text{min}}^\infty \cup P_s^\infty < P < \infty\}$.

(M3) $n \in \{n|n_s \leq n < \infty\} \cap P \in \{P|0 < P < \infty\}$.

Noticing that the range $0 < P \leq P_d^\infty$ in the first case contains some portion of bistable state for $f(a)$ and $g(a)$ as shown in Fig. 4-5a. However, since we only consider motion that start from initial conditions $a_0 = A$ and $\dot{a}_0 = 0$ and with constant applied load, that part will not be reached and is thus excluded from our discussion.

In Fig. 4-6, we show an example satisfying the condition (M1) with applied load $P = 2.40\mu > P_s^\infty$. In this case, the system has monostable behaviors both quasi-statically and dynamically. It is shown that all oscillatory motion eventually dissipate and the body transfers to the second stable branch with $a \simeq 22.09A$, which is identical to the quasi-static value. On the other hand, if we apply $0 < P < P_d^\infty$, the final solution will converge to the first stable branch. As we mentioned above, the transient motion are distinct for different values of $\tau$, which does not influence the final result since there is only one admissible long-time solution.
4.2.2 Bistable responses

Instead of having single long-time behavior as discussed above, there exist a situation where the long-time quasi-static response bifurcates such that the system has two possible final solutions given one of the following conditions:

(B1) \( n \in \{n|0 < n < n_d\} \cap P \in \{P|P_d^{\infty} < P \leq P_s^{\infty}\} \).

(B2) \( n \in \{n|n_d \leq n < n_s\} \cap P \in \{P|f_{\min} \leq P \leq P_s^{\infty}\} \).

For the first condition (B1), we consider the instantaneous application of \( P = 2.35\mu \) and plot results in Fig. 4-7. It is observed that for relatively small \( \tau \), the final solutions stay at the second stable quasi-static branch \( a \simeq 20.62A \), while for relatively large \( \tau \) final solutions converge to the first stable branch \( a \simeq 5.85A \). By looking at the trajectories, a narrow region is observed between two fixed points of the system. This region has lower kinetic energy as indicated by the elastic envelope. When \( \tau \) is sufficiently large, for example \( \tau = 20\bar{\ell} \), early motion is accompanied with more loss of kinetic energy and then attracted by the first stable branch. In contrast when \( \tau \) is relatively small, the dashpot is more responsive to current motion and therefore results in faster transition to oscillate around the second stable branch. However, after some dissipation during the process, the body is trapped in that regime and unable to cross back. The cases with \( \tau = 1.57\bar{\ell} \) and \( 1.58\bar{\ell} \) clearly indicate that a bifurcation point exists in the range \( 1.57\bar{\ell} < \tau < 1.58\bar{\ell} \). This bifurcation does not always exist as we will discuss next.

For the second case (B2), we take an intermediate value \( n = 0.05 \) such that \( f(a) \) has local extrema while \( g(a) \) is monotonic as shown in Fig. 4-5b. By selecting \( P = 2.49\mu \) and numerically integrating with various \( \tau \) ranging from \( 0.4\bar{\ell} \) to \( 3.0\bar{\ell} \) as shown in Fig. 4-8, we observe that small \( \tau \) will not necessarily yield final solutions on the second stable branch. This can be explained by noticing the disappearance of narrow region in the elastic envelope, where the loss of total energy does not prevent the system from returning to the first stable branch. Interestingly, in this case the final solution can be the first or second stable branch. However this phenomenon vanishes
Figure 4-7: Bifurcation in the long-time behavior for Mooney-Rivlin spheres with $0 < n < n_d$ under instantaneous application of $P_d^\infty < P < P_s^\infty$. This representative case has radii ratio $B/A = 10$ with elastic parameter $n = 0.04$ and viscoelastic parameters $\alpha = 0.5$ under application of $P = 2.35\mu$.

when $\tau$ becomes sufficiently large such that the system stabilizes before approaching the second stable branch, for example when $\tau = 3.0\bar{t}$. 
Figure 4-8: Bifurcation in the long-time behavior for Mooney-Rivlin spheres with $n_d < n < n_s$ under instantaneous application of $f_{\text{min}}^\infty < P < f_{\text{max}}^\infty$. This representative case has radii ratio $B/A = 10$ with elastic parameter $n = 0.05$ and viscoelastic parameters $\alpha = 0.5$ under application of $P = 2.49\mu$. 
Chapter 5

Dynamic Motion under Time-Varying Load

In this chapter, we will study dynamic motion under time-varying load. We examine natural frequencies of the system by investigating free oscillation with considerations of material behaviors and the geometry. Then the free oscillation is extended to describe subsequent motion after sudden removal of the load. Finally, we parametrically study dynamic responses under harmonic excitations with various amplitudes and frequencies around the natural frequency.

5.1 Free oscillation and natural frequencies

In order to estimate the natural frequency of oscillation, we release the body from a pre-deformed state, namely from $a_0 \neq A$ and $\dot{a}_0 = 0$. Then according to energy considerations, the relation between $a$ and $\dot{a}$ is the form

$$\frac{3\rho}{2} \frac{a^4 \dot{a}^2}{a^3 - A^3} \left( \frac{1}{a} - \frac{1}{b} \right) + g(a) = g(a_0), \quad (5.1)$$

with $g(a)$ derived previously in (3.28), which provides the phase portrait of free oscillation. The period of oscillation can be calculated by integrating along the closed
Figure 5-1: Time histories and phase portraits for neo-Hookean spheres with radii ratio $B/A = 10$ released from in-equilibrium states. Solid and dash lines indicate responses released from $a_0 = 0.50A$ and $a_0 = 0.75A$, respectively.

trajectory $\Gamma$

$$T = \oint_{\Gamma} \frac{da}{\dot{a}}. \quad (5.2)$$

For example, we plot time histories and phase portraits for $a_0 = 0.50A$ and $0.75A$ in Fig. 5-1, which have periods that are approximately equal to $2.87\bar{t}$ and $2.95\bar{t}$, respectively. It is shown that each closed trajectory with different amplitudes has a distinct period, which demonstrates the nonlinearity in this system.

Moreover, we can calculate natural periods for $0 < a_0 < 1$ with different geometries and material parameters as shown in Fig. 5-2. In Mooney-Rivlin spheres, the system becomes stiffer than neo-Hookean spheres due to the additional elastic term, therefore the natural period decreases when released from the same $a_0$, for example, blue curves denoting $n = 1$ shift downwards compared to the black ones with $n = 0$. Notice that under the same initial stretch $a_0$, the period of oscillation is higher for larger radii ratio $B/A$, this is explained by the fact that the increment of mass surpasses the increment of elastic stiffness.

In the presence of viscosity, the free oscillation will eventually dissipate and the body recovers its undeformed configuration, as shown in Fig. 5-3. This could be used to understand the phenomenon in the experiment [31], where the damped oscillation of a cavity triggered upon releasing from a pre-deformed state is observed.
Figure 5-2: Variation of natural periods with different initial deformations \( a_0 \) and geometries \( B/A \). Black lines indicate the neo-Hookean material \( (n = 0) \) and blue lines indicate the Mooney-Rivlin material \( (n = 1) \).

![Graph showing variation of natural periods with different initial deformations and geometries.]

Figure 5-3: Time histories and phase portraits for free oscillation with viscous dissipation released from in-equilibrium states. The sphere has radii ratio \( B/A = 10 \) and is made of viscoelastic neo-Hookean material with \( \alpha = 0.5 \) and \( \tau = 1\ell \).

![Graph showing time histories and phase portraits.]

(a) Time history

(b) Phase portrait
5.2 Sudden removal of external load

In many circumstances the load only lasts for some interval of time. For example, consider the passage of a finite-time pressure wave, or in more extreme cases a blast wave. It is important for us to investigate the subsequent motion after the external load is removed.

Representative time history and trajectory of the process are plotted in Fig. 5-4, where the instantaneous load $P = 2.0 \mu$ is applied to a neo-Hookean sphere with radii ratio $B/A = 10$ for $2\bar{t}$. Upon disappearance of the external load, as indicated by circles, the elastic restoring force generates negative acceleration and decreases the velocity. As the velocity becomes zero, the inner radius reaches its maximum and starts to decrease, followed by free oscillation as shown by the solid curve. The dash line indicates the path that would be followed if the load was continuously applied. This can actually be interpreted as a similar problem to the free oscillation, instead we start from a different initial state. At the moment of releasing $t = t_r$, we obtain new initial conditions where $a_r = a(t_r)$ and $\dot{a}_r = \dot{a}(t_r)$ and the boundary condition $P = 0$. A zoomed plot of intersections on trajectories, including various remove time $t_r$, is plotted in Fig. 5-5.

This statement is also valid for other previous loading conditions, for example, the
Figure 5-5: Sudden removal of load $P = 2\mu$ on neo-Hookean spheres with radii ratio $B/A = 10$ at various moments.

harmonic excitation as shown in Fig. 5-6. Dash lines represent elastic response under continuing harmonic excitation $P = \mu \sin(2\pi t/\bar{t})$. Solid lines represent the motion with removal of the harmonic load at $t = 10\bar{t}$ as indicated by circles. The closed trajectory is identical to free oscillation released from $a_0/A \simeq 0.83$.

In the presence of viscosity, we are able to make predictions given $(a_r, \dot{a}_r)$ along with the hoop stretch of dashpot at the inner radius $\xi_{a}(t_r)$. We can determine the stretch rate of dashpot by the evolution equation and calculate the subsequent motion. In Fig. 5-7 we plot results for same conditions as in Fig. 5-4 except now the material is viscoelastic. Fig. 5-7c illustrates the three-dimensional phase trajectory with the $z$-axis denoting the hoop stretch $\xi_{a}$, where the black curve is the trajectory of unbounded expansion and the blue curve is the trajectory of subsequent motion after removal of load.

We have shown that subsequent motion after removing external force can be determined given initial conditions at the releasing point. For elastic materials, the sphere keeps on oscillating freely. While in viscoelastic materials, the free oscillation will eventually be damped. In summary, this problem can be understood as finding a free oscillation trajectory that goes through the releasing phase point $(a_r, \dot{a}_r)$. 

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Figure 5-6: Sudden removal of harmonic excitation with \( P_a = \mu \) and \( T = \bar{\ell} \) on neo-Hookean spheres with radii ratio \( B/A = 10 \). Removal time is \( t_r = 10\bar{\ell} \) as indicated by red circles.

### 5.3 Harmonic excitations

We now consider the harmonic load described by frequency and amplitude in the form

\[
P(t) = P_a \sin(\omega t).
\]

We will separate our discussion into two parts with relatively small and large amplitudes, respectively.

#### 5.3.1 Small amplitude excitations

Notice that natural periods shown in Fig. 5-2 vary slowly and tend to be constant when oscillation amplitudes are small, which is similar to the linear oscillator where the natural period is independent of amplitude. We examine this regime by applying harmonic excitations with the relatively small magnitude \( P_a = 0.01\mu \).

Taking a neo-Hookean sphere with radii ratio \( B/A = 10 \) for example, the natural frequency gives period 2.982\( \bar{\ell} \) at small amplitude oscillation, we may apply the harmonic excitation with this period and plot its time history in Fig. 5-8. It is shown that at beginning the system behaves like resonance of a linear oscillator, where the
Figure 5-7: Sudden removal of load $P = 2\mu$ at $t_r = 2\bar{t}$ on viscoelastic ($\alpha = 0.5$, $\tau = 1\bar{t}$) neo-Hookean spheres with radii ratio $B/A = 10$. 
amplitude increases linearly along time with the slope indicated by dash lines. However, the amplitude arrives at the maximum $a_{\text{max}} \approx 1.29A$ and starts to decrease since the natural frequency changes as the amplitude gets higher.

By sweeping over a series of applied periods and recording the maximum amplitude of each entire response, we can study variations of maximum amplitude under different applied periods. Specifically for the viscoelastic material with two elastic limits at instantaneous and long-time behaviors, we plot the maximum amplitudes versus applied periods for two shear moduli in Fig. 5-9. For instantaneous behaviors, it is observed that the peak of variation appears at lower applied periods due to the increased shear modulus. Accounting for viscoelasticity with corresponding shear moduli, variations of maximum amplitude with respect to applied periods are plotted in Fig. 5-10, where a shift between two peaks is shown for different values of $r$. Therefore, the viscous effects can be summarized as follows: $\alpha$ determines the distance between the two peaks and $\tau$ determines the dominant peak of the viscoelastic response.

Figure 5-8: Oscillation excited by harmonic load with small amplitude $P_0 = 0.01\mu$ and period $T = 2.982\tilde{t}$. 


Figure 5-9: Maximum amplitudes versus applied periods for neo-Hookean spheres with radii ratio $B/A = 10$ under $P_a = 0.01\mu$. The instantaneous shear modulus is $\mu_0 = (1 + \alpha)\mu$ and the long-time shear modulus is $\mu_\infty = \mu$.

Figure 5-10: Variation of maximum amplitudes under harmonic excitations in the presence of viscosity. We take neo-Hookean spheres with radii ratio $B/A = 10$ for example here, the load amplitude is $P_a = 0.01\mu$. 
5.3.2 Large amplitude excitations

Now we consider relatively large amplitude excitations with $P_a = 0.5\mu$ and $1.0\mu$, where nonlinear effects are considerable and lower limits of responses could approach zero. For neo-Hookean spheres with radii ratio $B/A = 10$, the natural period turns out to be of the order $3\bar{f}$ as shown in Fig. 5-2. We plot time histories of the sphere subjected to harmonic excitations of different applied periods in Fig. 5-11.

As expected, the applied period $T = 3\bar{f}$ yields much higher oscillation amplitudes than other calculated cases. The curves of time history become sharper, especially when lower limits are close to zero. However, for other applied periods, the response amplitudes are relatively small and two force magnitudes generate more similar response patterns. If the force magnitude is further increased, even far away from the natural frequency, the response becomes highly nonlinear and similarities disappear. For example in Fig. 5-11f, we plot the time history of the response under $P_a = 2\mu$ and $T = 10\bar{f}$. The series of oscillation starts from $t = 40\bar{f}$ corresponds to wavy patterns of the same interval under smaller $P_a$ as shown in Fig. 5-11e. In this case the higher magnitude of applied load pushes the inner radius towards zero, thus generating high acceleration.
Figure 5-11: Time histories for neo-Hookean spheres with radii ratio $B/A = 10$ under different applied periods.
Chapter 6

Concluding Remarks & Future Work

6.1 Concluding remarks

In this research, we comprehensively investigate the spherically symmetric motion and instabilities in incompressible hollow spheres under dynamic load.

For the case of instantaneously applied load, we employ energy arguments to determine dynamic stability thresholds. Instead of focusing on limits at thin membranes or unbounded bodies, we consider arbitrary thicknesses to cover the entire geometric spectrum. In neo-Hookean spheres where indefinite expansion is permitted, it is shown that for a finite radii ratio $B/A$ (bounded bodies), the dynamic stability threshold $P_d$ is smaller than the quasi-static value $P_s$. Specifically, the ratio $P_d/P_s$ shows significant sensitivity to the geometric parameter $B/A$. A maximum reduction in dynamic load bearing capacity of about ten percent is reached at the membrane limit with $B/A \to 1$. On the other hand, for an infinite medium with $B/A \to \infty$, the quasi-static and dynamic stability thresholds become identical, which coincides the conclusions in [23, 7].

The response of Mooney-Rivlin spheres is studied by adding an extra elastic term in the formulation, which prevents indefinite expansion in both quasi-static and dynamic cases. Depending on the additional material parameter and the geometry, quasi-static and dynamic loading-expansion curves can either be monotonic or non-monotonic, where only the later case admits the snap-through instability. Similarly
it is shown that \( P_d \leq P_s \) if they exist. For the infinite medium, both quasi-static and dynamic load curves are monotonically increasing and the system is always stable. At the membrane limit, the critical values (in quasi-static and dynamic settings) of the additional parameter that separate monotonic and non-monotonic behaviors are obtained, which agrees with arguments in [1, 39]. As we have mentioned in the introduction, radial motion at both the membrane and the infinite limits has been extensively studied. Our study aims to fill up the gap between two extreme cases \((B/A \to 1 \text{ and } \infty)\) and provide a rather complete view, which lays a foundation for more complicated problems.

Viscous dissipation is also taken into account by considering a parallel Maxwell element in the constitutive model. For dynamic motion induced by instantaneously applied constant load, the viscous effects stabilize both indefinite expansion and snap-through instabilities in neo-Hookean and Mooney-Rivlin materials, respectively. It is shown that load with magnitude above quasi-static threshold, with respect to long-time shear modulus, will generate corresponding instabilities, which serve as the upper bound on the stability limit. In most loading scenarios, it is shown that the final long-time response is independent of transient motion. However, quite surprisingly, we find that under a certain range of applied load a bifurcation appears and multiple long-time solutions exist. In this case, the final equilibrium state of the body is shown to be highly sensitive to the viscoelastic time scale.

Finally, we examine the natural frequency of the system and the dynamic responses under harmonic excitations. The natural frequency is examined from our solution by releasing the system from a pre-deformed stationary state. Due to nonlinearity, the magnitude of the initial deformation influences the frequency of free oscillation. For small amplitude excitations near the natural frequency, the resonant response appears but the solution remains bounded due to the nonlinearity. The influence of viscosity is studied by considering the variation of maximum response amplitudes against applied periods. It is shown that the peak of viscous response appears inbetween two elastic limits (instantaneous and long-time behaviors). Large amplitude excitations is briefly discussed by examining the time histories, where extremely high accelerations are
observed when the cavity is pushed to the collapsing state $a \to 0$.

6.2 Future work

Although the inspiration for this work comes from the expansion of liquid-filled cavity triggered by internal liquid cavitation, we have not included the fluid part into the formulation in this study. Hence a fully coupled fluid-solid model remains to be established in future work. If the fluid is taken into account, additional viscous effects may further help stabilize the system, and the added inertia may influence the frequency of oscillatory motion.

Material failure criteria are necessary to describe real world observations. Although loss of stability may ultimately lead to damage, it cannot be captured by the elastic/viscoelastic modeling considered in this thesis. Additionally, significantly large deformations around the inner radius could be reached even without loss of stability, which can be a source of damage in realistic settings. By incorporating certain failure criteria, this model could be further developed to understand material damage during dynamic motion, especially when snap-through and indefinite expansion occur. More interestingly, localized material softening in damaged areas could lower the stability thresholds of the system and result in catastrophic failure.
Bibliography


Appendix A

Thermodynamical consistency

Given the energy density function

\[ W(\lambda, \xi) = \frac{\mu}{2} \left( \frac{1}{\lambda^4} + 2\lambda^2 - 3 \right) + \frac{\alpha \mu}{2} \left( \frac{\xi^4}{\lambda^4} + 2\frac{\lambda^2}{\xi^2} - 3 \right) + n \frac{\mu}{2} \left( 2\frac{1}{\lambda^2} + \lambda^4 - 3 \right) + n \frac{\alpha \mu}{2} \left( 2\frac{\xi^2}{\lambda^2} + \frac{\lambda^4}{\xi^4} - 3 \right), \]  

(A.1)

where \( \lambda \) is the total hoop stretch and \( \xi \) is the hoop stretch of viscous dashpot which serves here as an internal variable. By taking partial derivatives of (A.1) with respect to \( \xi \) we obtain

\[ \frac{\partial W}{\partial \xi} = \frac{\alpha \mu}{2} \left( 4\frac{\xi^3}{\lambda^4} - 4\frac{\lambda^2}{\xi^3} \right) + n \frac{\alpha \mu}{2} \left( 4\frac{\xi}{\lambda^2} - 4\frac{\lambda^4}{\xi^5} \right). \]  

(A.2)

Recalling that our evolution equation is defined as

\[ \dot{\xi} = \frac{\partial \xi}{\partial t} = \frac{1}{\tau} \frac{(\lambda^3 - \xi^3)}{\xi^2}. \]  

(A.3)

To obtain the rate of dissipation in the material unit we subtract between the power invested by the external load and the elastic energy absorbed in the material.
The only remaining non-trivial term is

\[
\dot{D} = -\frac{\partial W}{\partial \xi} \frac{\partial \xi}{\partial t} = -\frac{2\alpha \mu}{\tau} \left[ \left( \frac{\xi^4}{\lambda^4} - \frac{\lambda^2}{\xi^2} \right) + n \left( \frac{\xi^2}{\lambda^2} - \frac{\lambda^4}{\xi^4} \right) \right] \frac{\lambda^3 - \xi^3}{\xi^3} 
\]

(A.4)

where for \( \zeta = \lambda/\xi \in (0, \infty) \). To maintain thermodynamic consistency, we require that \( \dot{D} \geq 0 \). After differentiation with respect to \( \zeta \) it is straightforward to show that for non-negative \( \alpha \mu/\tau \) and \( n \) the global minimum of the above function is zero and appears at \( \zeta = 1 \).