A Combinatorial Flag Space

by

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Submitted to the Department of Mathematics
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Abstract

The main idea of the proof will be to fix a rank \( k \) oriented matroid \( M \), and consider \((1, 2; M)\), that is the set of pairs of oriented matroids \((N^1, N^2)\) of ranks 1 and 2 respectively with \( N^1 \) a strong image of \( N^2 \) and \( N^2 \) a strong image of \( M \).

Consider also a rank \( k \) coordinate oriented matroid \( M_c \) obtained by deleting all but the elements of one basis in \( M \).

There are now two projection maps, \( \pi_1 : (N^1, N^2) \to N^1 \) and \( \pi_2 : (N^2, N^1) \to N_1 \), and the deletion map taking \( M \) to \( M_c \), which induces maps from \((1, 2; M)\) to \((1, 2; M_c)\) and from \((2; M)\) to \((2; M_c)\) and from \((1; M)\) to \((1; M_c)\).

\( \pi_1 \) and \( \pi_2 \) are then seen to be essentially fibrations and the deletion map a fiber map between these. It is known that the deletion map induces a homotopy equivalence on both fibers, and also from \((1; M)\) to \((1; M_c)\). Thus deletion induces a homotopy equivalence from \((1, 2; M)\) to \((1, 2; M_c)\) and from \((2; M)\) to \((2; M_c)\). Finally the homotopy types of \((2; M_c)\) and \((1, 2; M_c)\) are known.

Most of the work in carrying out the above proof comes in showing that \( \pi_1 \) is essentially a fibration.

This is achieved through the introduction of some intermediate objects denoted hairs, which are pairs \((X, N^1)\) with \( X \) being an initial segment of \( N^2 \) thought of as a set of covectors, beginning at \( N^1 \) and continuing around.

\( \pi_2 \) is then written as a composition of a number of cut maps \( \Psi_k \), each of which either leaves a hair alone, or removes the elements farthest from \( N^1 \).

All but the last of these \( \Psi_2 \) is seen to be a homotopy equivalence while \( \Psi_2 \) is seen to be essentially a fibration.

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## Contents

0.1 Notation ................................................................. 5

0.2 Hairs ................................................................. 6
   0.2.1 Hair Facts ...................................................... 7
   0.2.2 Hair Notations ................................................. 9
   0.2.3 Proofs of hair facts .......................................... 9
   0.2.4 A partial order on hairs .................................... 16
   0.2.5 Initial Facts ................................................ 17
   0.2.6 Proof of Initial Facts ....................................... 17
   0.2.7 Wrapping Facts ............................................... 19
   0.2.8 Proofs of wrapping facts ................................... 19
   0.2.9 Cuts ............................................................. 22
   0.2.10 Hair Cut Proofs ............................................. 23
   0.2.11 EI .............................................................. 28
   0.2.12 Proofs of EI ................................................ 29

0.3 Hairs in Oriented Matroids ......................................... 31
   0.3.1 Short Facts .................................................. 31
   0.3.2 k-hairs ....................................................... 32
   0.3.3 Poset Lemmas ................................................ 33
   0.3.4 Cutting in an oriented matroid ............................. 34
   0.3.5 A Fiber Map ................................................ 34
   0.3.6 Rank 2 Oriented Matroids ................................... 35
0.1 Introduction

This thesis will consider a combinatorial analog to the real flag manifolds. Denote by $G(i_1 < i_2 < \ldots < i_k < n)$ the space $SO(n)/SO(i_1) \times SO(i_2 - i_1) \times \ldots SO(n - i_k)$. This is the real oriented flag manifold whose points are configurations of an oriented $i_1$-space contained in an oriented $i_2$-space ... contained in a fixed $n$-space. Similarly denote by $GI(\ldots )$ the real unoriented flag manifold (defined similarly but replacing $SO$ with $O$). A point in $G(i_1 < \ldots < n)$ may be represented (though not uniquely) as an orthogonal $n$ by $n$ matrix where the first $i_r$ rows span the $i_r$ dimensional space of the flag with the chosen orientation. This representation is just a choice of representative in $SO(n)$ from the first definition. Donote the set $\{1, \ldots, n\}$ by $[n]$. Definition: An oriented matroid of rank $k$ on $n$ elements is a map $M$ from ordered $k$ subsets of $[n]$ to $\{0, +, -\}$ satisfying:

1. $M$ is not the 0 map

2. $M$ is antisymmetric (here $-0 = 0, -+ = -, and -- = +$)

3. $M$ satisfies the exchange axiom. That is, $M(i_1, ..., i_k)M(j_1, ..., j_k) = M(j_r, i_2, ..., i_k)M(j_1, ..., j_r, i_1, ..., j_k)$ for some choice of $r$ depending on the subsets $\{i_k\}$ and $\{j_k\}$.

Folkman and Lawrence have shown that a rank $k$ oriented matroid on $n$ elements is equivalent to the following data: To each index in $[n]$ assign either 0 or a map $h_i$ from
the oriented (k-2) sphere into the oriented (k-1) sphere satisfying the following conditions:

1. each map $h_i$ commutes with the antipode maps on the spheres

2. each $h_i$ is a tame embedding

3. the intersection of the images of any set of these maps is homeomorphic to a sphere

The image of each $h_i$ will divide its complement into 2 balls. For each $h_i$ label one of these halves by + and the other by −.

Two such pictures M and N will represent the same oriented matroid if there is a homeomorphism from the (k-1) sphere to itself carrying the image of the map $h_i$ associated to the index i by M to the image of the map $k_i$ associated to the index i by N, which also preserves the choice of signs.

To get a sign from such a picture, and an ordered k-subset of [n], say $(i_1, ..., i_k)$, if any of the indices $i_r$ is associated to 0, take the sign 0. Otherwise take the intersection of the k positive balls associated to the images of $h_{i_1}, ..., h_{i_k}$. If this intersection is not a (k-1) simplex, assign 0. Otherwise this intersection will be a simplex whose faces are labeled by $i_1, ..., i_k$. Assign + if this ordering induces a positive orientation on the simplex, and − otherwise.

Example: Drawn below is only one hemisphere of a 2 sphere corresponding to a rank 3 oriented matroid on 6 elements.
This picture corresponds to the following sign function:

\[ M(123) = + \quad M(124) = 0 \quad M(125) = - \quad M(126) = - \quad M(134) = 0 \quad M(135) = + \quad M(136) = + \quad M(145) = 0 \quad M(146) = 0 \quad M(156) = + \quad M(234) = 0 \quad M(235) = + \quad M(236) = 0 \quad M(245) = 0 \quad M(246) = 0 \quad M(256) = + \quad M(345) = 0 \quad M(346) = 0 \quad M(356) = - \quad M(456) = 0 \]

Extend the function by antisymmetry. (ie \( M(132) = - \ldots \)).

Let \( M \) and \( N \) be oriented matroids of the same rank on the same set of elements. \( N \) is said to be a weak image of \( M \) if they give the same sign for subsets on which \( N \) is nonzero. That is, \( N(i_1, \ldots, i_k) = \) either 0 or \( M(i_1, \ldots, i_k) \). Denote this situation by \( M \rightarrow - \rightarrow N \).

Let \( M \) be a rank \( n \) oriented matroid. If \( \phi \) is a tame embedding of a \( k \)-sphere into the cell complex for \( M \) which commutes with the antipode map and whose image intersects the images of the \( h_i \)'s in spheres then the cell complex obtained by intersecting the cell complex of \( M \) with the image of \( \phi \) is an oriented matroid, say \( N \), called a strong image of \( M \). Denote this situation by \( M \Longrightarrow N \).

Now define a poset \( E(i_1 < \ldots < i_k; M^n) \) from these oriented matroids. The underlying set of \( E \) will be the set of diagrams: \( M^n \Longrightarrow M^k \Longrightarrow \ldots \Longrightarrow M^2 \Longrightarrow M^1 \) where \( M^r \) is a rank \( i_r \) oriented matroid. To define the partial ordering say that a sequence \( N \) is a sequence \( M \) if there is the following diagram:

\[
\begin{array}{c}
M^n \Longrightarrow M^k \Longrightarrow \ldots \Longrightarrow M^2 \Longrightarrow M^1 \\
\downarrow & \downarrow & \ldots & \downarrow & \downarrow \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vee & \vee & \ldots & \vee & \vee \\
M^n \Longrightarrow N^k \Longrightarrow \ldots \Longrightarrow N^2 \Longrightarrow N^1
\end{array}
\]

Note that the map \( I \) which just exchanges + and - is a fixed point free involution.
the poset \( E \).

Define \( EI(i_1 < ... < i_k, M^n) \) to be the poset resulting from \( E(i_1 < ... < i_k, M^n) \) by identifying \( M^n \Longrightarrow M^k \Longrightarrow ... \Longrightarrow M^2 \Longrightarrow M^1 \) with \( IM^n \Longrightarrow IM^k \Longrightarrow ... \Longrightarrow IM^2 \Longrightarrow IM^1 \).

There is a map from \( G(i_1 < ... < i_k < n) \) to oriented matroids defined as follows. Choose a representative of an element of \( G(i_1 < ... < i_k < n) \). This gives a map from ordered \( i_r \) subsets of \([n]\) to \( \{0, +, -\} \) by taking the sign of the determinant of the minor corresponding to the top \( i_r \) rows and the columns of the subset. This is clearly nontrivial, and antisymmetric. It turns out also to be an oriented matroid (that it satisfies the exchange axiom is clear from the Plucker relations). This oriented matroid is indendant of the choice of representative. In fact these oriented matroids, call them \( M^1, ..., M^k \), form a sequence: \( M^k \Longrightarrow ... \Longrightarrow M^1 \), and hence give a map from \( G \) to \( E(i_1 < i_2 < ... < i_k; M) \) where \( M \) is the coordinate oriented matroid of rank \( n \). Call this map \( p \).

The map \( p \) can also be constructed for the Folkman Lawrence picture. The maps of \((k-2)\) spheres into the \((k-1)\) sphere arise by considering the first \( i_r \) elements of a column to be a vector in \( k \)-space and taking the \((k-2)\) sphere to be the intersection of its orthogonal complement with the unit sphere. The positive side is then the direction the vector points. If the \( k \) vector is 0 then 0 is assigned to the index instead of a sphere map.

For any poset \( P \) its geometric realization is defined to be the following simplicial complex. To each chain of \( P \), \( p_0 < ... < p_k \), associate a \( k \) simplex, with each vertex associated to a different element \( p_r \) of the chain. Now glue any subchain's simplex onto the boundary of this \( k \) simplex along the face associated to its elements. Thus the realization of the face lattice of a simplicial complex is exactly the barycentric subdivision of the original simplicial complex. Denote this topological space by \( DP \).

There is a bijection between rank \( k \) matroids on \( n \) elements and rank \( n-k \) ones, given by assigning to a set \( (i_1 < ... < i_{n-k}) \) the sign associated to its complement in the order \( (j_1 < ... < j_k) \). This map induces an isomorphism of posets between \( EI(i_1 < ... i_k < n) \) and \( EI(n - i_k < ... < n - i_1 < n) \). This in turn induces a homeomorphism between the
two realizations.

One might guess that $DE(i_1 < ... < i_k; M^n)$ is homotopy equivalent to $G(i_1 < ... < i_k < n)$.

This guess is true and relatively easy for the cases of projective planes, Grassmannians of 2 planes in n space, and the tautological circle bundles over the latter with $M^n$ the coordinate oriented matroid, (that is, the cases $(1; M^n)$, $(2; M^n)$, $(1 < 2; M^n)$, and their duals). In fact in all these cases there is actually a homeomorphism between the two spaces.

If the inverse images of the points in E under the map p formed a cell decomposition of G and the notion of weak image agreed with the notion of containment in closure, then the general theory of CW complexes would give the desired homotopy equivalence. This is in fact the case for the above examples.

Folkman and Lawrence have also shown that there is a homeomorphism for the case $(1; M^n)$ for any oriented matroid $M^n$, while Richter-Gebert has shown that $(3; M^n)$ need not be connected.

This thesis shows that that while the homeomorphism is lost, there is still a homotopy equivalence for the cases of $(2; M^n)$ and $(1 < 2; M^n)$ with arbitrary $M^n$.

These spaces arise as the fibers over points in a construction by Gelfand and MacPherson for computing the Pontrjagin classes for combinatorial vector bundles via a local formula.

This thesis also contains a new definition for oriented matroids which is analogous to the rank function definition for matroids.
0.1.1 Idea of proof

The main idea of the proof will be to fix a rank $k$ oriented matroid $M$, and consider $(1, 2; M)$, that is the set of pairs of oriented matroids $(N^1, N^2)$ of ranks 1 and 2 respectively with $N^1$ a strong image of $N^2$ and $N^2$ a strong image of $M$.

Consider also a rank $k$ coordinate oriented matroid $M_c$ obtained by deleting all but the elements of one basis in $M$.

There are now two projection maps, $\pi_1 : (N^2, N^1) \to N^2$ and $\pi_2 : (N^2, N^1) \to N^1$, and the deletion map taking $M$ to $M_c$, which induces maps from $(1, 2; M)$ to $(1, 2; M_c)$ and from $(2; M)$ to $(2; M_c)$ and from $(1; M)$ to $(1; M_c)$.

$\pi_1$ and $\pi_2$ are then seen to be essentially fibrations and the deletion map a fiber map between these. It is known that the deletion map induces a homotopy equivalence on both fibers, and also from $(1; M)$ to $(1; M_c)$. Thus deletion induces a homotopy equivalence from $(1, 2; M)$ to $(1, 2; M_c)$ and from $(2; M)$ to $(2; M_c)$. Finally the homotopy types of $(2; M_c)$ and $(1, 2; M_c)$ are known.

Most of the work in carrying out the above proof comes in showing that $\pi_2$ is essentially a fibration.

This is achieved through the introduction of some intermediate objects denoted hairs, which are pairs $(X, N^1)$ with $X$ being an initial segment of $N^2$ thought of as a set of covectors, beginning at $N^1$ and continuing around.

$\pi_2$ is then written as a composition of a number of cut maps $\Psi_k$, each of which either leaves a hair alone, or removes the elements farthest from $N^1$.

All but the last of these ($\Psi_2$) is seen to be a homotopy equivalence while $\Psi_2$ is seen to be essentially a fibration.
0.2 Notation

This section introduces notation which is used throughout this thesis. Oriented matroids will be thought of as sets of covectors with elements labeled 1 through n. Thus every oriented matroid will be a subset of $\binom{[n]}{\pm_0-}$ which is just a rank n coordinate oriented matroid.

In this notation, if $x$ is a covector then $\overline{x}$ is the set of all elements, $x^0$ is the set of elements which are 0 on $x$, $x^+$ is the set of nonzero elements with signs, and $x^-$ is the same set as $x^+$ only with the opposite signs. $x + y$ is the set signed elements obtained by adding vectors $v_x$ and $v_y$ where $v_x$ is 0 where $x$ is 0, 1 where $x$ is + and -1 where $x$ is −.

$[n] = \{1, \ldots, n\}$
$\pm[n] = [n] \times \{+, 0, -\}$
$\overline{\sigma} : \{+, 0, -\} \rightarrow \{+, 0, -\}$ with $\overline{\sigma}^+ = -$ and $\overline{\sigma}^0 = 0$ and $\overline{\sigma}^- = +$
$\sigma = id \times \overline{\sigma} : \pm[n] \rightarrow \pm[n]$

Remark. $\sigma$ and $\overline{\sigma}$ are essentially minus.

$e^* = (e, s) \in \pm[n]$

$\binom{[n]}{\pm_0-} = \{x | x \subseteq \pm[n] \text{ and } \pi_1 : x \rightarrow [n] \text{ is a bijection} \}$

Remark. This is the set of covectors of the coordinate oriented matroid of rank n.

$\binom{[n]}{\pm_0-}^* = \binom{[n]}{\pm_0-} \backslash \{[n] \times \{0\}\} = \binom{[n]}{\pm_0-} \backslash \{\emptyset\}$

Remark. This is just the nonzero covectors of the coordinate oriented matroid of rank n.

If $x$ and $y$ are subsets of $\pm[n]$ then:

$\overline{x} = \pi_1^{-1} \pi_1 x$

$x^0 = x \cap Q$

$x^+ = x \backslash Q = \sigma x^-$

$x + y = (x \cap y) \cup (x \cap \overline{y}^0) \cup (\overline{x^0} \cap y) \cup (x^+ \cap y^-)^0$

$x \uplus y = (x + y) \cup x \cup y$

$x \rightarrow y = x^+ \cup (\overline{x^0} \cap y)$

Remark. This operation is just composition of covectors.
If \( \{x, y, z\} \subseteq (\text{\textsuperscript{1}n})^{\circ} \) then \( \{x + y, x \circ y\} \subseteq (\text{\textsuperscript{1}n})^{\circ} \) and \( x + x = x \cup x = x \circ x = x \) and \( (x \circ y) \circ z = x \circ (y \circ z). \)

### 0.3 Hairs

This section begins with the definition of a hair which is the main technical tool used in this thesis, being the objects used to interpolate between \( N^1 \) and the pair \((N^2, N^1)\) of respectively rank 2 and rank 1 oriented matroids with \( N^1 \) a strong image of \( N^2 \). The rest of the section contains a partial ordering on hairs and a variety of properties of hairs and their ordering.

This definition is analogous to the covector definition of a rank 2 oriented matroid. HD(0) is implicit for the oriented matroid. HD(1) is a restricted version of the composition axiom. HD(2) is the rank 2 exchange axiom. HD(3) is required to assure the proper large scale structure without the negative axiom.

**Definition:** If \( X_r \in X \subseteq (\text{\textsuperscript{1}n})^{\circ} \) then \((X_r, X)\), often written \( X \), is a hair if:

**HD(0)** \( |X| \geq 3 \)

**HD(1)** for all \( \{w, u\} \subseteq X \) and \( v \in X^* = X \setminus \{X_r, \sigma X_r\} \), there is \( \{v \circ \sigma u, w \circ u\} \subseteq X \)

**HD(2)** for all \( \{u, v\} \subseteq X \) and \( e^* \in u^+ \cap v^- \) there is a unique \( w \in X \) with both \( e^0 \in w \) and either \( w \subseteq u \cup v \) or \( \{u, v\} = \{X_r, \sigma X_r\} \)

**HD(3)** If \( \{X_r, \sigma X_r\} \not\subseteq \{u, v, w\} \subseteq X \) then there is \( z \in \{u, v, w\} \) so that:

(a) \( w \subseteq u \cup v \iff w = z \)

(b) if \( w \in \{X_r, \sigma X_r\} \) and \( w \subseteq u \cup v \) then \( w \in \{u, v\} \)

If \((X_r, X)\) and \((Y_r, Y)\) are hairs then \((X_r, X) \subseteq (Y_r, Y)\) means \( X_r = Y_r \) and \( X \subseteq Y \).

Write this simply as \( X \subseteq_h Y \).

For instance, in the example given in the introduction, the pair \((X_r, X)\) with \( X_r = \)
$+0+0-$ and $X = \{+0+0--, +--+0--, 0--+00-, --+0+, --+0+0, --+0++\}$
is a hair.

### 0.3.1 Hair Facts

This subsection states 8 basic properties of hairs.

If $X$ is a hair then:

**HF(1)** if $\{u, v, w\} \subseteq X$ with $e^0 \in u \cap v$ and $e^s \in w^+$ then either $u = v$ or $\{u, v\} = \{X_r, \sigma X_r\}$

Remark. HF(1) asserts that $X$ either lives in or crosses transversally the embedded sphere associated to each element.

**HF(2)** if $\{u, v\} \subseteq X^*$ then $u \subseteq v \cup X_r \iff v \subseteq u \cup \sigma X_r$

Remark. HF(2) asserts that the interior of $X$ forms a straight line segment headding from $X_r$ toward $\sigma X_r$.

**HF(3)** $\leq_X$, which is defined in HN(2), the next subsection is a total order on $X$ with $X_r$
minimal and if $\sigma X_r \in X$ then $X_m = \sigma X_r$

**HF(4)** if $\{X_r, \sigma X_r\} \not\subseteq \{u, v, w\} \subseteq X$ then $(v \subseteq u \cup w) \iff (u \leq_X v \leq_X w$ or $w \leq_X v \leq_X u)$

Remark. HF(4) asserts that the total ordering on $X$ is locally computable.

**HF(5)** if $e \in [n]$ then one of the following holds:

(a) for some $s \in \{+, 0, -\}$ and for all $x \in X$, have $e^s \in x$.

Remark. If $s=0$ this is the case where $X$ lives in $e$. If $s$ is $+$ or $-$ this is the case where $X$ does not touch $e$.

(b) $e^0 \in X_r \cap \sigma X_r$, and for some $s \in \{+, -\}$ and for all $x \in X^*$, have $e^s \in x$

Remark. This is the case where $X$ starts out in $e$ and leaves immediately.
(c) for some \( v \in X^* \) and for some \( s \in \{+,-\} \) have \( e^0 \in v \) and for all \( x \in X \setminus \{v\} \) have
\[
(e^s \in x^+ \text{ if } x \leq_X v) \text{ and } (e^s \in x^- \text{ if } v \leq_X x)
\]
Remark. This is the case where \( X \) crosses \( e \) transversally at the point \( v \) in \( X \).

HF(6) if \((u, v) \in X \times X\) then one of the following holds:

(a) \( v \leq_X (u \xrightarrow{\sigma} v) <_X u \) and if \( u \neq \sigma X_r \) then \( u <_X (u \xrightarrow{\sigma} \sigma v) \)

(b) \( u <_X (u \xrightarrow{\sigma} v) \leq_X v \) and if \( u \neq X_r \) then \( (u \xrightarrow{\sigma} \sigma v) <_X u \)

(c) \( u \xrightarrow{\sigma} v = u \xrightarrow{\sigma} \sigma v = u \) and if \( u \notin \{v, \sigma v\} \) then there is \( w \subseteq u \cup v \) with \( (w, u) \)
satisfying 6(a) or 6(b) and \( u^0 = \cap \)

Remark. HF(6) asserts that if \( u \) is a nongeneric element in \( X \) then composition
with any other element of \( X \) gives an adjacent element in the total ordering of \( X \).

HF(7) if \( u \in (\frac{\text{[n]}_{\text{[n]}}}{\text{[n]}}) \) then:

(a) \( \{x \supseteq u^+\} \) is an interval in \( \leq_X \) order

Remark. HF(7a) asserts that the intersection of \( X \) with the open star of any cell
is an interval.

(b) \( \{x \subseteq u^+\} \) is an interval in \( \leq_X \) order and if \( \{x \subseteq u^+\} \neq \phi \) then there is a unique
\( x \in X \) with \( \{x \subseteq u^+\} = \{x \subseteq x^+\} = \{w, \overrightarrow{w} = x = \overleftarrow{y}, y\} \)
where \( \overrightarrow{w} \) and \( \overleftarrow{y} \) are defined in the next subsection HN(6)

Remark. HF(7b) asserts that the intersectin of \( X \) with any closed cell is just a
closed cell of \( X \).

HF(8) if \( X_m \neq \sigma X_r \) then \( X_m^0 = \cap \)
where \( X_m \) and \( X^0 \) are defined in the next subsection, HN(3) and HN(1)

Remark. HF(8) asserts that the maximal element of \( X \) is either generic or \( \sigma X_r \).

**0.3.2 Hair Notations**

This subsection states some notations related to, and in some cases used in, the hair fact
from the previous subsection with the same number.
HN(1) $X^0 = \{x^0 | x \in X\}$

$\bigcup_{x \in X} x^0 = \text{the elements which } X \text{ touches}$

$X^* = X \setminus \{X_r, \sigma X_r\} = \text{the interior of } X$

$\bigcap_{x \in X} x^0 = \text{the elements containing } X$

Notice that $X^0 = v^0 \cap u^0$ for any $\{u, v\} \subseteq X$ with $u \not\in \{v, \sigma v\}$

HN(2) if $\{u, v\} \subseteq X$ then $u \leq_X v$ if either

(a) $\sigma X_r \neq u \subseteq v \cup X_r$ or

(b) $X_r \neq v \subseteq u \cup \sigma X_r$

HN(3) $X_m \geq_X x$ for all $x \in X$ so $X_m$ is the maximal element of X

HN(6) $\frac{X}{u} = \frac{u^d}{u^d} = \frac{id}{id} = u \rightarrow v$ if $(u, v)$ satisfies 6(b) and $\frac{X}{u} = \frac{u^d}{u^d} = \frac{id}{id} = u$ otherwise

$v <_X u$ if $v <_X u$ and there is no $w$ with $v <_X w <_X u$

Remark. HN(6) gives various convenient notations for the generic elements adjacent to $u$ in $X$.

HN(7) $\{X \unlhd u^+\} = \{x \in X | x^+ \supseteq u^+\}$ for any $u \in (\mathfrak{H}_{+0}^-)$

$\{X \unlhd u^+\} = \{x \in X | x^+ \subseteq u^+\} = \{X \subseteq X_u^+\}$ for any $u \in (\mathfrak{H}_{+0}^-)$ and some $X_u \in X$

0.3.3 Proofs of hair facts

This subsection contains proofs of the 8 facts stated in subsection 3.1 (hair facts).

HF1

(1) : If $\{u, v\} \subseteq \{X_r, \sigma X_r\}$ then $u = v$ or $\{u, v\} = \{X_r, \sigma X_r\}$.

If $u \rightarrow \sigma w = v \rightarrow \sigma w \in \{X_r, \sigma X_r\}$ and $w \in \{X_r, \sigma X_r\}$ then:
By the previous sentence and $e^0 \in u \setminus w$ have $u \in X^*$.

By the previous sentence and HD(1) and $w \in \{X_r, \sigma X_r\} \cap X$ have $u \xrightarrow{\sigma} \sigma w \in X$.

By $\{u \xrightarrow{\sigma} \sigma w, w\} \subseteq \{X_r, \sigma X_r\}$ and $u \not\in \{X_r, \sigma X_r\}$ have $\{u \xrightarrow{\sigma} \sigma w, w\} = \{X_r, \sigma X_r\}$.

By the previous sentence and $e^* \in w^+ \cap (u \xrightarrow{\sigma} \sigma w)^-$ and $e^0 \in u \cap v$ and $\{w, u \xrightarrow{\sigma} \sigma w, u, v\} \subseteq X$ and HD(2) have $u = v$.

If $\{u, v\} \not\subseteq \{X_r, \sigma X_r\}$ and $\{w, u \xrightarrow{\sigma} \sigma w, v \xrightarrow{\sigma} \sigma w\} \not\subseteq \{X_r, \sigma X_r\}$ then by $(a \in \{b, \sigma b\} \implies b \xrightarrow{\sigma} a = b)$ can choose $u \in \{u, v\}$ with $u \not\in \{X_r, \sigma X_r\}$ and $\{w, u \xrightarrow{\sigma} \sigma w\} \not\subseteq \{X_r, \sigma X_r\}$.

By the previous sentence and HD(1) and $w \in X$ and $u \in X^*$ have $u \xrightarrow{\sigma} \sigma w \in X$.

By $e^0 \in v$ and $e^* \in w^+ \cap (u \xrightarrow{\sigma} \sigma w)^-$ have $v \not\in \{w, (u \xrightarrow{\sigma} \sigma w), \sigma w, \sigma(u \xrightarrow{\sigma} \sigma w)\}$.

By the previous 3 sentences have $\{w, u \xrightarrow{\sigma} \sigma w, u, v\} \subseteq X$ and $\{X_r, \sigma X_r\} \not\subseteq \{w, u \xrightarrow{\sigma} \sigma w, u, v\}$.

By $e^* \in (w^+ \cup (u^+ \cup v^+ \cup (u \xrightarrow{\sigma} \sigma w)^+)) \cap ((u \xrightarrow{\sigma} \sigma w)^- \cup (u^+ \cup v^+ \cup w^-))$ have $w \not\subseteq u \cup \cup \cup (u \xrightarrow{\sigma} \sigma w)$ and $(u \xrightarrow{\sigma} \sigma w) \not\subseteq u \cup \cup \cup w$.

By the previous 3 sentences and HD(3a) have $u \cup v \subseteq w \cup (u \xrightarrow{\sigma} \sigma w)$.

By the previous sentence and $\{w, u \xrightarrow{\sigma} \sigma w, u, v\} \subseteq X$ and $e^0 \in u \cap v$ and $e^* \in w^+ \cap (u \xrightarrow{\sigma} \sigma w)^-$ and HD(2) have $u = v$.

**HF2**

If $u = v$ then $u \subseteq v \cup X_r$ and $v \subseteq u \cup \cup \sigma X_r$.

If $u \neq v$ then:

If $u \subseteq v \cup z$ with $z \in \{X_r, \sigma X_r\}$ then $v = (v \cap \sigma z) \cup (v^+ \cap z^+) \cup (v^+ \cap \sigma z) \cup v^0$ and $(v \cap \sigma z) \subseteq \sigma z \subseteq u \cup \cup \sigma z$.

By $u \subseteq v \cup z$ and $\{u, v, z\} \subseteq (v \cap v^0)$ have $(z^+ \cap v^+) \subseteq u^+ \subseteq u \cup \cup \sigma z$.

By $u \not\subseteq \{z, \sigma z\}$ and $\{u, v, z\} \subseteq X$ and HF(1) have $z^0 \cap u^0 = X^0 \subseteq v^0$ and hence $\sigma^0 \cup \sigma z \subseteq \sigma^0$.

15
By \( \{u, v\} \subseteq (\frac{[n]}{+[o]-}) \) and \( u \subseteq v \cup z \) have \( u \cap \overline{z} \subseteq (v \cap \overline{z}) \cup \overline{z} \) and hence \( u^+ \cap \overline{z} \subseteq v^+ \cap \overline{z} \) and so \( u^- \cap v^+ \cap \overline{z} = \phi \).

By the previous 2 sentences have \( v^+ \cap \overline{z} = (u^+ \cap \overline{z} \cap v^+) \cup (u^- \cap \overline{z} \cap v^+) \cup (u^0 \cap \overline{z} \cap v^+) \subseteq u \cup \phi \cup \overline{v}^0 \).

By the previous sentence and \( v^+ \cap \overline{v}^0 = \phi \) have \( (v^+ \cap \overline{z}) \subseteq u \subseteq u \cup z \).

By \( u \subseteq v \cup z \) have \( u \cap \overline{v}^0 \subseteq (z \cap \overline{v}) \cup \overline{v}^0 \) and hence \( u^+ \cap \overline{v}^0 \subseteq z^+ \cap \overline{v}^0 \).

By the previous sentence and \( u \in (\frac{[n]}{+[o]-}) \) have \( v^0 \subseteq u \cup \sigma z \).

By the previous eight sentences \( v \subseteq u \cup \sigma z \).

**HF3**

If \( \{u, v\} \subseteq X^* \) then by HF(2) have HN(2a) iff HN(2b).

By \( \{u, v\} \subseteq X^* \) and HD(3a) and HD(3b) have one of \( u \subseteq v \cup X_r \) or \( v \subseteq u \cup X_r \) holding and hence \( u \leq_X v \) or \( v \leq_X u \) holds.

By \( \{u, v\} \subseteq X^* \) and HF(2) and HD(3a) have \( (u \leq_X v \text{ and } v \leq_X u \text{ iff } u = v) \).

If \( \{u, v, w\} \subseteq X^* \) and \( u \leq_X v \leq_X w \) then by HF(2) have \( u \subseteq v \cup X_r \subseteq w \cup X_r \) and hence \( u \leq_X w \).

Thus \( \leq_X \) is a total ordering on \( X^* \).

If \( v \in X \) then \( X_r \subseteq X_r \cup v \) and hence \( X_r \leq_X v \).

If \( v \in X \) and \( v \leq_X X_r \) then \( v \subseteq X_r \cup X_r = X_r \) and by \( \{v, X_r\} \subseteq (\frac{[n]}{+[o]-}) \) have \( v = X_r \).

If \( v \in X \) then either \( \sigma X_r \not\subseteq X \) or \( (\sigma X_r \subseteq \sigma X_r \cup v \text{ and hence } v \leq_X \sigma X_r) \).

If \( v \in X \) and \( \sigma X_r \leq_X v \) then \( v \subseteq \sigma X_r \cup \sigma X_r = \sigma X_r \) and by \( \{v, \sigma X_r\} \subseteq (\frac{[n]}{+[o]-}) \) have \( v = \sigma X_r \).

Thus \( \leq_X \) is a total order on \( X \) with \( X_r \) as minimal element and if \( \sigma X_r \in X \text{ then } \sigma X_r \) is the maximal element.
If \( \{u,w\} \subseteq X^* \) then:

\( \implies \) Assume \( u \leq_X v \leq_X w \).

By HF(3) and \( \{u,w\} \subseteq X^* \) and \( u \leq_X v \leq_X w \) have \( v \in X^* \).

By \( \{u,v,w\} \subseteq X^* \) and HF(2) and \( u \leq_X v \leq_X w \) have \( v \subseteq u \uplus \sigma X_r \) and \( v \subseteq w \uplus X_r \).

By the previous sentence and \( X_r \in (\frac{\text{ln}w}{\text{ln}u} \uplus \text{ln}w) \) have \( v^+ \subseteq (u^+ \cup X_r^-) \cap (w^+ \cup X_r^+) \subseteq u^+ \cup w^+ \cup (X_r^+ \cap X_r^-) = u^+ \cup w^+ \).

By \( v \in X^* \) and \( \{u,v,X_r,w\} \subseteq X \) and HF(1) have \( v^0 \cap X_r^0 = X^0 \subseteq u^0 \cap w^0 \).

By the previous sentence and \( v \subseteq (u \uplus \sigma X_r) \cap (w \uplus X_r) \) have \( v^0 \subseteq X_r^0 \cup u^0 \cup w^0 \cup (u^+ \cap w^-)^0 \subseteq X_r^0 \cup (u \uplus w) \) and hence \( v^0 \subseteq u \uplus w \).

By the previous three sentences have \( v = v^+ \cup v^0 \subseteq u^+ \cup w^+ \cup u \uplus w = u \uplus w \).

\( \implies \) Assume \( v \subseteq u \uplus w \).

By HF(3) and exchange of \( u \) and \( w \) one of: \( w \leq_X u \leq_X v \) or \( v \leq_X u \leq_X w \) or \( u \leq_X v \leq_X w \) must hold.

If \( w \leq_X u \leq_X v \) or \( v \leq_X u \leq_X w \) then by \( \implies \) for \( \{u,v\} \subseteq X^* \) and \( \{u,w\} \subseteq X^* \) have \( u \subseteq v \uplus w \).

By the previous sentence and \( v \subseteq u \uplus w \) and \( \{u,w\} \subseteq X^* \) and HD(3a) have \( u = v \) and hence either \( w \leq_X v \leq_X u \) or \( u \leq_X v \leq_X w \).

If \( \{u,w\} \not\subseteq X^* \) then:

\( \implies \) Assume \( u \leq_X v \leq_X w \).

If \( X_r = w \) or \( \sigma X_r = u \) then by \( \implies \) for \( \{u,v\} \subseteq X^* \) and \( \{u,w\} \subseteq X^* \) have \( u = v = w \) and hence \( v \subseteq u \uplus w \).

If \( \sigma X_r = w \) and \( \{u,v\} \subseteq X^* \) then by HF(2) and \( u \leq_X v \) have \( v \subseteq u \uplus \sigma X_r = u \uplus w \).

If \( X_r = u \) and \( \{w,v\} \subseteq X^* \) then by HF(2) and \( v \leq_X w \) have \( v \subseteq w \uplus X_r = u \uplus w \).

\( \implies \) Assume \( v \subseteq w \uplus u \).
If \((u = X, v = \sigma X)\) and \(v \not\in \{X, \sigma X\}\) then by HN(2) and HF(3) and \(v \subseteq u \cup w\) have \(u \leq_X v \leq_X w\).

If \((u = X, v = \sigma X)\) and \(v \in \{X, \sigma X\}\) then by \(\{X, \sigma X\} \not\subseteq \{u, v, w\} \subseteq X\) and HF(3) have either \(X = u = v \leq_X w\) or \(u \leq_X v = w = \sigma X\).

\section*{HF5}

If \(e^0 \in X^0\) then HF(5a) holds for \(s = 0\).

If \(e^0 \in X_r \setminus X^0\) then:

By the previous sentence and HF(1) and for all \(x \in X^*\) have \(e^0 \not\in x\).

If \(e^* \in u^+ \cap v^-\) and \(\{u, v\} \subseteq X^*\) then by HD(2) can choose \(w \in X\) with \(e^0 \in w \subseteq u \cup v\).

By the previous sentence and HF(4) have \(u \leq_X w \leq_X v\) or \(v \leq_X w \leq_X u\).

By the previous sentence and \(\{u, v\} \subseteq X^*\) and \(w \in \{X, \sigma X\}\) and HF(3) have a contradiction.

Thus if \(e^0 \in X_r \setminus X^0\) 5(b) holds for some \(s \in \{+, -\}\).

If \(e^0 \in v \in X^*\) and \(e^0 \not\in X^0\) then:

By the previous sentence and HF(1) and for every \(y \in X \setminus \{v\}\) have \(e^0 \not\in y\).

If \(e^* \in u^+ \cap y^-\) and \(\{X, \sigma X\} \not\equiv \{u, y\} \subseteq X\) then by HD(2) there is a unique \(e^0 \in w \subseteq u \cup y\).

By the previous sentence and \(e^0 \in v \in X^*\) and HF(1) have \(v = w \subseteq u \cup w\).

By the previous sentence and \(\{X, \sigma X\} \not\subseteq \{u, v, w\}\) and HF(4) have \(u \leq_X v \leq_X w\) or \(w \leq_X v \leq_X u\).

By the previous four sentences and \(e^* \in X_r \cup \sigma X_r\) and HF(3) have HF(5c) holding for some \(s \in \{+, -, \}\) and \(v\) as above.

If for every \(x \in X\) have \(e^0 \not\in x\) then: If \(\{u, v\} \subseteq X\) with \(e^* \in u^+ \cap v^-\) then by HD(2) have \(e^0 \in w \in X\) and hence a contradiction.

Thus by the previous sentence 5(a) holds for some \(s \in \{+, -\}\).
HF6

By $u \xrightarrow{\sigma} v \subseteq u \cup v$ and $\{X_r, \sigma X_r\} \subseteq \{u, v, u \xrightarrow{\sigma} v\}$ implies that $\{X_r, \sigma X_r\} = \{u, v\}$ and HF(3) and HF(4) have $u \xrightarrow{\sigma} v$ is $\leq_X$-between $u$ and $v$.

If $w$ is $\leq_X$-between $u$ and $u \xrightarrow{\sigma} v$ then by HF(4) have $w \subseteq u \cup (u \xrightarrow{\sigma} v)$ and hence $u^+ \subseteq w^+ \subseteq (u \xrightarrow{\sigma} v)^+$. By the previous sentence and HF(1) have $|\{u, w, u \xrightarrow{\sigma} v\}| < 3$.

If $u \xrightarrow{\sigma} v \neq u$ then $u \xrightarrow{\sigma} \sigma v = u \xrightarrow{\sigma} (u \xrightarrow{\sigma} \sigma v) \notin \{u, u \xrightarrow{\sigma} v\}$.

Thus by the previous four sentences if $u \xrightarrow{\sigma} v \neq u$ then either HF(6a) or HF(6b) holds.

If $u \xrightarrow{\sigma} v = u$ and $u \notin \{v, \sigma v\}$ then by HF(1) and $u^0 = u^0 \cap v^0$ have $u^0 = X^0$.

If $u \xrightarrow{\sigma} v = u$ and $v \xrightarrow{\sigma} u = v$ and $u <_X v$ then $u^0 = v^0 = X^0$, but $u \neq v$.

If $e^* \in u^+ \cap v^-$ and $u <_X v$ then by HF(5c) there is a contradiction.

Thus if $u <_X v$ then $v \xrightarrow{\sigma} u \neq v$ and hence HF(6a) or HF(6b) holds.

HF7

(a) If $y \leq_X v \leq_X w$ and $\{y, w\} \subseteq \{X \supseteq u^+\}$ then:

If $\{X_r, \sigma X_r\} \neq \{y, w\}$ then by $y \leq_X v \leq_X w$ and HF(4) and HF(3) have $v \subseteq y \cup w$.

By the previous sentence and $\{y, w\} \subseteq (\frac{\|y\|}{\|v\|})$ have $y^+ \cap w^+ \subseteq v^+$.

By the previous sentence and $v \in X$ and $\{y, w\} \subseteq \{X \supseteq u^+\}$ have $u^+ \subseteq y^+ \cap w^+ \subseteq v^+$ and hence $v \in \{X \supseteq u^+\}$.

If $\{X_r, \sigma X_r\} = \{y, w\}$ then by $\{y, w\} \subseteq \{X \supseteq u^+\}$ have $u^+ \subseteq y^+ \cap w^+ = \emptyset$ and hence $u = \emptyset$.

By the previous sentence have $\{X \supseteq u^+\} = X$ and hence $v \in \{X \supseteq u^+\}$.

(b) If $y \leq_X v \leq_X w$ and $\{y, w\} \subseteq \{X \subseteq u^+\}$ then:

If $\{X_r, \sigma X_r\} \neq \{y, w\}$ then by $y \leq_X v \leq_X w$ and HF(4) and HF(3) have $v \subseteq y \cup w$.

By the previous sentence and $\{y, w\} \subseteq \{X \subseteq u^+\}$ have $v^+ \subseteq y^+ \cup w^+ \subseteq u^+$ and hence $v \in \{X \subseteq u^+\}$.
If \( \{X, \sigma X\} = \{y, w\} \) then by \( u \in \left(\frac{\text{in}}{+_{0-}}\right) \) and \( y^+ \cup w^+ = X^+ \cup X^- \subseteq z^+ \) have \( X = \emptyset \).

By the previous sentence and \( X \subseteq \left(\frac{\text{in}}{+_{0-}}\right) \), there is a contradiction.

If \( \{X \subseteq u^+\} \neq \emptyset \) then:

For every \( \{y, v\} \subseteq \{X \subseteq u^+\} \) with \( y^0 = v^0 \), by \( y^+ \subseteq u^+ \) and \( u \in \left(\frac{\text{in}}{+_{0-}}\right) \) have \( y = v \).

For every \( \{y, v\} \subseteq \{X \subseteq u^+\} \), by \( y^+ \subseteq u^+ \) and HD(1) have \( (y \rightarrow v)^+ \subseteq u^+ \) and hence \( y \rightarrow v \in \{X \subseteq u^+\} \).

By HF(1) for every \( \{y, v\} \in X \) with \( y \notin \{v, \sigma v\} \) have \( (y \rightarrow v)^0 = X^0 \).

By the previous three sentences and \( \{X \subseteq u^+\} \) an \( \leq X \)-interval have for every \( \{y, v\} \subseteq \{X \subseteq u^+\} \) that \( y \rightarrow v = v \rightarrow y \in \{X \subseteq u^+\} \).

If \( |\{X \subseteq u^+\}| = 1 \) then \( \{X \subseteq u^+\} = \{x\} = \{X \subseteq x^+\} \) and \( x^0 = X^0 \) so \( \overline{x} = \overline{x} = x \).

If \( |\{X \subseteq u^+\}| > 1 \) then:

For all \( \{y \neq w\} \subseteq \{X \subseteq u^+\} \) by \( y \notin \{w, \sigma w\} \) and the previous three sentences have \( y^+ \subseteq (y \rightarrow w)^+ \supseteq w^+ \).

By the previous sentence have that for all \( z \in \{X \subseteq u^+\} \) get \( z^+ \subseteq (y \rightarrow w)^+ \) and hence \( \{X \subseteq u^+\} \subseteq \{X \subseteq (y \rightarrow w)^+\} \).

By \( (y \rightarrow w) \in \{X \subseteq u^+\} \) and the previous sentence have \( z^+ \subseteq (y \rightarrow w)^+ \subseteq u^+ \) for all \( z \in \{X \subseteq (y \rightarrow w)^+\} \).

By the previous two sentences have \( \{X \subseteq u^+\} = \{X \subseteq (y \rightarrow w)^+\} \).

If \( \{y, v\} \subseteq X \) with \( \{X \subseteq v^+\} = \{X \subseteq v^+\} \) then have \( v^+ \subseteq y^+ \) and \( y^+ \subseteq v^+ \) and hence \( v = y \).

By the previous sentence and \( X \subseteq \left(\frac{\text{in}}{+_{0-}}\right) \) have that there is a unique \( x \in X \) with \( \{X \subseteq u^+\} = \{X \subseteq x^+\} \).

Denote this \( x \in X \) by \( X_u \).
If $X_m \neq \sigma X_r$ then:

By HD(0) and HF(3) there is $X_r <_X w <_X X_m$.

By the previous sentence and $X_m \neq \sigma X_r$ and HD(1) have $X_m \overset{\circ}{\rightarrow} \sigma w \in X$.

By the previous sentence and HF(6) have $X_m \leq_X X_m \overset{\circ}{\rightarrow} \sigma w$ and hence $X_m = X_m \overset{\circ}{\rightarrow} \sigma w$.

By the previous three sentences and HF(1) have $X_m^0 = X_m^0 \cap (\sigma w)^0 = X_m^0 \cap w^0 = X^0$.

### 0.3.4 A partial order on hairs

This subsection gives the definition of the partial order on hairs used in section 4 to realize sets of hairs as simplicial complexes. This partial order interpolates between the weak order on rank one oriented matroids and the weak order on pairs $(N^2, N^1)$ as before. WD(1) is exactly the weak order on $N^1$. WD(2) is the restriction of the weak order on $N^2$ to the subset $X$. WD(3) is needed to avoid degenerate weak maps between hairs which could not be extended to ones between full pairs of oriented matroids, and is stated in a way which makes $\leq$ obviously a partial ordering.

If $X$ and $Y$ are hairs then $X \leq Y$ if:

**WD(1)** $X^+_r \subseteq Y^+_r$

**WD(2)** for all $y \in Y$ have $\{X \subseteq y^+\} \neq \phi$

**WD(3)** for all $y \in Y$ and $x \in \{X \subseteq y^+\}$ and $\tau \in \{\sigma, id\}$ with $\tau \sigma X^+_r \not\subseteq y^+$ have $[y]^\tau_Y \subseteq [x]^\tau_X$.

**Notation:** For all $u, v \subseteq \pm [n]$ let $\frac{u}{v}$ denote $(\overline{(u \cup v)^0} \cap u) \cup u^0$.

If $X$ is a hair and $x \in X$ and $\tau \in \{\sigma, id\}$ let $[x]^\tau_X$ denote $\frac{x}{\tau X^+_r}$.

**Remark.** Thus $[x]^id_X$ is the set of all signs for elements in which $X$ lives together with the signs of all elements on $\overline{x}$ which differ on $X_r$.
0.3.5 Initial Facts

This subsection states two technical points used in subsection 3.7 to deal with WD(3) in proving basic facts about this partial order: \( \leq \).

If \( X \) is a hair then:

IF(1)(a) if \( x \in X^* \) then \( [x]^r_X = [x]^r_X \Pi ((x^0 \setminus X^0) \cap \overline{\tau x}) \)

(b) if \( x_2 \in X \) and \( x_1 \in X^* \) and \( x_1^+ \subseteq x_2^+ \) then \( [x_2]^r_X \subseteq [x_1]^r_X \)

(c) if \( \{x_1, x_2\} \subseteq X \) and \( x_1 \leq_X x_2 \) then (either \( [x_1]^r_X \subseteq [x_2]^r_X \) or \( x_2 = \sigma X_r \)) and (either \( [x_1]^r_X \supseteq [x_2]^r_X \) or \( x_1 = X_r \))

IF(2) If \( x_1^+ \subseteq y_1^+ \) and \( x_2^+ \subseteq y_2^+ \) then \( \frac{x_1}{x_2} \supseteq \frac{y_1}{y_2} \).

0.3.6 Proof of Initial Facts

This subsection contains proofs of the two facts stated in the previous subsection.

IF1

(a) By HF(6) and \( x \in X^* \) have \( \overline{\tau x} = x \zeta \sigma X_r \).

By the previous sentence have \( [x]^r_X = [x]^r_X = ((\overline{\tau X_r} \cup (x \zeta \sigma X_r))^0 \cap (x \zeta \sigma X_r)) \cup (x \sigma \sigma X_r)^0 \).

By \( x \in X^* \) and HF(1) have \( \overline{\tau x} = X^0 \).

By the previous two sentences have \( [x]^r_X = ((\overline{\tau X_r} \cup x)^0 \cap x^+) \cup X^0 \Pi (x^0 \cap X^+^+) = ((\overline{\tau X_r} \cup x)^0 \cap x^+) \cup X^0 \Pi ((x^0 \setminus X^0) \cap \overline{\tau x}) \).

By the previous three sentences have \( \overline{[\tau x]^r_X} = \overline{[\tau x]^r_X} = ((\overline{\tau X_r} \cup (x \zeta \tau X_r))^0 \cap (x \zeta \tau X_r)) \cup X^0 \Pi \phi \).

22
(b) If $x_1^+ \subseteq x_2^+$ and $\{x_1, x_2\} \subseteq X$ and $x_1 \in X^*$ then by HF(6) have $x_2 \in \{\overline{x_1}, x_1, \overline{x_1}\}$.

If $x_2 = x_1$ then clearly $[x_2]_X^+ \subseteq [x_1]_X^+$.

If $x_2 = \overline{x_1}$ then $\overline{x_2} = \overline{x_1}$ and hence $[x_2]_X^+ \subseteq [x_1]_X^+$.

If $x_2 = \overline{x_1}$ and $x_1 \in X^*$ then by IF(1a) have $[x_2]_X^+ = \overline{[x_1]_X^+} \subseteq [x_1]_X^+$.

(c) If $\{x_1, x_2\} \subseteq X^*$ and $x_1 \leq_X x_2$ then by HF(6) there is a sequence $(y_1, \ldots, y_k) \subseteq X$ with $\{\overline{x_1}, \overline{x_1}\} \subseteq \{\overline{y_1}, \overline{y_1}, \overline{y_2}\}$ and $\{\overline{x_2}, \overline{x_2}\} \subseteq \{\overline{y_{k-1}}, \overline{y_k}, \overline{y_k}\}$ and $y_i^+ \subseteq y_{i-1} \cap y_{i+1}$.

By the previous sentence and IF(1b) have $[y_{i-1}]_X \subseteq [y_i]_X$ and $[y_i]_X^+ \supseteq [y_{i+1}]_X^+.$

IF2

By $x_1^+ \subseteq y_1^+$ and $x_2^+ \subseteq y_2^+$ have $y_1^0 \subseteq x_1^0$ and $(y_1 \cup y_2)^0 \subseteq (x_1 \cup x_2)^0$.

By the previous sentence have $\overline{\overline{y}_1} = ((y_1 \cup y_2)^0 \cap y_1) \cup \overline{y}_1 \subseteq ((x_1 \cup x_2)^0 \cap y_1) \cup \overline{y}_1 \subseteq (\overline{(x_1 \cup x_2)^0} \cap \overline{x_1} \cup \overline{y}_1 \subseteq ((x_1 \cup x_2)^0 \cap \overline{x_1}) \cup \overline{x_1} = \overline{x_2}$.

0.3.7 Wrapping Facts

This subsection states four useful properties of the partial order $\leq$ on hairs.

If $X \leq Y$ then:

WF(1) $Y^0 \subseteq X^0$

Remark. WF(1) asserts that if $X$ is contained in some element $e$ then so are all less generic hairs.

WF(2) $Z \subseteq_h X$ implies that $X \leq Z$

Remark. WF(2) asserts that a subhair is more generic.

WF(3) $\leq$ is a partial order on the set of hairs

WF(4) $y_1 \leq_Y y_2$ implies that $X_{y_1} \leq_X X_{y_2}$

Remark. WF(3) asserts that HN(7) and WD(2) give an order preserving map from any
hair to any less generic one. This map is the identity inclusion in case the more generic is a subhair.

0.3.8 Proofs of wrapping facts

This subsection contains proofs of the 4 properties stated in the previous subsection.

WF1

By the definition of $[x]_X$ have $[x]_X^0 = \hat{X}^0$.
By the previous sentence and WD(1) and WD(3) have $\hat{Y}^0 = [y]^0_Z \subseteq [x]_X^0 = \hat{X}^0$.

WF2

By $Z \subseteq X$ have $Z_r = X_r$ and hence $X_r^+ \subseteq Z_r^+$ so WD(1) holds for $X \leq Z$.
For all $z \in Z \subseteq X$ by $z \in X$ and $z^+ \subseteq z^+$ have WD(2) holding for $X \leq Z$.
For all $z \in Z$ and $x \in X$ with $x^+ \subseteq z^+$ by IF(1b) have $[z]_Z^r = [z]^r_X \subseteq [x]_X^r$.
Thus WD(3) holds for $X \leq Z$.

WF3

If $X \leq Y \leq Z$ then:
By the previous sentence and WD(1) have $X_r^+ \subseteq Y_r^+ \subseteq Z_r^+$ and hence WD(1) holds for $X \leq Z$.
By the first sentence and WD(2) have that for all $z \in Z$ there is $x \in X$ and $y \in Y$ with $x^+ \subseteq y^+ \subseteq z^+$ and hence WD(2) holds for $X \leq Z$.
By $\sigma X_r^+ \subseteq \sigma Y_r^+$ and $y^+ \subseteq z^+$ have that (if $\sigma X_r^+ \not\subseteq z^+$ then $\sigma X_r^+ \not\subseteq y^+$ and $\sigma Y_r^+ \not\subseteq z^+$).
By the previous sentence and WD(3) have that if $x \in X$ and $z \in Z$ with $\sigma X_r^+ \not\subseteq z^+$
then \([x]_Y \supset [y]_Y \supset [z]_Y\).

By the previous four sentences have \(X \subseteq Z\).

If \(X \leq Y \leq X\) then:

By the previous sentence and WD(1) have that \(X_r^+ \subseteq Y_r^+ \subseteq X_r^+\) and hence \(X_r = Y_r\).

By \(X \leq Y \leq X\) if \(x \in X\) with \(x^0 \neq X^0\) then by WD(2) there are \(x_1^+ \subseteq y^+ \subseteq x^+\) with \(x_1 \in X\) and \(y \in Y\).

By the previous sentence and HF(1) have \(x_1 = x\) and hence \(x = y \in Y\).

If \(x \in X\) with \(x^0 \neq X^0\) then by HF(6) there is \(x_1 \in X\) with \(x_1^0 \neq X^0\) and \(x \in \{x_1 \rightarrow X_r, x_1 \leftarrow \sigma X_r\}\).

By the previous sentence and HD(1) and \(x_1 \in Y\) and \(Y_r = X_r\) have \(x \in Y\) and hence \(X = Y\).

**WF4**

Assume \(y_1 <_Y y_2\) and \(X_{y_1} = x_1\) and \(X_{y_2} = x_2\) then:

If \(y_1^+ \subseteq y_2^+\) then by HF(6) have \(y_2 = y_1\).

By \(x_1^+ \subseteq y_1^+ \subseteq y_2^+ \supseteq x_2^+\) and \(\{X \subseteq y_1^+\} = \{X \subseteq x_2^+\}\) have \(x_1^+ \subseteq x_2^+\) and hence \(x_2 \in \{x_1, x_1, x_1\}\).

If \(x_1 = x_2\) then \(x_1 \leq X x_2\).

If \(x_1 \neq x_2\) then:

By the previous sentence and \(x_2^+ \not\subseteq x_1^+\) and \(\{X \subseteq y_1^+\} = \{X \subseteq x_1^+\}\) have \(x_2^+ \not\subseteq y_1^+\).

By the previous sentence and \(y_1^+ \subseteq y_2^+ \supseteq x_2^+\) choose \(e^0 \in y_1 \setminus x_2\).

If \(x_1 \in X^\ast\) then:

By the previous sentence and \(e^0 \in y_1^0 \subseteq x_1^0\) and \(\cap Y^0 \subseteq X^0 \not\cap e^0\) and \(X_r^+ \subseteq Y_r^+\) and HF(5) have \(e^* \in y_2^+ \cap Y_r^\ast\) and hence \(e^* \in x_2^+ \cap X_r^\ast\) and so \(x_2 \geq X x_1\).

If \(x_1 = X_r\) then by HF(3) have \(x_1 \leq X x_2\).

If \(x_1 = \sigma X_r\) then:

By \(X_r \neq \emptyset\) and the previous sentence and \(x_1^+ \subseteq y_1^+ \subseteq y_2^+\) have \(X_r^+ \not\subseteq y_2^+ \cup y_1^+\).
By the previous sentence and $e^0 \in y_1 \setminus \overleftarrow{N}^0$ and WD(3) have $e^9 \in [y_1]_{\overleftarrow{Y}} \subseteq (x_1)_{\overrightarrow{X}}$.

By the previous sentence and $e^0 \in X_r \setminus \overleftarrow{N}^0$ and HF(5) have $e^9 \in x$ for all $x \in X^*$.

By the previous two sentences and $x_1 \neq x_2$ and $(x_2^+ \subseteq y_2^+ \supseteq x_1^+$ implies $x_1 \neq \sigma x_2$) and $x_2^+ \subseteq y_2^+$ have $e^9 \in y_2^+$.

By the previous sentence and $e^9 \in [y_1]_{\overleftarrow{Y}}$ and HF(5) have $y_2 \leq_Y y_1$ and hence a contradiction.

If $y_2^+ \subseteq y_1^+$ then by HF(6) have $y_1 = \overrightarrow{y}_2$.

By $x_2^+ \subseteq y_2^+ \subseteq y_1^+ \supseteq x_1^+$ and $\{X \subseteq y_1^+\} = \{X \subseteq x_1^+\}$ have $x_2^+ \subseteq x_1^+$ and hence $x_1 \in \{\overrightarrow{x}_2, x_2, \overleftarrow{x}_2\}$.

If $x_1 = x_2$ then $x_1 \leq_X x_2$.

If $x_1 \neq x_2$ then:

By the previous sentence and $x_1^+ \not\subseteq x_2^+$ and $\{X \subseteq y_2^+\} = \{X \subseteq x_2^+\}$ have $x_1^+ \not\subseteq y_2^+$.

By the previous sentence and $y_2^+ \subseteq y_1^+ \supseteq x_1^+$ choose $e^0 \in y_2 \setminus x_1$.

If $x_2 \notin \{X_r, \sigma X_r\}$ then:

By the previous sentence and $e^0 \in y_2 \setminus \overleftarrow{N}^0 \subseteq x_2 \setminus \overleftarrow{N}^0$ and $X_r^+ \subseteq Y_r^+$ and HF(5) have $e^9 \in y_1^+ \cap Y_r^+$ and hence $e^9 \in x_1^+ \cap X_r^+$ and so $x_1 \leq_X x_2$.

If $x_2 = X_r$ then by $X_r \neq \emptyset$ and $x_2^+ \subseteq y_2^+ \subseteq y_1^+$ have $\sigma X_r^+ \not\subseteq y_1^+ \cup y_2^+$.

By the previous sentence and $e^0 \in y_2 \setminus \overleftarrow{N}^0$ and WD(3) have $e^9 \in [y_2]_{\overleftarrow{Y}} \subseteq [x_2]_{\overrightarrow{X}}$.

By the previous sentence and $e^0 \in X_r \setminus \overleftarrow{N}^0$ and HF(5) have $e^9 \in x^+$ for all $x \in X^*$.

By previous three sentences and $x_1 \neq x_2$ and $(x_1^+ \subseteq y_1^+ \supseteq x_2^+$ implies $\overrightarrow{x}_1 \neq \sigma x_2$) and $x_2^+ \subseteq y_2^+$ and $e^9 \in [y_2]_{\overleftarrow{Y}} \subseteq [x_2]_{\overrightarrow{X}}$ and HF(5) have $e^9 \in x_1^+ \subseteq y_1^+$.

By the previous sentence and $e^0 \in y_1$ and $e^9 \in \overrightarrow{y}_1 \cap y_2$ and HF(5) have $y_2 \leq_Y y_1$ and thus a contradiction.

If $x_2 = \sigma X_r$ then by HF(3) have $x_1 \leq_X x_2$. 

26
0.3.9 Cuts

This subsection defines a relation on hairs and states some properties of this relation. This will be used in section 4. This relation is in fact the inverse to a function $\Psi$ defined on sufficiently long hairs which is used to construct all but the final cut map mentioned in the outline.

If $X$ is a hair: Define $\Psi^{-1}X = \{ Y | X \subset_h Y \text{ and for no } Z \text{ is } X \subset_h Z \subset_h Y \} = \{ Y | \Psi Y = X \}$.

Remark. The previous line defines both $\Psi^{-1}$ and $\Psi$.

HC(1) If $Y \in \Psi^{-1}X$ and $\{u, v\} \subset X$ then:
(a) $X^0 = Y^0$
(b) $u \leq_X v$ iff $u \leq_Y v$
(c) $u <_X v$ iff $u <_Y v$
(d) $\frac{x}{u} = \frac{y}{x}$ and $u = u$

Remark. HC(1) notes that the ordering of subhair is the same as that of the hair.

HC(2) If $u \in (\frac{[n]}{+0^-})$ and $u \rightarrow X_r = X_m \neq \sigma X_r$ then:
(a) for all $x \in X$ have $u \rightarrow x = X_m$ and $u \rightarrow \sigma x = u \rightarrow \sigma X_r$
(b) for all $x \in X$ have $x \rightarrow u = x \rightarrow X_m = \frac{x}{u} = x \rightarrow (u \rightarrow \sigma X_r)$
(c) for all $x \in X$ have $u^0 \cap x^0 = X^0$
(d) for all $\{x, y\} \subset X$ have $y \subset x \cup X_m$ iff $y \subset x \cup u$ iff $y \subset x \cup (u \rightarrow \sigma X_r)$

Remark. HC(2) gives a set of extensions of a hair and shows that they are also hairs.

HC(3)(a) if $\sigma X_r \in X$ then $\Psi^{-1}X = \phi$
(b) if $\sigma X_r \in X_m^+$ then $\Psi^{-1}X = \{ X \cup \{ \sigma X_r \} \}$
(c) if $\sigma X_r \in X_m^+$ then $\Psi^{-1}X = \{ X \cup \{ u, u \rightarrow \sigma X_r \} | u \rightarrow X_r = X_m \neq u \}$
(d) if $|X| \geq 5$ then $\Psi X = X \setminus \{ x \subset X_m^+ \}$.  

Remark. HC(3) is the main point of this section and uses 1 and 2 in it’s proof.
HC(3)(abc) give a nice description of the set of hairs which cut to X. This description will enable the computation in section 4 that the inverse image of a point under a cut map is contractible. HC(3)(d) describes the effect of cutting a hair.

0.3.10 Hair Cut Proofs

This subsection contains proofs of the 3 properties stated in the previous subsection.

HC1

(a) By \(|X| \geq 3\) choose \(\{u, v\} \subseteq X \subseteq Y\) with \(u \notin \{v, \sigma v\}\).

By the previous sentence and HF(1) have \(u^0 \cap v^0 = X^0 = Y^0\).

(b) By HN(2) and \(X_r = Y_r\) and \(\sigma X_r = \sigma Y_r\) have \(u \leq_X v\) iff \(u \leq_Y v\).

(c) By HF(6) and \(\{u, v\} \subseteq X \subseteq Y\) have \(u <_X v\) iff \(u <_Y v\).

(d) If \(u \notin \{X_r, \sigma X_r\} = \{Y_r, \sigma Y_r\}\) then by HF(6) have that \(\overrightarrow{u}\) and \(\overleftarrow{u}\) depend only on \(v <_X u\) which is equivalent to \(v <_Y u\) by HC(1c).

If \(u = X_r = Y_r\) then by HD(0) have \(X_r <_X v\) and \(\overrightarrow{x} \overrightarrow{v} = \overrightarrow{u} = u\) and \(\overleftarrow{u} = u = u \overset{\sigma}{\rightarrow} v\).

If \(u = \sigma X_r = \sigma Y_r\) then by HD(0) and HF(3) have \(v <_X \sigma X_r\) and by HC(1c) have \(v <_Y \sigma X_r = \sigma Y_r\).

By the previous sentence and HF(6) have \(\overrightarrow{u} = \overrightarrow{u} = u\) and \(\overleftarrow{u} = u = u \overset{\sigma}{\rightarrow} v\).
HC2

(a) By HF(3) and $x \in X$ have $X_r \leq_X x \leq_X X_m$.

By $u \rightarrow X_r = X_m$ have $u \rightarrow X_r = u \rightarrow X_m = u \rightarrow (X_r \cup X_m)$.

By HF(4) and $X_m \neq \sigma X_r$ and $X_r \leq_X x \leq_X X_m$ have $x \subseteq X_r \cup X_m$.

By the previous two sentences and $\{u, x, X_m\} \subseteq (\frac{\text{ln}}{\text{lo}})$ have $X_m = u \rightarrow (x \cup X_m) \supseteq u \rightarrow x \in (\frac{\text{ln}}{\text{lo}})$ and hence $X_m = u \rightarrow x$.

By $u \rightarrow x = u \rightarrow X_r$ have $u \rightarrow \sigma x = u \rightarrow \sigma X_r$.

(b) By HC(2a) have $x \rightarrow X_m = x \rightarrow (u \rightarrow x) = x \rightarrow u$.

By $X_m \neq \sigma X_r$ and HF(8) have $X_m = X_m$.

By the previous two sentences and HF(6) have $x \rightarrow X_m = x$.

By HF(1) have $x \rightarrow \sigma X_r = x$.

(c) By $X_m \neq \sigma X_r$ and HF(8) and $u \rightarrow x = X_m$ have $X_0 = X_m = u \cap x$.

(d) If $y \subseteq x \cup X_m$ then by HC(2a) have $y \subseteq x \cup X_m = x \cup (u \rightarrow x) \subseteq x \cup u$.

If $y \subseteq x \cup u$ then by HC(2b) have $y \subseteq x \cup u \subseteq x \cup (u \rightarrow \sigma x) = x \cup (u \rightarrow \sigma X_r)$.

If $y \subseteq x \cup (u \rightarrow \sigma X_r) = (x \cup u^+) \cup (x \cup (u^0 \cap \sigma X_r))$ then:

By HC(2a) have $u \rightarrow y = u \rightarrow x$ and hence $(u^0 \cap y) = (u^0 \cap x)$.

By the previous two sentences have $y = (y \cap u^+) \cup (y \cap u^0) \subseteq (x \cup u^+) \cup (x \cap u^0) \subseteq x \cup (u \rightarrow X_r) = x \cup X_m$.

HC3

(a) Assume $\sigma X_r \in X$ and $X \subseteq Y$.

By the previous sentence and HF(3) have $X_m = Y_m = \sigma Y_r = \sigma X_r$.

By the previous sentence and HC(1c) and $X_r = Y_r$ have $X = Y$.
(b) \( \geq \):

Assume \( \sigma X_r \subset X_m \) and set \( Y = X \cup \{ \sigma X_r \} \).

By \( |Y| = |X| + 1 \geq 43 \) have HD(0) for \( Y \).

For all \( \{ w, u \} \subset Y \) and \( v \in Y^* = X^* \):

- If \( \sigma X_r \not\in \{ w, u \} \) then \( \{ w, v, u \} \subset X \) and by HD(1) for \( X \) have \( \{ v \overset{\circ}{\rightarrow} \sigma u, w \overset{\circ}{\rightarrow} u \} \subset X \subset Y \).

- If \( u = \sigma X_r \neq w \) then \( \sigma u = X_r \) and by HD(1) for \( X \) have \( v \overset{\circ}{\rightarrow} \sigma u \in X \subset Y \).

  If \( w \in X^* \) then by HD(1) for \( X \) and \( u = \sigma X_r \) have \( w \overset{\circ}{\rightarrow} u \in X \subset Y \).

  If \( w \not\in X^* \) then by \( w = X_r \) and \( u = \sigma X_r \) have \( w \overset{\circ}{\rightarrow} u = X_r = Y_r \in Y \).

- If \( w = \sigma X_r \neq u \) then by \( \{ u, v \} \subset X \) and HD(1) for \( X \) have \( v \overset{\circ}{\rightarrow} \sigma u \in X \subset Y \).

  If \( u = X_r \) then \( w \overset{\circ}{\rightarrow} u = \sigma X_r \in Y \).

  If \( u \in X^* \) then by \( X_m \neq \sigma X_r \) and \( u \in X^* \) and HF(6b)(6c) have \( X_r \overset{\circ}{\rightarrow} u = X_r \overset{\circ}{\rightarrow} X_m \).

  By the previous sentence and \( \sigma X_r^+ \subset X_m^+ \) have \( \sigma X_r \overset{\circ}{\rightarrow} u = \sigma X_r \overset{\circ}{\rightarrow} X_m = X_m \in X \subset Y \).

- If \( u = w = \sigma X_r \) then \( w \overset{\circ}{\rightarrow} u = \sigma X_r \in Y \) and by HD(1) for \( X \) have \( v \overset{\circ}{\rightarrow} \sigma u = v \overset{\circ}{\rightarrow} X_r \in X \subset Y \).

By the previous seven sentences HD(1) holds for \( Y \).

If \( \{ u, v \} \subset Y \) with \( e^* \in u^+ \cap v^- \) then:

Either \( \{ u, v \} \subset X \) and there is a unique \( w \in X \) satisfying HD(2),
In which case by HF(6) have \( w \neq X_r \) and so by HF(1) have \( e^0 \notin X^0_r = \sigma X_r^0 \).

Or \( e^* \in v^+ \cap \sigma X_m^- \) and then:

By the previous sentence and \( \sigma X^+_r \subseteq X^+_m \) have \( e^* \in v^+ \cap X_m^- \).

By the previous sentence and the previous case and \( e^0 \notin X_r \) have a unique \( w \in Y \) with \( e^0 \in w \subseteq v \cup X_m \).

By the previous sentence either \( v = X_r \) or by HD(1) for \( Y \) and \( \sigma X_r \in Y \) and \( v \in Y^* \) and \( \sigma X^+_r \subseteq X^+_m \) and HF(1) have \( \sigma X_r \neq \sigma X_r \overset{*}{\rightarrow} v \in Y \).

By the previous sentence if \( v \neq X_r \) by \( e^{**} \in \sigma X_r \overset{*}{\rightarrow} v \in Y \) and \( e^* \in v \) and \( e^0 \in w \) and HF(6) have \( w \subseteq v \cup (\sigma X_r \overset{*}{\rightarrow} v) \subseteq \sigma X_r \cup v \).

By the previous eight sentences have HD(2) for \( Y \).

If \( \{X_r, \sigma X_r\} \not\subseteq \{y, v, w\} \subseteq X \) then:

If \( \sigma X_r \not\subseteq \{y, v, w\} \) then by HD(3) for \( X \) HD(3) holds \( Y \).

If \( w = \sigma X_r \) then by \( X_r \not\subseteq \{y, v\} \) and HF(2) and HF(3) have HD(3) for \( Y \).

(\( \subseteq \)):

Assume \( Y \in \Psi^{-1}X \) and \( \sigma X_r \subseteq X_m \) then:

Denote by \( y \cdot_Y X_m \).

By the previous two sentences and HF(6) and HF(8) and HC(1d) have \( y^+ \subseteq X^+_m \).

By \( y \neq X_m \) and \( y^+ \subseteq X^+_m \) have \( e^0 \in y \setminus X_m \).

By \( \sigma X^+_r \subseteq X^+_m \) and \( e^0 \in y \setminus X_m \) and HF(5) have some \( X_r \subseteq_X w \subseteq_X X_m \) with \( e^0 \in w \in X \).

By \( w \neq y \notin X \) and the previous two sentences and HF(1) have \( w = \sigma y = X_r \).

(c) (\( \supseteq \)):

Assume \( \sigma X_r \not\subseteq X_m \) and set \( Y = X \cup \{u, u \overset{*}{\rightarrow} \sigma X_r\} \) with \( u \overset{*}{\rightarrow} X_r = X_m \neq u \).

By \( |Y| \geq |X| \geq 3 \) have HD(0) for \( Y \).

By \( u \overset{*}{\rightarrow} X_r = X_m \neq \sigma X_r \) and HC(2a)(2b) and HD(1) for \( X \) have HD(1) for \( Y \).

If \( \{u', v\} \subseteq Y \) and \( e^* \in u'^+ \cap v^- \) then:
• If \( \{u', v\} \subseteq X \) then there is a unique \( w \in X \) with \( e^0 \in w \subseteq u' \cup v \) then:
  
  For all \( y \in Y \) with \( e^0 \in y \) by \( u \xrightarrow{0} X_r = X_m \neq \sigma X_r \) and HC(2c) have \( y = w \).

• If \( u' \in X \) and \( v \notin X \) and \( e^* \in X_m^- \) then again \( w \ni e^0 \) is unique and by HC(2d) have
  
  \( w \subseteq u' \cup X_m \) and hence \( w \subseteq u' \cup v \).

• If \( u' \in X \) and \( v \notin X \) and \( e^* \in X_m^+ \) then:
  
  By \( X_m = u \xrightarrow{0} X_r \) and \( v \in \{u, u \xrightarrow{0} \sigma X_r\} \) have \( e^* \in X_r^+ \).
  
  By the previous sentence and HF(5) have \( e^0 \notin z \in X \) and hence \( u \ni e^0 \) is unique.
  
  By the previous two sentences and HC(2a) have \( u = v \xrightarrow{0} u' \) and hence \( u \subseteq v \cup u' \).
  
  By the previous eight sentences and HF(8) have HD(2) for \( Y \).

If \( \{X_r, \sigma X_r\} \not\subseteq \{y, v, w\} \subseteq Y \) then:

• If \( \{y, v, w\} \subseteq X \) then by HD(3) for \( X \) have HD(3) for \( Y \).

• If \( \{y, v\} \subseteq X \) but \( w \in Y \setminus X \) then by HC(2d) and HD(3) for \( \{u, v, X_m\} \subseteq X \) have
  
  HD(3) for \( Y \).

• If \( y \in X \) and \( \{v, w\} \subseteq Y \setminus X \) then by HC(2a) have \( u \xrightarrow{0} X_r = u \xrightarrow{0} \sigma y \neq u \) and
  
  hence only \( u \subseteq y \cup u \xrightarrow{0} \sigma y \).

Thus by the previous four sentences and HD(3) holding when \( |\{y, v, w\}| < 3 \) have HD(3) for \( Y \).

(\( \subseteq \)):

Assume \( Y \in \Psi^{-1}X \) and \( \sigma X_r \not\subseteq X_m \).
By the previous sentence and HC(1c) have \( Y_m \neq X_m \) so choose \( y \cdot \geq Y X_m \).

By the previous sentence and HF (6) and HF(8) and \( y \neq \sigma X_r \) have \( y \circ \sigma X_r = X_m \).

(d) : By HC(3a)(3b)(3c) have HC(3d).

0.3.11 EI

The cutting maps will be shown to be homotopy equivalences using lemma 1 of section 4. The idea of this lemma is to use the fact that the fibers over vertices are contractible, as seen by HC(3) and then to see that the projection of the fiber over any simplex to that of one of its facets is a homotopy equivalence.

This subsection contains a nice description of the fiber of this latter map which will be used to check that this is contractible.

EI: If \( \Psi X \leq Y \) or \( \sigma X_r \in X \leq Y \) then:

\( \text{EI(1) if } \sigma Y^+ \subseteq Y_m \text{ then } \Psi X Y = \{ Y \cup \{ \sigma Y_r \} \}. \)

\( \text{EI(2) if } \sigma Y^+ \not\subseteq Y_m \text{ and } x = \max_x (X \subseteq Y_m) \text{ then } \Psi X Y = \{ Y \cup \{ u, u \circ \sigma Y_r \} | u \circ \sigma x \circ \}
Y_r = Y_m \neq u \}. \)

0.3.12 Proofs of EI

This subsection contains proofs of the two statements in the previous subsection.
EI1

(⊆) By HC(3b) have $\Psi_\mathcal{O} X Y \subseteq \Psi^{-1} Y = \{ Y \cup \{ \sigma Y_r \} \}$.

(⊇) If $Y' = Y \cup \{ \sigma Y_r \}$ have:

(WD1) By WD(1) for $X \leq Y$ have $Y_r^{++} = Y_r^+ \supseteq X_r^+$ and hence WD(1) holds for $X \leq Y'$.

(WD2) If $\Psi X \leq Y$ then by WD(2) and $\sigma X_r^+ \subseteq \sigma Y_r^+ \subseteq Y_m^+$ have $u \in (X \subseteq Y_m^+)$. By the previous sentence and HC(3b) and $\mathcal{O} \subseteq X_0^0$ have $\sigma X_r^+ \subseteq \mathcal{O}^+$ and hence $\sigma X_r \in X$.

By the previous two sentences and either $\Psi X \leq Y$ or $\sigma X_r \in X$ and $\sigma X_r^+ \subseteq \sigma Y_r^+$ and WD(2) for $X \leq Y$ have WD(2) for $X \leq Y'$.

(WD3) By $\sigma Y_r^+ \subseteq \sigma Y_r^+$ and $[\sigma Y_r]_Y = [\sigma Y_r]_Y = [Y_m]_Y$ and $(X \subseteq \sigma Y_r^+) \subseteq (X \subseteq Y_m^+)$ and WD(3) for $X \leq Y$ have WD(3) for $X \leq Y'$.

EI2

(⊆) By HC(3b) have $\Psi_\mathcal{O} X Y \subseteq \Psi^{-1} Y = \{ Y \cup \{ u, u \not\rightarrow \sigma Y_r \} | u \not\rightarrow Y_r = Y_m \neq u \}$.

If $Y' = Y \cup \{ u, u \not\rightarrow \sigma Y_r \} \in \Psi_\mathcal{O} X Y$ then:

By the previous sentence and WD(2) and $u^+ \subseteq Y_m^+$ have $v \in (X \subseteq u^+) \subseteq (X \subseteq Y_m^+)$. By the previous sentence and $u \geq \mathcal{O} Y_m$ and WF(4) have $X_u \geq X Y_m$.

By the previous two sentences and HF(7) and $x = \max \geq X (X \subseteq Y_m)$ have $x^+ \subseteq u^+$. By the previous sentence have $\Psi_\mathcal{O} X Y = \{ Y \cup \{ u, u \not\rightarrow \sigma Y_r \} | u \not\rightarrow Y_r = Y_m \neq u$ and $x^+ \subseteq u^+ \} \subseteq \{ Y \cup \{ u, u \not\rightarrow \sigma Y_r \} | u \not\rightarrow \sigma x \not\rightarrow Y_r = Y_m \neq u \}$. 

34
(2) For every $Y' = Y \cup \{u, u \circlearrowright \sigma Y_r\}$ with $u \circlearrowright \sigma x \circlearrowright Y_r = Y_m \neq u$ have:

(WD1) By WD(1) for $X \leq Y$ have $Y_r^+ = Y_r^+ \supseteq X_r^+$ and hence WD(1) holds for $X \leq Y'$.

(WD2) By $x^+ \subseteq Y_m^+$ and $(u \circlearrowright \sigma x)^+ \subseteq Y_m^+$ have $x^+ \subseteq u^+ \subseteq (u \circlearrowright \sigma Y_r)^+$ and hence WD(2) holds for $X \leq Y'$.

(WD3) By IF(1b) and $x^+ \subseteq \bar{u}^+$ and WD(3) for $X \leq Y \ni \bar{u}$ have $[u]_r^y, \subseteq [u]_x^y, = [\bar{u}^y, \subseteq [\bar{x}]_x^y.$

If $x^+ \subseteq \bar{u}^+$ then by WD(3) for $X \leq Y$ have $[u]_r^y, = [\bar{u}]_y, \ni (\underline{x^0} \setminus \hat{Y}^0) \cap Y_r^-$ and

$[x]_x = [\bar{x}]_x = \bar{y}_x \ni (\underline{x^0} \setminus \hat{Y}^0) \cap X_r^-$ and

$[u]_r^y, = [\bar{u}]_y, \ni (\underline{x^0} \setminus \hat{Y}^0) \cap Y_r^+$ and

$[x]_x = [\bar{x}]_x \ni (\underline{x^0} \setminus \hat{Y}^0) \cap X_r^+.$

By the previous sentence and $x^+ \subseteq u^+$ and $X_r^- \subseteq Y_r^-$ and $\bar{Y} \subseteq X^0$ and $\bar{u}^y, \subseteq [x]_x$ have

$[u]_r^y, = [\bar{u}]_y, \ni [x]_x = [\bar{x}]_x.$

If $x^+ \subseteq \bar{u}^+$ then by the previous two sentences and WF(4) and $\bar{u} \leq Y_r \bar{u}$ and HF(7) have

$x^+ \subseteq \bar{u}^+$ and hence by WD(3) for $X \leq Y$ have $[\bar{u}]_y, \subseteq [\bar{x}]_x.$

By the previous sentence and $x^+ \subseteq \bar{u}^+ \cap \bar{u}^+$ and $X_r^- \subseteq Y_r^-$ have $\bar{x^0} \cap \bar{u}^0 \cap X_r^- = \phi.$

By the previous sentence have $((\underline{x^0} \setminus \hat{Y}^0) \cap Y_r^-) \cap ((\underline{x^0} \setminus \hat{Y}^0) \cap X_r^-) = \phi.$

By the previous five sentences have $[u]_y, = [\bar{u}]_y, \subseteq [\bar{x}]_x.$

If $x^+ \subseteq \bar{u}^+$ then by $[u]_r^y, \subseteq [x]_x$ and $((\underline{x}^0 \setminus \hat{X}^0) \cap X_r^+) = \phi$ have $[u]_r^y, \cap ((\underline{x^0} \setminus \hat{X}^0) \cap X_r^+) = \phi$ and hence $[\bar{u}^y, \subseteq [\bar{x}]_x.$

By HF(7b) and $x^+ \subseteq u^+ \subseteq \bar{u}^+ \cap \bar{u}^+$ have $\{X_u^-, X_u, X_\bar{u}^+\} \subseteq \{\bar{x}, x, \bar{x}\}.$

By the previous ten sentences and WD(3) for $X \leq Y$ and $Y' \setminus Y = \{u, \bar{u}\}$ have WD(3) for $X \leq Y'.$

35
0.4 Hairs in Oriented Matroids

In this section oriented matroids will finally appear, and the main results of this thesis will be stated.

Define a hair $X$ to be short if $|X| \leq 4$.

Define a hair $X$ to be long if $\sigma X_\tau \subseteq X$.

Remark. Long hairs are equivalent to pairs $(N^2, N^1)$ of oriented matroids with $N^1$ a strong image of $N^2$. Short hairs are the image of all but the last cut map, and the set of all short hairs has the same homotopy type as the set of all long hairs.

0.4.1 Short Facts

This subsection contains the basic properties of short hairs.

SF(1) $X_\tau \subseteq X \subseteq (\frac{m+1}{4}+\frac{n-1}{2})$ is a short hair iff $X = \{X_\tau, X_\tau \circ X_1 = X_1 \circ X_\tau, X_1, X_1 \circ \sigma X_\tau\}$ with $X_\tau^+ \not\subseteq X_1$.

Remark. SF(1) is a useful characterization of a short hair for the proofs of SF(2) and SF(3).

SF(2) If $X$ is a short hair and $Y_\tau^+ \supseteq X_\tau^+$ then:

(a) if $Y_\tau^+ \cap X_1^+ \neq \phi$ then $\Psi^{-1}_{\geq X} Y_r = \{(Y_r, Y_1, Y_1, Y_1) | Y_1 \circ \sigma X_1 \circ \sigma X_\tau \circ Y_r \subseteq Y_\tau \circ X_1 \cup (Y_r \circ X_1)^0 \text{ and } Y_1 \not\subseteq Y_r \cup Y_1^0\}$.

(b) if $Y_\tau^+ \cap X_1^+ = \phi$ then $\Psi^{-1}_{\geq X} Y_r = \{(Y_r, Y_1, Y_1, Y_1) | Y_1 \circ \sigma X_1 \circ Y_r \subseteq Y_r \circ X_1 \cup (Y_r \circ X_1)^0 \text{ and } Y_1 \not\subseteq Y_r \cup Y_1^0\}$.

SF(3) If $Y$ is a short hair and $X_\tau^+ \subseteq Y_\tau$ then:
(a) if \( X^+ \not\subseteq Y_1 \) then \( \Psi^{-1}_{\leq Y} X_r = \{ \{X_r, \overline{X}_1, X_1, \overline{X}_1\} | X_1^+ \subseteq Y_1\}. \)

(b) if \( X^+ \subseteq Y_1 \) then \( \Psi^{-1}_{\leq Y} X_r = \{ \{X_r, \overline{X}_1, X_1, \overline{X}_1\} | X_1^+ \not\subseteq Y_1 \cup (X_1^0 \cap \overline{Y}^+_1 \cap \overline{Y}^+_1)\}. \)

Remark. SF(2) and SF(3) are descriptions of the two objects which must be contractible to use lemma 2 to see that the final cut map is a quasifibration.

The above facts follow from an easy but somewhat tedious check of the definitions.

### 0.4.2 k-hairs

This subsection introduces the notion of a length \( k \) hair, or a k-hair, the set of which is the \( k \)-th intermediate step between short and long hairs.

If \( k \in [n] \) then a hair \( X \) is a k-hair if either

\( \sigma X_r \in X \) and \( |X| \leq 2k - 2 \) or

\( 2k - 1 \leq |X| \leq 2k \)

Denote by \( H \) the poset of hairs and by \( H_k \) the subposet of k-hairs.

Notice that short hairs are equivalent to 2-hairs and long hairs are equivalent to \( n \)-hairs.

For \( k > 2 \), denote by \( \Psi_k : H_k \to H_{k-1} \) the map with \( \Psi_k(X) = X \) in case KD(1) and \( \Psi_k(X) = \Psi X \) in case KD(2).

Denote by \( \Psi_2 : H_2 \to (\frac{[n]}{40^-}) \) the map with \( \Psi_2(X) = X_r \).

Remark. These are the oft promised cut maps.
0.4.3 Poset Lemmas

This subsection contains two lemmas about poset maps essentially equivalent to [Q] ThmA and ThmB, and used to show that the cut maps give a quasifibration.

Lemma 1: Given posets $P$ and $Q$, and a map $f : P \to Q$, with:

1. $f$ is order preserving
2. for $q \in Q$, $f^{-1}q$ is contractible
3. for $p \in P$ and $fp \leq q \in Q$, $f^{-1}q \cap P_{\geq p}$ is contractible

then $f : P \to Q$ is a weak homotopy equivalence on the posets with the order topology, and induces a homotopy equivalence between their simplicial realizations.

Proof (Bjorner): Check that $f^{-1}Q_{\geq q}$ is contractible for all $q \in Q$. Consider the restriction of the identity $\phi : f^{-1}Q \to f^{-1}Q_{\leq q}$. Note that $\phi^{-1}(f^{-1}(Q_{\leq q})_{\geq p}) = f^{-1}Q \cap P_{\geq p}$ which is contractible by 2. Thus $\phi$ is a homotopy equivalence by [Q] ThmA. But, $f^{-1}q$ is contractible by 1. Thus done by [Q] ThmA.

Lemma 2: Given posets $P$ and $Q$, and a map $f : P \to Q$, with:

1. $F$ is order preserving
2. for $p \in P$ and $fp \leq q \in Q$, $f^{-1}q \cap P_{\geq p}$ is contractible
3. for $p \in P$ and $fp \geq q \in Q$, $f^{-1}q \cap P_{\leq p}$ is contractible

then $f : P \to Q$ is a quasifibration on the posets with the order topology, and induces a quasifibration between their simplicial realizations.

Proof: Fix a chain $C = (q_0 \leq q_1 \leq ... \leq q_r)$ in $Q$. Denote by $C' = (q_1 \leq q_2 \leq ... \leq q_r)$ and by $C'' = (q_0 \leq q_1 \leq ... \leq q_{r-1})$. Denote by $f^{-1}C = \{D\}D$ is a chain in $P$ and
\( fD = C \). Denote by \( \pi' : f^{-1}C \to f^{-1}C' \) and by \( \pi'' : f^{-1}C \to f^{-1}C'' \) the projection maps. By Lemma 1 and properties 1 and 2, \( \pi' \) induces a homotopy equivalence. By Lemma 1 and properties 1 and 3, \( \pi'' \) induces a homotopy equivalence. Thus by [Q] the corollary to ThmB, \( f \) induces a quasifibration.

### 0.4.4 Cutting in an oriented matroid

This subsection introduces the oriented matroid \( M \) into this thesis, and restricts the hairs and cut maps to subsets of \( M \).

Fix \( M \) to be the set of covectors of a rank \( d \geq 2 \) oriented matroid. Fix a cocircuit \( c \in M \). Denote by \( H_k(M)\{X \in H_k|X \subset M\} \) and by \( H_k^c(M)\{X \in H_k(M)|X_r = c\} \). Denote by \( \Psi_k \) the restriction of \( \Psi_k \) to \( H_k(M) \) or to \( H_k^c(M) \).

Note [BLSWZ] that if \( \{A, B\} \subset M \) with \( A \not\in \{B, \sigma B\} \) and \( B^0 \subseteq A^0 \) then \( \{C \in M|C \supseteq B = A \neq C\} \) realized with the usual covector partial order is contractible.

By the previous note and Lemma 1 and HC(3) and EI(1) and EI(2) have that \( \Psi_k \) for \( k > 2 \) induces a homotopy equivalence between \( H_k(M) \) and \( H_{k-1}(M) \) and also between \( H_k^c(M) \) and \( H_{k-1}^c(M) \).

Note [FL] that if \( N \) is the covectors of a rank \( d \) oriented matroid obtained by deleting \( e \) from \( M \) and \( S \subseteq N \) then the realization of \( S \) is homotopy equivalent to the realization of \( \{C \in M|C \setminus e \in S\} \).

By the previous two notes and Lemma 2 and SF(2) and SF(3) have that \( \Psi_2 \) induces a quasifibration of \( H_k(M) \) over \( M \).

### 0.4.5 A Fiber Map

This subsection contains one of the two theorems of this thesis.

Choose a basis \( S \) in \( M \) with \( |S \cap e^+| = 1 \) and define \( \rho : H_n(M) \to H_n((\frac{S}{+0^-})) \) by deletion, and \( \rho : M \to (\frac{S}{+0^-}) \) by deletion.

Define \( \Psi(M) : H_n(M) \to M \) by \( \Psi(M) = \Psi_n \Psi_{n-1} \Psi_{n-2} \ldots \Psi_2 \).
By the previous subsection $\Psi(M)$ and $\Psi((\frac{S}{+0-}))$ are quasifibrations. Notice that $\rho\Psi((\frac{S}{+0-})) = \Psi(M)\rho$.

By Folkman-Lawrence [FL] the deletion map induces a homotopy equivalence between the realizations of $M$ and $(\frac{S}{+0-})$.

Define $\tau : H^c_2(M) \to H^c_{n-1}((\frac{S}{+0-}))$ so that $\Psi_n\Psi_{n-1}...\Psi_3\tau = \rho$.

Notice that by the above notes and [Q] ThmA $\tau$ induces a homotopy equivalence between $H^c_2(M)$ and $H^c_{n-1}((\frac{S}{+0-}))$ and hence $\rho$ induces a homotopy equivalence between $H^c_n(M)$ and $H^c_{n-1}((\frac{S}{+0-}))$.

Thus by the previous five sentences and the long exact sequence in homotopy for a quasifibration:

**Theorem:** $H_n(M)$ is homotopy equivalent to $H_n((\frac{S}{+0-}))$ which is homotopy equivalent to the real flag variety of lines in planes in $d$-space.

**0.4.6 Rank 2 Oriented Matroids**

This subsection contains the oriented matroid equivalent of the Grassmannian tautological circle bundle projection.

If $X$ and $Y$ are long hairs then $X \sim Y$ if

(RD1) $X \cup \sigma X = Y \cup \sigma Y$ and

(RD2) either (exactly one of $X_r \in Y$ and $Y_r \in X$) or $(X_r, X) \in \{(Y_r, Y), (\sigma Y_r, \sigma Y) = \sigma Y\}$.

If $X$ is a long hair then the $\sim$ equivalence class $[X]_\sim$ is called a rank 2 oriented matroid.

If $M$ and $N$ are rank 2 oriented matroids then $M \leq N$ if there are $X \in M$ and $Y \in N$ with $X \leq Y$.  

40
0.4.7 Rank 2 Facts

This subsection contains basic facts about the map in the previous subsection.

(MF1) If $Y$ is a long hair and $y \in Y \cup \sigma Y$ then there is a unique $[Y]_y = (y, X) \in [Y]_\sim$

(MF2) If $Y$ is a long hair and $\{x, y\} \subseteq Y$ then $[Y]_y \leq [Y]_x$ iff $y^+ \subseteq z^+$

(MF3) If $M$ and $N$ are rank 2 oriented matroids and $Y \in N \geq M$ then
\[ \{X \in M | Y \geq X\} = \{(x, N_x) | x \in (M \subseteq Y^r)\} \]

(MF4) If $M$ and $N$ are rank 2 oriented matroids and $X \in M \leq N$ then
\[ \{Y \in N | X \leq Y\} = \{(y, N_y) | y \in (N \supseteq X^r)\} \]

0.4.8 Grassmannians

This subsection contains this thesis' second theorem.

Define $\chi(X) = [X]$. Notice that by MF(3) and MF(4) and HF(7) and Lemma 2 have that $\chi$ is a quasifibration.

By Folkman-Lawrence [FL] have that $\rho$ induces a homotopy equivalence on the fibers, while by a preceding section, $\rho$ induces a homotopy equivalence on the total spaces. Thus by the long exact sequence in homotopy, $\rho$ induces a homotopy equivalence on the base spaces. Thus:

Theorem: The space of all rank 2 strong images of $M$ is homotopy equivalent to the grassmannian of real 2-planes in d-space.
0.5 OM as Rank Function

This is a rank function definition for oriented matroids analogous to the rank function definition for matroids given in White's book.

0.5.1 Definition

Call $B_{2n}$ the boolean algebra (that is the set of subsets partially ordered by inclusion) of the set $\{1, 2, ..., n\} \times \{-1, 1\}$, denoted $[n]_2$.

Denote by $[n]$ the set $\{1, 2, ..., n\}$.

Take $i : [n]_2 \rightarrow [n]_2$. Extend this to an order preserving involution $i : B_{2n} \rightarrow B_{2n}$. Call $r : B_{2n} \rightarrow \{b, 0, 1, 2, ...\}$ a rank function if:

1) $r(\phi) = 0$

2) $r([n]_2) \neq b$

3) for any $S \in B_{2n}$, with $S \cap \{e, ie\} = \phi$, the small diamond

\[
\begin{array}{c}
\text{r}(S \cup \{e, ie\}) \\
/ \\
r(S \cup \{e\}) \quad r(S \cup \{ie\}) \\
\backslash / \\
r(S)
\end{array}
\]

has one of the following forms for some integer $m$, depending on $S$:

\[
\begin{array}{ccccccc}
m & m+1 & b & b \\
/ & / & / & / & / & / & / \\
m & m & b & b & b & m & b & b \\
\backslash & / & \backslash & / & \backslash & / & \backslash & / \\
m & m & m & m & m & m & b
\end{array}
\]
Define $H_r = \{ S \in B_{2n} | r(S) = b \text{ and } r(T[\subset S]) \in \{0, b\} \}$.
Define $C_r = \{ S \in B_{2n} | r(S) = b \text{ and } r(T[\subset S]) = 0 \} \subset H$.
Define $D_r = \{ U \in B_{2n} | r(U) = 1 \text{ and } r(V[\subset U]) \neq 1 \}$.
For $K \subseteq B_{2n}$, $T \in B_{2n}$ write $K \preceq_T$ for $\{ S \in K | S \subseteq T \}$.

0.5.2 Claims

Claim 1: If $S \not\in H_r$ and $r(S) \neq 0$, then there is $T \subseteq S$ with $r(T) = 1$.

Proof: Take $R \subseteq S \not\in H_r$ with $r(R) = m \not\in \{0, b\}$. For $e \in R \setminus iR$ get:

\[
\begin{align*}
    r(R \cup \{ie\}) & \quad ? \quad ? \\
    r(R) & \quad r(R \cup \{ie\}\{e\}) = m \quad ? \quad m \quad ? \\
    r(R\{e\}) & \quad ? \quad m \\
\end{align*}
\]

so $r(P\{e\}) = r(R)$. For $\{e, ie\} \subseteq R$ get:

\[
\begin{align*}
    r(R) & \quad m \quad m \quad m \\
    r(R\{e\}) & \quad r(R\{ie\}) = ? \quad ? \quad = m \quad m \quad \text{or} \quad b \quad b \\
    r(R\{e, ie\}) & \quad ? \quad m \quad m - 1
\end{align*}
\]

so $r(R\{e, ie\}) \in \{r(R), r(R) - 1\}$. Induct on $|R|$.

Claim 2: If $r(S) \neq 0$, and $r(S\{e\}) = 0$, then $S \in H_r$.

Proof: Assume not. By condition 3, $r(S) = b$. By Claim 1 take $T \subset S \not\in H_r$ with $r(T) = 1$, then $r(T\{e\}[\subset S\{e\})] = 0$, contradicting condition 3.

Claim 3: If $S \in H_r$, and $\{e, ie\} \cap S = \phi$, then $S \cup \{e\} \in H_r$.

Proof: Assume not. By condition 3, $r(S \cup \{e\}) = r(S) = b$. By Claim 1 take $T \subset S \cup \{e\} \not\in H_r$ with $r([ie \not\in T]) = 1$,
\[ r(T \cup \{ie\}) \]

get: \( r(T) \quad r(T \cup \{ie\}\{e\}) = 1 \quad ? \), \( r(T\{e\}[\subseteq S \in H_r]) \) not 1
contradicting condition 3.

Claim 4: For \( S \in C_r, S \cap iS = \phi \).

Proof: Assume not. Have some \( \{e, ie\} \subseteq S \in C \).
\( r(S[\in C]) \) \( b \)
Thus get \( r(S\{e\}) \quad r(S\{e\}) = 0 \quad 0 \),
\( r(S\{e, ie\}) \) 0
contradicting condition 3.

Thus every \( S \in C \) is a signed subset of \([n]\).

Claim 5: For \( U \in D, U = iU \).

Proof: Assume not. Take \( e \in U\backslash iU \). Then \( ie \not\in U \).
\( r(U \cup \{ie\}) \) ?
Get: \( r(U[\in D]) \quad r(U \cup \{ie\}\{e\}) = 1 \quad ? \),
\( r(U\{e\}) \) not 1
contradicting condition 3.

Claim 6: For \( \{e, ie\} \subseteq U \in D \), \( r(U\{e, ie\}) = 0 \), while \( r(U\{e\}[\in H]) = b \).
\( r(U[\in D]) \)
Proof: Get: \( r(U\{e\}) \quad r(U\{e\}) = \) not 1 not 1 = \( b \) \( b \)
\( r(U\{e, ie\}) \) not 1 0
, by condition 3. By Claim2 \( U\{e\} \in H \).

Claim 7: For \( T \in C_{U[\in D]} \), \( T \cup iT = U \).

Proof: Assume not. As by Claim5 \( U = iU \), \( T \cup iT \subseteq U \). Have \( e \in U \backslash (T \cup iT) \).
So \( \{e, ie\} \subseteq U \setminus (T \cup iT) \). By Claim 6 \( r(U \setminus \{e, ie\}) = 0 \).
Thus \( r(T[\subseteq U \setminus \{e, ie\}]) = 0 \), contradicting \( r(T[\in C]) = b \).

Thus by Claim 4 and Claim 7, \( T[\in C_{U[\in D]}] \) is just a choice of signs for the set \( U/i^2 \), of order \( \frac{|U|}{2} \).

Claim 8: For \( U \in D \), there is an \( S \in C \) such that for \( V \subseteq U \)

\[
r(V) = \begin{cases} 
1 & V = U \\
 b & S \subseteq V \neq U \text{ or } iS \subseteq V \neq U \\
 0 & \text{otherwise}
\end{cases}
\]

Proof: Note that it is sufficient to show that \( C_{U[\in D]} = \{S, iS\} \), for some \( S \).

By Claim 6 \( r(U \setminus \{e\}) = r(U \setminus \{ie\}) = b \), so there is

some \( T \in C_{U \setminus \{e\}} \), and some \( R \in C_{U \setminus \{ie\}} \).

But by Claim 7, \( T \cup iT = U \), so \( ie \in T \neq R \), and hence \( |C_U| \geq 2 \).

Thus it is sufficient to show that for \( T, R \in C_{U[\in D]} \), \( T = R \) or \( T = iR \).

By Claim 7 it is enough to show that \( T \cup R = T \) or \( U \).

Assume not. Take \( e \in T \setminus R \). Then by Claim 7, \( ie \in R \setminus T \).

\[
r(T \cup R[\subseteq U \in D])
\]

Not 1

\[
r(T \cup R \setminus \{e\}) \quad r(T \cup R \setminus \{ie\}) = ? \quad ?
\]

Get:

\[
| \quad r(T \cup R \setminus \{e, ie\}[\subseteq U \setminus \{e, ie\}]) \quad | \quad 0 \quad |
\]

 contradicting condition 3.

Thus \( C_{U[\in D]} = \{S, iS\} \).

Claim 9: For \( S \in C \), \( S \cup iS \in D \).

Proof: Note that \( r(S \cup iS) \notin \{0, b\} \), as \( (S \cup iS) = i(S \cup iS) \), and hence by conditions 2 and 3, \( r(S \cup iS) \neq b \), and \( r(S \cup iS) \geq r(S[\in C]) = b \).
Thus can choose $F \in D_{SU}$. It suffices to show $F = S \cup iS$. Assume not.

Choose $Q \in C_{SU} \setminus \{S\}$, with $Q \cup S$ minimal.
(Possible as by Claim8 $|C_F \subseteq C_{SU}| \geq 2$.)

Choose $u \in S \setminus F$. (Possible, as by Claim5, $F = iF \subset S \cup iS$.)

Choose $e \in Q \setminus S$. Thus $ie \in S$.

Note that $r(Q \cup S \setminus \{u, e\}) = 0$. Assume not. Take $Y \in C[sgQ \cup S \setminus \{u, e\}]$.

Now $Y \cup S \subseteq Q \cup S \setminus \{e\} \subset Q \cup S$, contradicting choice of $Q$.

Thus if $r(Q \cup S \setminus \{u\}) \neq 0$, then by Claim2, $Q \cup S \setminus \{u\} \in H$,

and as $iu \notin Q \cup S$, by Claim3 $Q \cup S \in H$.

If $r(Q \cup S \setminus \{u\}) = 0$ then by Claim2, $Q \cup S \in H$.

Thus in either case get:

\[
\begin{array}{cccccc}
  r(Q \cup S[\in H]) & b & b \\
r(Q \cup S \setminus \{ie\}) & r(Q \cup S \setminus \{e\}) & = & b \text{ or } 0 & b \text{ or } 0 & = & b & b \\
| & r(Q \cup S \setminus \{e, ie\}) & | & b \text{ or } 0 & | & b & | \\
r(Q[\in C]) & r(S[\in C]) & b & b & b & b \\
\end{array}
\]

Thus there must be an element $Y \in C[sgQ \cup S \setminus \{e, ie\}]$,

with $Y \cup S \subseteq Q \cup S \setminus \{e\} \subset Q \cup S$, contradicting choice of $Q$.

0.5.3 Facts

Fact 1: For a labeling $r$ of $B_2$, $C_r$ are the cocircuits of a rank $r([n]_2)$ oriented matroid on the set $[n]$.

First note that, as mentioned after Claim4 $C$ is a set of signed subsets of $[n]$.

(C0) $\emptyset \notin C$

Check. By condition 0, $r(\emptyset) = 0 \neq b$.

(C1) $C = iC$

Check. Take $S \in C$. By Claim9, $S \cup iS \in D$. Thus by Claim8,
\{S, iS\} = C_{SU}S \subseteq C.

(C2) For \(S, T \in C\), if \(S \cup iS \subseteq T \cup iT\), then \(S = T\) or \(S = iT\).

Check. By Claim9, \(T \cup iT \in D\), so by Claim8, \(\{S, iS, T, iT\} \subseteq C_{TU}T = \{T, iT\}\).

(C3) For \(S, T \in C\), \(S \neq iT\), and \(e \in S \cap iT\), \(C_{SU}T\{e, ie\} \neq \emptyset\).

Check. It suffices to check that \(r(S \cup T\{e, ie\}) \neq 0\).

But if \(r(S \cup T\{e, ie\}) = 0\), get:

\[
\begin{array}{cccc}
  r(S \cup T) & ? & 1 \\
  r(S \cup T\{e\}) & r(S \cup T\{e\}) & ? & ? & = b & b \\
  | & | & | & 0 & | & 0 & | \\
  r(S[e \in C]) & r(T[e \in C]) & b & b & b & b \\
\end{array}
\]

so it suffices to check that \(r(S \cup T) \neq 1\).

If \(S \cup T \in D\) then by Claim8 \(S = iT\).

Assume \(r(S \cup T) = 1\), but \(S \cup T \notin D\). Choose \(F \in D_{SU}T\).

Either \(S \not\subseteq F\) or \(T \not\subseteq F\). Say \(S \not\subseteq F\). As in the proof of Claim9:

Choose \(Q \in C_{SU}S\{S\}\), with \(Q \cup S\) minimal. Choose \(u \in S \setminus F\) and \(e \in Q \setminus S\).

Note that as above, \(r(Q \cup S\{u, e\}) = 0\), and hence \(Q \cup S \in H\).

Thus get:

\[
\begin{array}{cccc}
  r(Q \cup S[e \in H]) & b & b \\
  r(Q \cup S\{e\}) & r(Q \cup S\{e\}) & b or 0 & b or 0 & = b & b \\
  | & | & | & b or 0 & | & b & | \\
  r(Q[e \in C]) & r(S[e \in C]) & b & b & b & b \\
\end{array}
\]

Thus there must be \(Y \in C_{[SG]QU}S\{e, ie\}\), with \(Y \cup S \subseteq Q \cup S\{e\} \subseteq Q \cup S\), contradicting the choice of \(Q\).

Fact 2: Given an oriented matroid \(M\) on the set \([n]\), the function
\( r : B_{2n} \rightarrow \{b, 0, 1, 2, \ldots\} \)

given by fixing an essential pseudohalfspace representation of \( M \) with halfspaces labeled by \([n]_2\), and \( e \cap ie \) being a pseudosphere, and taking

\[ r(S) = \begin{cases} 
  b & \text{if the intersection of the pseudohalfspaces } [n]_2 \backslash S \text{ (denoted } I_S \text{) is a ball} \\
  \dim(I_S) + 1 & \text{if } I_S \text{ is a sphere}
\end{cases} \]

is a rank function.

Check 1): \( I_\emptyset = \emptyset \) by essentiality, so \( r(\emptyset) = 0 \).

Check 2): \( I_{[n]_2} \) = the whole sphere, so \( r([n]_2) = \text{rank of } M \neq b \).

Check 3): Note that for any small diamond

\[
\begin{array}{cc}
S \cup \{e, ie\} \\
/ & \backslash \\
S \cup \{e\} & S \cup \{ie\} \\
\backslash & / \\
S
\end{array}
\]

\( I_{S \cup \{e, ie\}} = I_{S \cup \{e\}} \cup I_{S \cup \{ie\}} \), and \( I_S = I_{S \cup \{e\}} \cap I_{S \cup \{ie\}} \).

Thus by excision and the fact that \( I_S \) is a ball or sphere

\[
\begin{array}{cc}
r(S \cup \{e, ie\}) \\
/ & \backslash \\
r(S \cup \{e\}) & r(S \cup \{ie\}) \\
\backslash & / \\
r(S)
\end{array}
\]
has one of the following forms for some integers \( m \), and \( n \), depending on \( S \), and \( e \):

\[
\begin{array}{cccccc}
  & m & m+1 & b & b & m \\
/ & \ / & \ / & \ / & \ / & \ / \\
m & n & b & b & m & b & m & b \\
/ & \ / & \ / & \ / & \ / & \ / & \ / & \ / \\
n & m & m & b & b
\end{array}
\]

A pseudosphere arrangement, however, is defined so that if \( I_S \) is a sphere, \( I_S \cap e \) is homeomorphic to \( I_S \cap ie \).

Thus only the 4 forms of condition 3 remain.

Note that \( C_r \) is exactly the set of cocircuits of \( M \).

Fact 3: \( C_r \) uniquely determines the rank function \( r \) on \( B_{2k} \).

Induct on \( k \). True for \( B_{2^0} = \{\phi\} \). Assume true for \( n < k \).

Call \( r_\phi \), the rank function \( r \) restricted to \( \{ S \in B_{2k} | \{e, ie\} \cap S = \phi \} \).

Note that \( r_\phi \) is a rank function on a set of size \( k - 1 \), with \( C(r_\phi) = C_{[k]_2 \setminus \{e, ie\}}(r) \).

Thus \( r(S) \) is uniquely determined for all \( S \) with \( S \cup iS \neq [k]_2 \).

Now for \( S \cup iS = [k]_2 \), induct on \( |S \cap iS| \).

By condition 3 \( r(S) \) is uniquely determined by \( r(S \setminus \{e\}) \), \( r(S \setminus \{ie\}) \), and \( r(S \setminus \{e, ie\}) \) for any \( e \in S \cap iS \), and further \( |S \setminus \{e\} \cap i(S \setminus \{e\})| = |S \cap iS| - 2 \).

Finally, take \( S \cup iS = [k]_2 \), and \( S \cap iS = \phi \). By Claim 3 \( r(S) = 0 \), or \( S \in H_r \).

If \( r(S \setminus \{e\}) = b \), for some \( e \in S \), then by condition 3 \( r(S) = b \).

Otherwise \( r(S \setminus \{e\}) = 0 \) for all \( e \in S \),
and \( r(S) = b \) if \( S \in C(r) \), while \( r(S) = 0 \) if \( S \not\in C(r) \).
0.5.4 Maps

Strong map: \( r_1 \Rightarrow r_2 \) if \( r_1(S) \neq b \Rightarrow r_1(S) \geq r_2(S) \neq b. \)

Weak map: \( r_1 \rightarrow r_2 \) if for \( S \subseteq T, r_1(T) \geq r_2(S) \) where \( n \geq b \) and \( n \leq b. \)
Bibliography


