ON WAVE INTERACTIONS: EXPLOSIVE RESONANT TRIADS AND CRITICAL LAYERS

by
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B. S., Applied Mathematics
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ABSTRACT

This thesis considers the phenomenon of explosive resonant triads in weakly nonlinear, dispersive wave systems. These are nearly linear waves with slowly varying amplitudes which become unbounded in finite time. It is shown that such interactions are much stronger than previously thought. These waves can be thought of as a nonlinear instability, in the sense that a weakly nonlinear perturbation to some system grows to such magnitudes that the behavior of the system is governed by strongly nonlinear effects. This may occur for systems which are linearly or neutrally stable. This is contrasted with previous resolutions of this problem, which treated such perturbations as being large amplitude, nearly linear waves. Analytical and numerical evidence is presented to support these claims.

These waves represent a potentially important effect in a variety of physical systems, most notably plasma physics. Attention here is turned to their occurrence in fluid mechanics. Here previous work is extended to include flow systems with continuously varying basic velocities and densities. Many of the problems encountered here will be found to be of a singular nature themselves, and the techniques for analyzing these difficulties will be developed. This will involve the concept of a critical layer in a fluid, a level at which a wave phase speed equals the unperturbed fluid velocity in the direction of propagation. Examples of such waves in this context will be presented.

Thesis Supervisor: Dr. D. J. Benney
Title: Professor of Applied Mathematics
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1. INTRODUCTION

Asymptotic methods represent one of the triumphs of applied mathematics, and have enjoyed much use and success in explaining a wide range of physical phenomena. Such techniques are typically employed in problems involving a very small or very large parameter, and their origins date back to the nineteenth century, though the methods to be employed here are rather more recent. They find great utility in the approximate evaluation of integrals and series, the solution of differential equations, and other topics to be discussed. The book by Carrier, Crook, and Pearson (1983) serves as a good source for these more classical applications. The more modern uses of asymptotic methods include such subjects as boundary layer theory and multiple-scale analysis, and were developed and popularized in the 1960's by a number of researchers in fluid mechanics, although the results were most likely known to some scientists around the same time. Bender and Orszag (1978) offer a good introduction to this field, as well the previous subjects. The application of these latter techniques to problems in fluid mechanics will be central to this thesis, but there are certainly a wide variety of fields where the ideas prove important.

Specifically, the focus here is on weakly nonlinear waves. Wave motion is an important field of physical study that arises in nearly every branch of science. While the relevance to fluid mechanics is most important here, the type of problems to be discussed have arisen in plasma physics, optics, and electrical engineering. If disturbances through some medium are linear, then a linear combination of these disturbances is itself a disturbance, or a solution to whatever (linear) equation governs the behavior of the system under consideration. This allows the superposition of waves to give quite general disturbances, and the study of linear waves provides a set of rich results. Such linear waves may be said to have infinitesimal amplitude, since nonlinear terms in the equations they satisfy are neglected. When even weak finite amplitude effects are retained, however, a wealth of interesting problems result. Other physical effects may be introduced as well. For example, for such waves on the surface of shallow water or with long wavelength, the equation governing the evolution of the free surface is the famous KdV equation:

\[ u_T = u_X + \varepsilon(u u_X + \sigma u_{XX}) \]  \hspace{1cm} (1.1)

upon suitable normalization, and where \( \varepsilon \) is a small parameter, usually considered positive. The notation for this throughout will be

\[ \varepsilon << 1 \]

Finite amplitude effects may also affect the frequency of a linear wave. The equation that governs the slowly varying amplitude of a traveling wave satisfies the so-called nonlinear Schrödinger equation:

\[ S_T + c_y S_X = i\varepsilon(S_{XX} + \sigma S^2 S^*) \]  \hspace{1cm} (1.2)

where \( * \) denotes complex conjugation. Both of these equations are derived from far more complicated equations by methods to be discussed. An extensive literature exists on (1.1)
and (1.2), as well as other so-called evolution equations. In particular, many of these equations may be solved by the method of inverse scattering, which has become a field in itself. The text by Ablowitz and Segur (1981) contains a thorough description of this method.

In each of the above equations, the temporal and spatial variables are assumed to be scaled so as to represent some sort of slow variations. When these characteristics, along with weak nonlinearity, are applied to the general theory of wave interactions, a fascinating and important field is opened up. The generic type of problem to which these methods apply involves partial differential equations (PDE’s) of the form

\[ u_t = Lu + \varepsilon Nu \]

where \( Lu \) and \( Nu \) represent respectively linear and nonlinear operators on \( u \), which could be a vector quantity. The nonlinearity is assumed to be at least quadratic to leading order. Both operators may involve the derivatives of \( u \) with respect to an arbitrary number of spatial variables \( x \), though physically there would be no more than three. The typical (first) approach to such problems to use the methods of so-called regular perturbation theory, and seek a solution in the straight-forward form of a perturbation series

\[ u \sim u_0 + \varepsilon u_1 + \cdots \]

The nonlinear problem is then broken into a sequence of linear ones, by equating coefficients of \( \varepsilon \). The leading order problem is the linearized equation

\[ u_{0t} = Lu_0, \]

Assuming homogeneity in the spatial variables, this equation has solutions of the following form:

\[ u_0 = S e^{i(k \cdot x - \omega t)} + * \]

If \( u \) is taken to be one-dimensional, the amplitude \( S \) is a constant scalar and the frequency \( \omega \) and wave number vector \( k \) must have some sort of functional dependence

\[ \omega = \omega(k) \]

known as the dispersion relation. In the case of several unknowns, this relation would be obtained from the eigenvalues of some matrix which depends on \( k \). If \( \text{Im} (\omega) = 0 \), these solutions are referred to as dispersive waves. They undergo no dissipation or growth, and a single such wave will merely propagate in the \( k \) direction with phase speed \( c = \omega/k \). Attention will be confined to such waves.

Higher order solutions such as \( u_1 \) will just represent “forced harmonics” of the linearized problem:
\[ u_{1t} = Lu_1 + N_0 u_0 \]

(where \( N_0 u_0 \) represents the leading order contribution to \( Nu \)) and too can be determined straight-forwardly. Difficulties arise, however, if this forcing term produces any terms which themselves are solutions to the linear equation. These terms result in so-called secular solutions which grow linearly in \( t \), and are analogous to the well-known resonant harmonic oscillator. The method of multiple scales (or equivalent methods known by different names; see Nayfeh (1973)) may be used to ensure a periodic solution on the fast time scale. Specifically, the amplitude \( S \) of \( u_0 \) is considered to be a slowly varying function of the variable

\[ T = \varepsilon t \]

which is treated independently throughout. Time derivatives then transform as

\[ u_t \to u_t + \varepsilon u_T \]

and the problem for \( u_1 \) becomes

\[ u_{1t} = Lu_1 + N_0 u_0 - u_0 T \]

and this new dependence on \( T \) is used to suppress any secular terms and so ensure a bounded solution.

In the context of our problem, these methods find great utility when the leading order solution is taken to be a combination of three waves (a triad)

\[ u_0 = S_1 e^{i(k_1 \cdot x - \omega_1 t)} + S_2 e^{i(k_2 \cdot x - \omega_2 t)} + S_3 e^{i(k_3 \cdot x - \omega_3 t)} + * \]

where the wavenumbers and frequencies satisfy

\[ k_1 + k_2 + k_3 = 0 \]
\[ \omega_1 + \omega_2 + \omega_3 = 0 \]

the so-called symmetric notation. Such a triad is said to be resonant, because owing to the quadratic nature of the nonlinearity, forcing terms proportional to

\[ S_2^* S_3^* e^{-i((k_2 + k_3) \cdot x - (\omega_2 + \omega_3) t)} \]

are produced, which of course satisfy the homogeneous equation. Once again, to obtain bounded, periodic solutions it is demanded that the amplitudes \( S_j \) be slowly varying functions of \( T \), and to suppress secular terms it is found they must satisfy equations of the following form:
\[ S_{1T} = \alpha_1 S_2^* S_3^* \]
\[ S_{2T} = \alpha_2 S_1^* S_3^* \]
\[ S_{3T} = \alpha_3 S_1^* S_2^* \]

(1.3)

As mentioned, these famous equations have been the basis of much research in fluid mechanics, as well as plasma physics and other fields. It is important then, and this is one of the issues to be addressed here, to determine whether a given wave system admits such behavior. Typically, the graphical technique illustrated in Figure 1.1 is the quickest way to answer this question. This approach will give triads in the non-symmetric notation

\[ k_1 + k_2 = k_3 \]
\[ \omega_1 + \omega_2 = \omega_3 \]

The symmetric notation may be obtained in an obvious fashion. Not all wave systems permit triads, but do admit similar four wave resonances or quartets. These are not the focus here, but do have some related properties that will be commented on later.

![Figure 1.1: Typical dispersion relation (solid line) and graphical construction of triads](image-url)
Typically, the interaction coefficients $\alpha_j$ are either purely real or purely imaginary, and in such a case these equations may be integrated exactly. This was first done in the context of fluid mechanics by Bretherton (1964), although previous researchers in optics and electronics, and later in plasma physics, independently arrived at these results as well. More will be said about the solutions to these equations later. In fluid mechanics, a number of researchers made important theoretical and experimental results regarding the occurrence of triads. Among these, Phillips (1960, 1961) did the pioneering work, and later McGoldrick (1965) and Simmons (1969) investigated triads in capillary-gravity wave systems. These interactions were later seen to exist among internal and internal-surface waves also; see Thorpe (1966). From the mathematical point of view, slow spatial dependence may be introduced, and the problem may be analyzed via inverse scattering; see for example Zakharov and Manakov (1973, 1975) or Kaup (1976). Furthermore, a small frequency mismatch may be incorporated, but such generalities will not be pursued here.

In addition, much experimental work has demonstrated the importance of these effects. McGoldrick again (1970), and Davis and Acrivos (1967) for internal waves, were just two sets of researchers among many in this aspect of the field. Good overviews of all of these results can be found in Phillips (1981), and more recently Hammack and Henderson (1993). Another excellent reference for this work, as well as many of the topics covered herein, is Craik (1985). Also, such interactions were found to be important in many other branches physics as well, particularly plasma physics. A good compendium here is Weiland and Wilhelmsson (1977). Of course, any branch of science dealing with dispersive wave behavior potentially exhibits triad phenomena, so it would be worth while to discuss the characteristics of (1.3), which will also lead into the main theme of this thesis.

As mentioned, the case when the interaction coefficients are purely real or imaginary is the most common, as well as being the only circumstance under which equations (1.3) are integrable. So it will be assumed without loss of generality that the $\alpha$’s are real; if not, the transformation $S_j \rightarrow i S_j$ will reduce the system to this form. If these coefficients are of differing signs, say two positive and one negative, then the amplitudes $S_j$ are just periodic, bounded functions (actually expressible in terms of elliptic functions), and the overall triad solutions are just slowly modulated, nearly linear waves. However, if these coefficients are all of one sign, say all positive, then the behavior is radically different. Except under certain very special initial conditions, these amplitudes will grow without bound, and in fact will blow up in finite time, becoming singular like

$$\frac{1}{T_0 - T}$$

for some $T_0$. Such triads are called explosive, for obvious reasons. For any physical problem, this is clearly unacceptable behavior, and so the triad equations cannot be valid near this point in time, and some previously neglected effects must be restored to get a satisfactory description of the system. Previously, the approach taken was essentially to retain higher order nonlinear terms in equations (1.3), which would be necessary to ensure a uniformly asymptotic solution to order $\epsilon^2$; in other words, a higher order analysis would reveal the need for a second time scale.
\[ \tau = \varepsilon^2 t \]

to suppress secularities in the problem, and treating time derivatives as

\[ S_T \to S_T + \varepsilon S_{\tau} \]

the triad equations would become

\[ S_{1T} = \alpha_1 S_2^* S_3^* + i \varepsilon S_1 (\beta_{11} |S_1|^2 + \beta_{12} |S_2|^2 + \beta_{13} |S_3|^2) \]

\[ S_{2T} = \alpha_2 S_1^* S_3^* + i \varepsilon S_2 (\beta_{21} |S_1|^2 + \beta_{22} |S_2|^2 + \beta_{23} |S_3|^2) \]

\[ S_{3T} = \alpha_3 S_1^* S_2^* + i \varepsilon S_3 (\beta_{31} |S_1|^2 + \beta_{32} |S_2|^2 + \beta_{33} |S_3|^2) \]  \hspace{1cm} (1.4)

valid to times \( t \) of order \( \varepsilon^{-2} \), and where the absolute values represent amplitudes of complex quantities. These cubic nonlinearities have the effect of preventing the blow up in finite time, and the amplitudes of solutions to equations (1.4) written in polar form exhibit the so-called “repeated stabilized explosions” of Figure 1.2.

![Graph](image-url)

**Figure 1.2:** Typical solution of (1.4); \( S = R \exp(i\theta) \)
This had previously been the last word on this subject. It is the contention of this thesis, however, that this interaction is much stronger than had been thought. In particular, explosive triads represent a nonlinear instability, in the sense that when such interactions occur, the problem can no longer be thought of as weakly nonlinear. It will be seen that in some sense, "all" of the higher order nonlinearities in equations (1.4) must be included near the blow-up point; they all become equally important. Of course, what this really means is that near the singularity, the behavior is governed by the full, nonlinear problem. Analytical arguments and numerical evidence will be provided for these statements, as well as analogies with similar phenomena that occur in certain resonant oscillator problems.

Furthermore, we wish to examine certain fluid mechanical systems in which such triads exist. Previously, these explosive triads had been primarily observed in plasma physics, but recent investigators in fluids have discovered them as well. Craik and Adam (1979), expanding work by Cairns on "negative energy" waves (1979), first noticed their existence in a three layer flow problem of piecewise constant velocity and density. Around the same time, various Russian researchers (Voronovich and Rybak (1980) and later Romanova and Shrira (1988)) deduced related results. Burns and Maslowe (1983) and Collins and Maslowe (1988) made contentions relevant to the appearance of explosive triads in meteorological-type flows. These results will be critiqued later. No other significant work, apparently, had been done along these lines involving flows with continuous velocity and stratification profiles until Becker and Grimshaw (1993) studied the problem from a Lagrangian point of view and gave necessary conditions for the appearance of explosive triads. Many of their results will be here derived in an alternate, Eulerian fashion. Also, an interpretation of negative energy waves will be made for continuously varying flows. More important, in such continuous flows so-called "critical layers", a level in the fluid domain where the wave speed equals the basic flow speed, prove paramount, and care must be exercised in dealing with such problems, which raises some questions concerning previous work in this area. Such waves present analytical as well as numerical difficulties, and they may be resolved by incorporating various physical effects. Here the work of Benney and Bergeron (1969) will prove most relevant, although different aspects of this problem are discussed in Drazin and Reid (1981). Specifically, the evaluation of the interaction coefficients must be done in a certain way, similar to Hadamard's (1923) method of finite parts, but with different interpretations depending on whether the stratification is continuous or not. Finally, examples of such triads in continuous flows will be presented, and the extension of the analysis to general, layered flows will be made.
2. EXPLOSIVE RESONANT TRIADS

As discussed in the introduction, the type of problems to be dealt here with fit into the general framework of weakly nonlinear PDE's or evolution equations of the following form:

\[ u_t = Lu + \varepsilon Nu \]

where \( \varepsilon \) is some small positive parameter, \( N \) is some (at least quadratic) nonlinear operator, and \( L \) is some linear operator that admits resonant triads. We will also be focusing on those triads of the explosive kind which blow up in finite time. First, some analytical results to explain the behavior of such systems will be provided, and then support for these claims will be made with numerical evidence from some specific examples.

2.1 ANALYTICAL RESULTS

The usual perturbation approach to problems involving a small parameter is to assume that the solution can be expressed as a regular perturbation series:

\[ u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \]

To leading order, then, it is found that

\[ u_{0t} = Lu_0 \]

The solution to this problem will be taken to be

\[ u_0 = S_1 e^{i(k_1 \cdot x - \omega_1 t)} + S_2 e^{i(k_2 \cdot x - \omega_2 t)} + S_3 e^{i(k_3 \cdot x - \omega_3 t)} + * \]

where \( \omega_j = \omega(k_j) \). Furthermore, it will be assumed that this set forms a resonant triad; ie,

\[ k_1 + k_2 + k_3 = 0 \]
\[ \omega_1 + \omega_2 + \omega_3 = 0 \]

Consequently, it is to be anticipated that secularities arise at the next order and so the amplitudes \( S_j \) are allowed to be functions of the slow variable \( T = \varepsilon t \). The problem to be solved at \( O(\varepsilon) \), then, is

\[ u_{1t} = Lu_1 + N_0 u_0 - u_0 T \]

where \( Nu = N_0 u_0 + \varepsilon N_1(u_0, u_1) + \cdots \). As has already been seen, in order to have periodic solutions, the amplitudes must satisfy equations of the following form:
\[ S_1 T = \alpha_1 S_2^* S_5^* \]
\[ S_2 T = \alpha_2 S_1^* S_3^* \]
\[ S_3 T = \alpha_3 S_1^* S_2^* \] (2.1)

The case where the interaction coefficients \( \alpha_j \) are either purely real or purely imaginary will be considered. If real, they are assumed to be all of one sign. (If imaginary, the real numbers \( i \alpha_j \) are assumed to be of one sign.) This is known to be the explosive case, where the solutions to (2.1) become infinite in finite time. If we let

\[ S_j = R_j(T) e^{i \theta_j(T)} \]

then it would be found that

\[ R_j \sim \frac{1}{T_0 - T} \]

as \( T \to T_0 \), for some \( T_0 \). \( \theta_j \) would tend to some multiple of \( 2\pi \). See Craik (1985) or Weiland and Wilhelmsson (1977) for details. At this stage, there are several ways of viewing the problem, each of which will now be examined.

### 2.1.1 Breakdown of the Perturbation Series

At this stage, secular-producing terms have been suppressed, and the forced, linear equation for \( u_1 \) may be solved. This will involve higher harmonics, terms proportional to

\[ S_1^2 e^{i 2(k_1 \cdot x - \omega_1 t)} \]

and so forth. Therefore, the amplitude of the \( O(\varepsilon) \) solution \( u_1 \) will also become singular near \( T_0 \), and it will blow up like

\[ \frac{1}{(T_0 - T)^2} \]

Now from the point of view of asymptotic analysis, the series solution for \( u \) is valid so long as each successive term in the series is "much smaller" than all preceding terms. Mathematically this means that we must have, for example, that

\[ \varepsilon^{n+1} u_{n+1} \ll \varepsilon^n u_n \] (2.2)

as \( \varepsilon \to 0^+ \) or

\[ \frac{\varepsilon u_{n+1}}{u_n} \to 0 \] (2.3)
as $\varepsilon \to 0^+$. Indeed, this was the whole rationale behind using the method of multiple scales to attack the problem in the first place. The original approach (a regular perturbation technique) would have resulted in (2.2), for $n = 0$, being false for those times $t$ of order of magnitude $\varepsilon^{-1}$. (Recall such resonant terms grow like $t$.) For times of order of magnitude of 1, the regular perturbation series would provide an acceptable asymptotic, and most likely adequate quantitative, description of the given system's behavior. But as time increases, it cannot be a valid asymptotic representation for the solution because it fails the basic test for such a series. For further discussion of these ideas, see Bender and Orszag (1978), Nayfeh (1973), or Kevorkian and Cole (1981). A similar viewpoint may be taken when the terms in the series become singular, but here notions associated with boundary layer theory as opposed to multiple scale analysis are more appropriate.

As an asymptotic solution, the series

$$u \sim u_0 + \varepsilon u_1 + \cdots$$

is valid so long as

$$\varepsilon u_1 << u_0$$

as $\varepsilon \to 0^+$. When these terms become comparable in an asymptotic sense, the series is said to break down, and the problem must be re-examined. That is the case here, however, for when

$$O\left(\frac{1}{T_0 - T}\right) = O\left(\frac{\varepsilon}{(T_0 - T)^2}\right)$$

or

$$T - T_0 = O(\varepsilon)$$

as $\varepsilon \to 0^+$, the first two terms of the perturbation series solution become disordered. Ironically, multiple scales techniques were appealed to in order to prevent one series from breaking down, and it results in another series failing! Once again though, for times not “close” (meaning not within order $\varepsilon$) to $T_0$ on the slow scale, or $T_0 / \varepsilon$ on the fast scale, the leading order, triad solutions will represent a satisfactory approximation (in the asymptotic and probably numerical sense) to the full solution.

However, when $T$ is within $O(\varepsilon)$ of $T_0$, this perturbation series is no longer acceptable, and the problem must be reexamined in this neighborhood. Drawing on the ideas of boundary layer theory, the new “slow” variable

$$\overline{T} = \frac{T - T_0}{\varepsilon}$$

is introduced in order to isolate the region under consideration, and the dependent variable $u$ is rescaled:
\[ v = \frac{u}{\varepsilon} \]

This reflects the fact that \( u \) becomes of order \( \varepsilon^{-1} \) when the perturbation series breaks down. Of course, this is not a standard boundary layer-type problem. We have another time scale in the problem, namely \( t \), and there is some question, as will shortly be seen, as to how to handle this variable near the blow-up point. We will later draw analogy with this problem to one analyzed by Kevorkian (1971) concerning the passage of a slowly-varying oscillator through resonance, which also involves a two-scale problem which breaks down on one of the time scales. Essentially though, the philosophy will be identical, and typical for these sorts of asymptotic problems: an “inner” problem will be identified, whose solution will be “matched” to some “outer” problem. If this can be done, then consistency ordinarily takes the place of a rigorous proof.

Now, under the proposed rescaling, the original problem, written as

\[ u_t + \varepsilon u_T = Lu + \varepsilon Nu, \]

becomes, taking \( v = v(t, T) \),

\[
\frac{v_t}{\varepsilon} + \frac{\varepsilon v_T}{\varepsilon^2} = \frac{Lv}{\varepsilon} + \varepsilon N\left(\frac{v}{\varepsilon}\right) \tag{2.4}
\]

If we take, as is typical for such problems,

\[ Nu = Qu + \varepsilon Cu + \cdots \tag{2.5} \]

where \( Qu, Cu \), etc., represent operators involving quadratic, cubic, etc., expressions in \( u \) and its spatial derivatives (so that the original problem comes from setting \( u \to \varepsilon u \) in a PDE with nonlinearity (2.5) for \( \varepsilon = 1 \)), the inner equation, the equation valid near the blow-up point, becomes

\[ v_t + v_T = Lv + Nu \tag{2.6} \]

ie, the full, nonlinear problem that was being analyzed by simplifications in the first place! Of course, if the nonlinearity were not of the form (2.5) but weaker, the nonlinearity in (2.6) would be the quadratic component of \( N \), which would still be a strongly nonlinear PDE. However, we shall focus on instances such as (2.6).

Apart from the fact that the full nonlinearity arises in the inner problem, owing to the nature of the blow-up, it may be noted that unlike the usual situation in two time scale problems, both time derivatives occur at the same order. The reason for this is that the “slow” inner variable \( T \) is not really slow at all, but in some sense a second fast scale:

\[ T = \frac{T - T_0}{\varepsilon} = t - \frac{T_0}{\varepsilon} \]
Therefore, a quandry would seem to have arisen: which, if not both, time scales is retained? It will be seen when some model equations are considered to simulate this behavior that essentially both time scales are present in the problem. The usual approach in this field, and really in most of applied mathematics, is to make an assumption and later verify it, and not to emphasize rigorous, a priori justifications. In that spirit, this inner problem, with its two fast time scales, will now be explained before discussing how these conclusions differ from previous work on the problem.

As a fully nonlinear problem, a solution to (2.6) will not just consist of a finite number of modes or, say, Fourier components like the leading order outer problem. In general, every mode must be present at the outset and accordingly we seek a solution to (2.6) of the form

\[ v = \sum_{p,q,r} A_{pqr}(\Omega) e^{i\theta_{pqr}} + * \]  

(2.7)

where the sum is taken over all integer combinations \( p, q, r \) and

\[ \theta_{pqr} = (p k_1 + q k_2 + r k_3) x - (p \omega_1 + q \omega_2 + r \omega_3) t \]

\( \omega = \omega(k) \) is obtained from the linear dispersion relation. For simplicity the focus shall be confined to one-dimensional waves only; of course the ideas generalize to the multi-dimensional case in an identical manner. In addition, it will be assumed that the nonlinearity generates no resonant mean or zero wave number terms (of the form \( A A^* \)). This is to preclude the appearance of such resonant terms in the leading order outer solution. When specific fluid flow problems with non-oscillatory vertical dependence are considered, such terms could arise in the outer problem, and they would need to be accounted for in the inner expansion, but they will be assumed nonresonant there as well. By substituting (2.7) in (2.6), an infinite system of coupled ordinary differential equations (ODE's) for the various amplitudes \( A_{ijk} \) results:

\[ \frac{dA_{pqr}}{dT} = i \delta_{pqr} A_{pqr} + N_{pqr}(A) \]  

(2.8)

where \( N_{pqr}(A) \) represents the coefficient of \( e^{i\theta_{pqr}} \) arising from the nonlinear interaction, and

\[ \delta_{pqr} = \omega(p k_1 + q k_2 + r k_3) - (p \omega_1 + q \omega_2 + r \omega_3) \]

If the nonlinearity \( Nu \) is simply a finite combination of polynomial terms in \( u \), the nonlinear interaction term in (2.8) may be expressed as some sort of convolution sum involving all of the amplitudes \( A_{ijk} \). So the nonlinearity consists of an infinite number of terms as well. If it is some sort of general function however, this simple interpretation is lacking, but it must still be thought of as the coefficient of \( e^{i\theta_{pqr}} \) in the nonlinear interaction. Numerically, it could be obtained from a Fourier decomposition. More will be said on this later.
Theoretically, then, this system would have to be solved subject to the terminal
(matching) condition that the amplitudes have a certain behavior as $\overline{T} \to -\infty$. What
this behavior is would be determined as follows. Essentially, $A_{110}, A_{101}, \text{and } A_{011}$ “agree”
with $S_1, S_2, \text{and } S_3$ near the blow-up point $T_0$. More rigorously, introduce the matching
variable

$$T_\eta = \frac{T - T_0}{\eta}$$

where

$$\epsilon << \eta << 1$$

as $\epsilon \to 0+$, and $T_\eta$ is fixed. Standard matching techniques are now performed, although
of course there is no hope of solving the inner problem analytically in this case, as it is an
infinite system of nonlinear ODE’s. At any rate, the outer solution would be expressed in
terms of $T_\eta$ by

$$T = T_0 + \eta T_\eta$$

which approaches $T_0$ as $\epsilon \to 0$, and the inner solution in terms of

$$\overline{T} = \frac{\eta}{\epsilon} T_\eta$$

which approaches $-\infty$ as $\epsilon \to 0+$ for $T < T_0$. Furthermore, following Kevorkian and Cole
(1981), the fast variable in both the inner and outer solution would also be transformed as

$$t = \overline{T}_0 + \frac{\eta}{\epsilon} T_\eta$$

where $\overline{T}_0 = T_0/\epsilon$. To effect asymptotic matching to some given order, it is demanded that
up to this order, the two expressions obtained above are identical (apart from transcen-
dentially small terms, of course). This will be reviewed again when certain model
problems are considered, but from a numerical point of view (the only apparent way to approach
the problem), one would have to pick the initial conditions for some truncated version
of the set of ODE’s governing the $A$’s, say at $\overline{T} = 0$ ($T = T_0$), in such a way that for
“large” negative times, the solutions tend to the amplitudes of the corresponding modes
of the outer problem for $T$ “close” to $T_0$. For nonlinearities consisting of a finite number
of polynomial terms in $u$ and its spatial derivatives, the resulting convolution described
above would have be appropriately truncated as well. For nonlinearities of a general, functional
form, some sort of Fourier decomposition would have to be used. For example, for
a given number of retained modes and a given spatial period (the numerical demand of
periodicity requires that the wavenumbers be commensurate), the equations (2.8) could
be modeled via finite differences or some other scheme and advanced in time, evaluating
the nonlinearity at the present time to determine the unknowns at the next time step.
To evaluate the nonlinearity at say, \( \bar{T} = \bar{T}_k \), we could determine \( u \) at this point, use say a fast Fourier transform to determine the coefficient of a desired Fourier mode \( e^{ikr} \), and multiply this coefficient by \( e^{i\omega(k)(\bar{T}_k+\bar{T}_0)} \) to obtain the coefficient of \( e^{i\phi(k)} \). This is a prohibitive task, and has not been attempted here. Nor have any rigorous results been established; such results are rather rare in dealing with perturbation problems. What will be done in subsequent sections, however, is to arrive at these notions in different ways, and to invoke analogies with related, but more tractable, problems involving oscillators. Furthermore, numerical evidence will be presented to support these claims, as well as some of the following statements.

The main conclusion to be drawn from these results is that near the blow-up point, the original perturbation expansion is invalid, and the equation in question must be reanalyzed near this point. It has been seen however, that near this point, due to the nature of the singularity, the nonlinear terms become the same order of magnitude as the linear terms. In most cases to be dealt with, the problem becomes fully nonlinear; that is to say, the methods of weakly nonlinear analysis are no longer applicable. This is very different from previous resolutions of this singularity, which, as mentioned in the introduction, involved retaining additional, cubic terms in the evolution equations for the amplitudes:

\[
S_1T = \alpha_1 S_2^* S_3^* + i \varepsilon S_1 (\beta_{11} |S_1|^2 + \beta_{12} |S_2|^2 + \beta_{13} |S_3|^2) \\
S_2T = \alpha_2 S_1^* S_3^* + i \varepsilon S_2 (\beta_{21} |S_1|^2 + \beta_{22} |S_2|^2 + \beta_{23} |S_3|^2) \\
S_3T = \alpha_3 S_1^* S_2^* + i \varepsilon S_3 (\beta_{31} |S_1|^2 + \beta_{32} |S_2|^2 + \beta_{33} |S_3|^2) \\
\]

(2.9)

where the \( \beta \)'s are typically real, and so the cubic terms above represent phase shifts, and have the effect of preventing the solution from blowing up.

These higher order corrections come from demanding uniformity in the series solution to \( O(\varepsilon^2) \). In other words, proceeding to this order in the perturbation expansion, terms proportional to

\[
S_1 S_2 S_2^* e^{i(k_1 x - \omega_1 t)}
\]

and others would be secular, so a time scale \( \tau = \varepsilon^2 t \) would need to be introduced to suppress them. Time derivatives would transform as

\[
u_t \rightarrow u_t + \varepsilon u_T + \varepsilon^2 u_\tau
\]

The wave amplitudes would be considered slowly varying functions of two variables, one “slower” than the other. Equivalently, one could take

\[
S_{jT} = Q_j(S_k) + \varepsilon P_j(S_k) + \cdots
\]

(2.10)

and choose the terms in this series to remove resonant terms at the required order. Of course, a multiple scale approach would also require “slower” time scales due to the appearance of secular terms at higher orders, thus producing more terms in the evolution
equations (2.9). In general, secularities appear at all orders. The system (2.9), then, must be thought of as a truncation of an asymptotic expansion itself, which will be seen to be an important point.

As mentioned in the introduction, the solutions to (2.9) exhibit the so-called “repeated stabilized explosions” of Figure 2.1, which displays the polar radius of the amplitude $S = Re^{i\theta}$. More will be said about the nature of these solutions in a subsequent section. However, according to the previous analysis, these additional terms cannot represent the behavior of the system near the blow-up point, as the problem becomes fully nonlinear there. (2.9) models weakly nonlinear waves which differ from linear waves in having slowly varying amplitudes. The inadequacies of (2.9) will also be seen later via a direct perturbation analysis of the equations, as well as by considering their exact solutions. Of course, it may be possible that the full, nonlinear problem and the solutions with amplitudes predicted by (2.9) have qualitatively similar behavior, and some examples of this will be seen later as well. Apart from this, though, they do not represent valid asymptotic descriptions of the system near $T_0$. It may be wondered what effect the amplitudes, as governed by (2.9), have on the relative importance of the terms in $t^\alpha$ perturbation series $u_0, u_1$, and even $u_2$. This will be discussed shortly. On the basis of the first two terms of this series becoming disordered, and the way in which the leading order solution becomes singular, the retention of extra terms in the evolution equations, as in (2.9), will not be sufficient, as these equations, meant to be valid to order $\varepsilon^2$, will be seen to break down for some time of order $\varepsilon$.

![Figure 2.1: Typical solution of (2.9); $S = Re^{i\theta}$.](image)

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Having mentioned alternate ways of viewing this problem, it should be said that the inclusion of cubic terms in the evolution equations for the amplitudes appears to have been accepted as a satisfactory resolution to the problem of explosive triads. Both Craik (1985) and Weiland and Wilhelmsson (1977) invoke this explanation, and Byers, et. al. (1971) compare an integral of (2.9) with the results of a computer simulation of certain plasma wave interactions. It is not clear, however, that they are consistently doing this, as certain terms in this conserved quantity (to be remarked upon later) should be small; specifically, those due to the cubic terms. This is not the case in this latter investigation; indeed, they determine the interaction coefficients from the “experimental” data, and it would appear that the cubic terms are of a larger order of magnitude than the quadratic ones! They claim good agreement with theory, but again, in light of the relative size of the parameters they are dealing with, it is not clear what they are observing. Interestingly, they claim a Fourier decomposition of their numerical data reveal the presence of frequencies that do not correspond to one of the original three triad modes. Presumably, as they do not exhibit this data, the size of these additional components was noteworthy. They give some physical arguments for this from the context of plasma physics, but it could also be indicative of what was argued above; namely, near the blow-up point, the problem becomes fully nonlinear, and all the Fourier modes (traveling waves) become important.

Hopefully by the end of this part of the thesis, it will be seen that these approaches are not correct, that as the leading order solution to the system becomes singular, (2.9) is inapplicable. It will now be considered how solutions to (2.9), thought of as a perturbation series itself, behave.

2.1.2 Evolution Equations as Series

As discussed in the last section, the evolution equations

\[
S_{1T} = \alpha_1 S_1^* S_3^* + i \varepsilon S_1 (\beta_{11} \vert S_1 \vert^2 + \beta_{12} \vert S_2 \vert^2 + \beta_{13} \vert S_3 \vert^2 )
\]

\[
S_{2T} = \alpha_2 S_2^* S_3^* + i \varepsilon S_2 (\beta_{21} \vert S_1 \vert^2 + \beta_{22} \vert S_2 \vert^2 + \beta_{23} \vert S_3 \vert^2 )
\]

\[
S_{3T} = \alpha_3 S_1^* S_2^* + i \varepsilon S_3 (\beta_{31} \vert S_1 \vert^2 + \beta_{32} \vert S_2 \vert^2 + \beta_{33} \vert S_3 \vert^2 )
\]

must be thought of as the truncation of another perturbation series (as opposed to the one which represents the solution to the original PDE). To ensure a solution uniformly valid to arbitrarily high order, additional terms would have to be included in (2.11) to eliminate the secularities that arise at these orders. So, one could seek a perturbation expansion

\[
S_j = S_{j0} + \varepsilon S_{j1} + \cdots
\]

(2.12)

to (2.11), and examine its behavior. It will be shown in this section that for the explosive triads under consideration, when the leading order solutions to (2.11) become infinite in finite time, every term in the series for which (2.11) is a truncation will be of the same order of magnitude near the blow-up point, and hence adding a single, cubic term to the triad equations cannot model the system as a whole.
As mentioned previously, the $\beta$'s are usually found to be real, and without loss of
generality it will be assumed that the $\alpha$'s are all equal to 1; the transformation $S_j \rightarrow i S_j$
will make the triad coefficients all real, $S_j \rightarrow -S_j$ will make them all positive, and

$$S_1 \rightarrow \frac{S_1}{|\alpha_2 \alpha_3|^{1/2}},$$

with cyclic permutations of the indices for the other amplitudes, will reduce the equations
to the desired form. Furthermore, the substitution

$$S_j = R_j e^{i \phi_j}$$

will be made, and define $\phi$ to be $\phi_1 + \phi_2 + \phi_3$. Expanding $R_j$ and $\phi$ in perturbation
series,

$$R_j = R_{j0} + \varepsilon R_{j1} + \varepsilon^2 R_{j2} + \cdots,$$

$$\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots,$$

it is found to leading order that

$$R_{10T} = R_{20} R_{30} \cos \phi_0$$

$$R_{20T} = R_{10} R_{30} \cos \phi_0$$

$$R_{30T} = R_{20} R_{10} \cos \phi_0$$

$$\phi_{0T} = -(\frac{R_{20} R_{30}}{R_{10}} + \frac{R_{10} R_{30}}{R_{20}} + \frac{R_{20} R_{10}}{R_{30}}) \sin \phi_0$$

As is well-known, the system of equations (2.13) can be integrated exactly; see Craik
(1985), for example. Three integrals of motions may be found for this system, which may
then be reduced to quadrature. For general initial conditions, the solution is expressible
in terms of Jacobi elliptic functions, and the blow-up point will in general be the root of
some transcendental equation. A frequency mismatch in the original set of triads may be
incorporated, but this does not eliminate the singularity. As using these general solutions
would not seem to offer much analytic tractability, the subsequent analysis will be confined
to the following special solutions of (2.13):

$$R_{10} = R_{20} = R_{30} = \frac{1}{T_0 - T}$$

and $\phi_0 = 0$. As this is the nature of the singularity in general, these special solutions will
exhibit the fundamental behavior of the system near the point of interest.

The equations at the next order are
\[ R_{11T} = (R_{20} R_{31} + R_{21} R_{30}) \cos \phi_0 - R_{20} R_{30} \phi_1 \sin \phi_0 \]
\[ R_{21T} = (R_{10} R_{31} + R_{11} R_{30}) \cos \phi_0 - R_{10} R_{30} \phi_1 \sin \phi_0 \]
\[ R_{31T} = (R_{20} R_{11} + R_{21} R_{10}) \cos \phi_0 - R_{20} R_{10} \phi_1 \sin \phi_0 \]
\[ \phi_{1T} = -\phi_1 \cos \phi_0 \left( \frac{R_{20} R_{30}}{R_{10}} + \frac{R_{10} R_{30}}{R_{20}} + \frac{R_{20} R_{10}}{R_{30}} \right) - \]
\[ \sin \phi_0 \left( \frac{R_{20} R_{31} + R_{21} R_{30}}{R_{10}} - \frac{R_{20} R_{30} R_{11}}{R_{10}^2} + \right. \]
\[ \frac{R_{10} R_{31} + R_{11} R_{30}}{R_{20}} - \frac{R_{10} R_{30} R_{21}}{R_{20}^2} + \]
\[ \frac{R_{20} R_{11} + R_{21} R_{10}}{R_{30}} - \frac{R_{20} R_{10} R_{31}}{R_{30}^2} \right) + \]
\[ \beta_1 R_{10}^2 + \beta_2 R_{20}^2 + \beta_3 R_{30}^2 \]

where \( \beta_j = \beta_{1j} + \beta_{2j} + \beta_{3j} \).

For the special case under consideration, these would reduce to

\[ R_{11T} = \frac{R_{31} + R_{21}}{T_0 - T} \]
\[ R_{21T} = \frac{R_{31} + R_{11}}{T_0 - T} \]
\[ R_{31T} = \frac{R_{11} + R_{21}}{T_0 - T} \]

(2.14)

\[ \phi_{1T} = -3 \frac{\phi_1}{T_0 - T} + \frac{\beta}{(T_0 - T)^2} \]

where \( \beta = \beta_1 + \beta_2 + \beta_3 \). One solution to the first three equations in (2.14) is \( R_{jT_0} \), since, for example,

\[ R_{10T} = R_{20} R_{30} \]

implies

\[ R_{10T_0T} = R_{20T_0} R_{30} + R_{20} R_{30T_0} \]

with similar results for the other two equations. Hence we could take

\[ R_{10} = R_{20} = R_{30} = \frac{-1}{(T_0 - T)^2} \]

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and find

\[ \phi_1 = \frac{\beta}{4(T_0 - T)} + C (T_0 - T)^3 \]

where \( C \) is a constant. It would be instructive, however, to seek the general solution to (2.14), as it will be needed when solving the next order problem. Note that the special solution above exhibits the same type of breakdown, from the point of view of the perturbation series, as in the previous section. Write the first three equations in (2.14) as

\[ R_{1T} = A R_1 \]  \hspace{1cm} (2.15)

where

\[ A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \frac{1}{T_0 - T} \]

Then the general solution to (2.15) may be written as

\[ R_1 = \exp(\int^T A \, d\xi) C \]  \hspace{1cm} (2.16)

To evaluate this exponential matrix, let

\[ A' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]

Then

\[ A = (A' - I) \frac{1}{T_0 - T} \]

where \( I \) is the 3 by 3 identity matrix. Then

\[ \exp(\int^T A \, d\xi) = \exp(\log(T_0 - T) (I - A')) \]

\[ = (T_0 - T) \exp(-A' \log(T_0 - T)) \]

Now to evaluate this exponential matrix, by the definition

\[ \exp(A) = I + \sum_{n=1}^{\infty} \frac{A^n}{n!} \]
for any matrix $A$, we have, since it may be easily verified that

$$A^{n} = 3^{n-1} A',$$

that

$$\exp(-A' \log(T_0 - T)) = I - \log(T_0 - T) A' + \frac{3}{2!} \log^2(T_0 - T) A' - \frac{9}{3!} \log^3(T_0 - T) A' + \cdots$$

$$= I + \frac{A'}{3} (1 - 3 \log(T_0 - T)) + \frac{9}{2!} \log^2(T_0 - T) - \frac{27}{3!} \log^3(T_0 - T) \cdots - 1)$$

$$= I + \frac{A'}{3} \left( \frac{1}{(T_0 - T)^3} - 1 \right)$$

Therefore, the general solution (2.16) for $R_1$ may be written as

$$R_1 = \begin{pmatrix} 2c_1 - c_2 - c_3 \\ -c_1 + 2c_2 - c_3 \\ -c_1 - c_2 + 2c_3 \end{pmatrix} \frac{T_0 - T}{3} + \frac{C}{3} \frac{1}{(T_0 - T)^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where $C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and $C = c_1 + c_2 + c_3$. $\phi$ remains unchanged. It should be noted that

the dominant behavior of $R_{j1}$ near $T_0$ remains $\frac{1}{(T_0 - T)^4}$, so that the series for $R_j$ becomes disordered when

$$T - T_0 = O(\varepsilon)$$

It would be advisable to determine the behavior of $R_{j2}$ near the blow-up, so as to determine whether the series might become disordered for values of $T$ further from $T_0$ than $O(\varepsilon)$, say $O(\varepsilon^{1/2})$. This will be seen not to be the case; the relevant scaling is as before.

To determine these second order effects, it is anticipated that the evolution equations for $S_j$ would include terms like

$$\varepsilon^2 S_2^* S_3^* (\alpha_{11} |S_1|^2 + \alpha_{12} |S_2|^2 + \alpha_{13} |S_3|^2)$$

$$\varepsilon^2 S_1^* S_3^* (\alpha_{21} |S_1|^2 + \alpha_{22} |S_2|^2 + \alpha_{23} |S_3|^2)$$

$$\varepsilon^2 S_2^* S_1^* (\alpha_{31} |S_1|^2 + \alpha_{32} |S_2|^2 + \alpha_{33} |S_3|^2)$$

where it is usual for the $\alpha_{ij}$’s to be real. Then the equations to be satisfied by $R_{j2}$ become
\[ R_{12T} = (R_{20} R_{32} + R_{21} R_{31} + R_{22} R_{30}) - \varphi_1 \sin \phi_0 (R_{20} R_{31} + R_{21} R_{30}) - \]
\[ \left( \frac{\phi_1^2}{2} \cos \phi_0 + \phi_2 \sin \phi_0 \right) R_{30} R_{20} + R_{20} R_{30} (\alpha_{11} R_{10}^2 + \alpha_{12} R_{20}^2 + \alpha_{13} R_{30}^2) \]
\[ R_{22T} = (R_{10} R_{32} + R_{11} R_{31} + R_{12} R_{30}) - \varphi_1 \sin \phi_0 (R_{10} R_{31} + R_{11} R_{30}) - \]
\[ \left( \frac{\phi_1^2}{2} \cos \phi_0 + \phi_2 \sin \phi_0 \right) R_{30} R_{10} + R_{10} R_{30} (\alpha_{21} R_{10}^2 + \alpha_{22} R_{20}^2 + \alpha_{23} R_{30}^2) \]  
\[ (2.17) \]
\[ R_{22T} = (R_{20} R_{12} + R_{21} R_{11} + R_{22} R_{10}) - \varphi_1 \sin \phi_0 (R_{20} R_{11} + R_{21} R_{10}) - \]
\[ \left( \frac{\phi_1^2}{2} \cos \phi_0 + \phi_2 \sin \phi_0 \right) R_{10} R_{20} + R_{20} R_{10} (\alpha_{31} R_{10}^2 + \alpha_{32} R_{20}^2 + \alpha_{33} R_{30}^2) \]

The equation for \( \phi \) would have an extra term

\[ -\varepsilon^2 \left( \sigma_1 \frac{R_2 R_3}{R_1} + \sigma_2 \frac{R_1 R_3}{R_2} + \sigma_3 \frac{R_2 R_1}{R_3} \right) \]

where \( \sigma_j = \alpha_{j1} R_1^2 + \alpha_{j2} R_2^2 + \alpha_{j3} R_3^2 \). Since for the special case second order effects from \( \phi \) do not enter into the equations for \( R_{j2} \), the equation for \( \phi_2 \) is simply a one-dimensional, forced ODE that can be integrated directly. It is a complicated equation, however, so \( \phi_2 \)'s behavior near \( T_0 \) will merely be commented on later.

The equations for \( R_{j2} \) may be written in the form

\[ R_{2j} = A \cdot R_2 + F \]  
\[ (2.18) \]

where \( A \) is the matrix from before and \( F \) is some vector representing the inhomogeneous terms in (2.17). For the special case,

\[ F_1 = R_{21} R_{31} - \frac{\phi_1^2}{2(T_0 - T)^2} + \frac{\alpha_1}{(T_0 - T)^4} \]
\[ F_2 = R_{11} R_{31} - \frac{\phi_1^2}{2(T_0 - T)^2} + \frac{\alpha_2}{(T_0 - T)^4} \]
\[ F_3 = R_{21} R_{11} - \frac{\phi_1^2}{2(T_0 - T)^2} + \frac{\alpha_3}{(T_0 - T)^4} \]

where \( \alpha_j = \alpha_{j1} + \alpha_{j2} + \alpha_{j3} \). To solve (8), make the transformation

\[ R_2 = \exp \left( \int T A \ d\xi \right) \]

which, since \( A \) and \( \int T A \ d\xi \) commute, results in
\[ \mathbf{D}_T = \exp(- \int^T A \, d\xi) \mathbf{F} \]

In a similar fashion to before, it is found that

\[ \exp(- \int^T A \, d\xi) = \frac{1}{3(T_0 - T)} \mathbf{B} + \frac{(T_0 - T)^2}{3} \mathbf{A}' \]

where

\[ \mathbf{B} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \]

Letting

\[ \mathbf{R}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (T_0 - T) + \frac{C}{(T_0 - T)^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \]

we would have

\[ \mathbf{D}_T = \left( \frac{1}{3(T_0 - T)} \mathbf{B} + \frac{(T_0 - T)^2}{3} \mathbf{A}' \right) \cdot \begin{pmatrix} a_2 & a_3 \\ a_1 & a_3 \\ a_1 & a_2 \end{pmatrix} (T_0 - T)^2 - C \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \frac{1}{T_0 - T} + \begin{pmatrix} C^2 + \alpha_1 \\ C^2 + \alpha_2 \\ C^2 + \alpha_3 \end{pmatrix} \frac{1}{(T_0 - T)^4} \]

\[ - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{\phi_1^2}{2(T_0 - T)^2} \]

\[ = \frac{2a_2a_3 + a_2^2}{2a_1a_3 + a_2^2} \frac{T_0 - T}{3} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \frac{C}{(T_0 - T)^2} + \begin{pmatrix} 2\alpha_1 - \alpha_2 - \alpha_3 \\ -\alpha_1 + 2\alpha_2 - \alpha_3 \\ -\alpha_1 - \alpha_2 + 2\alpha_3 \end{pmatrix} \frac{1}{3(T_0 - T)^5} \]

\[ + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{(a_2a_3 + a_1a_3 + a_1a_2)(T_0 - T)^4}{3} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{3C + \alpha_1 + \alpha_2 + \alpha_3}{3(T_0 - T)^2} \]

\[ - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{\phi_1^2}{2} \]

Recall, that after this is integrated, it must be multiplied by
\[ \exp(\int^T A \, d\xi) = \frac{T_0 - T}{3} B + \frac{1}{3(T_0 - T)^2} A' \]

This lengthy calculation does not need to be performed; we need only obtain the most singular term in \( R_2 \) as \( T \to T_0 \). It might appear that the term in \( D_T \) proportional to \( \frac{1}{(T_0 - T)^2} \) would integrate to a inverse fourth power, and be multiplied by an inverse second power to contribute a singularity like \( \frac{1}{(T_0 - T)^3} \). However, the effect of multiplying this term by the matrix \( A' \) is to eliminate it. It can then be seen that the worst singularity to second order is \( \frac{1}{(T_0 - T)^3} \), which means that the third and second terms in the perturbation series become disordered in the same way that the first and second do; that is, when

\[ T - T_0 = O(\epsilon) \]  \hspace{1cm} (2.19)

It can also be seen, by considering the equation satisfied by the second order phase, that \( \phi_2 \) blows up like \( \frac{1}{(T_0 - T)^2} \).

Hence it is seen that the original perturbation series becomes invalid for times satisfying (2.19). At this stage we must reexamine equations (2.11), or their polar form. The radii \( R_j \) would need to be rescaled by

\[ \bar{R}_j = \frac{R}{\epsilon} \]

and a new time scale

\[ \bar{T} = \frac{T - T_0}{\epsilon} \]

introduced. However, it should be noted that under the required scaling, the time derivative becomes order \( \epsilon^{-2} \), as do the quadratic terms, cubic terms, as well as the quartic terms. Presumably, all of the higher order terms in the evolution equations (2.11) which are lost in a truncation become this large as well. Which means that the inclusion of just the cubic terms is insufficient. All the neglected terms are most likely important. Of course, then a weakly nonlinear problem is no longer being dealt with; the phases are no longer slowly varying either, and the original PDE from which (2.11) was deduced must be reconsidered.

The above analysis was performed for a special set of solutions to the leading order equations; namely when the polar amplitudes \( R_j \) are all equal and the total phase \( \phi \) is zero. The general solution can be expressed in terms of Jacobi elliptic functions, but any analytical analysis of the higher order equations appears hopeless. However, it is to be expected that the results, namely that higher order solutions break down in such a manner as to make the problem fully nonlinear near the blow-up point, remain valid. An inspection of the solutions given by Craik (1985) indicates that near the blow-up point, all explosive solutions behave like
\[ R_j \sim \frac{1}{T_0 - T} \]

and the total phase tends to a multiple of 2\( \pi \); ie, sin\( \phi \) tends to 0 as \( T \to T_0 \).

Finally it would be desired to next examine the problem from the point of view of the exact solutions of the truncated equations (2.11); ie, investigating the problem that is uniformly valid to \( O(\epsilon^2) \).

### 2.1.3 Consideration of the Exact, Second Order Equations

As described earlier, in order to suppress secularities that arise at \( O(\epsilon^2) \), the evolution equations for the triad amplitudes must satisfy

\[
\begin{align*}
S_{1T} &= S_1^* S_3^* + i \epsilon S_1 (\beta_{11} |S_1|^2 + \beta_{12} |S_2|^2 + \beta_{13} |S_3|^2) \\
S_{2T} &= S_2^* S_3^* + i \epsilon S_2 (\beta_{21} |S_1|^2 + \beta_{22} |S_2|^2 + \beta_{23} |S_3|^2) \\
S_{3T} &= S_1^* S_2^* + i \epsilon S_3 (\beta_{31} |S_1|^2 + \beta_{32} |S_2|^2 + \beta_{33} |S_3|^2)
\end{align*}
\]

(2.20)

or, by introducing the polar variables again,

\[
\begin{align*}
R_{1T} &= R_2 R_3 \cos \phi \\
R_{2T} &= R_1 R_3 \cos \phi \\
R_{3T} &= R_2 R_1 \cos \phi \\
\phi_T &= -\left(\frac{R_2 R_3}{R_1} + \frac{R_1 R_3}{R_2} + \frac{R_2 R_1}{R_3}\right) \sin \phi \\
&\quad + \epsilon (\beta_1 R_1^2 + \beta_2 R_2^2 + \beta_3 R_3^2)
\end{align*}
\]

(2.21)

where \( \beta_j = \beta_{1j} + \beta_{2j} + \beta_{3j} \). It is demanded that equations (2.11) be satisfied so that to second order, the term \( \epsilon^2 u_2 \) in the original perturbation series remains of order \( \epsilon^2 \). However, although equations (2.20) prevent blow-up in finite time, the solutions nonetheless become quite large; large enough, in fact, to disorder the original perturbation series once again, and in the same manner as before.

The equations (2.21) may be integrated exactly for \( \epsilon \) nonzero. The details are in Weiland and Wilhelmsson (1977), but essentially we let

\[ r(T) = R_j(T)^2 - R_j(0)^2 \]

for \( j = 1, 2, 3 \), the so-called Manley-Rowe relations, and make use of the further conserved quantity

\[ \Gamma = R_1 R_2 R_3 \sin \phi + \frac{\epsilon}{4} (\beta_1 R_1^4 + \beta_2 R_2^4 + \beta_3 R_3^4) \]
Note that for \( \epsilon = 0 \), \( \sin \phi \) tends to zero as the \( R_j \) blow up. This constant of motion \( \Gamma \) may be used to reduce the system to the single ODE

\[
\frac{d r}{dT} = \pm 2 [(r + R_1(0)^2)(r + R_2(0)^2)(r + R_3(0)^2) - \Gamma^2 \\
+ \frac{\epsilon \Gamma}{2} \sigma - \frac{\epsilon^2}{16} \sigma^2]^{1/2}
\]

where

\[
\sigma = \beta_1 (r + R_1(0)^2)^2 + \beta_2 (r + R_2(0)^2)^2 + \beta_3 (r + R_3(0)^2)^2
\]

and the \( \pm \) denotes the sign of \( \cos \phi \). This could be rewritten as

\[
\frac{1}{2} \left( \frac{d r}{dT} \right)^2 - \Pi(r) = 0
\]

where \( \Pi(r) \) is a quartic polynomial in \( r \). In this form, similar to a potential-type problem, we can see that the solution will always oscillate between some values determined by the roots of this polynomial for \( \Pi > 0 \). For \( \epsilon = 0 \), this (now cubic) polynomial will be positive for large values of \( r \), and so as \( r \) becomes large it will satisfy

\[
\frac{d r}{dT} \sim \pm 2 r^{3/2}
\]

or

\[
r \sim \frac{1}{(T_0 - T)^2}
\]

as \( T \) approaches some \( T_0 \). For \( \epsilon \) nonzero, however small, this polynomial will be negative for large \( r \) owing to the sign of it's quartic term. Hence blow up will never occur in this case, but there will be oscillation between the two largest roots of the polynomial. One root must be the large root introduced by the small quartic term. However, a dominant balance argument reveals that this root must be of order \( \epsilon^{-2} \), which means at this point the original amplitudes \( R_j \) must be of order \( \epsilon^{-1} \). This in turn implies that the second order terms which arise from the quadratic nonlinearities must be of order \( \epsilon^{-2} \). So, although periodicity has been maintained through the removal of secular terms, the solution still becomes quite large; large enough, it is seen, to disorder the perturbation series as before. From the analysis of the last section, we can infer that this disordering occurs near the blow-up point for the quadratic evolution equations; ie, the triad equations without the higher order terms. This is true because a perturbation analysis of (2.21) would proceed exactly as before, and away from the blow-up point, the cubic terms in the evolution equations are negligible. The equations would need to be rescaled and reanalyzed near \( T = T_0 \), and again both the quadratic and cubic terms would be equally important. These, of course, are the only nonlinear terms present in this case. For \( T - T_0 = O(\epsilon) \), the leading order
solution would again be of order $\varepsilon^{-1}$. So viewed in this light, the retention of simply the cubic terms in the equations for the wave amplitudes is insufficient; regardless of how many of a finite numbers of terms are retained in these equations, the solutions become large enough to render the original perturbation expansion invalid, and in such a way that the problem becomes fully nonlinear near the blow-up point.

To close this section on analytic approaches to the problem, comparisons to certain oscillator problems will be made. These exhibit similar behavior and more can be said analytically about such problems near a singular point than is possible here.

2.1.4 Analogy with Passage Through Resonance Problem

In this section analogies will be drawn between the aspects of explosive triads and certain problems first studied by Kevorkian (1971, but see Kevorkian and Cole, 1981, for a more comprehensive discussion). To be specific, comments will be made on some features of the following equation:

$$\frac{d^2 y}{dt^2} + \mu^2(T) y = \alpha \cos(t + \beta) \quad (2.22)$$

where again $T = \varepsilon t$, $\varepsilon$ is some small parameter, and $\mu$ is some function which near some point $T_0$ has the following behavior:

$$\mu(T) = 1 + a_1 (T - T_0) + O((T - T_0)^2)$$

as $T \to T_0$. Thus, intuitively it would seem that near this point in time, the forcing term becomes resonant, and from a perturbation point of view difficulties may be anticipated. This is indeed the case. While this problem could certainly be handled via a WKB-type approach, it is instructive to analyze it by multiple scale techniques, as they will reveal the salient features of the problem as well as be more applicable to nonlinear problems.

Without going into the many subtleties of this problem, the main points will be discussed. As is typical with such inhomogeneous problems, a two-scale approach would seek a solution of the form

$$y \sim y_0(t^+, T) + \varepsilon y_1(t^+, T) + \cdots \quad (2.23)$$

where the new "fast" time is given by

$$\frac{dt}{dt^+} = \frac{1}{\mu(T)}$$

Again, there are many fine points to this problem, but the most interesting result is the fact that due to the forcing, the leading order solution will have, in addition to a homogeneous solution, a term like

$$\frac{\alpha}{\mu^2 - 1} \cos(t + \beta)$$
The second order solution $y_1$ will then be found to have a term like

\[ \frac{-4 \alpha \mu \mu'}{\left(\mu^2 - 1\right)^3} \sin(t + \beta) \]

Thus, the perturbation series (2.23) will become disordered, and hence invalid, when

\[ T - T_0 = O(\varepsilon^{1/2}) \]

in which regime $y$ becomes of order $O(\varepsilon^{-1/2})$. Kevorkian's approach, as here in dealing with the explosive triads, is to study the problem in this vicinity by introducing the "inner" variable

\[ \overline{T} = \frac{T - T_0}{\varepsilon^{1/2}} \]

and rescaling the dependent variable as

\[ y = \frac{\overline{y}}{\varepsilon^{1/2}} \]

The new dependent variable $\overline{y}$ is then assumed to a function of $t$ and $\overline{T}$. The second time derivative, taken as

\[ \frac{d^2 y}{dt^2} + \varepsilon \frac{d^2 y}{dT dt} + \varepsilon^2 \frac{dy^2}{dT^2} \]

becomes

\[ \frac{1}{\varepsilon^{1/2}} \left( \frac{d^2 \overline{y}}{dt^2} + \varepsilon^{1/2} \frac{d^2 \overline{y}}{dT dt} + \varepsilon \frac{d^2 \overline{y}}{dT^2} \right) \]

so to leading order the problem becomes

\[ \frac{d^2 \overline{y}}{dt^2} + \overline{y} = 0, \]

just a simple harmonic oscillator. As far as this problem is concerned, it may be solved to higher orders, and then a matching is performed between this solution as $\overline{T} \to -\infty$ and the "outer" (original) solution as $T \to T_0^-$. This may all be confirmed using the method of stationary phase in conjunction with variation of parameters, using the WKB technique to obtain the homogeneous solution. These are details which need not concern us, but some important points will be noted.

First, although this equation does not exhibit the blow-up in finite time in the way that the full triad equations do, it does share the feature of having a leading order solution which becomes unbounded in finite time. As before, this outer solution is invalid near this point, and the problem must be reanalyzed here.
Second, the nature of the blow-up is not the same, and in some sense weaker than in the case of explosive triads. The leading order inner equation is linear, and near \( T = T_0 \), the problem is still a "traditional" two-time scale problem. So here matters are considerably simpler. The "inner" problem can actually be solved as in any typical multiple scale problem, and the matching to the outer solution may be accomplished. Of course, the actual matching is rather difficult, but a uniformly valid solution near (and if desired, past) the singularity (known here as the passage through resonance) may be obtained.

But this turns out to be the more relevant point. Nowhere is the form of such a solution proved. Rather, a form is assumed, and it's consistency demonstrated. That is the attitude taken with the inner problem in the explosive triad case. A form of the solution is taken, and a consistent, if nearly intractable, matching procedure is laid out.

This problem is invoked so as to give added plausibility to the method for dealing with explosive triads. For here is a problem which also in some sense exhibits blow-up in finite time, and a matching technique similar to that above (though in this case possible to do directly) is used to study the problem near this blow-up time. Further more, it should be noted that the outer solution for the passage through resonance breaks down when \( T - T_0 \) is of order \( \varepsilon^{1/2} \) or equivalently when

\[
t - \frac{T_0}{\varepsilon} = O(\varepsilon^{-1/2})
\]

In other words, breakdown occurs on the fast time scale relatively "far away" from the singular point. In contrast, for the explosive triads, break down on the fast scale occurs when

\[
t - \frac{T_0}{\varepsilon} = O(1),
\]

or relatively "close" to the singularity. Thus, the outer solution in the passage through resonance problem should fail to be an "accurate" approximation to the problem as a whole well-before the singular point is approached. The outer solution for the explosive triads, however, should be a "good" approximation for the full problem for times close to the blow-up point.

Of course, from the point of view of asymptotic analysis, it impossible to quantify these statements. In the next section, numerical demonstration of these distinctions is offered, as well as further numerical support for the contentions laid out in this part of the thesis.

Before doing this, a few further comments are in order concerning the extent of the relation between these resonant oscillators and explosive triads in dispersive wave systems. It had been hoped that a nonlinear counterpart to (2.22) would share the feature of having two fast time derivatives appear at the same order in the inner problem, in the hopes that the inner problem here might be more tractable and offer support for some of the contentions above. Alternatively, the inhomogeneity in the frequency might be taken to have a different (possibly nonanalytic) form near \( T_0 \). This was found not to be the case. For consider the equation

28
\[
\frac{d^2y}{dt^2} + \mu^2(T)y + \varepsilon y^3 = \alpha \cos(t + \beta) \tag{2.24}
\]

also from Kevorkian and Cole (1981), but where \(\mu(T)\) is now assumed to behave like

\[
\mu(T) = 1 + a_1 |T - T_0|^\alpha + \cdots
\]

as \(T \to T_0\). To mimic the triad problem, it would required that outer perturbation series solution to (2.24) becomes disordered when

\[
T - T_0 = O(\varepsilon)
\]

A similar analysis as before reveals that near \(T_0\), the leading order outer term \(y_0\) in an expansion for \(y\) becomes singular like

\[
\frac{1}{\mu^2 - 1}
\]

The most singular term in \(y_1\) could now be proportional to either

\[
\frac{1}{(\mu^2 - 1)^4}
\]
or

\[
\frac{\mu'}{(\mu^2 - 1)^3}
\]
depending on the behavior of \(\mu\) near this point. Based on this assumed behavior as \(T \to T_0\),

\[
(T_0 - T)^{-\alpha}
\]

must be compared to

\[
\varepsilon \left((T_0 - T)^{-4\alpha}, (T_0 - T)^{-2\alpha - 1}\right) \tag{2.25}
\]

If \(\alpha > \frac{1}{2}\), the first element in the ordered pair (2.25) is dominant, and breakdown occurs when

\[
T - T_0 = O(\varepsilon^{\frac{1}{2\alpha}})
\]

which could never be made \(O(\varepsilon)\), as this would require \(\alpha = \frac{1}{3}\), in violation of the assumption \(\alpha > \frac{1}{2}\). If \(0 < \alpha < \frac{1}{2}\), so the second element in (2.25) is dominant, the series would become invalid when
\[ T - T_0 = O(\varepsilon^\alpha + \varepsilon) \]

Here to obtain the desired effects it would required that \( \alpha = 0 \), in which case we no longer have a slowly varying oscillator; the leading order problem is resonant from the start. So in this case, the inner problem the passage through resonance is again an ordinary two time scale problem. That is to say, the inner problem will consist again of a slowly varying, nearly linear oscillator. Of course, the matching may prove quite strenuous, but that is not the issue here.

In addition, this search was conducted on problems of the following sort:

\[
\frac{d^2 x}{dt^2} + a^2(T) x = \varepsilon y^2
\]

\[
\frac{d^2 y}{dt^2} + b^2(T) y = 2\varepsilon y x,
\]

also from Kevorkian and Cole (1981), where

\[
a(T) = 2 b_0 + a_1 (T - T_0) + \cdots
\]

\[
b(T) = b_0 + b_1 (T - T_0) + \cdots
\]

for some \( T_0 \), in the hopes of modeling the so-called degenerate triads to be studied numerically in the next section. Again, though, this problem was found to be quite different from the explosive triads to which it bears superficial similarity. The problem may require reanalysis near some singular point, but near this point the solution remains weakly non-linear, if large relative to the outer solution. It remains slowly varying as well, as opposed to the explosive triads which essentially possess two fast time scales near the blow-up point.

Absent this concrete analytical comparison, attention is now turned to some numerical studies of this problem, which will confirm the arguments made throughout this section, as well as quantify some of the asymptotic notions alluded to above.

### 2.2 Numerical Results

In this part of the thesis results of numerical simulations of resonant explosive triads will be examined. A model PDE which exhibits such behavior will be solved numerically, and the divergence from previously predicted results, ie, modeling the problem by retaining a finite number of terms in the amplitude evolution equations, will be established. Later, a family of model PDE’s which can be analyzed analytically will also be studied numerically to show how such systems behave near the blow-up point.

#### 2.2.1 A Model PDE

In this section the behavior of the following PDE will be studied:
\[ u_t = -7u_x - 5u_{3x} - u_{5x} + \frac{\sin(\varepsilon^2 u_x u_{xx})}{\varepsilon} \]  

(2.26)

In the above, \( \varepsilon \) is once again some small parameter, and \( u_{nx} \) denotes \( n \) derivatives of \( u \) with respect to \( x \). The equation written in the above form can be thought of as coming from a fully nonlinear PDE for which we seek solutions of order \( \varepsilon \). The reason the nonlinearity has been chosen as a sine will be commented on later.

The above equation has been selected for several reasons. First, the linearized dispersion relation

\[ \omega = 7k - 5k^3 + k^5 \]

permits triads, as can be g gleaned from Figure 2.2.

![Figure 2.2: Linear dispersion relation for (2.26).](image)

Second, the nonlinearity will admit explosive triads. This can be seen from the fact that to leading order this nonlinear term becomes

\[ u_x u_{xx} = \frac{1}{2} \left( u_x^2 \right)_x \]

and so it is a simple matter to show that for a leading order perturbation solution made up of triads \( (k_j, \omega_j) \), the amplitudes \( S_j \) must satisfy
\[ S_{1T} = -i K S_2^* S_3^* \]
\[ S_{2T} = -i K S_1^* S_3^* \]
\[ S_{3T} = -i K S_2^* S_1^* \]

where \( K = k_1 k_2 k_3 \). Since the interaction coefficients are all the same imaginary number, the transformation \( S \rightarrow i S \) changes this system into one with interaction coefficients all of one sign, so these will become singular in finite time. Furthermore, the dispersion relation allows so-called degenerate triads, where two members of the set are identical. In this case, the leading order solution would consist of the two waves with wave numbers and frequencies \((k_1, \omega_1), (k_2, \omega_2)\) where

\[ k_2 = 2k_1 \]
\[ \omega_2 = 2\omega_1 \]

and the evolution equations for the amplitudes would take the form

\[ S_{1T} = S_1^* S_2^* \]
\[ S_{2T} = S_1^{*2} \]

under the appropriate transformation. Now these equations for degenerate triads have the same properties as the regular triad equations; they may be solved in the same way and in particular blow up in finite time if the interaction coefficients are of one sign (which is assumed to be the case). For simplicity, and also to ensure periodicity in the numerical spatial domain, these degenerate triad interactions will be studied, both numerically for the full PDE, and in comparison to the exact results for the truncated evolution equations. The disagreement between the two results will demonstrate the points made in the preceding part of this thesis.

From the linear dispersion relation, we will have degenerate triads when

\[ \omega(2k) = 2\omega(k) \]

or when \( k = 1, \omega = 3 \). The leading order solution is taken to be

\[ u_0 = S_1 e^{i(x-3t)} + S_2 e^{i^2(x-3t)} + * \]

where * denotes the complex conjugate of the preceding expression. In order to suppress secularities at next order, \( S_1, S_2 \) must satisfy

\[ S_{1T} = 2i S_1^* S_2 \]
\[ S_{2T} = -i S_1^{*2} \]

(2.27)
With the substitution \( S_1 = s_1, S_2 = s_2^* \) these become

\[ s_{1T} = 2i s_1^* s_2^* \]
\[ s_{2T} = i s_1^* \]

These equations are of the explosive type. Indeed, in the following numerical analysis, the following exact solutions to the above will be used:

\[ s_1 = \frac{i}{\sqrt{2}} \frac{1}{T_0 - T} \]
\[ s_2 = \frac{-i}{2} \frac{1}{T_0 - T} \]

for some \( T_0 \), in the initial conditions for solving (2.26). This implies that the leading order solution becomes

\[ u_0 = -\sqrt{2} \frac{\sin \theta}{T_0 - T} - \frac{1}{T_0 - T} \sin 2\theta \quad (2.28) \]

where \( \theta = x - 3t \). Taking \( T_0 = 1 \), so that blow-up will occur when \( t = 1/\epsilon \), (2.26) was solved numerically with initial conditions \( u(x,0) = u_0(x,0) + \epsilon u_1(x,0) \), where \( u_1 \) is the \( O(\epsilon) \) solution (to be seen shortly). The method used was based on the pseudospectral scheme of Fornberg and Whitham (1978) for such problems. Spatial derivatives were evaluated by decomposing \( u \) into Fourier components and using efficient transform techniques, and the time derivative was handled implicitly using a second-order Runge-Kutta technique derived by Waleffe (1993). This method is described in detail in one of the appendices. It has been used it to integrate (2.26), and the results compared with the predictions of the weakly nonlinear theory; that is, with the solution whose amplitudes are governed by the triad equations (2.27) with cubic correction terms.

In Figure 2.3 the solution to (2.26) is displayed for \( \epsilon = 0.1 \) and \( x = 0 \). 64 Fourier modes in \( x \) have been taken. The solid line represents the numerical solution to the full, nonlinear problem, and the dashed line represents the two-term asymptotic solution based on the removal of secularities to order \( \epsilon \); that is to say, the leading order outer solution (2.28) plus the next term in the expansion.

It is seen that the outer solution becomes increasingly inaccurate near the blow-up point at \( t = 10 \) (or \( T = 1 \)). This is of course not surprising. However, in Figure 2.4 the following function has been plotted in a neighborhood of the blow-up point:

33
Figure 2.3: Comparison of exact (numerical) solution of (2.26) (solid line) and asymptotic solution (2.28) (dashed line) for $\varepsilon = 0.1$ and $x = 0$. The blow-up point is at $T_0 = 1$ on the slow scale or $1/\varepsilon = 10$ on the fast scale.

Figure 2.4: Weakly nonlinear solution of (2.26) based on cubic amplitude equations (2.29).
\[ S_1 e^{i\theta} + S_2 e^{i2\theta} + * \]

along with the full solution, where the \( S_j \) satisfy

\[ S_{1T} = 2iS_1^* S_2 - i\varepsilon \frac{3}{10} S_1 S_2 S_2^* \]
\[ S_{2T} = -i S_1^2 - i\varepsilon (\frac{3}{10} S_2 S_1 S_1^* + \frac{8}{45} S_2^2 S_2^*) \]  (2.29)

also referred to as the cubic equations, and obtained by suppressing secularities at second order and using

\[ \sin(\varepsilon^2 u_x u_{xx}) = \varepsilon^2 u_x u_{xx} - \frac{\varepsilon^6}{6} u_x^3 u_{xx} + \cdots \]

\[ u_1 = \frac{-S_1}{20} S_2 e^{i(\theta_1 + \theta_2)} - \frac{S_2^2}{90} e^{i2\theta_2} + * \]

Considerable disagreement is seen between Figure 2.3, which includes the “exact” solution, and Figure 2.4, which indicates the kind of behavior such systems were previously thought to have. The reason the solutions based on the cubic equations (2.29) were not graphed over the entire range of time under consideration is that the amplitudes, while remaining bounded, become quite large. This can be seen in Figures 2.5 and 2.6, which display the amplitudes \(|S_1|\) and \(|S_2|\). The asymptotic approximation to the full solution would be some combination of sinusoids which oscillate between these two envelopes, and so the solution would become highly oscillatory in between the peaks as in Figure 2.4. However, Figures 2.7 and 2.8 show the asymptotic solution plotted against the exact solution in neighborhoods excluding the blow-up point. As predicted, even the solutions based on (2.29) become inaccurate as the blow-up point is approached, as seen in Figure 2.7. In Figure 2.8, which displays the “tail” of the asymptotic solution and the actual solution, the divergence is quite clear. Beyond the blow-up point, the asymptotic solution ceases to be a valid description of the full solution. In light of the previous part of the thesis dealing with analytical aspects of this problem, it would be thought that the solution obtained from (2.29) would be of order \( \varepsilon^{-1} \) near the blow-up point, when Figures 2.5 and 2.6 would seem to suggest that this solution becomes much larger. However, this is an artifice due to the presence of small coefficients when (2.29) is reduced to quadrature. For introducing

\[ S_j = R_e e^{i\phi_j} \]

and \( \phi = 2\phi_1 - \phi_2 \), we find that

35
Figure 2.5: Polar solution of (2.29) for $\epsilon = 0.1$.

Figure 2.6: Polar solution of (2.29) for $\epsilon = 0.1$. 

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Figure 2.7: Comparison of exact solution of (2.26) (solid line) and asymptotic solution based on cubic amplitude equations (2.29) (dashed line), for $\varepsilon = 0.1$ and $x = 0$.

Figure 2.8: Same results as in Figure 2.7, only beyond the blow-up point.
\[ R_{1T} = 2 R_1 R_2 \sin \phi \]
\[ R_{2T} = R_1^2 \sin \phi \]
\[ \phi_T = (4 R_2 + \frac{R_1^2}{R_2}) \cos \phi + \varepsilon \left( \frac{3}{10} R_1^2 - \frac{19}{45} R_2^2 \right) \]

so, using the initial conditions \( R_1(0) = 1/\sqrt{2}, R_2(0) = 1/2, \phi = \pi/2 \), it is seen that

\[ \frac{R_1^2}{4} - \frac{R_2^2}{2} = 0 \]
\[ R_1^2 R_2 \cos \phi + \varepsilon \left( \frac{3}{80} R_1^4 - \frac{19}{180} R_2^4 \right) = \frac{\varepsilon}{360} \]

so letting

\[ x = \frac{R_2^2 - R_2(0)^2}{2} = \frac{R_1^2 - R_1(0)^2}{4} \]

it is found similarly to before that

\[ \left( \frac{dx}{dT} \right)^2 = 4 \left( 4 x + R_1(0)^2 \right) ^2 \left( 2 x + R_2(0)^2 \right) \]
\[ -\varepsilon^2 \left( \frac{1}{360} - \frac{8}{45} x^2 - \left( \frac{3}{10} R_1(0)^2 - \frac{19}{45} R_2(0)^2 \right) x - \frac{3}{80} R_1(0)^4 + \frac{19}{180} R_2(0)^4 \right)^2 \]

The solution \( x \) will oscillate between the two largest roots of the above polynomial. Three of the roots correspond to the unbounded case when \( \varepsilon = 0 \); namely, \( -R_2(0)^2 / 2 \) and \( -R_1(0)^2 / 4 \). Actually these are equal in this case, so this triple root represents a fixed point for the single-term triad equations. The solution would start at \( x = 0 \) (recall the definition of \( x \)), increase in magnitude until it became as large as the fourth root, then decrease and tend to the largest of the other roots, and begin the process anew. This fourth root, introduced due to the higher order effects, is of order \( \varepsilon^{-2} \) and is approximately

\[ \frac{32}{\left( \frac{8}{45} \right)^2 \varepsilon^2} = 101,250. \]

which would predict that \( R_1 \) would become as large as about 640., and \( R_2 \) as large as about 450., both quantitatively verified numerically in the graphs.

The same results for \( \varepsilon = 0.05 \), when the blow-up point is at \( t = 20 \), are in Figures 2.9-12. Again, there is substantial disagreement between the two approaches to the problem. The actual solution does become large as the blow-up point is approached, but not nearly as large as the solutions based on the cubic evolution equations become. Apart from the huge disparity in magnitudes, the "exact" solution, by which it is meant the result of a numerical integration, does not exhibit the repeated, stabilized explosions predicted.
Figure 2.9: Exact solution of (2.26) (solid line) and asymptotic solution (2.28) (dashed line) for $\varepsilon = 0.1$ and $x = 0$. Again, the blow-up point is at $T_0 = 1$ on the slow scale or $1/\varepsilon = 20$ on the fast scale.

Figure 2.10: Asymptotic solution of (2.26) based on cubic evolution equations (2.29).
Figure 2.11: Exact solution of (2.26) (solid) compared with asymptotic solution (dashed) based on cubic evolution equations (2.29) for $\varepsilon = 0.1$ and $x = 0$.

Figure 2.12: Same as Figure 2.11 beyond the blow-up point.
by the evolution equations obtained by retaining cubic nonlinearities. The solution does
not become peaked in any way, nor is it highly oscillatory near the blow-up point. But
this is as predicted in the previous part; cubic nonlinearities are insufficient; “all” of the
nonlinearities are equally important near the blow-up point, and beyond. Note nothing
is being said about the true behavior of equation (2.26) near the blow-up point. Indeed,
this is impossible as the governing behavior near this point is determined by full nonlinear
equation; that is to say, the equation obtained by setting $\varepsilon = 1$.

Before commenting on how close to the blow-up point the weakly nonlinear (outer)
solution remains an accurate approximation to the full problem, which will involve com-
parisons again with the oscillator problems from the last part, a few words on our choice
of nonlinearity above are in order. Initially, the nonlinearity

$$u_x u_{xx}$$

was used, as it will have the same properties as the sine if it remains small enough. However,
numerical results seemed to indicate that the full nonlinear problem, not just it’s weakly
nonlinear approximation, blew up in finite time in the presence of explosive triads. This
blow-up occurred near the blow-up time for the triads. However, the result differed from
what equations like (2.29) would have predicted. Of course, this particular model problem
may simply just be pathological, but the point is that the equation was not being modeled
by the triad equations with cubic terms, as was previously thought. Near the blow-up
point, the fully nonlinear equation governs the system. Indeed, the numerical runs on
this problem with $\varepsilon = 1$ seemed to become singular in finite time as well. But again, no
predictions are being made about the true nature of such equations, only what does not
happen.

The above special solutions were chosen so as to facilitate comparison with exact
analytical solutions. The same qualitative behavior, however, was observed for different
initial conditions (in which case the analytical comparisons would have involved elliptic
functions; eg, to determine where the blow-up time is).

A few more points may be gleaned from the numerics. The analytic work of the
previous section predicted that the perturbation series obtained via the triad equations
would be valid, that is to say an accurate approximation to the problem in question, as
long as $T - T_0$ was much larger than $\varepsilon$. Alternatively, it could be said that when $t - T_0/\varepsilon$
becomes order 1, the perturbation series is no longer an acceptable model for our problem.
In contrast, the resonant oscillators we discussed for comparison had outer solutions which
broke down when $T - T_0$ was of order $\varepsilon^{1/2}$, or when $t - T_0/\varepsilon$ was of order $\varepsilon^{-1/2}$. This
would seem to say that the outer solution in the case of explosive triads would “last longer”
than the outer solution for passage through resonance problems, and hence in some sense
be a better approximation. This is all in an asymptotic sense; no quantification has been
made of the above statements. But consider the problem from before

$$\frac{d^2 y}{dt^2} + \mu^2(T) y = \cos t$$

(2.30)

where
\[ \mu(T) = 1 + a_1 (T - T_0) \]

It was seen that the leading order solution to this problem, valid away from \( T = T_0 \), is

\[
y_0 = \frac{\cos t}{2 a_1 (T - T_0) + a_1^2 (T - T_0)^2} + \rho_0 (1 - a_1 T_0)^{1/2} \frac{\cos (t^+ + \phi_0)}{(1 + a_1 (T - T_0))^{1/2}}
\]

(see Kevorkian and Cole (1981) for details) where \( \rho_0 \) and \( \phi_0 \) are constants and

\[
t^+ = \frac{1}{\varepsilon} \int_0^T \mu(s) \, ds = \frac{1}{\varepsilon} \left( T + \frac{a_1}{2} T^2 - a_1 T_0 T \right)
\]

With the appropriate initial conditions, (2.30) was solved numerically, and a comparison was made with the asymptotic form (2.31). The results, for \( T_0 = 1, a_1 = 0.5, \rho_0 = 1, \phi_0 = 0, \) and \( \varepsilon = 0.1 \) and 0.05 are in Figures 2.13 and 2.14.

Again the solid lines denote the numerical solution of the full equations, and the dashed lines represent the asymptotic solutions. It can be seen that the outer solution does diverge from the true (numerical) not just near \( T_0 \) (here 1) but well before it also. Furthermore, the distance from the singular point at which the asymptotic solution begins to give inaccurate predictions increases as \( \varepsilon \) decreases. In contrast, the asymptotic solution to the triad equations provides a very accurate answer to within a few time units of the blow-up point, and the distance at which this approximation breaks down does not appear to vary with \( \varepsilon \). Of course, this is just what is predicted by the analytical analysis above. Thus, the numerical results give credence to asymptotic statements which were not quantified. Admittedly, the numerics does not quantify these issues precisely either, as no definition of “breakdown” has been offered. For example, halving \( \varepsilon \) should result in the outer perturbation series becoming invalid at a distance \( \sqrt{2} \) times as great as before. In our plots, it might seem reasonable to take as the breakdown point approximately 13 for \( \varepsilon = 0.05 \) and 28 for \( \varepsilon = 0.025 \). This would result in a breakdown ratio of 12/7 \( \approx 1.7 \), which is qualitatively close to the theoretical result. In contrast, the asymptotic solution in the case of the explosive (degenerate) triads seem to become inaccurate about 2 time units away from the blow-up point both for \( \varepsilon = 0.1 \) and 0.05. Again though, no absolute definition has been given for when a perturbation series becomes disordered on the basis of numerics.

The goals of this section were to demonstrate that when a weakly nonlinear system exhibits explosive triads, the dynamics are not governed by weakly nonlinear theory near the point of explosion. Weakly nonlinear theory does give accurate results away from this point, but near it, the behavior is not determined by retaining higher order terms in
Figure 2.13: Comparison of exact (numerical) solution of (2.30) (solid) with asymptotic solution (2.31) for $\varepsilon = 0.05$.

Figure 2.14: Same as Figure 2.14 for $\varepsilon = 0.025$. Note that the asymptotic solution becomes a worse approximation a further distance from the resonant point $t = 1/\varepsilon$ as $\varepsilon$ decreases.
the amplitude evolution equations. Near this point, the behavior is fully nonlinear. In addition, an attempt has been made to quantify some of the asymptotic results of the first part. The validity of certain oscillator problems which possess similar behavior as explosive triads has been confirmed. On this basis, it is felt that extensions to explosive triads are valid as well, although the inner problem in this case cannot be done analytically.

In the next section, a class of model PDE’s which can be analyzed analytically will be examined to give further credence to these claims.

2.2.2 Further Model PDE’s

In this section systems of the following form will be considered:

\[ u_t = Lu + \epsilon u^* v f_1(\epsilon^2 |u|^2, \epsilon^2 |v|^2) + u_x g_1(\epsilon^2 |u|^2, \epsilon^2 |v|^2) \]

\[ v_t = Lv + \epsilon u^2 f_2(\epsilon^2 |u|^2, \epsilon^2 |v|^2) + v_x g_2(\epsilon^2 |u|^2, \epsilon^2 |v|^2) \]  \hspace{1cm} (2.32)

where

\[ f_j(x,y) \sim 1 + a_j x + b_j y + \cdots \]

\[ g_j(x,y) \sim c_j x + d_j y + \cdots \]  \hspace{1cm} (2.33)

for \( x \) and \( y \) near 0, and \( L \) is any linear partial differential operator which permits explosive degenerate triads of the type described above.

Though admittedly artificial, this system will capture many of the aspects of more realistic problems where explosive triads arise.

The above system has exact solutions of the following form:

\[ u = S_1 e^{i\theta} \]

\[ v = S_2 e^{i2\theta} \]

where \( \theta = k x - \omega t \) represents an admissible linear phase and the amplitudes satisfy

\[ S_{1t} = \epsilon S_1^* S_2 f_1(\epsilon^2 |S_1|^2, \epsilon^2 |S_2|^2) + i k S_1 g_1(\epsilon^2 |S_1|^2, \epsilon^2 |S_2|^2) \]

\[ S_{2t} = \epsilon S_2^* f_2(\epsilon^2 |S_1|^2, \epsilon^2 |S_2|^2) + 2 i k S_2 g_2(\epsilon^2 |S_1|^2, \epsilon^2 |S_2|^2) \]  \hspace{1cm} (2.34)

Armed with the exact solution, the effects of neglecting various terms compared with the full solution, as would have been done with previous perturbation approaches, can be studied. Essentially, what is being said is that neglecting certain terms in an ODE can give varying results, depending not only on which terms are neglecting, but also in terms of the true solution obtained without any truncation. This is hardly a radical proposition, but it was a rather overlooked point as far as explosive triads had gone.

For a perturbation approach to (2.32) we would have
\[ u_0 = S_1 e^{i\theta} \]
\[ v_0 = S_2 e^{12\theta} \]
as before, but we will think of the amplitudes \( S_j \) as some slowly varying functions whose behavior will be determined by the removal of secular terms at particular orders in the perturbation expansion, as opposed to the solution of some system of ODE's we can find exactly. Using the expansions (2.33) for the nonlinear terms, we would have that

\[ S_{1T} = S_1^* S_2 \]
\[ S_{2T} = S_1^2 \]

which represents an explosive (degenerate) triad. As has been stated, the previous resolution of these solutions' singular behavior was to include the next higher order terms in the evolution equations (2.34), or equivalently suppressing secularities at order \( \epsilon^2 \) and introducing a time scale \( \tau = \epsilon^2 t \). This would give

\[ S_{1T} = S_1^* S_2 + \epsilon i k S_1 (c_1 |S_1|^2 + d_1 |S_2|^2) \]
\[ S_{2T} = S_1^2 + \epsilon i 2 k S_2 (c_2 |S_1|^2 + d_2 |S_2|^2) \]

(2.36)

which of course is simply a truncation of the full equations (2.34). We would not expect solutions to (2.36) to represent solutions to (2.34) accurately, at least not for all time. As an example, one need only consider modeling the equation

\[ \frac{dx}{dt} = x \sin(\epsilon^2 x^2) \]

with a three-term Taylor series approximation of it's nonlinearity:

\[ \frac{dx}{dt} = \epsilon^2 x^3 - \epsilon^7 x^7/6 + \epsilon^{11} x^{11}/120 \]

By inspecting plots of these nonlinearities in Figure 2.15 for \( \epsilon = 0.1 \) (again, solid lines represent exact quantities, dashed lines represent approximations), it is seen that the two ODE's above have similar behaviors for moderate values of \( t \), but eventually they have completely different characteristics. One solution has a fixed point, while the other becomes unbounded. Of course just keeping two terms in the Taylor series would give a solution with a fixed point, but at a different location the true value. Which is just to say that by keeping any number of terms in the truncation of the amplitude equations (2.34), which a weakly nonlinear approach would do, can give varying types of behavior. Of course, one dimensional solutions either tend to some value or become unbounded, but presumably in higher dimensions a wide variety of situations may arise.

For various nonlinearities, system (2.32) has been integrated, and compared with the result predicted by weakly nonlinear theory; that is to say, the near-linear wave solution with amplitudes governed 

\[ \text{by various high order amplitude equations such as (2.36). The} \]

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results are displayed in Figures 2.16 and 2.18 at $x = 0$ when two terms are retained in the evolution equations (2.36), with solid lines obtained from an integration of full equations and dashed lines representing asymptotic solutions. In each case, $\varepsilon = 0.1$. The actual wave amplitudes in comparison with a two term truncation of their governing ODE's are plotted in Figures 2.17 and 2.19. In Figures 2.16 and 2.17,

\[
    f_1(x, y) = \cos x \\
    f_2(x, y) = \cos y \\
    g_1(x, y) = e^{-(x+y)} - 1 \\
    g_2(x, y) = e^{-2(x+y)} - 1
\]

and in Figures 2.18 and 2.19,

\[
    f_1(x, y) = f_2(x, y) = 1 \\
    g_1(x, y) = -x - y + x^2 \\
    g_2(x, y) = -2(x + y) - xy
\]

In each, the leading order linear behavior of the respective functions, which determines the structure of the evolution equations (2.36), is the same. These two examples are typical of the cases observed. For the initial conditions used, the blow-up point turned out to be $\approx 8$. 

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Figure 2.16: Comparison of exact (numerical) solution (solid) of (2.32) with asymptotic solution (dashed) based on (2.36) for $\varepsilon = 0.1$. The choice of nonlinearities is discussed on page 46.

Figure 2.17: Amplitudes of the wave solutions. Solid lines denote solutions based on (2.34), dashed lines denoted solutions based on the truncation (2.36).
Figure 2.18: Same as in Figure 2.16 with different choice of nonlinearities (see page 46).

Figure 2.19: Same as in Figure 2.18 with different nonlinearities.
As can be seen, the actual solution and weakly nonlinear solution do differ. This is not surprising, as the exact solution (exact in the sense that a set of PDE’s may be reduced to a set of ODE’s) is known, and as argued, it would not be expected that an approximation of this exact solution would model the system well for all time. Of course, by retaining a different (finite) number of terms in the evolution equations (2.34), different kinds of behaviors can be obtained. Only one kind of behavior is valid, however, and that is when “all” of terms are retained; i.e., when the full ODE’s determine the amplitudes. This particular model is deceptively simple, but it illustrates the shortcomings of the previous approaches to the problem of explosive triads which occur in much more complicated systems. A few comments on these numerical results will now be made.

It will be noted that widely varying behaviors were not observed for the nonlinearities studied. 4 by 4 systems are being dealt with here, and it might be expected that chaotic solutions appear. Such solutions were not detected here, though that is not at all the purpose of this exercise. Indeed, the “exact” wave amplitudes often exhibit qualitatively the peaked solutions characteristic of the amplitude equations truncated at second order as in (2.36). However, they are clearly not the same. It is not being proposed that the simple models studied in this section are in any way indicative of more realistic systems. However, it might be conjectured that some of the physical systems in which explosive triads occur, such as in plasma physics, may have a fully nonlinear behavior which is similar to the repeated stabilized explosions exhibited by systems like (2.36). Any work done in this area might then have led the researchers involved to conclude that the truncations of weakly nonlinear theory are indeed valid. No search for such work was undertaken here, but it might remain an interesting project.

Finally, it should be noted that the truncated evolution equations obtained from equations like (2.34) may be solved perturbatively as in the previous, analytical part. Such series involved would become invalid as before, and again an inner equation which is the full equation is obtained. While such equations may not be solvable analytically either, it should be noted that the matching procedure is better understood here than for the PDE’s studied above. For in this case, the PDE’s studied here, with the types of solutions under consideration, are really ODE’s in some sense. For in the overall perturbation solution to (2.32), it would be found that \((u_1, v_1) = (u_2, v_2) = \cdots = 0\), and so to match the overall inner solution to the outer solution, only wave amplitudes need be considered, whereas for the kinds of PDE’s in the analytical part, the inner solution will consist of an infinite number of waves. But as was assumed there, in the current case one would look for inner amplitudes of the form \(\bar{S}_j(\bar{T})\) where \(\bar{T} = (T - T_0)/\varepsilon\). The \(\bar{S}_j\) satisfy the full equations (2.34), and some solution of these equations would need to be found which, as \(\bar{T} \to -\infty\), matches the outer solutions \(S_j\) as \(T \to T_0\).

Both analytical and numerical evidence has been presented to support the claims that explosive resonant triads are a much stronger interaction than previously thought. It would be useful, then, to determine whether such waves exist in a given system. In the next part of this thesis, this question, as well as other issues, will be addressed concerning problems in fluid mechanics.

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3. EXAMPLES FROM FLUID MECHANICS

In the previous part of this thesis, explosive resonant triads in general, weakly non-linear wave systems were discussed. In particular, such interactions were shown to be quite strong, much more so than previously thought. They therefore represent a potentially important effect in physical systems in which they occur. Whether or not a given system admits such waves, then, is an important question. In this part of the thesis an attempt is made to address this and other questions.

Of course, the system must admit triads to begin with, and so it is necessary to first determine the linear dispersion relation. For problems in fluid mechanics, the motion is usually not periodic in one direction (the vertical), and is often confined to be between finite solid boundaries. This non-oscillatory structure introduces several complications to the problem. While the governing equations may be written in the form of evolution equations studied in the first half, and solutions periodic in one (or two) directions sought, the non-periodic dependence requires additional analysis. Again however, the frequency will depend upon the wavenumber(s). As is well-known, at leading order some sort of eigenvalue problem results; namely, a homogeneous differential equation involving some parameters, with homogeneous boundary conditions, is obtained, and in general such equations have solutions only for certain values of the parameters. This parametric dependency implicitly gives the dispersion relation. As will be seen, such eigenvalue problems can be singular, and it is precisely these cases that will be relevant to the existence of explosive triads. Furthermore, the resonance in this case will result from an inhomogeneous boundary value problem, which will only have a solution under certain conditions. These conditions determine the amplitude equations of the last section. The higher order problems are typically singular as well, and how such problems are addressed will be discussed. While this will draw on previous results, it will be shown what role they play when “non-singular” methods (such as the Fredholm Alternative) must be modified when applied to singular problems of the sort to be encountered. This will be necessary when the search for explosive triads is undertaken.

As the conditions that such waves exist is that the coefficients in the leading order triad equations are all of one sign (assuming the appropriate transformations have been made), the first step in answering this question is to calculate these coefficients, or to at least be able to determine their signs for a given triad. As mentioned above, this would involve standard techniques to determine whether certain inhomogeneous boundary value problems have solutions. This will turn out to imply that explosive triads can only arise in singular problems where so-called critical layers are involved. These coefficients will be obtained by an Eulerian approach to the equations of motion. These will be contrasted with previous, Lagrangian approaches which overlooked some of the points raised above. It will be seen how the existence of explosive triads may be gleaned from the graph of the dispersion relation for such systems.

Finally, the occurrence of such waves in a fluid mechanical context involving continuously varying velocity profiles will be exhibited, and the results generalized to layered problems involving such velocity and density profiles.
3.1 EULERIAN VS. LAGRANGIAN ANALYSIS

In this section it will be shown how the interaction coefficients in the triad equations are derived and simplified for some fluid flow systems. The analysis will be undertaken from the Eulerian formulation as opposed to the Lagrangian viewpoint. The Eulerian framework views macroscopic properties such as velocity and density as being continuous functions of time and space, while the Lagrangian approach views the fluid as constituent particles, whose displacements are considered to be the main dependent variables. The Eulerian approach is more familiar and perhaps more accessible, but the results to be derived here were previously obtained via the Lagrangian formulation. While the Lagrangian approach is quite elegant, it is somewhat lacking in simplicity relative to the Eulerian viewpoint. In addition, as it will be seen that the problems under consideration involve ODE's with regular singular points, it is not quite clear how to consistently modify the Lagrangian methods in such scenarios. It would certainly be problematic numerically as well, and these issues will be discussed later, after the relevant quantities in question have been obtained.

3.1.1 Interaction Coefficients

In this section, the interaction coefficients will be derived and simplified for two types of inviscid, continuous shear flows: continuously stratified flows between flat plates, and constant density flows with a free surface. Actually, later in the thesis general, layered, interfacial flows with continuously varying velocity and density profiles will be considered, but this will draw on techniques developed here. Furthermore, the analysis will here be confined to two dimensions; an appendix will contain the analysis for three dimensional flows. As a further simplification, for the stratified flows to be considered here the Boussinesq approximation will be made for simplicity. This is not necessary, and once again generalizations may be found in appendix B.

So consider $O(\varepsilon)$ perturbations to a two-dimensional fluid with unperturbed velocity in the $x$ direction given by $(\bar{u}(y), 0)$, basic (scaled) density $\bar{\rho}(y)$, and between solid boundaries at $y = -1$ and $1$. By scaled density, it is meant that in the Boussinesq approximation, the basic density is taken to be

$$\rho_0 (1 + \sigma \bar{\rho})$$

where $\sigma$ is considered extremely small; negligible throughout, in fact, except in the inertial terms. In other words,

$$\sigma << 1, \varepsilon; \; g\sigma = O(1)$$

The equations of motion, the Euler equations, become
\[ u_x + v_y = 0 \]
\[ u_t + \overline{u}u_x + \overline{v}v + \varepsilon(uu_x + vu_y) = -p_x \quad (3.1) \]
\[ v_t + \overline{u}v_x + \varepsilon(uv_x + vv_y) = -p_y - g\rho \]
\[ \rho_t + \overline{u}\rho_x + \overline{v}\rho_y + \varepsilon(u\rho_x + v\rho_y) = 0 \]

where \((u, v)\) are respectively the velocity perturbations in the horizontal \((x)\) and vertical \((y)\) directions, and \(\rho\) is the scaled density perturbation. Each of the dependent variables may now be expanded in a perturbation series as before. This perturbation problem can be reduced at each stage to a single equation for the vertical velocity component. For example, if solutions of the form

\[ v_0 = S \phi(y) e^{i(kz-\omega t)} + * \]

are sought, then the celebrated Taylor-Goldstein equation is obtained:

\[ L_{TG}\phi \equiv \phi'' - k^2 \phi - \frac{k\overline{\omega}'}{\omega} \phi + \frac{k^2 JN^2(y)\phi}{\omega^2} = 0 \quad (3.2) \]

where \(\overline{\omega} = k\overline{\omega} - \omega\) is the negative local Doppler-shifted frequency and \(JN^2(y) = -g\overline{\rho}'(y)\) is the square of the Brunt-Väisälä frequency. In addition, since for an inviscid fluid there can be no motion normal to any solid boundaries, the boundary conditions

\[ \phi(-1) = 0, \; \phi(1) = 0 \]

are imposed. This eigenvalue problem determines the dispersion relation \(\omega = \omega(k)\). At the next order, there is again a forced linear problem as before, only now in the presence of triads a boundary value problem results, and some sort of solvability condition must be imposed.

To be specific, if to leading order

\[ v_0 = S_1\phi_1(y)e^{i(k_1z-\omega_1t)} + S_2\phi_2(y)e^{i(k_2z-\omega_2t)} + S_3\phi_3(y)e^{i(k_3z-\omega_3t)} + * \quad (3.3) \]

where the above triad is resonant in the so-called symmetric fashion discussed in the introduction, then at the next order, \(v_1\) will have a term like

\[ iS_2^* S_3^* \psi(y)e^{i(k_1z-\omega_1t)} \]

where \(\psi\) satisfies

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\[ L_{TG}\psi_1 = k_1 S_1 T \phi_1 \left( \frac{u''}{\omega_1^2} - 2Jk_1 \frac{N^2}{\omega_1^3} \right) - \]
\[ \frac{k_1}{\omega_1} \left( -\frac{\phi_2}{k_2} (\phi_3'' - k_3^2 \phi_3) - \frac{\phi_3}{k_3} (\phi_2'' - k_2^2 \phi_2) \right) + \]
\[ \frac{\phi_2}{k_3} (\phi_3'' - k_3^2 \phi_3)' + \frac{\phi_3}{k_2} (\phi_2'' - k_2^2 \phi_2)' \] -
\[ \frac{k_2}{\omega_2} \left( \frac{N^2 \phi_2 \phi_3 k_3}{k_2 \omega_2} + \frac{N^2 \phi_3 \phi_2 k_2}{k_3 \omega_3} \right) - \]
\[ \phi_2 \left( \frac{N^2 \phi_3}{\omega_3} \right)' - \phi_3 \left( \frac{N^2 \phi_2}{\omega_2} \right)' S_2^* S_3^* \] (3.4)

or more compactly

\[ L_{TG}\psi_1 = d_1 S_1 T + N_1 S_2^* S_3^* \]

It still must be that \( \psi_1 \) vanishes at the boundaries, and since the homogeneous part of the above differential equation has a solution which satisfies the boundary conditions (namely, \( \phi_1 \)), the Fredholm alternative (formally) gives conditions under which a solution exists. Since the Taylor-Goldstein equation is (formally) self-adjoint, each side of equation (3.4) is multiplied by \( \phi_1 \), and integrated from \( y = -1 \) to \( y = 1 \). With the appropriate integration by parts, it is found that

\[ \int_{-1}^{1} \phi_1 L_{TG}\psi_1 \, dy = 0 \]
so it must be that

\[ \gamma_1 S_1 T = -\Gamma_1 S_2^* S_3^* \] (3.5)

where

\[ \gamma_1 = \int_{-1}^{1} \phi_1 d_1 \, dy \]
\[ \Gamma_1 = \int_{-1}^{1} \phi_1 N_1 \, dy \]

Similar formulas hold for the other two equations in the triad set. Indeed, if the symmetric description of triads is used:

\[ \theta_1 + \theta_2 + \theta_3 = 0, \]
where the notation $\theta = kx - \omega t$ will be used throughout, then any two of the interaction coefficients $\Gamma_j/\gamma_j$ may be obtained from the other by a cyclic permutation of the indices. Disguised by the above notation is the fact that these coefficients are given by extremely complicated integrals, so they may not appear to be of much use. However, influenced by the work of Becker and Grimshaw (1993), the transformation

$$\eta = \frac{\phi}{\bar{\omega}},$$

(3.6)

the leading order local particle displacement eigenfunction, proves useful (not necessary here, but crucial in three dimensional extensions). This quantity may be thought of as $i$ times the leading order solution to the following relation:

$$\eta_t + u \eta_x = v$$

at $y = \epsilon \eta$, which certainly has familiar physical meaning for a free surface, but may be interpreted analogously for streamlines in a stratified flow; see Yih (1977). With this, and some judicious integrations by parts (to be described in appendix B), it can be shown that

$$\Gamma_j = k_j^2 \Gamma$$

where

$$\Gamma = \int_{-1}^{1} \left[ -\bar{u}' \left( \frac{\eta_1' \eta_2 \eta_3' (k_1 \bar{\omega}_2 + k_2 \bar{\omega}_1)}{k_1 k_2} \right) + \frac{\eta_1' \eta_2 \eta_3 (k_1 \bar{\omega}_3 + k_3 \bar{\omega}_1)}{k_1 k_3} + \frac{\eta_1 \eta_2' \eta_3' (k_3 \bar{\omega}_2 + k_2 \bar{\omega}_3)}{k_2 k_3} + J N^2 \left( \frac{\eta_1' \eta_2 \eta_3 \bar{\omega}_1 (k_3 \bar{\omega}_2 + k_2 \bar{\omega}_3)}{k_1 \bar{\omega}_2 \bar{\omega}_3} \right) + \frac{\eta_1 \eta_2' \eta_3 \bar{\omega}_2 (k_3 \bar{\omega}_1 + k_1 \bar{\omega}_3)}{k_2 \bar{\omega}_1 \bar{\omega}_3} + \frac{\eta_1 \eta_2 \eta_3' \bar{\omega}_3 (k_1 \bar{\omega}_2 + k_2 \bar{\omega}_1)}{k_3 \bar{\omega}_2 \bar{\omega}_1} \right] dy$$

(3.7)

and

$$\gamma_j = -2 \int_{-1}^{1} (k_j \bar{u} - \omega_j) \left( \eta_j'^2 + k_j^2 \eta_j^2 \right) dy$$

(3.8)

so the determination of the signs of the interaction coefficients, which determines whether the triads are explosive or not, comes down to evaluating the quantity $\gamma_j$ for each wave in

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the triad set. This is essentially one of the conclusions of Becker and Grimshaw (1993),
but it should be noted the full interaction coefficients given above are considerably simpler
than the ones obtained there, when the simplification to a Boussinesq fluid is made.

The above analysis will now be made for flows with an interface. Consider a flow of
finite or infinite expanse, with an interface at \( y = \varepsilon \eta \). The basic (undisturbed) velocity is
\( \overline{u}_1(y) \) for \( y > 0 \) and \( \overline{u}_2(y) \) for \( y < 0 \). In addition, the density will be assumed constant in
each region, \( \rho_1 \) for \( y > 0 \) and \( \rho_2 \) for \( y < 0 \). This scenario may obviously be specialized to
wind-driven shear flows (\( \overline{u}_2 = 0 \)) or free-surface flows (\( \overline{u}_1 = 0 \)). The equations of motion
are as before the Euler equations in each region, and the equations governing perturbations
to these basic profiles are:

\[
\begin{align*}
{u}_{jz} + {v}_{jy} &= 0 \\
{u}_{jt} + \overline{u}_j{u}_{jx} + \overline{u}_j'{v}_j + \varepsilon(\overline{u}_j{u}_{jx} + {v}_j{v}_{jy}) &= \frac{-p_{jx}}{\rho_j} \\
{v}_{jt} + \overline{u}_j{v}_{jx} + \varepsilon(\overline{u}_j{v}_{jx} + {v}_j{v}_{jy}) &= \frac{-p_{jy}}{\rho_j}
\end{align*}
\] (3.9)

where the subscripts denote region 1 \((j = 1, y > \varepsilon \eta)\) or region 2 \((j = 2, y < \varepsilon \eta)\). There are
the boundary conditions that \( v_1(1) = 0 \) and \( v_2(-1) = 0 \) for solid walls at \( \pm 1 \) or if the flow
is unbounded, \( v_1 \to 0 \) as \( y \to \infty \) and \( v_2 \to 0 \) as \( y \to -\infty \). In addition, at the interface
a kinematic and dynamic “boundary” condition must be imposed. This amounts to the
requirement that the particle displacement be continuous above and below the interface,
and that the pressure be continuous across the interface as well:

\[
\begin{align*}
\eta_t + \overline{u}_j \eta_x + \varepsilon u_j \eta_z &= v_j \\
-\rho_1 g \eta + p_1 &= -\rho_2 g \eta + p_2
\end{align*}
\] (3.10)

for \( j = 1, 2 \), at \( y = \varepsilon \eta \). In general, surface tension effects will be included and then the
dynamic boundary condition becomes

\[
-(\rho_1 - \rho_2) g \eta + p_1 - p_2 = \frac{T \eta_{xx}}{(1 + \varepsilon^2 \eta_z^2)^{3/2}}
\] (3.11)

at \( y = \varepsilon \eta \). Once again, a perturbation series solution may be sought, and at each stage
a single equation may be obtained for the vertical velocity component \( v_j \). The governing
ODE’s are not coupled, and if solutions of the following sort are taken:

\[
v_{j0} = S_j \phi_j(y) e^{i(kz - \omega t)} + *
\]

then the eigenfunctions \( \phi_j \) each satisfy the well-known Rayleigh equation:

\[
L_R \phi_j \equiv \phi_j'' - k^2 \phi_j - \frac{k \overline{u}_j'' \phi_j}{\overline{\omega}_j} = 0
\] (3.12)
where again $\tilde{\omega}_j = k \tilde{u}_j - \omega$. The appropriate boundary conditions at, say, $\pm \infty$ (the case to be focused on here) are taken, and the coupling occurs through the interfacial conditions:

$$i \eta = \frac{S_1 \phi_1}{\tilde{\omega}_1} = \frac{S_2 \phi_2}{\tilde{\omega}_2},$$

$$-g \rho_1 \phi_1 S_1 + g \rho_2 \phi_2 S_2 + \frac{\rho_1}{k^2} (\tilde{\omega}_1 \phi'_1 - k \tilde{u}_1 \phi_1) S_1 - \frac{\rho_2}{k^2} (\tilde{\omega}_2 \phi'_2 - k \tilde{u}_2 \phi_2) S_2$$

$$= T_1 k^2 \frac{\phi_1}{\tilde{\omega}_1} S_1 - T_2 k^2 \frac{\phi_2}{\tilde{\omega}_2} S_2$$

(3.14)

each at $y = 0$, and where $T = T_2 - T_1$. In the above,

$$u_{j0} = \frac{i}{k} S_j \phi'_j e^{i\theta} + *$$

$$p_{j0} = \frac{-i \rho_j}{k^2} (\tilde{u}_j \phi'_j - k \tilde{u}_j \phi_j) S_j e^{i\theta} + *$$

have been used. In the above, the $S_j$'s will eventually be thought of as slowly varying functions of time, but as far as the linear eigenvalue problem goes, the above conditions (3.13), (3.14) are a set of homogeneous linear equations, whose determinant must be zero for a nontrivial solution to exist. This would define the eigenvalue or dispersion relation. For example, at some suitably large value of $y$, say $M$, the boundary conditions may without loss of generality taken to be $(\phi_1, \phi'_1) \sim (e^{-ky}, -ke^{-ky})$. The condition at $-\infty$ may be handled similarly. Based on these initial conditions, the respective Rayleigh equations are then integrated from $\pm M$ to 0, and the aforementioned determinant evaluated. Thus $\omega$ could be varied until a root is found. For the cases to be considered shortly, one of the $\tilde{u}_j$'s will be set equal to zero, in which case one of the above Rayleigh equations may be solved exactly, and a boundary condition for the other eigenfunction alone at $y = 0$ may be obtained. Hence the above formulation seems unnecessarily complicated, but it will be seen to be quite useful when generalizations are made to multi-layered flows.

In particular, for a surface-shear flow, $\tilde{u}_1 = 0$ so $\phi_1 = e^{-ky}$ and the boundary condition for $\phi_2$ can be reduced to

$$(\tilde{u}_2 - c) \phi'_2 - \tilde{u}_2 \phi_2 = \frac{g + \sigma k^2}{\tilde{u}_2 - c} \phi_2 - \rho_d \frac{g + ke^2}{\tilde{u}_2 - c} \phi_2$$

(3.15)

at $y = 0$, where $\rho_d = \rho_1/\rho_2$ is the density ratio (typically quite small for the air-water interfaces to be studied) and $\sigma = T/\rho_2$. At the lower boundary, $\phi = 0$. Similarly, for wind-driven flows, where $\tilde{u}_2 = 0$ so $\phi_2 = e^{ky}$, one would obtain

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\[(\bar{u}_1 - c) \phi_1' - \bar{u}_1 \phi_1 = -\rho_d^{-1} g + \sigma k^2 - k c^2 \phi_1 \quad (3.16)\]

at \(y = 0\). At the upper boundary, \(\phi = 0\) again. These will be returned to later, but first the interaction coefficients for the case of triads will be examined.

As mentioned, generalizations will be made later. In the way of instruction, consider the surface-shear case with \(\rho_d = 0\), so the problem simply becomes a free-surface problem. If the leading order solution is once again taken to consist of triads, a forced Rayleigh equation, whose homogeneous counterpart has a solution, results as before, but now there are additional boundary terms owing to the free surface condition. Here it is perhaps useful to introduce a perturbation stream function, as the approach will offer clues when manipulating three-dimensional and more complicated flows:

\[\psi_y = u, \quad -\psi_x = v\]

The \(O(\varepsilon)\) problem for the vertical velocity will require a term like \(\phi_{11} S_2^* S_3^* e^{i\theta_1}\) arising not only from the equations of motions but also from the boundary conditions:

\[L_R \phi_{11} = -\gamma_1 \frac{\phi''_1 - k_1^2 \phi_1}{\omega_1} + \]

\[
\frac{1}{\omega_1} (k_2 \phi_2 (\phi''_3 - k_3^2 \phi_3) + k_3 \phi_3 (\phi''_2 - k_2^2 \phi_2)) -
\]

\[k_3 \phi_2 (\phi''_3 - k_3^2 \phi_3)' - k_2 \phi_3 (\phi''_2 - k_2^2 \phi_2)' \quad (3.17)\]

with

\[
\bar{\omega}_1 \eta_{11} + k_1 \phi_{11} = 
\]

\[
(\frac{k_1 k_3 \phi'_2 \phi_3}{\bar{\omega}_3} + \frac{k_1 k_2 \phi'_3 \phi_2}{\bar{\omega}_2} - \frac{\bar{u} k_1 k_2 k_3 \phi_2 \phi_3}{\bar{\omega}_2 \bar{\omega}_3}) + \frac{\gamma_1 k_1 \phi_1}{\bar{\omega}_1} \quad (3.18)\]

and

\[p_{11} - (g \rho + T k_1^2) \eta_{11} =
\]

\[-\rho k_2 k_3 \phi_2 \phi_3 (\frac{\bar{\omega}_3}{\bar{\omega}_2} + \frac{\bar{\omega}_2}{\bar{\omega}_3}) \quad (3.19)\]

with (3.18) and (3.19) imposed at \(y = 0\). In the above, \(\eta_{11}\) and \(p_{11}\) represent respectively the second order free surface displacement and pressure eigenfunctions. It has been anticipated that, after some sort of solvability condition has been deduced, a relationship of the following form will be obtained:

\[S_1 T = i \gamma_1 S_2^* S_3^* \quad (3.20)\]
with a cyclic permutation of the indices producing the other equations in the triad set. From the equations of motion, the second order pressure perturbation can be written

\[
p_{11} = -\frac{\rho}{k_1} (\frac{\omega_1}{\omega_1} \phi_{11} - k_1 \bar{u}' \phi_{11} +
\]

\[
k_1 \phi_2 \phi_3' - k_1 k_2 k_3 \phi_2 \phi_3 - \frac{k_2 k_3 \bar{u}'' \bar{\omega}_1 \phi_2 \phi_3}{\omega_2 \omega_3} + \gamma_1 \phi_1')
\]

Using the expression for \(\eta_{11}\) and using the second form of the pressure in the dynamic boundary condition, the boundary condition on \(\phi_{11}\) can be obtained:

\[
\phi'_{11} - \left(\frac{k_1 \bar{u}'}{\omega_1} - k_1^2 \left(\frac{g + \sigma k_1^2}{\omega_1^2}\right)\right) \phi_{11} =
\]

\[
- \frac{I}{\omega_1} - \frac{k_1 H}{\omega_1 \rho} - \frac{k_1 \gamma_1}{\omega_1} \left(\frac{g + \sigma k_1^2}{\omega_1^2}\right) \phi_{11}'
\]

(3.21)

at \(y = 0\), and where

\[
G = \frac{1}{\omega_1} \left(\frac{k_1 k_3 \phi_2 \phi_3}{\omega_3} + \frac{k_1 k_2 \phi_2 \phi_3}{\omega_2} - \frac{\bar{u}' k_1 k_2 k_3 \phi_2 \phi_3}{\omega_2 \omega_3} \right) + \frac{\gamma_1 k_1 \phi_1}{\omega_1^2}
\]

\[
H = -\rho k_2 k_3 \phi_2 \phi_3 \left(\frac{\bar{\omega}_3}{\omega_2} + \frac{\bar{\omega}_2}{\omega_3}\right)
\]

\[
I = k_1 \phi_2' \phi_3' - k_2 k_3 \phi_2 \phi_3 \left(k_1 + \frac{\bar{u}'' \bar{\omega}_1}{\omega_2 \omega_3}\right)
\]

So the solvability condition that determines \(\gamma_1\) involves not just an inhomogeneous ODE, but inhomogeneous boundary conditions as well. Of course, \(\phi_{11} = 0\) at the other boundary, be it at say \(-1\) or \(-\infty\). The general type of problem here, then, can be written as

\[
L \phi = f
\]  
\(\alpha \phi' + \beta \phi = g\)  

(3.22)  
(3.23)

at \(y = 0\) and \(\phi = 0\) at \(y = -\infty\). If there is a solution to the problem

\[
L \psi = 0
\]  
\(\alpha \psi' + \beta \psi = 0\)

(3.24)  
(3.25)

at \(y = 0\) and \(\psi = 0\) at \(y = -\infty\), then by multiplying (3.22) by \(\psi\), integrating by parts, and using (3.23-3.25), the following solvability condition is obtained:

\[
\int_{-\infty}^{0} \psi f \, dy = \frac{\psi(0) g}{\alpha}
\]

(3.26)
Note that this boundary value problem is self-adjoint. \( \gamma_1 \) may now be determined. The equation for \( \phi_{11} \) is multiplied by \( \phi_1 \), the Rayleigh equation is used to eliminate second derivatives of any eigenfunctions in the forcing term, and the last two terms in the nonlinear contribution to the forcing are integrated by parts. The resulting boundary terms do not vanish, as they did in the previous stratified case. These terms can be simplified to

\[
\frac{k_2 k_3 \phi_1 \phi_2 \phi_3 \bar{u}''}{\bar{\omega}_2 \bar{\omega}_3},
\]
evaluated at \( y = 0 \), and those arising from the solvability condition not proportional to \( \gamma_1 \) are

\[
K \phi_1 \phi_2 \phi_3 \left( \frac{\bar{u}''}{k_1 \bar{\omega}_1 \bar{\omega}_2 \bar{\omega}_3} - \frac{\bar{u}'}{W} - \left( \frac{1}{\bar{\omega}_1} + \frac{1}{\bar{\omega}_2} + \frac{1}{\bar{\omega}_3} \right) + \right.
\]
\[
- \frac{g \bar{u}'}{W} \left( \frac{k_1}{\bar{\omega}_1} + \frac{k_2}{\bar{\omega}_2} + \frac{k_3}{\bar{\omega}_3} \right) - \frac{\sigma g K}{W^2} \left( \frac{k_1^2 + k_3^2}{k_2} \frac{\bar{\omega}_2}{k_2} + \frac{k_1^2 + k_2^2}{k_3} \frac{\bar{\omega}_3}{k_3} + \frac{k_2^2 + k_3^2}{k_1} \frac{\bar{\omega}_1}{k_1} \right) - \frac{\sigma^2 K^3}{W^2} \left( \frac{\bar{\omega}_2}{k_2^3} + \frac{\bar{\omega}_3}{k_3^3} + \frac{\bar{\omega}_1}{k_1^3} \right) \tag{3.27}
\]

where \( W = \bar{\omega}_1 \bar{\omega}_2 \bar{\omega}_3 \) and \( K = k_1 k_2 k_3 \), and the above expression is evaluated at \( y = 0 \). After the term arising from the integration by parts is canceled, the above quantity is easily checked to be the same for each wave in the set; ie, the other interaction coefficients are obtained by a cyclic permutation of indices for one coefficient, and this operation leaves this quantity unchanged. As before, the integrated expression is invariant, and almost identical to that obtained in the stratified case (with \( J = 0 \) of course) so the interaction coefficients \( \gamma_j \) may once again be obtained from an expression of the form

\[
\gamma_j I_j = \Gamma
\]

for \( j = 1, 2, 3 \). In this case,

\[
I_j = (\phi_j^2 \frac{k_j^2}{\bar{\omega}_j^2} (g + \sigma k_j^2) + \frac{\phi_j \phi_j'}{\bar{\omega}_j}) \bigg|_{y=0}^{y=0} - \int_{-\infty}^{0} \frac{\bar{u}'' k_j \phi_j^2}{\bar{\omega}_j^2} dy \tag{3.28}
\]

Similarly to before, if the transformation \( \zeta = k \phi / \bar{\omega} \) is made, this quantity becomes

\[
\frac{2}{k_j^2} \int_{-\infty}^{0} \bar{\omega}_j (\zeta_j'^2 + k_j^2 \zeta_j^2) dy \tag{3.29}
\]

and the previous conclusions hold true in this case as well; namely, that the signs of the interaction coefficients for a symmetric resonant triad depend only on a single quantity
which involves only one of the waves for each coefficient. A similar result holds true for the
wind-driven shear as well, and as will be seen, for more general, layered profiles. The point
of this analysis is not to demonstrate any particular algebraic prowess. Nor is it to prove
that analyzing the problem from the Eulerian viewpoint is superior to the Lagrangian
viewpoint; in either case the interaction coefficients are complicated expressions. Rather,
the point is to demonstrate that the same conclusions may be obtained through the more
familiar Eulerian framework, and furthermore, it will be seen in a later section how these
results may be generalized to more complex flows. Without these initial forays into the
problem, such an analysis might not have been attempted due to its apparently formidable
nature. Moreover, it should be noted that any of the above integrals are potentially
quite singular, if the phase speed \( c = \omega/k \) falls within the range of \( \overline{u} \). Such cases will
prove important, and it will prove more convenient to analyze this situation from an
Eulerian approach. Indeed, it is not even clear how to modify the Lagrangian approach to
accomodate this case.

With these results, several conclusions may be drawn regarding the determination of
explosive triads. These will now be discussed.

### 3.1.2 Relation to the Dispersion Curve

Recall that the quantity which determines the sign of the interaction coefficients in
the case of a stratified flow considered above is, from (3.8),

\[
\gamma_j = -2 \int_{-1}^{1} (k_j \overline{u} - \omega_j) (\eta_j'^2 + k_j^2 \eta_j^2) \, dy
\]

where \( \eta = \phi/\overline{\omega} \). Note that the Taylor-Goldstein equation, written in terms of \( \eta \), becomes

\[
(\overline{\omega} \eta')' + k^2 (J N^2 - \overline{\omega}^2) \eta = 0 \quad (3.30)
\]

so a dispersion relation may be (formally) defined by

\[
D(k, \omega) \equiv \int_{-1}^{1} (\overline{\omega}^2 (\eta'^2 + k^2 \eta^2) - k^2 J N^2 \eta^2) \, dy = 0 \quad (3.31)
\]

This is derived by multiplying (3.30) by \( \eta \) and integrating by parts; if \( \eta \) and \( \omega \) are
a valid eigenfunction and eigenvalue, the boundary terms vanish and (3.31) is obtained.
This is also obtained by Becker and Grimshaw via a Lagrangian formulation, although
their results are expressed in terms of the local Doppler-shifted frequency; ie, \( \overline{\omega} \rightarrow -\overline{\omega} \)
here. Their equivalent contention, however, that

\[
\gamma = \frac{\partial D}{\partial \omega}
\]

is not quite true. Extra analysis is required since \( \eta \) is itself a function of \( \omega \), as well as \( k \).
It can be seen that (formally)
\[
\frac{\partial D}{\partial \omega} = -2 \int_{-1}^{1} \bar{\omega} (\eta'^2 + k^2 \eta^2) \, dy + \\
2 \int_{-1}^{1} (\bar{\omega} \eta' \eta'_\omega + k^2 (\bar{\omega}^2 - J N^2) \eta \eta'_\omega) \, dy
\]

(3.32)

However, from the governing equation for \( \eta \),
\[
\bar{\omega}^2 \eta'' + 2 k \bar{\omega} \eta'_\omega + k^2 (J N^2 - \bar{\omega}^2) \eta \eta'_\omega = 2 ((\bar{\omega} \eta')' - k^2 \bar{\omega} \eta)
\]

(3.33)

so that upon multiplying (3.33) by \( \eta \), integrating by parts, and using the fact that \( \eta(\pm 1) = 0 \) implies \( \eta'_\omega(\pm 1) = 0 \), it is found that
\[
\int_{-1}^{1} (\bar{\omega}^2 \eta' \eta'_\omega + k^2 (J N^2) \eta \eta'_\omega) \, dy
\]
\[
= 2 \int_{-1}^{1} \bar{\omega} (\eta'^2 + k^2 \eta^2) \, dy
\]

(3.34)

so that
\[
\frac{\partial D}{\partial \omega} = 2 \int_{-1}^{1} \bar{\omega} (\eta'^2 + k^2 \eta^2) \, dy
\]

(3.35)

or negative of what would have been obtained from formally differentiating with respect to \( \omega \), as appears to have been done in Becker and Grimshaw (1993). In other words,
\[
\gamma = -\frac{\partial D}{\partial \omega}
\]

(3.36)

Also, an alternative dispersion relation could be defined by directly using the TG equation:
\[
D(k, \omega) \equiv \int_{-1}^{1} (\phi'^2 + k^2 \phi^2 + \left(\frac{k u''}{\bar{\omega}} - \frac{k^2 J N^2}{\bar{\omega}^2}\right) \phi^2) \, dy = 0
\]

If the above analysis is repeated, the same result is obtained; namely, that an actual \( \omega \) derivative is the opposite sign from what a formal differentiation gives, ignoring the dependence of \( \phi \) on \( \omega \):
\[
\frac{\partial D}{\partial \omega} = -\int_{-1}^{1} \left(\frac{k u''}{\bar{\omega}^2} - \frac{2 k^2 J N^2}{\bar{\omega}^3}\right) \phi^2 \, dy
\]
Similar results hold true for $\frac{\partial D}{\partial k}$ as well. For the free surface flow also analyzed in the preceding section, if the following dispersion relation is defined:

$$D(k, \omega) \equiv \int_{-\infty}^{0} \bar{\omega}^2 (\eta'^2 + k^2 \eta^2) dy - \bar{\omega}^2 \eta \eta'\bigg|_{y=0} = 0,$$  \tag{3.37}

then

$$\frac{\partial D}{\partial \omega} = -2 \int_{-\infty}^{0} \bar{\omega}(\eta'^2 + k^2 \eta^2) dy +$$

$$2 \int_{-\infty}^{0} \bar{\omega}^2 (\eta' \eta'_\omega + k^2 \eta \eta'_\omega) dy + 2 \bar{\omega} \eta \eta'\bigg|_{y=0} - \bar{\omega}^2 (\eta \eta'_\omega + \eta' \eta_\omega)\bigg|_{y=0}$$  \tag{3.38}

As before, by (formally) differentiating the transformed Rayleigh equation with respect to $\omega$, it is found that

$$\bar{\omega}^2 \eta''_\omega + 2 k \bar{u}' \bar{\omega} \eta'_\omega - k^2 \bar{\omega}^2 \eta_\omega = 2((\bar{\omega} \eta')' - k^2 \bar{\omega} \eta)$$  \tag{3.39}

The boundary condition

$$\eta' = \frac{k^2 (g + \sigma k^2)}{\bar{\omega}^2} \eta$$  \tag{3.40}

at $y = 0$ implies

$$\eta'_\omega = \frac{k^2 (g + \sigma k^2)}{\bar{\omega}^2} \eta_\omega + 2 \frac{k^2 (g + \sigma k^2)}{\bar{\omega}^3} \eta$$  \tag{3.41}

at $y = 0$. The previous approach now yields

$$\int_{-\infty}^{0} \bar{\omega}^2 (\eta' \eta'_\omega + k^2 \eta \eta_\omega) dy =$$

$$= 2 \int_{-\infty}^{0} \bar{\omega}(\eta'^2 + k^2 \eta^2) dy + k^2 (g + \sigma^2 k^2) \eta \eta_\omega\bigg|_{y=0}$$  \tag{3.42}

so that once again

$$\frac{\partial D}{\partial \omega} = 2 \int_{-1}^{1} \bar{\omega}(\eta'^2 + k^2 \eta^2) dy$$  \tag{3.43}

and then in this case

$$\gamma = \frac{1}{k^2} \frac{\partial D}{\partial \omega}$$  \tag{3.44}
the main point being that the sign is the opposite of what might have been expected. These are not important corrections, as they do not affect the following conclusions. For from

$$\frac{\partial D}{\partial \omega} \omega'(k) + \frac{\partial D}{\partial k} = 0$$

(3.45)

it can be seen that $\gamma$ will change sign along the dispersion curve at a point with vertical slope, or infinite group velocity, as long as $\frac{\partial D}{\partial k}$ is nonzero there. Actually, the manner in which the group velocity $C_g$ becomes infinite is important. Essentially, the dispersion relation must be “parabolic” near this point. For if the point where $D_\omega = 0$ is denoted by $(k^*, \omega^*)$, then locally the dispersion relation may be described by

$$\frac{1}{2} D_{\omega \omega} \delta \omega^2 + D_{k}^* \delta k = 0$$

(3.46)

where asterisks (*) signify evaluation at the point $(k^*, \omega^*)$. For $D_\omega$ (and hence $\gamma$) to change sign across this point, $D_{\omega \omega}^*$ must be nonzero, and so locally the dispersion curve is a parabola. If $D_{\omega \omega} = 0^*$ then the vertical structure is say, like that of a cube root function and $\gamma$ would not change sign. If $D_{k}^* = 0$, the local behavior would be obtained from retaining all the quadratic terms in (3.46); solving this equation would give two roots, the local slope of two branches of the dispersion relation. This point can be thought of as a bifurcation point, and so following Ioss and Joseph (1981), such points where $\gamma$ changes sign can be classified. At any rate, the dispersion curve can be divided into segments with the same sign of $\gamma$, which will be helpful in determining whether explosive triads exist.

It needs first to be reviewed how the existence of triads, explosive or not, may be ascertained graphically. Consider Figures 3.1 and 3.2, which contain plots of

$$\omega^2 = k + \frac{k^3}{2}$$

(3.47)

which may be thought of as the dispersion relation for $k > 0$ of waves on the surface of deep water, when the effects of surface tension are included. By redrawing the dispersion curve with it’s origin on some point on the dispersion curve, a point of intersection with the original curve may or may not be found. If such a point exists, then clearly the wavenumber/frequency vector $\theta = (k, \omega)$ satisfies

$$\theta_1 + \theta_2 = \theta_3$$

The previous analysis has been done with the “symmetric” notation for triads, namely taking a triad to be a set of waves whose wave numbers and frequencies satisfy

$$k_1 + k_2 + k_3 = 0$$

$$\omega_1 + \omega_2 + \omega_3 = 0$$

(3.48)
Figure 3.1: Typical dispersion relation.

Figure 3.2: Graphical construct for determining the existence of triads.
Graphically, however, it is taken to mean

\[ k_1 + k_2 = k_3 \]
\[ \omega_1 + \omega_2 = \omega_3 \]

This distinction should be kept in mind when using the dispersion curve to detect explosive triads. Since typically \( \omega(-k) = -\omega(k) \), the symmetric and non-symmetric cases may be related by the transformation \( k_3 \rightarrow -k_3, \omega_3 \rightarrow -\omega_3 \).

Recall from (3.8) the definition of \( \gamma \), the quantity which determines the signs of the interaction coefficients, in the case of a stratified shear flow:

\[
\gamma = -2 \int_{-1}^{1} (k \bar{u} - \omega)(\eta^2 + k^2 \eta^2) \, dy
\]

Adopting the convention that waves will be described by positive wave numbers (not the way the formulas of the preceding section were derived), it is seen that \( \gamma \) will be positive for any waves whose phase speed exceeds the maximum value \( \bar{u}_{max} \) of the basic shear, and negative for any waves whose phase speed is less than the minimum value \( \bar{u}_{min} \) of the basic shear. For waves whose phase speed lies within the range of \( \bar{u} \), the sign of \( \gamma \) is indeterminate without an actual calculation. There are other difficulties in this case which will need to be addressed, but for now such waves will be called critical layer modes, and those waves with phase speeds outside the range of \( \bar{u} \) will be termed regular modes. The reason for this terminology will be clear shortly.

Graphically, then, the scenario of Figure 3.3 would result: By the symmetric approach \( \gamma \) needs to be the same sign for each wave for explosive triads to be present, so the graphic approach would imply that an explosive interaction would have the wave of largest wavenumber with a \( \gamma \) of opposite sign from the other two waves, since the substitution \( k_3 \rightarrow -k_3, \omega_3 \rightarrow -\omega_3 \) describes the triad symmetrically and has the effect of changing the sign of \( \gamma \).

It would be interesting to know if explosive triads exist in an interaction involving only regular modes. The answer is no, and may be seen from Figure 3.4.

Any regular mode will lie outside the sector bounded by the lines \( \omega = \bar{u}_{max} k \) and \( \omega = \bar{u}_{min} k \). Call this the basic sector for the set of regular modes. In graphically determining triads, then, the basic sectors may be redrawn along with the dispersion curves. Since regular modes must exist outside of these sectors, there is no way, say a positive \( \gamma \) curve could be redrawn off of another positive \( \gamma \) curve and intersect a negative \( \gamma \) curve. Any such second curve would have to lie above the upper edge of it’s basic sector, and also the lower edge as well. But this lower edge is parallel to the lower edge of the basic sector of the original dispersion curve; hence the second curve drawn cannot intersect a negative \( \gamma \) branch of the original dispersion curve; it can only (potentially) intersect another positive \( \gamma \) branch, and so the resulting triad will not be explosive. Also, there is no way for this second positive \( \gamma \) branch to intersect a negative \( \gamma \) critical layer branch either.
Figure 3.3: Sectors separating regular modes from critical layer modes, and the sign of $\gamma$ for particular waves.

Figure 3.4: As regular modes can never cross these sectors, an explosive interaction must consist of at least two critical layer modes. See the text.
Therefore, it must be that at least two waves participating in this interaction which have phase speeds within the range of $\bar{u}$; i.e., at least two of the waves must be critical layer modes. By the previous statements, such an interaction would have to fit one of these patterns:

$$CL(\pm) + R(\pm) = CL(\mp)$$
$$CL(\pm) + CL(\pm) = CL(\mp)$$

where the notation refers to either critical layer (CL) or regular (R) modes, and the sign of $\gamma$ for that wave ($\pm$). But if this occurs, an inspection of the Taylor-Goldstein equation (or it's better known homogeneous counterpart, the Rayleigh equation) reveals a differential equation with a regular singular point; namely, where

$$\bar{u}(y_c) = \frac{\omega}{k} = c$$

The part of the fluid where $y = y_c$ is known as a critical layer (hence the name above), and as they will be important to this thesis, a brief digression to explain what they mean is appropriate.

### 3.1.3 Relevance of Critical Layers

When the linear stability equations of hydrodynamic stability theory have singular points, a Frobenius series solution will consist of two linearly independent solutions which behave locally near the critical layer like

$$\phi_a(y) = (y - y_c)^{\alpha_1} (1 + a_1 (y - y_c) + \cdots)$$
$$\phi_b(y) = (y - y_c)^{\alpha_2} (1 + a_2 (y - y_c) + \cdots)$$

for a stratified flow, where $\alpha_{1,2}$ satisfy

$$\alpha (\alpha - 1) + J_c = 0$$

and

$$J_c = \frac{J N_c^2}{\bar{u}_c^2}$$

is known as the critical Richardson number. The subscript $c$ will denote evaluation at $y = y_c$. For a nonstratified flow, there are the perhaps more familiar solutions

$$\phi_a = (y - y_c) + a_1(y - y_c)^2 + \cdots$$
$$\phi_b = 1 + b_1(y - y_c)^2 + \cdots + \frac{\bar{u}_c''}{\bar{u}_c} \phi_a \log(y - y_c)$$

(3.51)
In either case, there is a branch point singularity which requires interpretation, and, as the velocity in the $x$ direction involves the derivatives with respect to $y$ of the above, there will be an infinite velocity at the critical layer. Clearly, these solutions are not physically valid near the critical layer, and effects that were neglected in their derivation must be appealed to in order to resolve this difficulty.

One candidate is viscosity. Lin (1955) did the pioneering work here (and also anticipated some of the results for a nonlinear critical layer) but the definitive reference for the results and ideas involved is Drazin and Reid (1981). The main result is that for $y < y_c$, $(y - y_c)$ must be interpreted as

$$e^{-i\pi \text{sgn}(\bar{u}_c)}(y_c - y)$$

There is said to be a phase shift of $-\text{sgn}(\bar{u}_c)\pi$ across the critical layer in this case. The other choice is nonlinearity, though it must be added that viscosity cannot be completely neglected in this case either. Here the ground-breaking work was done by Benney and Bergeron (1969), and later extended by Maslowe and Kelley (1971), Haberman (1972), and Benney and Maslowe (1975).

The basic idea is to re-examine the full equations of motion in the vicinity of the critical layer, specifically when

$$y - y_c = O(\varepsilon^{\frac{1}{4}})$$

where $\varepsilon$ is a measure of the nonlinearity. The ideas of boundary layer theory are then applied. The leading order "outer" solution is taken to be some linear combinations (possibly different) of the Frobenius solutions above and below the critical layer. The above scaling in $y$ reflects the way in which this outer series becomes disordered. The "inner" variable

$$Y = \frac{y - y_2}{\varepsilon^{1/2}}$$

is introduced, and the inner solution (which is strongly nonlinear) near $y = y_c$ is then matched to the outer solution. It is found in this case that there is no phase shift; ie, $\log(y - y_c)$ would be interpreted as

$$\log |y - y_c|$$

With the logarithm treated this way, the inner solution tends to the same linear combination of the Frobenius solutions as $Y \rightarrow \pm \infty$.

This approach leads to a new class of waves. They are found in an inverse fashion, by specifying the phase speed (or equivalently, the location of the critical layer) and then finding those wave numbers which not only satisfy the boundary conditions, but also give the proper behavior near $y = y_c$. For example, for a given wave speed, and hence a known location of the critical layer, the Rayleigh equation may be integrated from points sufficiently close to the critical point on either side of it to whatever boundaries are being

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considered. The "initial" conditions are derived from the same linear combination of the Frobenius solutions above and below the critical layer, with the logarithm interpreted appropriately. Since the Rayleigh equation is linear, the results of an integration using a linear combination of some initial conditions will be the same linear combination obtained from an integration based on these initial conditions taken separately. Hence demanding the two boundary conditions be satisfied amounts to solving a homogeneous linear system, whose determinant will vanish only for certain values of the wavenumber. In this manner, the dispersion relation for nonlinear critical layer modes may be obtained. Furthermore, the same results hold true in the stratified case as well; there is no phase shift for a nonlinear critical layer mode, and the dispersion relation may be determined similarly. With these notions explained, further difficulties due to critical layers will now be discussed.

3.1.4 Dilemma of Singular Integrals

It is clear from the previous sections that critical layers will play a central role in this thesis. Their interpretation differs depending on whether they are taken to be of the viscous or nonlinear sort. Nonlinearity will be considered the dominant effect here. As described in the preceding section, these critical layer modes may be calculated, and so at least this part of the problem may be constructed. However, in determining the sign of $\gamma$, there are additional problems. An inspection of any of the integrals needed to determine the interaction coefficients reveals that they are potentially divergent, and that they may not exist even in a principal value sense. Therefore, a formal application of the Fredholm alternative may be invalid, so the previous approach of monitoring a single quantity along the dispersion curve may no longer hold true. However, it turns out that with the proper modification, these results hold true, but some of these modifications are major, at least in the case of a continuously stratified fluid. First, these integrals will be examined to verify their singularity.

Many of the integrals in question were expressed in terms of the particle displacement eigenfunction $\eta = \phi / \bar{\omega}$. If for ease of notation $y_c$ is taken to zero, by considering the Frobenius series solutions (3.50) and (3.51) this function can be shown to have the following behavior as $y \to y_c \pm$:

$$
\eta \sim \pm (c_1 |y|^{\alpha_1 - 1} + c_2 |y|^{\alpha_1} + d_1 |y|^{\alpha_2 - 1} + d_2 |y|^{\alpha_2} + \cdots)
$$

$$
\eta' \sim (\alpha_1 - 1) c_1 |y|^{\alpha_1 - 2} + (\alpha_2 - 1) d_1 |y|^{\alpha_2 - 2} + \alpha_1 c_2 |y|^{\alpha_1 - 1} + \alpha_2 d_2 |y|^{\alpha_2 - 1} + \cdots
$$

for a stratified flow, and

$$
\eta \sim \frac{c_1}{y} + c_2 \log |y| + c_3 y \log |y| + \cdots,
$$

$$
\eta' \sim -\frac{c_1}{y^2} + \frac{c_2}{y} + c_3 \log |y| + \cdots
$$

for a homogeneous flow. In the above, the $c_j$ and $d_j$ are some constants, and the $+$ is taken for $y > 0$ and the $-$ for $y < 0$. The integrand in the expression for $\Gamma$ in (3.7)
will have singular behavior near y = 0 arising from a linear combination of the terms \((\eta', \eta' \tilde{\omega}, \eta)\), and \(J(\eta', \eta, \eta / \tilde{\omega})\). Of course, as there must be at least two critical layers participating in the kinds of interactions of interest here, there would be another point where the integrand becomes singular, but the same problems will occur there so this need not be considered separately. For the stratified case, note that for \(\alpha_j\) real, \(0 < \alpha_{1,2} < 1\) and these Frobenius exponents are symmetrically located about 1/2; otherwise \(\alpha_j\) has a real part of 1/2. Then while \(|y|^{\alpha_j-1}\) is certainly integrable, the function \(|y|^{\alpha_j-2}\) is not integrable even in the principal value (PV) sense owing to its evenness. For the homogeneous case, the term \(1/y^2\) is not PV-integrable either. For the important quantity \(\gamma\), it proves simpler to analyze the integrals written in terms of the eigenfunctions \(\phi\). In this case, the singularities behave like \(\phi^2/\tilde{\omega}^2\) or \(\eta^2\) and \(J\phi^2/\tilde{\omega}^3\) or \(J\eta^2/\tilde{\omega}\). The homogeneous case again has the non-integrable term \(1/y^2\) and the heterogeneous case has terms like \(\pm |y|^{2\alpha_j-3}\) which are PV-integrable. However, it will be seen that in the way in which such integrals must be interpreted, such terms would not arise in a PV-integrable fashion. On the other hand, \(|y|^{2(\alpha_1-1)}\), if \(\alpha_1\) is taken to be the smaller of the two \(\alpha\)'s, is not PV-integrable since it is even. Therefore, the calculation of the interaction coefficients from these integrals is not possible, even when the somewhat intuitive approach of appealing to principal values is used. This would seem unfortunate, because when coupled with the results above concerning the dispersion relation, a relatively simple way of determining whether explosive triads exist in a given system appears invalid. As stated, the Fredholm alternative, as it stands, cannot be applied to such problems. It would seem that each interaction coefficient has to be calculated separately for a particular triad, using a similar approach to solvability conditions taken in Benney and Maslowe (1975), a purely numerical procedure. The general, more analytical method is lost. However, the Fredholm alternative can be modified to handle such situations, so that the above approach remains applicable. This is the topic of the next section.

3.2 MODIFICATION OF THE SOLVABILITY CONDITIONS

The coefficients in the equations governing the amplitudes of various weakly nonlinear wave interactions such as resonant triads generally arise from requiring that some inhomogeneous boundary value problem has a solution. In the previous section, it was seen that for the problems under consideration, these ODE's may be singular, and so a formal application of the Fredholm alternative may be invalid, since the resulting integrals may not exist even in a principal value sense. In this section, this dilemma will be resolved, enabling such problems i.e. fluid mechanics to be solved. The method, however, will be seen to quite general in its application to such singular boundary value problems. It was realized after this analysis was done that the method is largely similar to Hadamard's method of finite parts (Hadamard, 1923). However, the method is here derived and applied to ODE's and boundary value problems in an independent way. Essentially, this method evaluates singular definite integrals by "subtracting off" or ignoring the behavior of the integrand near the singular point. For example, it would perform integrations in the following naive way:
\[
\int_{-1}^{1} \frac{dy}{y^2} = -\frac{1}{y}\bigg|_{-1}^{1} = -2
\]

However, there will be seen to be significant differences in the way such integrals are interpreted in the stratified and non-stratified cases. Ultimately, the formal results of the previous section will remain valid, with suitable modifications. In addition, some model problems which verify the technique will be submitted. As stated, the analysis is quite general, and so the question of how to define an adjoint to such singular problems will now be addressed. Later, variation of parameters will be used to confirm the resulting conclusions.

### 3.2.1 Modified Adjoint Analysis

Fredholm's alternative theorem applies to problems of the sort

\[ L\phi_I = f \quad (3.52) \]

where \( L \) is some linear differential operator with given boundary conditions at say \( y = \pm 1 \), and some sort of inner product \( \langle \cdot, \cdot \rangle \) relevant to the problems at hand has been defined. If the homogeneous counterpart to the above:

\[ L\phi = 0, \quad (3.53) \]

with the same boundary conditions, has a solution, then a necessary condition that the inhomogeneous problem has a solution is that

\[ \langle \phi^\dagger, f \rangle = 0 \quad (3.54) \]

where \( \phi^\dagger \) is the adjoint eigenfunction of the adjoint operator \( L^\dagger \); ie,

\[ \langle \psi, L\phi \rangle = \langle L^\dagger \psi, \phi \rangle \]

and

\[ L^\dagger \phi^\dagger = 0 \]

with the adjoint boundary conditions. Typically, for example,

\[ \langle f, g \rangle = \int_{-1}^{1} f^* g \, dy \quad (3.56) \]

Ordinarily, then, the adjoint operator is determined by integration by parts, which produces boundary terms involving \( \phi \) and \( \phi^\dagger \). The adjoint boundary conditions are then
chosen to eliminate these terms, and in such a way that no information concerning \( \phi \) other than the original boundary conditions is required; the problem is said to be made as general as possible. Then the solvability condition becomes

\[
\int_{-1}^{1} \phi^t f \, dy = 0
\]  

(3.57)

As already seen, though, such an approach would be invalid for the types of problems under consideration owing to their singular nature. But the whole utility of the method is to determine if an inhomogeneous problem is solvable, using only knowledge about the homogeneous solution and the inhomogeneity, so it would be desirable to extend the method to the kind of problems of interest here. When the problem is singular, as has been seen some interpretation must be given to the singularity, such as defining a critical layer to be viscous or nonlinear. This must be done a priori. Therefore, it would seem appropriate to interpret any singularities arising in the adjoint problem in a consistent way. As a first resolution to such a problem with a singular point at \( y_c \), one could define the inner product to be

\[
<f, g> = \int_{-1}^{y_c^-} f^* g \, dy + \int_{y_c^+}^{1} f^* g \, dy
\]  

(3.58)

where the \( \pm \) denote infinitesimal approaches to \( y_c \), and seek the proper notion of an adjoint for this problem. This approach, while incomplete, gives an adjoint eigenfunction and boundary conditions for the problem. The appropriate jump conditions across the critical layer, assumed already known for the original problem, must be obtained by other methods.

So consider first the Rayleigh equation

\[
L \phi = \phi'' - k^2 \phi - \frac{\overline{u}'' \phi}{\overline{u} - c} = 0
\]

with \( \phi(\pm 1) = 0 \) and a nonlinear critical layer at \( y = y_c \), where \( \overline{u}(y_c) = c \). Denote the adjoint \( \phi^t \) by \( \psi \). Then, by familiar manipulations, it is seen that

\[
<\psi, L\phi> = -[\psi^*, \phi'] + [\psi'^*, \phi] + <L\psi, \phi>
\]

where [ ] denotes the jump across \( y = y_c \) and as usual \( \psi \) is taken to be zero at the boundaries as well. Since \( L^t = L \), \( \psi \) also satisfies the Rayleigh equation, and so it consists of some linear combination of the two solutions \( \phi_{a,b} \):

\[
\psi = a'_1 \phi_a + b'_1 \phi_b, \; y > y_c
\]

\[
\psi = a'_2 \phi_a + b'_2 \phi_b, \; y < y_c
\]  

(3.59)

where
\[
\phi_a = (y - y_c) + \cdots
\]
\[
\phi_b = 1 + \cdots + \frac{\bar{u}_c''}{\bar{u}_c'} \phi_a \log |y - y_c|
\]

and the relationship between \((a_1, b_1)\) and \((a_2, b_2)\) must be determined. In addition,
\[
\phi = a \phi_a + b \phi_b
\]
is assumed to satisfy both the boundary conditions, and to have no phase shift across the critical layer, so
\[
\phi' \sim c_1 + b \frac{\bar{u}_c''}{\bar{u}_c'} \log |y - y_c|, \quad y \to y_c
\]
where \(c_1 = a + b \bar{u}_c'' / \bar{u}_c\). So, if the jump across the critical layer is taken in a principal value sense (ie, evaluating the relevant quantities at \(y = y_c \pm \delta\) as \(\delta \to 0+)\),
\[
[\psi^* \phi'] = (b_1' - b_2')^* \left( c_1 + b \frac{\bar{u}_c''}{\bar{u}_c'} \log \delta \right)
\]
and
\[
[\psi'^* \phi] = b (c_1' - c_2')^* + (b_1' - b_2') \left( \frac{\bar{u}_c''}{\bar{u}_c'} \right)^* \log \delta
\]
each as \(\delta \to 0+\), and where \(c_j' = a_j' + b_j' \bar{u}_c'' / \bar{u}_c\). Therefore, the “boundary” terms from the integration by parts become
\[
-c_1 (b_1' - b_2')^* + b (c_1' - c_2')^*
\]
\[
= b (a_1' - a_2') - a (b_1' - b_2') \tag{3.60}
\]

Now \(a\) and \(b\) are not linearly independent since \(\phi = 0\) at \(y = \pm 1\). Indeed, this defines the dispersion relation:
\[
\phi_a(1) \phi_b(-1) - \phi_a(-1) \phi_b(1) = 0 \tag{3.61}
\]

Suppressed in the above notation is the fact that the functions involved depend on \(k\) and \(c\), and given \(y_c\) this equality will only hold true for certain values of the wavenumber. Furthermore, since \(\psi\) has been taken to satisfy the boundary conditions also, it must be that
\[
b_1' = -\frac{\phi_a(1)}{\phi_b(1)} a_1'
\]
and

\[ b'_2 = -\frac{\phi_a(-1)}{\phi_b(-1)} a'_2 \]

But then clearly the terms arising from the jump conditions vanish identically. It would seem that this analysis is inconclusive; there does not appear to be way to make any definite statement about, in particular, how \( a'_1 \) and \( a'_2 \) are related, which would tell whether there is a phase shift across the critical layer or not. It will be seen in the next section how the adjoint must be interpreted. However, if the boundary conditions at \( y = \pm 1 \) are ignored for the moment, and the behavior of \( \phi \) near the critical layer taken simply to be the same above it as below it (so \( a \) and \( b \) are now independent), then the jump terms will vanish only if \( a'_1 = a'_2 \) and \( b'_1 = b'_2 \); this would seem to imply that if only the local behavior of the adjoint near the critical layer is of interest, then it too should be the same linear combination of the Frobenius solutions on either side of the critical layer. In other words, there is no phase shift for the adjoint either. This ad hoc analysis will be shown true in the next section.

As an aside, in the case of a viscous critical layer, \( \phi \) would be interpreted as

\[ \phi = a_1 \phi_a + b_1 \phi_b, \quad y > y_c \]
\[ \phi = a_2 \phi_a + b_2 \phi_b, \quad y < y_c \]

with \( b_2 = b_1 \) and \( a_2 = a_1 - i s \pi b_1 \overline{u''_c}/\overline{u'}_c \), where \( s = \text{sgn}(\overline{u'}_c) \). Of course, there is no \( \phi \) with this local behavior satisfying the boundary conditions; the imaginary part of \( \phi(-1) \) is not necessarily zero. Therefore, it does not make sense to even speak of an adjoint eigenfunction in this case as there is no eigenfunction. However, if once again the main concern is the local behavior, then proceeding as before, it would now be found that

\[
- [\psi^*, \phi'] + [\psi'^*, \phi] = -b'_1 c_1 + b'_2 c_2 + b_1 c'_1 - b_2 c'_2
\]

\[
= -c_1 (b'_1 - b'_2)^* + b_1 (c'_1 - c'_2 + i s \pi b'_2 \frac{\overline{u''_c}}{\overline{u'_c}})^*
\]

\[
= -a_1 (b'_1 - b'_2)^* + b_1 (a'_1 - a'_2 + i s \pi b'_2 \frac{\overline{u''_c}}{\overline{u'_c}})^* 
\]

(3.62)

so that this expression vanishes when \( b'_1 = b'_2 \) and \( a'_2 = a'_1 + i s \pi b'_1 \overline{u''_c}/\overline{u'_c} \); recall only the local behavior of the “eigenfunction” is being considered, so \( a_1 \) and \( b_1 \) are treated as linearly independent. Thus, the phase shift for the adjoint eigenfunction is the negative of the phase shift for the eigenfunction. If calculations involving viscous critical layers required that, say, contours of integration be deformed “under” the critical layer, calculations involving the adjoint would have to go “over” the critical layer. Again, these are remarks in passing; the focus here will be on nonlinear critical layers, for which the problem is self-adjoint in the sense mentioned.
For the stratified case, the analysis would proceed in a similar fashion. Again, the adjoint would satisfy (in this case) the Taylor-Goldstein equation away from the critical layer, and again it could be expressed as

\[
\psi = a'_1 \phi_a + b'_1 \phi_b, \quad y > y_c \\
\psi = a'_2 \phi_a + b'_2 \phi_b, \quad y < y_c
\]

where now

\[
\phi_a = |y - y_c|^\alpha_1 (1 + \cdots) \\
\phi_b = |y - y_c|^\alpha_2 (1 + \cdots)
\]

Of course the eigenfunction \( \phi \) not only satisfies the boundary conditions but can be written as

\[
\phi = a \phi_a + b \phi_b
\]

throughout the region. Using the same \( \pm \) notation from the last section,

\[
\phi' \sim \pm(a \alpha_1 |y - y_c|^{-\alpha_1 - 1} + b \alpha_2 |y - y_c|^{-\alpha_2 - 1}), \; y \to y_c \pm
\]

Then with the same inner product as before, the "boundary" terms arising from the integration by parts would become (it is to be anticipated that \( a'_j \) and \( b'_j \) are real)

\[
-b \alpha_2 (a'_1 + a'_2) - a \alpha_1 (b'_1 + b'_2) + a \alpha_2 (b'_1 + b'_2) + b \alpha_1 (a'_1 + a'_2) \\
= (\alpha_2 - \alpha_1) (a (b'_1 + b'_2) - b (a'_1 + a'_2)) \tag{3.63}
\]

Just as before, this expression vanishes identically. However, if again only the local behavior is being considered, it should be required that \( a'_2 = -a'_1 \) and \( b'_2 = -b'_1 \). In other words, the adjoint eigenfunction is the odd extension (to leading order) of the eigenfunction. Of course, these arguments concerning the focus on local behavior are somewhat spurious; both local and global behavior for these singular problems is needed. It will be seen in the next section that these notions hold true, however. If the viscous problem is again considered (which again has no solution satisfying the boundary conditions), the same conclusion as before would be reached: the phase shift changes sign for the adjoint relative to the eigenfunction.

The problem defining the adjoint comes from the fact that the operator \( L \) is formally self-adjoint. Therefore, solutions to the boundary value problem can be found above and below the critical layer:

\[
\phi = a \phi_a + b \phi_b
\]

throughout, or

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\[ \phi = a\phi_a + b\phi_b, \quad y > y_c, \]
\[ \phi = -a\phi_a - b\phi_b, \quad y < y_c \]

could both be valid adjoints, because all that is known about such a function at this point is that it satisfies the governing ODE above and below \( y_c \) and satisfies the boundary conditions. As has been seen, the jump conditions terms identically vanish. Again, it will be seen in the next section which choice to make for the adjoint’s local behavior.

### 3.2.2 Resolution Using Variation of Parameters

The goal of this section is to show how the Fredholm alternative theorem is modified for singular problems. Recall in the last section, a first step toward this was an attempt to define an adjoint problem in such. This analysis proved inconclusive, unless heuristic arguments were appealed to. These issues will now be resolved. Again boundary value problems of the sort

\[ L\psi = f, \]

where \( L \) is considered formally self-adjoint, are being considered. Actually, presently \( L \) will be considered to of the form

\[ L\psi = \psi'' + F(y)\psi, \quad (3.64) \]

which commonly occurs in hydrodynamic stability theory. Later more general self-adjoint operators will be considered. If the inner product of the section 3.2.1 is used, it would be found that

\[-[\phi\psi'] + [\phi'\psi] = \langle \phi, f \rangle \]
\[ = \int_{-1}^{y_c-} \phi f \, dy + \int_{y_c+}^{1} \phi f \, dy \quad (3.64) \]

Although the singular (critical) point(s) have been excluded from the integrations above (it’s really a limiting process), these integrals may not exist even in a principal value sense. Fortunately, their singular behavior is exactly balanced by the singular behavior of the jumps on the left, and th’: will provide the way of interpreting these divergent expressions.

Variation of parameters may be used to verify that the two sides of the above equation have identical singular behavior as \( y \to y_c \pm \), and so heuristically these integrals would be evaluated by “subtracting off the singular parts”. Such an approach is implicit in the following freshman way of evaluating

\[ \int_{-1}^{1} \frac{dy}{y^2} = -\frac{1}{y}\bigg|_{-1}^{1} = -2 \]
Variation of parameters will now be used to argue this point a bit more convincingly.

Consider first the case of critical layers in a homogeneous flow, so $L$ is the Rayleigh operator. Take the solution to the inhomogeneous problem to be

$$
\psi = -\phi_a \int_{-1}^{y} \frac{\phi_b f}{W} \, dy + \phi_b \int_{-1}^{y} \frac{\phi_a f}{W} \, dy + \beta_1 \phi_H, \ y < y_c
$$

$$
\psi = -\phi_a \int_{1}^{y} \frac{\phi_b f}{W} \, dy + \phi_b \int_{1}^{y} \frac{\phi_a f}{W} \, dy + \beta_2 \phi_H, \ y > y_c
$$

(3.65)

where

$$
\phi_H = a \phi_a + b \phi_b
$$

satisfies the boundary conditions and the nonlinear critical layer condition, and the Wronskian

$$
W = \phi_a \phi_b' - \phi_a' \phi_b = -1
$$

both above and below the critical layer. This will prove to be an important point. Now the above expression for $\psi$ satisfies both the differential equation and the boundary condition, and so may appear acceptable. However, the question of whether it has the proper behavior near the critical layer needs to be addressed.

First some notation needs to be defined. If

$$
\int_{1}^{y} (g(\xi) - S^+(\xi)) \, d\xi
$$

exists as $y \to y_c+$, define the singular part (SP) of

$$
\int_{1}^{y} g(\xi) \, d\xi
$$

as $y \to y_c+$ to be

$$
S^+(y)
$$

and so

$$
\int_{1}^{y} g(\xi) \, d\xi \sim S^+(y) + C_+,
$$

(3.66)

where $C_+$ is a constant, as $y \to y_c+$. Similarly, the statement

$$
\int_{-1}^{y} g(\xi) \, d\xi \sim S^-(y) + C_-
$$

(3.77)
as \( y \to y_c^- \) can be defined. Using this notation, the behavior of \( \psi \), the solution of the inhomogeneous equation in question, can be expressed as \( y \to y_c^- \) (needed for the jump conditions in the solvability criterion), as

\[
\psi \sim \phi_a S_b^- - \phi_b S_a^- + (\beta_1 a + D_1) \phi_a + (\beta_1 b - C_1) \phi_b, \ y \to y_c^- \\
\psi \sim \phi_a S_b^+ - \phi_b S_a^+ + (\beta_2 a + D_2) \phi_a + (\beta_2 b - C_2) \phi_b, \ y \to y_c^+ 
\]

(3.68)

where \( S_a^\pm \) and \( S_b^\pm \) represent the asymptotic behavior in the sense above of the integrals involving respectively \( \phi_a \) and \( \phi_b \), and \( C_j \) and \( D_j \) are the corresponding constants of integration, which in general may be different above and below the critical layer. Then the solvability condition (3.64),

\[
-[\phi_H \psi'] + [\phi'_H \psi] = <\phi_H, f>
\]

\[
= \int_{-1}^{y_c^-} \phi_H f \, dy + \int_{y_c^+}^{1} \phi_H f \, dy,
\]

can be shown to reduce to

\[
(a S_a^- + b S_b^-)_{y \to y_c^-} + a C_1 + b D_1 \\
-(a S_a^+ + b S_b^+)_{y \to y_c^+} - a C_2 - b D_2 =
\]

\[
a \int_{-1}^{y_c^-} \phi_a f \, dy + b \int_{-1}^{y_c^-} \phi_b f \, dy +
\]

\[
a \int_{y_c^+}^{1} \phi_a f \, dy + b \int_{y_c^+}^{1} \phi_b f \, dy
\]

(3.69)

So first of all it is seen that the singular behavior in (3.69) does indeed cancel out as \( y \to y_c^\pm \). Now in the situations of interest here, \( \psi \) will be thought of as part of a higher order term in some perturbation series, whose leading order term contains \( \phi \). Typically in perturbation problems, such homogeneous solutions are not included in the higher order terms, because by redefining \( \epsilon \), the perturbation parameter, these homogeneous terms can always be removed from higher order. For this to be possible for singular problems, it must be demanded that \( \psi \) as written in (3.65) consist of the same linear combination of \( \phi_a, \phi_b \) above and below the critical layer:

\[
D_1 + \beta_1 a = D_2 + \beta_2 a \\
-C_1 + \beta_1 b = -C_2 + \beta_2 b
\]

If this is true, i.e., the problem has a solution, then

\[
a(C_1 - C_2) + b(D_1 - D_2) = ab(\beta_1 - \beta_2) + ab(\beta_2 - \beta_1) = 0
\]

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\[ \int_{-1}^{y_c^-} \phi_H f \, dy + \int_{y_c^+}^{1} \phi_H f \, dy - SP = 0, \]  

(3.70)

where the "SP" represents any singular parts in the above integrals, becomes the modified Fredholm alternative. Additionally, this settles the question of how to define the adjoint problem. If the correspondence between the regular Fredholm alternative is to be retained, then the adjoint must be taken to be equal to the eigenfunction itself, so the problem is indeed self-adjoint when the behavior near the critical layer is taken into account.

The above may appear to be a relatively innocuous conclusion, but when the stratified case is considered, a very different result is obtained. An important point to notice is that the Wronskian for this system changes sign across the critical layer, owing to the evenness of the linearly independent solutions involved:

\[
W(\phi_a, \phi_b) = \alpha_2 - \alpha_1, \quad y > y_c
\]

\[
W(\phi_a, \phi_b) = \alpha_1 - \alpha_2, \quad y < y_c
\]

so that now

\[
\psi \sim \frac{\phi_a}{\alpha_2 - \alpha_1} S_b^- - \frac{\phi_b}{\alpha_2 - \alpha_1} S_a^- + (\beta_1 a + \frac{D_1}{\alpha_2 - \alpha_1}) \phi_a + (\beta_1 b - \frac{C_1}{\alpha_2 - \alpha_1}) \phi_b, \quad y \rightarrow y_c^- \\
\psi \sim -\frac{\phi_a}{\alpha_2 - \alpha_1} S_b^+ + \frac{\phi_b}{\alpha_2 - \alpha_1} S_a^+ + (\beta_2 a - \frac{D_2}{\alpha_2 - \alpha_1}) \phi_a + (\beta_2 b + \frac{C_2 f}{\alpha_2 - \alpha_1}) \phi_b, \quad y \rightarrow y_c^+
\]

Therefore, if the above analysis is repeated with the same inner product, it would be found that the solvability condition (3.64) becomes

\[
(a S_a^- + b S_b^-)_{y \rightarrow y_c^-} + a C_1 + b D_1 \\
-(a S_a^+ + b S_b^+)_{y \rightarrow y_c^+} - a C_2 - b D_2 = \\
a \int_{-1}^{y_c^-} \phi_a f \, dy + b \int_{-1}^{y_c^-} \phi_b f \, dy + \\
a \int_{y_c^+}^{1} \phi_a f \, dy + b \int_{y_c^+}^{1} \phi_b f \, dy
\]

(3.72)

just as before, so again the singular terms from the jump conditions balance the singular terms from the integrals. However, if once again it is demanded that \( \psi \) consist of the same linear combination of \( \phi_{a,b} \) above and below the critical layer, it would be required that
\[(\beta_2 - \beta_1) a = D_1 + D_2\]
\[(\beta_2 - \beta_1) a = -C_1 - C_2\]

When used in the solvability condition (3.72), this does not seem to be of much use, as the constants \(C_j\) and \(D_j\) cannot be eliminated (If these could be determined, there would be no need for a solvability condition). However, it should be recalled from the last section that heuristic arguments were put forth that the adjoint in the stratified problem should be defined as follows:

\[
\phi^\dagger \sim (y - y_c)^\alpha, \ y \to y_c^+ \\
\phi^\dagger \sim -(y_c - y)^\alpha, \ y \to y_c^-
\]
in other words, the adjoint would become the odd extension (to leading order) of the eigenfunction. This function certainly satisfies the Taylor-Goldstein equation away from the critical layer, and if a linear combination of the Frobenius solutions can be found which vanishes at the boundaries, then this same linear combination vanishes there even if these Frobenius solutions are defined as above; ie, the adjoint satisfies the boundary conditions as well. Making use of this second choice for the adjoint, it follows that

\[
-\{\phi_H \psi'\} + \{\phi'_H \psi\} = <\phi^\dagger_H, f >
\]

\[
= -\int_{-1}^{y_c^-} \phi_H f \, dy + \int_{y_c^+}^{1} \phi_H f \, dy
\]

(3.73)

where \(\{F\} \) denotes \(F|_{y \to y_c^+} + F|_{y \to y_c^-}\). Then (3.73) reduces to

\[
-(a S_a^- + b S_b^-)_{y \to y_c^-} - a C_1 - b D_1 \\
-(a S_a^+ + b S_b^+)_{y \to y_c^+} - a C_2 - b D_2 = \\
-a \int_{-1}^{y_c^-} \phi_a f \, dy - b \int_{-1}^{y_c^-} \phi_b f \, dy + \\
a \int_{y_c^+}^{1} \phi_a f \, dy + b \int_{y_c^+}^{1} \phi_b f \, dy
\]

(3.74)

and now the critical layer condition eliminates the remaining terms above. Therefore, the modified alternative becomes

\[
-\int_{-1}^{y_c^-} \phi_H f \, dy + \int_{y_c^+}^{1} \phi_H f \, dy - SP = 0,
\]

(3.75)

and in particular, the quantity \(\gamma\), which determines the sign of the triad interaction coefficients, would be evaluated as
\[
\gamma_j = -2 \left( -\int_{-1}^{y_e^-} (k_j \bar{u} - \omega_j) (\eta_j^2 + k_j^2 \eta_j^2) \, dy + \int_{y_e^+}^{1} (k_j \bar{u} - \omega_j) (\eta_j^2 + k_j^2 \eta_j^2) \, dy \right) - \text{SP} \tag{3.76}
\]

Now while the above integrals are always negative, when the "singular part" is subtracted off, the sign of the resulting quantity is a priori unknown. One need only consider the "freshman" example to see an example of this.

As some validation of the above results, some model problems are worth considering. For instance, the following boundary value problem

\[
x^2 y'' + x y' - y = 1 + \frac{\gamma}{x^2} \tag{3.77}
\]

with \(y(\pm 1) = 0\), has a singular homogeneous solution which satisfies the boundary conditions:

\[
y_h = c \left( x - \frac{1}{x} \right)
\]

There are no branch points in this case which require interpretation, but the problem is nonetheless problematic at \(x = 0\), and it will be seen that the heuristic notion of "subtracting off or neglecting the singular part" when dealing with the integrals arising from a solvability condition holds true. For, by variation of parameters, it is easily found that the full solution to (3.77) can be written as

\[
y = a x + \frac{b}{x} - 1 + \frac{\gamma}{3 x^2}
\]

Imposing the boundary conditions requires

\[
a + b = 1 - \frac{\gamma}{3}
\]

\[
a + b = -1 + \frac{\gamma}{3}
\]

For there to be a solution, then, \(\gamma = 3\). A formal application of the Fredholm alternative theorem would require that

\[
\int_{-1}^{1} \left( \frac{1}{x} + \frac{\gamma}{x^3} \right) (x - \frac{1}{x}) \, dx = 0
\]

or, ignoring the singularity at \(x = 0\),

\[
4 - \frac{4}{3} \gamma = 0
\]

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so again $\gamma = 3$. To mimic the stratified case above, consider

$$y'' + \frac{J}{x^2} y = \frac{\gamma}{x^3} + \frac{1}{x^5}$$

(3.78)

with boundary conditions $y(\pm 1) = 0$. The homogeneous equation has solutions of the form $y = x^\alpha$, where $\alpha$ satisfies

$$\alpha (\alpha - 1) + J = 0$$

so these solutions will in general have an algebraic branch point singularity. If they are interpreted in the nonlinear critical layer way, the homogeneous equation has the solution

$$y = c(|x|^\alpha_1 - |x|^\alpha_2)$$

The general solution to (3.78) is found to be

$$y = a|x|^{\alpha_1} + b|x|^{\alpha_2} + \frac{\gamma}{(1 + \alpha_1)(1 + \alpha_2)} \frac{1}{x} + \frac{1}{(3 + \alpha_1)(3 + \alpha_2)} \frac{1}{x^3}$$

which will satisfy the boundary conditions only if

$$\gamma = \frac{(1 + \alpha_1)(1 + \alpha_2)}{(3 + \alpha_1)(3 + \alpha_2)}$$

By applying the modified Fredholm alternative derived above, it would be required that

$$- \int_{-1}^{0-} \left( \frac{\gamma}{x^3} + \frac{1}{x^5} \right) \left( (-x)^{\alpha_1} - (-x)^{\alpha_2} \right) dx + \int_{0+}^{1} \left( \frac{\gamma}{x^3} + \frac{1}{x^5} \right) \left( x^{\alpha_1} - x^{\alpha_2} \right) dx - \text{SP} = 0$$

where the $\pm$ denote infinitesimal approaches to $x = 0$, and SP denotes the singular parts of the integrals, defined as before. This quantity is equivalent to

$$\int_{0+}^{1} \left( \frac{\gamma}{x^3} + \frac{1}{x^5} \right) \left( x^{\alpha_1} - x^{\alpha_2} \right) dx - \text{SP} = 0$$

owing to the evenness of the integrand. Note a straightforward application of Fredholm, even accounting for the singularities properly, would result in the identity $0 = 0$ and so reveal no information regarding the solvability of the system. The above reduces to

$$\gamma = \frac{(\alpha_1 - 2)(\alpha_2 - 2)}{(\alpha_1 - 4)(\alpha_2 - 4)}$$
which is the same as before, since \( \alpha_1 + \alpha_2 = 1 \). So the modified alternative is verified here as well. As a final illustration, and one in which branch point singularities arise but the non-stratified modification is needed, consider

\[
(x y')' - \frac{y}{4x} = \frac{\gamma}{x^2} + \frac{1}{x^4}
\]  

(3.79)

with \( y(\pm 1) = 0 \) again. Here, \( y_H = c(|x|^{1/2} - |x|^{-1/2}) \), and

\[
y = a|x|^{1/2} + b|x|^{-1/2} + \frac{4}{3} \gamma \frac{1}{x} + \frac{4}{35} \frac{1}{x^3}
\]

and the boundary conditions imply \( \gamma = -3/35 \). The non-stratified, modified Fredholm alternative becomes

\[
\int_{-1}^{1} \left( \frac{\gamma}{x^2} + \frac{1}{x^4} \right) (|x|^{1/2} - |x|^{-1/2}) \, dx = 0
\]

or again \( \gamma = -3/35 \). Note that if the integrals had been broken up as before, since the integrand is now even, a trivial statement would have resulted. The difference between this example and the previous one is how the respective Wronskians behave. Essentially, the Wronskian does not change sign in this example: \( W = -1/x \). In explaining this, the above analysis will be generalized to problems of the following form:

\[
(r(y) \psi')' + q(y) \psi = f(y)
\]  

(3.80)

For example, the equation governing the linear stability of a non-Boussinesq stratified flow is of this form. In the above, \( r(y) \) possibly approaches 0 as \( y \to y_c \). Again, the solution is required to satisfy \( \psi(\pm 1) = 0 \), and it assumed the homogeneous counterpart has some solution \( a\phi_a + b\phi_b \). In this case, the Wronskian \( W \) satisfies

\[
W = C \exp\left(- \int \frac{r'}{r} \, dy \right) = \frac{C}{|r|}
\]

but in general this constant \( C \) will be allowed to differ across the critical layer:

\[
W = \frac{C_a}{r}, \quad y > y_c
\]

\[
W = \frac{C_b}{r}, \quad y < y_c
\]  

(3.81)

Then by variation of parameters, the general solution to the inhomogeneous equation (3.80) may again be written

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\[ \psi = -\frac{\phi_a}{C_b} \int_{-1}^{y} \phi_b \, f \, dy + \frac{\phi_b}{C_b} \int_{-1}^{y} \phi_a \, f \, dy + \beta_1 \phi_H, \quad y < y_c \]

\[ \psi = -\frac{\phi_a}{C_a} \int_{1}^{y} \phi_b \, f \, dy + \frac{\phi_b}{C_a} \int_{1}^{y} \phi_a \, f \, dy + \beta_2 \phi_H, \quad y > y_c \]  \hspace{1cm} (3.82)

Proceeding as before, the solvability condition will involve the following “jump” terms:

\[ r \left( \phi_H \psi' + \phi_H' \psi \right) \bigg|_{y = y_c+} \pm r \left( -\phi_H \psi' + \phi_H' \psi \right) \bigg|_{y = y_c-} \]  \hspace{1cm} (3.83)

where the \( \pm \) reflects the fact that sometimes the adjoint implicit in the above comes from an odd extension of the eigenfunction across the critical layer. With the previous notation,

\[ \psi \sim -\frac{\phi_a}{C_b} S_b^- + \frac{\phi_b}{C_b} S_a^- + (\beta_1 a - \frac{D_1}{C_b}) \phi_a + (\beta_1 b + \frac{C_1}{C_b}) \phi_b, \quad y \to y_c^- \]

\[ \psi \sim -\frac{\phi_a}{C_a} S_b^+ + \frac{\phi_b}{C_a} S_a^+ + (\beta_2 a - \frac{D_2}{C_a}) \phi_a + (\beta_2 b + \frac{C_2}{C_a}) \phi_b, \quad y \to y_c^+ \]  \hspace{1cm} (3.84)

Then the above “jump” terms reduce to

\[ \pm (-a \, S_b^- - b \, S_a^- - a C_b (\beta_1 b + \frac{C_1}{C_b}) + b C_b (\beta_1 a - \frac{D_1}{C_b})) \bigg|_{y = y_c-} + \]

\[ (-a \, S_b^+ - b \, S_a^+ - a C_a (\beta_2 b + \frac{C_2}{C_a}) + b C_a (\beta_2 a - \frac{D_2}{C_a})) \bigg|_{y = y_c+} \]  \hspace{1cm} (3.85)

Clearly, the singular terms due to the inhomogeneity will always cancel with the singularities due to \( < \phi_H^T, f > \) regardless of how the adjoint is defined. In addition, the requirement that the inhomogeneous solution (3.82) consist of the same linear combination of the homogeneous solution above and below the critical layer requires:

\[ (\beta_2 - \beta_1) a = \frac{D_2}{C_a} - \frac{D_1}{C_b} \]

\[ (\beta_2 - \beta_1) b = \frac{C_1}{C_b} - \frac{C_2}{C_a} \]

For the type of problems under consideration, it is assumed that any singularities are regular; ie,

\[ r = r_0 + r_1 (y - y_c) + r_2 (y - y_c)^2 + \cdots \]

\[ q = \frac{q_0}{(y - y_c)^2} + \frac{q_1}{y - y_0} + q_2 \cdots \]
near \( y = y_c \). If \( r_0 \neq 0 \), the singularities have algebraic branch point singularities, and the Wronskian merely changes sign across the critical layer; \( C_b = -C_a \). Then the above condition becomes

\[
(\beta_2 - \beta_1) a = \frac{D_2 + D_1}{C_a} \\
(\beta_2 - \beta_1) b = -\frac{C_1 + C_2}{C_a}
\]

Thus, if the + is taken in the jump conditions (3.83), the singular terms will cancel in a principal value sense. Similarly, the − should be taken when the Wronskian does not change sign. This will be the case when \( r_0 = 0 \), as in (3.79), or when \( r = \text{const.} \) and \( q_0 = 0 \), for the Rayleigh equation.

In the case of a non-Boussinesq stratified flow, the relevant equation is

\[
\phi'' - k^2 \phi - \frac{k \bar{u}''}{\bar{\omega}} \phi - \frac{g k^2 \bar{\rho}'}{\bar{\rho} \omega^2} \phi + \frac{\bar{\rho}'}{\bar{\rho}} (\phi' - \frac{k \bar{u}'}{\bar{\omega}}) \phi = 0 \tag{3.86}
\]

with of course \( \phi(\pm 1) = 0 \). So here \( r = \bar{\rho}'/\bar{\rho} \), which is nonsingular at the critical layer. Hence the analysis is nearly identical to the Boussinesq case; the Frobenius solutions have the same local behavior. If again

\[
W(\phi_a, \phi_b) \rightarrow \pm 1
\]

as \( y \rightarrow y_c \), then \( C_a = \bar{\rho}_c \) and \( C_b = -\bar{\rho}_c \). So the solvability condition would be interpreted as before, for Boussinesq fluids.

It should be noted with any singular integrals evaluated in the appropriate way, all the previous results concerning the evaluation of the triad coefficients remain true. For example, consider the dispersion relations in terms of integrals. Multiplying the Rayleigh equation by \( \phi \) and performing an integration around \( y_c \) gives

\[
-\{\phi \phi'\} - \int_{-1}^{y_c-} \left( \phi'^2 + \left( k^2 + \frac{k \bar{u}''}{\bar{\omega}} \right) \phi^2 \right) dy - \int_{y_c+}^{1} \left( \phi'^2 + \left( k^2 + \frac{k \bar{u}''}{\bar{\omega}} \right) \phi^2 \right) dy = 0
\]

and as before, the “jump” terms vanish, and the dispersion relation remains unchanged from the formal approach in section 3.1.2.

For the stratified case, however, manipulating the Taylor-Goldstein results in

\[
-\{\phi \phi'\} + \int_{-1}^{y_c-} \left( \phi'^2 + \left( k^2 + \frac{k \bar{u}''}{\bar{\omega}} - \frac{J k^2 N^2}{\omega^2} \right) \phi^2 \right) dy \\
- \int_{y_c+}^{1} \left( \phi'^2 + \left( k^2 + \frac{k \bar{u}''}{\bar{\omega}} - \frac{J k^2 N^2}{\omega^2} \right) \phi^2 \right) dy = 0
\]

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where again the \{\} denotes the sum of the enclosed quantity across \(y = y_c\). These terms cancel, as has been seen, so the dispersion relation changes from the formal approach by the changing of the sign of the integral over \([-1, y_c]\). If the non-stratified interpretation, so the sum across the critical layer becomes the difference, then these terms would become infinite. These results would not change if the particle displacement eigenfunction \(\eta = \phi/\omega\) is used, because in the stratified case, while \(\phi_a\) and \(\phi_b\) are, to leading order, even across the critical layer, \(\eta_a\) and \(\eta_b\) are odd, and so again the terms \{\} would vanish while [\] would become infinite. For the homogeneous case, for \(y = y_c \pm \delta\),

\[
\eta_a \sim 1 + c_1 \delta
\]

\[
\eta_b \sim \frac{1}{\delta} + c_2 \delta + (b_1 + b_2 \delta) \log \delta
\]

and it can easily be checked that the difference across the critical layer vanishes; the logarithmic singularity has the same behavior behavior across the critical layer, but so does it’s derivative. So, the interpretation is unchanged.

In addition, any integrations by parts that were performed to simplify the interaction coefficients will remain the same, because any such manipulation done to an integrand can be done to the integrand’s singular behavior:

\[
\int (u v' - \text{Sing}(u v')) dy = u v - \text{Sing}(u v) - \int (u' v - \text{Sing}(u' v)) dy \quad (3.87)
\]

Of course, when there is more than one singular point present, the analysis is readily extended by simply breaking the region of integration into multiple sections. Finally, the group velocity \(C_g\) may be calculated from

\[
\omega' = -\frac{D_k}{D\omega}
\]

again, with the resulting integrals interpreted properly. It may be easily checked, then, that the results of Liu and Benney (1981) for the group velocity of waves in a stratified flow hold true with the proper modification in the presence of critical layers, excluded from their analysis.

It should be mentioned in passing that the issue of evaluating the singular integrals in the interaction coefficients was brought up briefly by Maslowe (1977), and addressed in more detail by Burns and Maslowe (1983) in studies on meteorological flows. In the latter paper, the work of Hadamard was explicitly mentioned. However, the latter work dealt with nonstratified meteorological flows, so an intuitive use of this method was correct. However, in stratified problems, the method would have to be modified as above. With these singularities properly addressed, the existence of such triads will now be investigated.
3.3 THE APPEARANCE OF CRITICAL LAYERS AND EXPLOSIVE TRIADS

The purpose of this section is two-fold. First, some results concerning (nonlinear) critical layers for certain fluid dynamical systems will be presented. For the flows in question, namely, stratified and interfacial flows, little previous work is known to have been done. Such waves will be found to exist, and some features of the numerical results will be explained analytically. Second, the question of whether explosive resonant triads exist in flows with continuously varying velocity and/or density profiles will be addressed. These waves are known to exist in flows with piecewise constant shears and densities. As seen in the last section, critical layers are essential for the appearance of explosive triads, so after the existence of critical layers has been established, the search for potentially explosive interactions will be undertaken by evaluating the quantity $\gamma$ from before at appropriate points on the dispersion curve. Of course, this quantity needs to be determined numerically, and the method of the last section for interpreting singular integrals needs to be used. It will be seen that for certain values of the physical parameters involved, explosive triads do exist, although in flows that are also linearly unstable.

3.3.1 Critical Layer Results

Three fluid systems already studied in this thesis will again be examined here. These are a stratified shear flow between flat plates, a free surface flow with a surface shear, and a wind-driven shear flow. The dispersion relations for linear wave perturbations to these basic flows are, like most such problems in fluid mechanics, obtained from an eigenvalue problem to some ODE with certain boundary conditions. These are the Taylor-Goldstein equation

$$\phi'' - k^2 \phi - \frac{\bar{u}'' \phi}{\bar{u} - c} + \frac{J N^2 \phi}{(\bar{u} - c)^2} = 0$$

with $\phi = 0$ at boundaries that can be at either $\pm 1$ or $\pm \infty$ for a stratified shear flow, and the Rayleigh equation

$$\phi'' - k^2 \phi - \frac{\bar{u}'' \phi}{\bar{u} - c} = 0$$

with appropriate boundary conditions. In either of the two cases to be studied here, the fluid will be considered to be of infinite expanse, so $\phi(-\infty) = 0$ for the surface shear case, and $\phi(\infty) = 0$ for the wind-driven shear case. The other boundary condition, imposed at the leading order free surface $y = 0$, is

$$(\bar{u} - c) \phi' - \bar{u}' \phi = \frac{g + \sigma k^2}{\bar{u} - c} \phi$$

(3.88)

for the surface shear, and

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\[(\bar{u} - c) \phi' - \bar{u}' \phi = -\rho_d^{-1} g + \sigma k^2 - k c^2 \bar{u} - c \phi \]  

(3.89)

for the wind-driven shear. For a given wavenumber \( k \), the wave speed \( c = \omega/k \) is the eigenvalue that permits nontrivial solutions to the above. Certainly the Taylor-Goldstein equation is well-studied; see Hazel (1972), and Drazin and Reid (1981) for more general results. The stability of a free surface, while less well-known, has also generated interest; see Yih (1972), Morland, Saffman, and Yuen (1991). However, unlike the case of a homogeneous shear flow between flat plates (or one boundary at infinity), as in Benney and Bergeron (1969, and hereafter referred to as BB), little work has been done concerning nonlinear critical layers in these flows. Maslowe (1973) is an exception. As a condition for explosive triads is the existence of critical layers of one sort or another, such an investigation will be undertaken here. To avoid the appearance of multiple critical layers, only monotone flows will be examined. Again, the nonlinear interpretation of critical layers will be adopted. This involves finding solutions to the above equations which not only satisfy the boundary conditions, but have the appropriate behavior near \( y = y_c \) as well. In terms of the Frobenius series solutions already encountered, this implies that

\[
\phi = a \phi_a + b \phi_b
\]

where

\[
\phi_a(y) = |y - y_c|^{\alpha_1} \left( 1 + a_1 (y - y_c) + \cdots \right)
\]
\[
\phi_b(y) = |y - y_c|^{\alpha_2} \left( 1 + a_2 (y - y_c) + \cdots \right)
\]

for the stratified case, with \( \alpha_{1,2} \) satisfying

\[
\alpha (\alpha - 1) + J_c = 0
\]

where \( J_c = J N_c^2 / \bar{u}_c^2 \), and

\[
\phi_a = (y - y_c) + a_1 (y - y_c)^2 + \cdots
\]
\[
\phi_b = 1 + b_1 (y - y_c)^2 + \cdots + \frac{\bar{u}''}{\bar{u}_c} \phi_a \log |y - y_c|
\]

for the non-stratified case. The coefficients in the above series are well-known (Miles (1961, 1963), Drazin and Reid (1981)), and need not be specified. Numerically, then, the relevant equations would be integrated from \( y = y_c \pm \delta \), where \( \delta \) is a small number, to the appropriate outer boundary, using as initial conditions the relevant Frobenius solutions: \( (\phi_a(y_c \pm \delta), \phi'_a(y_c \pm \delta)) \) and \( (\phi_b(y_c \pm \delta), \phi'_b(y_c \pm \delta)) \). Typically the above series are truncated at three terms. Each initial condition gives a value for respectively \( (\phi_a, \phi'_a) \) and \( (\phi_b, \phi'_b) \) at the relevant boundary, and the same linear combination of these quantities must satisfy both boundary conditions, which, when written as a system of linear equations, must
have a vanishing determinant. Given $y_c$ and hence $c$, this gives an implicit nor-linear equation for $k$ which can be solved by Newton's method, and so the dispersion relation is obtained in this inverse fashion. Of course, from a numerical point of view, if one (or both) of the boundaries is at $\pm \infty$, it is advisable to integrate in from the boundaries to the critical layer. For the boundary at infinity, some suitably large or negatively large value is used, and decaying exponentials are taken as initial conditions. The values on either side of the critical layer are required to have the proper behavior, determined from the asymptotics above. This can be done straightforwardly, although in practice in determining the dispersion relation it was found that the integration could be performed out to the boundaries without appreciable loss of accuracy. Certainly the eigenfunctions should be found by an inward integration, though. These approaches will be spelled out in the appendix on the numerics.

Consider first the stratified case. Results for the following profile will be presented:

$$
\bar{u} = \tanh y, \quad N^2 = \text{sech}^2 y
$$

the famous Holmboe profile. The Holmboe profile over $-\infty < y < \infty$ gives rise to known analytic solutions for $c = 0$; for a collection of other exact solutions to the Taylor-Goldstein equation, see Drazin and Howard (1966). This profile was also studied for a physical domain of $-1 < y < 1$, although in this case no known analytic solutions are known. This case will provide the numerical results, as no critical layer modes were found on the infinite domain. A variety of other profiles, selected without regard for physical plausible, were also studied, but the case above provides fairly generic results. For example, the Garcia profile

$$
\bar{u} = \tanh y, \quad N^2 = \text{sech}^2 y \ tanh^2 y
$$

was found to have critical layer modes on the unbounded domain, but none on the bounded domain. In a previous section, the Richardson number $J$ was taken to be $\sigma g$ where $g$ is acceleration due to gravity and $\sigma$ is a measure of the stratification. More generally, when the equations of motion are non-dimensionalized, $J$ is taken to be $\sigma g L/U$, where $L$ and $U$ are respectively measures of the length and velocity scales of the system. Therefore $J$ will be thought of here as a dimensionless parameter that characterizes the problems at hand. The differences between the bounded and unbounded cases were largely quantitative. The following plots (Figures 3.5-10) are fairly typical of the numerical runs performed.

For the Holmboe profile, it should be noted that $c = 0$ represents a nonlinear critical layer mode for all $k$, and therefore two branches of the dispersion relation intersect at some point on the $k$ axis. This is essentially because the Frobenius solutions are both even in these cases, so over the physical domain considered, any combination of these functions with the proper critical layer behavior that satisfies one boundary condition automatically satisfies the other. For example, the Taylor-Goldstein equation for the Holmboe profile becomes

$$
\phi'' - k^2 \phi + 2 \text{sech}^2 y \phi + \frac{J \phi}{\sinh^2 y} = 0
$$
Figure 3.5: Stratified critical layer branch for $J=0.75$

Figure 3.6: Dispersion relation for $J=0.75$
Figure 3.7: Stratified critical layer branch for $J=1.25$

Figure 3.8: Dispersion relation for $J=1.25$
Figure 3.9: Stratified critical layer branch for $J=1.75$

Figure 3.10: Dispersion relation for $J=1.75$
a Frobenius series solution would have only powers of $y^{\alpha+2n}$ for $n = 0, 1, 2 \ldots$ Indeed, the
general situation that would arise from steady perturbations to say, an odd velocity profile
and even Brunt-Väisälä frequency squared is

$$\phi'' + \left( \frac{a}{y^2} + b + cy^2 + \cdots \right) \phi = 0$$

where $a \neq 0$. Seeking a Frobenius series of the form

$$\phi = y^\alpha (a_0 + a_1 y + a_2 y^2 + \cdots)$$

would imply

$$\frac{\alpha(\alpha - 1)a_0}{y^2} + \frac{(\alpha + 1)\alpha a_1}{y} + (\alpha + 2)(\alpha + 1)a_2 + \cdots$$

$$= -\left(\frac{a}{y^2} + b + cy^2 + \cdots\right)(a_0 + a_1 y + a_2 y^2 + \cdots) \quad (3.90)$$

By comparing powers of $y^{-2}$ and $y^{-1}$ it can be seen that

$$\alpha(\alpha - 1) + a = 0,$$

the indicial equation, and that $a_1 = 0$. It is easy to see that the coefficient of any odd power
on the right-hand side of (3.90) will be equated to a linear recursion involving only odd-indexed $a_i$'s. Since $a_1 = 0$, all the odd-indexed coefficients are zero, and only even powers are present in the expansion. Thus, barring any other singularities in the problem, the two
linearly independent solutions of the Taylor-Goldstein equation are also even, and if the branch point singularity is interpreted as a nonlinear critical layer; ie, as $|y|^{\alpha}$ for $y < 0$, then as stated above it is always possible to satisfy identical boundary conditions at $y = \pm 1$ or $y = \pm \infty$. Thus there are a continuum of such modes for $c = 0$, and the ramifications of this have not been investigated. In particular, this would seem to suggest that there can be no weakly nonlinear discrete solutions with phase speed 0; a wave proportional to $e^{ikx}$ would produce resonant terms $e^{2ikx}$ at the next order, for any $k$, as would any linear superposition of waves. As all these waves must then be present at leading order with slowly varying amplitudes, it would appear the leading order solution would have to be taken in the form of a Fourier integral. The above conclusions seem to contradict the work of Maslowe (1973), who predicted there are no steady, nonlinear critical layers apart from
the well-known exact solutions

$$\phi = (\text{sech}y)^\alpha |\tanh y|^{1-\alpha} \quad (3.91)$$

with $J = \alpha(\alpha - 1)$, for the unbounded case. Furthermore, none of the other, nonsteady
modes detected by Maslowe were found here. This is away from the current topic, but it is
believed (after a private communication) that in this work, an improper connection across
the critical layer was used. Also, aspects of the later work of Collins and Maslowe (1988),
investigating degenerate triads of the form \((k, 0), (2k, 0)\), are called into question for this reason.

There are other features of these results worth commenting on. As \(J\) increases, the number of critical layer modes increases also. It was also found in other runs that more regular modes appear as well. In many cases, critical layer modes did not exist below a certain Richardson number. It is well-known (Drazin and Reid (1981), Miles (1961, 1963)) that if the Richardson “function” \(\mathcal{J} = JN^2/\bar{u}^2\) is less than \(1/4\) throughout the fluid, the flow is linearly stable. This is a sufficient condition for stability; If \(\mathcal{J}\) is greater than \(1/4\) anywhere in the fluid, the flow may or may not be stable. So these flows become potentially unstable, but their stability was not studied. It is not the central issue here, and will prove (for this case) to be unimportant. However, see Hazel (1972) for such work.

Before turning attention to triad interactions, some features of these plots are worth commenting on. For small \(k\), a perturbation analysis may be performed on the Taylor-Goldstein equation. Letting

\[
F(y, c) = -\frac{\bar{u}'}{\bar{u} - c} + \frac{JN^2}{(\bar{u} - c)^2}
\]  

(3.92)

and expanding \(\phi\) and \(c\) in series in \(k^2\):

\[
\phi = \phi_0 + k^2 \phi_1 + \cdots,
\]

\[
c = c_0 + k^2 c_1 + \cdots,
\]

it is found that

\[
\phi_0'' + F(y, c_0) \phi_0 = 0
\]

\[
\phi(\pm 1) = 0
\]

Attention will be confined to the regular modes. Solving this boundary value problem gives the long wave speed \(c_0\). At next order,

\[
\phi_1'' + F(y, c_0) \phi_1 = \phi_0 - F_1(y, c_0) \phi_0 c_1
\]

\[
\phi(\pm 1) = 0
\]

where

\[
F_1(y, c_0) = -\frac{\bar{u}''}{(\bar{u} - c_0)^2} + \frac{2JN^2}{(\bar{u} - c_0)^3}
\]  

(3.93)

Applying the Fredholm alternative to this problem implies

\[
c_1 = \frac{\int_{-1}^{1} \phi_0^2 dy}{\int_{-1}^{1} F_1(y, c_0) \phi_0^2 dy}
\]  

(3.94)
Although the bounded case is under consideration, it is clear the unbounded case would be handled no differently. The denominator of this expression is essentially the quantity $\gamma$ previously discussed. Once again, this integral may be simplified by the substitution $\bar{c}_0 = \bar{u} - c$ and $\zeta_0 = \phi_0/\bar{c}_0$. Denoting the denominator by $D$, it is found that

$$D = 2 \int_{-1}^{1} \bar{c}_0 \zeta_0^2 \, dy$$  \hspace{1cm} (3.95)

Therefore, $c_1 < 0$ if $c_0 > \bar{u}_{\text{max}}$ and $c_1 > 0$ if $c_0 < \bar{u}_{\text{min}}$. Since $\omega \sim c_0 k + c_1 k^3$ as $k \to 0$, these results are confirmed by the plots. The upper branch “bulges” outward, and the lower branch inward, though these features are best seen in the plots of $c$ vs $k$, as opposed to the dispersion relation itself.

Now consider large $k$ regular modes. Here, a WKB-approach is appropriate, and solutions of the form

$$\phi \sim \exp(k \psi)$$

are sought. The Taylor-Goldstein equation becomes

$$k^2 \psi'' + k \psi''' - k^2 - \frac{\bar{u}''}{\bar{u} - c} + \frac{J N^2}{(\bar{u} - c)^2} = 0$$ \hspace{1cm} (3.96)

Letting $\psi$ and $c$ have the expansions:

$$\psi \sim \psi_0 + \frac{\psi_1}{k} + \cdots,$$

$$c \sim c_0 + \frac{c_1}{k} + \cdots,$$

the following equations are obtained:

$$\psi_0' \psi_2 - 1 = 0 \Rightarrow \psi_0 = \pm y$$

$$2 \psi_0' \psi_1' = 0 \Rightarrow \psi_1 = C$$

$$2 \psi_0' \psi_2 = -F(y, c_0) \Rightarrow \psi_2 = \mp \frac{1}{2} \int_{0}^{y} F(\xi, c_0) \, d\xi$$

$$2 \psi_0' \psi_3' + \psi_2'' = -c_1 F_1(y, c_0) \Rightarrow$$

$$\psi_3 = \frac{1}{4} F_0(y, c_0) \mp \frac{c_1}{2} \int_{0}^{y} F_1(\xi, c_0) \, d\xi$$

where $C$ is a constant and $F$ and $F_1$ are defined above. Since the Taylor-Goldstein equation is invariant under the transformation $k \to -k$, $c \to c$, by symmetry it is anticipated that $c_1 = 0$, though this does not affect results to the order to be considered. Hence,
\[ \phi \sim C_1 \exp(ky - \frac{f}{2k}) + C_2 \exp(-ky + \frac{f}{2k}) \]  
\( (3.97) \)

as \( k \to \infty \), where \( C_{1,2} \) are constants and

\[ f = \int_{0}^{y} \left( -\frac{\bar{u}''}{\bar{u} - c_0} + \frac{JN^2}{(\bar{u} - c_0)^2} \right) d\xi \]  
\( (3.98) \)

This expression is valid throughout \(-1 < y < 1\) (or of course \( -\infty < y < \infty \)) in the absence of critical layers. The boundary conditions \( \phi(\pm 1) = 0 \) require

\[ C_1 \exp(k - \frac{f(1)}{2k}) + C_2 \exp(-k + \frac{f(1)}{2k}) = 0 \]

\[ C_1 \exp(-k - \frac{f(-1)}{2k}) + C_2 \exp(k + \frac{f(-1)}{2k}) = 0 \]

Of course, the second term in the first equation and the first term in the second equation are exponentially small as \( k \to \infty \), so the only way a nontrivial (at least one of \( C_{1,2} \) nonzero) solution can be obtained is for \( f \to -\infty \) as \( y \to 1 \) and \( f \to \infty \) as \( y \to -1 \). But as \( f \) is derived from an integral, this expression's integrand must become singular at the endpoints, if only regular modes are under consideration. This means, then, that \( c_0 \) equals \( \bar{u}(\pm 1) \). That is, there are two values of \( c_0 \) that allow the boundary conditions to be satisfied, at least in an asymptotic sense. If \( C_1 = 0 \), then \( \phi \) satisfies the boundary condition at \( y = 1 \) asymptotically for large \( k \). However, to satisfy the boundary condition at \( y = -1 \) without taking \( C_2 = 0 \) requires that \( c_0 = \bar{u}(-1) \), so \( f(y) \) becomes infinite like

\[ -\frac{JN^2(-1)}{\bar{u}^2(-1)} \frac{1}{y + 1} \]

and then \( \phi \to 0 \) as \( y \to -1+ \) for large \( k \). Similarly, if \( C_2 = 0 \), then the boundary condition at \( y = -1 \) is automatically satisfied (asymptotically), and the the condition at \( y = 1 \) requires \( c_0 = \bar{u}(1) \), so \( \phi \to 0 \) as \( y \to 1- \). Admittedly, this is a rather heuristic argument, and the question of what order the limits \( k \to \infty \) and \( y \to \pm 1 \) is taken could be raised. However, these results are borne out by the numerics, and will be considered valid. Of course, in these cases, the series for \( \psi \) is no longer valid near \( y = \pm 1 \), so the original equation would have to be re-analyzed near these points. However, if the problem does not become singular in this way, a nontrivial solution to the boundary value problem cannot be found, as \( \phi \) will grow exponentially at one of the boundaries.

It might be asked whether large \( k \) critical layer modes exist. It can be seen from the plots that they do in this case. In the non-stratified cases (between flat plates) considered by BB, they do not. Following this work's analysis, if there is a critical layer, then for \( J \neq 0 \) it should be noted that \( \psi_2 \) and \( \psi_3 \) blow up like \( 1/(y - y_c) \) and \( 1/(y - y_c)^2 \), respectively, as \( y \to y_c \). This suggests a rescaling in the Taylor-Goldstein equation of
\[ y - y_c = \frac{\overline{y}}{k} \]

By symmetry, it is expected that the term \( c_1 \) in the expansion for \( c \) be zero. If \( \bar{u}''_c / \bar{u}'_c = O(1) \), the leading order equation for \( \phi \) near the critical layer for large \( k \) becomes

\[ \overline{\phi''} - \overline{\phi} + \frac{JN^2}{\bar{u}'_c^2 \overline{y}^2} \overline{\phi} = 0 \quad (3.99) \]

where \( \overline{\phi} \) is the leading order term in an expansion in \( 1/k \) for \( \overline{\phi} = \phi(y_c + \overline{y}/k) \). The solution to this equation must have two properties:

i) It must match, as \( \overline{y} \to \pm \infty \) to the limit of the WKB (outer) solution as \( y \to y_c \pm \).

ii) Assuming \( 1/k \) is large compared with the critical layer scaling, it must be able to match to some consistent critical layer solution as \( \overline{y} \to 0 \pm \).

If \( c_0 = \bar{u}(y_c) \), the WKB solution can be written as

\[ \phi = C_1 \exp(-k(y - y_c) + \frac{\psi}{k}), \text{ region 1} \]

\[ \phi = C_2 \exp(k(y - y_c) - \frac{\psi}{k}), \text{ region 2} \]

where region 1 denotes \( y_c < y < 1 \) and \( y - y_c \gg O(1/k) \) and region 2 denotes \( -1 < y < y_c \) and \( y - y_c \gg O(1/k) \). \( \psi \) is taken to be

\[ \psi = \frac{1}{2} \int_1^y F(\xi, c_0) \, d\xi, \text{ region 1} \]

\[ \psi = \frac{1}{2} \int_{-1}^y F(\xi, c_0) \, d\xi, \text{ region 2} \]

Although the above solution is invalid when \( y - y_c = O(1/k) \), it satisfies the boundary conditions asymptotically and represents a valid WKB expansion for \( \phi \) away from \( y = y_c \). The expression for \( \phi \) has been written in terms of \( y - y_c \) to facilitate the matching process, which will now be sketched.

Introduce the two matching variables

\[ y_\eta = (y - y_c) \eta \]

above the critical layer, and

\[ y_\zeta = -(y - y_c) \zeta \]

below the critical layer. It is required that \( k \gg \eta, \zeta \gg 1 \), and that \( y_\eta, \zeta \) are fixed, positive quantities. For \( |\overline{y}| \) large,
\[ \overline{\phi}_0 \sim A e^{\overline{y}} + B e^{-\overline{y}} \]

where the constants \( A \) and \( B \) may differ above and below the critical layer, and it is anticipated that matching above the critical layer will require \( A = 0 \) above and \( B = 0 \) below. In terms of the intermediate variable \( y_\eta, \overline{y} = k y_\eta / \eta \), which tends to \( \infty \) as \( k \to \infty \). Then above the critical layer,

\[ \overline{\phi}_0 \sim B e^{-\frac{k}{\eta} y_\eta} \]

and similarly, below the critical layer,

\[ \overline{\phi}_0 \sim A' e^{-\frac{k}{\zeta} y_\zeta} \]

In terms of the matching variables, the outer solution becomes

\[ \phi \sim C'_1 \left( \frac{y_\eta}{\eta} \right)^a \exp \left( -\frac{k y_\eta}{\eta} + b \frac{\eta}{y_\eta} \right) \quad (3.100) \]

above the critical layer, and

\[ \phi \sim C'_2 \left( \frac{y_\zeta}{\zeta} \right)^{-a} \exp \left( -\frac{k y_\zeta}{\zeta} + b \frac{\zeta}{y_\zeta} \right) \quad (3.101) \]

below the critical layer. In the above, \( C'_{1,2} \) are related to \( C_{1,2} \) by

\[ C'_1 = C_1 \exp \left( \int_1^{y_e} \left( F_0(\xi, c_0) - \frac{a}{y - y_c} - \frac{b}{(y - y_c)^2} \right) dy \right) - a \log |1 - y_c| + \frac{b}{1 - y_c} \quad (3.102) \]

with a similar formula for \( C'_2 \). In the above, \( a \) and \( b \) simply come from a Laurent expansion of \( F_0(y, c_0) \) about \( y = y_c \), and need not be explicitly written down. Clearly, matching is possible, as the second term in each exponential tends to zero as \( \zeta, \eta \to \infty \) and if it is further assumed that \( \log \eta \) and \( \log \zeta \) are both much less than \( k \) as \( k \to \infty \), the multiplicative factors involving these expressions go to 1, and matching can facilitated by taking \( B = C'_1 \), \( A' = C'_2 \).

Thus, matching to the (outer) WKB solution is possible. Now it must be asked if the solutions of (3.99),

\[ \overline{\phi}''_0 - \overline{\phi}_0 + \frac{J N_c^2}{u_c^2 y^2} \overline{\phi}_0 = 0, \]

are compatible with the requirements of a nonlinear critical layer; that is, can solutions of the above be matched, as \( \overline{y} \to 0 \pm \), to some critical layer solution of the Taylor-Goldstein equation? It has been seen that such solutions have the asymptotic form
\[ \phi \sim a |y - y_c|^{\alpha_1} + b |y - y_c|^{\alpha_2} \]
as \( y \to y_c \pm \), where the \( \alpha \)'s satisfy

\[ \alpha (\alpha - 1) + J_c = 0 \]

and \( J_c = J N_c^2 / \bar{u}_c^2 \). Near \( \bar{y} = 0 \), \( \bar{\phi}_0 \) has behavior of the form \( \bar{y}^\alpha \), where \( \alpha \) satisfies the above quadratic equation. If such solutions are interpreted properly, they do have the same leading order behavior as nonlinear critical layer solutions to the original equation. This was not true for the cases considered by BB. The difference here is the evenness of the Frobenius solutions across the critical layer. It is consistent to find solutions \( \bar{\phi}_0 \) which decay exponentially as \( \bar{y} \to \pm \infty \) and which exhibit the proper behavior near \( \bar{y} = 0 \). It has not been proven that such modes exist, only that their existence is plausible. Of course, examples have been shown here where they do exist, but nothing in the way of prediction has been presented. Nor has anything been said about the speed of such modes, unlike in the case of large \( k \) regular waves, which necessarily must move at the speed of \( \bar{u} \) at the walls. In BB, it is shown that these waves do not exist in a homogeneous flow unless \( \bar{u}_c^2 / \bar{u}_c = O(k) \), and for the cases studied here, this condition cannot be satisfied, as it would imply

\[ -2 \tanh y_c = k \beta \]

which is clearly impossible. Attention will now be turned to the nonstratified, interfacial problems.

As mentioned, the problems to be studied here are

\[ \phi'' - k^2 \phi - \frac{\bar{u}''}{\bar{u} - c} \phi = 0 \]

with the boundary conditions (3.88),

\[ (\bar{u} - c) \phi' - \bar{u}' \phi = \frac{g + \sigma k^2}{\bar{u} - c} \phi \]
at \( y = 0 \), and \( \phi(-\infty) = 0 \) for the surface shear, and (3.89),

\[ (\bar{u} - c) \phi' - \bar{u}' \phi = -\rho_d^{-1} g + \sigma k^2 - k c^2 \frac{g + \sigma k^2}{\bar{u} - c} \phi, \]
at \( y = 0 \), and \( \phi(\infty) = 0 \) for the wind-driven shear. The parameters \( g \) and \( \sigma \) will be positive in the physical cases. The problem may be non-dimensionalized by introducing a typical velocity scale \( U \) (say the wind speed) and a typical length scale \( L \) (say the extent over which the shear effectively acts). Then \( g \) and \( \sigma \) may be thought of as the non-dimensional parameters which characterize the problem, although it is perhaps more intuitive to think
of gravity and surface tension as fixed and to vary the characteristic velocity and length scale. Again, the results are fairly generic and so only a few basic shears will be considered here. In the cases studied, $U$ was thought of as either the shear speed at the surface (in the surface shear case) or the free stream speed far from the surface (for the wind-driven flows). In addition, it was assumed that the shear does not vary appreciably far from the surface, so $L$ will be small in some sense. Physically realistic parameters will be introduced later, but typical dispersion relations appear in Figures 3.11-3.14. In Figures 3.13 and 3.14 the curves continue in to the origin. In each figure, $g$ and $\sigma$ were taken to be 10 and 50, respectively, or in terms of physical parameters, $U$ and $L$ both equaled 1. There was little qualitative difference between the two cases, so in the following analysis they will be interchanged freely. However, the wind-driven shear case is the physically more interesting one. It should be noted in passing that there were also few qualitative differences between the modes obtained here and the ones in BB, for a variety of basic flows. However, there can be no regular modes in the situations studied in BB (by the Howard semi-circle theorem) as there can be here.

Following the analysis above, some features of these plots will now be explained for the surface shear case. Similar results hold true for the wind-driven shear case. A large $k$ analysis would proceed as follows. Let

$$\phi = e^S$$

where

$$S \sim k S_0 + S_1 + \frac{S_2}{k} + \cdots$$

as $k \to \infty$. The expansion for $c$ will be deduced from the following analysis. The equation for $\phi$ and boundary condition at $y = 0$ become

$$S'' + S' \frac{k^2}{u - c} = 0 \quad (3.103)$$

and

$$S' = \frac{\overline{u}'}{u - c} + \frac{g}{(u - c)^2} + \frac{\sigma k^2}{(u - c)^2} \quad (3.104)$$

at $y = 0$. Thus again it is found that $S_0 = \pm y$, $S_1$ is a constant, and

$$\pm 2 S'_2 = \frac{\overline{u}''}{u - c}$$

so in general

$$\phi \sim C_1 \exp(ky + \frac{1}{2k} \int_{-\infty}^{-y} \frac{\overline{u}''}{u - c} d\xi) + C_2 \exp(-ky - \frac{1}{2k} \int_{-\infty}^{-y} \frac{\overline{u}''}{u - c} d\xi) \quad (3.105)$$
Figure 3.11: Critical layer branch for free surface flow with $\overline{u} = e^y$

Figure 3.12: Dispersion relation for free surface flow with $\overline{u} = e^y$
Figure 3.13: Critical layer branch for $\bar{u} = 1/(1 - y)$

Figure 3.14: Dispersion relation for $\bar{u} = 1/(1 - y)$
as $k \to \infty$. For $\phi \to 0$ as $y \to -\infty$, $C_2$ must be zero. $\phi(-1) = 0$ would demand the same condition for large $k$. The free surface condition implies

$$k + \frac{1}{2k} \frac{\overline{u}''}{\overline{u} - c} = \frac{\overline{u}'}{\overline{u} - c} + \frac{g}{(\overline{u} - c)^2} + \frac{\sigma k^2}{(\overline{u} - c)^2}$$

at $y = 0$. A dominant balance would seem to require that

$$c - \overline{u}(0) \sim \pm \sigma^{1/2} k^{1/2}$$

(3.106)

as $k \to \infty$ for $\sigma \neq 0$, or

$$c - \overline{u}(0) \sim \pm \frac{g^{1/2}}{k^{1/2}}$$

(3.107)

as $k \to \infty$ for $g \neq 0$. This heuristic analysis is supported by numerics; see Figures 3.15 and 3.16. As in section 1.2, solid lines will denote "exact" (ie, numerically integrated) solutions and dashed lines will represent asymptotic solutions. Again $g = 10$ and $\sigma = 50$. Of course, the expansion for $S$ would potentially have to involve powers of $k^{-1/2}$, but the primary interest here was to determine the behavior of the dispersion relation. For small $k$, further analysis is necessary. It proves useful to consider $\omega$, not $c$, as the phase speed appears numerically to become quite large as $k \to 0$; in fact, numerically like $k^{-1/2}$. Therefore consider

$$\phi'' - k^2 \phi - \frac{k \overline{u}'' \phi}{k \overline{u} - \omega} = 0$$

with

$$\phi' = \frac{k \overline{u}'}{k \overline{u} - \omega} \phi + \frac{g k^2 + \sigma k^4}{(k \overline{u} - \omega)^2} \phi$$

at $y = 0$ and $\phi(-\infty) = 0$. Matching techniques will prove relevant here, as opposed to a strict dominant balance. The numerics would lead to the suspicion that $\omega \sim \omega_s k^\nu$ as $k \to 0$, where $0 < \nu < 1$, so $c \sim \omega_s k^{\nu - 1} \to \infty$ for $k \ll 1$. So consider

$$\phi'' - k^2 \phi - \frac{k^{1-\nu} \overline{u}'' \phi}{k^{1-\nu} \overline{u} - \omega_s} = 0$$

(3.108)

with

$$\phi' = \frac{k^{1-\nu} \overline{u}'}{k^{1-\nu} \overline{u} - \omega_s} \phi + \frac{g k^{2(1-\nu)}}{(k^{1-\nu} \overline{u} - \omega_s)^2} \phi + O(k^{4-2\nu})$$

(3.109)
at \( y = 0 \). Now the first term \( \phi_0 \) in an expansion for \( \phi \) in powers of \( k \) would satisfy

\[
\phi_0'' = 0 \Rightarrow \phi_0 = ay + b
\]

and the boundary conditions imply that \( a \) and \( b \) are both 0. Hence it is to be anticipated this zeroth order solution is not valid at one of the boundaries, presumably \( y = -\infty \). Indeed, the asymptotic condition \( \phi \sim Ae^{ky} \) cannot be satisfied for \( k = 0 \) unless \( A = 0 \). So first an “outer” expansion valid for “moderately-sized” \( y \) will be sought, and matched to a solution valid for (negatively) large \( y \). This will give the proper scaling for \( \omega \); ie, \( \nu \). An expansion of \( \phi \) and \( \omega \), in powers of \( k^{1-\nu} \) gives

\[
\phi_0'' = 0 \Rightarrow \phi_0 = ay + b
\]

\[
\phi_0' = 0, y = 0, \Rightarrow a = 0
\]

At \( O(k^{1-\nu}) \),

\[
\phi_1'' + \frac{\bar{u}''}{\omega_0} b = 0
\]

\[
\phi_1' = -\frac{\bar{u}' b}{\omega_0}, y = 0
\]

so

\[
\phi_1 = -\frac{\bar{u} b}{\omega_0} + c
\]

At \( O(k^{2(1-\nu)}) \),

\[
\phi_2'' + \frac{\bar{u}''}{\omega} \left(c - \frac{b \omega_1}{\omega_0}\right) = 0
\]

\[
\phi_2' = -\frac{\bar{u}'}{\omega} \left(c - \frac{b \omega_1}{\omega_0}\right) + \frac{g b}{\omega_0^2}, y = 0
\]

so

\[
\phi_2 = -\frac{\bar{u}}{\omega} \left(c - \frac{b \omega_1}{\omega_0}\right) + \frac{g b}{\omega_0^2} y + d
\]

If \( \bar{u} \) is considered to vanish exponentially as \( y \rightarrow \), and \( c \neq 0 \), it can now be seen that this series for \( \phi \) becomes disordered when \( y = O(k^{-(1-\nu)}) \). This suggests the rescaling

\[
y = \frac{\bar{y}}{k^{1-\nu}}
\]

The Rayleigh equation becomes, to leading order,
Figure 3.15: Large wavenumber behavior of phase speed for free surface case; $\overline{u} = e^y$; solid lines denote exact solutions and dashed lines denote asymptotic solutions.

Figure 3.16: Large wavenumber behavior of phase speed for free surface case; parameters identical to those in Figure 3.15 but $\sigma = 0$. 

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\[ k^2(1-\nu) \phi'' - k^2 \phi = 0 \] (3.113)

Requiring \( \phi \) to vanish exponentially as \( y \to -\infty \), \( \nu \) would have to equal 1, which contradicts the numerical evidence that \( c \) blows up as \( k \to 0 \). Hence it must be demanded that \( c = 0 \) and then the series for \( \phi \) breaks down when \( y = O(k^{-2(1-\nu)}) \). With the obvious rescaling, the "inner" equation, valid for large (negative) \( y \), is

\[ k^{4(1-\nu)} \phi'' - k^2 \phi = 0 \] (3.113)

Exponentially decaying solutions for \( \phi \) now requires \( \nu = 1/2 \), and so to leading order

\[ \phi = A e^y \]

Matching requires \( A = b \) and \( gb/\omega_0^2 = 1 \), all of which are confirmed by the numerics in Figure 3.17. (The condition at \( -\infty \) was taken as \( \phi \sim e^{ky} \) for these runs.) In addition, Figure 3.18 shows the actual eigenfunction for small \( k \), and the three term series based on (3.110-112); good agreement is attained for \( y \) not large.

In the cases studied in BB, large wavenumber critical layer modes do not exist unless

\[ \frac{\bar{u}'_{\nu}}{u'_{\nu}} = \beta k \]

for some \( \beta = O(1) \). This can be seen as follows. As in the stratified case, introduce the variable

\[ \bar{y} = k(y - y_c) \]

Under this transformation, the Rayleigh equation becomes

\[ \bar{\phi}'' - \left(1 + \frac{1}{k^2} \frac{\bar{u}''_{\nu}(y_c + k^{-1}\bar{y})}{\bar{u}(y_c + k^{-1}\bar{y}) - c}\right) \bar{\phi} = 0 \] (3.114)

where \( \bar{\phi} \) is the eigenfunction near the critical layer. If \( \bar{u}'_{\nu}/\bar{u}'_{\nu} = O(1) \), the leading order equation is

\[ \bar{\phi}''_0 - \bar{\phi}_0 = 0 \]

where the primes denote derivatives with respect to the inner variable. The solutions to the above must be decaying exponentials, to enable matching as described above for the stratified case. But such solutions have behavior near the critical layer which is inconsistent with the type of behavior such solutions must have; namely, that their derivatives be even across the critical layer. Therefore, any high wavenumber critical layer modes must satisfy
the above condition in terms of \( \beta \), and it would appear the ideas developed in BB carry over to this case as well. However, the existence of large \( k \) solutions for \( y_c \) small cannot be ruled out. For if \( y_c \to 0 \) as \( k \to \infty \), then the boundary condition at the interface clearly becomes singular, due to the presence of \( \bar{u}(0) - \bar{u}(y_c) \) in denominators. It is not entirely clear how to deal with critical layers near boundaries, but it will be seen in the next section that under certain parameter regimes, for \( y_c \) small, large \( k \) modes can exist.

Furthermore, the case of limiting values of the wavenumber, that is, those wavenumbers for which the phase speed approaches the free stream speed or speed at infinity (ie, \( y_c \to \pm \infty \)), may also be handled similarly. With the substitution \( \tilde{y} = y - y_c, \tilde{\phi} = \phi(\tilde{y} + y_c) \), the Rayleigh equation becomes

\[
\tilde{\phi}'' - k^2 \tilde{\phi} - \frac{\bar{u}''(\tilde{y} + y_c)}{\bar{u}(\tilde{y} + y_c) - \bar{u}(y_c)} \tilde{\phi} = 0 \tag{3.115}
\]

and is studied for \( y_c \) large, negatively or positively as the case may be. The proper behavior is demanded near the critical layer (\( \tilde{y} = 0 \)), and the boundary condition at the free surface (\( \tilde{y} = -y_c \)) is becomes

\[
\tilde{\phi}' = \left( \frac{\bar{u}'(0)}{\bar{u}(0) - c} + \frac{g + \sigma k^2}{(\bar{u}(0) - c)^2} \right) \tilde{\phi} \tag{3.116}
\]

As \( y_c \to -\infty \), say, this boundary condition is to be imposed at \( \tilde{y} = \infty \). But if bounded solutions to the Rayleigh equation (3.115) will decay like \( e^{-k\tilde{y}} \), it would seem such solutions would not necessarily satisfy the condition (3.116). Of course, it could be argued that this condition is satisfied in an asymptotic sense, for if a decaying exponential is evaluated in (3.116), the trivial statement \( 0 = 0 \) results. Adopting this interpretation, and replacing the boundary condition at \( \infty \) by \( \tilde{\phi} \sim e^{-k\tilde{y}} \), the limiting value of \( \approx 0.75 \) is the same for the surface shear \( \bar{u} = e^y \) here as it is in BB for \( \bar{u} = 1 - e^{-y} \). Under the above substitution, the Rayleigh equation becomes

\[
\tilde{\phi}'' - (k_{lim}^2 - \frac{e^{\tilde{y}}}{e^{\tilde{y}} - 1}) \tilde{\phi} = 0
\]

to be solved subject to \( \tilde{y} \to \pm \infty \) and the proper critical layer behavior at \( \tilde{y} = 0 \). The transformation \( y \to -y \) reduces this to an identical problem in BB, and hence the equal limiting values. This is confirmed in the plots. For the algebraically decaying profiles here, the limiting wavenumber is 0, as can be seen by the way the dispersion curve “loops” around. This can be explained by the fact that as \( y_c \to -\infty \), since the second derivative of the basic shear now decays faster than the shear itself, the governing equation, in the limit \( y_c \to -\infty \), becomes

\[
\tilde{\phi}'' - k_{lim}^2 \tilde{\phi} = 0
\]

if \( k_{lim} \) is \( O(1) \), which as said before, cannot have the proper behavior near \( \tilde{y} = 0 \). Also, there can be no large \( k \) modes here, as the condition
Figure 3.17: Small wavenumber behavior of phase speed for free surface case; $\bar{u} = e^y$; again, exact solutions are solid lines and asymptotic solutions are dashed.

Figure 3.18: Comparison of eigenfunction and asymptotic behavior in free surface case for small wavenumbers; $\bar{u} = e^y$, $k = 0.05$
\[
\frac{\dddot{u}_c}{u_c'} = \beta k
\]

cannot be satisfied, as is easily checked. Hence it is to be expected that \( k_{\text{lim}} \to 0 \) in this case.

To repeat, this approach for determining limiting values of \( k \) replaces the interfacial condition by a zero boundary condition. To the extent that exponentially decaying solutions are being sought, this is consistent. However, it will be seen in the next section that other limiting values of \( k \) are possible, under situations to be described. The above are just sketches of how to analyze such problems. More details will be given when specific examples are considered. With these comments made, attention may now be turned to finding explosive triads in such systems.

### 3.3.2 Appearance of Explosive Triads

A search for explosive resonant triads will now be undertaken. Recall that for such interactions to occur, at least two critical layer modes must be involved. It is also anticipated that the quantity \( \gamma \) which determines the sign of the interaction coefficients will not change sign along a branch of the dispersion relation unless a point of infinite group velocity is crossed (and the dispersion curve is locally parabolic there). The quantity \( \gamma \) was evaluated numerically for chosen waves, using the ideas of section 3.2.2. That is, first the singular behavior of the integrand was determined. Although the theory only requires that non-integrable terms be subtracted off, in practice it is necessary to subtract off all of the singularities, such as \( \log(y - y_c) \). Furthermore, it proves advisable to perform the calculations in terms of the eigenfunction \( \phi \) rather than the the mean particle displacement \( \eta \) owing to the latter's stronger singularity. The integrand is only known from a numerical integration, and the integral is evaluated by the trapezoidal rule using the nodal values on a nonuniform mesh (from an adaptive ODE integrator). This method is applied to the integrand with it's singular part subtracted off for \( y > y_c \), and again for \( y < y_c \). The two results are added in the case of a homogeneous flow, and subtracted for a heterogeneous flow, as seen in the section 3.2.2. The details are again in the numerical appendix. From the results of previous sections, it is known that \( \gamma \) can only change sign along a given branch of the dispersion curve at a point of infinite slope or group velocity, as long as the local behavior of this curve is parabolic there. If such situations do not occur, by choosing a wave on a given branch and evaluating \( \gamma \) there, the sign of \( \gamma \) for all waves on that branch will be known. This was of course checked for several waves on a given branch and found to be true. At least two critical layer modes must participate in such interactions, one of which must have an oppositely signed \( \gamma \) from the other two waves.

However, of all cases studied in the stratified case, including ones not presented here, the critical layer modes were found to have the same sign of \( \gamma \). No points of infinite group velocity were detected in any of these numerical runs except for the Holmboe profile; that is, no other dispersion relations were found with vertical slope at some point. A necessary condition that \( \gamma \) change sign along a branch is that such a point exist. Due to a initial programming error, these integrals were evaluated the way the non-stratified case would
be, by not reversing the sign of the integrand below the critical layer. A sign change was detected in $\gamma$ along a branch without infinite group velocity, contradicting the theoretical results of section 3.1.3. After this error was corrected, this sign change no longer occurred, giving credence to the technique developed in section 3.2.2. It might be thought that $\gamma$ would undergo a change of sign along the critical layer branch that intersects the $k$ axis for the Holmboe profile, due to the infinite group velocity there. However this is not the case numerically, and can be anticipated analytically. First, this point represents the intersection of two branches, and so if the dispersion relation is thought of as in section 3.1.3:

$$D(k, \omega) = 0$$

with $\gamma \propto D_\omega$, then $\gamma$ can only change sign at a point if $D_k \neq 0$ there, as has been explained. This is not the case here, since two branches intersect. The local behavior at this point, where $D_\omega = 0$, is not

$$D_{\omega\omega} \delta \omega^2 + D_k \delta k = 0$$

but rather

$$D_{\omega\omega} \delta \omega^2 + \frac{1}{2} D_{k\omega} \delta k \delta \omega + D_{kk} \delta k^2 = 0$$

which locally gives the slope of two branches of the dispersion relation. Furthermore, as $\gamma$ is directly proportional to

$$\int_{-1}^{1} \left( \frac{u''}{(\bar{u} - c)^2} + \frac{2 J N^2}{(\bar{u} - c)^3} \right) \phi^2 dy,$$

for regular modes, and since if $(\phi(y, k), c)$ are an eigenfunction/ eigenvalue pair then so is $(\phi(-y, k), -c)$ for the profiles under consideration, it can be seen by a simple substitution that $\gamma$ is of opposite sign on the upper and lower regular mode branches. However, for critical layer modes, $\gamma$ involves integrals of the same integrand above, only interpreted as

$$- \int_{-1}^{y_{c-}} + \int_{y_{c+}}^{1} - \text{SP}$$

using notation from before. As the factor multiplying $\phi^2$ is here odd, and $\phi$ on the lower branch is related to $\phi$ on the upper branch by the transformation $y \rightarrow -y$, it can be seen that $\gamma$ is indeed the same sign on the upper and lower critical layer branches.

So again, the work of Collins and Maslowe (1988), which purported to investigate explosive degenerate triads in the Holmboe profile, is questioned. No mention is made how the singular integrals in the interaction coefficients are evaluated. In an earlier paper by one of these authors (Burns and Maslowe, 1983), such integrals in a homogeneous flow were handled in the way derived here by appealing to the work of Hadamard (1923); namely,
the singular parts were subtracted off near the singularity. If the integrals in the stratified case were evaluated in this "intuitive" way, by not reversing the sign of the integral over \(-\infty < y < y_c\), then the results for these coefficients would seem suspect. At any rate, the search here proved futile, so attention will now be turned to the interfacial problems. These are the more physically interesting problems anyway.

For the cases examined, studied numerically in the same way as the stratified case was above (with the integrals interpreted differently, of course), the only cases encountered thus far with two critical layer branches were algebraically decaying shears, as explained above. All the other cases only had one critical layer branch for the values of the parameters considered. These were presumably linearly stable, as the none of the regular modes exhibited the "gaps" characteristic of dispersion relations with imaginary parts. The above examples were largely for illustrative purposes, with little worry given to the physicality of the parameter values chosen. Nonetheless, it would be interesting to know if explosive triads exist in these flows. The only candidates are algebraically decaying flows, as they were the only ones with two critical layer branches in the range of parameters studied above. When the basic shear is

\[
\bar{u}(y) = 1 - \frac{1}{1 + y}
\]

the dispersion relation looks much like Figure 3.14 for similar ranges of the parameters. There is an upper critical layer loop and a lower one. As will be seen in the next section when the derivation of the interaction coefficients is generalized, the quantity which determines the sign of these coefficients is

\[
\gamma = \left( \frac{2 \phi'}{\bar{u} - c} - \frac{\bar{u}' \phi}{(\bar{u} - c)^2} + \frac{2 \rho_d c k \phi}{(\bar{u} - c)^2} \right) \phi \bigg|_{y=0} + \int_{0}^{\infty} \frac{\bar{u}'' \phi^2}{(\bar{u} - c)^2} \, dy
\]  

(3.117)

where the density ratio is taken to be that of water to air: \(\rho_d = \rho_w / \rho_a \approx 10^3\). It will be seen that this quantity is always positive for upper-branch regular modes (\(\bar{u}_{\text{max}} > c\)) and negative for lower-branch regular modes (\(\bar{u}_{\text{min}} < c\)). Adopting the technique for dealing with singular integrals in this case, it is found that the upper loop of the critical layer modes has \(\gamma > 0\) and the lower loop has \(\gamma < 0\). The quantity \(D_k\) from section 3.1 has also been evaluated, and it is confirmed that this quantity does not change sign across the point of infinite group velocity. However, it should be noted that this case does not give rise to explosive triads, because there is no way, say, a positive \(\gamma\) branch may be redrawn off of another positive \(\gamma\) branch and intersect a negative \(\gamma\) branch. The exponentially decaying profiles are the more realistic, anyway, and it will prove necessary to vary the parameters of the problem to produce another critical layer branch to potentially participate in explosive interactions.

In searching for explosive triads, then, it will prove useful to adopt a more physically realistic approach. With that in mind, attention will be confined to the wind-driven shear case, and the gravity \(g\) and density-scaled surface tension \(\sigma\) will be held constant.
Additionally, it will be assumed that the shear varies appreciably only near the interface, so the length scale \( L \) will be taken as some physically small amount, say of the order of a millimeter. Then the parameter to be varied is the velocity scale \( U \), which will taken as the free stream speed or speed far from the surface. Then in cgs units, \( g \approx 10, \sigma = T/\rho_w \approx 50 \), and the length scale of the problem will be taken as \( L = 0.5 \) or \( 5 \). Of course, the former is more physically relevant, if such flows are thought to vary over length scales on the order of millimeters, but the latter will offer a comparison with some analytical results. Throughout this analysis the basic flow will be taken as

\[
\bar{u}(y) = 1 - e^{-y},
\]

or \( U\bar{u}(y/L) \) in dimensional units. In terms of the scaling, \( g \rightarrow gL/U^2 \) and \( \sigma \rightarrow \sigma/(LU^2) \).

Figures 3.19-24 exhibit the behavior of the dispersion relation for increasing \( U \). It should be noted that while the lower branch of the dispersion relation barely changes, the upper branch moves closer to the line \( \omega = k \), which is the "barrier" between regular and critical layer modes. When it crosses over this line, presumably an instability is present, as this is characteristic behavior of such modes. This was not checked, though. Interestingly, however, at this stage a critical layer mode appears in such a way that the regular mode seems to "pass through" the dividing line \( \omega = k \). A few comments on this behavior are in order.

Recall the boundary condition at the interface \( y = 0 \) is

\[
(u - c) \phi' - \bar{u} \phi = -\rho_d^{-1} \frac{g + \sigma k^2 - k c^2}{u - c} 
\]

(3.118)

If as before, the limiting value of \( k \), that this, the value of \( k \) for which in this case \( y_c \rightarrow \infty \), is sought, the substitution \( \tilde{y} = y - y_c \) is made and the Rayleigh equation becomes as in BB

\[
\tilde{\phi}'' - (k_{\text{lim}}^2 + \frac{e^{-\tilde{y}}}{1 - e^{-\tilde{y}}}) \tilde{\phi} = 0
\]

(3.119)

with the boundary conditions \( \tilde{\phi} \rightarrow 0 \) as \( \tilde{y} \rightarrow \infty \) and

\[
\tilde{\phi} = (-1 - \rho_d^{-1}(g + \sigma k^2 - k)) \phi
\]

(3.120)

as \( \tilde{y} \rightarrow -\infty \), for \( y_c \rightarrow \infty \). If as before exponentially decaying behavior for \( \tilde{\phi} \) is required, then

\[
\tilde{\phi} \sim A e^{-k\tilde{y}}, \tilde{y} \rightarrow \infty
\]

\[
\tilde{\phi} \sim A e^{(k^2+1)^{1/2}\tilde{y}}, \tilde{y} \rightarrow -\infty
\]

Now if as before, the boundary condition at \( -\infty \) is thought of being satisfied in an asymptotic sense, since the decaying exponential behavior makes both sides of (3.119)
Figure 3.19: Dispersion relation for wind-driven shear; $\bar{u} = U e^{-y}$, $U = 25$

Figure 3.20: $U = 31.8$
Figure 3.21: $U = 35$

Figure 3.22: $U = 40$
Figure 3.23: $U = 75; \sigma \approx 0$

Figure 3.24: $U = 100; \sigma \approx 0$
vanish, then again this boundary condition is replaced by the requirement that $\tilde{\phi}$ simply tend to 0 as $\tilde{y}$ goes to $-\infty$, and as in BB the limiting wavenumber is found to be $\approx 0.75$. This of course takes into account the condition that $\tilde{\phi}$ have the proper critical layer behavior at $\tilde{y} = 0$. If, however, the above asymptotic form is substituted into the boundary condition, it must be that

$$ (k^2 + 1)^{1/2} = -1 - \rho_d^{-1}(g + \sigma k^2 - k) $$

(3.121)

Solutions to (3.120) may not yield the proper behavior at $\tilde{y} = 0$. If they do, then since in the cases considered, $\rho_d$ is large, $k_{\text{lim}}$ satisfies

$$ k_{\text{lim}} = \frac{1 \pm (1 - 4 g \sigma)^{1/2}}{2 \sigma} $$

(3.122)

which can be confirmed from the plots; these are the points of intersection of both the regular and singular modes with the line $\omega = k$, or $c = 1$. This holds for $\sigma \neq 0$ of course. When the length scale is taken on the order of centimeters (ie, $L = 5$), $\sigma$ is much smaller than $g$, and in some sense is effectively 0. In this case, $k_{\text{lim}}$ would have to be equal to $g$, which is also confirmed by the plots in Figures 3.23 and 3.24.

In any case, it can be seen in such cases that a second critical layer branch appears, and so the possibility of explosive triads occurring is raised. This will be discussed momentarily, but first let it be said that the above analysis predicts when the regular modes intersect the dividing line $\omega = k$ and presumably become unstable. It can make no prediction as to whether the critical layer modes “continue” off of these modes, as they do in the plots. As mentioned, such modes must behave like a nonlinear critical layer at $\tilde{y} = 0$, which must be determined numerically. Based on this conjecture, for the more physical cases when $L = 0.5$ and for the values of $g$ and $\sigma$ used, the value of the critical wind speed when this behavior occurs is

$$ U_{\text{crit}} = (2 g L)^{1/2} \approx 31.6 $$

There is indeed seen to some sort of transition in the graphs near this value. For the less physical examples with $L = 5$, such behavior would always occur, but as will be seen, for explosive interactions to occur, the limiting wavenumber must be fairly close to the “original” critical layer branch, and of course this wavenumber decreases as $U$ increases.

Using the method developed here for evaluating singular integrals, it is found that explosive triads do exist in these cases. These are shown in Figure 3.25, where branches with waves having negatively signed $\gamma$ are hatched. Of course, these flows are potentially unstable, so it is not clear how important these interactions are. More will be said about this later. In all these figures, $\gamma$ is positive along the upper regular mode branch and negative along the lower regular mode branch. The original critical layer branch (which really separates) remains negative throughout, and the second critical layer branch (the “passing through” of the upper regular mode) is positive. Of course, this second critical layer branch really consists of two parts.
Figure 3.25: Dispersion relation for wind-driven shear; $\bar{u} = U e^{-y}$, $U = 40$. Segments with $\gamma < 0$ are hatched.

Figure 3.26: An explosive interaction.
Although the branches appear to intersect one another, in fact the local behavior is hyperbolic. This was confirmed numerically; $D_o$ changes sign across this point, while $D_k$ does not, indicating a point of vertical slope. In addition, away from this point, $D_k$ changed sign while $D_o$ did not, indicating a point of zero slope that would be required to make the curves conform to such a shape (a skewed hyperbola). In Figure 3.26, an explosive interaction is shown. It should be clear that the lower branch of the regular mode may be redrawn off the small upper segment of the critical layer branch near the limiting wavenumber 0.75. This regular branch will intersect a positive $\gamma$ critical layer branch (the "passing through" of the upper regular mode, so to speak). Thus symmetrically written, this triad will involve waves with negative $\gamma$ and so be explosive. The lower branch of the "original" critical layer branch may be redrawn off itself, as in Figure 3.26. If close enough to the turning point, it will also intersect the same "new" critical layer branch from the previous statements, again leading to an interaction of waves with negative $\gamma$ and hence an explosive triad.

Thus, a fluid dynamical system with a continuously varying basic shear has been produced which exhibits explosive resonant triads. Unfortunately, these flows may be unstable. However, if the instability is weak, the finite-time singularity of the system may be dominant. It seems quite an enormous project, though, to perform a full numerical simulation on these interfacial problems, which are rotational. It would of course be interesting to have a physical system to perform such a comparison on, but that was not the purpose of this project. Rather, as far as fluid mechanics is concerned, it has been shown how to evaluate the parameters associated with such interactions, and numerical evidence has supported the theory here. Furthermore, the interesting aspect of the regular modes "becoming" critical layer modes may offer hints in other problems as to when explosive triads may occur, if this phenomenon does not represent too strong a linear instability. Some of the ideas developed will now be generalized in way of closing.
4. MISCELLANEOUS RESULTS

In this part of the thesis, a collection of results pertaining to explosive triads will be presented. The main topic will be the extension of the previous analysis to general, layered flows; that is to say, flows which have piecewise continuous velocity and density profiles. In addition, a physical interpretation will be given to such waves in these fluid systems. Finally, comments will be made concerning the strength of these interactions in weakly (linearly) unstable systems.

4.1 GENERAL LAYERED FLOWS

In this section perturbations to the following flow configuration will be considered:

\[
\begin{align*}
\bar{u}_1, \rho_1 & \quad y = d_0 \\
\bar{u}_j, \rho_j & \quad d_1 \\
\bar{u}_{j+1}, \rho_{j+1} & \quad d_j \\
\bar{u}_n, \rho_n & \quad d_{N-1} \\
\end{align*}
\]

where either of the boundaries \(d_0\) or \(d_N\) may be infinite in extent. In particular, the presence of resonant triads will be assumed. It has already been seen that the existence of explosive triads in some simple interfacial flows, though still difficult to analyze, can be established relatively simply. The signs of the interaction coefficients in the triad equations depend on a single quantity for each wave in the set, and changes in the sign of this quantity may be discerned from the dispersion relation. It will be seen in these more complex flows that these results still hold true, though they will appear in a slightly more complicated form. Essentially, the results for a single layer may be "connected" to the others.
So assume the unperturbed interfaces are at \( y = d_i \) for \( i = 0, \ldots, N \), and that the velocity is unidirectional in each layer with basic shear in the \( x \) direction \( \overline{u}_i(y) \) for \( d_{i-1} < y < d_i \), and constant density \( \rho_i \) in each such layer \( i \). In appendix B, these results will be extended to three dimensions, when the basic flow is in both the \( x \) and \( z \) directions, and the density in each layer is nonconstant. All of these quantities vary with \( y \) only, of course. In addition, interfacial tension effects will be incorporated by assigning a quantity \( T_i \) to each layer; the tension across each interface \( y = d_i \) will be defined to be \( T_{i+1} - T_i \). If a perturbation to the interface at \( y = d_i \) is denoted by \( \eta_i \), and the velocity components and pressure in layer \( i \) denoted by \( (u_i, v_i) \) and \( p_i \) respectively, then adopting the perturbation approach of section 3.1.1, once again

\[
v_{j0} = S_j \phi_j e^{i\theta} + *
\]

where \( \theta = kx - \omega t \) and \( \phi_j \) satisfies the Rayleigh equation in layer \( j \):

\[
L_R \phi_j \equiv \phi_j''' - k^2 \phi_j - \frac{k \overline{u}_j''}{\overline{w}_j} \phi_j = 0
\]

(4.1)

with \( \overline{w}_j = k \overline{u}_j - \omega \). Then

\[
u_{j0} = \frac{i}{k} S_j \phi_j e^{i\theta} + *
\]

\[
p_{j0} = -\frac{\rho_j}{k^2} (\overline{w}_j \phi_j' - k \overline{u}_j \phi_j) S_j e^{i\theta} + *
\]

and the boundary conditions that define the interface and the dispersion relation are

\[
i \eta_j = \frac{S_j \phi_j}{\overline{w}_j} = \frac{S_{j+1} \phi_{j+1}}{\overline{w}_{j+1}}
\]

(4.2)

and

\[
-\frac{g \rho_j \phi_j S_j}{\overline{w}_j} + \frac{g \rho_{j+1} \phi_{j+1} S_{j+1}}{\overline{w}_{j+1}} + \frac{\rho_j}{k^2} (\overline{w}_j \phi_j' - k \overline{u}_j \phi_j) S_j - \frac{\rho_{j+1}}{k^2} (\overline{w}_{j+1} \phi_{j+1}' - k \overline{u}_{j+1} \phi_{j+1}) S_{j+1}
\]

\[
= T_j k^2 \frac{\phi_j}{\overline{w}_j} S_j - T_{j+1} k^2 \frac{\phi_{j+1}}{\overline{w}_{j+1}} S_{j+1}
\]

(4.3)

both at \( y = d_j \). These are respectively the kinematic and dynamic conditions at the interface; see equations (3.13) and (3.14). If again a triad is taken at leading order; ie,

\[
v_{j0} = S_j^1 \phi_j e^{i\theta_1} + S_j^2 \phi_j e^{i\theta_2} + S_j^3 \phi_j e^{i\theta_3} + *
\]

where \( \theta_i = k_i x + \omega_i t \) for \( i = 1, 2, 3 \) and
\[ \theta_1 + \theta_2 + \theta_3 = 0 \]

then it would be anticipated that the amplitudes \( S_j^1 \) satisfy equations of the form

\[ S_{jT}^1 = \gamma_j^1 S_j^{2*} S_j^{3*} \quad (4.4) \]

with again the other two equations in the set resulting from a cyclic permutation of the indices in (4.4). This would be required to suppress the secularity resulting from the term (for example)

\[ -i S_j^{2*} S_j^{3*} \phi_{j11}^1 e^{i\Theta_i} \]

arising at \( O(\varepsilon) \) in the expansion for \( v_j \). The coefficient \( \gamma_j^1 \) would be chosen so that the following system has a solution:

\[ L_R \phi_{j11}^1 = -\frac{\gamma_j^1}{\overline{\omega_j}^1} (\phi_j^{1'''} - k_1^2 \phi_j^1) + \frac{N(\phi_j^2, \phi_j^3)}{\overline{\omega_j}^1}, \quad (4.5) \]

where \( \overline{\omega_j}^i = k_i \overline{u_j} - \omega_i \) and \( N \) is some nonlinear function of its arguments, to be displayed shortly. The distinction between superscripts and exponents should be clear. Equation (4.5) is to be solved subject to the following boundary conditions at \( y = d_j \):

\[ -\eta_{j11}^1 = S_j^{2*} S_j^{3*} (\phi_{j11}^{1} + f_j^1 + \gamma_j^1 g_j^1) = S_{j+1}^{2*} S_{j+1}^{3*} (\phi_{(j+1)11}^{1} + f_{j+1}^1 + \gamma_{j+1}^1 g_{j+1}^1) \quad (4.6) \]

\[ S_j^{2*} S_j^{3*} (\mu_j^1 \phi_{j11}^{1'} + \nu_j^1 \phi_{j11}^1) - S_{j+1}^{2*} S_{j+1}^{3*} (\mu_{j+1}^1 \phi_{(j+1)11}^{1'} + \nu_{j+1}^1 \phi_{(j+1)11}) \]

\[ = S_j^{2*} S_j^{3*} (m_j^1 + \gamma_j^1 n_j^1) - S_{j+1}^{2*} S_{j+1}^{3*} (m_{j+1}^1 + \gamma_{j+1}^1 n_{j+1}^1) \quad (4.7) \]

where the expressions underlying any variables above will also be revealed presently. Let

\[ S_j = S_j^{1*} S_j^{2*} S_j^{3*} \quad (4.8) \]

Although it is known from chapter 3 that critical layers play a major role in these problems and that any integrals arising from a solvability condition must be interpreted appropriately, it is was shown there that formal results still hold true. Therefore, any applications of the Fredholm alternative or integration by parts will be done without regard to rigor, knowing that as long as the final result is handled properly, these manipulations are justified. Then multiplying (4.5) by \( \phi_j^1 \) and integrating from \( y = d_{j-1} \) to \( d_j \), it is found that
\[
(\phi_j^1 \phi_j^{1'11} - \phi_j^{1'} \phi_j^{111})|_{d_{j-1}}^{d_j} + \gamma_j^1 \int_{d_{j-1}}^{d_j} k_1 \frac{u_j'' \phi_j^{12}}{\omega_j^{12}} \, dy \\
- \int_{d_{j-1}}^{d_j} \frac{\phi_j^1}{\omega_j^1} N(\phi_j^2, \phi_j^3) \, dy = 0
\] (4.9)

Essentially, the trick now is to multiply the inhomogeneous kinematic condition (4.6) by the homogeneous dynamic condition (4.3), and the inhomogeneous dynamic condition (4.7) by the homogeneous kinematic condition (4.2), and subtract the two. This will assist in eliminating the inhomogeneous solution from the solvability condition. Thus, at \( y = d_j \),

\[
S_j \left( \frac{\mu_j^1}{\omega_j^1} \phi_j^1 \phi_j^{1'} \phi_j^{111} + \frac{\nu_j^1}{\omega_j^1} \phi_j^1 \phi_j^{111} \right) - S_{j+1} \left( \frac{\mu_{j+1}^1}{\omega_{j+1}^1} \phi_{j+1}^1 \phi_{j+1}^{1'} \phi_{j+1}^{111} + \frac{\nu_{j+1}^1}{\omega_{j+1}^1} \phi_{j+1}^1 \phi_{j+1}^{111} \right) \\
= S_j \left( m_j^1 + \gamma_j^1 n_j^1 \right) \frac{\phi_j^1}{\omega_j^1} - S_{j+1} \left( m_{j+1}^1 + \gamma_{j+1}^1 n_{j+1}^1 \right) \frac{\phi_{j+1}^1}{\omega_{j+1}^1}
\] (4.10)

\[
S_j \left( \frac{\mu_j^1}{\omega_j^1} \phi_j^{1'} \phi_j^{11} + \frac{\nu_j^1}{\omega_j^1} \phi_j^1 \phi_j^{111} \right) - S_{j+1} \left( \frac{\mu_{j+1}^1}{\omega_{j+1}^1} \phi_{j+1}^{1'} \phi_{j+1}^{11} + \frac{\nu_{j+1}^1}{\omega_{j+1}^1} \phi_{j+1}^1 \phi_{j+1}^{111} \right) \\
= -S_j \left( f_j^1 + \gamma_j^1 g_j^1 \right) \left( \mu_j^1 \phi_j^{1'} + \nu_j^1 \phi_j^1 \right) + S_{j+1} \left( f_{j+1}^1 + \gamma_{j+1}^1 g_{j+1}^1 \right) \left( \mu_{j+1}^1 \phi_{j+1}^{1'} + \nu_{j+1}^1 \phi_{j+1}^1 \right)
\] (4.11)

Then subtracting (4.10) and (4.11), one obtains

\[
S_j \frac{\mu_j^1}{\omega_j^1} (\phi_j^1 \phi_j^{1'11} - \phi_j^{1'} \phi_j^{111})|_{d_j} - S_{j+1} \frac{\mu_{j+1}^1}{\omega_{j+1}^1} (\phi_{j+1}^1 \phi_{j+1}^{1'11} - \phi_{j+1}^{1'} \phi_{j+1}^{111})|_{d_j} \\
= S_j (A_j^1 + B_j^1)|_{d_j} - S_{j+1} (A_{j+1}^1 + B_{j+1}^1)|_{d_j}
\] (4.12)

where once again the notation will be revealed shortly. In way of further notation, introduce the following:

\[
\int_{d_{j-1}}^{d_j} \frac{k_1 u_j'' \phi_j^{12}}{\omega_j^{12}} \, dy \equiv L_j^1
\] (4.13)

\[
\int_{d_{j-1}}^{d_j} \frac{\phi_j^1}{\omega_j^1} N(\phi_j^2, \phi_j^3) \, dy \equiv M_j^1
\] (4.14)

Now, by multiplying all such conditions (4.9) by \( \mu_j^1/\omega_j^1 \), summing for \( j = 1, 2, \ldots N \), and using (4.12), it is found that

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\[ S_1 ((A_1^1 + B_1^1)|_{d_1} + \frac{\mu_1^1}{\omega_1^1} (\gamma_1^1 L_1^1 - M_1^1)) + S_2 ((A_2^1 + B_2^1)|_{d_2} + \frac{\mu_2^1}{\omega_2^1} (\gamma_2^1 L_2^1 - M_2^1)) + \cdots \]

\[ + S_N ((A_N^1 + B_N^1)|_{d_{N-1}} + \frac{\mu_N}{\omega_N} (\gamma_N^1 L_N^1 - M_N^1)) = 0 \quad (4.15) \]

Yet more notation is to be introduced. Using the kinematic condition (4.12), write

\[ S_j^k = s^k_j S^k_{j+1} \]

where

\[ s^k_j = \frac{\omega_j}{\omega_{j+1}} \frac{\phi_{j+1}^k}{\phi_j^k} \]

for \( j = 1, \ldots, N - 1 \) and \( j = 1, 2, 3 \). Then the triad equations

\[ S_{jT}^1 = \gamma_j^1 S^2_{j} S^3_{j} \]
\[ S_{(j+1)T}^1 = \gamma_{j+1}^1 S^2_{j+1} S^3_{j+1} \]

can be used to imply that

\[ \gamma_{j+1}^1 = \frac{s^2_j s^3_j \gamma_j^1}{s^1_j} \quad (4.16) \]

It should kept in mind that the wave amplitudes \( S_j^k \) are not independent, but related via the interfacial conditions (4.2) and (4.3). Finally, note that

\[ S_{j+1} = \frac{S_j}{s_j^1 s_j^2 s_j^3} \quad (4.17) \]

With these relations, and the notation above manipulated, a single equation for \( \gamma_1^1 \) will be obtained, which will give the crucial information regarding the sign of the interaction coefficients. To this end, we have the following:

\[ \mu_j^1 = \frac{\rho_j \omega_j^1}{k_1^2} \]
\[ \nu_j^1 = -\rho_j \left( \frac{g}{\omega_j^1} + \frac{\bar{u}_j^1}{k_1} \right) - \frac{T_j k_1^2}{\omega_j^1} \]

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\[ f^j_1 = \frac{1}{\omega^j_1} \left( -\frac{k_2 \phi^2_2 \phi^3_2}{k_3 \omega^2_1} - \frac{k_3 \phi^3_2 \phi^3_2}{k_2 \omega^3_1} + \frac{\bar{u}^j_1 k_2 \phi^2_2 \phi^3_1}{\omega^2_1 \omega^3_1} + \frac{\bar{u}^j_1 k_3 \phi^2_2 \phi^3_1}{\omega^3_1 \omega^3_1} \right) \]
\[ \left( -\frac{\phi^2_2 \phi^3_2}{\omega^2_1} - \frac{\phi^3_2 \phi^2_2}{\omega^3_1} \right) \]
\[ g^1_j = -\frac{\phi^1_2}{\omega^2_1} \]
\[ m^1_j = (\rho_j g + T_j k^2_1) f^1_1 - \frac{\rho_j}{k_1} \left( \frac{\phi^{2'}_2 \phi^{3'}_2}{k_2} + \frac{\phi^{2'}_2 \phi^{3'}_2}{k_3} - \frac{\phi^{3'}_2 \phi^{3''}_2}{k_3} + \frac{\phi^{2''}_2 \phi^{3}_2}{k_2} \right) \]
\[ -\rho_j \left( \frac{\phi^2_2 \phi^3_2 \omega^2_1}{\omega^3_1} + \frac{\phi^2_2 \phi^3_2 \omega^3_1}{\omega^2_1} \right) \]
\[ n^1_j = -\frac{\rho_j \phi^1_2}{k^2_1} + (g \rho_j + T_j k^2_1) g^1_j \]

and then

\[ A^1_j = (m^1 + \gamma^1_j n^1_j) \frac{\phi^1_2}{\omega^1_2} \]
\[ B^1_j = (f^1_1 + \gamma^1_j g^1_1) \left( \mu^1_j \phi^1_2 + \nu^1_j \phi^1_2 \right) \]

all imposed at interfaces, of course. The analysis proceeds much as in section 3.1.1. \( N(\phi^2_2, \phi^3_2) \) simply arises from the nonlinear terms in the Euler equations, as in section 3.1.1:

\[ N(\phi^2_2, \phi^3_2) = -k_1 \left( \frac{\phi^2_2}{k_2} (\phi^{3''}_2 - k^2_2 \phi^2_2) + \frac{\phi^{3'}_2}{k_3} (\phi^{2''}_2 - k^2_2 \phi^2_2) \right) \]
\[ -\frac{\phi^2_2}{k_3} (\phi^{3''}_2 - k^2_2 \phi^2_2)' + \frac{\phi^3_2}{k_2} (\phi^{2''}_2 - k^2_2 \phi^2_2)' \]

(4.18)

As alluded to in section 3.1.1 (more details will be given in appendix B when generalizations to three dimensions are discussed) in the integral defining \( M^1_j \), the Rayleigh equation is used to eliminate second derivatives of the eigenfunctions \( \phi \) and the last two terms in (4.18) are integrated by parts. Then, much as in section 3.1.1, it can be shown that

\[ M^1_j = -k_1 \left( \frac{\phi^1_2 \phi^2_2 \phi^3_2 \bar{u}^{1''}_2}{\omega^2_1 \omega^3_1} \right) \bigg|_{d_j}^0 - k^2_1 M_j \]

(4.19)

where \( M_j \) is the same for each wave in the triad:
\[ M_j = \int_{d_{j-1}}^{d_j} \bar{u}_j'' \left( \frac{\phi_j^1 \phi_j^2 \phi_j^3}{k_1 \omega_j^3 \omega_j^3} + \frac{\phi_j^2 \phi_j^1 \phi_j^3}{k_2 \omega_j^1 \omega_j^3} + \frac{\phi_j^3 \phi_j^1 \phi_j^2}{k_3 \omega_j^2 \omega_j^1} \right) dy \]

The expressions evaluated at interfaces may be simplified in a straightforward, if tedious, manner. In particular, the terms not involving \( \gamma_j^1 \) reduce to

\[ \rho_j \left( \frac{\phi_j^1 \phi_j^2 \phi_j^3}{k_1 k_2 \omega_j^2} + \frac{\phi_j^2 \phi_j^3 \phi_j^1}{k_2 k_3 \omega_j^1} - \bar{u}_j' \left( \frac{\phi_j^1 \phi_j^2 \phi_j^3}{k_1 \omega_j^3 \omega_j^3} + \frac{\phi_j^2 \phi_j^3 \phi_j^1}{k_2 \omega_j^1 \omega_j^3} + \frac{\phi_j^3 \phi_j^1 \phi_j^2}{k_3 \omega_j^2 \omega_j^1} \right) \right) + \frac{\bar{u}_j''}{\omega_j^2} \left( \frac{\phi_j^1 \phi_j^2 \phi_j^3}{k_1 \omega_j^3 \omega_j^3} + \frac{\phi_j^2 \phi_j^3 \phi_j^1}{k_2 \omega_j^1 \omega_j^3} + \frac{\phi_j^3 \phi_j^1 \phi_j^2}{k_3 \omega_j^2 \omega_j^1} \right) \]

\[ \equiv \rho_j \left( B_j - \frac{\phi_j^1 \phi_j^2 \phi_j^3 \bar{u}_j''}{k_1 \omega_j^2 \omega_j^3} \right) \quad (4.20) \]

evaluated between \( y = d_{j-1} \) and \( d_j \). Using \( \mu_j^1 / \omega_j^1 = \rho_j / k_1^2 \), it is found that

\[ (A_j^1 + B_j^1) |_{d_{j-1}}^{d_j} + \frac{\mu_j^1}{\omega_j^1} \left( \gamma_j^1 L_j - M_j^1 \right) = \gamma_j^1 \alpha_j^1 + \rho_j \beta_j \quad (4.21) \]

where

\[ \beta_j = M_j + B_j |_{d_{j-1}}^{d_j} \quad (4.22) \]

and

\[ \alpha_j^1 = -\rho_j \left( \frac{2 \phi_j^1 \phi_j^1'}{k_1^2 \omega_j^1} - \frac{\bar{u}_j' \phi_j^1}{k_1 \omega_j^1} \right) |_{d_{j-1}}^{d_j} + \frac{\rho_j}{k_1} \int_{d_{j-1}}^{d_j} \frac{\bar{u}_j'' \phi_j^1}{\omega_j^1} dy \quad (4.23) \]

Finally, letting

\[ \sigma_j = s_j^1 s_j^2 s_j^3 \quad (4.24) \]

the solvability condition (4.15) becomes

\[ \gamma_j^1 \alpha_j^1 + \rho_1 \beta_1 + \frac{1}{s_j^1 s_j^2 s_j^3} \left( \gamma_j^1 \frac{s_j^2 s_j^3}{s_j^1} \alpha_j^2 + \rho_2 \beta_2 \right) + \frac{1}{s_j^1 s_j^2 s_j^3} \frac{1}{s_j^1 s_j^2 s_j^3} \left( \gamma_j^1 \frac{s_j^2 s_j^3}{s_j^1} \alpha_j^3 + \rho_3 \beta_3 \right) + \cdots = 0 \]

or

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\[
\gamma_1 \left( \alpha_1 \frac{\alpha_2}{s_1^{1/2}} + \frac{\alpha_3}{s_1^{1/2} s_1^{1/2}} + \cdots + \frac{\alpha_N}{s_1^{1/2} \cdots s_N^{1/2}} \right) \\
+ \rho_1 \beta_1 + \frac{\rho_2 \beta_2}{\sigma_1} + \frac{\rho_3 \beta_3}{\sigma_1 \sigma_2} + \cdots + \frac{\rho_N \beta_N}{\sigma_1 \cdots \sigma_{N-1}} = 0 \tag{4.25}
\]

and so once again it is seen that the sign of \(\gamma_1\) (and similarly for \(\gamma_2\) and \(\gamma_3\)) depends only on wave 1. Furthermore, by again introducing the particle displacement eigenfunction \(\zeta_j^1 = \phi_j^1/\omega_j^1\), it can be shown that

\[
k_1 \int_{d_{j-1}}^{d_j} \frac{\bar{u}_j^1 \phi_j^{12}}{\omega_j^1} \, dy = -2 \int_{d_{j-1}}^{d_j} \omega_j^1 (\zeta_j^{12} + k_1^2 \zeta_j^{12}) \, dy + \left( \frac{2 \phi_j^1 \phi_j^1}{\omega_j^1} - \frac{k_1 \bar{u}_j^1 \phi_j^{12}}{\omega_j^{12}} \right)_{d_{j-1}}^{d_j} \tag{4.26}
\]

by letting \(k_1 \bar{u}_j'' = \omega_j^{1''}\) and integrating by parts so as to eliminate derivatives of \(\omega_j^1\) from the problem. Hence

\[
\alpha_j^1 = -\frac{2 \rho_j}{k_1^2} \int_{d_{j-1}}^{d_j} \omega_j^1 (\zeta_j^{12} + k_1^2 \zeta_j^{12}) \, dy \tag{4.27}
\]

Since, from the kinematic condition (4.2) and the definition of \(s_j^1, \zeta_{j+1}^1 = s_j^1 \zeta_j^1\) at \(y = d_j\), the quantity which determines the sign of \(\gamma_1\) may now be written as

\[
-\frac{2}{k_1^2} \int_{d_0}^{d_N} \bar{\rho} \omega^1 (\zeta^{12} + k_1^2 \zeta^{12}) \, dy
\]

where \((\bar{\rho}, \bar{\omega}^1) = (\rho_j, \omega_j^1)\) when \(d_{j-1} < y < d_j\), and \(\zeta^1\) may be thought of as the normalized particle displacement eigenfunction; normalized in the sense that it is continuous at interfaces. Of course,

\[
\zeta^1 = \frac{\zeta_j^1}{\prod_{k=1}^{j-1} s_k^{1/2}}
\]

for \(d_{j-1} < y < d_j\). Hence it is seen that the results deduced in section 3.1.1 carry over to more general flows. That is, the sign of the interaction coefficients depends only on the quantity \(\gamma\) above for each wave in the triad. Furthermore, if the Rayleigh equation in each layer is written in terms of the particle displacement eigenfunction,

\[
(\bar{\omega}_j^{12} \zeta_j^{1'})' - \bar{\omega}_j^{12} \zeta_j^1 = 0
\]

and multiplied by \(\rho_j \zeta_j^1\), integration by parts from \(y = d_{j-1}\) to \(d_j\) yields

\[
- \int_{d_{j-1}}^{d_j} \rho_j \bar{\omega}_j^{12} (\zeta_j^{12} + k_1^2 \zeta_j^{12}) \, dy + \rho_j \bar{\omega}_j^{12} \zeta_j^1 \zeta_j^{1'}|_{d_{j-1}}^{d_j} = 0 \tag{4.28}
\]
In terms of $\zeta$, the interfacial conditions become

$$S_j^1 \zeta_j^1 = S_{j+1}^1 \zeta_{j+1}^1 \quad (4.29)$$

$$\frac{\rho_j}{k_1^2} \bar{\omega}_j^{12} \zeta_j^1 S_j^1 - \frac{\rho_{j+1}}{k_1^2} \bar{\omega}_{j+1}^{12} \zeta_{j+1}^1 S_{j+1}^1 = (g \rho_j + T_j k_1^2) \zeta_j^1 S_j^1 - (g \rho_{j+1} + T_{j+1} k_1^2) \zeta_{j+1}^1 S_{j+1}^1 \quad (4.30)$$

each imposed at $y = d_j$. Multiplying (4.28) by $S_j^{12}$ and adding over $j$, with the conditions (4.29) and (4.30) it is found that

$$- \int_{d_0}^{d_N} \rho S^{12} \bar{\omega}^{12} (\zeta_1^1 + k_1^2 \zeta_1^{12}) dy + \sum_{j=0}^{N} k_1^2 (g \rho_j + T_j k_1^2) S^{12} \zeta_1^{12} \big|_{d_{j-1}}^{d_j} = 0 \quad (4.31)$$

and this may be viewed as the dispersion relation for the system. In (4.31), quantities such as $\rho$ should be viewed as $\rho_j$ when $d_{j-1} < y < d_j$. Since $S_j^1 = s_1^1 s_2^1 \cdots s_j^1$, it should be clear that $\gamma_1^1$ may again be related to an $\omega_1$-derivative of the dispersion relation, and so the results concerning the location on the dispersion curve of sign changes of $\gamma$ carry over as well. In appendix B, some of these results will be generalized further to three dimensions. A possible physical interpretation of these waves will now be offered.

### 4.2 POSSIBLE PHYSICAL INTERPRETATION

In Craik and Adam (1979) and Weiland and Wilhelmsson (1977) so-called negative energy waves prove to have an important connection to explosive resonant triads. By negative energy waves, it is meant that the presence of such waves reduces the total energy of the system in which they occur. Cairns (1979) puts forth a relationship between an appropriate dispersion relation for a given system and the energy of waves in that system. By considering the role the dynamic (pressure) condition plays in both the determination of the dispersion relation and the energy of waves across an interface separating flows of constant shear and density, this energy is found to be

$$\frac{1}{4} \omega \frac{\partial D}{\partial \omega} S S^*$$

where $D(k, \omega) = \wedge$ is some appropriate dispersion relation for the system, and $S$ is the wave amplitude. As stated in this paper, this approach is not apparently applicable to continuously-varying flows. Grimshaw and Becker (1993), in terms of a Lagrangian analysis, offer the definition of the "pseudo-energy" of a wave:

$$E = \langle \frac{\partial \xi}{\partial t} \frac{\partial L}{\partial \xi_t} - L \rangle$$

$$= -\frac{1}{2} S S^* \frac{\omega}{k^2} \int_{-1}^{1} \rho_0 \bar{\omega} (\eta^2 + k^2 \eta^2) dy$$

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adapting notation from section 3.1.1. In the above, $<>$ denotes a horizontal average, $L$ is the Lagrangian of the system, and $\xi$ is a Lagrangian variable (a particle displacement). Again, $\eta$ is the particle displacement eigenfunction. The above integral should be recognized from section 3.1.2 as proportional to $D_\omega$. Again there seems to be a connection between $D_\omega$, which determines the sign of the triad interaction coefficients, and wave energy. It is not entirely clear how to extend these notions when an Eulerian approach is being taken, as here. However, following Lighthill (1978), the wave energy per unit volume, averaged over the horizontal, may be taken to be

$$\rho_0 \left( \frac{1}{2} < u^2 + v^2 > + \frac{1}{2} J N^2(y) < \zeta^2 > \right)$$

where $<>$ denotes the horizontal average, $u$ and $v$ are respectively the horizontal and vertical velocity components, and $\zeta$ is again the local vertical displacement of a material line. $N$ is of course the Brunt-Väisälä frequency. A Boussinesq fluid has been assumed, so $\rho_0$ can be thought of as the underlying density of the fluid, constant except when involved in inertial terms. Expressing these quantities in terms of the eigenfunctions $\phi$, this wave energy density becomes

$$\rho_0 \left( \frac{\phi'^2}{k^2} + \phi^2 + \frac{J N^2}{\omega^2} \phi^2 \right) \tag{4.32}$$

When multiplied by $SS^*$ (where $S$ is the wave amplitude) and integrated over the domain in question, this expression gives the energy $E_f$ imparted to the system by the waves. Integration by parts gives

$$E_f = \frac{SS^*}{k^2} \int_{-1}^{1} \rho_0 \left( -\frac{k \w''}{\omega} + \frac{2 J N^2 k^2}{\omega^2} \right) \phi^2 \, dy \tag{4.33}$$

for a stratified flow between flat plates. This is an Eulerian concept, as it gives the energy of the fluid without regard to it's constituent parts; i.e., the waves themselves. The waves that have been dealt with throughout are really velocity fields, which give the velocity at points in the fluid, thought of as a continuum. It would be useful to assign some meaning to the notion of the energy of the waves themselves, as in Cairns (1979). Of course, when the flow has piecewise constant shear and stratification, the two notions (energy of the flow vs. energy of the waves) are essentially equivalent, as the wave motion can only be along interfaces in this case. In the continuously varying cases, there is motion throughout, in the sense that streamlines (or isopycnal lines) in the unperturbed flow will exhibit wave motion. It is proposed here to adopt some notions from the Lagrangian approach and define the wave energy density $\tilde{E}$ in terms of $\zeta^2$, since locally this can be thought of as a wave amplitude. This was the point of integrating the energy density (4.32) by parts. Then from (4.32),

$$\tilde{E} = \rho_0 \frac{k^2}{k^2} \left( -k \w'' \omega + 2 J N^2 k^2 \right) \zeta^2 \tag{4.34}$$
Following Whitham (1974), define the wave action density $\tilde{I}$ by dividing the wave energy density (4.34) by the local Doppler shifted frequency $-\bar{\omega}$:

$$\tilde{I} = \frac{\rho_0}{k^2} (k \bar{u}'' - \frac{2 J N^2 k^2}{\bar{\omega}}) \zeta^2$$

The total wave action $I$ would be obtained by integrating $\tilde{I}$ over the domain in question. Then the wave energy $E_w$ would be defined as $I$ multiplied by the wave frequency $\omega$:

$$E_w = SS^* \rho_0 \frac{\omega}{k^2} \int_{-1}^{1} (k \bar{u}'' - \frac{2 J N^2 k^2}{\bar{\omega}}) \zeta^2 dy$$

$$= SS^* \rho_0 \frac{\omega}{k^2} \int_{-1}^{1} \left( \frac{k \bar{u}''}{\bar{\omega}^2} - \frac{2 J N^2 k^2}{\bar{\omega}^3} \right) \phi^2 dy = - \frac{\omega}{k^2} D_\omega SS^*$$  \hspace{1cm} (4.35)

from (3.36). Note that if the shear were absent ($\bar{u} = 0$) then the wave energy and the fluid energy would be the same; $E_w = E_w$. Recall from section 3.1.1 that the triad equations in this case can be written

$$S_{1T} = - \frac{k_1^2}{\gamma_1} S_{2}^* S_{3}^*$$

$$S_{2T} = - \frac{k_2^2}{\gamma_2} S_{1}^* S_{3}^*$$

$$S_{3T} = - \frac{k_3^2}{\gamma_3} S_{2}^* S_{1}^*$$

Thus it can be seen that the wave energy (4.35) for this system is conserved;

$$\frac{\omega_1 \gamma_1}{k_1^2} S_1 S_1^* + \frac{\omega_2 \gamma_2}{k_2^2} S_2 S_2^* + \frac{\omega_3 \gamma_3}{k_3^2} S_3 S_3^* = \text{const.}$$  \hspace{1cm} (4.36)

Alternatively, if a dispersion relation for this systems is defined as in section 3.1.2:

$$D(k, \omega) \equiv \frac{1}{k^2} \int_{-1}^{1} \rho_0 (\phi'^2 + k^2 \phi^2 + \left( \frac{k \bar{u}''}{\bar{\omega}} - \frac{k^2 J N^2}{\bar{\omega}^2} \right) \phi^2) dy = 0$$  \hspace{1cm} (4.37)

then again following Whitham (1974), it could be anticipated that to leading order, any Lagrangian for this system would take the form

$$L = D(k, \omega) SS^*$$  \hspace{1cm} (4.38)

so that the energy of waves in such systems would be
\[ \omega L_\omega = -SS^* \frac{\omega}{k^2} \int_{-1}^{1} \rho_0 \left( \frac{k u''}{\omega^2} - \frac{2 JN^2 k^2}{\omega^3} \right) \phi^2 dy = -\frac{\omega}{k^2} S S^* \] (4.39)

which by the triad equations, is again a conserved quantity for the three waves. It can be verified that this quantity has the dimensions of energy. This is admittedly a somewhat ad-hoc analysis. It represents an attempt to apply ideas well-understood from a Lagrangian point of view to an Eulerian framework. The point being made here is that the energy of the waves themselves must be distinguished from the energy they impart to the fluid as a whole. If this distinction is accepted, then the role of negative energy waves becomes clear. For it is seen above that this wave energy is proportional to \( \omega \gamma \), with a positive constant of proportionality. In treating triads by the graphic convention, all the frequencies were positive and the condition that explosive triads result from an interaction was that the wave of largest frequency had a \( \gamma \) with different sign than the \( \gamma \)'s of the other two waves. It can be seen that this condition translates into this largest frequency wave having an energy of differing sign than the energy of the other two. Admittedly, no physical arguments went into deducing these "energies", only adaptation of previous results. Thus the proper constant of proportionality has not been determined, so it can't be said whether these waves have negative or positive energy. However, the importance of relative sign differences should be clear.

Again, the presence of shear is crucial. Explosive interactions cannot occur in conservative systems. For consider the general evolution equation

\[ u_t + Lu = Nu \] (4.40)

studied by Benney (1977), where \( L \) is a linear operator permitting only dispersive behavior (ie, only odd spatial derivatives are present) and \( N \) is a nonliner operator such that both \( Nu \) and \( uNu \) are expressible in divergence form. Therefore, the integrals over the physical domain of both \( u \) (mass, say) and \( u^2 \) (eg, energy) are conserved quantities for this system. If the weakly nonlinear counterpart of the above permits triads, then since by hypothesis \( Nu = (N_1 u)_x \), the triad equations must be of the form

\[ S_1 T = i k_1 \Gamma_{23} S_2^* S_3^* \]
\[ S_2 T = i k_2 \Gamma_{13} S_1^* S_3^* \]
\[ S_3 T = i k_3 \Gamma_{21} S_2^* S_1^* \]

Since \( uNu = (N_2 u)_x \) by assumption, by multiplying (4.40) by \( u \) and comparing mean (nonoscillatory) terms, it can be seen that

\[ k_1 \Gamma_{23} + k_2 \Gamma_{13} + k_3 \Gamma_{21} = 0 \]

from which it is clear that explosive triads are not possible, if the symmetric convention for triads is adopted: \( k_1 + k_2 + k_3 = 0 \). For then it is impossible for the imaginary parts
of the interaction coefficients to be of one sign and conservation of energy to be satisfied as well.

These results are certainly open to contention, but is felt these wave energies do have physical significance as far as the waves themselves are concerned, if not for the fluid as a whole. It should be mentioned that these ideas certainly carry over to interfacial situations, with the interfacial conditions properly accounted for. With these notions discussed, this chapter will close with a discussion of explosive triads in linearly unstable systems.

4.3 RELEVANCE OF EXPLOSIVE TRIADS IN UNSTABLE SYSTEMS

In section 3.3, examples of explosive triads in continuously varying flow systems were observed, which were possibly unstable. It might be wondered, then, how relevant explosive triads are to such systems. The waves participating in this interaction were all stable, so if these were the only waves perturbing the system, the explosive effects would certainly be the only instability present. However, physically it is not expected that such waves would be the only ones present, that general initial conditions might consist of some unstable components. Of course, if perturbations to some system exhibit strong exponential growth, then it is to be anticipated that a weakly nonlinear theory would not be relevant. If a weakly unstable component were part of a leading order solution which also consisted of triads, the time scales over which the two kinds of solutions render the overall perturbation series invalid must be compared. If the unstable term grew like, say $e^{et}$, where $e$ is a measure of the nonlinearity, then presumably second order terms would grow like $e^{2et}$ and the perturbation series would become disordered when $t = O(\log e/e)$. This must be compared to the time scale within which the explosive triads cause the series to break down; this was seen from chapter 1 to be when $t - T_0/e = O(e)$ for some $T_0$. It is unlikely that any quantitative statements may be made concerning which effect causes the series to break down "faster". But it seems plausible that if $T_0$ is not "too large", then before the weak exponential growth may disorder the series, the explosive triad effects may prove more important. That is to say, by the time the triad solution breaks down, the weakly unstable solution may still represent an acceptable approximation, and so it would be anticipated that explosiveness is a more important effect than linear instability.

If an otherwise explosively unstable triad consisted of slightly linearly unstable waves as well, the finite-time singularity might still dominate. For consider the set of equations

$$
S_{1T} = g_1 S_1 + \alpha_1 S_2^* S_3^*
$$

$$
S_{2T} = g_2 S_2 + \alpha_2 S_1^* S_3
$$

$$
S_{3T} = g_3 S_3 + \alpha_3 S_2^* S_1^*
$$

(4.41)

If the $\alpha$'s are such that explosive triads are permitted for $g_j = 0$, then for $g_j$ small a perturbation approach to equations (4.41) could be taken, as in section 2.1.2, and it would be anticipated that these equations would need to be rescaled near the blow-up point to
bring the exponential growth terms into the analysis at leading order. If the $g$'s are equal, say to $g$, then equations (4.41) may be solved exactly. The transformation

$$ A_j = S_j e^{gT} $$

reduces (4.41) to

$$ A_{1T} = \alpha_1 e^{gT} A_2^* A_3^* $$
$$ A_{2T} = \alpha_2 e^{gT} A_1^* A_3^* $$
$$ A_{3T} = \alpha_3 e^{gT} A_2^* A_1^* $$

(4.42)

and the change of variable

$$ \tau = \frac{e^{gT} - 1}{g} $$

(so $\tau = 0$ when $T = 0$) brings (4.41) into the form

$$ A_{1\tau} = \alpha_1 A_2^* A_3^* $$
$$ A_{2\tau} = \alpha_2 A_1^* A_3^* $$
$$ A_{3\tau} = \alpha_3 A_2^* A_1^* $$

(4.43)

Under the current assumptions, this system blows up at say, $\tau = \tau_0$, so the original system (4.41) becomes singular when

$$ T = T_0 = \frac{\log(g \tau_0 + 1)}{g} $$

which always falls into the physical domain $T > 0$ if the growth rate $g$ is positive. If $g < 0$, so the system is damped, then there will still be blow up in physical time for (4.41) if

$$ 0 < 1 + g \tau_0 < 1 $$

or if the damping $g$ is greater than $-1/\tau_0$. For smaller damping than this, it is to expected that the exponential decay eventually governs the system. Numerical runs on (4.41) for nonequal $g$'s led to similar results, though it is impossible to verify the results analytically in this case. This is quite a simple model; the effect of higher order terms in (4.41) was not studied, and while it was shown in chapter 2 that such terms are insufficient to truly model systems with explosive triads, it is not known what effect the inclusion of exponential growth has. In the case of equal growth rates, the amplitudes behave like

$$ \frac{e^{gT}}{T_0 - T} $$

(4.44)
near the blow-up point $T_0$. Presumably, from the point of view of a perturbation series as in chapter 2, any leading order behavior like (4.44) would give rise to second order effects like

$$\frac{e^{2gT}}{(T_0 - T)^2}$$  \hspace{1cm} (4.45)

and the condition that a perturbation series became disordered would become

$$\frac{T_0 - T}{e^{gT}} = O(\epsilon)$$  \hspace{1cm} (4.46)

and the competing effects of exponential growth and singularity in finite time can be seen from this relation. If the growth rate $g$ is not too large, then (4.46) will be satisfied for $T$ close to $T_0$, and it could be expected that explosive effects dominate. If the growth rate is sufficiently large, though, then the denominator of (4.46) could potentially become large enough that (4.46) is satisfied without $T$ being near $T_0$. It could also be argued that if the blow-up point $T_0$ is not too large, then the exponential growth factor will not become too large over the time range required for (4.46) to be satisfied. These are all heuristic notions; no quantitative statements about which effect is more important have been put forth. Rather, the point here is to demonstrate how explosive effects could be important in linearly unstable systems. This of course assumes that the original system under consideration is only weakly unstable. Similar statements hold true for the weakly damped case as well.

A summary of results for equations such as (4.41) is discussed in Craik (1985). Included there are cases where some of the waves are damped and some grow. There appear to be a wealth of possible behaviors. Numerical experiments undertaken here reveal that such systems may possess solutions which consist of amplitudes that blow-up in finite time along with amplitudes that remain bounded. Again, it is not the intent to fully analyze (4.41); this does not appear to be possible analytically. Rather, the point is to demonstrate that the kind of singularity for the neutral systems may still be present in those with some sort of non-dispersive aspects.
5. CONCLUSIONS

This thesis has considered the phenomenon of explosive resonant triads, in both general dispersive wave systems and in particular examples from fluid mechanics. These waves were seen to represent a much stronger interaction than previously thought. In particular, retaining an additional term in the evolution equations for the amplitudes of such waves is insufficient; the behavior of systems exhibiting explosive triads is not governed by weakly nonlinear theory. As these wave amplitudes become large, weakly nonlinear theory breaks down, and the equation(s) governing the system at this stage are strongly (or fully) nonlinear. It is impossible to make any predictions about the long-time behavior of such systems, as this is in general determined by a nonlinear partial differential equation. This is to be contrasted with previous theories, which predicted that these system would undergo recurring “bursts” of large increases in amplitude followed by sharp decreases in amplitude. Near the blow-up point, in general all wave harmonics will become important, as opposed to the three original members of the triad set remaining dominant.

In fluid systems with continuously varying velocity and density profiles, it was seen how the interaction coefficients in the triad evolution equations could be simplified, and how the determination of explosive triad existence came down to analyzing the linearized dispersion relation of the system. Critical layers proved important to this problem; their presence is a necessary condition for explosive triads to exist. This confirmed and generalized previous work, but the problem was approached from the perhaps more accessible viewpoint of Eulerian as opposed to Lagrangian analysis. In addition, the singularities associated with critical layers were more readily handled from this framework, and techniques for dealing with singular boundary value problems in general were developed. Finally, examples of explosive triads in continuously varying fluids were put forth.

There remain further directions this work could take. It was initially hoped that not only would explosive triads be found in the fluid systems described, but that such systems would be relatively accessible to a full numerical simulation. Such simulations were carried out for a series of model problems, but it would be interesting to obtain a physical system where such an analysis could be performed. As the only examples uncovered were in interfacial flows, such a project seems prohibitive at the moment. Perhaps other physical systems which admit wave triads would be more amenable to this analysis. Furthermore, the examples presented here were likely linearly unstable, casting some doubt on the strength, if not importance, of these explosive interactions. As demonstrated in chapter 4, more complicated flows may be analyzed in a fairly straightforward manner, so it would be interesting to know if continuous versions of the multi-layered flows considered by Craik and Adam (1979) or Romanova and Shrir (1988) are linearly stable and support explosive triads, as do the piecewise constant velocity flows of the former work, or the piecewise linear flows of the latter.

Preliminary calculations by the author suggest that the coefficients of resonant quartet evolutions equations may be simplified in the manner of chapter 2 and discussed in appendix B. This would imply that the signs of these interaction coefficients, which as in the case of triads can lead to unbounded solutions, could be similarly analyzed along dispersion curves. By the analysis undertaken here, it would seem that explosive quartets
could represent a very strong interaction in the absence of triads.

Finally, it could be argued that a more satisfactory physical explanation of these waves could be put forth. The statements in this direction made in section 4.2 were based on an analogy with discrete systems that are well-understood from a Lagrangian basis. The notion of wave energy (as opposed to fluid energy) put forth by Cairns (1979) and in plasma physics (Weiland and Wilhelmsson (1977)) is well-defined, and could perhaps be pursued further for the continuous systems studied here. In particular, the role of the shear in these explosive interactions has not been fully explained. Of course, this question as it relates to linearly unstable systems is far from fully understood, so it may be ambitious to ask such questions as they pertain to the nonlinear instabilities observed here. However, there may be some deeper physical mechanism here worthy of study. A more basic question might be the co-existence of explosive triads and unstable linear modes in the problems modeling wind-driven flows. As the generation of waves by wind is itself a field with many open questions, perhaps the discovery of explosive triads here may offer insights to these problems.
APPENDIX A: NUMERICAL METHODS

In this appendix, the various numerical methods used throughout will be briefly discussed. As stated in section 2.2, any PDE’s were integrated by a pseudo-spectral decomposition in $x$ as in Fornberg and Whitham (1978). The advancement in time was carried out with an implicit, second order Runge-Kutta technique. This method was derived by Waleffe, who also proposed the scheme to the author (1993). Applied to equations of the form

$$u_t = Lu + Nu$$

where again $L$ and $N$ are respectively linear and nonlinear operators on $u$ (which may be a vector quantity), the integration proceeds in two steps:

$$v = u_n + \frac{k}{2} Nu_n + \frac{k}{2} Lv$$

$$u_{n+1} = u_n + k Nv + \frac{k}{2} (Lu_n + Lu_{n+1}) \quad (A.1)$$

where $k$ is the time step and $u_n$ denotes the numerical value of $u$ after the $n$-th time step. The above scheme may be shown to be of second-order accuracy in a straightforward manner. The above scheme was implemented by applying it to each component of a Fourier decomposition of $u_n$, considering real and imaginary parts separately. Of course, this decomposition was performed via the fast Fourier transform (FFT). Again, the overall method used was quite standard, with the exception of the time-stepping technique (A.1)

The critical layer results were obtained by an inward shooting method. Of course, as mentioned in section 3.3, one could simply take as initial conditions $\phi_{a}^{\pm} = \phi_{a}(y_c \pm \delta)$ and $\phi_{b}^{\pm} = \phi_{b}(y_c \pm \delta)$, with the derivatives determined analytically also, where $\phi_{a,b}$ are the two Frobenius solutions about $y = y_c$ with the nonlinear critical layer interpretation (ie, no algebraic branch cuts) and $\delta$ is some appropriately small number. As a nonlinear critical layer solution must consist of the same linear combination of these two solutions above and below the critical layer, the boundary conditions (say for a stratified flow) require that

$$A \phi_{a}(1) + B \phi_{b}(1) = 0$$
$$A \phi_{a}(-1) + B \phi_{b}(-1) = 0 \quad (A.2)$$

where, for example, $\phi_{a}(1)$ denotes the value of $\phi$ at $y = 1$ based on an integration with initial conditions $(\phi_{a}, \phi'_{a})$ at $y = y_c + \delta$ as deduced from the Taylor series expansions. Then (A.2) is only satisfied for those wavenumbers for which

$$\phi_{a}(1) \phi_{b}(-1) - \phi_{a}(-1) \phi_{b}(1) = 0 \quad (A.3)$$

Newton’s method is used to solve (A.3). This is all standard, and indeed is found to work well even when one of the boundaries is at $\pm \infty$, at least as far as determining
the eigenvalue. However, as the eigenfunction is needed to perform the integrations which define
the interaction coefficients, it is advisable to integrate into the critical layer from the
boundaries, as any outward integration will most likely excite an exponentially growing
mode. In general if the relevant initial conditions are imposed at boundary 1 above the
critical layer and boundary 2 below the critical layer, and the relevant equation (Taylor-
Goldstein or Rayleigh) integrated to respectively $y_c \pm \delta$, then the two relations result:

\begin{align}
A \phi_a^+ + B \phi_b^+ &= CL_1 \\
A \phi_a'^+ + B \phi_b'^+ &= CL_2
\end{align}

(A.4)

and

\begin{align}
A \phi_a^- + B \phi_b^- &= DL_3 \\
A \phi_a'^- + B \phi_b'^- &= CL_4
\end{align}

(A.5)

where $C$ and $D$ are constants multiplying the conditions at the boundaries; one of them
is arbitrary. For example, if $\phi \to 0$ as $y \to \infty$, then the behavior of $\phi$ at $\infty$ is $\phi \sim Ce^{-ky}$,
$\phi' \sim -Cke^{-ky}$ as $y \to \infty$. The initial condition on $\phi$ at some appropriately large value
of $y$, say $M$, is taken to be $(e^{-kM}, -ke^{-kM})$. As the governing equation is linear, any
other solution with the proper behavior for large $y$ must be some multiple of the solution
resulting from this initial condition; ie, with $C = 1$. Then, (A.4) is solved for $A$ and $B$
in terms of $C$, which are then substituted into (A.5). The determinant of the resulting
homogeneous system involving $C$ and $D$ is set equal to zero, and again an implicit equation
results which determines the eigenvalue. For a specific behavior at $\infty$, $C$ is known, so $D$
may be determined when the eigenvalue is known. Thus the proper eigenfunction behavior
at the boundaries is determined, and the governing equation may be integrated to within
a suitably small neighborhood of the critical layer to give the eigenfunction over the rest
of the domain.

Once the eigenfunction is known, the integrals which define the interaction coefficients
may be evaluated. Actually, only the quantity $\gamma$ from section 3.1 need be calculated, as
this expression determines the sign of the interaction coefficients, necessary to establish
the existence of explosive triads. For example, in the homogeneous case, an integral like

$$\int_0^\infty \frac{\phi'^2}{(\bar{u} - c)^2} dy$$

needs to be calculated. Treating $\phi$ as $A\phi_a + B\phi_b$ (where it was explained above how to
determine $A$ and $B$) and using Taylor series, the singular behavior of the above integrand
may be determined. As mentioned in section 3.3, the theory developed in section 3.2
only requires the non-integrable behavior be subtracted off, but in practice it was found
any singular behavior must be subtracted off from the integrand. By singular it is meant
simply any behavior of the function which does not approach a limit as $y \to y_c$. Sub-
tracting off from the integrand any integrable behavior does not affect any of the results
from section 3.2; it would merely amount to changing the definition of the constants of integration involved. Then, given a suitable cutoff parameter \( c_0 \), the following integral may be evaluated:

\[
\int_{y_c+\delta}^{c_0} \left( \frac{\phi^2}{(u - c)^2} - SP' \right) dy = \int_{y_c+\delta}^{c_0} \frac{\phi^2}{(u - c)^2} dy + SP|_{y=y_c+\delta} - SP|_{y=c_0} \quad (A.6)
\]

The reason a cutoff parameter is introduced is that logarithmic singularities which are non-integrable over infinite domains result, so the above technique is not valid over the entire domain in question. Of course, numerically an infinite domain is never under consideration, but it seems advisable to take the above precaution. Now the left-hand side of (A.6) may be evaluated numerically, as \( \phi \) is known on some numerical grid determined by an adaptive, fourth order Runge-Kutta integration of the equation in question (here, the Rayleigh equation). Then the trapezoidal rule on this grid is used to evaluate this (convergent) integral. The value of the integrated singular part \( SP \) at \( y = c_0 \) is added to this integral, which gives the first two terms of the right-hand side of (A.6). This is what is required; the integral minus it's singular part. The other integral, over (in this case) \( 0 < y < y_c - \delta \) may be handled similarly, and the manner in which other specifics are handled (interfacial terms, boundaries at \( -\infty \)) should be clear.
APPENDIX B: SIMPLIFICATION OF THE INTERACTION COEFFICIENTS

In section 3.1 it was seen that interaction coefficients involve integrals of complicated expressions. It was stated there that the integral involving the three eigenfunctions $\phi_1, \phi_2, \phi_3$ could be reduced to a quantity which, apart from a possible factor of $k_j^2$, was invariant under a cyclic permutation of the indices 1, 2, and 3. Indeed, for the case of a free surface flow, it was alluded to that this involved integrating certain terms by parts and eliminating certain derivatives via the governing equations (eg, Rayleigh’s equation). These examples were two dimensional. In this appendix, it will be shown how this technique may be extended to three dimensions, and to flows with general horizontal basic flows and non-Bousinesq effects. So consider first a three dimensional stratified flow between solid boundaries at $y = \pm 1$ with basic (unperturbed) density $\bar{\rho}(y)$ and velocity $(\bar{u}(y), 0, \bar{w}(y))$. Taking as solutions to the three-dimensional Euler equations perturbations to this flow yields

\[
\bar{\rho}(u_t + \bar{u}u_x + \bar{w}u_z + \bar{v}v) + \varepsilon \bar{\rho} N_1u + \varepsilon \rho L_1u = -p_x \quad (B.1)
\]

\[
\bar{\rho}(v_t + \bar{u}v_x + \bar{v}v_z + \varepsilon \bar{\rho} N_2v + \varepsilon \rho L_2v = -p_y - g \rho \quad (B.2)
\]

\[
\bar{\rho}(w_t + \bar{u}w_x + \bar{w}w_z + \bar{w}v) + \varepsilon \bar{\rho} N_3w + \varepsilon \rho L_3w = -p_z \quad (B.3)
\]

\[
u_x + v_y + w_z = 0 \quad (B.4)
\]

\[
\rho_t + \bar{u}\rho_x + \bar{w}\rho_z + \bar{v}'v + \varepsilon N_4\rho = 0 \quad (B.5)
\]

where primes denote derivatives with respect to $y$ and the expressions $L$ and $N$ represent respectively linear and nonlinear operators and can be easily determined; for simplicity, only a few will be listed here. For example,

\[
L_1u = u_t + \bar{u}u_x + \bar{w}u_z + \bar{u}v \quad \text{(B.1)}
\]

\[
N_1u = uu_x + uv_y + wu_z \quad \text{(B.2)}
\]

The above may be reduced to a single equation involving only the vertical velocity $v$ to leading order; essentially, the horizontal Laplacian of the pressure $p$ is found from (B.1) and (B.2) using continuity, (B.4), to eliminate $u$ and $w$ to leading order. A $y$ derivative of this equation is taken, equated to the horizontal Laplacian of (B.2) and finally (B.5) is used to eliminate $\rho$ to $O(1)$. The result is

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\[ \bar{p} \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{w} \frac{\partial}{\partial z} \right)^2 \nabla^2 v - g \bar{p}_y \nabla^2 v + \bar{p} \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{w} \frac{\partial}{\partial z} \right) (\bar{u}' v_x + \bar{w}' v_z) \]

\[ + \bar{p}' \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{w} \frac{\partial}{\partial z} \right)^2 v_y - \bar{p}' \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{w} \frac{\partial}{\partial z} \right) (\bar{u}' v_x + \bar{w}' v_z) = -\epsilon \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{w} \frac{\partial}{\partial z} \right) (\bar{p} (\nabla^2 N_2 v - (N_1 u)_{xy} - (N_3 w)_{xy}) \]

\[ \nabla^2_1 (\rho L_2 v) - (\rho L_1 u)_{xy} - (\rho L_3 v)_{xy} - \bar{p}' ((N_1 u)_{x} + (N_3 w)_{x}) + \epsilon \bar{g} \nabla^2_1 N_4 \rho \]  (B.6)

where \( \nabla^2_1 \) denotes the horizontal Laplacian:

\[ \nabla^2_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \]

As before, if a perturbation approach to (B.6) were taken, with triads at leading order, it would be fairly clear how to pick off the terms from the right-hand side of (B.6) which would contribute resonant terms at \( O(\epsilon) \). Since it is known here that the Fredholm alternative may be used with singular integrals interpreted properly, a solvability condition may be obtained. The interaction coefficients would again be of the form \( \Gamma_j / \gamma_j \), where \( \gamma_j \) arises from terms on the left-hand side of (B.6) associated with slow temporal effects, and \( \Gamma_j \) arises from the secular terms on the right. It was seen in section 3.1 how \( \gamma_j \), which will involve only the eigenfunction \( \phi_j \), may be simplified, and the question of whether \( \Gamma_j \) is again equal to an invariant quantity (apart from possibly a factor of \( k^2_j \)) remains. It turns out that it does, so the results from two dimensions hold true here as well. In two dimensions, in several cases (see sections 3.1 and 4.1) it was seen that a judicious use of integration parts was sufficient. Also, no transformations were necessary; the vertical velocity eigenfunction \( \phi \) could be used throughout. In three dimensions, it proved necessary to do the manipulations in terms of the particle displacement eigenfunction \( \eta = \phi / \bar{w} \). Even with this transformation, these manipulations prove to be algebraically immense, and in fact, a Mathematica™ program was used to do these calculations.

Following previous notation, secular terms proportional to \( e^{i\theta} \) will arise on the right-hand due to the quadratic interaction of any terms involving the subscripts 2 and 3. Once again, an interaction equation of the form

\[ \gamma_1 S_{1T} = \Gamma_1 S^*_{2} S^*_{3} \]

is expected, with a cyclic permutation of the indices giving the other two equations in the set. Essentially, \( \Gamma \) is obtained by multiplying the secular terms on the right-hand side of (B.6) by \( \phi / \bar{w}^2 \), where in this case

\[ \bar{w} = \alpha \bar{u} + \beta \bar{w} - \omega \]
where \((\alpha, \beta)\) is the perturbation wavenumber vector. It proved necessary to consider the following nonlinear terms in (B.6) as

\[
\nabla_2^2 N_2 v - (N_1 u)_{xy} - (N_3 w)_{yz} = \\
(u v_{xx} + v v_{xy} + w v_{zz} - w_x v_z + w_z v_x \\
- u u_{yx} - v u_{yy} - w u_{zy} + w_x u_y - w_y u_x) + \\
+ (u v_{xx} + v v_{xy} + w v_{zz} + u_x v_z - u_z v_x \\
- u w_{yz} - v w_{yy} - w w_{zy} - w_x u_y + w_y u_x) 
\]

(B.7)

were continuity (B.4) has been appropriately used. In addition, the following terms were expressed as

\[
N_4 = (u \rho)_x + (v \rho)_y + (w \rho)_z 
\]

(B.8)

When (B.7) and (B.8) are used in (B.6), which is multiplied by \(\phi/\omega^2 = \eta/\overline{\omega}\) in the solvability condition, the 2nd, 7th, 12th, and 17th terms in (B.7) are integrated by parts. For example, secular terms due to the 2nd and 7th terms would arise as

\[
\overline{\rho} \frac{v_1}{\omega_1} (v_2 (v_{3x} - u_{3y})_y + v_2 (v_{3x} - u_{3y})_y) 
\]

the subscripts denote which wave is the contribution. This expression would be integrated by parts to give

\[
-(\overline{\rho} \frac{v_1}{\omega_1} v_2) (v_{3x} - u_{3y}) - (\overline{\rho} \frac{v_1}{\omega_1} v_3) (v_{2x} - u_{2y}) 
\]

Also, the 2nd term in (B.8) would be handled as

\[
-\left(\frac{v_1}{\omega_1^2}\right) v_2 \rho_3 + v_3 \rho_2 
\]

The terms \((\rho L_1 u)_{xy}\) and \((\rho L_3 w)_{yz}\), when multiplied by \(\phi/\omega^2\), would also be integrated in this manner; namely, by taking the \(y\) derivative off of them and placing it on the multiplicative factor. For example,

\[
\frac{v_1}{\omega_1^2} (\rho L_1 u)_{xy} \rightarrow -\left(\frac{v_1}{\omega_1^2}\right)_y (\rho L_1 u)_z 
\]

For the rest of (B.6), the secular nonlinearities are determined first; these are then treated as proportional to \(e^{i\theta_1}\) when any derivatives are taken. Finally, it was found necessary to remove \(\overline{u''}\) and \(\overline{w''}\) from the problem through integration by parts. Any terms like \(\overline{u''}\overline{w''}\) were treated as \(\frac{1}{2} \overline{u'^2}\) in an integration by parts. After all this was done and the result simplified by Mathematica\textsuperscript{TM}, \(\Gamma_1\) was found to be of the form \(k_1^2 \Gamma\), as in the two-dimensional case, so the sign of the interaction coefficients again depends only on \(\gamma_1\).
REFERENCES


