CALENDERING OF AN ELASTIC-VISCOUS MATERIAL

by

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Calendering of an Elastic-viscous Material
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Abstract

Calendering is the term applied to the process of rolling bulk plastics
into sheets. This process is widely used in industry for the production of
plastic sheets and linoleum. In this process as well as any large-scale
industrial process there is always the desire to have a given piece of equip-
ment produce the largest amount of its product possible. For this reason
an analytical approach to the problem seemed useful.

A study of the flow of an elastic-viscous material that is being
calendered is undertaken. The solution of this problem is known for the
case of a viscous material. Advantage is taken of the viscous solution to
predict the behavior of an elastic-viscous material whose properties are
"almost" viscous. The problem is broken into two parts, as follows:

1. A stress-strain rate equation is chosen that can give suitable
predictions for simple types of flows while remaining simple
enough to work with. This equation is given below.

\[(1 + a J_2) \dot{\varepsilon}_{ij} = \frac{S_{ij}}{2\eta} + \frac{\dot{S}_{ij}}{2\eta}\]

\[J_2 = \text{second invariant of the strain rate tensor}\]
\[\varepsilon_{ij} = \text{strain}\]
\[S_{ij} = \text{reduced stress}\]
\[\eta = \text{viscosity}\]
\[G = \text{shear modulus}\]
\[a = \text{parameter selected to match material in question}\]

2. An approximate solution of the calendering problem is obtained and
calculated results are given to determine the effect of several
parameters. (A formal perturbation solution is given in the Appendix.)
This solution indicates the effect of elasticity on the calendering of
a material which is very close to being viscous. This solution can be
used as a guide in selecting suitable speeds of operation for calendering
when the material to be calendered exhibits elastic as well as viscous
behavior.

Thesis Supervisor: E. Crowan
Title: Professor of Mechanical Engineering
Cambridge, Massachusetts
December 17, 1954

Professor J. H. Keenan
Chairman, Departmental Committee on Graduate Students
Department of Mechanical Engineering
Cambridge 39, Massachusetts

Dear Professor Keenan:

In partial fulfillment of the requirements for the degree of Doctor of Science in Mechanical Engineering, I, herewith, submit my thesis entitled, "Calendering of an Elastic-viscous Material".

Very truly yours,

Paul R. Paslay
ACKNOWLEDGEMENT

The author wishes to express his thanks to Professor G. F. Carrier of Harvard University who suggested the subject of this thesis and offered continued help to the author throughout the entire investigation.
INTRODUCTION

Calendering is the term applied to the process of rolling bulk plastic into sheets. The process finds wide application in such industries as the manufacture of linoleum and the production of plastic sheets. A diagram of the process is shown below.

The bulk material that will be considered in this thesis is a heated plastic. The properties of this heated plastic are those of an elastic-viscous material.

From a manufacturing point of view the speed of production from a set of rolls should be the maximum allowable for the production of a sufficiently high-quality sheet. One primary factor controlling the speed of manufacture is the force exerted by the material to force the rolls apart. The smaller the force is between the rolls, the lighter the rolling equipment may be. To decrease the force between the rolls the material may be heated (thereby reducing the viscosity of the material). In order that the material may retain its shape after rolling, the rolls are cooled.
If the material is above some temperature when it enters the rolls, one of two undesirable things may happen. The material may blister below the surface of the rolled sheet; this leaves an undesirable appearance and is not acceptable. The other thing that may happen is that part of the sheet may tear away from the rest and stick to the roll as shown below.

Both the blistering and tearing are the result of a common cause. This is the fact that the energy dissipated and the cooling accomplished in the rolling process produces a non-uniform temperature distribution across the sheet. Professor M. Finston* has shown that for a viscous material, taking into account the thermal conductivity, the position of the peak temperature of the sheet leaving the rolls is under the surface of the sheet. This explains both the blistering and peeling phenomena.

The groundwork for the calendering problem was laid by Dr. R. E. Gaskell** in 1950. Dr. Gaskell considered both Newtonian and non-Newtonian materials for the case of viscous flow neglecting momentum changes. The boundary conditions Dr. Gaskell used were that the velocity of the material in contact with the rolls be the same as the velocity of the surface of the rolls (i.e., no slipping between roll and material) and that the velocity profile of the material leaving the rolls be uniform. For the Newtonian material Dr. Gaskell's result was that the velocity profile was composed of a uniform velocity equal to the roll surface velocity plus a parabolic distribution that disappeared.

* Reference 2  ** Reference 1
at the surface. Let $V_0$ be the roll surface velocity, $\alpha$ the position along the passage and $y$ the distance from the center of material to a general point of the material.

Then:

$$\text{velocity of } (x,y) = V_0 + K(\alpha)(1 - \frac{y^2}{y_s^2})$$

$$\frac{\partial (\text{velocity at } (x,y))}{\partial y} = -2K(\alpha)\frac{y}{y_s^2}$$

where $y_s$ is the distance from the center of the material to the surface of the roll ($y_s$ is a function of $\alpha$) and $K(\alpha)$ a function of $\alpha$ determined so that there is equal mass flow across each section. The shear strain rate (and shear stress as well) are linear functions of $y$ for a prescribed value of $\alpha$. The material is taken to be incompressible. Dr. Gaskell determined pressure distributions as well, considering normal stresses large as compared to shear stresses.

Professor M. Finston* extended Dr. Gaskell's solution by allowing the viscosity to be a function of temperature. Professor Finston carried out a perturbation solution around constant viscosity, that is, he allowed only

* Reference 2
small variations of viscosity. Let $\eta$ be viscosity at temperature $T$ and $\eta_0$ viscosity at temperature $T_0$ then Professor Finston's viscosity-temperature relation may be written as the following,

$$\eta = \eta_0 \left(1 - \delta \left(\frac{T - T_0}{T_0}\right)\right)$$

He then carried out a heat transfer solution of this problem. Professor Finston determined pressure, shear stress and velocity profiles as well as temperature distributions. The prediction of the point of maximum temperature in the cross section from Professor Finston's calculation agreed very well with the position of blisters formed in actual experiments. Professor Finston considered normal stress in each direction at a point to be equal, an incompressible material and he neglected momentum changes.

In 1953 Professor G. F. Carrier* discussed the determination of the temperature distribution of the calendering problem, after the velocity profile and pressure distribution are obtained. This work involves the manipulation of the conservation of energy equation to obtain a suitable approach to the heat transfer aspects of this problem.

The results obtained by Dr. Gaskell and Professor Finston compare favorably with experimental results, except in the case where the material exhibits elastic as well as viscous behavior. This thesis was undertaken as an attempt to shed light on what happens to an elastic viscous material when the material is calendered. A stress strain relation is chosen that gives suitable results for some of the simpler tests and an approximate solution is carried out for the calendering process.

*Reference 3
ANALYSIS

Determination of a Stress-Strain Rate Equation

In order to obtain a result suitable for engineering purposes for a problem of the sort treated here it is often necessary to make simplifying assumptions regarding the behavior of the material and the geometry of the problem. The first such simplification to be encountered here has to do with the stress-strain law to be used. To describe the behavior of an Hookean-elastic, Newtonian-viscous material one might heuristically proceed as follows for an incompressible material.

Let

\[ e_{ij} = \text{strain} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

where

\[ u_i = \text{displacement in } i^{th} \text{ direction} \]

and

\[ x_i = i^{th} \text{ cartesian coordinate} \]

Also let

\[ S_{ij} = \text{reduced stress} = \sigma_{ij} - \delta_{ij} \sigma \]

where

\[ \sigma_{ij} = \text{stress} \]

\[ \delta_{ij} = \text{Kronecker delta} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \]

and

\[ \sigma = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = -p. \]

Other notation used:

\[ t = \text{time} \]

\[ G = \text{shear modulus} \]

\[ \eta = \text{viscosity}. \]

Then calculating elastic and viscous strains separately one obtains:

\[ (e_{ij})_{\text{elastic}} = \frac{S_{ij}}{2G} \]

\[ e_{11} + e_{22} + e_{33} = 0 \]

Hooke's law

(1)  (2)
and \[ (\dot{\varepsilon}_{ij})_{\text{viscous}} = \frac{S_{ij}}{2\nu} \] \) \[ e_{11} + e_{22} + e_{33} = 0 \] \)

Newtonian Viscous Material.

Taking the time derivative of equation (1) yields:

\[ (\dot{\varepsilon}_{ij})_{\text{elastic}} = \frac{S_{ij}}{2G} \]

Now since the strain rates add linearly, one writes

\[ (\dot{\varepsilon}_{ij})_{\text{elastic}} + (\dot{\varepsilon}_{ij})_{\text{viscous}} = (\dot{\varepsilon}_{ij})_{\text{total}} \]

and, therefore,

\[ \dot{\varepsilon}_{ij} = \frac{S_{ij}}{2\nu} + \frac{\dot{S}_{ij}}{2G} \]

A material which satisfies the stress–strain equation (6) is called a Maxwell material. Before discussing the physical interpretation of equation (6) two remarks are appropriate, first,

\[ \dot{\varepsilon}_{ij} = \frac{1}{2} \frac{d}{dt} \left( \frac{\partial u_{ij}}{\partial x_{j}} + \frac{\partial u_{ji}}{\partial x_{i}} \right) = \frac{1}{2} \left( \frac{\partial \dot{u}_{ij}}{\partial x_{j}} + \frac{\partial \dot{u}_{ji}}{\partial x_{i}} \right) \]

where \( \dot{u}_{i} = u_{i}^{t} \) component of velocity

and, therefore,

\[ \dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_{ij}}{\partial x_{j}} + \frac{\partial u_{ji}}{\partial x_{i}} \right). \]

The second observation is that the stress–strain law intrinsically relates the deformation (and its rates of change) to the stress acting on a given sample of the material, that is to say, it is the Lagrangian stress which must appear in the basic stress–strain law. However, in one flow problem, the stress one must discuss is the Eulerian stress. This implies that, although the \( \dot{S}_{ij} \) of equation (6) is merely the partial derivative \( \frac{\partial (S_{ij})^{t}}{\partial x} \) where \( S_{ij}^{t} \) is the Lagrangian stress, it can be anticipated that the corresponding operation on the Eulerian stress will be more involved. The determination of the correct form for \( \dot{S}_{ij} \) has been discussed by Fromm (4,5) and the following equations are derived by him:
\[ \dot{S}_{ij} = \frac{\partial S_{ij}}{\partial t} + u_1 \frac{\partial S_{ij}}{\partial x_1} + u_2 \frac{\partial S_{ij}}{\partial x_2} + u_3 \frac{\partial S_{ij}}{\partial x_3} + \omega_{12} \Pi_{12} + \omega_{31} \Pi_{31} + \omega_{23} \Pi_{23} \]

where

\[ \omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = \text{rotation at a point} \]

\[ \Pi_{ij} = \begin{bmatrix} \sigma_{x_i} & \sigma_{y_i} & \sigma_{z_i} \\ \sigma_{y_i} & -\sigma_{x_i} & \sigma_{z_i} \\ \sigma_{z_i} & \sigma_{z_i} & -\sigma_{x_i} \end{bmatrix} \]

and

\[ \Pi_{23} = \begin{bmatrix} 0 & \sigma_{xy} & -\sigma_{xz} \\ -\sigma_{xy} & 2\sigma_{yz} & \sigma_{yy} \\ \sigma_{xz} & -\sigma_{yz} & -2\sigma_{yy} \end{bmatrix} . \]

A simple example will show that the rotation of the element must be taken into account. A bar subjected to a constant uniform tension in the \( x - y \) plane is rotated at constant angular velocity \( \dot{\theta}_{xy} = \omega \) (see sketch below). The stress does not depend on time, therefore, the time derivative of the stress tensor is zero. This may be computed as follows:

\[ \sigma_{xx} = \frac{\sigma}{2} \left( 1 + \cos 2\omega t \right) \]
\[ \sigma_{yy} = \frac{\sigma}{2} \left( 1 - \cos 2\omega t \right) \]
\[ \sigma_{xy} = \frac{\sigma}{2} \left( \sin 2\omega t \right) \]
Thus, as expected, the rotation of an element of material must be taken into account in the time derivative.

One method of considering the physical interpretation of equation (6) is to determine what results are predicted by the equation for several simple problems. If the results predicted by equation (6) are in accord with what one expects for the simple cases, one may hope that reasonable results can be obtained for more complicated cases that are not so easily substantiated by experiment. Three cases are considered below.
Case 1. Consider a bar with a constant uniform uniaxial tension applied in the direction.

In this case:

\[
e_{ij} = \begin{cases} 
    e_{xx} & 0 & 0 \\
    0 & -\frac{1}{3}e_{xx} & 0 \\
    0 & 0 & -\frac{1}{3}e_{xx}
\end{cases}
\]

\[
S_{ij} = \begin{cases} 
    \frac{2}{3}\sigma_{xx} & 0 & 0 \\
    0 & -\frac{1}{3}\sigma_{xx} & 0 \\
    0 & 0 & -\frac{1}{3}\sigma_{xx}
\end{cases}
\]

\[
\dot{S}_{ij} = 0
\]

and, therefore,

\[
e_{xx} = \frac{\sigma_{xx}}{G}
\]

This case degenerates into the viscous case which is perfectly reasonable.

Case 2. Consider a bar on which is imposed a prescribed uniform uniaxial strain in the \( x \) direction at \( t = 0 \). The problem is to find the variation of stress with time.

For \( t > 0 \), \( \dot{e}_{ij} = 0 \) and \( \sigma_{xx} \) is function of time only. All other stress components are zero.

Now

\[
\frac{1}{G} \left\{ \begin{align*}
\frac{2}{3}\sigma_{xx} & 0 & 0 \\
0 & -\frac{1}{3}\sigma_{xx} & 0 \\
0 & 0 & -\frac{1}{3}\sigma_{xx}
\end{align*} \right\} + \frac{1}{\frac{\nu}{2}} \left\{ \begin{align*}
\frac{2}{3}\sigma_{xx} & 0 & 0 \\
0 & -\frac{1}{3}\sigma_{xx} & 0 \\
0 & 0 & -\frac{1}{3}\sigma_{xx}
\end{align*} \right\} = 0
\]

or

\[
\frac{d\sigma_{xx}}{dt} + \frac{G}{\nu} \sigma_{xx} = 0.
\]
Solving the equation with the initial condition that $\sigma_{\alpha\alpha} = \sigma_{\alpha\alpha_0}$ at $t = 0^+$ yields:

$$\sigma_{\alpha\alpha} = \sigma_{\alpha\alpha_0} e^{-\frac{t}{\eta_G}}. \quad (10)$$

This result shows that the stress decreases exponentially with time, a situation which one might expect.

**Case 3.** The following case is due to H. Fromm.* Determine the shear-stress versus rate-of-strain curve for the shearing of the material between two infinite parallel plates in a steady process.

![Diagram of shear stress](image)

Because of equilibrium the shear stress and the shear-strain rate are uniform throughout the material, therefore, we may define the following:

$$\dot{\varepsilon}_{\alpha\alpha} = 0, \dot{\varepsilon}_{yy} = 0, \dot{\varepsilon}_{u2} = 0$$

$$\dot{\varepsilon}_{y2} = 0, \dot{\varepsilon}_{x2} = 0$$

$$\dot{\varepsilon}_{xy} = \frac{1}{\eta} \dot{\gamma} = \omega xy.$$  

Also

$$\frac{\partial S_{ij}}{\partial t} = 0, \frac{\partial S_{ij}}{\partial \alpha} = 0, \frac{\partial S_{ij}}{\partial y} = 0, \frac{\partial S_{ij}}{\partial \alpha} = 0.$$  

Now

$$\{0 \quad \dot{\gamma} \quad 0\} = \frac{1}{\eta} \dot{\gamma} \left\{2\sigma_{xy} \quad \sigma_{yy} - \sigma_{xx} \quad 0\right\} + \frac{1}{2\eta} \left\{\sigma_{xx} \quad \sigma_{yy} - \frac{1}{2}\sigma_{zz} \quad \sigma_{xy} \quad 0\right\}$$

and by simplifying one obtains,

$$\frac{\sigma_{xy}}{G} = \frac{n}{G} \frac{\dot{\gamma}}{1 + \left[ \frac{\dot{\gamma}}{\eta} \right]^2} \quad (11)$$

$$\frac{\sigma_{yy}}{G} = -\frac{\sigma_{xx}}{G} = \left[ \frac{n}{G} \frac{\dot{\gamma}}{1 + \left[ \frac{\dot{\gamma}}{\eta} \right]^2} \right]^2. \quad (12)$$

*Reference 5*
A plot of \( \frac{G_{xy}}{G} \) versus \( n \) is shown in Figure 1. The questionable point of this result is that the shear stress reaches a maximum for a finite shear strain rate and that each shear stress below this maximum is associated with two strain rates. This result is not in accord with what one expects based on the available experimental data. To eliminate this difficulty one might assume variations in viscosity or shear modulus with strain rate or might alter the stress-strain equation. This result would not be too disturbing if one knew that the shear strain rates would be sufficiently low so that the calculated shear stress would not reach the peak indicated in Figure 1.

Unfortunately, for the calendering problem the primary interest is to be focused on shear stress and shear strain rate close to the surface of the material. The shear-stress and shear-strain rates are highest at the surface of the material as it is being calendered. If equation (6) could be modified to overcome the difficulty encountered in case three without unduly disturbing the results of cases one and two, then one might hope that the result obtained by the modified equation would be representative of the physical picture.

In modifying equation (6) care must be exercised not to alter the equation in such a way that if the equation were transformed to another Cartesian coordinate system \( \mathbf{x}', \mathbf{y}', \mathbf{z}' \) that the meaning of the equation would be changed. In other words, these equations are for an isotropic material and should not change meaning when different Cartesian coordinates are used. The most obvious way to modify such an equation is by use of invariants of either strain or stress. For the problem to be considered in this thesis the following stress-strain rate relation was chosen:
\[(1 + \alpha J_2) \ddot{e}_{ij} = \frac{S_{ij}}{2\eta} + \frac{\dot{S}_{ij}}{2G} \tag{13}\]

where \(J_2 = \) second invariant of the strain rate tensor
\[= \frac{1}{8} \left[ (\dot{e}_{xx})^2 + (\dot{e}_{yy})^2 + (\dot{e}_{zz})^2 + (\dot{e}_{xy})^2 + (\dot{e}_{xz})^2 + (\dot{e}_{yz})^2 \right]
\]
and \(\alpha = \) an arbitrary parameter.

In order to demonstrate the meaning of equation (13), the results of cases one, two and three are given below where equation (13) has replaced equation (6).

Case 1.
\[\sigma_{xx} = \frac{1}{3\eta} \left(1 - \frac{3\alpha}{4}(\dot{e}_{xx})^2\right) \dot{e}_{xx}. \tag{14}\]

This equation is plotted in Figure 2 for various values of .

Case 2.
\[\sigma_{xx} = \sigma_{xx0} e^{\frac{-t}{\eta G}}. \tag{15}\]

(same result as with equation (6)).

This equation is plotted in Figure 3. The equation is independent of \(\alpha\).

Case 3.
\[\frac{\sigma_{xy}}{G} = \frac{[1 + \frac{9}{4}(\dot{e}_{xx})^2]^{-\frac{1}{2}}}{[1 + \frac{12}{5}(\dot{e}_{xy})^2]} \tag{16}\]
\[-\frac{\sigma_{yx}}{G} = \sigma_{yy} = \frac{[1 + \frac{9}{4}(\dot{e}_{xx})^2]^{-\frac{1}{2}}}{[1 + \frac{12}{5}(\dot{e}_{xy})^2]} \tag{17}\]

Equations (16) and (17) are plotted in Figures 4 and 5 respectively for various ratios of \(\frac{\alpha}{\eta G^2}\).

Figure 4 shows that by manipulating the parameter \(\alpha\) in equation (13), the shear stress-strain rate curve can be adjusted to be in better accord with what one would expect. Reasonable values of \(\frac{\alpha}{\eta G^2}\) would be \(\frac{\alpha}{\eta G^2} > 0\), \(\alpha\) should be adjusted to agree with the material under consideration.
Experiments show that $a$ could be adjusted to be in good agreement with many materials (see example reference 6 where curves are given for non-vulcanized rubber or reference 7 where results are shown for a 35% polystyrene solution). Before the stress-strain equation (equation 13) proposed here is used for a specific case results of several tests should verify the suitability of this equation. The experimental results of the three cases considered above should furnish sufficient data to judge the suitability of equation 13. Experimental investigations require great skill to avoid serious errors, for example, when concentric cylinders are rotated with respect to each other with the material to be tested between the cylinders there is a strong tendency for slip between the material and the cylinder. In the case of concentric cylinders, roughing of the cylinder to prevent slip often requires irregularities in the surface large enough to disrupt the flow pattern which invalidates the test.

If a material and geometrical configuration are chosen so that

$$\frac{n}{\sigma}(e_{ic})<<1$$

(where $e_{ic}$ is a characteristic strain rate of the system) then by referring to Figure 2 or equation (14) one may see that the results are substantially unchanged. The cases to be considered for the calendering process fall into the class of materials and geometry where $\frac{n}{\sigma} e_{ic}$ is small compared to one.
SOLUTION OF CALENDERING PROBLEM

Having chosen a suitable stress-strain rate equation, the calendering problem may now be considered. The following notation will be used:

- $u_0$ is the roll surface velocity.
- $x_0$ is the value of $x$ where the material leaves the rolls.
- $z_0$ is one-half the minimum gap between the rolls.
- $t(x)$ is one-half the thickness of the sheet at various values of $x$.
- $u$ is the component of velocity in the $x$ direction.
- $v$ is the component of velocity in the $y$ direction.
- $R$ is the radius of each of the rolls.

The subscript $o$ refers to specified velocities or distances.

A complication that immediately presents itself is that the value of $x_0$ must be arbitrarily prescribed. The real condition to be applied here is that the material leaves the roll when the stress pulling the material away from the roll equals the adhesion between the roll and the material. The determination of $x_0$ must be left to experiment as the present status of the theoretical work does not allow a theoretical prediction of the position of the material leaving the rolls. Approximate values of $x_0$ have been reported by Dr. R. E. Gaskell* as follows:

$$x_0 = \sqrt{\frac{Rt_0}{2}}$$

This value of $x_0$ makes it possible to determine solutions for comparative

* In reference 2 Professor M. Fisston mentions values of $x$ which were contained in a summary report to the Armstrong Cork Company by Dr. R. E. Gaskell.
purposes by a more precise value of $x_o$ would be required for an actual specific problem.

This solution assumes as both Dr. Gaskell's and Professor Finston's solution do, that there is no slipping between the material and the rolls. One anticipates that this solution will be similar to Dr. Gaskell's and Professor Finston's solutions so that normal stresses will be large compared to shear stress.

This means that moderate values of the coefficient of friction (less than 1) will prevent slip between the roll and material.

Now it is possible to deduce certain important results from the geometry of the calendering process. The solution will be restricted to configurations that comply with the following restrictions (most actual calendering processes will be included in the solution).

1. The thickness variation $2(t(t_o) - t_o)$ is of the order of magnitude of the minimum thickness $2t_o$. This means that $\frac{t(t_o) - t_o}{t_o}$ is of no higher order than one.

2. The passage length (\approx 2x_o) is long compared to the thickness $2t_o$.

3. The axial length of the rolls is long compared with the length of the passage (\approx 2x_o). This permits the approximation of plane strain. The plane strain approximation will be good except at the ends of the roll (edges of the sheet).

The implications of statement 3 are that:

\[ e_{xx} = e_{yy} = e_{zz} = 0 \]
\[ s_{zz} = 0, \quad \rho = -\sigma_{zz} = -\frac{\sigma_{xx} + \sigma_{yy}}{2} \]
\[ \tau_{xz} = \tau_{yz} = 0. \]

Since an incompressible material has been assumed from the outset, the velocity profiles at the various sections may be roughly pictured as follows:
Revising the rough shape of the velocity profile and statement 1, one may say that \[ \frac{u_{\text{max}} - u_0}{u_0} \] is of order \[ \frac{\Delta u_{\text{max}}}{t_{\text{max}}} \]. In the equations that follow \[ \rho = O(\delta) \] should be read “\[ \rho \] is of the order of \[ \delta \].”

Now \[ \frac{u_{\text{max}} - u_0}{u_0} = O(1) \]

and from this the following statements may be obtained:

\[ \left. \frac{\partial u}{\partial x} \right|_{\text{max}} = O\left(\frac{u_0}{x_0}\right) \quad \text{\( u = u_{\text{max}} \text{ when } x = 0, y = 0 \); and } u = u_0 \text{ at } x = x_0 \] 

\[ \left. \frac{\partial u}{\partial y} \right|_{\text{max}} = O\left(\frac{u_0}{x_0}\right) \quad \text{\( u = u_{\text{max}} \text{ when } y = 0, x = 0 \); and } u = u_0 \text{ at } y = y_0 \].

Now referring to the conservation of mass equation (written for an incompressible material):

\[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \text{or } \left. \frac{\partial u}{\partial y} \right|_{\text{max}} = O\left(\frac{\partial u}{\partial x} \right|_{\text{max}} = O\left(\frac{u_0}{x_0}\right). \]

Then \[ u_{\text{max}} = O\left(\frac{u_0}{x_0}\right) \text{ so that } \left. \frac{\partial u}{\partial x} \right|_{\text{max}} = O\left(\frac{u_0}{x_0}\right). \]

From the above order of magnitude study and statement 2 \( (\frac{x_0}{\delta} \ll 1) \) some consequences are:

\[ u_{\text{max}} \gg \delta_{\text{max}} \]

\[ \left. \frac{\partial u}{\partial y} \right|_{\text{max}} \gg \frac{\partial u}{\partial x} \right|_{\text{max}} \]

\[ \left. \frac{\partial u}{\partial y} \right|_{\text{max}} \gg \left. \frac{\partial u}{\partial y} \right|_{\text{max}} \]

\[ \left. \frac{\partial u}{\partial y} \right|_{\text{max}} \gg \left. \frac{\partial u}{\partial y} \right|_{\text{max}} . \]
The stresses may be analyzed in a similar way. This solution is intended to be for a material and geometry that exhibit primarily viscous behavior so that for an order of magnitude study we may write:

\[ \tau_{\text{max}} = o \left( \eta \frac{\partial u}{\partial y}_{\text{max}} \right) = o \left( \eta \frac{u_0}{\xi_0} \right) \]

and

\[ \frac{\partial \tau}{\partial x} \bigg|_{\text{max}} = o \left( \eta \frac{u_0}{\xi_0} \right) \quad (\tau = 0 \text{ when } x = x_0; \tau = \tau_{\text{max}} \text{ at } x = 0, y = 0) \]

\[ \frac{\partial \tau}{\partial y} \bigg|_{\text{max}} = o \left( \eta \frac{u_0}{\xi_0^2} \right) \quad (\tau = 0 \text{ when } y = 0; \tau = \tau_{\text{max}} \text{ at } x = 0, y = \xi_0). \]

The remaining stresses are \( \sigma_x + p \) and \( \sigma_y + p \) which reduce to a single variable in the following way:

\[ \sigma_x + p = \sigma_x - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \]

\[ \sigma_y + p = \sigma_y - \frac{\sigma_x - \sigma_y}{2} = \frac{\sigma_y - \sigma_x}{2} \]

so that

\[ \sigma_x + p = - (\sigma_y + p) = \frac{\sigma_x - \sigma_y}{2}. \]

Both of the stresses \( \sigma_x + p \) and \( \sigma_y + p \) are a result of shearing stresses acting on a rotating element for this problem so that one would anticipate that:

\[ (\sigma_x + p)_{\text{max}} = o \left( \tau_{\text{max}} \right) = o \left( \frac{u_0}{\xi_0} \right) \]

\[ \frac{\partial (\sigma_x + p)}{\partial x} \bigg|_{\text{max}} = o \left( \frac{\partial \tau}{\partial x}_{\text{max}} \right) = o \left( \frac{\eta}{\xi_0} \frac{u_0}{\xi_0} \right) \]

and

\[ \frac{\partial (\sigma_x + p)}{\partial y} \bigg|_{\text{max}} = o \left( \frac{\partial \tau}{\partial y}_{\text{max}} \right) = o \left( \frac{\eta}{\xi_0^2} \right). \]

The order of magnitude of \( \sigma_x + p \) will have to be checked to see if this condition is satisfied in the solution.

The above equations make suitable simplifications so that an approximate solution is possible. Before considering the equations, however, there is a limitation on the stress-strain rate equations. This limitation is that
where is a characteristic strain rate of the system. For the calendering problem the following restriction is imposed:

\[ \frac{\nu}{G} \left( \frac{\partial u}{\partial y} \right)_{\text{max}} = o(1) \quad \Rightarrow \quad \frac{\nu}{G} \frac{\partial u}{\partial x} = o\left(\frac{\nu}{G}\right) \ll 1. \]

The stress-strain rate equations (with the implications of statement 3) are now written out for examination.

\[ \left[ 1 + \frac{\nu}{G} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right)^2 \right) \right] \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) & 0 \\ \frac{\partial u}{\partial y} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) & 0 \\ \frac{\partial u}{\partial z} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \begin{align*}
&= \frac{1}{2G} \begin{bmatrix} \sigma_{xx} + \nu \sigma_{yy} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} + \nu \sigma_{xx} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz} + \nu \sigma_{xx} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} + \frac{1}{2G} \begin{bmatrix} \frac{\partial (\sigma_{xx} + \nu \sigma_{yy})}{\partial x} \\ \frac{\partial (\sigma_{xx} + \nu \sigma_{yy})}{\partial y} \\ \frac{\partial (\sigma_{xx} + \nu \sigma_{yy})}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} \\
&\quad + \frac{1}{2G} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} + \frac{1}{2G} \left( \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \right) \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} \end{align*} \]

The scheme to be followed is that the maximum order of each term in each stress-strain equation is to be written out and then only the highest order terms will be retained, thus, leading to simplified equations.

First taking the x x stress-strain rate equation:

\[ \begin{align*}
\left[ 1 + \frac{\nu}{G} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right)^2 \right) \right] \frac{\partial u}{\partial x} &= \frac{\sigma_{xx} + \nu \sigma_{yy}}{2G} \\
&\quad + \frac{1}{2G} \left[ \frac{\partial}{\partial x} \left( \sigma_{xx} + \nu \sigma_{yy} \right) + \nu \frac{\partial}{\partial y} \left( \sigma_{xx} + \nu \sigma_{yy} \right) + \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \right) \sigma_{xx} \right].
\end{align*} \]

The orders of the respective terms are:

\[ \begin{align*}
\left[ 1 + \alpha \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] \right] \frac{\alpha u}{\alpha x} &= \frac{\alpha u}{\alpha x} \\
&\quad + \left( \frac{\nu}{G} \frac{\alpha u}{\alpha y} \right) \left[ \frac{\alpha u}{\alpha x} + \frac{\alpha u}{\alpha y} + \frac{\alpha u}{\alpha z} \right] - \frac{\alpha u}{\alpha z}.
\end{align*} \]
Recalling that \( \alpha = \Omega(\frac{r}{\xi}) \), \( \frac{r}{\xi} \ll 1 \) and \( \left( \frac{K}{C} \frac{\gamma}{\xi} \right) = O(1) \)

the highest order terms turn out to be:

\[
0 = \frac{\sigma_x + p}{2\eta} - \frac{\gamma}{2\varepsilon} \frac{du}{dy} \quad \text{or} \quad \sigma_x + p = \frac{\varepsilon}{C} \frac{du}{dy} \gamma. \tag{19}
\]

\( \sigma_x + p \) is of the order of \( \gamma \), as anticipated.

Taking the \( xy \) stress-strain rate equation,

\[
\frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{r}{2\gamma} \\
+ \frac{1}{2\varepsilon} \left[ u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 v}{\partial y^2} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \left( \sigma_y - \sigma_x \right) \right]
\]

the orders of the respective terms are:

\[
\left\{ \frac{1}{2} + \alpha \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)^2 \right] \right\} \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) = \frac{u_o}{\varepsilon} \\
+ \left( \frac{r}{2\gamma} \right) \left[ \frac{u_o}{\varepsilon} + \frac{u_o}{\varepsilon} + \frac{u_o}{\varepsilon} - \frac{u_o}{\varepsilon} \right].
\]

The highest order terms of this equation turn out to be:

\[
\frac{1}{2} \left( 1 + \frac{\alpha}{2} \left( \frac{\partial u}{\partial y} \right)^2 \right) \frac{\partial u}{\partial y} = \frac{r}{2\gamma} + \frac{\sigma_x + p}{2\varepsilon} \left( \frac{\partial u}{\partial y} \right). \tag{20}
\]

Substituting equation (19) into equation (20) yields:

\[
\left( 1 + \frac{\alpha}{2} \left( \frac{\partial u}{\partial y} \right)^2 \right) \frac{\partial u}{\partial y} = \frac{r}{2} \left( 1 + G \left( \frac{\partial u}{\partial y} \right)^2 \right). \tag{21}
\]

The stress-strain rate equations have reduced to the following:

\[
\left( 1 + \frac{\alpha}{2} \left( \frac{\partial u}{\partial y} \right)^2 \right) \frac{\partial u}{\partial y} = \frac{r}{2} \left( 1 + \frac{\xi}{C} \left( \frac{\partial u}{\partial y} \right)^2 \right) \tag{21}
\]

and \( \sigma_x + p = -(\sigma_y + p) = \frac{r}{C} \frac{\partial u}{\partial y} \gamma. \)

Next consider the continuity equation,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{22}
\]
This equation states that when $U$ has been determined then $V$ may be determined.

The only equations that remain are the equilibrium equations. If kinetic energy terms are neglected these equations may be written as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial T}{\partial y} = 0$$  \hspace{1cm} (23)

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial T}{\partial x} = 0.$$  \hspace{1cm} (24)

The equilibrium equations may be simplified as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial T}{\partial y} = 0$$  \hspace{1cm} (23)

$$\frac{\partial T}{\partial x} = \frac{\partial (\sigma_x + p)}{\partial x} + \frac{\partial T}{\partial y}.$$  \hspace{1cm} (24)

This may be approximated by:

$$\frac{(\Delta p)^x}{x_0} = \frac{\text{change in pressure along passage}}{\text{length of passage}} = O\left(\frac{(\Delta (\sigma_x + p))_{\text{max}} + \frac{u_0 x_0}{c_0}}{x_0}\right).$$

$(\Delta p)^x$ is of the order $\Delta (\sigma_x + p)_{\text{max}} + \frac{u_0 x_0}{c_0}$.

Also:

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial T}{\partial x} = 0$$  \hspace{1cm} (24)

$$\frac{\partial T}{\partial y} = \frac{\partial (\sigma_y + p)}{\partial y} + \frac{\partial T}{\partial x} = -\frac{\partial (\sigma_y + p)}{\partial x} + \frac{\partial T}{\partial x}.$$  \hspace{1cm} (24)

This may be approximated by:

$$\frac{(\Delta p)^y}{c_0} = \frac{\text{change of } p \text{ across passage}}{\text{width of passage}} = O\left(\frac{-\Delta (\sigma_y + p)_{\text{max}} + \frac{u_0 x_0}{c_0}}{c_0}\right).$$

$(\Delta p)^y$ is of the order $\Delta (\sigma_y + p)_{\text{max}} + \frac{u_0 x_0}{c_0}$.

Since $\Delta (\sigma_x + p)_{\text{max}}$ is of the order $T_{\text{max}}$ one finds that $(\Delta p)^x \gg (\Delta p)^y$.

The above illustrates that the variation of $p$ with $x$ is much larger than the variation of $p$ with $y$. Now we make the approximation that $p = \text{function of } x \text{ only } = f(x)$.  \hspace{1cm} (25)
By disregarding higher order terms in equation (23) one obtains

\[ \frac{\partial \tau}{\partial y} = f'(x) \]

\[ \tau = f'(x) y + \text{constant.} \]

\[ \tau = 0 \quad \text{when} \quad y = 0 \quad \text{(from symmetry).} \]

and, therefore,

\[ \tau = f'(x) y = \eta F \frac{y}{\varepsilon(x)} \quad (26), \text{where} \quad F = \frac{\varepsilon(x) f'(x)}{\eta}. \quad (27) \]

Substituting equation (26) into equation (31) there results the following:

\[ \left(1 + \frac{\eta}{\varepsilon(x)} \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} = F \frac{y}{\varepsilon(x)} \left[1 + \frac{\eta^2}{\varepsilon(x)} \left(\frac{\partial u}{\partial y}\right)^2\right]. \quad (28) \]

At this point the results of this approximate procedure are collected.

\[ \left(1 + \frac{\eta}{\varepsilon(x)} \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} = F \frac{y}{\varepsilon(x)} \left[1 + \frac{\eta^2}{\varepsilon(x)} \left(\frac{\partial u}{\partial y}\right)^2\right] \quad (28) \]

\[ \tau = \eta F \frac{y}{\varepsilon(x)} \quad (26) \quad F = \frac{\varepsilon(x) f'(x)}{\eta} \quad (27) \]

\[ \phi = f'(x) \quad (25) \quad \sigma_x + \rho = -(\sigma_y + \rho) = \frac{k}{C} \tau \frac{\partial u}{\partial y}. \quad (19) \]

Overall conservation of mass yields a further restriction:

\[ \int_0^* \mu dy = \text{constant} = u_0 \varepsilon(x_0). \quad (25). \]

A numerical procedure may be set up to solve these equations. The procedure is as follows:

1. The geometry, speed, \( \frac{y}{C} \) and \( \alpha \) must be selected. \( \frac{y}{C} \), \( \frac{\varepsilon}{C} \) must not be of order greater than one.

2. Arbitrarily select a value for \( F \) (equation 26). This may be done, to start with, as a value of \( F \) occurring in the viscous solution. Now with this value of \( F \) and a value of \( \frac{\partial u}{\partial y} \) (chosen arbitrarily) equation 28 may be solved. For the selected value of \( F \) calculate several values of \( \frac{y}{\varepsilon(x)} \) (enough to give a sufficiently accurate curve of \( \frac{\partial u}{\partial y} \)).
versus \( \frac{y}{E(x)} \) in the region \( 0 < \frac{y}{E(x)} < 1 \).

3. With the \( \frac{du}{dy} \) variation with \( \frac{y}{E(x)} \) for a prescribed \( \alpha \), \( \frac{u}{E(x)} \) and \( F \) a numerical integration may be started with the condition that
\[
\alpha = \alpha_0 \quad \text{at} \quad \frac{y}{E(x)} = 1.
\]
This determines \( \frac{u}{E(x)} \) at every point along the section. \( \alpha \) is not known yet since \( \epsilon(x) \) is unknown. Since \( \frac{du}{dy} \) versus \( \frac{y}{E(x)} \) is nearly linear (the viscous case is linear), the trapezoidal rule furnishes a sufficiently accurate method of numerical integration.

4. \( \frac{u}{E(x)} \) may now be plotted against \( \frac{y}{E(x)} \). If the area under this curve is obtained (by Simpson's rule or some other appropriate means) then \( \epsilon(x) \) may be determined (from equation 29). One method which is convenient and accurate is to plot \( \frac{u}{E(x)} - \frac{u_0}{E(x)} \) versus \( \frac{y}{E(x)} \).

\[
\frac{u}{E(x)} - \frac{u_0}{E(x)}
\]
is the variation of the velocity profile from the uniform velocity profile. If the area under the \( \frac{u}{E(x)} - \frac{u_0}{E(x)} \) versus \( \frac{y}{E(x)} \) is determined and designated by \( A \) then from continuity (equation 29).

\[
A \left( \epsilon(x) \right)^2 + 2u_0 \epsilon(x) - 2u_0 \epsilon(x_0) = 0
\]

This quadratic may be solved for \( \epsilon(x) \) and the value of \( \alpha \) corresponding to this value of \( F \) may be determined as well.

5. Steps 1 through 4 are carried out for enough values of \( F \) to obtain a sufficiently accurate plot of \( F \) versus \( \alpha \). \( F \) is the maximum shear stress for the corresponding values of \( \alpha \) so that \( \tau_{\max} \) versus \( \alpha \) is known. Also \( f(x) \) in equation (25) may be found as a functions of \( \alpha \) by using equation (27). A numerical integration may be used to find \( f(x) \) for each value of \( x \). The boundary condition to be used is that \( f(x) = 0 \) at \( x = x_0 \).

The scheme outlined above was used to find the solution of a problem with fixed geometry, viscosity and speed of rolling but with variable \( G \).
The numerical values used for the geometry were:

Radius of rolls = 14" *

21o = .100"

Roll surface speed = 100 sec and \( \kappa_0 = .446\) .

Other parameters used were:

\[
\frac{\mu_0}{\kappa_0} \frac{n}{G} = \sqrt{1.1}, \sqrt{1.5}, 1.0
\]

(of order 1).

For each value of \( \frac{\mu_0}{\kappa_0} \frac{n}{G} \), the following values of \( a \) were used:

\( a = 0 \), \( a = \frac{n}{G} \), \( a = 2.5 \frac{n}{G} \), \( a = 4 \frac{n}{G} \).

The results obtained are tabulated in Tables I to X. Plots of all results are shown for \( a = 0 \) and \( \frac{\mu_0}{\kappa_0} \frac{n}{G} \) in Figures 6 through 13. Plots of \( \frac{\partial u}{\partial y} \) and \( \frac{u}{G(a)} - \frac{u_0}{G(a)} \) versus \( \frac{y}{G(a)} \) for \( F = 500 \) are shown for the remaining cases in Figures 14 through 22. The dotted line on the curve represents the viscous solution for the same value of \( F \). Pressure distributions and \( T_{\text{max}} \) distributions for the remaining cases are shown in Figures 23 through 38. Values given in the tables for the force pushing the rolls apart is the force exerted in the passage \( (-\kappa_0 \leq \kappa \leq \kappa_0) \) only. For velocity distributions and positions in the passage for values of \( F \) below these tabulated the viscous solution is used.

Since \( \frac{n}{G} \), \( \frac{\mu_0}{\kappa_0} \), \( a \frac{\mu_0}{\kappa_0} \), and \( \frac{\kappa_0}{x_0} \) are small, a perturbation procedure seems in order. This is carried out in the Appendix. Unfortunately, the most conveniently chosen parameters restrict the solution far more than the approximate procedure given in this section.

* These values are reported by Fiston in Reference 2, to be appropriate.
CONCLUSIONS

Figures 14 through 22 illustrate clearly the effect of replacing a viscous material by an elastic-viscous material of the type considered. The most important result shown here is the variation in $\frac{du}{dy}$ from the viscous case at the surface of the material. The indication from these calculations is that $\frac{du}{dy}$ varies across the section in a different manner (in most cases) from the viscous case. The percent difference between $\frac{du}{dy}|_{\text{max}}$ for the elastic viscous case and the viscous case was from 0 to 14.6 for equal values of maximum shearing stress. The percent difference depends on the value of $a$ and the speed of operation. The percent difference goes up as the speed is increased and down as $a$ is increased.

The difference in variation in $\frac{du}{dy}$ across the section for an elastic-viscous material as compared to a viscous material explains the failure of the viscous solution in making suitable predictions regarding the behavior of an elastic viscous material. The calculations carried out are to indicate how the elastic-viscous solution varies from the viscous solution. The approximate method included here certainly indicates which direction the variables go from the viscous variables if a suitable value of $a$ is known.

The results of the approximate solution indicate that the pressure and shear stress drop as $G$ is lowered. This is because the viscosity is held constant and the lowering of $G$ corresponds to a reduction of stiffness of the material. In other words, the material may accommodate a given load by elastic as well as viscous deformation so that the load necessary for a prescribed time dependent deformation will in general go down as $G$ decreases. Therefore, these results show a slight decrease in total lateral force when $G$ decreases (maximum decrease of 7.5% for these calculations).
The parameter $\frac{u_0}{k} \frac{R}{G}$ is seen to be an important parameter in this problem. This is not too surprising. $\frac{\partial u}{\partial y}|_{\text{max}}$ and $\frac{n}{G}$ are certainly two major parameters and the solution would depend strongly on some combination of these since they are related through the stress-strain rate equation. $\frac{u_0}{k}$ is the order of $\frac{\partial u}{\partial y}|_{\text{max}}$.

The reader should note that this approximate solution does not allow the determination of the elastic recovery after the material has left the rolls.

This solution only includes the solution of the velocities and the stresses. The associated heat transfer problem was not undertaken but for values of $a < \frac{n^2}{6}$ one would anticipate an increase of peak temperature over the viscous case because the maximum shear stress is larger for the elastic viscous material.

This solution should be of use to manufacturers who use the calendering process. The knowledge of how the variables of the problem change when the material has elastic as well as viscous properties may be used as a guide in selecting the speed of rolling, the temperature of the bulk to be rolled and the amount of cooling of the rolls necessary.
BIBLIOGRAPHY


LIST OF TABLES

I  Tabulated values for  \( a = 0 \) and  \( \frac{\mu_0 H}{E_0} = \sqrt{1.1} \)

II  \( a = 0 \)  \( \frac{\mu_0 H}{E_0} = \sqrt{5} \)

III  \( a = 0 \)  \( \frac{\mu_0 H}{E_0} = 1.0 \)

IV  \( a = \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = \sqrt{1.1} \)

V  \( a = \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = \sqrt{5} \)

VI  \( a = \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = 1.0 \)

VII  \( a = 2.5 \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = \sqrt{1.1} \)

VIII  \( a = 2.5 \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = \sqrt{5} \)

IX  \( a = 2.5 \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = 1.0 \)

X  \( a = 4 \frac{r^2}{E_0} \)  \( \frac{\mu_0 H}{E_0} = \sqrt{1.1}, \sqrt{5}, 1.0 \)
**TABLE I**

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For \( \alpha = 0 \)

\[
\frac{\phi}{L} \cdot \begin{array}{c c c}
600 & 700 & 800 \\
\phi(x) & 0.0518 & 0.0509 & 0.0503 \\
\alpha & 0.218 & 0.162 & 0.080 \\
\end{array}
\]

\[
\frac{\phi}{L} \cdot \begin{array}{c c c c c c}
\kappa & \phi \left( \frac{1}{L} \right) & \phi \left( \frac{1}{L} \right) & \phi \left( \frac{1}{L} \right) \\
0.446 & 9190 & 9190 & 9190 \\
0.40 & 9140 & 9130 & 9140 \\
0.35 & 8960 & 8940 & 8970 \\
0.30 & 8640 & 8610 & 8660 \\
0.25 & 8190 & 8140 & 8230 \\
0.20 & 7630 & 7560 & 7630 \\
0.15 & 6970 & 6890 & 7050 \\
0.10 & 6230 & 6130 & 6330 \\
0.05 & 5430 & 5320 & 5540 \\
0 & 4600 & 4480 & 4710 \\
0.05 & 3760 & 3650 & 3870 \\
0.10 & 2960 & 2870 & 3060 \\
0.15 & 2220 & 2140 & 2300 \\
0.20 & 1570 & 1500 & 1630 \\
0.25 & 1000 & 960 & 1050 \\
0.30 & 650 & 530 & 580 \\
0.35 & 240 & 220 & 250 \\
0.40 & 60 & 50 & 60 \\
0.446 & 0 & 0 & 0 \\
\end{array}
\]

Lateral force on rolls = \( \frac{9180}{\text{sec}} \)
TABLE II

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a=0$, \( \frac{u_0}{2C} = \sqrt{5} \)

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| $\frac{\partial^2 u}{\partial t^2}$ (in sec$^{-2}$) | | | | | | |
| 0 | 151 | 202 | 254 | 307 | 363 | 418 |
| 0.1 | 144 | 200 | 252 | 304 | 358 | 414 |
| 0.2 | 145 | 194 | 244 | 295 | 348 | 402 |
| 0.3 | 137 | 184 | 232 | 280 | 330 | 382 |
| 0.4 | 127 | 170 | 214 | 259 | 305 | 354 |
| 0.5 | 113 | 152 | 192 | 232 | 273 | 317 |
| 0.6 | 97 | 130 | 164 | 198 | 234 | 272 |
| 0.7 | 77 | 104 | 131 | 159 | 187 | 218 |
| 0.8 | 55 | 73 | 92 | 112 | 133 | 155 |
| 0.9 | 29 | 39 | 49 | 60 | 71 | 83 |
TABLE II (continued)

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For $a = 0$ and $\frac{c_0}{c_0} = \sqrt{3}$,

Lateral force on roll $\frac{\eta}{\tau} = 4140 \text{ in}^2 \text{sec}$
### TABLE III

Numerical solution of equations 14, 25, 26, 27, 28 & 29

For \( \alpha = 0 \) \hspace{1cm} \frac{\psi}{\xi} \xi = 1.0

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Lateral force on rolls = \( 4030 \text{ in}^2 \text{ sec}^{-1} \)
TABLE IV
Numerical solution of equations 19, 25, 26, 27, 28 & 29
For $a = \frac{\nu}{3}$  $\frac{\nu}{4} = \sqrt{\frac{3}{x}}$

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Lateral force on rolls $= 4180 \text{ lb/sec}$

This case is very close to the viscous case
TABLE V
Numerical solution of equations 19, 25, 26, 27, 28 & 29

For \( \alpha = \frac{n^2}{G} \), \( \frac{\mu_0}{\epsilon} \frac{\delta}{\epsilon} = \sqrt{5} \)

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Lateral force on rolls \( \eta = 4190 \text{ in} \cdot \text{sec}^{-1} \)
TABLE VI
Numerical solution of equations 19, 25, 26, 27, 28 & 29
For \( a = \frac{L}{D} \quad \frac{L}{C} = 1.0 \)

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\( \frac{\partial^2 u}{\partial y^2} + \frac{1}{\lambda} \)

<p>| 0 | 151 | 203 | 256 | 315 | 368 | 428 |
| .1 | 150 | 201 | 253 | 312 | 364 | 424 |
| .2 | 145 | 195 | 246 | 303 | 354 | 412 |
| .3 | 136 | 185 | 233 | 288 | 336 | 392 |
| .4 | 127 | 171 | 216 | 267 | 312 | 364 |
| .5 | 119 | 153 | 193 | 239 | 279 | 327 |
| .6 | 97 | 131 | 165 | 206 | 240 | 281 |
| .7 | 77 | 104 | 132 | 165 | 192 | 226 |
| .8 | 55 | 74 | 94 | 119 | 137 | 161 |
| .9 | 29 | 39 | 50 | 61 | 73 | 86 |
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Lateral force on rolls \( \beta \) = 4120 \( \text{in/s} \)
TABLE VII

Numerical solution of equations 19, 25, 26, 27, 28 & 29

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Lateral force on roller $= 4240 \text{ in/ft}$.
**Table VIII**

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For \( \alpha = 2.5 \frac{m}{s^2} \quad \frac{\mu}{\rho_0} \frac{v_0}{c} = \sqrt{5} \)

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**Lateral forces on rolls**

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Lateral force on rolls \(= 4340 \frac{\text{in}^2}{\text{sec}} \frac{1}{\eta} \)
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<th>( \varepsilon (\alpha) )</th>
<th>( \alpha )</th>
<th>( \varepsilon \left( \frac{\varepsilon}{\alpha} \right) )</th>
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For \( \alpha = 4 \frac{y}{y} \)

For all values of \( \frac{u_0 y}{L_0 G} \)

\[
\frac{dy}{dy} = F \left( \frac{y}{\alpha} \right) \]

\[
u = u_0 \left( 1 - \frac{y}{y_0} \right) F \frac{\varepsilon}{\alpha} \]
TABLE X (continued)

when \( \frac{V}{G} = 0 \) this is the viscous case
for other values of \( \frac{V}{G} \)

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Lateral force on rolls
= 4210 lb*

Lateral force on rolls
= 1310 lb*

Lateral force on rolls
= 1360 lb*
1. Plot of $\frac{1}{\sqrt{e}}$ versus $\frac{1}{\sqrt{F}}$ for Maxwell Fluid between parallel plates.

2. Tension Test with Constant Stress.

3. Relaxation Test with Constant Strain.

4. Fluid between Two Parallel Plates $\frac{1}{\sqrt{e}}$ versus $\frac{1}{\sqrt{F}}$.

5. Fluid between Two Parallel Plates $\frac{1}{\sqrt{e}}$ and $\frac{1}{\sqrt{F}}$ versus $\frac{1}{\sqrt{F}}$.

6. $\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{\epsilon}$ versus $\frac{1}{\sqrt{F}}$ for $a = 0$, $\frac{u_0}{\epsilon} = 1$ and $F = 200$.

7. $F = 300$.

8. $F = 400$.


10. $F = 600$.

11. $F = 700$.

12. $\frac{1}{\sqrt{e}}$ versus for $a = 0$ and $\frac{u_0}{\epsilon} = 1$.

13. $\frac{1}{\sqrt{e}}$ and $\frac{u_0}{\epsilon} |_{\text{max}}$ versus $x$ for $a = 0$ and $\frac{u_0}{\epsilon} = 1$.

14. $\frac{1}{\sqrt{e}}$ and $\frac{u-u_0}{\epsilon}$ versus $\frac{1}{\sqrt{F}}$ for $a = 0$ and $\frac{u_0}{\epsilon} = 1$.

15. $\frac{1}{\sqrt{e}}$ for $a = 0$ and $\frac{u_0}{\epsilon} = \sqrt{.5}$

16. $a = \frac{1}{\sqrt{F}}$ and $\frac{u_0}{\epsilon} = \sqrt{.1}$

17. $a = \frac{1}{\sqrt{F}}$ and $\frac{u_0}{\epsilon} = \sqrt{.5}$

18. $a = \frac{1}{\sqrt{F}}$ and $\frac{u_0}{\epsilon} = 1$

19. $a = 2.5$ and $\frac{u_0}{\epsilon} = \sqrt{.1}$

20. $a = 2.5$ and $\frac{u_0}{\epsilon} = \sqrt{.5}$

21. $a = 2.5$ and $\frac{u_0}{\epsilon} = 1$

22. $\frac{1}{\sqrt{e}} |_{\text{max}}$ versus $x$ for $a = 0$ and various values of $\frac{u_0}{\epsilon}$

23. $\frac{1}{\sqrt{e}}$ for $a = 0$

24. $a = \frac{1}{\sqrt{F}}$

25. $a = 2.5\frac{1}{\sqrt{F}}$

26. $a = 4 \frac{1}{\sqrt{F}}$ and all values of $\frac{u_0}{\epsilon}$
27. \( \frac{L}{h} \) and \( \frac{c_n}{h} \) max. versus \( x \) for \( a = 0 \) and \( \frac{\pi}{a} = \sqrt{1.1} \)

28. 

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36. 

37. 

38. 

\( a = 4 \) and \( \frac{\pi}{a} = 1 \)
Figure 1
Plot of \( \frac{\tau}{\gamma} \) Versus \( \frac{\gamma}{\gamma_0} \) for Maxwell Fluid Between Parallel Plates
Figure 2

Tension Test with Constant Stress

\[
\frac{\sigma_{xx}}{9\eta} \quad \text{versus} \quad \dot{\varepsilon}_{xx} \quad \quad (1+\alpha J) \dot{\varepsilon}_{ij} = \frac{\dot{S}_{ij}}{2G} + \frac{S_{ij}}{2\eta}
\]
Figure 3

Relaxation Test in Tension with Constant Strain

$$(1 + \alpha J_2) \dot{\varepsilon}_{ij} = \frac{\dot{\sigma}_{ij}}{2\sigma} + \frac{\dot{\varepsilon}_{ij}}{2\varepsilon}$$

Normal Stress at Time = t
Normal Stress at Time = 0 versus Time $\frac{t}{\eta/\sigma}$

-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0

$\frac{\sigma_{xx}}{\sigma_{xx_0}}$

all values of $\alpha$

$\frac{t}{\eta/\sigma}$

0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
Fluid between Two Parallel Plates

Shear Stress versus Shear Strain Rate

\[(1 + \sigma J) \dot{\varepsilon} = \frac{\dot{\gamma} \dot{\gamma}}{2G} + \frac{5\dot{\gamma} \dot{\gamma}}{2\eta} \]
Fluid between Two Parallel Plates

$\left(1 + a \frac{J_2}{J_1}\right) \varepsilon_0 = \frac{5 \varepsilon_0}{2} + \frac{5 \varepsilon_0}{2}$

Figure 5

Normal Strain versus Shear Strain Rate

$G/\mu$

$G^*$

$\frac{G^*}{G}$

$\frac{\mu}{G}$

$a = 0$

$a = 0.5$

$a = 1.0$

$a = 2.0$

$a = 1.5$

$a = 1.2$

$a = 1.6$

$a = 2.0$

$a = 1.6$

$a = 1.0$

$a = 0.5$

$a = 0.0$
Figure 6

Dotted line corresponds to viscous solution for some value of $p$.

\[ \frac{u - u_0}{2} \]

versus \( y \), for \( a = 0 \) and

\[ y = \frac{t}{\sqrt{4 \pi}} \]

with \( a = 1 \).

\[ \frac{\partial u}{\partial y} \]

and

\[ \frac{\partial u}{\partial \rho} \]

versus \( \frac{y}{t} \) for \( \rho = 200 \).

\[ x = 0.374 \]
Figure 7

\[ \frac{du}{dy} \text{ and } \frac{\kappa - \kappa_0}{\kappa} \text{ versus } \frac{y}{\epsilon} \]

for \( \alpha = 0 \) and \( \frac{\kappa_0}{\kappa} \frac{y}{\epsilon} = 1 \)

\( F = 300 \)
\( x = 0.342 \)

Dotted line corresponds to viscous solution for same value of \( F \).
Figure 8

\[ \frac{\partial u}{\partial y} \text{ and } \frac{u - u_0}{c} \text{ versus } \frac{y}{t} \text{ for } a = 0 \text{ and } \frac{u_0}{c} \frac{x}{t} = 1 \]

\( F = 400 \)
\( x = 0.300 \)

Dotted line corresponds to viscous solution for same value of \( F \).
Figure 9

\[ \frac{\partial u}{\partial y} \text{ and } \frac{u - u_0}{\varepsilon} \text{ versus } \frac{y}{\varepsilon} \]

for \( a=0 \) and \( \frac{u_0}{\varepsilon} = 1.0 \)

\[ F = 500 \]
\[ x = 0.252 \]

Dotted line corresponds to viscous solution for same value of \( F \).
Figure 10

\[ \frac{du}{dy} \quad \text{and} \quad \frac{u-u_0}{\varepsilon} \quad \text{versus} \quad \frac{y}{\varepsilon} \quad \text{for} \quad a = 0 \quad \text{and} \quad \frac{u_0}{\varepsilon} = \frac{\varepsilon}{\varepsilon} = 1 \]

- \( P = 600 \)
- \( x = 0.198 \)

Dotted line corresponds to viscous solution for same value of \( P \).
Figure 11

\( \frac{\partial u}{\partial y} \) and \( \frac{u-u_0}{e} \) versus \( \frac{y}{\pi} \) for \( \alpha = 0 \) and \( \frac{u_0}{\sigma} \frac{\kappa}{\kappa} = 1.0 \)

\( P = 700 \)
\( x = .123 \)

Dotted line corresponds to viscous solution for same value of \( P \).
Figure 12

\[ \frac{T_{\text{max}}}{R} \text{ versus } x \text{ for } a=0 \text{ and } \frac{\mu_0}{c_0} = 1 \]
Figure 14

\[ \frac{\partial u}{\partial y} \text{ and } \frac{u - u_0}{c} \text{ versus } \frac{c}{c} \]

for \( \alpha = 0 \) and \( \frac{u_0}{c} \frac{c}{c} = \sqrt{1} \)

F 800
x .080

Dotted line corresponds to viscous solution for same value of F.
Figure 15

\( \frac{du}{dy} \) and \( \frac{u-u_0}{\varepsilon} \) versus \( \frac{y}{\varepsilon} \) for \( \alpha = 0 \) and \( \frac{u_0}{\varepsilon} \frac{\beta}{\kappa} = \sqrt{5} \)

\begin{align*}
F & = 800 \\
x & = 0.046
\end{align*}

Dotted line corresponds to viscous solution for same value of F.
Figure 16

\[ \frac{\partial u}{\partial y} \text{ and } \frac{u - u_0}{c} \text{ versus } \frac{y}{\xi} \]

for \( \alpha = \frac{v}{c} \) and \( \frac{u_0}{\xi} \frac{y}{\xi} = \sqrt{1} \)

F = 800

x = 0.082

Dotted line corresponds to viscous solution for same value of F.
Figure 17

\[ \frac{\partial u}{\partial y} \text{ and } \frac{u - u_0}{\tau} \text{ versus } \frac{y}{\varepsilon} \text{ for } a = \frac{\eta^2}{\varepsilon^3} \text{ and } \frac{u_0}{\tau} \frac{y}{\varepsilon} = \sqrt{5} \]

F = 800
x = 0.0611

Dotted line corresponds to viscous solution for same value of F.
Figure 18

\[ \frac{\partial u}{\partial y} \text{ and } \frac{u - u_0}{\tau} \text{ versus } \frac{y}{\tau} \text{ for } a = \frac{\tau}{\alpha}, \text{ and } \frac{u_0}{\tau} \frac{\eta}{\tau} = 1 \]

\[ F = 800 \]

\[ x = 0 \]

Dotted line corresponds to viscous solution for same value of F.
Figure 19

\( \frac{\partial u}{\partial y} \) and \( \frac{u - u_0}{\zeta} \) versus \( \frac{y}{\zeta} \) for \( \alpha = 25 \frac{m}{\zeta} \) and \( \frac{u_0}{\zeta} \frac{K}{\zeta} = 1 \)

\[ F = 800 \]
\[ x = 0.095 \]

Dotted line corresponds to viscous solution for same value of \( F \).
Figure 20

\( \frac{\partial u}{\partial y} \) and \( \frac{u - u_0}{\eta} \) versus \( \frac{y}{\delta} \) for \( \alpha = 2s \frac{\eta^2}{\delta} \) and \( \frac{u_0}{\eta} \frac{\eta}{\delta} = \sqrt{s} \)

\( F = 800 \)

\( x = .082 \)

Dotted line corresponds to viscous solution for same value of \( F \).
Figure 21

Dotted line corresponds to viscous solution for same value of $P$. 

\[ P = 800 \]
\[ x = \frac{0.64}{2} \]

\[
\frac{\nu}{\nu_0} = 1 - \frac{2}{2} \frac{\nu_0}{\nu} \frac{\nu}{\nu_0} = 1
\]

\[
\frac{\mu - \mu_0}{\mu_0} \quad \text{versus} \quad \frac{y}{\rho}
\]

\[
\frac{\rho}{\rho_0} \quad \text{versus} \quad \frac{y}{\rho}
\]
Figure 22

\( \frac{du}{dy} \) and \( \frac{u-u_0}{c} \) versus \( \frac{y}{c} \) for \( \alpha = 1.8^\circ \) and all values of \( \frac{u_0}{c} \).

\[ F = 800 \]
\[ x = 0.095 \]

Dotted line corresponds to viscous solution for same value of \( F \).
Figure 23

\( \frac{T_{max}}{n} \) versus \( x \) for \( a = 0 \) and various values of \( \frac{u_0}{c_0} \frac{n}{c_0} \)

The numbers refer to values of \( \frac{u_0}{c_0} \frac{n}{c_0} \).
Figure 24

\( \frac{T_{\text{max}}}{\eta} \) versus \( x \) for \( a = \frac{\eta^3}{\delta^2} \) and various values of \( \frac{\delta}{\delta_0} \)

The numbers refer to values of \( \frac{\delta}{\delta_0} \)
Figure 25

$\frac{T_{\text{max}}}{\eta}$ versus $x$ for $\alpha = 2.5 \frac{u_0}{c}$ and various values of $\frac{u_0}{c}$

Numbers refer to values of $\frac{u_0}{c}$. $\sqrt{5}$, $\sqrt{1}$, and 1.0.
Figure 26

\[
\frac{T_{\text{max}}}{k} \quad \text{versus} \quad x \quad \text{for} \quad a = \frac{n^2}{c^2} \quad \text{and all values of} \quad \frac{u_0}{c} \quad \frac{k}{c}
\]
Figure 27

$\frac{\sigma_{yy}}{n}$ and $-\frac{\sigma_{yy}}{n_{\text{max}}}$ versus $x$ for $a = \sigma$ and $\frac{u_{0}}{c_{0}} = \sqrt{1}$
Figure 29

\( \frac{P}{h} \) and \(-\frac{Q_y}{h}\) versus \(x\) for \(a = 0\) and \(\frac{\mu_e \eta}{\rho} = 1\)
Figure 31

\[ \frac{p}{\eta} \quad \text{and} \quad -\frac{\delta_{1}}{\max} \quad \text{versus} \quad x \quad \text{for} \quad \alpha = \frac{\beta^2}{\sigma} \quad \text{and} \quad \frac{\alpha_0}{\sigma_0} \quad = \sqrt{5} \]
Figure 32

\[ \frac{\sigma_{x}^{2}}{\sigma} \text{ and } -\sigma_{x} |_{\text{max}} \text{ versus } x \text{ for } \alpha = \frac{\mu_{x}}{\sigma_{x}} \text{ and } \frac{\mu_{x}}{\sigma_{x}} = \lambda \]
Figure 33

$\frac{P}{L}$ and $G_y$ vs. $x$ for $v = 2.5$ and $2.5 k_x = 0.1$
Figure 3.4: Graph of $x$ versus $y$ for $a = 2.5/\sqrt{\alpha^2}$ and $y = 0.5\sqrt{\alpha}$. The graph shows the relationship between $x$ and $y$ under these conditions.
Figure 36

\[ \frac{p}{n} \text{ and } -\frac{\sigma_n}{n} \text{ max versus } x \text{ for } a = \frac{k^2}{c^2} \text{ and } \frac{\omega_0}{c_0} \frac{n}{c} = \sqrt{1} \]
Figure 37

$-\frac{q_x}{x_{\text{max}}} \text{ versus } x \text{ for } a = 4 \frac{h}{c} \text{ and } \frac{a}{\pi^2} \frac{x_2}{c} = \sqrt{.5}$
APPENDIX

Perturbation Procedure

Before undertaking the formal perturbation, the boundary conditions may be simplified by converting the equations to bipolar coordinates.

\[ \eta = \log \frac{r}{\eta_i}, \quad \xi = \theta_1 - \theta_2 \]
\[ x = \frac{a \sinh \eta}{\cosh \eta - \cosh \xi}, \quad y = \frac{a \sin \xi}{\cosh \eta - \cosh \xi} \]

Then lines of \( \eta = \text{constant} \) are shown below.

Now the two rolls may be presented by \( \eta = \text{constant} \equiv \eta_0 \) lines in the orthogonal coordinates. The equations to be transformed are:

\[ \frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial y} = 0 \]  \text{continuity}

\[ \left\{ \begin{array}{l}
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \\
\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x} = 0
\end{array} \right. \]  \text{equilibrium}

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \]
\[
(1 + \alpha \psi_k) \left( \frac{\partial u_x}{\partial x} \right) = \frac{\sigma_{xx} + \tau}{2 G} + \frac{u_x \frac{\partial (\sigma_{xx} + \tau)}{\partial x} + u_y \frac{\partial (\sigma_{xx} + \tau)}{\partial y}}{2 G} + \frac{(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y})}{\lambda} \gamma_{xy}
\]

\[
\frac{1}{2} \left( (1 + \alpha \psi_k) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) \right) = \frac{\sigma_{yy}}{2 G} + \frac{u_x \frac{\partial T_{xx}}{\partial x} + u_y \frac{\partial T_{yy}}{\partial y}}{2 G} + \frac{1}{2} \left( \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} \right)^2 (\sigma_{xy} - \sigma_{yy})
\]

Following conventional means the equation in \( \eta \xi \) coordinates are:

\[
\frac{\partial}{\partial \eta} \left( \frac{a \frac{\partial u_x}{\partial \eta}}{\cosh \eta - \cos \theta} \right) + \frac{\partial}{\partial \xi} \left( \frac{a \frac{\partial u_x}{\partial \xi}}{\cosh \eta - \cos \theta} \right) = 0 \quad \text{continuity}
\]

\[
\frac{\partial \sigma_{xx}}{\partial \eta} + \frac{\partial T_{xx}}{\partial \xi} + \frac{\sin \theta}{\cosh \eta - \cos \theta} \left( \frac{\sigma_{xx} - \sigma_{yy}}{\cosh \eta - \cos \theta} \right) - 2 \frac{\cosh \eta}{\cosh \eta - \cos \theta} \frac{T_{xy}}{\cosh \eta - \cos \theta} = 0 \quad \text{equilibrium}
\]

\[
\frac{\partial \sigma_{yy}}{\partial \eta} + \frac{\partial T_{yy}}{\partial \xi} - \frac{\sin \theta}{\cosh \eta - \cos \theta} \left( \frac{\sigma_{yy} - \sigma_{xx}}{\cosh \eta - \cos \theta} \right) - 2 \frac{\cosh \eta}{\cosh \eta - \cos \theta} \frac{T_{xy}}{\cosh \eta - \cos \theta} = 0 \quad \text{equilibrium}
\]

\[
J_2 = \frac{1}{2} \frac{(\sinh \eta + \sin \theta)}{a^2} u_x^2 + \frac{\sin \theta}{a^2} \sinh \eta u_y u_x + \frac{1}{2} \frac{(2 \sinh \eta + \sin \theta)}{a^2} u_y^2 + \frac{(\cosh \eta - \cos \theta)}{a^2} (u_y \sinh \eta + u_y \sin \theta) \left( \frac{\partial u_x}{\partial \eta} + \frac{\partial u_y}{\partial \xi} \right) - 2 \frac{(\cosh \eta - \cos \theta)}{a^2} (u_y \frac{\partial u_x}{\partial \eta} \sinh \eta + u_y \frac{\partial u_y}{\partial \eta} \sin \theta) + \frac{1}{2} \frac{a}{a^2} \left( \left( \frac{\partial u_x}{\partial \eta} + \frac{\partial u_y}{\partial \xi} \right)^2 + \left( \frac{\partial u_x}{\partial \eta} \right)^2 + \left( \frac{\partial u_y}{\partial \eta} \right)^2 \right)
\]

\[xx \quad \text{stress strain equation}\]
\[
\begin{align*}
(1 + \varepsilon J_0) \left\{ \frac{\sinh \gamma}{a(\cosh \gamma - \cosh \eta)} \left( u_y (1 - \cosh \gamma \cosh \eta) + u_\eta \sinh \gamma \sinh \eta \right) \\
+ \frac{\sinh \gamma \sinh \eta}{a(\cosh \gamma - \cosh \eta)} \left( \frac{\partial u_y}{\partial \xi} \sinh \gamma \cosh \eta - \frac{\partial u_\eta}{\partial \eta} (1 - \cosh \gamma \cosh \eta) \right) \\
- \frac{(1 - \cosh \gamma \cosh \eta)}{a(\cosh \gamma - \cosh \eta)} \left( \frac{\partial u_y}{\partial \xi} \sinh \gamma \cosh \eta - \frac{\partial u_\eta}{\partial \eta} (1 - \cosh \gamma \cosh \eta) \right) \right\} =
\end{align*}
\]

\[
\frac{(\sinh \gamma \sinh \eta - (1 - \cosh \gamma \cosh \eta) \frac{dz}{d\xi})}{4 \left( \cosh \gamma - \cosh \eta \right)^2} \left( \frac{u_y}{\cosh \eta} \frac{\partial (\gamma - \eta)}{\partial \xi} + u_\eta \frac{\partial (\gamma - \eta)}{\partial \eta} \right)
\]

\[
+ \frac{\sinh \gamma}{2a(\cosh \gamma - \cosh \eta)} \frac{(\gamma - \eta)}{\cosh \eta} \left( u_y \frac{\partial (\gamma - \eta)}{\partial \xi} - u_\eta \frac{\partial (\gamma - \eta)}{\partial \eta} \right)
\]

\[
+ \frac{\sinh \gamma \sinh \eta \left( 1 - \cosh \gamma \cosh \eta \right)^2}{2 \left( \cosh \gamma - \cosh \eta \right)^2} \frac{\partial u_y}{\partial \xi} \left( \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial \eta} - \frac{\partial}{\partial \eta} \frac{\partial \xi}{\partial \xi} \right)
\]

\[
+ \frac{\sinh \gamma \sinh \eta \left( 1 - \cosh \gamma \cosh \eta \right)}{2 \left( \cosh \gamma - \cosh \eta \right)} \left( \frac{\partial u_y}{\partial \xi} - \frac{\partial u_\eta}{\partial \eta} \right)
\]

x - y stress strain equation

\[
\begin{align*}
\frac{1 + \varepsilon J_0}{2a(\cosh \gamma - \cosh \eta)} \left\{ u_y (\sinh \gamma \cosh \eta \cosh \gamma + (1 - \cosh \gamma \cosh \eta) \cos \gamma \sinh \eta) \\
+ u_\eta (\sinh \gamma \cosh \gamma \cosh \eta - (1 - \cosh \gamma \cosh \eta) \sin \gamma \cosh \eta) \right\} \\
+ \left( \sinh \gamma \sinh \eta - (1 - \cosh \gamma \cosh \eta) \right) \left( \frac{\partial u_y}{\partial \xi} + \frac{\partial u_\eta}{\partial \eta} \right) \\
+ 2 \left( \sinh \gamma \sinh \eta \right) \left( \frac{\partial u_y}{\partial \xi} - \frac{\partial u_\eta}{\partial \eta} \right) \right\} =
\end{align*}
\]

\[
\frac{(\sinh \gamma \sinh \eta - (1 - \cosh \gamma \cosh \eta) \frac{dz}{d\xi})}{2 \left( \cosh \gamma - \cosh \eta \right)^2} \left( \frac{u_y}{\cosh \eta} \frac{\partial (\gamma - \eta)}{\partial \xi} + u_\eta \frac{\partial (\gamma - \eta)}{\partial \eta} \right)
\]

\[
+ \frac{\sinh \gamma \sinh \eta \left( 1 - \cosh \gamma \cosh \eta \right)^2}{2 \left( \cosh \gamma - \cosh \eta \right)} \left( u_y \sinh \gamma \sinh \eta - u_\eta \sin \gamma \cosh \eta \right)
\]

\[
+ \frac{\sinh \gamma \sinh \eta \left( 1 - \cosh \gamma \cosh \eta \right)}{2 \left( \cosh \gamma - \cosh \eta \right)} \left( \frac{\partial u_y}{\partial \xi} - \frac{\partial u_\eta}{\partial \eta} \right)
\]
Before the perturbation can be carried out the equations must be written out in dimensionless form. The coordinates, velocities and stresses are collected in the following groups:

\[
\frac{\eta}{\eta_0} , \frac{\theta}{\theta_0}, \frac{u}{U} , \frac{\alpha}{U} , \frac{T_\theta}{\nu U} , \frac{\partial}{\partial \theta} , \frac{\partial}{\partial \tau} , \frac{\delta}{\delta \theta}
\]

For the class of problem considered here \( \frac{\eta}{\eta_0} , \frac{\theta}{\theta_0} \), and \( \eta_0, \theta_0 \) are small. Because of this we hope the following expansions are permissible.

\[
\eta_0 (\eta, \theta, \eta_0, \theta_0, \epsilon) = \eta_0 (\eta, \theta) + \eta_0 \Delta \eta_0 (\eta, \theta) + \epsilon \Delta \eta_0 (\eta, \theta) + \epsilon^2 \Delta \eta_0 (\eta, \theta) + \epsilon^3 \Delta \eta_0 (\eta, \theta) + \epsilon^4 \Delta \eta_0 (\eta, \theta) + \epsilon^5 \Delta \eta_0 (\eta, \theta) + \epsilon^6 \Delta \eta_0 (\eta, \theta) + \epsilon^7 \Delta \eta_0 (\eta, \theta) + \epsilon^8 \Delta \eta_0 (\eta, \theta) + \epsilon^9 \Delta \eta_0 (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0} \]

\[
\tau_\theta (\eta, \theta, \eta_0, \theta_0, \epsilon) = \tau_\theta (\eta, \theta) + \tau_\theta \Delta \eta_0 (\eta, \theta) + \epsilon \Delta \tau_\theta (\eta, \theta) + \epsilon^2 \Delta \tau_\theta (\eta, \theta) + \epsilon^3 \Delta \tau_\theta (\eta, \theta) + \epsilon^4 \Delta \tau_\theta (\eta, \theta) + \epsilon^5 \Delta \tau_\theta (\eta, \theta) + \epsilon^6 \Delta \tau_\theta (\eta, \theta) + \epsilon^7 \Delta \tau_\theta (\eta, \theta) + \epsilon^8 \Delta \tau_\theta (\eta, \theta) + \epsilon^9 \Delta \tau_\theta (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0}
\]

\[
\frac{\partial}{\partial \theta} \left( t_\theta (\eta, \theta, \eta_0, \theta_0, \epsilon) = \frac{\partial}{\partial \theta} (t_\theta (\eta, \theta) + \epsilon \Delta t_\theta (\eta, \theta) + \epsilon^2 \Delta t_\theta (\eta, \theta) + \epsilon^3 \Delta t_\theta (\eta, \theta) + \epsilon^4 \Delta t_\theta (\eta, \theta) + \epsilon^5 \Delta t_\theta (\eta, \theta) + \epsilon^6 \Delta t_\theta (\eta, \theta) + \epsilon^7 \Delta t_\theta (\eta, \theta) + \epsilon^8 \Delta t_\theta (\eta, \theta) + \epsilon^9 \Delta t_\theta (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0} \right) \]

\[
\frac{\delta}{\delta \theta} \left( t_\theta (\eta, \theta, \eta_0, \theta_0, \epsilon) = \frac{\delta}{\delta \theta} (t_\theta (\eta, \theta) + \epsilon \Delta t_\theta (\eta, \theta) + \epsilon^2 \Delta t_\theta (\eta, \theta) + \epsilon^3 \Delta t_\theta (\eta, \theta) + \epsilon^4 \Delta t_\theta (\eta, \theta) + \epsilon^5 \Delta t_\theta (\eta, \theta) + \epsilon^6 \Delta t_\theta (\eta, \theta) + \epsilon^7 \Delta t_\theta (\eta, \theta) + \epsilon^8 \Delta t_\theta (\eta, \theta) + \epsilon^9 \Delta t_\theta (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0} \right) \]

\[
\tau_\theta (\eta, \theta, \eta_0, \theta_0, \epsilon) = \tau_\theta (\eta, \theta) + \epsilon \Delta \tau_\theta (\eta, \theta) + \epsilon^2 \Delta \tau_\theta (\eta, \theta) + \epsilon^3 \Delta \tau_\theta (\eta, \theta) + \epsilon^4 \Delta \tau_\theta (\eta, \theta) + \epsilon^5 \Delta \tau_\theta (\eta, \theta) + \epsilon^6 \Delta \tau_\theta (\eta, \theta) + \epsilon^7 \Delta \tau_\theta (\eta, \theta) + \epsilon^8 \Delta \tau_\theta (\eta, \theta) + \epsilon^9 \Delta \tau_\theta (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0}
\]

\[
\frac{\partial}{\partial \theta} \left( t_\theta (\eta, \theta, \eta_0, \theta_0, \epsilon) = \frac{\partial}{\partial \theta} (t_\theta (\eta, \theta) + \epsilon \Delta t_\theta (\eta, \theta) + \epsilon^2 \Delta t_\theta (\eta, \theta) + \epsilon^3 \Delta t_\theta (\eta, \theta) + \epsilon^4 \Delta t_\theta (\eta, \theta) + \epsilon^5 \Delta t_\theta (\eta, \theta) + \epsilon^6 \Delta t_\theta (\eta, \theta) + \epsilon^7 \Delta t_\theta (\eta, \theta) + \epsilon^8 \Delta t_\theta (\eta, \theta) + \epsilon^9 \Delta t_\theta (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0} \right) \]

\[
\frac{\delta}{\delta \theta} \left( t_\theta (\eta, \theta, \eta_0, \theta_0, \epsilon) = \frac{\delta}{\delta \theta} (t_\theta (\eta, \theta) + \epsilon \Delta t_\theta (\eta, \theta) + \epsilon^2 \Delta t_\theta (\eta, \theta) + \epsilon^3 \Delta t_\theta (\eta, \theta) + \epsilon^4 \Delta t_\theta (\eta, \theta) + \epsilon^5 \Delta t_\theta (\eta, \theta) + \epsilon^6 \Delta t_\theta (\eta, \theta) + \epsilon^7 \Delta t_\theta (\eta, \theta) + \epsilon^8 \Delta t_\theta (\eta, \theta) + \epsilon^9 \Delta t_\theta (\eta, \theta) + \epsilon^{10} \frac{\eta_0}{\eta_0} \frac{\theta_0}{\theta_0} \right) \]
\[ \sigma_y(\eta, \psi, \theta, \epsilon) = \sigma_0(\eta, \psi) + \eta_0 \tau_{y0}(\eta, \psi) + \epsilon \tau_{y0}(\eta, \psi) + \eta_0 \epsilon \tau_{y0}(\eta, \psi) + \eta_0 \epsilon^2 \tau_{y0}(\eta, \psi) + \cdots \]

\[ \sigma_z(\eta, \psi, \theta, \epsilon) = \sigma_0(\eta, \psi) + \eta_0 \tau_{z0}(\eta, \psi) + \epsilon \tau_{z0}(\eta, \psi) + \eta_0 \epsilon \tau_{z0}(\eta, \psi) + \eta_0 \epsilon^2 \tau_{z0}(\eta, \psi) + \cdots \]

where \( \eta_0 = \text{value of } \eta \text{ on roll}, \quad \epsilon = \frac{u}{c} \quad \text{and} \quad \epsilon = a \left( \frac{u_0}{c_0} \right)^2 \)

By substituting the expansions into the original equations and expanding the \( \cosh \eta \) and \( \sinh \eta \) in their respective series, a set of equations in powers of \( \eta \), \( \epsilon \) and \( \epsilon \) are obtained. If equal powers are equated we obtain sets of equations which may be solved. The first of these sets of equations are coefficients which are not multiplied by \( \eta \), \( \epsilon \) or \( \epsilon \). These are:

\[ \frac{\partial u_0}{\partial \xi} + \eta_0 \frac{\partial u_0}{\partial \eta} = \frac{\sin \psi}{1 - \cos \psi} \quad u_0 \quad \text{continuity} \]

\[ \frac{\partial \sigma_{y0}}{\partial \xi} + \frac{\partial \tau_{y0}}{\partial \eta} = 0 \quad \text{\& equilibrium} \]

\[ \sigma_{y0} = \frac{\rho u}{\alpha} \quad f(\xi) \quad \text{\& equation} \]

\[ \sigma_{z0} = \sigma_0 \quad \text{\& stress strain} \]

\[ \tau_{z0} = \frac{\epsilon}{\alpha} \left( 1 - \cos \psi \right) \frac{\partial u_0}{\partial \eta} \quad \text{\& stress strain} \]

This set of equations may be solved if the following boundary conditions are imposed.
\[ T_{\eta_0} = 0 \quad \text{when} \quad \eta = 0 \]

\[ u_{\eta_0} = U \quad \text{when} \quad \eta = \pm \eta_0 \]

\[ \frac{2 \alpha_u U_0 \eta_0}{1 - \cos^2 \gamma_1} = 2 \int_{\gamma_1}^{\eta_0} \frac{\alpha_u U_0}{\cosh \gamma - \cos \gamma} \, d\gamma \quad \text{\(\gamma_1 = \text{entrance}\)} \]

\[ \int_{\gamma_1}^{\eta_0} \frac{\alpha u_0}{\cosh \gamma - \cos \gamma} \bigg| \frac{d\eta_2}{d\gamma} \bigg| = 0 \quad \text{\(\gamma = \gamma_1\)} \]

\[ u_{\eta_1} = 0 \quad \text{when} \quad \eta_2 = 0 \]

**The solution is:**

\[ u_{\eta_0} = \frac{3}{2} U \left( \frac{1 - \cos^2 \gamma_1 - 1}{1 - \cos^2 \gamma_1} \right) + U \]

\[ u_{\eta_1} = \frac{U \sin \beta}{2 (1 - \cos \gamma_1)} \left( \frac{\eta_2}{\eta_0} - \eta_2 \right) \]

\[ T_{\eta_1} = -3 \frac{U \eta_1}{\alpha_0 (1 - \cos \gamma_1)} \left( \frac{\eta_2}{\eta_0} - 1 \right) \]

\[ \sigma_{\eta_0} = \sigma_{\eta_0} = 3 \frac{U \eta_1}{\alpha_0 \eta_0^2} \left\{ \frac{1}{(1 - \cos \gamma_1)} \left[ \frac{3}{2} \left( \frac{\eta_1}{\gamma_1} \right) - 2 \left( \sin \beta - \sin \gamma_1 \right) \right. \right. \]

\[ \left. \left. + \frac{1}{4} \left( \sin 2 \beta - \sin 2 \gamma_1 \right) - \left( \frac{\eta_1}{\gamma_1} \right) + \left( \sin \gamma_1 - \sin \beta \right) \right] \right\} \]

For the remainder of the solutions the boundary conditions are:

\[ T_{\eta_n} = 0 \quad \text{when} \quad \eta = 0 \]

\[ u_{\eta_n} = 0 \quad \text{when} \quad \eta = \pm \eta_0 \]

\[ \int_{\gamma_1}^{\eta_0} \frac{u_{\eta_n}}{\cosh \gamma - \cos \gamma} \, d\gamma = 0 \]

\[ \int_{\gamma_1}^{\eta_0} \frac{\alpha u_{\eta_n}}{\cosh \gamma - \cos \gamma} \bigg| \frac{d\eta_2}{d\gamma} \bigg| = 0 \]

\[ u_{\eta_n} = 0 \quad \text{when} \quad \eta_2 = 0 \]
The equations and resulting solutions are given on the following pages. In cases where the normal stresses are more easily obtained by a numerical integration rather than by substitution into the integrated form, the equation has been left as an integral (for example see § order equation).
Zero Order Equations

\[ \frac{\partial u_{0}}{\partial \xi} + \nu \frac{\partial u_{1}}{\partial \eta} = \frac{\sin \theta}{1 - \alpha \cos \theta} u_{0} \]

continuity

\[ \frac{\partial \sigma_{0}}{\partial \xi} + \frac{\partial \tau_{0}}{\partial \eta} = 0 \]

equilibrium

\[ \tau_{20} = \frac{\pi u}{\alpha \eta_{0}} f(\theta) \]

equilibrium

\[ \tau_{20} = \sigma_{0} \]

stress strain

\[ \tau_{0} = \frac{4}{\alpha} (1 - \alpha \cos \theta) \frac{\partial u_{0}}{\partial \xi} \]

stress strain
Zero Order Solution

\[ u_{o} = \frac{3U}{2} \left( 1 - \cos \theta - 1 \right)(1 - \frac{x^2}{\eta_0^2}) + u \]

\[ u_{2r} = \frac{U}{2} \left( \frac{\sin \theta}{1 - \cos \theta} \right)(\frac{x^2}{\eta_0^2} - \frac{\eta}{\eta_0}) \]

\[ \gamma_{o} = -\frac{34U}{a \eta_0} \left( 1 - \cos \theta_{i} \right)(1 - \cos \theta) \frac{x}{\eta_0} \]

\[ \sigma_{o} = \sigma_{2r} = \frac{34U}{a \eta_0^2} \left( \frac{1}{1 - \cos \theta} \left( \frac{1}{2} (\theta - \theta_{i}) - 2(\sin \theta - \sin \theta_{i}) \right) \right) \]

\[ + \frac{34U}{a \eta_0^2} \left( \frac{1}{1 - \cos \theta_{i}} \left( \frac{1}{2} \sin 2 \theta - \frac{1}{2} \sin 2 \theta_{i} \right) - (\theta - \theta_{i}) \right) \]

\[ + \frac{34U}{a \eta_0^2} (\sin \theta - \sin \theta_{i}) \]
2nd Order Equations

\[ \frac{du_{\theta}}{\partial \theta} + \nu_0 \frac{du_{\phi}}{\partial \phi} = \frac{\sin \phi}{1 - \cos \phi} u_{\phi}, \]

continuity

\[ \frac{du_{\phi}}{\partial \phi} + \frac{d}{d z} \tau_{\phi z} = 0 \]

equilibrium

\[ \sigma_{\phi} = \frac{\nu_0 \nu}{\alpha \phi^2} f(\phi) \]

equilibrium

\[ \sigma_{\theta} = \sigma_{\phi} \]

stress strain

\[ \tau_{\phi z} = \frac{\nu}{\alpha} (1 - \cos \phi) \frac{du_{\phi}}{d z} \]

stress strain

2nd Order Solution

trivial solution
$$\frac{\partial u_{13}}{\partial y} + r_0 \frac{\partial u_{33}}{\partial n} = \frac{\sin \theta}{1 - \cos \theta} u_{33}$$
continuity

$$\frac{\partial \sigma_{33}}{\partial y} + \frac{\partial \tau_{33}}{\partial n} = 0$$
equilibrium

$$\sigma_{33} = \sigma_{33}^* = \frac{2Y}{a \eta_0} f(\beta)$$
equilibrium
stress strain

$$\tau_{33} = \frac{Y}{a} (1 - \cos \beta) \frac{\partial \sigma_{33}}{\partial n}$$
stress strain

2 Order Solution

trivial solution
Order Equations

\[ \frac{\partial u_{*2}}{\partial \xi} + \eta \frac{\partial u_{*2}}{\partial \eta} = \frac{\alpha \sin \theta}{(1 - \alpha \cos \theta)} \ u_{*2} \] continuity

\[ \frac{\partial \sigma_{*2}}{\partial \xi} + \frac{\partial \tau_{*1*2}}{\partial \eta} = 0 \] equilibrium

\[ \sigma_{*2} = \frac{2 \mu \nu}{\alpha \eta_0^2} f(\xi) \] equilibrium

\[ \sigma_{*2} = \sigma_{*2} \] stress strain

\[ \tau_{*2*7} = \frac{\mu}{\alpha} \ (1 - \alpha \cos \theta) \left( \frac{\partial u_{*2}}{\partial \xi} + \frac{\nu}{8 \beta^2} \left( \frac{\partial u_{*2}}{\partial \eta} \right)^3 \right) \] stress strain
\[ u_{\psi_2} = \frac{U}{160} (1 - \cos \phi)^2 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right) \left( 135 \frac{h^7}{\zeta^5} - 162 \frac{h^6}{\zeta^4} + 27 \right) \]

\[ u_{\psi_1} = -\frac{U}{160} \sin \phi (1 - \cos \theta) \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right)^2 \left[ \frac{4}{1 - \cos \theta} - 1 \right] \left( 27 \frac{h^5}{\zeta^3} - 54 \frac{h^4}{\zeta^2} + 27 \frac{h^3}{\zeta} \right) \]

\[ T_{\theta \phi} = -\frac{g_1 + \epsilon}{40a_{\gamma_0}} (1 - \cos \theta)^2 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right)^3 \frac{h}{\zeta} \]

\[ \sigma_{\psi_2} = \sigma_{\psi_1} = \frac{g_1 + \epsilon}{40a_{\gamma_0}} \int_{\phi_1}^{\phi_2} (1 - \cos \phi)^2 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right)^3 d\phi \]
\[ \frac{\partial \gamma_{20}}{\partial \eta} + \frac{\partial \gamma_{10}}{\partial \eta} = \frac{\sin \theta}{1 - \cos \theta} \gamma_{20} + \frac{\eta}{\eta_0 (1 - \cos \theta)} \gamma_{10} - \frac{\eta^2}{2 \eta_0^2 (1 - \cos \theta)} \left( \frac{\partial \gamma_{20}}{\partial \eta} + \frac{\partial \gamma_{10}}{\partial \eta} \right) \]

Continuity

\[ \frac{\partial \sigma_{20}}{\partial \eta} + \frac{\partial \sigma_{10}}{\partial \eta} = \frac{\sin \theta}{1 - \cos \theta} \left( \sigma_{20} - \sigma_{10} \right) + \frac{2 \eta}{\eta_0^2 (1 - \cos \theta)} \gamma_{10} - \frac{\eta^2}{2 \eta_0^2 (1 - \cos \theta)} \left( \frac{\partial \sigma_{20}}{\partial \eta} + \frac{\partial \sigma_{10}}{\partial \eta} \right) \]

Equilibrium

\[ \frac{\partial \sigma_{20}}{\partial \eta} = \frac{2 \sin \theta}{\eta_0^2 (1 - \cos \theta)} \gamma_{10} - \frac{1}{\eta_0} \frac{\partial \gamma_{10}}{\partial \eta} \]

Equilibrium

\[ 2 \sin \theta \left( \gamma_{20} + \eta \frac{\partial \gamma_{20}}{\partial \eta} \right) - 2 (1 - \cos \theta) \gamma_{10} \frac{\partial \gamma_{10}}{\partial \eta} = \frac{\alpha \eta_0}{2 \gamma} \left( \sigma_{20} - \sigma_{10} \right) \]

Stress Strain

\[ \frac{1}{2} (1 - \cos \theta) \left[ \frac{\eta}{\eta_0} \sin \theta + \frac{\eta}{\eta_0} \cos \theta (1 - \cos \theta) \right] \gamma_{20} - \sin \theta (1 - \cos \theta) \gamma_{10}, \]

\[ - (1 - \cos \theta) \left( \frac{\partial \gamma_{10}}{\partial \eta} + \frac{\eta}{\eta_0} \frac{\partial \gamma_{10}}{\partial \eta} \right) + 2 \sin \theta (1 - \cos \theta) \frac{\eta}{\eta_0} \left( \frac{\partial \gamma_{20}}{\partial \eta} - \frac{\eta}{\eta_0} \frac{\partial \gamma_{10}}{\partial \eta} \right) \]

\[ + \frac{2 \eta_0}{2^2} \gamma_{10} \frac{\partial \gamma_{10}}{\partial \eta} = \frac{-\alpha \eta_0}{2} (1 - \cos \theta)^2 \gamma_{10}, \]

\[ + \frac{\alpha \eta_0}{2} \left( \frac{\eta^2}{\eta_0^2} \sin \theta + \frac{\eta^2}{\eta_0^2} \cos \theta (1 - \cos \theta) - \frac{1}{2} \frac{\eta^2}{\eta_0^2} (1 - \cos \theta) \right) \gamma_{20}, \]

\[ + \frac{\alpha \eta_0}{2} \sin \theta (1 - \cos \theta) \left( \sigma_{20} - \sigma_{10} \right) \]

Stress Strain
$\eta^2$ Order Solution

$$U_{\eta_1} = \frac{U}{16(1-\eta_0)} \left( \frac{\eta^4}{\eta_0} - \frac{5}{5 \eta_0^2} + \frac{\lambda}{5} \right) (-82 \cos \theta - 39 + \frac{1-\eta_0^6}{1-\eta_0^6} (102 \cos \theta + 55))$$

$$U_{\eta_{10}} = \frac{U \sin \theta}{80(1-\eta_0)^2} \left( \frac{\eta^5}{\eta_0^2} - \frac{2 \eta^3}{\eta_0} + \frac{\eta^2}{\eta_0} \right) (-82 \cos \theta - 160 + \frac{1-\eta_0^6}{1-\eta_0^6} (157))$$

$$+ \frac{U \sin \theta}{16(1-\eta_0)^2} \left( \frac{\eta^5}{5 \eta_0^2} - \frac{\eta^3}{3 \eta_0^2} \right) (-4 + 12 \frac{1-\eta_0^6}{1-\eta_0^6}) - \frac{U \sin \theta}{6(1-\eta_0)^2} \frac{\eta^3}{\eta_0}$$

$$\eta_{\theta_{10}} = \frac{4U}{40 \alpha \eta_0} \left( 20 - 20 \cos \theta + \frac{1-\eta_0^6}{1-\eta_0^6} (-80 + 250 \cos \theta) \right) \frac{\eta^3}{\eta_0^2}$$

$$+ \frac{4U}{40 \alpha \eta_0} \left( 199 + 157 \cos \theta + \frac{1-\eta_0^6}{1-\eta_0^6} (-235 - 557 \cos \theta) \right) \frac{\eta^2}{\eta_0}$$

$$\sigma_{\eta_1} = \frac{34U \sin \theta}{2 \alpha \eta_0^2} \left( \frac{\eta^2}{\eta_0} \right) - \frac{4U (297 \sin \theta + 199 \eta)}{40 \alpha \eta_0} + \frac{4U (153 \sin \theta + 164 \eta)}{80 (1-\eta_0^6) \alpha \eta_0}$$

$$- \frac{4U (317 \sin \theta \cos \theta + 199 \eta)}{80 (1-\eta_0^6) \alpha \eta_0} + \frac{4U \sin \theta (3-\eta_0^6)}{2 \eta_0} - 2 \left( \frac{1-\eta_0^6}{2 \eta_0} \right)$$

$$- \frac{4U \sin \theta}{2 \eta_0} + \frac{4U (297 \sin \theta \cos \theta + 199 \eta)}{40 \alpha \eta_0} - \frac{4U (153 \sin \theta + 164 \eta)}{80 (1-\eta_0^6) \alpha \eta_0}$$

$$+ \frac{4U (317 \sin \theta \cos \theta + 199 \eta)}{80 (1-\eta_0^6) \alpha \eta_0}$$

$$\sigma_{\eta_1} - \sigma_{\theta_1} = \frac{4U}{\alpha \eta_0} \left( 4 \sin \theta + 2 \sin \eta \left( \frac{3-\eta_0^6}{1-\eta_0^6} - 2 \right) \left( 1-3 \frac{\eta_0^6}{\eta_0^2} \right) \right)$$
\[ \frac{\partial u_{66}}{\partial \xi} + \eta_0 \frac{\partial u_{66}}{\partial \eta} = \frac{\sin \theta}{1 - \eta_0 \phi} u_{66} \] continuity

\[ \frac{\partial \sigma_{66}}{\partial \xi} + \frac{\partial \sigma_{66}}{\partial \eta} = 0 \] equilibrium

\[ \sigma_{66} = \frac{a_2 \psi}{a \eta_0} f(\xi) \] equilibrium

\[ \sigma_{66} = \sigma_{77} \] stress strain

\[ 4 (1 - \eta_0 \phi) \eta_0 \frac{\partial u_{66}}{\partial \eta} = \alpha \eta_0 \gamma_{012} - \frac{a_0^2}{40} (1 - \eta_0 \phi) (\sigma_{66} - \sigma_{77}) \frac{\partial u_{66}}{\partial \eta} \] stress strain

also from \( \eta_0 \varepsilon \) equations (since \( \sigma_{66} - \sigma_{77} \) is unknown)

\[ \sigma_{66} - \sigma_{77} = \gamma_{01} (1 - \eta_0 \phi) \frac{\partial u_{66}}{\partial \eta} \] stress strain
\[ U_{\xi_\epsilon} = \frac{27U}{16} (1-\cos \xi)^2 \left( \frac{1-\cos \xi}{1-\cos \xi_1} - 1 \right) \left( \frac{\eta}{\eta_0} - \frac{2}{5} \frac{\eta_1^3}{\eta_0^3} + \frac{1}{5} \right) \]

\[ U_{\eta_1} = -\frac{27U}{16} \sin \xi (1-\cos \xi) \left( \frac{1-\cos \xi}{1-\cos \xi_1} - 1 \right) \left( \frac{\eta}{\eta_0} - \frac{2}{5} \frac{\eta_1^3}{\eta_0^3} + \frac{1}{5} \right) \]

\[ \eta_{\eta_1} = -\frac{81}{20} \frac{U}{\eta_0} (1-\cos \xi)^3 \left( \frac{1-\cos \xi}{1-\cos \xi_1} - 1 \right)^3 \frac{\eta}{\eta_0} \]

\[ \sigma_{\xi_\epsilon} = \sigma_{\eta_\epsilon} = \frac{81}{20} \frac{U}{\eta_0^2} \int_{\xi_1}^{\xi} (1-\cos \xi)^3 \left( \frac{1-\cos \xi}{1-\cos \xi_1} - 1 \right)^3 d\xi \]
\[ \sigma_{zz} = \frac{\rho u}{a \varepsilon_{\text{G}}^2} f(\delta) \]

\[ \sigma_{zz} = \sigma_{zz} \]

\[ \frac{\partial u_{zs}}{\partial \delta} + \frac{\rho_0}{\varepsilon_{\text{G}}^2} \frac{\partial u_{zs}}{\partial \varepsilon_{\text{G}}} = \frac{\rho_0 u_{zs}}{1 - \cos \theta} \]

continuity

\[ \frac{\partial \sigma_{zz}}{\partial \delta} + \frac{\partial \sigma_{zz}}{\partial \varepsilon_{\text{G}}} = 0 \]

equilibrium

\[ n_0 (1 - \cos \theta) \frac{\partial u_{zs}}{\partial \varepsilon_{\text{G}}} + \frac{3 \varepsilon_{\text{G}}^2}{8 a^2} (1 - \cos \theta)^3 \left( \frac{\partial u_{zs}}{\partial \varepsilon_{\text{G}}} \right)^2 \left( \frac{\partial u_{zs}}{\partial \varepsilon_{\text{G}}} \right) = \frac{\alpha n_0}{4} T_{yy} \]

stress strain
\[ U_{K_5} = -U \left( 1 - \cos \phi \right)^4 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right) \left( \frac{2430}{850 \gamma_0} \frac{n^6}{\gamma_0} - \frac{2187}{850 \gamma_0} \frac{n^7}{\gamma_0} - \frac{2187}{850 \gamma_0} \frac{n^8}{\gamma_0} + \frac{10371}{1280} \right) \]

\[ U_{K_6} = U \sin \phi (1 - \cos \phi)^3 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 3 \right) \left( \frac{2430}{850 \gamma_0} \frac{n^6}{\gamma_0} - \frac{2187}{850 \gamma_0} \frac{n^7}{\gamma_0} - \frac{729}{850 \gamma_0} \frac{n^8}{\gamma_0} + \frac{19371}{1280} \right) \]

\[ \sigma_{K_1} = \frac{4374}{850 \gamma_0} \frac{4U}{\alpha \gamma_0} \left( 1 - \cos \phi \right)^5 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right)^5 \frac{\theta}{\gamma_0} \]

\[ \sigma_{K_5} = \sigma_{K_6} = -\frac{4374}{850 \gamma_0} \frac{4U}{\alpha \gamma_0} \int_{K_5} \left( 1 - \cos \phi \right)^5 \left( \frac{1 - \cos \phi}{1 - \cos \theta} - 1 \right)^5 d\phi \]
Order Equations

\[ \frac{\partial u_{f_0}}{\partial \xi} + \nu \frac{\partial u_{n_{r_1, r_2}}}{\partial \eta} = \frac{\sin \theta}{1 - \cos \theta} \frac{u_{f_0}}{1 - \cos \theta} \]
continuity

\[ \frac{\partial \sigma_{f_0}}{\partial \xi} + \frac{\partial \sigma_{n_{r_1, r_2}}}{\partial \eta} = \frac{\sin \theta}{1 - \cos \theta} (\sigma_{f_0} - \sigma_{n_{r_1, r_2}}) \]
equilibrium

\[ \tau_{f_0} = \frac{\varepsilon \nu}{v_0^2} f(\theta) \]
equilibrium

\[ \sigma_{f_0} - \sigma_{n_{r_1, r_2}} = \frac{(1 - \cos \theta)(\sigma_{f_0} - \sigma_{n_{r_1, r_2}})}{U} \frac{d u_{f_0}}{d \eta} \]
stress strain

\[ \varepsilon_0 (1 - \cos \theta) \frac{d u_{f_0}}{d \eta} = \frac{\alpha_0}{4} \frac{\tau_{f_0}}{\nu} \tau_{f_0} \frac{d u_{f_0}}{d \eta} - \frac{\alpha_0}{1 - \cos \theta} \frac{\tau_{f_0}}{\nu} \tau_{f_0} \frac{d u_{f_0}}{d \eta} \]

\[ + \frac{\alpha_0}{4 U} \tau_{f_0} \frac{d u_{f_0}}{d \eta} + \frac{\alpha_0}{4 U} (1 - \cos \theta)(\sigma_{f_0} - \sigma_{n_{r_1, r_2}}) \frac{d u_{f_0}}{d \eta} \]

\[ + \frac{\alpha_0}{2 U} (\alpha_0 (1 - \cos \theta) - \frac{U}{2} \sin \theta u_{f_0}) \frac{d \sigma_{f_0}}{d \eta} \]
stress strain
\[ u_{q_0} = U \sin \theta \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( -\frac{9}{8} + \frac{9}{16} \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) \left( \frac{\eta_0^3}{\eta^2} - \frac{6 \eta_0^2}{5 \eta^2} + \frac{1}{3} \right) \]

\[ u_{q_{14}} = U (1 + \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( \frac{9}{8} + \frac{9}{16} \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) \left( \frac{\eta_0^5}{\eta^2} - \frac{2 \eta_0^3}{5 \eta^2} + \frac{\eta_0}{5 \eta^2} \right) \]

\[ - \frac{9}{16} U (1 + \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) \left( \frac{\eta_0^5}{\eta^2} - \frac{2 \eta_0^3}{5 \eta^2} + \frac{\eta_0}{5 \eta^2} \right) \]

\[ - U \cos \theta \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( -\frac{9}{8} + \frac{9}{16} \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) \left( \frac{\eta_0^5}{\eta^2} - \frac{2 \eta_0^3}{5 \eta^2} + \frac{\eta_0}{5 \eta^2} \right) \]

\[ \eta_{q_{14}} = \frac{94U}{a \rho_0} \sin \theta (1 - \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( -3 \frac{1 - \cos \theta}{1 - \cos \theta_0} + 1 \right) \frac{\eta_0^3}{\eta^2} \]

\[ + \frac{4U}{a \eta_0} \sin \theta (1 - \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( \frac{27}{10} + \frac{9}{10} \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) \frac{\eta_0}{\eta^2} \]

\[ \frac{\partial \eta_q}{\partial \gamma} - \frac{\partial \eta_0}{\partial \gamma} = \frac{94U}{a \eta_0^2} (1 - \cos \theta)^2 \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \frac{2 \eta_0}{\eta^2} \]

\[ \eta_q = \frac{94U}{a \rho_0 \eta_0^2} \left[ \int \left\{ (1 - \cos \theta)^2 \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right)^2 - \sin \theta (1 - \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( -3 \frac{1 - \cos \theta}{1 - \cos \theta_0} + 1 \right) \right\} d\theta \right] \]

\[ - \frac{4U}{a \rho_0} \left[ \int \sin \theta (1 - \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( \frac{27}{10} + \frac{9}{10} \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) d\theta \right] \]

\[ - \frac{3U}{a \rho_0^2} \left[ \int \left\{ (1 - \cos \theta)^2 \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right)^2 - \sin \theta (1 - \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( -3 \frac{1 - \cos \theta}{1 - \cos \theta_0} + 1 \right) \right\} d\theta \right] \]

\[ + \frac{4U}{a \rho_0^2} \left[ \int \sin \theta (1 - \cos \theta) \left( \frac{1 - \cos \theta}{1 - \cos \theta_0} - 1 \right) \left( \frac{27}{10} + \frac{9}{10} \frac{1 - \cos \theta}{1 - \cos \theta_0} \right) d\theta \right] \]
\begin{align*}
\frac{\partial u_7}{\partial \xi} + \nu_0 \frac{\partial u_{13}}{\partial \eta} &= \frac{\sin \theta}{1 - \cos \theta} u_{13} & \text{continuity} \\
\frac{\partial \sigma_{11}}{\partial \xi} + \frac{\partial \sigma_{13}}{\partial \eta} &= 0 & \text{equilibrium} \\
\sigma_{11} &= \frac{2\mu \nu_0}{\alpha \nu_0^2} f(\xi) & \text{equilibrium} \\
\sigma_{13} &= \sigma_{11} & \text{stress strain} \\
\nu_0 \left(1 - \cos \theta\right) \frac{\partial u_7}{\partial \eta} + \frac{3\nu_0^3}{8\nu_0^2} \left(1 - \cos \theta\right)^3 \left(\frac{\partial u_{10}}{\partial \eta} \right) \left(\frac{\partial u_{13}}{\partial \eta} \right) &= \frac{\alpha \nu_0}{2} \sigma_{11} & \text{stress strain} \\
\end{align*}

\text{\textsection 6 Order Solution}

trivial solution
66 Order Equations

\[ \frac{\partial \sigma_{zz}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} = \frac{\partial \sigma_{zz}}{\partial \theta} \]

continuity

\[ \frac{\partial \sigma_{zz}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} = 0 \]

equilibrium

\[ \sigma_{zz} = \frac{\partial \sigma_{zz}}{\partial z} = f(\theta) \]

equilibrium

\[ \sigma_{zz} = \sigma_{zz} \]

stress strain

\[ \eta_0 (1 - \cos^2 \phi) \frac{\partial u_{zz}}{\partial \eta} + \frac{\eta_0^2}{4 \mu^2} (1 - \cos^2 \phi) \left( \frac{\partial u_{zz}}{\partial \eta} \right)^2 \frac{\partial \sigma_{zz}}{\partial \eta} = \frac{a \eta_0}{2} \tau_{zz} \]

stress strain

66 Order Solution

trivial solution
BIOGRAPHY

The author was born on May 9, 1931. He received his elementary and high school training in New Orleans, Louisiana. In 1950 he received his Bachelor of Science degree in Mechanical Engineering from Louisiana State University, Baton Rouge, Louisiana and in 1952 his degree of Master of Science in Mechanical Engineering from The Rice Institute, Houston, Texas.

The author enrolled at M.I.T. in 1952 and became a Teaching Assistant in Mechanical Engineering. During the academic year 1953-1954 he was the Dupont Fellow in the Mechanical Engineering Department.

"Design of Shrink Fits" by Paul Paslay and Robert Plunkett, Transactions of the ASME, October 1953, is the author's only publication.