Abstract

This thesis consists of three chapters on international economics. The first chapter explores the implications of the large increase in cross-border holdings of financial assets for monetary policy and capital controls. I study an open economy model with nominal rigidities, incomplete markets, and assets denominated in home and foreign currency. I develop an approximation method that allows me to characterize the optimal policy sharply. The planner trades-off stabilizing output gaps with creating insurance via cross-country balance-sheet effects. Perhaps surprisingly, as insurance considerations become more important, home-currency positions become larger, and the excess-return volatility of home-currency assets actually decreases, rather than increases as one would expect with fixed ad hoc portfolios. Capital controls are not called for by the approximate solution, i.e., private portfolio decisions are approximately efficient. In my baseline calibration, the welfare gains from the optimal policy are 1.5 times larger than those from inflation-targeting.

The second chapter, joint with Ludwig Straub, develops a theory of foreign exchange interventions for small open economies. In the model, the central bank can implement nonzero spreads between home- and foreign-currency bonds by managing its portfolio due to financial frictions that limit arbitrage by the private sector. Nonzero spreads are costly as they allow foreign intermediaries to make carry-trade profits. Optimal interventions balance these costs with terms of trade benefits. The optimal policy gives rise to a smooth path for the spread, relying on credible promises of future interventions (forward guidance). By contrast, we find smoothing exchange rates aggressively is not optimal since it invites costly speculation. We conclude with a multi-country extension of our model.

The third chapter, joint with Juan Carlos Hallak, studies the relevance of uncertainty and experimentation as a central feature of exporter dynamics. We show that a standard model without these features cannot explain two key facts of exporter dynamics: the strikingly low survival rates one year after entering a foreign market, and the novel fact that re-entrants in export markets are more likely to survive than first-time entrants. We develop a tractable model with experimentation that can explain these facts. We also provide support for the main mechanism of the model by exploiting variation in the degree of uncertainty across products and markets.

Thesis Supervisor: Iván Werning
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I consider myself very fortunate to have had Iván Werning as my advisor. I have always tried to look for beauty in economic models, and Iván taught me where to look and how to look. In his classes, by drawing connections across models I had never seen before and by endowing me with a very large toolkit to tackle economic problems. In my own research, by making innumerable suggestions that helped me understand my own models at a substantially deeper level. Very often he would ask me questions that at face value seemed rather technical, but that after long examination revealed to be incredibly insightful. My thesis benefited from these kinds of comments to a large extent. Iván has also been very supportive at a personal level. Writing a thesis is hard, and Iván always had a word of encouragement when I needed one.

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I dedicate my thesis to my family. Throughout my life my parents have created a very stimulating intellectual environment that led to me getting admitted to MIT and ultimately producing this thesis. Special thanks to my father, who made me love macroeconomics.
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Chapter 1

Monetary Policy, Capital Controls, and International Portfolios

In the past two decades, there has been a large increase in cross-border holdings of financial assets, making currency movements important sources of capital gains and losses. In this context, monetary policy can enhance risk-sharing across countries by influencing exchange rates. Furthermore, the strength of this channel depends on the portfolio the country holds, giving rise to a potential rationale for capital controls. To shed light on these issues, I study an open economy model with nominal rigidities, incomplete markets, and assets denominated in home and foreign currency. I develop an approximation method that allows me to characterize the optimal policy explicitly. I show that optimal monetary policy is a weighted average of an inflation target and an insurance target and characterize the optimal weight sharply. Perhaps surprisingly, as insurance considerations become more important, home-currency positions become larger, and the excess-return volatility of home-currency assets actually decreases, rather than increases as one would expect with fixed ad hoc portfolios. In addition, I find that private portfolio decisions in small open economies are approximately efficient so that differential capital controls on foreign- vs. home-currency assets are not called for by the approximate solution. In my baseline calibration, the welfare gains from the optimal policy are 1.5 times larger than those from inflation-targeting.

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1.1 Introduction

The size of international balance sheets has increased dramatically in the past two decades (Lane and Milesi-Ferretti, 2007). A recent literature has argued that this financial integration has significant implications for the dynamics of a country's net foreign asset position, as movements in asset prices create sizeable capital gains and losses. Today, these *valuation effects* are often of comparable magnitude to current-account fluctuations (Gourinchas and Jeanne, 2013; Lane and Milesi-Ferretti, 2007; Tille and van Wincoop, 2010). The goal of this paper is to study the implications of financial integration for optimal monetary policy and capital controls.

The analysis is motivated by two observations. The first observation is that monetary policy and capital controls can be used to influence exchange rate movements, which are one of the most important sources of asset price fluctuations in open economies (Lane and Shambaugh, 2010b). For example, tightening monetary policy and taxing savings typically leads to a stronger currency, increasing the real value of home-currency bonds. Thus, by increasing the returns of the country's international portfolio in bad times, and decreasing them in good times, central bank policies can improve the hedging properties of the portfolio; that is, they can play an *insurance* role.¹

The second observation is that the country's international portfolio is a key determinant of the strength of the insurance channel. When agents have sizeable positions in home-currency bonds, exchange rate movements can be very powerful as a means of completing markets. This has two implications. First, there is a two-way feedback between monetary policy and portfolio choice, as positions depend on agents' expectations of monetary policy. Second, capital controls taxing the composition of international portfolios may be desirable, as agents do not internalize the effect of their portfolio choice on the ability of the central bank to provide insurance. Indeed, the presence of incomplete markets and nominal rigidities guarantees this will be the case (Geanakoplos and Polemarchakis, 1986, Farhi and Werning, 2016). However, there is little guidance as to how important these taxes may be.

The main contribution of this paper is to characterize optimal monetary policy and capital controls in a model that allows for the previous considerations. From an economic standpoint, this requires: (i) extending the typical open economy macroeconomic model used for optimal policy analysis, where either markets are complete and there is no insurance role, or there is a single asset and there is no role for portfolio choice;² and (ii) developing new tools to study optimal policy in these richer environments, where the standard linear-quadratic framework cannot immediately be applied due to the indeterminacy of the portfolio at the steady state. To this end, I extend a canonical open economy model by allowing the home country to trade multiple assets with the rest-of-the-world. I assume these assets are insufficient to span the whole state space (i.e., markets are incomplete) and the return of some of these assets depends on monetary policy. I overcome

¹It is well-understood that monetary policy can play an insurance role in environments with incomplete markets via terms-of-trade manipulation (Obstfeld and Rogoff, 2002, Corsetti, Dedola and Leduc, 2010). I assume the terms-of-trade are exogenous to focus on the role of gross positions, which is less well-understood.

²An important exception is the work of Benigno (2009a), Benigno (2009b), and Senay and Sutherland (2017). There are important differences with these papers, which I discuss in detail at the end of this section.
the indeterminacy of the steady-state portfolio by showing how the perturbation approach in Judd and Guu (2001) employed in positive analysis can be used to extend the linear-quadratic normative framework in Benigno and Woodford (2012). Using this new approximation method, I provide a sharp characterization of the solution and comparative statics.

The main results in this paper arise from the interaction between exchange rate management and international portfolio choice. To illustrate the forces at play in the simplest possible way, I start with a two-period open economy model where agents have an endowment of tradable goods and produce nontradable goods. In the first period, agents only trade financial assets. In the second period, the state of the world is realized, agents produce, honor their financial obligations, and consume. The model has two key ingredients. First, like in the canonical model, there are nominal rigidities (price stickiness). This ingredient gives rise to the traditional demand-management role for monetary policy, concerned with undoing the distortions associated with price stickiness. The second ingredient is the availability of home- and foreign-currency bonds that can be traded internationally. This ingredient gives rise to the insurance channel discussed above, and a nontrivial portfolio problem.

In this environment, I study the problem of a planner that maximizes the utility of home households under commitment. She has two set of tools: monetary policy and capital controls. Monetary policy is a state-contingent exchange rate rule. Capital controls are taxes on financial assets. Since there is no consumption in the first-period, this baseline model is essentially static so there is a single portfolio tax. In the dynamic model I discuss later, capital controls also include a savings tax.

My approximation method allows me to get closed form solutions for the optimal monetary policy, portfolio, and capital controls. These solutions are valid for small disturbances around the nonstochastic steady state. Monetary policy is characterized to first-order, the portfolio is characterized at the steady state ("zero-order"), and the portfolio tax is characterized to second-order. In terms of monetary policy, I show that the optimal policy is a weighted average of two targets: a demand-management target and an insurance target. The former is the exchange rate that would be required to attain a zero output gap and no price dispersion. The latter is the exchange rate that would be required to replicate the transfer the planner would desire under complete markets. For example, if the home country experiences a negative endowment shock and has home-currency debt, the insurance target would imply a depreciation. The optimal weight depends on the outstanding portfolio. When positions are large in absolute value, providing insurance is cheap: only a small exchange rate movement is needed to replicate the desired transfer. By the same token, restoring

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3 The approach is also related to the perturbation approach in Devereux and Sutherland (2011) and Tille and van Wincoop (2010), who use it to characterize the competitive equilibrium. I extend the analysis to optimal policy problems.

4 In Appendix A.2.3, I consider a general asset structure with an arbitrary number of assets that are allowed to load arbitrarily on endogenous variables and shocks. All the results go through in this richer environment.

5 In Appendix A.2.1, I study the problem from the point of view of a supranational authority that takes into account foreigners' welfare. I show that this increases the importance of the insurance motive, leading to more levered portfolios and a lower volatility of the exchange rate.
production efficiency could be very costly, since the required exchange rate movement would create a large transfer of wealth that may be undesirable. As a result, the optimal weight on the insurance target increases with the size of the position. This captures a form of fear-of-floating due to currency mismatches.\footnote{In this paper, the relevant measure of the mismatches is the foreign-currency value of home-currency debt, different from the standard fear-of-floating, which refers to home-currency value of foreign-currency debt as in Lane and Shambaugh (2010a) and Bénétrix, Lane and Shambaugh (2015). The latter does not lead \emph{per se} to a deviation from price stability in my model, which features a representative agent, exogenous terms-of-trade, and no borrowing constraints (unlike the work by Caballero and Krishnamurthy (2004) and Ottonello (2015), among others).}

In terms of the portfolio, I show that its optimal sensitivity to the value of the home-currency is higher when the insurance motive is more relevant. This is a direct consequence of the fact that a larger position reduces the cost of providing insurance ex post and makes it harder to correct production inefficiencies, as discussed above. The relative importance of the insurance motive depends on both structural characteristics of the economy, such as risk aversion and the degree of price flexibility, and the stochastic properties of the shocks. For example, shocks to the terms-of-trade typically matter relatively more for the insurance motive compared to nontradable productivity shocks. If the former are more frequent, positions will be larger.

One perhaps surprising implication of the interaction between monetary policy and portfolio choice is that the optimal degree of exchange-rate volatility actually \emph{decreases} with the insurance motive. By contrast, if the portfolio is constrained, volatility increases when insurance considerations are more important. To understand this result, consider first the case with a fixed portfolio. For the reasons described above, an increase in the importance of the insurance motive dampens the response aimed at correcting production inefficiencies but exacerbates exchange rate movements to provide insurance. If the constrained portfolio is optimal, this \emph{composition effect} leads to higher exchange rate volatility. When the portfolios are endogenous, there is an opposing force: the insurance motive also induces a larger position, which not only lowers the incentives to correct production inefficiencies even further, but also decreases the required exchange rate movement to provide insurance. This last force dominates if home has home-currency liabilities, which is typically the case in the data.

In terms of capital controls, the model provides two potential rationales for taxing portfolios. First, if the foreign demand of home-currency bonds is not perfectly elastic there is a terms-of-trade externality: Agents overinsure on their own, so the planner finds it optimal to put a tax that pushes positions towards zero.\footnote{The planner can manipulate the intratemporal price of consumption across states, i.e., the stochastic discount factor. The logic is the same as in Costinot, Lorenzoni and Werning (2014), except it manifests across states rather than time.} More interestingly, there is another motive related to the presence of incomplete markets and nominal rigidities. To isolate this second motive, suppose the foreign demand for home-currency bonds is perfectly elastic (i.e., a small open economy). In this environment, Farhi and Werning (2016) show that taxes are desirable to correct pecuniary and aggregate-demand externalities. Perhaps surprisingly, I show that taxes are zero in the approximate solution, i.e., they are at most third-order. In other words, private portfolio decisions...
are approximately efficient. The key observation behind this result is that eliminating production inefficiencies in this economy is feasible, i.e., there is divine coincidence: output gaps and price dispersion can be closed simultaneously. As a result, the economy only experiences booms and recessions because the planner is trying to provide insurance. Formally, this implies that output gaps are, to first-order, proportional to social marginal utility. Furthermore, the wedge between social and private marginal utility is proportional to the output gap. These observations imply that social and private marginal utilities are proportional to one another, which is enough to establish the asymptotic optimality of the private portfolio decision as risk vanishes.

I then study a dynamic version of this economy. This extension serves three goals. First, I study the robustness of the results and show that they all generalize. The only qualification is that, if home-currency bonds are long-lived, results on volatility hold for the excess-return of the bond, rather than the exchange rate. Second, I characterize new features of the solution. Unlike the static model, where the planner had a single possibility to engineer an excess return (i.e., creating an output gap), now the planner has more options: she can promise either current or future output gaps and inflation and may try to manipulate tradable consumption over time. The planner then solves a cost-minimization problem among these tools. I show that capital controls on the total size of capital flows are desirable, although their composition is still efficient. In other words, the planner wants to tax financial assets, but not differentially so. I also show their effectiveness is higher when bonds have a shorter duration. Indeed, if home-currency bonds are perpetuities and prices are perfectly rigid, optimal savings taxes are exactly zero. Third, I introduce ingredients that are important for the quantitative exercise. Of particular relevance are two shocks that play a key role in the calibrated model: (i) world-interest-rate shocks, which create a large demand for insurance, and (ii) liquidity shocks on home-currency bonds, which introduce noise in the return of the home-currency bond.

To conclude, I explore the quantitative implications of the optimal policy for monetary policy, observed portfolios, capital controls, and welfare. I compare it to the benchmark demand-management policy (strict inflation-targeting). I calibrate the model to Canada, a prototype small open economy. I find that the optimal policy increases external home-currency debt from 16% to 23%, which translates into an ex post weight on the insurance target of 8%. This changes the variance decomposition of the excess-returns of home-currency bonds: the contribution of liquidity shocks decreases from around 80% to 70%, with a mirror increase in the contribution of interest-rate shocks. This, however, does not translate into noticeable changes in overall volatility. Concerning capital controls, I find a limited role for savings taxes and an important role for portfolio taxes, due to the limited size of the foreign-investor base in home-currency bond markets. This also implies a high value of cooperation, as I show in Section 1.5. Regarding welfare, I compute the gain in consumption-equivalents of moving from an economy without home-currency bonds to an economy with home-currency bonds

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8 There are two main rationales for savings taxes. First, they can be used to move the exchange rate without creating an output gap. Second, they correct the misvaluation of tradable goods during booms and recessions.

9 This result is sensitive to the cost of inflation. Reducing the average duration of prices from a year to three quarters leads to an insurance weight of 25%.
and flexible prices (the first-best). Then, I compute how much of these welfare gains are achieved by each policy. I find that, while inflation-targeting attains 12% of the benefits, the optimal policy attains 17% - almost 1.5 times as much. Portfolio endogeneity is quantitatively important for this result: gains would only be 15% if home-currency debt were fixed at the calibrated value (15%).

**Related literature** This paper contributes to a large literature exploring deviations of optimal monetary policy from inflation-targeting in open economies, surveyed by Corsetti, Dedola and Leduc (2010). In particular, my analysis is related to the idea that, in incomplete markets, monetary policy plays an insurance role (Obstfeld and Rogoff, 2002; Corsetti, Dedola and Leduc, 2010). However, in most of the literature, insurance is improved via terms-of-trade manipulation and not through asset positions; formally, the solution is typically approximated around a symmetric steady state with a zero net-foreign-asset position. I abstract from this insurance channel by focusing on a small open economy that faces exogenous terms-of-trade. Instead, the insurance role in this paper is linked to the size of gross positions and is, therefore, closest to the work of Benigno (2009a), Benigno (2009b), and Senay and Sutherland (2017). The first two papers study optimal monetary policy in a New Keynesian open economy model with home- and foreign-currency bonds. Importantly, in those papers the steady-state portfolio is exogenous. By contrast, in the present paper the portfolio is endogenous, which is key for my results. Senay and Sutherland (2017) present a rich two-country New Keynesian open economy model with two nominal bonds and equities in firms from both countries. They allow portfolios to be endogenous, but they do not study fully optimal policy. Instead, they focus on a limited set of policy rules, and optimize numerically over the parameters of such rule. By contrast, I study fully optimal monetary policy and characterize it analytically. None of these papers study capital controls.

Farhi and Werning (2016) develop a general theory for the joint problem of optimal monetary policy and macroprudential policy and provide several applied examples. In one of their applications, they discuss a static small-open economy with home- and foreign-currency debt. They note that there is generally a trade-off between insurance and demand-management and provide a formula for portfolio taxes, pointing out that they are generally nonzero. My analysis and results confirm these observations, but also provides a sharper characterization of both monetary policy and macroprudential policy in this context. Indeed, using my approximation, I am able to show precisely how monetary policy is a weighted average of two targets, characterize the weight and the debt positions, and, somewhat surprisingly, show that while portfolio taxes are generally non-zero, they are zero in an approximate sense around the steady state. My paper also extends the study of these issues to a dynamic setting to provide a quantitative analysis.

Another strand of literature studies monetary policy and portfolio choice between nominal and real debt in environments without commitment. In a seminal paper, Bohn (1988) demonstrates

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10Devereux and Sutherland (2008) also stress the role of gross portfolio positions. However, they focus on a case where the optimal policy can replicate complete markets, so inflation-targeting is still optimal.

11Chang and Velasco (2006) also present an open economy model with home and foreign-currency bonds but they only compare two extreme exchange rate regimes (inflation-targeting and a fixed exchange rate).
the optimality of issuing nominal debt to minimize distortionary taxation, despite the inflationary incentives created by lack of commitment. Bohn (1990) argues that foreign-currency debt may lower home-currency debt issuance, since unlike indexed debt it may have some desirable hedging properties. More recently, motivated by the increase in the share of government debt issued in home-currency in emerging markets during the past twenty years, a recent literature has revisited the trade-off between incentives (due to lack of commitment) and insurance in home-currency-bond issuance; see Du, Pflueger and Schreger (2017), Engel and Park (2017), and Ottonello and Perez (2017). The present paper has a more normative focus and studies optimal monetary policy, assuming full commitment, as well as optimal capital controls, which these papers abstract from. This leads to two key results about optimal policy—lower volatility of the exchange rate and zero portfolio taxes, up to second-order—that have no counterparts in previous studies.

My paper is also related to a closed-economy literature that studies the potential of monetary policy to complete markets with nominal assets in environments with commitment; see Schmitt-Grohe and Uribe (2004), Siu (2004), Lustig, Sleet and Yeltekin (2008) and Sheedy (2014). In these papers, a similar trade-off between demand-management and insurance emerges, but insurance takes place between the government and the private sector, or between borrowers and savers. In addition, my analysis emphasizes the role of exchange rate movements, and the portfolio decision between home- and foreign-currency bonds, which is absent from these studies.

From a methodological perspective, this paper makes a contribution to the recent literature on portfolio choice within dynamic stochastic general equilibrium models (Devereux and Sutherland, 2011; Evans and Hnatkovska, 2012; Tille and van Wincoop, 2010). These papers are positive, aiming to approximate the competitive equilibrium given a policy rule. I extend these methods to tackle normative questions. Following the same steps in Benigno and Woodford (2012), I derive an approximate problem around an arbitrary steady-state portfolio that is linear-quadratic in all the remaining endogenous variables. Then, I use the perturbation approach in Judd and Guu (2001) on the first-order conditions of the nonlinear planning problem and show they coincide with the first-order conditions of the approximate problem, including the first-order conditions with respect to the steady-state portfolio. The validity of the procedure depends on the availability of taxes. Otherwise, one needs an additional quadratic constraint.

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12Du, Pflueger and Schreger (2017) document that governments in countries where the home-currency bond has worse hedging properties issue a larger share of their debt in home-currency. They show that the interaction between limited commitment and risk-averse foreigners can rationalize their empirical findings. Engel and Park (2017) study an endowment small-open-economy model in which the government can borrow in both home- and foreign-currency subject to two commitment frictions: strategic default and debasement. They show that a higher cost of inflation leads to endogenously looser limits to borrowing in home-currency and that this can account for the surge in home-currency borrowing in recent decades. Ottonello and Perez (2017) study a small open economy model with tradable and nontradable goods where the government does not default but cannot commit to future policy. They show that lack of commitment can rationalize the currency-composition of government debt and the procyclicality of home-currency-debt shares for a sample of emerging markets.

Layout The paper is organized as follows. In Section 1.2, I present a static version of the model and derive the planning problem. In Section 1.3, I characterize the optimal policy in this setting. In Section 1.4, I extend the model to a dynamic setting and characterize the optimal policy in this context. In Section 1.5, I calibrate the model and explore the quantitative importance of the channels emphasized in the paper. Section 1.6 concludes. Appendix C.1 contains all proofs and detailed derivations, Appendix A.2 contains some additional extensions of the model, and Appendix A.3 contains additional sensitivity exercises for the quantitative section.

1.2 Static model

In this section, I present a two-period version of the model. Henceforth, I refer to this version of the model as “static” because agents only trade financial assets in the first period. Section 1.2.1 presents the setup. Section 1.2.2 presents the planning problem and discusses the main trade-offs the planner faces. The optimal policy is analyzed in Section 1.3.

1.2.1 Set up

There are two periods, 0 and 1. At $t=0$, agents trade financial assets. At $t=1$, the state of the world is realized, agents produce, honor their financial obligations, and consume. There are two final goods (tradables and nontradables), and a continuum of varieties of intermediate inputs, which are used to produce nontradables.

Home households There is a continuum of households in the home country, maximizing a standard utility function

$$
E u(C_T, C_N, L; \xi) \tag{1.1}
$$

where $C_T$ is tradable consumption, $C_N$ is nontradable consumption, $L$ is labor, and $\xi$ is a vector of shocks. The function $u$ is assumed increasing and strictly concave in consumption and leisure $(-L)$.

Agents have access to two assets in period 0: a bond with a fixed payment in home-currency $B$ and a bond with a fixed payment in foreign currency $B^*$. Since there is no consumption at $t = 0$, it is without loss of generality to normalize the exchange rate at $t = 0$ and the return of the foreign currency asset $R^*$ to 1. I denote the return of the home-currency asset $R$. The budget constraint at $t = 0$ is

$$(1 + \tau_B)B + B^* = T_0,$$

where $\tau_B$ is an ad-valorem tax on home bonds and $T_0$ is a lump-sum transfer from the central bank. I assume positions are bounded by a large constant $K$, i.e., $|B| \leq K$.

At $t = 1$, agents receive a tradable endowment $Y_T(\xi)$, which can be interpreted as the product of tradable output and the terms-of-trade as in Mendoza (1995). They also work, collect profits,
honor their financial obligations, and consume both goods. The budget-constraint at \( t = 1 \) is

\[ E_s C_{Ts} + P_{Ns} C_{Ns} = E_s Y_T(\xi_s) + (1 + \tau_L) W_s L_s + \Pi_{Ns} + \Pi_{Is} + RB + E_s B^* + T_s \]

where \( E_s \) is the nominal exchange rate, \( P_{Ns} \) is the price of nontradables, \( W_s \) is the nominal wage, \( \tau_L \) is a labor subsidy, \( \Pi_{Ns} \) and \( \Pi_{Is} \) are profits from nontradable and intermediate good producers, respectively, and \( T_s \) are lump-sum transfers from the central bank. I use the convention that a higher exchange rate means a more depreciated currency and normalize the international price of the tradable good to 1. Optimization over labor and tradable and nontradable consumption yields

\[ \frac{u_N(s)}{u_T(s)} = \frac{P_{Ns}}{E_s}. \]  
(1.2)

\[ (-u_L(s))/u_T(s) = (1 + \tau_L) W_s / E_s. \]  
(1.3)

where \( u_N(s), u_T(s), \) and \( u_L(s) \) are the first-derivatives with respect to nontradables, tradables, and labor, respectively. Asset optimization yields a no arbitrage condition,

\[ E \left[ (1 + \tau_B)^{-1} R E_s^{-1} - 1 \right] u_T(s) = 0. \]  
(1.4)

**Nominal rigidities** I introduce nominal rigidities in the form of sticky prices in the production of nontradable intermediate goods. There is a continuum of varieties \( i \in [0, 1] \), which can be aggregated into a composite \( Y_i \) that can be used for production,

\[ Y_{Is} = \left( \int_0^1 Y_{Is}(i) \frac{\eta-1}{\eta} \, di \right)^{\frac{\eta}{\eta-1}}. \]  
(1.5)

Production of the intermediate input is linear in labor,

\[ Y_{Is}(i) = L_s(i). \]  
(1.6)

In each product market \( i \) prices are set by a monopolistically competitive firm who faces a demand given by

\[ Y_{Is}(i) = (P_{Is}(i)/P_{Is})^{-\eta} Y_{Is}. \]  
(1.7)

A share \( \phi \) of firms has their nominal price fixed at \( \tilde{P}_T \) and supply any amount of output that is required, while a share \( 1 - \phi \) can optimize its price state by state.\(^{14}\) Firms that are able to optimize set a constant mark-up over the marginal cost (the wage),

\[ P_{Is}(i) = \frac{\eta}{\eta - 1} W_s. \]  
(1.8)

\(^{14}\)I could allow the former group to set the average price. This would only complicate the exposition, without changing the results. In the dynamic model of section 1.4 I show all the results are robust to standard Calvo price-setting.
Combining (1.3) and (1.8), one can write the ideal intermediate input price index \( P_{Is} \) as

\[
P_{Is} = \left( \phi F_{I}^{1-\eta} + (1 - \phi) \frac{\eta}{\eta - 1} \frac{1}{\eta - 1 + \eta} \left( \frac{-u_L(s)}{\eta} \right)^{1-\eta} E_{s}^{1-\eta} \right)^{1-\eta}.
\]

(1.9)

**Foreign households** A measure \( m \in \mathbb{R}_+ \cup \infty \) of foreign households may participate in home-currency asset markets. Each household is endowed with \( \{Y^*(\xi_s)\} \) units of the tradable good. Using asset market clearing conditions, this leads to a no-arbitrage condition given by

\[
\mathbb{E} \left[ (RE_{s}^{-1} - 1)u^*(Y^*(\xi_s)) - \frac{1}{m} (RE_{s}^{-1} - 1)B \right] = 0.
\]

(1.10)

When \( m = \infty \), the small open economy takes the stochastic discount factor as given. In this case, I say there is perfect financial integration in home-currency markets. When \( m < \infty \), there is limited participation and the home economy has market power in home-currency bond markets. Alternatively, one may interpret the case \( m < \infty \) as a large economy whose actions affect the world’s stochastic discount factor.

**Final production** Firms have access to an increasing and concave production function \( F(Y_{Is}; \xi_s) \). They maximize profits, which are given by \( \Pi = P_{Ns}F(Y_{Is}; \xi_s) - P_{Is}Y_{Is} \). Firm optimization implies

\[
P_{Ns}F_Y(Y_{Is}; \xi_s) = P_{Is}.
\]

(1.11)

where \( F_Y \) is the derivative with respect to intermediate inputs.

**Central bank** The central bank in the economy has three tools: monetary policy, capital controls, and the labor subsidy. Monetary policy is a state-contingent exchange rate policy rule \( \{E_s\}_s \). Since some prices are fixed in home currency, this instrument allows the central bank to determine the equilibrium price of intermediate inputs in foreign-currency, which in turn affects the level of employment.\(^{15}\) Capital controls in this model are represented by the portfolio tax \( \tau_B \). This instrument allows the central bank to effectively control the balance sheet of the country vis-a-vis the rest-of-the-world.\(^{16}\) The proceeds are then rebated to home households through lump-sum taxes,

\[
T_0 = \tau_B B.
\]

(1.12)

\(^{15}\)I could have also stated monetary policy as a money-supply rule. I follow the literature and consider a cashless economy, avoiding an explicit modelling of money demand. This allows me to focus on the demand-management motive, which is more relevant for most economies due to low inflation.

\(^{16}\)Equivalently, I could have the government be the only one allowed to borrow in foreign markets in home-currency bonds, which reflects the situation is many emerging economies (Du and Schreger (2015)). However, capital controls may still be desirable in these countries to control positions in other assets that are traded by the private sector and are exposed to the stance of monetary policy, such as equity and foreign direct investment (see Appendix A.2.3).
Finally, the labor subsidy allows the planner to obtain production efficiency at the steady-state. These subsidies are also rebated lump-sum to households.\footnote{Note that assuming the central bank balances its budget period-by-period is without loss of generality, since agents have access to the same set of assets (i.e., there is Ricardian equivalence).}

\begin{equation}
\tau_L W_s L_s + T_s = 0.
\end{equation}

The monetary authority announces the monetary and tax policies at the beginning of time, \textit{before} agents engage in bond trading, and is assumed to be perfectly credible.

\textbf{Goods and labor market clearing} Replacing profits, labor income and the \( t = 0 \) budget constraint into the \( t = 1 \) budget constraint, I obtain

\begin{equation}
C_{Ts} = Y_{Ts} + (RE_s^{-1} - 1)B.
\end{equation}

The market clearing condition for nontradables and labor are given by

\begin{equation}
C_{Ns} = F(Y_s^I, \zeta_s)
\end{equation}

\begin{equation}
L_s = \int_0^1 L_s(i)di
\end{equation}

Next, I formally define a competitive equilibrium in this economy.

\textbf{Definition 1.1. Given a policy \( \{E_s, \tau_B, T_0, \tau_L, T_s\} \), an allocation \( \{C_{Ts}, C_{Ns}, L_s, L_s(i), Y_s^I(i), Y_s^I, B\} \) together with prices \( \{P_N, P_I, P_I(i), W_s\} \) and a home-currency bond return \( R \) is a competitive equilibrium if and only if they solve (1.2)-(1.16).}

\subsection{Planning problem}

The planner in the economy is the central bank, who chooses a state-contingent exchange rate and capital controls to maximize the utility of home households. To simplify the exposition, we set \( \tau_L = \frac{1}{\eta - 1} \), which is optimal at the steady state (i.e., when \( \xi_s \equiv 0 \)), and do not discuss it any further as part of the policy mix.\footnote{Given that there is price dispersion, optimizing over \( \tau_L \) would introduce an additional constraint. This is without loss of generality for the approximate solution discussed in Section 1.3.} Next, I simplify the problem to obtain a reduced set of implementability conditions that describe the set of equilibria that can be attained through different policies \( \{E_s, T_0, \tau_B\} \). Given \( C_{Ts} \) and \( E_s \), one can find \( \{C_{Ns}, L_s, L_s(i), Y_s(i), Y_s, W_s, P_I(i), P_I, P_N\} \) that solve (1.2), (1.3), (1.5) - (1.9), (1.11), (1.15) and (1.16).\footnote{This system of equations can be reduced state by state to a system of two equations in \( (Y_s^{fix}, Y_s^{fix}) \), which I assume has at least one solution. The latter can be guaranteed in a neighborhood of the steady state using the implicit function theorem.} Furthermore, the planner may use \( \tau_B \) to satisfy (1.4) and \( T_0 \) to satisfy (1.12). Using these observations, I obtain the following implementability result.
Lemma 1.1. An allocation for tradable consumption \( \{C_{Ts}\} \), an exchange rate policy \( \{E_s\} \), a home-currency position \( B \) and a home-currency bond return \( R \) form part of an equilibrium if and only if they solve (1.10) and (1.14).

As in Farhi and Werning (2016), I define the following indirect utility function:

\[
V(C_{Ts}, E_s; \xi_s) = \max_{\{C_{Ns}, L_s, L_s(i), Y_s(i), Y_s, W_s, P_{Ts(i)}, P_{Ts}, P_{Ns}\}} u(C_{Ts}, C_{Ns}, \int_0^1 L_s(i) di; \xi_s)
\]

subject to (1.2), (1.3), (1.5) – (1.9), (1.11), (1.15), and (1.16)

Using this definition, the planner’s problem can be formulated as follows.

Problem 1.1. The planner’s problem is to choose \( \{C_{Ts}, E_s, B\} \) to maximize

\[
\mathbb{E}V(C_{Ts}, E_s; \xi_s)
\]

subject to

\[
C_{Ts} = Y_T(\xi_s) + (RE_s^{-1} - 1)B
\]

\[
\mathbb{E}\left[(RE_s^{-1} - 1)u''(Y^*(\xi_s) - \frac{1}{m}(RE_s^{-1} - 1)B)\right] = 0
\]

Before tackling this problem, it is useful to study a simpler benchmark with complete markets.

Problem 1.2. When markets are complete, the planner’s problem is to choose \( \{T_s, E_s\} \) to maximize

\[
\mathbb{E}V(Y_T(\xi_s) + T_s, E_s; \xi_s)
\]

subject to

\[
\mathbb{E}\left[T_s u''(Y^*(\xi_s) - \frac{1}{m}T_s)\right] = 0
\]

Under complete markets the transfer of wealth in each state of the world \( T_s \) is decoupled from monetary policy \( E_s \). This implies the exchange-rate has a single role in this economy: closing the output gap and ensuring there is no price dispersion. This is the traditional demand-management role of monetary policy. Regarding transfers, the solution replicates the laissez-faire competitive equilibrium with flexible prices when \( m = \infty \). When \( m < \infty \), the solution is not the laissez-faire competitive equilibrium due to the presence of terms-of-trade externalities. Although the country is a price-taker in tradable good markets, it can influence the stochastic-discount-factor of the foreigners that participate in home-bond markets (i.e., the state prices). The solution follows from a reinterpretation of the results in Costinot, Lorenzoni and Werning (2014) across states instead of over time: The planner wants the state price to be high when it is relatively rich. This implies she wants foreigners to be relatively poor in those states, i.e., she wants less than full insurance. In Section 1.3.4 we provide an explicit solution for the tax, which reflects this motive.
The key difference between the complete-markets benchmark and the full problem is that in the latter the exchange rate \( \{E_s\} \) and the transfers \( \{T_s\} \) are linked by the relationship \( T_s = (RE_s^{-1} - 1)B \). As a result, the exchange rate plays an additional insurance role, given by the desire to replicate some transfers \( \{T_s\} \). When prices are flexible, the planner can perfectly replicate the transfers from the complete markets solution since the exchange rate plays no demand-management role. When prices are sticky, there is a trade-off between both objectives of monetary policy. In the solution, the planner balances both forces so many states of the world feature nonzero output gaps and deviations from the complete markets transfer. Thus, portfolio taxes are generically nonzero even if \( m = \infty \): the presence of ex post output gaps and pecuniary externalities due to incomplete markets implies agents fail to internalize the proper value of a unit of tradable goods (Farhi and Werning (2016)).

### 1.3 Optimal Policy

In this section, I study the optimal policy in the static model. I start with a brief description of the solution method in Section 1.3.1, which may be skipped without loss of continuity. I then present the main theoretical results. First, I characterize the optimal monetary policy and the optimal portfolio (Section 1.3.2). Second, I derive the implications for exchange rate volatility (Section 1.3.3). Beyond being interesting in its own right, this result illustrates the importance of portfolio endogeneity. Third, I characterize the optimal capital controls, i.e., the optimal portfolio tax (Section 1.3.4). Section 1.3.5 concludes with a simple example that illustrates the results.

Appendix A.2 contains extensions of the model. Appendix A.2.1 revisits the problem from the point of view of a supranational authority that internalizes the effect of the optimal policy on foreigners' welfare. Appendix A.2.2 considers an environment with equity on nontradable firms, instead of a nominal asset. This alternative asset structure illustrates the differences between a nominal asset and a real asset with returns that are sensitive to monetary policy. Appendix A.2.3 generalizes all the results in this section to an environment with a multiple assets that load arbitrarily on endogenous variables and shocks. Appendix A.2.5 presents the problem without capital controls. Proofs for the results in the main text can be found in Appendix C.1.

#### 1.3.1 An approximation for small risks

The standard linear-quadratic framework (Benigno and Woodford (2012)) cannot be used in environments with portfolio choice. The problem is that at the riskless steady-state all assets are perfect substitutes, which makes the portfolio indeterminate and prevents the application of the implicit function theorem. However, it is well-known that a perturbation approach can still be used to find approximate solutions to the first-order conditions (Judd and Guu (2001); Devereux and Sutherland (2011); Tille and van Wincoop (2010)). These approaches rely either on a bifurcation theorem, or on a higher-order approximation of the no-arbitrage equations.\(^{20}\) In this paper, I show that deriving a linear-quadratic approximation around an arbitrary steady-state portfolio,

\(^{20}\)Both approaches are equivalent to the order of approximation I work in this paper.
and then maximizing over the approximation point, is correct when the planner has access to a portfolio tax, which is the case in the current setting. Formally, the first-order conditions of the approximate problem coincide with the result of the perturbation approach in Judd and Guu (2001) to the original nonlinear first-order conditions. The main added benefit of using the linear-quadratic framework rather than just a perturbation approach is that it allows one not only to check locally the second-order conditions, but also to pick the best local solution when there is more than one. This is important because the solution of equilibrium objects such as the exchange rate is typically nonlinear in the steady-state portfolio, which in turn implies there are multiple portfolios that satisfy the first-order conditions. I prove a more general version of this result that holds in a dynamic setting with forward and backward-looking constraints and multiple assets, as in Benigno and Woodford (2012), in my companion paper (Fanelli (2017)).

Let \( \epsilon \) denote the amount of risk in the economy, i.e., \( \xi_s = \epsilon u_s \) where \( u_s \) is a random variable with compact support. I am interested in the limit \( \epsilon \to 0 \). The next lemma provides a “purely quadratic” second-order expansion of the objective function around a steady state with some arbitrary steady-state portfolio \( \tilde{B} \). This expression is obtained by combining a second-order expansion of the objective, the budget constraint, and the foreign Euler equation. Lowercase letters denote log-deviations from the steady state.

**Lemma 1.2.** When \( \epsilon \to 0 \),

\[
EV(\{e_s, \tilde{B}\}) = -k_0 E \left[ \frac{1}{2} (\tilde{B}e_s + T_s)^2 + \frac{1}{2} \chi((1 + \mu \tilde{B})e_s - e_s^{dm}(0))^2 \right] + \text{t.i.p.} + O(\epsilon^3) \tag{1.18}
\]

where \( k_0 > 0 \), \( \chi \geq 0 \) and \( \mu \) are constants, and t.i.p. stands for “terms independent of policy”. \( T_s \) is the transfer of tradable goods the planner would desire if prices were flexible and \( e_s^{dm}(0) \) is the exchange rate that would close the output gap if \( \tilde{B} = 0 \). The objects \( e_s^{dm}(0), T_s, k_0, \mu \) and \( \chi \) are specified in Appendix A.1.1.

The next result shows that the solution to this problem yields a bifurcation point for the portfolio and a local linear approximation to the behavior of all remaining endogenous variables and is, thus, equivalent to a perturbation approach. Importantly, this result relies on the availability of the portfolio tax. Without it, one would need to add an additional quadratic constraint.

**Proposition 1.1.** (LQ equivalence to perturbation) Suppose \( u, u^* \) and \( F \) are locally analytic functions around the steady state. Then, maximizing (1.18) with respect to \( \{e_s\} \) and \( \tilde{B} \) yields a

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21 Without a tax, the problem has two degrees of indeterminacy at the steady state: the Lagrange multiplier on the home no-arbitrage condition and the portfolio. This implies that one needs to consider an additional quadratic constraint in the approximate problem (see Appendix A.2.5).

22 It is important to remember not to drop terms that involve only the steady-state portfolio and no other endogenous variables; i.e., terms one would call “independent of policy” in the original Benigno and Woodford (2012) setup.

23 See Appendix A.2.5 for an example where the additional constraint is needed.

24 By analytic I mean they are infinitely differentiable, and their infinite Taylor expansion coincides with the actual function in a neighborhood of the approximation point. This is required to apply the bifurcation theorem in Judd and Guu (2001).
linear approximation of a solution to the first-order conditions of problem 1.1 around $(\bar{B}, \epsilon = 0)$ for \{es\} and a bifurcation point of the system $B$.

**Remark 1.1.** The term $\frac{\partial B}{\partial \epsilon}$ in the welfare loss function $V$ is irrelevant for welfare to second-order.

One important feature of the loss function derived in lemma 1.2 is that being able to change the portfolio as risk increases does not matter for welfare to a second-order of approximation - a result pointed out by Samuelson (1970) in the context of a standard portfolio problem. Intuitively, this is a consequence of the fact that assets are perfect substitutes not only in the steady state, but also to first-order, since agents behave as if they were risk neutral. This implies that one does not need to know how the portfolio varies with risk to characterize the optimal exchange rate to first-order. It also implies one can only pin down the steady-state portfolio to this order of approximation.

### 1.3.2 Optimal monetary policy and optimal portfolio

In this section, I characterize the optimal monetary policy and the optimal portfolio. I do so in two steps. First, I solve the *inner problem* (i.e., the solution conditional on a steady-state portfolio) in Section 1.3.2. Second, I use the results from the inner problem to solve the *outer problem* (i.e., the optimal portfolio) in Section 1.3.2.

**The inner problem: Optimal monetary policy conditional on a portfolio**

In this log-linear framework, the two objectives of monetary policy discussed in section 1.2.2 (insurance and demand-management) can be described by exchange-rate targets. The insurance target $e_{s}^{in}$ replicates the desired transfers of tradable goods $T_s$,

$$e_{s}^{in}(\bar{B}) = -\frac{1}{\bar{B}}T_s + O(\epsilon^2)$$

while the demand-management target $e_{s}^{dm}(\bar{B})$ closes the output gap (and price dispersion),

$$e_{s}^{dm}(\bar{B}) = \frac{1}{1 + \mu \bar{B}} e_{s}^{dm}(0) + O(\epsilon^2). \tag{1.19}$$

Note that to compute the solution to the approximate problem, one only needs the first-order behavior of $T_s$ and $e_{s}^{dm}(0)$. These are complicated linear functions of the shocks, explicitly solved for in Appendix A.1.1. In Section 1.3.5, I present an example where they take a simple form.

One important feature of these exchange-rate targets is that they depend on the outstanding portfolio. Consider first the insurance target. Any exchange rate movement of size $\epsilon$ creates a transfer of tradable goods of $-\epsilon \bar{B}$. Thus, a more sizeable position requires a smaller exchange rate movement to replicate a desired transfer. Next, consider the demand-management target. The key parameter in this case is $\mu$, which captures the effect of wealth on the demand-management exchange rate: When agents have an additional wealth of $\epsilon$, the exchange rate must move $\mu \epsilon$
to restore production efficiency. Typically $\mu < 0$, which implies home-currency debt dampens exchange rate volatility under demand-management targeting. To see this, suppose the exchange rate would depreciate if $B = 0$, i.e., $e_{s}^{dm}(0) > 0$. When home agents are holding home currency debt ($B < 0$), the depreciation makes them richer. This typically leads to an increase in nontradable demand, which mitigates the required depreciation to restore production efficiency. The opposite is true if home agents are holding home-currency assets.

**Proposition 1.2.** (Optimal monetary policy) Consider an economy with small risks, i.e., $\epsilon \to 0$. Then,

$$e_{s}^{opt}(\bar{B}) = \frac{X}{\chi + f(\bar{B})^2} e_{s}^{dm}(\bar{B}) + \frac{f(\bar{B})^2}{\chi + f(\bar{B})^2} e_{s}^{in}(\bar{B}) + O(\epsilon^2). \quad (1.20)$$

where $f(\bar{B}) = \frac{\bar{B}}{\mu + \bar{B}}$. The parameter $\chi$ satisfies the following properties:

1. (Price flexibility) If a marginal exchange rate depreciation leads to a positive output gap, then there exists $\tilde{\phi} \in (0, 1]$ such that for $\phi < \tilde{\phi}$ $\chi$ increases with $\phi$ while for $\phi > \tilde{\phi}$ $\chi$ decreases with $\phi$. When $\phi = 0$, $\chi = 0$. Furthermore, $\chi$ is increasing and $\tilde{\phi}$ is weakly decreasing in the elasticity of substitution across varieties $\eta$ when $\phi > 0$.

2. (Risk aversion) It is decreasing in the planner’s absolute risk-aversion to movements in tradable-good consumption $\hat{\gamma}$ (computed in Appendix A.1.1), increasing in the measure of foreigners $m$, and, provided $m < \infty$, decreasing in absolute foreign risk aversion $\gamma^*$. 

Proposition 1.2 shows that the optimal monetary policy conditional on the portfolio is a weighted average of both exchange rate targets. The weight has two main components: an exogenous one, controlled by the parameter $\chi$, and an endogenous one $f(B)$, which depends on the portfolio. First, consider the exogenous component $\chi$. One of its key determinants is the degree of price flexibility. When more firms optimize their price (i.e., when $\phi$ decreases), there are two opposing effects. On the one hand, more firms are able to reduce any production inefficiencies that an exchange rate movement may create, i.e., the sensitivity of the output gap to monetary policy decreases. On the other hand, higher flexibility increases price dispersion if most firms cannot adjust their prices. Proposition 1.2 shows the first force dominates when price flexibility is high (i.e., $\chi$ decreases with price flexibility) while the latter sometimes dominates when price flexibility is low (high $\phi$). The second region is more likely to exist when the elasticity of substitution across varieties is high, since this increases the cost of price dispersion. This result on price flexibility relies on the fact that the asset with returns that depend on monetary policy is a nominal asset. In Appendix A.2.2, I study a case where agents may sell equity in nontradable firms, instead of nominal bonds. Expression (1.20) still holds, but $\chi$ now increases with price flexibility. The intuition is simple: when prices are completely flexible, the return on equity is independent from monetary policy. It is only because there are nominal rigidities that the planner may manipulate the return on equity.

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25If $u$ is Greenwood–Hercowitz–Huffman (GHH) or separable in labor, and the composite between tradables and nontradables is CES, $\mu \leq 0$ and devaluations are expansionary.

26As long as $B \mu > -1$. A sufficient condition for this case is provided in proposition 1.3.
The second natural determinant of $\chi$ is risk aversion. As one would expect, if the home economy is more risk-averse to movements in tradable goods, the planner should place a larger weight on the insurance motive. This is captured by a parameter $\gamma$, computed in Appendix A.1.1, which depends not only on overall risk-aversion but also on the ability of the economy to substitute a lack of tradable goods with nontradable consumption and production. More interestingly, when foreign capital becomes more scarce ($m$ is lower or foreign risk aversion $\gamma^f$ is higher), the planner realizes that any additional volatility in the excess returns of home-currency bonds is penalized by foreigners with a lower relative price on the country’s liabilities. As a result, the planner is less willing to let the exchange rate float freely to accommodate any changes in aggregate demand and, instead, prefers to dampen exchange rate volatility.

**Corollary 1.1.** The weight on insurance $\omega(\bar{B}) = f(\bar{B})^2 / (\chi + f(\bar{B})^2)$ increases with the size of the portfolio if and only if $B < -\mu^{-1}$.

The other key determinant of the weight is the outstanding portfolio, which is endogenous. Consider first the case where the output gap is independent from tradable consumption ($\mu = 0$). When positions are small, providing insurance is very costly: any given transfer requires large movements in the exchange rate, which may significantly distort production efficiency. Conversely, attempting to use the exchange rate to close output gaps and achieve no price dispersion when positions are large generates sizeable transfers of tradable goods, which may be undesirable from an insurance perspective. In sum, the optimal weight on the insurance motive increases with the size of the position.

When $\mu \neq 0$, the size of the position is not conceptually the most adequate measure of sensitivity to monetary policy. To see why that is, suppose the planner was willing to create a “deviation” consisting of a 1% depreciation with respect to the demand-management target. If $B < 0$, such a deviation would also make agents richer. As discussed above, this would in turn appreciate the demand-management target, implying that the actual exchange rate movement, and resulting transfer, would be smaller than 1%. Indeed, the transfer would be $\bar{B}(1 + \mu\bar{B})^{-1}%$. This notion of “sensitivity to monetary policy” is robust to other asset market structures. In Appendix A.2.3 I show that with multiple assets that load arbitrarily on endogenous variables and shocks, one may still define a sufficient statistic that plays the same role of $f(\bar{B})$.

In Section 1.4 I show equation (1.20) generalizes to the dynamic model. Thus, once I have calibrated that model in Section 1.5, I can compute the weights in proposition 1.2 to measure the importance of the insurance motive vis-a-vis the demand-management motive in the data.

**The outer problem: Optimal portfolio**

The exchange rate targets depend on the outstanding portfolio. Choosing the portfolio optimally gives the planner a tool to mitigate the trade-off between insurance and demand-management as much as possible. Indeed, the linearity of the model implies that the planner could perfectly align
the targets if there were a single shock by choosing:

\[ f(\bar{B}) = \frac{\bar{B}}{1 + \mu \bar{B}} = -\frac{\sigma_{Te^{dm}(0)}}{\sigma_{e^{dm}(0)}} = -\frac{T_s}{e^{dm}(0)}. \]

The sensitivity of the portfolio to monetary policy \( f(\bar{B}) \) depends on the size of the required transfer, and its sign depends on the covariance. As long as \( \mu \) is not strong enough when the covariance between \( e \) and \( T \) is positive (so that \( \bar{B} < -\mu^{-1} \)), one may derive similar implications for the actual portfolio. Note that in this case, markets are "locally complete", i.e., the planner replicates the desired transfers to first-order. This is no longer the case when markets are locally incomplete. The next lemma solves the optimal portfolio in closed-form for the general case.

**Lemma 1.3.** (Optimal policy portfolio) The optimal portfolio is given by

\[ B_{op} = -\frac{1}{2} \frac{\sigma_T^2 - \chi \sigma_{e^{dm}(0)}^2 - 2\chi \mu \sigma_{T^{dm}(0)} + \sqrt{(\sigma_T^2 - \chi \sigma_{e^{dm}(0)}^2)^2 + 4 \chi (\sigma_{T^{dm}(0)})^2}}{(1 - \mu^2 \chi) \sigma_{T^{dm}(0)} + \mu (\sigma_T^2 - \chi \sigma_{e^{dm}(0)}^2)}. \] (1.21)

Using the solution for the optimal portfolio, the next proposition derives comparative statics results.

**Proposition 1.3.** Suppose that either \( \mu \sigma_{T^{dm}(0)} > 0 \) or \( \mu \sigma_{T^{dm}(0)} < 0 \) and the following holds,

\[ (1 - \mu^2 \chi) \sigma_{T^{dm}(0)} + \mu (\sigma_T^2 - \chi \sigma_{e^{dm}(0)}^2) > 0. \] (1.22)

Then, under the optimal policy \( B_{op} \) satisfies \( B_{op} < -\mu^{-1} \), and:

(i) positions become larger (in absolute value) when the insurance motive becomes more important (i.e., when \( \sigma_T^2 / \sigma_{e^{dm}(0)}^2 \) increases or \( \chi \) decreases)

(ii) a decrease in the covariance between the insurance and the demand-management targets

- \( |\sigma_{T^{dm}(0)}| / \sigma_{e^{dm}(0)} \) - makes positions smaller (in absolute value) if and only if the demand-management motive is more important than the insurance motive, i.e., if \( \chi \sigma_{e^{dm}(0)}^2 > \sigma_T^2 \). Conversely, i.e., if \( \chi \sigma_{e^{dm}(0)}^2 < \sigma_T^2 \), it makes positions larger (in absolute value)

(iii) positions have the opposite sign of \( \sigma_{T^{dm}(0)} \):

(iv) positions become smaller (in absolute value) when \( m \) decreases

If condition (1.22) does not hold, the result still holds in terms of the “sensitivity to monetary policy”, i.e., \( f(B_{op}) = \frac{B_{op}}{1 + \mu B_{op}} \).

Proposition 1.3 shows how the planner resolves the trade-off when there are multiple shocks. The main result is that the sensitivity of the portfolio to monetary policy, which typically maps to a larger position,\(^{27}\) increases with the importance of insurance considerations. Thus, if the economy

\(^{27}\)The mapping from the sensitivity of monetary policy \( f(\bar{B}) \) to actual portfolios \( \bar{B} \) depends qualitatively on whether the condition (1.22) holds. The “reversal” case may only arise if the feedback effect \( \mu \) is strong, the correlation between \( e_{e^{dm}(0)} \) and \( T_s \) is small and negative, and shocks create a large demand for insurance. This case is especially unlikely to arise in the dynamic model, where the feedback effect \( \mu \) is significantly weaker (see Section 1.4).
faces shocks that create relatively large insurance needs \( \sigma_{T/\sigma_{e_{dm}(0)}} \), or creating transfers is not very costly (low \( \chi \)), gross positions are large (part i). This reflects that a more sizeable position reduces the cost of providing insurance ex post but increases the cost of using the exchange rate for demand-management. Furthermore, when correlation is imperfect the planner must prioritize one objective (part ii): a lower correlation pushes the planner towards larger positions (if insurance dominates) or smaller ones (if demand-management dominates). The alignment of the targets is reflected by part (iii): if the economy needs insurance when the exchange rate is weak \( \sigma_{T_{e_{dm}(0)}>0} \), then she borrows in home-currency \( B<0 \). Finally, part (iv) shows that when there is a small foreign-investor base in home-currency markets, the planner avoids large positions, which are very expensive.

Replacing the optimal portfolio in (1.20) completes the characterization of the optimal monetary policy. Using the full solution, the next corollary provides some comparative statics.

**Corollary 1.2.** The optimal insurance weight \( \omega(B_{op}) \) increases with the importance of the insurance motive (i.e., when \( \sigma_{T/\sigma_{e_{dm}(0)}} \) increases or \( \chi \) decreases). It also increases with the measure of foreigners \( m \).

A smaller \( \chi \) leads to a larger insurance weight overall, since it leads to both a larger insurance weight conditional on the portfolio (proposition 1.2), and a larger portfolio (proposition 1.3). This suggests that portfolio endogeneity may be important quantitatively, as it amplifies the effect of structural features that affect \( \chi \), such as price flexibility and risk-aversion. Furthermore, in contrast to economies without a portfolio problem, risks now matter to first-order for the optimal monetary policy through their effect on the portfolio. For example, if shocks create a large demand for insurance, positions are sizeable, and the ex post weight on the insurance target increases. Finally, note that considering the portfolio endogeneity is also key to get the right sign for some comparative statics. For example, while a smaller measure of foreigners leads to a larger weight on insurance for a fixed portfolio, insurance is less important once the portfolio adjusts. This last result is also useful to think about the cooperative solution. In Appendix A.2.1, I show that the solution under cooperation is isomorphic to the decentralized solution with twice as many foreigners. Thus, an immediate implication of corollary 1.2 is that cooperation increases the size of the portfolio and the weight on the insurance motive.

### 1.3.3 Implications for exchange rate volatility

It is often argued that using nominal assets to complete markets would imply excessive volatility in nominal quantities (Siu (2004)). In this section, I show that while this intuition is justified in an environment with exogenous portfolios, optimal portfolio choice may endogenously lead to environments with high leverage and low volatility in nominal quantities. To clarify the importance of portfolio endogeneity, I compare the baseline economy to an economy in which the planner cannot optimize its position because it is already at its upper bound, i.e., \( |\tilde{B}| = \tilde{K} \).

**Proposition 1.4.** (*Optimal exchange rate volatility*). Consider an economy with small risks \( \epsilon \rightarrow 0 \).
(i) Suppose the portfolio decision is constrained and, as a result, is unresponsive to marginal changes in risks or parameter values (i.e., \( |\tilde{B}| = \tilde{K} \)). Furthermore, assume \( \tilde{B}\mu > -1 \). Then, exchange rate volatility \( \sigma^2_e/\sigma^2_{edm(0)} \) increases with the importance of the insurance motive (i.e., \( \chi \) decreases or \( \sigma^2_T/\sigma^2_{edm(0)} \) increases, keeping \( \sigma_{Teedm(0)} \) constant)

(ii) Suppose \( \mu B > 0 \) and the optimum \( B \) is interior. Then, exchange rate volatility \( \sigma^2_e/\sigma^2_{edm(0)} \) decreases with the importance of the insurance motive (i.e., \( \chi \) decreases or \( \sigma^2_T/\sigma^2_{edm(0)} \) increases, keeping \( \sigma_{Teedm(0)} \) constant). If \( \mu B < 0 \), the result is ambiguous.

There are two main forces at play that shape the optimal degree of exchange rate volatility. First, there is a composition effect conditional on the portfolio. When the importance of insurance increases, the planner responds by increasing exchange rate volatility after shocks that create a demand for insurance. This is the standard channel considered in the closed-economy literature, which naturally leads to the conclusion that nominal quantities should become volatile. On the other hand, the planner exhibits more “fear-of-floating”: afraid of the transfers of wealth a freely-floating exchange-rate regime may create, she decides to dampen exchange rate volatility. This effect is typically absent in the closed-economy literature, where a constant inflation rate is taken as the benchmark for proper demand management. Despite these two opposing effects, part (i) of proposition 1.4 establishes that overall exchange rate volatility increases if the portfolio is not locally endogenous, i.e., if it is constrained at the current level \( B \). 28

Second, there is a novel effect through the portfolio. Consider first the case with \( \mu = 0 \). When insurance becomes more important, the planner chooses higher leverage. This implies that smaller exchange rate movements create larger transfers, which both lower the cost of providing insurance and increase the cost of letting the exchange rate float to stabilize aggregate demand. This force tends to lower exchange rate volatility. When \( \mu < 0 \) and \( B < 0 \), depreciations coincide with positive transfers, which dampens the volatility of the demand-management target, adding another force towards lower overall volatility. In contrast, if \( B > 0 \), depreciations coincide with negative transfers, which increase the volatility of the demand-management target. Part (ii) of proposition 1.4 states that as long as the feedback through \( \mu \) is not destabilizing (i.e., \( \mu \tilde{B} \leq 0 \)), the dampening effect through the portfolio is strong enough to overturn the composition effect described above.

1.3.4 Optimal capital controls

The previous sections have shown that portfolio choice endogeneity is key to alleviate the trade-offs involved in monetary policy design. A natural question is whether the private sector on its own would pick the right portfolio. Farhi and Werning (2016) showed that the answer is negative: Since there is imperfect stabilization of aggregate demand in some states of the world, and pecuniary externalities due to incomplete markets, agents do not properly internalize the value of a unit

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28 For the result with respect to \( \chi \) it is important that the portfolio is optimal (subject to the bound), and that it is in the “regular” region where wealth effects have a limited strength (\( \tilde{B}\mu > -1 \)). If the portfolio were entirely suboptimal this may not be true. For example, if there were only nontradable productivity shocks in the context of the example economy of section 1.3.5 and \( B \neq 0 \), volatility would decrease with the importance of insurance since only the dampening effect is present.
of tradable goods, which in turn implies they choose the wrong portfolio. The next proposition provides a qualification in the context of the present model.

**Proposition 1.5.** *(Asymptotic portfolio taxes)* Consider an economy with small risks \(\epsilon \to 0\). Then, optimal portfolio taxes \(\tau_B\) are given by

\[
\tau_B = \gamma_s m^{-1} \hat{B} \sigma_s^2 + O(\epsilon^3). 
\]  

(1.23)

When \(m \to \infty\), the only externalities in the economy are aggregate demand externalities (due to nominal rigidities) and pecuniary externalities (due to incomplete markets). Proposition 1.5 shows that in this case taxes are asymptotically of third-order. In other words, portfolio decisions are asymptotically efficient. What is the intuition behind this result? Even to first-order, private marginal utility and social marginal utility differ in every state. Indeed, in this environment, one can show that the gap between the social and the private marginal utility is proportional to the output gap, as agents do not internalize the effect of their choice on the planner's ability to create insurance. The key observation is that output gaps are themselves deviations generated with the sole intent and purpose of creating insurance. In other words, output gaps are proportional to social marginal utility under the optimal policy. As a result, private marginal utility and social marginal utility are proportional to one another and, since the planner sets the covariance of the exchange rate and social marginal utility to zero, the covariance of the exchange rate and private marginal utility is also zero. The key assumption that makes this result hold is the fact that *divine coincidence* holds in this model: If the planner wanted to she could always close the output gap and price dispersion at the same time. In contrast, if there were more than one output gap (i.e., an export and domestic sector), or there were markup shocks, taxes would be nonzero to second-order. This last effect, however, is unrelated to market incompleteness per se: In those environments the planner would like to put portfolio taxes even with complete markets. Furthermore, the result is not driven by the simple asset market structure with two bonds (I show it generalizes to much more general environments in Appendix A.2.3) nor by the absence of dynamics (I show it generalizes to a dynamic setting in Section 1.4).\(^{29}\) The only difference in those complicated environments is that there are more wedges between private and social marginal utility (i.e., in addition to the output gap). However, the value of each wedge is proportional to the value of the insurance they create (to first-order). As a result, they are all proportional to social marginal utility and the result, once again, follows. Note that the fact that *portfolio decisions* are asymptotically efficient does not mean *savings decisions* are, as I will show in Section 1.4.

As discussed in Section 1.2.2, the model has an additional rationale for taxes when \(m < \infty\). In this case, the home economy behaves as a monopolist in its home-currency bond market and has an incentive to distort the stochastic discount factor to increase its price. This implies taxing home-currency debt if \(\hat{B} < 0\) or taxing home-currency assets if \(\hat{B} > 0\). By curbing the demand of home agents, the planner manages to increase the value of the country's international investment

\(^{29}\)It also generalizes to a multi-asset dynamic model (see Appendix A.2.4).
position. Unlike aggregate demand externalities, which increase the size of the pie, this motive is clearly mercantilistic, i.e., it implies a redistribution from foreigners to home agents. In Appendix A.2.1, I show that this tax would be zero in the cooperative solution. Finally, note that this motive is also unrelated to market incompleteness, as discussed in Section 1.2.2.

1.3.5 An example

I illustrate the results of the previous section with a simple example. I set \( u = \ln(C_T^{\alpha}C_N^{1-\alpha} - \frac{\alpha}{1+\psi} L^1 + \psi) \) and \( F = Z_s Y_s \), where \( Z_s \) is a productivity shock. The parameter \( \alpha \) is an index of openness in this economy and I calibrate steady state values accordingly: \( Y_{Tss} = \alpha \) and \( Z_{ss} = 1 - \alpha \). Beyond standard productivity shocks, the example economy is also affected by endowment shocks \( \{Y_{Ts}\} \), which capture in reduced form innovations to the “terms-of-trade” of a country, i.e., shocks to the world prices of the country’s exports. These shocks are often considered as one of the key drivers of business cycles in open economies (Fernández, Schmitt-Grohé and Uribe (2017)). In the quantitative model in Section 1.5 I also consider other shocks that have been stressed by the literature, such as interest-rate shocks and shocks to the uncovered interest rate parity (UIP) condition. To make the figures, I set \( \alpha = 0.55, \psi = 2, \eta = 6, \gamma^* = 1, \) and \( m = \infty \).

Optimal monetary policy and optimal portfolio

Inner problem I begin by computing the exchange rate targets. Figure 1-1 shows the response of the economy after a positive nontradable productivity shock (left panel) and a positive endowment shock (right panel) under a demand-management policy (solid-blue line), an insurance policy (dashed-red), the optimal policy when prices are relatively flexible \( \phi = 0.1 \) (dotted-dashed-yellow), and the optimal policy when prices are relatively rigid \( \phi = 0.9 \) (dotted-purple line) for different home-currency positions. I set \( \alpha = 0.55, \psi = 1, \eta = 6, \gamma^* = 1, \) and \( m = \infty \).
shock (right panel). Consider first the insurance target (dashed-red line). A simplifying feature of this parametrization is that in equilibrium agents want to consume the same amount of tradables regardless of their consumption of nontradables.\(^\text{30}\) As a result, the planner does not desire any transfer of tradables after \(Z\) shocks. In contrast, because they are risk-averse, home agents dislike volatility in their tradable consumption. Hence, when the endowment is low, the planner depreciates if home agents have home-currency debt and vice versa,

\[
\tau_s = -\frac{\alpha}{1 + 2m^{-1}\tilde{\gamma}^{-1}\gamma^*}yTs + O(\epsilon^2) = \epsilon_s^\text{in}(\tilde{B}) = \left(\frac{\alpha}{1 + 2m^{-1}\tilde{\gamma}^{-1}\gamma^*}\right)\frac{1}{\tilde{B}}yTs + O(\epsilon^2) \tag{1.24}
\]

where \(\tilde{\gamma} = 1/\alpha\), which is absolute risk aversion in the tradable good. Away from unitary elasticity between tradable and nontradables, \(\tilde{\gamma}\) would depend on the elasticity of substitution, increasing with the degree of complementarity.

Next, consider the demand-management target (solid-blue line) and suppose \(\tilde{B} = 0\). Under flexible prices, a positive nontradable productivity shock would require an increase in employment and a decrease in the foreign-currency price of intermediate inputs. To replicate this price movement, the planner needs to depreciate the exchange rate. When \(\tilde{B} \neq 0\), this is not the end of the story: the depreciation triggers a negative transfer of tradables (if \(\tilde{B} > 0\)). Agents would then be more poor, which implies that replicating the flexible price allocation requires a further depreciation of the currency. In other words, \(\tilde{B} > 0\) triggers a destabilizing wealth effect. Conversely, when \(\tilde{B} < 0\), this wealth effect is stabilizing. This effect is captured by \(\mu\),

\[
\mu = -\frac{1}{\alpha} \frac{\psi}{\psi + \alpha}.
\]

Its strength depends on openness (a more open economy faces a smaller proportional change in tradable consumption), and on the disutility of labor (if it were linear then increasing production would not require higher intermediate input prices). Note that, since an endowment shock is in fact a wealth transfer, its effect on the demand-management target is also given by \(\mu\). In sum, the demand-management target is given by

\[
e_s^\text{dm}(\tilde{B}) = \frac{1}{1 - \frac{1}{\alpha} \frac{\psi}{\psi + \alpha} \tilde{B}} \left(\frac{1 - \alpha}{\psi + \alpha} z_s - \frac{\psi}{\psi + \alpha} yTs\right) + O(\epsilon^2) \tag{1.25}
\]

To complete the characterization of the inner problem, I now compute the insurance weight \(\omega(\tilde{B})\) described in Section 1.3.2. To do so, I first need to compute the parameter \(\chi\),

\[
\chi = \frac{1}{1 + 2m^{-1}\tilde{\gamma}^{-1}\gamma^*} \frac{\alpha \phi (1 - \alpha)(\alpha + \psi)(\phi + (\alpha + \psi)(1 - \phi)\eta)}{(\alpha + \psi + (1 - \alpha - \psi)\phi)^2},
\]

which is monotonic in \(\phi\) if \(\eta < 2\) and hump-shaped otherwise with a critical threshold of \(\tilde{\phi} = \frac{(\alpha + \psi)\eta}{(1 + \alpha + \psi)\eta - 2}\). Furthermore, note that \(\chi\) becomes zero when the economy is completely closed or

\(^{30}\)This case is usually studied with separable utility. It can be shown that with the proposed steady state parametrization the result is also true for the Greenwood–Hercowitz–Huffman (GHH) utility I specified.
open \((\alpha \in \{0, 1\})\). Since I assumed production in the tradable sector was not affected by the price rigidity, the output gap becomes irrelevant when \(\alpha = 1\). On the other hand, when \(\alpha \to 0\) exchange rate movements create transfers which, in proportion to the steady state level of tradable consumption, are very large. Note this does not imply the insurance weight goes to 1 under the optimal policy, since desired transfers also scale with openness so the portfolio is also affected. Figure 1-1 computes the optimal exchange rate conditional on a portfolio \(\tilde{\mu}\) for a case relatively flexible \((\phi = 0.1\), dotted-dashed-yellow line) and relatively sticky \((\phi = 0.9\), dotted-purple line) prices. As expected from proposition 1.2, the optimal policy lies closer to the demand-management target when positions are closer to zero and when the country issues home-currency debt, due to the endogenous feedback effect through \(\mu\). Note that in the depicted example, the “low” \(\phi\) policy is sufficiently low such that high price flexibility actually implies a larger weight on the insurance motive.

**Outer problem** To find the optimal portfolio, I need to compute the covariance structure of the targets in an economy without home bonds. This yields,

\[
\begin{align*}
\sigma^2_T &= \frac{\alpha}{1 + m^{-1} - 1} \sigma^2_y \\
\sigma^2_{\epsilon_{\text{em}(0)}} &= \frac{1}{(\psi + \alpha)^2}((1 - \alpha)^2 \sigma^2_Z + \psi^2 \sigma^2_y - (1 - \alpha)\psi \sigma_{yz}) \\
\sigma_{\tau_{\text{em}(0)}} &= \frac{1}{\psi + \alpha} \frac{1}{1 + 2m^{-1} - 1} (\psi \alpha \sigma^2_y - \alpha (1 - \alpha) \sigma_{yz}).
\end{align*}
\]
First, consider the case of uncorrelated shocks ($\sigma_{yz} = 0$). Panel (a) in Figure 1-2 illustrates how the optimal portfolio (x axis) varies as I increase the standard deviation of $y$ from 0% to 10% (y axis) while keeping the volatility of nontradable productivity shocks at 10%. I consider the cases with perfect integration in home-currency-bond markets ($m = \infty$, red and blue lines) and with limited participation ($m = 1$, green and black lines). Since the exchange rate depreciates when the economy has a low endowment of tradables ($\sigma_T e^{dm}(0) > 0$), the planner chooses a short position in home-currency bonds. The larger the endowment shocks, the shorter the position. This allows the planner to provide insurance more cheaply, i.e., by using smaller exchange rate movements. On the other hand, a smaller home-currency market (low $m$) induces the planner to choose smaller positions and insure less overall to avoid paying a large premium on its debt. In either case, the approximation (red and black dotted lines) is very close to the true solution in this model (blue and green lines).

Next, suppose $y$ and $z$ are correlated. When they are perfectly negatively correlated, the exchange rate depreciates when the tradable endowment is low. Thus, the optimal portfolio is short home-currency bonds. In contrast, when they are perfectly positively correlated, there are opposing effects on the exchange rate. I assume $y$ shocks are four times less volatile than $z$ shocks ($\sigma_y = 2.5\%$), which implies the exchange rate depreciates when the tradable endowment is high. As a result, the optimal portfolio is long home-currency bonds. When correlation is imperfect, the response of the optimal portfolio depends on the relative importance of the insurance vis-a-vis the demand-management motive. Figure 1-2 shows how the optimal portfolio varies as I increase the correlation between $y$ and $z$ shocks from $-1$ to $1$ in a case with low flexibility ($\phi = 0.9$, green and black lines) and in a case of high flexibility ($\phi = 0.1$, red and blue lines). To make the plot easier to interpret, I show the implied correlation between the desired transfers $T$ and the demand-management exchange rate $e^{dm}(0)$ in the y axis rather than the correlation between fundamentals (they are one-to-one in the example). When flexibility is low (red and blue lines), the parametrization lies in the region where the demand-management objective dominates ($\chi \sigma_{e^{dm}(0)}^2 > \sigma_T^2$) and the planner decides to reduce leverage. In contrast, when flexibility is high (green and black lines), the parametrization lies in the region where the insurance objective dominates ($\chi \sigma_{e^{dm}(0)}^2 < \sigma_T^2$), and the planner increases leverage to reduce the cost of providing insurance.
Figure 1-3: Exchange rate volatility. I decompose the total standard deviation of the exchange rate (solid-blue line) into the contribution of $y$ shocks (red-dashed line) and $z$ shocks (dotted-yellow line) as the parameter $\chi$ varies. Shocks are uncorrelated. On the left, the portfolio is constrained at some level $\hat{K}$, which is suboptimally low for the plotted values $\chi$. On the right, the portfolio is at its optimal value. The standard deviation is in percentage points. These figures were computed using the approximation. Results are very similar in the nonlinear model.

Implications for exchange rate volatility

Figure 1-3 illustrates how exchange rate volatility changes with the importance of insurance ($\chi$) in the example economy (I keep $\sigma_z = 10\%$ and $\sigma_y = 2.5\%$). In panel (a), I keep the portfolio fixed at a level that would be optimal with the highest plotted $\chi$ ($\approx 20$). This case illustrates the composition effect: when the planner places a higher weight on the insurance target, it dampens the response after nontradable productivity shocks (dotted-yellow line) and takes a more active stance against endowment shocks (dashed-red line). Overall volatility (solid-blue line) increases, driven by the large response to endowment shocks. Panel (b) plots the response when the portfolio decision is unconstrained. Nontradable productivity shocks are now dampened even further, due to the larger portfolio. For endowment shocks, the effect is now ambiguous: Since the portfolio is larger, smaller exchange-rate movements are necessary to achieve the same amount of insurance and, in addition, the demand-management target becomes more stable. In this parametrization, I find a hump-shaped response, with volatility due to $y$ shocks increasing when portfolios are relatively

31Note that the volatility level only matters for the nonlinear model; for the approximation only relative volatilities matter.

32I observe the largest difference when the model yields large positions, since the objective function then becomes very flat. Even then, the difference is small; the approximation yields $B = -1.92$ vs $B = -1.89$ in the nonlinear model. I found similar results for other parameter values.

33When the correlation becomes zero, the problem becomes bang-bang. When demand-management dominates, the planner prefers to remain in autarky ($\hat{B} = 0$). In contrast, when insurance dominates the planner chooses the other corner, which is given by $\hat{B} = -\mu^{-1}$. More rigorously, I approach this value as the correlation goes to 0, since when $\hat{B} = \mu^{-1}$, the equilibrium is only well-defined if the exchange rate is fixed. Furthermore, when correlation is low and positive, the economy is in a “reversal” region not covered by proposition 1-2 and $\hat{B} > -\mu^{-1}$, which in turn implies demand-management responses change signs.
small and decreasing thereafter. Furthermore, since $\mu \hat{B} \geq 0$, overall volatility decreases with the strength of the insurance motive.

**Optimal capital controls**

Figure 1-4 plots the optimal tax in the example economy when there is perfect integration ($m = \infty$) and when there is limited participation ($m = 1$). I keep the covariance structure as in the previous section and vary the overall amount of risk $\epsilon$ in the economy from 0 to 20%. I compare the full nonlinear solution (solid-blue line) with the approximated solution (dashed-red line) given by equation (1.23). When $\epsilon \to 0$, the tax becomes zero in every case, since assets are perfect substitutes in the limit. When $m = \infty$, the approximation predicts a zero tax while the nonlinear model predicts a very small subsidy in home current assets. In contrast, when $m = 1$, the implied subsidy is two orders of magnitude larger. Finally, note that the approximation tracks very closely the true nonlinear solution.

### 1.4 Dynamic model

In this section, I generalize the model to a dynamic setting. I present the model in Section 2.6.1. Section 1.4.2 characterizes the optimal policy and is divided into two parts. Section 1.4.2 establishes the robustness of all the results in terms of the excess returns of home-currency bonds. Section 1.4.2 studies the implementation of these excess returns in terms of monetary policy and savings taxes, which allows me to deliver new results. Appendix A.2.4 shows the robustness of the results.

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34 The underlying portfolio is $\hat{B} = -0.6$. 
in an extension with multiple assets that load on endogenous variables and shocks.

1.4.1 Setup

Home agents solve

\[
\max_{E_1} \mathbb{E}_{-1}\{\sum_{t=0}^{\infty} \beta^t u(C_{t+1}, C_{Nt}, L_t; \xi_t)\}
\]  

(1.26)

subject to

\[
C_{t+1} + E_t^{-1} P_{t+1} C_{Nt} + (1 + \tau_t^*) NF A_t + \tau_t B_t = Y_{t+1} + E_t^{-1} \Pi_{t+1} + E_t^{-1} \Pi_{Nt} + (1 + \tau_L) W_t L_t
\]

(1.27)

\[
+ R_{t-1}^* NFA_{t-1} + RR_t B_{t-1} + T_t
\]

where \(NFA_t = B_t^* + B_t\) is the net foreign asset position of the country (with both asset positions measured in foreign currency), \(\tau_t^*\) is a savings tax, and \(RR_t\) are the realized excess returns of the home-currency bond. I allow home-currency bonds to have a long duration by assuming they pay an initial coupon of \(\delta\) units of home currency that decays at rate \(1 - \delta\), as in Hatchondo and Martinez (2009). Excess returns on home-currency bonds are given by

\[
RR_t = \{(1 + \psi_{t-1}) R_{t-1} (\delta E_t^{-1} E_{t-1} + (1 - \delta) E_t^{-1} E_{t-1} R_{t-1}^{-1}) - R_{t-1}^* \}
\]

(1.28)

where \(\psi\) is a shock to the liquidity service of home-currency bonds, similar to Lahiri and Végh (2003).\(^{35}\) This shock introduces noise in the return of home-currency bonds, and is meant to capture in reduced form disturbances to the arbitrage between bonds in the spirit of portfolio balance models la Kouri (1976). It will play an important role in the quantitative section, where they will allow the model to match the volatility of the nominal exchange rate. I assume these shocks are symmetric across agents, i.e., they do not reflect heterogeneous beliefs.\(^{36}\) Furthermore, note that while this shock is, indeed, an uncovered interest rate parity (UIP) shock, it is different from the traditional one considered in the literature with a single foreign-currency bond (see, for example, Kollmann (2001)), which is captured in the framework by an interest-rate shock \(R_t^*\). Crucially, the \(R_t^*\) shock affects the savings decision while the \(\psi_t\) shock affects the portfolio decision.

I assume foreign agents that access home-currency bond markets are marginally indifferent between saving and consuming if they do not participate,

\[
u^*(Y_{t+1}^*) = \beta^* R_t^* \mathbb{E}_{t+1} u^*(Y_{t+1}^*),
\]

(1.28)

where \(R^*\) is unaffected by the decision of participating foreigners.\(^{37}\) Optimization yields the

\(^{35}\) Equivalently, I could have introduced a taste shock on the holdings of home bonds in the utility function (with a normalization such that taste shocks are still symmetric across agents).

\(^{36}\) The framework cannot accommodate first-order differences in beliefs, since then positions would become unbounded as \(\epsilon \to 0\).

\(^{37}\) For this reason, the case with \(m \to 0\) is no longer be interpreted as a large economy. Relaxing this assumption would imply the economy has an additional terms-of-trade manipulation motive as it would try to influence the world interest rate in its favor, exactly as in Costinot, Lorenzoni and Werning (2014).
no-arbitrage condition,

$$\mathbb{E}_{t-1} RR_t u'(Y_t^* + R_{t-1}^* B^{f*}_{t-1} - B_t^* - RR_t m^{-1} B_{t-1}) = 0$$

where $B^{f*}$ is the savings decision of foreigners participating in home-currency markets.

Finally, staggered price-setting is modeled by making the identity of the $1 - \phi$ share of reoptimizing firms stochastic (i.e., Calvo price-setting). The rest of the model is the same as before.

1.4.2 Optimal policy

**Characterization in terms of excess returns**

The key economic quantity to characterize the optimal policy with long bonds is the realized *excess return* of home-currency bonds. This is not a property of the dynamic model per se; rather, it is a consequence of breaking the perfect correlation between excess returns and contemporaneous exchange rate movements that arises with short-bonds.\(^{38}\) Define, by analogy to the static model, the demand-management target as the excess return that is consistent with a zero output gap and no savings taxes,

$$rr_{t}^{dm}(\hat{B}_{t-1}) = \frac{1}{1 + \mu \hat{B}_{t-1} rrm(0)} + O(\epsilon^2), \quad (1.29)$$

where $rr_{t}^{dm}(0)$ is the excess return in an economy without home-currency bonds but with foreign-currency bonds. Similarly, define the insurance excess return as the one that creates the transfer of goods the planner would desire if prices were flexible ($\mathcal{T}_t$),

$$rr_{t}^{in}(\hat{B}_{t-1}) = \frac{1}{\hat{B}_{t-1}} T_t + O(\epsilon^2). \quad (1.30)$$

The next lemma shows that welfare can still be written as a function of deviations from these two targets.

**Lemma 1.4.** *In the limit $\epsilon \to 0$, the optimal excess returns of home-currency bonds at $t$ and the optimal portfolio at $t - 1$ solve*

\[
\mathcal{W} = \max_{\{rr_t, \hat{B}_{t-1}\}} -k_0 \mathbb{E}_{t-1}\left\{ \frac{1}{2}\hat{B}_{t-1}^2 (rr_t - rr_t^{in}(\hat{B}_{t-1}))^2 + \frac{1}{2} \chi (1 + \mu \hat{B}_{t-1})^2 (rr_t - rr_t^{dm}(\hat{B}_{t-1}))^2 \right\} + \text{t.i.p.} + O(\epsilon^3) \quad (1.31)
\]

*The objects $rr_{t}^{dm}(0)$, $T_t$, $k_0$, $\mu$ and $\chi$ are specified in Appendix A.1.2.*

Lemma 1.4 shows that the problem of finding the optimal first-order behavior of the excess return of home-currency bonds and the steady-state optimal portfolio is essentially a static problem that is isomorphic to the one in Section 1.3, stated in terms of the excess returns of home-currency

---

\(^{38}\)In Appendix A.2.2, I show an example in the context of the static model. In that example, agents can trade equity in the nontradable sector. There, too, the important economic quantity is the realized excess return.
bonds, rather than the exchange rate. The portfolio problem is separable over time: future portfolios do not affect the path of expected variables to first-order. This implies that it is without loss of generality to shut-down uncertainty from \( t+1 \) onwards (a manifestation of certainty equivalence) to study the optimal distribution of endogenous variables at \( t \) and the portfolio chosen at \( t-1 \).

Lemma 1.4 immediately implies that the following results from Section 1.4.2 generalize to the dynamic environment: (i) the optimal excess returns are a weighted average of the targets implied by the two motives - equations (1.29) and (1.30), (ii) the optimal portfolio satisfies an equation analogous to (1.21), and, (iii) the volatility of excess returns decreases with the importance of insurance if the portfolio is endogenous but it increases if it is exogenous (when \( \mu B \geq 0 \)). Furthermore, proposition 1.3.4 shows the optimal portfolio tax result also generalizes to the dynamic model. In other words, no portfolio taxes are necessary in the absence of terms-of-trade manipulation (\( m = \infty \)), or with cooperation. The intuition is the same as before: excess return deviations are proportional to the value of insurance.\(^{39}\)

**Proposition 1.6.** Propositions 1.2, 1.3, and 1.4 generalize to the dynamic environment in terms of excess returns \( r_{rs} \). (the static model is a special case after using \( r_{rs} = -e_s \)). The portfolio tax formula generalizes to \(^{40}\)

\[
\tau_B = (1 - \beta)\gamma^*_s m^{-1} B E_{t-1} r_{rs}^2 + O(\varepsilon^3).
\]

Finally, note that while the reduced form representation is the same, \( \chi \) is now a more complicated object. Whereas in the static model there is a single way of creating an excess return, in the dynamic model the planner has more tools, such as promises of future exchange rate movements. \( \chi \) now contains the information on how to choose among the many instruments the planner has to deliver any given excess returns \( r_{rs} \) at a minimum cost, a problem I study in Section 1.4.2. Another important difference with the static model is that now agents only consume a share \( 1 - \beta \) of any additional transfer. This has two important consequences. First, the wealth effect on the demand-management target is now much weaker: \( \mu \) is now multiplied by \( \beta^{-1}(1 - \beta) \). Second, it significantly reduces the impact of a transfer on marginal utility. This is clearly seen in the extreme with rigid prices (\( \phi = 1 \)) and short-bonds. In this case, \( \chi \) is divided by \( \beta^{-1}(1 - \beta) \). Naturally, to the extent that shocks may be close to permanent, insurance desirability (\( T \)) also increases, which offsets the smaller effect of a transfer by increasing the size of the desired transfer and the optimal portfolio.

**Remark 1.2.** The feedback effect from transfers to equilibrium demand-management exchange rates is much weaker: \( \mu_{\text{dynamic}} = \beta^{-1}(1 - \beta)\mu_{\text{static}} \). In addition, if prices are rigid \( \phi = 1 \) and bonds are short, \( \chi_{\text{dynamic}} = \beta(1 - \beta)^{-1}\chi_{\text{static}} \).\(^{41}\)

\(^{39}\)In Appendix A.2.4 I show that all these results also generalize to an economy with one asset sensitive to monetary policy and multiple exogenous assets. If there are multiple assets that are sensitive to monetary policy, I can show that portfolio taxes are still zero if \( m = \infty \) and monetary policy also has a similar weighted average representation. However, unlike the static model, the portfolio can no longer be solved in closed form and propositions 1.3 and 1.4 do not seem to carry over. A simple algorithm is provided to solve this case.\(^{40}\) The \( 1 - \beta \) reflects the fact that foreigners can also borrow in their own currency, smoothing bad shocks. If they lived for only period, then the formula would hold with a 1 instead of \( 1 - \beta \), exactly the same as in the static model.\(^{41}\) If I had a share \( \phi \) of "flexible" firms and the rest setting prices one period in advance, then this result would be
Implementation of excess returns

In this section, I study the “continuation” problem, which involves finding the optimal value of endogenous variables at $t$ and the path for expected endogenous variables from $t + 1$ onwards after the state of the world at $t$ is realized. In other words, I take the promised excess return $rr_0$ as given and ask: What is the optimal path that delivers this excess return? By certainty equivalence, this is a deterministic problem. The answer to this question pins down the cost of replicating $rr_0$, i.e., it pins down $\chi$. I first describe the optimal path for monetary policy and then the implications for the path of savings taxes.

Monetary Policy  The implementation in terms of monetary policy is now more complicated: the planner needs to specify a full path for the exchange rate.$^{42}$ To streamline the discussion, consider the response of the economy after an innovation at time $t = 0$ and suppose the economy was at a steady-state at $-1$. Combining the firms’ first-order condition and consumers’ demand for nontradables I obtain:

$$
\Delta e_t = \pi_{It} + \beta(1 - \beta)^{-1} \mu \Delta c_{Tt} + k_c \Delta \xi_t + k_\text{ex} \Delta x_t + O(\epsilon^2)
$$

(1.32)

where $x_t$ is the output gap, $\pi_{It}$ is intermediate-input inflation, and $k_\text{ex}$ and $k_c'$ are constants ($k_c'$ is a vector). To find the demand-management exchange rate target, set $x_t = \pi_{It} = 0$ and plug in the solution for $c_{Tt}$ (from the consumer’s Euler and the budget constraint). This yields a path $\{e_t^\text{dm}(\vec{B})\}$. By contrast, the insurance exchange target consistent with $rr^\text{in}_0$ is not uniquely defined if bonds are long. Indeed, any $\{\Delta e_t\}$ path that satisfies

$$
\beta rr^\text{in}_0 = -\epsilon_0 - \sum_{t=1}^{\infty} \beta^t (1 - \delta)^t (\Delta e_t + \pi_{It-1} + \psi_{t-1}) + O(\epsilon^2)
$$

is admissible, as long as one can find a path $\{x_t, \pi_{It}, c_{Tt}\}$ that satisfies equation (1.32), the budget constraint, a transversality condition for bonds, and the dynamic Phillips curve

$$
\pi_{It} = \kappa x_t + \beta E_t \pi_{It+1} + O(\epsilon^2)
$$

where $\kappa > 0$ is a constant. Among these many potential paths, I define the insurance exchange rate target as the path $\{e_t^\text{in}\}$ that delivers $rr^\text{in}_0$ at the minimum cost. This implies that, to find such path, the planner needs to solve a cost-minimization problem, choosing the optimal paths of $\{c_{Tt}, x_t, \pi_{It}, e_t\}$ that minimize consumption distortions and production inefficiency. To characterize the solution to this problem it will be convenient to define the costly exchange rate deviations $\{\tilde{e}_t\}$ still hold with short bonds.

$^{42}$Since the trilemma holds in the economy, monetary policy can also be restated in terms of an interest rate path and a long-run level of the exchange rate.
associated with an exchange rate path \(\{e_t\}\) and an excess return \(rr_0\) as:

\[
\tilde{e}_t(B) = e_t(B) - \{e_t^{dm}(0) + \mu \tilde{B} rr_0\}.
\]

Whenever \(\tilde{e} \neq 0\), the planner must either distort consumption (i.e., put a savings tax) or production (i.e., have a nonzero output gap). Naturally, the demand-management policy has no deviations: \(e_t^{dm} \equiv 0\). In contrast, when \(\tilde{B} rr_0^{dm} \neq \mathcal{T}_0\), the planner needs to create deviations \((\tilde{e}_t \neq 0)\) to provide the desired insurance. The next proposition characterizes the optimal behavior of these deviations.

**Proposition 1.7. (Optimal monetary policy II)** Let \(\tilde{e}_t(B) = e_t(B) - \{e_t^{dm}(0) + \mu \tilde{B} rr_0\}\) denote the costly exchange rate deviations.

1. If prices are rigid, then:

\[
\tilde{e}_t = -\tilde{k}_{e1}(1 - \delta^t)\{(1 + \mu \tilde{B})rr_0 - rr_0^{dm}(0)\} + \tilde{k}_{e2}\{(1 + \mu \tilde{B})rr_0 - rr_0^{dm}(0)\} + O(e^2)
\]

where \(\tilde{k}_{e1} > 0\) and \(\tilde{k}_{e2} \geq 0\). Furthermore, \(\text{sign}(\tilde{e}_0) = \text{sign}(-\{(1 + \mu \tilde{B})rr_0 - rr_0^{dm}(0)\})\). If \(\mu = 0\), then \(\tilde{k}_{e2} = 0\).

2. Suppose \(k_{ex} > 0\) (expansionary devaluations) and suppose inflation is relatively more costly than the output gap: \(\kappa \lambda_x k_{ex} - \lambda_x > 0\), where \(\lambda_x\) and \(\lambda_x\) are the costs of (squared) output gaps and inflation, respectively.\(^{43}\) If bonds are short, then:

\[
\Delta \tilde{e}_t = \mathcal{P}(k_{ex}) R_x^{-1}\{(1 + \mu \tilde{B})rr_0 - rr_0^{dm}(0)\} + O(e^2) \quad \forall t \geq 2
\]

\[
\Delta \tilde{e}_1 = \tilde{k}_{e3}\{(1 + \mu \tilde{B})rr_0 - rr_0^{dm}(0)\} + O(e^2)
\]

\[
\tilde{e}_0 = -\tilde{k}_{e4}\{(1 + \mu \tilde{B})rr_0 - rr_0^{dm}(0)\} + O(e^2)
\]

where \(R_x\) is the optimal decay rate of inflation, \(\tilde{k}_{e3} > 0\) and \(\tilde{k}_{e4} > 0\) are constants and \(\mathcal{P}(k_{ex})\) is a continuous and monotonic function of \(k_{ex}\) with \(\mathcal{P}(0) > 0\) and \(\lim_{k_{ex} \to \infty} \mathcal{P}(k_{ex}) < 0\).

3. In either of the cases above, setting \(rr_0 = rr_0^{dm}\) yields the deviations corresponding to the insurance target. Furthermore, once the targets have been defined, the optimal exchange rate has the same weighted average representation as before (even when \(\delta < 1\) and \(\phi < 1\)), i.e.,

\[
e_t^{op} = \frac{B^2}{\chi(1 + \mu \tilde{B})^2} e_t^{in}(B) + \frac{\chi(1 + \mu \tilde{B})^2}{B^2} e_t^{dm}(\tilde{B}) + O(e^2)
\]

Proposition 1.7 shows that the optimal exchange rate has the same weighted average representation as excess returns if one defines the insurance target as the one that minimizes the cost of providing the transfer. More interestingly, it characterizes the deviations \(\{\tilde{e}_t\}\) in two special cases: with rigid intermediate-input prices, and with short bonds. When prices are rigid, the planner spreads the adjustment in proportion to the amount of debt maturing in each period: If she wants to deliver a positive excess return, she creates a persistent overvaluation of the exchange rate, which

---

\(^{43}\)If \(u\) is GHH or separable, the composite between tradables and nontradables is CES, and \(F = Y_{t+1}^{\alpha_F}\), then \(k_{ex} = \frac{\alpha_F}{1 - \alpha_F + \phi - \gamma \alpha_F} > 0\) so devaluations are expansionary.
decays at the same rate as bonds if $\mu = 0$. If $\mu < 0$ the planner brings consumption forward, which implies the exchange rate eventually settles at a more depreciated level. In other words, the exchange rate appreciates on impact and then slowly returns back to its demand-management level (given the new long-run net foreign asset position). When bonds are short, the contemporaneous exchange rate is immediately determined by the desire to create an excess return. When devaluations are expansionary, the planner induces a recession and deflation today, followed by a boom and inflation tomorrow.44 This immediately implies an appreciation at $t = 0$ followed by a depreciation at $t = 1$. After $t = 1$, the boom and inflation slowly subside, with opposing implications on the exchange rate path. Thus, depending on the sensitivity of the output gap to exchange rate movements, the $t = 1$ exchange rate may overshoot (if $k_{ex}$ is large) or converge monotonically (if $k_{ex}$ is small) to its new long-run level. While the general case can also be written in closed form, it is easy to see from these examples how a variety of cases may arise.

**Savings taxes** Proposition 1.8 characterizes savings taxes in the cases with either rigid prices or short bonds.

**Proposition 1.8.** (Savings taxes) Suppose the planner wants to create an excess return of $rr_0$ at $t = 0$ and suppose there are no further shocks. Then:

1. If prices are rigid ($\phi = 1$), savings taxes decay at rate $1 - \delta$

$$\tau_{B^t} = -\tilde{K}_0(\delta)(\delta\mu - \tilde{K}_1k_{ux}^{-1}k_{ux})(1 - \delta)^{(1 + \mu\bar{B})rr_0 - rr_0^{dm}(0)} \tag{1.33}$$

where $\tilde{K}_0 > 0$, $\tilde{K}_1 > 0$ are constants, $k_{ux}$ captures the reaction of private marginal utility to the output gap ($k_{ux} > 0$ implies agents overvalue tradable goods in booms).45 When $\delta = 0$, $\tilde{K}_0(\delta) = 0$.

2. If bonds are short ($\phi = 1$), then savings taxes from $t \geq 1$ are given by

$$\tau_{B^t} = -\tilde{k}_1k_{ux}R_{t-1}R_1$$

where $R_1$ is the optimal decay rate of inflation after $t = 1$ and $\tilde{k}_1 > 0$. At $t = 0$,

$$\tau_{B^0} = -\tilde{k}_2\mu(1 + \beta^{-1}(1 - \beta)k_{ex}rr_0 - rr_0^{dm}(0)) + k_{ux}\Delta x_1$$

where $\tilde{k}_2 > 0$. If $k_{ex} > 0$ and $\kappa\lambda_{ex}k_{ex} - \lambda_x > 0$, then $\Delta x_1 > 0$ and $\pi_1 > 0$.

Like the exchange rate, savings taxes decay at the same rate as bonds when prices are rigid. However, since savings taxes are only useful to the extent that the adjustment is unevenly split across periods, the level of the tax is very small if debt is very long. Indeed, if debt is a perpetuity, the optimal tax is zero. The two terms inside the first parenthesis in equation (1.33) reflect the two

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44The latter depends on inflation being relatively more costly than the output gap. (Otherwise the planner may promise deflation in the continuation, see Appendix A.1.2)

45If $u$ is GHH, the composite between tradables and nontradable is CES, and $F = Y_{is}^{op}$, then $k_{ux} = \frac{1 - a}{\rho} > 0$. If $u$ is separable, $k_{ux} = (\gamma - \rho^{-1})(1 - \alpha)\alpha_F$, so it depends on whether tradables and nontradables are Edgeworth complements or substitutes (i.e., if $\gamma \rho > 1$, they are substitutes and $k_{ux} > 0$).
reasons why savings taxes are useful. The first term reflects that when $\mu \neq 0$ manipulating tradable consumption allows the planner to move the exchange rate at no cost in terms of production efficiency. For example, suppose the planner wants to create a positive excess return on home-currency bonds. Then, if $\mu < 0$, the planner taxes savings, which boosts consumption and appreciates the exchange rate. The second term reflects that agents do not value tradable goods properly. Although state-by-state this valuation mistake is proportional to the value of insurance if $m = \infty$ and, hence, no portfolio taxes are required, the strength of the externality varies across periods. If agents undervalue tradable goods in recessions ($k_{ux} > 0$), the planner has an incentive to boost savings in booms and tax them in recessions. In the example discussed above, where the planner creates a persistent yet declining overvaluation, the planner has an additional incentive to tax savings.

With staggered prices and short bonds, the only incentive to tax savings after $t = 1$ is the aggregate demand externality. Consider the case with $k_{ux} > 0$ and suppose the planner wants to increase the return on home assets. Recall from the earlier discussion this induced a recession at $t = 0$ followed by a boom from $t = 1$ onwards. When agents undervalue tradables in recessions ($k_{ux} > 0$), the planner wants to tax savings at $t = 0$ and subsidize them at $t = 1$ to correct the aggregate demand externality. Furthermore, at $t = 0$ she also wants to tax savings to increase consumption, which has the added benefit of appreciating the exchange rate (if $\mu < 0$).

1.5 Quantitative Analysis

In this section, I evaluate the quantitative importance of the theoretical results presented in Sections 1.3 and 1.4. It is divided into three parts. Section 2.4.1 briefly describes the calibration strategy. Section 1.5.2 presents the implications of the optimal policy for the statistics that played a key role in the analysis and welfare. As a benchmark, I also present the results under a demand-management-targeting policy, which is equivalent to strict (intermediate-input) inflation targeting. Section 1.5.3 studies the sensitivity of the results to the presence of liquidity shocks and the cost of inflation, and shows that they play an important role.

Appendix A.3 provides additional sensitivity analysis and shows how the results change when savings taxes are not available.46
Table 1.1: Parameter values and shocks

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
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<tr>
<td>A. Structural parameters</td>
<td></td>
<td>A. Structural parameters</td>
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<td></td>
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<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>0.99</td>
<td>$\phi$</td>
<td>Probability of not adjusting prices</td>
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<td>$\gamma$</td>
<td>Home risk aversion</td>
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<td>$\eta$</td>
<td>Elasticity of substitution (varieties)</td>
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<tr>
<td>$\gamma^*$</td>
<td>Foreign risk aversion</td>
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<td>$\eta$</td>
<td>Bond depreciation</td>
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</tr>
<tr>
<td>$\alpha$</td>
<td>Tradable share</td>
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<td>$m$</td>
<td>Measure of foreigners</td>
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</tr>
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<td>$\nu^{-1}$</td>
<td>Frisch elasticity</td>
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<td>$\phi_i$</td>
<td>Reaction to CPI inflation</td>
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</tr>
<tr>
<td>$\rho$</td>
<td>Elasticity of substitution</td>
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<td>$\rho_i$</td>
<td>Smoothing coefficient</td>
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<td>B. Shocks</td>
<td></td>
<td>B. Shocks</td>
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<td></td>
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</tr>
<tr>
<td>$\sigma_z$</td>
<td>Productivity s.d.</td>
<td>0.47%</td>
<td>$\rho_\psi$</td>
<td>Liquidity service persistence</td>
<td>0.79</td>
</tr>
<tr>
<td>$\sigma_{p^*}$</td>
<td>Terms-of-trade s.d.</td>
<td>0.2%</td>
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<td>Correlation: $z$ and $p^*$</td>
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</tr>
<tr>
<td>$\sigma_{r^*}$</td>
<td>World interest-rate s.d.</td>
<td>0.23%</td>
<td>$corr(\epsilon_t^x,\epsilon_t^{r^*})$</td>
<td>Correlation: $z$ and $r^*$</td>
<td>-0.13</td>
</tr>
<tr>
<td>$\sigma_{y^*}$</td>
<td>Foreigners’ output s.d.</td>
<td>0.53%</td>
<td>$corr(\epsilon_t^x,\epsilon_t^{y^*})$</td>
<td>Correlation: $z$ and $y^*$</td>
<td>0.41</td>
</tr>
<tr>
<td>$\sigma_\psi$</td>
<td>Liquidity service s.d.</td>
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<td>$corr(\epsilon_t^{p^<em>},\epsilon_t^{r^</em>})$</td>
<td>Correlation: $p^<em>$ and $r^</em>$</td>
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</tr>
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<td>$\rho_z$</td>
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<td>$\rho_{p^*}$</td>
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<td>$corr(\epsilon_t^{p^<em>},\epsilon_t^{y^</em>})$</td>
<td>Correlation: $r^<em>$ and $y^</em>$</td>
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<tr>
<td>$\rho_{y^*}$</td>
<td>World output persistence</td>
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</table>

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1.5.1 Calibration

Table 1.1 lists the parameter values and stochastic processes used in the baseline calibration. A period in the model is one quarter. Flow utility is assumed to take a standard separable form,

\[ u = \frac{1}{1 - \gamma} C_t^{1 - \gamma} - \frac{1 - \alpha}{1 + \nu} N_t^{1 + \nu} \]

\[ C_t = (\alpha \rho^e C_T^\rho + (1 - \alpha) \rho^e C_N^\rho)^{\rho^e} \]

I adopt standard values for the discount factor (0.99), risk aversion (2), and the Frisch elasticity of labor supply (2/3). For simplicity, I assume nontradable production is linear in intermediate inputs and set the elasticity of substitution between tradable and nontradable goods at \( \rho = 0.74 \), following Mendoza (1992), who estimates it in a sample of 13 industrial countries. I assume that intermediate good producers do not reoptimize each period with probability 0.75, and set the elasticity across varieties \( \eta \) to 6, as in Gali and Monacelli (2005).

For the remaining parameters and stochastic processes, I use data from Canada, which I take as a prototype small open economy. I assume that the monetary authority follows a simple Taylor rule,

\[ i_t = (1 - \rho_i)(\beta^{-1} - 1) + \rho_i i_{t-1} + (1 - \rho_i) \phi_i \pi_t \]

where \( \pi_t \) is CPI inflation, \( \phi_i = 1.5 \) - a standard value - and \( \rho_i = 0.84 \), which is the estimated persistence of the 3-month Canadian treasury bill rate over the sample period 1997:1-2016:4. I classify as nontradable sectors those with a very low export share: construction and services related to real estate services, public administration, education, health services and professional and scientific services. This leads to a share of tradables in output (\( \alpha \)) of 55%. Furthermore, the net foreign asset position is balanced (i.e., \( NF_A_{ss} = 0 \)).

I assume productivity shocks are perfectly correlated across sectors due to lack of reliable data on sectoral output at a quarterly frequency (i.e., \( Y_T = Z_t \) and \( Y_{NT} = Z_t Y_{Tt} \)). I also allow for terms-of-trade shocks, implying total tradable income is \( P_t Y_T \) (leaving the foreign-currency price of imports fixed at 1). In order to calibrate the stochastic properties of the exogenous driving forces, I fit AR(1) processes to (log) labor productivity (\( z \)), the (log) terms of trade in Canada (\( p^* \)), (log) U.S. real seasonally-adjusted output (\( y^* \)) and the U.S. 3 month treasury bill rate deflated by the U.S. CPI (\( r^* \)), using quarterly HP-filtered data (except for \( r^* \)) over the sample period 1997:1-2016:4.\(^48\)

As is well known, calibrating these shocks alone would predict too little exchange rate volatility.

---

\(^{46}\)I provide additional sensitivity exercises with respect to the elasticity of substitution between tradable and nontradable goods, risk-aversion, the degree of openness, the duration of home-currency bonds, and the discount factor.

\(^{47}\)I also explored the predictions of a model with GHH utility, as in Section 1.3. Results with respect to the outcomes of interest are similar but the performance worsens in other dimensions: it predicts too high consumption and output volatility.

\(^{48}\)Note that the autocovariance of \( y^* \) is irrelevant in the model; what matters is the interest rate. Implicitly, in writing equation (1.28) I am assuming there are discount factor shocks \( \{\beta^*_t\} \) that make the foreign Euler equation hold.
For this reason, I add an AR(1) liquidity shock $\psi_t$ which, as explained in the previous section, adds noise to the return of holding home-currency bonds. I calibrate it to match the exchange rate volatility in the period ($\sigma_{\Delta e} = 0.036$), while keeping the persistence of the 3-month Canadian treasury bill rate unaffected. This yields $\sigma_\psi = 0.0092$ and $\rho_\psi = 0.79$. This shock also moves the model closer to the data in other respects, generating a negative Fama coefficient and a low $R^2$ in a Fama regression. I pick the measure of intermediaries $m$ to match the size of home-currency debt liabilities in 2012 ($-15\%$), the last year I have access to data from Bénétrix, Lane and Shambaugh (2015).

Finally, since I lack data on the duration of home-currency external debt, I choose $\delta$ to match a duration of 4.85 years, which corresponds to the average duration of Canadian government debt between 1997 and 2010.49

1.5.2 Baseline model

Implications for key statistics

In this section, I explore the importance of the theoretical results presented in the previous sections. The first result stated that optimal monetary policy is a weighted average of a demand-management target and an insurance target. The first row in panel A of Table 1.2 shows that the planner places a nontrivial weight on the insurance motive of around 8% (column 3), although demand-management is still the most important consideration for monetary policy. The second result stated that the optimal portfolio is larger under the optimal policy than under demand-management. Despite the modest weight on the insurance motive, row 2 in Table 1.2 shows the portfolio is indeed significantly more sizeable: home-currency debt under the optimal policy is 23%, compared to only 16% under demand-management targeting (column 2). The third result stated that excess returns are less volatile under the optimal policy. While true, this result does not seem to be quantitatively relevant (row 3). This, however, masks an important composition effect. Rows 4-6 compute the volatility of excess returns driven by interest-rate $r^*$ (row 4), liquidity service $\psi$ (row 5), and world output $y^*$ (row 6) shocks.50 The optimal policy shifts the volatility: it smooths the response after $\psi$ shocks and increases it after interest-rate $r^*$ and $y^*$ shocks. Lowering the exposure to $\psi$ shocks, which are essentially noise for agents, allows them to take large positions at a low cost. This increases the effectiveness of policy after the shocks that create the majority of the demand for insurance: interest-rate shocks (for home agents) and world output shocks (for foreigners).

Panel B shows the implications for the policy instruments. Consider first the portfolio tax $\tau_B$ (row 1). Since home agents hold home-currency debt, theory implies that the planner should put a subsidy on home-currency assets (i.e., $\tau_B < 0$). In fact, the subsidy is quantitatively relevant:

49 The data is from OECD statistics and the series was discontinued after 2010. In a sample of emerging markets, Du and Schreger (2015) report an average McCauley duration of home-currency government debt held by foreigners of 5 years.

50 The remaining shocks explain a tiny amount of the volatility. Furthermore, note that since shocks are correlated this is not a strict variance decomposition.
Table 1.2: Results in baseline model.

<table>
<thead>
<tr>
<th></th>
<th>Calibrated Demand</th>
<th>Management</th>
<th>Optimal</th>
<th>Optimal: fixed B</th>
<th>Optimal: Cooperation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Taylor rule</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A. Home-currency bond positions and excess returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega$</td>
<td>0%</td>
<td></td>
<td>7.75%</td>
<td>3.57%</td>
<td>22.1%</td>
</tr>
<tr>
<td>$\bar{B}$</td>
<td>-15.0%</td>
<td>-16.0%</td>
<td>-22.8%</td>
<td>-15.0%</td>
<td>-58.0%</td>
</tr>
<tr>
<td>$\sigma(r\epsilon):total$</td>
<td>6.12%</td>
<td>3.79%</td>
<td>3.72%</td>
<td>3.82%</td>
<td>3.42%</td>
</tr>
<tr>
<td>$\sigma(r\epsilon):r^{*}$</td>
<td>2.75%</td>
<td>1.58%</td>
<td>1.99%</td>
<td>1.91%</td>
<td>2.26%</td>
</tr>
<tr>
<td>$\sigma(r\epsilon):\psi$</td>
<td>5.72%</td>
<td>3.44%</td>
<td>3.15%</td>
<td>3.32%</td>
<td>2.56%</td>
</tr>
<tr>
<td>$\sigma(r\epsilon):y^{*}$</td>
<td>0%</td>
<td>0%</td>
<td>0.35%</td>
<td>0.24%</td>
<td>0.70%</td>
</tr>
<tr>
<td>B. Policy instruments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau_B/risk premium$</td>
<td>-81.1%</td>
<td>-104%</td>
<td>0%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma(\tau^{*})$</td>
<td>0%</td>
<td>0.03%</td>
<td>0.02%</td>
<td>0.06%</td>
<td></td>
</tr>
<tr>
<td>$\sigma(e): total$</td>
<td>3.59%</td>
<td>1.48%</td>
<td>1.60%</td>
<td>1.58%</td>
<td>1.70%</td>
</tr>
<tr>
<td>$\sigma(e):r^{*}$</td>
<td>1.85%</td>
<td>1.48%</td>
<td>1.60%</td>
<td>1.59%</td>
<td>1.64%</td>
</tr>
<tr>
<td>$\sigma(e):\psi$</td>
<td>3.34%</td>
<td>0.06%</td>
<td>0.19%</td>
<td>0.11%</td>
<td>0.49%</td>
</tr>
<tr>
<td>$\sigma(e):y^{*}$</td>
<td>0%</td>
<td>0%</td>
<td>0.15%</td>
<td>0.10%</td>
<td>0.29%</td>
</tr>
<tr>
<td>C. Welfare gains (% of first-best)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gains</td>
<td>11.9%</td>
<td>16.9%</td>
<td>15.0%</td>
<td>41.3%</td>
<td></td>
</tr>
</tbody>
</table>

Note: In column 4, the portfolio is fixed at -15% while the remaining columns it is optimally chosen by the planner. The portfolio is normalized by annual gdp. Every other variable is expressed in quarterly units. The portfolio tax $\tau_B$ is normalized by the second-order risk-premium on home-currency bonds, which is positive. Welfare gains are measured by how much of the welfare gap between the first-best (a model with flexible prices) and an economy without home bonds ($\bar{B} = 0$) economy is achieved by each policy: $rac{\text{welf}(\text{policy}) - \text{welf}(\bar{B}=0)}{\text{welf}(\text{firstbest}) - \text{welf}(\bar{B}=0)} \%$. It is even larger than the steady-state risk-premium. This large subsidy reflects that the model predicts a low penetration of foreigners into home-currency bond markets to match the observed portfolios. Next, consider saving taxes $\tau^{*}$ (row 2). These taxes are very small, with a standard deviation of around 3 basis points. In fact, I show in Appendix A.3 that solving the optimal policy without the savings taxes yields quantitatively very similar results.

One recurring theme in this paper is that the endogeneity of portfolios matters. To illustrate this, column 4 in Table 1.2 presents the results if the portfolio were fixed at the calibrated 15% of home-currency debt over GDP. In this case, the policy deviates noticeably less from demand-management. Indeed, the weight on the insurance target is more than halved.

Finally, column 5 in Table 1.2 shows the results from the point of view of a supranational authority. Theory indicates that when there is limited foreign participation in home-currency markets (i.e., with $m < \infty$) coordination increases the importance of the insurance motive, since the cooperative planner internalizes the hedging benefits accrued to foreigners (see Appendix A.2.1).

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51 The risk premium is a second-order phenomenon, like the portfolio tax.

52 This result is explained by the long calibrated duration of the bonds. In Appendix A.3 I show that savings taxes are important when bonds have a duration of one year.
Table 1.3: Sensitivity analysis

<table>
<thead>
<tr>
<th></th>
<th>Baseline</th>
<th>( \psi_t \equiv 0 )</th>
<th>( \eta = 11 )</th>
<th>( \phi = .8 )</th>
<th>( \phi = 2/3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{B} ): Demand-management</td>
<td></td>
<td>-16.0%</td>
<td>-10.1%</td>
<td>-16.0%</td>
<td>-15.6%</td>
</tr>
<tr>
<td>( B ): Optimal Policy</td>
<td></td>
<td>-22.8%</td>
<td>-10.8%</td>
<td>-19.6%</td>
<td>-19.4%</td>
</tr>
<tr>
<td>( \omega )</td>
<td></td>
<td>7.75%</td>
<td>14.8%</td>
<td>3.60%</td>
<td>4.1%</td>
</tr>
<tr>
<td>Welfare: Demand-management</td>
<td></td>
<td>11.9%</td>
<td>67.7%</td>
<td>11.9%</td>
<td>12.2%</td>
</tr>
<tr>
<td>Welfare: Optimal Policy</td>
<td></td>
<td>16.9%</td>
<td>72.2%</td>
<td>14.5%</td>
<td>15.2%</td>
</tr>
</tbody>
</table>

Note: Column (2) turns off \( \psi \) shocks, column (3) increases the elasticity of substitution across varieties, and columns (4) and (5) change the frequency of price adjustment. In every case except (2), \( m \) and \( \sigma_\psi \) are re-calibrated to match the exchange volatility and a portfolio of \(-15\%\) over annual GDP. In (2), only \( m \) is re-calibrated to match the portfolio. Welfare gains are measured as the steady-state-consumption-equivalent gains under such a policy with respect to the demand-management \( \hat{B} = 0 \) economy as a share of the total potential gains under the first-best:

\[
\frac{\text{welf}(\text{policy}) - \text{welf}(\hat{B} = 0)}{\text{welf(first best)} - \text{welf}(\hat{B} = 0)} \%
\]

Given that the model requires very limited foreign access to home-currency bond markets to match observed positions, the difference between the decentralized and the cooperative solution is substantial: the weight on the insurance target and the portfolio increase by almost a factor of three, which in turn imply a noticeable decrease in the volatility of home-currency excess returns.

Welfare gains

How effective is the optimal policy in completing markets? To answer this question, I compute the welfare gains (in consumption equivalents) of moving from an economy without bonds to an economy with these bonds and flexible prices (i.e., the first best). I then compute what share of these gains are attained in the economy with sticky prices under different policies. Panel C shows the results. The optimal policy improves significantly over the demand-management policy, attaining 17% instead of 12%, a gain almost 1.5 times as large. Portfolio endogeneity is again quantitatively important for this result: If the portfolio were fixed at 15%, gains would only be 15%. Finally, the cooperative solution attains a much larger share of the gains, reflecting the limited participation of foreigners in home-currency markets.

1.5.3 Sensitivity analysis

No liquidity shocks

In this section, I turn off liquidity shocks and recalibrate the measure of foreigners to match observed portfolios. Column 2 in Table 1.3 shows the results. Column 1 reproduces the results in the baseline model to facilitate the comparison. The most important difference is that the trade-off between demand-management and insurance is significantly weakened. This can be seen by computing the correlation between the desired transfers \( (T) \) and the excess-returns when \( \hat{B} = 0 \). While the

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53 The absolute size of the gains is tiny: the first-best gains are only 0.067% steady-state consumption equivalents. This is a manifestation of the well-known fact that the welfare gains from completing markets in standard macroeconomic models are very small (Lucas (1987)), which is exacerbated in this context by limited participation by foreigners.
correlation is $-0.35$ with liquidity shocks, without them the correlation is much higher at $-0.82$ (not shown in table). One reason why the exchange rate targets are so correlated is that the model significantly underestimates the volatility of exchange rates under the calibrated Taylor rule, with the remainder tightly linked to fundamentals.$^{54}$ The high correlation implies that portfolios under demand-management targeting and the optimal policy are very close to one another, differing by less than a percentage point. By the same token, the demand-management policy already achieves 67.7% of the potential gains of completing markets, with the optimal policy only increasing this number to 72.2%.

Although policies are on average similar, the insurance weight is actually higher (15%) since foreign participation in home-currency markets is even more restricted, which implies the planner is now heavily penalized with a low home-bond price if excess returns are volatile. To understand why this is the case, note that given that the model underpredicts the volatility of excess returns in the absence of liquidity shocks, agents would like to choose a very large position if there were perfect intermediation ($m = \infty$). As a result, very limited foreign participation in home-currency markets is necessary to match the observed portfolios. Although the planner would be willing to dampen returns in order to lower costs, the very high correlation implies this trade-off hardly ever materializes: the demand-management target and the insurance target are often aligned.

**Cost of inflation**

Next, I explore the sensitivity of the results to the cost of inflation. First, I lower the mark-up in intermediate inputs to 10% ($\eta = 11$), which increases the cost of price dispersion. Column 3 in Table 1.3 shows the results. Clearly, this parameter is important for the quantitative results. Doubling the equilibrium mark-up of intermediate good producers essentially halves the importance of insurance, as measured by its implications for the portfolio, the weight, and welfare.

Second, I vary the share of firms that are able to update their prices. The results suggest this is also a critical parameter to determine the relative importance of the insurance motive. Assuming that firms update their prices on average every 3 quarters implies the portfolio more than doubles in size compared to demand-management targeting, implying an insurance weight of almost 25%. The planner is also able to attain a much larger share of the welfare gains of completing markets. Conversely, assuming firms update their prices every 5 quarters halves the importance of the insurance motive.

1.6 Conclusion

I developed a framework to study optimal monetary policy and capital controls in open economies with incomplete markets and portfolio choice. I presented three main results. First, I showed that monetary policy can be described by an exchange rate rule that is a weighted average of two targets: a demand-management/inflation target, concerned with the traditional role of “undoing” nominal

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$^{54}$Exchange rate volatility in this model is essentially explained by $\tau^*$. 

rigidities, and an insurance target, which is the exchange rate that would be required to replicate
the transfer the planner would desire under complete markets. Second, I showed that positions in
home-currency become larger when insurance considerations are more important, which in turn
leads to a higher ex post weight on the insurance target. To lower the cost of such large positions,
the planner strongly dampens the volatility of the exchange rate when insurance is not needed (an
example of “fear-of-floating”). Perhaps surprisingly, when portfolios are endogenous this effect is
so strong that overall volatility under the optimal policy is lower than under inflation-targeting.
By contrast, with constrained portfolios the effect is weaker and overall volatility would actually
be higher than under inflation-targeting, illustrating the importance of modeling portfolio choice.
Finally, I showed that in a small open economy portfolio decisions are approximately efficient
despite the presence of aggregate demand externalities (due to nominal rigidities) and pecuniary
externalities (due to incomplete markets), so no capital controls on the composition of capital flows
are necessary in the approximate solution.

In this paper, I focused on the trade-off between insurance and demand-management, abstracting
from other relevant macroeconomic forces. However, the methodology I develop is widely applicable
and can be used to explore optimal policy in other environments with portfolio choice and incomplete
markets. It is also simple computationally, even with multiple assets (see Appendix A.2.4), so it
could be applied to richer macroeconomic models than the one presented in this paper. Doing so
could deliver new interesting results on capital controls, as well as a reappraisal of the quantitative
importance of the insurance channel of monetary policy.
Chapter 2

A Theory of Foreign Exchange Interventions

This paper develops a theory of foreign exchange interventions in a small open economy with limited capital mobility. Home and foreign bond markets are segmented and intermediaries are limited in their capacity to arbitrage across markets. As a result, the central bank can implement nonzero spreads by managing its portfolio. Crucially, spreads are inherently costly, over and above the standard costs from distorting households’ consumption profiles. The extra term is given by the carry-trade profits of foreign intermediaries, is convex in the spread—as more foreign intermediaries become active carry traders—and increasing in the openness of the capital account—as foreign intermediaries find it easier to take larger positions. Optimal interventions balance these costs with terms of trade benefits. We show that they lean against the wind of global capital flows to avoid excessive currency appreciation. Due to the convexity of the costs, interventions should be small and spread out, relying on credible promises (forward guidance) of future interventions. By contrast, excessive smoothing of the exchange rate path may create large spreads, inviting costly speculation. Finally, in a multi-country extension of our model, we find that the decentralized equilibrium features too much reserve accumulation and too low world interest rates, highlighting the importance of policy coordination.

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This chapter is joint with Ludwig Straub. We are grateful to Iván Werning for continuous guidance and support. We also would like to thank Manuel Amador, Javier Bianchi, Ricardo Caballero, Andrés Fernández, Adrien Verdelhan, and participants at the MIT Macro Lunch, the MIT Sloan Asset Pricing Reading Group, the 2015 RIDGE conference in Montevideo, the 2017 Seminar on Financial Volatility and Foreign Exchange Intervention of the Central Bank of Perú, the 2017 CEPR-SNB-Bol conference “Foreign Exchange Market Intervention: Conventional or Unconventional Policy?”, the Barcelona GSE Summer Forum, the Society for Economic Dynamics Conference, and the “International Finance and Macroeconomics” and “Macroeconomics Within and Across borders” groups in the NBER Summer Institute for many useful comments. Ludwig Straub appreciates support from the Alfred P. Sloan Foundation, the CME Group Foundation, Fidelity Management & Research, and the MFM initiative.
2.1 Introduction

Foreign exchange interventions are among the most important macroeconomic policy tools, yet among the least understood. Countries mainly use them for two purposes: To manage their exchange rate, when relying on monetary policy alone is infeasible or undesired, and to accumulate reserves as insurance against sudden stops. Both roles are of crucial importance. There is ample evidence that many countries intervene to dampen exchange rate volatility, slow down exchange rate adjustments or lean against capital flows.\(^1\) And reserve accumulation has gone so far that now $12$ trillion, or 80\% of US GDP, are being saved by the world's central banks. The effect of this reserve hoarding on global imbalances, world interest rates and exchange rates can hardly be overstated.

In light of the popularity of foreign exchange interventions among policymakers, it might almost come as a surprise that there is relatively little guidance from theory on how they should be conducted. Although there has been important progress in recent years (see Cavallino, 2018 and Liu and Spiegel, 2015), the answer to many important questions remains incomplete. When are foreign exchange interventions desirable? How costly are they and what is the right, welfare-relevant way to measure these costs? How should interventions be designed to maximize their effectiveness, and how does that depend on the specific goal of the intervention? What are the implications of the increasingly common usage of interventions for the world economy? Should countries coordinate their interventions?

In this paper, we propose a tractable and microfounded framework that speaks to all these questions. We base our analysis on a canonical real small open economy model augmented with limited capital mobility, in which the country faces endowment and interest rates shocks. As is well-known in this type of model,\(^2\) the economy has market power as exporter of its endowment, generating a desire for terms-of-trade management. In the model's financial markets, domestic and foreign intermediaries can arbitrage between domestic and foreign bond markets, but arbitrage is limited due to a fixed transaction cost and position limits. Under these conditions, a portfolio balance channel emerges: changes in the portfolio of the central bank induce short-lived interest rate spreads—that is, exchange rate adjusted or “UIP” spreads\(^3\)—between domestic and foreign bonds, as in Kouri (1976), Branson and Henderson (1985), or more recently Gabaix and Maggiori (2015). We analyze this model through the lens of the small open economy's central bank as social planner and ask: How should it optimally manage its holdings of foreign bonds?

Our first contribution is to show that this central bank planning problem can be entirely framed in terms of the UIP spreads that the central bank's portfolio choice generates. The logic is straightforward: If a country seeks to depreciate its exchange rate, it sells home bonds and

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\(^1\) This holds for emerging markets and advanced economies alike. For emerging markets, see for example the “fear of floating” literature around Calvo and Reinhart (2002), Levy-Yeyati and Sturzenegger (2005) and McKinnon and Schnabl (2004), among many others. Among advanced economies, recent interventions have for example been conducted by Denmark, Switzerland, or the Czech Republic to depreciate their respective currencies. See Section 2.2 below for more examples.

\(^2\) See e.g. Costinot, Lorenzoni and Werning (2014), or Farhi and Werning (2012, 2014).

\(^3\) UIP is short for the uncovered interest parity condition which is satisfied if the expected excess return from investing in local bonds is exactly zero.
purchases foreign ones, generating a positive UIP spread. Crucial to our analysis, UIP spreads are inherently costly, over and above the standard costs from distorting (domestic) households’ consumption profiles. The reason is intuitive: UIP spreads invite foreign intermediaries to take profitable carry trade positions, and hence the country as a whole is losing money at an amount equal to the carry trade profits made by foreign intermediaries. These additional costs are naturally convex in the level of the spread—as more foreign intermediaries become active carry traders when spreads are higher—and increasing in the openness of the capital account—as foreign intermediaries then find it easier to take larger positions. It is worth stressing that our costs are the welfare-relevant costs identified by our model and apply to the whole country; as such, our costs can, and often will, be different from the central banks’ own quasi-fiscal cost of holding reserves, which do not include costs incurred by the country’s private economy from nonzero UIP spreads.4

The formulation of the planning problem in terms of UIP spreads highlights an interesting connection with the recent literature on optimal capital controls (see, for example, Farhi and Werning, 2014). In this literature, the planner typically chooses proportional taxes on capital flows, which also manifest themselves as UIP spreads. The crucial difference to this literature is that in our model, the planner faces an extra cost from nonzero UIP spreads, coming from the carry-trading activities of foreign intermediaries. Indeed, we show that in the limit of zero private capital mobility (financial autarky), our planning problem becomes essentially analogous to one of optimal capital controls. This seems to suggest that capital controls and foreign exchange interventions are substitutes. Yet, this is not the case: Precisely in the limit of zero capital mobility, capital controls are meaningless. Instead, as we explain below, our theory suggests that capital controls and foreign exchange interventions are complements: the former enhance the effectiveness of the latter.

Our second contribution is to fully characterize the optimal policy. We find that it should lean against the wind of capital inflows by implementing positive UIP spreads, and thus depreciate the exchange rate (vice versa when capital flows reverse direction). When faced with positive endowment or wealth shocks, e.g. productivity boosts in the export sector or natural resource discoveries, the central bank should intervene to avert any excessive exchange rate movements and replicate the frictionless equilibrium (zero UIP spreads).

For intermediate degrees of capital mobility, the additional cost term delivers several new insights about the optimal design of foreign exchange interventions. First, interventions should be smooth. Since nonzero spreads lead to costly carry trade activities and households are forward-looking, it is undesirable to adjust the spread immediately after a shock hits. Second, the convexity of the cost term generates a desire for spreading out interventions over time. In particular, this includes promising future interventions, a form of “forward guidance” of foreign exchange interventions. As a consequence of this, interventions should be highly inertial, possibly lasting significantly longer than the shock itself. This, however, leads to natural and novel type of time inconsistency: After the shock has subsided the central bank would like to revert to zero UIP spreads, if allowed to

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4 For a recent study on the quasi-fiscal cost of reserve holdings, see Adler and Mano (2016).
reoptimize. Thus, central bank credibility turns out to be an essential input into a successful conduct of intervention policies. To sum up, we find that foreign exchange interventions should be small, frequent, persistent and credible. None of these properties can be obtained in the special case with no private capital mobility.

Our lessons are not limited to the canonical model in which management of a country’s terms-of-trade is the underlying motive. We present three extensions that each exhibit a different rationale to conduct foreign exchange interventions. In the first extension, we explain how an economy with an exchange rate peg and sticky prices in the domestic good optimally intervenes to smooth the path of output gaps.\footnote{Countries that face(d) this kind of problem are for example advanced economies during the Bretton Woods era, or European countries that peg to the Euro such as Denmark or the Czech Republic. Even Switzerland can be counted into this category during the time they had in place an exchange rate floor, which was effectively a peg, from 2011 to 2015. Our analysis abstracts from an effective lower bound on policy rates and instead stresses that foreign exchange interventions can help a pegging country regain some control over its interest rate more generally. For an analysis with a binding zero lower bound see Amador et al. (2016).} We show that in this case the planner still leans against the wind of capital inflows but for a different reason: By accumulating reserves it is able to raise the domestic interest rate and shift household spending to the future in order to avoid excessive consumption of the domestic good in the present. The qualitative properties of interventions—size, frequency, persistence and credibility—go through unaltered. In our second extension, we allow for taste shocks in intermediaries’ demand for home bonds. This allows us to capture, albeit in reduced form, aspects such as liquidity premia or heterogeneous beliefs. Analyzing this case is particularly important given that many central banks allegedly intervene when the exchange rate moves “away from fundamentals”. In such a scenario, there is an incentive for the home country to behave as a monopolist in the supply of its own bond, optimally selling, but not fully accommodating, foreigners’ demand for the home bond. Unlike before, the central bank can now make profits, behaving as a speculator in the sense of Friedman (1953). Aside from this profit opportunity, we show that the dynamic properties of optimal interventions are still qualitatively the same as before. Finally, as our third extension, we present an economy which is pursuing a “managed float” policy in which the exchange rate is required to follow a smooth path. We show that the slow exchange rate adjustments at the core of this kind of policy—which seems to be very common among EMEs—may cause significant costs by creating large UIP spreads, which invite foreign intermediaries to enter carry trades against the central bank.

Our third contribution is to characterize the positive and normative consequences of widespread foreign exchange interventions for the international monetary system. We embed our baseline model in a world composed of two continua of small open economies: a continuum of “emerging market economies” (EMEs), which are subject to limited capital mobility as before; and a continuum of “advanced economies” (AEs), which have perfect capital mobility. We hit AEs with a savings shock to capture recent trends like population aging or debt overhang. We show that in response to the capital inflows from AEs, EMEs engage in “reserve wars”, as each EME tries to manipulate its terms-of-trade in its favor—an effort which turns out to be self-defeating in the world equilibrium. Interestingly, such behavior by EMEs leads to public flows flowing upstream and private flows...
flowing downstream, which is what the evidence in Gourinchas and Jeanne (2013) and Aguiar and Amador (2011) suggests. In addition, such reserve wars depress the world interest rate, making the incentive for other EMEs to intervene even stronger. We explain that for reasonable calibrations, both AEs and EMEs would be better off if interventions were ruled out altogether. In fact, the model suggests that even a “unilateral” move by EMEs to coordinate their interventions would significantly reduce their volumes, possibly all the way to zero, again with welfare gains for both AEs and EMEs.

**Literature** Our paper is part of a nascent literature that incorporates a portfolio balance channel into a general equilibrium framework to study foreign exchange interventions. Like this paper, Cavallino (2018) and Liu and Spiegel (2015) embed imperfect intermediation across borders within a standard New Keynesian model like the one considered in this paper, and study optimal policy. Cavallino (2018) studies the optimal response to nonfundamental capital flow shocks. He solves the optimal policy analytically and shows interventions lean-against-the-wind. By contrast, we also study the response to fundamental shocks. While the optimal policy still leans-against-the-wind, we show that the spread on the return between home and foreign-currency bonds is smooth, even when the shock itself is not, which is not true for nonfundamental shocks. In addition, we show that limited financial integration benefits the planner for fundamental shocks, giving rise to a complementarity with capital controls. Liu and Spiegel (2015) study numerically the jointly optimal response of taxes on financial assets, foreign exchange interventions, and monetary policy to fundamental shocks, and also find interventions lean against the wind. We focus on a real model where the only tool is foreign exchange interventions and characterize the solution tightly in this environment. None of these papers study the solution without commitment, an economy without terms-of-trade manipulation and fixed exchange rates, or coordination of interventions in a multi-country setting. In other related work, Chang and Velasco (2017) build a model with borrowing constraints on the financial sector and show that foreign exchange interventions may be useful when those constraints are binding. Amador et al. (2016) consider an environment similar to ours to study foreign exchange interventions in the zero lower bound. They find interventions could be useful to mitigate the recession. Benes et al. (2015), Blanchard, Filho and Adler (2015), Devereux and Yetman (2014) and Ostry et al. (2012) study the effects of interventions but lack a fully microfounded model. Gabaix and Maggiori (2015) study the effect of small foreign exchange interventions, but lack a fully microfounded model.⁶

As we mentioned in the introduction, our paper is related to the burgeoning literature on optimal capital controls.⁷ In an environment similar to ours but with perfect financial markets, Farhi and Werning (2012, 2014) find that optimal capital controls are used to lean against the wind after interest rate shocks while they are not used against endowment shocks. In our baseline model, we

⁶Devereux and Yetman (2014) has a microfounded new Keynesian model but their modeling of capital immobility is ad hoc, which makes the results somewhat harder to interpret. In particular, the model does not predict a cost of sterilization, which is a feature of microfounded models.

⁷See, among many others, Bianchi (2011); Magud, Reinhart and Rogoff (2011); Farhi and Werning (2012, 2014); Heathcote and Perri (2016).
derive an analogous result for foreign exchange interventions but show that the additional costs associated with foreign exchange interventions are crucial for the optimal policy. In a model with two large economies, Costinot, Lorenzoni and Werning (2014) show that a country might want to use capital controls to depress the international interest rate if the endowment is growing over time. This is related to some of our results in Section 2.6.

There is a large literature documenting EMEs’ reserve accumulation in the past decades. The main goal of these papers has been to quantify the contribution of different potential explanations, such as building buffers against sudden stops or “neo-mercantilist” strategies of real exchange rate undervaluation. However, most papers do not tackle the question of how the country actually manages to manipulate the net foreign asset position of the country to achieve this objectives. Closest to us in this literature is Jeanne (2012), which emphasizes the interaction of public and private flows. That paper allows domestic households to access foreign markets subject to a transaction cost, which allows the planner to costlessly manipulate the UIP within certain limits. However, since capital is otherwise perfectly mobile, there is no region in which the planner balances “costs” and “benefits” of foreign exchange interventions.

Finally, our study of a world equilibrium with reserve accumulation in Section 2.6 is related to Obstfeld (2011), who emphasizes the dangers of currency wars through reserve accumulation and its consequences for the global interest rate. Models of low global interest rates are also put forth by Coeurdacier, Guibaud and Jin (2015) and Caballero and Farhi (2017). In our case, lower interest rates are a consequence of reserve accumulation and an increasing share of EMEs in world markets (see Section 2.6).

2.2 Background on Foreign Exchange Interventions

Foreign exchange interventions are defined as changes in a central bank’s holdings of reserve assets, where reserve assets are defined as “those external assets that are readily available to and controlled by, the monetary authorities for meeting balance of payments financing needs, for intervention in exchange markets to affect the currency exchange rate, and for other related purposes” (IMF, 2011). In this section, we briefly describe the history of such interventions, followed by a discussion of the most commonly cited benefits and costs of interventions. Finally, we summarize the debate on the optimal way of trading off costs and benefits. Throughout this section, we will both draw on existing papers as well as establish new facts. A reader mainly interested in our theoretical analysis may skip this section.

2.2.1 A brief history

Foreign exchange interventions have been a key part of the international monetary system in the last century. During the times of the gold standard, and after its collapse, during the Bretton Woods
system, interventions have routinely been used to “break” the trilemma and generate some degree of monetary independence (Bordo, Humpage and Schwartz, 2015). After the collapse of Bretton Woods, exchange rates were allowed to float, with one of the promises being that this would reduce the need of large scale foreign exchange interventions. Yet, this promise soon turned out to be false as advanced economies continued regular, and often coordinated, interventions until the turn of the century. For example, as the United States struggled with a strong dollar, the five most important central banks negotiated the “Plaza Accord” in September 1985, following which they engaged in massive interventions to depreciate the dollar. In the 1980s and 1990s, European countries’ central banks intervened to limit exchange rate volatility in the context of the European Exchange Rate Mechanism, a fixed exchange rate system which preceded the introduction of the Euro. Furthermore, Japan was intervening heavily until 2004, mostly in order to moderate the appreciation of the Yen. After that, however, interventions by advanced economies did begin to fall out of favor—at least until a few years ago, when, faced with the limitations of monetary policy at very low interest rates, some European countries, such as Switzerland or Denmark, have resorted once again to foreign exchange interventions in order to achieve their policy objectives.

In contrast with the recent decline among advanced economies, interventions have become a very important policy tool for many emerging market economies (EMEs). Since the famous “sudden stop” episodes of the 1990s, one of the main objectives of EMEs’ interventions has been to build a “war chest” of reserves to insure against sudden stops. As a result, EMEs often engage in reserve-accumulation strategies when their current level of reserves is perceived to be inadequate. Furthermore, unlike the early interventions by advanced economies after the collapse of Bretton Woods, these reserve accumulation programs are conducted unilaterally by each EME central bank and generally tend to rely on comparisons with peers, raising concerns about coordination failures and amplification of reserve hoarding behavior IMF (2011). In addition, Obstfeld (2011) emphasizes the spillovers excessive reserve accumulation may have on the world equilibrium through the world interest rate. We address these issues in Section 2.6.

This type of policies has drastically altered the landscape of the international monetary system. Figure 2-1 plots in Panel (a) the reserve holdings of EMEs and AEs, relative to world GDP, and in Panel (b) EMEs’ and AEs’ shares of world GDP. Before the 1990s, most of global reserves were owned by AEs—as one would expect given their dominant share of world GDP. Since then, however, the rate of reserve accumulation by EMEs has been staggering. As of 2011, EMEs’ reserves increased to 9% of world GDP, twice those held by AEs, despite the fact that EMEs only account for half as much GDP (Panel (b)).

---

9 A sudden stop is a quick reversal in capital flows that leads to a shortage of international liquidity.
10 For example, in 2011 Chile started a reserve accumulation program to raise its reserves from 13.3% to 17% of GDP within a single year.
11 The lion share of this increase of course came from China, which held reserves worth 4% of world GDP in 2011. Nevertheless, even without China, EME reserves over world GDP still rose eightfold between 1990 and 2011, compared to twelvefold when China is included.
2.2.2 Benefits and costs of interventions

Over the years, policymakers have intervened based on a wide variety of different reasons. In this subsection, we briefly summarize those reasons and the costs associated with foreign exchange interventions.

Benefits. The policy debate around foreign exchange interventions has identified the following three broad reasons why a country might engage in foreign exchange interventions: (i) exchange rate management, either to reduce exchange rate volatility, to smooth out exchange rate adjustments over time, or to improve the terms of trade; (ii) reserve accumulation, to insure against future sudden stops; (iii) regaining some monetary independence despite fixed exchange rates. Our baseline model in Section 2.3 features a canonical terms of trade management motive, finding that the central bank leans against the wind.

Actual policy implementation in the data seems to share this property with the optimal policy. Chang (2007) documents that many EME central banks have indeed been leaning against the wind of private capital flows—even those countries that appear to follow a policy of inflation targeting. Figure 2-2 shows a similar picture. Panel (a) plots the time series of quarterly net private flows into a large sample of 50 EMEs (blue, dashed) and flows into reserves (red, solid), aggregated over the countries in our sample. The comovement is striking, and not driven by aggregation: 49 of 50 in our sample show a similarly strong positive comovement, with an average correlation of 0.51. It is particularly interesting that there is not just a strong positive correlation between the two lines, but their volatilities are also of similar size. Figure 2-2(b) shows the two volatilities in a scatter plot across the same set of countries. While volatilities do not necessarily exactly line up along the 45
Figure 2-2: Note: Panel (a) shows the aggregate reserve flows over aggregate GDP and aggregate net private capital flows over aggregate GDP. Panel (b) compares the volatility of reserve flows over GDP with the volatility of net private capital flows over GDP across countries. The time period underlying both plots is from 1990:1 to 2008:4 and the sample consists of 50 emerging markets. The signs in Panel (a) are such that positive reserve flows reflect reserve accumulation and positive private capital flows reflect inflows. The data is from the IMF’s Balance of Payments statistics. Details on this figure can be found in Appendix B.1.2.

degree line (dashed), the volatility of reserves is significant at around one half of the volatility of net private flows (red, solid). By comparison, the volatility of reserves in the US is only 3.5% of the volatility of net private capital flows.

In Section A.2 below, we also discuss how our model can be used to analyze different motives for interventions than to manage the terms of trade.

Costs. There are many ways economists have been measuring the costs associated with foreign exchange interventions.\(^{12}\) Our model takes a particularly practical stance on this issue: The relevant economic costs are transfers that the country implicitly pays to foreign arbitrageurs running successful carry trades against central bank interventions. Critical to this story is that there is an empirical connection between UIP spreads and foreign exchange interventions. To provide evidence that there is, we combine data from Lustig, Roussanov and Verdelhan (2011) with the IMF’s Balance of Payments Statistics and run the following OLS regression,\(^{13}\)

\[
UIP_{spread_{it}} = \alpha_i + \delta_t + \beta ResFlows_{it} + X_{it} + \epsilon_{it}
\]

where \(\alpha_i\) and \(\delta_t\) are country and time fixed effects; and \(X_{it}\) is a set of controls, in which we include past reserve flows as well as the past interest rate spread (which is well known to predict excess returns, see e.g. Lustig, Roussanov and Verdelhan (2011)). The outcomes, in Table 2.1, show a

\(^{12}\)For example, Adler and Mano (2016) measures the costs as quasi-fiscal costs the central bank incurs when it invests in low-yielding assets, selling high-yielding ones.

\(^{13}\)We thank Adrien Verdelhan for sharing his data with us.
Table 2.1: Do increases in reserves coincide with positive wedges in the uncovered interest parity condition (UIP)? Quarterly reserve flows are taken from the IMF’s Balance of Payments statistics, merged with quarterly UIP wedges and interest rate spreads computed based on data from Lustig, Roussanov and Verdelhan (2011). Details on the table can be found in Appendix B.1.3. Note: Standard errors, corrected for heteroskedasticity, are in brackets. * p < 0.1, ** p < 0.05, *** p < 0.01

<table>
<thead>
<tr>
<th></th>
<th>UIP wedge</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>Res. flows / GDP (t)</td>
<td>0.723**</td>
</tr>
<tr>
<td></td>
<td>(0.34)</td>
</tr>
<tr>
<td>Res. flows / GDP (t-1)</td>
<td>0.117</td>
</tr>
<tr>
<td></td>
<td>(0.20)</td>
</tr>
<tr>
<td>Int. rate spread (t-1)</td>
<td>2.258*</td>
</tr>
<tr>
<td></td>
<td>(1.16)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.353***</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
</tr>
<tr>
<td>Year dummies</td>
<td>No</td>
</tr>
<tr>
<td>Country FE</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>1528</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.011</td>
</tr>
</tbody>
</table>

|                      | (2)       |
| Res. flows / GDP (t) | 0.713**   |
|                      | (0.33)    |
| Res. flows / GDP (t-1)| 0.148    |
|                      | (0.19)    |
| Int. rate spread (t-1)| 1.931*   |
|                      | (1.11)    |
| Constant             | 0.321**   |
|                      | (0.14)    |
| Year dummies         | No        |
| Country FE           | Yes       |
| Observations         | 1528      |
| R-squared            | 0.011     |

|                      | (3)       |
| Res. flows / GDP (t) | 0.666**   |
|                      | (0.30)    |
| Res. flows / GDP (t-1)| 0.215    |
|                      | (0.21)    |
| Int. rate spread (t-1)|         |
|                      |          |
| Constant             | -0.250    |
|                      | (0.36)    |
| Year dummies         | No        |
| Country FE           | Yes       |
| Observations         | 1528      |
| R-squared            | 0.033     |

|                      | (4)       |
| Res. flows / GDP (t) | 0.535**   |
|                      | (0.24)    |
| Res. flows / GDP (t-1)|         |
|                      |          |
| Int. rate spread (t-1)|         |
|                      |          |
| Constant             | -5.100*** |
|                      | (0.73)    |
| Year dummies         | Yes       |
| Country FE           | Yes       |
| Observations         | 1528      |
| R-squared            | 0.163     |

strong significant relationship between reserve purchases and positive UIP spreads, that is, expected excess returns from investing in the intervening country’s bond market.

2.2.3 Empirical evidence on effectiveness

Convincing empirical evidence about the effectiveness of foreign exchange interventions is very rare. This is mainly due to a simple simultaneity problem: Interventions influence exchange rates but also respond to shocks to exchange rates. One of the more solid papers that attempts to go around this issue is Kearns and Rigobon (2005), exploiting a “natural experiment”, in which Japan and Australia changed their foreign exchange policy for arguably exogenous reasons. They show that in both countries, interventions seem to have a significant effect on the exchange rate, with most of the effect occurring during the day of the intervention.

It is worth pointing out that, despite the shortage of convincing formal evidence on interventions, policymakers’ own experiences with them seem to suggest that they are indeed effective. For example, a survey by the BIS shows that 90% of the respondents believe were their foreign exchange interventions were completely or partially successful.

One of the least controversial points regarding the effectiveness of interventions, is that the effectiveness should be tightly linked to the (in)ability of capital to flow freely across borders. This is important for our theory since the assumption of limited capital mobility is at the core of our analysis. It is important to notice that such limited capital mobility need not necessarily come from limits to arbitrage in the financial sector: Canales-Kriljenko (2003) documents that central
banks in EMEs are large players in their foreign exchange markets, partially due to consciously taken policy measures designed to inhibit free cross-border capital flows.\textsuperscript{14} As already previewed in the introduction, this points to a \textit{complementarity} between capital controls and macroprudential measures on the one side and foreign exchange interventions on the other.\textsuperscript{15}

### 2.2.4 Implementation

There are roughly three important degrees of freedom in the implementation of foreign exchange interventions: Frequency vs. size, rules vs. discretion, and exchange rate rules vs. quantity-based rules.\textsuperscript{16} We discuss each of them in turn.

**Frequency vs size.** Countries are divided as to the optimal size and frequency of interventions. For example, Kearns and Rigobon (2005) document that Australia and Japan abandoned their small and frequent interventions in favor of large and infrequent ones in an effort to maximize their impact on the exchange rate. In contrast, other countries have had a very persistent presence in foreign exchange markets. Adler and Tovar (2011) document that Brazil and Uruguay intervened two-thirds of the days between 2004 and 2010.

**Rules vs discretion.** Another source of debate refers to whether interventions should be secret or public information. In a well-known survey of EME central banks, Canales-Kriljenko (2003) documents that about one half of the respondents carry out their interventions in secret. Among advanced economies, public interventions are more common. However, the historically largest intervener—the Bank of Japan—has frequently favored secret interventions. Note, however the line separating secret and public interventions becomes blurred in shallower markets, where the central bank’s presence hardly goes undetected.

**Exchange rate rules vs quantity-based rules.** In addition, even if one agrees that transparency and predictability are desirable, there is heterogeneity among countries regarding the kind of rules that they implement. On the one hand, some countries follow quantity-based rules. For example, Chile’s reserve accumulation program of 2011 consisted of buying USD 12 billion in pre-announced daily amounts at an average of USD 50 million per day. On the other hand, some countries follow exchange-rate based rules, mostly aimed at smoothing the exchange rate path. For example,

\textsuperscript{14}In a sample of 90 countries, he finds that 36\% have surrender requirements, 90\% have some form of position limits, 50\% prohibit usage of foreign currency for some domestic transactions and in 45\% both legs of foreign exchange transactions are settled at the central bank. Mohanty and Berger (2013) confirms this observation in a more recent survey.

\textsuperscript{16}Arguably, whether to intervene in spot or forward markets is another degree of freedom. However, most, i.e. 70-80\%, of the interventions are carried out in spot markets, since forward markets are generally more illiquid. There are some notable exceptions, such as Brazil.
Colombia had a rule that authorized the central bank to auction put options up to a specific amount whenever the exchange rate fell more than 5% below its average of the previous 20 days.

**Perspective of our model.** Our model provides simple yet powerful guidance on these questions. If the goal is to minimize the aforementioned costs coming from foreign arbitrageurs running carry trades (and hence speculating) against central bank interventions, then interventions should be frequent but small in size and pre-announced. Moreover, following a quantity-based rule is generally found to be closer to the optimal policy than an exchange-rate based rule that guarantees smooth exchange rate movements.

Frequent but small interventions are powerful due to two reasons: First, they span over a significant time period, so they are likely to affect interest rates for longer, amplifying the initial response of economic agents, especially when interventions are pre-announced. Second, the relatively small size of any specific intervention is associated with relatively minor UIP spreads and hence limits the room for foreign arbitrageurs to take advantage of the central bank action. Exchange rate rules are found to do the exact opposite: As we show in Section A.2, by slowing down the exchange rate adjustment, central banks invite foreign arbitrageurs to take bets and trade against the central banks’ interventions.

### 2.3 Baseline Model

In this section, we present our baseline model. The model is a real small open economy (SOE) model in continuous time. The model is stylized as we strive to focus on the two essential model ingredients. These are on the one hand a finitely elastic foreign demand for home bonds that allows the home central bank to change home interest rates via a portfolio balance channel, and on the other a terms-of-trade management motive, which gives the central bank a reason for such interventions. We first describe the model and then discuss the equilibrium dynamics without interventions. Optimal interventions are then characterized in great detail in Section 2.4.

#### 2.3.1 Model setup

There are four agents in our model: Domestic households and a domestic central bank, and, foreign and domestic intermediaries. In line with our SOE assumption we also introduce an export demand curve of foreign households. There are two goods markets (a “home good” and a “foreign good”) as well as two asset markets (“home bonds” and “foreign bonds”). Throughout, we use “home”

---

17 In an empirical setting, our model would predict that the exchange rate should respond the moment the policy is announced. Interestingly, Tapia and Tokman (2004) finds that the announcement of intervention in Chile in 2001 had a large and significant effect on the exchange rate.

18 This is similar to papers by Lahiri and Végh (2003), Gabaix and Maggiori (2015), and Liu and Spiegel (2015) among others.

19 This is similar to the recent literature on capital controls by Farhi and Werning (2012, 2014) and Costinot, Lorenzoni and Werning (2014), among others, which is based on the framework by Gali and Monacelli (2005).
and "domestic", as well as "UIP spread" and "UIP wedge" interchangeably. We start by describing domestic households and the two goods markets.\textsuperscript{20}

**Households.** There is a continuum of households in the home country, maximizing a common utility function \( \int_0^\infty e^{-\rho t} \log(c_t)dt \), with \( c_t \) being a consumption bundle defined as \( c_t = \kappa^{1-\alpha} c_{Ht}^{1-\alpha} c_{Ft}^{\alpha} \). Here, \( c_{Ht} \) and \( c_{Ft} \) denote home's consumption of home and foreign goods, respectively, and \( \kappa \equiv (1 - \alpha)^{-(1-\alpha)\alpha - \alpha} > 0 \) is a positive normalization constant. Throughout our analysis, we normalize the foreign good's price to 1 and refer to that numeraire as "dollars". The relative price of the home good is denoted by \( p_t \). The per-period dollar budget constraint of the household is given by

\[
\dot{b}_{Ht} = p_{t} y_{Ht} + y_{Ft} - p_{t} c_{Ht} - c_{Ft} + r_{t} b_{Ht} + t_{t} + \pi_{t}, \tag{2.1}
\]

where \( y_{Ht} \) is home's endowment of the home good, \( y_{Ft} \) is home's endowment of the foreign good, \( b_{Ht} \) is the households' position in home bonds, \( t_{t} \) are transfers from the central bank, and \( \pi_{t} \) are profits from domestic financial intermediaries. Both \( t_{t} \) and \( \pi_{t} \) are specified below. We denote by \( q_{t} \equiv p_{t}^{-1} \) the country’s real exchange rate, following the convention that high values correspond to depreciated exchange rates. Here, domestic households are only allowed to trade home bonds with a real interest rate of \( r_{t} \). Later, we introduce financial intermediaries, who may access both home and foreign bond markets. We wish to stress that in this environment, domestic households’ own a nontrivial share of the home good and exhibit home bias in their preferences. These two assumptions are essential in generating the terms of trade management motive in our environment.

Maximizing utility subject to this budget constraint yields the following Euler equation,

\[
\frac{\dot{c}_t}{c_t} = r_{t} - \rho + \frac{\dot{q}_{t}}{q_{t}}. \tag{2.2}
\]

Finally, home’s total dollar expenditure is given by \( q_{t}^{-1} c_{t} = p_{t} c_{Ht} + c_{Ft} \), which henceforth we denote by \( \theta_{t} \equiv q_{t}^{-1} c_{t} \) due to its prominent role the analysis to come. The optimal demand for home and foreign goods is then

\[
c_{Ht} = (1 - \alpha) \frac{\theta_{t}}{p_{t}} \tag{2.3}
\]

\[
c_{Ft} = \alpha \theta_{t}.
\]

By symmetry, foreign’s demand for home goods is

\[
c_{Ht}^* = \alpha \frac{c_{Ft}^*}{p_{t}} \tag{2.4}
\]

where in the following we assume foreign’s consumption \( c_{Ft}^* \) to be equal to 1.

\textsuperscript{20}This presumes trade taxes are infeasible, as is standard in this literature, so terms-of-trade management is a second best tool. See, e.g. Costinot, Lorenzoni and Werning (2014).
Foreign intermediaries. There are two types of intermediaries in our model: Foreign intermediaries and domestic intermediaries. Both types of intermediaries will behave similarly but they differ in their ownership structure, which will play a key role in our analysis. We describe foreign intermediaries in detail in this section and their domestic counterparts in the next.

The key ingredient in our model that makes foreign exchange interventions effective is a finite elasticity of demand for home bonds. As a result, a change in the portfolio of the central bank has an effect on the expected return of domestic assets \( r_t \) relative to their foreign counterpart \( r_t^* \), i.e. the UIP wedge. Backus and Kehoe (1989) pointed out that these portfolio balance effects are muted in general equilibrium in a frictionless world in which Ricardian equivalence holds, as any actions by the central bank would be perfectly undone by the private sector. We break away from this result by modeling limited asset market participation, in the spirit of Bacchetta and Van Wincoop (2010) and Gabaix and Maggiori (2015). In particular, we assume that there exists a continuum of intermediaries owned by foreigners, labeled by \( j \in [0, \infty) \), which can trade in both foreign and domestic bond markets. Foreign intermediaries’ investment decisions are subject to three important restrictions.

First, each intermediary is subject to a net open position limit \( X > 0 \). Second, we follow Alvarez, Atkeson and Kehoe (2009) in assuming that intermediaries face heterogeneous participation costs. In particular, each intermediary \( j \) active in the domestic bond market at time \( t \) is obliged to pay a participation cost of exactly \( j \).

Putting these two ingredients together, intermediary \( j \) optimally invests an amount \( x_{jt} \), solving

\[
\max_{x_{jt} \in [-X,X]} x_{jt} (r_t - r_t^*) - 1_{\{x_{jt} \neq 0\}} j.
\]

Intermediary \( j \)'s present value of net profits conditional on investing is \( X |r_t - r_t^*| \), so investing is optimal for all intermediaries \( j \in [0, j] \) with the marginal intermediary \( \bar{j} \) given by \( \bar{j} = X |r_t - r_t^*| \). This gives an aggregate investment volume of

\[
b_t = \bar{j} X \cdot \text{sign} (r_t - r_t^*).
\]

Defining \( \Gamma_F \equiv (X^2)^{-1} \) and substituting \( \bar{j} \), we obtain

\[
b_{It} = \frac{1}{\Gamma_F} [r_t - r_t^*]. \quad (2.5)
\]

Equation (2.5) embodies that the foreign intermediaries’ demand for home bonds has a finite

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21Since our model is deterministic, there is no difference between realized and expected returns. Hence, any UIP violation would also lead to a violation of the covered interest parity (CIP). In reality, foreign exchange interventions deal with assets of different risk characteristics. We deal with this in an extension with risk premium shocks in Section A.2.

22It is worth noting that, as discussed in Section 2.2, many emerging market central banks in fact do impose position limits on intermediaries’ investments as a form of capital controls, hence artificially decreasing \( X \).

23It is straightforward to relax this assumption of linear costs to a more general monotonic function \( f(i) \). We show in Appendix B.5.2 that assuming linear costs is inessential for our results.
elasticity to the return spread.\textsuperscript{24} This equation is crucial to our analysis because it implies that changes in $b_{it}$, as for example induced by foreign exchange interventions, can indeed affect home interest rates. The key parameter in (2.5) is the inverse demand elasticity $\Gamma_F$.

When $\Gamma_F$ is large, e.g. if position limits $X$ are very small, intermediation is obstructed as evidenced by both, small levels of $b_{it}$ and a small sensitivity of $b_{it}$ to the interest rate spread. The case where $\Gamma_F = \infty$ corresponds to financial autarky. In this case, $b_{it} = 0$, which shuts down any sort of private financial intermediation in this baseline model. When $\Gamma_F$ is small, e.g. if position limits $X$ are very high, this leads to an equilibrium with less imperfect intermediation. In fact, if position limits were infinite, $\Gamma_F$ would equal to zero and we would recover the infinite elasticity, $r_t = r_t^*$. We call this case the frictionless economy.

**Home intermediaries.** In addition to foreign intermediaries, we shall also assume there is a similar continuum of home intermediaries generating an analogous bond demand schedule,

$$b_{it}^H = \frac{1}{\Gamma_H} [r_t - r_t^*].$$

(2.6)

There are two main differences between domestic and foreign intermediaries. First, domestic intermediaries are allowed to have a different elasticity $\Gamma_H$, which can be anywhere in $(0, \infty]$. Second, home intermediaries are owned by the representative household at home, whereas foreign intermediaries are owned by foreign households. To compute intermediaries’ profits we need to take a stance on the way transaction costs are being paid. To keep the model tractable, we assume that domestic intermediaries pay the transaction costs as transfers to each other, ultimately reaching domestic households through profits.\textsuperscript{25} This means, domestic intermediaries’ total profits are just given by their total revenues, and so households receive the following stream of per-period profits (in dollars),

$$\pi_t = b_{it}^H (r_t - r_t^*).$$

(2.7)

**Central bank.** The home central bank is the home country’s social planner in our model. It chooses a foreign exchange intervention (FXI) policy $\{b_{Gt}, b_{Gt}^*, t_t\}$ consisting of home bond investments $b_{Gt}$, foreign bond investments $b_{Gt}^*$, and transfers $t_t$ to home households, subject to the central bank budget constraint\textsuperscript{26}

$$b_{Gt} + b_{Gt}^* = r_t b_{Gt} + r_t^* b_{Gt}^* - t_t.$$ 

(2.8)

\textsuperscript{24}Strictly speaking, $\Gamma_F$ is only a semi-elasticity. For simplicity, we abuse the terminology and call it an “elasticity” in the remainder of this paper.

\textsuperscript{25}It would make absolutely no difference if transaction costs would directly flow to households, it would just require an additional term in the representative household’s budget constraint.

\textsuperscript{26}Note we implicitly assumed that the relevant interest rate for marginal changes of reserves is $r_t^*$. One might argue that negative levels of $b_{Gt}$ should be associated with a different, higher interest rate. In reality however, reserves are (almost) always positive and so marginal changes in reserves are associated with the foreign interest rate on savings, $r_t^*$. 

67
The central bank’s FXI policy must also ensure that the country satisfies a no-Ponzi condition,

\[ \lim_{t \to \infty} e^{-\int_0^t r_s \, ds} \text{nfa}_t = 0 \]  

(2.9)

where \( \text{nfa}_t \equiv \text{b}_H + \text{b}^*_G + \text{b}_G \) is the net foreign asset position of the country. Note that in this economy, it is without loss to set \( \text{b}^*_G + \text{b}_G = 0 \) due to the availability of transfers between the central bank and households.

**Competitive equilibrium.** The model is closed with a goods market clearing condition,

\[ c_H + c^*_H = y_H \]  

(2.10)

and a bond market clearing condition,

\[ \text{b}_H + \text{b}_I + \text{b}^*_H + \text{b}_G = 0. \]  

(2.11)

We can now formally define a competitive equilibrium in this environment.

**Definition 2.1.** Given initial debt positions \((\text{b}_H^0, \text{b}_I^0, \text{b}^*_H^0, \text{b}_G^0)\), paths for shocks \(\{y_H, y_F, r, r^*_t\}\), and a central bank FXI policy \(\{\text{b}_G, \text{b}^*_I, r_t\}\), an allocation \(\{c_t, c_H, c_F, c^*_H, \text{b}_H, \text{b}_I, \text{b}^*_H, \pi_t\}\) together with prices \(\{q_t, r_t\}\) is a competitive equilibrium iff they solve (2.1)–(2.11).

Next, we characterize the competitive equilibrium, with the goal to derive “implementability conditions” that describe the set of competitive equilibria that can be attained through different FXI policies. Substituting consumption demands (2.3) and (2.4) into the goods market clearing condition (2.10) gives us an expression for the dollar value of the endowment of home goods,

\[ q_t^{-1/(1-\alpha)} y_H = (1-\alpha)\theta_t + \alpha. \]  

(2.12)

Using the households’ dollar budget constraint (2.1), we then obtain

\[ b_H = \alpha(1-\theta_t) + y_F + r_t b_H + t_t + \pi_t. \]  

(2.13)

Here, the policy variable \(t_t\) can be eliminated after adding the central bank’s budget constraint (2.8), which allows us to rewrite the households’ budget constraint as a country-wide budget constraint,

\[ \text{nfa}_t = \alpha(1-\theta_t) + y_F + (r_t - r^*_t)(b_H + b_G) + r^*_t \text{nfa}_t + \pi_t \frac{b^*_H(r_t - r^*_t)}{b^*_H(r_t - r^*_t)}. \]  

(2.14)

In this equation, policy variable \(b_G\) can be expressed as \(-b_H - b_I - b^*_H\) using home bond market clearing (2.11), where intermediaries’ bond demand \(b_I\) is given by (2.5). Then, the country-wide
budget constraint (2.14) simplifies to

$$n_{fa_t} = \alpha (1 - \theta_t) + y_{Ft} + r^n_{t} n_{fa_t} - \frac{1}{\Gamma_F} (r_t - r^*_t)^2.$$  \( (2.15) \)

Up to the last term, equation (2.15) is nothing more than a standard open economy budget constraint. It implies that home's net foreign asset position improves if the trade balance (net exports) \(\alpha (1 - \theta_t) + y_{Ft}\) is large, or interest income \(r^n_{t} n_{fa_t}\) from existing foreign assets is large. The last term, however, is new. It captures the costs the country incurs if the interest rate spread \(r_t - r^*_t\), which is the same as a UIP deviation in our context, is different from zero.

Why does the country face costs from UIP deviations? Suppose the spread \(r_t - r^*_t\) is positive. This invites foreign intermediaries to come in and take a position \(b_{It} = \frac{1}{\Gamma_F} (r_t - r^*_t)\) in the domestic bond market, taking home revenues

$$b_{It} \cdot (r_t - r^*_t) = \frac{1}{\Gamma_F} (r_t - r^*_t)^2.$$  \( (2.16) \)

These carry trades represent economic costs to home as they are paying a premium to foreign investors over and above the world interest rate \(r^*_t\). Naturally, the costs increase when foreign intermediaries become more elastic to the UIP wedge, that is, when \(\Gamma_F\) is lower. In that case, for a given UIP wedge, intermediaries take larger positions, generating larger costs. Vice versa, if there are no active foreign intermediaries, \(\Gamma_F = \infty\), the country does not incur any costs.

Noticeably, the costs in (2.16) are independent of the degree of domestic intermediation \(\Gamma_H\). The reason is straightforward: While domestic intermediaries, similar to foreign ones, take a position that is proportional to the UIP wedge, their revenues do not leave the country and are instead rebated to domestic households.\(^{27}\)

Next we study the set of equilibria that are implementable by choosing a given path of foreign exchange interventions. For this result and the remainder of the paper, we introduce as notation for the UIP wedge \(\tau_t \equiv r_t - r^*_t\).

Rewriting the budget constraint (2.15) in present value terms we obtain the following implementability result.

**Proposition 2.1 (Implementability conditions.).** Suppose \(\Gamma_F > 0\) and \(\Gamma_H > 0\). Let \(\theta_t = q_{t}^{-1} c_{t}\) be the dollar value of home consumption and \(\tau_t = r_t - r^*_t\) be the "wedge" in the uncovered interest parity (UIP) condition. Then, given an initial net foreign asset position \(n_{fa_0}\) and shocks \(\{y_{Ht}, y_{Ft}, r^*_t\}\), the paths \(\{c_t\}\) and \(\{q_t, \tau_t\}\) are part of a competitive equilibrium iff the corresponding \(\{\theta_t, \tau_t\}\) solve the

\(^{27}\)Even though the assumption that domestic intermediaries' revenues entirely enter the representative household's budget constraint seems like a strong one, the model can easily cope with less extreme situations where only a fraction of those revenues fall to agents that enter the government's welfare considerations. In that case, one would merely relabel "domestic" and "foreign" intermediaries as "those who enter the government's welfare considerations" and "those who do not".
following two conditions: The Euler equation,

\[ \frac{\dot{\theta}_t}{\theta_t} = \tau_t^* + \tau_t - \rho \]  

(2.17a)

and the country-wide present value budget constraint,

\[ \int_0^\infty e^{-\int_0^t \tau_s^* ds} \left[ \alpha(\theta_t - 1) + y_F t + \frac{1}{\Gamma_F} \tau_t^2 \right] dt = nfa_0. \]  

(2.17b)

Proposition 2.1 gives us a simple characterization of the set of competitive equilibria as it is commonly used in models of optimal Ramsey taxation (see, e.g., Lucas and Stokey, 1983 or Chari and Kehoe, 1999). A key difference with this literature, however, is that the planner in our model does not choose a path of taxes, but rather an FXI policy as defined above. The importance of Proposition 2.1 is that it shows that setting FXI policies—which are paths of asset positions—in fact is equivalent to setting wedges \( \tau_t \) in the UIP condition—which behave like taxes.

As a side remark, we would like to stress that in addition to the costs \(-2\) coming from foreign intermediation, setting a path of nonzero UIP wedges \( \tau_t \) is, of course, already "costly" in that it distorts the consumption choices of domestic households. This will be the reason a planner in our economy only cares to deviate from \( \tau_t = 0 \) if there is an additional reason, like managing the terms of trade, that makes such deviations beneficial. There, the costs \( \frac{1}{\Gamma_F} \tau_t^2 \) coming from foreign intermediation will be an additional resource cost that the country incurs, and that, as it turns out, critically changes the optimal policy.

A simple corollary of Proposition 2.1 is that the set of implementable allocations is independent of the degree of domestic intermediation.

**Corollary 2.1.** In Proposition 2.1 the set of implementable allocations is independent of the degree of domestic intermediation \( \Gamma_H \) (as long as \( \Gamma_H > 0 \)).

We next set up the planning problem of choosing the optimal FXI policy.

### 2.3.2 Planning problem

We think of the central bank as the home economy's social planner. Thus, the central bank maximizes the welfare of domestic households across all competitive equilibria it can possibly implement using foreign exchange interventions. Domestic households' utility is given by \( \int_0^\infty e^{-\mu t} \log c_t dt \). Since the dollar value of home consumption, \( \theta_t = q_t^{-1} c_t \), is slightly more convenient to use, we express utility in terms of \( \theta_t \) and state the planning problem as

\[ \max_{\{\theta_t, \tau_t\}} \int_0^\infty e^{-\mu t} \{ \log \theta_t - (1 - \alpha) \log ((1 - \alpha) \theta_t + \alpha) \} dt \]  

(2.18)

subject to the two implementability conditions (2.17a) and (2.17b).

\[ ^{28} \text{See Appendix B.2.2 for a derivation.} \]
In the planning problem (2.18), the freedom of setting different FXI policies is completely embodied in the choice of the UIP wedge $\tau_t$. When the central bank desires to raise consumption in period $t$ relative to the next, it lowers $\tau_t$. Such a policy would then be implemented by selling reserves and purchasing home bonds, which, due to a finitely elastic foreign demand function, affects the domestic interest rate $r_t$ and thus $\tau_t$.

One possibility for the central bank in this baseline model is to set $\tau_t = 0$ in all periods, in which case it implements an allocation that would prevail if $\Gamma_F$ were equal to zero, that is, it “undoes” the imperfect intermediation friction. This is clearly possible in this model since the central bank can freely access both bond markets, and thus it may always create the right kind of bond supply to ensure that $\tau_t = \tau_t^*$, in which case foreign intermediaries’ positions $b^*_Gt$ are zero. We wish to emphasize that, however, there is no simple relationship between $\tau_t$ and the country’s reserve position $b^*_t$ in this baseline model. In particular, $\tau_t = 0$ does not necessarily correspond to zero reserves, and the relationship between $\tau_t$ and reserves more generally depends on $\Gamma_F$ and $\Gamma_H$.\footnote{Even though $\Gamma_H$ does not enter the planning problem directly, it turns out to matter for the reserve accumulation policy that implements the optimal paths for $\theta_t$ and $\tau_t$.} We discuss how a $\tau_t = 0$ policy can be implemented in Section 2.3.4.

As a benchmark, we now characterize the first best allocation. We define this to be the optimum to the planning problem (2.18) subject only to the resource constraint (2.17b).\footnote{There are other ways to define “first best” here. For instance, one could allow the planner to set optimal tariffs on exports of the home good. In that case, however, the SOE can extract an unlimited amount of resources from the rest of the world, so this alternative definition is rather meaningless.} For ease of notation, we abbreviate the planner’s per-period objective as $V(\theta) \equiv \log \theta - (1 - \alpha) \log ((1 - \alpha)\theta + \alpha)$.

**Lemma 2.1 (First best.).** When only the resource constraint (2.17b) is binding, the optimal (first best) allocation $\{\theta_t, \tau_t\}$ and the corresponding shadow resource cost $\lambda$ satisfy (i) $\tau_t = 0$, (ii) the implicit equation,

$$e^{-\rho t}V'(\theta_t) = e^{-\int_0^t \tau_s^* ds} \lambda \alpha$$

for each $t \geq 0$, and (iii) the resource constraint (2.17b).

Lemma 2.1 states the obvious: Absent any incentive compatibility conditions, the planner equates the marginal utility of (dollar) consumption in any period $t$ to the corresponding resource cost. Still, Lemma 2.1 will prove to be a useful benchmark later on.

A general advantage of writing the planning problem in terms of the UIP spread $\tau_t$ is that it provides us with a convenient link to the large literature on optimal capital controls. We now explore this link.

### 2.3.3 Connection to literature on capital controls

Our planning problem (2.18) is related to a recent literature on capital controls (see, e.g., Bianchi (2011); Farhi and Werning (2012, 2014); Heathcote and Perri (2016); Jeanne (2012)). In that literature, capital controls are typically modeled as a proportional tax on capital flows, which
directly induces a spread in the uncovered interest parity equation, i.e. a spread between \( r_t \) and \( r_t^* \)—just like in our model, where foreign exchange interventions induce such a spread. The key difference with our framework is the additional cost term \( \frac{1}{\Gamma F} r_t^2 \) in the resource constraint, capturing net losses from foreign intermediation.

To better understand the connection to the literature on capital controls, consider a model with the same real structure as our paper but frictionless financial markets. Suppose the planner is allowed to pick a path for taxes on capital inflows, which immediately show up as UIP wedges \( \tau \). Then, the planning problem in that economy,\(^{31}\) casted in terms of UIP wedges \( \tau \), is exactly the same as our planning problem, except for the extra resource costs \( \frac{1}{\Gamma F} r_t^2 \). Indeed, our planning problem and the one described above become formally equivalent when \( \Gamma_F = \infty \). As will become clear in Section 2.4, the new term will shape the response of optimal policy and deliver several new insights.

However, we would like to stress that the economic interpretation is very different. Capital flow taxes are powerful to the extent that they shape the response of the private sector. Therefore, they are ineffective in the absence of private intermediation, which is precisely when FXI policies are most effective. The converse is true when private intermediation is frictionless. More generally, the two tools can be viewed as complements: capital controls may effectively put sand in the wheels of private intermediation, which increases \( \Gamma_F \) and thereby relaxes the planner’s FXI problem.\(^{32}\)

### 2.3.4 Zero-reserves and zero UIP wedge allocations

Before we move on to study how the central bank should optimally use foreign exchange interventions, we analyze two specific implementable allocations: the “laissez-faire” equilibrium, where the central bank keeps reserves at zero, and a \( r_t = 0 \) economy, where the central bank undoes the financial friction. These two allocations will help gauge the optimal interventions we study in the next section. In both cases, we characterize the response of the economy to three perfect foresight shocks: one to the world interest rate \( r_t^* \), one to the endowment of the home good \( y_{Ht} \), and one to the endowment of the foreign good \( y_{Ft} \).\(^{33}\) We choose the signs of the shocks so that they imply a positive response of home consumption. To be consistent with our exercise in Section 2.4 below, we use the same simple calibration for the graphs shown in this section. In order to avoid repetition, we refer the reader to Section 2.4.1 for details on calibrated parameter values and a discussion of our calibration strategy.

**Negative shock to the world interest rate \( r_t^* \).** Figure 2-3 illustrates the effects of a negative, 3-period-long shock to \( r_t^* \) on the two allocations. First, consider the laissez-faire allocation, the solid blue line. There, the shock increases foreigners’ demand for home bonds, which pushes down \( r_t \), but less than one-for-one with the shock to \( r_t^* \) due to the finite demand elasticity. This means, \( r_t \) rises

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\(^{31}\)Farhi and Werning (2014) analyze a version of this planning problem that includes labor.

\(^{32}\)This may explain why policymakers often put taxes on both inflows and outflows (Fernández et al., 2016) or put in place position limits on foreign exchange positions (BIS, 2005, 2013).

\(^{33}\)Since this is a deterministic economy, we refer to “shocks” as the deterministic response of an economy previously in steady state to a change in the deterministic path of a parameter such as \( r_t^* \), \( y_{Ht} \) or \( y_{Ft} \).
Figure 2-3: Response of the laissez-faire economy (blue, solid) and the $r_t = 0$ economy (red, dashed) to a negative $r^*$ shock in a deterministic model economy calibrated to Brazil, with $\Gamma_F = 10$. The shock lowers $r^*$ by 2% for $t \in [0, 3]$.

over time (Panel (a)). The lower rate $r_t$ then has the following consequences. It leads to higher domestic consumption $c_t$, an appreciated real exchange rate (Panel (b)) and a worsening of the net foreign asset position. As shown in Panel (c), reserves are zero by design.

The response of the $r_t = 0$ economy (dashed red line) is quite different from that. First, as a direct consequence of $r_t = 0$, interest parity holds, $r_t = r^*_t$. Therefore, the shock to $r^*_t$ is passed through to $r_t$ one-for-one. This leads to a more pronounced uptick in consumption and hence a more pronounced real exchange rate appreciation (Panel (a)), compared to the laissez-faire economy. Finally, note that since zero spreads reduce intermediation profits to zero, the central bank must do all the intermediation, selling reserves to reflect the country’s desire to borrow (Panel (c)).

Why does the central bank need to intervene in the same direction as the shock in order to achieve $r_t = 0$? Notice that any friction inhibiting the free flow of international capital always exhibits the following two properties: First, there is the “elasticity” property, discussed in detail above: By taking a certain foreign exchange position, the central bank has the ability to influence domestic real interest rates and exchange rates. But there is also an “underreaction” property: In response to shocks to the foreign interest rate $r^*_t$, the level of domestic interest rates underreacts and moves less than one for one, giving rise to a positive UIP spread $r_t > 0$ (Panel (a)). The real exchange rate underreacts as well, since the expected UIP spreads are all positive. It is precisely this “underreaction” aspect of limited capital mobility that seems to suggest reserve decumulation is necessary to achieve $r_t = 0$ after a negative $r^*_t$ shock.

Seemingly contrary to our model’s prediction, policymakers believe that domestic real interest rates $r_t$ are more likely to overreact rather than underreact to global liquidity shocks, absent any intervention. However, in practice, global liquidity shocks do not only affect $r^*$ but also the risk

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$^{34}$To check the prediction of our model we also conducted a (preliminary) analysis of the response of various countries' UIP spreads to identified US monetary policy shocks. We could not find any statistically significant evidence for a nonzero response of UIP wedges.
and liquidity properties of local bonds vis-a-vis foreign bonds. At the end of Section 2.5.2 we discuss how shocks with these features that comove with $r^*$ shocks might eliminate any "underreaction", without affecting the "elasticity" aspect.\footnote{In a recent paper Engel (2016) makes a related point, arguing that multiple comoving shocks are necessary to explain the covariance between interest rate differentials and UIP spreads.} For expositional clarity, in Section 2.4’s plots we show the reserve purchases or sales the central bank needs to make at the optimum relative to any interventions it might have to conduct (if any) to achieve $\tau_t = 0$. This avoids confusion as to whether paths for reserves are determined by the "underreaction" aspect or rather by the optimal policy itself.

**Shocks to endowments $y_{Ht}$ and $y_{Ft}$.** Assuming $r_t^* = \rho$, it follows immediately that—for any path $\{y_{Ht}, y_{Ft}\}$—a constant path of dollar consumptions $\{\theta_t\}$ achieves the first-best outcome described in Lemma 2.1. Note that when $\theta_t$ is constant, equation (2.12) directly links higher values for $y_{Ht}$ to a depreciated real exchange rate in a way that exactly compensates $y_{Ht}$, leaving the exported value in dollars, and hence the current account, constant. This is a direct consequence of assuming Cole-Obstfeld preferences. This implies that reacting to $y_{Ht}$ does not actually require any action by the central bank.

In contrast, $y_{Ft}$ shocks require an active portfolio management by the central bank. After a positive $y_{Ft}$ shock, the central bank needs to accumulate reserves in order to save on behalf of households, undoing the financial friction. In the laissez-faire equilibrium, households can only save by paying a premium, which lets dollar expenditure become procyclical. In other words, the laissez-faire equilibrium has inefficient real exchange rate fluctuations in response to wealth shocks $\{y_{Ft}\}$.

As we have just seen, for $y_{Ht}$ and $y_{Ft}$ shocks, the first best is the optimal policy, as there is no terms-of-trade management motive in this case. This is not the case for world interest rate shocks $r_t^*$, in response to which we know study the optimal foreign exchange interventions.

## 2.4 Optimal foreign exchange interventions

In this section, we present our main results about the normative behavior of foreign exchange interventions. The fundamental trade-off that determines the optimal use of foreign exchange intervention in our model is between two forces. The first is the desire to minimize the additional cost term (2.16). The second is a terms-of-trade management motive, which appears when the economy is subject to $r_t^*$ shocks. More generally, however, our analysis should carry over to any sort of "macroeconomic stabilization" motive, which might lead the central bank to use foreign exchange interventions as a second best tool to influence the exchange rate or home interest rates. We explore this trade-off in three steps in this section. In Section 2.4.1 we briefly introduce the calibration underlying the plots shown in this section and the previous one. Section 2.4.2 characterizes the optimal intervention policy without any motive to manage the terms-of-trade. In this case, the planner would like best not to distort private consumption decisions. And in a final step in
Section 2.4.3, we include our terms-of-trade management motive and study the optimal policy under the full trade-off. Before this, we briefly introduce the calibration underlying the plots in this section and the previous one.

2.4.1 Calibration

For our illustrative simulations, we calibrate the model parameters to Brazil. For the discount rate we pick $\rho = 0.075$, corresponding to the average 5yr treasury yield from 2000–2015 plus the average J.P. Morgan EMBI+Brazil return over the same time period. For the openness $\alpha$ of the economy, we choose $\alpha = 0.15$ matching a 15% imports to GDP ratio in 2013. We normalize $y_H$ to 1 and $y_F$ to 0. Our results do not really depend on the initial net foreign asset position, so for simplicity, we set it to zero at the beginning of our figures. To get an idea of the relative size of domestic compared to foreign intermediation, we note that Brazil’s domestic banks operated balance sheets roughly five times the size of foreign banks’ subsidiaries in Brazil. Interpreting balance sheet size as rough proxy for portfolio constraints (corresponding to $X$ in our microfoundation), this leads us to calibrate $\Gamma_H/\Gamma_F = 5^2$.

Unfortunately, there is no easy way to calibrate $\Gamma_F$. Therefore, we provide three values for $\Gamma_F$, $\Gamma_F \in \{1, 10, \infty\}$, wherever it does not clutter up figures; elsewhere we use a single value, mostly $\Gamma_F = 10$, as illustration. Notice that when varying $\Gamma_F$, we also vary $\Gamma_H$ according to our calibration of the ratio $\Gamma_H/\Gamma_F$. As will become clear below, choices for $\Gamma_F$ of 1 or 10 approximately imply that a real exchange rate depreciation of 1% for one year requires a peak accumulation of reserves relative to GDP of 1.5% for $\Gamma_F = 10$, and of 7% for $\Gamma_F = 1$. This seems to be in the ballpark of empirical estimates. Kearns and Rigobon (2005) use structural breaks in the intervention policies of two advanced economies, Japan and Australia, to identify the effectiveness of interventions. Converting their findings into this context reveals that the same 1% depreciation requires a reserve accumulation of 4% over GDP for both economies. Equivalent numbers for emerging market economies are most certainly lower than these. De Gregorio (2013) mentions that practitioners in those countries often use a reserve accumulation of 1-2% (over GDP) as benchmark.

Finally, we consider in our plots $r^*$ shocks that temporarily lower $r^*$ by 2% for 3 years before they return to the steady state value of 7.5%.

2.4.2 Optimal interventions without terms-of-trade management

We can shut off the macro stabilization motive in our planning problem (2.18) in two ways. First, we can set $\alpha = 1$. In this case, the country loses home bias for its own good and hence the central bank loses the ability to influence the price of its own good by reallocating consumption over time. Thus, there is no longer a motive to manage the terms-of-trade. Second, it turns out that there is also no such motive when the world interest rate is constant and equal to home’s discount factor, i.e. $r^*_w = \rho$ at all times. We now investigate both of these cases, showing that in both of them the optimal UIP wedge $\tau_t$ is equal to zero. After that, we discuss the implications for reserves $b^*_G$ and actual interventions.
Proposition 2.2. Suppose $\alpha = 1$ or $r_t^* = \rho$ for all $t \geq 0$. Then, the optimal allocation coincides with the first best. In particular, $\tau_t = 0$ at all times $t$.

Proposition 2.2 identifies two cases for which the planner sees no need for nonzero UIP wedges and hence chooses not to distort the economy. The arguments behind the two cases are distinct. When $\alpha = 1$, the home economy has no home bias and therefore its consumption is unable to affect the real exchange rate. Therefore, interventions are completely ineffective in this case. When $r_t^* = \rho$ at all times, even if $\alpha$ is possibly less than 1 or there are endowment shocks $\{y_{Ht}\}$, home’s consumption $\theta_t$ is constant over time, and so are home’s exports. Thus, there is no reason to manipulate the terms of trade over time.

In the next subsection, we explore deviations from this “neutrality” result. In particular, when $\alpha < 1$ and the economy faces $r_t^*$ shocks, it turns out that the central bank has a macroeconomic stabilization motive and generally finds it optimal to implement nonzero UIP wedges $\tau_t$. For the remainder of Section 2.4, we set $\alpha < 1$.

2.4.3 Optimal interventions with terms-of-trade management

In this subsection, we study the solution to the planning problem (2.18) in the presence of a motive to manage the terms of trade. This motive is also at the core of many papers on capital controls (see e.g. Costinot, Lorenzoni and Werning, 2014 or Farhi and Werning, 2012, 2014). In Section 2.5.1 below, we study an alternative motive for intervention, based on fixed exchange rates and sticky prices.

We focus on $r^*$ shocks since we have already seen in the previous section that time-varying $\{r_t^*\}$ is crucial for an intervention motive. Specifically, we refer to paths $\{r_t^*\}$ such that $r_t^* > \rho$ for all $t \in [0, T)$ and $r_t^* = \rho$ thereafter as positive interest rate shocks; and to paths $\{r_t^*\}$ such that $r_t^* < \rho$ for all $t \in [0, T)$ and $r_t^* = \rho$ thereafter as negative interest rate shocks. We assume that $\{r_t^*\}$ is integrable throughout this section. As before, we first characterize the optimal foreign exchange intervention policy in terms of the path of induced UIP wedges $\{\tau_t\}$. Subsequently, we discuss the implications for reserves and exchange rates, relative to the “undoing” benchmark, $\tau_t = 0$ for all $t$ (see the discussion in Section 2.3.4).

The benchmark of financial autarky, $\Gamma_F = \infty$

We begin the analysis by studying the special case of our model in which the private sector is in financial autarky ($\Gamma_F = \infty$). This is useful to isolate the motive for intervention by the central bank.

Proposition 2.3. Suppose $\Gamma_F = \infty$. Then, the optimal intervention after a positive interest rate shock hits is to set $\tau_t < 0$ for any $t \in [0, T)$ and $\tau_t = 0$ thereafter. In particular, the central bank only intervenes during the time of the interest rate shock. Analogously, $\tau_t > 0$ for $t \in [0, T)$ and $\tau_t = 0$ thereafter in response to a negative interest rate shock.
Figure 2-4: Optimal intervention after a negative $r^*$ shock in a deterministic model economy calibrated to Brazil, for various degrees of capital market imperfection $\Gamma_F$. The shock lowers $r^*$ by 2% for $t \in [0,3]$. The results are relative to the zero UIP wedge economy ($\tau_t = 0$).

Similar to Costinot, Lorenzoni and Werning (2014) our model embeds a terms-of-trade management motive: Individual agents do not internalize the effect of their consumption decisions on the price of the exported good. This effect is nonzero as a result of the assumption of home bias. To fix ideas, suppose the foreign interest rate is temporarily low at time $t$, $r_t^* < \rho$. Since this implies that exports are relatively low at time $t$ (borrowing against future income), the planner would like to lower the export price, or equivalently depreciate the real exchange rate. Setting a positive UIP wedge, $\tau_t > 0$ then reduces current consumption, which in turn, achieves the desired real exchange rate depreciation.

This is also what we see in Figure 2-4, which shows the economy’s reaction to the simple negative interest rate shock described in our calibration in Section 2.4.1. The red line shows the response in case of financial autarky, $\Gamma_F = \infty$. It is evident that the UIP wedge jumps up as the shock hits and back down to zero as the shock fades (Panel (b)), thereby reallocating domestic demand into the future and depreciating the real exchange rate (Panel (c)). The economy executes this intervention by accumulating additional reserves during the period of the shock (Panel (d)). Such a policy is often referred to as “leaning against the wind” of international capital flows.

**Intervention smoothing**

Compared to the special case of $\Gamma_F = \infty$, studying optimal policy with intermediate degrees of capital mobility delivers three key new insights. Given that setting wedges is costly, one may expect that the optimal policy would lie somewhere between the $\Gamma_F = 0$ solution, that is $\tau_t = 0$, and the $\Gamma_F = \infty$ solution characterized in Proposition 2.3. The following result shows this intuition is fundamentally wrong.

**Proposition 2.4 (Smoothing.).** Suppose $\Gamma_F \in (0, \infty)$. Then, at the optimum, $\tau_t$ is continuous in $t \in (0, \infty)$, with $\tau_0 = 0$. 
Proposition 2.4 highlights a property of the model that is only present for intermediate degrees of capital mobility: the central bank chooses a smooth path for \( \tau_t \).\(^{36}\) Contrast this with the \( \Gamma_F = \infty \) solution: There, \( \tau_t \) jumps whenever \( r_t^* \) jumps. The reason for the "smoothing" result is very natural: With \( \Gamma_F \in (0, \infty) \), each deviation of \( \tau_t \) away from zero incurs convex costs (2.16). Hence, it is optimal to spread out interventions over time, optimally making interventions small and long lived, rather than large and short lived.

This result follows from a helpful lemma. To state the lemma, we introduce

\[
T_t = e^{-\int_0^t r^*_s ds} \frac{\lambda}{\Gamma_F} \left( e^{-pt} V'(\theta_t) - \text{marg resource cost at time } t \right) - \text{marg utility at time } t
\]

as the deviation of time \( t \) (dollar) consumption \( \theta_t \) from first best levels. \( T_t > 0 \) whenever consumption \( \theta_t \) is too large relative to first best, and \( T_t < 0 \) whenever \( \theta_t \) is too small relative to first best.\(^{37}\) Here, \( V(\theta) \) is the planner's per-period objective. Using this notation, the lemma can be stated.

**Lemma 2.2.** Suppose \( \Gamma_F \in (0, \infty) \). Let \( V(\theta) \) be the planner’s per period objective, as defined before Lemma 2.1. Then, under the optimal foreign exchange intervention policy, the interest rate spread \( \tau_t \) satisfies the following first order condition

\[
e^{-\int_0^t r^*_s ds} \frac{2}{\Gamma_F} \tau_t = \int_0^t T_s ds
\]

(2.19)

Lemma 2.2 is a straightforward consequence of the first order conditions of the planning problem and immediately implies Proposition 2.4. First, since the integrals of \( r^*_t \) and \( T_t \) are continuous functions, it follows that \( \tau_t \) is continuous. And second, since the right hand side of (2.19) is zero at \( t = 0 \), it must be that \( \tau_0 = 0 \).

The first order condition (2.19) has a useful intuition. Suppose the planner increased \( \tau_t \) by a marginal unit. Increased carry trades by foreign intermediaries would then consume an extra \( \frac{2}{\Gamma_F} \) of the economy’s resources, valued at shadow price \( \lambda \). This is captured as the marginal cost term on the left hand side of (2.19). However, such an intervention would also lower \( \theta_s \) in all previous periods \( s < t \), due to the forward looking nature of the Euler equation. In each such period \( s \), it saves one unit of resources and increases marginal utility, whose joint effect on utility is precisely captured by \( T_s \). This explains the right hand side of (2.19).

Panel (b) of Figure 2-4 illustrates our “intervention smoothing” result. Even though the path for the interest rate shock \( \{r^*_t\} \) is discontinuous, and in stark contrast with the optimal UIP wedges when \( \Gamma_F = \infty \) (the red line), for finite positive values of \( \Gamma_F \) the optimal UIP wedges are continuous and start at zero. Their sign is the same as the one for \( \Gamma_F = \infty \), so here again, the planner leans against the wind and accumulates reserves (Panel (c)) to depreciate the real exchange rate (Panel

\(^{36}\)This is reminiscent of a number of “tax smoothing” results in the optimal taxation literature, spawned by Barro (1979).

\(^{37}\)Notice that this is not a mathematically rigorous statement since \( \lambda \) here is not the same as in the first best problem. \( T_t \) still turns out to be a very useful object.
It is worth pointing out that the reason for why the lowest value for $\Gamma_F$, $\Gamma_F = 1$, is associated with the largest accumulation of reserves comes from the fact that we calibrate the ratio $\Gamma_H/\Gamma_F$ to the data, and so lower values for $\Gamma_F$, capturing more foreign intermediation, automatically lead to lower values for the $\Gamma_H$ as well, capturing more domestic intermediation. This leads to two countervailing forces: Lower $\Gamma_F$ pushes for less aggressive and smoother interventions, while lower $\Gamma_H$ means the central bank needs to accumulate more reserves to achieve a given spread $\pi$.

The fact that in Panel (b) of Figure 2-4, optimal UIP wedges for $\Gamma_F < \infty$ are still positive well beyond the end of the shock at $t = 3$ is the subject of our next subsection.

**Forward guidance**

The smoothness of the intervention $\pi$ has two interesting indirect consequences: The first one, described in this subsection, could be described as “FXI forward guidance”. Since interventions are smoothed out over time, the planner in fact has an interest in promising to keep intervening—that is, creating nonzero wedges $\pi_t \neq 0$—even at times $t > T$, after the shock subsided. We formalize this in the following result.

**Proposition 2.5.** Suppose $\Gamma_F \in (0, \infty)$. Then, after a positive interest rate shock, $\pi_t < 0$ at all times $t$ (including $t > T$). Analogously, after a negative interest rate shock, $\pi_t > 0$ at all times $t$ (including $t > T$).

To see the intuition behind this result, consider the first order condition (2.19). While the marginal cost of a marginal intervention at time $t$ is born at that current time due to increased private carry trade activity, the marginal benefits $T_s$ of influencing future interest rates accrue at all times $s \leq t$ before $t$. In that sense, the logic is analogous to forward guidance in a New Keynesian model (see, e.g., Eggertsson and Woodford 2003 or Werning, 2011), where marginal benefits of low rates after the zero lower bound stops binding also propagate back in time through the Euler equation.

This result also speaks to the ongoing debate over whether the likely channel through which foreign exchange interventions work is a portfolio balance channel or some kind of signaling channel. While the core of our model consists of a portfolio balance channel—foreign intermediaries only imperfectly react to the interest rate spread $r - r^*$, well in the spirit of the old portfolio balance literature (see, e.g. Kouri, 1976, Branson and Henderson (1985) or Kenen (1987))—our microfoundations and rational expectations weave a natural signaling channel into our model. Since future interventions are effective through a portfolio balance channel in the future and agents are forward-looking, signaling future interventions has the power to affect agents’ actions today. In Panel (b) of Figure 2-4, the forward guidance aspect of optimal interventions is clearly visible. For finite, positive $\Gamma_F$, the optimal UIP wedges are not only smooth over time, but also stretch well into the future, beyond the period of the shock.
Time inconsistency

Clearly, in contrast with the direct effect through current portfolio choices, the effectiveness of signaling future interventions critically depends on the credibility of the central bank. This naturally opens the door to problems of credibility and time inconsistency. We formulate this in the next proposition.

**Proposition 2.6.** Suppose $\Gamma_F \in (0, \infty)$. The optimal policy is time-inconsistent and re-optimization at any time $t_0 \geq T$ yields $\tau_t = 0$ for all $t > t_0$. Moreover, a planner without any commitment power can only achieve the no-intervention outcome, $\tau_t = 0$ at all times $t$.

The argument behind the first part of Proposition 2.6 is quite straightforward. We already saw above that in an environment without shocks (see Lemma 2.2) the optimal UIP wedge is zero at all times. The time after a shock has faded, that is, $t > T$ for the interest rate shocks we have focused on in this section, is precisely such a time of no more shocks. Yet the optimal policy as described in Proposition 2.5 requires nonzero UIP wedges in all periods. The planning problem thus is time inconsistent.

The second part is more involved. Even if a planner without any commitment power will set $\tau_t = 0$ for all $t > T$, why would he do so during the time of the shock as well? The answer to this question lies in the fact that even during the time of the shock, interventions derive their effect from affecting earlier consumption decisions, see e.g. the first order condition in (2.19). Yet, since those consumption decisions are in the past, a planner without any commitment power does not take them into account and instead chooses as optimal policy $\tau_t = 0$.

The time inconsistency issue raises the question of how much the effectiveness of interventions depends on the credible signaling of future interventions. We explore this question in our simulations in Figure 2-5. Here we compare a full commitment (FC) policy with $\Gamma = 10$ to a limited commitment (LC) policy, where the central bank can only credibly commit to interventions until the shock fades at time $T = 3$. We see in Panel (b) that the LC planner uses the limited commitment to promise strong interventions during the time of the shock, to make up for the interventions the FC planner promises after $t = T$. While the increased interventions until $t = T$ are costly to the LC planner, they do achieve almost exactly the same extent of real exchange rate depreciation as the FC planner. Panel (d) shows that the LC planner accumulates more reserves than the FC counterpart until $t = T$. At that time, however, when the UIP wedge $\tau_t$ drops to zero and home as well as foreign intermediaries close their carry trade positions, the LC planner balances this by repatriating a large fraction of the accumulated reserves.

We should note that the type of time inconsistency is different from the standard time inconsistency coming from the tendency to depreciate one’s (real or nominal) exchange rate when foreigners hold domestic local currency bonds. Since we analyzed our model in terms of domestic bonds measured in dollars, this type of time inconsistency does not appear in our setup and hence does not get entangled with the novel type of time inconsistency that we describe in Proposition 2.6. Of course, were we to denote initial positions in terms of the local price index, the more standard time
Figure 2-5: Comparing interventions with full commitment to interventions with limited commitment (only commitment until \( t = 3 \)). The deterministic model economy is calibrated to Brazil with \( \Gamma_F = 10 \). The shock lowers \( r^* \) by 2\% for \( t \in [0,3] \). The results are relative to the zero UIP wedge economy (\( \tau_t = 0 \)).

In sum, Sections 2.4.3 and 2.4.3 highlight that foreign exchange interventions are more powerful when they are coupled with signaling and when the central bank has at least some amount of commitment power.

2.5 Extensions

In this section, we present and discuss three extensions to our baseline model. In a first one, in Section 2.5.1, we introduce a model with an alternative motive for interventions. There, the central bank faces constraints on exchange rate movements—to make it stark we assume a fixed exchange rate—and yet seeks to use foreign exchange interventions to stabilize the output gap. We find that all our previous analytical results go through in this economy. In Section 2.5.2 we enrich our model somewhat to include liquidity and risk premia, at least in some abstract form. We show that when a country sells “safe haven” bonds, it might actually earn money from foreign exchange interventions, rather than pay for them. Finally, Section 2.5.3 presents an economy which is pursuing a “managed float” policy in which the real exchange rate is required to follow a smooth path. We show that this kind of policy, which resonates well with many exchange-rate based rules of EMEs, may significantly backfire, inviting costly speculation.

In Appendix B.5 we provide two additional extensions, one on whether the time inconsistency of our baseline model can be fixed by the planner itself using assets of multiple maturities a la Lucas and Stokey (1983); and the other generalizing intermediaries’ asset demand to nonlinear demand schedules.
2.5.1 Sticky prices and fixed exchange rates

So far, our analysis focused on foreign exchange interventions driven by a terms-of-trade management motive. One possible interpretation of that model is that there is a monetary authority in the background choosing the nominal interest rate to close the output gap at all times. To see this, consider a simple extension in which the home good is actually produced with a simple technology \( y_{Ht} = n_t \), and the household experiences disutility of labor given by \( v(n_t) \), so preferences are given by

\[
\int_0^\infty e^{-\rho t} \{ \log(c_t) - v(n_t) \} \, dt.
\]

(2.20)

In addition, to have a meaningful monetary policy problem, assume the home currency price of the home good is fixed at \( P_{Ht} = 1 \) and the nominal exchange rate is given by \( e_t \). This implies that the dollar value of \( y_{Ht} \) is given by

\[
e_t^{-1} y_{Ht} = (1 - \alpha) \theta_t + \alpha c^*,
\]

(2.21)

where we re-introduced \( c^* \) from Section 2.3.1. Compare this to the flexible price allocation: There, output is given as \( y'_{Ht} = n'_t \), pinned down jointly with the flexible price exchange rate \( e_t = e'_t \) by combining (2.21) with

\[
v'(n'_t) = \theta_t^{-1} (e'_t)^{-1}.
\]

Thus in our previous analysis, we can imagine that monetary policy is implementing \( e_t = e'_t \) at all times. In this sense, the output gap objective takes priority over the real exchange rate objective. In this subsection, we explore the polar opposite: We assume that—for some unmodeled reason—the monetary authority has some exchange rate objective \( e_t \). To make it stark, we assume a fixed exchange rate regime, \( e_t = \bar{e} \), and ask: How can the planner use foreign exchange interventions to regain some monetary independence and mitigate the impact on the domestic economy? Examples of interventions of this sort arguably include recent interventions by Euro neighbors like Denmark, Switzerland, or the Czech Republic, which try to fend off appreciations and at the same time avoid being pushed into the zero, or effective, lower bound for interest rates.

In a first step, we ask which allocations can be implemented by central bank policies. Fortunately, it is straightforward to show that, in fact, when stated in terms of \( \{ \theta_t, r_t \} \), the same implementability conditions as in Proposition 2.1 continue to hold in this economy. The reason is that sticky prices let labor supply and the home endowment depend on \( \theta_t \), yet neither enters the implementability conditions. They do, however, enter the objective function. Replacing labor with (2.21) in the

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38 We use the convention that lower values of \( e_t \) reflect a more appreciated home currency.

39 Here, we take a shortcut and loosely describe monetary policy as choosing a path for the nominal exchange rate. This can be made more formal by assuming that there is a nominal interest rate \( i_t \) such that \( r_t = i_t + e_t \). This interest rate can then be implemented using a standard interest rate rule. Note that, according to the standard definition, our interventions in the baseline model are not fully sterilized: The nominal interest rate automatically adjusts in response to interventions to replicate the flexible price allocation, and is therefore not constant.

40 Strictly speaking, with \( c^* \neq 1 \), the term \( \alpha(\theta_t - 1) \) in (2.17b) needs to be replaced by \( \alpha(\theta_t - c^*) \) but everything else is unchanged.
utility function (2.20) and following the same steps as before, we find the planning problem to be

$$\max_{\{\theta_t, \tau_t\}} \int_0^\infty e^{-\rho t} \{\log \theta_t - (1 - \alpha) \log((1 - \alpha)\theta_t + \alpha c^*) - v((1 - \alpha)\theta_t + \alpha c^*)\} \, dt$$

(2.22)

subject to (2.17a) and (2.17b).

The crucial difference to our previous planning problem is the objective function. The reason for this is that the rationale for intervening has changed. Before, the central bank was intervening to manage the country’s terms of trade. Now a second rationale emerges: Regaining monetary independence despite the fixed exchange rate. Suppose the world interest rate decreases temporarily. The flexible exchange rate response would be to let the currency appreciate today and depreciate in the future. Since this is impossible with a fixed exchange rate, the economy experiences a boom and a subsequent recession. In this situation, by accumulating reserves and hence generating a positive UIP spread, the planner is able to shift expenditure into the future. This mitigates both the boom and the subsequent recession.

To see whether this “leaning against the wind” property, as well as our other results in Section 2.4, carry over to this fixed exchange rate environment, notice that the planning problem (2.22) is almost unchanged: It still involves a strictly concave per period objective function and the maximization is subject the exact same constraints. Therefore, the results in Lemmas 2.1 and 2.2 and Propositions 2.2–2.6 carry over to this alternative environment, one for one. In particular, optimal interventions in this fixed exchange rate environment are still small, frequent, persistent and credible. This example illustrates that while the reason for intervening may differ across applications, the way interventions are implemented does not. In this sense, the results of Section 2.4 are robust.

2.5.2 Safe havens

While the fact that our analysis is deterministic has several advantages, in particular in terms of clarity and tractability, it lacks risk or “safety” premia. In recent years, some advanced economies, most notably Switzerland, have conducted foreign exchange interventions in a setting where domestic bond yields are typically lower than the world interest rate, despite the interventions. This raises the question of whether these interventions are costly to the central bank at all, or if they might actually benefit from them. Here, by cost we mean our cost term $1/\Gamma_F \tau^2$ which can never be negative (i.e. a profit) in our analysis without risk premia so far. In this subsection, we add a simple modification to our existing model that seeks to capture the key implications of risk premia for foreign exchange interventions, without giving up our tractable deterministic framework.

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41To see this most clearly, one may set $c^* = 0$. In that case, there is no “terms-of-trade management” motive and, yet, the planner would like to intervene to stabilize the output gap.


43We also would like to highlight a different but highly complementary perspective by Amador et al. (2016) to understanding Swiss interventions, using CIP rather than UIP violations: If one interprets the risk of the Swiss
The reason why risk is important to study in our context is that agents in the domestic economy might value bonds using a different stochastic discount factor than foreign intermediaries.\textsuperscript{44} To capture this idea in our deterministic model, we now explore what happens if intermediaries perceive an additional benefit of $\xi t$ for each additional unit of the domestic bond held.\textsuperscript{45}

\[ b^H_{it} = \frac{1}{\Gamma_H} (r_t + \xi_t - r^*_t) \quad \text{and} \quad b^F_{it} = \frac{1}{\Gamma_F} (r_t + \xi_t - r^*_t). \]

These equations only affect the previous intermediary bond demands (2.5) and (2.6) but leave our definition of competitive equilibrium otherwise unchanged. In particular, intermediary profits are still given by $b^j_H (r_t - r^*_t)$ for intermediary $j = H, F$. While it turns out that, as before, domestic intermediaries' bond demand does not enter the implementability conditions, the new foreign intermediaries' demand function changes the costs of UIP wedges $\tau_t = r_t - r^*_t$ to

\[ \tau_t b^F_{it} = \frac{1}{\Gamma_F} \tau_t^2 + \frac{1}{\Gamma_F} \xi_t r_t = \frac{1}{\Gamma_F} \left( \frac{\xi_t}{2} \right)^2 + \frac{1}{\Gamma_F} \xi_t^2 / 4. \]  \hspace{1cm} (2.23)

In this environment, it is evident that not all interventions are costly. If—as one could argue applies to safe havens like Denmark or Switzerland—the domestic bond is considered a particularly safe asset to investors, captured by a positive $\xi_t$, then intervening in the foreign exchange market to generate positive UIP wedges $\tau_t$ can leave the country with a net profit. The intuition for this is straightforward: In the case of $\xi_t > 0$, the country is the sole producer of an asset that outside investors value higher than the asset’s producer (the home country). Thus, it can supply the market with the asset and charge a premium in form of a positive $\tau$ for it. This generates profits.

Apart from the fact that interventions can profitable, (2.23) also reveals that the myopically optimal UIP wedge $\tau_t$ is no longer zero, and instead equal to $-\xi_t / 2$. That is, the planner now seeks to smooth out $\tau_t + \xi_t / 2$ over time, rather than $\tau_t$, since those are the deviations from the myopically optimal policy. This is essentially the key difference to our previous analysis. Figure 2-6 illustrates the optimal intervention in response to a $\xi_t$ shock, again relative to the non-intervention economy. It can be seen that qualitatively, the plots look very similar to their counterparts in Figure 2-4.

Intermediary preference shocks are also useful in a different way. In Section 2.3.4 we argued that limited capital mobility plays two roles in our analysis. On the one hand, they determine the elasticity with which reserve flows are able to affect domestic interest rates. On the other, they determine the "UIP costs" which with reserve flows are able to affect domestic interest rates. On the other, they determine the "UIP costs" which with reserve flows are able to affect domestic interest rates.

\textsuperscript{44}Risks that symmetrically affect all bond market participants (intermediaries and the central bank) can be captured by movements in $r^*_t$ in our model.

\textsuperscript{45}A simple microfoundation of these benefits would be a latent risk that materializes with some Poisson intensity $\lambda \to 0$ and intermediaries that are Knightian to different degrees. See Caballero and Farhi (2017) for a microfoundation of safety along those lines in a deterministic economy.
Figure 2-6: Optimal intervention after a positive $\xi_t$ shock to intermediaries' preferences for home bonds. The deterministic model economy is calibrated to Brazil, with $\Gamma_F = 1$. The shock increases $\xi_t$ by 2% for $t \in [0, 3]$. The results are relative to the zero UIP wedge economy ($\tau_t = 0$).

cause *underreaction* of domestic interest rates in the laissez-faire competitive equilibrium in response to shocks to the world interest rate $r^*_t$. The preference shocks $\xi_t$ can help disentangle the elasticity role—which we are ultimately interested in—from the underreaction. To provide a clean argument, consider a situation where $\xi_t$ only affects home intermediaries. When $\xi_t$ increases as $r^*_t$ falls, or in other words intermediaries associate a smaller risk premium with domestic bonds, it is possible that changes in $r^*_t$ are passed through one-for-one to changes in $r_t$ (rather than less than one-for-one). In this case, both the equilibrium UIP spread and equilibrium reserve holdings are exactly zero. In this interpretation, since home intermediaries' demands do not affect the planning problem, Figures 2-4 and 2-5 represent the actual holdings of reserves.

### 2.5.3 Smooth exchange rates vs. smooth UIP wedges

There is ample evidence by now that some emerging market policymakers, especially East-Asian ones, seem to conduct policies aimed at smoothing *exchange rates*, at short to medium horizons.\(^{46}\) This is sometimes referred to as a “managed float” and should not be confused with the kind of “intervention smoothing” policy that we found to be optimal in our model: Here, *UIP wedges* $\tau_t$ are smooth, while exchange rates jump initially, albeit by less than without intervention if the optimal policy is employed. This is the case even though an exchange rate that is smooth at $t = 0$ would certainly lie in the space of implementable allocations. This raises the natural question of why our optimal policy problem did not select such a “managed float” allocation.

To explore this question, we simulate the optimal policy under an additional ad-hoc “smooth exchange rate” constraint, namely that the initial real exchange rate $q_0$ be the same as in the steady state, $q_0 = q_{ss}$.\(^{47}\) Since the real exchange rate in the model moves one-to-one with dollar

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\(^{46}\)See the voluminous “fear of floating” literature, e.g., Calvo and Reinhart (2002), Levy-yeyati and Sturzenegger (n.d.) or McKinnon and Schnabl (2004).

\(^{47}\)To be clear, by “smooth exchange rate” we mean continuity of the exchange rate, not any kind of differentiability. Furthermore, note we can interpret this a smooth nominal exchange rate requirement, together with the commitment
Figure 2-7: Comparing the optimal policy (solid) to a “managed float” policy (dashed) Panel (a) shows the UIP wedge, which in the model is proportional to carry trade activity by foreigners. Panel (b) shows the unnormalized real exchange rate, not the one relative to the competitive equilibrium. Panel (c) shows the country’s reserve positions. The deterministic model economy is calibrated to Brazil, with $\Gamma_F = 10$. The $r_t^*$ shock hits for $t \in [0, 3]$.

We thus solve our planning problem (2.18) subject to (2.17a), (2.17b) and (2.24). It is worth stressing that when $\Gamma_F \in (0, \infty)$ this additional constraint does not make the problem trivial, in the sense that it is impossible to approximate the solution of our original problem arbitrarily closely. The reason for this comes from the fact that this would require infinitely large, and infinitely costly, UIP wedges $\tau_t$ for $t$ close to zero.

We plot the optimal managed float intervention in Figure 2-7 and compare it to our unconstrained optimal policy, computed without (2.24) as constraint. Panel (b) shows the two economies’ real exchange rate paths. It is clearly visible that while the unconstrained optimal policy features a sharp and sudden exchange rate appreciation (solid), the managed float policy generates a much slower appreciation that stretches even beyond the first year of the shock (dashed). As a consequence of the slow and predictable appreciation, the country must bear the cost of an excessive UIP spread (Panel (a)), inviting a significant amount of carry trade activity both by foreign and by domestic intermediaries. To implement this policy, the domestic central bank acts as a “shock absorber”: it accumulates reserves at a rapid pace initially, and then slowly decumulates them over the years.

The reader may wonder why the central bank trying to implement a peg in Section 2.5.1 did not encounter such heightened carry-trade activity in the beginning of the intervention period. The reason for this is that in Section 2.5.1 we allowed for a non-zero output gap while in this subsection we assumed the output gap is zero at all times, i.e. the central bank is choosing one among the to pick a flex-price allocation, as explained in 2.5.1.
flex-price allocations. Thus, this subsection emphasizes that smoothing the exchange rate path may be very costly if domestic objectives are perceived by agents to be more important than external objectives. Put differently, a policymaker seeking to implement a smooth exchange rate path may avoid speculation by foregoing some domestic stability, as in Section 2.5.1.

2.6 Reserve wars

So far, we have analyzed the optimal policy of a small open economy (SOE) against a passive rest of the world. Now, we explore the strategic interaction among different SOEs' foreign exchange interventions and their effects on the rest of the world. For this purpose, we consider a world with two kinds of countries: Advanced economies (AE), which are assumed to have frictionless financial markets, and emerging markets (EME), which are SOEs like the one described in Section 2.3. For ease of exposition, we focus on a two period version of our model.

The structure of this section is as follows. In Subsection 2.6.1, we describe the model. Then, we characterize the world equilibrium in a decentralized setting where each EME central bank chooses its own foreign exchange policy taking as given the actions of the other EMEs. Finally, we consider the world equilibrium where foreign exchange policies are set by a single “EME cooperative planner”, who can be interpreted as a stand-in for closer central bank cooperation.

2.6.1 Setup

The world consists of an interval $[0, 2]$ of small open economies and exists for two periods $t = 0, 1$. Economies $i \in [0, 1]$ are “advanced economies” (AE) while economies $i \in (1, 2]$ are “emerging market economies” (EME). Economies within each of the two regions are identical in all respects. A typical SOE $H \in [0, 2]$ has the same preferences as the SOE in Section 2.3 except for being two-period lived, that is, $U = \ln(C_0) + (1 + \rho_j)^{-1} \ln(C_1)$ with $C_t = \kappa c_{Ht}^{1-\alpha} c_{Ft}^\alpha$, where the discount factor $\rho_j$ is allowed to depend on the region $j = AE, EME$. Here, $c_{Ht}$ denotes consumption of $H$’s own good, and $c_{Ft}$ denotes consumption of a common foreign good. We assume that the foreign good is a composite good given by $c_{Ft} = c_{AEt} + c_{EMEt}$, where $c_{AEt}$ and $c_{EMEt}$ are themselves aggregates of varieties produced in each region,

$$\ln(c_{AEt}) = \int_{i \in [0,1]} \ln(c_t) di \quad \text{and} \quad \ln(c_{EMEt}) = \int_{i \in (1,2]} \ln(c_t) di.$$

This market structure captures that EMEs compete more with one another than with AEs in world good markets and allows us to obtain clean benchmark results. We normalize the price of the composite foreign good to 1 in each period. We assume that each SOE is endowed with its own good: EMEs with $\chi \in [0, 1]$ units of their own good and AEs with $1 - \chi$ units of their own good. Henceforth, we will exploit symmetry and label with a star “*” variables from a typical AE, and without stars variables from a typical EME.

EMEs are characterized by imperfect financial mobility. Residents in any given EME are only
allowed to trade a bond in the jurisdiction of their own country, paying gross interest rate \(1 + r\) (as before in units of the foreign good which we call “dollars”). In addition, there is a continuum of intermediaries located in AEs who can freely access each of the AEs’ bond markets, which pay a common interest rate \(r^*\) in units of the foreign good \(c_F\). They can trade in each of the EMEs’ bond markets but only up to a position limit \(X\sqrt{x}\), and only after paying an idiosyncratic transaction cost. Following the same steps as in Section 2.3, this leads to a finitely elastic demand function for each EME bond, \(b_I = \chi^{-1}(r - r^*)\). As in the previous sections, central banks are assumed to have perfect access to AEs’ bond markets as well as their own bond market, but not other EMEs’ bond markets. We define \(\theta_t \equiv \chi^{-1}P_tC_t\) and \(\theta^*_t \equiv (1 - \chi)^{-1}C^*_t\) as the (normalized) dollar consumptions of EMEs and AEs. As before, the UIP wedge is defined as \(\tau = \frac{1 + r}{1 + r^*} - 1\).

We next characterize and define a competitive equilibrium in this economy. The home market clearing for a typical EME is

\[
(1 - \alpha)p_{Ht}^{-1}\theta_t + p_{Ht}^{-1}C_{EMEt} = 1
\] (2.25)

where \(C_{EMEt}\) is the world demand for the EME aggregate good, again normalized by the endowment \(\chi\). Similarly, in AEs,

\[
(1 - \alpha)p_{Ht}^{-1}\theta^*_t + p_{Ht}^{-1}C_{AEt} = 1.
\] (2.26)

The market clearing condition for the composite foreign good yields,

\[
\alpha\chi\theta_t + \alpha(1 - \chi)\theta^*_t = \chi C_{EMEt} + (1 - \chi)C_{AEt}.
\] (2.27)

In what follows, we focus on a symmetric equilibrium with positive consumption of both aggregate goods, \(P_{EMEt} = P_{AEt} = PFt = 1\). By symmetry we mean that every EME central bank finds it optimal to carry out the same foreign exchange policy, implying that

\[
p_{Ht} = p_{Ht}^* = 1.
\]

Symmetry allows us to simplify (2.25), (2.26) and (2.27), yielding

\[
(1 - \alpha)\theta_t + C_{EMEt} = (1 - \alpha)\theta^*_t + C_{AEt} = 1.
\] (2.28)

and

\[
\chi\theta_t + (1 - \chi)\theta^*_t = 1.
\] (2.29)

---

48 We normalize \(X\) by \(\sqrt{x}\) to make sure the limit where EMEs are small, \(x \to 0\), is well-defined.

49 If central banks could access other EMEs bond markets, they would behave like arbitrageurs, exploiting opportunities generated by other central banks. This may generate another motive for central bank coordination, independent of the one we focus on. In any event, our results would go through as long as trading in other EME bonds by central banks is costly.

50 This will always occur in equilibrium if discount factors are not too different. Note, however, that the following analysis would still be true if one of the aggregate goods had negative consumption, in the interpretation that then extra supply of that good would be created, using some of the other aggregate good as inputs.
In each SOE, the optimal solution of the consumers’ problem implies the following Euler equations,
\[
\begin{align*}
\theta_1 &= \frac{(1 + r^*) (1 + \tau)}{1 + \rho_{EME}} \theta_0 \quad \text{and} \quad \theta^*_t = \frac{1 + r^*}{1 + \rho_{AE}} \theta^*_0.
\end{align*}
\] (2.30)

Finally, consolidating the consumer’s budget constraint with the central bank’s yields the countrywide budget constraint,
\[
\alpha \theta_0 - C_{EME0} + \frac{1}{1 + r^*} (\alpha \theta_1 - C_{EME1}) + \frac{1}{\Gamma_F} \tau^2 = 0,
\]
which using (2.28) implies
\[
(\theta_0 - 1) + \frac{1}{1 + r^*} (\theta_1 - 1) + \frac{1}{\Gamma_F} \tau^2 = 0. \quad (2.31)
\]

For simplicity, we take initial net foreign asset positions to be zero.

We are ready to formally define a symmetric world competitive equilibrium in this economy. For ease of exposition and brevity, we state the foreign exchange policy directly in terms of \( \tau \).

**Definition 2.2.** A symmetric world competitive equilibrium given an EME central bank foreign exchange policy \( \tau \), is an allocation \( \{ \theta_t, \theta^*_t, C_{EMEt}, CAEt \}_{t=0,1} \) together with an interest rate \( r^* \), such that equations (2.28) – (2.31) hold.

This defines a competitive equilibrium given a (symmetric) set of EME foreign exchange policies. We now characterize the competitive equilibrium that occurs if EME central banks play a Nash equilibrium in their choice of foreign exchange policies. For our numerical illustrations, we stick with our baseline calibration and assume that \( \Gamma_F = 10 \) and \( \alpha = 0.15 \). Interpreting one period as the equivalent of 3 years, so that our 2-period model captures the same kind of shock as before, we set \( 1 + \rho_{EME} = 1.075^3 \). We re-calibrate \( \rho_{AE} \) as we vary \( \chi \) such that that the annualized response of the world interest rate when \( \tau = 0 \) is 5.5%. All results below will be stated in annualized terms.

**Noncooperative world equilibrium.** The typical EME central bank maximizes its own welfare taking two objects as given: the world interest rate \( r^* \) and foreign expenditure levels \( \{ C_{EMEt} \} \). Proceeding exactly like in equation (2.18), we find that the problem of an individual EME central bank is
\[
\max \sum_{t=0}^{1} (1 + \rho_{EME})^{-t} \left\{ \ln(\theta_t) - (1 - \alpha) \ln((1 - \alpha)\theta_t + C_{EMEt}) \right\} \quad (2.32)
\]
subject to
\[
\alpha \theta_0 - C_{EME0} + \frac{1}{1 + r^*} (\alpha \theta_1 - C_{EME1}) + \frac{1}{\Gamma_F} \left( \frac{\theta_1 (1 + \rho_{EME})}{\theta_0 (1 + r^*)} - 1 \right)^2 = 0. \quad (2.33)
\]
This is a two-period version of the problem analyzed in Section 2.4, where we had $C_{EME1} = \alpha$ and $\chi = 1$. In (2.33) we also substituted out $\tau = \frac{\theta_1(1+\rho_{EME})}{\theta_0(1+r^*)} - 1$. The intuition behind this problem is the same as before: When the interest rate $r^*$ is lower than the EME discount rate $\rho_{EME}$, or foreign expenditure $C_{EME1}$ is higher at $t = 0$ than at $t = 1$, the EME central bank optimally accumulates reserves and implements a positive UIP wedge $\tau$. It is worth noting that the IC constraint (2.33) is a non-linear function of $\theta_0$ and $\theta_1$ for any $\Gamma_F \in (0, \infty)$. In Appendix B.6.1 we prove that it is without loss to relax (2.33) as inequality and that this inequality constraint describes a convex, bounded set in $(\theta_0, \theta_1) \in \mathbb{R}^2_{++}$. Thus, whenever $C_{EME1} > 0$ for all $t$ (to ensure the concavity of the objective) this is a well-behaved concave maximization problem with a convex constraint.

We next characterize the central bank Nash equilibrium, in which each central bank solves (2.32) taking $\{C_{EME1}\}$ and $r^*$ as given, but in equilibrium $\{C_{EME1}\}$ is pinned down by (2.28), and $r^*$ is pinned down by (2.29) and (2.30). To simplify the proofs, we assume that EMEs are small in the following formal result ($\chi = 0$). For $\chi > 0$, we verified numerically that the proposition still holds.

**Proposition 2.7** (Reserve wars.). Assume imperfect capital mobility, $\Gamma_F \in (0, \infty)$, and that emerging markets are small, $\chi = 0$. If emerging market central banks choose their foreign exchange policy in a non-cooperative way, then it holds that:

1. There exists a unique Nash equilibrium.

2. In the Nash equilibrium, emerging markets accumulate reserves and the UIP wedge $\tau$ is strictly positive.

3. Compared to a no-intervention world, capital flows more upstream towards the advanced economies, driven by reserves, while private capital flows more downstream towards emerging markets.

4. Welfare of emerging markets is lower than without intervention.

Proposition 2.7 describes four key properties of the Nash equilibrium. We discuss them in turn. First, the existence of a unique Nash equilibrium follows because for $\chi = 0$, foreign exchange interventions in the model are strategic substitutes: When EMEs choose to accumulate reserves, they depreciate their real exchange rates leading to higher total consumption of the EME good $C_{EME,0}$ in the first period. Because the EME good is a Cobb-Douglas aggregate of all EMEs, this also raises the first period demand for any single EME, hence calling for a more appreciated real exchange rate to exert monopoly power. Notice that, by contrast, for larger values of $\chi$, a force for complementarity emerges: Then, more reserve accumulation lowers world real interest rates even more, causing even stronger capital inflows into every single EME, and raising their desire to counter that with more reserve accumulation.

In the unique Nash equilibrium, the UIP wedge $\tau$ is strictly positive since advanced economies are attempting to save, $\rho_{AE} < \rho_{EME}$, causing private capital to flow into EMEs. The reserve accumulation by EMEs then pushes public funds upstream, while the positive UIP wedge lets
intermediaries take larger downstream positions. This explains parts 2 and 3 of Proposition 2.7. We illustrate these outcomes in Figure 2-8 as function of the overall size of EMEs $\chi$ (red, solid line). Panel (a) shows that the equilibrium UIP wedges are positive throughout, and rise with $\chi$ as the feedback loop through lower world interest rates kicks in (Panel (b)). Analogously, reserves are positive and increase with $\chi$ relative to GDP, as shown in Panel (c).

In general equilibrium, interventions are self-defeating: Even if all EMEs accumulate reserves to depreciate their $t = 0$ real exchange rate, this does not happen. Foreign demand is infinitely elastic if $\chi = 0$, fixing the real exchange rate at 1, as can be inferred from equation (2.28). This means that, in contrast to their intended purpose, interventions cause welfare losses for emerging markets. In addition, if $\chi > 0$, interventions might also reduce the welfare of advanced economies by depressing the world interest rate. Put together, these kinds of noncooperative reserve wars can cause welfare losses for all countries. We illustrate this in Figure 2-9 as function of the overall size of EMEs $\chi$ (red, solid line). As Panel (a) shows, welfare of EMEs suffers due to the competitive devaluations up until EMEs are so large that their effect on the world interest rate compensates for the welfare losses associated with the devaluations. In Panel (b) we see that welfare of AEs rises ever so slightly for small $\chi$ due to intermediary profits from carry trades against emerging markets and then rapidly falls below zero as AEs try to save in an increasingly low interest rate environment.

**Cooperation of emerging market central banks** The self-defeating nature of interventions suggest that there may be gains from policy coordination among EME central banks. This is what we consider next. The world equilibrium can now be regarded as the outcome of a planning problem in which a single “EME planner” maximizes the objective (2.32) subject to the IC constraint (2.33), but now takes into account the endogeneity of $\{C_{EME_t}\}$ and $r^*$, coming from equilibrium conditions (2.28), (2.29) and (2.30). In the case where $\chi = 0$, we can prove the following result, standing in stark contrast with the noncooperative outcome.
Proposition 2.8 (Central bank cooperation.). Assume imperfect capital mobility, $\Gamma_F \in (0, \infty)$, and that emerging markets are small in total, $\chi = 0$. If emerging market central banks cooperate, then it is optimal for emerging markets not to accumulate any reserves, implying a zero UIP wedge $\tau = 0$.

We illustrate the contrast between the Nash equilibrium outcome (red line, solid) and the cooperative solution (blue line, dashed) in Figures 2-8 and 2-9 as function of the overall size of EMEs $\chi$. Figure 2-8 compares the equilibrium UIP wedge, the world interest rate and total reserves position as fraction of GDP. When $\chi = 0$, internalizing that competitive devaluations are self-defeating and with no possibility of manipulating the world interest rate in its favor, the cooperative planner sets $\tau = 0$. As $\chi$ increases, the planner boosts savings in an attempt to lower the interest rate thereby increasing $\tau$. Panel (b) shows that the cooperative planner believes that EMEs’ reserve wars let the world interest rate fall too low in the Nash equilibrium. Put differently, the Nash equilibrium has reserve over-accumulation even from the point of view of the cooperative planner (Panel (c)).

Figure 2-9 shows the welfare of EMEs and AEs under policy cooperation. Naturally, the welfare of EMEs is now always larger than with a $\tau = 0$ policy or with the Nash equilibrium policy, since both are feasible policies. Interestingly, although the cooperative policy is still of the “beggar-thy-neighbor” type—EMEs manipulate the interest rate in their favor at the expense of AEs—AEs are better off under this partial degree of cooperation than in the Nash equilibrium for reasonable values of $\chi$. In other words, even partial cooperation can make everyone better-off with respect to decentralized foreign exchange interventions.\footnote{The only exception is when $\chi$ is small enough that foreigners are better-off as a result of carry-trade profits.} Finally, note that given that international transfers are infeasible, a $\tau$ lying between this “partial” cooperative solution and $\tau = 0$ may be an indirect way of transferring resources to AEs for a global planner. It should be noted, however, that we abstracted from heterogeneity in initial NFA positions, which is important in reality to assess the consequences of world interest rate movements.
2.7 Conclusion

Foreign exchange interventions are one of the most important policy tools for many countries around the world. Yet, many debates regarding their usefulness and the best implementation design persist. We believe this is partly due to the lack of a unified framework to analyze the optimal design jointly with the macroeconomic rationales behind interventions. In this paper, we provided such a framework. At the core of our model lies the assumption of limited capital mobility, which gives rise to a general equilibrium portfolio balance channel. We showed that interventions essentially manage a path of UIP spreads, and that each nonzero UIP spread represents a cost to the economy, coming from foreign intermediaries’ carry trade activity. These costs, which are naturally convex as larger spreads invite further speculation, lie at the heart of our optimal policy design. In a nutshell, they make it optimal to spread out interventions.

Our findings pick a clear side in the debate. Interventions should be small and frequent to avoid inviting significant speculation. Furthermore, they should be highly inertial and pre-announced to maximize the impact on the contemporaneous exchange rate. Interventions were found to be more powerful if the monetary authority is more credible, as this allows it to spread out interventions even further into future, minimizing the overall cost of generating an exchange rate response today. Finally, we also showed that the optimal policy is better approximated by a quantity rule rather than a smooth exchange rate rule. In the case of the latter, speculative costs may become prohibitively costly if the monetary authority tries to close the output gap at the same time.

Our unified framework allowed us to derive these “micro” features of optimal interventions and at the same time to analyze the macroeconomic motives for interventions. We found that interventions lean against the wind after global interest rate shocks—either for a terms-of-trade manipulation motive or a “output gap stabilization” motive, and serve a market-making role after large commodity shocks. In addition, since our framework is embedded in a standard macroeconomic model, we also used the model to tackle the important question about the degree to which intervention policies should be coordinated across countries. We made the point that coordination is essential to avoid wasteful competitive devaluations and reserve over-accumulation. Such reserve over-accumulation was shown to have important amplification effects on the fall of the world interest rate, hurting advanced economies. As a result, committing to replicate a world with free capital mobility led to a strict Pareto improvement over the Nash equilibrium.

We believe there are several avenues for future research. Using a richer model with a realistic calibration seems necessary for a more serious quantification of the importance of the channels stressed in this paper. For example, one may add the friction of limited capital mobility to a medium-scale version of a standard New Keynesian dynamic stochastic general equilibrium model and estimate it. In addition, it may be interesting to use such a structural model to back out an estimation of the foreign exchange intervention reaction function and compare it to the model-predicted optimal policy.

In addition, there is still much progress to be made even from a purely theoretical side. For example, our paper assumes that there is Ricardian Equivalence between the central bank and
domestic households. Yet this is certainly a strong assumption. More realistically, in a model with heterogeneous households and/or firms one could imagine that interventions have important redistributive effects that could both amplify or mitigate the effectiveness of interventions.
Chapter 3

Export Survival with Uncertainty and Experimentation

Two central facts characterize the dynamics of firm exports. One is the known fact that export survival rates are strikingly low one year after entering a foreign market. The other is the novel fact that re-entrants in export markets are more likely to survive than first-time entrants. Traditional models of exporter dynamics cannot explain these two facts. In this paper, we develop a tractable model of exporter dynamics that can explain them by introducing uncertainty and experimentation. The model delivers analytical predictions on survival probabilities upon entry in a foreign market. We test the main mechanism of the model by exploiting variation in the degree of uncertainty across products and markets. The results support the relevance of uncertainty and experimentation as a central feature that characterize exporter dynamics.

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3.1 Introduction

Both developed and developing countries display a variety of policies and dedicated agencies aimed at helping firms establish a sustained presence in foreign markets. Underlying these policies is the view that increasing the set of domestic firms capable of achieving sustained exports is key to foster aggregate export growth and, potentially, economic development. Recent evidence by Eaton et al. (2008), Freund and Pierola (2010), and Lederman, Rodríguez-Clare and Xu (2011) for Colombia, Peru, and Costa Rica, respectively, supports this view by showing that a considerable fraction of aggregate exports in a given year is accounted for by firms that were not exporting a few years earlier. However, while new entries in foreign markets can potentially have a relevant long run impact, this potential is usually unrealized as most export incursions do not become established export businesses. In fact, about two thirds of firms that make an incursion in a specific foreign market do not continue to export to that market in the subsequent year (Eaton et al. (2014); this paper). The reasons for such short spells are not yet well understood. In particular, little is known about what determines export survival upon entry in a foreign market.

Only recently has a growing literature started to uncover empirical regularities about the more general dynamic process of firm exports, of which export survival is one of its salient manifestations. The regularities tend to mimic analogous patterns long identified in the (domestic) firm dynamics literature. For example, new exporters, like new firms, are smaller, tend to grow faster conditional on survival, and are less likely to survive (Eaton et al. (2008), Arkolakis (2016)), while the size distribution of export sales, as the distribution of firm sales, resembles a Pareto distribution (Eaton, Kortum and Kramarz (2011)). Notwithstanding the similarities, two facts distinctly characterize the dynamics of firm exports. One of these facts, emphasized in recent work, is that the survival rate of export incursions is strikingly low in the first year after entry – particularly lower than the survival rates of domestic firms – and drops further only modestly in subsequent years (e.g., Eaton et al. (2008), Ruhl and Willis (2017)). The second fact, which is novel, is that re-entrants in export markets are more likely to survive than first time entrants to those markets. These two facts describe central features of exporter survival. As such, they are also central to characterize, more generally, the dynamics of firm exports. The distinguishing nature of these facts suggests that standard models of firm dynamics might not be appropriate to explain exporter dynamics, which might be characterized by distinct ingredients. We show that these facts also set tight constraints on the class of models that can explain them and thus are critical in guiding the construction of a relevant theory of exporter dynamics.

We build a theoretical model of exporter dynamics guided by these two facts. The estimated model can explain these facts as well as other relevant facts that have been the focus of previous work. The main feature of the model is the existence of uncertainty about foreign market profitability that can only be resolved by actively exporting (Segura-Cayuela and Vilarrubia (2008), Freund and Pierola (2010), Albornoz et al. (2012), Nguyen (2012), Eaton et al. (2014)). As a result, firms experiment under losses to resolve this uncertainty. The model is flexible yet it is parsimonious, and exhibits a number of tractable features that we exploit to obtain analytical results on survival
probabilities. Those results help us estimate the model and derive predictions that we contrast with the data. In order to establish the key role that uncertainty and experimentation play in the dynamics of exports, we follow a two pronged approach. First, we show that other models often used in the literature which do not include uncertainty and experimentation are unable to generate both facts as a joint prediction. Second, we test the central mechanism of the model by exploiting hypothesized variation in the degree of uncertainty by product and distance to the destination. The implied predictions of uncertainty variation on survival probabilities are confirmed by the data in most cases.

We model a simple uncertainty and experimentation mechanism embedded in a theoretical framework with otherwise standard elements. A firm’s operating profit in an export market is initially determined by an idiosyncratic time-varying component that follows a Geometric Brownian Motion (GBM) and a constant and idiosyncratic market-specific component. Operation in a foreign market requires that firms pay a continuous, constant, and idiosyncratic fixed cost while firms are allowed to enter and exit the market freely, particularly since there are no sunk costs.

The uncertainty and experimentation mechanism operates as follows. Before entering a foreign market, firms are uncertain about their potential profitability. This uncertainty can only be resolved by actively exporting. Artopoulos, Friel and Hallak (2013) argue that adapting products and marketing practices to match foreign market tastes and ways of doing business is critical for long run export success. Our model postulates that firms are uncertain about the extent to which they will be able to match those foreign tastes and business practices. Hence, they are willing to experiment to find this out by initially exporting at a loss. Specifically, the model includes a multiplicative shock to operating profits with Poisson arrival rate, which increases profits in expected value. The firm knows the parameters of the distribution where the shock comes from and the Poisson arrival rate but is uncertain about the particular realization of both random processes. In other words, it is uncertain about how much and how fast profits will jump. In this environment, the firm enters the foreign market even when operating profits are lower than fixed costs in the expectation of eventually improving performance and justifying the initial investment. Once it has received the shock, however, the firm only stays active if operating profits are higher than fixed costs as there is no further uncertainty to resolve. In the empirical section, we parametrize this shock with a Pareto distribution with scale parameter 1.

A key analytical result is that the probability of survival upon entry at any given horizon is independent of the firm-specific profitability shifters and fixed costs. Firms time their entry and exit decisions as a function of these heterogeneous parameters precisely in a way that cancels out their potential impact on survival probabilities once we condition on entry. Hence, those probabilities are identical across firms and can be obtained without information on the firm-specific parameters or the probability distribution that generates them as they only depend on common parameters. This is one of the main advantages of focusing on survival upon entry. Since observed survival rates average the realization of a common probability across firms, we can use them as empirical counterparts of those theoretical predictions to estimate the common parameters of the model.
Although these are only a small subset of all the parameters, they alone determine some of the most important features of the dynamics of firm exports. These features include not only those related to export survival but also those related to export growth.

The fact that the survival rate of export incursions is strikingly low cannot be easily accounted for by standard models. This notion is illustrated by a special case of our model where we shut down the uncertainty and experimentation mechanism. In order to fit the survival rate in the first year, this special case needs to set such a negative trend on the GBM process (relative to its volatility) that it severely underpredicts survival rates at later horizons. In contrast, uncertainty and experimentation arise as a natural explanation for the observed survival patterns. The low initial survival rate of export incursions is an expected outcome of the experimentation process. In turn, deaths can occur disproportionately during the first year as long as the resolution of uncertainty occurs sufficiently fast. This uncertainty and experimentation mechanism also explains why re-entrants exhibit higher survival rates than entrants. Since a large fraction of re-entrants have already resolved their uncertainty during their initial export spell, their re-entry decision is not driven by an intention to experiment. Thus, they are more conservative to enter and as a result survive more.

We estimate the model using firm-level customs data of exports from Peru for the period 1993-2009. We calculate survival rates one to five years after entry both for entrants and for re-entrants. These ten moments are used to estimate the parameters of the model with the Simulated Method of Moments (SMM). Before performing the estimation we develop a correction in the theoretical survival probabilities that accounts for the mismatch between a model set in continuous time and data recorded over discrete time periods. One correction is the “partial-year” effect emphasized by Berthou and Vicard (2015) and Bernard et al. (2017), which deals with the fact that firms may enter the export market at different points along the year. The second correction is the “re-entry” effect, which deals with the possibility that a firm may be out of the export market at the time of computing the instantaneous probability but re-enter it during the relevant discrete (calendar) year. Although the latter effect has been neglected so far both in the firm dynamics and in the exporter dynamics literatures, it is the one with the largest impact on predicted survival probabilities.

The estimated model predicts quite closely the survival rates of export incursions. The predictions are slightly below the survival rates in the data by an average of three percentage points over the first five horizons. The parameter estimates also indicate that uncertainty is resolved notably fast. A firm that continuously exports has a 56.5% probability of receiving the multiplicative shock in less than a month. The shock also has a considerable dispersion (0.54), which justifies the willingness of firms to experiment in foreign markets in the hope of benefiting from a good realization of this random variable. The estimated model also predicts the qualitative fact that survival rates are higher for re-entrants than for entrants. However, it overpredicts the gap. While in the data the average survival rate of re-entrants over the first five horizons is 0.12 percentage points higher than for entrants, the model predicts a 0.19 percentage point difference between the two average rates.

We use hypothesized variation in uncertainty across products and markets to test our uncertainty
and experimentation mechanism. Specifically, the model predicts that survival probabilities should be lower the higher is the variance of the shock, which is a measure of the degree of uncertainty that can be resolved by exporting. First, we postulate that the degree of uncertainty should be higher in the case of differentiated products, where firms need to adapt to idiosyncratic tastes and find distributors that help propel sales (Artopoulos, Friel and Hallak (2013)). Consistent with the predictions of the model, survival rates are lower for entrants in differentiated products than for entrants in homogenous products. Second, since a fraction of re-entrants have resolved their uncertainty, this prediction should hold attenuated in the case of re-entrants. We find that the gap between survival rates of re-entrants of differentiated and homogenous products is indeed lower but not as much lower as predicted. Third, we postulate that in the case of differentiated products, the degree of uncertainty over export market profitability should increase with the distance to the destination. Consistent with the model predictions, we find that survival rates in differentiated products are lower the farther away is the destination market. Finally, though this prediction should also be attenuated in the case of re-entrants, we find that survival rates decrease with distance as much for re-entrants as for entrants. Examined together, the evidence of this section is consistent with the main predictions of the uncertainty and experimentation mechanism of our model though not with all of its subtler implications. As a whole, we find it strongly supportive of the notion that uncertainty and experimentation are central components in the dynamics of firm exports.

Our choice of a GBM to model the evolution of the profitability process is made primarily for analytical simplicity. This choice is nevertheless not unjustified since the evolution of export sales has been found to be a highly persistent process (Roberts and Tybout (1997), Das, Roberts and Tybout (2007)). In any event, we also consider a broader class of models and show that, in the absence of uncertainty and experimentation, they are unable to match the two distinguishing facts that we highlight. In particular, as in Arkolakis (2016) we postulate a more general model that embeds as special cases a GBM and a Geometric Ornstein-Uhlenbeck process expanded with a trend in its long-run value. While there exist parameter configurations in this encompassing model that can explain the survival rates of export incursions at various horizons, they are unable to account for the higher survival rates of export re-entrants. First, to fit the decreasing decay in survival rates it is necessary that the process has a negative long-run trend. Second, since re-entrants tend to be older firms, they reach the re-entry point (when they do) farther away from the long run value and thus mean-revert more strongly. As a result, they survive less, not more, than entrants.

This paper is connected to several strands of literature. The oldest and most influential literature on exporter dynamics has focused on the hysteresis implications of exporting sunk costs (Baldwin and Krugman (1989), Dixit (1989), Alessandria and Choi (2007), Impullitti, Irarrazabal and Opromolla (2013)). In this paper, we show that despite their starring role in the literature, sunk costs do not appear to be necessary to deliver predictions on export survival that can fit the data. On the contrary, they only exacerbate the predictive shortcomings of models that do not include uncertainty and experimentation. A more recent literature develops methods to structurally estimate sunk costs (Das, Roberts and Tybout (2007), Morales, Sheu and Zahler (2017)). Our results suggest that
such methods may yield sunk-cost estimates that are largely determined by assumptions about the relationship between domestic and foreign profitability. We generate predictions that do not require making assumptions on this relationship.

Except for our inclusion of uncertainty and experimentation, this paper follows closely the work of Arkolakis (2016). In particular, we also model a GBM process and allow for free exit and re-entry in foreign markets. However, we do not include market penetration costs (Arkolakis (2010)), which are critical in that work and in Eaton, Kortum and Kramarz (2011) to generate the observed size distribution of exporters. We show that, in the absence of sunk costs, market penetration costs do not change the survival predictions of standard models and hence their shortcomings in matching the facts that we document. Furthermore, our uncertainty and experimentation mechanism could be an alternative explanation for the observed deviations from Pareto in the lower tail of the exporter distribution since it reduces the size of new, usually small, exporters.

The essence of our uncertainty and experimentation mechanism has already been postulated in various forms in previous studies of exporter dynamics (Segura-Cayuela and Vilarrubia (2008), Freund and Pierola (2010), Albornoz et al. (2012), Nguyen (2012)). We build on this literature by embedding this mechanism in a more general framework that can deliver a wider set of quantitative and qualitative predictions. In this goal, this paper complements Eaton et al. (2014) by sacrificing a relevant dimension in newer foreign transaction databases – relationships with distributors on the importer side – in the sake of parsimony and tractability. Finally, one additional contribution to this literature is that we test for the relevance of uncertainty and experimentation in exporter dynamics and show the limitations of models do not account for these features.

This paper is also related to work specifically oriented to explain exporter survival (Békés and Muraközy (2012); Albornoz, Fanelli and Hallak (2016)). While our model and theoretical results emphasize exporter survival, we hope to make a contribution to a broader literature on exporter dynamics.1

The rest of the paper is organized as follows. Section 3.2 describes the two distinguishing facts about exporter survival that we emphasize in this paper. Section 3.3 sets up the model and derives predictions on survival probabilities. Section 3.4 estimates the model, compares its predictions with the data, and discusses why standard models in the literature cannot explain the two facts. Section 3.5 tests for the uncertainty and experimentation mechanism of the model by looking at its implications across products and markets. Section 3.6 provides concluding remarks.

### 3.2 Two central facts about exporter survival

A vast amount of literature has established a number of facts about patterns of firm dynamics related to their survival (e.g., Mansfield (1962), Evans (1987), Dunne, Roberts and Samuelson (1988, 1989)), growth rates (e.g., Hart and Prais (1956), Mansfield (1962), Evans (1987), Hall

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1Preliminary results suggest our model is also able to explain most of the relevant facts on the dynamics of firm exports that the literature has highlighted and that motivated work on alternate models. We hope to include these results in the next version of this paper.
(1987), Dunne, Roberts and Samuelson (1989), Davis and Haltiwanger (1992)), and size distribution (Simon and Bonini (1958), Cabral and Mata (2003), Luttmer (2007)). A more incipient strand of literature has recently uncovered analogous patterns in the dynamics of firm exports. For example, smaller and younger exporters, like smaller and younger domestic firms, are less likely to survive and display higher growth rates conditional on survival (Eaton et al. (2008), Berthou and Vicard (2015), Arkolakis (2016)). Also, the upper tail of the size distribution of export sales resembles a Pareto (Eaton, Kortum and Kramarz (2011), Arkolakis (2016)). In spite of the notable similarities, two facts uniquely distinguish exporter dynamics. The first is that the survival profile (i.e. the line connecting survival rates at different horizons) of export entrants is low and flat. The second is that the survival profile of export re-entrants is higher than the survival profile of entrants. This section describes these two facts and discusses how they guide our search for a parsimonious model of exporter dynamics that can explain them.

First, we briefly discuss some definitions and basic data issues. We employ firm-level customs data from Peru for the period 1993-2009 graciously provided to us by the Trade and Integration Unit of the World Bank Research Department. Our dataset covers all export transactions from Peru between 1993 and 2009 by firm, destination country (i.e. export market), and year. We define an export “incursion” as the first entry of a firm in an export market. The “survival rate” \( S_T \) is the proportion of incursions that are active in the corresponding export market \( T \) years after entry. We follow an incursion up to five years. Hence, the “survival profile” includes the set of survival rates \( \{ S_T \}_{T=1,\ldots,5} \). Since we do not observe data before 1993, we only consider incursions starting in 1997 to minimize the chances of falsely identifying as incursions export instances with an antecedent before 1993. Also, since we track survival up to five years after entry, we restrict the sample to incursions starting no later than 2004. Our definition of survival does not impose consecutive activity as an exporter up to \( T \). Thus, an incursion that exited at \( T = 2 \) but is active at \( T = 3 \) after re-entering the market is considered a survivor in the latter horizon.

If the firm does not maintain a continuous presence in the market during all consecutive years after the incursion, subsequent entries are defined as “re-incursions” or “re-entries”. We define an export re-entry as the start of a new spell of exports to a destination by a firm that has exported to that destination in the past but has not done so in the previous year. Re-incursions may also be instances of survival for the original incursion. In the example above, the survival status of the firm at \( T = 4 \) and \( T = 5 \) is taken into account both for the final years of the survival profile of entrants and for the first two years of the survival profile of re-entrants.

**Fact 1:** The exporter survival profile is low and flat

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2 The dataset was collected by this unit as part of their efforts to build the Exporter Dynamics Database. Details of its construction are described in the Annex of Cebeci et al. (2012).

3 For example, incursions in 1997 would be false if the firm exported in the past but not in the last four years. Using the latest years in our database, we find the proportion of incursions that have exported in the past but not in the last four years to be 8.4%. As we consider incursions in later years, false incursions will arise only after a longer period of inactivity. For example, the proportion of false incursions is 3%(1%) when we firms are inactive for 7(10) years. Averaging across incursions in all years, we estimate the proportion of false incursions to be 3.3%.
Figure 3-1: Two key facts of exporter dynamics

Figure 3-1a shows the survival profile of export incursions in our dataset (red solid line). A striking feature of this profile is the low survival rates it displays. Only 35.8% of Peruvian export incursions are still active one year after entry. Five years after entry, the survival rate is 17.7%. Low survival rates are not specific to Peru. Using data from the Exporter Dynamic Database, Cebeci et al. (2012) report that the average and median one-year survival rates across 38 countries are both 43%. Another salient feature of the survival profile is the flat slope after $T = 1$. In contrast to the vast fraction of firms that exits just after entering the export market, further increases in the fraction of non-survivors at longer horizons are considerably more gradual. As a reference, Figure 3-1a displays the domestic survival profile, which is the survival profile of firms as production units (blue dotted line). We denote it “domestic” since all producing firms, except for a negligible fraction of them, sell in the domestic market. Compared to exporter survival rates, domestic survival rates are substantially higher. The first year after entry, 77.9% of U.S. firms in an entry cohort are still in operation. Five years after entry, the survival rate is 49.1%.

The features of the exporter survival profile depicted in Figure 3-1a are not driven by composition. To control for other covariates, we can obtain the survival profile from a regression framework. First, we regress the survival status of incursions in each of the first five horizons on horizon dummies. This exercise is equivalent to simply calculating the survival rate per horizon as we did in the figure. The results of this regression are displayed in column 1 of Table C.1. Then, we add a set of

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4 Splitting the sample into developed and developing countries, the average survival rate is 43% for each of the two groups. For Peru, they find a survival rate of 44%. Their reported rates are higher because they are calculated by previously merging all destinations into one aggregate export market.

5 Domestic survival rates are computed using the number of firms by entry cohorts reported in the Business Dynamics Statistics (BDS) constructed by the Bureau of the Census. For comparison with export survival rates, we only consider tradable-firm producers (agriculture, mining, and manufacturing) in entry cohorts 1997-2004. The survival profile is almost unaffected if we include only manufacturing firms or firms in all remaining sectors.

6 Domestic and exporter survival rates are not strictly comparable. While domestic survival rates capture persistence as an employer, exporter survival rates capture persistence as a seller in a specific market.
fixed effects by product (2-digit Harmonized System), destination country, and year (i.e. the year corresponding to the survival status). We can see in column 2 that adding these flexible controls in all three dimensions has a negligible impact on the estimated survival rates.

The fact that exporter survival rates are notoriously low has already been emphasized in the literature. Freund and Pierola (2010), Albornoz et al. (2012), Nguyen (2012), and Eaton et al. (2014) provide a plausible explanation for this fact. If export profitability has an uncertain component that can only be resolved by being actively exporting, firms have incentives to export as an experiment to resolve their uncertainty. Thus, export entry is consistent with low survival rates since firms are betting on a relatively unlikely outcome. This is also the core mechanism operating in our model. As long as firms resolve their uncertainty sufficiently fast, this mechanism can explain both features of the exporter survival profile. It is low at early horizons because firms soon find that exporting is not a profitable activity. It is flat because firms that resolve their uncertainty favorably are less likely to exit afterwards.

As an additional reference, Figure 3-la also displays the best prediction of a special case of our model—the benchmark model—where this source of uncertainty is removed (green dashed line). As we can see in the figure, the benchmark model is unable to predict the exporter survival profile observed in the data as it predicts too high survival rates early upon entry together with too low survival rates at longer horizons. Despite the specificity of this special case, its inability to fit the exporter survival profile captures a broader implication of standard firm and exporter dynamics models whenever profitability follows a persistent process. These models have difficulty explaining low survival rates at early horizons without also predicting a steep survival profile.

Fact 1 has been key to motivate recent work on uncertainty and experimentation in models of exporter dynamics. Nevertheless, it is the novel fact we present next that, combined with fact 1, makes a substantially stronger case for the relevance of such models.

Fact 2: The survival profile is higher for re-entrants than for (first-time) entrants

It is frequent that firms temporarily cease to export only to re-enter the same market later on. In our dataset, the total instances of re-entry represent 26.4% of total incursions. Figure 3-1b compares the survival profile of re-entrants with the profile for (first-time) entrants displayed in Figure 3-1a. Re-entrants have uniformly higher survival rates. Most of the difference already takes place in the first year after entry, when the survival rate is 46.0% for re-entrants versus 35.8% for entrants. Over longer horizons, this gap is preserved with only slight changes. Like fact 1, fact 2 is not driven by composition either. Columns 3 and 4 of Table C.1 display analogous results including re-incursions. In column 3, we simply include horizon dummies for re-entrants, which delivers the survival rates depicted in Figure 3-1b. In column 4, we include a full set of dummies by product,
destination, and year. Again, we find that these controls for composition do not substantially affect the survival profiles depicted in the figure.\textsuperscript{10}

Fact 2 has no corresponding analog in the firm dynamics literature. As a matter of fact, we are not aware of any study that has computed re-entrant domestic survival rates. A likely reason is that instances of domestic re-entry are much more infrequent than in the case of exports and are typically either dismissed as nuisance or tinkered with assuming they are due to measurement error.\textsuperscript{11}

An appealing explanation for the higher survival rates of re-entrants arises naturally from the experimentation mechanism described above. Firms that exit and re-enter have already resolved their uncertainty and hence do not enter to experiment. As their (re-)entry decisions are made with more accurate information about their potential profitability, they tend to survive longer. This is indeed how our model explains fact 2. Furthermore, in Section 3.4.6 we show that a broad class of models of exporter dynamics is unable to explain facts 1 and 2 jointly in the absence of uncertainty and experimentation.

### 3.3 The model

#### 3.3.1 Set up

Firms go through two stages in their lifetime as exporters in a given market. At first, they are \textit{inexperienced} and earn flow profits

\[ \pi_i(\theta_t) = \begin{cases} \kappa \theta_t - F & \text{if export at } t \\ 0 & \text{otherwise} \end{cases} \]

(subindex \(i\) is for inexperienced) where \(\theta_t\) is a time-varying index of profitability, \(\kappa\) is a profitability shifter and \(F\) is a fixed cost. We allow firms to be heterogeneous in \(\kappa\) and \(F\), as well as in their particular trajectory \(\{\theta_t\}\). However, we assume all firms have the same law of motion for \(\theta_t\), which for analytical tractability we assume to be a geometric brownian motion (GBM),\textsuperscript{12}

\[ d \log \theta_t = \mu dt + \sigma dZ_t. \tag{3.1} \]

\textsuperscript{10}We note that since the horizon dummies sum up to a constant, like the different sets of fixed effects, there is a degree of freedom to set the level of the survival profile at any arbitrary level by choosing an appropriate normalization of the fixed effects. To ease readability, we choose normalizations that leave the coefficient on the horizon dummies at similar levels as the observed survival rates. In any event, those normalizations do not affect the decay of the survival profile.

\textsuperscript{11}Due to how "entry" is defined in standard firm dynamic databases (Baldwin, Beckstead and Girard (2002)), recorded re-entry instances might be spurious. For example, the BDS reports that re-entry instances represent 7% of incursions. However, since the database only includes firms with at least one employee in its payroll, a large fraction of this percentage probably comes from transitions in and out of employer status (Jarmin and Miranda (2002)).

\textsuperscript{12}The profitability parameter \(\theta_t\) can be microfounded as the combination of random processes for demand and productivity jointly determined by a multivariate GBM in a stationary competition environment with CES preferences. See Luttmer (2007).
In other words, we assume that $\mu$, $\sigma$, and the initial level of the process $\theta$ are common across firms. We assume that the firm's discount factor satisfies $r > \mu + \frac{1}{2}\sigma^2$ so that expected profits are finite. Furthermore, to guarantee the existence of a stationary distribution, we follow Arkolakis (2016) and assume that the mass of firms that are born each instant grows at rate $g_B > 0$.

Since all firms are born with $\theta$, $\kappa$ is an index of initial profitability in the market. For example, a high value of $\kappa$ may capture a prior understanding of demand characteristics in the export market that allows the firm to make product adaptations that match their idiosyncratic characteristics (Artopoulos, Friel and Hallak (2013)). This parameter may also capture an advantage in communicating or conducting transactions with foreign agents at lower variable trade costs, e.g. due to family ties. The fixed expenses $F$ represent the costs incurred in activities such as sustaining a distribution network and conducting marketing efforts in the foreign market, which are paid on a continual basis while exporting.

For inexperienced firms, exporting yields additional benefits beyond receiving flow profits. Inexperienced firms know that their current profitability level in the export market is only transient and they will eventually become *experienced* if they keep exporting. More specifically, while exporting, inexperienced firms become experienced with intensity $\lambda$. An experienced firm earns flow profits

$$\pi_e(\theta_t; \psi) = \begin{cases} \psi \kappa \theta_t - F & \text{if export at } t \\ 0 & \text{otherwise} \end{cases}$$

where $\psi$ is the new profitability component that separates an experienced firm from an inexperienced firm. The new component $\psi$ intends to capture the fact that by engaging in the exporting activity the firm might acquire fine-grained knowledge about the tastes and needs of consumers or reconfigure its distribution network by finding more suitable partners (Eaton et al. (2014)).

A key feature of our model is that $\psi$ is unknown ex-ante by inexperienced firms. Those firms only know the distribution of $\psi$, which we assume is common across firms and satisfies $E(\psi) \geq 1$, implying that being experienced is desirable in expectation. The possible sources of uncertainty are various. One of them stems from the need to adapt products to satisfy demand idiosyncrasies in foreign markets (Artopoulos, Friel and Hallak (2013), Eaton et al. (2014)). Firms may be uncertain about the extent to which their product adaptations match those idiosyncrasies and experiment in the market to figure this out. Another source of uncertainty stems from the need to match with distributors that will exert effort to push their products in the destination market (Artopoulos, Friel and Hallak (2013), Eaton et al. (2014)). Firms may also be uncertain about their ability to find such distributors.

Note that we do not include entry sunk costs in the model, so firms may exit and re-enter markets freely. While this is an assumption made for simplicity, we argue later that sunk costs are

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13 We could also assume an exogenous death rate $\delta > 0$. The only difference is that this parameter would directly affect the prediction of the probability of survival of first-time entrants while $g_B$ does not.

14 One could expect $\lambda$ to increase with the length of the exporting experience or the sales volume. This, however, would imply a substantial loss of tractability.

15 Alternatively, we could have modelled $\psi$ affecting fixed costs rather than operating profits. This decision is inconsequential for the purpose of explaining facts related to exporter survival.
neither necessary to obtain the qualitative predictions of the model nor do they help improve its quantitative predictions.

Finally, note that we presented the setup for a generic market. In doing so, we implicitly assumed that while the exogenous part of profitability may be correlated across markets \((\theta_t)\), its endogenous part \((\psi)\) is independent across markets.\(^{16}\) In other words, there are no complementarities in entry decisions across markets. While we think exploring these complementarities is interesting, they are outside the scope of this paper and are left for future research.

### 3.3.2 Entry and exit decisions

It will prove convenient to work with normalized profits defined as \(\tilde{\theta}_t \equiv \frac{\theta_t}{\psi}\). By Ito’s Lemma, \(\tilde{\theta}_t\) is a GBM with the same parameters as \(\theta_t\). Let \(y_e \in \{0, 1\}\) be an indicator function that takes the value of one if the firm exports when its status is \(x = i, e\). The firm’s problem is to maximize its discounted expected profits by choosing an exporting policy \(\{y_e(\tilde{\theta}_t), y_i(\tilde{\theta}_t)\}_{t=0}^{\infty}\). We will solve this problem in two steps. Since \(x = e\) is an absorbing state, we first solve for the optimal policy of an experienced firm \(\{y^*_e(\tilde{\theta}_t; \psi)\}\). Then, we solve for the optimal policy \(\{y^*_i(\tilde{\theta}_t)\}\) of an inexperienced firm taking into account that once it becomes experienced it will follow policy \(\{y^*_e(\tilde{\theta}_t; \psi)\}\).

#### The experienced firm

An experienced firm receives profits given by \(\pi_e(\tilde{\theta}_t; \psi) = \psi \kappa \theta_t - F = F(\psi \tilde{\theta}_t - 1)\) if it exports and 0 otherwise. The value of an experienced firm \((V_e)\) at \(t = 0\) is the solution to the following problem:

\[
V_e(\tilde{\theta}_0; \psi) = \sup_{\{y_e(\tilde{\theta}_t)\}} \mathbb{E} \left( \int_0^\infty e^{-rt} F(\psi \tilde{\theta}_t - 1) y_e(\tilde{\theta}_t) dt \right)
\]

subject to (3.1) with \(\tilde{\theta}_0\) given, where \(r\) is the discount rate.

Suppose a firm follows any constant policy \(y_e \in \{0, 1\}\) during an interval of time \([t, t + \tau]\). Exploiting the stationarity of the problem, we can write the problem recursively as

\[
V_e(\tilde{\theta}_t; \psi) = \max_{y_e \in \{0, 1\}} \mathbb{E} \left( \int_0^\tau e^{-rs} F(\psi \tilde{\theta}_{t+s} - 1) y_e ds + e^{-r\tau} V_e(\tilde{\theta}_{t+\tau}; \psi) \right).
\]

Taking the limit \(\tau \to 0\) and rearranging we obtain

\[
rV_e(\tilde{\theta}; \psi) dt = \max_{y_e \in \{0, 1\}} \left\{ F(\psi \tilde{\theta} - 1) y_e \right\} dt + \mathbb{E} \left( dV_e(\tilde{\theta}; \psi) \right) \tag{3.2}
\]

where due to the stationarity of the problem we drop subscript \(t\). This equation says that the return of the firm is the sum of the instantaneous profit flow plus the expected appreciation. Since future profitability is independent from the firm’s actions and there are no exit or re-entry costs,
the exporting decision only depends on whether current profits are non-negative. Thus, the firm’s optimal policy is simply \( y^*_e (\bar{\theta}; \psi) = 1 \) if \( \bar{\theta} \geq \frac{1}{\psi} \) and \( y^*_e (\bar{\theta}; \psi) = 0 \) if \( \bar{\theta} < \frac{1}{\psi} \).

**The inexperienced firm**

First, we make a technical assumption so that the inexperienced firms’ problem is well-defined: we assume that the distribution of \( \psi \) is such that \( E\psi \left( V_e (\bar{\theta}; \psi) \right) \) satisfies a polynomial growth condition.\(^{17}\) Let \( t \) denote the (random) time at which a firm becomes experienced. Given that this event occurs with intensity \( \lambda \) only if the firm exports, the probability density function (p.d.f) of \( t \) depends on the export policy. At time \( t = 0 \), this density is given by \( \lambda y_i (\bar{\theta}_t) e^{-\int_0^t \lambda y_i (\bar{\theta}_s) ds} \), where the exponent term captures the probability that the shock did not take place until \( t \) and \( \lambda y_i (\bar{\theta}_t) \) is the instantaneous arrival rate. Then, the inexperienced firm’s problem can be written as

\[
V_i (\bar{\theta}_t) = \sup_{\{y_t (\bar{\theta}_t)\}} \left( E \int_0^\infty \left[ \int_0^t e^{-ru} F (\bar{\theta}_u - 1) y_i (\bar{\theta}_u) du + e^{-ru} E \left( V_e (\bar{\theta}_t; \psi) \right) \right] \lambda y_i (\bar{\theta}_t) e^{-\int_0^t \lambda y_i (\bar{\theta}_s) ds} dt \right)
\]

subject to (3.1) with \( \bar{\theta}_0 \) given. Fixing a time \( t \) at which the firm receives the shock, the term in square brackets in (3.3) captures the expected discounted profits, which consist of the discounted stream of net profit flows \( F (\bar{\theta}_u - 1) du \) accumulated during export periods up to \( t \) and the discounted expected value of being an experienced firm. Note that by exporting the firm may become experienced sooner, which is always desirable because it implies a higher profit flow on average.

Manipulating (3.3), we can rewrite the inexperienced firm’s problem as\(^{18}\)

\[
V_i (\bar{\theta}_t) = \sup_{\{y_t (\bar{\theta}_t)\}} E \left( \int_0^\infty e^{-(r+\lambda)y_i} \left\{ F (\bar{\theta}_t - 1) + \lambda E\psi V_e (\bar{\theta}_t) \right\} y_i (\bar{\theta}_t) dt \right)
\]

subject to (3.1) and \( \bar{\theta}_0 \) given. Consider a firm that follows any constant policy \( y_t \in \{0, 1\} \) during an interval of time \([t, t + \tau]\). Exploiting the stationarity of problem (3.4) we can write it recursively as

\[
V_i (\bar{\theta}_t) = \max_{y_t \in \{0, 1\}} E \left( \int_0^\tau e^{-\lambda y_i} \left\{ F (\bar{\theta}_{t+s} - 1) + \lambda E\psi V_e (\bar{\theta}_{t+s}; \psi) \right\} y_i ds + e^{-(r+\lambda)y_i} \tau V_i (\bar{\theta}_{t+\tau}) \right).
\]

Taking the limit \( \tau \to 0 \) and rearranging, we obtain

\[
rV_i (\bar{\theta}) dt = \max_{y_t \in \{0, 1\}} \left\{ F (\bar{\theta} - 1) + \lambda \left( E\psi V_e (\bar{\theta}; \psi) - V_i (\bar{\theta}) \right) \right\} y_i dt + E \left( dV_i (\bar{\theta}; \psi) \right).
\]

The term in brackets in equation (3.6) clarifies the potential trade-off involved in the firm’s exporting decision. On the one hand, by exporting there is a chance that the firm will become experienced.

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\(^{17}\)We say that \( f : [0, \infty) \to \mathbb{R} \) satisfies a polynomial growth condition if there exist \( M > 0 \) and \( \nu > 0 \) such that \( |f (\theta)| \leq M (1 + \theta^\nu) \).

\(^{18}\)Distribute the term \( \lambda y_i (\bar{\theta}_t) e^{-\lambda \int_0^t y_i (\bar{\theta}_s) ds} \) inside the parenthesis and note that

\[
\int_0^\infty \left( \int_0^t e^{-ru} \int_0^s y_i (\bar{\theta}_u) F (\bar{\theta}_u - 1) y_i (\bar{\theta}_u) du \right) dt = \int_0^\infty e^{-ru} \int_0^s y_i (\bar{\theta}_s) ds F (\bar{\theta}_u - 1) y_i (\bar{\theta}_u) du.
\]
Accordingly, the term $\lambda \left( E_\psi V_e (\hat{\theta}; \psi) - V_i (\hat{\theta}) \right)$ captures the benefits of experimentation, which are always non-negative since profits are on average higher for an experienced firm. In fact, when $\hat{\theta}^* \geq 1$ and experimentation is meaningful (i.e. $\psi$ is not identical to 1), these benefits are strictly positive. Thus, inexperienced firms unambiguously prefer to export when $\hat{\theta}^* \geq 1$. On the other hand, when $\hat{\theta} < 1$ the first term becomes negative, i.e. $F \left( \hat{\theta} - 1 \right) < 0$. Thus, when $\hat{\theta} < 1$ the firm faces a trade-off: by exporting it earns the possibility of becoming experienced at the cost of incurring a loss. The following proposition shows that there exists a region $\left( \hat{\theta}^*, 1 \right)$ where firms choose to experiment.

**Proposition 3.1.** (a) There exists an optimal policy characterized by a threshold $\hat{\theta}^* \in [0, 1]$ such that if $\hat{\theta} < \hat{\theta}^*$, the firm does not export while if $\hat{\theta} \geq \hat{\theta}^*$, the firm exports. This policy is the unique piecewise continuous optimal policy. Furthermore, if the distribution of $\psi$ is not degenerate at 1, then $\hat{\theta}^* < 1$. (b) $\hat{\theta}^*$ solves $\pi_i \left( \hat{\theta}^* \right) + \lambda \left( E_\psi V_e \left( \hat{\theta}^*; \psi \right) - V_i \left( \hat{\theta}^* \right) \right) = 0$.

**Proof.** See Appendix C.1.1.

Proposition 3.1 states that there exists a threshold $\hat{\theta}^* \leq 1$ such that the firm exports iff $\hat{\theta} \geq \hat{\theta}^*$. In fact, this result holds in a more general setup than assuming a GBM and a multiplicative shock. In Appendix C.1.1, we specify a set of sufficient conditions such that the firm follows a threshold strategy. The key condition, which is satisfied in our setup, is:

$$\frac{d\lambda E_\psi (\max \{ \pi_e (\theta_t; \psi), 0 \})}{d\theta_t} > \lim_{dt \to 0} \left\{ \frac{d}{d\theta_t} \left[ e^{rt} \pi_i (\theta_t) - \pi_i (\theta_t) \right] \right\}.$$ 

This condition says that the expected flow benefits of becoming experienced should increase faster than the costs of experimenting today rather than tomorrow (recall $\pi_i < 0$ in the relevant region).

Proposition 3.1 also states that optimality at $\hat{\theta}^*$ requires that:

$$F \left( \hat{\theta}^* - 1 \right) + \lambda \left( E_\psi V_e \left( \hat{\theta}^*; \psi \right) - V_i \left( \hat{\theta}^* \right) \right) = 0. \quad (3.7)$$

Appendix C.1.2 shows that (3.7) can be solved to obtain

$$\hat{\theta}^* = 1 + \lambda \left( \frac{2}{J + \tilde{J}} \right) \left[ \int_{\hat{\theta}^*}^{\infty} \left( \hat{\theta}_z \right) \beta_1 \left( E_\psi (\max \{ \psi z - 1, 0 \}) - (z - 1) \right) \frac{dz}{z} \right] + \int_0^{\hat{\theta}^*} \left( \hat{\theta}_z \right) \beta_2 \left( E_\psi (\max \{ \psi z - 1, 0 \}) \right) \frac{dz}{z} = 0 \quad (3.8)$$

where $J = \sqrt{\mu^2 + 2\sigma^2}$, $\tilde{J} = \sqrt{\mu^2 + 2 \left( \frac{r + \lambda}{\sigma^2} \right)}$, $\beta_1 = -\frac{\mu + J}{\sigma^2} > 1$ and $\beta_2 = -\frac{\mu - \tilde{J}}{\sigma^2} < 0$.

The intuition for (3.8) is as follows. First, note that for any GBM we can write the solution

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19 This rules out cases in which there is a region for $\theta$ where experienced-firm profits are relatively high but inexperienced-firm losses from exporting are strongly decreasing in $\theta$, inducing firms to wait, and another region in which inexperienced-firm losses from exporting are flat in $\theta$ and experienced-firm profits are low but high enough so that firms want to export.
as an integral of the flow over states $z$ multiplied by a "weight" for that state.\textsuperscript{20} The weight represents the length of time the process spends in each state, taking into account the proper discounting. For states with $z > \tilde{\theta}^*$, the correct discount – which is reflected in $\beta_1$ – is $r + \lambda$ since the inexperienced firm becomes experienced at rate $\lambda$. Since in that region the inexperienced firm exports, the integrand is the difference between the (expected) flow profits of an experienced firm and that of an inexperienced firm. Note $\beta_1 > 0$ since larger $z$ are less likely and therefore more heavily discounted. For states $z < \tilde{\theta}^*$, only some experienced firms export. Hence, we only have the (expected) flow profits of an experienced firm. The proper discount, which is reflected in $\beta_2$, is now $r$ since an inexperienced firm remains inexperienced in this region. Note that $\beta_2 < 0$, reflecting that when $z < \tilde{\theta}^*$ lower states are less likely and, thus, more heavily discounted.

Equation (3.8) shows that our uncertainty and experimentation model preserves a tractable property of simpler continuous-time models as we only need to solve one equation in one unknown to characterize the whole strategy of the firm.\textsuperscript{21,22} While this is special to the GBM framework, we can still allow an arbitrary distribution for $\psi$, $F$ and $\kappa$. Furthermore, note that $F$ and $\kappa$ do not appear in (3.8). Hence, $\tilde{\theta}^*$ does not depend on these parameters. In other words, $\tilde{\theta}^*$ is proportional to $\kappa$ and $\frac{1}{F}$. Intuitively, the firm "undoes" the effect of $\kappa$ and $F$ by timing its entry decision: a low-$\kappa$ firm will wait longer until $\theta$ is large enough to perfectly offset the effect of $\kappa$. This property of the model is going to be very important in the next section and in the empirical exercise.

Let us recap the alternative export trajectories a firm can exhibit. The firm is originally inexperienced and stays away from the market as long as normalized profits are below $\tilde{\theta}^*$. As soon as $\tilde{\theta}$ crosses this entry threshold it starts to export. The purpose of entering the market is initially to experiment, albeit incurring losses, in the expectation of resolving the uncertainty with respect to their profitability shifter $\psi$. Eventually, one of three events might occur: (a) $\tilde{\theta}$ might decrease and cross $\tilde{\theta}^*$ from above, in which case the firm will stop exporting; (b) $\tilde{\theta}$ might increase above 1, in which case it will turn losses into profits; (c) the firm might receive the $\psi$ shock. In the last case, the firm will keep exporting only if the shock has been sufficiently large to generate a net profit flow (i.e. if $\psi \geq \frac{1}{\tilde{\theta}}$).

### 3.3.3 The probability of survival

Henceforth, we assume that all firms are born inactive in the export market, ie. $\frac{\kappa \tilde{\theta}}{F} < \tilde{\theta}^*$. Normalizing the entry time to $t = 0$, the firm enters the foreign market with $\tilde{\theta}_0 = \tilde{\theta}^*$. Since $\tilde{\theta}_t$ is a GBM,

$$\ln \tilde{\theta}_T = \ln \tilde{\theta}^* + \mu T + \sigma Z_T$$

\textsuperscript{20}The weight here is $\left( \frac{2}{\mu + \lambda} \right) \left( \frac{\tilde{\theta}^*}{\mu} \right)^{\frac{\lambda}{\mu}}$ for $\tilde{\theta} > \tilde{\theta}^*$ and $\left( \frac{2}{\mu + \lambda} \right) \left( \frac{\tilde{\theta}^*}{\mu} \right)^{\frac{\lambda}{\mu}}$ for $\tilde{\theta} < \tilde{\theta}^*$. This is a property of GBM processes (see Stokey (2009)).

\textsuperscript{21}By contrast, note that solving for the optimal strategy in discrete-time dynamic models requires the computationally-intensive procedure of iterating on the Bellman equation.

\textsuperscript{22}$\pi_\theta$ and $\pi_\psi$ being both linear in $\theta$ (or, equivalently, $\psi$ being multiplicative) is not important for this result. In Appendix C.1.2 we show that with general profit functions $\pi_\theta(\theta)$ and $\pi_\psi(\theta; \psi)$ the problem can still be reduced to one equation in one unknown, as long as the conditions in Proposition 3.1, specified in Appendix C.1.1, hold.
is a SBM where \( Z_T \) is a standard normal random variable. Defining \( x_T = \frac{\ln \tilde{\theta}_T - \ln \tilde{\theta}^*}{\sigma} \),

\[
x_T = \frac{\mu}{\sigma} T + Z_T.
\]

First, note that, while a firm is inexperienced, it is active iff

\[
\ln \tilde{\theta}_T > \ln \tilde{\theta}^* \Leftrightarrow x_T > 0 \Leftrightarrow \frac{\mu}{\sigma} T + Z_T > 0. \quad (3.9)
\]

Thus, the likelihood of this event depends only on \( \frac{\mu}{\sigma} \). Since an exporter becomes experienced at an intensity governed by \( \lambda \), the likelihood of being experienced at any point in time depends only on \( \frac{\mu}{\sigma} \) and \( \lambda \).

Second, define \( \tilde{\psi} \equiv \left( \frac{\psi}{\psi_m} \right)^{\frac{1}{\lambda}} \) for some \( \psi_m > 0 \). \( \tilde{\psi} \) will later be useful to compare different distributions of \( \psi \) that differ only in a scale parameter \( \psi_m \). Note that, while a firm is experienced, it is active iff

\[
\ln \psi + \ln \tilde{\theta}_T > 0 \Leftrightarrow \ln \tilde{\psi} + \ln \psi_m \tilde{\theta}^* \Leftrightarrow x_T > 0 \Leftrightarrow \ln \tilde{\psi} + \ln \psi_m \tilde{\theta}^* + \frac{\mu}{\sigma} T + Z_T > 0 \quad (3.10)
\]

Let \( \tilde{\Psi} \) denote the set of parameters that characterize the distribution of \( \tilde{\psi} \). Thus, the likelihood of this event depends only on \( \frac{\mu}{\sigma}, \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \) and \( \tilde{\Psi} \).

Let \( y(t) \) be an indicator function that takes the value of 1 if the firm is an exporter at \( t \). Putting both parts together, we see that knowing \( \Upsilon = \left\{ \mu, \lambda, \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma}, \tilde{\Psi} \right\} \) is sufficient to determine the likelihood of any given trajectory of \( \{y(t)\} \). Thus, the expectation of any function of those trajectories also depends on \( \Upsilon \). The following proposition formally states this result:

**Proposition 3.2.** Take any function \( f: A \to \mathbb{R} \) with \( A \subset P(\{y(t)\}_{t=0}^{\infty}) \). Then, \( E(f) \) depends solely on \( \Upsilon \). In particular, the probability of survival of an entrant, at horizon \( T \) (\( p_T \)), depends only on \( \Upsilon \).

**Proof.** Since \( A \) is a set of subsets of export trajectories \( \{y(t)\}_{t=0}^{\infty} \), and the likelihood of each trajectory \( \{y(t)\}_{t=0}^{\infty} \) only depends on \( \Upsilon \), the likelihood of an event \( a \in A \) only depends on \( \Upsilon \). Since \( f \) takes different values depending only on which event \( a \in A \) occurs, it follows that \( E(f) \) depends only on \( \Upsilon \). For example, in the case of the probability of survival, take \( f = y(T) \) and apply the result.

As we will discuss later, Proposition 3.2 shows that using only information on entrant survival allows us to identify only a subset of the parameters of the model. Furthermore, the following corollary shows that the probability of survival of incumbents also depends only on this combination of parameters,

**Corollary 3.1.** The probability of survival of incumbents depends only on \( \Upsilon \).
Proof. Since incumbents are just a weighted sum of entrants of different ages and the probability of survival for entrants of a given age falls within the scope of Proposition 3.2, it immediately follows that the probability of survival of incumbents is also a function of $T$.

Another consequence of Proposition 3.2 is the following,

**Corollary 3.2.** $p_T$ is independent of $\kappa$ and $F$.

**Proof.** By Proposition 3.2, $p_T$ only depends on $T$. From equation (3.8) it follows that $\tilde{\theta}^*$ is independent of $\kappa$ and $F$. Thus, $p_T$ is also independent of $\kappa$ and $F$.

Corollary 3.2 is a key result. It establishes that the probability of survival of an export incursion is independent of $\kappa$ and $F$ and hence only depends on parameters that are common across firms.\(^{23}\) The main implication of this result is that all firms entering a given market have the same probability of survival $T$ periods after entry. The strength of this prediction is achieved despite a substantial amount of heterogeneity in the model allowed for by heterogeneous profit shifters ($\kappa$) and fixed costs ($F$) across firms and markets. Heterogeneity in $\kappa$ allows the model to affect the likelihood of any entry sequence into foreign markets. This parameter, however, does not provide any information about the probability of survival in the market once it has entered it.

Since entry profits are given by $\pi_o \sim F$ (the common factor of proportionality is $\tilde{\theta}^*$), heterogeneity in $F$ also implies heterogeneous sales at the time of entry. For example, if sales are a constant proportion of profits, entry sales will also be proportional to fixed costs. Thus, the model has a degree of freedom left to rationalize the shape of the size distribution of entrants – and potentially the size distribution of incumbents – by adjusting accordingly the distribution of fixed costs. Most results in this paper do not depend on specific assumptions about this distribution, which we do not need to impose. The two implications of Corollary 3.2 highlight an advantage of focusing on entrant survival since we can obtain sharp predictions on observables without sacrificing flexibility over firm-specific parameters we have little information about.

Next, we will compute $p_T$. Since the firm can only receive shock $\psi$ while it exports, it will be useful to define the occupation time $s$ as the total length of time the stochastic process $x_t$ spends above 0 between $t = 0$ and $t = T$:

$$s = \int_0^T 1_{x_t \geq 0} dt$$

where $1_{x_t \geq 0}$ is an indicator function for the event $\{x_t \geq 0\}$. Given an occupation time $s$, the probability of not receiving the shock between 0 and $T$ is $P(\text{no jump} | s) = e^{-\lambda s}$.

Denote by $\omega_T(s, x)$ the joint density of an occupation time of $s$ (between 0 and $T$) and $x_T = x$. Then, we can express the joint probability of not receiving the shock until $T$ and $x_T = x$ as

$$P(\text{no jump}, x_T = x) = \int_0^T e^{-\lambda s} \omega_T(s, x) ds$$

\(^{23}\)An analogous result with respect to heterogeneous market-specific profitability shifters is obtained in Albornoz, Fanelli and Hallak (2016) in a framework without experimentation and with homogenous fixed costs.
while the analogous joint probability for the case in which the firm receives the shock is

\[ P(\text{jump}, x_T = x) = \int_0^T (1 - e^{-\lambda s}) \omega_T(s, x) ds. \]

Conditional on \( x_T \), the probability of survival of an experienced firm at \( T \) is \( P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \bar{\theta}^*)}{\sigma} \right) \).

Now we have all the required elements to compute \( p_T \). Taking into account that the \( \psi \) shock and the process \( x_t \) are mutually independent, we can decompose this probability into two terms. If \( x_T \leq 0 \) then the firm will only survive if it has received the shock \( \text{and} \) the shock was sufficiently large. If \( x_T \geq 0 \) then two things may happen: (a) if the firm has received the shock, then survival depends on the magnitude of the shock; (b) if the firm has not received the shock, then it will be exporting at \( T \) since it finds it profitable (in expected terms) to keep waiting for the shock. Thus, \( p_T \) can be written as:

\[
P_T = \begin{cases} 
\int_{-\infty}^{0} \int_{s=0}^{T} \left(1 - e^{-\lambda s}\right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \bar{\theta}^*)}{\sigma} \right) \omega_T(s, x) ds dx \\
\int_{0}^{\infty} \int_{s=0}^{T} \left(1 - e^{-\lambda s}\right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \bar{\theta}^*)}{\sigma} \right) + e^{-\lambda s} \omega_T(s, x) ds dx.
\end{cases}
\]

Pechtl (1999) shows that \( \omega_T(s, x) \) has the following closed form solution:

\[
\omega_T(s, x) = \begin{cases} 
\exp \left\{ -\frac{(\mu x)^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} \left( \int_{T-s}^{\infty} \frac{x(T-t)^{3/2}}{\sqrt{2\pi(t-T)}} dt \right) \right\} & \text{if } x \geq 0 \\
\exp \left\{ -\frac{(\mu x)^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} \left( \int_{T-s}^{\infty} \frac{x(T-t)^{3/2}}{\sqrt{2\pi(t-T)}} dt \right) \right\} & \text{if } x < 0.
\end{cases}
\]

Therefore, given a parametrization of the distribution of \( \psi \) (we do this in the next section) equation (3.11) is easy to compute numerically.

The introduction of uncertainty and experimentation allows the model to explain fact 1. Given \( \mu \) and \( \sigma \), a model with uncertainty about \( \psi \) can consistently predict lower survival rates over a finite horizon than a “benchmark model” without experimentation (in which case \( P(\psi = 1) = 1 \)) and hence help explain the low survival profile exhibited by fact 1. This result is established in the following proposition:

**Proposition 3.3.** Define \( p_{BT} \) as the probability of survival at horizon \( T \) in the benchmark model. Then, given \( (\mu, \sigma, r) \), any distribution of \( \psi \) with unbounded support, and any interval \([T, \bar{T}]\) with \( T > 0 \) and \( \bar{T} < \infty \), we can pick \( \bar{\lambda} \) such that for any \( \lambda > \bar{\lambda} \), \( p_T < p_{BT} \) for \( T \in [T, \bar{T}] \).

**Proof.** See Appendix C.1.3.

The intuition of this result is the following. Firms are willing to enter the foreign market with a low value \( \bar{\theta}^* \) to experiment and resolve their uncertainty regarding \( \psi \). Since there are no sunk costs, the only cost of this strategy is the accumulation of losses until the firm becomes experienced. Lured by potential future profits, firms are willing to bet on this uncertain event even if only a good
shock will justify their permanence as exporters. Furthermore, for a sufficiently high probability
that the jump occurs fast (a high \( \lambda \)) the firm will be willing to accept a very low survival probability
in exchange of the chance of getting a good \( \psi \) draw.\(^{24}\) This is the main insight of the proposition.
In addition to explaining a low export survival profile for any set of dynamic parameters \((\mu, \sigma)\), this
mechanism can also explain the flatness of the survival profile observed in the data. While in the
short run the uncertainty regarding \( \psi \) implies many firms die because of the shock, firms that do
receive the \( \psi \) shock are placed far away from the threshold. Hence, they take longer to exit than
firms in the benchmark model. In contrast, as we discuss in the next section, the benchmark model
can only match a low survival rate at a specific horizon by also predicting a steep survival profile.

The introduction of uncertainty can also help explain fact 2. This is established by a simple
corollary to Proposition 3.3:

**Corollary 3.3.** Define \( p_{RT} \) as the probability of survival over horizon \( T \) for a re-entrant to the
market. Then, given \((\mu, \sigma, r)\), any distribution of \( \psi \) with unbounded support, and any interval \([T, \bar{T}]\)
with \( T > 0 \) and \( \bar{T} < \infty \), we can pick \( \lambda \) such that for any \( \lambda > \bar{\lambda} \), \( p_{RT} > p_T \) for \( T \in [T, \bar{T}] \).

*Proof.* There are two classes of re-entrants. Re-entrants that have not received the shock re-enter
and exit at the same thresholds as first-time entrants. Hence, they have their same probability
of survival \((p_T)\). Re-entrants that have received the shock enter and exit at \( \theta^* = \frac{1}{\psi} \). Although
this threshold is different from the entry and exit thresholds of firms in the benchmark model, the
fact that entry and exit thresholds coincide in both cases implies that they have identical survivalprobabilities \((p_{BT})\). Denoting by \( \rho_s > 0 \) the probability that a re-entrant has received the shock,
the survival probability for a re-entrant is \( p_{RT} = (1- \rho_s)p_T + \rho_s p_{BT} \). Using Proposition 3.3, this
directly implies that \( p_{RT} > p_T \).

This result is driven by the fact that a fraction of re-entrants already knows \( \psi \) and hence is only
willing to re-enter when they can make positive profits. As a result, they enter with a higher value
of \( \theta_t \) and thus are more likely to survive. The role of uncertainty to explain fact 2 is much more
transparent here since in this case the predictive failure of the benchmark model is qualitative. In
the absence of uncertainty, entrants and re-entrants are predicted to have the same probability of
survival.

While the firm is uncertain about the value of \( \psi \), it also knows that \( E(\psi) \geq 1 \). However, the
results of Proposition 3.3 and Corollary 3.3 are driven by the fact that \( \psi \) is ex-ante unknown. To see
why, consider a family of distributions \( \Psi \) that satisfy \( \psi = \psi_m \hat{\psi} \), i.e. they are identical up to a scale
parameter \( \psi_m \). Hence, \( E(\psi) \geq 1 \) can be decomposed into \( \psi_m \geq 1 \) and \( E(\hat{\psi}) = 1 \). The following
proposition states that the probability of survival goes up with \( \psi_m \). Thus, the fact that \( E(\psi) \geq 1 \)
could not alone generate Proposition 3 and Corollary 3 if the distribution of \( \hat{\psi} \) were degenerate.

\(^{24}\)The role of the unbounded support for the distribution of \( \psi \) is to make the threshold \( \tilde{\theta}^* \) arbitrarily small when
\( \lambda \rightarrow \infty \). This is a sufficient condition to make the entrant probability of survival arbitrarily small in that limiting case.
Proposition 3.4. Let $\psi = \psi_m \tilde{\theta}$. Then, the probability of survival at any horizon increases with $\psi_m$.

Proof. See Appendix C.1.4.

Proposition 3.2 establishes that survival only depends on $\psi_m \tilde{\theta}^*$. In fact, it follows immediately from (3.10) that the probability of survival increases with this product. The key step in the result is proving that $\psi_m \tilde{\theta}^*$ increases with $\psi_m$ (note that $\tilde{\theta}^*$ is a decreasing function of $\psi_m$). Recall that $\tilde{\theta}^*$ is independent of $\kappa$, i.e. if $\kappa$ increases then $\theta^*$ decreases exactly such that $\kappa \theta^*$ stays constant. An increase in $\psi$, realization by realization, is similar to an increase in $\kappa$ except during the experimentation period. Thus, the firm does not fully offset the larger $\psi$ with a smaller $\tilde{\theta}^*$. If the distribution of $\psi$ were degenerate (i.e. deterministic), then $E(\psi) \geq 1$ would imply that $\psi_m > 1$. Thus, in this case the previous result implies that firms in the full model would survive at least as much as firms in the benchmark model, which would contradict fact 2. It would also imply that in an uncertain world making the shock more attractive realization by realization (ie. more learning by exporting) would worsen the ability of the model to explain the data.

3.4 Estimation

In this section, we estimate the model parametrizing the shock with a Pareto distribution. We describe the data and the estimation strategy, and we discuss the ability of the model to explain facts 1 and 2. This section also discusses the extent to which alternate models often used in the literature that do not account for uncertainty and experimentation can explain these two facts.

3.4.1 Survival probabilities under time aggregation

Our model is set in continuous time and hence assumes that firms make export decisions at every instant in time. The data, however, record transactions over discrete time periods. This mismatch introduces a time aggregation problem. We describe here how we correct for it.

For expositional transparency, let us focus on the “benchmark model” (i.e. the special case without uncertainty where either $\alpha \to \infty$ or $\lambda \to 0$). In the benchmark model, the decision to export does not bear dynamic consequences. Hence, the firm exports whenever it makes positive profits ($\tilde{\theta}^* = 1$). For the normalized process $x_t$, this implies that entry an exit thresholds coincide at $x_t = 0$. Given these thresholds and the fact that $x_t$ follows a SBM with drift $\tilde{\mu} = \frac{\mu}{\sigma}$ and diffusion parameter 1, the probability of survival at instant $T$ after entry is simply given by:

$$p_{BT} = \Phi \left( \tilde{\mu} \sqrt{T} \right).$$

(3.12)

This probability depends only on $\tilde{\mu}$, not on the individual values of $\mu$ and $\sigma$.

Equation (3.12) predicts an “instantaneous” probability, i.e. the probability that a firm entering at $t = 0$ is still active at $t = T$. For example, for a firm that entered on January 1st of calendar
year 0 this formula provides the probability that it exports on January 1st of calendar year $T$. However, this is not how we observe the data. First, calendar year $T$ will report positive exports even if the firm was not exporting at $t = T$ (January 1st) as long as it re-entered anytime between that instant and $t = T + 1$. We call this the “re-entry effect”. Second, the firm could have first entered the market at any instant $t \in [0, 1)$ (e.g. August 14th) rather than at $t = 0$ (January 1st). In that case, the relevant horizon to compute survival in calendar year $T$ is shorter. This is the “partial year effect” emphasized by Berthou and Vicard (2015) and Bernard et al. (2017). Thus, time aggregation requires two adjustments to make $P_{BT}$ a proper theoretical counterpart of survival rates as we observe them in the data.

Denote by $P_{BT}(\tau)$ the probability of survival adjusted for the re-entry effect for an incursion made at $\tau \in [0, 1)$. Assuming a uniform density for the time of entry throughout the year, we can account for the partial year effect by computing $P_{BT} = \int_0^1 P_{BT}(\tau)d\tau$. In turn, $P_{BT}(\tau)$ consists of two parts. The first is the instantaneous component. It captures the probability that a firm that entered the market at $t = \tau$ is actively exporting in the instant $t = T$. This event will happen if $X_T \geq 0$. The second part is the re-entry component. It captures the probability that the firm is not exporting at $t = T$ but does it at some point during calendar year $T$. This event will occur if $X_T < 0$ but $x_t \geq 0$ for some $t \in [T, T + 1)$. Denote by $a$ the first instant in time, starting from $t = T$, that $x_t \geq 0$. This is known as the “first passage time” (FPT) and can be defined as $a = \inf \{t : t \geq T \text{ s.t. } x_t \geq 0\}$. Then, the two parts of $P_{BT}(\tau)$ can be written as:

$$P_{BT}(\tau) = P(a = T) + P(T < a \leq T + 1).$$

(3.13)

Computing the instantaneous part just requires modifying the formula in equation (3.12) to account for the fact that the relevant time horizon is $T - \tau$ rather than $T$. Thus, $P(a = T) = \Phi \left( \frac{\bar{\mu}\sqrt{T - \tau}}{\sigma} \right)$. Computing the re-entry part requires that we appeal to known formulas for the FPT of a SBM. We omit the resulting formula but note that it can be derived as a mathematical expression that can be solved up to integrals that need to be numerically computed. Suffice it to say here that the re-entry component is also only a function of $\bar{\mu} = \frac{\mu}{\sigma}$. Thus, so are $P_{BT}(\tau)$ and $P_{BT}$.

Adjusting for the re-entry effect and the partial-year effect has a considerable impact on survival probabilities. To assess the quantitative importance of these two adjustments, in Table 3.1 we report, for values of $\bar{\mu}$ ranging from $-0.1$ to $-0.9$: (a) the instantaneous probability of survival without adjustment (i.e. calculated according to (3.12)); (b) the instantaneous probability of survival adjusted for the partial year effect assuming uniform entry; (c) the probability of survival adjusted only for the re-entry effect; and (d) the probability of survival adjusted for both the partial year effect and the re-entry effect. Panel A reports these probabilities at horizon $T = 1$ while panel B reports them at horizon $T = 5$. As we can see in the table, the combined impact of the two effects is substantial. For example, in the case of $\bar{\mu} = -0.5$, accounting for both effects more than doubles.

---

Using Peruvian export data at a monthly frequency, Bernard et al. (2017) show that actual export entry throughout the year is close to uniform.
Table 3.1: Adjustments of survival probabilities for time aggregation effects

<table>
<thead>
<tr>
<th>Fixed parameters</th>
<th>-0.1</th>
<th>-0.3</th>
<th>-0.5</th>
<th>-0.7</th>
<th>-0.9</th>
</tr>
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<tbody>
<tr>
<td>( \frac{\mu}{\sigma} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Instantaneous</td>
<td>0.46</td>
<td>0.38</td>
<td>0.31</td>
<td>0.24</td>
<td>0.18</td>
</tr>
<tr>
<td>Partial-year correction</td>
<td>0.47</td>
<td>0.42</td>
<td>0.37</td>
<td>0.32</td>
<td>0.28</td>
</tr>
<tr>
<td>Re-entry correction</td>
<td>0.71</td>
<td>0.62</td>
<td>0.52</td>
<td>0.43</td>
<td>0.33</td>
</tr>
<tr>
<td>Fully-adjusted</td>
<td>0.79</td>
<td>0.73</td>
<td>0.66</td>
<td>0.59</td>
<td>0.52</td>
</tr>
</tbody>
</table>

Panel A: One year horizon

<table>
<thead>
<tr>
<th>Panel B: Five year horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instantaneous</td>
</tr>
<tr>
<td>Partial-year correction</td>
</tr>
<tr>
<td>Re-entry correction</td>
</tr>
<tr>
<td>Fully-adjusted</td>
</tr>
</tbody>
</table>

Note: Reported figures survival probabilities computed as described in Section 3.4.3.

the survival prediction when \( T = 1 \) while it increases it by 68% when \( T = 5 \). The table also shows that the re-entry effect is more important than the partial-year effect. In the case of \( \bar{\mu} = -0.5 \), the re-entry effect alone accounts for 60% of the total adjustment when \( T = 1 \) while it accounts for 77% when \( T = 5 \).

While the exposition in this section focused on the predictions for survival of the benchmark model, analogous adjustment for re-entry and partial-year effects can be made on the predictions for survival of the full model derived in Section 3.3.3. Although they can also be derived as mathematical expressions that can be solved up to integrals that need to be numerically computed, the estimation of the full model, described next, simulates those adjusted probabilities.

3.4.2 Parameter identification

Proposition 3.2 states that survival probabilities only identify particular combinations of the model's parameters, \( \Upsilon = \left\{ \frac{\ln(\psi_m \delta^x)}{\sigma}, \bar{\mu}, \bar{\Psi}, \lambda \right\} \). For parsimony, we specify \( \psi \) to follow a Pareto distribution with location parameter \( \psi_m \) and scale parameter \( \alpha \). This, in turn, implies that \( \bar{\psi} = \left( \frac{\psi}{\psi_m} \right)^{\frac{1}{\sigma}} \) follows a Pareto distribution with scale parameter 1 and shape parameter \( \sigma \alpha \). In other words, \( \bar{\Psi} = \{ \sigma \alpha \} \).

Although the model has six unknowns, \( \Phi = \{ \mu, \sigma, \alpha, \psi_m, \lambda, r \} \), \( \Upsilon \) has only four elements. In fact, using the threshold equation (3.8) we can vary \( r \) or \( \psi_m \), and change \( \sigma \) while keeping \( \alpha \sigma \), \( \lambda \) and \( \frac{\bar{\mu}}{\sigma} \) constant in order to keep \( \ln(\psi_m \delta^x) \) constant. In other words, without more information we cannot separately identify \( r \), \( \psi_m \) and the level of the parameters of the profitability process \( \alpha \), \( \mu \) and \( \sigma \). Put differently, we do not need to identify these parameters to obtain predictions on survival probabilities.

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Table 3.2: Identification

Fixed parameters

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
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<td>$\mu$</td>
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<td>-0.1</td>
<td>-0.15</td>
</tr>
<tr>
<td>$\sigma$</td>
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<td>0.28</td>
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<td>$\alpha$</td>
<td>6.9</td>
<td>3.7</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Predicted survival probabilities

Panel A: Entrants

<table>
<thead>
<tr>
<th>Year</th>
<th>0.372</th>
<th>0.357</th>
<th>0.357</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 2</td>
<td>0.226</td>
<td>0.213</td>
<td>0.213</td>
</tr>
<tr>
<td>Year 3</td>
<td>0.197</td>
<td>0.182</td>
<td>0.185</td>
</tr>
<tr>
<td>Year 4</td>
<td>0.177</td>
<td>0.166</td>
<td>0.167</td>
</tr>
<tr>
<td>Year 5</td>
<td>0.161</td>
<td>0.150</td>
<td>0.152</td>
</tr>
</tbody>
</table>

Panel B: Re-entrants

<table>
<thead>
<tr>
<th>Year</th>
<th>0.628</th>
<th>0.617</th>
<th>0.627</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 2</td>
<td>0.463</td>
<td>0.459</td>
<td>0.462</td>
</tr>
<tr>
<td>Year 3</td>
<td>0.388</td>
<td>0.386</td>
<td>0.385</td>
</tr>
<tr>
<td>Year 4</td>
<td>0.334</td>
<td>0.332</td>
<td>0.332</td>
</tr>
<tr>
<td>Year 5</td>
<td>0.294</td>
<td>0.292</td>
<td>0.292</td>
</tr>
</tbody>
</table>

Note: Reported survival probabilities are computed using the parameters specified in the corresponding column.
Now focus on $\bar{\mu} = \bar{\theta} \sigma$. In the benchmark model, equation (3.12) shows that survival probabilities depend only on this ratio. In the full model, although this ratio is only one of four arguments in $\Upsilon$, the levels of $\mu$ and $\sigma$ are still only weakly identified. The identification problem is illustrated in Table 3.2. First, fixing $\mu = -0.1$, $\psi_m = 1$ and $r = 0.1$, we estimate $\sigma$, $\alpha$, and $\lambda$ by the Simulated Method of Moments (SMM). This is in fact our baseline estimation, discussed below, which we perform using survival rates of entrants and re-entrants in the first five horizons. Alternatively, we fix $\mu = -0.05$ and $\mu = -0.15$, set $\sigma$ to maintain the same ratio $\bar{\theta} \sigma$, and estimate $\alpha$ and $\lambda$ by the SMM with the same moments. The identification problem is manifest in the comparison across columns. Despite the wide variation of $\mu$ and $\sigma$ across the three parameter configurations, they all deliver similar survival predictions.

Given the identification scope of survival probabilities, we set $\mu = -0.10$, $r = 0.10$ and $\psi_m = 1$ and estimate $\{\sigma, \alpha, \lambda\}$. Note that the assumption that the scale parameter of the Pareto distribution is 1 implies that the firm will always improve profitability when the shock arrives. In this sense, we can interpret it as a learning shock. We think this is a reasonable assumption as firms are likely to go through a learning period while operating in a foreign market.

### 3.4.3 Estimation strategy

We simulate the model by generating 10000 artificial export profit trajectories. An export profit trajectory consists of three independent random components: a GBM trajectory for $\tilde{\theta}_t$, a realization of the learning shock $\psi$, and a Poisson process governing its arrival. In the computer, we can only simulate an approximation to a continuous process – both for the GBM and the Poisson processes – by discretizing the time space. We artificially create calendar years and divide each in $N = 1000$ intervals (each represented by its middle point) to make this approximation as precise as possible subject to computing constraints. Since all predictions of the model can be expressed in terms of normalized profits and since all firms first enter at $\tilde{\theta}_0 = \tilde{\theta}^*$, we generate 10000 trajectories for $\tilde{\theta}_t$ that start at this threshold value. We assume that firms’ entrance is uniformly distributed along the unit interval during year 0. To gain simulation precision minimizing loss in computer efficiency, each simulation is used 1000 times by sliding the entry time during the first year along each of its 1000 intervals.

For each simulation, we track whether $\psi \tilde{\theta}_t \geq 1$ for an experienced firm or $\tilde{\theta}_t \geq \tilde{\theta}^*$ for an inexperienced firm in each of the 1000 intervals of each calendar year. Whenever a firm satisfies this condition in at least one interval of a calendar year, we consider it to have survived in that year. For entrants, we only need to keep track of survival status in the next five years after the start of the simulation process. The survival probabilities $\{P_T\}_{T=1,\ldots,5}$ are computed as the proportion of surviving firms in each of the first five years. Computing the survival probabilities of re-entrants ($P_{RT}$) is more involved. As discussed in Section 3.3.3, this probability is a weighted average of the survival probability of entrants, $P_T$, which applies to inexperienced re-entrants, and the survival probability in the benchmark model, $P_{BT}$, which applies to experienced re-entrants. While both of these probabilities are easier to compute, we still need the probability that a re-entrant is experienced.
to apply appropriate weights (the proportion of experienced re-entrants is 82% when we simulate the model with the parameters of the baseline estimation). Furthermore, as this probability varies with age, we also need the age distribution of re-entrants in the steady state. Hence, we simulate survival probabilities of re-entrants by running each simulation up to $T = 31$ and tracking survival status for five periods after each instance of re-entry.

The estimated parameters minimize the following objective function:

$$
\hat{H}(\sigma, \alpha, \lambda) = \left[ S - \hat{P}(\sigma, \alpha, \lambda) \right]'W \left[ S - \hat{P}(\sigma, \alpha, \lambda) \right]
$$

(3.14)

where $\hat{P}(\sigma, \alpha, \lambda)$ is the simulated vector of survival probabilities, $S$ is the vector of observed survival rates, and $W$ is a block diagonal weighting matrix. The matrix $W$ has two blocks, $\hat{W}_E$ and $\hat{W}_R$, where $\hat{W}_{j=E,R}$ are the inverse matrices of the sample analogs of the variance-covariance matrices $E \left[ (S_j - E(S_j)) (S_j - E(S_j))' \right]$. The standard errors we report for the estimated parameters are preliminary and incorrectly-estimated statistics that need to be taken with caution. First, we have yet not accounted for the various sources of simulation error. Second, we have computed the variance-covariance matrix as if $W$ were the efficient weighting matrix, which is not the case. For this reason, we report them only as indicative but refrain at this point from making comments on their values. We are currently working on these issues.

3.4.4 Descriptive statistics

Table 3.3 provides descriptive statistics about exporters and incursions in our dataset (left panel) and about the macroeconomic environment in Peru (right panel) during the sample period. The first column details the number of exporters each year. The second column details the number of instances, which are any active firm-market-year combination. The third and fourth columns display the number of incursions and re-entries, respectively. Note that an exporting firm might not have made an incursion during the sample period. In total, during the sample period (1997-2004) we identify 34,830 incursions by 13,664 unique firms and 9205 re-incursions by 3090 unique firms. The first four columns in the left panel display evidence of growing exporting activity in Peru during the sample period. The last column shows the survival rate for each cohort of incursions in a two-year horizon. The survival rate hovers around an average of 26.6%.

The right panel of the table displays summary indicators of the macroeconomic performance of Peru. The information is provided for an expanded period that includes both the sample years used to identify incursions (1997-2004) and the years used to compute survival (2005-2009). The first column of this panel shows a strong positive trend for aggregate exports in Peru. Similarly, the second column shows a strong positive trend in the evolution of GDP, particularly in the later years of the sample. The last column displays the evolution of the real exchange rate, which exhibits an accumulated depreciation of 29% during the period 1996-2002 followed by an accumulated appreciation of 16% during the period 2002-2009. While our model does not account for changes over time in potential export profitability due to changes in the real exchange rate, by focusing
Table 3.3: Descriptive Statistics

<table>
<thead>
<tr>
<th>Firms and Entries</th>
<th>Survival rate of incursions (%)</th>
<th>Macro Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of:</td>
<td>Total Exports (US$ mill.)</td>
</tr>
<tr>
<td></td>
<td>Firms</td>
<td>Instances</td>
</tr>
<tr>
<td>1997</td>
<td>3,775</td>
<td>9,859</td>
</tr>
<tr>
<td>1998</td>
<td>3,563</td>
<td>9,116</td>
</tr>
<tr>
<td>1999</td>
<td>3,895</td>
<td>10,425</td>
</tr>
<tr>
<td>2000</td>
<td>4,016</td>
<td>11,347</td>
</tr>
<tr>
<td>2001</td>
<td>4,347</td>
<td>11,536</td>
</tr>
<tr>
<td>2002</td>
<td>4,685</td>
<td>12,042</td>
</tr>
<tr>
<td>2003</td>
<td>5,094</td>
<td>13,171</td>
</tr>
<tr>
<td>2004</td>
<td>5,467</td>
<td>14,318</td>
</tr>
<tr>
<td>1997-2009 Total</td>
<td>34,842</td>
<td>91,814</td>
</tr>
</tbody>
</table>

Note: Left panel based on Peruvian customs dataset (World Bank). First two columns of right panel based on INEI. Real exchange rate multiplies nominal exchange rate by US PPI (BLS) and divides it by Peruvian CPI (INEI).
on averages over a time period that includes both appreciation and depreciation of the domestic currency we hope to capture patterns in the data that approximate those that would arise in a fully stable macroeconomic environment.

3.4.5 Results

The top part of Table 3.4 displays the estimation results. The estimate of \( \sigma \) is 0.279. Given \( \mu = -0.10 \), this estimate implies that \( \tilde{\mu} = -0.359 \). This value is not very different from the method-of-moments estimates of \(-0.0279\) and \(-0.270\) obtained by Luttmer (2007) and Arkolakis (2016), respectively, for this ratio. The similarity might seem striking given that those two papers estimate \( \tilde{\mu} \) based on domestic survival rates. However, our model generates lower exporter survival rates primarily due to the uncertainty and experimentation mechanism rather than imposing a more negative trend in the profitability process.\(^2\) The estimate for the parameter of the Poisson process is \( \lambda = 10 \). This is a very high value. It implies that a firm that continuously exports has a 56.5\% probability of receiving the learning shock in less than a month. Finally, \( \tilde{\alpha} = 3.736 \), which implies a standard deviation of 0.54 for the multiplicative shock \( \psi \).

The second part of Table 3.4 compares the data with the model predictions. A visual representation of the same information is provided in Figure 3-2. The model does a good job predicting the survival rates of entrants. In particular, it predicts a survival profile that is both low and flat. The average absolute discrepancy between data and predictions is three percentage points, with the largest discrepancy in the second year (27\% in the model versus 21\% in the data). The model predictions are less accurate in the case of re-entrant survival rates. In this case, the average

\(^{26}\)This result also suggest that our model could potentially provide a unifying framework for understanding both domestic and export survival.
Table 3.4: SMM Estimation results

<table>
<thead>
<tr>
<th>Fixed parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>-0.1</td>
</tr>
<tr>
<td>$r$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimated parameters</th>
<th>Coefficient</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>0.279</td>
<td>0.0005</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>9.993</td>
<td>0.1830</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>3.736</td>
<td>0.0352</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Survival probabilities</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Entrants</td>
<td></td>
</tr>
<tr>
<td>Model</td>
<td>Data</td>
</tr>
<tr>
<td>Year 1</td>
<td>0.36</td>
</tr>
<tr>
<td>Year 2</td>
<td>0.21</td>
</tr>
<tr>
<td>Year 3</td>
<td>0.19</td>
</tr>
<tr>
<td>Year 4</td>
<td>0.17</td>
</tr>
<tr>
<td>Year 5</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Panel B: Re-entrants

| Year 1 | 0.61 | 0.46 |
| Year 2 | 0.44 | 0.39 |
| Year 3 | 0.37 | 0.35 |
| Year 4 | 0.32 | 0.33 |
| Year 5 | 0.29 | 0.31 |
### Table 3.5: SMM estimation results (Benchmark model)

<table>
<thead>
<tr>
<th>Estimated parameter</th>
<th>Coefficient</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu/\sigma$</td>
<td>-0.67</td>
<td>0.002</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Survival probabilities: Entrants</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 1</td>
<td>0.594</td>
<td>0.358</td>
</tr>
<tr>
<td>Year 2</td>
<td>0.354</td>
<td>0.266</td>
</tr>
<tr>
<td>Year 3</td>
<td>0.242</td>
<td>0.223</td>
</tr>
<tr>
<td>Year 4</td>
<td>0.176</td>
<td>0.196</td>
</tr>
<tr>
<td>Year 5</td>
<td>0.130</td>
<td>0.177</td>
</tr>
</tbody>
</table>

discrepancy between data and model predictions is five percentage point with a particularly large underprediction in the first year (46% in the model versus 61% in the data). Nevertheless, the model is still able to deliver the qualitative fact that re-entrants survive more than entrants. It is interesting to notice that the difficulty of the model to explain the low survival rate of re-entrants also explains why there is a relatively large discrepancy for entrants in period 2. The following trade-off arises. Setting a higher $\lambda$, the model can achieve a better fit for the survival profile of entrants by flattening its slope between periods 1 and 2. However, a higher $\lambda$ also implies a larger proportion of experienced re-entrants and hence even higher predictions for their survival in the earliest periods. A potential hypothesis that might explain the quantitative discrepancy between predictions and data in the case of re-entrants is that the resolution of uncertainty takes places in “steps” rather than in one event. This alternative structure for the resolution of uncertainty might generate survival predictions for re-entrants that are closer to those for entrants. In the sake of parsimony, we leave such extension of our model for future research.

#### 3.4.6 Alternate models without uncertainty and experimentation

In this section, we discuss why other models that do not include uncertainty and experimentation are unable to explain facts 1 and 2. First, we estimate the benchmark model and extensions that include sunk costs and an exogenous death rate. Then, we estimate another version of the benchmark model where we substitute a mean-reverting Geometric Ornstein-Uhlenbeck (GOU) process for the GBM profitability process we have been assuming so far. Finally, we nest the GBM and GOU processes by expanding the latter allowing long-run profitability to (potentially) drift downwards over time.

**Brownian motion** The benchmark model is the starting point in the class of models that assumes a GBM profitbility process but does not assume uncertainty and experimentation. It is easy to see why this model is not a useful alternative. Since entrants and re-entrants enter and exit at $\bar{\theta}^* = 1$, in both cases survival probabilities are identical. Therefore, this model is unable to explain
fact 2. In fact, the benchmark model is also unable to explain fact 1. The green-colored dashed line in Figure 3-1a (discussed in Section 3.2) corresponds to the best prediction of the benchmark model. This prediction is obtained by estimating the model with the SMM using only the survival profile of entrants \( \{S_t\}_{t=1,\ldots,5} \). In this case, the only parameter to estimate is the ratio \( \tilde{\mu} = \frac{\hat{\mu}}{\gamma} \). Table 3.5 shows that the estimate of this parameter is \( \hat{\mu} = -0.67 \), which is much more negative than the estimate for this ratio that we obtain in the full model. The table also presents the predicted survival rates using this estimate. These are the predictions depicted in Figure 3-1a. We can see that the model overpredicts survival rates at short horizons while it underpredicts them at longer horizons. In unreported results, we have also included a constant exogenous death rate in the benchmark model and included it as an additional parameter to estimate. The SMM estimation delivers a value of 0 for this parameter. Thus, this exogenous proportional death rate cannot help explain the disproportionate amount of exit during the first year.

Next, we extend the benchmark model to include sunk costs \( (S) \). Sunk costs have been the focus of most theoretical and empirical work on exporter dynamics (Baldwin and Krugman (1989), Dixit (1989), Roberts and Tybout (1997), Alessandria and Choi (2007), Das, Roberts and Tybout (2007), Impullitti, Irarrazabal and Opromolla (2013), Morales, Sheu and Zahler (2017)). More specifically, we assume that when firms switch from non-exporter to exporter in a given market for the first time, they need to pay a sunk cost \( S \). Given that incurring the sunk cost is an irreversible decision, firms only enter with profitability above the exit threshold, which raises the probability of survival at short horizons relative to longer ones. This is exactly the opposite of what is needed to fit the observed survival profile. Hence, sunk costs do not help explain fact 1.

Finally, although we have assumed that sunk costs only need to be paid the first time a firm exports, a straightforward extension is to assume that a firm needs to pay a fraction \( \phi \in [0, 1] \) of the original sunk cost in subsequent export experiences. This would generate an inaction region \( [\theta_{FT}, \tilde{\theta}_{FT}] \) such that firms start exporting when \( \theta \geq \tilde{\theta}_{FT} \) and stop exporting when \( \theta \leq \theta_{FT} \) during their first export experience while it would similarly generate an inaction region for re-entrants \( [\theta_{RE}, \tilde{\theta}_{RE}] \) with \( \theta_{RE} = \tilde{\theta}_{FT} \) and \( \theta_{RE} \leq \tilde{\theta}_{FT} \) (with equality iff \( \phi = 1 \)). If \( \phi = 1 \), then (first-time) entrants and re-entrants have equal survival probabilities. In the more general case of \( \phi < 1 \), re-entrants survive less than entrants as their inaction region is smaller. This prediction goes against explaining fact 2.\(^{27}\) Hence, this extension of the benchmark model is also qualitatively unable to explain this fact.\(^{28}\)

**More general diffusions** Most empirical work on export dynamics has modelled the logarithm of operating profits as a mean-reverting process. To accommodate this possibility, we change the specification of the profitability process to the continuous-time analog of an AR1 in logarithms, a

---

\(^{27}\)In fact, if some entrants were allowed to be born above the entry threshold then, since re-entrants necessarily enter the market at the threshold, the counterfactual prediction would arise even with \( \phi = 1 \).

\(^{28}\)Note that fact 2 may explain why Das, Roberts and Tybout (2007) find that sunk-costs fully depreciate \( (\phi = 1) \). By having entrants and re-entrants pay similar sunk costs, the estimated model might want to minimize the failure of the sunk-cost model to explain fact 2.
Geometric Ornstein-Uhlenbeck process (GOU):

$$d\theta_t = \eta \left( \log \left( \bar{\theta} \right) - \log \left( \theta_t \right) \right) \theta_t dt + \sigma \theta_t dW_t.$$ 

As is usual in the exporter dynamics literature, we assume that the parameters that govern the law of motion of profitability (i.e. $\eta$, $\log \left( \bar{\theta} \right)$ and $\sigma$) are common across firms after controlling for observable characteristics. We also allow for an exogenous death rate $\delta > 0$, which in the context of a stationary process becomes necessary to generate the downward sloping profile of survival probabilities, particularly at long horizons.

Although this specification of the profitability process might improve the ability of the model to fit fact 1, since all entries and exits take place at the same threshold level $\theta^*_{\text{GOU}}$, the model still predicts the same survival probability for entrants and re-entrants. Thus, it is also unable to explain fact 2. This result is in fact more general. Assume that $\theta$ follows any general diffusion,

$$d\theta_t = \mu \left( \theta_t \right) dt + \sigma \left( \theta_t \right) dZ_t, \quad (3.15)$$

that satisfies that if $\theta' > \theta''$, then $F(\theta_{t+s}|\theta') \leq F(\theta_{t+s}|\theta'')$ (a first-order-stochastic-dominance condition) and regularity conditions such that there is a unique solution to $(3.15)$ and the value functions are bounded. Then, the optimal policy will still be characterized by a unique entry and exit threshold both for entrants and for re-entrants and hence produce the identical survival probabilities. As a result, the inability to explain fact 2 generalizes to this broader class of models. Furthermore, introducing entry sunk costs $S \geq 0$ and re-entry sunk costs $\phi S \geq 0$, as analyzed in the case of a GBM, just worsens the problem by inducing lower survival rates for re-entrants.

Estimated and calibrated models of exporter dynamics (Alessandria and Choi (2007), Das, Roberts and Tybout (2007), Ruhl and Willis (2017)) also have two additional features that are worth considering. A first feature is that while we assume that firms are born below the threshold $\theta^*$, these models assume that firms are born with a profitability taken from the stationary distribution (as implied for example by an AR1). As discussed earlier, having entrants be born above the exit threshold only worsens the inability of this model to explain fact 2. A second feature of those models is allowing for idiosyncratic random processes – i.i.d. across firms and over time – for fixed and sunk costs. However, given that firms in these models are born with the stationary distribution, the composition of the pool of first-time entrants is exactly the same as the pool of re-entrants (when $\phi = 1$; if $\phi < 1$ as usual the pool of re-entrants has lower sunk-costs), so entrants and re-entrants have the same probability of survival.

Finally, following Arkolakis (2016), we extend the GOU model to include a deterministic trend in long run profitability. More specifically, $\theta$ now follows

$$d\log \left( \theta_{a,t} \right) = \eta \left( \log \left( \bar{\theta} \right) + \mu a - \log \left( \theta_{a,t} \right) \right) da + \mu da + \sigma dW_a$$

29The first-order stochastic dominance condition is a natural and common assumption in the literature that prevents strange cases in which the firm is very profitable today but does not pay the sunk cost because it knows this high profitability will be the cause of low profitability tomorrow.
where \( a \) is the age of the firm and \( t \) is calendar time.\(^{30}\) As in Luttmer (2007) and Arkolakis (2016), the negative trend captures the fact that older technologies become obsolete, implying that we need to keep track of the age of the firm. This process nests the GBM and GOU processes we have considered so far. If \( \eta = 0 \), then we obtain a GBM. If \( \eta > 0 \) but \( \mu = 0 \), then we obtain a GOU. Note also that if \( \sigma = 0 \), so that the process is deterministic, firm profits drift downwards at log-rate \( \mu \). This flexible specification is able to fit the survival profile of entrants (fact 1). In addition, the deterministic trend generates heterogeneity in survival probabilities across entrants since older firms have lower long run profitability and, thus, survive less as they mean-revert more rapidly. Thus, it also generates a composition effects that may explain fact 2. However, setting parameters that allow the model to match fact 1, the composition effect goes in the opposite direction of explaining fact 2. Since re-entrants tend to be older firms, they re-enter further away from their long-term profitability and hence tend to survive less than entrants.

### 3.5 Mechanisms

We have presented two facts about exporter survival and developed a model with uncertainty and experimentation in export markets that naturally explains them. We have also shown that the most representative exporter dynamics models that do not include these elements are unable to explain the two facts simultaneously. In this section, we provide further evidence of the relevance of uncertainty and experimentation in the dynamics of firm exports by associating variation in the parameter \( \alpha \) to observed characteristics of products and markets.

When \( \alpha \) is lower, the distribution of \( \psi \) has fatter tails. Furthermore, both the mean and the variance of the distribution of \( \psi \) increase. While the variance may be associated with uncertainty, the fact that the mean increases is associated with learning-by-exporting. Proposition 3.4 helps us separate both effects. Suppose we lowered \( \alpha \) while varying \( \psi_m \) in order to keep the mean of the distribution constant, i.e. \( \psi_m = \frac{\alpha-1}{\alpha} \). It can be checked that the variance of the shock decreases with \( \alpha \) even after the compensating \( \psi_m \) effect (provided the variance exists, i.e. \( \alpha > 2 \)). Next, note that by Proposition 3.4, decreasing \( \psi_m \) lowers the probability of survival. In other words, if we lowered \( \alpha \) alone and the probability of survival decreased, then correcting for the effect of the mean with \( \psi_m \) can only exacerbate the effect on the probability of survival. Henceforth, we ignore the effect on the mean and talk about decreasing \( \alpha \) as intensifying uncertainty but it should be clear that the effect of this change on survival probabilities would be stronger if we corrected for the mean.

Since profits are increasing and convex in \( \psi \), it is unsurprising that the threshold \( \bar{\theta}^* \) goes up with \( \alpha \). Consider a decrease in \( \alpha \) such that the variance increases. Conditional on a realization of \( \psi \) the firm will survive less. On the other hand, a larger variance of \( \psi \) implies that there are more firms that receive a realization of the shock from the upper tail of the distribution, who are less likely to exit later on. Since the estimated GBM process has a negative drift, this effect is relatively

\(^{30}\)We set \( \delta = 0 \) for this exercise.
Table 3.6: Effect of $\alpha$ on survival probabilities

<table>
<thead>
<tr>
<th>Fixed parameters</th>
<th>-0.1</th>
<th>-0.1</th>
<th>-0.1</th>
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</thead>
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<tr>
<td>$\mu$</td>
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<td>0.279</td>
<td>0.279</td>
<td>0.279</td>
<td>0.279</td>
</tr>
<tr>
<td>$\sigma$</td>
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<td>9.4</td>
</tr>
<tr>
<td>$\lambda$</td>
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<td>10.0</td>
<td>10.0</td>
<td>10.0</td>
<td>10.0</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2.144</td>
<td>1.072</td>
<td>0.536</td>
<td>0.268</td>
<td>0.134</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Survival probabilities</th>
<th>$\text{Pareto std. dev.}$</th>
<th>Normalized Pareto std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Entrants</td>
<td>2.144</td>
<td>4</td>
</tr>
<tr>
<td>Year 1</td>
<td>0.262</td>
<td>0.172</td>
</tr>
<tr>
<td>Year 2</td>
<td>0.090</td>
<td>0.095</td>
</tr>
<tr>
<td>Year 3</td>
<td>0.213</td>
<td>0.166</td>
</tr>
<tr>
<td>Year 4</td>
<td>0.536</td>
<td>0.263</td>
</tr>
<tr>
<td>Year 5</td>
<td>0.943</td>
<td>0.290</td>
</tr>
</tbody>
</table>

| Panel B: Re-entrants    | 4                            | 1/4                         |
| Year 1                  | 0.510                       | 0.570                       |
| Year 2                  | 0.353                       | 0.403                       |
| Year 3                  | 0.593                       | 0.386                       |
| Year 4                  | 0.250                       | 0.332                       |
| Year 5                  | 0.220                       | 0.245                       |

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more relevant in the long run, where only firms that have benefited from a very good draw of \( \psi \) survive. Although the net effect is ambiguous, we show with simulations that the “threshold effect” dominates. That is, a lower (higher) \( \alpha \) implies a lower (higher) probability of survival. This result is displayed in Table 3.6. Across columns from left to right, we maintain all parameters in their estimated values but set \( \alpha \) so that the dispersion of the learning shock decreases from four times to a quarter of its estimated value. Accordingly, the predicted survival rates increase uniformly as \( \alpha \) increases.

We do not observe \( \alpha \). However, the magnitude of this parameter can be linked to observable characteristics of products and export destinations. Since \( \alpha \) governs the variance of the shock, its magnitude captures the degree of uncertainty about the component of export market profitability that can be resolved by experimenting.\(^{31}\) As discussed in Section 3.3, possible sources are the need to adapt products to satisfy foreign demand idiosyncrasies and the need to match with distributors that will make efforts to propel sales of the firm’s products. It is reasonable to assume that these sources of uncertainty are more relevant for differentiated products than for homogeneous products. Thus, we expect a lower \( \alpha \), and hence a lower \( P_T \), for the former type of products.

To test this implication, we classify all incursions in our database in either of two categories, differentiated or homogeneous, following Rauch (1999).\(^{32}\) We first map export data classified at the Harmonized System 10-digit level into Rev.2 SITC 4-digit categories using the Conversion Tables from the United Nations Statistics Division. Then, we map the latter categories into one of our two categories.\(^{33}\) Finally, we identify the category with the largest value of exports in the year of

---

\(^{31}\)The remaining component, i.e. uncertainty about the future trajectory of the GBM process, remains unresolved after the shock.

\(^{32}\)We merge homogeneous and referenced-priced categories in Rauch (1999) into only one “homogeneous” category.

\(^{33}\)The mapping from SITC to Rauch leaves 5.74% of the instances unclassified. We reduce this proportion to 2.33% by arbitrarily assigning unclassified SITC 4-digit categories the classification of similar SITC 4-digit categories. Of the remaining unclassified instances, 60% are transactions without reported HS-code.
entry and assign the incursion to that category. There are 20,907 differentiated incursions and 13,120 homogenous incursions in our database. For each of these categories, the survival profile is displayed in Figure 3-3a. Consistent with the hypothesis of a lower $\alpha$ for differentiated products, these products display uniformly lower survival rates. For example, the survival rate is more than six percentage points lower in the first year after entry and more than four percentage points lower in the fifth year. Similar results are obtained in a regression framework, where we can perform inference and control for other covariates. We regress the survival status of each incursion-horizon combination on dummies for horizon and a dichotomous variable for differentiated products. The results, displayed in column 1 of Table C.2, show that the survival rate of differentiated products is significantly lower than for homogeneous products. We also interact the differentiated dummy with horizon dummies. Column 2 shows that the survival rate of incursions in differentiated products is lower at all horizons. Columns 3 and 4 show that these results are not an artifact of composition effects. The results are very similar when we replicate the regressions in the first two columns by adding a full set of product (2-digit HS), destination, and year fixed effects.

Since a fraction of re-entrants in differentiated products has already resolved their uncertainty, we should expect a smaller gap between re-entrant survival rates in differentiated versus homogeneous products than between entrant survival rates in the two types of products. Figure 3-3b displays re-entrant survival profiles in these two cases (for reference we also include survival profiles for entrants). As predicted, the gap is smaller for re-entrants than for entrants. The prediction is formally tested by performing a diff-in-diff estimation. We regress the exporting status of each incursion-horizon and re-incursion-horizon combination on entrant horizon dummies, re-entrant horizon dummies, a dummy for differentiated products, and an interaction dummy for re-entrants in differentiated products. We cluster standard errors by firm-destination allowing for correlation between incursions and re-incursions at any horizon. Consistent with the prediction of a smaller gap for re-entrants in differentiated versus homogeneous products, we find a positive and significant (at the 10% level) estimated coefficient on the interaction term (column 5). A similar result is obtained by interacting the differentiated product dummy with horizon dummies (column 6) and by adding a full set of product, destination, and year fixed effects (columns 7 and 8). We note, however, that given the high proportion of experienced re-entrants in the baseline estimation, the predicted magnitude of the gap is substantially larger than is observed in the data. This result is in line with the quantitative difficulty of the model to match the survival rates of re-entrants discussed in Section 3.4.5. Re-entrants exhibit survival patterns that are considerably more similar to those for entrants than predicted by the model.

The uncertainty surrounding export market profitability could also be hypothesized to vary according to distance to the destination. In the first place, neighboring countries are more likely to have similar income levels and thus share similar consumption patterns. In the second place, even controlling for income levels, demand idiosyncrasies are more likely to coincide the closer are the exporter and the importer. In the third place, less distant countries are more likely to have a more similar business culture that facilitates communication with distributors and anticipation.
of their actions. As a result, we could expect a higher degree of uncertainty about export market profitability in more distant destinations. Setting a smaller $\alpha$ for those destinations, the model predicts lower survival probabilities in those cases. To assess this prediction, we divide export destinations into three groups according to their distance from Peru. Short distance destinations are those with a distance smaller than 3440 km. Medium distance destinations are those with a distance between 3440 km. and 10100 km. Long distance destinations are those with a distance above 10100 km. The cut-offs are chosen so that each distance group has an equal number of incursions. Figure 3-4a displays the survival profile for each distance group in the case of differentiated goods. We can see that the profile is uniformly lower the farther away is the destination. For example, one year after entry the survival rate for the long distance group is six percentage points lower than for the short distance group while five years after entry the survival rate for the former group is five percentage points lower. Figure 3-4b displays analogous information in the case of homogeneous products, where it is unclear whether distance should matter. In this case, we do not see lower survival rates in more distant destinations. If anything, the opposite seems to be the case.

These results also arise in a regression framework where in addition to controlling for other covariates we can also control for distance as a continuous variable. The results are displayed in Table C.3. In column 1 we only control for horizon fixed effects, a fixed effect for differentiated products, and the interaction of distance with fixed effects for differentiated and homogenous products, respectively. In accordance with the graphical results, we see that distance decreases survival rates in the case of differentiated products but increases them in the case of homogenous products. Column 2 presents the results of analogous regression interacting the variables above with horizon dummies. While the coefficient on the interactions for differentiated products are uniformly negative and significant at every horizon, the magnitude and statistical significance of the positive coefficients for analogous interactions in the case of homogenous products are more
disparate. Similar results are obtained when we reproduce these regressions including the usual set of product, destination, and year fixed effects.

In the case of differentiated products, where distance matters, we should also expect a smaller gap for re-entrants across distance groups. Figure 3-5 displays the relevant survival profiles (survival profiles for entrants are also included for reference). In this case, the prediction is not borne by the data. The gap in survival rates across distance groups does not dwindle for re-entrants. Regression analysis confirms this result. Column 1 of Table C.4 reports the result of regressing survival status of entrants and re-entrants on horizon dummies, a dummy for re-entrants, a control for distance, and this control interacted with the re-entrant dummy. The interest is in the coefficient on the latter interaction. The estimated sign is opposite to the prediction albeit not significantly different from zero. Similar results are obtained when all three terms are interacted with horizon dummies (column 2) and when we include the full set of fixed effects (columns 3 and 4). Once again, re-entrants display survival patterns that are more similar to those of entrants than predicted by the model.

In sum, the results of this section show that reasonable assumptions about how $\alpha$ varies across products and destinations yield predictions that are consistent with the data in most cases. We regard those results as supportive of the notion that uncertainty and experimentation are crucial features in the dynamics of firm exports.

### 3.6 Concluding remarks

This paper develops a model of exporter dynamics with uncertainty and experimentation. The model is parsimonious and has tractable features that allow us to obtain analytical results on survival probabilities. Those results explain two central facts about export survival in foreign markets that existing models that neglect uncertainty and experimentation are unable to account for. The first fact is that the survival profile of export entrants is low and flat. The second is that re-entrants to foreign markets display higher survival rates than first-time entrants. Based on the analytical
results we derive, we can estimate the parameters of the model and derive quantitative predictions on these two facts. The importance of uncertainty and experimentation in exporter dynamics is further supported by evidence that exploits hypothesized variation in the degree of uncertainty about foreign market profitability across products and distance to the destination.

The paper also makes a methodological contribution to the literature on firm and exporter dynamics by proposing a correction for the mismatch between a model set in continuous time and data recorded in discrete-time periods. This correction has a substantial impact on the model predictions. Conceptually, the correction is more general than our specific application here since the source of mismatch arises even in discrete time models. In particular, it should be applied whenever there is a discrepancy between the frequency at which firms make decisions and the frequency at which the data are recorded. In addition to correcting survival predictions, analogous adjustment could be made to correct for other key variables in dynamic models such as the amount of sales and their growth. We are currently working on developing such corrections.

While focusing on export survival, we hope to contribute to a broader literature that attempts to characterize the main features in the dynamics of firm exports. A next natural step would be to study the implications of our theoretical framework for other moments analyzed in the exporter dynamics literature. We are currently working in this direction. Preliminary results are promising and will hopefully be included in future versions of this paper.
Appendix A

Appendix to Chapter 1

A.1 Appendix: Proofs and derivations

A.1.1 Proofs for section 1.3

Lemma 1.2

First, I derive a log-linear approximation to the output gap in this economy. Second, I derive an equation linking the exchange rate to the output gap, tradable consumption and shocks. Third, I compute the static Phillips curve linking the output gap and intermediate-input prices. Finally, I derive a second-order approximation to welfare around the riskless steady state \((\bar{B}, \epsilon = 0)\). I use undercapitalized letters for log deviations from the steady state for all variables except \(B\). I use \(\bar{B}\) to denote the steady-state value of the home-currency position and \(B_\epsilon\) to denote \(\frac{\partial B}{\partial \epsilon}\). For simplicity, I only consider the following shocks: a tradable endowment shock \(\{Y_{TS}\}\), a nontradable productivity shock \(\{Z_s\}\), a general taste shock \(\{t_s\}\) that premultiplies flow utility, and a shock to foreigners' endowment \(\{Y_{Ts}\}\).

Output gap I compute the output gap between the actual output and the output that would arise if prices were flexible, conditional on having the same level of tradable consumption. First, I need to compute the level of employment at the flexible price allocation \(l_{flex}^{1}\), which solves \(u_N F_Y + u_L = 0\) (set \(\phi = 0\) in equation (1.9)). Let \(Z\) denote the shock to the final nontradable production function, \(i.e. F(Y_s, Z_s)\). To a first-order approximation,

\[
L_{ss1}^{flex} = -\frac{F_Y u_{NN} F_Z + u_N F_Y Z + u_N L F_Z}{F_Y u_{NN} + 2 F_Y u_{NL} + u_N F_{YY} + u_L} Z_{ss} z_s
- \frac{F_Y u_{TN} + u_L}{F_Y u_{NN} + 2 F_Y u_{NL} + u_N F_{YY} + u_L} C_{Tss} C_{Ts}.
\]

Define the output gap as \(x_s = F^{-1} F_Y (y_1 s - y_{1s}^{flex})\) and let \(\Delta \equiv \int_0^1 (P_{ts}^{i(i)})^{-\eta} di - 1\) denote intermediate input price dispersion deviations. Since production at the steady state is efficient, \(\Delta\) is 0 to first-order.
Thus,

$$x_s = \frac{F^{-1} F_Y}{F_Y F_\nu F_{\nu} + 2 F_Y F_{\nu} + u_N F_{\nu} + u_{LL}} \{ (F^2 F_{\nu} + 2 F_Y F_{\nu} + u_N F_{\nu} + u_{LL}) L_{ss} l_s \} \text{ (A.1)}$$

$$+ (F_Y F_{\nu} F_Z + u_N F_{\nu} F_Y + u_{LL} F_{\nu}) Z_{ss} \phi_s + (F_Y u_T + u_{TL}) C_{Tss} C_{Ts} \}$$

**Exchange rate sensitivity to the output gap** Log-linearizing the equation

$$u_N(C_{Tss}, F(Y_s, Z_s), L_s) F_Y(Y_s, Z_s)/u_T(C_{Tss}, F(Y_s, Z_s), L_s) = P_{ls}/E_{ls},$$

which comes from the first-order condition of nontradable good producers and consumer optimization, yields

$$\phi_c C_{Tss} C_{Ts} + \phi_z Z_{ss} Z_{ss} + \phi_l L_{ss} l_s = p_{ls} - e_s$$

where

$$\phi_c = u^{-1} u_{TT} - u^{-1} u_{TT}$$
$$\phi_z = u^{-1} u_{NN} F_Z + F^{-1} F_{YZ} - u^{-1} u_{TT} F_Z$$
$$\phi_l = -u^{-1} u_T F_Y - u^{-1} u_{TL} + u^{-1} u_{NN} F_Y + u^{-1} u_{NL} + F^{-1} F_{YY}$$

Or, in terms of the output gap,

$$e_s = p^i_s + k_{ec} C_{Tss} C_{Ts} + k_{ez} Z_{ss} Z_{ss} + k_{ex} x_s \text{ (A.2)}$$

where

$$k_{ex} = -F F^{-1} F_Y$$
$$k_{ec} = -\phi_c + \phi_l F_Y u_{NN} + 2 F_Y u_{NN} + u_N F_{YY} + u_{LL}$$
$$k_{ez} = -\phi_z + \phi_l F_Y u_{NN} F_Z + u_N F_{YY} + u_{LL} F_Z$$

In general, $k_{ex} > 0, k_{ec} < 0, k_{ez} > 0$, although it depends on the application (and, more generally, on the source of nominal rigidities).

**Static Phillips curve** Let $P^{flex}_{ls}$ denote the price of firms who optimize. To first-order,

$$P^{flex}_{ls} = e_s - u^{-1} u_{TT} C_{Tss} C_{Ts} - u^{-1} u_{TT} (F_Y L_{ss} l_s + F_Z Z_{ss} z_s) - u^{-1} u_{TL} L_{ss} l_s$$

$$+ u^{-1} u_{TL} C_{Tss} C_{Ts} + u^{-1} u_{NL} (F_Y L_{ss} l_s + F_Z Z_{ss} z_s) + u^{-1} u_{LL} L_{ss} l_s \text{ (A.3)}$$
Log-linearizing (1.9),
\[ p_{Is} = \frac{1 - \phi}{\phi} (p_{Is}^{\text{flex}} - p_{Is}) \]  
(A.4)

Using (A.3) and (A.2), I obtain
\[ p_{It} = \kappa x_t \]  
(A.5)

where
\[ \kappa = \frac{1 - \phi}{\phi} \lambda_x F^{-1} u_N^{-1} > 0 \]
\[ \lambda_x = -\frac{F^2 Y u_{NN} + F_Y u_{NL} + u_N F_Y + u_{NL} F_Y + u_{LL}}{F^2 - F_Y^2} > 0 \]

**Welfare loss around (\( \bar{B}, 0 \))** Following steps analogous to those in Gali and Monacelli (2005), I find that to second-order price-dispersion deviations (\( \Delta \)) are given by
\[ \Delta_s = \frac{\eta}{2} [\phi(-p_{Is})^2 + (1 - \phi)(p_{Is}^{\text{flex}} - p_{Is})^2] + O(\varepsilon^3). \]

Using (A.4), I can rewrite this as
\[ \Delta_s = \frac{\eta}{2} \frac{\phi}{1 - \phi} p_{Is}^2 + O(\varepsilon^3). \]  
(A.6)

To simplify, I assume taste shocks take the form \( \exp(t_s)u(C_{Ts}, C_{Ns}, L_s) \). Using (A.6), I find that a second-order approximation to the utility flow yields
\[ u_s = u_T C_{Ts} c_{Ts} + \frac{1}{2} (u_{TT} C_{Ts}^2 + u_T C_{Ts}) c_{Ts}^2 + u_T N F_Z Z_{ss} z_s c_{Ts} + (u_{TN} F_Y + u_{TL}) C_{Ts} L_{ss} c_{Ts} l_s \\
+ t_s u_T C_{Ts} c_{Ts} + (u_N F_Y Z + u_N N F_Y F_Z + u_{NL} F_Z) Z_{ss} z_s l_s \\
+ \frac{1}{2} (u_N F_Y Y + u_N N F_Y^2 + 2 u_{NL} F_Y + u_{LL}) L_{ss}^2 z_s^2 + u_{LL} L_{ss} \frac{\eta}{2} \frac{\phi}{1 - \phi} p_{Is}^2 + t.i.p. + O(\varepsilon^3) \]  
(A.7)

Define:
\[ \lambda_x \equiv -u_{LL} L_{ss} \eta \frac{\phi}{1 - \phi} > 0 \]
\[ V_{11} \equiv u_{TT} - F^2 Y^{-2} \lambda_x^{-1} (F_Y u_{TN} + u_{TL})^2 < 0 \]
\[ V_{1z} \equiv u_{TN} F_Z - F^2 Y^{-2} \lambda_x^{-1} (F_Y u_{NN} F_Z + u_N F_Y Z + u_{NL} F_Z) (F_Y u_{TN} + u_{TL}). \]

Then, using (A.1) I can rewrite welfare as
\[ u_s = u_T C_{Ts} c_{Ts} + t_s u_T C_{Ts} c_{Ts} + \frac{1}{2} (V_{11} C_{Ts}^2 + u_T C_{Ts}) c_{Ts}^2 + V_{1z} Z_{ss} z_s c_{Ts} - \frac{1}{2} \lambda_x x_s^2 \]
\[ - \frac{1}{2} \lambda_x p_{Is}^2 + t.i.p. + O(\varepsilon^3) \]
A second-order approximation to the budget constraint yields

\[ C_{Tss}c_{Tss} + \frac{1}{2} C_{Tss}c_{Tss}^2 = Y_{Tss}y_s + \frac{1}{2} Y_{Tss}y_s^2 + \bar{B}(r - e_s) + \frac{1}{2} \bar{B}(r - e_s)^2 + (r - e_s)B_e + O(e^3) \]

A first-order approximation of the foreign Euler equation yields

\[ \mathbb{E}(r - e_s) + O(e^2) = 0 \]

while a second-order approximation yields

\[ \mathbb{E}(r - e_s) + \frac{1}{2} \mathbb{E}(r - e_s)^2 - \mathbb{E}\gamma_s^*(r - e_s)(y_s^* - \frac{1}{m} \bar{B}(r - e_s)) + O(e^3) = 0, \]

where I assumed \( Y_{Tss}^* = 1 \). Noting that I only need the know the first order behavior of \( r - e_s \) to compute \( (r - e_s)B_e \) and that \( B_e \) is determined ex ante, I get \( \mathbb{E}(r - e_s)B_e = 0 \) so this term vanishes from the approximated welfare function. Combining the second-order approximation to the budget constraint with the second-order approximation to the foreign Euler equation (A.26) I obtain

\[ \mathbb{E}V = \mathbb{E}[u_T \bar{B}(r - e_s)\gamma_s^*y_s^* - u_T(\gamma_s^*/m)\bar{B}^2(r - e_s)^2 + u_Tc_{Tss}c_{Tss}] \\
+ \frac{1}{2} V_{11}C_{Tss}c_{Tss}^2 + V_{11}Z_{ss}C_{Tss}z_{ss}c_{Tss} - \frac{1}{2} \lambda \bar{x}^2 - \frac{1}{2} \lambda \bar{p}^2 + t.i.p. + O(e^3) \]

Note that in the absence of shocks, \( e_s = 0 \). Since shocks are mean zero, I obtain using the foreign Euler that \( r = 0 \). Thus, using a first-order approximation to the budget constraint and to the foreigners’ Euler equation, the previous expression simplifies to

\[ \mathbb{E}V = \mathbb{E}[u_T \bar{B}e_s\gamma_s^*y_s^* - u_T(\gamma_s^*/m)\bar{B}^2e_s^2 + \frac{1}{2} V_{11}(Y_{Tss}y_s - \bar{B}e_s)^2] \\
+ (V_{11}Z_{ss}z_s + u_Te_s)(Y_{Tss}y_s - \bar{B}e_s) - \frac{1}{2} \lambda \bar{x}^2 - \frac{1}{2} \lambda \bar{p}^2 + t.i.p. + O(e^3) \]

Define the “desired transfer” \( T_s \) as

\[ T_s = \frac{1}{1 - 2m^{-1}u_TV_{11}^{-1}\gamma_s^*} \{-Y_{Tss}y_s - V_{11}^{-1}V_{11}Z_{ss}z_s - V_{11}^{-1}u_Te_s + V_{11}^{-1}u_T\gamma_s^*y_s^*\}. \]

Then, I can rewrite (A.8) as

\[ \mathbb{E}V = \mathbb{E}[\frac{1}{2} (1 - 2m^{-1}V_{11}^{-1}\gamma_s^*)V_{11}(\bar{B}e_s + T_s)^2 - \frac{1}{2} \lambda \bar{x}^2 - \frac{1}{2} \lambda \bar{p}^2 + t.i.p. + O(e^3) \]

Next, I define the “demand-management” exchange rate \( e_s^{dm}(B) \), which closes the output gap with no price dispersion efficiency-loss when the portfolio is \( \bar{B} \),

\[ e_s^{dm} = \frac{1}{1 + kec\bar{B}}(kecY_{Tss}y_s + kecZ_{ss}z_s). \]
This implies

\[ e_s - e''_s(\tilde{B}) = \frac{1}{1 + k_{ee}\tilde{B}^2}(k_{ex}x_s + p_l) \]

Using this relationship, the static Phillips curve (A.5) and the exchange rate function (A.2), I can rewrite (A.9) as

\[ EV = \mathbb{E}\left[ \frac{1}{2}(1 - 2u_Tm^{-1}V_{11}^{-1}\gamma_{ss}^*)V_{11}(\tilde{B}e_s + T_s)^2 - \frac{1}{2}(1 + k_{ee}\tilde{B}^2)^2(\frac{\lambda_x + \kappa^2\lambda_\pi}{(\kappa + k_{ex})^2})(e_s - e''_s) \right] \]

\[ + t.i.p. + O(\varepsilon^3) \]

Defining

\[ e''_s(\tilde{B}) = -\frac{1}{\tilde{B}}T_s \]

\[ \tilde{\gamma}_{ss} = -u_T^{-1}V_{11} > 0 \]

\[ \chi = \frac{u_T^{-1}}{\tilde{\gamma}_{ss} + 2m^{-1}\gamma_{ss}^*}(\frac{\lambda_x + \kappa^2\lambda_\pi}{(\kappa + k_{ex})^2}) > 0 \]

\[ k_0 = u_T(\tilde{\gamma}_{ss} + 2m^{-1}\gamma_{ss}^*) > 0 \]

I obtain the expression (1.18) in lemma 1.2.

**Proposition 1.1**

I first find a perturbation of the first-order conditions using a bifurcation theorem. Then, I derive our linear quadratic approximation and show it implies the same FOC for \( \tilde{B} \). Equivalence conditional on \( \tilde{B} \) is a well-known result so I do not prove it (Benigno and Woodford (2012)).

I show a more general version of the result here. To define a useful class of problems, I isolate some endogenous variables from the rest: the excess return on an asset \( rr^j_s \in \mathbb{R}^S \), its expected return \( R^j \in \mathbb{R}_+ \) and the portfolio \( \theta \in R^{j+1} \), where \( j \) indexes the asset with 0 being a reference asset. The root of the indeterminacy at the steady state comes from two conditions, which I will assume hold in our problem: First, I assume the portfolio by itself has no direct effect on utility or the constraints: its only effect is indirect through the transfers it creates \( rr^j_s \theta \). Second, there is a no-arbitrage constraint on each asset that implies that in the steady state \( rr^j_s = 0 \) \( \forall j, s \).

Let \( \pi \) denote the objective function, \( F \) denote other constraints on the problem, \( X^j \) denote the determinants of the equilibrium excess returns of asset \( j \) and \( M \) determine the stochastic discount factor of the agent that prices the asset. All functions are assumed to be locally analytic, which is necessary to apply the bifurcation theorem in Judd and Guu (2001). The class of problems I
consider take the following form:

$$\max \mathbb{E} \pi(y_s, rr_s^j, \xi_s) \text{ subject to}$$

$$F(y_s, \xi_s, rr_s^j) = 0$$

$$R^j X^j(y_s, \xi_s, T_s) - X^0(y_s, \xi_s, rr_s^j) - rr_s^j = 0 \ \forall j$$

$$\mathbb{E}_0 rr_s^j M(y_s, \xi_s, rr_s^j) = 0 \ \forall j.$$ 

where $y_s$ are other endogenous variables, and $\xi_s$ are shocks,

$$\xi_s = \epsilon u_s$$

where \{u_s\} is a random variable with compact support. For example, in our problem above, $\pi = \omega V(C_{T_s}, E_s^{-1})$, $y_s = \{C_{T_s}, E_s^{-1}\}$, $\xi_s = \{\iota_s, Z_s, Y_s, Y^s\}, \theta = B$, $X^0 = 1$, $X^1 = E_s^{-1}$, $F = Y_{T_s} - C_{T_s} + Brr_s$, $M = u^*(Y_s^* - Brr_s)$. Naturally, I could also have written the problem including more equilibrium objects and more constraints $F$.

Let $\lambda_s^k$ denote the multiplier on constraint $F_k$, $\nu$ denote the multiplier on the definition of excess returns, and $\varphi \theta$ denote the multiplier on the no-arbitrage conditions. Following Benigno and Woodford (2012), I derive the following quadratic expansion to the objective with respect to $e$ around an arbitrary steady-state portfolio $\theta_0$

$$\mathbb{E}_0 \pi(y_s, \xi_s, rr_s^j) = \frac{1}{2} \mathbb{E}_0 \sum_s \lambda_s^k \{ \tilde{y}_s^j D_{yy} F \cdot \tilde{y}_s + 2 \tilde{\xi}_s^j D_{\nu \xi} F \cdot \tilde{\xi}_s + 2 \theta_0 \tilde{r}_s D_{y \nu} F \cdot \tilde{y}_s \}$$

$$+ \frac{1}{2} \mathbb{E}_0 \{ \tilde{y}_s^j D_{yy} \xi \cdot \tilde{y}_s^j + 2 \tilde{\xi}_s^j D_{\nu \xi} \xi \cdot \tilde{\xi}_s^j + 2 \theta_0 \tilde{r}_s D_{y \nu} \xi \cdot \tilde{r}_s \}$$

$$+ \frac{1}{2} \mathbb{E}_0 \{ \tilde{y}_s^j D_{y \nu} \xi \cdot \tilde{y}_s^j + 2 \tilde{\xi}_s^j D_{\nu \xi} \xi \cdot \tilde{\xi}_s^j + 2 \theta_0 \tilde{r}_s D_{y \nu} \xi \cdot \tilde{r}_s \} + (\text{t.i.p.} + O(e^3))$$

and a linear approximation to the constraints,

$$D_y F \cdot \tilde{y}_s + D_{\xi} F \cdot \tilde{\xi}_s + \theta_0 D_T F \tilde{r}_s = O(e^2) \quad \text{(A.11)}$$

$$\mathbb{E}_0 \tilde{r}_s^j = O(e^2) \quad \text{(A.12)}$$

$$R^j D_y X \cdot \tilde{y} + R^j D_{\xi} X \cdot \tilde{\xi}_s + \theta_0^j R^j D_T X \cdot \tilde{r}_s^j - \tilde{r}_s^j = O(e^2) \quad \text{(A.13)}$$

where $\tilde{x} = x - \bar{x}$ with bars denoting steady-state values. In the context of our particular model, applying this procedure and substituting in the constraints yields (1.18). The next theorem states that solving this problem yields a correct linear-quadratic approximation, in the sense that it yields a linear-approximation to optimal policy in terms of $(y_s, R, rr_s)$ and a bifurcation point $\theta_0$. To prove the result, I show that using the first-order conditions of the nonlinear problem and then using a bifurcation theorem yields the same answer.
Using bifurcation theorem The problem is

$$\max \mathbb{E} \pi(y_s, \xi_s, r_{rs}\theta)$$

subject to:

$$F(y_s, \xi_s, r_{rs}\theta) = 0$$
$$RX(y_s, \xi_s, r_{rs}\theta) - 1 - r_{rs} = 0$$
$$\mathbb{E}_0 r_{rs}g(y_s, \xi_s, r_{rs}\theta) = 0$$

Let $\varphi\theta$ denote the multiplier on the last equation and $\mu_{s}\theta$ denote the multiplier on the second equation. The FOC yield

$$D_y \pi(y_s, \xi_s, r_{rs}\theta) + \lambda'_{s} D_y F(y_s, \xi_s, r_{rs}\theta) + \theta r_{rs} D_y g(y_s, \xi_s, r_{rs}\theta)$$
$$-R \mu_{s} D_y X(y_s, \xi_s, r_{rs}\theta) = 0$$
$$\mathbb{E}_0 \mu_{s} X(y_s, \xi_s, r_{rs}\theta) = 0$$
$$\{D_T \pi(y_s, \xi_s, r_{rs}\theta) + \lambda'_s D_T F(y_s, \xi_s, r_{rs}\theta) + \theta r_{rs} D_T g(y_s, \xi_s, r_{rs}\theta)$$
$$+ R \theta \mu_{s} D_T X(y_s, \xi_s, r_{rs}\theta)\} + \mu_{s} + \varphi g(y_s, \xi_s, r_{rs}\theta) = 0$$
$$F(y_s, \xi_s, r_{rs}\theta) = 0$$
$$r_{rs} - (RX(y_s, \xi_s, r_{rs}\theta) - 1) = 0$$
$$\mathbb{E}_0 r_{rs}g(y_s, \xi_s, r_{rs}\theta) = 0$$

Clearly at the steady state $r_{rs} = 0$ and $\mu = 0$. This implies that at the steady state,

$$\{D_T \pi(y_s, \xi_s, r_{rs}\theta) + \lambda'_s D_T F(y_s, \xi_s, r_{rs}\theta)\} = -\varphi g(y_s, \xi_s, r_{rs}\theta).$$

I can rewrite the last equation as

$$\mathbb{E}_0 [r_{rs}' \{D_T \pi(y_s, \xi_s, r_{rs}\theta) + \lambda'_s D_T F(y_s, \xi_s, r_{rs}\theta)$$
$$+ \varphi r_{rs} D_T g(y_s, \xi_s, r_{rs}\theta) + R \theta \mu_{s} r_{rs} D_T X + \varphi g(y_s, \xi_s, r_{rs}\theta)\}] = 0. \quad (A.14)$$

Furthermore, I can apply the IFT on the first five equations to obtain $y_s(\theta, \epsilon), x(\theta, \epsilon), r_{rs}(\theta, \epsilon),$ $\lambda(\theta, \epsilon), \varphi(\theta, \epsilon)$. It is easy to check the first derivative of these objects with respect to $\theta$ is zero (i.e.
the steady state values of the other variables does not depend on \( \theta \). Let

\[
H(\theta, \epsilon) = E_0[rr_s\{D_T \pi(y_s(\theta, \epsilon), \xi_s(\epsilon), rr_s(\theta, \epsilon) \theta) + \lambda'_s(\theta, \epsilon) D_T F(y_s(\theta, \epsilon), \xi_s(\epsilon), rr_s(\theta, \epsilon) \theta)
+ \varphi(\theta, \epsilon) \theta rr_s(\theta, \epsilon) D_T g(y_s(\theta, \epsilon), \xi_s(\epsilon), rr_s(\theta, \epsilon) \theta)
+ \theta R(\theta, \epsilon) \mu_s(\theta, \epsilon) D_T X + \varphi(\theta, \epsilon) g(y_s(\theta, \epsilon), \xi_s(\epsilon), rr_s(\theta, \epsilon) \theta)\}]
\]

First, I show there is a singularity. Note:

\[
\frac{\partial H}{\partial \theta} = E_0(\frac{\partial rr_s}{\partial \theta})\{D_T \pi + \lambda'_s D_T F + \varphi \theta rr_s D_T g + R \theta \mu_s D_T X + \varphi g\}
+ rr_s\{(D_T y + \lambda'_s D_T y F + \varphi \theta rr_s D_T y g + R \theta \mu_s D_T y X + \varphi D_T y)\frac{\partial y}{\partial \theta} + \theta \mu_s D_T y \frac{\partial R}{\partial \theta}
+ (\theta D_T T + \theta \lambda'_s D_T T F + \varphi \theta^2 rr_s D_T T g + \varphi \theta D_T T g + \theta R \mu_s D_T T X + \theta \varphi D_T g)\frac{\partial rr_s}{\partial \theta}
+ \frac{\partial \lambda'_s}{\partial \theta} D_T F + \varphi \theta rr_s D_T g + \mu_s rr_s D_T X + (rr_s D_T g + g)\frac{\partial \varphi}{\partial \theta} + RD_T X \frac{\partial \mu_s}{\partial \theta}\}
\]

Ignoring terms preceded by \( D_T \pi + \lambda'_s D_T F + \varphi \theta rr_s D_T g + R \theta \mu_s rr_s D_T X + \varphi g \}, \mu_s and rr_s, which will be zero anyway at the steady state. and using that the derivatives wrt \( \theta \) are also zero, this immediately yields \( \frac{\partial^2 H}{\partial \theta \partial \epsilon} = 0 \) so I have a singularity. I use the “dividing by \( \epsilon \) trick” to solve the singularity,

\[
\hat{H}(\theta, \epsilon) = \begin{cases} 
\frac{H(\theta, \epsilon)}{\epsilon} & \text{if } \epsilon \neq 0 \\
\frac{\partial H}{\partial \epsilon} & \text{if } \epsilon = 0 
\end{cases}
\]

Since \( H(\theta, 0) = 0 \forall \theta \), \( H = \epsilon \hat{H} \). Then I can rewrite the old FOC equation as

\[
E_0 \hat{H}(\theta, \epsilon) = 0.
\]

To solve this, I use the following bifurcation theorem in Judd and Guu (2001),

**Theorem A.1. (Bifurcation Theorem).** Suppose \( H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( H \) is analytic for \((x, \epsilon) \) in a neighborhood of \((x_0, 0)\), and \( H(x, 0) = 0 \forall x \in \mathbb{R} \). Furthermore, suppose that

\[
H_x(x_0, 0) = 0, H_{x\epsilon}(x_0, 0), H_{x\epsilon} \neq 0.
\]

Then \((x_0, 0)\) is a bifurcation point and there is an open neighborhood \( \mathcal{N} \) of \((x_0, 0)\) and a function \( h(\epsilon), h(\epsilon) \neq 0 \) for \( \epsilon \neq 0 \), such that \( h \) is analytic and \( H(h(\epsilon), \epsilon) = 0 \) for \((h(\epsilon), \epsilon) \in \mathcal{N} \).

Let’s check the conditions for the bifurcation theorem. First, note that to compute \( \frac{\partial H}{\partial \theta} \) at \( \epsilon = 0 \), I need to compute \( \frac{\partial^2 H}{\partial \theta \partial \epsilon} \), which I already showed is zero. In addition, it is clear that if I take another derivative wrt \( \epsilon \) the answer will not be zero. I just need to find the new bifurcation points, which I
find by solving $\frac{\partial H}{\partial \epsilon} |_0 = \frac{\partial^2 H}{\partial \epsilon^2} |_0$:

$$\frac{\partial H}{\partial \epsilon} = E_0(\frac{\partial r_s}{\partial \epsilon}) \{D_T \pi + \lambda_s' D_T F + \varphi r_s D_T g + R \mu_s D_T X + \varphi g\}$$

$$+ r_s \{(D_T y + \lambda_s' D_T y F + \varphi r_s D_T y g + R \mu_s D_T y X + \varphi D_y g) \frac{\partial y}{\partial \epsilon} + \mu_s D_T y X \frac{\partial R}{\partial \epsilon}$$

$$+ (D_T \xi + \lambda_s' D_T \xi F + \varphi r_s D_T \xi g + R \mu_s D_T \xi X + \varphi D_\xi g) \frac{\partial \xi}{\partial \epsilon}$$

$$+ \theta(D_T T + \lambda_s' D_T T F + \varphi r_s D_T T g + \varphi D_T g + \theta R \mu_s D_T T X + \varphi D_T g) \frac{\partial r_s}{\partial \epsilon}$$

$$+ D_T F \frac{\partial \lambda_s'}{\partial \epsilon} + (r_s D_T g + g) \frac{\partial \varphi}{\partial \epsilon} + RD_T X \frac{\partial \mu_s}{\partial \epsilon}\}
$$

Ignoring the terms preceded by $rr_s$ and $\{D_T \pi + \lambda_s' D_T F + \varphi r_s D_T g + \varphi g\}$ which are zero at the steady state, I compute $\frac{\partial^2 H}{\partial \epsilon^2}$:

$$\frac{\partial^2 H}{\partial \epsilon^2} = 2E_0[\frac{\partial r_s}{\partial \epsilon}] \{(D_T y + \lambda_s' D_T y F + \varphi r_s D_T y g + R \mu_s D_T y X + \varphi D_y g) \frac{\partial y}{\partial \epsilon} + \mu_s D_T y X \frac{\partial R}{\partial \epsilon}$$

$$+ (D_T \xi + \lambda_s' D_T \xi F + \varphi r_s D_T \xi g + R \mu_s D_T \xi X + \varphi D_\xi g) \frac{\partial \xi}{\partial \epsilon}$$

$$+ \theta(D_T T + \lambda_s' D_T T F + \varphi r_s D_T T g + \varphi D_T g + \theta R \mu_s D_T T X + \varphi D_T g) \frac{\partial r_s}{\partial \epsilon}$$

$$+ D_T F \frac{\partial \lambda_s'}{\partial \epsilon} + (r_s D_T g + g) \frac{\partial \varphi}{\partial \epsilon} + RD_T X \frac{\partial \mu_s}{\partial \epsilon}\}
$$

Noting that $rr_s = 0$ and $\mu_s = 0$ at the steady state,

$$\frac{\partial^2 H}{\partial \epsilon^2} = 2E_0[\frac{\partial r_s}{\partial \epsilon}] \{(D_T y + \lambda_s' D_T y F + \varphi D_y g) \frac{\partial y}{\partial \epsilon}$$

$$+ (D_T \xi + \lambda_s' D_T \xi F + \varphi D_\xi g) \frac{\partial \xi}{\partial \epsilon}$$

$$+ \theta(D_T T + \lambda_s' D_T T F + 2\varphi D_T g) \frac{\partial r_s}{\partial \epsilon}$$

$$+ D_T F \frac{\partial \lambda_s'}{\partial \epsilon} + g \frac{\partial \varphi}{\partial \epsilon} + RD_T X \frac{\partial \mu_s}{\partial \epsilon}\}
$$

Welfare expansion approach  Starting with the objective function,

$$E_0 \pi(y_s, \xi_s, rr_s \theta) = \pi + E_0 \{D_T \pi \cdot \tilde{y}_s + D_\xi \pi \cdot \tilde{\xi}_s + \theta D_T \cdot \pi \cdot r_s + rr_s D_T \pi \cdot \tilde{\theta}$$

$$+ \frac{1}{2} \{E_0 \{g_s' D_T y \pi \cdot \tilde{y}_s + 2\xi_s D_T \pi \cdot \tilde{\xi}_s + 2 \theta r_s D_T \pi \cdot \tilde{\theta} D_T \cdot \pi \cdot \tilde{\xi}_s + 2 \theta r_s \tilde{\theta} D_T \cdot \pi \cdot D_\xi \pi \cdot \tilde{\xi}_s$$

$$+ \theta^2 r_s D_T T \cdot \pi \cdot r_s + 2 \tilde{\theta} D_T \cdot \pi \cdot \tilde{r}_s + \tilde{\theta} rr_s D_T T \cdot \pi \cdot rr_s \tilde{\theta} + O(\epsilon^3)\}$$

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Using that \( rr_s = 0 \),

\[
E_0 \pi(y_s, \xi_s, rr_s, \theta) = E_0 \{ D_y \pi \cdot \bar{y}_s + \theta D_T \cdot \pi \cdot \bar{r} r_s \\
+ \frac{1}{2} E_0 \{ \dot{y}_s' D_{yy} \pi \cdot \bar{y}_s + 2 \dot{\xi}_s' D_y \xi \cdot \bar{y}_s + 2 \theta \dot{r} r_s D_y \pi \cdot \bar{y}_s \\
+ 2 \theta \dot{r} r_s D_{\xi} \pi \cdot \bar{\xi}_s + \theta^2 \dot{r} r_s D_T \pi \cdot \bar{r} r_s + 2 \theta D_T \cdot \pi \cdot \bar{r} r_s \} + O(\epsilon^3) + \text{tip}
\]

Next, I approximate the \( F \) constraints,

\[
E_0 F(y_s, \xi_s, rr_s, \theta) = E_0 \{ D_y F \cdot \bar{y}_s + D_\xi F \cdot \bar{\xi}_s + \theta D_T F \cdot \bar{r} r_s \\
+ \frac{1}{2} E_0 \{ \dot{y}_s' D_{yy} F \cdot \bar{y}_s + 2 \dot{\xi}_s' D_y \xi F \cdot \bar{y}_s + 2 \theta \dot{r} r_s D_y T F \cdot \bar{y}_s \\
+ \dot{\xi}_s' D_\xi F \cdot \bar{\xi}_s + 2 \theta \dot{r} r_s D_{\xi} T F \cdot \bar{\xi}_s \\
+ \theta^2 \dot{r} r_s D_{TT} F \cdot \bar{r} r_s + 2 \theta D_T F \cdot \bar{r} r_s \} + O(\epsilon^3)
\]

and the \( g \) constraint,

\[
E_0 \Omega r r_s g(y_s, \xi_s, rr_s, \theta) = E_0 \{ g(y_s, \xi_s, rr_s, \theta) \bar{r} r_s + E_0 \{ \dot{r} r_s D_y g \cdot \bar{y}_s + \dot{r} r_s D_\xi g \cdot \bar{\xi}_s \\
+ \theta \dot{r} r_s D_{TT} g \cdot \bar{r} r_s \} + O(\epsilon^3)
\]

Recall that at the steady state,

\[
D_y \pi(y_s, \xi_s, rr_s, \theta) + \lambda_y' D_y F(y_s, \xi_s, rr_s, \theta) = 0 \\
D_T \pi(y_s, \xi_s, rr_s, \theta) + \lambda_{\xi} D_T F(y_s, \xi_s, rr_s, \theta) + \varphi g(y_s, \xi_s, rr_s, \theta) = 0
\]

Then,

\[
D_y \pi \cdot \bar{y}_s + \theta D_T \cdot \pi \cdot \bar{r} r_s = -\lambda_y' D_y F \cdot \bar{y}_s - \theta D_T \pi \cdot \bar{r} r_s - \theta \lambda_{\xi} D_T F \cdot \bar{r} r_s - \theta \varphi g \cdot \bar{r} r_s
\]

Using this I can get rid of the linear terms,

\[
E_0 \pi(y_s, \xi_s, rr_s, \theta) = \frac{1}{2} E_0 \sum \lambda_x' \{ \dot{y}_s' D_{yy} F \cdot \bar{y}_s + 2 \dot{\xi}_s' D_y \xi F \cdot \bar{y}_s + 2 \theta \dot{r} r_s D_y T F \cdot \bar{y}_s \\
+ \dot{\xi}_s' D_\xi F \cdot \bar{\xi}_s + 2 \theta \dot{r} r_s D_{\xi} T F \cdot \bar{\xi}_s \\
+ \theta^2 \dot{r} r_s D_{TT} F \cdot \bar{r} r_s + 2 \theta D_T F \cdot \bar{r} r_s \}
\]

\[
+ \frac{1}{2} \theta \varphi E_0 \{ \dot{r} r_s D_y g \cdot \bar{y}_s + \dot{r} r_s D_\xi g \cdot \bar{\xi}_s + 2 \theta \dot{r} r_s D_{TT} g \cdot \bar{r} r_s \}
\]

\[
+ \frac{1}{2} E_0 \{ \dot{y}_s' D_{yy} \pi \cdot \bar{y}_s + 2 \dot{\xi}_s' D_y \xi \pi \cdot \bar{y}_s + 2 \theta \dot{r} r_s D_y T \pi \cdot \bar{y}_s \\
+ 2 \theta \dot{r} r_s D_{\xi} T \pi \cdot \bar{\xi}_s + \theta^2 \dot{r} r_s D_{TT} \pi \cdot \bar{r} r_s + 2 \theta D_T \pi \cdot \bar{r} r_s \} + \text{tip}
\]
Using the $g$ constraint to first order yields $\mathbb{E}\dot{r}r_s = 0$ and since $\bar{\theta}$ is predetermined,

$$
\mathbb{E}_0\pi(y_s, \xi_s, r_s, \theta) = \frac{1}{2} \mathbb{E}_0 \sum \lambda_s^k \{ \dot{y}_s D_{yy} F \cdot \dot{y}_s + 2 \dot{\xi}_s D_{y\xi} F \cdot \dot{\xi}_s + 2\theta \dot{r}_s D_{yT} F \cdot \dot{y}_s \\
+ 2\theta \dot{r}_s D_{\xi T} F \cdot \dot{\xi}_s \\
+ \theta^2 \dot{r}_s D_{T T} F \cdot \ddot{r}_s \}
$$

$$
+ \frac{1}{2} \theta \varphi \mathbb{E}_0 \{ \dot{r}_s D_y g \cdot \dot{y}_s + \dot{r}_s D_{\xi T} F \cdot \dot{\xi}_s + 2\theta \dot{r}_s D_{T T} g \cdot \ddot{r}_s \}
$$

$$
+ \frac{1}{2} \mathbb{E}_0 \{ \dot{y}_s D_{yy} \pi \cdot \dot{y}_s + 2\dot{\xi}_s D_{y\xi} \pi \cdot \dot{\xi}_s + 2\theta \dot{r}_s D_{yT} \pi \cdot \dot{y}_s \\
+ 2\theta \dot{r}_s D_{\xi T} \pi \cdot \dot{\xi}_s + \theta^2 \dot{r}_s D_{T T} \pi \cdot \ddot{r}_s \}
$$

+ tip

The objective is to maximize this then subject to the first-order constraints,

$$
D_y F \cdot \dot{y}_s + D_{\xi T} F \cdot \dot{\xi}_s + \theta D_T F \cdot \ddot{r}_s = 0
$$

$$
\mathbb{E}_0\dot{r}_r_s = 0
$$

$$
RD_y X \cdot \dot{y} + RD_{\xi T} X \cdot \dot{\xi} + \theta RD_T X \cdot \ddot{r}_s - \dot{r}_s = 0
$$

The FOC wrt $\theta$ yields

$$
FOC = \mathbb{E}_0 \{ \sum \lambda_s^k \{ \dot{r}_s D_{\xi T} F \cdot \dot{\xi}_s + \theta \dot{r}_s D_{T T} F \cdot \ddot{r}_s + \dot{r}_s D_y F \cdot \dot{y}_s \} \\
+ \varphi \{ \dot{r}_s D_y g \cdot \dot{y}_s + \dot{r}_s D_{\xi T} F \cdot \dot{\xi}_s + 2\theta \dot{r}_s D_{T T} g \cdot \ddot{r}_s \} \\
+ \{ \dot{r}_s D_y T \pi \cdot \dot{y}_s + \dot{r}_s D_{\xi T} \pi \cdot \dot{\xi}_s + \theta \dot{r}_s D_{T T} \pi \cdot \ddot{r}_s \} \\
+ \lambda_s D_T F \cdot \ddot{r}_s + \mu_s D_T X \cdot \ddot{r}_s \}
$$

Rearranging,

$$
FOC = \mathbb{E}_0 \{ \sum \lambda_s^k \{ \dot{y}_s D_{yy} F^k + D_y g \} \cdot \dot{y}_s \\
+ (D_{\xi T} \pi + \sum \lambda_s^k D_{\xi T} F^k + D_{\xi T} \pi) \cdot \dot{\xi}_s \\
+ \theta (\sum \lambda_s^k D_{T T} F + 2\varphi D_{T T} g + D_{T T} \pi + \lambda_s D_T F + R\mu_s D_T X) \ddot{r}_s \}
$$

which coincides with the other method.

**Proposition 1.2**

Differentiating the expression in equation (1.18) with respect to $e_s$ yields

$$
\bar{B}(\bar{B}e_s + \bar{T}_s) + \chi(1 + \bar{B}\mu)((1 + \bar{B}\mu)e_s - e_s(0)) = 0
$$

In terms of $e_s^{in}(B)$ and $e_s^{dm}(B),$

$$
(\bar{B}^2 + \chi(1 + \bar{B}\mu)^2)e_s - \bar{B}^2 e_s^{in}(B) - \chi(1 + \bar{B}\mu)^2 e_s^{dm}(B) = 0.
$$
Rearranging yields the desired expression.

To show part (i), note

\[ \kappa^2 \lambda_\pi = \tilde{k}_0 \eta \frac{1 - \phi}{\phi} \]
\[ \kappa = \tilde{k}_1 \frac{1 - \phi}{\phi} \]

for some positive constants \( \tilde{k}_0 \) and \( \tilde{k}_1 \). This allows us to write

\[ \chi = k'\left( \frac{\lambda_\pi + \tilde{k}_0 \eta \phi}{(k_1 \phi + k_{ex})^2} \right) \]

for some positive constant \( k' \). Differentiating \( \chi \) with respect to \( \phi \),

\[ \chi = \frac{k'}{(k_1 \phi + k_{ex})^3} (-2\tilde{k}_1 (\lambda_\pi + \tilde{k}_0 \eta \phi) + \tilde{k}_0 \eta (k_1 \phi + k_{ex})) \]

Since depreciations are assumed expansionary, \( k_{ex} > 0 \), the derivative is positive iff

\[ -2\tilde{k}_1 (\lambda_\pi + \tilde{k}_0 \eta \phi) + \tilde{k}_0 \eta (k_1 \phi + k_{ex}) \geq 0 \]

Solving,

\[ \phi \leq \frac{\tilde{k}_0 k_{ex} \eta - 2\tilde{k}_1 \lambda_\pi}{k_1 \tilde{k}_0} \]

This immediately translates into a threshold in terms of \( \phi \) such that \( \chi \) is increasing in \( \phi \) if \( \phi \leq \tilde{\phi} \).

Since \( \tilde{\phi} \) increases with \( \eta \), \( \phi \) decreases with \( \eta \). For \( \tilde{\phi} \) to be meaningful, I need \( \tilde{\phi} < 1 \) or

\[ \tilde{\phi} > 0 \Leftrightarrow \eta > \frac{1}{\tilde{k}_0} k_{ex}^{-1} 2\tilde{k}_1 \lambda_\pi. \]

Part (ii) follows from differentiating \( \chi \) with respect to \( m, \gamma^* \) and \( \tilde{\gamma} \).

**Lemma 1.3**

Let \( \mathcal{W}(B) = V(B; \{e_{s}^{op}(B)\}) \) where \( e_{s}^{op}(B) \) is the optimal exchange rate policy. Replacing the optimal exchange rate inside \( \mathbb{E}V \) and simplifying, we obtain

\[ \mathcal{W}(\tilde{B}) = \frac{k_0/2}{\chi(1 + \mu \tilde{B})^2 + \tilde{B}^2} \{(1 + \mu \tilde{B})^2 \chi^2 \sigma_{edm(0)}^2 + \tilde{B}^2 \sigma_{T}^2 - 2\tilde{B} \chi (1 + \mu \tilde{B}) \sigma_{Tedm(0)} \} + t.i.p. + O(\epsilon^3) \]

(A.15)
The first derivative with respect to $\tilde{B}$ yields

$$\frac{\partial \mathcal{W}(B)}{\partial \tilde{B}} = \frac{k_0 \chi}{(\chi(1 + \mu\tilde{B})^2 + \tilde{B}^2)^2} \left\{ (\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 + \mu(\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 - \chi\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 - \chi\mu\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)})B^2 \right. \\
+ (\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 - \chi\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 - 2\chi\mu\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}B - \chi\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 \left. \right\} \quad (A.16)$$

Solving the quadratic and picking the local maximum yields the result.

**Proposition 1.3**

It is convenient to prove these results to work directly with the expression in equation (A.16). For expositional reasons, I prove part (iii) first together with the remark that $B_{op}\mu > -1$.

(iii) $B < -\mu^{-1}$ First, doing algebra it can be shown that the determinant of the quadratic inside the bracket is given by $(\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 - \chi\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2)^2 + 4\chi\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 > 0$.

I prove the case of $\mu < 0$. The case $\mu > 0$ is analogous. I need to consider several subcases. Consider first the case $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)} > 0$. If $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 > \chi + (\chi\mu - \mu^{-1})\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$, the quadratic is convex. Since the intercept is negative, this implies $B_{op} < 0$. Next, suppose $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 < \chi + (\chi\mu - \mu^{-1})\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$ since $\mu < 0$, $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 < \chi + 2\chi\mu\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$. Thus, the quadratic is concave and increasing at $B = 0$. Since the intercept is negative, this implies $B_{op} > 0$. I call this the “reversal” case since $\text{sign}(\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}) = \text{sign}(B_{op})$. The condition in (1.22) rules this case out.

Next, consider the case $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)} < 0$. If $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 > \chi + (\chi\mu - \mu^{-1})\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$, then the quadratic is concave. Since the intercept is positive, this implies $B_{op} > 0$. Next, suppose $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 < \chi + (\chi\mu - \mu^{-1})\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$. Since $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)} < 0$ and $\mu < 0$, $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 < \chi + 2\chi\mu\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$. Thus, the quadratic is convex and decreasing at $B = 0$. Since the intercept is positive, $B_{op} > 0$. Note there is no “reversal” case when $\text{sign}(\mu) = \text{sign}(\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)})$.

Finally, I show that $1 + \mu B_{op} > 0$ under the condition in (1.22). I prove the case of $\mu < 0$; the case $\mu > 0$ is analogous. I only need to show this for the case $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)} < 0$, since $B_{op} < 0$ in the admissible range for $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)} > 0$ under the condition in (1.22). Evaluating (A.16) at $B = -\mu^{-1}$ yields

$$\mathcal{W}'(-\mu^{-1}) = (-V_{11})\chi\mu^2 \sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2 < 0$$

Since the intercept is positive, it follows that $B_{op} \in (0, -\mu^{-1})$.

(i) I apply the implicit function theorem. Note:

$$\frac{\partial B}{\partial \sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2} \propto \frac{(\mu B_{op} + 1)B_{op}}{(-\mathcal{W}'(B))}$$

Since under the condition in (1.22), $\text{sign}(-\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}) = \text{sign}(B)$ and $1 + \mu B_{op} > 0$, $|B_{op}|$ increases with $\sigma_{\mathcal{T}_{e\mathcal{d}m}(0)}^2$. 145
With respect to $\chi$, 
\[
\frac{\partial B}{\partial \chi} \propto - (1 + \mu B_{op}) B_{op} + (1 + \mu B_{op})^2 \sigma_{Te^{dm}(0)} \sigma_{Te^{dm}(0)} + (-W''(B))^{-1} \frac{\partial B}{\partial m} \propto - (1 + \mu B_{op}) B_{op} + (1 + \mu B_{op})^2 \sigma_{Te^{dm}(0)} \sigma_{Te^{dm}(0)} + (-W''(B))^{-1} \frac{\partial B}{\partial m}.
\]

Since under the condition in (1.22), $\text{sign}(-\sigma_{Te^{dm}(B)}) = \text{sign}(B)$ and $1 + \mu B_{op} > 0$, $|B_{op}|$ decreases with $\chi$.

(ii) By the implicit function theorem, and given the condition in (1.22) I can write

\[
\frac{\partial |B|}{\partial |\sigma_{Te^{dm}(0)}|} \propto \frac{1 + \mu B_{op} |B_{op}|}{(-W''(B))^{-1}} \sigma_{Te^{dm}(0)}^2 (\sigma_{Te^{dm}(0)}^2 - \chi)
\]

Thus $|B_{op}|$ increases (decreases) with $|\sigma_{Te^{dm}(0)}|$ if $\sigma_{Te^{dm}(0)}^2 < (>) \chi$.

(iv) Define

\[
T_s^\infty = (1 + 2m^{-1} \gamma^{-1} \gamma_{ss}^*) T_s,
\]

\[
\chi^\infty = (1 + 2m^{-1} \gamma^{-1} \gamma_{ss}^*) \chi,
\]

Then

\[
W'(B) \propto \{(1 + 2m^{-1} \gamma^{-1} \gamma_{ss}^*) \sigma_{Te^{dm}(0)} + \mu (\sigma_{Te^{dm}(0)}^2 - (1 + 2m^{-1} \gamma^{-1} \gamma_{ss}^*)^2 \chi^\infty \sigma_{Te^{dm}(0)}^2} - (\chi^\infty \mu \sigma_{Te^{dm}(0)}^2) B_{op}^2 + (\sigma_{Te^{dm}(0)}^2 - (1 + 2m^{-1} \gamma^{-1} \gamma_{ss}^*)^2 \chi^\infty \sigma_{Te^{dm}(0)}^2 - 2 \chi^\infty \mu \sigma_{Te^{dm}(0)}^2) B_{op} - \chi^\infty \sigma_{Te^{dm}(0)}^2 \}
\]

Then, using the implicit function theorem and

\[
\frac{\partial B}{\partial m} \propto \sigma_{Te^{dm}(0)} B_{op} + (1 + B_{op} \mu) B_{op} \chi^\infty \sigma_{Te^{dm}(0)}^2 \frac{\partial B}{\partial m} \propto \sigma_{Te^{dm}(0)} B_{op} + (1 + B_{op} \mu) B_{op} \chi^\infty \sigma_{Te^{dm}(0)}^2 \frac{\partial B}{\partial m} + (-W''(B_{op}))^{-1} \frac{\partial B}{\partial m}.
\]

Since under the condition in (1.22), $\text{sign}(-\sigma_{Te^{dm}(B)}) = \text{sign}(B_{op})$ and $1 + \mu B_{op} > 0$, $\frac{\partial B}{\partial m}$ decreases with $m$ if $B_{op} < 0$ and increases with $m$ when $B_{op} > 0$.

Finally, we show that if the condition in equation (1.22) does not hold, the result still holds in terms of the sensitivity $f(\bar{B})$. To see this, note that welfare can be written as

\[
W(f(\bar{B})) = \frac{1}{2 + f(\bar{B})^2} \left\{ \chi^2 \sigma_{Te^{dm}(0)}^2 + f(\bar{B})^2 \sigma_{Te^{dm}(0)}^2 - 2 f(\bar{B}) \chi \sigma_{Te^{dm}(0)}^2 \right\} + \text{t.i.p.} + O(\epsilon^3)
\]

This is isomorphic to the previous problem with $\mu = 0$, which satisfies the condition in (1.22).

Corollary 1.1

The weight clearly decreases with $\chi$ conditional on $f(\bar{B})$. Furthermore, proposition 1.3 implies $f(\bar{B})$ decreases with $\chi$ and increases with $\sigma_{Te^{dm}(0)}^2$ and $\omega$ increases in $f(\bar{B})$. These observations
immediately imply the first two results.

The last observation regarding $m$ requires more work. Using $\chi = h(m)\chi^\infty$ as in the proof of proposition 1.3, one may write the insurance weight as

$$\omega = \frac{h(m)^{-1} f(B)^2}{h(m)^{-1} f(B)^2 + \chi^\infty} = \frac{\tilde{B}^2}{\tilde{B}^2 + \chi^\infty},$$

where $\tilde{B} = (\sqrt{h(m)})^{-1} f(B)$. Using $\mathcal{T} = h(m)\mathcal{T}^\infty$, the first derivative with respect to the sensitivity $f(B)$ yields

$$\frac{\partial \mathcal{W}(f(B))}{\partial f(B)} = \frac{k_0 h(m)^2}{(\chi + f(B)^2)^2} \{\sigma_{\mathcal{T}^\infty e_{dm}(0)} \tilde{B}^2 + (h(m)\chi^\infty \sigma_{e_{dm}(0)}^2 \tilde{B} - \chi^\infty \sigma_{\mathcal{T}^\infty e_{dm}(0)} \}$.

Thus,

$$\frac{\partial \mathcal{W}(f(B))}{\partial \chi} = \frac{1}{2} h'(m) \tilde{B} \{h(m)^{-\frac{1}{2}} \sigma^2_{\mathcal{T}^\infty} + 3\chi^\infty \sigma^2_{e_{dm}(0)} \}.$$

Recall $h'(m) > 0$, so by the implicit function theorem: $\frac{\partial \tilde{B}}{\partial \chi} > 0$ if $\tilde{B} > 0$ and $\frac{\partial \tilde{B}}{\partial \chi} < 0$ if $\tilde{B} < 0$. Thus, $\frac{\partial \mathcal{W}}{\partial \chi} > 0$.

**Proposition 1.4**

i. Using (1.20), I see that the variance of the optimal exchange rate is given by

$$\sigma^2_e = \frac{1}{(\chi(1 + \mu B_{op})^2 + B_{op}^2)^2} \{\chi^2(1 + \mu B_{op})^4 \sigma^2_{e_{dm}(B_{op})} + B_{op}^2 \sigma^2_{\mathcal{T}^\infty} - 2 B_{op} \chi(1 + \mu B_{op})^2 \sigma_{\mathcal{T}^\infty e_{dm}(B_{op})} \}.$$

Note that $e_{dm}(B_{op})$ is the exchange rate policy under a demand-management policy if agents are holding $\tilde{B} = B_{op}$ (rather than the optimal portfolio corresponding to the demand-management policy $\tilde{B} = B_{dm}$). Using that $e_{dm}(\tilde{B}) = (1 + \mu \tilde{B})^{-1} e_{dm}(0)$, I can rewrite this as

$$\frac{\sigma^2_e}{\sigma^2_{e_{dm}(0)}} = \frac{1}{(\chi(1 + \mu B_{op})^2 + B_{op}^2)^2} \{\chi^2(1 + \mu B_{op})^4 + B_{op}^2 \sigma^2_{\mathcal{T}^\infty} / \sigma^2_{e_{dm}(0)} \}.$$

Clearly, if $B_{op} = K$, an increase in $\sigma^2_{\mathcal{T}^\infty} / \sigma^2_{e_{dm}(0)}$ leads to higher volatility, $\sigma^2_e / \sigma^2_{e_{dm}(0)}$.

The derivative with respect to $\chi$ yields

$$\frac{\sigma^2_e}{\sigma^2_{e_{dm}(0)}} = \frac{2 B_{op}(1 + \mu B_{op})}{\chi(1 + \mu B_{op})^2 + B_{op}^2} \{B_{op}^2 \sigma_{\mathcal{T}^\infty e_{dm}(0)} / \sigma^2_{e_{dm}(0)} + B_{op}(1 + \mu B_{op}) / (\sigma^2_{\mathcal{T}^\infty} / \sigma^2_{e_{dm}(0)}) - \chi \}.$$

Since $B_{op}$ is optimal it must be that if $B_{op}$ is positive, the term in brackets is positive and, conversely, if $B_{op}$ is negative, the term in brackets is negative (it has the same sign as the FOC). Under the condition in (1.22), $1 + \mu B_{op} > 0$, so $\sigma^2_e / \sigma^2_{e_{dm}(0)}$ decreases with $\chi$. 

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ii and iii) In an interior optimum,

\[ B_{\text{op}}^{2} \sigma_{T_{\text{edm}}(0)} + B_{\text{op}}(1 + \mu B_{\text{op}})(\sigma_{T}^{2}/\sigma_{e_{\text{edm}}(0)}^{2} - \chi) - \chi(1 + \mu B_{\text{op}})^{2}(\sigma_{T_{\text{edm}}(0)}/\sigma_{e_{\text{edm}}(0)}^{2}) = 0 \]

\[
\frac{(1 + \mu B_{\text{op}})^{2}}{(\chi(1 + \mu B_{\text{op}})^{2} + B_{\text{op}}^{2})^{2}}(\chi^{2}(1 + \mu B_{\text{op}})^{2} + B_{\text{op}}^{2}(\sigma_{T}^{2}/\sigma_{e_{\text{edm}}(0)}^{2}))
\]

\[-2B_{\text{op}} \chi(1 + \mu B_{\text{op}})(\sigma_{T_{\text{edm}}(0)}/\sigma_{e_{\text{edm}}(0)}^{2}) = \frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}} \]

Letting \( \tilde{B} = f(B) = (1 + \mu B)^{-1}B \), one may rewrite this as

\[
\frac{\sigma_{T_{\text{edm}}(0)}}{\sigma_{e_{\text{edm}}(0)}^{2}} \tilde{B}^{2} + (\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} - \chi) \tilde{B} - \frac{\sigma_{T_{\text{edm}}(0)}}{\sigma_{e_{\text{edm}}(0)}^{2}} = 0
\]

\[
\frac{1}{(\chi + \tilde{B}^{2})^{2}}(\chi^{2} + \tilde{B}^{2} \frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} - 2\chi \tilde{B} \frac{\sigma_{T_{\text{edm}}(0)}}{\sigma_{e_{\text{edm}}(0)}^{2}}) = \frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}}
\]

It can be verified using Mathematica that plugging in the correct root from the first equation into the second equation and computing the derivative with respect to \( \frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} \) and \( \chi \) yields

\[
\frac{\partial}{\partial \frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}}} - \frac{\partial}{\partial \frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}}} = -\frac{(\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} + \chi)(\frac{\sigma_{T_{\text{edm}}(0)}}{\sigma_{e_{\text{edm}}(0)}^{2}})^{2}}{\left(\left(\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}}\right)^{2} - 2\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} \chi + 4\left(\frac{\sigma_{T_{\text{edm}}(0)}}{\sigma_{e_{\text{edm}}(0)}^{2}}\right)^{2} \chi + \chi^{2}\right)^{3/2}} < 0, \tag{A.18}
\]

and

\[
\frac{\partial}{\partial \chi} \frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}} = \frac{2\left(\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}}\right) - 1}{\left(\left(\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}}\right)^{2} - 2\frac{\sigma_{T}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} \chi + 4\left(\frac{\sigma_{T_{\text{edm}}(0)}}{\sigma_{e_{\text{edm}}(0)}^{2}}\right)^{2} \chi + \chi^{2}\right)^{3/2}} > 0, \tag{A.19}
\]

Finally, note:

\[
\frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} = \frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}} \frac{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}}{\sigma_{e_{\text{edm}}(0)}^{2}} = \frac{\sigma_{e}^{2}}{\sigma_{e_{\text{edm}}(B_{\text{op}})}^{2}} \left(\frac{1}{1 + \mu B_{\text{op}}}\right)^{2}.
\]

I know the first term decreases by the results in (A.18) and (A.19). In addition, I know that \( B_{\text{op}} \) becomes larger in absolute value with a larger importance of the insurance motive (\( \uparrow \sigma_{T}^{2}/\sigma_{e_{\text{edm}}(0)}^{2} \) or \( \downarrow \chi \)). Thus, if \( \mu B_{\text{op}} \) is nonnegative, a larger portfolio \( |B_{\text{op}}| \) makes the second term weakly smaller, which in turn shows that overall volatility strictly decreases. In contrast, if \( \mu B_{\text{op}} < 0 \) the effect on volatility is ambiguous.
Proposition 1.5

I start by deriving an approximation to the consumers’ Euler equation. Expanding the first-order condition of the consumer with respect to tradable consumption yields

$$ u_{Tt} + u_{TT}C_{Tss}C_{Ts} + u_{TN}F_ZZ_{ss}z_s + (u_{TN}F_Y + u_{TL})L_{ss}l_s = u_T\lambda_s + O(\epsilon^2), $$

where $\lambda_s$ is the first-order expansion of the multiplier on the budget constraint. Rewriting this in terms of the output gap,

$$ u_{Tt} + V_{11}C_{Tss}C_{Ts} + V_{12}Z_{ss}z_s + u_{Tt} + FF_{YY}^{-1}(u_{TN}F_Y + u_{TL})x_s = u_T\lambda_s + O(\epsilon^2) \quad (A.20) $$

Next, note that to first-order $\tau^B$ must be zero. Otherwise, portfolio positions would become unbounded. Expanding the no-arbitrage condition of home agents to second order,

$$ E[r - e_s + (r - e_s)^2 + \lambda_s(r - e_s)] = \tau^B + O(\epsilon^3) $$

and expanding the no-arbitrage condition of foreigners to second-order,

$$ E[(r - e_s) + (r - e_s)^2 - \gamma_{ss}(y_s^* - \frac{1}{m}\tilde{B}(r - e_s))(r - e_s)] + O(\epsilon^3) = 0 $$

where I used $C_{ss}^* = Y_{ss}^* = 1$. Then,

$$ E[(\lambda_s + \gamma_{ss}y_s^* - m^{-1}\tilde{B}(r - e_s))(r - e_s)] = \tau^B + O(\epsilon^3). \quad (A.21) $$

For future reference, note that the planner’s objective can be rewritten as

$$ EV = -k_0\mathbb{E}\left[\frac{1}{2}(\tilde{B}e_s + T_s^\infty)^2 + \frac{1}{2} \chi^\infty((1 + \mu\tilde{B})e_s - e_s^{dm}(0))^2 - u_T\gamma_{ss}^*m^{-1}V_{11}^{-1}\tilde{B}^2e_s^2\right] + t.i.p + O(\epsilon^3) $$

where I used the definitions of $T_s^\infty$ and $\chi^\infty$ given in the proof of part (iv) of proposition 1.3.

Next, note that equation (A.20) can be rewritten as

$$ -V_{11}(\tilde{B}e_s + T_s^\infty) + FF_{YY}^{-1}(u_{TN}F_Y + u_{TL})x_s = u_T(\lambda_s + \gamma_{ss}^*y_s^*) + O(\epsilon^2) $$

Using (A.2),

$$ x_s = k_{ex}^{-1}(1 + \mu\tilde{B})\{e_s - (1 + \mu\tilde{B})^{-1}e_s^{dm}(0)\} + O(\epsilon^2) $$

I can write

$$ -V_{11}(\tilde{B}e_s + T_s^\infty) + \tilde{k}\{(1 + \mu\tilde{B})e_s - e_s^{dm}(0)\} = u_T(\lambda_s + \gamma_{ss}^*y_s^*) + O(\epsilon^2) $$
for some constant \( \tilde{k} \). I also know that the planner picks \( e_s \) to solve

\[-\tilde{B}V_{11}(\tilde{B}e_s + \tau_s^\infty) + 2m^{-1}\gamma_{ss}\tau_s^\infty \tilde{B}^2 + (1 + \mu \tilde{B})V_{11}\chi_s^\infty((1 + \mu \tilde{B})e_s - e_s(0)) = O(\epsilon^2)\]

Then,

\[\tilde{k}' \left(-V_{11}(\tilde{B}e_s + \tau_s^\infty) - 2um^{-1}\gamma_{ss}e_s \tilde{B}\right) = u_T(\lambda_s + \gamma^*_s y^*_s - 2m^{-1}\gamma_{ss}e_s \tilde{B}) + O(\epsilon^2)\]

The planners’ FOC implies

\[E e_s \{V_{11}(\tilde{B}e_s - \tau_s) + 2m^{-1}\gamma_{ss} \tilde{B}e_s\} = O(\epsilon^3)\]

Thus,

\[E e_s (\lambda_s + \gamma^*_s y^*_s - 2m^{-1}\gamma_{ss}e_s \tilde{B}) = O(\epsilon^3)\]

Comparing this to equation (A.21) implies the result.

A.1.2 Proofs for section 1.4

I first present some preliminary computations required for the proofs in Section A.1.2. I then use the results to prove lemma 1.4 in Section A.1.2, both for the case with savings taxes covered in the main text, and the case without them, which I discuss in Appendix A.3. Section A.1.2 presents the generalization of the results in the static model (proposition 1.6). Section A.1.2 proves the exchange rate result (proposition 1.7) and Section 1.8 proves the results for savings taxes (proposition 1.8)

Model solution and preliminary computations

First, I derive the Phillips curve. Second, I derive a second-order approximation to the welfare loss around the riskless steady state around some portfolio \( \{B_t, \epsilon = 0\} \). Note that given a path \( \{B_t\} \)-I have a standard linear-quadratic problem, which I can solve using the certainty equivalent property. Third, I solved the relaxed (continuation) problem under flexible prices. Fourth, I show how this relaxed problem allows me to write the welfare function in a familiar form. Fifth, I will rewrite the bond-pricing equation in a way that makes it easier to solve the planning problem. Sixth, I derive the excess returns of home-currency bonds in a savings-only \( (B = 0) \) economy.

Deriving the Phillips’ curve  The FOC with respect to \( P_{1t}(i) \) for intermediate-good producer \( i \) yields

\[E_t \sum_{s=0}^{\infty} (\beta \phi)^s t_s Y_{1t+s} \left\{ E_s^{-1}u_{Tt+s}(1 - \eta)(\frac{P_{1t}(i)}{P_{1t+s}})^{1-\eta} - \eta(1 - \tau)P_{1t}(i)^{-1}(\frac{P_{1t}(i)}{P_{1t+s}})^{-\eta}u_{Lt+s} \right\} = 0\]

\[E_t \sum_{s=0}^{\infty} (\beta \phi)^s t_s u_{Tt+s}(\frac{P_{1t}(i)}{P_{1t+s}})^{-\eta}Y_{1t+s} \left\{ \frac{P_{1t}(i)}{E_{t+s}} + \frac{u_{Lt+s}}{u_{Tt+s}} \right\} = 0\]
Log-linearizing around steady state and rearranging,

\[ p^*_{t+1} - (1 - \beta \phi) \{ \epsilon_t - u^{-1}_T u_{TT} Y_{Tss} c_{Tt} - u^{-1}_T u_{TN} (F_Y L_{ss} l_t + F_Z Z_{ss} z_t) - u^{-1}_T u_{LL} L_{ss} l_t \\
+ u^{-1}_L u_{TL} C_{Tss} c_{Tt} + u^{-1}_L u_{NL} (F_Y L_{ss} l_t + F_Z Z_{ss} z_t) + u^{-1}_L u_{LL} L_{ss} l_t \} + \beta \phi \epsilon_{t+1} p^*_{t+1} = 0 \quad (A.22) \]

where \( p^*_{t+1} \) is the wage optimizers at \( t \) set. The evolution of the aggregate intermediate-input price index is given by

\[ P_{it} = [\phi P_{it-1} + (1 - \phi) P_{it}^{*\eta}]^{1/(1-\eta)} \]

Log-linearize around zero (intermediate-input-price) inflation,

\[ \pi_{it} = \frac{1 - \phi}{\phi} (p^*_{it} - p_{it}) \]

Replacing in (A.22),

\[ \pi_{it} = \frac{1 - \phi}{\phi} (1 - \beta \phi) \{ \epsilon_t - p_{it} - u^{-1}_T u_{TT} Y_{Tss} c_{Tt} - u^{-1}_T u_{TN} (F_Y L_{ss} l_t + F_Z Z_{ss} z_t) - u^{-1}_T u_{LL} L_{ss} l_t \\
+ u^{-1}_L u_{TL} C_{Tss} c_{Tt} + u^{-1}_L u_{NL} (F_Y L_{ss} l_t + F_Z Z_{ss} z_t) + u^{-1}_L u_{LL} L_{ss} n_{t+s} \} + \beta \epsilon_{t+1} \pi_{it+1} \quad (A.23) \]

The firms' FOC is now given by

\[ k_{cc} C_{Tss} c_{Tt} + k_{ez} Z_{ss} z_t + k_{ex} x_t = \epsilon_t - p_{it}, \]

Replacing in (A.23) and simplifying,

\[ \pi_{it} = \kappa x_t + \beta \epsilon_{t+1} \pi_{it+1} \]

where

\[ \kappa = \frac{1 - \phi}{\phi} (1 - \beta \phi) \lambda_x F^{-1} u^{-1}_N > 0. \]

**Welfare loss around steady-state path \( \{ B_t \} \)** Proceeding as in the static model and using the result in Woodford (2003) (ch.6),

\[ \sum \beta^t \text{Var}(\hat{p}_{it}(i)) = \frac{\phi}{(1 - \phi)(1 - \beta \phi)} \sum \beta^t \pi^2_{ft} \]

where \( \hat{p}_{it}(i) = p_{it}(i) - p_{it} \), I obtain

\[ U = \mathbb{E}_0 \sum \beta^t \{ u_T C_{Tss} c_{Tt} + \frac{1}{2} (V_{11} C_{Tss}^2 - u_T C_{Tss}) c_{Tt}^2 + (V_{12} Z_{ss} z_t + u_T \xi_t) C_{ss} c_{Tt} \\
- \frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_{11} \pi^2_{ft} \} + \text{t.i.p.} + O(\epsilon^3) \quad (A.24) \]
where \[ \lambda_x = -\frac{\eta \phi}{(1 - \beta \phi)(1 - \phi)} u_L. \]

The constants \( V_{11}, V_{1Z}, w_T \) and \( \lambda_x \) are the same as in the static model.

A second-order approximation to the budget constraint - equation (1.27) - yields

\[
C_{Tss}c_{Tt} + \frac{1}{2} C_{Tss}c_{Tt}^2 + b_t^* - (\beta^{-1} b_{t-1}^* + \beta^{-1} B_{ss}^* r_{t-1}^* + \frac{1}{2} \beta^{-1} B_{ss}^* r_{t-1}^* + \beta^{-1} h_{t-1}^* r_{t-1}^*) = \]
\[
Y_{Tss}y_T + \frac{1}{2} Y_{Tss}y_T^2 + B_{t-1}rr_t + B_{et-1}rr_t + O(\varepsilon^3). \tag{A.25}
\]

A second order approximation to the foreign no-arbitrage condition yields

\[
E_{t-1} \{ r_{t+1} - \gamma_{ss}^r r_{t+1}(y_{t+1}^* + \beta^{-1} b_{t+1}^* - b_{t+1}^* - \frac{1}{m} B_{t+1}rr_t) + O(\varepsilon^3) = 0,
\]

Furthermore, since for foreigners \( r^* \) and \( y^* \) shocks are exactly compensated by \( \beta^* \) shocks, they always save a constant share of their income,

\[ b_{t+1}^* = b_{t+1}^* - \beta m^{-1} B_{t+1}rr_t, \]

where I used that at the steady state \( \beta = \beta^* \). Let \( \tilde{x}_t = x_t - E_{t-1} x_t \) denote the innovation for a generic variable \( x \). Then,

\[
E_{t-1} \{ r_{t+1} - \gamma_{ss}^r r_{t+1}(\tilde{y}_{t+1}^* - \frac{1}{m} \tilde{B}_{t+1}rr_t) + O(\varepsilon^3) = 0, \tag{A.26}
\]

Combining (A.25) and (A.26), and replacing in (A.24),

\[
U = E_0 \left\{ \beta^t \left\{ \beta^{-1} u_T b_{t-1}^* - b_t^* \right\} + u_T B_{t-1}rr_t \gamma_{ss}^* \tilde{y}_{t+1}^* - u_T B_{t-1}^2 \frac{1 - \beta}{m} \gamma_{ss}^r r_{t+1}^* + u_T \beta^{-1} b_{t+1}^* r_{t+1}^* + \frac{1}{2} V_{11} Y_{Tss}c_{Tt}^2 + (V_{1Z} Z_{ss} z_t + u_T \xi_t) C_{Tss} c_{Tt} - \frac{1}{2} \lambda_{x} x_t^2 - \frac{1}{2} \lambda_{\pi} \pi_t^2 \right\} + t.i.p. + O(\varepsilon^3)
\]

Using the transversality condition for bonds,

\[
U = \beta^{-1} u_T b_{t+1}^* + E_0 \left\{ \beta^t \left\{ u_T B_{t-1}rr_B t^* \gamma_{ss}^* \tilde{y}_{t+1}^* - u_T B_{t-1}^2 \frac{1 - \beta}{m} \gamma_{ss}^r r_{t+1}^* + u_T \beta^{-1} b_{t+1}^* r_{t+1}^* + \frac{1}{2} V_{11} Y_{Tss}c_{Tt}^2 + (V_{1Z} Z_{ss} Y_{Tss} z_t + u_T \xi_t) C_{Tt} - \frac{1}{2} \lambda_{x} x_t^2 - \frac{1}{2} \lambda_{\pi} \pi_t^2 \right\} + t.i.p. + O(\varepsilon^3)
\]
This is already purely quadratic. Using a first-order approximation to the budget constraint,

\[
U = u_T b_{t-1}^* + E_0 \left[ \sum \beta^t \{ u_T \bar{B}_{t-1} r t \gamma_{ss} y_{t-1} - u_T \bar{B}_{t-1}^2 \frac{1 - \beta}{m} \gamma_{ss} r_{t-1}^2 + u_T \beta^{-1} b_{t-1}^* r_{t-1}^* \right. \\
+ \frac{1}{2} V_{11}(Y_{Tss} y_t + \bar{B}_{t-1} r t + \beta^{-1} B_{ss}^* r_{t-1}^* - b_{t}^* + \beta^{-1} b_{t-1}^*)^2 \\
+ \left( V_{12} Z_{ss} z_t + u_T \xi_t \right) (Y_{Tss} y_t + \bar{B}_{t-1} r t + \beta^{-1} B_{ss}^* r_{t-1}^* - b_{t}^* + \beta^{-1} b_{t-1}^*) \\
- \frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_\pi \pi_{t-1}^2 \right] + t.i.p. + O(\varepsilon^3)
\]

(A.27)

The approximate problem (given \{\bar{B}_t\}) is to maximize (A.28) subject to

\[
\kappa x_t + \beta \pi_{t+1} = \pi_t \\
\beta^{-1}(r_{t-1} - \Delta e_t - \beta(1 - \delta)r_t - r_{t-1}^*) = r_{t-1}
\]

By the certainty equivalent property (conditional on \{\bar{B}_t\}), it is without loss of generality to consider the case in which the economy is at a steady state at \(t = -1\), receives a shock at \(t = 0\) and no further shocks from \(t = 1\) onwards. In other words, from \(t \geq 1\) onwards the second equation is a bond pricing equation,

\[
\gamma_{ss} r_{t-1}^* = \Delta e_t + \beta(1 - \delta)r_t - r_{t-1}^*.
\]

**A relaxed problem: Flexible prices** Consider the relaxed problem

\[
W_{R1} = \max_{\{b_t^*\}_t=0} \left[ u_T \bar{B} r 0 \gamma_{ss} y_0^* - u_T \bar{B}_{t-1}^2 \frac{1 - \beta}{m} \gamma_{ss} r_0^2 \right. \\
+ \frac{1}{2} V_{11}(Y_{Tss} y_0 + \bar{B} r 0 - b_0^*)^2 + \left( u_T t_0 + V_{12} Z_{ss} z_0 \right) (Y_{Tss} y_0 + \bar{B} r 0 - b_0^*) \\
+ \sum_{t=1}^{\infty} \beta^t \left( \frac{1}{2} V_{11}(Y_{Tss} y_t + \beta^{-1} B_{ss}^* r_{t-1}^* - b_t^* + \beta^{-1} b_{t-1}^*)^2 + u_T b_t^* r_t^* \right. \\
+ \left. \left( V_{12} Z_{ss} z_t + u_T t_t \right) (Y_{Tss} y_t + \beta^{-1} B_{ss}^* r_{t-1}^* - b_t^* + \beta^{-1} b_{t-1}^*) \right] + t.i.p. + O(\varepsilon^3)
\]

with \(\bar{B}\) and \(rr_0\) given. Note that this is the problem the planner would solve if prices were flexible.

For \(t \geq 1\), the FOC with respect to \(b_t^*\) yields

\[
k_x \xi_{t+1} + V_{11} b_{t+1}^{R*} = (1 + \beta^{-1})V_{11} b_t^{R*} - \beta^{-1} b_{t-1}^* + k_{L, \xi} \xi_t
\]

where

\[
k_x = [-V_{11} Y_{Tss}, -V_{12} Z_{ss}, -V_{11} \beta^{-1} \rho_{rr}^{-1} B_{ss}^*, -u_T, 0] \\
k_{L, \xi} = [-V_{11} Y_{Tss}, -V_{12} Z_{ss}, -V_{11} \beta^{-1} \rho_{rr}^{-1} B_{ss}^* + u_T, -u_T, 0]
\]

and \(\xi = [y_T, z, r^*, t, \psi]\). Assuming shocks follow an VAR1, \(\xi_{t+1} = V_{\xi} \xi_t\), with the world interest rate
following an AR1 for simplicity. The solution is given by

\[ b_t^R = b_{t-1}^R - \beta(kL - k_\xi V) \epsilon_t \]

At \( t = 0 \),

\[ u_T r_0^* - V_{11} (\beta \tilde{B} r_0 + Y_{Tss} y_0 - b_0^*) - V_{11} Z_{ss} z_0 - u_T v_0 \]

\[ + V_{11} (Y_{Tss} y_1 + \rho_{ss}^{-1} \beta^{-1} B_{ss}^* r_1^* - b_1^* + \beta^{-1} b_0^*) + V_{11} Z_{ss} z_1 + u_T v_1 = 0, \]

i.e., the same except that \( \beta^{-1}(\rho_{ss}^{-1} B_{ss}^* r_0^*) \) is missing. Since \( \rho_{ss}^{-1} B_{ss}^* r_0^* \) enters like \( b_{-1}^* \), I get

\[ b_0^R = \beta \tilde{B} r_0 - \rho_{ss}^{-1} B_{ss}^* r_0^* - \beta(kL - k_\xi V) \epsilon_t \]

If all shocks are AR1 with no cross-lag terms,

\[ b_0^R = \beta \tilde{B} r_0 + \frac{\beta(1 - \rho_z)}{(1 - \beta \rho_z)} V_{11} Z_{ss} z_0 + \frac{\beta(1 - \rho_\xi)}{(1 - \beta \rho_\xi)} u_T v_0 + \frac{\beta(1 - \rho_\nu)}{(1 - \beta \rho_\nu)} Y_{Tss} y_0 \]

\[ - \frac{\beta}{1 - \beta \rho_\nu} V_{11} r_0^* + \frac{1 - \beta}{1 - \beta \rho_r} B_{ss}^* r_0^*. \] (A.29)

**Back to the original problem**  Let \( b_0^{dm}(0) \) denote the optimum in a savings-only economy (demand-management with no savings taxes), and \( \tilde{b}_0 = b_0^* - b_0^R = b_0^* - b_0^*(0) - \beta \tilde{B} r_0 \). Using the solution for \( b_0^{dm}(0) \) and imposing a transversality condition on \( \{\tilde{b}_t\} \), I can rewrite the welfare loss function as

\[ U = \frac{1}{2} (1 - \beta) V_{11} (\tilde{B} r_0 - \mathcal{T}_0)^2 + \frac{1}{2} V_{11} \tilde{b}_0^2 - \frac{1}{2} \lambda_\sigma x_0^2 - \frac{1}{2} \lambda_\pi \pi_0^2 \]

\[ + \sum_{t} \beta^t \left\{ - \frac{1}{2} \lambda_\sigma x_t^2 - \frac{1}{2} \lambda_\pi \pi_t^2 + V_{11} (-\tilde{b}_t + \beta^{-1} \tilde{b}_{t-1})^2 \right\} + t.i.p. + O(\epsilon^3) \]

where

\[ \mathcal{T}_0 = -\frac{1}{1 - \beta} \frac{1}{2m^{-1} u_T V_{11}^{-1} \gamma_{ss}} \{ Y_{Tss} \tilde{y}_0 + V_{11}^{-1} V_{12} Z_{ss} \tilde{z}_0 + V_{11}^{-1} u_T v_0 + V_{11}^{-1} u_T \epsilon_{ss}^* \tilde{y}_0^* - b_0^*(0) \} \]

If all shocks are AR1, then after replacing \( b_0^*(0) \) we obtain

\[ \mathcal{T}_0 = -\frac{1}{1 - \beta} \frac{1}{2m^{-1} u_T V_{11}^{-1} \gamma_{ss}} \{ \frac{1 - \beta}{1 - \beta \rho_y} Y_{Tss} \tilde{y}_0 + \frac{1 - \beta}{1 - \beta \rho_z} V_{11}^{-1} V_{12} Z_{ss} \tilde{z}_0 + \frac{1 - \beta}{1 - \beta \rho_\xi} V_{11}^{-1} u_T v_0 \]

\[ + \frac{V_{11}^{-1} u_T \epsilon_{ss}^* \tilde{y}_0^* + \frac{\beta}{1 - \beta \rho_r} V_{11}^{-1} u_T \tilde{r}_0^* + \frac{1 - \beta}{1 - \beta \rho_r} B_{ss}^* \tilde{r}_0^*} {1 - \beta \rho_r} \}

Note this is the transfer that would be required to complete markets, starting from an economy that has access to the foreign bond \( B^* \). I wrote it in terms of the innovations \( \tilde{z} \) to emphasize that I only obtained the levels because I assumed the economy was at the steady state at \( t = -1 \); what requires insurance between \(-1\) and \( 0 \) is obviously only the *innovations*.  

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Rewriting bond pricing constraint  After replacing the exchange rate depreciation using equation (A.2), the bond-pricing constraint \( (t \geq 1) \) and excess returns constraint \( (t = 0) \) are given by

\[
\beta^{-1}(-k_{ex}x_0 - \pi_{I0} - k_{ez}Z_{ss}z_0 - k_{ec}(Y_{T,ss}y_0 + \bar{B}rr_0 - b_0^{*}) - \beta(1 - \delta)r_0) = rr_0
\]

\[
k_{ex}\Delta x_t + \pi_{I1} + k_{ez}Z_{ss}\Delta z_t + k_{ec}\Delta c_{Tt} + \beta(1 - \delta)r_t + r_{t-1}^{*} - \psi_{t-1} = r_{t-1}
\]

Note the second equation at \( t = 1 \) becomes

\[
k_{ex}\Delta x_1 + \pi_{I1} + k_{ez}Z_{ss}\Delta z_1 + k_{ec}(\Delta y_1 + \beta^{-1}b_0^{*-} - \bar{B}rr_0 + \beta^{-1}B_{ss}^{*}r_{0}^{*-} - b_1 + b_{0})
\]

\[
+ \beta(1 - \delta)r_1 + r_{0}^{*} - \psi_{0} = r_0
\]

and for \( t \geq 2 \),

\[
k_{ex}\Delta x_t + \pi_{I1} + k_{ez}Z_{ss}\Delta z_t - k_{ec}(\Delta y_{Tt} + \beta^{-1}\Delta b_{t-1}^{*} + \beta^{-1}B_{ss}^{*}\Delta r_{t-1}^{*-} - \Delta b_{t}^{*})
\]

\[
+ \beta(1 - \delta)r_t + r_{t-1}^{*} - \psi_{t-1} = r_t
\]

Making \( \tilde{b} \) appear and writing in terms of the savings-only economy (using \( b_{t}^{*}\text{dm}(0) = b_{t}^{*R} - \beta\bar{B}rr_0 \))

\[
-k_{ex}x_0 - \pi_{I0} - k_{ez}Z_{ss}z_0
\]

\[
-k_{ec}(Y_{T,ss}y_0 + (1 - \beta)\bar{B}rr_0 - \tilde{b}_0^{*} - b_{0}\text{dm}(0)) - \beta(1 - \delta)r_0 = \beta rr_0
\]

\[
k_{ex}\Delta x_1 + \pi_{I1} + k_{ez}Z_{ss}\Delta z_1 + k_{ec}(\Delta y_1 + \beta^{-1}(\tilde{b}_0^{*} + b\text{dm}(0)) + 
\]

\[
\beta^{-1}B_{ss}^{*}r_{0}^{*-} - (\tilde{b}_1 - \tilde{b}_0) - (b_{1}\text{dm}(0) - b_{0}\text{dm}(0)) + \beta(1 - \delta)r_1 + r_{0}^{*} - \psi_{0} = r_0
\]

\[
k_{ex}\Delta x_t + \pi_{I1} + k_{ez}Z_{ss}\Delta z_t - k_{ec}(\Delta y_{Tt} + \beta^{-1}(\Delta \tilde{b}_{t-1}^{*} + \Delta b_{t-1}\text{dm}(0)) + 
\]

\[
\beta^{-1}B_{ss}^{*}\Delta r_{t-1}^{*-} - (\Delta \tilde{b}_{t}^{*} + \Delta b_{t}\text{dm}(0))) + \beta(1 - \delta)r_t + r_{t-1}^{*} - \psi_{t-1} = r_{t-1}
\]

Next let \( rr_{0}\text{dm}(0) \) and \( \{r_{t}\text{dm}(0)\} \) solve

\[
\beta^{-1}(-k_{ex}Z_{ss}z_0 - k_{ec}(Y_{T,ss}y_0 - b_{0}\text{dm}(0)) - \beta(1 - \delta)r_0) = rr_{0}\text{dm}(0)
\]

\[
k_{ex}Z_{ss}\Delta z_1 + k_{ec}(\Delta y_1 + \beta^{-1}b_{0}\text{dm}(0) + \beta^{-1}B_{ss}^{*}r_{0}^{*-} - (b_{1}\text{dm}(0) - b_{0}\text{dm}(0))
\]

\[
+ \beta(1 - \delta)r_{1}\text{dm}(0) + r_{0}^{*} - \psi_{0} = r_{0}\text{dm}(0)
\]

\[
k_{ex}Z_{ss}\Delta z_t - k_{ec}(\Delta y_{Tt} + \beta^{-1}\Delta b_{t-1}\text{dm}(0) + \beta^{-1}B_{ss}^{*}\Delta r_{t-1}^{*-} - \Delta b_{t}\text{dm}(0)) + 
\]

\[
\beta(1 - \delta)r_{t}\text{dm}(0) + r_{t-1}^{*} - \psi_{t-1} = r_{t-1}\text{dm}(0)
\]

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Then, I can rewrite (A.31)-(A.33) as

\[-k_{ex} x_0 - \pi_{I0} + k_{ec} \bar{b}_0^* - \beta(1 - \delta)(r_0 - r_{0}^{dm}(0)) = \beta(1 + \beta^{-1}(1 - \beta)k_{ec})r r_0 - rr_{0}^{dm}(0)\]

\[k_{ex} \Delta x_1 + \pi_{I1} + k_{ec}(\beta^{-1} \bar{b}_0^* - (\bar{b}_1 - \bar{b}_0)) + \beta(1 - \delta)(r_1 - r_{1}^{dm}(0)) = r_0 - r_{0}^{dm}(0)\]

\[k_{ex} \Delta x_t + \pi_{I0} + k_{ec}(\beta^{-1} \Delta \bar{b}_{t-1}^* - \Delta \bar{b}_t^*) + \beta(1 - \delta)(r_t - r_{t}^{dm}(0)) = r_{t-1} - r_{t-1}^{dm}(0)\]

Defining

\[\bar{r}_0 = r_0 - r_{0}^{dm}(0) + k_{ex} x_0 - \beta^{-1} k_{ec} \bar{b}_0^*\]

\[\bar{r}_t = r_t - r_{t}^{dm}(0) + k_{ex} x_t - \beta^{-1} k_{ec} \Delta \bar{b}_t^*\]

I get

\[\beta^{-1}(-(1 - \beta(1 - \delta))k_{ex} x_0 - \pi_{I0} - \delta k_{ec} \bar{b}_0^* - \beta(1 - \delta) \bar{r}_0) = (1 + \beta^{-1}(1 - \beta)k_{ec})r r_0 - rr_{0}^{dm}(0)\]

\[k_{ex}(1 - \beta(1 - \delta))x_t + \pi_{I0} + \beta(1 - \delta) \bar{r}_t - \delta k_{ec} \Delta \bar{b}_t^* = \bar{r}_{t-1}\]

\textbf{Rewriting Euler equation constraint}  
Next, we derive a convenient reformulation of the Euler equation, which will be useful to solve the case without capital controls. The Euler equation is given by

\[FF^{-1}(u_{TN}F_Y + u_{TL})\Delta x_{t+1} + V_{11}C_{Ts} \Delta c_{Tt+1} + u_T \Delta u_{t+1} + V_{1Z} Z_{ss} \Delta z_{t+1} - \bar{r}_{t}^* = 0\]

Using the budget constraint,

\[FF^{-1}(u_{TN}F_Y + u_{TL})\Delta x_{1} + V_{11}(Y_{Ts} \Delta y_1 + \beta^{-1} B_{ss}^* \bar{r}_0^* + \beta^{-1} b_{0}^* + \bar{B} r r_0 - \Delta b^*_1) + u_T \Delta u_1 + V_{1Z} Z_{ss} \Delta z_1 - \bar{r}^*_0 = 0\]

\[FF^{-1}(u_{TN}F_Y + u_{TL})\Delta x_{t+1} + V_{11}(Y_{Ts} \Delta y_{t+1} + \beta^{-1} B_{ss}^* \Delta r_{t}^* + \beta^{-1} \Delta b_{t+1}^* - \Delta b^*_t) + u_T \Delta u_{t+1} + V_{1Z} Z_{ss} \Delta z_{t+1} - \bar{r}_{t}^* = 0 \forall t \geq 1\]

The relaxed $b^R$ defined above solves

\[V_{11}(Y_{Ts} \Delta y_1 + \beta^{-1} B_{ss}^* \bar{r}_0^* + \Delta b^*_1 + \beta^{-1} b_{0}^* + \bar{B} r r_0) + u_T \Delta u_1 + V_{1Z} Z_{ss} \Delta z_1 - \bar{r}^*_0 = 0\]

\[V_{11}(Y_{Ts} \Delta y_{t+1} + \beta^{-1} B_{ss}^* \Delta r_{t}^* + \Delta b_{t+1}^* + \beta^{-1} b_{t+1}^R) + u_T \Delta u_{t+1} + V_{1Z} Z_{ss} \Delta z_{t+1} - \bar{r}_{t}^* = 0 \forall s \geq 1\]

The reason this is true is, essentially, that if prices were flexible no capital controls would be required. Substracting,

\[FF^{-1}(u_{TN}F_Y + u_{TL})\Delta x_{1} + V_{11}(\Delta \bar{b}_1 + \beta^{-1} b_0^*) = 0\]

\[FF^{-1}(u_{TN}F_Y + u_{TL})\Delta x_{t+1} + V_{11}(\Delta \bar{b}_{t+1} + \beta^{-1} \Delta b_{t+1}^*) = 0 \forall s \geq 1\]
Implied excess returns on home-currency bonds in $\tilde{B} = 0$ economy Solving forward,

$$\beta r r_0(0) = \left(k^B - \beta(1 - \delta)k^B_{L,\xi}\right) \left(I_{dim(\xi)} - \beta(1 - \delta)V_{\xi}\right)^{-1} \tilde{\xi}_0 + k_{ec}\rho^{-1}_{\tilde{\xi}} B_{ss}^* \tilde{r}_0^*$$

$$+ k_{ec}b_0^{*dm}(0) + k_{ec}\beta(1 - \delta)((k_{L,\xi} - k_{\xi}V_{\xi})V_{11}\left(I_{dim(\xi)} - \beta(1 - \delta)V_{\xi}\right)^{-1} \tilde{\xi}_0$$

where

$$k^B_{\xi} = [-k_{ec}Y_{Ts}, -k_{ez}Z_{ss}, -k_{ecz}\rho^{-1}_{\tilde{\xi}} B_{ss}^*]$$

$$k^B_{L,\xi} = [-k_{ec}Y_{Ts}, -k_{ez}Z_{ss}, -k_{ecz}\rho^{-1}_{\tilde{\xi}} B_{ss}^* + 1, 1]$$

**Lemma 1.4**

**With capital controls** If capital controls are available, the problem is

$$W_{t \geq 1} = \max_{(b_t, x_t, \pi_t)} \left\{ \frac{1}{2} \left(1 - \beta\right)V_{11}^{-1} (\tilde{B}rr_0 - T_s)^2 + \frac{1}{2} V_{11}^{\tilde{b}_t - \beta^{-1} B_{ss}^*(-1)^2} + t.i.p. + O(\epsilon^3) \right\}$$

s.t.

$$\kappa x_t + \beta \pi_{t+1} = \pi_t$$

$$\beta^{-1}(-(1 - \beta(1 - \delta))k_{ex}x_0 - \pi_{t0} + \delta k_{ecz}b_0^* - \beta(1 - \delta)\tilde{r}_0) = (1 + \beta^{-1}(1 - \beta)k_{ec})rr_0$$

$$- rr_0^{*dm}(0)$$

$$(1 - \beta(1 - \delta))k_{ex}x_t + \pi_{It} - \delta k_{ecz}(\tilde{b}_t - \tilde{b}_{t-1}) + \beta(1 - \delta)\tilde{r}_t = \tilde{r}_{t-1}$$

I will first solve the problem from $t \geq 1$ onwards,

$$W_{t \geq 1} = \max \sum \beta^t \left\{ -\frac{1}{2} \lambda x_t^2 - \frac{1}{2} \lambda \pi_{t+1}^2 + \frac{1}{2} V_{11}(-\tilde{b}_t + \beta^{-1} \tilde{b}_{t-1})^2 \right\} + t.i.p. + O(\epsilon^3)$$

s.t.

$$\kappa x_{t+s} + \beta \pi_{It+s+1} = \pi_{It+s}$$

$$(1 - \beta(1 - \delta))k_{ex}x_{t+s} - \delta k_{ecz}(\tilde{b}_t - \tilde{b}_{t-1}) + \pi_{It+s} + \beta(1 - \delta)\tilde{r}_{t+s} = \tilde{r}_{t+s-1}$$
where $\tilde{b}_0$, $\tilde{r}_0$ and $\pi_{f1}$ are given. Let $\phi_{1t}$ denote the multiplier on the first constraint and $\phi_{2t}$ denote the multiplier on the second constraint. FOC:

$$-\lambda_x x_t = -\kappa \phi_{1t} + k_{ex}(1 - \beta(1 - \delta)) \phi_{2t}$$

$$\lambda_x \pi_{f1} + \phi_{1t+1} + \phi_{2t+1} = \phi_{1t}$$

$$(1 - \delta) \phi_{2t} = \phi_{2t+1}$$

$$V_{11} \tilde{b}_{t+1} + \beta k_{ec} \delta \phi_{2t+1} = (1 + \beta^{-1}) V_{11} \tilde{b}_t - V_{11} \beta^{-1} \tilde{b}_{t-1} + \delta \phi_{2t}$$

plus the constraints. There are two unit roots inside the unit circle, a unit root root and three outside, which reflects that the bond position has a unit root. I pick $\phi_{11}$, $\phi_{21}$ and $\tilde{b}_t^*$ to kill the exploding roots. Finally, I write

$$W_{t \geq 1} = \frac{1}{2} W_{\pi \pi_{11}} + \frac{1}{2} W_{\tilde{r} \tilde{r}^2} + \frac{1}{2} W_{\tilde{b} \tilde{b}^2} + W_{\pi \pi_{11}} \tilde{r}_0 + W_{\tilde{r} \tilde{r}} \tilde{b}_0 + W_{\pi \pi_{11}} \tilde{b}_0$$

and compute these constants by using the envelope theorem and then matching coefficients. It can be shown that $W_{\pi \pi} = W_{rr} = 0$ and $W_b = \beta^{-2} V_{11}(1 - \beta)$. This reflects that whether bonds are initially higher or lower, they do not make it more or less costly to satisfy the constraint since they have a unit root. The planner does use, however, deviations to lower the cost of the promise (by tilting $\tilde{b}$ path). The remaining constants are explicit but very complicated.\(^1\)

I can then use these expressions as the continuation values at $t = 0$. The problem at $t = 0$ is given by

$$W = \max \left\{ \frac{1}{2} (1 - \beta) V_{11} \gamma_{ss} - \frac{1}{2} \lambda_x x_0^2 - \frac{1}{2} \lambda_{\pi \pi_{11}} \tilde{r}_0^2 + \frac{1}{2} W_{\pi \pi_{11}} \tilde{r}_0 + \frac{1}{2} W_{\pi \pi_{11}} \tilde{r}_0 \right\} + t.i.p. + O(\epsilon^3)$$

s.t.

$$\lambda_x x_0 + \beta \pi_{11} = \pi_{f0}$$

$$-(1 - \beta(1 - \delta)) k_{ex} x_0 - \pi_{f0} + \delta k_{ec} \tilde{b}_0 - \beta(1 - \delta) \tilde{r}_0 = \beta \{(1 + k_{ec} \beta^{-1}(1 - \beta) \bar{B}) r_{00} - r_{00}^{dm}(0)\}$$

The FOC are

$$-\lambda_x x_0 + \kappa \phi_{10} - (1 - \beta(1 - \delta) k_{ex} \phi_{20} = 0$$

$$-\lambda_{\pi \pi_{11}} \phi_{10} - \phi_{20} = 0$$

$$W_{\pi \pi_{11}} + W_{\pi \pi_{f1}} + \phi_{10} = 0$$

$$W_{\pi \pi_{11}} + W_{\pi \pi_{r0}} + (1 - \delta) \phi_{20} = 0$$

$$(V_{11} + \beta W_{bb}) \tilde{b}_0 + \delta k_{ec} \phi_{20} = 0$$

\(^1\)I find them using the symbolic toolbox in MATLAB.
plus the constraints. This is just a linear system that is easy to solve. Note that the multiplier on the second constraint contains the information on how costly it is to deviate from a demand-management policy. Solving and using the envelope theorem, I may write

\[
W = \{ \frac{1}{2} \frac{(1 - \beta)V_{11}}{1 - 2m^{-1}u^T V_{11}^{-1} \gamma_{ss}} (\bar{B}r_0 - \mathcal{T})^2 - \frac{1}{2} \beta^2 \tilde{\phi}((1 + k_{ec}\beta^{-1}(1 - \beta)\bar{B})r_0 - r_{0}^{dm}(0))^2 \} \\
+ t.i.p. + O(\varepsilon^{3})
\]

Thus, defining \( \chi = -\frac{1-2m^{-1}u^T V_{11}^{-1} \gamma_{ss}}{(1 - \beta)V_{11}} \beta^2 \tilde{\phi} > 0 \) and \( \mu = \beta^{-1}k_{ec}(1 - \beta) \), I can write the welfare function as desired,

\[
W = \frac{(1 - \beta)V_{11}}{1 - 2m^{-1}u^T V_{11}^{-1} \gamma_{ss}} \left\{ \frac{1}{2} (\bar{B}r_0 - \mathcal{T})^2 + \frac{1}{2} \chi((1 + B\bar{B})r_0 - r_{0}^{dm}(0))^2 + t.i.p. + O(\varepsilon^{3}) \right\}
\]

**Without savings taxes (with portfolio taxes)**

Let \( \tilde{c}_t = \beta^{-1}b_t - \tilde{b}_{t-1} \). I have

\[
W = \max \left\{ \frac{1}{2} \frac{(1 - \beta)V_{11}}{1 - 2m^{-1}u^T V_{11}^{-1} \gamma_{ss}} (\bar{B}r_0 - \mathcal{T})^2 + \frac{1}{2} V_{11} \tilde{b}_0^2 - \frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_\pi \pi_t^2 \\
+ \sum \beta^t \left\{ -\frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_\pi \pi_t^2 + \frac{1}{2} V_{11} \tilde{c}_t^2 \right\} + t.i.p. + O(\varepsilon^{3}) \right\}
\]

s.t.

\[
\kappa x_t + \beta \pi_{It+1} = \pi_t \\
\beta^{-1}(-k_{ex}x_0 - \pi_{f0} - k_{ec}\tilde{c}_{Tt} + \beta(1 - \delta)(r_0 - t_0^{dm}(0)) = (1 + \mu B)r_0 - r_{0}^{dm}(0) \\
k_{ex} \Delta x_t + \pi_{It} + k_{ec}\Delta \tilde{c}_{Tt} + \beta(1 - \delta)(r_t - r_{t}^{dm}(0)) = (r_{t-1} - r_{t-1}^{dm}(0)) \\
k_{ux} \Delta x_{t+1} + k_{uc} \Delta \tilde{c}_{Tt+1} = 0
\]

where

\[
k_{ux} \equiv F F_{Y}^{-1}(u_{TN} F_{Y} + u_{TL}) \\
k_{uc} \equiv V_{11} C_{Tss}.
\]

Note the last constraint implies

\[
k_{uc} \tilde{c}_{Tt} + k_{ux} x_t = \nu \ \forall t
\]

where \( \nu \) is a choice variable. Using this insight, I rewrite the problem as

\[
W = \max \left\{ \frac{1}{2} \frac{(1 - \beta)V_{11}}{1 - 2m^{-1}u^T V_{11}^{-1} \gamma_{ss}} (\bar{B}r_0 - \mathcal{T})^2 + \frac{1}{2} V_{11} \tilde{b}_0^2 - \frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_\pi \pi_t^2 \\
+ \sum \beta^t \left\{ -\frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_\pi \pi_t^2 + \frac{1}{2} V_{11} k_{uc}^2 (\nu - k_{ux} x_t)^2 \right\} + t.i.p. + O(\varepsilon^{3}) \right\}
\]

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\[ \kappa x_t + \beta \pi t_{t+1} = \pi t \]
\[ \beta^{-1}(- (k_{ex} - k_{ec} k_{uc}^{-1} k_{ux}) x_0 \]
\[ - \pi_{t0} - k_{ec} k_{uc}^{-1} \nu - \beta (1 - \delta)(r_0 - r_{t0}^{dm}(0)) = (1 + \mu \bar{B}) r_0 - r_{t0}^{dm}(0) \]
\[ (k_{ex} - k_{ec} k_{uc}^{-1} k_{ux}) \Delta x_t + \pi t_{t} + \beta (1 - \delta)(r_t - r_{t}^{dm}(0)) = r_{t-1} - r_{t-1}^{dm}(0) \]

Next define \( \tilde{r}_t = r_t - r_t^{dm}(0) - (k_{ex} - k_{ec} k_{uc}^{-1} k_{ux}) x_t \). Thus,
\[ \kappa x_t + \beta \pi t_{t+1} = \pi t \]
\[ \beta^{-1}(- (1 - \beta(1 - \delta))(k_{ex} - k_{ec} k_{uc}^{-1} k_{ux}) x_0 \]
\[ - \pi_{t0} - k_{ec} k_{uc}^{-1} \nu - \beta (1 - \delta) \tilde{r}_0 = (1 + \mu \bar{B}) r_0 - r_{t0}^{dm}(0) \]
\[ (1 - \beta(1 - \delta))(k_{ex} - k_{ec} k_{uc}^{-1} k_{ux}) x_t + \pi t_{t} + \beta (1 - \delta) \tilde{r}_t = \tilde{r}_{t-1} \]

The continuation problem is given by
\[ W_{t \geq 1} = \max \{ \sum \beta^t \{- \frac{1}{2} \lambda x_t^2 - \frac{1}{2} \lambda \pi t_{t}^2 + \frac{1}{2} V_{11} k_{uc}^{-2} (\nu - k_{ux} x_t)^2 \} + t.i.p. + O(\epsilon^3) \]

s.t.
\[ \kappa x_t + \beta \pi t_{t+1} = \pi t \]
\[ (1 - \beta(1 - \delta))(k_{ex} - k_{ec} k_{uc}^{-1} k_{ux}) x_t + \pi t_{t} + \beta (1 - \delta) \tilde{r}_t = \tilde{r}_{t-1} \]
\[ \nu_{t+1} = \nu_t \]

with \( (\pi t, \tilde{r}_{t-1}, \nu_t) \) given. The FOC yield
\[ -(\lambda_x - V_{11} k_{uc}^{-2} k_{ux}^{-1}) x_t - V_{11} k_{uc}^{-2} k_{ux}^{-1} \nu_t + \kappa \phi t_{t} - (1 - \beta(1 - \delta))(k_{ex} + k_{ec} k_{uc}^{-1} k_{ux}) \phi t_t = 0 \]
\[ \lambda t \pi t_{t+1} + \phi t_{t+1} - \phi t_{t+1} = 0 \]
\[ (1 - \delta) \phi t_t - \phi t_{t+1} = 0 \]

I solve by picking \( (\phi_{t0}, \phi_{20}) \) to kill the exploding roots. Finally, I write
\[ W_{t \geq 1} = \frac{1}{2} W_{\pi \pi}(\pi t_{1})^2 + W_{\nu \nu}(\pi t_{1})^2 + \frac{1}{2} W_{\nu \nu} \nu^2. \]

To find these constants, let \( s_t = [\pi t; \nu_t] \) and write \( x_t = [x_t; \pi t; \nu_t] = k[\pi t; \tilde{r}_{t-1}; \nu_t] \) and \( s_{t+1} = V_s s_t \).
\[ s_t \big| W_{t \geq 1} = \sum \beta^t s_{t}^t (V_{s}^t)^t k' A_w k V_{s}^t s_{t} \]
where $A_w$ is a negative definite matrix that represents the previous loss function. Solving,

$$vec(W_{t \geq 1}) = (I - \beta(V_0)' \otimes (V_0)')^{-1} vec(k' A_w k).$$

Next, note that using the relation

$$b_t = \beta^{-1} b_{t-1} + k_u^{-1} k_u x_t - k_u^{-1} \nu_t$$

I find $b_0^*$ that is consistent with the promised $\pi_0$, $\nu_0$ and the transversality condition. To do so, write this as a system,

$$[b_t; \pi_{It+1}; \tilde{\pi}_t; \nu_{t+1}] = K[b_{t-1}; \pi_{It}; \tilde{\pi}_{t-1}; \nu_t].$$

and find $b_0^*$ to kill the exploding root. This yields

$$b_0^* = k_{br} \tilde{r}_0 + k_{bu} \nu_1 + k_{b\pi} \pi_{I1}.$$

At $t = 0$,

$$W = \max\left\{ \frac{(1-\beta)V_{11}}{2 - 2m^{-1} u^T V_{11}^{-1} \gamma_{ss}} (\tilde{B} r_{01} - \gamma)^2 + \frac{1}{2} V_{11} \tilde{b}_0 - \frac{1}{2} \lambda x x_0 - \frac{1}{2} \lambda \pi \pi_0^2 + \frac{1}{2} \beta W_{\pi \pi I1} \tilde{r}_0 + \beta W_{\pi \nu} \pi_{I1} \nu + \frac{1}{2} \beta W_{\pi \tau} \tilde{r}_0^2 + \beta W_{\tau \nu} \nu_0 \nu + \frac{1}{2} \beta W_{\nu \nu} \nu^2 + t.i.p. + O(\epsilon^3) \right\}$$

s.t.

$$\begin{align*}
\kappa x_0 + \beta \pi_{I1} &= \pi_{I0} \\
\beta^{-1}\{- (1 - \beta(1-\delta))(k_{ex} - k_{ec} k^{-1} u^T k_u) x_0 \\
- \pi_{I0} - k_{ec} k_{uc}^{-1} \nu - \beta(1-\delta) \tilde{r}_0} &= (1 + \mu \tilde{B}) r_{01} - rr_0^{dm}(0) \\
- k_{uc} \tilde{b}_0 + k_u x_0 &= \nu \\
k_{br} \tilde{r}_0 + k_{bu} \nu_1 + k_{b\pi} \pi_{I1} &= b_0^* 
\end{align*}$$

FOC,

$$\begin{align*}
-\lambda x x_0 + \kappa \phi_1 - (1 - \beta(1-\delta))(k_{ex} - k_{ec} k^{-1} u^T k_u) \phi_2 + k_{ux} \phi_3 &= 0 \\
-\lambda \pi \pi_0 - \phi_1 - \phi_2 &= 0 \\
W_{\pi \pi I1} + W_{\pi \tau} \tilde{r}_0 + W_{\pi \nu} \nu + \phi_1 + k_{b\pi} \phi_4 &= 0 \\
W_{\pi \tau} \pi_{I1} + W_{\tau \tau} \tilde{r}_0 + W_{\tau \nu} \nu + (1-\delta) \phi_2 + k_{b\pi} \phi_4 &= 0 \\
W_{\pi \nu} \pi_{I1} + W_{\tau \nu} \tilde{r}_0 + W_{\nu \nu} \nu - \beta^{-1} k_{ec} k_{uc} \phi_2 - \beta^{-1} \phi_3 + k_{bu} \phi_4 &= 0 \\
V_{11} \tilde{b}_0 - k_{uc} \phi_3 + \beta^{-1} \phi_4 &= 0
\end{align*}$$

This is just a linear system that is easy to solve. Note that the multiplier on the second constraint contains the information on how costly it is to deviate from a demand-management policy. Solving
and using the envelope theorem, I may write

$$\mathcal{W} = \{ \frac{1}{2} (1 - \beta) V_{11}(\tilde{B} r r_0 - \mathcal{T})^2 - \frac{1}{2} \beta^2 \tilde{\phi} ((1 + k_{ec} \beta^{-1}(1 - \beta) \tilde{B}) r r_0 - r r_0^{dm}(0))^2 \} + t.i.p. + O(\epsilon^3)$$

Thus, defining $\chi = -\frac{1 - 2m^{-1} u T V_{11}^{-1} \gamma_{ss}^* \beta^2 \tilde{\phi}}{(1 - \beta) V_{11}} > 0$ and $\mu = \beta^{-1} k_{ec}(1 - \beta)$, I can write the welfare function as desired,

$$\mathcal{W} = \frac{(1 - \beta) V_{11}}{1 - 2m^{-1} u T V_{11}^{-1} \gamma_{ss}^*} \mathbb{E} \{ \frac{1}{2} (\tilde{B} r r_0 - \mathcal{T})^2 + \frac{1}{2} \chi ((1 + \mu \tilde{B}) r r_0 - r r_0^{dm}(0))^2 \} + t.i.p. + O(\epsilon^3)$$

Note the only difference is that the multiplier $\tilde{\phi}$ will now be larger because it is a more restricted problem.

**Proposition 1.6**

Propositions 1.2 and 1.4 are trivial given Lemma 1.4. Next, I prove the generalization of proposition 1.5.

Combining the home and foreign Eulers to second order,

$$\mathbb{E}[(\lambda_0 + \gamma_{ss}^* y_0^* - (1 - \beta)m^{-1} \tilde{B} r r_0) r r_0] = \tau^B + O(\epsilon^3). \quad (A.34)$$

where $\lambda_0$ is still the multiplier on the budget constraint.

In the dynamic model private marginal utility still satisfies

$$u_T t_0 + V_{11} C_{T ss} c_{T 0} + V_{12} Z_{ss} z_0 + F F_Y^{-1}(u_T N F_Y + u_T L) x_0 = u_T \lambda_0 + O(\epsilon^2)$$

Using the budget constraint,

$$u_T t_0 + V_{11} (Y_{T ss} y_0 + \tilde{B} r r_0 - b_0^R) + V_{12} Z_{ss} z_0 + F F_Y^{-1}(u_T N F_Y + u_T L) x_0 = u_T \lambda_0 + O(\epsilon^2)$$

Using the solution from the “relaxed” problem discussed above, I can rewrite this as

$$u_T t_0 + V_{11} (Y_{T ss} y_0 + (1 - \beta) \tilde{B} r r_0 - b_0^{R'}) - V_{11} \tilde{b}_0 + F F_Y^{-1}(u_T N F_Y + u_T L) x_0 = u_T \lambda_0 + O(\epsilon^2),$$

Using the solution for $b_0^{R'}$, I obtain,

$$(1 - \beta) V_{11}(\tilde{B} r r_0 - \mathcal{T}_0) - V_{11} \tilde{b}_0 + F F_Y^{-1}(u_T N F_Y + u_T L) x_0 = u_T \lambda_0 + u_T \gamma_{ss}^* y_0^* + O(\epsilon^2)$$

Regardless of whether there are capital controls or not, I know from solving the problem in proposition 1.4, that I can write

$$\tilde{b}_0 = \tilde{k}_b \{(1 + \mu \tilde{B}) r r_s - r r_s^{dm}(0)\} + O(\epsilon^2)$$

$$x_0 = \tilde{k}_x \{(1 + \mu \tilde{B}) r r_s - r r_s^{dm}(0)\} + O(\epsilon^2)$$
for some constants $\tilde{k}_b$ and $\tilde{k}_x$, so
\[(1 - \beta)V_{11}(\bar{B}rr_s - \tau_s^\infty) + \tilde{k}_{rr}\{(1 + \mu\bar{B})rr_s - rr_s^{dm}(0)\} = u_T \lambda_0 + u_T y_s^0 + O(\epsilon^2)\]
for some constant $\tilde{k}_{rr}$. I also know that the planner picks $rr_s$ to solve
\[-\bar{B}V_{11}(\bar{B}rr_s - \tau_s^\infty) + 2m^{-1}\gamma_{ss}^*rr_s\bar{B}^2 + (1 + \mu\bar{B})V_{11}\chi_s^\infty((1 + \mu\bar{B})e_s - e_s(0)) = O(\epsilon^2)\]
Then,
\[\tilde{k}'\left(-V_{11}(\bar{B}e_s + \tau_s^\infty) - 2u_Tm^{-1}\gamma_{ss}^*e_s\bar{B}\right) = u_T(\lambda_s + \gamma_{ss}^*y_s^* - 2(1 - \beta)m^{-1}\gamma_{ss}^*rr_s\bar{B}) + O(\epsilon^2)\]
for some constant $\tilde{k}'$. The planners' FOC implies
\[\bar{r}_{rrs}\{V_{11}(\bar{B}e_s - \tau_s) + 2m^{-1}\gamma_{ss}^*\bar{B}rr_s\} = O(\epsilon^3)\]
so
\[\tilde{r}^B = (1 - \beta)m^{-1}\gamma_{ss}^*\bar{B}\bar{r}_{rr} + O(\epsilon^3)\]
Note that if foreigners were “short-lived” the $1 - \beta$ would vanish. In that case, their Euler equation would look exactly the same as in the static model. The fact that they can save lowers their risk aversion by $(1 - \beta)^{-1}$.

**Proposition 1.7**

**Rigid prices and long bonds** With rigid prices, the continuation problem is:

\[
\mathcal{W} = \max_{\{b_t, x_t, \pi_t\}} \left\{ \frac{1}{2} \frac{(1 - \beta)V_{11}(\bar{B}rr_0 - \tau_0)^2}{2 - 1 - 2m^{-1}u_1V_{11}^{-1}\gamma_{ss}^* - 1} \right\} + \frac{1}{2} V_{11}\tilde{v}_0^2 - \frac{1}{2} \lambda_x x_t^2 + \sum \beta^t\left\{ -\frac{1}{2} \lambda_x x_t^2 + \frac{1}{2} V_{11}(-\bar{b}_t + \beta^{-1}\bar{b}_{t-1})^2 \right\} + t.i.p. + O(\epsilon^3)
\]
s.t.
\[
\beta^{-1}(-(1 - \beta(1 - \delta))k_{ex}x_0 + \delta k_{ec}\bar{b}_0^* - \beta(1 - \delta)\bar{r}_0) = (1 + \beta^{-1}(1 - \beta)k_{ec})rr_0 - rr_0^{dm}(0)
\]
\[(1 - \beta(1 - \delta))k_{ex}x_t - \delta k_{ec}(\bar{b}_t - \bar{b}_{t-1}) + \beta(1 - \delta)\bar{r}_t = \bar{r}_{t-1}.
\]
From $t \geq 1$ onwards,

\[
\mathcal{W} = \max_{\{b_t, x_t, \pi_t\}} \left\{ \sum \beta^t\left\{ -\frac{1}{2} \lambda_x x_t^2 + \frac{1}{2} V_{11}(-\bar{b}_t + \beta^{-1}\bar{b}_{t-1})^2 \right\} + t.i.p. + O(\epsilon^3)
\]
The solution is

\[
\begin{align*}
\tilde{b}_t &= \tilde{b}_{t-1} - \beta \delta k_{ec} V_{11}^{-1} \tilde{K}_0 \tilde{v}_{t-1} \\
\phi_1 &= \tilde{K}_0 \tilde{v}_{t-1} \\
x_t &= -\lambda_x^{-1}(1 - \beta(1 - \delta))k_{ex} \tilde{K}_0 \tilde{v}_{t-1} \\
\tilde{v}_t &= (1 - \delta)\tilde{v}_{t-1}
\end{align*}
\]

where

\[
\tilde{K}_0 = k_{ex}^{-2} \lambda_x \frac{(1 - \beta(1 - \delta)^2)}{(1 - \beta(1 - \delta)^2 - \lambda_x k_{ec}^2 V_{11}^{-1} \beta \delta^2 k_{ex}^2)} > 0.
\]

Since there is no inflation, one may write, for \(t \geq 1\)

\[
\tilde{e}_t = k_{ex} x_t + k_{ec}(\beta^{-1} \tilde{b}_{t-1} - \tilde{b}_t) + O(\varepsilon^2)
\]

Replacing,

\[
\tilde{e}_t = (\lambda_x^{-1} k_{ex}^{-2} - k_{ec}^2 V_{11}^{-1})(1 - \beta(1 - \delta))\tilde{K}_0 \tilde{v}_{t-1} + (1 - \beta)k_{ex}^2 V_{11}^{-1} \tilde{K}_0 \tilde{v}_0.
\]

Noting the problem at \(t = 0\) is the same with \(rr_0 - \frac{1}{1 + \mu \bar{B}} rr_0^{dm}(0)\) instead of \(-\tilde{r}_0\),

\[
\tilde{e}_t = -(\lambda_x^{-1} k_{ex}^{-2} - k_{ec}^2 V_{11}^{-1})(1 - \beta(1 - \delta))\tilde{K}_0 (1 - \delta) \{ (1 + \mu \bar{B})rr_0 - rr_0^{dm}(0) \} + (1 - \beta)k_{ex}^2 V_{11}^{-1} \tilde{K}_0 \{(1 + \mu \bar{B})rr_0 - rr_0^{dm}(0) \}.
\]

Defining \(\tilde{k}_1 = (\lambda_x^{-1} k_{ex}^{-2} - k_{ec}^2 V_{11}^{-1})(1 - \beta(1 - \delta))\tilde{K}_0 > 0\) and \(\tilde{k}_2 = -(1 - \beta)k_{ex}^2 V_{11}^{-1} \tilde{K}_0 \geq 0\). Note \(\mu = 0\) if and only if \(k_{ec} = 0\). Finally, note at \(t = 0\), \(\tilde{e}_t\) can be rewritten as

\[
\tilde{e}_0 = -(\lambda_x^{-1} k_{ex}^{-2} (1 - \beta(1 - \delta)) - \beta \delta k_{ec}^2 V_{11}^{-1}) \tilde{K}_0 \{(1 + \mu \bar{B})rr_0 - rr_0^{dm}(0) \} + O(\varepsilon^2),
\]

so \(\tilde{e}_0\) has the same sign as \(-\{(1 + \mu \bar{B})rr_0 - rr_0^{dm}(0)\}\).

**Calvo and short bonds** The problem is

\[
\mathcal{W} = \max_{\{b_t, x_t, \pi_t\}} \left\{ \frac{1}{2} \frac{(1 - \beta)V_{11}}{2 - 2m^{-1}u_t V_{11}^{-1} \gamma s} (\bar{B}rr_0 - T_s)^2 + \frac{1}{2} V_{11} \tilde{b}_0^2 - \frac{1}{2} \lambda x_0^2 - \frac{1}{2} \lambda \pi t_0^2 \\
+ \sum \beta^t \{- \frac{1}{2} \lambda x_t^2 - \frac{1}{2} \lambda \pi t_1^2 + \frac{1}{2} V_{11} (-\tilde{b}_t + \beta^{-1} \tilde{b}_{t-1})^2 \} + t.i.p. + O(\varepsilon^3) \right\}
\]

s.t.

\[
\begin{align*}
\beta^{-1}(-k_{ex} x_0 + k_{ec} \tilde{x}_0) &= (1 + \beta^{-1}(1 - \beta)k_{ec})rr_0 - rr_0^{dm}(0) \\
\beta \pi_{t+1} + \kappa x_t &= \pi_t
\end{align*}
\]

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The continuation problem is a standard disinflation problem (clearly $\tilde{b}_t = \tilde{b}_{t-1}$ is the solution for savings part):

$$W_{t \geq 1} = \sum \beta^t \left\{ -\frac{1}{2} \lambda_t x_t^2 - \frac{1}{2} \lambda_t \pi_t^2 \right\} + t.i.p. + O(\epsilon^3)$$

s.t.

$$\beta \pi_{t+1} + \kappa x_t = \pi_t$$

Solving this,

$$x_t = \kappa \lambda_t^{-1} \phi \pi_t$$

$$\pi_{t+1} = \frac{1}{\beta} \left( 1 - \kappa^2 \lambda_t^{-1} \phi \right) \pi_t$$

Note $1 - \kappa^2 \lambda_t^{-1} \phi < \beta$. Thus, from $t \geq 2$ onwards,

$$\Delta \tilde{e}_t = \pi_{It} + k_{ex} \Delta x_t$$

$$= \mathcal{P}(k_{ex}) \pi_{It-1}.$$  

where

$$\mathcal{P}(k_{ex}) = (1 - k_{ex} \kappa \lambda_t^{-1} \phi) \frac{1}{\beta} (1 - \kappa^2 \lambda_t^{-1} \phi) + k_{ex} \kappa \lambda_t^{-1} \phi$$

This is clearly increasing in $k_{ex}$ with $\mathcal{P}(0) = 1$ and $\mathcal{P}(\infty) < 0$.

The problem at 0 is given by

$$W = \max_{\{b_t, x_t, \pi_t\}} \left\{ \frac{1}{2} \frac{(1 - \beta)V_{11}}{1 - 2 \lambda_t^{-1} u_1 V_{11}^{-1} \gamma_{\pi}} (\bar{B} r_{t} - \tau_0)^2 + \frac{1}{2} \beta^{-1} V_{11} \bar{b}_0^2 - \frac{1}{2} \lambda_t x_0^2 - \frac{1}{2} \lambda_t \pi_t^2 \right\} + t.i.p. + O(\epsilon^3)$$

s.t.

$$\beta \pi_{I1} + \kappa x_0 = \pi_0$$

$$(-k_{ex} x_0 - \pi_{I0} + k_{ex} \bar{b}_0^*) = \beta \{(1 + \beta^{-1}(1 - \beta)k_{ex})r_{t0} - r_{t0}^{\text{cm}}(0)\}$$
Let \( \nu \) denote the multiplier on the promise-keeping constraint (i.e., the second one). The solution is:

\[
\begin{align*}
\pi_{t1} &= \frac{\kappa \lambda_x^{-1} k_{ex} - \lambda_x^{-1}}{\beta \phi^{-1} + \kappa^2 \lambda_x^{-1} + \lambda_x^{-1}} \nu_0 \\
x_0 &= \frac{\kappa \lambda_x^{-1} + k_{ex}(\beta \phi^{-1} + \lambda_x^{-1})}{\lambda_x(\beta \phi^{-1} + \kappa^2 \lambda_x^{-1} + \lambda_x^{-1})} \nu_0 \\
\pi_{t0} &= \frac{(\kappa \lambda_x^{-1} k_{ex} + \beta \phi^{-1} + \kappa^2 \lambda_x^{-1})}{\lambda_x(\beta \phi^{-1} + \kappa^2 \lambda_x^{-1} + \lambda_x^{-1})} \nu_0 
\end{align*}
\]

Note at \( t = 0, \) if \( \nu > 0, \) I get \( x_0 < 0 \) and \( \pi_{t0} < 0 \) unambiguously when \( k_{ex} > 0. \) This implies \( \hat{\ell}_0 < 0. \) Depending on how large \( k_{ex} \) is, the continuation may feature inflation or deflation.

At \( t = 1, \)

\[\Delta \hat{\ell}_1 = \pi_{t1} + k_{ex} \Delta x_1\]

If \( k_{ex} \) is large, \( \pi_{t1} > 0 \) and \( x_1 > 0. \) Thus, \( \Delta \hat{\ell}_1 > 0. \) If \( k_{ex} \) is small, \( \pi_{t0} < 0 \) and \( x_1 < 0. \)

**Proposition 1.8**

**Rigid prices and long bonds**  Savings taxes from \( t \geq 1, \) are

\[
\tau_{B^*t} = -V_{11}(\Delta \tilde{\ell}_{t+1} - \Delta \tilde{\ell}_t) + k_{ux} \Delta x_{t+1} \\
= -\beta \delta k_{ec} \tilde{K}_0(\Delta \tilde{\ell}_t) - k_{ux} \lambda_x^{-1}(1 - \beta(1 - \delta))k_{ex} \tilde{K}_0(\Delta \tilde{\ell}_t) \\
= (\beta \delta k_{ec} - \lambda_x^{-1} k_{ux}(1 - \beta(1 - \delta))k_{ex}) \tilde{K}_0 \delta \tilde{\ell}_{t-1}
\]

Thus, it decays at rate \( 1 - \delta. \) Since, \( \tilde{\ell}_t = (1 - \delta) \tilde{\ell}_{t-1}, \)

\[\tau_{B^*t} = (1 - \delta)^t \tau_{B^1}\]

At \( t = 0 \) it’s the same (there’s no asymmetry). Thus,

\[
\tau_{B^*0} = -\tilde{K}_0 \beta \delta(\beta \delta k_{ec} - k_{ex} k_{ux} \lambda_x^{-1})\{(1 + \mu)rr_0 - rr_{0}^{dm}(0)\}
\]

Since \( \tilde{K}_0 < 0, \) and usually \( k_{ex} > 0, k_{ux} > 0, k_{ec} < 0, \) it has the opposite sign of \( \{(1 + \mu)rr_0 - rr_{0}^{dm}(0)\}, \) i.e, if she creates a positive excess return, she taxes savings.

**Calvo pricing and short bonds**  Savings tax (\( t \geq 1):\)

\[k_{ux} \Delta x_{t+1} = \tau_{B^*t}\]

\[k_{ux} \left( \frac{1}{\beta}(1 - \kappa^2 \lambda_x^{-1}) - 1 \right) \pi_{It} = \tau_{B^*t}\]

So, if the economy is experiencing disinflation (\( \pi_{It} > 0, \) \( \Delta x < 0 \) and \( \tau_{B^*} < 0 \) (subsidize savings).
At $t = 0$, the savings tax is given by
\[
\tau_{B*} = V_1 b_0^* + k_{ux} \Delta x_{t+1}
\]
and
\[
b_0^* = -\beta V_1^{-1} \nu_0 k_{cc},
\]
which has the same sign as $\mu$.

## A.2 Appendix: Extensions

### A.2.1 Cooperation

In the previous sections, the utility of the home household was the relevant welfare metric. As a result, the planner found it optimal to manipulate the stochastic discount factor of the foreigners that participate in home-currency bond markets to redistribute wealth to home agents. In this section, I revisit the optimal policy from the point of view of a supranational authority, which would also consider foreigners’ welfare. More specifically, the welfare metric is now

\[
E\left(\lambda^H u(C_{Ts}, C_{Ns}, L_s, \xi_s) + m\lambda^F u^*(C_{Ts}, \xi_s)\right)
\]

where $\lambda^H$ and $\lambda^F$ are fixed at their steady state values, implying the supranational authority does not seek to redistribute wealth ex ante.

**Proposition A.1.** \((\text{Cooperation})\) Let $\chi^{nc}(m)$ and $\mathcal{T}^{nc}(m)$ denote the parameter $\chi$ and the desired transfers in the noncooperative solution when the measure of foreigners is $m$. Then, in the cooperative solution,

\[
\chi^{coop}(m) = \chi^{nc}(2m) \quad \mathcal{T}^{coop}(m) = \mathcal{T}^{nc}(2m).
\]

In other words, it is as if the model had twice the number of foreigners. Furthermore, the optimal capital control \(i.e.,\) the portfolio tax is zero to second-order,

\[
\tau_B = O(\epsilon^3).
\]

Proposition A.1 shows the connection with the decentralized solution. Since the planner now internalizes the benefits of insuring foreigners, she will desire larger transfers $\mathcal{T}$. In addition, since she realizes that the additional risk-premium from insurance increases foreigners welfare, she will be less afraid to allow the exchange rate to float given a position $B$, so $\chi$ increases. Noting that the cooperative solution is analogous to increasing $m$, proposition 1.3 and corollary 1.2 immediately imply that positions $|B|$ and the insurance weight $\omega$ are larger in the cooperative solution. Finally,
since the global planner is not trying to redistribute wealth in expectation, there is no need for capital controls.

**Corollary A.1.** Under the same conditions as in proposition 1.3, the portfolio $|B|$ and the insurance weight $\omega$ are larger under cooperation (if $m < \infty$).

**Proof of Proposition A.2.1**

I prove the result for the static model to simplify the exposition, but it is immediate that the result extends to the dynamic model. Expanding the welfare loss function, we obtain the following new term,

$$\text{new term} = \lambda_F m \left( u_1^* C^* s^* + \frac{1}{2} u_1^{11} C^* s^2 + u_1^* C^* s c_s^2 \right) + O(\epsilon^3)$$

A second-order approximation to foreigners’ budget constraint yields

$$C^* s^2 + u_1^* C^* s^2 = Y_s^* y_s^* + \frac{1}{2} Y_s^* y_s^2 - m^{-1} \bar{B} \bar{r} s - m^{-1} \bar{B}^2 \bar{r}^2 - m^{-1} B_c r s$$

Replacing,

$$\text{new term} = m u_T \left( Y_s^* y_s^* + \frac{1}{2} Y_s^* y_s^2 - m^{-1} \bar{B} \bar{r} s - m^{-1} \bar{B}^2 \bar{r}^2 - \frac{1}{2} \gamma_{ss}^* C^* s^2 \right) + O(\epsilon^3)$$

where $\gamma_{ss}^* \equiv -u_1^{*-1} u_{11}^*$. Using a first-order approximation of foreigners’ budget constraint to get rid of $c_s^2$,

$$\text{new term} = u_T \left( -\bar{B} \bar{r} s - \bar{B}^2 \bar{r}^2 + \gamma_{ss}^* Y_s^* y_s^* \bar{B} \bar{r} s - \frac{1}{2} \gamma_{ss}^* m^{-1} \bar{B}^2 \bar{r}^2 \right) + O(\epsilon^3)$$

Thus, the new welfare loss function is given by

$$EV = E[u_T \bar{B} (r - e_s) y_s^* + \frac{1}{2} u_T (\gamma_{ss}^*/m) \bar{B}^2 (r - e_s)^2 + u_T T_s c_s T_s$$

$$+ \frac{1}{2} V_{11} C^2_T s^2 \bar{c}_T s + V_{12} Z_s C_T s \bar{c}_T s - \frac{1}{2} \lambda z^2 - \frac{1}{2} \lambda_{\pi} x^2 + t.i.p. + O(\epsilon^3)$$

The only difference is the $\frac{1}{2}$ in front of the term $u_T (\gamma_{ss}^*/m) \bar{B}^2 (r - e_s)^2$. Thus, the model is isomorphic to a model with having twice as many foreigners in the decentralized solution. There is one difference, however: the private utility still has the original $m$. It follows immediately from the proof of propositions 1.5 and 1.6 that $\tau_B = O(\epsilon^3)$ regardless of $m$.

**A.2.2 Model with equity in nontradables**

In this Appendix, I consider an extension of the example economy in Section (1.3.5) that allows for non-unitary elasticity of substitution between tradables and nontradables, as well as decreasing
returns to scale in nontradable production:

\[ u(s) = \ln\left((\alpha \frac{1}{\rho} C_{T_s} + (1 - \alpha) \frac{1}{\rho} C_{N_s})^{\frac{\rho}{1 - \rho}} - \frac{1 - \alpha}{1 + \psi} N_s^{1+\psi}\right) \]

\[ F = Z_s Y_{I_s}^{\alpha_F} \]

Dividends on nontradable firms are then given by,

\[ D_s = (1 - \alpha_F) E_s^{-1} P_{N_s} Z_s Y_{I_s}^{\alpha_F} \]

To a first-order approximation,

\[ div_s = (1 - \rho^{-1}) x_s + (z_s + \alpha_F y_{flex}^{I_s} + p_{N_s}^{flex}) \]

The elasticity of substitution is the key parameter governing the dependence of equity on monetary policy. On the one hand, a depreciation boosts employment, which increases profits. On the other hand, it also increases the supply of nontradables goods, depressing their price. If tradables and nontradables are complements, the price decreases significantly, and profits (in foreign currency) decrease, as in Ottonello (2015). If they are substitutes, the opposite is true: profits increase with a depreciation. In the knife-edge of \( \rho = 1 \), equity is independent from monetary policy and thus, absent nominal bonds, the optimal policy would be demand-management (i.e., inflation targeting). Note that, given that countries are short their own equity, and typically are debtors in foreign currency, the total exposure will be larger with \( \rho < 1 \) since in that case, the return on both assets moves in the same direction with monetary policy.

Replacing the flexible price allocation \( y_{flex}^{I_s} \),

\[ div_s = (1 - \rho^{-1}) x_s + \frac{1}{\alpha_F \alpha + \rho (\psi + 1 - \alpha_F)} \{(\rho - 1)(\psi + 1) z_s + (\alpha_F \alpha + (\psi + 1 - \alpha_F)) c_{T_s}\} \]

A similar argument that the one given for monetary policy explains that the effect of \( z_s \) on profits depends on \( \rho \). Regardless of \( \rho \), higher tradable consumption pushes up demand for nontradable goods, unambiguously increasing profits.

Assume the return on equity is the dividend plus noise,

\[ r_{eq}^{eq} = div_s + \nu_s \]

and write

\[ r_{eq}^{eq} = k_{eq}^{\nu} x_s - \mu^{eq} c_{T_s} + k_{eq}^{\xi} \xi_s \]

where \( \xi_s = [z_s, \nu_s] \) collects the shocks.²

²Noise is introduced to prevent the planner from approximating the first-best arbitrarily closely when \( \rho = 1 \).
The welfare function is still given by

\[
EV = E_0 \left[ \frac{1}{2} (1 - 2m^{-1} V_{11}^{-1} \gamma_s) V_{11} (B^{eq} r_{eq} + \tau_s)^2 - \frac{1}{2} (\lambda_\pi + \lambda_\pi \kappa^2) x_s^2 \right] + O(\epsilon^3) + t.i.p. \]  

(A.35)

Replacing,

\[
EV = -k_0 E_0 \left[ \frac{1}{2} (B r_{rs} - \tau_s)^2 + \chi (1 + \mu^{eq} B)^2 (r_{rs} - \frac{1}{1 + \mu^{eq} B} r_{rs}^{dm}(0))^2 \right] + O(\epsilon^3) + t.i.p.
\]

where

\[
\chi = \frac{\lambda_\pi + \kappa^2 \lambda_\pi}{k_{rrx} V_{11}} = \frac{\lambda_\pi + \kappa^2 \lambda_\pi}{(1 - \rho^{-1})^2 V_{11}}.
\]

Using that \( \kappa = \tilde{k} (1 - \tilde{\phi}) \) for some positive constant \( \tilde{k} > 0 \), it is immediate that price flexibility decreases the ability of monetary policy to provide insurance when the economy features real assets (indeed, \( \chi \to \infty \) when \( \phi \to 0 \)). By contrast, with nominal bonds, we had \( k_{rrx} = -k_{ex} - \kappa \). The \( \kappa^2 \) in the denominator that appears in that case is what formally drives the difference in the results. The next remark summarizes this “real-asset” case.

**Remark A.1.** In an economy where no asset loads on nominal quantities, i.e., in an economy with only real assets, higher price flexibility decreases the importance of the insurance motive (i.e., \( \chi \) increases).

### A.2.3 General asset structure: Static model

The reader may wonder at this point whether the results are driven by the very special asset structure considered in the previous sections. To address this concern, I now consider a general asset structure. To save notation, I assume there is still a risk-free asset in foreign-currency with a return normalized to one. Let \( \Theta \in \mathbb{R}^N \) denote the remaining \( N \) assets in the economy. I allow these assets’ returns to depend on any of the equilibrium variables and the shocks. After log-linearizing, the excess-return can be written in reduced form as

\[
r_{rs} = k_{rrx} x_s - \mu c_{T_s} + k_{rrx} \xi_s + O(\epsilon^2)
\]

where \( k_{rrx}, \mu \in \mathbb{R}^N \) and \( k_{rrx} \in \mathbb{R}^{N+S} \). I assume \( N < S \) so that markets are still locally incomplete. Premultiplying by \( \Theta' \), assuming \( \Theta' k_{rrx} \neq 0 \), and solving yields

\[
x_s = \frac{1 + \Theta' \mu}{\Theta' k_{rrx}} (\Theta' r_{rs} - \Theta' \frac{\Theta'}{1 + \Theta' \mu} r_{rs}^{dm}(0)). \]  

(A.36)

Although there are many assets, the planner still has a single way of deviating from demand-management - by creating output gaps. This suggests this economy will still have a “sufficient
"statistic" for the sensitivity to monetary policy. Indeed, this sufficient statistic is given by

\[ B \equiv \frac{\Theta' k_{rrx}}{1 + \Theta' \mu}. \]  

(A.37)

**Lemma A.1.** Suppose \( B \neq 0 \).\(^3\) Then, the objective function can be written as

\[ \mathbb{E}V(e, \Theta) = -k_0 \mathbb{E}_0 \left[ \frac{1}{2} (\Theta'rr_s - T_s)^2 + \frac{1}{2} \chi \frac{B}{\chi + B^2} (\Theta'rr_s - \frac{\Theta'}{1 + \Theta' \mu} rr_s B(0))^2 \right] + t.i.p + O(\epsilon^3) \]

Taking the first-order condition with respect to \( \tilde{r}r_s = \Theta'rr_s \), I find that the total return of the portfolio is still a weighted average of the "demand-management" target and the "insurance target",

\[ \tilde{r}r_s = \frac{B^2}{B^2 + \chi} (\Theta'rr_s)^{\text{in}} + \frac{\chi}{\chi + B^2} (\Theta'rr_s)^{\text{dm}} + O(\epsilon^2) \]  

(A.38)

where

\[ (\Theta'rr_s)^{\text{in}} = T_s \]

\[ (\Theta'rr_s)^{\text{dm}} = \frac{1}{1 + \Theta' \mu} \Theta'rr_s(0) \]

and \( rr_s(0) \) is the return in financial autarky. Replacing back, I obtain

\[ \mathbb{E}V(e_s, \tilde{\Theta}, B) = -\frac{1}{2} \frac{k_0 \chi}{B^2 + \chi} \mathbb{E}_0 [T_s^2 + (\tilde{\Theta}'rr_s^{\text{dm}}(0))^2 - 2T_s \tilde{\Theta}'rr_s^{\text{dm}}(0)] + t.i.p + O(\epsilon^3) \]  

(A.39)

where \( \tilde{\Theta}' \equiv (1 + \mu' \Theta)^{-1} \Theta' \) is a convenient rotation to cancel out the wealth effect.

**Proposition A.2.** The optimal portfolio \( \tilde{\Theta} \) conditional on some sensitivity to monetary policy \( B \) is given by

\[ \tilde{\Theta} = k_{\Theta_0} + k_{\Theta B} B + O(\epsilon) \]

(A.40)

where

\[ k_{\Theta_0} = \left( \Sigma_{rrdm}^{-1}(0) - \Sigma_{rrdm}^{-1}(0) k_{rrx} (k_{rrx}^{\prime} \Sigma_{rrdm}^{-1}(0) k_{rrx})^{-1} k_{rrx}^{\prime} \Sigma_{rrdm}^{-1}(0)) \right) \sigma_{rrdm}(0) \]

\[ k_{\Theta B} = \Sigma_{rrdm}^{-1}(0) k_{rrx} (k_{rrx}^{\prime} \Sigma_{rrdm}^{-1}(0) k_{rrx})^{-1}. \]

Note that the necessary transfer may now be smaller than before: Even without any exposure to monetary policy, the planner may choose the portfolio to diversify away some risk. For example, if she can sell claims to the tradable endowment, then she does not need to use monetary policy, which is a costly source of insurance, against these shocks. This is captured by \( k_{\Theta_0} \). Furthermore, even if assets are uncorrelated with the desired transfers, they may still be useful to hedge the "undesirable transfers" created by the nominal asset. This last effect is captured by \( k_{\Theta B} \). Define the

---

\(^3\)If \( B = 0 \), then feasibility implies \( \Theta'rr_s - \frac{\epsilon^r}{1 + \Theta' \mu} rr_s B(0) \). Thus, this objective function is continuous in \( B \) at \( B = 0 \).
realized excess-return of the portfolio that is sensitive to monetary policy as

$$rr_s^B = k'B^rr_s.$$  \hfill (A.41)

Then, (A.38) can be expressed in a more familiar form,

$$rr_s^B = \frac{B^2}{B^2 + \lambda}rr_s^{B,\text{in}} + \frac{\lambda}{B^2}rr_s^{B,\text{dm}}$$  \hfill (A.42)

with

$$rr_s^{B,\text{in}} = B^{-1}(T_s - k'\sigma_0^B r_s(0))$$

$$rr_s^{B,\text{dm}} = k'B^rr_s^{dm}(0)$$

Note that the linearity of the model implies all endogenous variables are weighted averages of their target values. In other words, the exchange rate is also a weighted average,

$$e_s^B = \frac{B^2}{B^2 + \lambda}e_s^{B,\text{in}} + \frac{\lambda}{B^2}e_s^{B,\text{dm}}.$$  \hfill (A.43)

Replacing (A.40),

$$\mathcal{V}(B) = \frac{1}{2}k_0^2 - B^2 \mathbb{E}[B^2 \sigma_f^2 + \lambda^2 \sigma_{rr}^2(0) + 2B\lambda \sigma_{rr}^B(0)] + t.i.p. + O(\epsilon^3)$$  \hfill (A.43)

where $\tilde{T}_s = T_s - k'\sigma_0^B r_s^{dm}(0)$. It is immediate from (A.42) and (A.43) that propositions 1.3 and 1.4 hold in this environment in terms of the exposure to monetary policy and volatility of the returns that are sensitive to it. If there is a single "exposed" asset, then it is immediate that the volatility of the excess-returns of such an asset, which may be called "home-currency" asset, is lower under the optimal policy. Finally, I show that the generality of the asset markets still does not justify a tax if $m < \infty$ or countries cooperate. Even though incomplete markets introduce pecuniary externalities, these are still proportional to the value of social utility asset-by-asset under the optimal policy, implying portfolio taxes are still not required.

**Proposition A.3.** Propositions 1.3 and 1.4 hold in terms of the "sensitivity" to monetary policy $B$ defined in (A.37) and its return defined in (A.41). In addition, if $m = \infty$ or countries cooperate, the optimal capital controls policy is

$$\tau^j = O(\epsilon^3) \forall j.$$  

**Proof of lemma A.1**

Following the same steps as in the proof of lemma 1.2 I obtain,

$$\mathbb{E}V = \mathbb{E}\left[\frac{1}{2} V_{11}(\Theta'rr_s - T_s)^2 - \frac{1}{2} \lambda_2 x_s^2\right] + t.i.p. + O(\epsilon^3)$$

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where $\tilde{\lambda}_x = \lambda_x + \kappa^{-2}\lambda_{\pi}$. Using equations (A.36) and (A.37) and replacing I obtain the desired expression.

**Proof of lemma A.2**

To prove this one needs to maximize (A.39) subject to $B = k_{rrx}'\tilde{\Theta}$. The FOC yields,

$$
\sum_{rrx} \tilde{\Theta} - \sigma_{rr}T - \lambda k_{rrx} = 0
$$

$$
k_{rrx}'\tilde{\Theta} = B
$$

Solving this using a formula for the inverse of a block matrix yields the desired result.

**Proof of proposition A.3**

I prove here the result on taxes. The remainder follow from the isomorphism of equations (A.42) and (A.43). To prove the no-tax result, take a few steps back and note that the planner maximizes

$$
U = E_0[u_T \sum_j \tilde{\Theta}_j^r r_j^s \gamma_{ss}^* y_s^* + u_T y_s C_{ss} c_{Ts} + \frac{1}{2} V_{11} C_{ss}^2 c_{Ts}^2 + V_{12} Z_{ss} Y_{ss} x_{ss} c_{Ts} - \frac{1}{2} \tilde{\lambda}_x x_s^2] + t.i.p. + O(\epsilon^3)
$$

subject to

$$
c_{Ts} = y_{Ts} + \tilde{\Theta}^r T_r
$$

$$
r_j^s = k_j^r r_j^s x_s - \mu_j(c_{Ts} - c_{Ts}(0)) + k_j^r \xi s
$$

The FOC yield

$$
u_T y_s + V_{11} C_{ss} C_{Ts} + V_{12} Z_{ss} Y_{ss} x_{ss} + \sum_j k_j^r r_j^s \lambda_j^r \gamma_{rrs} = u_1 \lambda_s
$$

$$
u_T \Theta^j (\lambda_s + \gamma_{ss}^* y_s^*) = \lambda_j^r
$$

$$\tilde{\lambda}_x x_s = \sum_j k_j^r r_j^s \lambda_j^r
$$

$$
Err_j^r (\lambda_s + \gamma_{ss}^* y_s^*) = 0
$$

Recall that private marginal utility is given by

$$
u_T y_s + V_{11} C_{ss} c_{Ts} + V_{12} Z_{ss} Y_{ss} x_{ss} + k_{ux} x_s = u_1 \lambda_s^priv
$$

Note here the two sources of externalities: $\sum_j k_j^r r_j^s \lambda_j^r$ captures pecuniary externalities due to incomplete markets,\(^4\) while $k_{ux} x_s$ captures the aggregate-demand externality due to nominal rigidities. However: (i) the second FOC implies the pecuniary externality is proportional to the social value of insurance asset-by-asset; (ii) the aggregate demand externality is purely endogenous.

\(^4\)They were still there in the two asset case, but they did not show up because I had directly substituted out consumption.
and is thus only there to relax pecuniary externalities. (i) and (ii) together imply that the strength of the externalities must be related to their social value \( \lambda_s + \gamma_{ss} y_s^* \). Thus, optimality of the portfolio implies

\[
\mathbb{E} r_j^t x_j = 0 \forall j
\]
\[
\mathbb{E} r_j^t \lambda_{r_j} = 0, \forall j
\]

which in turn implies

\[
\mathbb{E} r_j^t (\lambda_{r_j} + \gamma_{ss} y_s^*) = 0 \forall j
\]

so \( \tau^j = O(\varepsilon^3) \) \( \forall j \).

### A.2.4 General asset structure: Dynamic model

In this section, I study an economy with multiple assets. Noting the “relaxed” continuation problem is the same as before, I obtain:

\[
\mathcal{W} = \max_{\{b_t, x_t, \pi_t\}} \left\{ \frac{1}{2} \left[ 1 - \frac{2}{1 - 2m^{-1} u_1 V_{11}^{-1} \gamma_{ss}} \left( \delta \bar{r}_{rr0} - \mathcal{J}_t \right)^2 + \frac{1}{2} V_{11}^2 \tilde{b}_t^2 - \frac{1}{2} \lambda_x x_0^2 - \frac{1}{2} \lambda_{\pi} \pi_0^2 \right] \right\}
\]

(A.44)

\[
+ \sum_{t} \beta^t \left\{ -\frac{1}{2} \lambda_x x_t^2 - \frac{1}{2} \lambda_{\pi} \pi_t^2 + \frac{1}{2} V_{11} (\tilde{b}_t + \beta^{-1} \tilde{b}_{t-1})^2 \right\} + t.i.p. + O(\varepsilon^2)
\]

I divide assets into two classes: “nominal” assets, which have a stationary price in home-currency such as a nominal bond, and “real” assets, which have a stationary price in foreign-currency. I rule out “mixed” assets, which would not have a stationary price in either currency. For tractability, I assume dividends of asset \( j \) decay at rate \( \delta^j \leq 1 \). For simplicity, I assume there is a risk-free asset in foreign currency, although this is not necessary.

The return on real assets is assumed to satisfy

\[
r_{t-1}^j = (1 - \beta (1 - \delta^j)) \left( k_{rrx}^j x_t + k_{rrc}^j c_t + k_{r\pi}^j \pi_t \right) + k_{rr\pi}^j \pi_{t-1} + \beta (1 - \delta^j) r_{t-1}^i - r_{t-1}^j.
\]

For these assets, define

\[
\tilde{r}_{t}^j = r_{t}^j + k_{rrc}^j \{(1 - \beta) \tilde{r}_{rr0} + \beta^{-1} (1 - \beta) \tilde{b}_t \} - r_{0, dm}^j (0).
\]

Then,

\[
(1 - \beta (1 - \delta^j)) k_{rrx}^j x_0 - k_{rr\pi}^j \pi_{t-1} - \delta^j k_{rrc}^j \tilde{b}_0^* - \beta (1 - \delta^j) \tilde{r}_{t-1}^j = \beta r_{t}^j - (1 - \beta) k_{rrc}^j (\tilde{r}_{rr0}) - \beta r_{0, dm}^j (0)
\]

\[
- (1 - \beta (1 - \delta^j)) k_{rrx}^j x_t + k_{rr\pi}^j \pi_t + \delta^j k_{rrc}^j \Delta \tilde{b}_t^* + \beta (1 - \delta^j) \tilde{r}_{t-1}^j = \tilde{r}_{t-1}^j
\]
The return on nominal assets is assumed to satisfy

\[ r^J_{t-1} = \Delta e_t + \beta(1 - \delta^J) r^J_t - \psi^J_{t-1} - r^*_t. \]

Define like before,

\[ \tilde{r}^J_0 = r^J_0 - r^{J, dm}_{0}(0) + k_{ex} x_0 - \beta^{-1} k_{ec} \tilde{b}^*_0 \]
\[ \tilde{r}^J_t = r^J_t - r^{J, dm}_{t}(0) + k_{ex} x_t - \beta^{-1} k_{ec} \Delta \tilde{b}_t \]

where \( r^{dm}_{0}(0) \) is again the return when the agent only holds risk-free short foreign-currency bonds \( B^* \). Then,

\[ -(1 - \beta(1 - \delta^J)) k_{ex} x_0 - \pi E_t - \delta k_{ec} \tilde{b}^*_0 - \beta(1 - \delta) \tilde{r}_0 = \beta \tilde{r}^J_0 + (1 - \beta) k_{ec} (\hat{\Theta'} r^J_0) - \beta \tilde{r}^{J, dm}_{0}(0) \]
\[ k_{ex}(1 - \beta(1 - \delta^J)) x_t + \pi E_t - \delta k_{ec} \Delta \tilde{b}_t + \beta(1 - \delta) \tilde{r}_t = \tilde{r}_{t-1} \]

Define \( k_{rrx} = -k_{ex}, k_{rrx} = -1, k_{rrc} = -k_{ec} \) for nominal assets. Note that, within real assets, there are two classes of assets: those sensitive to monetary policy and savings taxes (i.e., either \( k^J_{rrc} \neq 0, k^J_{rrx} \neq 0 \), or \( k^J_{rrc} \neq 0 \)) and those that are not (\( k^J_{rrc} = k^J_{rrx} = k^J_{rrc} = 0 \)). All nominal assets are clearly sensitive. Let \( J_1 \) denote the set of “sensitive” assets, and \( J_2 \) denote the set of “insensitive” assets. For all \( j \in J_2 \), it must be that \( \tilde{r}_t^J \equiv 0 \).

The next proposition shows that taxes are still zero in this environment. However, unless there is only one asset sensitive to monetary policy (i.e., \( \#J_1 = 1 \)), there is no longer one “sufficient statistic” so propositions (1.3) and (1.4) do not carry over. While the solution is not explicit when \( \#J_1 > 1 \), the solution solves a polynomial ensuring all possible candidates are taken into account, providing an algorithm for this case.

**Proposition A.4.** Let \( \Theta^{J_1} \) denote the positions in assets in \( J_1 \) (i.e., sensitive to monetary policy) and \( \Theta^{J_2} \) denote the positions in assets in \( J_2 \) (i.e., insensitive to monetary policy). Furthermore, let \( \Sigma_{J_1} \) denote the excess-returns of assets in \( J_1 \) in a savings-only economy, \( \Sigma_{J_1, J_2} \) their covariance, and \( \Sigma_{J_1 T} \) the covariance with the desired transfers. Also, define

\[ \tilde{\Theta}' \equiv \Theta'(I_{J_1} \times J_1 + \mu \Theta'^{J_1})^{-1}. \]

The optimal weight on the insurance motive is

\[ \omega = \frac{\tilde{\Theta}'^{I_{J_1} J_1} \chi^{-1} \tilde{\Theta}^{J_1}}{1 + \tilde{\Theta}'^{I_{J_1} J_1} \chi^{-1} \tilde{\Theta}^{J_1}} \]

where \( \chi \) is now a symmetric positive definite matrix \( \#J_1 \times \#J_1 \). The optimal portfolio on assets that are insensitive to monetary policy \( \Theta^{J_2} \) is given by:

\[ \tilde{\Theta}^{J_2} = -\Sigma_{J_2}^{-1} \Sigma_{J_2 J_1} \tilde{\Theta}^{J_1} + \Sigma_{J_2}^{-1} \Sigma_{J_1 T J_2}. \]
The optimal portfolio on assets that are sensitive to monetary policy $\tilde{\Theta}^{J_1}$ solves

$$W = -\frac{1}{2} \frac{k_0}{1 + \tilde{\Theta}^{J_1}\chi^{-1}\tilde{\Theta}^{J_1}} \left[ \sigma_T^2 + \tilde{\Theta}^{J_1}\tilde{\Sigma}_J \tilde{\Theta}^{J_1} - 2\tilde{\Sigma}_{\tau J_1} \tilde{\Theta}^{J_1} \right] + t.i.p + O(\varepsilon^3)$$

where $\chi$ is now a symmetric positive definite matrix $\#J_1 \times \#J_1$ and:

$$\tilde{\sigma}_T^2 = \sigma_T^2 - \Sigma_{J_2} \Sigma_{J_1}^{-1} \Sigma'_{J_2}$$
$$\tilde{\Sigma}_{J_1} = \Sigma_{J_1} - \Sigma_{J_2} \Sigma_{J_2}^{-1} \Sigma'_{J_1 J_2}$$
$$\tilde{\Sigma}_{\tau J_1} = \Sigma_{\tau J_1} - \Sigma_{\tau J_2} \Sigma_{J_2}^{-1} \Sigma'_{J_1 J_2}.$$

If there is only one asset sensitive to monetary policy (i.e., if $\#J_1 = 1$), then propositions (1.3) and (1.4) carry over to this environment. Regardless of $\#J_1$, if $m = \infty$ optimal portfolio taxes are given by

$$\tau^j = O(\varepsilon^3).$$

**Proof & algorithm for solving $\tilde{\Theta}$**

The continuation problem is to maximize (A.44) subject to

$$\kappa x_t + \beta \pi_{It+1} = \pi_{It}$$
$$\beta^{-1} \left\{ (1 - \beta(1 - \delta^j)) k_{\tau x} x_0 + k_{\tau x}^{\pi} \pi_0 - \delta^j (\tilde{b}_0^* - \beta(1 - \delta^j) \tilde{b}_0^\dagger) \right\}$$
$$+ \beta^{-1} (1 - \beta) k_{\tau c} (\Theta' r_0^J) + r_{J, dm}(0) = \tau^J_j \forall j \in J_1$$
$$-(1 - \beta(1 - \delta^j)) k_{\tau x} x_t + \delta^j k_{\tau c} \Delta \tilde{b}_t^* + \beta(1 - \delta^j) \tilde{r}_t^J = \tau^J_{t-1} \forall j \in J_1$$

I solve this problem the same way as before. First, I take the $t \geq 1$ problem with $\tilde{b}_0^*, \tilde{r}_0$ and $\pi_{It}$ taken as given. Then, I use the solution to solve the $t = 0$ problem.

$$W = -k_0(\tilde{\Theta}' r_0 - \tau_0)^2 - \frac{1}{2} (r_{J_1}^J + \mu \Theta' r_0 - r_{J_1, dm}(0))' \chi (r_{J_1}^J + \mu \Theta' r_0 - r_{J_1, dm}(0))$$
$$+ t.i.p + O(\varepsilon^3)$$

where $\chi$ is now a $J_1 \times J_1$ symmetric positive-definite matrix and $\mu = -\beta^{-1}(1 - \beta) k_{\tau c}^{J_1}$. This can be rewritten as

$$W = -k_0 \left\{ (\tilde{\Theta}' r_0 - \tau_0)^2 + (r_{J_1}^J - \tilde{r}_{J_1}^J)'(I + \mu \Theta^{J_1})' \chi (I + \mu \Theta^{J_1})(r_{J_1}^J - \tilde{r}_{J_1}^J) \right\} + t.i.p + O(\varepsilon^3)$$

where

$$\tilde{r}_{J_1}^J = (I + \mu \Theta^{J_1})^{-1}(-\mu \Theta^{J_1} r_{J_1}^J + r_{J_1, dm}(0)).$$

where I assume $(I + \mu \Theta^{J_1})^{-1}$ is invertible, which is necessary for the equilibrium to be well-defined. (This is equivalent to $B \neq -\mu^{-1}$ in the two-asset model). Next, we choose $r_{J_1}^J$ to minimize this.
expression subject to $\Theta'r_0 = \bar{T}$. Let $\nu_0$ denote the multiplier. The FOC yields

$$(I + \mu \Theta^{J_1'})'\chi(I + \mu \Theta^{J_1'})(r^{J_1}_0 - \bar{r}^{J_1}_0) + \nu_0 \Theta^{J_1} = 0$$

Solving for the multiplier,

$$\nu_0 = -\frac{1}{\Theta^{J_1'}\chi^{-1}\Theta^{J_1}}(\Theta^{J_1}r^{J_1}_0 - \bar{\Theta}^{J_1'}r^{J_1}_0, dm(0) + \mu \bar{\Theta}^{J_2'}r^{J_2}_0)$$

so

$$(I + \mu \Theta^{J_1'})(r^{J_1}_0 - \bar{r}^{J_1}_0) = \frac{\chi^{-1}\bar{\Theta}^{J_1}}{\Theta^{J_1'}\chi^{-1}\Theta^{J_1}}(\Theta^{J_1}r^{J_1}_0 - \bar{\Theta}^{J_1'}r^{J_1}_0, dm(0) + \mu \bar{\Theta}^{J_2'}r^{J_2}_0)$$

Replacing in the objective and simplifying,

$$W = -\frac{1}{2} k_0 \left\{ (\bar{T} - T_s)^2 + \frac{1}{\Theta^{J_1'}\chi^{-1}\Theta^{J_1}}(\bar{T} - \hat{\Theta}' r^{dm}_0(0))^2 \right\} + t.i.p + O(\epsilon^3)$$

The FOC with respect to $\bar{T}$ yields

$$\bar{T} = \frac{\hat{\Theta}^{J_1'}\chi^{-1}\hat{\Theta}^{J_1}}{1 + \hat{\Theta}^{J_1'}\chi^{-1}\hat{\Theta}^{J_1}} T_s + \frac{1}{1 + \hat{\Theta}^{J_1'}\chi^{-1}\hat{\Theta}^{J_1}} \hat{\Theta}' r^{dm}_0(0).$$

Note that using the same arguments as in the two-asset dynamic model, we see that every endogenous variable can be written as a weighted average, i.e.,

$$e_t = \frac{\hat{\Theta}^{J_1'}\chi^{-1}\hat{\Theta}^{J_1}}{1 + \hat{\Theta}^{J_1'}\chi^{-1}\hat{\Theta}^{J_1}} e_t^{in} + \frac{1}{1 + \hat{\Theta}^{J_1'}\chi^{-1}\hat{\Theta}^{J_1}} e_t^{dm}$$

where once again $e_t^{in}$ is the one that minimizes the cost. The deviation can be computed from the paths for $x_t$ and $\pi_{ft}$ computed above:

$$e_t^{in} = e_t^{in} - \{ e_t^{dm}(0) + k_{ec} T_s \}.$$  

Repeating back,

$$W = -\frac{1}{2} k_0 \chi \frac{E_0 [T_s^2 + (r_s^{dm}(0)'\hat{\Theta})^2 - T_s r_s^{dm}(0)'\hat{\Theta}] + t.i.p + O(\epsilon^3)$$

Solving for the optimal "insensitive" assets yields

$$\hat{\Theta}^{J_2} = -\Sigma_{j_2}^{-1} \Sigma_{j_2}^{J_2} \hat{\Theta}^{J_1} + \Sigma_{j_2}^{-1} \Sigma_{j_2}' \bar{T} J_2.$$

Replacing back,

$$W = -\frac{1}{2} k_0 \chi \frac{[\hat{\Theta}^{J_2}_T + \hat{\Theta}^{J_1'} \Sigma_{j_2} \hat{\Theta}^{J_1} - 2 \Sigma_{j_2} \hat{\Theta}^{J_1}] + t.i.p + O(\epsilon^3)$$
where
\[
\hat{\sigma}_T^2 = \sigma_T^2 - \Sigma J_2 \Sigma J_1 \Sigma J_2' \\
\hat{\Sigma}_{J_1} = \Sigma J_1 - \Sigma J_1 J_2 \Sigma J_1 J_2' \\
\hat{\Sigma}_{T J_1} = \Sigma T J_1 - \Sigma T J_2 \Sigma J_1 J_2'.
\]

This does not have a closed-form solution if \#\(J_1\) > 1. However, there are finite solutions that can be compared. To see this, define \(B^2 = \hat{\Theta} J' \chi^{-1} \hat{\Theta} J\) and solve the problem conditional on a “sensitivity” \(B\). This yields
\[
\hat{\Sigma}_{J_1} \hat{\Theta} J - \hat{\Sigma}_{T J_1} - \lambda \chi^{-1} \hat{\Theta} J = 0 \\
\hat{\Theta} J' \chi^{-1} \hat{\Theta} J = B^2.
\]
Solve for \(\hat{\Theta}\),
\[
\hat{\Theta} J' = (\hat{\Sigma}_{J_1} - \lambda \chi^{-1})^{-1} \hat{\Sigma}_{T J_1}
\]
and replace to obtain an equation in \(\lambda\),
\[
\hat{\Sigma}'_{T J_1} (\hat{\Sigma}_{J_1} - \lambda \chi^{-1})^{-1} \chi^{-1} (\hat{\Sigma}_{J_1} - \lambda \chi^{-1})^{-1} \hat{\Sigma}_{T J_1} = B^2.
\]
Note this can be written as
\[
\frac{\mathcal{P}_1(\lambda)}{\mathcal{P}_2(\lambda)^2} = B^2
\]
where \(\mathcal{P}_1(\lambda)\) is a polynomial of degree \((\#(J_1) - 1)^2\) and \(\mathcal{P}_2(\lambda)\) is a polynomial degree \#(\(J_1\)). Thus, there are at most max\{\((\#(J_1) - 1)^2\), \#(\(J_1\))\} solutions which need to be checked. Using this and then maximizing over \(B^2\) one can compute the optimal portfolios. Unfortunately, \(\hat{\Theta}\) is nonlinear in \(B\) if \#\(J_1\) > 1, so propositions (1.3) and (1.4) do not carry over.

**No portfolio tax** Next, I show that the result on taxes does carry over. I have:
\[
\mathcal{W} = \max \left\{ \frac{1}{2} (1 - \beta) V_{11} (\Theta' r r_0 - \mathcal{T})^2 + \frac{1}{2} V_{11} \beta^2_0 - \frac{1}{2} \lambda x x^2_0 - \frac{1}{2} \lambda \pi \pi_0^2 \right. \\
+ \sum_{t \geq 1} \beta^t \left\{ -\frac{1}{2} \lambda x x^2_t - \frac{1}{2} \lambda \pi \pi_t + \frac{1}{2} V_{11} \bar{c}_{T t} \right\} + t.i.p. + O(\epsilon^3) \right\}
\]
s.t.
\[
\kappa x_t + \beta \pi_{I t+1} = \pi_{I t} \\
\beta^{-1} (k_{r x} x_0 + k_{r r r} \pi_{I 0} + k_{r r e} \bar{c}_{T 0} + \beta (1 - \delta) (r_{0} - r_{0}^{dm} (0)) = r_{r}^0 + k_{r r c} \Theta' r r_0 - r_{r}^{dm} (0) \\
k_{c x} \Delta x_t + \pi_{I t} + k_{c c} \Delta \bar{c}_{T t} + \beta (1 - \delta) (r_{t} - r_{t}^{dm} (0)) = (r_{t-1} - r_{t-1}^{dm} (0))
\]
The first-order condition with respect $E'$ yields

$$E_{rr0} \left( V_{11}(\Theta'rr0 - T_s) + \sum_{m \in J_1} k_{rrc}^m \phi'^m \right) = 0.$$ 

The first-order condition with respect to $rr_j$ yields

$$\phi^j + \Theta^j \sum_{m \in J_1} k_{rrc}^m \phi'^m = -\Theta^j V_{11}(\Theta'rr0 - T).$$

Note:

$$\phi^j + \Theta^j \sum_{m \in J_1} k_{rrc}^m \phi'^m = -\Theta^j V_{11}(\Theta'rr0 - T)$$

$$k_{rrc}^j \phi^j + k_{rrc}^j \Theta^j \sum_{m \in J_1} k_{rrc}^m \phi'^m = -k_{rrc}^j \Theta^j V_{11}(\Theta'rr0 - T)$$

$$\sum_{m \in J_1} k_{rrc}^j \phi^j + \sum_{m \in J_1} k_{rrc}^j \Theta^j \sum_{m \in J_1} k_{rrc}^m \phi'^m = - \sum_{m \in J_1} k_{rrc}^j \Theta^j V_{11}(\Theta'rr0 - T)$$

$$\sum_{m \in J_1} k_{rrc}^m \phi'^m = - \frac{\sum_{m \in J_1} k_{rrc}^j \Theta^j}{1 + \sum_{m \in J_1} k_{rrc}^j} V_{11}(\Theta'rr0 - T)$$

Thus,

$$\frac{1}{1 + \sum_{m \in J_1} k_{rrc}^j} E_{rr0} \left( \Theta'rr0 - T \right) = 0.$$

Recalling that

$$rr_0^j + \mu \Theta'rr0 - rr_0^j dm(0) \propto \Theta'rr0 - \Theta'rr0 dm(0) \propto \Theta'rr0 - T$$

implies that the two sources of externalities introduced by the planner are once more proportional to the value of insurance,

$$x_t \propto \Theta'rr0 - T$$

$$\tilde{b}_t \propto \Theta'rr0 - T$$

so the result still follows.

### A.2.5 No capital controls

I will show how the results change in the static model. Clearly, the availability of portfolio taxes only matters when $m < \infty$. The dynamic model can be solved analogously (the continuation problem is the same). Without capital controls, one needs to keep track of the home no-arbitrage equation. Proceeding as in the proof of proposition 1.5, one obtains

$$\mathbb{E} \left[ \bar{B}e_s \left( (\bar{B} + T_s^s) + k_{ux} \chi((1 + \mu \bar{B})e_s - e_s^{dm}(0)) \right) \right] = O(\varepsilon) \quad \text{(A.45)}$$
where
\[ T_s^{priv} = \frac{1 - u_1V_{11}^{-1}2\eta B}{1 - u_1V_{11}^{-1}2\eta B} T_s = \tilde{k}(m)T_s \]
is the transfer desired by home agents.

The problem without portfolio taxes is to maximize
\[
EV(\{e_s, B\}) = -k_0E\left[ \frac{1}{2}(Be_s + T_s)^2 + \frac{1}{2}\chi((1 + \mu B)e_s - e_s^{dm}(0))^2 \right] + t.i.p. + O(\epsilon^3) \quad (A.46)
\]
subject to (A.45), which is an additional quadratic constraint. Ignoring this constraint would lead to a solution \( \bar{B} \) that is not feasible as \( \epsilon \to 0 \). Formally, one may see that the multiplier on the Home no-Arbitrage condition is also indeterminate at the steady state. The planner’s FOC with respect to the portfolio and the home-no arbitrage condition are the two set of conditions that together allow one to pin down these values at the steady state. Let \( \bar{\eta} \) denote the multiplier. The Lagrangian is given by,
\[
EV(\{e_s, B\}) = -k_0E\left[ \frac{1}{2}(Be_s + T_s)^2 + \frac{1}{2}\chi((1 + \mu B)e_s - e_s^{dm}(0))^2 \right] - \bar{\eta}Be_s \left( (Be_s + \bar{k}(m)T_s) + kux\chi((1 + \mu B)e_s - e_s^{dm}(0)) \right) + t.i.p. + O(\epsilon^3)
\]
Conditional on the multiplier \( \bar{\eta} \) and the position \( \bar{B} \), one may still write the optimal exchange rate as a weighted average,
\[
e_s = \frac{(1 + 2\bar{\eta})\bar{B}^2}{(1 + 2\bar{\eta})\bar{B}^2 + (1 + 2\frac{\bar{\eta}\bar{B}}{1 + \mu B}kux)\chi} e_s^{in}(\bar{\eta}, \bar{B}) + \frac{1 + 2\frac{\bar{\eta}\bar{B}}{1 + \mu B}kux}{(1 + 2\bar{\eta})\bar{B}^2 + (1 + 2\frac{\bar{\eta}\bar{B}}{1 + \mu B}kux)\chi} e_s^{dm}(\bar{\eta}, \bar{B})
\]
where
\[
e_s^{in}(\bar{\eta}, \bar{B}) = B^{-1}\frac{1 + \bar{k}(m)\bar{\eta}}{1 + 2\bar{\eta}} T_s
\]
\[
e_s^{dm}(\bar{\eta}, \bar{B}) = (1 + \mu \bar{B})^{-1}(\frac{1 + \frac{\bar{\eta}\bar{B}}{1 + \mu B}kux}{1 + 2\frac{\bar{\eta}\bar{B}}{1 + \mu B}kux}) e_s^{dm}(0).
\]
Note that the targets depend on the multiplier: the planner now internalizes that his monetary policy may lead agents to pick undesirable positions. One can then replace back in (A.46), maximize over \( \bar{B} \), get \( \bar{B}(\bar{\eta}) \) and then look for a fixed point of the home no-arbitrage equation (A.45) to find \( \bar{\eta} \). If there is more than one solution for \( \bar{\eta} \), one may then compare them using the welfare function. It is also important to check the second-order conditions of the inner problem with respect to the exchange rate.

Clearly, this problem is significantly less tractable than the case with capital controls when \( m < \infty \). Perhaps surprisingly, one can show that when \( \chi \to 0 \) (i.e., flexible prices), the solution
Table A.1: Sensitivity analysis.

<table>
<thead>
<tr>
<th></th>
<th>$\rho = 0.4$</th>
<th>$\rho = 1.5$</th>
<th>$\alpha = 0.4$</th>
<th>$\alpha = 0.7$</th>
<th>$\gamma = 10$</th>
<th>$\beta = 0.98$</th>
<th>1 year bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{B}$: Inflation-targeting</td>
<td>-16.8%</td>
<td>-15.2%</td>
<td>-16.6%</td>
<td>-15.6%</td>
<td>-20.4%</td>
<td>-19.3%</td>
<td>-33.5%</td>
</tr>
<tr>
<td>$\bar{B}$: Optimal Policy (savings tax)</td>
<td>-25.0%</td>
<td>-21.3%</td>
<td>-22.2%</td>
<td>-26.3%</td>
<td>-33.8%</td>
<td>-32.7%</td>
<td>-58.4%</td>
</tr>
<tr>
<td>$\bar{B}$: Optimal Policy (no savings tax)</td>
<td>-23.0%</td>
<td>-20.9%</td>
<td>-20.9%</td>
<td>-25.4%</td>
<td>-31.8%</td>
<td>-31.5%</td>
<td>-41.0%</td>
</tr>
<tr>
<td>$\omega$: savings tax weights</td>
<td>9.79%</td>
<td>6.36%</td>
<td>6.19%</td>
<td>13.4%</td>
<td>12.1%</td>
<td>11.3%</td>
<td>43.3%</td>
</tr>
<tr>
<td>$\omega$: no tax weights</td>
<td>6.93%</td>
<td>5.92%</td>
<td>4.57%</td>
<td>12.0%</td>
<td>9.83%</td>
<td>9.97%</td>
<td>11.3%</td>
</tr>
<tr>
<td>Welfare:</td>
<td>12.9%</td>
<td>10.5%</td>
<td>12.6%</td>
<td>11.2%</td>
<td>11.2%</td>
<td>9.93%</td>
<td>30.7%</td>
</tr>
<tr>
<td>Inflation-targeting</td>
<td>18.9%</td>
<td>14.7%</td>
<td>16.8%</td>
<td>18.8%</td>
<td>18.1%</td>
<td>16.4%</td>
<td>52.2%</td>
</tr>
<tr>
<td>Welfare: Optimal Policy (savings tax)</td>
<td>17.4%</td>
<td>14.4%</td>
<td>15.9%</td>
<td>18.1%</td>
<td>17.0%</td>
<td>15.8%</td>
<td>37.3%</td>
</tr>
<tr>
<td>Welfare: Optimal Policy (no savings tax)</td>
<td>18.9%</td>
<td>14.7%</td>
<td>16.8%</td>
<td>18.8%</td>
<td>18.1%</td>
<td>16.4%</td>
<td>52.2%</td>
</tr>
</tbody>
</table>

Note: Columns (1) and (2) change the elasticity of substitution between tradable and nontradable goods, columns (3) and (4) change the share of tradables in consumption, column (5) changes risk-aversion, column (6) changes the discount factor, and column (7) modifies the duration of bonds to 1 year. In every case I re-calibrated $m$ and $a_4$ to match the exchange volatility and a portfolio of −15% over annual GDP. Welfare gains are measured as the steady-state-consumption-equivalent gains under such a policy with respect to the demand-management $\bar{B} = 0$ economy as a share of the total potential gains under the first-best:

$$\frac{\text{welf}(\text{policy}) - \text{welf}(\bar{B}=0)}{\text{welf}(\text{first-best}) - \text{welf}(\bar{B}=0)} \%.$$

converges to the cooperative solution.5

A.3 Appendix: Quantitative analysis

In this section, I provide additional sensitivity analysis and show how the results change with the availability of savings taxes.

A.3.1 Savings taxes and bond duration

In all the previous experiments, I found that the taxes were very small. This suggests their availability is not very important for the optimal policy. Table 1.3 confirms these results: portfolios, weights, and welfare are in general quite similar if they are not available. The analysis in Section 1.4 suggests this could be tightly related to the duration of the bonds. Column 7 in Table 1.3 shows the results when I assume home-currency bonds have a duration of 1 year rather than 4.85 years. Savings taxes are now much more important: While the insurance weight without them is only 11%, once savings taxes are allowed, the weight increases almost four-fold to 43%. Accordingly, agents expand their positions from 34% to 58% and the planner achieves 52% of the potential welfare gains from

5One must be careful with randomization in these environments without capital controls. While the cooperative solution solves the approximate problem when prices are flexible, I found that with CRRA the planner may approximate the solution with capital controls arbitrarily closely by randomizing and putting a vanishing probability on $c = 0$. (The argument relies on higher order derivatives, so it does not show up if one approximates the problem first, i.e., with quadratic utility randomization is not optimal).
completing markets, compared to only 37% without savings taxes. Finally, note that welfare gains under demand-management targeting are now larger, which is a result of the higher correlation between returns and transfers implied by short bonds (they are less sensitive to liquidity shocks).

A.3.2 Other sensitivity analysis

I start by varying the complementarity between tradable and nontradable goods (columns 1 and 2 in table A.1). I adopt two values, which correspond to the bounds on the estimates in the literature (see Akinci (2011) for a survey): $\rho = 0.4$ and $\rho = 1.5$. A lower elasticity of substitution decreases the pass-through of the exchange rate to the output gap, which lowers the cost of providing insurance. In this range, however, the effects are modest: portfolios, insurance weights, and welfare gains vary only a few percentage points from one extreme to the other. More interestingly, a low elasticity of substitution also increases the effects of savings’ manipulation on bond returns since shifts in tradable consumption create large movements in demand for nontradable goods. As a result, the lower the elasticity, the more effective capital controls are.

Next, I vary openness increasing and decreasing it by 15 pp (columns 3 and 4). In very open economies, the inefficiencies affect a smaller share of the economy in our model. Put differently, deviating from demand-management to provide insurance is less costly from a welfare perspective because the planner cares less about the output gap and price dispersion compared to smoothing tradable consumption. As a result, the weight on insurance increases. Indeed, when tradables represent 70% of the economy, the insurance weight almost doubles, reaching over 13%. Accordingly, the planner achieves a larger share of the insurance gains of being able to issue home-currency debt.\(^6\)

Next, I vary risk aversion (column 5). I set $\gamma = 10$ - the upper bound of the range considered by Mehra and Prescott (1985). A higher risk aversion naturally makes insurance more important. However, while it increases the weight to around 12%, the realized share of the gains compared to demand-management is only slightly larger.

Finally, I change the discount factor (column 6). For illustrative purposes, I set $\beta=0.98$, which is very low for a model at the quarterly frequency. Ceteris paribus the shocks, a higher discount factor implies transfers become more valuable, as their present value increases. It has a similar effect to risk aversion: while it increases the weight to 11%, the realized share of the gains is only slightly larger compared to demand-management.

\(^6\)For this it becomes important where one introduces the price-rigidity. This result is likely to be less sensitive to openness if there is also stickiness in the tradable sector.
Appendix B

Appendix to Chapter 2

B.1 Data used in Section 2.2

B.1.1 Time series of reserve holdings

The two plots in Figure 2-1 show data from the 2011 revision of Lane and Milesi-Ferretti (2007). On the left, we aggregate paths for reserves over world GDP by the identifier “income_class”, which categorizes countries into low, middle, and high income. On the left, we show the shares of world GDP by “income_class”.

B.1.2 Time series and scatter plots of reserve flows

To construct Figure 2-2, we use IMF’s quarterly Balance of Payments statistics from 1990:1 to 2008:4 and restrict the sample to only include emerging markets, that is, countries with “income_group” identifier of 2 or 3. We also keep Israel and South Korea since they only recently became recognized as advanced economies. The final sample is an unbalanced panel of 50 emerging markets. We focus on three variables: quarterly reserve flows (FX reserve flows “BFRAFX”), quarterly private capital flows (net financial account minus reserve assets, “BF” minus “BFRA”), (annualized) trend GDP.

For Figure 2-2(a), we then aggregate reserve flows and private capital flows across countries and plot their ratios with trend GDP. For Figure 2-2(b) we compute ratios over (trend) GDP by country and plot the standard deviations of reserve flows over GDP vs. the standard deviations of private capital flows over GDP.

B.1.3 Reserve flows and UIP wedges

To construct Table 2.1, we first create a quarterly version of the UIP wedge data from Lustig, Roussanov and Verdelhan (2011) by summing their monthly excess return measure over the months of each quarter. We also use their data on interest rate differentials, averaged for each quarter. This data is then merged the IMF’s quarterly Balance of Payments statistics (this time across all countries and available times). We use the same measure of reserve flows over trend GDP as
B.2 Proofs for Section 2.3

B.2.1 Implementability conditions

This section proves Proposition 2.1. It requires two directions. We start by showing that (2.17a) and (2.17b) are necessarily satisfied if \( \{c_t, q_t, r_t\} \) belong to a competitive equilibrium with interest rate shocks \( \{r^*_t\} \). The paragraph below Definition 2.1 already showed that the flow version (2.15) of the present value budget constraint (2.17b) holds along a competitive equilibrium. (2.17a) follows directly from the Euler equation (2.2) and the definitions of \( \theta_t \) and \( \tau_t \).

Now, consider the reverse direction: Given paths \( \{\theta_t, \tau_t\}, \{r^*_t, Y_{H_t}\} \), and an initial net foreign asset position \( nfa_0 \) that satisfy (2.17a) and (2.17b), can we always find a competitive equilibrium consisting of initial debt positions \( (b_{HO}, b_{IO}, b_{GO}, b^*_G) \), a central bank FXI policy \( \{b_{GT}, b^*_G, t_t\} \) and an allocation \( \{c_t, c_{H_t}, c_{F_t}, c^*_{H_t}, b_{HT}, b_{It}, b^*_H\} \) with prices \( \{q_t, r_t\} \) such that (2.1)–(2.11) hold?

We first construct the equilibrium objects and then check optimality conditions. We can take the initial debt positions to be \( b_{HO} = nfa_0 \). Moreover, we define for any \( T > 0 \)

\[
b_{HT} = \int_T^\infty \exp(\int_t^s \tau_s^* ds) \left[ \alpha(\theta_t - 1) + \frac{1}{\Gamma_F} \tau_t^2 \right] dt, \quad (B.1)
\]

and thus construct \( b_{It} = \frac{1}{\Gamma_F} \tau_t^* b_{It} \), \( b^*_H = \frac{1}{\Gamma_H} \tau_t \), \( b^*_G = -b_{GT} = b_{HT} + b_{It} + b^*_H \), and \( \tau_t = b_{HT}^*(r_t - r^*_t) \) for each \( t \geq 0 \). Transfers are defined to be \( t_t = r_t b_{GT} + r^*_t b^*_G \). We let the real exchange rate be defined by \( y_{Ht} = (1 - \alpha) \theta_t + \alpha \) and let consumption paths be \( c_{Ft} = \alpha \theta_t \) and \( c_{Ht} = (1 - \alpha) \frac{y_{Ht}}{(1 - \alpha) \theta_t + \alpha} \). This concludes our construction of a candidate equilibrium. We move on to checking the equilibrium conditions.

The Euler equation (2.2) is equivalent to (2.17a). Equations (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), and (2.11) hold by construction. It is straightforward to check that the home good market clears—that is, equation (2.10) holds—given our definitions for \( c_{Ht} \) and \( c^*_{Ht} \). Finally, reversing the steps in equations (2.12)–(2.15) shows that the differential (flow) version of the (B.1) (which is exactly (2.15)) implies the budget constraint (2.1).

B.2.2 Simplifying the planner’s objective

In this section we derive the simplified objective function used in (2.18) from the original per period utility \( \log c_t \). Using (2.12), we can express

\[
c_t = q_t \theta_t = \left( \frac{y_{Ht}}{(1 - \alpha) \theta_t + \alpha} \right)^{1-\alpha} \theta_t
\]

which then yields a per period utility of

\[
\log c_t = \log \theta_t - (1 - \alpha) \log \left( (1 - \alpha) \theta_t + \alpha \right) + (1 - \alpha) \log y_{Ht}.
\]
The $y_{Ht}$ term in this expression is exogenous so it is without loss for our planning problem to drop it. This gives us the objective in (2.18).

**B.2.3 First best**

Here, we prove Lemma 2.1 that characterizes the first best allocation, that is, the optimal allocation when the Euler implementability condition (2.17a) does not bind. In that case the first order conditions with respect to $\theta_t$ and $\tau_t$ of the planning problem (2.18) read

$$
\tau_t = 0 \quad \text{and} \quad e^{-\rho t}V'(\theta_t) = e^{-\int_0^t r^*_s ds} \lambda \alpha,
$$

using the notation $V(\theta)$ for the planner’s per period objective, as described in Lemma 2.1, and calling $\lambda$ the multiplier on the resource constraint (2.17b). Due to the Inada conditions of this problem, the first order conditions are necessary. This proves Lemma 2.1.

**B.3 Proofs for Section 2.4**

**B.3.1 Model without macro stabilization motive**

Here, we provide a proof to Proposition 2.2. Consider first the case where $\alpha = 1$. Let $\{\theta_t\}, \lambda$ be the first best path of dollar consumption and the corresponding first best shadow value of resources, respectively. We now show that the (2.17a) does not bind, that is, it is satisfied for the first best $\theta_t$ with $\tau_t = 0$. Lemma 2.1 describes $\theta_t$ as the solution to the first order condition

$$
e^{-\rho t}V'(\theta_t) = e^{-\int_0^t r^*_s ds} \lambda \alpha,
$$

where, with $\alpha = 1$, $V'(\theta_t) = 1/\theta_t$. Log-differentiating this first order condition yields $\dot{\theta}_t/\theta_t = r^*_t - \rho$ which is exactly what we needed to show.

Now consider the case where $\alpha$ is allowed to be less than 1, but $r^*_t = \rho$. Again, let $\{\theta_t\}, \lambda$ be as in the first best allocation, satisfying (B.2). Since $r^*_t$ is constant, this implies that $\theta_t$ is constant too, trivially satisfying the Euler equation $\dot{\theta}_t/\theta_t = r^*_t - \rho = 0$. This concludes our proof of Proposition 2.2.

**B.3.2 Financial autarky, $\Gamma_F = \infty$**

Before we prove Proposition 2.3, we first derive very generally, for any $\Gamma_F > 0$, the (necessary) first order conditions to the planning problem (2.18) in the following lemma, a version of Lemma 2.2.

**Lemma B.1.** Suppose $\Gamma_F \in (0, \infty)$. Let $V(\theta)$ be the planner’s per period objective, as defined before Lemma 2.1, and let $\bar{T}_t(\lambda, \theta_t)$ be defined by

1. $\bar{T}_t$ is the current value equivalent of $T_t$, defined before Lemma 2.2.

$$
\bar{T}_t(\lambda, \theta_t) = \lambda - e^{\int_0^t (r^*_u - \rho) du} V'(\theta_t).
$$

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Then, under the optimal foreign exchange intervention policy, the interest rate spread \( \tau_t \) and the (dollar) consumption \( \theta_t \) satisfy the following first order condition

\[
\dot{\tau}_t = r^*_t \tau_t + \frac{\Gamma_F}{2\lambda} \tilde{T}_t(\lambda, \theta_t) \tag{B.3a}
\]

\[
\frac{\dot{\theta}_t}{\theta_t} = r^*_t - \rho + \tau_t \tag{B.3b}
\]

where the derivative of \( \tau_t \) exists whenever \( r^*_t \) does not jump. When \( \Gamma_F = \infty \), \( \tilde{T}_t(\lambda, \theta_t) = 0 \), instead of (B.3a).

**Proof.** The current value Hamiltonian of the planning problem (2.18) with \( \Gamma_F \in (0, \infty) \) is given by

\[
H(\theta, \tau, \lambda, \mu, t) = e^{-\int_0^t (\rho-r^*_s)ds}V(\theta) - \lambda\alpha(\theta - 1) - \frac{1}{\Gamma_F} \tau^2 + \mu(r^* + \tau - \rho).
\]

This is an optimal control problem with a subsidiary condition, as in Gelfand and Fomin (1963). The state variable is \( \theta \) and has a free initial value \( \theta_0 \). \( \theta \) has costate \( \mu \). As before, \( \lambda \) is the multiplier on the resource constraint (2.17b), and \( \tau \) is the control variable. Notice that when \( \lambda > 0 \), this is a strictly concave problem with a unique maximizer \( \{\theta_t, \tau_t\} \) and unique multipliers \( \{\mu_t, \lambda\} \), \( \lambda \), satisfying the first order conditions. And indeed \( \lambda \) has to be positive, or else shadow cost to scaling \( \theta_t \) up indefinitely (which would not violate the Euler equation (2.17a)) is zero or even negative, which is not resource feasible according to (2.17b).

The first order condition for \( \tau \) is simply \( \mu_t = \lambda_1^{\frac{1}{\Gamma F}} \tau_t \), and the costate equation for \( \theta \) is \( \dot{\mu}_t = r^*_t \mu_t + \tilde{T}_t(\lambda, \theta_t) \). Putting these two equations together yields (B.3a). Equation (B.3b) is just the Euler equation (2.17a).

When \( \Gamma_F = \infty \), the Hamiltonian is

\[
H(\theta, \tau, \lambda, \mu, t) = e^{-\int_0^t (\rho-r^*_s)ds}V(\theta) - \lambda\alpha(\theta - 1) + \mu(r^* + \tau - \rho)
\]

implying that the costate \( \mu_t \) is equal to zero at all times, \( \mu_t = 0 \) and hence that \( \tilde{T}_t(\lambda, \theta_t) = 0 \).

With Lemma B.1 under the belt, we now approach Proposition 2.3. For concreteness, we prove the proposition for positive interest rate shocks, that is, we assume that there is some \( T > 0 \) such that \( r^*_t > \rho \) for all \( t \in [0, T] \) and \( r^*_t = \rho \) thereafter. We show that, at the optimum, \( \tau_t < 0 \) for all \( t \in [0, T) \) and \( \tau_t = 0 \) for \( t \geq T \), with a jump at \( t = T \).

Define as an auxiliary object, \( \tilde{\theta}_t \equiv e^{\int_0^t (\rho-r^*_s)ds} \theta_t \). The necessary conditions now become \( \frac{\dot{\tilde{\theta}}_t}{\tilde{\theta}_t} = \tau_t \) and

\[
0 = \tilde{T}_t(\lambda, e^{\int_0^t (\rho-r^*_s)ds} \tilde{\theta}_t) = \lambda - \frac{1}{\tilde{\theta}_t} + \frac{(1-\alpha)^2}{(1-\alpha)\tilde{\theta}_t + \alpha e^{\int_0^t (\rho-r^*_s)ds}}.
\]

\footnote{It is also possible to prove directly that the constraint set of the planning problem is convex, analogous to the steps in Appendix B.6.1.}
which can be simplified to
\[
\frac{\lambda}{\alpha} \frac{\dot{\tau}_t}{\dot{\theta}_t} = 1 + \frac{1}{\dot{\theta}_t e^{\int_0^t (r^*_u - \rho) du} + \frac{\alpha}{1-\alpha}}.
\]
(B.4)

For any \( t < T \), this equation has a unique solution for \( \dot{\theta}_t \) that is strictly decreasing in \( t \). Therefore, \( \tau_t < 0 \) for \( t < T \). For \( t \geq T \), there is no explicit time dependence in (B.4) and \( \dot{\theta}_t \) is constant, which means \( \tau_t = 0 \) for \( t \geq T \). At \( t = T \), \( \tau_t \) jumps since the only time varying element of (B.4), \( \int_0^t (\rho - r^*_u) du \), has a kink at \( t = T \). This concludes our proof of Proposition 2.3.

### B.3.3 Intervention smoothing

Lemma 2.2 is a special case of Lemma B.1 in the previous subsection. In particular, (2.19) is an integral version of (B.3a). Moreover, as explained in the text below Lemma 2.2, Proposition 2.4 is a direct consequence of Lemma 2.2.

### B.3.4 Forward guidance

This section contains the proof of Proposition 2.4.3. As in Section B.3.2, we will only provide a proof for positive interest rate shocks \( \{r^*_t\} \). The case of negative interest rate shocks is analogous.

We start with the first order conditions from Lemma B.1, (B.3a) and (B.3b), for the case where \( \Gamma_F \in (0, \infty) \). Again, as in Section B.3.2, we work in terms of \( \dot{\theta}_t \), leading to the first order conditions
\[
\dot{\tau}_t = r^*_t \tau_t + \frac{\Gamma_F}{2\lambda} \tilde{\tau}_t (\lambda, e^{\int_0^t (r^*_u - \rho) du} \dot{\theta}_t)
\]
\[
\dot{\theta}_t = \dot{\theta}_t \tau_t.
\]

This is a 2-dimension system of ODEs, with initial condition \( \tau_0 = 0 \) and the terminal condition that \( (\tau_t, \dot{\theta}_t) \) converge to the unique steady state given by \( \tau^{ss} = 0 \) and
\[
\tilde{\tau}_T (\lambda, e^{\int_0^T (r^*_u - \rho) du} \dot{\theta}^{ss}) = 0.
\]

Since the system is stationary after \( t = T \), the state at \( t = T \), \( (\tau_T, \dot{\theta}_T) \), has to lie on the stationary system’s stable arm. Figure B-1 illustrates the phase diagram and its stable arm. The green line that then merges into the red line depicts the shape of the optimal trajectory that we are trying to determine mathematically.

To do this, it turns out to be helpful to define the path \( \{\dot{\theta}_t^{\infty}\} \) as the solution to \( \tilde{\tau}_t (\lambda, e^{\int_0^t (r^*_u - \rho) du} \dot{\theta}_t^{\infty}) = 0 \), for all \( t \geq 0 \). In a first step, we show that it can never be that \( \dot{\theta}_t > 0 \) and \( \dot{\theta}_t > \dot{\theta}_t^{\infty} \) for any \( t > 0 \). In Figure B-1, this would be a state \( (\tau_t, \dot{\theta}_t) \) that lies to the top right of the time-\( t \) \( \tau \)-locus. In such a case, for any \( s > t \), both \( \dot{\theta}_s \) and \( \dot{\tau}_s \) are positive and bounded away from zero, and hence the state \( (\dot{\theta}_t, \tau_t) \) would diverge to \( \infty \). Translating the divergence back to \( \theta_t \), this would mean that the growth

\[\text{We call this } \dot{\theta}_t^{\infty} \text{ since, given a certain } \lambda, \text{ it corresponds to the optimal path for } \dot{\theta}_t \text{ when } \Gamma_F = \infty. \text{ This is also why } \dot{\theta}_t^{\infty} \text{ is well-defined, continuous and piece-wise differentiable for each } t.\]
Figure B-1: Describing the optimal policy in the state space for $(\tau, \theta)$.

rate of $\theta_t$, $\dot{\theta}_t/\theta_t = r^*_t - \rho + \tau_t$, diverges to infinity, violating the resource constraint (2.17b).

Second, consider the possibility that for some $t > 0$,

$$(\hat{\theta}_t, \tau_t) \in \{(\hat{\theta}, \tau) \mid \tau \geq 0 \text{ and } \dot{\theta} \leq \dot{\theta}^\infty_t \} = \Theta_t.$$ 

Given $\dot{\theta}^\infty_t$ is decreasing in $t$, if $(\hat{\theta}_t, \tau_t) \in \Theta_t$, then $(\hat{\theta}_s, \tau_s) \in \Theta_s$ for any $s < t$ as well. In particular $(\hat{\theta}_t, \tau_t) \in \Theta_0$. Given no path satisfying the ODEs can ever enter $\Theta_0$ (that is, $\Theta_0$, is a “source” in the vector field sense), it must hold that $(\hat{\theta}_0, \tau_0) \in \text{int}\Theta_0$ (the interior of $\Theta_0$). This contradicts the fact that $\tau_0 = 0$. Together, these two steps prove that $\tau_t \geq 0$ is impossible for any $t > 0$. This concludes the proof of the proposition.

B.3.5 Time inconsistency

This section proves the results claimed in Proposition 2.6. When there is re-optimization at any time $t_0 \geq T$, there is no more interest rate shock to $r^*$ after $t_0$. Thus, Proposition 2.2 applies to this re-optimization problem, making it optimal to set $\tau_t = 0$ for all $t \geq t_0$.

Consider now the case of no commitment at all. For ease of notation, call $b_t$ Home’s net foreign asset position $nfa_t$ and let $v(t, b_t)$ be the no-commitment planner’s current time-$t$ value function with current net foreign asset position $b_t$. The Hamilton-Jacobi-Bellman equation for the time-$t$ planner is then

$$r^*_tv(t, b) = \max_{\tau} V(\theta) + v_t(t, b) + v_\theta(t, b) \left( \alpha(1 - \theta) + r^*_tb - \frac{1}{\Gamma_F} \tau^2 \right).$$

The first order condition for $\tau$ immediately yields the desired result, $\tau_t = 0$. 

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B.4 Proofs for Section A.2

B.4.1 Fixed exchange rate economy

In this section, we list the results we proved in the home-bias economy and how their proof can be modified to also apply to the fixed exchange rate economy. To do this, define the per period objective as follows,

\[ V(\theta) \equiv \log \theta_t - (1 - \alpha) \log((1 - \alpha)\theta_t + \alpha c^*) - v((1 - \alpha)\theta_t + \alpha c^*). \]

Further, we assume that the multiplier on the resource constraint is positive, \( \lambda > 0 \). Lemma 2.1 and Propositions 2.2—2.6 go through word by word as in the baseline model, given our assumption of a positive multiplier \( \lambda \).

B.5 Additional extensions

B.5.1 Lucas-Stokey and long term bonds

In a seminal contribution, Lucas and Stokey (1983) show how a planner without commitment power can achieve the full-commitment solution by essentially implementing a specific composition of household asset holdings across certain asset classes. Even though the asset classes are linearly dependent and hence redundant, they react different to policy changes, which lets a planner today select asset classes that would "punish" future planners upon deviations.

In our setup, we allow all agents—households, intermediaries and the central bank—to trade Arrow securities in addition to the existing short-term bonds. As in Lucas and Stokey (1983) the central bank can now control its asset composition across all asset classes. Crucially, however, this will not pin down the country's asset composition since households are free to trade with intermediaries in whatever asset classes they like. Therefore, in our model, the central bank cannot implement a specific country's asset composition. In this section, we sketch out a natural version of our economy with multiple types of bonds and show that they do not help the planner overcome his time consistency problem. As in the main body of our paper, we restrict our attention to the case where domestic bonds are also measured in dollars. This lets us focus on our novel type of time inconsistency, different from the standard exchange rate based time inconsistency.\(^5\)

Let \( \pi_{t,s}^* \) denote the state price density for an international dollar payment at time \( s \), measured at time \( t \). That is, if a country promises a stream of payments \( x_{t,s} \) in the foreign bond market at time \( t \), and has no other external assets or liabilities, its time-\( t \) net foreign asset position is

\[ n_{fa} = \int_t^\infty \pi_{t,s}^* x_{t,s} ds. \]

Similarly, \( \pi_{t,s} \) denotes the state price density for a promised payment in the

\(^4\)Compared to our baseline model, the per period objective \( V(\theta) \) is no longer monotonically increasing in the fixed exchange rate economy. This means, under certain, arguably contrived, circumstances, it can occur that more resources hurt the economy. To focus on the "normal" case, we assume that \( \lambda > 0 \), which is always the case if the fixed exchange rate \( \overline{Z} \) is sufficiently low or the initial labor wedge sufficiently close to zero.

\(^5\)Here, we mean the tendency of a re-optimizing planner with external local currency liabilities to depreciate the currency.
domestic bond market at time $s$, measured at time $t$. Note that $\pi_{t,s}^* = e{\int_t^s r_s^* du}$ and $\pi_{t,s} = e{\int_t^s r_u du}$.

To see the link to our previous short-term bonds, a single outstanding domestic short-term bond promises payments $x_{t,s} = r_s$ so that its value is exactly 1.

We allow all domestic and foreign bond market participants to trade all Arrow securities with each other, including short-term bonds. We choose the following notation: For each possible short term bond position in our previous notation $X_t = b_{Ht}, b_{It}, b_{Ht,s}, b_{Gt,s}, b_{*t,s}^*$ we allow positions in Arrow securities denoted by $b_{Ht,s}, b_{It,s}, b_{Ht,s}^*, b_{Gt,s}, b_{*t,s}^*$. In this notation, domestic bond market clearing requires that

$$b_{Ht,s} + b_{It,s} + b_{Ht,s}^* + b_{Gt,s} = 0$$

in addition to the short term bond market clearing condition (2.11). The main effect of the different Arrow securities in our model is that it creates an indeterminacy in intermediaries and households’ positions. Using a natural extension of the intermediaries’ maximization problem to this setup gives intermediary demand functions

$$b_{It} + \int_t^\infty \pi_{t,s} b_{It,s} ds = \frac{1}{\Gamma_F} (r_t - r_t^*) \quad \text{and} \quad b_{It}^H + \int_t^\infty \pi_{t,s} b_{It,s}^H ds = \frac{1}{\Gamma_H} (r_t - r_t^*).$$

Evidently, the demand functions do not pin down intermediary positions uniquely.

In this economy, the country’s time $t$ implementability conditions are given by (2.17a) and

$$\int_t^\infty e^{-\int_t^s r_u^* du} \left[ \alpha (\theta - \frac{1}{\Gamma_F})^2 \right] ds = nfa_t,$$

where the net foreign asset position is now given by

$$nfa_t = b_{Ht} + b_{*t}^* + b_{Gt} + \int_t^\infty \pi_{t,s} (b_{Ht,s} + b_{Gt,s}) ds + \int_t^\infty \pi_{t,s}^* b_{Gt,s}^* ds.$$

Without loss of generality due to Ricardian equivalence between the central bank and domestic households, we let the central bank choose foreign and domestic bond portfolios whose values sum to zero. Then,

$$nfa_t = b_{Ht} + \int_t^\infty \pi_{t,s} b_{Ht,s} ds.$$

Here, $\pi_{t,s}$ is affected by central bank policies. So in order to fix a time inconsistency problem, the central bank would have to be able to control the positions $\{b_{Ht}, b_{Ht,s}\}$. Yet, for any change in the composition of the central bank position $\{b_{Gt}, b_{Gt,s}\}$ (leaving the total value of the position unchanged), there exist changes in intermediary portfolio compositions such that the household positions $\{b_{Ht}, b_{Ht,s}\}$ as well as the equilibrium quantities and prices $\{c_t\}$ and $\{q_t, r_t\}$ do not change. This makes it impossible for the central bank to commit themselves to future interventions through this mechanism.
### B.5.2 Nonlinear intermediary demands

In our baseline model, intermediaries’ demand functions for domestic bonds are linear functions of the UIP wedge \( \tau_t \). In this extension, we explore the implications of nonlinear demand schedules

\[
\begin{align*}
    b_{It} &= g^F(\tau_t) \\
    b_{It}^H &= g^H(\tau_t)
\end{align*}
\]

(B.5)

where \( g^F, g^H \) are strictly increasing and differentiable functions defined for all reals, with \( g^k(0) = 0 \) and \( g'^k(0) = \frac{1}{1_k} \), for \( k = F, H \). Such general demand schedules can be microfounded the same way we microfounded the linear demand schedules in Section 2.3.1, just with a more general transaction function: one where \( f(j) \) is strictly increasing and continuous, with \( f(0) = 0 \). Using the nonlinear schedules (B.5), instead of the linear demands (2.5) and (2.7), Definition 2.1 defines the right notion of competitive equilibrium.

The key difference to the linear demand model is that now costs are no longer exactly quadratic, rather only locally so, given by

\[
b_{It}\tau_t = g^F(\tau_t)\tau_t.
\]

The new resource constraint implementability condition is then given by

\[
\int_{0}^{\infty} e^{-\int_{0}^{t} \tau^* ds} \left[ \alpha(\theta_t - 1) + \tau_t g^F(\tau_t) \right] dt = nfa_0.
\]

To ease notation in this subsection, we introduce \( G(\tau) \equiv g^F(\tau)\tau \). To ensure the associated planning problem with such an implementability condition is well defined, we further assume that \( G(\tau) = \tau g^F(\tau) \) is strictly convex in \( \tau \). Since \( \tau g^F(\tau) \) is always locally convex around \( \tau = 0 \), this essentially restricts \( g^F(\tau) \) to not become too flat for large positive or negative values of \( \tau \).

Next, we argue that under these conditions, despite nonlinear demand functions, our key results remain true. Lemma 2.1 and Proposition 2.2 and their proofs go through without any changes. Lemma B.1 in the appendix still holds when (B.3a) is replaced by

\[
\dot{\tau}_t = n^* \frac{G'(\tau_t)}{G''(\tau_t)} + \frac{1}{G''(\tau_t)} \ddot{T}_t(\lambda, \theta_t).
\]

(B.6)

Proposition 2.3 trivially holds as well since it treats the special case where \( \Gamma_F = \infty \), that is translated into this notation, \( g^F(\tau) = 0 \) for all \( \tau \). The first order condition in Lemma 2.2 is still the integral equivalent of (B.6) and reads

\[
e^{-\int_{0}^{t} \tau^* ds} \lambda G'(\tau_t) = \int_{0}^{t} \tau_s ds,
\]

with the exact same interpretation as in Lemma 2.2. Based on this lemma, Proposition 2.4 carries over as well. We speculate that Proposition 2.5 goes through unchanged (there is no intuitive reason why not) but we have not worked out the formal proof. The partial commitment part of Proposition 2.6 holds conditional on Proposition 2.5 being correct, while the no-commitment part can be proven using the same argument as before.
B.6 Proofs for Section 2.6

B.6.1 Convexity of the planning problem

To show that planning problem (2.32) is well-behaved, first note that the objective is strictly concave and strictly increasing in \((\theta_0, \theta_1)\) as before. Therefore, if the constraint set were bounded and convex, the unique maximizer would necessarily lie on the constraint set’s boundary. We now prove that the inequality version of (2.33),

\[
B(\theta_0, \theta_1) \equiv (\alpha \theta_0 - C_{EME0}) + \frac{1}{1 + r^*} (\alpha \theta_1 - C_{EME1}) + \frac{1}{\Gamma_F} \left( \frac{\theta_1 (1 + \rho_{EME})}{\theta_0 (1 + r^*)} - 1 \right)^2 \leq 0.
\]

is indeed bounded and convex, thus validating our relaxation.

Boundedness is straightforward as the intermediary cost term \(1/\Gamma_F(\ldots)^2\) is always bounded from below by zero. For convexity, consider two points \(\theta = (\theta_0, \theta_1)\) and \(\theta' = (\theta'_0, \theta'_1)\) in \(\mathbb{R}_{++}^2\), and choose \(\lambda \in [0, 1]\). Define \(\theta^\lambda = \lambda \theta + (1 - \lambda) \theta'\). We now prove that \(\lambda \mapsto B(\theta^\lambda)\) is a quasi-convex function, implying that \(B(\theta^\lambda) \leq \max\{B(\theta), B(\theta')\} = 0\) as desired.

\(B(\theta^\lambda)\) consists of two linear terms and a third, non-linear term. The nonlinear term is

\[
\frac{1}{\Gamma_F} \left( \frac{(1 + \rho_{EME})}{\lambda \theta_0 + (1 - \lambda) \theta'_0} \right) \left( \frac{(1 + \rho_{EME})}{\lambda \theta_1 + (1 - \lambda) \theta'_1} - 1 \right)^2
\]

which is a composition of a convex function \(f(z) = \frac{1}{\Gamma_F} \left( \frac{(1 + \rho_{EME})}{(1 + r^*)} z - 1 \right)^2\) and a function \(g(\lambda) = \frac{\lambda \theta_1 + (1 - \lambda) \theta'_1}{\lambda \theta_0 + (1 - \lambda) \theta'_0} \). Here, \(g\) is either strictly monotone or constant. Either way, the composition \(f(g(\lambda))\) is quasiconvex, implying the result. Therefore, the constraint set \(B(\theta_0, \theta_1) \leq 0\) is convex.

B.6.2 Characterizing the Nash equilibrium

Part 1

This subsection provides a proof for Proposition 2.7. We start with part 1 and rewrite the planning problem in a slightly more convenient form, by defining \(\kappa_t = \frac{(1 + \rho_{EME})}{(1 + r^*)} \), \(\hat{\theta}_t \equiv \kappa_t \theta_t\), and \(\hat{C}_{EME} \equiv \kappa_t C_{EME}\). We also abbreviate \(\rho = \rho_{EME}\) and \(\hat{C} = \hat{C}_{EME}\) and let the per period objective be defined as

\[
V(\hat{\theta}, \hat{C}) \equiv \log \hat{\theta} - (1 - \alpha) \log((1 - \alpha) \hat{\theta} + \hat{C})
\]

and the constraint function be defined as

\[
B(\hat{\theta}_0, \hat{\theta}_1, \hat{C}_0, \hat{C}_1) = \sum_{t=0} (1 + \rho)^{-t} \left( \alpha \hat{\theta}_t - C_t \right) + \frac{1}{\Gamma_F} \left( \frac{\hat{\theta}_1}{\hat{\theta}_0} - 1 \right)^2.
\]
As before, $\hat{\theta}$ and $\hat{C}$ are complements in the utility function, that is $V_{\hat{\theta},\hat{C}} > 0$. Using this notation, (2.32) becomes
\[
\max \sum_{t=0}^{1} (1 + \rho)^{-t} \left\{ V(\hat{\theta}_t, \hat{C}_t) - \alpha \log \kappa_t \right\} \tag{B.7}
\]
subject to
\[
B(\hat{\theta}_0, \hat{\theta}_1, \hat{C}_0, \hat{C}_1) \leq 0,
\]
where we relaxed the constraint as inequality, applying the result from Appendix B.6.1. In that section we also mentioned that (for any $\hat{C}_t > 0$) the optimum choice for $(\hat{\theta}_0, \hat{\theta}_1)$ lies on a downward sloping, convex arc of the constraint set. On that arc, it holds that the marginal rate of substitution, $MRS(\hat{\theta}_0, \hat{\theta}_1) \equiv \frac{B_{\theta_0}}{B_{\theta_1}}$ is weakly increasing in $\hat{\theta}_0$ and weakly decreasing in $\hat{\theta}_1$. Notice that $MRS(\hat{\theta}_0, \hat{\theta}_1)$ is independent of $\{\hat{C}_t\}$. In a (pure strategy) Nash equilibrium, the demands $\hat{C}_t$ are then determined by
\[
\hat{C}_t = \kappa_t - (1 - \alpha)\hat{\theta}_t \tag{B.8}
\]
and the equilibrium interest rate is pinned down by AEs’ discount rate, $r^* = \rho_{AE}$ since $\chi = 0$. As a third equilibrium condition, we need the resource constraint to hold, which, having substituted in (B.8) becomes
\[
\overline{B}(\hat{\theta}_0, \hat{\theta}_1) \equiv \sum_{t=0}^{1} (1 + \rho)^{-t} \left( \hat{\theta}_t - \kappa_t \right) + \frac{1}{\Gamma_F} \left( \frac{\hat{\theta}_1}{\hat{\theta}_0} - 1 \right)^2 \leq 0. \tag{B.9}
\]
The Euler equation of (B.7) reads
\[
V_{\hat{\theta}}(\hat{\theta}_0, \hat{C}_0) = MRS(\hat{\theta}_0, \hat{\theta}_1)V_{\hat{\theta}}(\hat{\theta}_1, \hat{C}_1)
\]
and substituting in the Nash equilibrium condition (B.8) we find
\[
V_{\hat{\theta}}(\hat{\theta}_0, 1 - (1 - \alpha)\hat{\theta}_0) = MRS(\hat{\theta}_0, \hat{\theta}_1)V_{\hat{\theta}}(\hat{\theta}_1, \kappa_1 - (1 - \alpha)\hat{\theta}_1). \tag{B.10}
\]
Using the properties of $V$ and $MRS$ mentioned above, it is immediate that the left hand side of this equation strictly decreases in $\hat{\theta}_0$, while the right hand side increases in $\hat{\theta}_0$ and falls with $\hat{\theta}_1$. This implicit equation thus describes a strictly increasing, continuous function $\hat{\theta}_1 = h(\hat{\theta}_0)$, defined for any $\hat{\theta}_0 > 0$. Notice that the solution to (B.10) must necessarily generate a positive $MRS(\hat{\theta}_0, h(\hat{\theta}_0))$, and hence positive derivatives $B_{\theta_0}$ and $B_{\theta_1}$.\(^6\) It is easy to see that the positivity of the two derivatives immediately implies that $h(\hat{\theta}_0)/\hat{\theta}_0 \to 1$ as $\hat{\theta}_0 \to 0$. A Nash equilibrium can be found as a solution to (B.9) with $\hat{\theta}_1 = h(\hat{\theta}_0)$ substituted in.

Before we show existence and uniqueness of such a solution, we establish a few helpful auxiliary results. First, note that $\overline{B}(\hat{\theta}_0, h(\hat{\theta}_0))$ is strictly increasing in $\hat{\theta}_0$. Suppose it were not: Then there
\(^6\) Notice that it can never be the case that both $B_{\theta_0}$ and $B_{\theta_1}$ are negative since in that case, there would have to exist some positive $\{\hat{C}_t\}$ such that the budget set $B \leq 0$ does not include any points close to zero. Yet, by scaling down any point $(\hat{\theta}_0, \hat{\theta}_1)$ in such a budget set, $(0,0)$ can always be approximated arbitrarily closely.
has to be a point \((\hat{\theta}_0, h(\hat{\theta}_0))\) where either \(B_{\hat{\theta}_0}\) (which is larger than \(B_{\hat{\theta}_1}\)) or \(B_{\hat{\theta}_1}\) (which is larger than \(B_{\hat{\theta}_0}\)) is negative, contradicting the positivity of \(MRS(\hat{\theta}_0, h(\hat{\theta}_0))\). Second, note that \(\overline{B}(\hat{\theta}_0, h(\hat{\theta}_0))\) approaches \(\infty\) as \(\hat{\theta}_0 \to \infty\) and approaches a negative number as \(\hat{\theta}_0 \to 0\). The former is straightforward since the quadratic cost term is bounded below by zero. The latter follows because \(h(\hat{\theta}_0)/\hat{\theta}_0 \to 1\) as \(\hat{\theta}_0 \to 0\), as explained above, and so the cost term vanishes as \(\hat{\theta}_0 \to 0\). The intermediate value theorem then proves that there exists a unique number \(\hat{\theta}_0\) such that \(\overline{B}(\hat{\theta}_0, h(\hat{\theta}_0)) = 0\). This proves the existence and uniqueness of the symmetric world equilibrium as defined in Definition 2.2.

Part 2

To show that \(\tau\) is positive, we now argue that \(h(\hat{\theta}_0) > \hat{\theta}_0\) for all \(\hat{\theta}_0\). Suppose this were not the case. Then, the right hand side of (B.10) can be bounded from above by

\[
MRS(\hat{\theta}_0, \hat{\theta}_1)V_{\hat{\theta}}(\hat{\theta}_1, \kappa_1 - (1 - \alpha)\hat{\theta}_1) \leq MRS(\hat{\theta}_0, \hat{\theta}_0)V_{\hat{\theta}}(\hat{\theta}_0, \kappa_1 - (1 - \alpha)\hat{\theta}_0) < V_{\hat{\theta}}(\hat{\theta}_0, 1 - (1 - \alpha)\hat{\theta}_0).
\]

This contradicts (B.10). Therefore, \(h(\hat{\theta}_0) > \hat{\theta}_0\) and hence \(\tau > 0\) in the Nash equilibrium.

Part 3

Compared to a no-intervention world, the EME consumption profile \((\hat{\theta}_0, \hat{\theta}_1)\) is tilted more towards the future, and hence EME's net foreign assets are larger after \(t = 0\). Positivity of the UIP wedge, \(\tau > 0\), however, means that private capital must flow more downstream, implying that the increase in net foreign assets is driven by rise of EME reserves.

Part 4

This follows from Proposition 2.8 below.

B.6.3 Characterizing the cooperative outcome

This section proves Proposition 2.8. Utilizing the notation introduced in Appendix B.6.2 above, the cooperative planning problem can be written as

\[
\max_{\theta_t} \sum_{t=0}^{1} (1 + \rho)^{-t} \{\log \hat{\theta}_t - \alpha \log \kappa_t\}
\]

subject to the total EME resource constraint (B.9) which takes into account the endogeneity of \(\{C_{EME}\}\). To analyze this problem, we guess and verify that it is without loss to relax the constraint (B.9) by setting \(\Gamma_F = \infty\). In that case, the solution is simply given by

\[
\hat{\theta}_0 = \hat{\theta}_1 = \frac{1 + (1 + \rho)\kappa_1}{2 + \rho}.
\]
This "relaxed" features that $\hat{\theta}_1/\hat{\theta}_0 = 1$, that is, $\tau = 0$. Therefore, relaxing (B.9) by setting $\Gamma_F$ to $\infty$ was without loss of generality, and we found the solution.
Appendix C

Appendix to Chapter 3

C.1 Proofs and Derivations

C.1.1 Proof of Proposition 3.1

We will prove the result under the following (more general) conditions,

Assumption C.1. \( E_\psi \pi_e (\psi, \theta) \geq \pi_i (\theta) \) \( \forall \psi, \theta \). \( \pi_e \) is continuous, \( \pi_i \) belongs to \( C^2 \) and both are weakly increasing in \( \theta \) \( \forall \psi \). \( \psi \) and \( \theta \) are independent.

Assumption C.2. Let \( h \equiv \lambda E_\psi (\max \{ \pi_e (\theta; \psi), 0 \}) - \lim_{dt \to 0} \left\{ E \left( e^{-r dt} \pi_i (\theta_{t+dt}) \right) - \pi_i (\theta_t) \right\} \). If \( \lambda > 0, h (\theta) \) is weakly increasing in \( \theta \). Furthermore, \( E \left[ \int_0^\infty e^{-rt} h (\theta_t) d\theta_t | \theta_0 \right] \) satisfies a polynomial growth condition.\(^1\)

Assumption C.3. There exists \( \bar{\theta} \) such that \( \forall \theta > \bar{\theta} \), flow profits are positive even for inexperienced firms, \( \pi_i (\theta) \geq 0 \).

Assumption C.4. The profitability process \( \{ \theta_t \} \) is assumed to follow a diffusion,

\[
d\theta_t = \mu_\theta dt + \sigma_\theta dZ_t \quad (C.1)
\]

where \( Z_t \) is a standard brownian motion. We assume \( \mu (\theta) \) and \( \sigma (\theta) \) are continuous functions of \( \theta \) that satisfy Lipschitz and growth conditions on \( \mu \) and \( \sigma \).\(^2\) Furthermore, if \( \theta'' > \theta' \), then \( F (\theta | \theta'') \gtrsim \text{FOSD} F (\theta | \theta') \).

\(^1\)We say that \( f : [0, \infty) \to \mathbb{R} \) satisfies a polynomial growth condition if there exist \( M > 0 \) and \( \nu > 0 \) such that \( |f (\theta)| \leq M (1 + \theta^\nu) \).

\(^2\)We say that \( \mu \) satisfies a Lipschitz condition if there exists \( k > 0 \) such that \( |\mu (\theta) - \mu (\theta')| \leq k |\theta - \theta'| \).

This ensures the existence of a strong solution to (3.15)
**Assumption C.5.** $E_{\psi}V_{e}$ satisfies a polynomial growth condition $\forall \theta$

Assumption C.1 is satisfied in the model in the text because $E(\psi) \geq 1$. Applying Ito’s Lemma to Assumption C.2 we get

$$h \equiv \lambda E_{\psi}e(\theta;\psi) + r\pi_{i}(\theta) - \frac{\mu_{\theta}}{\lambda} \frac{d\pi_{i}(\theta)}{d\theta} - \frac{1}{2} \frac{d^{2}\pi_{i}}{d\theta^{2}}$$

In the model in the text,

$$h \equiv E_{\psi} \left[ \max \left\{ \psi\tilde{\theta} - 1, 0 \right\} \right] + \frac{r}{\lambda} (\tilde{\theta} - 1) - \frac{\mu\tilde{\theta}}{\lambda}$$

which is clearly increasing in $\tilde{\theta}$ (recall $r - \mu > 0$). Furthermore Assumption C.3 is satisfied taking $\tilde{\theta} = \frac{E}{\kappa}$ and Assumption C.4 is satisfied by the GBM assumption ($\mu_{\theta} = \mu\theta$ and $\sigma_{\theta} = \sigma\theta$). Finally Assumption C.5 is satisfied by assumption. In the Pareto case, for example, it can be shown that

$$E_{\psi}V_{e} \left( \tilde{\theta} \right) = \begin{cases} \frac{2}{(\alpha-1)(\beta_{1}-\alpha)(\alpha-\beta_{2})\sigma^{2}} \tilde{\theta}^{\alpha} - \frac{A_{e1}}{\beta_{1}-\alpha} \tilde{\theta}^{\beta_{1}} & \text{if } \tilde{\theta} < 1 \\ \frac{A_{e2}}{\alpha_{e2}} \tilde{\theta}^{\beta_{2}} + \frac{\alpha_{e2}}{(r-\mu)(\alpha-1)\tilde{\theta} - \frac{1}{r}} & \text{if } \tilde{\theta} \geq 1 \end{cases}$$

for some constants $A_{e1}$ and $A_{e2}$, which clearly satisfies a polynomial growth condition.

First, we prove the following result,

**Lemma C.1.** Exporting is optimal for an inexperienced firm when $\theta > \tilde{\theta}$.

*Proof.* Exporting while $\theta > \tilde{\theta}$ yields additional flow profits $\pi_{i}(\theta) \geq 0$ in $[\tilde{\theta}, +\infty)$ if the firm remains inexperienced and increases the odds of becoming experienced, which increases profits in expectation by $E_{\psi} \left( \max \left\{ \pi_{e}(\theta;\psi), 0 \right\} \right) - \max \left\{ \pi_{i}(\theta), 0 \right\} \geq 0 \ \forall \theta$. Hence, exporting is optimal in this region.

D

Define $\pi^{EE}(\theta) \equiv E_{\psi} \left( \max \left\{ \pi_{e}(\theta;\psi), 0 \right\} \right)$. Note that the flow benefits of exporting ($W$) are given by

$$W = \pi_{i} + \lambda (V_{e} - V_{i}).$$

Since $y_{i}$ is piecewise continuous, $V_{i}$ is continuous. Given that $\pi_{i}$ and $V_{e}$ are continuous, this implies $W$ is continuous. Assuming an indifferent firm exports, a firm will export iff $W \geq 0$. By Assumption C.1 and the possibility of inaction we know that $0 \leq V_{i}(\theta) \leq V_{e}(\theta) < \infty \ \forall \theta$. Moreover, since $W$ is continuous and $\pi_{e}$ and $\pi_{i}$ are continuous, by the Feynman-Kac Theorem we know that $V_{i}, V_{e} \in C^{2}$ and, thus, $W \in C^{2}$. Hence, $V_{e}$ and $V_{i}$ satisfy the following Hamilton-Jacobi-Bellman equations,

$$rV^{E} = \pi^{EE} + \mu_{\theta} \frac{dV_{e}}{d\theta} + \frac{1}{2} \frac{d^{2}V_{e}}{d\theta^{2}} \ \forall \theta$$

(C.2)
\[(r + \lambda) V_i = \pi_i + \lambda V_e + \mu \frac{dV_i}{d\theta} + \frac{1}{2} \sigma^2 \frac{d^2V_i}{d\theta^2} \text{ when } W(\theta) \geq 0 \quad (C.3)\]

\[r V^I = \mu \frac{dV^I}{d\theta} + \frac{1}{2} \sigma^2 \frac{dV^I}{d\theta} \text{ when } W(\theta) < 0 \quad (C.4)\]

Next, subtract (C.12) and (C.13) from (C.2) to obtain,

\[(r + \lambda) (V_e - V_i) = \pi^{EE} - \pi_i + \mu \left( \frac{dV_e}{d\theta} - \frac{dV_i}{d\theta} \right) + \frac{1}{2} \sigma^2 \left( \frac{d^2V_e}{d\theta^2} - \frac{d^2V_i}{d\theta^2} \right) \text{ when } W(\theta) \geq 0 \quad (C.5)\]

\[r (V_e - V_i) = \pi^{EE} + \mu \left( \frac{dV_e}{d\theta} - \frac{dV_i}{d\theta} \right) + \frac{1}{2} \sigma^2 \left( \frac{d^2V_e}{d\theta^2} - \frac{d^2V_i}{d\theta^2} \right) \text{ when } W(\theta) < 0 \quad (C.6)\]

Rewrite (C.5) and (C.6) in terms of \( W \) to obtain

\[\left( \frac{r + \lambda}{\lambda} \right) (W - \pi_i) = \pi^{EE} - \pi_i + \mu \frac{dW}{d\theta} - \frac{d\pi_i}{d\theta} + \frac{1}{2} \sigma^2 \left( \frac{d^2W}{d\theta^2} - \frac{d^2\pi_i}{d\theta^2} \right) \text{ when } W(\theta) \geq 0 \]

\[\left( \frac{r}{\lambda} \right) (W - \pi_i) = \pi^{EE} + \mu \left( \frac{dW}{d\theta} - \frac{d\pi_i}{d\theta} \right) + \frac{1}{2} \sigma^2 \left( \frac{d^2W}{d\theta^2} - \frac{d^2\pi_i}{d\theta^2} \right) \text{ when } W(\theta) < 0 \]

where we used the fact that \( \pi_i \in \mathcal{C}^2 \). Rearranging,

\[(1 + \frac{r}{\lambda}) W = \pi^{EE} + \frac{r}{\lambda} \pi_i - \frac{\mu}{\lambda} \frac{d\pi_i}{d\theta} - \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2\pi_i}{d\theta^2} + \mu \frac{dW}{d\theta} + \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2W}{d\theta^2} \text{ when } W(\theta) \geq 0 \]

\[(1 + \frac{r}{\lambda}) W = W + \pi^{EE} + \frac{r}{\lambda} \pi_i - \frac{\mu}{\lambda} \frac{d\pi_i}{d\theta} - \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2\pi_i}{d\theta^2} + \mu \frac{dW}{d\theta} + \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2W}{d\theta^2} \text{ when } W(\theta) < 0.\]

Define \( h \equiv \pi^{EE} + \frac{r}{\lambda} \pi_i - \frac{\mu}{\lambda} \frac{d\pi_i}{d\theta} - \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2\pi_i}{d\theta^2} \), which is exactly Assumption C.1 after applying Ito's Lemma. We can rewrite this as

\[(1 + \frac{r}{\lambda}) W = W1_{W < 0} + \pi^{EE} + \frac{r}{\lambda} \pi_i - \frac{\mu}{\lambda} \frac{d\pi_i}{d\theta} - \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2\pi_i}{d\theta^2} + \mu \frac{dW}{d\theta} + \frac{1}{2} \frac{\sigma^2}{\lambda} \frac{d^2W}{d\theta^2} \quad (C.7)\]

By assumptions C.2 and C.5, we know that \( W \) and \( h \) satisfy a polynomial growth condition. Furthermore, we know that \( W \) is continuous. Hence, by Feynman-Kac theorem (Duffie, Appendix E, p.344), the unique solution that satisfies a polynomial growth condition to (C.7) is given by

\[W(\theta_0) = E \left( \int_0^\infty e^{-(1+\frac{\lambda}{\lambda})t} \left\{ W(\theta_t) 1_{W(\theta_t) < 0} + h(\theta_t) \right\} d\theta_t | \theta_0 \right).\]

The solution requires \( W \geq 0 \) for \( \theta \geq \tilde{\theta} \). Thus, \( W \) solves

\[W(\theta) = E \left( \int_0^\infty e^{-(1+\frac{\lambda}{\lambda})t} \left\{ W(\theta_t) 1_{W(\theta_t) < 0 \cap \theta_t < \tilde{\theta}} + h(\theta_t) \right\} d\theta_t | \theta_0 \right) \quad (C.8)\]

\[W(\theta) \geq 0 \text{ for } \theta_t \geq \tilde{\theta} \quad (C.9)\]
First, we solve (C.8) disregarding (C.9),

$$W'(\theta) = E \left( \int_0^\infty e^{-\left(1+\frac{r}{k}\right)t} \left\{ W(\theta_t) 1_{W(\theta_t)<\theta} + h(\theta_t) \right\} d\theta_t | \theta_0 \right)$$  \hfill (C.10)

**Lemma C.2.** There is a unique continuous solution $\bar{W}$ to (C.10).

**Proof.** Define the operator $T : C(X) \to C(X)$ as the RHS on (C.10) restricted to $[0, \bar{\theta}]$, where $C$ is the space of continuous and bounded functions. Note that $T$ is well-defined in the sense that if $f \in C$, $Tf \in C$.

Next, we show that $T$ satisfies monotonicity and discounting:

(i) Monotonicity. Take $f \geq g$. Then,

$$Tf(\theta_0) = E\left[ \int_0^\infty e^{-\left(1+\frac{r}{k}\right)t} \{f(\theta_t) 1_{f(\theta_t)<\theta} + h(\theta_t)\} d\theta_t | \theta_0 \right] \geq E\left[ \int_0^\infty e^{-\left(1+\frac{r}{k}\right)t} \{g(\theta_t) 1_{g(\theta_t)<\theta} + h(\theta_t)\} d\theta_t | \theta_0 \right]$$

The first step uses $f \geq g$ while the second step uses the fact that if $f(z) < 0 \Rightarrow g(z) < 0$ so $g(z) 1_{g(z)<\theta} = g(z) 1_{f(z)<\theta} + g(z) 1_{f(z)\geq\theta} 1_{g(z)<\theta} \leq g(z) 1_{f(z)<\theta}$.

(ii) Discounting. Take $a > 0$. Then,

$$T(f(\theta_0) + a) = E\left[ \int_0^\infty e^{-\left(1+\frac{r}{k}\right)t} \{(f(\theta_t) + a) 1_{f(\theta_t)+a<\theta} + h(\theta_t)\} d\theta_t | \theta_0 \right]$$

Since $r > 0$ by Assumption C.5, the result follows. Thus, by Blackwell’s theorem $T : C(X) \to C(X)$ is a contraction. Since $\bar{W}_{\{0,\bar{\theta}\}} \in C(X)$, by the contraction mapping theorem there exists a unique continuous $\bar{W} : [0, \bar{\theta}] \to \mathbb{R}$ that solves (C.10). Given this, $\bar{W}(\theta)$ for $\theta > \bar{\theta}$ is uniquely defined from (C.10).

Next, we show that $W = \bar{W}$.

**Lemma C.3.** $W = \bar{W}$

**Proof.** Let $V_i$ be the value function associated with the export strategy 'export iff $\bar{W} \geq 0$ or $\theta \geq \bar{\theta}$'. Note that $\bar{W} = \pi_i + \lambda (V_e - V_i)$. Since $V_i \leq V_e \leq V_e$, it follows that $\bar{W} (\theta) \geq \pi_i (\theta) \geq 0 \ \forall \theta \geq \bar{\theta}$. Hence, $\bar{W}$ is a solution of (C.8). Note that since $\bar{W}$ is unique, there can be no other continuous solution.

**Lemma C.4.** $W$ is weakly increasing
Proof. Take some weakly increasing function \( f \) and apply \( T \) for \( \theta \in [0, \bar{\theta}] \),

\[
Tf (\theta) = E \left( \int_0^\infty e^{-\frac{1+t}{2}} \left\{ f(\theta_t) I_{f(\theta_t) < 0 \cap \theta_t < \bar{\theta}} + h(\theta_t) \right\} d\theta_t | \theta_0 \right)
\]

Since \( f(z) \, 1_{f(z) < 0 \cap \theta < \bar{\theta}} + h(z) \) is weakly increasing and \( \theta \) has the FOSD property, \( Tf \) is also weakly increasing. Since the space of bounded, continuous and weakly increasing functions is complete, \( \tilde{W} \) is also weakly increasing in \([0, \bar{\theta}]\). By Lemma 3, \( W(\theta) \) is weakly increasing in \([0, \bar{\theta}]\). Since \( W \geq 0 \) for \( \theta \geq \bar{\theta} \), (C.8) immediately implies \( W \) is weakly increasing also for \( \theta \geq \bar{\theta} \).

Now we are ready to prove the main result,

**Proposition.** The unique piecewise continuous optimal strategy features a threshold \( \theta^* \) for \( \theta < \theta^* \) not exporting is optimal while for \( \theta > \theta^* \) exporting is optimal.

Proof. Since \( W \) is continuous, \( V_i \in C^2 \) everywhere and satisfies the following HJB,

\[
rV^I = \max \{ W, 0 \} + \mu \frac{dV^I}{d\theta} + \frac{1}{2} \sigma^2 \theta^2 \frac{d^2V^i}{d\theta^2}.
\]

If \( \lambda = 0 \), then \( W = \pi_i \) and by A1 the result follows. If \( \lambda > 0 \), then by Lemma 3 \( W \) is weakly increasing, so it follows that there exists a unique \( \theta^* \in [0, \bar{\theta}] \) such that \( W \geq 0 \) for \( \theta > \theta^* \) and \( W < 0 \) for \( \theta < \theta^* \).

Note that since \( \theta^* \leq \bar{\theta} = \frac{E}{\kappa}, \bar{\theta}^* \leq 1 \). Furthermore, since \( W(\bar{\theta}^*) = 0 \) and \( E \psi V_e - V_i > 0 \) when \( \bar{\theta} = 1 \) and \( \psi \neq 1 \), it follows that \( \bar{\theta}^* < 1 \) in this case.

**C.1.2 Derivation of the threshold equation (3.8)**

In the GBM case, the HJB equations become

\[
rE_\psi (V_e) = \pi^E (\theta_t) + \mu \frac{dE_\psi V_e}{d\theta} + \frac{1}{2} \sigma^2 \frac{d^2E_\psi V_e}{d\theta^2}
\]

(C.11)

\[
(r + \lambda) V_i = \pi_i + \lambda E_\psi V_e + \mu \frac{dV_i}{d\theta} + \frac{1}{2} \sigma^2 \frac{d^2V_i}{d\theta^2} \text{ when } \theta > \theta^*
\]

(C.12)

\[
rV^I = \mu \frac{dV_i}{d\theta} + \frac{1}{2} \sigma^2 \frac{d^2V_i}{d\theta^2} \text{ when } \theta < \theta^*
\]

(C.13)

Define \( \Delta V \equiv E_\psi (V_e) - V_i \). Substracting (C.12) and (C.13) from (C.11) yields

\[
(r + \lambda) \Delta V = \pi^E (\theta) - \pi_i + \mu \frac{d\Delta V}{d\theta} + \frac{1}{2} \sigma^2 \frac{d^2\Delta V}{d\theta^2} \text{ when } \theta > \theta^*
\]

(C.14)

\[
r \Delta V = \pi^E (\theta) + \mu \frac{d\Delta V}{d\theta} + \frac{1}{2} \sigma^2 \frac{d^2\Delta V}{d\theta^2} \text{ when } \theta < \theta^*
\]

(C.15)
When $\theta > \theta^*$, the solution to (C.14) is given by

$$\Delta V(\theta) = \frac{1}{J} \left[ \int_{\theta}^{\infty} \left( \frac{\theta}{z} \right) \tilde{\beta}_1 \left( \pi^{EE}(z) - \pi_1(z) \right) \frac{dz}{z} + \int_{\theta}^{\theta^*} \left( \frac{\theta}{z} \right) \tilde{\beta}_2 \left( \pi^{EE}(z) - \pi_1(z) \right) \frac{dz}{z} \right] + C_{1U} \theta^{\tilde{\beta}_1} + C_{2U} \theta^{\tilde{\beta}_2}$$

where

$$\tilde{J} = \sqrt{\left( \mu - \frac{1}{2} \sigma^2 \right)^2 + 2(r + \lambda) \sigma^2} \geq \left| \mu - \frac{1}{2} \sigma^2 \right|$$

$$\tilde{\beta}_1 = \frac{- \left( \mu - \frac{1}{2} \sigma^2 \right) + \tilde{J}}{\sigma^2} > 1$$

$$\tilde{\beta}_2 = \frac{- \left( \mu - \frac{1}{2} \sigma^2 \right) - \tilde{J}}{\sigma^2} < 0$$

and $C_{1U}$ and $C_{2U}$ are unknown constants. Using the transversality condition, $C_{1U} = 0$.

Note the derivative wrt $\theta$ is

$$\frac{d\Delta V}{d\theta} = \frac{1}{J} \left[ \tilde{\beta}_1 \int_{\theta}^{\infty} \left( \frac{\theta}{z} \right) \left( \pi^{EE}(z) - \pi_1(z) \right) \frac{dz}{z} + \tilde{\beta}_2 \int_{\theta}^{\theta^*} \left( \frac{\theta}{z} \right) \left( \pi^{EE}(z) - \pi_1(z) \right) \frac{dz}{z} \right]$$

When $\theta < \theta^*$, the solution to (C.15) is given by

$$\Delta V(\theta) = \frac{1}{J} \left[ \int_{\theta}^{\theta^*} \left( \frac{\theta}{z} \right) \beta_1 \pi^{EE}(z) \frac{dz}{z} + \int_{0}^{\theta} \left( \frac{\theta}{z} \right) \beta_2 \pi^{EE}(z) \frac{dz}{z} \right] + C_{1D} \theta^{\beta_1} + C_{2D} \theta^{\beta_2}$$

where

$$J = \sqrt{\left( \mu - \frac{1}{2} \sigma^2 \right)^2 + 2\sigma^2} \geq \left| \mu - \frac{1}{2} \sigma^2 \right|$$

$$\beta_1 = \frac{- \left( \mu - \frac{1}{2} \sigma^2 \right) + J}{\sigma^2} > 1$$

$$\beta_2 = \frac{- \left( \mu - \frac{1}{2} \sigma^2 \right) - J}{\sigma^2} < 0$$

and $C_{1D}$ and $C_{2D}$ are unknown constants. Using the initial condition $\Delta V(0) = 0, C_{2D} = 0$.

\[^3\text{See formula 5.24. in Stokey (2008).}\]
Note the derivative wrt \( \theta \) is

\[
\frac{d\Delta V}{d\theta} = \frac{1}{J} \left[ \beta_1 \int_{\theta}^{\theta^*} \left( \frac{\theta}{z} \right) \beta_1 \pi_{EE} (z) \frac{dz}{z} + \beta_2 \int_{\theta}^{\theta^*} \left( \frac{\theta}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z} + \beta_1 C_{1D} \theta^{\beta_1} \right]
\]

We have three unknowns, \( C_{1D}, C_{2U} \) and \( \theta^* \). Using the fact that \( \Delta V \) is C1 at \( \theta^* \),

\[
\frac{1}{J} \left[ \int_{\theta^*}^{\infty} \left( \frac{\theta^*}{z} \right) \delta_1 \left( \pi_{EE} (z) - \pi_i (z) \right) \frac{dz}{z} \right] + C_{2U} \theta^* \beta_2 = \frac{1}{J} \left[ \int_{0}^{\theta^*} \left( \frac{\theta^*}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z} \right] + C_{1D} \theta^{\beta_1}
\]

\[
\frac{1}{J} \left[ \beta_1 \int_{\theta^*}^{\infty} \left( \frac{\theta^*}{z} \right) \delta_1 \left( \pi_{EE} (z) - \pi_i (z) \right) \frac{dz}{z} \right] + \beta_2 C_{2U} \theta^* \beta_2 = \frac{1}{J} \left[ \beta_2 \int_{0}^{\theta^*} \left( \frac{\theta^*}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z} \right] + \beta_1 C_{1D} \theta^{\beta_1}.
\]

Next, multiply the first equation by \( \beta_1 \) and substract the second equation to obtain,

\[
\left( \frac{\beta_1 - \tilde{\beta}_1}{J} \right) \int_{\theta^*}^{\infty} \left( \frac{\theta^*}{z} \right) \delta_1 \left( \pi_{EE} (z) - \pi_i (z) \right) \frac{dz}{z} + \left( \beta_1 - \tilde{\beta}_2 \right) C_{2U} \theta^* \beta_2 = \left( \frac{\beta_1 - \tilde{\beta}_2}{J} \right) \int_{0}^{\theta^*} \left( \frac{\theta^*}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z}
\]

Rearranging,

\[
C_{2U} = \frac{\theta^* - \tilde{\beta}_2}{\beta_1 - \tilde{\beta}_2} \left( \frac{\beta_1 - \tilde{\beta}_2}{J} \right) \int_{0}^{\theta^*} \left( \frac{\theta^*}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z} + \left( \frac{\beta_1 - \tilde{\beta}_1}{J} \right) \int_{\theta^*}^{\infty} \left( \frac{\theta^*}{z} \right) \delta_1 \left( \pi_{EE} (z) - \pi_i (z) \right) \frac{dz}{z}
\]  \hspace{1cm} (C.16)

Since \( \pi_{EE} - \pi_i \geq 0 \) and \( \tilde{\beta}_1 \geq \beta_1 \), it follows that \( C_{2U} \geq 0 \).

Next, multiply the first equation by \( \tilde{\beta}_2 \) and substract the second equation to obtain,

\[
\left( \frac{\tilde{\beta}_2 - \tilde{\beta}_1}{J} \right) \int_{\theta^*}^{\infty} \left( \frac{\theta^*}{z} \right) \delta_1 \left( \pi_{EE} (z) - \pi_i (z) \right) \frac{dz}{z} = \left( \frac{\tilde{\beta}_2 - \tilde{\beta}_1}{J} \right) \int_{0}^{\theta^*} \left( \frac{\theta^*}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z} + \left( \tilde{\beta}_2 - \beta_1 \right) C_{1D} \theta^* \beta_1
\]

Rearranging,

\[
C_{1D} = \frac{\theta^* - \beta_1}{\beta_1 - \tilde{\beta}_1} \left( \frac{\tilde{\beta}_1 - \tilde{\beta}_2}{J} \right) \int_{\theta^*}^{\infty} \left( \frac{\theta^*}{z} \right) \delta_1 \left( \pi_{EE} (z) - \pi_i (z) \right) \frac{dz}{z} + \left( \tilde{\beta}_2 - \tilde{\beta}_1 \right) \int_{0}^{\theta^*} \left( \frac{\theta^*}{z} \right) \beta_2 \pi_{EE} (z) \frac{dz}{z}
\]  \hspace{1cm} (C.17)
The remaining equation is the fact that by continuity the conjecture can only be true if at the threshold the firm is indifferent between exporting and not exporting, i.e. \( \pi_i(\theta^*) + \lambda \Delta V(\theta^*) = 0 \),

\[
\pi_i(\theta^*) + \frac{1}{J^2} \lambda \left[ \int_{\theta^*}^{\theta^*+} \frac{\hat{\beta}_1}{z} \left( \pi^E(z) - \pi_i(z) \right) \frac{dz}{z} + C_{2U}(\theta^*) \right] = 0
\]

Substituting in (C.16),

\[
\pi_i(\theta^*) + \frac{1}{J} \int_{\theta^*}^{\theta^*+} \left( \frac{\hat{\beta}_1}{z} \right) (\pi^E(z) - \pi_i(z)) \frac{dz}{z} + \left( \frac{1}{J} \right) (\frac{\hat{\beta}_1 - \hat{\beta}_2}{\beta_1 - \beta_2}) \int_{0}^{\theta^*} \left( \frac{\hat{\beta}_2}{z} \right) (\pi^E(z) - \pi_i(z)) \frac{dz}{z} = 0
\]

Simplifying,

\[
\pi_i(\theta^*) + \frac{\lambda}{\beta_1 - \beta_2} \left[ \left( \frac{\hat{\beta}_1 - \hat{\beta}_2}{\beta_1 - \beta_2} \right) \frac{1}{J} \int_{\theta^*}^{\theta^*+} \left( \frac{\hat{\beta}_1}{z} \right) (\pi^E(z) - \pi_i(z)) \frac{dz}{z} + \frac{1}{J} (\beta_1 - \beta_2) \int_{0}^{\theta^*} \left( \frac{\hat{\beta}_2}{z} \right) (\pi^E(z) - \pi_i(z)) \frac{dz}{z} \right] = 0.
\]  

(C.18)

Next, note

\[
\begin{align*}
\beta_1 - \beta_2 &= \frac{2J}{\sigma^2} \\
\hat{\beta}_1 - \hat{\beta}_2 &= \frac{2\hat{J}}{\sigma^2} \\
\hat{\beta}_1 - \hat{\beta}_2 &= \frac{J + \hat{J}}{\sigma^2}
\end{align*}
\]

Thus,

\[
\pi_i(\theta^*) + \lambda \left( \frac{2}{J + \hat{J}} \right) \left[ \int_{\theta^*}^{\theta^*+} \left( \frac{\hat{\beta}_1}{z} \right) (\pi^E(z) - \pi_i(z)) \frac{dz}{z} + \int_{0}^{\theta^*} \left( \frac{\hat{\beta}_2}{z} \right) (\pi^E(z) - \pi_i(z)) \frac{dz}{z} \right] = 0.
\]

As suggested in the text, this equation shows that the model boils down to one equation in one unknown even if \( \psi \) is not multiplicative. For the case in the text, note \( \pi^E = E_{\psi} \left[ \max \left\{ \psi \frac{\kappa z}{\hat{F}} - 1, 0 \right\} \right] \) and \( \pi_i = \kappa \theta - F \).

Replacing,

\[
\kappa \theta - F + \lambda \left( \frac{2}{J + \hat{J}} \right) \left[ \int_{\theta^*}^{\theta^*+} \left( \frac{\hat{\beta}_1}{z} \right) (E_{\psi} \left[ \max \left\{ \psi \kappa z - F, 0 \right\} \right]) \frac{dz}{z} \right] = 0
\]

In terms of \( \tilde{\theta} \) and redefining \( z = \frac{\kappa z}{\hat{F}} \),

\[
\tilde{\theta} - 1 + \lambda \left( \frac{2}{J + \hat{J}} \right) \left[ \int_{\tilde{\theta}^*}^{\tilde{\theta}^*+} \left( \frac{\hat{\beta}_1}{z} \right) (E_{\psi} \left[ \max \left\{ \psi z - 1, 0 \right\} \right]) \frac{dz}{z} \right] = 0.
\]

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C.1.3 Proof of Proposition 3.3

First, we show that when $\lambda \to \infty$, $\tilde{\theta}^*(\lambda) \to 0$. Recall the threshold equation,

$$
\tilde{\theta}^*(\lambda) - 1 + \lambda \left( \frac{2}{J + J} \right) \left[ \int_0^\infty \frac{\tilde{\theta}^*(\lambda)}{z} \frac{\partial \tilde{\theta}^*(\lambda)}{z} (E_\psi [\max (\psi z - 1, 0]) - (z - 1)) \frac{dz}{z} 
+ \int_0^\infty \frac{\tilde{\theta}^*(\lambda)}{z} \frac{\partial \tilde{\theta}^*(\lambda)}{z} E_\psi [\max (\psi z - 1, 0)] \frac{dz}{z} \right] = 0.
$$

Since $\frac{\lambda}{J+J} \to \infty$, it must be that

$$
\lim_{\lambda \to \infty} \left[ \int_0^\infty \frac{\tilde{\theta}^*(\lambda)}{z} \frac{\partial \tilde{\theta}^*(\lambda)}{z} (E_\psi [\max (\psi z - 1, 0]) - (z - 1)) \frac{dz}{z} 
+ \int_0^\infty \frac{\tilde{\theta}^*(\lambda)}{z} \frac{\partial \tilde{\theta}^*(\lambda)}{z} E_\psi [\max (\psi z - 1, 0)] \frac{dz}{z} \right] = 0.
$$

Note that for any $\theta^*$, the first term goes to 0. Then,

$$
\lim_{\lambda \to \infty} \left[ \int_0^\infty \frac{\tilde{\theta}^*(\lambda)}{z} \frac{\partial \tilde{\theta}^*(\lambda)}{z} E_\psi [\max (\psi z - 1, 0)] \frac{dz}{z} \right] = 0.
$$

Since $h > 0 \forall \psi > M$, $E_\psi [\max (\psi z - 1, 0)] > 0 \forall z \neq 0$. Then,

$$
\lim_{\lambda \to \infty} \theta^*(\lambda) = 0.
$$

Next, recall the formula for $p_T$,

$$
p_T = \begin{cases} 
\int_{-\infty}^0 \int_{s=0}^T \left( 1 - e^{-\lambda s} \right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln (\psi m \tilde{\theta}^*)}{\sigma} \right) \omega_T (s, x) ds dx \\
\int_0^\infty \int_{s=0}^T \left( 1 - e^{-\lambda s} \right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln (\psi m \tilde{\theta}^*)}{\sigma} \right) + e^{-\lambda s} \right) \omega_T (s, x) ds dx
\end{cases}
$$

Note that $p_{BT}$ is given by

$$
p_{BT} = \int_0^\infty \int_{s=0}^T \omega_T (s, x) ds dx
$$

Subtracting $p_{BT}$ from $p_T$,

$$
p_T - p_{BT} = \int_{-\infty}^0 \int_{s=0}^T \left( 1 - e^{-\lambda s} \right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln (\psi m \tilde{\theta}^*)}{\sigma} \right) \omega_T (s, x) ds dx \\
- \int_0^\infty \int_{s=0}^T \left( 1 - e^{-\lambda s} \right) \left( 1 - P \left( \ln \tilde{\psi} > -x_T - \frac{\ln (\psi m \tilde{\theta}^*)}{\sigma} \right) \right) \omega_T (s, x) ds dx
$$
Next, pick some $\tilde{z} > 0$ and rewrite the previous expression as

$$p_T - p_{BT} = \int_{-\infty}^{\tilde{z}} \int_{s=0}^{T} \left( 1 - e^{-\lambda s} \right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{0}^{\tilde{z}} \int_{s=0}^{T} \left( 1 - e^{-\lambda s} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{\tilde{z}}^{\infty} \int_{s=0}^{T} \left\{ \left( 1 - e^{-\lambda s} \right) \left( 1 - P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \right) \right\} \omega_T(s, x) \, ds \, dx$$

Next, pick some $\tilde{s} < T$ and note

$$p_T - p_{BT} < \int_{-\infty}^{\tilde{z}} \int_{s=0}^{T} \left( 1 - e^{-\lambda s} \right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{0}^{\tilde{z}} \int_{s=\tilde{s}}^{T} \left( 1 - e^{-\lambda s} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{\tilde{z}}^{\infty} \int_{s=\tilde{s}}^{T} \left\{ \left( 1 - e^{-\lambda s} \right) \left( 1 - P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \right) \right\} \omega_T(s, x) \, ds \, dx$$

Next, note that since $1 - e^{-\lambda s} \to 1$ when $\lambda \to \infty$, then $\forall \epsilon_1 \in (0, 1)$, $\exists \lambda_1$ such that for $\lambda > \lambda_1$, $1 - e^{-\lambda s} > 1 - \epsilon_1 \forall s > \delta$. Hence, for $\lambda > \lambda_1$,

$$p_T - p_{BT} < \int_{-\infty}^{\tilde{z}} \int_{s=0}^{T} \left( 1 - e^{-\lambda s} \right) P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{0}^{\tilde{z}} \int_{s=\delta}^{T} \left( 1 - e^{-\lambda s} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{\tilde{z}}^{\infty} \int_{s=\delta}^{T} \left\{ \left( 1 - e^{-\lambda s} \right) \left( 1 - P \left( \ln \tilde{\psi} > -x_T - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \right) \right\} \omega_T(s, x) \, ds \, dx$$

Since $1 - e^{-\lambda s} < 1$,

$$p_T - p_{BT} < \int_{-\infty}^{\tilde{z}} \int_{s=0}^{T} P \left( \ln \tilde{\psi} > -x - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \omega_T(s, x) \, ds \, dx - \int_{0}^{\tilde{z}} \int_{s=\delta}^{T} \left( 1 - e^{-\lambda s} \right) \omega_T(s, x) \, ds \, dx$$

$$- \int_{\tilde{z}}^{\infty} \int_{s=\delta}^{T} \left\{ \left( 1 - e^{-\lambda s} \right) \left( 1 - P \left( \ln \tilde{\psi} > -x - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) \right) \right\} \omega_T(s, x) \, ds \, dx$$

Next, note $\forall \epsilon > 0, \tilde{z} > 0 \exists \lambda_2$ such that $\forall \lambda > \lambda_2$, $P \left( \ln \tilde{\psi} > -\tilde{z} - \frac{\ln(\psi_m \tilde{\theta}^*)}{\sigma} \right) < \epsilon$ (this uses the fact that for all $M$ there exists $\lambda$ such that $\forall \lambda > \lambda$, $\theta^* < M$). Next, pick $\epsilon_2 > 0$ and corresponding $\lambda_2$ such that

$$\epsilon_2 < (1 - \epsilon_1) \inf_{T \in \mathbb{Z}, T} \left( \frac{\int_{-\infty}^{\tilde{z}} \int_{s=\delta}^{T} \omega_T(s, x) \, ds \, dx}{\int_{-\infty}^{\tilde{z}} \int_{s=0}^{T} \omega_T(s, x) \, ds \, dx} \right). \tag{C.19}$$

Since $\delta < T$, the numerator is strictly positive $\forall T$. Since $T > 0$, the denominator is also strictly positive. Since $\omega$ is continuous in $T$, the integrals are continuous functions of $T$ and, since the denominator is always strictly positive, the quotient of the integrals is continuous in $T$. Hence, given that $[T, \tilde{T}]$ is compact, the
infimum is attained and, thus, \( \inf_{t \in [T, T]} \left( \frac{\int_0^T \int_{s=0}^{t} \omega(s, z; t) ds dz}{\int_{s=0}^{T} \int_{s=0}^{T} \omega(s, z; t) ds dz} \right) > 0 \). Hence, \( \epsilon_2 \) is well-defined.

Picking \( \hat{\lambda}_3 > \max \{ \hat{\lambda}_1, \hat{\lambda}_2 \} \) and noting
\[
\mathbb{P} \left( \ln \tilde{\psi} > -z - \frac{\ln (\psi_m \tilde{\theta}^*)}{\sigma} \right) \leq \mathbb{P} \left( \ln \tilde{\psi} > -\hat{z} - \frac{\ln (\psi_m \tilde{\theta}^*)}{\sigma} \right) < \epsilon_2
\]
for \( z \leq \hat{z} \), implies
\[
p_T - p_{BT} < \epsilon_2 \int_{-\infty}^{\hat{z}} \int_{s=0}^{T} \omega_T(s, x) ds dx - (1 - \epsilon_1) \int_{-\infty}^{\hat{z}} \int_{s=0}^{T} \omega_T(s, x) ds dx
\]
\[
- \int_{\hat{z}}^{\infty} \int_{s=0}^{T} \left\{ (1 - e^{-\lambda s}) \left( 1 - \mathbb{P} \left( \ln \tilde{\psi} > -e^{-\lambda s} \right) \right) \right\} \omega_T(s, x) ds dx
\]

From the definition of \( \epsilon_2 \) it follows that
\[
\epsilon_2 \int_{-\infty}^{\hat{z}} \int_{s=0}^{T} \omega_T(s, z) ds dz < (1 - \epsilon_1) \int_{-\infty}^{\hat{z}} \int_{s=0}^{T} \omega_T(s, z) ds dz
\]
\[
\forall T \in [T, T]. \text{ Hence, picking } \hat{\lambda}_3 = \max \{ \hat{\lambda}_1, \hat{\lambda}_2 \},
\]
\[
p_T - p_{BT} < - \int_{\hat{z}}^{\infty} \int_{s=0}^{T} \left\{ (1 - e^{-\lambda s}) \left( 1 - \mathbb{P} \left( \ln \tilde{\psi} > -e^{-\lambda s} \right) \right) \right\} \omega_T(s, x) ds dx,
\]
which immediately implies \( p_T - p_{BT} < 0 \).

### C.1.4 Proof of Proposition 3.4

Define \( \tilde{\theta} = \psi_m \tilde{\theta} \) and \( \tilde{\psi} = \frac{\psi}{\psi_m} \) and rewrite equation (3.8),
\[
\frac{1}{\psi_m} \tilde{\theta} - 1 + \lambda \left( \frac{2}{J + \tilde{J}} \right) \left[ \int_{0}^{\infty} \frac{\tilde{\theta} \left( \frac{\tilde{\theta}^*}{\psi_m z} \right) \tilde{\psi} \left( E \max \{ \psi_m \tilde{\psi} z - 1 \} - (z - 1) \right) dz}{z} \right] = 0
\]
Let \( \hat{z} \equiv \psi_m z \). Then,

\[
\begin{align*}
+\lambda \left( \frac{2}{J + \hat{J}} \right) & \left[ \int_{\hat{\theta}}^{\infty} \left( \frac{\hat{\theta}}{\hat{J}} \right)^{\hat{\beta}_1} \left( E \max \left\{ \hat{\psi} z - 1, 0 \right\} - \frac{1}{\psi_m} \hat{z} + 1 \right) \frac{d\hat{z}}{\hat{J}} \right] + \frac{1}{\psi_m} \hat{\theta} - 1 \\
+\lambda \left( \frac{2}{J + \hat{J}} \right) & \left[ \int_{\hat{\theta}}^{\infty} \left( \frac{\hat{\theta}}{\hat{J}} \right)^{\hat{\beta}_1} \left( E \max \left\{ \hat{\psi} z - 1, 0 \right\} + 1 \right) \frac{d\hat{z}}{\hat{J}} \right] = 0
\end{align*}
\]

Since \( 1 - \lambda \frac{2}{J + \hat{J}} \frac{1}{\hat{\beta}_1 - 1} \geq 0 \), the LHS decreases with \( \psi_m \).

Changing the dummy of integration to \( m = \frac{\hat{z}}{\hat{\theta}^*} \),

\[
\begin{align*}
\frac{1}{\psi_m} \hat{\theta} \left( 1 - \lambda \frac{2}{J + \hat{J}} \frac{1}{\hat{\beta}_1 - 1} \right) + \lambda \left( \frac{2}{J + \hat{J}} \right) & \left[ \int_{1}^{\infty} m^{-\hat{\beta}_1} \left( E \max \left\{ \hat{\psi} m \hat{\theta}^* - 1, 0 \right\} + 1 \right) \frac{dm}{\hat{J}} \right] = 0
\end{align*}
\]

The first derivative wrt \( \hat{\theta}^* \) yields

\[
\begin{align*}
\frac{1}{\psi_m} \left( 1 - \lambda \frac{2}{J + \hat{J}} \frac{1}{\hat{\beta}_1 - 1} \right) + \lambda \left( \frac{2}{J + \hat{J}} \right) & \left[ \int_{1}^{\infty} m^{-\hat{\beta}_1} \frac{dE}{\max \left\{ \hat{\psi} m \hat{\theta}^* - 1, 0 \right\}} \frac{dm}{\hat{J}} \right] > 0.
\end{align*}
\]

Hence, the LHS increases with \( \hat{\theta}^* \). Thus, by the implicit function theorem, \( \frac{d\hat{\theta}^*}{d\psi_m} > 0 \).
C.2 Facts in a Regression Framework

Table C.1: Facts 1 and 2 controlling for composition

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<th>(4)</th>
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<td>Entrants</td>
<td>Re-entrants</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.361*** (0.0886)</td>
<td>0.358*** (0.00257)</td>
<td>0.360*** (0.0744)</td>
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<td>0.263*** (0.0885)</td>
<td>0.266*** (0.00237)</td>
<td>0.262*** (0.0744)</td>
</tr>
<tr>
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<td>0.222** (0.0885)</td>
<td>0.223*** (0.00223)</td>
<td>0.218*** (0.0745)</td>
</tr>
<tr>
<td>Year 4</td>
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<td>0.195** (0.0885)</td>
<td>0.195*** (0.00212)</td>
<td>0.188** (0.0745)</td>
</tr>
<tr>
<td>Year 5</td>
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<td>0.178** (0.0886)</td>
<td>0.177*** (0.00204)</td>
<td>0.169** (0.0745)</td>
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<td>yes</td>
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<td>0.315</td>
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Clustered errors by firm-destination in parenthesis.

***p < 0.01, **p < 0.05, *p < 0.1.
Table C.2: Effect of type of product on survival

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<td>-0.0634***</td>
<td>-0.0547***</td>
<td>-0.0644***</td>
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<td>(0.00388)</td>
<td>(0.00375)</td>
<td>(0.00387)</td>
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<td>-0.0645***</td>
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<td>-0.0621***</td>
<td>-0.0717***</td>
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<td>Year 3*Diff.</td>
<td>-0.0508***</td>
<td>-0.0593***</td>
<td>-0.0507***</td>
<td>-0.0602***</td>
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<td>(0.00484)</td>
<td>(0.00457)</td>
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<tr>
<td>Year 4*Diff.</td>
<td>-0.0526***</td>
<td>-0.0615***</td>
<td>-0.0531***</td>
<td>-0.0630***</td>
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<tr>
<td></td>
<td>(0.00452)</td>
<td>(0.00465)</td>
<td>(0.00437)</td>
<td>(0.00449)</td>
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<tr>
<td>Year 5*Diff.</td>
<td>-0.0454***</td>
<td>-0.0538***</td>
<td>-0.0429***</td>
<td>-0.0523***</td>
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<td>Re-entr.*Diff</td>
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<td>0.0157*</td>
<td>0.0157*</td>
<td>0.0175**</td>
<td>0.0175**</td>
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<td>no</td>
<td>no</td>
<td>yes</td>
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</tr>
<tr>
<td>Year FE</td>
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<td>170,135</td>
<td>170,135</td>
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<tr>
<td>R-squared</td>
<td>0.265</td>
<td>0.265</td>
<td>0.272</td>
<td>0.297</td>
<td>0.297</td>
<td>0.303</td>
<td>0.303</td>
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</table>

All regressions include horizon dummies. Regressions in columns (5)-(8) also include the interaction of horizon and re-entrant dummies. They are omitted to save space, since the previous results are roughly unchanged. Clustered errors by firm-destination in parenthesis.

***p < 0.01, **p < 0.05, *p < 0.1.
### Table C.3: Effect of distance on survival

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<th>(1)</th>
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<tbody>
<tr>
<td>log(dist)*Diff.</td>
<td>-0.0254***</td>
<td>-0.0253***</td>
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<tr>
<td></td>
<td>(0.00289)</td>
<td>(0.00289)</td>
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</tr>
<tr>
<td>log(dist)*Homog.</td>
<td>0.00836**</td>
<td>0.00828**</td>
<td></td>
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<tr>
<td></td>
<td>(0.00377)</td>
<td>(0.00377)</td>
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<tr>
<td>Year 1*Diff.*log(dist)</td>
<td>-0.0343***</td>
<td>-0.0340***</td>
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<td></td>
<td>(0.00424)</td>
<td>(0.00424)</td>
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<tr>
<td>Year 2*Diff.*log(dist)</td>
<td>-0.0246***</td>
<td>-0.0247***</td>
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<tr>
<td></td>
<td>(0.00387)</td>
<td>(0.00387)</td>
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<tr>
<td>Year 3*Diff.*log(dist)</td>
<td>-0.0206***</td>
<td>-0.0206***</td>
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<tr>
<td></td>
<td>(0.00366)</td>
<td>(0.00367)</td>
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</tr>
<tr>
<td>Year 4*Diff.*log(dist)</td>
<td>-0.0197***</td>
<td>-0.0197***</td>
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<tr>
<td></td>
<td>(0.00347)</td>
<td>(0.00347)</td>
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<tr>
<td>Year 5*Diff.*log(dist)</td>
<td>-0.0277***</td>
<td>-0.0275***</td>
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<td></td>
<td>(0.00336)</td>
<td>(0.00336)</td>
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<tr>
<td>Year 1*Homog.*log(dist)</td>
<td>-0.00244</td>
<td>-0.00284</td>
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<td></td>
<td>(0.00536)</td>
<td>(0.00536)</td>
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<tr>
<td>Year 2*Homog.*log(dist)</td>
<td>0.0168***</td>
<td>0.0170***</td>
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<tr>
<td></td>
<td>(0.00496)</td>
<td>(0.00497)</td>
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<tr>
<td>Year 3*Homog.*log(dist)</td>
<td>0.00824*</td>
<td>0.00825*</td>
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<tr>
<td></td>
<td>(0.00472)</td>
<td>(0.00472)</td>
<td></td>
<td></td>
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<tr>
<td>Year 4*Homog.*log(dist)</td>
<td>0.00984**</td>
<td>0.00974**</td>
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<tr>
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<td>(0.00454)</td>
<td>(0.00454)</td>
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<tr>
<td>Year 5*Homog.*log(dist)</td>
<td>0.00937**</td>
<td>0.00920**</td>
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<tr>
<td></td>
<td>(0.00438)</td>
<td>(0.00439)</td>
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<tr>
<td>Year FE</td>
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<td>yes</td>
<td>yes</td>
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<td>Observations</td>
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<td>168,315</td>
<td>168,315</td>
<td>168,315</td>
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<td>R-squared</td>
<td>0.267</td>
<td>0.268</td>
<td>0.268</td>
<td>0.268</td>
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</table>

All regressions include horizon dummies, a differentiated good dummy, and the interaction between a differentiated good dummy and horizon dummies (omitted). Clustered errors by firm-destination in parenthesis.

***p < 0.01, **p < 0.05, *p < 0.1.
Table C.4: Effect of distance on survival for differentiated goods only

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<td>Dependent variable: Survival status</td>
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<tr>
<td>log(dist)*Re-ent.</td>
<td>-0.00917</td>
<td>-0.00902</td>
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<td></td>
<td>(0.00728)</td>
<td>(0.00729)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Year 1*log(dist)*Re-ent.</td>
<td>-0.00578</td>
<td>-0.00560</td>
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<td></td>
<td>(0.00986)</td>
<td>(0.00987)</td>
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<tr>
<td>Year 2*log(dist)*Re-ent.</td>
<td>0.00107</td>
<td>0.00121</td>
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<td></td>
<td>(0.00972)</td>
<td>(0.00972)</td>
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<tr>
<td>Year 3*log(dist)*Re-ent.</td>
<td>-0.0102</td>
<td>-0.0101</td>
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<td></td>
<td>(0.00948)</td>
<td>(0.00948)</td>
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<td></td>
</tr>
<tr>
<td>Year 4*log(dist)*Re-ent.</td>
<td>-0.0130</td>
<td>-0.0126</td>
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<tr>
<td></td>
<td>(0.00944)</td>
<td>(0.00944)</td>
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<tr>
<td>Year 5*log(dist)*Re-ent.</td>
<td>-0.0179*</td>
<td>-0.0180*</td>
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<td>(0.00933)</td>
<td>(0.00933)</td>
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<td>0.277</td>
<td>0.277</td>
<td>0.277</td>
</tr>
</tbody>
</table>

All regressions include horizon dummies, a re-entrant dummy, and log(distance), which are omitted. In addition, columns (2) and (4) include the interaction of horizon dummies with log(distance), which are also omitted. Clustered errors by firm-destination in parenthesis.

***p < 0.01, **p < 0.05, *p < 0.1.
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