INTERFERENCE REJECTION IN FM RECEIVERS

by

ELIE JOSEPH BAGHDADY

B. A., American University of Beirut
(1961)

S. M., Massachusetts Institute of Technology
(1964)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June, 1966

VOLUME I

Chapters One and Two
Pages 1-159
INTERFERENCE REJECTION IN FX RECEIVERS

by

ELIE JOSEPH BAGHDADY

B. A., American University of Beirut (1951)

S. M., Massachusetts Institute of Technology (1954)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1956

Signature of Author.

Department of Electrical Engineering May 14, 1956

Certified by: . . .

Thesis Supervisor

Accepted by: . . .

Chairman, Department of Graduate Students
INTERFERENCE REJECTION IN FM RECEIVERS
by
Elie Joseph Baghdady
Submitted to the Department of Electrical Engineering on May 16, 1956 in partial fulfillment of the requirements for the degree of Doctor of Science

ABSTRACT

A recognition of the fact that interference is suppressible if it is noticeably distinguishable from the type of modulation normally expected in the pertinent parameter of the resultant signal passed by the i-f amplifier, leads to a new role for the amplitude limiter in an FM receiver. By spreading out the significant spectrum which is necessary for the reproduction of the extraneous frequency modulation caused by the interference, the limiter makes it possible for a filter to reject an important portion of that spectrum without noticeably affecting the spectrum carrying the message specifications. The interference that survives a first such treatment may be subjected to several cycles of limiting followed by narrow-band filtering, until what is left of the interference spectrum after the last operation is either insignificant, or indistinguishable from the desired message-bearing spectrum by being essentially confined within the minimum permissible bandwidth after a limiter. This operation may also be viewed as preparing the resultant signal, by amplitude limiting, for the action of a sluggish filter, which deforms the original extraneous modulation into a less disturbing one, by refusing to follow that modulation through quasi-stationary states. The same operation may be profitably repeated until the fundamental distinguishing features of the residual extraneous modulation become almost indistinguishable from the features characterizing a message modulation. The most important distinguishing feature of any frequency modulation, as far as interference filtering is concerned, is found to be the maximum rate at which the instantaneous frequency is varied during a cycle of the modulation.

In Chapter One the conditions under which important portions of the interference spectrum may be filtered out without upsetting the desired average frequency of the resultant signal are explored. A detailed study of this spectrum leads to a criterion which enables the minimum permissible bandwidth of an ideal filter which follows the limiter, to be determined. The minimum necessary extent of the linearity of the FM-to-AM conversion characteristic is then determined as a function of the limiter bandwidth used.

In Chapter Two, the effect upon the interference spectrum, of a repeated cycle of amplitude limiting and spectrum filtering is explored in detail. This is followed by a
study of the effect of this scheme upon the minimum permissible limiter bandwidth after the later limiters, the necessary extent of the linearity of the discriminator characteristic, the maximum permissible RC time constant of a grid-bias limiter and of the discriminator output circuit, and the amplitudes of the Fourier components of a detected frequency disturbance pattern.

In Chapter Three, a study of the conditions for the validity of the instantaneous frequency approach to the solution of steady-state problems, based upon the work of Carson and Fry and of van der Pol, is presented and the results are applied to the interference problem, and to the problem of reproducing instantaneous frequency pulses and square waves. A study of the concept of capture ratio follows, and is followed by a critical reexamination of the "widebanding" theory of Arguimbau and Grenlund accounting for the successful achievement of capture ratio near 0.98, in a laboratory receiver. This older theory is found incomplete and inadequate, and a new theory is proposed and some experimental evidence presented.

In Chapter Four, a new theory is proposed for the effect of feedback around the limiter upon the interference rejection ability of an FM receiver. The new theory shows that, in direct contradiction with an earlier inadequate theory and feedback proposal by R.W. Wilmotte, inverse feedback cannot improve the capture ability of the receiver. If the bandwidth of the feedback spectrum is comparable with the extent of the significant interference spectrum, then at no feedback angle will the feedback affect the instantaneous frequency disturbances caused by the interference in a manner that is favorable for improvement in the capture conditions. If the feedback spectrum is only a portion of the significant interference spectrum, inverse feedback will cause a deterioration in the capture conditions by effectively decreasing the amplitude difference between the stronger signal and equivalent interfering signal, while positive feedback will improve the capture performance by decreasing the effective ratio of weaker-to-stronger signal amplitude and improving the limiting threshold. The positive feedback operation is shown to perform best under conditions favorable to self-oscillation in the absence of an input signal. Under these conditions, the scheme will exhibit a significant degree of background noise quieting. Detailed analysis of the limiter-with-positive-feedback scheme, viewed either as an oscillator, or purely as a feedback loop having a non-linear element and no initial self-oscillation, shows that both points of view are helpful and lead to equivalent results. The condition on the phase shift off the center of the feedback bandwidth that will insure the locking of the self-oscillation energy to the input signal frequency is determined and shown to be equivalent to the condition necessary for the establishment of a feedback steady state under the same phase shift conditions around the loop.

Thesis Supervisor: Jerome J. Wiesner
Title: Professor of Electrical Engineering
ACKNOWLEDGEMENT

The author is gratefully indebted to Professor Jerome B. Wiesner, for critical supervision of the thesis; Professor Ernst A. Guillemin, for providing critical listening in many informal discussions of the work; and Professor Peter Elias for reading and helpful criticism of the manuscript.
# TABLE OF CONTENTS

## CHAPTER ONE

The Limiter and Discriminator Bandwidth Requirements

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.1</td>
<td>Instantaneous Frequency and The Problem of Interference Rejection</td>
<td>3</td>
</tr>
<tr>
<td>I.2</td>
<td>The Two-Path Interference Spectrum After Limiting</td>
<td>14</td>
</tr>
<tr>
<td>I.3</td>
<td>A Useful Theorem</td>
<td>34</td>
</tr>
<tr>
<td>I.4</td>
<td>A Criterion for Interference Rejection</td>
<td>46</td>
</tr>
<tr>
<td>I.5</td>
<td>The Minimum Permissible Limiter Bandwidths</td>
<td>52</td>
</tr>
<tr>
<td>I.6</td>
<td>Discriminator Bandwidth Requirements</td>
<td>71</td>
</tr>
</tbody>
</table>

References ............................................................................. 104a

## CHAPTER TWO

The Effect of Cascading Narrow-Band Limiters

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.1</td>
<td>Capture Conditions at the Output of the Second Limiter</td>
<td>109</td>
</tr>
<tr>
<td>II.2</td>
<td>Upper Bounds on the Limiter and Discriminator Time Constants</td>
<td>132</td>
</tr>
<tr>
<td>II.3</td>
<td>Effect Upon Harmonic Structure of Detected Spike Trains</td>
<td>153</td>
</tr>
<tr>
<td>II.4</td>
<td>Concluding Remark</td>
<td>159</td>
</tr>
</tbody>
</table>

## CHAPTER THREE

The Theory of Quasi-Stationary Analysis and Applications to Interference Problems

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>III.1</td>
<td>Filter Response to Modulated Waves</td>
<td>135</td>
</tr>
<tr>
<td>III.2</td>
<td>Applications to the Interference Problem</td>
<td>181</td>
</tr>
<tr>
<td>III.3</td>
<td>The Capture Ratio: Concept and Measurement</td>
<td>204</td>
</tr>
<tr>
<td>III.4</td>
<td>Critique and Reexamination of the &quot;Widebanding&quot; Theory for Improved Capture Performance</td>
<td>216</td>
</tr>
</tbody>
</table>

References ............................................................................. 229a
CHAPTER ONE

THE LIMITER AND DISCRIMINATOR BANDWIDTH REQUIREMENTS
INTRODUCTION

Essential to the interference rejection ability of a frequency-modulation receiver is the use of the proper bandwidths in its nonlinear sections. The weaker of two competing signals (whose amplitude may approach the amplitude of the stronger signal within arbitrary limits) can be suppressed by a frequency-modulation (FM) receiver if, other requirements being met, the limiter and discriminator bandwidths exceed certain minimum permissible values. In this chapter a brief survey is first made of the problem of interference rejection and the FM receiver design requirements set by previous investigators to achieve it. This is followed by a study of the spectrum of the amplitude-limited resultant of two carriers differing in amplitude as well as in frequency. The properties exhibited by the spectral components lead to a simple criterion for interference suppression when only certain portions of the spectrum are passed by an ideal filter that follows the limiter. The criterion is tacitly based upon the assumption that it is the message carried by the stronger signal that we desire to get through, although the conditions for reliable capture of the weaker signal will receive a later treatment. The interference rejection criterion is then used to calculate the minimum bandwidths required after the limiter in order to preserve the interference rejection ability of the receiver for capture ratios up to 0.98.

A narrow-band filter after the limiter will in general distort the pattern of the instantaneous frequency perturbations caused by the interference. This distortion will vary with the bandwidth of the filter and with the position of the stronger of the two signals relative to the center frequency of the filter, as well as with the frequency of the weaker signal relative to the stronger one. The configurations leading to the largest instantaneous frequency deviations (from the desired average frequency) with various values of limiter bandwidth are therefore studied to determine the
corresponding minimum necessary ranges over which the FM-to-AM detection characteristic of the receiver must be linear. The results of this study will reveal how the first stage of band-pass limiting will modify the character of the resultant signal passed. They will also show how the minimum requirement in discriminator bandwidth will vary with the value of limiter bandwidth used. The effect upon other design considerations such as the time-constant requirements of the networks involved are taken up in another chapter.

The investigation of this chapter is carried out on a steady-state basis in terms of an ideal limiter followed by an ideal bandpass filter. The desirability of either adapting the results arrived at here to the case of simple parallel-tuned and coupled circuits, or making use of the experience and insight gained from the idealized solution to develop a steady-state treatment employing such filters, will be met in another chapter also.

Finally, the significance of the results of this chapter in relation to the possibility of reducing the required discriminator bandwidth to that of the intermediate frequency by cascading a sufficient number of limiter stages is also touched upon, but will still remain the subject of a detailed study in Chapter Two.

Throughout our discussions here, the concept of instantaneous frequency is used frequently and our understanding of it is fully exploited. A brief statement of the mathematical description of this useful concept, together with a brief survey of the problem of interference rejection, the requirements imposed in receiver design, and the bearing of "narrow-band limiting" upon these requirements are taken up first. The term frequency is loosely used to mean "angular frequency". When cyclic frequency is meant, it will be specifically stated.
I. INSTANTANEOUS FREQUENCY AND THE PROBLEM OF INTERFERENCE REJECTION

The mathematical operations leading to the unambiguous formulation of the exceedingly useful concept of instantaneous frequency have been the subject of much discussion. Elaborate mathematics, using, among other means, such integral transforms as Fourier's and Hilbert's, has been harnessed for the purpose. But it is significant that in almost all the publications of those who have used it effectively and productively in physics and in electrical communications (such as Helmholtz, Rayleigh, Carson, van der Pol, Armstrong, etc.) the characteristic features in introducing the concept and utilizing it have almost invariably been simplicity and straightforwardness. Although this concept undoubtedly is now so well appreciated as to need no special clarification, it seems pertinent to start out a discussion of the type we have ahead of us by a statement of how it is mathematically described in most problems of practical interest.

The two most significant (and useful) ways of introducing the concept of instantaneous frequency (keeping in close touch with its practical significance) follow. The first is best stated in the form:

(a) If the real time function \( f(t) \) describing the signal, or the vibration, is reducible to the forms \( A(t)\cos\phi(t) \) or \( A(t)\sin\phi(t) \), both of which are clearly included in the complex function

\[
F(t) = A(t)e^{j\phi(t)}
\]  

where \( A(t) \) and \( \phi(t) \) are real functions of time (and \( f(t) \) is the real or imaginary part of \( F(t) \)), and further, if \( A(t) \) contains none of the zeros of \( f(t) \), then \( \phi(t) \) is by definition the "instantaneous phase angle" of \( f(t) \), and \( (d/dt)\phi(t) \) is by definition the "instantaneous frequency". The amplitude function \( A(t) \) is the "instantaneous amplitude".
(\leq)

This choice of definitions is unique and unambiguous in almost all the situations of practical interest in sinusoidal-carrier modulation. For in most such situations $A(t)$ is bounded and usually not called upon to contribute to the zeros of the signal; the unmodulated carrier frequency is usually much larger than the extent of the significant spectra of the modulating functions in $A(t)$ and $\phi'(t)$, and the extent of the frequency swings about the mean unmodulated carrier frequency is usually a small fraction of that frequency.

The second definition essentially counts the density of zero crossings per unit interval of time. In a period of $2\pi/\omega_0$ seconds, for instance, the sinusoidal signal $\cos \omega_0 t$ has two zeros. Therefore, in every second, this sinusoid has $\omega_0/\pi$ zeros, and the (angular) frequency may be said to be equivalent to the number of zeros in a time interval of $\pi$ seconds. When these notions are extended to the case of a time function $f(t)$, the definition reads: (Ref. 3):

(b) The instantaneous frequency of $f(t)$ is defined at the time $t$ as the ratio of the number of zeros of $f(t)$ in the interval of time between $t - \tau/2$ and $t + \tau/2$ to $\tau/\pi$, or as the mean density of zero crossings averaged over $\tau/\pi$ seconds.

The two definitions yield the same result for an ordinary sinusoidal-carrier frequency-modulated signal, but the first one is the more common one and the one we will apply in our computations. The second definition was found (by Stumpers) more suitable for use in the analysis of frequency-modulation noise at arbitrary levels.

Most of the signals that will be analyzed in our study will consist of a superposition of several sinusoidally-varying time functions having different frequencies and amplitudes. The quickest (as well as the most elegant)
way of achieving the reduction of the sum to the form indicated in the first statement of the definition of instantaneous frequency is the following.

1. Replace each sinusoidal component of amplitude $A_n(t)$ and phase angle $\phi_n(t)$ by the corresponding complex function indicated in Eq. 1, with the understanding that only the real or the imaginary part of this function (as the case may be) is the quantity of physical significance.

2. Represent each complex function $F_n(t) = A_n(t)\exp[j\phi_n(t)]$ thus obtained, by a directed rotating line (henceforth called phasor) in an Argand diagram, using an arbitrary reference axis (labeled the "axis of reals") for the measurement of the phase angle $\phi_n(t)$. The rotation of the phasors is conventionally positive if it is counterclockwise.

3. Add the representative component phasors vectorially to obtain their resultant. The amplitude and phase functions of this resultant will then be those of the resultant signal.

Analytically, the addition indicated in step 3 leads to

$$F(t) = \sum_{n=1}^{k} A_n(t)e^{j\phi_n(t)}$$

$$= A(t)e^{j\phi(t)}$$

where

$$A(t) = \sqrt{[\text{Re}F(t)]^2 + [\text{Im}F(t)]^2}$$

and

$$\phi(t) = \text{Im} \left[ \text{Im}F(t) \right]$$

Let us now apply the above ideas to the case of two-signal interference (Refs. 1 and 2).

Consider that two carriers of relative strengths 1 and $a$ (where $a<1$) and of frequencies $p$ and $p+r$ rad/sec
FIG. 1

FIG. 2
fall within the linear passband of a frequency-modulation receiver. The signals are supposed to be unmodulated in amplitude or frequency or, at worst, to have a frequency modulation so slow relative to the difference frequency \( r \) that the signal frequencies are not appreciably changed during a period of \( 2\pi/r \) sec.

At the input to the first limiter stage, the resultant signal will be, if time is counted from the instant at which the two signals are momentarily in phase,

\[
f(t) = \cos pt + a \cos(p + r)t.
\]

The corresponding complex function of time is

\[
F(t) = e^{jpt} + ae^{j(p+r)t} = e^{jpt} \left[ 1 + ae^{jrt} \right].
\]

Figure 1 is a phasor diagram representation of the linear superposition of the carriers. The instantaneous phase of the resultant is \( \Phi = pt + \Theta \); so, the instantaneous frequency of the resultant signal is \( d\Phi/dt = p + d\Theta/dt \).

Clearly, \( d\Theta/dt \) represents the instantaneous deviation of the frequency of the resultant signal from that of the stronger signal. In essence, the most important step toward achieving interference rejection is to make the instantaneous frequency deviation of the resultant signal from the desired frequency \( p \) average out to zero, over one period of the difference frequency \( r \), at every point in the receiver prior to FM-to-AM conversion. This latter process must then be such that the average direct voltage level at the output of the discriminator corresponds to that dictated by the desired frequency \( p \). If \( r \) is beyond the audible range, then the preceding requirements are necessary and sufficient since the interference will not pass through the de-emphasis and audio filters. If, however, \( r \) is audible, then those requirements (though necessary) will not insure complete rejection of the interference, although it can be shown that through special design,
as well as with the help of the de-emphasis circuit and
the audio filter, the disturbance that can get through can
be greatly reduced, if not effectively eliminated. This
question will be taken up in greater detail in Chapter Two.

From Fig. 1 we have

\[
\frac{d\Theta}{dt} = \frac{d}{dt} \text{Im} \left[ \ln(1 + ae^{j\omega t}) \right] = \text{Re} \left[ \frac{rae^{j\omega t}}{1 + ae^{j\omega t}} \right]
\]

or

\[
\frac{d\Theta}{dt} = r \frac{a \cos \omega t + a^2}{1 + 2a \cos \omega t + a^2}
\]  \hspace{1cm} (2)

A plot of \( \frac{d\Theta}{dt} \) vs. \( t \) is shown in Fig. 2 for \( a = 0.8 \).

From Fig. 2 we find that the instantaneous frequency
deviation caused by the presence of the weaker signal is
such that the frequency of the resultant signal lingers
near the average of the two carrier frequencies, \( p + r/2 \),
during a large fraction of the difference-frequency cycle,
attaining a maximum of \( p + ar/(1 + a) \), and then dips to
a sharp minimum of \( p - ar/(1 - a) \) at \( t = \pi/r \). This cycle
of instantaneous variation recurs \( r/2\pi \) times per second.
Over one such complete cycle, the average phase angle of
the resultant signal is exactly the phase angle of the
stronger signal, no net phase change being acquired from
the instantaneous deviations in frequency. This means that
the areas enclosed by the instantaneous frequency deviation
curve, above and below the frequency \( p \), are exactly equal.
Thus the average frequency of the resultant signal, over
one period of the difference frequency \( r \), is exactly the
frequency of the stronger signal.

In addition to the instantaneous deviations in
frequency, the interference also causes instantaneous
amplitude variations, with a ratio of maximum to minimum
amplitude of \( (1 + a)/(1 - a) \). The instantaneous amplitude
and frequency variations of the resultant signal arise
simultaneously, and, so long as no nonlinearities in response are encountered, the resultant signal will still be the result of a linear superposition of two signals, and the spectrum of the resultant will continue to be the sum of the spectra of the component signals. This will be true throughout the linear stages of the receiver, up to the first limiter stage, and in those stages, the passband need not exceed the frequency range in which the desired signal may be expected to fall.

However, when the resultant is passed through the limiter, the amplitude variations are completely eliminated, leaving behind the large excursions in instantaneous frequency. The spectrum, after limiting, is spread out with an "infinite" number of components on both sides of the frequency \( p \) of the stronger signal (and of harmonics of \( p \)). Thus, it becomes necessary to re-examine the bandwidth requirements after limiting, so that the average frequency of the signal at the input to the discriminator will still be the frequency of the stronger signal, as is required for the capture of this signal. The specification of the discriminator bandwidth should also be studied in relation to its possible dependence upon the value of limiter bandwidth used. It is with these questions that the present chapter is chiefly concerned.

The work of Arguimbau and Granlund (Refs. 1 and 2) at the Research Laboratory of Electronics, has indicated that interference, with arbitrary values of \( a \) in the range \( 0 < a < 1 \), can be suppressed at the output if the receiver is designed to have the following characteristics:

(a) In the linear sections, namely, the stages preceding the limiter–discriminator section, the bandwidth should be sufficient to accommodate the desired stronger signal over the whole range of its frequency variations. Furthermore, these linear stages must have a constant gain
over the whole passband to preserve the relative magnitudes of the signals that are passed; this gain should fall very steeply at the skirts to effect essentially complete rejection outside the passband and secure excellent selectivity.

(b) Since a frequency-modulation receiver should be completely insensitive to amplitude changes, the linear stages should be followed by a perfect rapid-acting limiter to cope with amplitude ratios of the order of \((1 + a)/(1 - a)\), (or 39/1 for \(a = 0.95\)) that may recur at a maximum rate equivalent to the intermediate-frequency (i-f) bandwidth in cycles/sec. If a capture ratio \(a\) (strength of weaker signal relative to the desired stronger signal) is desired, it is clear that the linear stages must provide enough gain to raise the value of the minimum amplitude \((1 - a)x\) (expected minimum signal strength) to the level necessary to drive the limiter. The discriminator section should also be sufficiently rapid-acting to handle the sharp changes in instantaneous frequency (that may recur at a maximum rate equivalent to the i-f bandwidth in cycles/sec) and still preserve the average output d-c level at the value dictated by the frequency \(p\).

(c) For the requirements in the bandwidths of the limiter and the discriminator sections, Argunbau and Granlund indicated that interference rejection will be fully achieved (with arbitrary values of \(a\)) if the interference frequency spikes are fully accommodated within a linear passband in the non-linear sections. If account is taken of the situation in which the stronger signal will have the higher frequency, then, from Fig. 2, the bandwidth required to accommodate the spikes is given by \([ (1 + a)/(1 - a) ] \cdot (BW)_{if}\) when \(r\) is assigned its maximum value of one i-f bandwidth, \((BW)_{if}\).

A plot of the required bandwidth as a function of \(a\), and as given by \([ (1 + a)/(1 - a) ] \cdot (BW)_{if}\), is presented in Fig. 16, Section V of the present chapter.
The keynote to interference rejection was thus seen to reside in the fast action of limiter and discriminator (to avoid diagonal clipping), and in the full accommodation of the instantaneous frequency excursions within limiter and discriminator linear passbands (to preserve the equality of the areas enclosed by the \( \frac{d\Theta}{dt} \) curve above and below the frequency \( p \)). The physical basis for such an argument may be traced (as will be shown in Chapter Three) to the behaviour of networks involving energy-storage elements when excited by variable-frequency sources. The response of such networks will follow a variable frequency excitation, through essentially stationary states, provided the bandwidth involved is much larger than the rate at which the excitation frequency is varied, still better, provided the static amplitude response characteristic is essentially a constant, or a linear function of frequency, over the whole range of the instantaneous frequency excursions of the excitation. Under such conditions, the dynamic response is readily evaluated from the static characteristics on an instantaneous frequency basis. Consequently with the FM-to-AM conversion characteristic of the discriminator linear over the range of the spikes (thus extending over a bandwidth that is much larger than the spike repetition rate for values of \( a > 0.8 \)) and with the associated low-frequency circuit time constants sufficiently low, we can plot the instantaneous detected output as a function of the instantaneous frequency on a static basis (in the same way in which we handle the static tube characteristics in low-frequency electronic circuit problems). However, if we deal with a relatively narrow-band discriminator, we have no assurance that we can plot the instantaneous detected output as a function of instantaneous frequency because the narrow-band detector is likely to be too sluggish to follow the rapid spike variations and, so, the quasi-static reasoning is likely to break down.
It becomes important to determine whether or not the bandwidth given by \( \frac{1+a}{1-a} \) is a necessary requirement in the nonlinear sections. This is contingent upon the role played by the limiter bandwidth in the over-all picture. Granlund (Ref. 1) performed a Fourier analysis of the resultant of two carriers after limiting with the intention of "determining whether or not the bandwidth specified by the extent of the spikes is a reasonable estimate of the extent of the spectrum after limiting. Thus the result was to be used as a guide in determining limiter and discriminator bandwidths". A good portion of our treatment in the next section will parallel Granlund's analysis, and some of his results (particularly the valuable tables of spectral amplitudes) will be repeated here for completeness.

Finally, aside from its being of considerable theoretical interest, the question of whether or not "wideband" limiting and detecting is a necessity has some important practical and economical implications in communicating by frequency modulation, and in frequency-modulation receiver design. Some of the more obvious considerations are:

a. Wide-band discriminators are more expensive to construct than the narrow-band types. This is also true of limiters.

b. Wide-band discriminators require critical adjustments that become more and more unreliable with time and with changes of ambient temperature and humidity.

c. Wide-band discriminators are considerably less efficient FM-to-AM converters than are the narrow-band types, and this can have detrimental effects upon the quality of reception at the low-modulation levels.

d. A narrow-band limiter yields a stronger signal at its output than a wide-band limiter does. Furthermore, the fact that the audio signal level is higher at the output of a narrow-band discriminator than at the output of a wide-band discriminator, decreases the demand on the
number of audio stages necessary to bring the signal strength up to the desired level at the loud-speaker.

   e. In frequency-modulation television, wide-bandning demands prohibitive band-widths to effect a reasonable degree of interference rejection.
Consider two frequency-modulated carriers of relative constant amplitudes \( a \) and \( A \), where \( a < 1 \), and having such frequencies that they fall simultaneously within the ideal intermediate-frequency (i-f) passband of the receiver. For simplicity, assume the modulation to be so slow that the frequencies of the modulated carriers (henceforth called signals) do not change appreciably during several cycles of the difference frequency. Thus let the instantaneous frequencies be momentarily \( p \) and \( p + r \) rad/sec, the former being that of the stronger signal. Consider the resultant wave to be passed through an ideal limiter that is followed by an ideal wide-band filter. A simple analysis will show that the structure of the unfiltered amplitude-limited resultant signal will include a fundamental carrier frequency of \( p \) rad/sec with associated sidebands, plus other carriers at harmonic frequencies of \( p \) (only odd harmonics with symmetrical limiting) that have associated sidebands also. The wide-band filter will thus be assumed to be sufficiently selective to make only the spectral components centered about the frequency \( p \) of significance, with \( r << p \), and with the harmonics of \( p \) and their associated sidebands completely rejected or negligible. Thus with the input (to the ideal selective limiter) described by \( A(t) \cos \Phi(t) \), the output signal will be

\[
e(t) = \cos \Phi(t) = \cos(pt + \Theta)
\]  

(3a)

where \( \Phi(t) \) and \( \Theta \) are as shown in Fig. 1.

The assumptions made strip the problem of unnecessary computational complexities and make it easier to "see the forest for the trees". In the light of standard FM practice, it is readily appreciated that the assumptions made correspond rather well to most practical situations of importance. Furthermore, the assumption of a slow modulation in comparison with the difference frequency is realistic since, with wide-band FM, the maximum allowable frequency deviation is
often much larger than the audio frequencies of importance, and so the difference frequency \( r \text{ rad/sec} \) will be supersonic most of the time. In a later discussion, the problem in which the difference frequency is within the audio range will be given special attention and the assumption regarding the relative frequencies of the modulation and the difference frequency will be reconsidered.

Going back to Eq. 3(a), we note that if we expand the cosine of the sum we get

\[
e(t) = \cos pt \cos \Theta - \sin pt \sin \Theta \quad (3b)
\]

From Fig. 1 we note that, with

\[
g(t) = (1 + 2a \cos rt + a^2)^{-1/2}, \quad \cos \Theta = g(t)(1 + a \cos rt) \text{ and } \sin \Theta = g(t) \cdot a \sin rt,
\]

so that

\[
e(t) = g(t) \left[ \cos pt + a(\cos pt \cos rt - \sin pt \sin rt) \right]
\]

or

\[
e(t) = g(t) \left[ \cos pt + a \cos(p + r) t \right] \quad (4)
\]

Equation 4 could have been written directly through normalizing the instantaneous phasor-amplitude scales in Fig. 1 by dividing by \((1 + 2a \cos rt + a^2)^{1/2}\). This result shows that the resultant constant-amplitude signal at the output of the ideal limiter may be expressed as the sum of two amplitude-modulated waves having the same carrier frequencies and the same instantaneous relative amplitudes as the two input signals. The resultant amplitude at any instant remains, of course, constant. Plots of the amplitude-modulating function \( g(t) \) appear in Fig. 3.

Next, if we note that the amplitude-modulating function

\[
g(t) = (1 + 2a \cos rt + a^2)^{-1/2}
\]

is an even periodic function of \( \phi = rt \), we can write

\[
g(t) = \sum_{n=0}^{\infty} \alpha_n \cos n\phi
\]

where

\[
\alpha_0 = \frac{1}{\pi} \int_0^\pi g(\phi/r) d\phi = \frac{1}{\pi} G_0(a)
\]

and

\[
\alpha_n = \frac{2}{\pi} \int_0^\pi g(\phi/r) \cos n\phi d\phi
\]
\[
g(x) = \frac{1}{\sqrt{1 + 2a \cos x + a^2}}
\]

\( x = \varphi 
\)

\( a = 0.9 
\)

\( a = 0.8 
\)

\( a = 0.7 
\)
\( (17) \)

\[ = \frac{(2/\pi) \int_{0}^{\pi} \frac{\cos n\phi}{\sqrt{1 + 2a \cos \phi + a^2}} \, d\phi}{\sqrt{1 + 2a \cos \phi + a^2}} \]

or \( a_n = \frac{(2/\pi) G_n(a)}{a^2} \)

where

\[ G_n(a) = \int_{0}^{\pi} \frac{\cos n\phi}{\sqrt{1 + 2a \cos \phi + a^2}} \, d\phi \]  \( (5) \)

Thus

\[ g(t) = (1 + 2a \cos rt + a^2)^{-1/2} \]

\[ = \left( \frac{1}{\pi} \right) G_0(a) + \left( \frac{2}{\pi} \right) \sum_{n=1}^{\infty} G_n(a) \cos nr \]

Substitution in Eq. 4 yields, after some trigonometric manipulations,

\[ e(t) = \left( \frac{1}{\pi} \right) \left[ G_0(a) + aG_1(a) \right] \cos pt \]

\[ + \left( \frac{1}{\pi} \right) \sum_{n=1}^{\infty} \left[ G_n(a) + aG_{n-1}(a) \right] \cos (p + nr)t \]

\[ + \left( \frac{1}{\pi} \right) \sum_{n=1}^{\infty} \left[ G_n(a) + aG_{n+1}(a) \right] \cos (p - nr)t \]

which can be expressed in the final form

\[ e(t) = \sum_{n=-\infty}^{\infty} A_n \cos (p - nr)t \]

\[ = \text{Re} \left[ e^{ipt} \sum_{n=-\infty}^{\infty} A_n e^{-jnrt} \right] \]

(6)

with the definitions

\[ A_0 = \left( \frac{1}{\pi} \right) \left[ G_0(a) + aG_1(a) \right] \]

\[ A_{-n} = \left( \frac{1}{\pi} \right) \left[ G_n(a) + aG_{n-1}(a) \right] \]

\[ A_n = \left( \frac{1}{\pi} \right) \left[ G_n(a) + aG_{n+1}(a) \right] \]

\[ (7) \]
The auxiliary function \( G_n(a) \) is readily recognized to be an elliptic integral. A fruitful analysis of the function \( G_n(a) \), well-known in celestial mechanics (Ref. 4), has been cited by Granlund (Ref. 1). For completeness, some of the steps involved in this analysis are outlined briefly here and the results important to our discussion are presented and expanded.

First, we note that \( G_n(a) \) can be expressed in the form

\[
G_n(a) = \frac{1}{2} \Re \int_{-\pi}^{\pi} \frac{e^{j n \phi} d\phi}{\sqrt{1 + 2a \cos \phi + a^2}}
\]

\[
= \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{j n \phi} d\phi}{\sqrt{1 + 2a \cos \phi + a^2}}
\]

since the contribution from the odd imaginary part vanishes. With \( z = e^{j \phi} \), \( G_n(a) \) reduces to

\[
G_n(a) = \frac{1}{2} \int \frac{z^{n-1/2}}{\sqrt{(1 + az)(a + z)}} \, dz
\]

where the path of integration is a complete circuit of the unit circle in the \( z \)-plane.

By a straightforward contour integration, we get

\[
G_n(a) = (-1)^n \int_0^a \frac{x^{n-1/2}}{\sqrt{(1 - ax)(a - x)}} \, dx \quad (8)
\]

where \( x \) is a dummy variable of integration.

Finally, the substitution of \( x = a \sin^2 \Theta \) yields \( G_n(a) \) in the form
\[ G_n(a) = 2(-a)^n \int_0^{\pi/2} \frac{\sin^{2n}\theta}{\sqrt{1 - a^2 \sin^2 \theta}} \text{ d}\theta \]  

or

\[ G_n(a) = (-a)^n \int_0^{\pi} \frac{\sin^{2n}\theta}{\sqrt{1 - a^2 \sin^2 \theta}} \text{ d}\theta \]  

The last integral on the right is tabulated (Refs. 5 and 8) and may be expressed as

\[ I_n = \int_0^{\pi} \frac{\sin^{2n}\theta}{\sqrt{1 - a^2 \sin^2 \theta}} \text{ d}\theta \]  

\[ = (1,2:n) \sum_{k=0}^{\infty} \frac{(1,2;k)(2n+1)2;2;k}{(2,2;n)(2n+2)2;2;k} a^{2k}, \quad a^2 < 1, \]  

where we have used the notation (Ref. 7)

\[ (m,d,v) = m[m + d][m + 2d]...[m + (v-1)d] = \frac{d^v \Gamma\left(\frac{m}{d} + v\right)}{\Gamma\left(\frac{m}{d}\right)}, \quad v=1,2, ... \]  

When simplified, the expression for \( I_n \) becomes

\[ I_n = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)\Gamma(k+n+1/2)}{\Gamma(k+1)\Gamma(k+n+1)} a^{2k} \]  

Substitution into Eq. 9(b) results in

\[ G_n(a) = (-a)^n \sum_{k=0}^{\infty} c_1(k,n) a^{2k} \]  

where

\[ c_1(k,n) = \frac{\Gamma(k+1/2)\Gamma(k+n+1/2)}{\Gamma(k+1)\Gamma(k+n+1)} \]
Similar expressions may be found for $A_n(a)$ and $A_{-n}(a)$ by substituting from Eqs. 9 into Eqs. 7. This yields the equations

$$A_n(a) = \frac{2}{\pi} (-a)^n \int_0^{\pi/2} \sin^{2n} \theta \sqrt{1 - a^2 \sin^2 \theta} \, d\theta$$  \hspace{1cm} (12)

and

$$A_{-n}(a) = \frac{2}{\pi} (-a)^n \left\{ \int_0^{\pi/2} \frac{\sin^{2n} \theta \, d\theta}{\sqrt{1 - a^2 \sin^2 \theta}} \right\} - \int_0^{\pi/2} \frac{\sin^{2(n-1)} \theta \, d\theta}{\sqrt{1 - a^2 \sin^2 \theta}}$$

or

$$A_{-n}(a) = \left( \frac{1}{\pi} \right) (-a)^n \left[ I_n - I_{n-1} \right]$$ \hspace{1cm} (13)

where $I_n$ is defined in Eq. 10(a). The integral

$$J_n = 2 \int_0^{\pi/2} \sin^{2n} \theta \sqrt{1 - a^2 \sin^2 \theta} \, d\theta$$

may be easily evaluated from another tabulated integral (Ref. 8) and the result may be reduced to the form

$$J_n = -\sum_{k=0}^{\infty} \frac{\Gamma(k-1/2)\Gamma(k+n+1/2)}{\Gamma(k+1)\Gamma(k+n+1)} a^{2k}, \quad a^2 < 1.$$  \hspace{1cm}

Compare this expression with the expression 10(c) for $I_n$. Substitution into Eq. 12 leads to

$$A_n(a) = \left( \frac{-1}{\pi} \right) (-a)^n \sum_{k=0}^{\infty} c_2(k,n) a^{2k}$$ \hspace{1cm} (14a)

where

$$c_2(k,n) \equiv \frac{\Gamma(k - 1/2)\Gamma(k + n + 1/2)}{\Gamma(k - 1)\Gamma(k + n + 1)}$$ \hspace{1cm} (14b)
The value of the expressions 11, 13 and 14 in the numerical evaluation of \( G_n(a) \), \( A_n(a) \) and \( A_n(a) \) is best brought out by studying the convergence properties of the infinite series involved, and by safely estimating the necessary number of terms required in each summation to meet a certain prescribed tolerance in the computed values of the desired functions. The details of this study will not be presented here. Only steps and results are outlined. In this study, Stirling's asymptotic formula for the Gamma function is first used to simplify the expression for \( C_1(k, n) \) and \( C_2(k, n) \) in Eqs. 11(b) and 14(b). There immediately follows that

\[
C_1(k, n) < \left[ k(k + n) \right]^{-1/2} \cdot e < e/k \tag{15a}
\]

and

\[
C_2(k, n) < \left[ k^3(k + n) \right]^{-1/2} \cdot e^2 < e^2/k^2 \tag{15b}
\]

for all integer \( n > 0 \), \( e \) being the base of natural logarithms. The series in Eq. 14 is thus seen to converge much more rapidly than that in Eq. 11, the latter converging, in fact, only for \( a < 1 \) (which is the only range of significance in our discussions). Estimated quite conservatively (through a rough estimate of the remainder), the number of terms, \( N \), that must be added to meet a prescribed tolerance, \( \epsilon \), in the computed value of the series, may be obtained from the formulas

\[
\epsilon = \frac{a^{2N + n}}{[N(N + n)]^{1/2}} \cdot \frac{a^2}{1 - a^2} \text{ for each sum in Eq. 13;}
\]

\[
\epsilon = \frac{a^{2N + n}}{[N^3(N + n)]^{1/2}} \cdot \frac{a^2}{1 - a^2} \text{ for Eq. 14.}
\]

In each case, the error, \( \epsilon \), is about \((1 - a^2)^{-1}\) times the first neglected term in the sum. Estimates for \( N \), computed from these expressions for various prescribed tolerances, \( \epsilon \), are plotted against \( a \) in Fig. 4 for the worst possible situation, namely that with \( n = 0 \).
\[ \varepsilon = \left[ \frac{a^2}{1-a^2} \right] \frac{a^{2N}}{N} \]

\[ \varepsilon = \left[ \frac{a^2}{1-a^2} \right] \frac{a^{2N'}}{N'^2} \]
The computation of the coefficients $C_1$ and $C_2$ is greatly facilitated by the availability of excellent tables of $\log T'(x)$ for $x$ ranging up to fairly large values (see for instance Ref. 9). Admittedly, some of the estimates shown in Fig. 4 are not quite encouraging, cognizant as we may be of the high degree of safety insured by these estimates.

For an alternative approach to the evaluation of $G_n(a)$, $A_n(a)$ and $A_{-n}(a)$ we go back to Eq. 8 and note that the substitution $x = au^2$ yields

$$G_n(a) = 2(-a)^n \int_0^1 \frac{u^{2n} du}{\sqrt{(1 - a^2 u^2)(1 - u^2)}}$$  \hspace{1cm} (16)

The elliptic integral on the right is of the general type represented by (Ref. 6)

$$I_k = \int \frac{u^k}{\sqrt{R(u)}} \, du$$

for which a recursion formula may be found in the following way. First the expression for $d/du \left[ u^{k-1} \sqrt{R(u)} \right]$ is formed, and then both sides are integrated between the limits 0 and 1. The result for $G_n(a)$ is given by (see also Ref. 1)

$$G_{n+1}(a) + \frac{2n}{2n + 1} \cdot \frac{(1 + a^2)}{a} G_n(a) + \frac{2n - 1}{2n + 1} G_{n-1}(a) = 0$$  \hspace{1cm} (17)

for $n > 1/2$.

The restriction on $n$ is inconsequential since, from Eq. 9(a), we have

$$G_0(a) = 2 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - a^2 \sin^2 \theta}} = 2 K(a)$$  \hspace{1cm} (18)

and

$$G_1(a) = (2/a) \left[ \int_0^{\pi/2} \sqrt{1 - a^2 \sin^2 \theta} \, d\theta - \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - a^2 \sin^2 \theta}} \right] = (2/a) \left[ E(a) - K(a) \right]$$  \hspace{1cm} (19)
where $K(a)$ and $E(a)$ are the complete elliptic integrals of the first and second kind.

Equations 17, 18 and 19, with the help of a table of complete elliptic integrals, reduce the task of computing $G_n(a)$, for any integer $n$, to a fairly systematic procedure. Granlund (Ref. 1) used ten-place tables of complete elliptic integrals to evaluate $G_0(a)$ and $G_1(a)$ as given by Eqs. 18 and 19 for several values of $a$. These, then, together with the recursion formula, Eq. 17, and the expressions for the spectral-component amplitudes given by Eqs. 7, were used to calculate and tabulate the latter up to reasonably large values of $n$. Those tables (pp. 43–44 of Ref. 1), reproduced and expanded to include the case $a = 0.35$, appear at the end of the present section.

Equation 17 is readily recognized as a linear difference equation with variable coefficients. The task of developing the general expression for $G_n(a)$ by solving this difference equation directly is not a pleasant undertaking. However, for large values of $n$, the coefficients become approximately constant, and the solution to the resulting constant-coefficient difference equation shows that $G_n(a)$ is asymptotically approximated by a constant times $a^n$. It may further be shown (using recursion formulas for $A_n$ and $A_{-n}$ derived with the help of Eqs. 17 and 7) that $A_n(a)$ and $A_{-n}(a)$ tend asymptotically to expressions also of the form (constant) $\times a^n$. This information is a useful background for the discussion of Section III.

It is convenient at this point to summarize the important properties displayed by the side-frequency components having the amplitudes $A_n$ and $A_{-n}$. First it is noted that the spectral component that has the frequency of the stronger of the two input signals is $A_0$. The component $A_{-1}$ has the frequency of the weaker signal. The amplitudes of the side-frequency components are not symmetrically distributed about the center frequency component $A_0$. This lack of symmetry
conforms to our physical expectations. For, on an instantaneous-frequency basis, the instantaneous frequency of the resultant signal (as portrayed in Fig. 2) places this signal on one side of the center frequency much longer than it does on the other. This means that the power in the composite signal will not be equally shared by the two sidebands. Since the instantaneous frequency of the composite signal lingers in the neighborhood of the mean of the two carrier frequencies (that is \( p + \frac{1}{2} r \) rad/sec) during the major portion of the difference-frequency cycle, more signal power should reside in each of the two components having frequencies closest to that frequency (namely \( A_0 \) and \( A_{-1} \) ) than in any of the other components. This is indeed confirmed by the computed values for the amplitudes. The magnitude of \( A_0 \) is larger than that of \( A_{-1} \), and this may be appreciated through noting that the instantaneous frequency of the composite signal always puts that signal on the \( A_0 \) side of the mean frequency \( (p + (1/2)r) \) rad/sec.

The choice of time reference made in the analysis of this section (namely, \( t = 0 \) when the two signals are in phase) has resulted in alternately positive and negative real values for the spectral amplitudes. The distribution of signs at \( t = 0 \) or \( 2m\pi/r \), where \( m \) is any integer, is such that the \( A_+^m \)'s alternate in sign starting with \( A_{+1} \) negative, \( A_0 \) and \( A_{-1} \) positive. However, it is readily seen that at \( t = q\pi/r \), where \( q \) is an odd integer, all the \( A_+^m \)'s line up in the same positive direction as \( A_0 \), while all the \( A_-^m \)'s line up in phase opposition to \( A_0 \).

Figure 5 shows the input and output spectra superimposed upon a plot of the instantaneous frequency deviation of the resultant signal from the frequency of the stronger signal for \( a = 0.8 \).

It is thus seen the limiting process, by eliminating the amplitude variations of the resultant, spreads out the resulting spectrum over an "infinite" band. The instantaneous frequency of the resultant signal after limiting (but with
essentially all of the significant side-frequencies centered about \( p \) rad/sec passed unaltered) appears as depicted by the spike trains of Figs. 2 and 5. However, it must be borne in mind that the amplitude of the resultant will remain substantially constant, and the instantaneous frequency variations will follow, essentially, the spike pattern given by Eq. 1, Section I, only if most, or all of the sideband components of significant strength (and centered about the frequency \( p \)) are passed by the filter that follows the limiter.

We shall next determine the effect of eliminating some, or most, of the significant sideband components upon the possibility of rejecting the disturbance arising from the simultaneous presence of the weaker signal. This will spotlight the important considerations to be taken into account in providing the necessary bandwidths in the limiter–discriminator sections for the preservation of the interference rejection ability of the FM receiver.

![Figure 5](image)
TABLES OF TWO-SIGNAL INTERFERENCE SPECTRUM
AT THE OUTPUT OF A LIMITER
(Reproduced, except for \(a = 0.85\), from Ref. 1, pp. 43-44)

\(a = 0.2\)

<table>
<thead>
<tr>
<th>n</th>
<th>(G_n(a))</th>
<th>(A_{+n})</th>
<th>(A_{-n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.17373</td>
<td>56950</td>
<td>0.98992</td>
</tr>
<tr>
<td>1</td>
<td>-0.31689</td>
<td>30122</td>
<td>-0.99848</td>
</tr>
<tr>
<td>2</td>
<td>0.04793</td>
<td>05440</td>
<td>0.01474</td>
</tr>
<tr>
<td>3</td>
<td>-0.00799</td>
<td>52557</td>
<td>-0.00245</td>
</tr>
<tr>
<td>4</td>
<td>0.00139</td>
<td>98339</td>
<td>0.00042</td>
</tr>
<tr>
<td>5</td>
<td>-0.00025</td>
<td>20885</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

\(\sum A_n = 0.00006 \ 64892\)

\(a = 0.3\)

<table>
<thead>
<tr>
<th>n</th>
<th>(G_n(a))</th>
<th>(A_{+n})</th>
<th>(A_{-n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.21609</td>
<td>72398</td>
<td>0.97710</td>
</tr>
<tr>
<td>1</td>
<td>-0.48810</td>
<td>10335</td>
<td>-0.14483</td>
</tr>
<tr>
<td>2</td>
<td>0.11025</td>
<td>67567</td>
<td>0.03245</td>
</tr>
<tr>
<td>3</td>
<td>-0.02761</td>
<td>90193</td>
<td>-0.00809</td>
</tr>
<tr>
<td>4</td>
<td>0.00725</td>
<td>86910</td>
<td>0.00249</td>
</tr>
<tr>
<td>5</td>
<td>-0.00196</td>
<td>14240</td>
<td>0.00006</td>
</tr>
</tbody>
</table>

\(\sum A_n = 0.00083 \ 67685\)

\(a = 0.4\)

<table>
<thead>
<tr>
<th>n</th>
<th>(G_n(a))</th>
<th>(A_{+n})</th>
<th>(A_{-n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.27999</td>
<td>97318</td>
<td>0.95871</td>
</tr>
<tr>
<td>1</td>
<td>-0.67029</td>
<td>12677</td>
<td>-0.18756</td>
</tr>
<tr>
<td>2</td>
<td>0.20256</td>
<td>32068</td>
<td>0.05584</td>
</tr>
<tr>
<td>3</td>
<td>-0.06777</td>
<td>18792</td>
<td>-0.01854</td>
</tr>
<tr>
<td>4</td>
<td>0.02377</td>
<td>35234</td>
<td>0.00647</td>
</tr>
<tr>
<td>5</td>
<td>-0.00857</td>
<td>13987</td>
<td>0.00029</td>
</tr>
</tbody>
</table>

\(\sum A_n = 1.00177 \ 39554\)
\[ a = 0.5 \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( G_n(a) )</th>
<th>( A_{+n} )</th>
<th>( A_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.37150 07096</td>
<td>0.93421 54575</td>
<td>0.25865 79046</td>
</tr>
<tr>
<td>1</td>
<td>-0.87315 25820</td>
<td>-0.22515 58510</td>
<td>-0.03347 20536</td>
</tr>
<tr>
<td>2</td>
<td>0.33142 07333</td>
<td>0.08337 99297</td>
<td>0.00851 81156</td>
</tr>
<tr>
<td>3</td>
<td>-0.13894 99174</td>
<td>-0.03451 73819</td>
<td>0.00269 10653</td>
</tr>
<tr>
<td>4</td>
<td>0.06102 07278</td>
<td>0.01504 20514</td>
<td>-0.0094 88514</td>
</tr>
<tr>
<td>5</td>
<td>-0.02752 94594</td>
<td>-0.00675 10369</td>
<td>0.00035 77251</td>
</tr>
<tr>
<td>6</td>
<td>0.01264 09032</td>
<td>0.00308 83458</td>
<td>-0.00143 11050</td>
</tr>
<tr>
<td>7</td>
<td>-0.00587 71571</td>
<td>-0.00143 18211</td>
<td>0.00005 75063</td>
</tr>
<tr>
<td>8</td>
<td>0.00275 79172</td>
<td>0.00067 04154</td>
<td>-0.00001 02311</td>
</tr>
<tr>
<td>9</td>
<td>-0.00130 34901</td>
<td>-0.00031 63008</td>
<td>0.00002 42224</td>
</tr>
<tr>
<td>10</td>
<td>0.00061 96033</td>
<td>0.00015 01323</td>
<td>-0.00001 02311</td>
</tr>
<tr>
<td>11</td>
<td>-0.00029 58978</td>
<td>-0.00007 16100</td>
<td>0.00000 44257</td>
</tr>
<tr>
<td>12</td>
<td>0.00014 18569</td>
<td>0.00003 42952</td>
<td>-0.00000 19391</td>
</tr>
<tr>
<td>13</td>
<td>-0.00006 82306</td>
<td>-0.00001 64807</td>
<td>0.00000 08588</td>
</tr>
<tr>
<td>14</td>
<td>0.00003 29099</td>
<td>0.00000 79429</td>
<td>-0.00000 03837</td>
</tr>
<tr>
<td>15</td>
<td>-0.00001 59127</td>
<td>-0.00000 38378</td>
<td>0.00000 01726</td>
</tr>
<tr>
<td>16</td>
<td>0.00000 77118</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \sum A_n = 0.99999 87240 \]

\[ a = 0.5 \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( G_n(a) )</th>
<th>( A_{+n} )</th>
<th>( A_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.50150 76058</td>
<td>0.90277 99276</td>
<td>0.31576 44262</td>
</tr>
<tr>
<td>1</td>
<td>-1.10890 13617</td>
<td>-0.25585 71623</td>
<td>-0.04992 27199</td>
</tr>
<tr>
<td>2</td>
<td>0.50850 39668</td>
<td>0.11282 68940</td>
<td>-0.04992 27199</td>
</tr>
<tr>
<td>3</td>
<td>-0.25674 63755</td>
<td>-0.05582 65749</td>
<td>0.01529 21943</td>
</tr>
<tr>
<td>4</td>
<td>0.13560 44105</td>
<td>0.02912 17643</td>
<td>-0.00567 07213</td>
</tr>
<tr>
<td>5</td>
<td>-0.07352 61496</td>
<td>-0.01565 78319</td>
<td>0.00249 44344</td>
</tr>
<tr>
<td>6</td>
<td>0.04055 93664</td>
<td>0.00858 49521</td>
<td>-0.00113 20129</td>
</tr>
<tr>
<td>7</td>
<td>-0.02264 82400</td>
<td>-0.00477 17658</td>
<td>0.00053 71097</td>
</tr>
<tr>
<td>8</td>
<td>0.01276 21591</td>
<td>0.00267 91634</td>
<td>-0.00026 31738</td>
</tr>
<tr>
<td>9</td>
<td>-0.00724 21982</td>
<td>-0.00151 59387</td>
<td>0.00013 21296</td>
</tr>
<tr>
<td>10</td>
<td>0.00413 28938</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \sum A_n = 0.99942 76771 \]
\[ a = 0.7 \]

\[
\begin{array}{cccc}
 n & G_n(a) & A_{+n} & A_{-n} \\
 0 & 3.69138 & 79968 & 0.86304 & 68833 \\
 1 & -1.40009 & 38939 & -0.27713 & 89930 \\
 2 & 0.75833 & 72415 & 0.14095 & 48384 \\
 3 & -0.44787 & 79380 & -0.07927 & 26736 \\
 4 & 0.27690 & 78411 & 0.04902 & 08188 \\
 5 & -0.17557 & 77096 & -0.03066 & 67364 \\
 6 & 0.11319 & 35084 & 0.01957 & 76500 \\
 7 & -0.07384 & 04385 & -0.01267 & 62485 \\
 8 & 0.04659 & 54706 & 0.00829 & 35002 \\
 9 & -0.03220 & 09692 & -0.00546 & 94786 \\
 10 & 0.02145 & 44079 & 0.00362 & 98539 \\
 11 & -0.01435 & 84080 & -0.00242 & 12943 \\
 12 & 0.00964 & 52682 & 0.00162 & 19453 \\
 13 & -0.00649 & 96812 & -0.00109 & 03493 \\
 14 & 0.00439 & 18231 & 0.00073 & 51859 \\
 15 & -0.00297 & 45264 & & \\
\end{array}
\]

\[ \sum A_n = 1.00190 \ 02387 \]

\[ a = 0.8 \]

\[
\begin{array}{cccc}
 n & G_n(a) & A_{+n} & A_{-n} \\
 0 & 3.99060 & 55554 & 0.81254 & 96100 \\
 1 & -1.79738 & 20863 & -0.28333 & 48346 \\
 2 & 1.12622 & 03322 & 0.16277 & 17893 \\
 3 & -0.76857 & 20932 & -0.10559 & 41474 \\
 4 & 0.54604 & 78694 & 0.07265 & 55225 \\
 5 & -0.39724 & 22671 & -0.05619 & 45793 \\
 6 & 0.29354 & 86956 & 0.03758 & 05077 \\
 7 & -0.21935 & 63826 & -0.02773 & 17860 \\
 8 & 0.16529 & 30092 & 0.02068 & 95656 \\
 9 & -0.12536 & 85272 & -0.01556 & 54878 \\
 10 & 0.09558 & 51314 & 0.01178 & 80316 \\
 11 & -0.07318 & 99224 & -0.00897 & 50382 \\
 12 & 0.05624 & 25105 & 0.00686 & 34248 \\
 13 & -0.04335 & 05320 & -0.00526 & 80037 \\
 14 & 0.03350 & 07627 & 0.00405 & 61305 \\
 15 & -0.02594 & 75660 & & \\
\end{array}
\]

\[ \sum A_n = 1.00188 \ 02967 \]
$$a = 0.85$$

<table>
<thead>
<tr>
<th>n</th>
<th>$G_n(a)$</th>
<th>$A_{+n}$</th>
<th>$A_{-n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.21987</td>
<td>0.78193</td>
<td>80350</td>
</tr>
<tr>
<td>1</td>
<td>-2.07486</td>
<td>-0.28261</td>
<td>30658</td>
</tr>
<tr>
<td>2</td>
<td>1.39650</td>
<td>0.16880</td>
<td>36615</td>
</tr>
<tr>
<td>3</td>
<td>-1.01905</td>
<td>-0.11534</td>
<td>70926</td>
</tr>
<tr>
<td>4</td>
<td>0.77258</td>
<td>0.08383</td>
<td>96327</td>
</tr>
<tr>
<td>5</td>
<td>-0.59903</td>
<td>-0.06311</td>
<td>69204</td>
</tr>
<tr>
<td>6</td>
<td>0.47146</td>
<td>0.04859</td>
<td>90592</td>
</tr>
<tr>
<td>7</td>
<td>-0.37503</td>
<td>-0.03800</td>
<td>98203</td>
</tr>
<tr>
<td>8</td>
<td>0.30073</td>
<td>0.03006</td>
<td>99857</td>
</tr>
<tr>
<td>9</td>
<td>-0.24267</td>
<td>-0.02399</td>
<td>71966</td>
</tr>
<tr>
<td>10</td>
<td>0.19680</td>
<td>0.01928</td>
<td>27864</td>
</tr>
<tr>
<td>11</td>
<td>-0.16026</td>
<td>-0.01558</td>
<td>06083</td>
</tr>
<tr>
<td>12</td>
<td>0.13095</td>
<td>0.01264</td>
<td>67479</td>
</tr>
<tr>
<td>13</td>
<td>-0.10732</td>
<td>-0.01030</td>
<td>45846</td>
</tr>
<tr>
<td>14</td>
<td>0.08818</td>
<td>0.00842</td>
<td>34301</td>
</tr>
<tr>
<td>15</td>
<td>-0.07261</td>
<td>-0.00690</td>
<td>48833</td>
</tr>
<tr>
<td>16</td>
<td>0.05990</td>
<td>0.00567</td>
<td>37914</td>
</tr>
<tr>
<td>17</td>
<td>-0.04950</td>
<td>-0.00467</td>
<td>20785</td>
</tr>
<tr>
<td>18</td>
<td>0.04097</td>
<td>0.00385</td>
<td>44224</td>
</tr>
<tr>
<td>19</td>
<td>-0.03395</td>
<td>-0.00318</td>
<td>51594</td>
</tr>
<tr>
<td>20</td>
<td>0.02817</td>
<td>0.00263</td>
<td>60282</td>
</tr>
</tbody>
</table>

$$\sum A_n = 1.00109 \ 78085$$
\( a = 0.9 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( G_n(a) )</th>
<th>( A_{+n} )</th>
<th>( A_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.56109</td>
<td>0.74592</td>
<td>55110</td>
</tr>
<tr>
<td>1</td>
<td>-2.46411</td>
<td>-0.27345</td>
<td>33054</td>
</tr>
<tr>
<td>2</td>
<td>1.78337</td>
<td>0.15923</td>
<td>60803</td>
</tr>
<tr>
<td>3</td>
<td>-1.39078</td>
<td>-0.12081</td>
<td>07532</td>
</tr>
<tr>
<td>4</td>
<td>1.12360</td>
<td>0.09211</td>
<td>80049</td>
</tr>
<tr>
<td>5</td>
<td>-0.92899</td>
<td>-0.07292</td>
<td>84863</td>
</tr>
<tr>
<td>6</td>
<td>0.77531</td>
<td>0.05914</td>
<td>47989</td>
</tr>
<tr>
<td>7</td>
<td>-0.65500</td>
<td>-0.04877</td>
<td>38870</td>
</tr>
<tr>
<td>8</td>
<td>0.55753</td>
<td>0.04071</td>
<td>56760</td>
</tr>
<tr>
<td>9</td>
<td>-0.47735</td>
<td>-0.03430</td>
<td>63100</td>
</tr>
<tr>
<td>10</td>
<td>0.41064</td>
<td>0.02911</td>
<td>77756</td>
</tr>
<tr>
<td>11</td>
<td>-0.35463</td>
<td>-0.02485</td>
<td>97281</td>
</tr>
<tr>
<td>12</td>
<td>0.30725</td>
<td>0.02132</td>
<td>71632</td>
</tr>
<tr>
<td>13</td>
<td>-0.26695</td>
<td>-0.01837</td>
<td>06010</td>
</tr>
<tr>
<td>14</td>
<td>0.23248</td>
<td>0.01587</td>
<td>84168</td>
</tr>
<tr>
<td>15</td>
<td>-0.20289</td>
<td>-0.01376</td>
<td>42143</td>
</tr>
<tr>
<td>16</td>
<td>0.17739</td>
<td>0.01196</td>
<td>21272</td>
</tr>
<tr>
<td>17</td>
<td>-0.15534</td>
<td>-0.01041</td>
<td>91580</td>
</tr>
<tr>
<td>18</td>
<td>0.13624</td>
<td>0.00909</td>
<td>30583</td>
</tr>
<tr>
<td>19</td>
<td>-0.11963</td>
<td>-0.00794</td>
<td>95898</td>
</tr>
<tr>
<td>20</td>
<td>0.10318</td>
<td>0.00696</td>
<td>07856</td>
</tr>
<tr>
<td>21</td>
<td>-0.09257</td>
<td>-0.00610</td>
<td>35244</td>
</tr>
<tr>
<td>22</td>
<td>0.08155</td>
<td>0.00535</td>
<td>86332</td>
</tr>
<tr>
<td>23</td>
<td>-0.07190</td>
<td>-0.00471</td>
<td>00743</td>
</tr>
<tr>
<td>24</td>
<td>0.06345</td>
<td>0.00414</td>
<td>43841</td>
</tr>
<tr>
<td>25</td>
<td>-0.05603</td>
<td>0.0034</td>
<td>92705</td>
</tr>
</tbody>
</table>

\[
\sum A_n = 1.00209 \ 68939
\]
\( a = 0.95 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( G_n(a) )</th>
<th>( A_{+n} )</th>
<th>( A_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.18002 24618</td>
<td>0.70201 44040</td>
<td>0.56973 82095</td>
</tr>
<tr>
<td>1</td>
<td>-3.13113 59644</td>
<td>-0.25469 63862</td>
<td>-0.16581 14861</td>
</tr>
<tr>
<td>2</td>
<td>2.45366 70194</td>
<td>0.16040 69563</td>
<td>0.02869 14479</td>
</tr>
<tr>
<td>3</td>
<td>-2.05235 12672</td>
<td>-0.11794 68883</td>
<td>-0.05710 70797</td>
</tr>
<tr>
<td>4</td>
<td>1.77032 65218</td>
<td>0.09325 43616</td>
<td>0.04032 83151</td>
</tr>
<tr>
<td>5</td>
<td>-1.55511 50572</td>
<td>-0.07687 16475</td>
<td>-0.03011 40200</td>
</tr>
<tr>
<td>6</td>
<td>1.38275 33202</td>
<td>0.06509 40585</td>
<td>0.02334 73825</td>
</tr>
<tr>
<td>7</td>
<td>-1.24026 76887</td>
<td>-0.05616 02396</td>
<td>-0.01859 81753</td>
</tr>
<tr>
<td>8</td>
<td>1.11982 64133</td>
<td>0.04911 98511</td>
<td>0.01512 18935</td>
</tr>
<tr>
<td>9</td>
<td>-1.01632 82631</td>
<td>-0.04341 19769</td>
<td>-0.01249 47943</td>
</tr>
<tr>
<td>10</td>
<td>0.92675 84374</td>
<td>0.03868 25148</td>
<td>0.01045 88274</td>
</tr>
<tr>
<td>11</td>
<td>-0.84708 81401</td>
<td>-0.03469 59015</td>
<td>-0.00884 87786</td>
</tr>
<tr>
<td>12</td>
<td>0.77593 44746</td>
<td>0.03128 87109</td>
<td>-0.00649 83237</td>
</tr>
<tr>
<td>13</td>
<td>-0.71435 58845</td>
<td>-0.02834 35952</td>
<td>-0.00555 40877</td>
</tr>
<tr>
<td>14</td>
<td>0.65822 30042</td>
<td>0.02577 38383</td>
<td>0.00582 71016</td>
</tr>
<tr>
<td>15</td>
<td>-0.60763 36928</td>
<td>-0.02351 37835</td>
<td>0.00490 07777</td>
</tr>
<tr>
<td>16</td>
<td>0.56185 58561</td>
<td>0.02151 26740</td>
<td>0.00428 98499</td>
</tr>
<tr>
<td>17</td>
<td>-0.52028 61025</td>
<td>-0.01973 05577</td>
<td>-0.00377 19655</td>
</tr>
<tr>
<td>18</td>
<td>0.48242 18182</td>
<td>0.01813 54894</td>
<td>0.00332 99226</td>
</tr>
<tr>
<td>19</td>
<td>-0.44783 94714</td>
<td>-0.01670 15704</td>
<td>-0.00288 54109</td>
</tr>
<tr>
<td>20</td>
<td>0.41617 88846</td>
<td>0.01540 75480</td>
<td>0.00262 24687</td>
</tr>
<tr>
<td>21</td>
<td>-0.38713 12119</td>
<td>-0.01423 57818</td>
<td>-0.00233 79808</td>
</tr>
<tr>
<td>22</td>
<td>0.36042 96679</td>
<td>0.01317 15044</td>
<td>0.00208 99922</td>
</tr>
<tr>
<td>23</td>
<td>-0.33684 22805</td>
<td>-0.01220 22887</td>
<td>-0.00187 29386</td>
</tr>
<tr>
<td>24</td>
<td>0.31316 61562</td>
<td>0.01131 73202</td>
<td>0.00168 22543</td>
</tr>
<tr>
<td>25</td>
<td>-0.29222 28907</td>
<td>-0.01050 76560</td>
<td>-0.00151 41643</td>
</tr>
<tr>
<td>26</td>
<td>0.27285 48587</td>
<td>0.00976 53593</td>
<td>0.00136 55267</td>
</tr>
</tbody>
</table>

\[ \sum A_n = 1.00541 \ 78050 \]
\[ a = 0.98 \]

<table>
<thead>
<tr>
<th>n</th>
<th>( G_n(a) )</th>
<th>( A_{+n} )</th>
<th>( A_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.04196 08912</td>
<td>0.66850 94960</td>
<td>0.60443 89257</td>
</tr>
<tr>
<td>1</td>
<td>-4.02222 07846</td>
<td>-0.23527 98956</td>
<td>-0.18834 64439</td>
</tr>
<tr>
<td>2</td>
<td>3.35006 85643</td>
<td>0.14679 01498</td>
<td>-0.10669 62204</td>
</tr>
<tr>
<td>3</td>
<td>-2.94787 11306</td>
<td>-0.10809 34049</td>
<td>-0.07238 29373</td>
</tr>
<tr>
<td>4</td>
<td>2.66151 60040</td>
<td>0.06611 88178</td>
<td>0.05364 30799</td>
</tr>
<tr>
<td>5</td>
<td>-2.43976 09782</td>
<td>-0.07184 73944</td>
<td>0.01493 28666</td>
</tr>
<tr>
<td>6</td>
<td>2.25923 03985</td>
<td>0.06176 86669</td>
<td>0.03397 03133</td>
</tr>
<tr>
<td>7</td>
<td>-2.10732 48451</td>
<td>-0.05423 30535</td>
<td>0.02823 47685</td>
</tr>
<tr>
<td>8</td>
<td>1.97647 62068</td>
<td>0.04836 04982</td>
<td>0.02392 53820</td>
</tr>
<tr>
<td>9</td>
<td>-1.86178 25785</td>
<td>-0.04363 96739</td>
<td>-0.02058 34363</td>
</tr>
<tr>
<td>10</td>
<td>1.75968 24486</td>
<td>0.03975 01985</td>
<td>0.01792 52596</td>
</tr>
<tr>
<td>11</td>
<td>-1.66837 09356</td>
<td>-0.03648 21314</td>
<td>-0.01576 75475</td>
</tr>
<tr>
<td>12</td>
<td>1.58546 83056</td>
<td>0.03369 16872</td>
<td>0.01398 64754</td>
</tr>
<tr>
<td>13</td>
<td>-1.50981 91312</td>
<td>-0.03127 69528</td>
<td>0.01249 54845</td>
</tr>
<tr>
<td>14</td>
<td>1.44036 70264</td>
<td>0.02916 35899</td>
<td>0.01123 23067</td>
</tr>
<tr>
<td>15</td>
<td>-1.37627 23537</td>
<td>-0.02729 60398</td>
<td>0.01015 10494</td>
</tr>
<tr>
<td>16</td>
<td>1.31685 64444</td>
<td>0.02563 19025</td>
<td>0.00921 71764</td>
</tr>
<tr>
<td>17</td>
<td>-1.26156 27018</td>
<td>-0.02413 82232</td>
<td>0.00840 42195</td>
</tr>
<tr>
<td>18</td>
<td>1.20992 86134</td>
<td>0.02278 89814</td>
<td>0.00769 16737</td>
</tr>
<tr>
<td>19</td>
<td>-1.16156 64458</td>
<td>-0.02156 33357</td>
<td>-0.00706 29670</td>
</tr>
<tr>
<td>20</td>
<td>1.11614 61517</td>
<td>0.02044 43850</td>
<td>0.00650 53908</td>
</tr>
<tr>
<td>21</td>
<td>-1.07338 59407</td>
<td>-0.01941 82678</td>
<td>-0.00600 83275</td>
</tr>
<tr>
<td>22</td>
<td>1.03304 25043</td>
<td>0.01847 34973</td>
<td>0.00556 31861</td>
</tr>
<tr>
<td>23</td>
<td>-0.99490 43886</td>
<td>-0.01760 04697</td>
<td>-0.00516 28865</td>
</tr>
<tr>
<td>24</td>
<td>0.95878 66145</td>
<td>0.01679 10861</td>
<td>0.00480 15431</td>
</tr>
<tr>
<td>25</td>
<td>-0.92452 63897</td>
<td>-0.01603 84642</td>
<td>-0.00447 42275</td>
</tr>
<tr>
<td>26</td>
<td>0.89197 96616</td>
<td>0.01533 67154</td>
<td>0.00417 67837</td>
</tr>
<tr>
<td>27</td>
<td>-0.86101 83154</td>
<td>-0.01468 07672</td>
<td>-0.00390 56834</td>
</tr>
<tr>
<td>28</td>
<td>0.83152 78828</td>
<td>0.01406 62248</td>
<td>0.00365 79147</td>
</tr>
<tr>
<td>29</td>
<td>-0.80340 56473</td>
<td>-0.01348 92594</td>
<td>-0.00343 08915</td>
</tr>
<tr>
<td>30</td>
<td>0.77655 90704</td>
<td>0.01294 65176</td>
<td>0.00322 23827</td>
</tr>
<tr>
<td>31</td>
<td>-0.75090 44753</td>
<td>-0.01243 50481</td>
<td>-0.00303 4526</td>
</tr>
<tr>
<td>32</td>
<td>0.72836 59382</td>
<td>0.01195 22426</td>
<td>-0.00285 34164</td>
</tr>
<tr>
<td>33</td>
<td>-0.70287 43475</td>
<td>-0.01149 57856</td>
<td>-0.00268 8504</td>
</tr>
<tr>
<td>34</td>
<td>0.68036 66034</td>
<td>0.01106 36141</td>
<td>-0.00253 83107</td>
</tr>
<tr>
<td>35</td>
<td>-0.65878 46332</td>
<td>-0.01065 36820</td>
<td>-0.00239 78070</td>
</tr>
<tr>
<td>36</td>
<td>0.63607 63017</td>
<td>0.01026 49329</td>
<td>0.00225 72811</td>
</tr>
<tr>
<td>37</td>
<td>-0.61319 19019</td>
<td>-0.00989 51898</td>
<td>0.00214 57519</td>
</tr>
<tr>
<td>38</td>
<td>0.59908 69859</td>
<td>0.00954 34804</td>
<td>0.00203 25981</td>
</tr>
<tr>
<td>39</td>
<td>-0.58071 96510</td>
<td>-0.00920 84944</td>
<td>-0.00192 70221</td>
</tr>
</tbody>
</table>

\[ \sum A_n = 0.99450 \times 10^8 \]
I.3 A USEFUL THEOREM

The mathematical formulation of the general criterion that we shall use in determining whether or not interference may still be suppressed when an ideal bandpass filter (that may reject major portions of the output spectrum) is inserted after the limiter, will prove to be largely an outcome of the previously demonstrated properties of the spectral amplitude components, $A_{in}$. The most important consequence of these properties may be appreciated by examining the behaviour of the locus described during a period of $2\pi/r$ seconds, by the end point of the phasor representing the resultant signal at the output of the filter. This behavior will be indicated by the following theorem.

Theorem I.

If at the output of the limiter, an ideal filter is inserted that will pass (a) an arbitrary number of components from both sidebands simultaneously; or (b) an arbitrary number of components from the upper sideband, along with $A_o$ only, then, over a period of $2\pi/r$ seconds, the terminus of the resultant phasor representing the signal at the output of the filter will cross a reference axis along which a phasor representing $A_o$ lies, only at $rt = mw$, where $m$ is an integer or zero.

The ideal filter referred to is one characterized by a constant amplitude response within, and sharp cut-offs at the edges of, the passband, in addition to a linear phase characteristic (or constant time delay) over the passband.

Figure 6 shows a phasor diagram illustrating the linear superposition of several spectral components that fall within the ideal filter passband. The plane of the figure may be imagined to be rotating clockwise with an angular velocity of $p$ rad/sec. This will leave $A_o$ stationary, and will result in the $n$th component phasor rotating with $nr$ rad/sec about
its origin.

We shall arbitrarily choose the line along which \( A_0 \) is laid as an axis of reference, and call the origin of \( A_0 \), 0. We want to show that the path traced by the point \( R \) during one complete cycle of the difference frequency \( r \) will cross the reference axis only at \( rt = 0 \) and \( rt = \pi \). This we shall do by first demonstrating a few helpful lemmas.

Let us first translate the assertion of the theorem into a more specific mathematical statement. To this end, we note from Fig. 6 that the locus of \( R \) crosses the reference axis only when \( Y \), the instantaneous length of the vertical component of the resultant phasor, vanishes. But

\[
\overline{OR} = \sum_{n=-N}^{M} A_n e^{-j n \phi}
\]

\[
= \sum_{n=-N}^{M} A_n \cos n \phi - j \sum_{n=-N}^{M} A_n \sin n \phi \tag{20}
\]

\[\phi = rt\]

\[\phi = rt\]

FIG. 6
where $\phi \equiv \pi t$, $N$ is the number of upper-sideband components passed, and $M$ the number of lower-sideband components passed. Therefore, the locus traced by $R$ crosses the reference axis for values of $\phi$ that are the roots of

$$Y = -\sum_{n=-N}^{M} A_n \sin n\phi$$

When components from both sidebands are passed, Eq. 2 can be reduced to the form

$$y = \frac{Y}{A_{-1} - A_1} = \sin \phi + \frac{A_{-2} - A_2}{A_{-1} - A_1} \sin 2\phi$$

$$+ \frac{A_{-3} - A_3}{A_{-1} - A_1} \sin 3\phi + \ldots,$$

the sum terminating with the term contributed by the last component (in either or both sidebands) passed.

From the tables in Section I.2, pp. 27-33, we find that the expression for $y$ can be rewritten as

$$y = \sin \phi - b_1 \sin 2\phi + b_2 \sin 3\phi - b_3 \sin 4\phi$$

$$+ \ldots + b_{q-1} \sin q\phi$$

(22a)

where

$$b_0 = 1, \quad b_1 = \frac{|A_{-2}| + |A_2|}{|A_{-1}| + |A_1|}, \ldots,$$

$$b_{q-1} = \frac{|A_{n-1}| + |A_n|}{|A_{-1}| + |A_1|}.$$  

(22b)

Similarly, we can show that expressions of the form of Eq. 22(a) exist for the special cases in which only upper-sideband components or only lower-sideband components are passed. For the former $b_{-(n-1)} = |A_{n-1}|/|A_{-1}|$; for the latter, $b_{+(n-1)} = |A_n|/|A_1|$.

By direct substitution from Eqs. 7, p.17, the expression for $b_{-n}$ takes the alternative useful form
\[ b_{\pm n} = (-1)^n \frac{G_n - G_{n+2}}{G_0 - G_2} \]  

(22c)

In terms of Eq. 22(a), Theorem I in effect states that, when the magnitude of the coefficient of \( \sin n\phi \) is given by either (a) \( b_{+n} \) or (b) \( b_{-n} \), then, in the interval \( 0 \leq \phi < 2\pi \), \( y \) will have zeros only at \( \phi = 0 \) and \( \phi = \pi \).

Admittedly, the zeros of the finite sum in Eq. 22(a) would best be placed in evidence by expressing this sum in some convenient closed form. However, any attempt on Eq. 22(a) in this direction is quickly discouraged by the formidable appearance of the expressions for the coefficients of the sine terms. On the other hand, the following lemmas are quite helpful.

**Lemma 1.** If a finite sum of harmonically-related sine terms (each weighted by an appropriate coefficient such that the \( n \)th term is given by \( \alpha_n \sin n\phi \)) is to vanish only at \( \phi = 0 \) or \( \phi = \pi \), in the range \( 0 \leq \phi < 2\pi \), then the zeros of the sum must not be produced by mutual cancellation among the terms, but only by the individual terms vanishing simultaneously and separately, and this can only be insured by special restrictions on the magnitudes of the weighting coefficients (\( \alpha_n \)'s).

The truth of this statement is best illustrated by considering Eq. 22(a). If only the first term in the sum is present, then \( y = \sin \phi \) with zeros at \( \phi = 0 \) and \( \phi = \pi \) only. If only the first two terms are present, then \( y = \sin \phi - b_1 \sin 2\phi \), whose zeros are those of \( \sin \phi \) only if \( b_1 \leq 1/2 \). If only the first three terms are present, then \( y = \sin \phi - b_1 \sin 2\phi + b_2 \sin 3\phi \), and with \( b_1 \) assigned its highest permissible value of 1/2, the zeros of \( y \) will be those of \( \sin \phi \) only if \( b_2 < 0.933 \). The illustration grows in complication as more terms are dragged in, but the pattern is obvious. The coefficients \( b_1, b_2, b_3, \ldots \) must obey certain restrictions on their magnitudes in order for the zeros of \( y \) to be identical with those of \( \sin \phi \); that is, in order for the zeros of \( y \) not to be brought about by the various component terms cancelling
one another, but only by the simultaneous vanishing of the individual terms.

Lemma 2. Given the two finite sums of harmonically-related sine terms

\[ y_1 = \sum_{n=1}^{q} a_n \sin n\phi \text{ and } y_2 = \sum_{n=1}^{q} b_n \sin n\phi \text{ in which } \]

\[ a_1 = b_1 \text{ but, otherwise, the terms of the first sum dominate } \]

those of the second sum (that is \( |a_n| \geq |b_n| \)), and corresponding coefficients (e.g., \( a_n \) and \( b_n \)) have the same sign.

If \( y_1 \) has zeros only where \( \sin \phi \) has zeros, then the zeros of \( y_2 \) must likewise be those of \( \sin \phi \) only.

Clearly, if the magnitudes of the various coefficients in the expression for \( y_1 \) are within the bounds imposed upon them by the condition that the zeros of \( y_1 \) be those of \( \sin \phi \), then with those same restrictions applying to the coefficients of \( y_2 \) (since \( a_1 = b_1 \) and \( a_n \) and \( b_n \) have same signs) and with \( |a_n| > |b_n|, n \neq 1 \), the magnitudes of the coefficients of \( y_2 \) are certainly within the proper bounds to insure that the zeros of \( y_2 \) be those of \( \sin \phi \).

Lemma 3. In the finite sum of sine terms given by Eq. 22(a), namely

\[ y = \sin \phi - b_1 \sin 2\phi + b_2 \sin 3\phi - b_3 \sin 4\phi + \ldots + (-1)^{q-1} b_{q-1} \sin q\phi \]

where

\[ b_{n-1} = \frac{|A_{n-1}(a)| + |A_n(a)|}{|A_{-1}(a)| + |A_1(a)|} \text{ (for case (a) of Theorem 1)} \]

and

\[ b_{n-1} = \frac{|A_{n-1}(a)|}{|A_{-1}(a)|} \text{ (for case (b) of Theorem 1)} \]

the \( n \)th coefficient, \( b_n (n \neq 0) \), is dominated by the corresponding coefficient \( (1/2)a_n \), in the similar sum
\[ S = \sin \phi - (1/2)a \sin 2\phi + (1/2)a^2 \sin 3\phi - (1/2)a^3 \sin 4\phi + \ldots + (-1)^{q-1}a^{q-1} \sin q\phi \]  

(23)

That is to say,
\[
\frac{|A_n(a)|}{|A_{n-1}(a)|} < (1/2)a^{n-1}
\]

(24a)

and
\[
\frac{|A_{n-1}(a)|}{|A_{n-1}(a)|} < (1/2)a^{n-1}
\]

(24b)

where, as before, \(a\) lies between 0 and 1, and \(n\) is a positive integer different from 1.

From Eqs. 22(c) and 11(a) it is readily seen that
\[
b_{+n} = f(n,a) \cdot a^n
\]

(25)

where \(f(n,a)\) is a complicated function with no factorable powers of \(a\). That \(f(n,a) < 1/2\) for all values of \(a\) and all values of \(n\), is quite obvious from the plots Figs. 7 and 8. An analytical demonstration is also possible, but it is too involved to be worthy of reproduction. Similar statements may also be made for \(b_{-n}\), but not for \(b_{+n}\), as is obvious from Figs. 7 and 8.

**Lemma 4.** The finite sum, \(S\), given by Eq. 23, has zeros only where \(\sin \phi\) has zeros, for all values of \(a\) between 0 and 1, and for all \(q = 1, 2, 3, \ldots\).

This is obviously true when \(S = \sin \phi\) only. It is true for \(S = \sin \phi - (1/2)a \sin 2\phi\) since the necessary restriction is that \((1/2)a \leq 1/2\); that is for \(a \leq 1\). When \(S\) is made up of the first three terms it can be readily shown to be true for all \(a < \sqrt{\pi/4}\), which includes the range \(a \leq 1\). Finally, Eq. 23 can be expressed in closed form as follows. First, write
\[
2S = \sin \phi + \sin \phi - a \sin 2\phi + a^2 \sin 3\phi + \ldots + (-a)^{q-1} \sin q\phi
\]

\[
= \sin \phi - (1/a) \sum_{n=1}^{q} (-a)^n \sin n\phi
\]

(23a)

If \(\sin n\phi\) is replaced by its value in terms of complex exponentials, and the standard formula for the sum of a finite geometric progression is used, it is readily established that
\[ 2S = \sin \phi + \frac{\sin \phi - (-a)^{q} \sin(q+1)\phi + a \sin \phi}{1 + 2a \cos \phi + a^2} \] (23b)

As \( q \) is increased, \( a^q \) approaches zero and the zeros of \( S \) become more and more obviously those of \( \sin \phi \). Therefore, we may conclude that Lemma 4 is true for all \( a < 1 \), and all positive integer \( q \). Another argument based on Eq. 23(b) and making use of phasors is also possible, but will not be carried out here.

The argument that proves Theorem I is now obvious. The sum in Eq. 22(a) must vanish only at the zeros of \( \sin \phi \). But, this sum is exactly similar to the sum in Eq. 23, in that they both are made up of the same number of harmonically-related weighted sine terms, the coefficient of the first term, \( \sin \phi \), is the same in both, and the coefficient of \( \sin n\phi \) has the same sign in both. Furthermore, by Lemma 4, the finite sum in Eq. 23 vanishes only where \( \sin \phi \) vanishes, and by Lemma 3 the \( n^{th} \) coefficient, \( (1/2)a^n \), in Eq. 23, dominates the \( n^{th} \) coefficient, \( b_n \), in Eq. 22(a) (only when \( b_n = b_{-n} \), or \( b_{-n} \); that is, for conditions (a) and (b) of Theorem I). Therefore, by Lemma 2, the finite sum in Eq. 22(a) can vanish only at the zeros of \( \sin \phi \), and this proves Theorem I.

In Fig. 9, plots of typical \( y \)'s are shown for arbitrarily chosen values of \( a, N, \) and \( M \) to illustrate the above demonstration, and perhaps provide an independent and more "engineering-like" demonstration by themselves.

In conclusion, the theorem cannot be extended to include the situation in which \( A_o \) is accompanied by lower-sideband components only, for \( a \) greater than approximately 0.69, and for all values of \( q \). The quoted upper limit on \( a \) may be read directly off the plot of \( b_{+1} \) in Fig. 7, since for \( a > 0.69 \), \( b_{+1} \) exceeds the maximum permissible value of 0.5. Furthermore, the plots of Figs. 7 and 8 show that \( b_{+n} \) cannot
\[ y = \frac{y}{A_1 - A_1} \]

Plots of \( y = \sin \varphi - b_1 \sin 2\varphi + b_2 \sin 3\varphi - \ldots (-1)^{n+1} b_{n-1} \sin n\varphi \)

For the case of both side bands simultaneously

\[
b_{n-1} = \frac{|A_{-n}| + |A_n|}{|A_1| + |A_{-1}|}
\]

Fig. 9(a)
Plots of \( y = \frac{\sin \varphi - b_1 \sin 2\varphi + b_2 \sin 3\varphi - \cdots - b_5 \sin 6\varphi}{1A_{-1}} \) for the case of upper-side-band components only

\[ b_{n-1} = \frac{|A_{n}|}{|A_{-1}|} \]

**FIG. 9(b)**
PLOTS OF $y = \sin \phi - b_1 \sin 2\phi$, OVER $\frac{1}{2}$ CYCLE, FOR CASE OF LOWER SIDE-BAND ONLY.

FIG. 9(c)

PLOTS OF $y = \sin \phi - b_1 \sin 2\phi + b_2 \sin 3\phi$, OVER $\frac{1}{2}$ CYCLE, FOR LOWER SIDE-BAND ONLY.

FIG. 9(d)
be said to be bounded by \((1/2)a^n\) for all \(n\) and all \(a \leq 1\),
and so the argument presented above does not apply. Actually
the most serious violation of the conditions for the above
argument is the fact that \(b_1\) does exceed 0.5 for \(a > 0.69\),
for otherwise the remaining coefficients \(b_n\) are not
large enough to exceed the more liberal bounds that apply
to them when \(b_1\) is within its own bounds. Indeed, for
the range \(a > 0.69\) for which \(b_{+1} < 0.5\) the corresponding
finite sum, Eq. 22(a), will have zeros at 0 or \(\pi\) only.
This is illustrated in Figs. 9(c) and 9(d). From these
plots we may also conclude that the theorem holds for all
values of \(a\) when \(q\) is odd, but only breaks down for even
values of \(q\) in the range \(0.69 < a < 1\).

The importance of Theorem I will best be appreciated
from the discussions of the following two sections.
I.4 A CRITERION FOR INTERFERENCE REJECTION

If the limiter bandwidth is narrowed down to pass only a portion of the power in each sideband, the interference will still be suppressed only if, over a period of \(2\pi/r\) sec, the average frequency of the resultant of the passed components is still equal to the frequency of the stronger of the two carriers. It is clear that the minimum value that the limiter bandwidth can have is equivalent to one intermediate-frequency (i-f) bandwidth. The conditions for this, or any other value of limiter bandwidth, to be permissible will now be determined.

At the output of the limiter, the component that has the frequency of the stronger signal is \(A_o\). From Fig. 5 we find that the average frequency of the resultant, \(\overline{OR}\), will be the frequency of \(A_o\) if and only if, over a period of \(2\pi/r\) sec, the net phase deviation, \(\theta\), is zero. It is readily appreciated that since the locus of the point \(R\) traces a closed path during a complete period of \(r\), the net value of the phase deviation \(\theta\) will be nil only if this closed path does not enclose the origin, \(O\).

Now, the closed path traced by \(R\) will enclose the origin, \(O\), if, at any instant of time, the resultant of the sideband components passed opposes \(A_o\) in phase and exceeds it in magnitude. Or, in terms of the resultant phasor \(\overline{OR}\), the locus of \(R\) will enclose the origin \(O\) only if \(\overline{OR}\) can assume a **negative real** value at any instant during the difference-frequency cycle. Obviously, \(\overline{OR}\) becomes real only when the path of the terminal point \(R\) crosses the axis along which \(A_o\) lies (this axis being chosen as the axis of reals). But Theorem I states that this can occur only when \(rt = 0\) or \(\pi\), and at no other instant during the cycle. Consequently, loci of \(R\) of the type shown in Fig. 10, for instance, are ruled out completely.

Now, the \(n^{th}\) upper and lower sideband components are \(A_{-n}e^{jnrt}\) and \(A_n e^{-jnrt}\). Furthermore, the tables of the
spectral amplitudes (pp.27-33) reveal that the spectral terms in each sideband alternate in sign, \(A_n\) and \(A\) being positive for \(n\) odd and even, respectively. As a consequence, we find that since (with \(\text{rt} = \pi\))

\[ e^{jn\pi} = e^{-jn\pi} = \begin{cases} -1, & \text{for } n \text{ odd} \\ +1, & \text{for } n \text{ even} \end{cases} \]

the distribution of signs is such that, at \(rt = \pi\), all the upper sideband components line up in phase opposition to \(A_0\), while all the lower sideband components line up in phase aiding \(A_0\).

Finally, at \(rt = 0\), \(e^{\pm j\text{rt}} = 1\), for all \(n\). Consequently, the components in each sideband are so oriented that every other component aids or opposes \(A_0\) directly, \(A_n\) and \(A\) aiding \(A_0\) for \(n\) odd and even, respectively. In this mutual cancellation among the terms, with \(A_{-1}\) heavily weighting the positively oriented components, it is very unlikely that the passed components will subtract from the magnitude of \(A_0\).

We conclude, therefore, that the only critical instant of time to consider, during a difference-frequency cycle, is that corresponding to \(t = \pi/r\). The following theorem can therefore be stated as the criterion for the loss or
preservation of the desired average frequency (and hence for the possibility of rejecting the interference) when the ideal limiter is followed by an ideal narrow-band filter.

**Theorem II.**

If arbitrary numbers \( N \) and \( M \) of upper and lower sideband components\(^\dagger\), respectively, fall within the passband of the ideal filter following the limiter, the average frequency of the resultant of all the passed components, including the component \( A_0 \), will be exactly the frequency of \( A_0 \) if and only if

\[
\sum_{n=-N}^{M} A_n e^{-jmn} = \sum_{n=-N}^{M} (-1)^n A_n < A_0, \ n \neq 0.
\]

This important inequality can also be expressed in the more convenient form

\[
\sum_{n=0}^{M} |A_n| > \sum_{n=1}^{N} |A_{-n}|.
\]

(26)

More formally, the proof of Theorem II may be carried out as follows.

At the output of the ideal filter, the resultant signal is given by

\[
e(t) = \sum_{n=-N}^{M} A_n \cos(p - nr)t.
\]

The corresponding complex function of time is

\[
E(t) = e^{jpt} \sum_{n=-N}^{M} A_n e^{-jnr}t
\]

\[
= e^{jpt} F(t), \text{ say.}
\]

(27)

Over a period of \( 2\pi/r \) sec, the net phase shift of \( E(t) \) is \( 2\pi p/r \) if and only if, over \( 2\pi/r \) sec, the complex function

\[
F(t) = \sum_{n=-n}^{M} A_n e^{-jnr}t
\]

\( \dagger\)Strictly speaking the situation \( N=0, M= \text{ even integer } \neq 0 \) is not covered by this theorem, but it will be handled separately later on.
introduces no net phase change.

For convenience, let us shift our time reference from the instant at which $rt = 0$ to the instant where $rt = \pi$. For this purpose we substitute $+\pi/r$ for $t$ to get

$$H(\tau) = F(\tau + \pi/r) = \sum_{n=-N}^{M} (-1)^n A_n e^{-jn\tau t}$$

or

$$H(\tau) = -\sum_{n=1}^{N} |A_{-n}| e^{jn\tau t} + \sum_{n=0}^{M} |A_n| e^{-jn\tau t}$$

Now let $z = e^{jn\tau t}$, to get

$$h(z) = -\sum_{n=1}^{N} |A_{-n}| z^n + \sum_{n=0}^{M} |A_n| z^{-n} \quad (28)$$

As $e^{jn\tau t}$ covers one complete cycle of variation over a period of $2\pi/r$ sec, $z$ traverses the unit circle in the $z$-plane once in the counterclockwise direction, and $h(z)$ traces some closed path $C^1$ in the $h(z)$-plane, as shown in Fig. 11. In tracing out $C^1$ in the counterclockwise direction, $h(z)$ will sustain a net phase shift given by $2\pi(Z-P)$, where $Z$ and $P$ are the numbers of zeros and poles of $h(z)$, within the unit circle in the $z$-plane, each zero or pole being counted in accordance with its multiplicity. But, from a well-known theorem in function theory (Ref. 10), if a function $f(z)$ is analytic, except for possible poles within and on a given contour, the number of times that the plot of $f(z)$ encircles the origin of the $h(z)$ plane in the counterclockwise direction while $z$ itself traverses a prescribed contour once in the counterclockwise direction, is equal to the number of zeros, $Z$, diminished by the number of poles, $P$, of $f(z)$ within the contour in the $z$-plane (each pole or zero being counted according to its multiplicity).

Therefore, if $h(z)$ is to acquire no net phase shift in tracing the path $C^1$ once, the quantity $Z-P$ must be zero, or, equivalently, the path $C^1$ must not encircle the origin.
of the \( h(z) \)-plane. This latter condition is also rather obvious from an examination of Fig. 11. It is also readily appreciated that if, while \( z \) traverses the unit circle and \( h(z) \) describes the path \( C^1 \), \( h(z) \) never assumes a negative real value, \( C^1 \) will never encircle the origin of the \( h(z) \)-plane.

Now, on the unit circle, Eq. 28 can be written in the form

\[
h(z) \big|_{z=1} = - \sum_{n=1}^{N} |A_{-n}| \cos n\phi + \sum_{n=0}^{M} |A_n| \cos n\phi \]

\[-j \left[ \sum_{n=1}^{N} |A_{-n}| \sin n\phi + \sum_{n=0}^{M} |A_n| \sin n\phi \right]
\]

and this is readily recognized to be equivalent to Eq. 20, Section I.3, with the reference axis shifted by \( \pi \) radians. The roots of the imaginary component of \( h(|z|=1) \) are then exactly the roots of Eq. 22(a), namely \( \phi = 0 \) and \( \phi = \pi \) in the range \( 0 \leq \phi < 2\pi \). Therefore \( h(|z|=1) \) becomes real only when \( z = 1 \) or \(-1\), corresponding to \( \phi = 0 \) and \( \phi = \pi \), and its real values are given by

\[
h(-1) = - \sum_{n=1}^{N} (-1)^n |A_{-n}| + \sum_{n=0}^{M} (-1)^n |A_n|
\]

\[= \sum_{n=-N}^{M} A_n\]

\[= A_0 + (|A_{-1}| - |A_1| - |A_{-2}|) + (|A_2| - |A_3|) + (|A_4| - |A_5|) + \ldots + (|A_{M-1}| - |A_M|) + (|A_{-3}| - |A_{-4}|) + (|A_{-5}| - |A_{-6}|) + \ldots + (|A_{N-1}| - |A_N|)\]

and

\[
h(1) = - \sum_{n=1}^{N} |A_{-n}| + \sum_{n=0}^{M} |A_n|
\]

(29)
It is readily ascertained, from the tables of the spectral amplitudes, that all of the terms within brackets in the expression for \( h(-1) \) are positive. Consequently, the obvious conclusion is that \( h(1) \) is the minimum real value that \( h(z) \) can assume on the unit circle. If this minimum real value is positive, \( h(z) \) will never become negative real for \( |z| = 1 \), and hence the path \( C' \) traced by \( h(z) \) in the \( h(z) \)-plane (as \( z \) traces the unit circle in the \( z \)-plane), will never encircle the origin of the \( h(z) \)-plane. From Eq. 30, the condition for the minimum real part, \( h(1) \), to be positive is

\[
\sum_{n=0}^{M} |A_n| > \sum_{n=1}^{N} |A_{-n}|
\]

(26)

This is recognized as the inequality stated in Theorem II, and it completes the formal proof of this theorem.

We will next apply the criterion of Theorem II to the determination of the minimum permissible values of limiter bandwidth for the suppression of the interference to remain possible.
I.5 THE MINIMUM PERMISSIBLE LIMITER BANDWIDTHS

At the outset, we recognize that with a narrow-band filter whose bandwidth can at best be equal to, but never less than, the i-f bandwidth, the possible configurations of accomodated side-frequency components resolve themselves into three different situations. First, there is the limiting situation in which only $A_0$ and an arbitrary number of lower sideband components are passed, to the complete exclusion of all of the upper sideband components. A second limiting situation arises when it is the lower sideband components that the ideal filter will not pass. The third situation arises when some components from both sidebands are simultaneously passed (along with $A_0$, of course). It is needless to point out that the remarkable simplification in the approach that the use of the concept of ideal filters makes possible, will be best manifested by the analysis that we are about to unfold. For instance, with an ideal filter, we are able to draw sharp lines of demarcation between the three possible situations, and thus reduce our problem to three simpler problems. The results and experience are not only needed for the analysis of Section I.6, but they also serve as an invaluable guide to a clearer understanding of the nature of the problem, and to the selection of actual design figures.

Case A. Consider first the situation in which only an arbitrary number, $M$, of lower sideband components is passed, along with $A_0$, while all of the upper sideband components fall outside the passband. Since this situation is not fully covered by Theorem I, it calls for special treatment.

At $t = \pi/r$, all of the lower sideband components line up in phase, aiding $A_0$. Thus the resultant phasor can never be negative at this instant of time. At $t = 0$, we have

$$F(0) = \sum_{n=0}^{M} A_n$$

$$= (|A_0| - |A_1|) + (|A_2| - |A_3|) + (|A_4| - |A_5|)$$

$$+ \ldots + (|A_{M-1}| - |A_M|).$$
All the terms in parentheses on the right are positive numbers, and so \(F(0)\) is also always positive real. This completes the check for odd values of \(M\), since this case is covered by Theorem I. However, for even values of \(M\), we must investigate the positive-realness of \(F(t)\) at an additional instant in the cycle, given by \(rt = \phi_1\), where \(0 < \phi_1 < \pi\). Here

\[
F(\phi_1/r) = |A_o| \cdot \left[ 1 - |A_1/A_o| \cos \phi_1 + |A_2/A_o| \cos 2\phi_1 - \ldots 
- |A_M/A_o| \cos M\phi_1 \right]
\]

It is a simple matter to show that the coefficients in the finite series in brackets are dominated by the corresponding coefficients in the series

\[
z(\phi) = 1 - a\cos \phi + a^2\cos 2\phi - a^3\cos 3\phi + \ldots + (-a)^n\cos n\phi + \ldots + a^M\cos M\phi
= \sum_{n=0}^{M} (-a)^n \cos n\phi
\]

If \(\cos n\phi\) is replaced by its value in terms of complex exponentials, and the resulting finite geometric progressions are summed in the usual manner, \(z(\phi)\) may be expressed in the closed form (with \(a^2 < 1\))

\[
z(\phi) = \sum_{n=0}^{M} (-a)^n \cos n\phi
= \frac{1 + a\cos \phi + a^{M+1} \cos(M + 1)\phi + a^{M+2}\cos M\phi}{1 + 2a \cos \phi + a^2}
\]

Since \(a < 1\), it is evident that as \(M\) gets large

\[
z(\phi) \rightarrow \frac{1 + a \cos \phi}{1 + 2a \cos \phi + a^2}
\]

and this quantity can never go negative for any real value of \(\phi\). For the lower values of \(M\), a close examination of the numerator in the expression for \(z(\phi)\) reveals that \(z(\phi)\) can never go negative.

\[\text{Actually the following argument (which is similar to the argument of Section I.3) is independent of } \phi \text{ and of whether } M \text{ is even or odd. It, alone, can therefore be used to establish Theorem III A without the help of Theorem I.}\]
We conclude that at no instant of time will any arbitrary number of lower sideband components produce a resultant that opposes $A_0$ in phase and exceeds it in magnitude. Furthermore, this holds for all $a < 1$. It follows that

**Theorem III A.**

If only $A_0$ and an arbitrary number of lower sideband components fall within the ideal filter passband, the average frequency of the resultant signal at the output of the filter is still the frequency of the stronger signal.

Granlund (Ref. 1) proved this theorem in the following way.

Over a period of $2\pi/r$ sec, the quantity

$$F(t) = \sum_{n=0}^{M} A_n e^{-jnrt}$$

must add no phase shift to the resultant signal. To show that, in fact, it does not, let $z = e^{-jrt}$ and write

$$f(z) = \sum_{n=0}^{M} A_n z^n.$$ 

As $e^{-jrt}$ covers one cycle of variation over a period $2\pi/r$, $z$ traverses the unit circle in the $z$-plane in the clockwise direction. Since $f(z)$ has no poles within the unit circle, the net phase change that $f(z)$ sustains while $z$ traverses the unit circle is simply $2\pi$ times the number of zeros of $f(z)$ within the unit circle, each zero being counted in accordance with its multiplicity. But $f(z)$ has as many zeros within the unit circle as

$$f(-z) = \sum_{n=0}^{M} (-1)^n A_n z^n$$

$$= \sum_{n=0}^{M} |A_n| z^n.$$ 

The zeros of a polynomial of this type (characterized by
positive real coefficients that decrease with \( n \), according to Hurwitz (Ref. 11), lie within the annular ring

\[
\left[ \frac{|A_n|}{|A_{n+1}|} \right]_{\text{MIN.}} < |z| < \left[ \frac{|A_n|}{|A_{n+1}|} \right]_{\text{MAX.}}
\]

\( n = 0, 1, 2, \ldots, M - 1 \).

Since \( |A_n| \) decreases monotonically with \( n \), this ring lies outside the unit circle, and so \( f(-z) \) has no zeros within the unit circle. This completes the proof of Theorem III A.

**Case B.** Consider next the situation in which only an arbitrary number, \( N \), of upper sideband components is passed, along with \( A_0 \) only, to the complete exclusion of the lower sideband components. This case is adequately covered by Theorem I, and hence Theorem II is directly applicable. Thus, for the average frequency of the resultant to be equal to the frequency of \( A_0 \) (and hence for the interference suppression to remain possible) the inequality

\[
A_0 > \sum_{n=1}^{N} |A_{-n}|
\]  

must be satisfied. This conclusion can also be reached in the following interesting manner. As before, we require that

\[
F(t) = \sum_{n=0}^{N} A_{-n} e^{jnrt}
\]  

shall not introduce any net phase shift over a period of \( 2\pi/r \) sec. If, for convenience, we substitute \( +\pi/r \) for \( t \) to shift the time reference from \( t = 0 \) to \( t = \pi/r \), we can write

\[
H(\tau) = F(\tau + \pi/r) = A_0 - \sum_{n=1}^{N} |A_{-n}| e^{jnrt}.
\]
If we set \( z = e^{j\omega t} \), we get

\[
h(z) = A_0 - \sum_{n=1}^{N} |A_{-n}|z^n
\]

(33)

As before, \( h(z) \) will acquire a net phase shift, as \( z \) traverses the unit circle once, if and only if \( h(z) \) has zeros within the unit circle (it obviously has no poles there). Such zeros can exist only if, for \( |z| \leq 1 \), the right-hand side of Eq. 33 vanishes. Since the summation term is analytic within, and on, the unit circle in the \( z \)-plane, we have from the principle of the maximum modulus (Ref. 10) that this term assumes its maximum value of

\[
\sum_{n=1}^{N} |A_{-n}|
\]

on the circle itself. Therefore, if

\[
A_0 > \sum_{n=1}^{N} |A_{-n}|
\]

\( h(z) \) cannot have any zeros within the unit circle.

In view of the complexity of the expressions for the \( A_n \)'s, the criterion is best applied graphically. Figure 12(a) shows a plot of the sum of all the tabulated \( A_n \) amplitudes (essentially all those of significance) over the whole range of \( a \) available from the tabulated spectral amplitudes. Superimposed on this plot are plots of \( A_0(a) \) and of \( \sum_{n=0}^{M} |A_n(a)| \) for several values of \( M \). Figure 12(b) shows an enlarged view of the region of intersections in Fig. 12(a). From these plots it is evident that the magnitude of \( A_0 \) is greater than the sum of the magnitudes of (effectively) all of the upper sideband components for \( a \leq 0.863 \); whence we state

**Theorem III B.**

If only \( A_0 \) and an arbitrary number of upper sideband components fall within the ideal filter passband, then the average frequency of the resultant signal at the output
Fig. 12 (b)
of the filter will still be the frequency, \( p \), of the stronger signal for values of \( a \leq 0.863 \).

For \( a > 0.863 \), the average frequency of the resultant signal is \( p + r \), the frequency of the weaker (interfering) signal, if more than a few upper sideband components are passed. This is illustrated in Fig. 13 by a plot of the path traced by the end point of the resultant phasor over a period of \( 2\pi/r \) sec, for \( a = 0.95 \), when only \( A_0 \), \( A_{-1} \), and \( A_{-2} \) are passed. The encirclement of the origin, \( O \), by the traced path signifies a gain of \( 2\pi \) radians, over the phase of \( A_0 \), by the resultant signal every \( 2\pi/r \) sec. The resultant has, therefore, an average frequency of \( p + r \) radians/sec.

Figure 12(b) shows also plots of \( \sum_{n=1}^{N} |A_{-n}(a)| \) for various values of \( N \). The intersections of these plots with the plot for \( A_0(a) \) determines up to what value of \( a \) a certain number, \( N \), of upper sideband components may be passed (along with \( A_0 \) only) before the desired average frequency, \( p \), is lost.

We may conclude, therefore, that the bandwidth of the limiter need not exceed the bandwidth of the i-f section (for interference rejection to remain possible) for capture ratios up to \( a = 0.863 \). For values of \( a > 0.863 \), bandwidths greater than that of the i-f section are required.

The minimum permissible limiter bandwidths, for \( a > 0.863 \), may be determined as follows.

As before, let \( N \) be the number of upper sideband components passed, and \( M \) be the number of lower sideband components passed.

(a) Let the worst situation that must be handled satisfactorily be one in which \( M = 0 \) and \( N = N_{\text{max}} \). Clearly, this implies that the situation where \( N = N_{\text{max}} + 1 \) can only arise if \( M = 1 \) arises simultaneously.

(b) Determine the minimum ideal filter bandwidth, in units of one i-f bandwidth, for which the situation in step (a) is the limiting situation. This can most conveniently be done by first drawing a diagram like the one in Fig. 14 (drawn for \( N_{\text{max}} = 4 \)). It is evident from such a diagram
that for the situation \( N = N_{\text{max}} \), \( M = 0 \) to arise, the difference frequency, \( r \), should be greater than some value, \( r \), given by \( r = \frac{1}{N_{\text{max}}} \). For this value of \( r \), the situation \( N = N_{\text{max}} + 1 \), \( M = 1 \) arises, and so the limiter bandwidth should be

\[
(BW)_{\text{lim}} = r(M + N) = r(2 + N_{\text{max}}) = (BW)_{\text{if}} \left[ 1 + 2/N_{\text{max}} \right] \tag{34}
\]

Clearly, this is the minimum limiter bandwidth required here, since smaller values of bandwidth will allow situations in which \( N > N_{\text{max}} \) and \( M = 0 \) to arise, while larger values will have limiting configurations in which \( N < N_{\text{max}} \) and \( M = 0 \), the value \( N = N_{\text{max}} \) arising only along with some non-zero \( M \).

(c) Determine from Fig. 12(b) up to what value of \( a \) the inequality

\[
A_0 > \sum_{n=1}^{N_{\text{max}}} |A_n|
\]

is satisfied. Then up to this value of \( a \), the minimum required limiter bandwidth is that found in step (b).

Table I is a summary of the results of calculations (carried out as just outlined) which cover the requirements for the range \( 0.863 < a < 0.937 \). These results are also plotted in Figs. 15 and 16. The transition in the requirements from one range of values of \( a \) to the next takes place in steps. This may be justified in the following way. Let \( a = a_{\text{max}} \) mark the end of a range in which the requirement is set by the configuration \( M = 0 \), \( N = N_{\text{max}} \). This means that immediately beyond \( a = a_{\text{max}} \) the requirement is set by the situation \( M = 0 \), \( N = N_{\text{max}} - 1 \). Since the ideal filter response is such that it will either pass, or completely reject, a spectral component in the neighborhood of its
cut-off frequencies, the transition from one region to the next must occur in a step.

It is evident from the preceding results, that for \( a > 0.937 \) the bandwidth of the limiter must be so chosen that at least one, or more, lower sideband components are passed at all instants of time, regardless of the value that the difference frequency, \( r \), may have, if interference is to be suppressed.

Case 2. Consider, finally, the situation in which components from both sidebands fall within the ideal filter passband. Configurations falling in this category will evidently decide the minimum limiter bandwidth requirements in the range \( a > 0.937 \), and, here, the criterion of Theorem II applies directly.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( N_{\text{max}} )</th>
<th>( \min \frac{(BW)}{lim} )</th>
<th>Required ( a )</th>
<th>( a_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td></td>
<td>0.937</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>1.2/3</td>
<td></td>
<td>0.906(5)</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>1.5</td>
<td></td>
<td>0.891</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>1.4</td>
<td></td>
<td>0.882</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>1 1/3</td>
<td></td>
<td>0.877</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>1 2/7</td>
<td></td>
<td>0.873</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>1.25</td>
<td></td>
<td>0.870(5)</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
<td>1 2/9</td>
<td></td>
<td>0.868(6)</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>1.2</td>
<td></td>
<td>0.867</td>
</tr>
<tr>
<td>0</td>
<td>11</td>
<td>1 2/11</td>
<td></td>
<td>0.865(6)</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>1 1/6</td>
<td></td>
<td>0.865(2)</td>
</tr>
</tbody>
</table>
A rough indication of the relative numbers of upper and lower sideband components that must be accommodated, in limiting situations, in order to preserve the possibility of suppressing the interference, is brought out by Table II. From the criterion of Theorem II, it is clear that the interference rejection ability is enhanced by the presence (at all times) of some lower sideband components within the ideal filter passband. The ratios N/M, presented in Table II, suggest that for \( a \leq 0.98 \), interference can always be rejected if, in the worst possible situation, the ideal filter bandwidth is sufficient to accommodate about twice as many upper sideband components as lower sideband components. This rule of thumb is helpful only in the range \( 0.937 < a \leq 0.98 \), where M must always be different from zero for interference rejection.

For a more careful determination of the minimum requirements in limiter bandwidth for the range \( a > 0.937 \), we first extend the reasoning used in Case B to the present

<table>
<thead>
<tr>
<th>Capture ratio ( a )</th>
<th>Number of M.S. components passed=( M )</th>
<th>Max. permissible number of U.S. components=( N )</th>
<th>( N/M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>14</td>
<td>3.5</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>N arbitrary</td>
<td></td>
</tr>
</tbody>
</table>
situation with \( M \neq 0 \). Thus, if the worst configuration of passed components is to arise only with \( M = M_{\text{min}} \) and \( N = N_{\text{max}} \), it is clear that the bandwidth employed must be so chosen that \( N \) can assume the value \( N_{\text{max}} + 1 \) only if \( M = M_{\text{min}} + 1 \) will arise simultaneously with that value. As is readily evident from a diagram similar to the one in Fig. 14, if for the worst situation to arise the difference frequency, \( r \), must exceed a limiting value of \( r_m \) rad/sec, then the necessary bandwidth is given by

\[
\frac{(BW)_{\text{lim}}}{(BW)_{\text{if}}} = 2 \frac{r_m}{(BW)_{\text{if}}} (M_{\text{min}} + 1) + 1
\]

But

\[
r_m = \frac{r_m (M_{\text{min}} + 1) + (BW)_{\text{if}}}{N_{\text{max}} + 1}
\]

or

\[
\frac{r_m}{(BW)_{\text{if}}} = \frac{1}{N_{\text{max}} - M_{\text{min}}}
\]

Therefore

\[
\frac{(BW)_{\text{lim}}}{(BW)_{\text{if}}} = \frac{N_{\text{max}} + M_{\text{min}} + 2}{N_{\text{max}} - M_{\text{min}}}
\]

(35)

From Table II we find that the situations in which \( M_{\text{min}} = 1, N_{\text{max}} = 3 \) and \( M_{\text{min}} = 2, N_{\text{max}} = 5 \), are both limiting situations that the limiter filter must handle. Both configurations require that the limiter bandwidth be three times the i-f bandwidth, as may be verified by direct substitution in Eq. 35. It is also readily appreciated that this value of bandwidth is, in fact, the limiter bandwidth required to make it impossible for the configuration \( M = 0, N = 2 \) to arise in the range \( a > 0.937 \). We further notice from Fig. 12(b) that the plot of
\[
\sum_{n=0}^{1} |A_n|
\]
intersects the plot of
\[
\sum_{n=1}^{3} |A_{-n}|
\]
at \(a = 0.9807\), while the intersection of
\[
\sum_{n=0}^{2} |A_n|
\]
with
\[
\sum_{n=1}^{5} |A_{-n}|
\]
occurs at a higher value for \(a\). Thus, in the range \(0.937 < a \leq 0.9807\), the configuration \(M = 1, N = 3\) is the most critical one. The minimum limiter bandwidth required in this range is therefore set, by this situation, at three times the i-f bandwidth.

For values of \(a\) just exceeding \(a = 0.9807\), \(N = 3\) should only arise with \(M = 2\). This corresponds to a limiting situation described by \(M_{\text{min}} = 1, N_{\text{max}} = 2\), and so requires a minimum limiter bandwidth of five i-f bandwidths. However, a little check reveals that with a limiter bandwidth of \(5(BW)_{1f}\), \(M = 2, N = 3\) can arise only when the difference frequency \(r = 1\) i-f bandwidth, whereas \(M = 1, N = 2\) cannot arise at all since \(r\) would have to be larger than one i-f bandwidth. But the situation \(M = 2, N = 4\) can arise here, and it is even more serious than \(M = 2, N = 3\). Consequently, the upper limit on \(a\), for the present range, is defined by \(M = 2, N = 4\).

It should be borne in mind that Eq. 35 above will give the correct answer only when the values corresponding to the worst (or limiting) situation are substituted for
$M_{\text{min}}$ and $N_{\text{max}}$. On the basis of the argument leading to this equation, the "worst situation" is such that $N_{\text{max}} + 1$ must simultaneously be accompanied by $M_{\text{min}} + 1$; that is, it should be possible through an infinitesimal change in the value of the difference frequency, $r$, to restore the situation $M_{\text{min}}$, $N_{\text{max}}$. Thus, for the calculation of the required limiter bandwidth immediately above $a = 0.9807$, substituting $M_{\text{min}} = 2$, $N_{\text{max}} = 4$ in Eq. 35 does not yield the right answer, since this configuration does not satisfy the indicated criterion.

The results of the preceding calculations are presented graphically in Figs. 15 and 16.

![Graph](image-url)
Up to this point, we have been carrying out the discussion in terms of the case in which the weaker signal has the higher frequency, \( p + r \). The results can be easily carried over to the case in which the weaker signal has the lower frequency. There, \( r \) is simply replaced by \(-r\), and so the line spectra that appeared in the upper sideband previously, will now form the lower sideband, while those that appeared in the lower sideband will now constitute the upper sideband. The steps in the previous analysis may thus be retraced with the terms "upper" and "lower" interchanged throughout. Therefore, the swapping of either relative signal strengths, or
relative signal frequencies, will not affect the conclusions reached in connection with the limiter bandwidth requirements.

In fact, it is to take care of such an alternative situation that the limiter-filter and the i-f amplifier amplitude response characteristics have been drawn centered about the same frequency throughout the analysis (see Fig. 14, for instance), and no effort has been made to allow the limiter selectivity to discriminate between the two sidebands.

In either situation, however, the effect that each of the two sidebands will have on the loss or preservation of the desired average frequency, is easily distinguishable. The sideband that is on the same side as the weaker signal (relative to the stronger signal), will always contribute the components that will try to offset the average frequency in favor of the frequency of the weaker signal. A physical feeling for this conclusion may also be acquired through a better appreciation of the basic relationship between the the constituent spectral components and the characteristics of the resultant wave. The degree of correlation that seems to exist between the amplitude distribution of the components, on the frequency scale, and the instantaneous frequency pattern of the resultant wave, has already been pointed out in our discussion of the spectrum. Inasmuch as the spectral components are basically the "building blocks" of the resultant signal, the components that tend to pull the instantaneous frequency of the resultant signal toward the frequency of the weaker signal must logically be those that lie on the same side relative to the frequency \( p \) (of the stronger signal) as the frequency of the weaker signal. The components in the opposite sideband provide the balancing necessary (to aid the component \( A_0 \)) to preserve the desired average value, \( p \), of the frequency of the resultant signal.

The preceding study embodies the first step in a novel switch in the basic approach to the limiter selectivity
problem, and in the philosophy of the limiter’s share of
the task of interference suppression in FM. The more impres-
sive aspects of this new line of thinking will be amply
covered by later discussions.
At this point, it suffices to say that the discussion
of this section emphasizes the minimum basic requirement
that the limiter filter must satisfy; namely, in whatever
way it may alter the spectrum delivered to it by the limiter,
it must always preserve the desired average instantaneous
frequency in the resultant signal it delivers to the succeeding
stages, by accommodating proper portions of each sideband.
The reason that the large values of limiter bandwidth formerly
prescribed and tested (Ref. 1) have enabled high capture
ratios (better than 0.95) to be achieved, is thus primarily
attributable to the fact that such bandwidths will allow a
considerable number of components from the helpful sideband
to be present within the passband at all times. Consequently,
the bandwidth values specified by the formula

\[(BW)_{\text{lim}} = \frac{1 - \frac{a}{1 - a}}{r},\]  

although found helpful, are not necessary for interference
rejection. Equation 36 is plotted in Fig. 16 for comparison
with the results of our computations.

These computations also emphasize the fact that the
desired average frequency of the resultant signal may be
preserved without necessarily providing the bandwidth value
dictated by the extent of the frequency spikes. In fact,
as far as the limiter bandwidth is concerned, it will become
apparent, later on, that there is no special significance
to be attached to the bandwidth value specified by Eq. 36.
Such a value will even be found to fall short of satisfying
the conditions for a physical filter to be able to follow
the instantaneous frequency of the resultant signal through
quasi-stationary states. The basic condition that the
limiter bandwidth must satisfy is merely to be able, in the
worst situations possible, to pass portions of the sidebands
that will result in a signal whose average instantaneous
frequency deviation from the frequency, \( p \), of the stronger signal is zero, over a period of the difference frequency, \( r \). Since, in general, only a finite number of significant components will be passed, the resultant signal at the output of the limiter filter will exhibit instantaneous variations in amplitude, as well as frequency. If an amplitude-insensitive discriminator follows the filter, the conversion of the instantaneous frequency pattern (of the resultant signal delivered by the filter), into a variable direct voltage level, is then achieved. The discriminator characteristic must therefore be linear over a frequency band that is sufficiently wide to accommodate the instantaneous variations in frequency (above and below the level corresponding to the frequency \( p \)), in order to preserve the average direct voltage level at the output, at the value dictated by the frequency of the stronger signal.
I.6 DISCRIMINATOR BANDWIDTH REQUIREMENTS

In general, the resultant signal at the output of the limiter narrow-band filter will exhibit instantaneous frequency, as well as amplitude, variations. The resultant signal will be of constant amplitude only if it is the sum of all the spectral components on both sides of the central component $A_0$. When the limiter bandwidth used is such that the bulk of the components of significant amplitude are always passed, the instantaneous amplitude variations will usually be insignificant. However, in the narrow-band limiter case, situations where only a few significant components are passed are quite likely at all times, and, in a sense, the resultant signal will behave as one would in the presence of multisignal interference.

The narrow-band limiter case will thus, in general, call for a limiter stage following the narrow-band filter, when amplitude-sensitive frequency detectors are employed. Even if this second stage of limiting may theoretically require a very wide bandwidth in order to deliver a constant-amplitude signal to the discriminator, it is significant to note (in anticipation of our results) that the combination of one narrow-band limiter, followed by a relatively wide-band limiter, will still serve the desirable purpose of reducing, by a sizable amount, the minimum discriminator bandwidth required, in addition to protecting an amplitude-sensitive discriminator from variations in the resultant signal amplitude. If we then assume that an amplitude-insensitive discriminator is used, such a device will only respond to the instantaneous frequency variations of the resultant signal at its input, and convert those variations into a variable direct voltage level. For the average value of this voltage level to correspond to that dictated by the frequency of the stronger signal, the instantaneous frequency swings must be accommodated fully over a linear range of the FM-to-AM conversion.
characteristic. The discriminator bandwidth required will, thus, be dictated by the maximum swing in instantaneous frequency to be expected.

At this point, the question may be raised as to why the discriminator FM-to-AM conversion characteristic must be linear (over the range of the maximum instantaneous frequency variations) and not some other type of curve which might perhaps produce (in response to the sharp and large instantaneous frequency deviations caused by the interference) such a conveniently distorted replica of the undesirable frequency variations as will minimize the overall effects of the disturbance. Such a curved characteristic might be considerably less expensive to construct and maintain than a straight-line characteristic, and by smoothing out the sharper variations in the incoming instantaneous frequency pattern, might even ease up the fast-action requirements on the amplitude-detecting parts of the circuit. Such a question, though at first reasonable, really overlooks some fundamental considerations that enter into the mechanism of FM-to-AM conversion. Fundamentally, this conversion takes place when the amplitude-limited FM wave is impressed upon a filter whose amplitude-vs-frequency steady-state characteristic varies with frequency. The type of variation with frequency that this steady-state characteristic must exhibit is dictated by the basic requirement that (at least over the range of the frequency excursions caused by the expected message) the resulting amplitude variations be linearly related to the instantaneous frequency variations of the message-bearing signal for final undistorted reproduction of the message. Fortunately, the linear variation of the steady-state amplitude characteristic with frequency also insures that the filter will follow the impressed variable-frequency excitation through stationary states over the entire extent of the linearity and so, no distortion due to FM transients can arise.
Under interference conditions, the same considerations apply, and now the extent of the linearity in the steady-state amplitude-vs-frequency characteristic must encompass the range of the maximum frequency deviations that the circuit must handle. The necessity of this requirement may be appreciated from the fact that it is not possible to produce an amplitude response characteristic which (is linear within, and) has such nonlinearity outside the message modulation band that it will translate an arbitrarily situated (and perhaps arbitrarily distorted) frequency spike pattern, into an amplitude variation that (despite the further distortion in the non-linear region, and lack of it in the linear region) will still average out to zero (over one cycle of the difference frequency) about a value corresponding to the level dictated by the frequency of the stronger signal. Moreover, a nonlinear steady-state amplitude characteristic always leads to deviation from the steady-state amplitude response which are larger the greater the degree of non-linearity and the higher the repetition rate, rate of change, and extent, of the instantaneous frequency variations of the excitation. The result is increased disturbance and no capture. The only characteristic that will satisfy all of the fundamental performance requirements, is, therefore, one which varies linearly with frequency over the entire range of instantaneous frequency excursions that must be handled successfully.

As a first step toward the determination of the variation of the minimum requirements in discriminator bandwidth as a function of the limiter bandwidth used, we shall study the variation of the instantaneous frequency pattern of the resultant signal with the number of significant components passed from each sideband. It will be noted that with every prescribed value of limiter bandwidth we may associate certain possible configurations of passed components. For each configuration, the number of components passed (along with the desired component, \(A_0\)) from each
sideband will, generally, depend upon the value of limiter bandwidth used, the difference frequency between the two signals, and the positions of the signals relative to the center frequency of the i-f passband. In view of the insight gained, thus far, about the effect of each sideband upon the character of the resultant signal, it will be appreciated that, of all the different possible configurations, a few can always be spotted, by inspection, and expected to produce instantaneous frequency deviations (from the frequency of $A_o$) of such magnitude as to require greater discriminator bandwidths than the remaining legion of possibilities. Among those few cases, the one configuration that will impose the greatest requirement in discriminator bandwidth can then be determined by direct computation. As a consequence, our problem is thus reduced to one of spotting the most critical situation that may arise with every value of limiter bandwidth proposed, and stating the value of discriminator bandwidth dictated by this case as the one that is demanded by the particular limiter bandwidth considered.

Accordingly, we now consider an arbitrary configuration of passed components, and determine by direct computation how the magnitude of the greatest resultant instantaneous deviation (from the frequency of the stronger signal) is affected by the number of components passed from each sideband. This may be done as follows.

As before, if $N$ and $M$ upper and lower sideband components, respectively, are passed, then the resultant phasor, $\overline{OR}$, in Fig. 6, will be given by

$$\overline{OR} = \sum_{n=-N}^{M} A_n e^{-jnRT}$$

The instantaneous phase angle, $\theta$, that $\overline{OR}$ makes with $A_o$, will be
\[ \dot{\theta} = \text{Im} \left\{ \ln \sum_{n=-N}^{M} A_n e^{-jnrt} \right\} \]  

(37)

The time derivative of \( \theta \) will therefore be the instantaneous deviation, from the frequency of the stronger signal, that the resultant signal will experience. Plots of

\[ \frac{d\theta}{dt} = \frac{y}{M-N} \]

for various values of \( M \) and \( N \), and for values of \( \theta \) between 0.8 and 0.95, appear in Figs. 17(a) through 17(c).

It is clear from the properties of the interference spectrum (and the discussion of Section I.5), that the situations in which only lower sideband components pass along with \( A_0 \) present no serious discriminator bandwidth problem at all, since the maximum deviations in instantaneous frequency that they engender are of comparatively small size (see also Fig. 17(c)). The most serious situations arise when only upper sideband components are passed. The presence of components from the lower sideband simultaneously with upper sideband components within the ideal filter pass-band results in reduced frequency spike magnitudes. These conclusions are clearly illustrated by the plots of Fig. 17, and will presently be reinforced by the expression next to be derived for the spike magnitude.

The frequency spikes occur at \( t = \pi/r \) sec or any odd multiple thereof. From Eq. 37 we therefore have

\[ \frac{d\theta}{dt} = \text{Im} \left\{ \frac{\sum_{n=-N}^{M} (-jnrt)A_n e^{-jnrt}}{\sum_{n=-N}^{M} A_n e^{-jnrt}} \right\} \]

or

\[ \frac{d\theta}{dt} = \text{Re} \left\{ \frac{-r \sum_{n=-N}^{M} nA_n e^{-jnrt}}{\sum_{n=-N}^{M} A_n e^{-jnrt}} \right\} \]

(38)
Fig. 17(a)
FIG. 17(c)
whence, at \( t = \pi / r \), and from the properties displayed by
the \( A_n \)'s we can write
\[
- \frac{d[\omega]}{dt} \bigg|_{t=\pi/r} = -r \frac{\sum_{n=1}^{M} n|A_n| + \sum_{n=1}^{N} n|A_{-n}|}{\sum_{n=0}^{M} |A_n| - \sum_{n=1}^{N} |A_{-n}|}
\]
where \([\Delta \omega] = \text{magnitude of frequency spike}\). This expression demonstrates clearly the effect that the components from each sideband will have upon the magnitude of the frequency spike. We finally have
\[
\lambda_{M-N} = \frac{[\omega]}{r} = \frac{\sum_{n=1}^{M} n|A_n| + \sum_{n=1}^{N} n|A_{-n}|}{\sum_{n=0}^{M} |A_n| - \sum_{n=1}^{N} |A_{-n}|}
\]
(39)

Table I presents values of \([\Delta \omega]/r\) for the more serious configurations to be encountered in the course of the present investigation. Given the magnitude of the frequency spike that arises with a given configuration of components, the discriminator bandwidth necessary to accommodate the whole spike pattern can be calculated readily. Thus, with due consideration of the possibility of an interchange of the relative signal magnitudes, or frequencies, the required discriminator bandwidth is given by
\[
(BW)_{\text{disc}} = 2[\Delta \omega] + r
\]
\[
= r\left[2 \frac{[\Delta \omega]}{r} + 1\right]
\]
(40)
where \( r \) is again the difference frequency between the two signals, in rad/second. Equation 40 can be written in the more convenient form
\[
\frac{(BW)_{\text{disc}}}{(BW)_{\text{if}}} = \frac{r}{(BW)_{\text{if}}} \cdot \beta_{N} = \delta \cdot \beta_{N}
\]
(41)
dictates the requirement in discriminator bandwidth.

Finally, values from Tables II and III have been used in conjunction with Eq. 43 to construct Table IV. The checked entries in Table IV have then been taken as the minimum discriminator bandwidths required with the corresponding specified values of limiter bandwidth. The results, as read off the checked entries, have then been plotted as shown in Fig. 18.

Next, before embarking on a study of the plots of Fig. 18, let us consider the case in which the limiter bandwidth used is just one i-f bandwidth, as this case deserves a special treatment. It is clear that this case applies only for $a \leq 0.863$. Here the situations in which only upper sideband components are passed along with $A_o$ will impose the greatest requirements in discriminator bandwidth; so, only such cases need be considered. Accordingly, in order for $N$ upper sideband components to pass, the difference frequency, $r$, must have a maximum value of $r_m = (BW)_{1f}/N$. Values of $[\Delta \omega]/(BW)_{1f}$ showing how the effective frequency spike magnitude changes with $N$ are presented in Table V. Also presented in Table V is a column showing which value of $N$ requires the most bandwidth in the discriminator. The values of $(BW)_{disc}/(BW)_{1f}$ for values of $a$ between $a = 0.8$ and $a = 0.85$, plotted against $N$, are also shown in Fig. 19.

As for $a = 0.85$, Table V and Fig. 19 show that the discriminator bandwidth required rises with $N$, until it peaks, for $N = 7$, at the value of $5.76(BW)_{1f}$, and then starts to decline. This ultimate decline is due to the fact that, beyond $N = 7$, the decrease in the maximum $r$ with $N$ wrests control of the requirement from the increase of $[\Delta \omega]$ with $N$. It is also readily seen, from Table V and Fig. 19, that, for all $a$'s up to $a = 0.84$, the case in which only one upper sideband component is passed imposes the value of discriminator bandwidth required.
\( M^{\lambda-N} = [\Delta w]/r \)

<table>
<thead>
<tr>
<th>( a )</th>
<th>0 ( \lambda_-1 )</th>
<th>0 ( \lambda_-2 )</th>
<th>0 ( \lambda_-3 )</th>
<th>0 ( \lambda_-4 )</th>
<th>0 ( \lambda_-5 )</th>
<th>1 ( \lambda_-1 )</th>
<th>1 ( \lambda_-2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.77506</td>
<td>1.2512</td>
<td>1.5349</td>
<td>1.7058</td>
<td>1.8105</td>
<td>0.8587</td>
<td>1.1508</td>
</tr>
<tr>
<td>0.8</td>
<td>1.2051</td>
<td>2.3860</td>
<td>3.3855</td>
<td>4.1736</td>
<td>4.7738</td>
<td>1.1156</td>
<td>1.6730</td>
</tr>
<tr>
<td>0.85</td>
<td>1.6014</td>
<td>3.8929</td>
<td>6.7234</td>
<td>9.9066</td>
<td>13.240</td>
<td>1.3099</td>
<td>2.1395</td>
</tr>
<tr>
<td>0.9</td>
<td>2.3357</td>
<td>9.3558</td>
<td>58.44</td>
<td></td>
<td></td>
<td>1.6009</td>
<td>2.9883</td>
</tr>
<tr>
<td>0.95</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \lambda_-3 )</th>
<th>2 ( \lambda_-2 )</th>
<th>2 ( \lambda_-3 )</th>
<th>2 ( \lambda_-4 )</th>
<th>2 ( \lambda_-5 )</th>
<th>3 ( \lambda_-4 )</th>
<th>3 ( \lambda_-5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.3124</td>
<td>1.2943</td>
<td>1.4325</td>
<td>1.5120</td>
<td>1.5600</td>
<td>1.6469</td>
</tr>
<tr>
<td>0.8</td>
<td>2.0573</td>
<td>1.7472</td>
<td>2.0435</td>
<td>2.2460</td>
<td>2.3886</td>
<td>2.3509</td>
</tr>
<tr>
<td>0.85</td>
<td>2.8052</td>
<td>2.1024</td>
<td>2.5715</td>
<td>2.9265</td>
<td>3.1990</td>
<td>2.9392</td>
</tr>
<tr>
<td>0.9</td>
<td>4.3935</td>
<td>2.6715</td>
<td>3.5127</td>
<td>4.2437</td>
<td>4.6876</td>
<td>3.9654</td>
</tr>
<tr>
<td>0.95</td>
<td>10.375</td>
<td>3.8704</td>
<td>5.9510</td>
<td>8.3612</td>
<td>11.1185</td>
<td>6.5735</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \lambda_-5 )</th>
<th>4 ( \lambda_-6 )</th>
<th>5 ( \lambda_-6 )</th>
<th>5 ( \lambda_-7 )</th>
<th>6 ( \lambda_-7 )</th>
<th>6 ( \lambda_-8 )</th>
<th>7 ( \lambda_-8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.8146</td>
<td>1.8410</td>
<td>1.9435</td>
<td>1.9597</td>
<td>2.0419</td>
<td>2.0521</td>
</tr>
<tr>
<td>0.8</td>
<td>2.6108</td>
<td>2.6923</td>
<td>2.8305</td>
<td>2.8870</td>
<td>3.0176</td>
<td>3.0577</td>
</tr>
<tr>
<td>0.85</td>
<td>3.2594</td>
<td>3.4136</td>
<td>3.5413</td>
<td>3.6525</td>
<td>3.7909</td>
<td>3.8736</td>
</tr>
<tr>
<td>0.9</td>
<td>4.3685</td>
<td>4.7001</td>
<td>4.7353</td>
<td>4.9813</td>
<td>5.0663</td>
<td>5.2578</td>
</tr>
<tr>
<td>0.95</td>
<td>7.1312</td>
<td>8.1812</td>
<td>7.6419</td>
<td>8.4368</td>
<td>8.1165</td>
<td>8.7457</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \lambda_-9 )</th>
<th>8 ( \lambda_-9 )</th>
<th>8 ( \lambda_-10 )</th>
<th>9 ( \lambda_-10 )</th>
<th>9 ( \lambda_-11 )</th>
<th>10 ( \lambda_-11 )</th>
<th>10 ( \lambda_-12 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>2.1231</td>
<td>2.1726</td>
<td>2.1770</td>
<td>2.2148</td>
<td>2.2176</td>
<td>2.2461</td>
</tr>
<tr>
<td>0.8</td>
<td>3.2056</td>
<td>3.3113</td>
<td>3.3226</td>
<td>3.4252</td>
<td>3.4437</td>
<td>3.5215</td>
</tr>
<tr>
<td>0.85</td>
<td>4.0753</td>
<td>4.2090</td>
<td>4.2576</td>
<td>4.3836</td>
<td>4.4217</td>
<td>4.5385</td>
</tr>
<tr>
<td>0.9</td>
<td>5.5239</td>
<td>5.6538</td>
<td>5.7766</td>
<td>5.9140</td>
<td>6.0147</td>
<td>6.1547</td>
</tr>
</tbody>
</table>
$$M \beta_N = 2_M \lambda_N + 1$$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$0\beta_1$</th>
<th>$0\beta_2$</th>
<th>$0\beta_3$</th>
<th>$0\beta_4$</th>
<th>$0\beta_5$</th>
<th>$1\beta_1$</th>
<th>$1\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>2.5501</td>
<td>3.5024</td>
<td>4.0698</td>
<td>4.4116</td>
<td>4.6212</td>
<td>2.7134</td>
<td>3.3012</td>
</tr>
<tr>
<td>0.8</td>
<td>3.4102</td>
<td>5.7720</td>
<td>7.7710</td>
<td>9.3472</td>
<td>10.5475</td>
<td>3.2312</td>
<td>4.3460</td>
</tr>
<tr>
<td>0.85</td>
<td>4.2028</td>
<td>8.7858</td>
<td>14.4468</td>
<td>20.813</td>
<td>27.480</td>
<td>3.6198</td>
<td>5.2790</td>
</tr>
<tr>
<td>0.9</td>
<td>5.6714</td>
<td>19.7136</td>
<td>117.88</td>
<td></td>
<td></td>
<td>4.2018</td>
<td>6.9766</td>
</tr>
<tr>
<td>0.95</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>5.2610</td>
<td>11.454</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$</th>
<th>$1\beta_3$</th>
<th>$2\beta_2$</th>
<th>$2\beta_3$</th>
<th>$2\beta_4$</th>
<th>$2\beta_5$</th>
<th>$3\beta_4$</th>
<th>$3\beta_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>3.6248</td>
<td>3.5886</td>
<td>3.8650</td>
<td>4.0240</td>
<td>4.1200</td>
<td>4.2938</td>
<td>4.3826</td>
</tr>
<tr>
<td>0.8</td>
<td>5.1146</td>
<td>4.4944</td>
<td>5.0870</td>
<td>5.4920</td>
<td>5.7773</td>
<td>5.7018</td>
<td>5.9502</td>
</tr>
<tr>
<td>0.85</td>
<td>6.6104</td>
<td>5.2048</td>
<td>6.1430</td>
<td>6.8530</td>
<td>7.3980</td>
<td>6.8784</td>
<td>7.3274</td>
</tr>
<tr>
<td>0.9</td>
<td>9.7870</td>
<td>6.3430</td>
<td>8.0254</td>
<td>0.4874</td>
<td>10.7527</td>
<td>8.9308</td>
<td>9.6694</td>
</tr>
<tr>
<td>0.95</td>
<td>22.470</td>
<td>8.7408</td>
<td>12.902</td>
<td>17.7224</td>
<td>23.2370</td>
<td>14.147</td>
<td>17.126</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$</th>
<th>$4\beta_5$</th>
<th>$4\beta_6$</th>
<th>$5\beta_6$</th>
<th>$5\beta_7$</th>
<th>$6\beta_7$</th>
<th>$6\beta_8$</th>
<th>$7\beta_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>4.6292</td>
<td>4.6820</td>
<td>4.8870</td>
<td>4.9194</td>
<td>5.0838</td>
<td>5.1042</td>
<td>5.2330</td>
</tr>
<tr>
<td>0.8</td>
<td>6.2216</td>
<td>6.3846</td>
<td>6.6610</td>
<td>6.7740</td>
<td>7.0352</td>
<td>7.1154</td>
<td>7.3530</td>
</tr>
<tr>
<td>0.85</td>
<td>7.5188</td>
<td>7.8272</td>
<td>8.0826</td>
<td>8.3050</td>
<td>8.5818</td>
<td>8.7472</td>
<td>9.0246</td>
</tr>
<tr>
<td>0.9</td>
<td>9.7370</td>
<td>10.400</td>
<td>10.4666</td>
<td>10.9626</td>
<td>11.1326</td>
<td>11.5156</td>
<td>11.7442</td>
</tr>
<tr>
<td>0.95</td>
<td>15.262</td>
<td>17.362</td>
<td>16.284</td>
<td>17.874</td>
<td>17.233</td>
<td>18.491</td>
<td>18.123</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$</th>
<th>$7\beta_9$</th>
<th>$8\beta_9$</th>
<th>$8\beta_10$</th>
<th>$9\beta_10$</th>
<th>$9\beta_11$</th>
<th>$10\beta_11$</th>
<th>$10\beta_12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>5.2462</td>
<td>5.3452</td>
<td>5.3540</td>
<td>5.4296</td>
<td>5.4352</td>
<td>5.4922</td>
<td>5.4960</td>
</tr>
<tr>
<td>0.8</td>
<td>7.4112</td>
<td>7.5226</td>
<td>7.6652</td>
<td>7.8504</td>
<td>7.8874</td>
<td>8.0426</td>
<td>8.0664</td>
</tr>
</tbody>
</table>
$$\delta_m = \delta_{\text{max}} = r_m/(\text{BW})_1f$$

<table>
<thead>
<tr>
<th>(BW)$_{1f}$</th>
<th>M</th>
<th>N</th>
<th>$\delta_m$</th>
<th>(BW)$_{1f}$</th>
<th>M</th>
<th>N</th>
<th>$\delta_m$</th>
<th>(BW)$_{1f}$</th>
<th>M</th>
<th>N</th>
<th>$\delta_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0</td>
<td>2</td>
<td>0.6</td>
<td>0</td>
<td>5</td>
<td>0.24</td>
<td>0</td>
<td>3</td>
<td>0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0</td>
<td>2</td>
<td>0.6875</td>
<td>0</td>
<td>4</td>
<td>0.3125</td>
<td>0</td>
<td>3</td>
<td>0.4177</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.667</td>
<td>0</td>
<td>2</td>
<td>0.6875</td>
<td>0</td>
<td>3</td>
<td>0.444</td>
<td>0</td>
<td>3</td>
<td>0.4177</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0.75</td>
<td>1</td>
<td>3</td>
<td>0.5</td>
<td>1</td>
<td>3</td>
<td>0.5625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.25</td>
<td>0</td>
<td>2</td>
<td>0.8125</td>
<td>1</td>
<td>3</td>
<td>0.5417</td>
<td>1</td>
<td>3</td>
<td>0.5625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>0</td>
<td>2</td>
<td>0.875</td>
<td>1</td>
<td>3</td>
<td>0.583</td>
<td>1</td>
<td>3</td>
<td>0.5625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>0</td>
<td>2</td>
<td>0.95</td>
<td>1</td>
<td>3</td>
<td>0.633</td>
<td>1</td>
<td>3</td>
<td>0.5625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.667</td>
<td>2</td>
<td>4</td>
<td>0.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.75</td>
<td>2</td>
<td>4</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.833</td>
<td>2</td>
<td>4</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.9167</td>
<td>2</td>
<td>4</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>0.75</td>
<td>3</td>
<td>5</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>0.8125</td>
<td>3</td>
<td>5</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>0.875</td>
<td>3</td>
<td>5</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>0.9375</td>
<td>3</td>
<td>5</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0.8</td>
<td>4</td>
<td>6</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0.85</td>
<td>4</td>
<td>6</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0.9</td>
<td>4</td>
<td>6</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0.9</td>
<td>4</td>
<td>6</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0.95</td>
<td>4</td>
<td>6</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0.95</td>
<td>4</td>
<td>6</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>0.833</td>
<td>5</td>
<td>7</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>0.875</td>
<td>5</td>
<td>7</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>0.9167</td>
<td>5</td>
<td>7</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>0.958</td>
<td>5</td>
<td>7</td>
<td>0.791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>0.857</td>
<td>6</td>
<td>8</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.5</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>0.883</td>
<td>6</td>
<td>8</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>0.929</td>
<td>6</td>
<td>8</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>0.964</td>
<td>6</td>
<td>8</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>0.875</td>
<td>7</td>
<td>9</td>
<td>0.806</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.5</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>0.906</td>
<td>7</td>
<td>9</td>
<td>0.806</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>0.9375</td>
<td>7</td>
<td>9</td>
<td>0.833</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>0.969</td>
<td>7</td>
<td>9</td>
<td>0.861</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>8</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>0.869</td>
<td>8</td>
<td>10</td>
<td>0.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.5</td>
<td>7</td>
<td>8</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>0.917</td>
<td>8</td>
<td>10</td>
<td>0.825</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>8</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>0.944</td>
<td>8</td>
<td>10</td>
<td>0.85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.5</td>
<td>7</td>
<td>8</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>0.972</td>
<td>8</td>
<td>10</td>
<td>0.875</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>8</td>
<td>9</td>
<td>1</td>
<td>8</td>
<td>10</td>
<td>0.9</td>
<td>9</td>
<td>11</td>
<td>0.818</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.5</td>
<td>8</td>
<td>9</td>
<td>1</td>
<td>8</td>
<td>10</td>
<td>0.925</td>
<td>9</td>
<td>11</td>
<td>0.841</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>9</td>
<td>1</td>
<td>8</td>
<td>10</td>
<td>0.95</td>
<td>9</td>
<td>11</td>
<td>0.864</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.5</td>
<td>8</td>
<td>9</td>
<td>1</td>
<td>8</td>
<td>10</td>
<td>0.975</td>
<td>9</td>
<td>11</td>
<td>0.887</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>9</td>
<td>11</td>
<td>0.909</td>
<td>10</td>
<td>12</td>
<td>0.833</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19.5</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>9</td>
<td>11</td>
<td>0.932</td>
<td>10</td>
<td>12</td>
<td>0.854</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>9</td>
<td>11</td>
<td>0.954</td>
<td>10</td>
<td>12</td>
<td>0.875</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.5</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>9</td>
<td>11</td>
<td>0.977</td>
<td>10</td>
<td>12</td>
<td>0.896</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>11</td>
<td>1</td>
<td>10</td>
<td>12</td>
<td>0.916</td>
<td>11</td>
<td>13</td>
<td>0.846</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(BW)_{lim}</td>
<td>M</td>
<td>N</td>
<td>(BW)_{disc}</td>
<td>M</td>
<td>N</td>
<td>(BW)_{disc}</td>
<td>M</td>
<td>N</td>
<td>(BW)_{disc}</td>
<td>M</td>
<td>N</td>
</tr>
<tr>
<td>------------</td>
<td>----</td>
<td>----</td>
<td>-------------</td>
<td>----</td>
<td>----</td>
<td>-------------</td>
<td>----</td>
<td>----</td>
<td>-------------</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2.55012</td>
<td></td>
<td></td>
<td>1.3786</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0</td>
<td>2</td>
<td>2.1890</td>
<td>0</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>0</td>
<td>2</td>
<td>2.550</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2.6268</td>
<td>1</td>
<td>3</td>
<td>1.8124</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.25</td>
<td>0</td>
<td>2</td>
<td>2.8487</td>
<td>1</td>
<td>3</td>
<td>1.964</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>0</td>
<td>2</td>
<td>3.0646</td>
<td>1</td>
<td>3</td>
<td>2.11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>0</td>
<td>2</td>
<td>3.3273</td>
<td>1</td>
<td>3</td>
<td>2.29</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3.3012</td>
<td>1</td>
<td>3</td>
<td>2.42</td>
<td>2</td>
<td>4</td>
<td>2.0120</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>1</td>
<td>2</td>
<td>3.3012</td>
<td>1</td>
<td>3</td>
<td>2.7186</td>
<td>2</td>
<td>4</td>
<td>2.2635</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3.3012</td>
<td>1</td>
<td>3</td>
<td>3.02</td>
<td>2</td>
<td>4</td>
<td>2.5150</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.46</td>
<td>1</td>
<td>2</td>
<td>3.3012</td>
<td>1</td>
<td>3</td>
<td>3.3012</td>
<td>2</td>
<td>4</td>
<td>2.7665</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>1</td>
<td>2</td>
<td>3.3012</td>
<td>1</td>
<td>3</td>
<td>3.323</td>
<td>3</td>
<td>5</td>
<td>2.6206</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>3.8650</td>
<td>2</td>
<td>4</td>
<td>3.018</td>
<td>3</td>
<td>5</td>
<td>2.8487</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>2</td>
<td>3</td>
<td>3.8650</td>
<td>2</td>
<td>4</td>
<td>3.6295</td>
<td>3</td>
<td>5</td>
<td>3.0678</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>3.8650</td>
<td>2</td>
<td>4</td>
<td>3.7725</td>
<td>3</td>
<td>5</td>
<td>3.2870</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.684</td>
<td>2</td>
<td>3</td>
<td>3.8650</td>
<td>2</td>
<td>4</td>
<td>3.8650</td>
<td>3</td>
<td>5</td>
<td>3.5115</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>4.2938</td>
<td>3</td>
<td>5</td>
<td>3.5061</td>
<td>4</td>
<td>6</td>
<td>3.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>3</td>
<td>4</td>
<td>4.2938</td>
<td>3</td>
<td>5</td>
<td>3.7252</td>
<td>4</td>
<td>6</td>
<td>3.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>4.2938</td>
<td>3</td>
<td>5</td>
<td>3.9433</td>
<td>4</td>
<td>6</td>
<td>3.5115</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.5</td>
<td>3</td>
<td>4</td>
<td>4.2938</td>
<td>3</td>
<td>5</td>
<td>4.1635</td>
<td>4</td>
<td>6</td>
<td>3.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.8</td>
<td>3</td>
<td>4</td>
<td>4.2938</td>
<td>3</td>
<td>5</td>
<td>4.2938</td>
<td>4</td>
<td>6</td>
<td>3.90</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>5</td>
<td>4.6292</td>
<td>4</td>
<td>6</td>
<td>4.0968</td>
<td>5</td>
<td>7</td>
<td>3.6896</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>4.6292</td>
<td>4</td>
<td>6</td>
<td>4.292</td>
<td>5</td>
<td>7</td>
<td>3.865</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.5</td>
<td>4</td>
<td>5</td>
<td>4.6292</td>
<td>4</td>
<td>6</td>
<td>4.485</td>
<td>5</td>
<td>7</td>
<td>4.038</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.9</td>
<td>4</td>
<td>5</td>
<td>4.6292</td>
<td>4</td>
<td>6</td>
<td>4.6292</td>
<td>5</td>
<td>7</td>
<td>4.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>6</td>
<td>4.8870</td>
<td>5</td>
<td>7</td>
<td>4.22</td>
<td>6</td>
<td>8</td>
<td>3.83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.5</td>
<td>5</td>
<td>6</td>
<td>4.8870</td>
<td>5</td>
<td>7</td>
<td>4.39</td>
<td>6</td>
<td>8</td>
<td>3.9864</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>6</td>
<td>4.8870</td>
<td>5</td>
<td>7</td>
<td>4.57</td>
<td>6</td>
<td>8</td>
<td>4.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.5</td>
<td>5</td>
<td>6</td>
<td>4.8870</td>
<td>5</td>
<td>7</td>
<td>4.74</td>
<td>6</td>
<td>8</td>
<td>4.26</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.9</td>
<td>5</td>
<td>6</td>
<td>4.8870</td>
<td>5</td>
<td>7</td>
<td>4.8870</td>
<td>7</td>
<td>9</td>
<td>4.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>7</td>
<td>5.0838</td>
<td>6</td>
<td>8</td>
<td>4.4662</td>
<td>7</td>
<td>9</td>
<td>4.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.5</td>
<td>6</td>
<td>7</td>
<td>5.0838</td>
<td>6</td>
<td>8</td>
<td>4.62</td>
<td>7</td>
<td>9</td>
<td>4.37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>7</td>
<td>5.0838</td>
<td>6</td>
<td>8</td>
<td>4.7852</td>
<td>7</td>
<td>9</td>
<td>4.52</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.5</td>
<td>6</td>
<td>7</td>
<td>5.0838</td>
<td>6</td>
<td>8</td>
<td>4.95</td>
<td>7</td>
<td>9</td>
<td>4.81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>8</td>
<td>5.2330</td>
<td>7</td>
<td>9</td>
<td>4.66</td>
<td>8</td>
<td>10</td>
<td>4.2832</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.5</td>
<td>7</td>
<td>8</td>
<td>5.2330</td>
<td>7</td>
<td>9</td>
<td>4.81</td>
<td>8</td>
<td>10</td>
<td>4.4171</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>8</td>
<td>5.2330</td>
<td>7</td>
<td>9</td>
<td>4.95</td>
<td>8</td>
<td>10</td>
<td>4.5509</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.5</td>
<td>7</td>
<td>8</td>
<td>5.2330</td>
<td>7</td>
<td>9</td>
<td>5.10</td>
<td>8</td>
<td>10</td>
<td>4.6848</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>8</td>
<td>9</td>
<td>5.3452</td>
<td>8</td>
<td>10</td>
<td>4.8186</td>
<td>9</td>
<td>11</td>
<td>4.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.5</td>
<td>8</td>
<td>9</td>
<td>5.3452</td>
<td>8</td>
<td>10</td>
<td>4.9525</td>
<td>9</td>
<td>11</td>
<td>4.57</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>9</td>
<td>5.3452</td>
<td>8</td>
<td>10</td>
<td>5.0863</td>
<td>9</td>
<td>11</td>
<td>4.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.5</td>
<td>8</td>
<td>9</td>
<td>5.3452</td>
<td>8</td>
<td>10</td>
<td>5.2202</td>
<td>9</td>
<td>11</td>
<td>4.82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>10</td>
<td>5.4296</td>
<td>9</td>
<td>11</td>
<td>4.94</td>
<td>10</td>
<td>12</td>
<td>4.58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19.5</td>
<td>9</td>
<td>10</td>
<td>5.4296</td>
<td>9</td>
<td>11</td>
<td>5.07</td>
<td>10</td>
<td>12</td>
<td>4.69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>10</td>
<td>5.4296</td>
<td>9</td>
<td>11</td>
<td>5.19</td>
<td>10</td>
<td>12</td>
<td>4.8090</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.5</td>
<td>9</td>
<td>10</td>
<td>5.4296</td>
<td>9</td>
<td>11</td>
<td>5.31</td>
<td>10</td>
<td>12</td>
<td>4.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>11</td>
<td>5.4922</td>
<td>10</td>
<td>12</td>
<td>5.03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{(BW)<em>{lim}}{(BW)</em>{if}}$</td>
<td>M</td>
<td>N</td>
<td>$\frac{(BW)<em>{disc}}{(BW)</em>{if}}$</td>
<td>M</td>
<td>N</td>
<td>$\frac{(BW)<em>{disc}}{(BW)</em>{if}}$</td>
<td>M</td>
<td>N</td>
<td>$\frac{(BW)<em>{disc}}{(BW)</em>{if}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>-----</td>
<td>-----</td>
<td>-----------------</td>
<td>-----</td>
<td>-----</td>
<td>-----------------</td>
<td>-----</td>
<td>-----</td>
<td>-----------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3.4102</td>
<td>0</td>
<td>4</td>
<td>2.9210</td>
<td>1.5</td>
<td>2</td>
<td>3.6075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4.3290</td>
<td>1</td>
<td>3</td>
<td>2.5523</td>
<td>2.25</td>
<td>2</td>
<td>4.6898</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>0</td>
<td>2</td>
<td>5.0650</td>
<td>1</td>
<td>3</td>
<td>2.98</td>
<td>2.8</td>
<td>0</td>
<td>5.4834</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4.3460</td>
<td>1</td>
<td>3</td>
<td>3.41</td>
<td>3.5</td>
<td>1</td>
<td>4.3460</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>4.3460</td>
<td>1</td>
<td>3</td>
<td>3.8360</td>
<td>4</td>
<td>1</td>
<td>4.3460</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>4.3460</td>
<td>1</td>
<td>3</td>
<td>4.26</td>
<td>4.1</td>
<td>1</td>
<td>4.3460</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>4.3460</td>
<td>1</td>
<td>3</td>
<td>4.3460</td>
<td>4.5</td>
<td>1</td>
<td>4.3460</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>5.0870</td>
<td>2</td>
<td>4</td>
<td>4.1190</td>
<td>5.5</td>
<td>2</td>
<td>5.0870</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>5.0870</td>
<td>2</td>
<td>4</td>
<td>4.4623</td>
<td>6</td>
<td>2</td>
<td>5.0870</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>5.0870</td>
<td>2</td>
<td>4</td>
<td>4.8055</td>
<td>6.41</td>
<td>2</td>
<td>5.0870</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>2</td>
<td>3</td>
<td>5.0870</td>
<td>2</td>
<td>4</td>
<td>5.1488</td>
<td>7</td>
<td>3</td>
<td>5.7018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>5.7018</td>
<td>3</td>
<td>5</td>
<td>4.7601</td>
<td>7.5</td>
<td>3</td>
<td>5.7018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>5.4018</td>
<td>3</td>
<td>5</td>
<td>5.3552</td>
<td>8.5</td>
<td>3</td>
<td>5.7018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>4</td>
<td>5.7018</td>
<td>3</td>
<td>5</td>
<td>5.6527</td>
<td>6.58</td>
<td>3</td>
<td>5.7018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.5</td>
<td>4</td>
<td>5</td>
<td>6.2216</td>
<td>4</td>
<td>6</td>
<td>5.32</td>
<td>9.5</td>
<td>4</td>
<td>6.2216</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>6.2216</td>
<td>4</td>
<td>6</td>
<td>5.853</td>
<td>10.5</td>
<td>4</td>
<td>6.2216</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.69</td>
<td>4</td>
<td>5</td>
<td>6.2216</td>
<td>4</td>
<td>6</td>
<td>6.12</td>
<td>11</td>
<td>5</td>
<td>6.6610</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>6</td>
<td>6.6610</td>
<td>5</td>
<td>7</td>
<td>5.81</td>
<td>11.5</td>
<td>5</td>
<td>6.6610</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>6</td>
<td>6.6610</td>
<td>5</td>
<td>7</td>
<td>6.29</td>
<td>12.5</td>
<td>5</td>
<td>6.6610</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>6</td>
<td>6.6610</td>
<td>5</td>
<td>7</td>
<td>6.53</td>
<td>12.76</td>
<td>5</td>
<td>6.6610</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7</td>
<td>7.0352</td>
<td>6</td>
<td>8</td>
<td>6.2601</td>
<td>13.5</td>
<td>6</td>
<td>7.0352</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7</td>
<td>7.0352</td>
<td>6</td>
<td>8</td>
<td>6.45</td>
<td>14</td>
<td>6</td>
<td>7.0352</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.5</td>
<td>6</td>
<td>7</td>
<td>7.0352</td>
<td>6</td>
<td>8</td>
<td>6.89</td>
<td>14.8</td>
<td>6</td>
<td>7.0352</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>8</td>
<td>7.3530</td>
<td>7</td>
<td>9</td>
<td>6.59</td>
<td>15.5</td>
<td>7</td>
<td>7.3530</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>8</td>
<td>7.3530</td>
<td>7</td>
<td>9</td>
<td>7.00</td>
<td>16.5</td>
<td>7</td>
<td>7.3530</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>8</td>
<td>7.3530</td>
<td>7</td>
<td>9</td>
<td>7.20</td>
<td>16.86</td>
<td>7</td>
<td>7.3530</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>8</td>
<td>9</td>
<td>7.3530</td>
<td>7</td>
<td>9</td>
<td>7.3530</td>
<td>17</td>
<td>8</td>
<td>7.3530</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>9</td>
<td>7.6226</td>
<td>8</td>
<td>10</td>
<td>6.8987</td>
<td>18.5</td>
<td>8</td>
<td>7.6226</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>9</td>
<td>7.6226</td>
<td>8</td>
<td>10</td>
<td>7.0903</td>
<td>19</td>
<td>9</td>
<td>7.8504</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>10</td>
<td>7.8504</td>
<td>9</td>
<td>11</td>
<td>7.17</td>
<td>19.5</td>
<td>9</td>
<td>7.8504</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>10</td>
<td>7.8504</td>
<td>9</td>
<td>11</td>
<td>7.35</td>
<td>20.5</td>
<td>9</td>
<td>7.8504</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>11</td>
<td>8.0425</td>
<td>10</td>
<td>12</td>
<td>7.39</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE IV C

\( a = 0.85 \)

<table>
<thead>
<tr>
<th>(BW)_{lim}</th>
<th>M</th>
<th>N</th>
<th>(BW)_{disc}</th>
<th>M</th>
<th>N</th>
<th>(BW)_{disc}</th>
<th>M</th>
<th>N</th>
<th>(BW)_{disc}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(BW)_{1f}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>2</td>
<td>5.76</td>
<td>1</td>
<td>2</td>
<td>5.2790</td>
<td>2</td>
<td>3</td>
<td>5.0967</td>
</tr>
<tr>
<td>1.5</td>
<td>0</td>
<td>2</td>
<td>5.4911</td>
<td>0</td>
<td>4</td>
<td>5.6941</td>
<td>4</td>
<td>3</td>
<td>5.0075</td>
</tr>
<tr>
<td>1.567</td>
<td>0</td>
<td>2</td>
<td>5.86</td>
<td>0</td>
<td>3</td>
<td>6.41</td>
<td>1</td>
<td>3</td>
<td>3.85</td>
</tr>
<tr>
<td>2.5</td>
<td>0</td>
<td>2</td>
<td>6.5894</td>
<td>1</td>
<td>3</td>
<td>3.3052</td>
<td>1</td>
<td>3</td>
<td>3.85</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>5.4790</td>
<td>1</td>
<td>3</td>
<td>4.41</td>
<td>2</td>
<td>4</td>
<td>3.4265</td>
</tr>
<tr>
<td>3.5</td>
<td>1</td>
<td>2</td>
<td>5.2790</td>
<td>1</td>
<td>3</td>
<td>5.51</td>
<td>2</td>
<td>4</td>
<td>3.8548</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>5.2790</td>
<td>1</td>
<td>3</td>
<td>6.060</td>
<td>2</td>
<td>4</td>
<td>4.2831</td>
</tr>
<tr>
<td>4.5</td>
<td>1</td>
<td>2</td>
<td>5.2790</td>
<td>1</td>
<td>3</td>
<td>6.060</td>
<td>2</td>
<td>4</td>
<td>4.7114</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>6.1430</td>
<td>2</td>
<td>4</td>
<td>5.1398</td>
<td>3</td>
<td>5</td>
<td>4.3964</td>
</tr>
<tr>
<td>5.5</td>
<td>2</td>
<td>3</td>
<td>6.1430</td>
<td>2</td>
<td>4</td>
<td>5.5881</td>
<td>3</td>
<td>5</td>
<td>4.7628</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>6.1430</td>
<td>2</td>
<td>4</td>
<td>5.9964</td>
<td>3</td>
<td>5</td>
<td>5.1292</td>
</tr>
<tr>
<td>6.17</td>
<td>2</td>
<td>3</td>
<td>6.1430</td>
<td>2</td>
<td>4</td>
<td>6.1430</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>2</td>
<td>3</td>
<td>6.1430</td>
<td>2</td>
<td>4</td>
<td>6.4247</td>
<td>3</td>
<td>5</td>
<td>5.4956</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>6.4878</td>
<td>3</td>
<td>5</td>
<td>6.8619</td>
<td>4</td>
<td>6</td>
<td>5.22</td>
</tr>
<tr>
<td>7.5</td>
<td>3</td>
<td>4</td>
<td>6.8784</td>
<td>3</td>
<td>5</td>
<td>6.2283</td>
<td>4</td>
<td>6</td>
<td>5.54</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>6.8784</td>
<td>3</td>
<td>5</td>
<td>6.5947</td>
<td>4</td>
<td>6</td>
<td>5.8704</td>
</tr>
<tr>
<td>8.39</td>
<td>3</td>
<td>4</td>
<td>6.8784</td>
<td>3</td>
<td>5</td>
<td>6.8784</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.5</td>
<td>3</td>
<td>4</td>
<td>6.8784</td>
<td>3</td>
<td>5</td>
<td>6.9610</td>
<td>4</td>
<td>6</td>
<td>6.19</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>5</td>
<td>7.5188</td>
<td>4</td>
<td>6</td>
<td>6.52</td>
<td>5</td>
<td>7</td>
<td>5.93</td>
</tr>
<tr>
<td>9.5</td>
<td>4</td>
<td>5</td>
<td>7.5188</td>
<td>4</td>
<td>6</td>
<td>6.8468</td>
<td>5</td>
<td>7</td>
<td>6.2288</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>7.5188</td>
<td>4</td>
<td>6</td>
<td>7.175</td>
<td>5</td>
<td>7</td>
<td>6.525</td>
</tr>
<tr>
<td>10.5</td>
<td>4</td>
<td>5</td>
<td>7.5188</td>
<td>4</td>
<td>6</td>
<td>7.50</td>
<td>5</td>
<td>7</td>
<td>6.82</td>
</tr>
<tr>
<td>10.53</td>
<td>4</td>
<td>5</td>
<td>7.5188</td>
<td>4</td>
<td>6</td>
<td>7.5188</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>6</td>
<td>8.0826</td>
<td>5</td>
<td>7</td>
<td>7.12</td>
<td>6</td>
<td>8</td>
<td>6.5604</td>
</tr>
<tr>
<td>11.5</td>
<td>5</td>
<td>6</td>
<td>8.0826</td>
<td>5</td>
<td>7</td>
<td>7.42</td>
<td>6</td>
<td>8</td>
<td>6.83</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>6</td>
<td>8.0826</td>
<td>5</td>
<td>7</td>
<td>7.72</td>
<td>6</td>
<td>8</td>
<td>7.107</td>
</tr>
<tr>
<td>12.5</td>
<td>5</td>
<td>6</td>
<td>8.0826</td>
<td>5</td>
<td>7</td>
<td>8.01</td>
<td>6</td>
<td>8</td>
<td>7.30</td>
</tr>
<tr>
<td>12.62</td>
<td>5</td>
<td>6</td>
<td>8.0826</td>
<td>5</td>
<td>7</td>
<td>8.0826</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>7</td>
<td>8.5818</td>
<td>6</td>
<td>8</td>
<td>7.6538</td>
<td>7</td>
<td>9</td>
<td>7.12</td>
</tr>
<tr>
<td>13.5</td>
<td>6</td>
<td>7</td>
<td>8.5818</td>
<td>6</td>
<td>8</td>
<td>7.92</td>
<td>7</td>
<td>9</td>
<td>7.38</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>7</td>
<td>8.5818</td>
<td>6</td>
<td>8</td>
<td>8.2005</td>
<td>7</td>
<td>9</td>
<td>7.62</td>
</tr>
<tr>
<td>14.5</td>
<td>6</td>
<td>7</td>
<td>8.5818</td>
<td>6</td>
<td>8</td>
<td>8.48</td>
<td>7</td>
<td>9</td>
<td>7.88</td>
</tr>
<tr>
<td>14.7</td>
<td>6</td>
<td>7</td>
<td>8.5818</td>
<td>6</td>
<td>8</td>
<td>8.5818</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>8</td>
<td>9.0246</td>
<td>7</td>
<td>9</td>
<td>8.13</td>
<td>8</td>
<td>10</td>
<td>7.6122</td>
</tr>
<tr>
<td>15.5</td>
<td>7</td>
<td>8</td>
<td>9.0246</td>
<td>7</td>
<td>9</td>
<td>8.39</td>
<td>8</td>
<td>10</td>
<td>7.8500</td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>8</td>
<td>9.0246</td>
<td>7</td>
<td>9</td>
<td>8.64</td>
<td>8</td>
<td>10</td>
<td>8.0879</td>
</tr>
<tr>
<td>16.5</td>
<td>7</td>
<td>8</td>
<td>9.0246</td>
<td>7</td>
<td>9</td>
<td>8.89</td>
<td>8</td>
<td>10</td>
<td>8.3258</td>
</tr>
<tr>
<td>16.75</td>
<td>7</td>
<td>8</td>
<td>9.0246</td>
<td>7</td>
<td>9</td>
<td>9.0246</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>8</td>
<td>9</td>
<td>9.4180</td>
<td>8</td>
<td>10</td>
<td>8.6637</td>
<td>9</td>
<td>11</td>
<td>8.05</td>
</tr>
<tr>
<td>17.5</td>
<td>8</td>
<td>9</td>
<td>9.4180</td>
<td>8</td>
<td>10</td>
<td>8.8016</td>
<td>9</td>
<td>11</td>
<td>8.28</td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>9</td>
<td>9.4180</td>
<td>8</td>
<td>10</td>
<td>9.0394</td>
<td>9</td>
<td>11</td>
<td>8.50</td>
</tr>
<tr>
<td>18.5</td>
<td>8</td>
<td>9</td>
<td>9.4180</td>
<td>8</td>
<td>10</td>
<td>9.2773</td>
<td>9</td>
<td>11</td>
<td>8.73</td>
</tr>
<tr>
<td>18.80</td>
<td>8</td>
<td>9</td>
<td>9.4180</td>
<td>8</td>
<td>10</td>
<td>9.4130</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>10</td>
<td>9.7672</td>
<td>9</td>
<td>11</td>
<td>8.95</td>
<td>10</td>
<td>12</td>
<td>8.44</td>
</tr>
<tr>
<td>19.5</td>
<td>9</td>
<td>10</td>
<td>9.7672</td>
<td>9</td>
<td>11</td>
<td>9.17</td>
<td>10</td>
<td>12</td>
<td>8.66</td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>10</td>
<td>9.7672</td>
<td>9</td>
<td>11</td>
<td>9.39</td>
<td>10</td>
<td>12</td>
<td>8.8704</td>
</tr>
<tr>
<td>20.5</td>
<td>9</td>
<td>10</td>
<td>9.7672</td>
<td>9</td>
<td>11</td>
<td>9.62</td>
<td>10</td>
<td>12</td>
<td>9.08</td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>11</td>
<td>10.077</td>
<td>10</td>
<td>12</td>
<td>9.29</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### TABLE IV D

When \( a = 0.9 \)

<table>
<thead>
<tr>
<th>((BW)_{lim}^{-1}f)</th>
<th>M</th>
<th>N</th>
<th>((BW)_{disc}^{-1}f)</th>
<th>M</th>
<th>N</th>
<th>((BW)_{lim}^{-1}f)</th>
<th>M</th>
<th>N</th>
<th>((BW)_{disc}^{-1}f)</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.567</td>
<td>0</td>
<td>2</td>
<td>13.1</td>
<td>0</td>
<td>3</td>
<td>52.34</td>
<td>1</td>
<td>3</td>
<td>4.8935</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>14.7</td>
<td>1</td>
<td>3</td>
<td>5.71</td>
<td>2</td>
<td>4</td>
<td>4.7437</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2.5</td>
<td>1</td>
<td>2</td>
<td>6.9766</td>
<td>1</td>
<td>3</td>
<td>6.53</td>
<td>2</td>
<td>4</td>
<td>5.3367</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3.28</td>
<td>1</td>
<td>2</td>
<td>6.9766</td>
<td>1</td>
<td>3</td>
<td>6.9766</td>
<td>2</td>
<td>4</td>
<td>5.9296</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3.5</td>
<td>1</td>
<td>2</td>
<td>6.9766</td>
<td>1</td>
<td>3</td>
<td>7.34</td>
<td>2</td>
<td>4</td>
<td>6.5226</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>8.0254</td>
<td>2</td>
<td>4</td>
<td>7.1156</td>
<td>3</td>
<td>5</td>
<td>5.9216</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>8.0254</td>
<td>2</td>
<td>4</td>
<td>7.7085</td>
<td>3</td>
<td>5</td>
<td>6.4151</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>5.77</td>
<td>2</td>
<td>3</td>
<td>8.0254</td>
<td>2</td>
<td>4</td>
<td>8.0254</td>
<td>3</td>
<td>5</td>
<td>6.9086</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>8.0254</td>
<td>2</td>
<td>4</td>
<td>8.3015</td>
<td>3</td>
<td>5</td>
<td>7.4021</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>6.5</td>
<td>3</td>
<td>4</td>
<td>8.9308</td>
<td>3</td>
<td>5</td>
<td>7.8955</td>
<td>4</td>
<td>6</td>
<td>6.9022</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>8.9308</td>
<td>3</td>
<td>5</td>
<td>8.3890</td>
<td>4</td>
<td>6</td>
<td>7.36</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>7.5</td>
<td>3</td>
<td>4</td>
<td>8.9308</td>
<td>3</td>
<td>5</td>
<td>8.8825</td>
<td>4</td>
<td>6</td>
<td>7.8002</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>8.9308</td>
<td>3</td>
<td>5</td>
<td>9.3759</td>
<td>4</td>
<td>6</td>
<td>8.23</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>8.05</td>
<td>4</td>
<td>5</td>
<td>9.7370</td>
<td>4</td>
<td>6</td>
<td>9.7370</td>
<td>5</td>
<td>7</td>
<td>9.00</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>8.5</td>
<td>4</td>
<td>5</td>
<td>9.7370</td>
<td>4</td>
<td>6</td>
<td>9.96</td>
<td>5</td>
<td>7</td>
<td>9.90</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>6</td>
<td>10.467</td>
<td>5</td>
<td>7</td>
<td>9.39</td>
<td>6</td>
<td>8</td>
<td>8.6367</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>9.5</td>
<td>5</td>
<td>6</td>
<td>10.467</td>
<td>5</td>
<td>7</td>
<td>9.79</td>
<td>6</td>
<td>8</td>
<td>8.99</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>6</td>
<td>10.467</td>
<td>5</td>
<td>7</td>
<td>10.2</td>
<td>6</td>
<td>8</td>
<td>9.3564</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>10.24</td>
<td>5</td>
<td>6</td>
<td>10.467</td>
<td>5</td>
<td>7</td>
<td>10.467</td>
<td>6</td>
<td>8</td>
<td>9.60</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>10.5</td>
<td>5</td>
<td>6</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>10.076</td>
<td>7</td>
<td>9</td>
<td>9.37</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>10.433</td>
<td>7</td>
<td>9</td>
<td>9.71</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>11.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>10.796</td>
<td>7</td>
<td>9</td>
<td>10.0</td>
<td>20</td>
<td>21</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>11.133</td>
<td>7</td>
<td>9</td>
<td>10.4</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>12.37</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>11.2</td>
<td>7</td>
<td>9</td>
<td>10.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>11.4</td>
<td>7</td>
<td>9</td>
<td>10.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>11.7</td>
<td>7</td>
<td>9</td>
<td>10.984</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.46</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.54</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.61</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.5</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.66</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>6</td>
<td>7</td>
<td>11.133</td>
<td>6</td>
<td>8</td>
<td>12.308</td>
<td>8</td>
<td>10</td>
<td>12.308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(BW)_{1f}</td>
<td>M</td>
<td>N</td>
<td>(BW)_{disc}</td>
<td>M</td>
<td>N</td>
<td>(BW)_{disc}</td>
<td>M</td>
<td>N</td>
<td>(BW)_{disc}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>-----</td>
<td>-----</td>
<td>-------------</td>
<td>-----</td>
<td>-----</td>
<td>-------------</td>
<td>-----</td>
<td>-----</td>
<td>-------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>11.454</td>
<td>1</td>
<td>3</td>
<td>15.0</td>
<td>2</td>
<td>4</td>
<td>8.8612</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>1</td>
<td>2</td>
<td>11.454</td>
<td>1</td>
<td>3</td>
<td>16.853</td>
<td>2</td>
<td>4</td>
<td>9.9689</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>11.454</td>
<td>1</td>
<td>3</td>
<td>18.7</td>
<td>2</td>
<td>4</td>
<td>11.077</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>2</td>
<td>3</td>
<td>12.902</td>
<td>1</td>
<td>3</td>
<td>20.600</td>
<td>2</td>
<td>4</td>
<td>12.184</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>12.902</td>
<td>2</td>
<td>4</td>
<td>13.292</td>
<td>3</td>
<td>5</td>
<td>10.276</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>2</td>
<td>3</td>
<td>12.902</td>
<td>2</td>
<td>4</td>
<td>14.399</td>
<td>3</td>
<td>5</td>
<td>11.132</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>12.902</td>
<td>2</td>
<td>4</td>
<td>15.507</td>
<td>3</td>
<td>5</td>
<td>11.988</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>2</td>
<td>3</td>
<td>12.902</td>
<td>2</td>
<td>4</td>
<td>16.615</td>
<td>3</td>
<td>5</td>
<td>12.845</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>14.147</td>
<td>3</td>
<td>5</td>
<td>13.701</td>
<td>4</td>
<td>6</td>
<td>11.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.26</td>
<td>3</td>
<td>4</td>
<td>14.147</td>
<td>3</td>
<td>5</td>
<td>14.147</td>
<td>4</td>
<td>6</td>
<td>12.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>3</td>
<td>4</td>
<td>14.147</td>
<td>3</td>
<td>5</td>
<td>14.557</td>
<td>4</td>
<td>6</td>
<td>13.022</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>14.147</td>
<td>3</td>
<td>5</td>
<td>15.413</td>
<td>4</td>
<td>6</td>
<td>13.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.5</td>
<td>3</td>
<td>4</td>
<td>14.147</td>
<td>3</td>
<td>5</td>
<td>16.270</td>
<td>4</td>
<td>6</td>
<td>13.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>5</td>
<td>15.262</td>
<td>4</td>
<td>6</td>
<td>14.5</td>
<td>5</td>
<td>7</td>
<td>12.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.5</td>
<td>4</td>
<td>5</td>
<td>15.262</td>
<td>4</td>
<td>6</td>
<td>15.192</td>
<td>5</td>
<td>7</td>
<td>14.046</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>15.262</td>
<td>4</td>
<td>6</td>
<td>15.92</td>
<td>5</td>
<td>7</td>
<td>14.044</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.5</td>
<td>4</td>
<td>5</td>
<td>15.262</td>
<td>4</td>
<td>6</td>
<td>16.6</td>
<td>5</td>
<td>7</td>
<td>14.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>6</td>
<td>16.284</td>
<td>5</td>
<td>7</td>
<td>15.3</td>
<td>6</td>
<td>8</td>
<td>13.868</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.5</td>
<td>5</td>
<td>6</td>
<td>16.284</td>
<td>5</td>
<td>7</td>
<td>16.0</td>
<td>6</td>
<td>8</td>
<td>14.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.76</td>
<td>5</td>
<td>6</td>
<td>16.284</td>
<td>5</td>
<td>7</td>
<td>16.284</td>
<td>6</td>
<td>8</td>
<td>15.024</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>6</td>
<td>16.284</td>
<td>5</td>
<td>7</td>
<td>16.6</td>
<td>6</td>
<td>8</td>
<td>15.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.5</td>
<td>5</td>
<td>6</td>
<td>16.284</td>
<td>5</td>
<td>7</td>
<td>17.2</td>
<td>6</td>
<td>8</td>
<td>15.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>7</td>
<td>17.233</td>
<td>6</td>
<td>8</td>
<td>16.180</td>
<td>7</td>
<td>9</td>
<td>14.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.5</td>
<td>6</td>
<td>7</td>
<td>17.233</td>
<td>6</td>
<td>8</td>
<td>16.8</td>
<td>7</td>
<td>9</td>
<td>15.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.91</td>
<td>6</td>
<td>7</td>
<td>17.233</td>
<td>6</td>
<td>8</td>
<td>17.233</td>
<td>7</td>
<td>9</td>
<td>16.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>7</td>
<td>17.233</td>
<td>6</td>
<td>8</td>
<td>17.335</td>
<td>7</td>
<td>9</td>
<td>16.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.5</td>
<td>6</td>
<td>7</td>
<td>17.233</td>
<td>6</td>
<td>8</td>
<td>17.9</td>
<td>7</td>
<td>9</td>
<td>16.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>8</td>
<td>18.123</td>
<td>7</td>
<td>9</td>
<td>17.0</td>
<td>8</td>
<td>10</td>
<td>15.855</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.5</td>
<td>7</td>
<td>8</td>
<td>18.123</td>
<td>7</td>
<td>9</td>
<td>17.6</td>
<td>8</td>
<td>10</td>
<td>16.351</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>8</td>
<td>18.123</td>
<td>7</td>
<td>9</td>
<td>18.1</td>
<td>8</td>
<td>10</td>
<td>16.846</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.04</td>
<td>7</td>
<td>8</td>
<td>18.123</td>
<td>7</td>
<td>9</td>
<td>18.123</td>
<td>8</td>
<td>10</td>
<td>17.342</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.5</td>
<td>7</td>
<td>8</td>
<td>18.123</td>
<td>7</td>
<td>9</td>
<td>18.6</td>
<td>8</td>
<td>10</td>
<td>17.342</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>8</td>
<td>9</td>
<td>18.963</td>
<td>8</td>
<td>10</td>
<td>17.837</td>
<td>9</td>
<td>11</td>
<td>16.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.5</td>
<td>8</td>
<td>9</td>
<td>18.963</td>
<td>8</td>
<td>10</td>
<td>18.333</td>
<td>9</td>
<td>11</td>
<td>17.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>9</td>
<td>18.963</td>
<td>8</td>
<td>10</td>
<td>18.828</td>
<td>9</td>
<td>11</td>
<td>17.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.14</td>
<td>8</td>
<td>9</td>
<td>18.963</td>
<td>8</td>
<td>10</td>
<td>19.324</td>
<td>9</td>
<td>11</td>
<td>18.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.5</td>
<td>8</td>
<td>9</td>
<td>18.963</td>
<td>8</td>
<td>10</td>
<td>19.863</td>
<td>9</td>
<td>11</td>
<td>18.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>10</td>
<td>19.759</td>
<td>9</td>
<td>11</td>
<td>18.5</td>
<td>10</td>
<td>12</td>
<td>17.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19.5</td>
<td>9</td>
<td>10</td>
<td>19.759</td>
<td>9</td>
<td>11</td>
<td>19.1</td>
<td>10</td>
<td>12</td>
<td>18.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>10</td>
<td>19.759</td>
<td>9</td>
<td>11</td>
<td>19.5</td>
<td>10</td>
<td>12</td>
<td>18.499</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.22</td>
<td>9</td>
<td>10</td>
<td>19.759</td>
<td>9</td>
<td>11</td>
<td>19.759</td>
<td>10</td>
<td>12</td>
<td>18.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>11</td>
<td>20.517</td>
<td>10</td>
<td>12</td>
<td>19.4</td>
<td>11</td>
<td>12</td>
<td>18.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The fact that, for all $a$'s up to $a = 0.84$, the bandwidth requirement is set by the case where only $A_0$ and $A_{-1}$ are passed, allows some interesting observations to be made. The observation was first made by Granlund (Ref. 1) that the tables of spectral amplitudes (reproduced in Section I.2) show a ratio, $A_{-1}/A_0$, always less than the corresponding $a$. In fact, for $a < 0.5$, the plot of $A_{-1}/A_0$ vs. $a$, shown in Fig. 20, shows that

$$\frac{A_{-1}(a)}{A_0(a)} \approx (1/2)a,$$

and for other values of $a$, $A_{-1}/A_0$ is always less than $a$, and approaches $a$ only as $a \to 1$. This latter fact may also be directly confirmed from an analysis in terms of the expressions presented in Section I.2. Thus, if the limiter bandwidth is such that only $A_0$ and $A_{-1}$ are passed in the worst situation (i.e. the situation that dictates the required discriminator bandwidth), then under the most critical condition, the resultant output will still be a superposition of two signals.
having the frequencies of the input signals, but now the ratio of weaker-to-stronger signal is lower than before. This means that a reduction of the effective \( a \) has been achieved through the process. Consequently, if the process of limiting and filtering (with one i-f bandwidth) is repeated often enough, it should be possible to reduce the relative strength of the interfering signal to a negligible value and, with it, reduce the required discriminator bandwidth to the value of one intermediate-frequency bandwidth.

The condition for the success of the cascading scheme, as just outlined, is seen to hinge upon the requirement that the configuration in which only \( A_0 \) and \( A_{-1} \) are passed must be the most serious one that can arise. From Fig. 19, we note that the configuration \( A_0, A_{-1} \) is the most critical one for all \( a \)'s up to approximately \( a = 0.84 \). Also, according to the results of Section I.5, with a limiter bandwidth value of one i-f bandwidth, the configuration \( A_0, A_{-1} \) will remain

\[ \frac{(BW)_{disc}}{(BW)_{if}} \]

\[ (BW)_{lim} = (BW)_{if} \]

\( N = \text{Number of Upper Sideband Components Passed} \)

\[ a = 0.85 \]

\( a = 0.84 \)

\( a = 0.83 \)

\( a = 0.82 \)

\( a = 0.8 \)

\[ \text{FIG. 19} \]
Number of limiter-filter stages

| stages | 1 | 2 | 3 | 4 | 5 | 6 | "infinitely" wide-band limiter |
|---------------------------------|

(BW)_{disc}/(BW)_{if} required for

<table>
<thead>
<tr>
<th>a = 0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.41</td>
</tr>
</tbody>
</table>

the most critical one for all values of \( a < 1 \) so long as the difference frequency, \( r \), is greater than one-half the i-f bandwidth. When \( r \) takes values that are equal to, or less than, one-half the i-f bandwidth, more than one upper sideband component will pass through, and (for \( a \) greater than 0.84) either the average frequency of the resultant signal at the output of the limiter filter will not always be equal to that of the stronger signal, or the most serious situation involves more upper sideband components than just \( A_{-1} \).

In accordance with the above scheme, the table presented above shows how the ratio \( (BW)_{disc}/(BW)_{if} \), required for \( a = 0.8 \), decreases with the number of cascaded limiter-filter stages. These numbers are also plotted in Fig. 21, where a similar plot for \( a = 0.7 \) is also shown.

Returning to the plots of Fig. 18, we note that two important observations are clearly brought out, which we shall now discuss in detail.

First, it is noticed that the minimum requirements on the discriminator bandwidth are always less than, but approach asymptotically, the values previously specified (Ref. 1) by the formula

\[
\frac{(BW)_{disc}}{(BW)_{if}} = \frac{1 + a}{1 - a} \tag{44}
\]

That in the limit, as the limiter bandwidth becomes very large, the minimum discriminator bandwidth required should approach the asymptotes shown dotted in Fig. 18, and specified by Eq. 44, is indeed perfectly plausible. For as the limiter bandwidth becomes very large, essentially all of the significant sideband components are passed, and the resultant
signal at the output of the limiter filter approaches the amplitude-limited value of the resultant of the two signals delivered by the intermediate-frequency amplifier to the ideal limiter. Since the ideal limiter action per se does not affect the instantaneous variations in the frequency of the resultant signal, the values specified by Eq. 44 become the limiting values approached as the limiter bandwidth becomes very large.

Associated with each of the broken curves of Fig. 18 is a smooth curve that may be passed through the values of \((BW)_{\text{disc}}/(BW)_{\text{if}}\) required at the odd integral values of \((BW)_{\text{lim}}/(BW)_{\text{if}}\) used. We will call these smooth curves the "envelope" curves of the broken-line plots in Fig. 18, since the latter tend to be bounded by these smooth curves as the values of the limiter bandwidth used grow large. Figure 22 shows the envelope curves superimposed upon the curves of Fig. 18 for illustration.

We can readily show that the envelope curves are rising exponentials. This exponential character of the curves is demonstrated very clearly by the semilogarithmic plots of Fig. 23. In this figure, the values of the deviation of each of the envelope curves from the corresponding asymptotic value for the curve, are plotted on a logarithmic scale against the values of limiter bandwidth for which they occur, measured on a linear scale. The accuracy with which the plotted points fall on straight lines is a curious check on our calculations. This curious and interesting coincidence enables us to develop a simple analytical expression relating close estimates (which are accurate only for odd integral values of \((BW)_{\text{lim}}/(BW)_{\text{if}}\) of the minimum discriminator bandwidth) required to follow prescribed values of the limiter bandwidth.

The straight lines of Fig. 23 have equations of the form

\[
\ln \left[ \frac{1 + a}{1 - a} - y(a, x) \right] = -k(a)x + \ln B(a)
\]

where

\[
x \equiv \frac{(BW)_{\text{lim}}}{(BW)_{\text{if}}}
\]
\[ y(a,x) \equiv \text{envelope value of (BW)_{dis}/(BW)_{if}} \]
\[ = \text{value of (BW)_{dis}/(BW)_{if} at odd integral values of } x, \]
\[ -k(a) = \text{slope of the straight line considered, and} \]
\[ \ln B(a) = \text{the vertical intercept of the straight line.} \]

Equation 45 may take the more convenient form
\[
y(a,x) = \frac{1 + a}{1 - a} - B(a)e^{-k(a)x} \]
\[ = \frac{1}{1 + a} \left[ 1 - \zeta(a)e^{-k(a)x} \right] \quad (46)\]

Equation 46 is the desired analytical expression for the envelope curves. Calculated values of the functions \( k(a) \) and \( \zeta(a) \)
appear in Table VI and are plotted in Figs. 24(a) and 24(b). For the values of \( a \) of interest, \( k(a) \) appears to satisfy the approximate expression
\[ k(a) = -0.395 \ln a \quad (47) \]
very closely, as revealed by Table VI and Fig. 24(a).

The values of \( \zeta(a) \) shown in this table (and plotted in Fig. 24(b)) are based on \( k(a) \) as given by Eq. 47, with the reasonable assumption that the small deviations in the computed values of \( k(a) \) from the values given by Eq. 47 may be attributed to small cumulative errors in the computations. Fig. 24(b) and Table VI show that the plotted values of \( \zeta(a) \) fit into a straight line given by
\[ \zeta(a) = 0.30a + 0.44 \quad (48) \]
rather closely.

**TABLE VI**

<table>
<thead>
<tr>
<th>( a )</th>
<th>( k(a) )</th>
<th>( 0.395 \ln a )</th>
<th>( \zeta(a) )</th>
<th>( 0.30a + 0.440 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.1401</td>
<td>0.1409</td>
<td>0.6500</td>
<td>0.650</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0879</td>
<td>0.0881</td>
<td>0.6791</td>
<td>0.680</td>
</tr>
<tr>
<td>0.85</td>
<td>0.0630</td>
<td>0.0642</td>
<td>0.6925</td>
<td>0.695</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0409</td>
<td>0.0416</td>
<td>0.7093</td>
<td>0.710</td>
</tr>
<tr>
<td>0.95</td>
<td>0.0210</td>
<td>0.0203</td>
<td>0.7345</td>
<td>0.725</td>
</tr>
</tbody>
</table>
Fig. 23
$k(a) = -0.395 \ln a$

$\zeta(a) = 0.300 + 0.44$

**Fig 24**
Equation 46 may be normalized into the form

$$\psi(a) = y(a,x) / \left[ \frac{1 + a}{1 - a} \right]$$

$$= 1 - \zeta(a)e^{-k(a)x} \quad (46 \ a)$$

which is plotted in Fig. 25. The second term on the right, in Eq. 46(a), gives the fractional amount by which the minimum required discriminator bandwidth has been reduced by passing the resultant two-path signal through an ideal limiter whose bandwidth is an odd integral multiple, \( x \), of the bandwidth of the intermediate frequency section.

In addition to the light it throws upon our results, Eq. 46 may be used to supplement the plots of Fig. 18 to extrapolate the correct values of \( y(a,x) \) at odd integral values of \( x \) lying beyond the range covered by the plots, and approximations to \( y \) at other values of \( x \). It could also be safely used for the same purpose if \( a \) is desired to assume values, ranging between 0.7 and 0.95, that are not covered by the plots of Fig. 18. Moreover, the degree of approximation with which Eqs. 47 and 48 seem to describe \( k(a) \) and \( \zeta(a) \) in the range \( 0.7 \leq a \leq 0.95 \), would seem to indicate that these expressions should also be useful for lower values of \( a \).

Now, at the odd integral values of \( x \), it is recalled that if \( [\Delta \omega] \) is the magnitude of the maximum deviation in the instantaneous frequency of the resultant signal (at the output of the ideal limiter filter), from the frequency of the stronger signal, then

$$y = \frac{(BW)_{\text{diso}}}{(BW)_{\text{1f}}} = 2 \frac{[\Delta \omega]}{(BW)_{\text{1f}}} + 1 \quad (49)$$

since, at those values of \( x \), \( \delta_m = r_m / (BW)_{\text{1f}} \), in Eq. 43, is unity. Thus, an expression for the magnitude of the frequency spikes that dictate the minimum required discriminator bandwidth at the odd integral values of \( x \) may be obtained.
by combining Eqs. 46 and 49. The result is

\[
\frac{[\Delta w]}{(BW)_{1f}} = \frac{1}{2} (y - 1) \\
= \frac{a}{1 - a} - \frac{1}{2}\left(\frac{1 + a}{1 - a}\right) \zeta(a)e^{-k(a)x}
\]  

Further normalization yields

\[
[\Delta \Omega] = \frac{\Delta w}{(BW)_{1f}} \cdot \frac{1 - a}{a} \\
= 1 - \frac{1 + a}{2a} \zeta(a)e^{-k(a)x}
\]  

Here, the second term on the right is the fractional amount by which the magnitude of the maximum deviation in the instantaneous frequency has been reduced, or "damped", by passing the resultant two-path signal through an ideal limiter whose bandwidth is an odd integer multiple, \(x\), of the bandwidth of the intermediate frequency. It is clear, also, that the same value of \([\Delta w]\) calculated at a particular odd integer \(x\) will also hold for the range of larger values of \(x\) covered by the horizontal segment that follows the particular odd \(x\) considered, in the plots of Fig. 18, since \(\delta_m = 1\) in those ranges too.

The second important observation brought to light by the plots of Fig. 18 is that cascading alternate stages of limiting and filtering should, in fact, reduce the requirement in discriminator bandwidth to smaller and smaller values. The calculations have considered the action of one stage of ideal limiting and filtering upon the resultant of two signals, but the results are indicative of the action of the same device when more than two sinusoids are present at its input. Clearly, if it were possible to do without a stage of amplitude limiting (as when an amplitude-insensitive discriminator is used), the requirements in discriminator bandwidth would be dictated by the ratio
(1 + a)/(1 - a), and those requirements would be the asymptotic values in the plots of Fig. 18. The reduction in requirement achieved by the action of the first stage of limiting and filtering upon the resultant of two sinusoids delivered to it by the intermediate-frequency amplifier would conceivably be duplicated (though, possibly, to a varying degree) by the action of the second stage of limiting and filtering upon the resultant of more than two sinusoids delivered to it, in turn, by the first stage, and so on. With enough cascaded stages, then, it should be possible to reduce the necessary discriminator bandwidth to that of the intermediate-frequency section. This important question will be pursued further, and in the necessary detail it deserves, in the next chapter.

Viewed in an alternative way, the plots of Fig. 18 show that the effective magnitude of the instantaneous frequency spike has been reduced (or the spike train has been effectively "damped") by passing the resultant signal through the limiter-filter stage. This "damping" action would conceivably be duplicated, (though, possibly, to a different extent) by further stages of bandpass limiting that may follow the first stage, on the spike train associated with the resultant signal appearing at the output of this first stage. Hence the justification of the cascading scheme. This viewpoint will also receive further attention later on.

We may now conclude that the function of a bandpass limiter, in frequency-modulation receiver design, may be viewed in a new light. For, in addition to eliminating interference and noise coming in as amplitude perturbations in the resultant signal, a bandpass limiter is seen to be effective in relaxing the bandwidth requirements on the frequency discriminator, to achieve rejection of frequency-modulation interference produced by signals that may approach
the desired carrier in strength. Furthermore, a sufficiently long chain of such limiters, each in turn reducing the effective interference in the resultant signal delivered to it, would, at least in theory, enable us to eliminate the interference completely, as the detailed discussion of this scheme will show in the next chapter.
REFERENCES


8. Ibid: Page 331, Nos. 93 and 94.


CHAPTER TWO

THE EFFECT OF CASCADING NARROW-BAND LIMITERS
INTRODUCTION

In Chapter One, we established that a process of limiting followed by ideal filtering using a bandwidth of one 1-f bandwidth, will, if repeated a sufficient number of times, reduce the relative intensity of the interference to any desired degree, for all values of initial ratio of weaker-to-stronger signal amplitude, at the input to the first limiter, that are smaller than about 0.84. The theoretical demonstration of the success of this scheme hinges upon the fact that the configuration made up of the spectral components at the frequencies p and p + r rad/sec represented the most adverse condition of interference at the output of each limiter filter under the specified conditions. Therefore the results of the spectral analysis at the output of the first limiter were directly applicable to the second and other limiters. However, for other ratios of weaker-to-stronger signal amplitude at the input to the first limiter, either the minimum permissible limiter bandwidths exceeded one 1-f bandwidth sufficiently to accommodate most unfavorable configurations involving more than just two spectral components, or such configurations were possible even with limiter bandwidths equal to or a little greater than one 1-f bandwidth. For these situations, the results obtained from a study of the effect of ideal narrowband limiting on the resultant of two sinusoids, cannot be used as direct evidence of similar interference reduction when several sinusoids are present at the input to the limiter. Nevertheless, we observed that the plots of Fig. 18, of Chapter One, demonstrated at least qualitative evidence that a second, (third, etc.), narrow-band limiter would generally yield an additional reduction of the interference for arbitrary values of the ratio of weaker-to-stronger signal amplitude, a, delivered to the first limiter by the intermediate-frequency amplifier, provided the proper value of first-limiter bandwidth is used in the range of a > 0.84. Before any quantitative evidence can be produced in the range a > 0.84, an analysis of the spectrum resulting from amplitude limiting the resultant of
more than two sinusoids must be carried out. This will be our starting point in the discussions of the present chapter.

To bring the present task to sharper focus, some obstacles relating to the specification of the number and relative amplitude, frequency and phase relationships of the sinusoids to be superimposed at the input of the limiter, must be overcome. The most general specifications would leave the amplitudes arbitrary, and the relative frequency and phase relationships random. But such an approach unduly complicates the problem by formulating it with unnecessary generality. Such specifications would be appropriate in the more general problem of multipath interference in which signals from more than two paths are accommodated simultaneously within the linear passband of the receiver. The problem of whether or not the effect of one stage of ideal narrow-band limiting on two-path interference (determined in Chapter One) will also be demonstrated under conditions of several-path interference (in which capture of one of these paths remains possible), although worthwhile, is not really what we are after here.

Basically, we are seeking to determine quantitatively what a scheme of cascaded narrow-band limiters will do to the interference arising from the simultaneous presence of only two carriers within the passband of the receiver. The most direct approach would, therefore, consider the interest to have shifted from the effect of the first narrow-band limiter upon the resultant of the two carriers, to a study of the subsequent modifications that a second stage of narrow-band limiting will inflict upon the resultant signal delivered by the first stage. The sinusoidal components that make up the signal at the input to the second limiter will therefore be considered to be a selection made by the first limiter filter from the spectrum analyzed in Section I.2. This decision fixes the relative amplitudes, frequencies, as well as initial phases associated with the assumed sinusoidal components, and therefore disposes of a major hurdle in the formulation of the problem.
We must next decide upon the particular configuration, or configurations, of spectral components passed by the first limiter-filter whose consideration will be most informative. We are compelled to make definite choices here because the nonlinear action of the limiter upon the resultant of a given configuration deprives us of the conveniences of linear superposition.

To start with, we may reduce the number of possibilities drastically by considering only the configuration of components which presents the most serious capture problem associated with an assumed value of first-limiter filter bandwidth. This is readily done by drawing upon the results of Section 1.6, where the most serious configurations associated with various values of limiter-filter bandwidth have been determined. Thus the pursuit of our main objective involves another restriction upon the generality of the approach; namely, we must, for definiteness, specify what value of bandwidth to associate with the first limiter, and then concentrate upon the action of the second limiter upon the most troublesome configuration that can arise with that value of bandwidth. It is recalled that the most serious configuration is the one that would demand the largest value of minimum required discriminator bandwidth if it were fed directly into an amplitude-insensitive discriminator. In terms of the concept of equivalent capture ratio, \( \rho \), introduced in Chapter Three to describe the capture conditions at the output of a narrow-band limiter stage, the most serious configuration that can be delivered by this stage is the one that yields the largest value of \( \rho \).

The only (and very) unfortunate part about restricting the choice of configuration is the fact that the same computational effort must be repeated as many times as the number of different values of bandwidth that we choose to associate with the first limiter, in order for numerical results to be secured. Furthermore, as the number of sinusoids necessary to specify any specific configuration
is increased, the computational labor is found to increase very quickly, and after the results of the simpler configurations have been determined, no additional information of fundamental significance will be gained about the importance of the cascading scheme from the consideration of the more complicated variations. Therefore we shall find it feasible to confine our interest to the simplest possible situations that can yield significant indications on the effectiveness of the cascading scheme in the suppression of interference, with a minimum of labor, and without running into the law of diminishing returns.

After a Fourier analysis of the output of the second limiter in response to an input made up of a few interesting configurations, the questions investigated will relate to the requirements in the second, and later, limiter bandwidths, and the bearing of the results upon the discriminator bandwidth requirements, the maximum permissible time constants in the grid circuit of a grid-bias limiter as well as in the output circuit of the discriminator, and the effect of narrow-band limiting upon the harmonic component amplitudes in the structure of detected spike patterns recurring at an audible rate.
II.1 CAPTURE CONDITIONS AT THE OUTPUT OF THE SECOND LIMITER

Our first objective, here, is best summarized in terms of the scheme shown in Fig. 1. The idealized i-f amplifier response is shown to accommodate two signal carriers whose properties have been described in Section I.2. The assumptions and notation of the preceding chapter carry over unaltered. The i-f amplifier delivers the resultant of the two signals to the first limiter stage which has a specified bandwidth

\[ W_{L1} = \left( \frac{(BW)_{\text{lim}}}{(BW)_{\text{if}}} \right)_{1} \].

This limiter delivers a resultant signal \( e_{12} \) the variation of whose composition and properties with \( W_{L1} \) is now well understood from the results of Chapter One. Our immediate task is to determine the spectral properties of \( e_{02} \) when the worst composition of \( e_{12} \) corresponding to any assumed value of \( W_{L1} \) is impressed at the input to the second limiter.

In Table I are presented various configurations which were found in the preceding chapter to represent the most adverse capture conditions possible with the associated values of \( W_{L1} \). Appropriately, only values of \( a \) in the range exceeding 0.84 are considered, since for values of \( a < 0.84 \) a workable cascading scheme has already been discussed quantitatively in which each of the limiters had only one i-f bandwidth.

![Fig. 1](image-url)

The simplest configuration of interest is seen to arise with a value of "first limiter bandwidth whose use was found to lead to the smallest minimum required discriminator bandwidth for \( a = 0.9 \), and the second smallest for \( a = 0.85 \).
### TABLE I

<table>
<thead>
<tr>
<th>a</th>
<th>( W_L )</th>
<th>Worst Configuration</th>
<th>( \delta_m )</th>
<th>Minimum Required ( (BW) )</th>
<th>( (BW) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \lim )</td>
<td>( M )</td>
<td>( N )</td>
<td>( r_{max} )</td>
<td>( \text{disc} )</td>
</tr>
<tr>
<td>0.85</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5.2790</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>6.1430</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>6.8784</td>
</tr>
<tr>
<td>0.9</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>6.9766</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>8.0254</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>8.9308</td>
</tr>
<tr>
<td>0.95</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2/3</td>
<td>15.0</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>3/4</td>
<td>13.292</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>14.147</td>
</tr>
</tbody>
</table>

This value of first limiter bandwidth is \( W_L = 3 \) and the associated worst configuration is \( M = 1, N = 2 \). It is unfortunate that this same configuration does not happen to represent the composition most unfavorable for capture associated with \( W_L = 3 \) for \( a = 0.95 \) also. The next simplest configuration which will, however, apply for all three values of \( a \) listed is \( M = 3, N = 4 \) which can arise with \( W_L = 7 \). Only these two configurations will be treated here to establish quantitative evidence for the effectiveness of at least two cascading schemes (differing perhaps only in the bandwidth value of the first limiter) in the minimization and suppression of the interference. Other configurations arising with other assumed values for the first limiter bandwidth may, if desired, be handled in an entirely similar manner out with increased labor and no additional fundamental information beyond what the presently chosen configurations will lead to.

The linear superposition of the components corresponding to either of the above two chosen configurations may best be illustrated by a phasor diagram similar to that
we may express $e_{o2}(t)$ in the form

$$e_{o2}(t) = \sum_{n=-\infty}^{\infty} B_n \cos(p - nr)t$$

(3)

where $B_0 = \alpha_0$

$$B_n = \frac{1}{2} (\alpha_n - \beta_n)$$

and

$$B_{-n} = \frac{1}{2} (\alpha_n + \beta_n).$$

We observe that since

$$e_{o2}(0) = \sum_{n=-\infty}^{\infty} B_n = \sum_{n=0}^{\infty} \alpha_n = 1$$

we have a convenient check on at least the computation of the values of $\alpha_n$.

It is readily appreciated from the experience with the much simpler two-signal problem treated in Section I.2, that the task of deriving useful expressions for $B_n$ and $B_{-n}$ is rather hopeless. Moreover, such an attempt is not even justified, since we are chiefly interested in the values for $a = 0.85$ and $a = 0.9$, and therefore much less effort is involved in evaluating the $B_n$'s by direct numerical analysis with the help of Fourier coefficient schedules. Such an evaluation has been carried out, and the results are presented in Table II A.

An examination of the spectral amplitudes, $B^{(n)}$, reveals that they possess properties much like those exhibited by the $A^{(n)}$'s in Chapter One. In particular, their signs alternate starting with $B^{(0)}$ and $B^{(1)}$ positive, and $B^{(1)}$ negative. There is also a qualitative correspondence between the distribution of relative amplitudes and the instantaneous frequency pattern of the resultant signal (shown in Fig. 2). It is a simple matter to show that statements similar to those of Theorems I through III B of Chapter One apply here directly. Finally, from Table III we find that not only have the (troublesome) upper sideband components shrunk in magnitude
of Fig. 6 of Chapter One. For the configuration \( M = 1, N = 2 \),
the resultant signal at the input to the second limiter is

\[
e_{12}(t) = \text{Re} \left[ e^{j\beta t} \sum_{n=-2}^{1} a_n e^{-jnrt} \right]
\]

\[
= \text{Re} \left[ A_0 e^{j\beta t} \left( 1 + be^{j\phi} - c e^{-j\phi} - de^{j2\phi} \right) \right] \tag{1}
\]

where

\[
\phi = rt, \quad b = \frac{|A_{-1}|}{A_0}, \quad c = \frac{|A_1|}{A_0}, \quad \text{and} \quad d = \frac{|A_{-2}|}{A_0}.
\]

Thus, we may write

\[
e_{12}(t) = A(t) \cos(pt + \Theta)
\]

in which

\[
A(t) = \sqrt{R^2(\phi) + I^2(\phi)}, \quad \Theta = \tan^{-1} \frac{I(\phi)}{R(\phi)},
\]

\[
R(\phi) = 1 + (b - c) \cos \phi - d \cos 2\phi
\]

and

\[
I(\phi) = (b + c) \sin \phi - d \sin 2\phi.
\]

In a manner similar to that in Section I.2, we associate
sufficient selectivity with the second limiter to reject all
harmonics of \( p \) and their associated sidebands so that the
signal at the output of this limiter (in the absence of
narrow-band filtering) may be described by

\[
e_{02}(t) = \cos(pt + \Theta) \tag{2}
\]

where, again, the constant amplitude has been assumed to be
unity for convenience. If we next write

\[
\cos \Theta = a_0 + \sum_{n=1}^{\infty} a_n \cos n\phi
\]

and

\[
\sin \Theta = \sum_{n=1}^{\infty} n \sin n\phi,
\]
\[ \sum_{n=-24}^{24} B_n = 0.99999998 \]

\[ \sum_{n=-24}^{24} B_n = 0.99999998 \]
TABLE II B

<table>
<thead>
<tr>
<th>n</th>
<th>( C_{+n} )</th>
<th>( C_{-n} )</th>
<th>( C_n )</th>
<th>( C_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.80017600</td>
<td>0.77188817</td>
<td>0.85</td>
<td>0.9</td>
</tr>
<tr>
<td>1</td>
<td>-0.29668850</td>
<td>0.45770003</td>
<td>-0.29507931</td>
<td>0.49127662</td>
</tr>
<tr>
<td>2</td>
<td>0.17980165</td>
<td>-0.08852647</td>
<td>0.18704689</td>
<td>-0.10308432</td>
</tr>
<tr>
<td>3</td>
<td>-0.123367232</td>
<td>0.01996077</td>
<td>-0.13524200</td>
<td>0.2896045</td>
</tr>
<tr>
<td>4</td>
<td>0.05582540</td>
<td>0.00953726</td>
<td>0.06953726</td>
<td>0.00261839</td>
</tr>
<tr>
<td>5</td>
<td>-0.03844722</td>
<td>0.01258463</td>
<td>-0.05113540</td>
<td>0.01696557</td>
</tr>
<tr>
<td>6</td>
<td>0.02660961</td>
<td>-0.00589528</td>
<td>0.03810971</td>
<td>-0.00873571</td>
</tr>
<tr>
<td>7</td>
<td>-0.01858683</td>
<td>0.00203819</td>
<td>-0.02833829</td>
<td>0.00390189</td>
</tr>
<tr>
<td>8</td>
<td>0.01207127</td>
<td>-0.00014574</td>
<td>0.02013156</td>
<td>-0.00143672</td>
</tr>
<tr>
<td>9</td>
<td>-0.00823326</td>
<td>0.00095336</td>
<td>-0.1494002</td>
<td>0.00197482</td>
</tr>
<tr>
<td>10</td>
<td>0.00576595</td>
<td>-0.00063900</td>
<td>0.01121969</td>
<td>-0.00136341</td>
</tr>
<tr>
<td>11</td>
<td>-0.00402425</td>
<td>0.00031986</td>
<td>-0.00845296</td>
<td>0.00082122</td>
</tr>
<tr>
<td>12</td>
<td>0.00278912</td>
<td>-0.00013728</td>
<td>0.00635138</td>
<td>-0.00048925</td>
</tr>
<tr>
<td>13</td>
<td>-0.00194982</td>
<td>0.00012896</td>
<td>-0.00448570</td>
<td>0.00039060</td>
</tr>
<tr>
<td>14</td>
<td>0.00137033</td>
<td>-0.00009157</td>
<td>0.00356223</td>
<td>-0.00027679</td>
</tr>
<tr>
<td>15</td>
<td>-0.00096552</td>
<td>0.00005484</td>
<td>-0.00278268</td>
<td>0.00017940</td>
</tr>
<tr>
<td>16</td>
<td>0.00068065</td>
<td>-0.00003042</td>
<td>0.00212306</td>
<td>-0.00010854</td>
</tr>
<tr>
<td>17</td>
<td>-0.00048101</td>
<td>0.00001987</td>
<td>-0.00162334</td>
<td>0.00006206</td>
</tr>
<tr>
<td>18</td>
<td>0.00034066</td>
<td>-0.00001097</td>
<td>0.00124342</td>
<td>-0.00004154</td>
</tr>
<tr>
<td>19</td>
<td>-0.00024160</td>
<td>0.00000291</td>
<td>-0.00095349</td>
<td>0.00001602</td>
</tr>
<tr>
<td>20</td>
<td>0.00017145</td>
<td>0.0000428</td>
<td>0.00073147</td>
<td>0.00005245</td>
</tr>
<tr>
<td>21</td>
<td>-0.00012169</td>
<td>-0.00001116</td>
<td>-0.00056099</td>
<td>0.00009028</td>
</tr>
<tr>
<td>22</td>
<td>0.00008029</td>
<td>0.00001914</td>
<td>0.00042954</td>
<td>0.00015322</td>
</tr>
<tr>
<td>23</td>
<td>-0.00009096</td>
<td>-0.00002927</td>
<td>-0.00032760</td>
<td>-0.00018454</td>
</tr>
<tr>
<td>24</td>
<td>0.00004269</td>
<td>0.00004269</td>
<td>0.000024784</td>
<td>0.000024784</td>
</tr>
</tbody>
</table>

\[
\sum_{n=-24}^{24} C_{-n} = 0.999999999 \\
\sum_{n=-24}^{24} C_{-n} = 1.000000001
\]
<table>
<thead>
<tr>
<th>n</th>
<th>$C_n$</th>
<th>$C_{-n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.73523438</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.28443725</td>
<td>0.53299857</td>
</tr>
<tr>
<td>2</td>
<td>0.18673803</td>
<td>0.12657495</td>
</tr>
<tr>
<td>3</td>
<td>-0.14094844</td>
<td>0.04735312</td>
</tr>
<tr>
<td>4</td>
<td>0.08551518</td>
<td>-0.01397191</td>
</tr>
<tr>
<td>5</td>
<td>-0.06744733</td>
<td>0.02464010</td>
</tr>
<tr>
<td>6</td>
<td>0.05406953</td>
<td>-0.01477437</td>
</tr>
<tr>
<td>7</td>
<td>-0.04351496</td>
<td>0.00877105</td>
</tr>
<tr>
<td>8</td>
<td>0.03424925</td>
<td>-0.00537343</td>
</tr>
<tr>
<td>9</td>
<td>-0.02778427</td>
<td>0.00480514</td>
</tr>
<tr>
<td>10</td>
<td>0.02275956</td>
<td>-0.00351308</td>
</tr>
<tr>
<td>11</td>
<td>-0.01872553</td>
<td>0.00247015</td>
</tr>
<tr>
<td>12</td>
<td>0.01542139</td>
<td>-0.00173406</td>
</tr>
<tr>
<td>13</td>
<td>-0.01277079</td>
<td>0.00128370</td>
</tr>
<tr>
<td>14</td>
<td>0.01061500</td>
<td>-0.00088292</td>
</tr>
<tr>
<td>15</td>
<td>-0.00884549</td>
<td>0.00054239</td>
</tr>
<tr>
<td>16</td>
<td>0.00738420</td>
<td>-0.00025589</td>
</tr>
<tr>
<td>17</td>
<td>-0.00617525</td>
<td>0.00001288</td>
</tr>
<tr>
<td>18</td>
<td>0.00517067</td>
<td>0.00021461</td>
</tr>
<tr>
<td>19</td>
<td>-0.00433220</td>
<td>0.00043566</td>
</tr>
<tr>
<td>20</td>
<td>0.00362936</td>
<td>0.00065852</td>
</tr>
<tr>
<td>21</td>
<td>-0.00303767</td>
<td>0.00089166</td>
</tr>
<tr>
<td>22</td>
<td>0.00253691</td>
<td>0.00114429</td>
</tr>
<tr>
<td>23</td>
<td>-0.00211029</td>
<td>0.00142522</td>
</tr>
<tr>
<td>24</td>
<td>0.00174374</td>
<td>0.00174374</td>
</tr>
</tbody>
</table>

$$\sum_{n=-24}^{24} C_{-n} = 1.00000001$$
### TABLE III

RELATIVE SPECTRAL AMPLITUDES AT OUTPUT OF:

**FIRST LIMITER**

| a  | n  | $|A_n/A_0|$ | $|B_n/B_0|$ | $|C_n/C_0|$ |
|----|----|------------|------------|------------|
| 0.85 | -1 | 0.615582  | 0.489775  | 0.57199920 |
|     | -2 | 0.149476  | 0.038199  | 0.11063375 |
|     | -3 | 0.068389  | 0.034537  | 0.02494547 |
|     | -4 | 0.038120  | 0.008429  | 0.01190016 |
|     | 1  | 0.361473  | 0.385025  | 0.37102900 |
|     | 2  | 0.215906  | 0.159684  | 0.22470263 |
|     | 3  | 0.147533  | 0.094305  | 0.15455640 |

| 0.9  | -1 | 0.700212  | 0.552505  | 0.63646087 |
|     | -2 | 0.185342  | 0.0666564 | 0.13354826 |
|     | -3 | 0.091429  | 0.046464  | 0.03761897 |
|     | -4 | 0.054664  | 0.016124  | 0.00539219 |
|     | 1  | 0.366596  | 0.398646  | 0.38228246 |
|     | 2  | 0.226881  | 0.186962  | 0.24232382 |
|     | 3  | 0.161961  | 0.119581  | 0.17520932 |

| 0.95 | -1 | 0.811576  |             | 0.72493695 |
|     | -2 | 0.236194  |             | 0.17215592 |
|     | -3 | 0.126338  |             | 0.06440548 |
|     | -4 | 0.081347  |             | 0.01900334 |
|     | 1  | 0.362808  |             | 0.38686609 |
|     | 2  | 0.228495  |             | 0.25398435 |
|     | 3  | 0.168012  |             | 0.19170545 |
relative to the desired component, \( B_0 \), but also the (helpful) lower sideband components have increased in relative amplitude. In the light of these facts, the insight gained in Chapter One leads us to expect significant reductions in the effectiveness of the interference to result from subjecting the new spectral distribution to the action of a narrow-band filter.

The first question that presents itself at this point is whether or not it is permissible to use a second-limiter filter bandwidth that is smaller than the bandwidth following the first limiter. To answer this question, we may feel tempted to make use of the spectral amplitudes, \( B_{\text{in}} \), at the output of the second limiter in the same way that we used the \( A_{\text{in}} \) amplitudes delivered by the first, to determine the minimum permissible limiter bandwidths. But if we remember that the \( B_{\text{in}} \) represent the spectral amplitudes at the output of the second limiter only when the configuration \( M = 1, N = 2 \) is delivered by the first limiter filter, the limitations on the usefulness of the \( B_{\text{in}} \) become immediately apparent. We now recall, from Chapter One, that the minimum permissible bandwidth after a limiter is equivalent to one i-f bandwidth only if, at least in theory, the situation in which the desired component having the frequency of the stronger signal, \( p \), along with all the sideband components on the same side as the weaker signal relative to \( p \), may be accommodated to the complete exclusion of all the helpful sideband components on the opposite side of the frequency \( p \), and still retain an average frequency for the resultant signal equal to \( p \). For such a situation to arise, the stronger signal in our analysis must lie infinitesimally to the right of the lower cutoff frequency, and \( r \) must be sufficiently small to allow all of the significant upper sideband components to pass. But if the two carriers delivered by the i-f take the necessary positions on the frequency scale for this limiting situation to arise, (and \( r \) is sufficiently small of course), it may be argued that the first limiter bandwidth of \( 3 (3W)_{\text{if}} \) (or any other value greater
than one \( \rightarrow (BW) \) if for that matter since \( R \) may be assumed as small as is necessary) will then be sufficiently wide to accommodate all of the significant sideband components on both sides of \( p \), with the result that it will deliver to the second limiter essentially the amplitude limited version of the resultant of the two input carriers without any significant instantaneous phase and frequency alterations. The spectrum at the output of the second limiter will then be described by the \( A_{\text{in}} \) discussed in detail in Chapter One, and the criterion for the permissibility of only one i-f bandwidth after the second limiter will be identical with that applying after the first limiter. The obvious conclusion may therefore be stated as the following theorem.

**Theorem I:**

The minimum permissible limiter bandwidth is theoretically equal to one i-f bandwidth for all values of \( a \leq 0.863 \) delivered by the i-f amplifier, regardless of whether this limiter happens to be the first, an intermediate one, or the last in a chain of limiters, and regardless of the value of bandwidth used in the limiters (if any) preceding that stage, provided such bandwidths exceed or equal one i-f bandwidth.

This theorem obviously states that the minimum requirements in a limiter's bandwidth are unaffected by the action of any narrow-band limiters, that may precede the one under consideration, upon the resultant signal passing through those limiters. The reason for this is of course the fact that under the conditions of the limiting situation dictating the minimum requirements (which situation is mainly of theoretical interest, of course) any allowance in the preceding limiter bandwidth for a finite nonzero extent in the passband below the lower cutoff frequency of the i-f amplifier will, by reason of its being nonzero, admit of a sufficiently smaller, but nonzero, \( r \) for which the effect of this preceding limiter filter upon the amplitude-limited resultant of the two input carriers will be insignificant.
Although the argument is carried out for a situation in which the preceding limiter (or limiters) has a bandwidth greater than the i-f bandwidth, the use of one i-f bandwidth in a preceding stage obviously imposes an a priori restriction upon the usefulness of the combined cascade of limiters to capture ratios (at the input to the first limiter in the chain) of less than 0.863, (since this preceding stage will either be the first in the chain or will be preceded by another stage or stages of wider bandwidth). In other words, the chain is no stronger than its first weakest link.

The determination of the variation of the minimum permissible value of second limiter bandwidth with a, for a > 0.863, is quite laborious computationally, and the result will vary with the bandwidth used after the first limiter. Only when the first limiter has such a bandwidth that it will always pass the entire significant spectrum centered about p will the requirements in the bandwidth of the second limiter vary with a exactly as the first limiter bandwidth did, because only then will the spectral amplitude distribution at the output of the second limiter be given by the A\textsubscript{in} of Chapter One. Since the extent of the significant (troublesome) A\textsubscript{in}(a) components is greater the closer a is to unity, the narrow-band limiting effect upon the character of the resultant signal with any given value of first limiter bandwidth will increase in significance with increase in a. This means that the effect of a narrow-band filter after the first limiter upon the minimum requirements in the bandwidth of the second limiter should be more noticeable in the range of a values that are closer to the maximum value of a which the first limiter filter can handle successfully. This maximum value of a will of course mark the limit for the usefulness of the combination of the two limiters in cascade since a loss of the desired average frequency will be introduced by the first limiter for larger values of a. Again the failure of the first link in the chain marks the failure of the chain.
As an illustration of these ideas let us return to the original question of whether or not one can use a bandwidth after the second limiter which is narrower than the $3(BW)_{1f}$ of the first limiter and still retain the usefulness of the combination for all $a$ up to 0.9807. It is recalled that the $a = 0.9807$ limit is set by the configuration $M = 1$, $N = 3$ upon the permissibility of $3(BW)_{1f}$ after the first limiter. The narrow-band limiter action in the first stage will be in evidence at the output of the second stage as a general decrease in the amplitudes of the upper sideband components relative to the desired component (at p rad/sec) and an increase in the relative amplitudes of the (helpful) lower sideband components. Using a procedure similar to that illustrated in Cases B and C of Section I.5, for $a > 0.863$, one proceeds to determine up to what value of $a$ (less than 0.9807) a certain number $N_{max}$ and $M_{min}$ of upper and lower sideband components, specified as indicated in Section I.6, may be accommodated within the second limiter filter pass-band without upsetting the desired average frequency of the resultant. In the present instance, this determination involves first determining the value of bandwidth for the second limiter for which $M = M_{min}$, $N = N_{max}$ is a permissible limiting situation, and then the configuration of sideband components that the first limiter filter must pass in order for $M = M_{min}$, $N = N_{max}$ to appear at the output of the second limiter filter using the just determined bandwidth. The amplitude-limited resultant of the appropriate configuration delivered by the first limiter is then Fourier analyzed to determine the amplitudes of the $M = M_{min}$, $N = N_{max}$ sideband components that make up the limiting configuration of the second limiter filter. From the results of the Fourier analysis one then determines the maximum value of $a$ for which the sum of the magnitudes of the $M_{min}$ components and the desired component exceeds the sum of the magnitudes of the $N_{max}$ upper sideband components. Up to this value of $a$, the second limiter bandwidth that will accommodate the
specified \( M = M_{\text{min}} \), \( N = N_{\text{max}} \) as a limiting configuration, is the minimum permissible value of bandwidth. Since the narrow-band limiter action in the first stage will decrease the magnitudes of the upper sideband components and increase those of the lower sideband components relative to the magnitude of the desired component, the maximum value of a resulting from the computation in terms of \( M = M_{\text{min}}, \ N = N_{\text{max}} \) at the output of the second limiter will be higher than the corresponding value at the output of the first limiter. Even though the computational task is somewhat simplified by \( M_{\text{min}} = 0 \) for most of the practically important values of a exceeding 0.863, the importance of the numerical results does not outweigh the labor necessary.

One consequence of the effect of narrow-band limiting in the first stage upon the amplitudes of the upper sideband components at the output of the second limiter is that when the bandwidth of this second limiter is made equal to the i-f bandwidth, the configuration in which only the desired component at \( p \) and the component at \( p + r \ (r = (BW)_{i-f}) \) are passed becomes the one that dictates the discriminator bandwidth requirement (i.e. the one that has the largest frequency spike magnitude) not only for all a's up to 0.84, but also for a's that can be made to close the gap between 0.84 and 0.863. It may be argued that the first limiter bandwidth need not exceed one i-f bandwidth in order for this to be achieved, but more than just two stages of narrow-band limiting may then be needed to abate the importance of the upper sideband components, relative to the desired component, as a approaches 0.863 closely.

With the second limiter bandwidth taken equal to one i-f bandwidth, however, it is clear that when the first limiter filter delivers its worst possible configuration with \( r = (BW)_{i-f} \), the corresponding configuration accommodated by the ideal filter following the second limiter will also represent the worst possible condition of interference for the overall two-limiter scheme. Fortunately, the latter
configuration happens to be the one in which only $B_0(a)$ at $p$ rad/sec (corresponding to the stronger of the two carriers delivered by the i-f) and $B_1(a)$ at $p + r$ rad/sec (corresponding to the weaker signal) are the only passed spectral components. Under the worst condition of interference, therefore, a scheme made up of two limiters in cascade in which the first limiter has three times the i-f bandwidth and the second has only one i-f bandwidth, will deliver at its output only two sinusoids corresponding to the two input sinusoids with the ratio of weaker-to-stronger signal amplitude reduced from its input value of $a$ to the value $B_{-1}(a)/B_0(a)$. Reference to Table III will show that for $a = 0.85$, this represents a reduction to about 0.49. Therefore, the indicated scheme will demonstrate the same effect upon the interference in the range of $a$ between about 0.84 and 0.863 as one stage of ideal narrow-band limiting using only one i-f bandwidth did in the range $a < 0.84$; namely, under the worst condition of two-signal interference at the input (which arises with $r = (BW)_{1f}$), the worst condition of interference at the output will also involve exactly two signals which will correspond to the input signals and will be separated in frequency by one i-f bandwidth, but the ratio of weaker-to-stronger signal amplitude at the output will be considerably smaller than that at the input. Starting with this output ratio of weaker-to-stronger signal amplitude (which will incidentally now be well within the range $a < 0.84$ in which one ideal narrow-band limiter having only the i-f bandwidth will be most effective in reducing the interference), one may retrace the argument established in Chapter One concerning the possibility of reducing the interference under its worst condition to any desired small value by cascading the necessary number of ideal narrow-band limiters each having a bandwidth equal to that of the intermediate frequency amplifier.

We now summarize this result in the following form.

**Theorem II:**

If a system of two or more cascaded ideal narrow-band
limiters, in which the first limiter has a bandwidth a few times greater than (perhaps three times) or equal to one i-f bandwidth, and the others have just one i-f bandwidth each, is incorporated in an FM receiver, then the most adverse condition of two-signal interference will arise at both the input and the output of the scheme when the difference frequency \( r = (BW)_{if} \). Under this condition of interference, the scheme will deliver at its output only two sinusoids, corresponding to the input carriers, with the ratio of weaker-to-stronger signal amplitude reduced from its input value of \( a \leq 0.863 \) to a value that can be made as small as desired by cascading the necessary number of narrow-band limiters.

Equivalently, this important theorem states that a scheme starting with an ideal narrow-band limiter having a bandwidth three times that of the i-f, followed by a sufficiently long chain of ideal narrow-band limiters each having one i-f bandwidth will reduce the minimum necessary discriminator bandwidth to essentially that of the i-f for all input values of \( a \) less than about 0.863. In Table IV we present the results of computations which show the speed with which the minimum required discriminator bandwidth decreases with the number of limiters used in this scheme when \( 3(BW)_{if} \) is used in the first stage. These results are also shown plotted in Fig. 3. Plots for \( a = 0.8 \) and \( a = 0.7 \) are also reproduced from Fig. 21 of Chapter One for ease of comparison.

<table>
<thead>
<tr>
<th>TABLE IV</th>
<th>CASCADING SCHEME</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Limiter Bandwidth = ( 3(BW)_{if} )</td>
<td></td>
</tr>
<tr>
<td>Second and Succeeding Limiters Each Has ( (BW) = (BW)_{if} )</td>
<td></td>
</tr>
<tr>
<td>Number of ideal narrow-band limiters</td>
<td>Minimum discriminator bandwidth required ( a = 0.85 )</td>
</tr>
<tr>
<td>1</td>
<td>5.2790</td>
</tr>
<tr>
<td>2</td>
<td>2.922</td>
</tr>
<tr>
<td>3</td>
<td>1.74</td>
</tr>
<tr>
<td>4</td>
<td>1.325</td>
</tr>
<tr>
<td>5</td>
<td>1.15</td>
</tr>
</tbody>
</table>
Consider next the situation in which the second limiter is given a bandwidth three times that of the i-f. It is clear that with this value of bandwidth, the configuration that will dictate the minimum required discriminator bandwidth is again $M = 1, N = 2$, as it was for the first limiter. The present case is best illustrated by means of the plots of Fig. 2. In these plots, $\Omega_0(t)$ is the instantaneous frequency variation of the resultant two-path signal delivered to the first limiter, over one period of the difference frequency, $r$, between the two carriers. If an amplitude-insensitive discriminator is used immediately following the i-f amplifier, the FM-to-AM conversion characteristic of the discriminator must be linear over the whole range of variation of $\Omega_0(t)$. However, if one stage of ideal narrow-band limiting is inserted between the i-f section and the amplitude-insensitive discriminator, the most serious variations in the instantaneous frequency of the resultant signal delivered to the discriminator will follow the curve denoted by $\Omega_1(t)$ when the ideal limiter bandwidth is three times the i-f bandwidth. The effect of cascading two such identical stages of narrow-band limiting between the i-f and the discriminator is that the most serious variations in resultant signal frequency will now follow the curve marked $\Omega_2(t)$.

Table V shows the values of minimum required discriminator bandwidth under the conditions of each of the plots of Fig. 2. These values are also plotted in Fig. 3. It is clear that the same type of action indicated by these results will also be exhibited by further stages of ideal narrow-band limiting in which each stage has a bandwidth of $3(BW)_{IF}$, until after a sufficient number of them has been cascaded, the minimum required discriminator bandwidth becomes essentially equal to that of the i-f. The choice of $3(BW)_{IF}$ for the bandwidth of each stage is inspired by the desire to investigate a scheme in which each limiter bandwidth meets
the minimum requirement for all \( a \)'s up to 0.9807. The choice of
the minimum permissible value for this range is also in
line with the basic aim at determining the greatest achievable
reductions in the required discriminator bandwidth. For,
with a scheme in which only two limiters are cascaded, the
first of which has a bandwidth of \( 3(BW)_{1f} \), it is clear that
as the bandwidth of the second limiter is increased from
its minimum permissible value to higher and higher values,
the minimum discriminator bandwidth required after this
second limiter increases from a small value toward a larger
value which would be dictated by the worst configuration
delivered by the first limiter in the absence of the second
one. The latter value is achieved when the second limiter
bandwidth becomes sufficiently large to accommodate all of
the spectral components of significance in the structure of
\( e_{02}(t) \).

Strictly speaking, the effectiveness of the cascading
scheme in which each limiter has a bandwidth of \( 3(BW)_{1f} \) has
so far been demonstrated only for values of \( a \) for which the
configuration \( M = 1, N = 2 \) is the most critical one at the
output of the first stage. With the help of Table IV of
Section I.6 this may be closely estimated to be the case for
all \( a \)'s up to about \( a = 0.91 \). Therefore the above is evidence
that at least up to this value of \( a \), a quantitative account
of this scheme has been provided. Although for higher values
of \( a \) the most adverse interference condition at the output
of the first limiter does not correspond to \( r = (BW)_{1f} \) (which
holds for the worst condition at the input) but to a smaller
value of \( r \), with \( M = 1, N = 3, \) up to 0.9807, we now have
little doubt that the scheme will exhibit similar reductions
in the over all relative importance of the interference
without the need of a separate computation starting with
\( N = 1, N = 3 \) at the input to the second limiter, for \( a \) in
the range \( 0.91 < a < 0.98 \).

As a final illustration, let us associate with the first
limiter a bandwidth seven times that of the 1-f. From Table I
TABLE V
CASCADING SCHEME

<table>
<thead>
<tr>
<th>Capture ratio</th>
<th>Frequency spike magnitude at the output of the:</th>
<th>Minimum required ((BW)<em>{disc}/(BW)</em>{if}) after (n) identical narrow-band limiters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1-(f) section</td>
<td>first narrow-band limiter</td>
</tr>
<tr>
<td>0.85</td>
<td>5.667r</td>
<td>2.1395r</td>
</tr>
<tr>
<td>0.9</td>
<td>9r</td>
<td>2.9883r</td>
</tr>
<tr>
<td></td>
<td>(\approx 3r)</td>
<td></td>
</tr>
</tbody>
</table>

It is evident that for this value of bandwidth, the worst configuration that the first limiter will deliver to the second one is given by \(M = 3\), \(N = 4\) for all three values of \(a\) listed. Therefore the present example will involve \(a = 0.95\) in a direct computation, and will also reinforce the various conclusions reached above, and illustrate others. It is observed from Table I that the worst condition of interference arises here, with \(r = (BW)_{if}\) at the input as well as at the output of the limiter.

With reference to Fig. 1, and with an analysis entirely analogous to the one carried out in deriving Eq. 3, it may be shown that when \(e_{12}\) is the resultant of the configuration \(M = 3\), \(N = 4\), \(e_{02}\) is expressible in the form

\[
e_{02}(t) = \sum_{n=-\infty}^{\infty} C_n \cos(p - nr)t
\]

where the \(C_n\)'s have been computed by numerical analysis. The values of the spectral amplitudes, \(C_{\pm n}\), at the output of the second limiter are tabulated in Table IIB for \(a = 0.85\), 0.9 and 0.95. These tables portray general properties of the spectral amplitudes that differ from those of \(A_{\pm n}\) or \(B_{\pm n}\) only in so far as the new instantaneous frequency pattern of the
resultant signal at the input to the second limiter requires a slight reshuffling in the amplitudes of some of the components. This reshuffling may, with methods already illustrated in Chapter One, be shown not to affect the validity of statements concerning conditions at the output of the second limiter that are similar to those stated in Theorems I through IIIB of Chapter One. In Table IIIB, as in Table IIA, peculiarities of the method of computation have affected some of the signs, as well as the values, of the spectral amplitudes for values of n in excess of about 17, so as to make them in some instances quite unreliable. But this is not disturbing because this range of n values has mainly been included to improve the accuracy of the computation of the lower-order components which are of major significance in this study.

Table III shows that, in the present instance, also, the amplitudes of the upper sideband components relative to the amplitudes of the upper sideband components relative to the amplitude of the desired component, \( C_0 \), have been decreased, while those of the lower sideband components have been increased. We now accept this effect to be characteristic of the action of the ideal narrow-band limiting process upon the resultant of two or more sinusoids that differ in frequency by an amount which is small compared with the frequency of any one of them, but whose amplitudes and initial phases are such that, with the various difference frequencies harmonically related, the average frequency of the resultant over a period of the fundamental of the difference frequencies is always equal to the frequency of the strongest one of the component sinusoids. Those components which by virtue of their initial phases and frequency specifications tend to help preserve the average frequency value at that of the strongest component, will generally have their amplitudes relative to that of the strongest component boosted, while those that tend to upset the average value will have their relative amplitudes diminished. This is at least true when the various sinusoids portray the
MINIMUM REQUIRED DISCRIMINATOR BANDWIDTH
IN UNITS OF ONE I-F BANDWIDTH

\[ W_L = \frac{(BW)_{\text{LM}}}{(BW)_{IF}} \]

NUMBER OF CASCADED NARROW-BAND LIMITERS

FIG. 3
properties of the spectral components centered about the frequency \( p \) (or any of its harmonics, using the proper selectivity) in the structure of the amplitude-limited resultant of two sinusoids of different amplitudes but slightly different frequencies.

Suppose we now cascade two ideal narrow-band limiters the first of which has seven times the i-f bandwidth. For the bandwidth of the second limiter we will first choose \( 3(BW)_{if} \) and later \( 7(BW)_{if} \). In all cases, the worst condition of interference will arise at the input to the scheme, as well as at the output of each stage and of the whole scheme, when \( r = (BW)_{if} \). Therefore, when \( W_{L2} = 3 \), the minimum discriminator bandwidth requirement will be dictated by the configuration \( M = 1, N = 2 \) at the output of the second limiter, and the values dictated when \( a = 0.85, 0.9 \) or \( 0.95 \) are presented in Table VI, and plotted in Fig. 3. Table VI presents also the minimum values required when \( W_{L2} = 7 \), in which case \( M = 3, N = 4 \) dictates the requirements. The decrease in discriminator bandwidth required, brought about by each scheme, suggests that if the first limiter with \( 7(BW)_{if} \) is followed by a chain of limiters in which every stage has \( 7(BW)_{if} \), or \( 3(BW)_{if} \), then a scheme capable of reducing the minimum discriminator bandwidth value to essentially that of the i-f is at hand, provided the proper number of narrow-band limiters is used. It is also clear that the minimum requirement in \( (BW)_{disc} \) will converge faster toward one \( (BW)_{if} \) when the first limiter is followed by limiters having \( 3(BW)_{if} \) rather than \( 7(BW)_{if} \), that is to say, fewer stages with \( 3(BW)_{if} \) each after the first would be needed to reduce the discriminator bandwidth requirement to a prescribed value, than with higher values of limiter bandwidth.

The above conclusions may now be summarized by the following theorem.

**Theorem III:**

Under conditions of two-signal interference, the minimum discriminator bandwidth requirement for all values of weaker-
TABLE VI
CASCADING SCHEME
FIRST LIMITER BANDWIDTH = 7(BW)_{1f}

<table>
<thead>
<tr>
<th>Capture Ratio</th>
<th>Minimum (BW)<em>{disc}/(BW)</em>{1f} Required</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>WL₂ Very Large</td>
</tr>
<tr>
<td>0.85</td>
<td>6.8784</td>
</tr>
<tr>
<td>0.9</td>
<td>8.9308</td>
</tr>
<tr>
<td>0.95</td>
<td>14.147</td>
</tr>
</tbody>
</table>

To-stronger signal amplitude ratio less than unity, delivered by the intermediate frequency amplifier, may be reduced to a value which is as close to one i-f bandwidth as is desired by cascading the necessary number of ideal narrow-band limiters having appropriately chosen bandwidths.
II.2 UPPER BOUNDS ON THE LIMITER AND DISCRIMINATOR TIME CONSTANTS

A basic requirement in receiver design to suppress multipath and cochannel disturbances is the use of the proper time constants in the limiter and discriminator circuits. The limiter circuit must be capable of following the sharp changes, in the envelope of the resultant signal, that may recur at a rate equivalent to the bandwidth of the i-f in cycles per second. At the input to the first limiter, the ratio of maximum-to-minimum amplitude for a capture ratio, \( a \) (ratio of peak value of interference to peak value of signal), is readily seen to be \((1 + a)/(1 - a)\). In the discriminator circuit, the output circuit across which the voltage level (varying with the instantaneous-frequency variations of the resultant signal impressed upon the discriminator) is taken, must be capable of following the detected instantaneous frequency pattern in order to avert the possibility of diagonal clipping and leave the variational control of the operation of the detecting diodes entirely in the hands of the amplitude of the hybrid signal (AM plus FM) delivered by the FM-to-AM converting section of the discriminator circuit. Here, also, the sharp changes in the detected voltage level may recur at a maximum rate of one i-f bandwidth in cycles/sec, and so an examination of the time constant requirement in the output circuit is necessary.

We have so far shown the action of a stage of ideal bandpass-limiting upon the disturbances arising from multipath and cochannel interference to result in a substantial reduction in the minimum necessary bandwidth requirements (for interference rejection) in the FM-to-AM conversion characteristic of the discriminator, plus certain reductions of limited significance in the limiter bandwidth requirements. We have also demonstrated that the same effect will be experienced with additional stages of bandpass limiters, and, thus, that cascading a sufficient
number of such limiters will successively reduce the minimum requirement in discriminator bandwidth to the bandwidth of the intermediate frequency. Since this effect may be looked upon as a reduction in the effective peak strength of the weaker signal relative to the stronger one, (as will be illustrated in Chapter Three by the concept of equivalent capture ratio) it is also reasonable to expect the successive reductions in the minimum required bandwidths to be accompanied by successive (a) increases in the maximum permissible values of the time constants in the discriminator and limiter circuits; and (b) decreases in the effective amplitudes of the audible harmonics in the structure of the detected instantaneous-frequency spike trains when the difference frequency between the two paths is audible.

In view of these disturbance-reducing characteristics, such a chain of bandpass limiters appears indispensable in any effective attempt to abate cochannel and multipath disturbances. The bearing of this scheme upon the performance of the laboratory receivers, designed and used in the Arguimbau-Granlund investigations at the Research Laboratory of Electronics, will be the subject of a forthcoming re-examination of those receivers and of the Arguimbau-Granlund theory accounting for that performance.

A. Limiter Time-constant Requirements:

Perhaps the only important limiter circuit that presents a time-constant problem, at present, is the grid-bias pentode limiter circuit shown in Fig. 4. A discussion of the operation of this circuit has been presented by Arguimbau in Ref. 2. It will suffice for our purposes, here, to recall that the operation of this limiter depends upon the control, by the instantaneous amplitude of the input signal, of an automatic self-rectified grid bias, which in turn controls the conduction angle and the height of the plate-current pulses, increasing the height and
decreasing the angle with increasing signal amplitude so as to keep the net charge delivered by each pulse to the plate tank condenser approximately constant. The condition that the dynamic self-bias must be exclusively controlled by the instantaneous amplitude of the input signal imposes necessary restrictions on the largest value of the RC product for the grid-leak-and-capacitor arrangement in the grid circuit — restrictions that are directly dictated (as will become apparent presently) by the reciprocal of the maximum ratio of time-rate of change of instantaneous amplitude to the value of instantaneous amplitude.

The mechanism of the operation in the grid circuit of the grid-bias limiter lends itself to a treatment that is very much like the usual analysis of the simple diode peak detector. In our problem, it is readily recognized, at the outset, that if the time constant \( R_g C_g \) is too high, the grid-bias variations will not follow but the slowest variations in the envelope value of the amplitude of the impressed signal; and if it is too low, the change in bias will not be great enough for effective smoothing.
of the amplitude variations. The time-constant considerations that must be taken into account if this circuit is to be used in a receiver designed to handle some specified capture ratio \( g \), resolve themselves into the following three items.

(a) The product \( R_g C_g \) must be sufficiently small to enable the circuit to follow the amplitude variations of a resultant two-path signal.

(b) \( C_g \) must be sufficiently large to by-pass \( R_g \) at the radio frequencies and offer an impedance at these frequencies that is much lower than the input impedance of the tube when the grid potential is on the negative swing in order for the intermediate frequency voltage to effectively appear between grid and cathode of the tube. If \( \omega_d \) = the i-f frequency, and \( C_{in} = \) the input capacity of the stage, the present requirements may be summarized by

\[
C_g R_g \gg \frac{1}{\omega_d} \text{ and } C_g \gg C_{in}.
\]

(c) \( R_g \) must be sufficiently large for the development of the necessary bias on the grid for effective smoothing of the envelope of the input wave. If the grid-to-cathode conduction resistance is \( r_g \), then we want \( R_g \gg r_g \).

For the determination of the requirement in (a), we note that if \( A(t) \) denotes the instantaneous amplitude of the resultant signal impressed at the input to the grid circuit, then, assuming that grid-to-cathode conduction occurs only when the grid goes positive, and that the conduction resistance is negligible compared to \( R_g \), we find that the grid current that flows, averaged over one radio-frequency cycle, is given by

\[
i_{av} = C_g \frac{dA(t)}{dt} + \frac{A(t)}{R_g} \geq 0
\]

since, by assumption, current can flow only from the grid to the cathode and not in reverse. It is readily appreciated that the input envelope and the bias voltage will keep
together provided the grid draws current for a short interval during each radio-frequency cycle, (which amounts to a process of sampling) and provided the fractional change in the envelope value during any one radio-frequency cycle is small. The latter is comfortably met with the assumption \( p \gg r \), explicitly understood throughout this study, whereas the conduction (or sampling) condition is met only if the average value of grid current over one radio-frequency cycle is positive, as indicated in Eq. 5.

It is of interest to note that the condition 5 may also be written in the form

\[- \frac{A(t)}{R_C g} \leq \frac{dA(t)}{dt} \equiv A'(t) \]  \hspace{1cm} (6)

The quantity on the left is recognized to be the negative of the magnitude of the rate at which the capacitor tends to discharge at the instant of time, \( t \), when the amplitude of the input signal is given by \( A(t) \) and the rate at which the amplitude is changing is \( A'(t) \). This condition therefore states that the magnitude of the rate at which the capacitor tends to discharge at any instant of time must always be greater than, or at worst equal to, the magnitude of the rate at which the amplitude is changing at that instant, in order for the self-rectified bias on the grid to follow the amplitude of the input signal. The problem could have thus been approached from this alternative, but entirely equivalent, point of view. Both pictures are needed for a thorough understanding of the situation; both points of view must be satisfied simultaneously on a purely physical basis; and both are seen to imply exactly the same inequality. The condition on the RC time constant of the grid circuit may thus be written

\[- \frac{A(t)}{A'(t)} \leq \frac{R_C g}{A'(t)} \]
The quantity on the right will of course be positive, and it will vary with time. Therefore, if the most unfavourable situation is to be met, the condition should read

\[
\frac{R_c g}{R_g g} \leq \begin{bmatrix} -\frac{A(t)}{A'(t)} \end{bmatrix}_{\text{min}}
\]  

(7)

Under two-signal interference conditions, we find from Fig. 1, Chapter One, that

\[
A(t) = \sqrt{1 + 2\cos rt + a^2}
\]

which, upon substitution into Eq. 5, yields after a straightforward simplification

\[
1 + a^2 = a\sqrt{4 + R_c g^2 r^2} \cos(rt + \psi) > 0
\]

(8)

where

\[
\psi = \tan^{-1}(1/2)(R_c g r).
\]

The worst condition arises when

\[
\cos(rt + \psi) = -1
\]

and this will be met if

\[
1 + a^2 > a\sqrt{4 + R_c g^2 r^2}
\]

whence, we want

\[
\frac{R_c g}{R_g g} \leq \frac{1 - a^2}{ar}
\]

(9)

The difference frequency, \( r \), can have a maximum value of

\[
r_{\text{max}} = 2\pi W_{\text{rf}}
\]
where $W_{if}$ denotes the intermediate-frequency bandwidth in cycles/sec. Therefore under the worst conditions, we must have

$$\tau_e = W_{if}Rg \leq \frac{1 - a^2}{2\pi a}. \quad (10)$$

The maximum permissible value of $\tau_e$, as given by the equality sign in Eq. 10, is shown in Fig. 5 plotted vs. $a$. The plotted values are also listed in Table VII. Multiplication by a scale factor appropriate to the value of $W_{if}$ used will convert the normalized values of time constant to read in microseconds.

For conditions at the input to a second limiter stage, which immediately follows the filter of the first limiter, the problem is greatly simplified if the concept of the ideal filter is used here, as is quite feasible. Thus, if we consider the filter in the plate circuit of the first grid-bias pentode limiter to be an ideal one, then the input signal to the next stage is simply described in terms of the worst configuration of sideband components that this filter will pass. The amplitude of the resultant of these components will not be constant since, in general, the components will include only a finite number of those having significant amplitudes at the output of the first limiter. If the limiter performance is also assumed ideal, then the results of the spectral analysis of the first chapter are directly applicable, and the experience gained there is useful. Thus if the first limiter filter is assumed to have a bandwidth equal to one $1-f$ bandwidth (as is permissible for all $a \leq 0.863$) then we know (from Section 1.6) that the configuration $A_0$, $A_{-1}$ will be the worst one possible for all $a$'s up to about 0.84, and so, for all such cases the result of Eq. 9 is directly applicable with $a$ replaced
TABLE VII

Maximum permissible values of normalized time constants

\[ T_f = \frac{fW_iR_C}{g} \]

in the grid circuit of the:

<table>
<thead>
<tr>
<th>a</th>
<th>( T_f )</th>
<th>( 2\varphi_1 )</th>
<th>( 3\varphi_1 )</th>
<th>( 2\varphi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.7639</td>
<td>1.5514</td>
<td>3.115</td>
<td>4.9694</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4828</td>
<td>1.0001</td>
<td>2.018</td>
<td>2.1361</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5342</td>
<td>0.7131</td>
<td>1.451</td>
<td>1.1548</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2387</td>
<td>0.5308</td>
<td>1.0936</td>
<td>0.7014</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1698</td>
<td>0.3994</td>
<td>0.8396</td>
<td>0.4520</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1160</td>
<td>0.2950</td>
<td>0.6380</td>
<td>0.2962</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0716</td>
<td>0.2042</td>
<td>0.4652</td>
<td>0.1877</td>
</tr>
<tr>
<td>0.85</td>
<td></td>
<td></td>
<td></td>
<td>0.1425</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0336</td>
<td></td>
<td></td>
<td>0.1004</td>
</tr>
<tr>
<td>0.95</td>
<td></td>
<td></td>
<td></td>
<td>0.0576</td>
</tr>
<tr>
<td>0.98</td>
<td>0.0064</td>
<td></td>
<td></td>
<td>0.0280</td>
</tr>
</tbody>
</table>

\[ W_L = \frac{(BW)_{lim}}{(BW)_{if}} \]

†
by $A_{-1}/A_0$. Since $(A_{-1}/A_0) < \xi$, it is readily recognized that the highest permissible value of the time constant $R C$ is larger at the input to the second limiter than it was at the input to the first — a decided improvement in the design conditions. The extent to which improvement has been achieved is readily seen from the curve for $2\tau_1$ shown in Fig. 5. Conditions at the input to the third limiter stage are similarly computed if the second stage is assumed identical to the first one, and so on. Figure 5 also shows a plot marked $3\tau$, which applies for the time constant at the input to the third limiter.

When the ideal filter following the first limiter is considered to have three times the 1-f bandwidth, values of $\xi$ up to 0.9807 may be considered. The computation is restricted here to $\xi = 0.85$ and $\xi = 0.9$ however. The critical configuration for this value of bandwidth is $M = 1$, $N = 2$ and the spectral analysis of Section II.1 of this chapter is directly useful for a further study of conditions at the input to a third limiter stage that may follow. Higher values of first-limiter bandwidth may be used, and the same reasoning, as before, applies to the conditions at the input to the second limiter. However, in the general case where the configuration $m \neq 0$, $N > 1$ is to be handled, the step that corresponds to the transition from Eq. 8 to Eq. 9, above, is not obvious by inspection of the expression corresponding to Eq. 8, but must be done through a process of minimization of the quantity $-A(t)/A'(t)$ as is indicated in formula 7.

With reference to Fig. 6 of Chapter One, we recall that the square of the amplitude function of the resultant of the configuration $M, N$ is given by

$$A^2(t) = \left[ \sum_{n=-N}^{M} A_n \cos nrt \right]^2 + \left[ \sum_{n=-N}^{M} A_n \sin nrt \right]^2$$

$$= \frac{F}{\sqrt{N-M}} (rt), \quad \text{say.} \quad (11)$$
With \( A(t) = \frac{1}{2}F^2(rt) \), and \( \Phi = rt \), we may write

\[
\frac{A(t)}{A'(t)} = \frac{(2/r) F(\Phi)}{F'(\Phi)}
\]

and the condition 3 becomes

\[
T \equiv rR_C g \leq \left[ -2 \frac{F(\Phi)}{F'(\Phi)} \right]_{\text{min}}
\]

(13)

When the first-limiter bandwidth used is three times the i-f bandwidth, the most unfavourable situation arises when \( M = 1, N = 2 \), and \( r = 2\pi W_{if} \). In this case,

\[
F_{L}(\Phi) = A_0^2 \left[ c_1 + a_1 \cos \Phi + \beta_1 \cos 2\Phi + \gamma_1 \cos 3\Phi \right]
\]

(14)

where \( c_1, a_1, \beta_1 \) and \( \gamma_1 \) are combinations of sums and products of the amplitudes, \( A_{in} \). It is thus readily shown that when expression 14 is substituted in 13, and the derivative of the resulting right-hand member is set equal to zero, the value of \( \Phi \) that gives a minimum is a root of a fifth-degree equation in \( \cos \Phi \). The root was determined graphically for various values of \( q \), and the results were used to determine the maximum permissible value of \( 2 \frac{L}{L} 3 \equiv W_{if} R_C g \)

(in the grid-circuit of the second limiter) when the first-limiter bandwidth is three times the i-f) as a function of \( q \). The resulting maximum permissible values of \( 2 \frac{L}{L} 3 \) are plotted in Fig. 5, also, for ease of comparison with the other normalized time constant curves. The numerical values are also presented in Table VII.

With larger values of first-limiter bandwidth, the variations in the amplitude of the most troublesome signal delivered to the second limiter, become less and less severe, and the upper bound on the permissible \( R_C g \) product at the input to the second stage becomes higher and higher. This is also brought out by the plots of \( A(t) \) for various
configurations shown in Fig. 6.

We may now conclude that although the action of a narrow-band limiter upon the character of the resultant signal is such as to effect only a partial reduction in the severity of the amplitude variations of the signal (instead of a complete smoothing out process), the partial abatement of the amplitude variation is nevertheless sufficient to show a significant increase in the upper bound on the permissible grid-circuit time constant. Eventually, with a sufficient number of cascaded narrow-band limiters, the resultant signal amplitude is ironed out to an essentially constant value, and the upper bound on the RC product in the later stages is sufficiently high to be of no importance in the design of these stages.

It may, finally, seem unnecessary to recall, before leaving this topic, that at least two other important considerations must be kept in mind in the choice of the grid-bias pentode limiter $R_e$ and $C_e$ values. These considerations have been briefly discussed at the beginning of this discussion.

B. Discriminator Time Constant Requirements:

The low-frequency output circuit of a discriminator may, in general, be reduced to the form shown in Fig. 7, which is more convenient for the present study. The detected voltage which is proportional to the instantaneous frequency variations of the signal at the discriminator input, appears across the equivalent RC combination shown. For some circuits, this RC combination is a reduced form of a slightly more elaborate connection involving two resistors and two capacitors; in others (notably the fast-acting discriminator circuit devised by Granlund) $C$ is the capacitor across which the output voltage of the discriminator is taken, and $R$ is the total equivalent resistance seen in parallel with $C$ and composed mainly of the low equivalent output resistance that $C$ sees looking back into the rest of the detector circuit. In any case, the time constant
of the equivalent combination shown, must be sufficiently low to enable the voltage across the capacitor, C, to follow the detected voltage. Failure of the voltage across the capacitor to follow the voltage variations dictated by the instantaneous frequency of the signal at the input to the discriminator, will result in the output low-frequency voltage having an average value (over one difference frequency cycle) which does not correspond to the value dictated by the frequency of the stronger signal at the input to the receiver. This obviously defeats our purpose, and the restrictions on the RC product for this loss of desired average voltage level not to arise will now be determined.

Let us consider first the situation in which the discriminator is either amplitude-insensitive and hence is not preceded by any limiters, or amplitude-sensitive but preceded by an "infinitely"wideband ideal limiter. In either case, if the discriminator is assumed to have a linear overall detection characteristic of unit slope, over the whole range of the instantaneous frequency variations of the input signal, then, under the two-path conditions described in Chapter One, the voltage waveform that the output RC combination must handle is given by
$$e(t) = r \frac{a \cos rt + a^2}{1 + 2a \cos rt + a^2}$$  \hspace{1cm} (15)$$

This waveform is superimposed upon a direct voltage component that corresponds to the level dictated by the frequency, \( p \), of the stronger of the two signals. If the average value of the output capacitor voltage (over a period of \( 2\pi/r \) secs) is to be maintained at the value dictated by the frequency, \( p \), the capacitor must, at every instant of time, tend to charge or discharge at a rate that is faster than (or at worst just as fast as) the rate at which the impressed waveform tends to change at that instant. This will insure sufficient rapidity of charging and discharging to enable the capacitor voltage to remain in step with the dictates of the instantaneous frequency of the resultant input signal, and thus will insure proper detector operation and lack of harmful diagonal clipping effects. In perfect analogy with the similar problem of the limiter time constant, the restriction to be imposed on the RC product is, from condition 3, above,

$$RC \leq \left[ \frac{-e(t)}{e'(t)} \right]_{\text{min}}, \quad e'(t) = \frac{d}{dt} e(t).$$  \hspace{1cm} (16)$$

A direct attempt to substitute from Eq. 15 into 16, however, meets with a frustration, since \( e(t) \) in Eq. 15 goes to zero when \( rt = \cos^{-1} (-a) \), and so it would appear that the only permissible value of RC is zero — an obvious absurdity. This absurdity is quickly resolved, however, if the expression used for \( e(t) \) is such that \( e(t) \) does not present the mathematical difficulty of going to zero at any instant during a difference-frequency cycle. The expression that must be used here, therefore, is the one that describes the deviations of the instantaneous frequency from the frequency of the stronger signal when the stronger signal is above the weaker signal in frequency. This amounts to a mathematically expedient change of notation, and does not affect the physical argument. With reference
to Fig. 2 of Chapter One, it is readily seen that this change of notation replaces the spike train described by the solid curve, in our consideration, by the one depicted by the dotted curve. Analytically, the dotted plot may be described by Eq. 15 if \( a \), there, is restricted to values greater than 1 (as is also indicated in Fig. 2 of Chapter One). However, in order to conform to our previous understanding that \( a \) throughout these discussions is always less than 1, our purposes will be served well if in Eq. 15 \( a \) is replaced by \((1/a)\). Accordingly,

\[
e_1(t) = r \frac{1 + a \cos rt}{1 + 2a \cos rt + a^2}.
\]

That Eq. 17 describes the desired spike train may also be shown by introducing a transformation that moves the origin from the point marked \( p \), on the vertical axis, in Fig. 2, Chapter One, to the point marked \( p + r \), and then following this by a transformation that replaces the original function by its mirror image about the new horizontal axis. The result of the two transformations combined is

\[
e_1(t) = r \left[ 1 - \frac{a \cos rt + a^2}{1 + 2a \cos rt + a^2} \right]
\]

\[
= r \frac{1 + a \cos rt}{1 + 2a \cos rt + a^2}
\]

which confirms Eq. 17.

We may, thus, proceed to show that, when the right-hand member of the condition 16 is evaluated by substitution from Eq. 17, and the derivative of the result is set equal to zero, the value of \( \Phi = rt \) that gives the minimum of the right-hand member of 16 turns out to be the root of a third degree equation in \( \cos \Phi \). This root was evaluated for various values of \( a \), and the results were used to determine the maximum permissible
value of $\gamma_{d_0} = W_{1f}R_C$ as a function of $a$. The computed values of $d_0$ appear in Table VIII, and are plotted in Fig. 8. The vertical scale in this figure may be calibrated to read in microseconds by multiplication by a scale factor appropriate to the value of $W_{1f}$ used.

When the discriminator is preceded by one stage of narrow-band limiting, the formulas of the preceding computation are directly applicable to the computation of the maximum permissible discriminator $R_C$ product when the limiter bandwidth is equal to one $i-f$ bandwidth. Here, since the most unfavorable conditions at the output of the limiter filter arise when only $A_0$ and $A_1$ pass, and $r = 2nW_{1f}$, it is only necessary to replace $a$ in those formulas by $(A_1/A_0)$. Thus, by direct computation, the maximum permissible values of $\gamma_{d_1} = W_{1f}R_C$ are as presented in Table VIII and plotted in Fig. 8. It is recalled from Chapter One that the conditions of the present situation apply only in the range $a < 0.84$; consequently the curve for $\gamma_{d_1}$ is only plotted for $a < 0.84$.

When two identical bandpass limiters, each having one $i-f$ bandwidth, precede the limiter, the maximum permissible values of $\gamma_{d_1} = W_{1f}R_C$ are those shown in Table VIII and plotted in Fig. 8.

We consider finally the situation in which the discriminator is preceded by a limiter whose bandwidth is three times that of the $i-f$. Here the instantaneous frequency pattern that accompanies the configuration $M = 1, N = 2$, with $r = (BW)_{1f}$, dictates the critical requirements. Simple as this configuration may seem, the computation in this case is extremely tedious. It is identical in principle to the approach used above, but now the value of $\phi = rt$ that leads to the minimum value of the right-hand member of condition 16 turns out to be a root of the equation
TABLE VIII

Maximum permissible values of normalized time constant $\tau_d = W_{1f} \cdot RC$ in the output circuit of the discriminator when the latter is preceded by $n$ narrow-band limiters, where:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\tau_{d_0}$</th>
<th>$1\tau_{d_1}$</th>
<th>$2\tau_{d_1}$</th>
<th>$1\tau_{d_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.553</td>
<td>3.1513</td>
<td>6.34</td>
<td>1.611</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7234</td>
<td>1.5282</td>
<td>3.10</td>
<td>0.8362</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4316</td>
<td>0.97742</td>
<td>2.00</td>
<td>0.5911</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2790</td>
<td>0.67079</td>
<td>1.427</td>
<td>0.4501</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1848</td>
<td>0.48147</td>
<td>1.062</td>
<td>0.3546</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1213</td>
<td>0.34546</td>
<td>0.801</td>
<td>0.2617</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0762</td>
<td>0.23971</td>
<td>0.593</td>
<td>0.1850</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0432</td>
<td>0.15238</td>
<td>0.413</td>
<td>0.1191</td>
</tr>
<tr>
<td>0.85</td>
<td></td>
<td></td>
<td></td>
<td>0.0899</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0185</td>
<td></td>
<td></td>
<td>0.0615</td>
</tr>
<tr>
<td>0.95</td>
<td></td>
<td></td>
<td></td>
<td>0.0332</td>
</tr>
<tr>
<td>0.98</td>
<td></td>
<td></td>
<td></td>
<td>0.0152</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ \sum_{n=0}^{11} \lambda_n \cos n\phi = 0 \]

in which the coefficients, \( \lambda_n \), are extremely involved. Naturally, the desired root was determined graphically, and from it, the maximum permissible values of \( S_d \) were determined and tabulated in Table VIII. Those values are also shown plotted in Fig. 8.

It is of interest to observe that a comparison of the curve for \( S_d \) with that for \( S_{d0} \) reveals that the percentage by which the upper bound on \( S_d \) has been raised by the action of one limiter having three times the i-f bandwidth is greatest for values of \( a \) in the neighborhood of \( a = 1 \), and decreases rapidly with decreasing values of \( a \). This may be readily explained by the fact that a limiter bandwidth of \( 3(BW)_{if} \) approaches more closely the order of bandwidth needed to reproduce the input two-signal frequency spike pattern with increase singly less distortion as the value of \( a \) is decreased. In terms of the spectrum, the limiting configuration \( M = 1, N = 2 \) (i.e. the most unfavorable configuration for \( (BW)_{lim} = 3(BW)_{if} \)) forms an increasingly important percentage of the significant spectral components at the output of the limiter as the value of \( a \) is made smaller. In fact, for \( a = 0.1 \), the lowest value considered in the plot, this configuration includes essentially all of the components of significant amplitude (as is readily ascertained by reference to the spectral tables in Chapter One), and so the resulting instantaneous frequency pattern (which dictates the curve \( S_d \)) is a slightly distorted version of the original input two-signal pattern (which dictates the curve \( S_{d0} \)).

The plots of Fig. 8 show clearly how the upper bound on the permissible time constant in the output circuit of the discriminator is raised by preceding the discriminator by a process of narrow-band limiting. The effect produced
by one narrow-band limiter will, clearly, be displayed by additional stages. After a sufficient number of these has been cascaded, the upper-bound on the maximum permissible time constant values becomes sufficiently high to have no important restraint on the design of the output circuit of the discriminator.
II.3 EFFECT UPON HARMONIC STRUCTURE OF DETECTED SPIKE TRAINS

In general, after the instantaneous-frequency variations are properly translated into instantaneous voltage variations (about the direct-voltage level dictated by the frequency of the stronger signal), those voltage variations are modified by the action of the de-emphasis and audio filters which follow the discriminator circuit. If the difference frequency between the two input carriers lies beyond the range of audibility (as it will do much of the time with long-distance communication, but less frequently with communication over shorter distances) the Fourier components of the recurrent spike train will all be filtered out by the two low-pass filters. This will effectively be the end of the disturbance caused by the presence of the weaker signal within the i-f passband. However, if the difference frequency between the two paths is audible, the component having the fundamental frequency of recurrence, plus a number of harmonics depending upon the position of this frequency in the audible spectrum, will pass through the low-pass filters, and will therefore disturb the output signal.

Note that two factors play more or less obvious roles in minimizing the importance of the disturbance that leaks through: the action of the two low-pass filters in rejecting most of the harmonic components, and the fact that the magnitude of the interference spikes (and hence the amplitude of each constituent Fourier component of the detected spike train) is directly proportional to the value of the difference frequency. Consequently, even though more undesired harmonics of the difference frequency are likely to get through the low-pass filters as this difference frequency assumes lower and lower values in the audio range, the amplitudes of the passed components will also be lower and lower.
A third factor that tends to minimize the importance of the audible disturbance is brought about by the effect of narrow-band limiting upon the shape and magnitude of the spike trains. This distortion of the extraneous modulation by the narrow-band filter after the limiter generally results in the Fourier components of the modified spike train having smaller amplitudes than their counterparts in the structure of the undistorted waveform. The effect gets increasingly more pronounced with increasing values of the difference frequency, \( r \), and the ratio of weaker-to-stronger signal \( a \). On the other hand the lower the value of the beat frequency, \( r \), or the ratio \( a \), the less significant this effect gets. The reason for this dependence of the importance of the effect upon the magnitude of the difference frequency, \( r \), is readily appreciated from the fact that as \( r \) gets lower, the number of significant sideband components accommodated within the limiter filter passband increases. With given constant value of \( a \), the instantaneous frequency variations of the resulting signal therefore tend to become an increasingly less distorted copy of the instantaneous frequency pattern of the amplitude-limited resultant of the two signals delivered by the i-f amplifier. When \( r \) gets well within the audio range, the effect will tend to be negligible, especially with the lower values of \( a \), even when the smallest permissible value of one (BW) if is used after the limiter. The instantaneous frequency variations introduced by the presence of the weaker signal will then tend to be quite indistinguishable (and hence less separable) in their characteristics (pertaining to spike magnitude, or maximum deviation, repetition rate, and maximum time rate of change during a cycle of the fundamental) from the variations which represent the desired message modulation. When \( r \) is held constant at some audible value, the distortion of the spike train by the narrow-
band filtering after the limiter will have an increasingly noticeable effect with increasing values of \( a \), and therefore the effect of this distortion upon the amplitudes of the harmonic components in the structure of the detected spikes (which will presently be illustrated), becomes increasingly significant, and vice versa.

To illustrate the above ideas, consider the situation in which the two input carriers accommodated within the i-f passband have constant amplitudes and frequencies, and are impressed directly upon an amplitude-insensitive discriminator (or a discriminator which is preceded by an ideal limiter which passes essentially all of the significant spectrum centered about the frequency of the stronger signal). The detection of the instantaneous frequency variations of the resultant signal results, at the output of the discriminator, is a voltage proportional to

\[
f(t) = \frac{r a^2 + a \cos rt}{1 + 2a \cos rt + a^2}
\]

\[
= -r \sum_{n=1}^{\infty} (-a)^n \cos nrt. \tag{18}
\]

Equation 18 shows that the amplitude of each harmonic varies directly with \( r \). As a function of \( a \), the amplitude of the fundamental component is directly proportional to \( a \), and the amplitude of the \( n \)th harmonic component relative to the amplitude of the fundamental component is given by \( a^{n-1} \). Plots of the relative amplitudes of the various harmonics, as compared with the amplitude of the fundamental, are shown in Figs. 9(a), (b) and (c), where each is marked \( a^n \) - curve.

When an ideal limiter of some specified bandwidth \( W_L = (BW)_{lim}/(BW) \) is inserted in the path of the resultant signal before it gets to the discriminator, we have found that the action of the narrow-band limiting process is such as to dampen out the fast and large excursions of the
instantaneous frequency of the resultant signal that goes through. The effect of the resulting modifications in the waveform of the instantaneous frequency variations upon the amplitudes of the harmonic components in the structure of this waveform is best studied quantitatively by direct Fourier analysis of a few typical and informative cases. The general character of this effect may also be anticipated on the basis of the concept of the equivalent capture ratio defined in Chapter Three. This concept recognizes the importance of the repetition frequency, \( r \), and the spike magnitude of the instantaneous frequency spike train of the resultant signal at the output of a narrow-band limiter, in providing a basis for comparing the capture conditions at that point with those elsewhere in the receiver. On this basis, the effect of narrow-band limiting upon the capture conditions is equivalent to a reduction in the equivalent capture ratio. Consequently, this reduction should generally result in a reduction of the relative amplitudes of the various harmonics in the structure of the instantaneous frequency waveform. This effect is illustrated by the plots of Figs. 9(a), (b) and (c). The examples chosen for illustration correspond to resultant signals made up of a number of lower sideband components \( M \), and a number of upper sideband components \( N \), these numbers being indicated in parenthesis in the order \((M,N)\). Each configuration \((M,N)\) is associated with the proper set of points through a pointing arrow. The various configurations indicated make up most troublesome resultants when associated with the limiter bandwidths, \( W_L \), whose values are indicated on the plots. When the indicated values of \( W_L \) are used, the values of \( r \) for which these configurations can arise may not be audible, depending upon the i-f bandwidth used. If smaller values of \( W_L \) are used, such as unity for \( a = 0.8 \) or \( a = 0.85 \), some of the indicated configurations may arise with audible values of \( r \). In any case, the choice of the indicated
configurations is dictated by convenience of illustration only.

The computations leading to the plots of Figs. 9(a), (b) and (c) have shown that the absolute values of the harmonic components are generally decreased by the narrow-band limiting effect below the corresponding values given by \( a^n \) in the absence of narrow-band limiting. Figs. 9(a), (b) and (c) illustrate the decrease in the relative amplitudes of the harmonics as compared with the corresponding fundamental component. These results illustrate the effect of one stage of narrow-band limiting — the first one. A second or later stage will usually have a less troublesome signal at its input to cope with than the first or earlier intermediate stages. Nevertheless, these later stages will exhibit the same type of effect so long as the instantaneous frequency variations caused by the interference differ from the type of expected message modulation in characteristics that enable a narrow-band filter to distinguish them.
II.4 CONCLUDING REMARK

In retrospect, it should be clear that the various effects attributed to the action of a narrow-band limiter are all closely tied together. They all stem from the fact that the extraneous modulation imposed by the presence of the undesired, but weaker, signal within the i-f pass-band, differs in features that make it distinguishable from the type of expected message modulation, especially under the most troublesome interference conditions. Generally speaking, these conditions prevail when the two carriers are furthest apart in frequency while their individual frequency modulations are slow. The most notable distinguishing characteristics of the extraneous modulation relate to its high fundamental repetition rate, its large spike magnitude, and its steep slope on the flanks of the spike. As the discussion of the next chapter will further clarify, these are features that make it difficult for all but the very wideband filters to follow the excitation carrying that modulation. Therefore, they offer a basis for the abatement of their disturbance value through filtering that would be narrow-band for them, but sufficiently wideband for the desired message modulation to reproduce this modulation without impairing the quality of the reception.

The various interference-reducing qualities that result from the use of narrow-band limiting are therefore all well-tied together, and once the stiffest limitation in a specific design has been met by the proper number of cascaded narrow-band limiters, the resulting chain should meet all the remaining considerations within a more or less safe margin.
INTERFERENCE REJECTION IN FM RECEIVERS

by

ELID JOSSEF BAGHDADY

B. A., American University of Beirut
(1951)

S. M., Massachusetts Institute of Technology
(1954)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June, 1958

VOLUME II
Chapters Three and Four
Pages 160-381
CHAPTER THREE
THE THEORY OF QUASI-STATIONARY ANALYSIS
AND APPLICATIONS TO INTERFERENCE PROBLEMS
INTRODUCTION

In the preceding chapters we presented a detailed study of the effect of passing the resultant of two sinusoidal carriers differing in amplitude and in frequency, through an idealized limiter followed by an idealized selective filter, upon the instantaneous frequency as well as amplitude perturbations inflicted upon the resultant signal at the input to the limiter by the presence of the weaker of the two signals within the passband of the i-f amplifier. This study embodied a switch in the basic approach to the question of limiter bandwidth requirement, and in the philosophy of the limiter's share of the task of interference suppression in FM reception. We now ask of the limiter stage to do more than just eliminate undesirable changes in the amplitude — we also require that it contribute to the abatement of the FM disturbances wrought by the presence of the weaker signal, as well, by decreasing the range and intensity of the extraneous instantaneous frequency variations of the resultant signal through a process of instantaneous frequency limiting, or perhaps quasi-limiting. Evidently the undesirable frequency changes must differ in the degree of their extent, rate of change, and rate of recurrence, any one of these or all combined, from those of the "garden variety" of changes that a usual message causes in the instantaneous frequency of the carrier, in order for the narrow-band limiting effect not to distort the desired message modulation as well. Granted these differences, the limiter proper in this new task prepares the resultant signal for the (frequency-limiting) treatment by eliminating the amplitude changes. This spreads out the significant spectrum, and the sluggish (narrow-band) filter, immediately following, performs the (frequency-limiting) operation on the instantaneous frequency of the resulting signal by refusing to follow
the more drastic frequency variations, or equivalently, by eliminating portions of the spread-out spectrum that owe their existence to the interfering signal and which are necessary for the undistorted reproduction of the instantaneous frequency spike pattern of the amplitude-limited resultant of the two input sinusoids. The minimum requirement of one i-f bandwidth for the limiter filter is calculated to meet the prerequisites of undistorted reproduction of the expected message modulation. The situation is here entirely analogous in philosophy to the well-known procedures for eliminating certain types of AM noise and interference of the impulsive variety. There, a clipper is introduced in the path of the input signal whose threshold exceeds the maximum value of instantaneous amplitude expected with the desired message amplitude modulation in order to leave this desired message modulation unaffected. The impulsive interference should therefore exceed the clipping threshold, and the duration of the individual impulses should be short compared with the period of a desired modulating frequency in order for the "noise-silencing" scheme to introduce noticeable differences between the resultant waveforms in the presence of the interference and in its absence. In other words, the modulation introduced by the interference must differ in a manner that can be distinguished by the circuitry in the path of the signal from the changes introduced by the desired message modulation in order for the abatement of the interference to remain possible through special scheming or design that cannot affect the desired message significantly.

In the present chapter the emphasis in the study of the effect of a linear narrow-band filter upon the amplitude-limited resultant of two sinusoids, will shift from the frequency domain, where in the preceding chapters it was studied on a spectrum basis, to the time domain where the spectrum will not be directly involved. Here
the filter will be viewed not as a selector of certain frequency bands, but as a system containing energy-storage elements, and therefore excercising a degree of inertia to fast, sharp, or large, (or all combined) changes in amplitude or in instantaneous frequency. Since we will be anxious to draw comparisons with the results reached in the preceding chapters on the spectrum basis, the amount of inertia that a filter will excercise in the presence of the type of signals that we will discuss will be conveniently expressed in terms of the filter bandwidth measured between half-power points. Since the range of our explorations will be severely hampered by the tedium and complexity of the mathematics, we shall concentrate on the simpler problems which promise a minimum of labor and a maximum of enlightenment. In particular we shall be concerned with an important question which bears heavily upon the Arguinbau-Granlund "widebanding" theory --- namely, how wide is the Arguinbau-Granlund "wideband"? More clearly stated, we want to determine the extent of the filter bandwidth after the limiter which is necessary for a close reproduction of the instantaneous-frequency spike trains produced by the superposition of two carriers which are arbitrarily close in relative amplitudes. It may be argued that the answer to this is immediately available from Figs 18 and 25 of Chapter One. But those figures are the results of a study involving idealized filters and the effect of the bandwidth of those filters upon the spectrum passed to the next stage of limiting, or to the discriminator. On the other hand, the Arguinbau-Granlund theory utilizes simple parallel-tuned LC circuits, or at worst a pair of critically-coupled tank circuits, and the estimated bandwidths are arrived at largely on an instantaneous-frequency basis. Therefore, it behooves us to re-examine our results in these very terms, before taking critical issue with the Arguinbau-Granlund "widebanding" theory.
The spectral approach of the preceding chapters is usually referred to as the Fourier method of solving FM as well as AM steady-state problems. The spectrum of the impressed signal is first determined, and the spectrum of the steady-state response of the filter is then obtained by multiplying each input spectral component by the value of the system function of the filter evaluated at the frequency of the input component considered. This approach is extremely reliable and is unbeatable for conceptual simplicity and straightforwardness, and the results and implications of any other approach must agree with the predictions of the Fourier method for a vote of confidence. But, unfortunately, when the number of significant spectral components that must be taken into account is large, the computation becomes extremely laborious, and the physical feeling and appreciation of the results may be lost in a maze of numbers. In this regard, the idealization of the filter amplitude and phase characteristics in the preceding chapters has gone a long way toward simplifying the computations involved. Contrary to prevalent thought, based on scrupulous enforcement of certain fundamental aspects of modern filter theory, this theory itself shows that the idealized filter characteristics assumed in the previous chapters may be approached quite closely by physical filters. Therefore, where these idealizations are helpful, their assumption is no unacceptable deviation from reality. Indeed, an idealized bandpass filter is as unrealistic as an idealized amplitude limiter, or an idealized linear amplifier, or resistor, or lossless network, or any other one of tens and scores of idealized concepts in engineering and science.

The conveniences of the idealized bandpass filter are not available to the discussions of the present chapter. In fact, for reasons that will be clarified in this chapter, the solution of steady-state FM problems
involving idealized bandpass filters by means of the instantaneous frequency concept is permissible only in a limited number of relatively uninteresting situations, even though the criteria for the applicability of the concept may appear to be satisfied. The same considerations, however, that will rule out this applicability, will provide a basis for comparing the significance of the term bandwidth as applied to the idealized filter with its significance in the case of non-idealized filters. The non-idealized filters (as far as system function is concerned) that will be discussed here are the high-frequency parallel-tuned model of a tank circuit, and the second-order bandpass Butterworth-type filter. For these filters the conditions for the applicability of the concepts of instantaneous frequency and instantaneous amplitude to the determination of the steady-state FM and AM response (as opposed to the Fourier spectral approach) will be determined for assumed sinusoidal modulations as well as for modulations of the type exhibited by the resultant of two sinusoids of different amplitudes and frequencies. These conditions will be expressed in terms of bandwidth requirements on the filters. The approach used is based fundamentally upon the work of Carson and Fry, and of van der Pol. The results will then guide first a reexamination of the conclusions of the preceding chapters, and perhaps recast those conclusions in a new frame of thinking; and, second, a reevaluation of the Arguimbau-Granlund "widebanding" theory.
III.1 FILTER RESPONSE TO MODULATED WAVES

At the basis of the present investigation lie two series expansions of the steady-state response of a filter to a sinusoidal carrier modulated in amplitude and in frequency. The first of these expansions was first presented by Carson and Fry (Ref. 1). Carson and Fry were also the first to explore the conditions for the validity of a quasi-stationary solution by presenting a second expression for the response of the network as a sum of the quasi-stationary term and a correction series depending on the derivatives of the sinusoidal steady-state system function evaluated at the frequency corresponding to the unmodulated carrier frequency of the excitation, and the time derivatives of the frequency-modulating time function of the excitation. The second expansion of interest to us was first derived by van der Pol (Ref. 2) and by Stumpers (Ref. 3). It differs from the second Carson and Fry expansion in that in addition to involving the time derivatives of the modulating function, the correction series appears explicitly in terms of the derivatives of the system function evaluated at the same value of instantaneous frequency as does the quasi-stationary term. In their publication, Carson and Fry did show that under conditions of large modulation index and slow modulation their second expansion could be approximated by the quasi-stationary term plus the first term of the correction series expressed in a form identical to that of the correction series of van der Pol and Stumpers. Therefore in recognition of the combined contribution of all of these pioneers, we propose to refer to the expansion of the FK steady-state response as a sum of the quasi-stationary solution plus the correction series of van der Pol and Stumpers, as the JFPS expansion.

The trend in our presentation will be toward adapting the development of the expansions and their properties
to the needs, requirements, and substantiation of the various concepts that are fundamental to the discussions of these chapters. We may reserve legitimate title of originality for many of the things said and done, particularly in what pertains to supplying sharply focused justifications, where such are needed, and emphasizing pertinent and helpful properties in order to enhance the appreciation of, and the confidence in, the results and conclusions. Our applications, on the other hand, are chosen to answer specific questions of fundamental interest to our main study, and not to illustrate all that could be done with these expansions.

Consider the differential equation relating the current through some branch in a linear, lumped, finite, passive and bilateral network to the voltage across the same, or a different, branch in the network, namely,

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} i(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} e(t) \tag{1}$$

This may also be written in the form

$$P(\frac{d}{dt})i(t) = Q(\frac{d}{dt})e(t). \tag{2}$$

For an excitation of the type

$$i(t) = e^{j \left[ pt + \theta(t) \right]} \tag{3}$$

express the steady-state response in the form

$$e(t) = E(t)e^{jpt}. \tag{4}$$

A substitution of 3 and 4 into 2 may readily be shown to result in
\[ Q(jp + d/dt)E(t) = P(jp + d/dt)e^{j\theta(t)} \]  
\( \text{or} \)
\[ E(t) = \frac{P(jp + d/dt)}{Q(jp + d/dt)} e^{j\theta(t)} = Z(jp + d/dt)e^{j\theta(t)} \]

where the impedance function, \( Z \), is the same rational function of the operator \( jp + d/dt \) as it is of the more familiar complex frequency variable \( s \). Assuming \( Z \) and \( \theta(t) \), as well as all of their derivatives, to be bounded and continuous everywhere, we may expand Eq. 6 in a Taylor's series in powers of \( d/dt \) in the form

\[ E(t) = \sum_{k=0}^{\infty} \frac{1}{k!} Z^{(k)}(jp) \frac{d^k}{dt^k} e^{j\theta(t)} \]

\[ = \left[ Z(jp) + Z'(jp)\frac{d}{dt} + \frac{1}{2} Z''(jp)\frac{d^2}{dt^2} + \ldots \right] e^{j\theta(t)}. \]

This is the first expansion, it is appropriately called Carson and Fry expansion. As is often the case with first derivations, the approach of Carson and Fry is not as simple as the above widely-adopted approach.

Let us now consider the excitation \( i(t) \) of Eq. 3 to be the amplitude-limited resultant of two or more sinusoids of the type appearing at the output of an idealized narrow-band limiter when the i-f amplifier delivers two carriers of relative amplitudes 1 and a (\(< 1\)), and of frequencies \( p \) and \( p + r \) rad/sec. Also let the function \( Z \) be the system function of the FM-to-AM converting network (or discriminator). Then, if the envelope \( E(t) \) is faithfully detected by an ideal linear amplitude detector of unit detection efficiency, the first term, \( Z(jp) \), in Eq. 7, represents the direct voltage level dictated by the frequency, \( p \), of the stronger of the two carriers delivered by the i-f amplifier, when this
frequency is considered to be essentially constant, or perhaps varying very slowly. The second term, \( iZ'(jp) \frac{d\theta}{dt} \), is the first-order distortion term due to the interference, and so on. It is clear that the system function \( Z \) must be some function of frequency for FM-to-AM conversion to be possible. It is also clear that the distortion due to the interference, as well as the distortion that a non-linear \( Z(jp) \) as a function of \( p \) will introduce into the desired relatively slow direct voltage level variation whose instantaneous value is given here by \( Z(jp) \) itself, will be minimized if \( Z(j\omega) \) is a linear function of \( \omega \) at least over the expected range of frequency variations. For under this condition, only the first two terms of the expansion \( 7 \) will be different from zero, and the detected output will consist of the slowly varying direct voltage level, \( Z(jp) \), dictated by the frequency \( p \), and the interference term \( Z'(jp) \frac{d\theta}{dt} = \frac{d\theta}{dt} \), if the slope of \( Z \) over its linear range is taken as unity.

This of course assumes that the network described by \( Z \) is capable of translating the instantaneous frequency deviations \( \frac{d\theta}{dt} \), from the frequency, \( p \), of the stronger signal (caused by the presence of the weaker signal within the i-f passband), into instantaneous envelope variations without any significant distortion of the waveshape of \( \frac{d\theta}{dt} \). Two questions immediately arise: the first concerns the desirability of this faithful translation of the waveshape of \( \frac{d\theta}{dt} \) from a frequency variation to an amplitude variation; the second concerns the explicit conditions on \( Z \) as a function of frequency that will insure this faithful translation. The first question is immediately disposed of if we recall that the average value of \( \frac{d\theta}{dt} \) over one cycle of the difference frequency between the two carriers delivered by the i-f, is exactly zero if the resultant of those two carriers is delivered to the discriminator through a limiter of proper bandwidth. Therefore if \( \frac{d\theta}{dt} \) is faithfully
reproduced and envelope-detected, the average value of

\[ Z(jp) + jZ'(jp)d\theta/dt \]

over one difference-frequency cycle is exactly \( Z(jp) \), the direct voltage level dictated by the frequency of the stronger signal. To insure this faithful reproduction of \( d\theta/dt \), the discriminator network must be capable of following the excitation of Eq. 3 through stationary states — that is, at every value of the instantaneous frequency \( p + d\theta/dt \), we should be able to approximate the instantaneous amplitude of the response, on a steady-state basis, by computing the value of \( |Z(jp + j\Delta \theta)| \) and multiplying it by the (constant) amplitude of the impressed excitation. The conditions that \( Z \) must satisfy in order for this quasi-stationary analysis to apply will be explored after the second more pertinent CFFS expansion has been derived. At this point we may point out that if \( Z(j\omega) \) is a continuous function possessing finite derivatives of all orders, then \( Z(j\omega) \) will satisfy the necessary requirements if \( Z(j\omega) \) is characterized, over the range of expected instantaneous frequency variations by an essentially linear, or flat, magnitude, and an essentially linear phase characteristic with a small time delay, although some of these or all of them collectively are not necessary conditions.

The expansion \( \gamma \) is helpful in understanding the discriminator action in still another way. If the excitation is described by

\[ i(t) = e^{j \left[ \omega_0 t + \Phi(t) \right]} \]

where \( \omega_0 \) is the unmodulated value of the impressed carrier frequency, and \( d\Phi/dt \) is some desired message modulation, then it is desirable in practice to use a balanced discriminator for which \( Z(j\omega_0) = 0 \). If in addition, a straight line FM-to-AM conversion characteristic is used, the
discriminator converts the impressed wave into

\[ e(t) = Z'(j\omega_o) \frac{d\phi}{dt} \cdot e^{j(\omega_o t + \phi(t))} \]

whereupon an ideal envelope detector of unit detection efficiency will yield a detected voltage given by

\[ |Z'(j\omega_o)| \frac{d\phi}{dt} \cdot \]

Of course, \(Z(j\omega)\) must maintain the same slope, \(Z'(j\omega_o)\), over the whole range of \(d\phi/dt\) for faithful reproduction. If we go back to the excitation of Eq. 3, we find that, with \(p\) varying much more slowly than \(d\phi/dt\), we can easily visualize the output

\[ Z(jp) + jZ'(j\omega_o) \frac{d\phi}{dt} \]

as being a slowly varying d-c component \(Z(jp)\) which is directly proportional to the deviation of \(p\) from \(\omega_o\) (which deviation is the desired information borne by the stronger signal), with an interference component

\[ Z'(j\omega_o) \frac{d\phi}{dt} \]

superimposed upon it.

For a physical impedance function \(Z(j\omega)\), Eq. 7 indicates that the FM-to-AM conversion characteristic is, for all practical purposes, sufficiently linear if it can be shown that for the intended detection of the modulation \(d\phi/dt\),

\[ (1/2) |Z''(jp) \frac{d^2\phi}{dt^2}| < |Z'(jp) \frac{d\phi}{dt}| \] (8)

It is readily ascertained that this condition must be satisfied in the more conservative form

\[ (1/2) \left| \frac{Z''(jp)}{Z'(jp)} \right|_{\text{max}} \cdot \frac{\phi''(t)}{\phi'(t)}_{\text{max}} |\phi(t)|_{\text{max}} \ll 1 \] (9)
since the frequency $p$ may coincide with the frequency
for which the ratio of the impedance derivatives is a
maximum. The implications set by the desirability of
a small value of $|Z''/Z'|$ max on the character of $Z$ are
clarified by setting $Z = A(\omega)e^{j\phi(\omega)}$ and evaluating the
indicated ratio. This led to

$$
\frac{Z''(j\omega)}{Z'(j\omega)} = 2\phi'(\omega) - \frac{A''(\omega)/A(\omega) + \phi'^2(\omega) + \phi''(\omega)}{\phi'(\omega) - j(A'(\omega)/A(\omega))}.
$$

(10)

It is clear from this expression that if $A(\omega)$
and $\phi(\omega)$ are linear functions of frequency, then the
ratio is decreased significantly. Thus set

$$
A(\omega) = m(\omega - \omega_0), \text{ and } \phi(\omega) = t_d(\omega - \omega_0).
$$

(11)

Substitution into Eq. 9 leads to

$$
\left|\frac{Z''(j\omega)}{Z'(j\omega)}\right| = t_d \left(1 + \frac{3}{1 + (\omega - \omega_0)^2 t_d^2}\right)^{1/2}
$$

which assumes, at $\omega = \omega_0$, a maximum value of

$$
\left|\frac{Z''(j\omega)}{Z'(j\omega)}\right|_{\text{max}} = 2t_d = 2\phi'(\omega).
$$

(12)

Evidently, therefore, the ratio $|Z''/Z'|$ for any discriminator
circuit, will be minimized if in addition to making $A(\omega)$
and $\phi(\omega)$ linear functions of frequency, the slope of
$\phi(\omega)$ (or the time delay $t_d$) is made as small as possible.
We note with great curiosity that it is the slope of $\phi(\omega)$
that sets the final limitation and not that of $A(\omega)$.

The condition on the linearity of $A(\omega)$ is too well-
known to deserve any special comment. However, as far
as the author knows, the conditions on the linearity
and slope of $\phi(\omega)$ have heretofore not been known or
appreciated in the FM discriminator art. We have reasons
to suspect that the conditions on the character of $\phi(\omega)$
may play an important role in the faithful FM-to-AM.
conversion of instantaneous frequency waveforms having steep edges (or edges with short rise time), as well as instantaneous frequency pulse-like waveforms.

We now turn to a second expansion, the CFFS expansion, which will be of prime interest to the following discussions. The derivation we shall first present and discuss is an adaptation of work published by Clavier (Ref. 4).

We again consider the network to be linear and passive, but we now characterize it by its unit impulse response, \( h(t) \). If the current source excitation is given by

\[
i(t) = e^{j[pt + \theta(t)]}
\]

and we denote the steady-state response voltage by

\[
e(t) = E(t)e^{j[pt + \theta(t)]}
\]

we have, from a well-known superposition integral,

\[
e(t) = \int_0^\infty h(\gamma) i(t-\gamma) d\gamma
= e^{j[pt + \theta(t)]} \int_0^\infty h(\gamma) e^{-j[p + \theta'(t)]} \phi(t, \gamma) d\gamma
\]

where

\[
\phi(t, \gamma) = e^{j[\theta(t-\gamma) - \theta(t) + \theta'(t)]}
\]

If \( \theta(t) \) and its derivatives of all orders are everywhere continuous and bounded, and if in addition \( \theta(t) \) does not vanish exponentially at \( \gamma = t \), we may expand \( \theta(t) \) in the Taylor's series

\[
\theta(t-\gamma) = \theta(t) - \theta'(t) + (1/2) \theta''(t) \gamma^2 - (1/3) \theta'''(t) \gamma^3 + \ldots
\]

which converges uniformly and absolutely for all values of \( \gamma \). Therefore

\[
\theta(t-\gamma) - \theta(t) + \theta'(t) = (1/2) \theta''(t) \gamma^2 - (1/3) \theta'''(t) \gamma^3 + \ldots
\]
and

\[ g(t, \tau) = e^{j[(1/2) \Theta''(t) \tau^2 - (1/3) \Theta''(t) \tau^3 + \ldots]} \]

\[ = 1 + j(1/2)\Theta''(t)\tau^2 - j(1/3!)\Theta''(t)\tau^3 \]

\[ + (1/4!)(j\Theta^1 - j\Theta^2)\tau^4 \]

\[ + (1/5!)(100^2\Theta^n - j\Theta^5) \]

\[ - (1/6!)(j15\Theta^n + 15\Theta^n \Theta^1 + 10\Theta^n \Theta^2 - j\Theta^6) \tau^6 \]

\[ + \ldots \]

(17)

This series is uniformly convergent. If every term of this uniformly convergent series is multiplied by the bounded function \( h(t) \) the result is also uniformly convergent, and consequently integrable term by term.

The function \( h(t) \) is bounded because the assumed network is passive, and therefore its response to a unit impulse excitation has a bounded envelope made up of decaying exponentials barring impulses at \( t = 0 \). We may therefore substitute the expansion of Eq. 17 for \( g(t, \tau) \) in Eq. 15, and integrate term by term. If in addition we substitute from Eq. 14 into Eq. 15 for \( e(t) \), let \( \omega_1 = p + d\Theta/dt \) the instantaneous angular frequency of the excitation, and recall that

\[ Z(j\omega_1) = \int_0^\infty h(\tau)e^{-j\omega_1 \tau} d\tau \]

and

\[ Z^{(n)}(j\omega_1) = \frac{d^n Z(j\omega_1)}{d(j\omega)^n} \]

\[ = (-1)^n \int_0^\infty h(\tau)e^{-j\omega_1 \tau} d\tau \]

we arrive at the expression

\[ E(t) = Z(j\omega_1) + \frac{1}{2} \Theta''(t) Z^{(2)}(j\omega_1) \]
This expansion for the envelope function of the steady-state response of the filter to the frequency-modulated excitation, in terms of the system function, $Z(j\omega_1)$ as a function of the instantaneous frequency \( \omega_1 \), and the derivatives of $Z(j\omega_1)$, and in terms of derivatives of the angle-modulating function $\Theta(t)$, shall henceforth be referred to as the CFPS expansion, in contrast with the first Carson and Fry expansion given by Eq. 7. The usefulness of the CFPS expansion hinges upon its convergence properties, and its convergence is guaranteed, as is evident from the crucial turning points of the above derivation, if $\Theta(t)$ as well as $Z(j\omega)$ are everywhere continuous and possess derivatives of all orders, and if $Z(j\omega)$ characterizes a linear passive network. The expansion 18 states that if $\Theta(t)$ and $Z(j\omega)$, combined, are such that the second and later terms are negligible compared with the first, then the envelope of the steady-state response of the filter to the excitation angle-modulated by $\Theta(t)$ is proportional to the value of the system function $Z(j\omega_1)$ as a function of the instantaneous frequency $\omega_1 = p + d\Theta/dt$. This means that if $\Theta(t)$ and $Z(j\omega)$ satisfy the requirements of the derivation, and if

$$\left| \frac{1}{2} \frac{d^2 \Theta(t)}{dt^2} \right| \cdot \frac{d^2 Z(j\omega_1)}{d(j\omega_1)^2} < < |Z(j\omega_1)|$$

(19)
then the steady-state response of the filter will be essentially given by

\[ e(t) = Z(j\omega_1(t))e^{j\int_0^t \omega_1(t)dt} \]  

(20)

which is formally the same as the expression for the steady-state response to a constant-frequency excitation. Condition 19 is a consequence of the uniform convergence of the expansion 18 as it stands, under the stated conditions. The filter will therefore be said to follow the excitation through stationary states, meaning that at any instant of time, \( t_1 \), one may approximate the envelope value of the steady-state response by evaluating the sinusoidal steady-state value of the system function at the value of instantaneous frequency \( \omega_1(t_1) \) as though, at that instant, the excitation were of constant frequency equal to \( \omega_1(t_1) \). The condition 19 for the validity of the quasi-stationary response 20, may equivalently be considered to be the condition for the applicability of the instantaneous frequency concept as a tool in evaluating the steady-state response of a filter. If we set

\[ Z(j\omega) = A(\omega)e^{j\phi(\omega)} \]  

(21)

we observe that

\[ \frac{Z''(j\omega)}{Z(j\omega)} = \phi''(\omega) - \frac{A''(\omega)}{A(\omega)} - j\left[ \phi''(\omega) + 2\frac{A'(\omega)}{A(\omega)} \phi'(\omega) \right] \]  

(22)

The magnitude of this quantity multiplied by \((1/2)d^2\theta/dt^2\) must at all times be much less than unity in order to meet condition 19 for quasi-stationary response. It is clear therefore that not only the frequency behavior of the amplitude characteristic \( A(\omega) \) will influence the importance of the second term relative to the leading term in the expansion 18, but also the behavior of the phase characteristic \( \phi(\omega) \). It is therefore not generally sufficient to restrict the amplitude characteristic to be flat (or linear) for
discriminators) over the desired range, unless the phase characteristic is simultaneously made linear and the time delay, $\phi'(\omega)$, is made small.

In the next section, condition 19 will be applied to a few examples of general practical interest, and of particular value to our main interference problem.

Stumpers has demonstrated an interesting property of the Carson and Fry and the CFFS expansions 7, and 18. Essentially, his demonstration shows that if the terms in each of these expansions are grouped into a series in ascending powers of the fundamental frequency, $r$, of the modulation $\theta(t)$, the resulting series is asymptotic for $r \to 0$. Stumpers refers to what we have called the CFFS expansion, as the "asymptotic expansion", although it only has this character for $r \to 0$ when arranged in ascending powers of $r$.

In his demonstration, Stumpers starts with the differential equation (1) relating the excitation $i(t)$ to the response $e(t)$, when the excitation is modulated simultaneously in amplitude and in frequency at a fundamental rate of $r$ rad/sec. We shall restrict the approach here to an angle-modulated excitation of the form

$$i(t) = e^{j[pt + \theta(rt)]}$$

If we set $\phi = rt, (d/dt) = r(d/d\phi)$, and

$$e(t), = E(\phi)e^{j[pt + \theta(\phi)]}$$

we may write corresponding to Eq. 5, above,

$$Q(jp + r \frac{d}{d\phi})E(\phi)e^{j\theta(\phi)}$$

$$= F(jp + r \frac{d}{d\phi})e^{j\theta(\phi)}$$

or

$$Q(jp + r \frac{d}{d\phi})E(\phi) = F(r, \phi)$$

(23)
where
\[ F(r, \phi) e^{i \theta(\phi)} = \sum_{k=0}^{n} a_k (j p + r \frac{d}{d \phi})^k e^{i \theta(\phi)}. \]

\( F(r, \phi) \) is, clearly, a polynomial in \( r \), and a periodic function of \( \phi \). The differential equation 19 may therefore be considered as parametric in \( r \), and consequently its particular integral (steady-state solution) may be quickly determined by a theorem of Perron (Ref. 5). This theorem (quoted for this purpose by Stumpers) states that Eq. 23 has a particular integral of the form
\[ E(\phi) = \sum_{k=0}^{\infty} r^k r_k(\phi) \tag{24} \]
which is asymptotic for \( r \rightarrow 0 \), and the functions \( r_k(\phi) \) remain to be determined to fit the requirements of Eq. 23. Formally, if the remainder after the sum of the first \( n + 1 \) terms in Eq. 24 is denoted by \( R_0(r) \), then the expansion in Eq. 24 is termed asymptotic for \( r \rightarrow 0 \) if
\[ \lim_{r \rightarrow 0} r^n R_n(r) = 0 \]
with \( n \) held fixed, even though for any fixed value of \( r \)
\[ \lim_{n \rightarrow \infty} |r^n R_n(\phi)| \rightarrow \infty. \]

Before digressing on properties of such expansions that are of interest to our discussions, we note that the final steps of the solution consist in the formal substitution of Eq. 24 into Eq. 23 and the equation of coefficients of like powers in \( r \). With \( \omega_1 = p + d \theta / dt \), and \( \Phi = rt \), the result is
\[ E(rt) = \Xi(j \omega_1) + j(1/2)r^2 \Theta''(rt) \Xi'(\omega_1) + j(1/2)r[2 \Theta'(rt) \Xi'(\omega_1) - (1/8) \Theta''(rt) \Xi''(\omega_1)] + \ldots. \tag{25} \]
This result is identical in substance with Stumpers' result; the two become identical in form if time derivatives of the frequency modulating function

\[ r \frac{d \theta(\tau t)}{d(\tau t)} = S(\tau t), \]

let us say, instead of \( \theta(\tau t) \), are used, and then

\[
E(\tau t) = Z(j\omega_1) + j(1/2)rS^1(\tau t)Z^{(2)}(j\omega_1) + r^2 \left[ j(1/6)S^2(\tau t)Z^{(3)}(j\omega_1) \right] + \ldots .
\]

which is Stumpers' asymptotic expansion. The statement made above about this expansion being the result of a rearrangement of the terms in the CPTS expansion is now clarified in the following manner. First \( \theta(t) \) is better described as \( \theta(\tau t) \) if it is periodic and of fundamental frequency \( r \), and therefore the \( n \)th time derivative of \( \theta(\tau t) \) may be written as \( r^n \theta(n)(\tau t) \). Now if \( \theta(\tau t) \) and \( Z(j\omega) \) satisfy the conditions of the derivation leading to Eq. 18, the series expansion in \( \lambda \) will be not only uniformly, but also absolutely convergent, in view of the properties of power series, and the argument justifying the term-by-term integration that introduced \( Z(j\omega) \) and its derivatives into the picture.

The absolute convergence enables us, therefore, to rearrange the terms without affecting the convergence of the series.

In particular, if terms of like powers of \( r \) are grouped together, the Stumpers' expansion will result, and the expansion may be considered to be an expansion about \( r = 0 \), which Stumpers' argument proves is asymptotic for \( r \to 0 \).

It is now recalled that asymptotic expansions have the peculiar property of their terms first diminishing in size with increasing \( n \), until a minimum is reached, after which, the terms again increase beyond limits. This property, however, does not impair their usefulness. Indeed, such expansions possess some remarkably useful properties (Ref.5).

Of interest to us are the following facts. First, for
sufficiently small values of $r$, the error incurred by summing only the first $n$ terms of the series is less than the first rejected term. The best approximation is therefore obtained when $n$ is so chosen that the $(n + 1)$th term is the smallest term in the expansion. When only the fundamental rate of the modulation, $r$, is varied, the predominance of the leading terms is improved as $r$ is decreased, a fact which is in complete agreement with our expectations. In the limit as $r \to 0$, only the first term, $Z(jw)$, will be different from zero, as static reasoning and steady-state a-c theory confirm. For a fixed value of $r$, which is different from zero, restrictions could be imposed upon $Z(jw)$ that will insure the predominance of the earlier terms. If only the first term, $Z(jw)$, is to be of importance in evaluating $E(rt)$, the envelope function of the response of the filter, then the condition

$$\left(\frac{1}{2}\right)r^2|\Theta''(rt)Z^{(2)}(jw_1)| \ll |Z(jw_1)|$$  \hspace{1cm} (27)

at all instants of time, which is equivalent to condition 19. This time, the same condition follows from the asymptotic properties of the expansion 18 with respect to $r \to 0$, when this expansion is rearranged in terms of ascending powers of $r$, as in Eq. 25. This condition is, therefore, best expressed in the form

$$\left(\frac{1}{2}\right)r^2\left|\frac{\Theta''(rt)Z^{(2)}(jw_1)}{Z(jw_1)}\right|_{\text{max}} \ll 1.$$  \hspace{1cm} (28)

When this condition is satisfied, and the envelope $E(rt)$ is described by the first term, $Z(jw_1)$, only, the error incurred is more negligible than is the second term in the expansion, compared with $Z(jw_1)$.

We shall now consider briefly an expansion for the AM steady-state response which is identical in form with expansion 7 above. The simplicity of the AM spectrum usually makes the steady-state solution by the Fourier approach a fairly manageable matter. However, if in Eq. 3, $e^{i\Theta(t)}$ is replaced by $X(t)$, and the same steps leading to Eq. 7 are retraced, the result is also Eq. 7 with $e^{i\Theta(t)}$ replaced by $X(t)$. An argument starting from the impulse response of the network and leading to the same result is also possible and fairly straightforward. In terms of this expansion, we may also discuss the conditions
that the network and the excitation combined must satisfy
in order for a quasi-static solution to be valid. Instead
of talking in terms of sideband clipping and waveform
distortion, we may talk in terms of filter sluggishness.
Clearly, the condition for the filter to follow the amplitude-
modulated excitation in a quasi-stationary manner may be
put in the form

\[ |Z(j\omega)M(t)| \gg |Z'(j\omega)M'(t)| \]  

(29)

where \( \omega \) is the carrier frequency of the wave, and does not
necessarily coincide with the center frequency of the filter.
Condition 29 must be satisfied in the more conservative
form

\[ \frac{|Z'(j\omega)|}{|Z(j\omega)|_{\text{max}}} \cdot \frac{|M'(t)|}{|M(t)|_{\text{max}}} \ll 1 \]  

(30)

When \( Z(j\omega) \) is expressed as in Eq. 21, we have

\[ \frac{Z'(j\omega)}{Z(j\omega)} = \Phi'(\omega) - j \frac{A'(\omega)}{A(\omega)} \]

which shows that the condition 30 for quasi-stationary
response implies conditions on \( \Phi(\omega) \) as well as \( A(\omega) \).
Here, it may be convenient to charge \( A(\omega) \) with the side-
band clipping part of the sluggishness and \( \Phi(\omega) \) with the
waveform distortion and delay part, provided \( \omega = \omega_0 \) and
\( A(\omega_0) \) is perfectly symmetrical about \( \omega_0 \). The responsibilities
are not so simply shared otherwise.
III.2 APPLICATIONS TO THE INTERFERENCE PROBLEM

It was stated in the preceding section that the instantaneous frequency method of computing the steady-state response of a filter to a variable-frequency excitation was permissible only when certain joint restrictions on the character of the phase modulating function \( \Theta(t) \) of the excitation and the system function \( Z(j\omega) \) of the filter were satisfied. It is recalled that this method assumes that the filter is able to follow the excitation in a quasi-stationary manner, and thus to respond effectively, at every instant, as though it were excited by a forcing sinusoid which maintained its frequency constant, at the pertinent value of instantaneous frequency, for a sufficiently long time to allow the filter response to settle to a steady-state condition. The degree to which the filter is able to respond in this manner, will guage the faithfulness with which the instantaneous frequency variations of the excitation will be reproduced in the response.

When the filter is able to follow the excitation through stationary states, the instantaneous frequency variations of the response will not be noticeably different from those of the excitation. A filter will be termed "wideband" relative to a prescribed frequency-modulated excitation, if this filter is sufficiently rapid-acting to follow the instantaneous frequency variations of the excitation through quasi-stationary states. The filter will be termed "narrow-band", if it is too sluggish to follow the instantaneous frequency variations of the excitation. The terms "wideband" (and sometimes "infinitely" wide) and "narrow-band" have been used to convey this very significance in the preceding two chapters. These terms have also been used by Arguibau and Granlund with the same connotation in their "wide-banding" proposal for the achievement of high capture ratios.

We shall presently consider two widely used band-pass filters, assume certain types of periodic frequency
modulations of practical interest, and determine explicitly what condition 19 means in terms of these filters and the assumed modulations.

We start by rewriting the condition in the form

$$\frac{1}{\epsilon} \left| \frac{E''(t)Z''(j\omega_1)}{Z(j\omega_1)} \right| \ll 1$$

(31)

at all instants of time. Clearly if the maximum instantaneous value of the quantity on the left satisfies this condition, then the condition will be satisfied throughout the modulation cycle. Also, $|E''(t)|$ and $|Z''(j\omega_1)/Z(j\omega_1)|$ will generally achieve their maximum instantaneous values at different instants of time. Therefore, the condition is more conservatively expressed in the form

$$\frac{1}{\epsilon} \left| E''(t) \right|_{\text{max}} \times \frac{\left| Z''(j\omega_1)/Z(j\omega_1) \right|_{\text{max}}}{\epsilon} = \epsilon' \ll 1.$$ 

(32)

This expression of the condition simplifies its application to specific situations considerably. The necessity of satisfying the condition in this form stems from the likelihood for $|E''(t)|$ and $|Z''/Z|$ to assume their maximum values simultaneously during the modulation cycle even though, in general they may not. It is of great interest to observe that the condition 32 involves the maximum rate at which the instantaneous frequency of the variable-frequency excitation is varied.

Consider the high-frequency model of a parallel-resonant circuit shown in Fig. 1. The impedance of this circuit as a function of the instantaneous frequency, $\omega_1$, is readily expressed in the form

$$Z(j\omega_1) = \frac{R}{1 + jQ_0 \left( \frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} \right)}$$

(33)
where $\omega_0 = 1/\sqrt{LC}$, and $Q_0 = \omega_0 CR$. If the assumptions $Q_0 \gg 1$ and $\omega_1 = \omega_0 (1 + \delta)$ where $\delta << 1$ are acceptable, we may write

$$\left| \frac{Z''}{Z} \right| = \frac{8}{(BW)^2} \cdot \frac{1 + \delta/2}{4Q_0 \delta^2 (\delta + 1) + 2\delta + 1}$$

which is readily shown to attain its maximum value at $\delta = 0$. Therefore

$$\left| \frac{Z''(j\omega_1)}{Z(j\omega_1)} \right|_{\text{max}} = \left| \frac{Z''(j\omega_0)}{Z(j\omega_0)} \right| = \frac{8}{(BW)^2} \tag{34}$$

where $(BW)$ = the bandwidth between half-power points in rad/sec. Relevant to expressing condition 31 in the form 32, we note that if

$$\omega_1(t) = \omega_0 + d\theta(t)/dt, \tag{35a}$$

then the instant of time at which the maximum value, Eq. 34, is achieved, is defined by the value of $t$, during a modulation cycle, at which

$$d\theta(t)/dt = 0, \tag{36a}$$

which is the instant of maximum (positive or negative) phase deviation.

If the frequency modulation is centered about a carrier frequency $p \neq \omega_0$, then

$$\omega_1(t) = p + d\theta/dt \tag{35b}$$

and the maximum value in Eq. 34 may or may not be achieved depending upon whether $\omega_1(t)$ can equal $\omega_0$ at any instant of time. If it can, the maximum is achieved when

$$\frac{d\theta}{dt} = \omega_0 - p. \tag{36b}$$

In the general case, the instants of time defined by 36(a) and 36(b) may or may not coincide with the instant, during
the modulation cycle, at which $|\theta''(t)|$ becomes a maximum, depending upon the functional dependence of $\theta(t)$ upon $t$.

Under assumptions of high $Q_0$ and small fractional deviations from the center frequency, Eq. 33 may also be put in the more convenient form

$$Z(j\Omega) = \frac{\alpha R}{\alpha + j\Omega}$$

where $\omega_1 - \omega_0 = \Omega$ is the instantaneous frequency deviation from the resonance frequency and $\alpha = RC/2 = (BW)/2$ is the damping factor of the circuit. The computation in terms of Eq. 37 quickly yields

$$\left|\frac{Z''}{Z}\right|_{\text{max}} = 8 \frac{\alpha}{(BW)^2}$$

at $\Omega = 0$, as before.

Actually, as may be shown by a more careful and tedious computation, the maximum value of the ratio $|Z''/Z|$ is attained at a frequency slightly below $\omega_0$, but the discrepancy is small for high values of $Q_0$.

The second type of bandpass filter that we shall consider is the two-pole Butterworth-type band-pass filter. This filter commonly takes the form of two critically-coupled (either magnetically or electrostatically) or cascaded stage-tuned (amplifiers using) high-$Q$ parallel resonant circuits. If the damping factor associated with each pole of the filter is denoted by $\alpha$, it may be shown that, to within a scale factor, the transfer impedance (or dimensionless voltage ratio if the filter is part of a linear amplifier), on the $j\omega$-axis, is expressible in the form

$$Z(j\Omega) = \frac{1}{2\alpha^2 - \Omega^2 + j2\alpha\Omega}$$
where \( \Delta \) is, as before, the deviation from the center frequency of the filter in rad/sec. In this case, it is readily shown that the ratio \(|Z''/Z|\) has a local minimum of \(1/\alpha^2\), at \( \Omega = 0 \), and an absolute maximum of

\[
\left| \frac{Z''(j\Omega)}{Z(j\Omega)} \right|_{\text{max}} = \frac{9/\sqrt{13}}{\alpha^2} = \frac{20}{(BW)^2},
\]

at \( \Omega_m = \pm (2/\sqrt{3})\alpha \), where \((BW) = (2 \sqrt{2})\alpha\) is the overall bandwidth of the filter between half-power points. This maximum value for the critically-coupled pair of parallel resonant circuits is seen to be about two and one half times the maximum value for the single parallel resonant circuit. It is seen to arise at two values of frequency which are within the half-power points, since

\[
\Omega_m = \pm 1.15\alpha = \pm 0.815 \times (BW)/2
\]

Corresponding to Eqs. 36(a) and 36(b) for the instants of time at which \(|Z''(j\omega_1)/Z(j\omega_1)|\) will achieve its maximum value, we note that if \(\omega_1(t)\) is given by Eq. 35(a), with \(\omega_0\) now equal to the center frequency of the filter described by Eq. 38, then

\[
\frac{d\theta}{dt} = \pm \frac{2}{\sqrt{3}} \alpha
\]

determines one such instant of time, whereas if \(\omega_1(t)\) is given by Eq. 5(b) such an instant is defined by

\[
\frac{d\theta}{dt} = \pm \frac{2}{\sqrt{3}} \alpha \pm \omega_0 - \phi.
\]

The above ratio may also be computed (though with more labor) for additional band-pass filters, such as an \(n\)-pole Butterworth type filter, and expressed in the form \(k/(BW)^2\). We propose to refer to the constant \(k\) as the
index of the filter. Since our interest in the present investigation centers on the two filters just discussed, the extension to other filters will not be pursued here.

We next turn to the phase-modulating function θ(t) and consider some interesting examples. The simplest such example is offered by

\[ θ(t) = δ \sin rt \]

(40)

where δ may be interpreted here as either a phase deviation independent of r, or as the ratio of frequency deviation, Δω, to the modulation frequency r. In the latter case

\[ δ = \frac{Δω}{r} \]

(41)

and is commonly referred to as the modulation index.

In either case

\[ |θ''(t)|_{max} = r^2 δ. \]

(42)

Substitution from Eqs. 34 and 42 into condition 32 yields

\[
\left( \frac{r}{(BW)/2} \right)^2 δ \ll 1
\]

(43)

which specifies the condition that the various parameters — (BW) of the simple parallel-resonant circuit, and r and δ of the sinusoidal modulation of Eq. 40 — must satisfy in order for the parallel-resonant circuit to follow the impressed wave whose unmodulated carrier frequency coincides with the resonant frequency of the circuit.

If δ is independent of r, a condition usually termed phase modulation, condition 43 states that the product of the square of the modulation frequency measured in unit of one half the filter bandwidth and the phase deviation, δ, must be negligible compared to unity. For the FM case, a substitution from Eq. 41 into 43 results in

\[
\frac{r}{(BW)/2} \cdot \frac{Δω}{(BW)/2} \ll 1
\]

(44)

which states that the product of the modulating frequency
and the maximum frequency deviation, when each is measured in units of one-half the filter bandwidth, must be negligible compared to unity, in order for a computation of the steady-state response of the filter on the instantaneous frequency basis to closely approximate the true response. An equivalent condition has first been published by van der Pol (Ref. 2). Although the unmodulated carrier frequency has thus far been assumed to coincide with the resonant frequency of the filter, it is clear that the carrier frequency can have any other value without affecting the argument. In particular, this carrier frequency may fall at a suitably chosen point on the sloping sides of the filter response for the purpose of converting frequency to amplitude changes.

The condition 44 is clearly to be satisfied for proper discriminator action — that is, if the instantaneous amplitude of the response is to be closely defined by the sinusoidal steady-state amplitude response characteristic of the filter. The condition insures this result whether the amplitude characteristic of the filter happens to be approximately linear over the swept range or not. If the FM-to-AM conversion is to be linear, additional, but obvious, precautions must be taken.

The corresponding condition for the critically-coupled pair is also found by substituting from Eqs. 39 and 42 into condition 32, to get

\[ 10 \left( \frac{r}{BW} \right)^2 \delta \ll 1 \]  \hspace{1cm} (45)

and

\[ 10 \frac{r}{\Delta \omega \left( \frac{BW}{3W} \right)^2} \ll 1 \]  \hspace{1cm} (46)

corresponding to 43 and 44.

The second \( \Theta(t) \) time function that we shall presently consider is of special interest to our discussion of the interference problem. We shall now assume that
\[ \omega_1(t) = p + \frac{d\theta}{dt} \]
\[ = p + a \frac{a + \cos rt}{1 + 2a \cos rt + a^2} \]

where \( \theta(t) = \tan^{-1} \frac{a \sin rt}{1 + a \cos rt} \)

is, with reference to Fig. 1 of Chapter One, the instantaneous phase deviation from the phase of the stronger of the two signals delivered by the i-f amplifier, due to the presence of the weaker signal. The instantaneous frequency expression given by 47 may be associated with the amplitude-limited resultant of the two signals, as given by Eq. 3(a) of Chapter One. We recall that this is the resultant of only the primary spectrum at the output of an ideal limiter — that is those significant spectral components centered about the frequency \( p \) only. The value of \( |\theta''(t)|_{\text{max}} \), together with Eqs. 34 and 39 when substituted into condition 32, will yield the condition that a single parallel resonant circuit, or a critically-coupled pair of resonant circuits must satisfy in order to be able to follow the excitation

\[ i(t) = \cos \left[ pt + \theta(t) \right] \]

and, therefore, have a response whose instantaneous frequency is closely approximated by Eq. 47. Since

\[ \frac{d^2\theta}{dt^2} = \frac{ar^2(1 - a^2) \sin rt}{(1 + 2a \cos rt + a^2)^2} \]

the value of \( \theta''(t) \) is a maximum at the maxima of

\[ W(a, \phi) = \frac{\sin \phi}{(1 + 2a \cos \phi + a^2)^2}. \]

If the maximum value of \( W(a, \phi) \) is denoted by \( W_{\text{max}}(a) \), then

\[ |\theta''(t)|_{\text{max}} = ar^2(1 - a^2) W_{\text{max}}(a). \]
Condition 32, therefore, becomes

\[
\frac{(BW)^2}{r^2} = \frac{1}{2} \frac{k}{\varepsilon} \cdot a(1 - a^2) W_{\text{max}}(a)
\]

where , defined in condition 32, is negligible compared to unity. If we set \((BW) = (BW)_{\text{lim}}\) and replace \(r\) by its maximum value of one \((BW)_{\text{if}}\) (assuming the i-f passband to be flat and steep-sided to relieve the ambiguity in the definition of its bandwidth, we may write

\[
\frac{(BW)_{\text{lim}}}{(BW)_{\text{if}}} = \left( \frac{1}{2} \frac{k}{\varepsilon} \right)^{1/2} \left[ a(1 - a^2) \right]^{1/2} \frac{1}{2} W_{\text{max}}(a) \quad (52)
\]

where the index of the filter

\[
k = \begin{cases} 
8 & \text{for a single parallel-resonant circuit} \\
20 & \text{for a two-pole Butterworth-type filter.}
\end{cases}
\]

The analytical expression for \(W_{\text{max}}(a)\) is given by

\[
W_{\text{max}}(a) = \sqrt[4]{\frac{\left(2/a^2\right)(1 + a^2)(1 + 34a^2 + a^4)^{1/2} - (1 + 10a^2 + a^4)^{1/2}}{3(1 + a^2) - (1 + 34a^2 + a^4)^{1/2}}}^{1/2}
\]

\[
W_{\text{max}}^{1/2}(a) \begin{cases} 
\rightarrow 1 & \text{as } a \rightarrow 0 \\
\rightarrow 1 & \text{as } a \rightarrow 1
\end{cases}
\]

Figure 2 shows a plot of

\[
B_{\text{lim}} = \frac{a(1 - a^2)}{\left(1 + a^2\right)^{1/2}} W_{\text{max}}(a) \quad (53)
\]

which, except for a vertical scale factor, is the value of \((BW)_{\text{lim}}/ (BW)_{\text{if}}\) normalized with respect to the ratio \((1 + a)/(1 - a)\).

This plot shows that \(B_{\text{lim}}\) rises steeply from zero for \(a = 0\), to 0.3 at \(a = 0.15\) and then tends to rise slowly thereafter. Between \(a = 0.15\) and \(a = 1\), \(B_{\text{lim}}\) rises from 0.38 to its limiting value of \(0.4029\) at \(a = 1\).
The vertical scale factor is, from Eq. 52, given by

\[ K = \left( \frac{1}{2} \frac{k}{\epsilon} \right)^{1/2} \]  \hspace{1cm} (54)

where the index of the filter, \( k \), is defined by the relation

\[ \left| \frac{Z''(j \omega_1)}{Z(j \omega_1)} \right|_{\text{max}} = \frac{k}{(BW)^2} \]  \hspace{1cm} (55)

for a given bandpass filter, \((BW)\) being the bandwidth of the filter between half-power points. It is also recalled that \( \epsilon \) is defined by condition 32, and is, therefore, a measure of the error involved in assuming that the filter is capable of following the instantaneous variations in the frequency of the impressed wave. The tolerance factor, \( \epsilon \), may be disregarded compared to unity if it is of the order of \( 1/10 \) or less. In Table I are listed values for the multiplier \( K \), Eq. 54, based upon a few reasonable choices for the tolerance \( \epsilon \).

The significance of the plot of Fig. 2 is now readily interpreted. The plot shows, for instance, that for the second term in the expansion 18 to be about one tenth of the first (or quasi-stationary) term, when \( \theta(t) \) is given by Eq. 48, the value of the filter bandwidth for any prescribed value \( a \), is given by

\[ (BW)_{\text{lim}} = K B_{\text{lim}} \frac{1 + a}{1 - a} (BW)_{\text{if}} \]  \hspace{1cm} (56)

where \( K = 6 \) for a parallel-resonant circuit,

\[ = 10 \] for a two-pole Butterworth-type filter.

For a parallel-resonant circuit, the product \( KB_{\text{lim}} \geq 2 \) for all \( a \geq 0.13 \) and \( \epsilon \leq 1/10 \). For a two-pole Butterworth-type circuit, the product \( KB_{\text{lim}} \geq 2 \) for all \( a \geq 0.05 \) and \( \epsilon \leq 1/10 \). In the range \( a > 0.5 \), \( B_{\text{lim}} \) is approximately constant and equal to 0.4. Thus, in this range, \( KB_{\text{lim}} = 1 \) for \( \epsilon = 15/25 \) with a parallel-resonant circuit, and for \( \epsilon = 8/5 \) with a
two-pole Butterworth-type circuit, and these figures represent excessive errors.

Alternatively, in the range $1 > a > 0.5$, the requirement that $\epsilon \leq 1/10$ can be met if

$$KB_{\text{lim}} > 2.5 \quad \text{for a parallel-resonant circuit},$$

and

$$KB_{\text{lim}} > 4 \quad \text{for a critically-coupled pair}.$$

These results clearly show that a filter bandwidth given by

$$\langle BW \rangle_{\text{lim}} = \frac{1 + a}{1 - a} \langle BW \rangle_{\text{if}} \quad (57)$$

is well within the narrow-band classification. A filter of this bandwidth, be it a simple tank circuit, or a combination of two parallel-resonant circuits yielding a maximally-flat amplitude characteristic, is clearly too sluggish to follow the impressed excitation whose instantaneous phase varies in accordance with $\Theta(t)$ of Eq. 48. This checks with the conclusions reached on a spectrum and ideal filter basis. In fact, reference to Fig. 25 of Chapter One reveals that a value of bandwidth of about four to five times $(1 + a)/(1 - a)$ i-f bandwidths is a close estimate of the ideal filter bandwidth necessary to accommodate essentially all of the spectral components of significance at the output to the ideal limiter. This corresponds rather well to the order of magnitude we have just found necessary for two common types of practical filters to follow closely the amplitude-limited resultant of the two carriers, particularly for the critically-coupled pair. The degree to which these filters are able to follow the variable-frequency excitation is a measure of the faithfulness with which the instantaneous frequency variations of the response will approximate those of the excitation. The practical importance of these results will be highlighted in the
TABLE I

<table>
<thead>
<tr>
<th>Type of filter</th>
<th>$k$</th>
<th>$1/\varepsilon$</th>
<th>Multiply vertical scale in Fig. 2 by</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-pole Butterworth (parallel-tuned circuit)</td>
<td>8</td>
<td>10</td>
<td>6.32</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Two-pole Butterworth</td>
<td>20</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>25.6</td>
<td></td>
<td>16</td>
</tr>
</tbody>
</table>

discussion of Section III.4; their theoretical significance is equally important.

An interesting significance may be attached to the product $KB_{lim}$. The instantaneous frequency of the assumed excitation, as given by Eq. 47, may deviate from the center frequency of the limiter filter by a maximum amount given by

$$\Delta f_{\text{max}} = \frac{1}{2} \cdot \frac{1 + \alpha}{1 - \alpha} (BW)_{1f}$$

assuming good alignment, and a sharply-defined i-f passband. In units of $\frac{1}{2}(BW)_{lim}$, this maximum possible instantaneous frequency deviation from the center of the limiter filter passband is given by

$$x_{\text{max}} = \frac{\Delta f_{\text{max}}}{(BW)_{lim}/2} = \frac{1}{KB_{lim}}$$

(58)

For a parallel-resonant circuit, Eq. 37 may be put in the form

$$Z/R = \frac{1}{1 + jx}$$

(59)

where $x$ is the frequency deviation from the resonant frequency in units of one half bandwidth. Therefore, for this filter,
the requirement that $K_{B_{\lim}} > 5/2$ implies that the resultant amplitude-limited signal delivered by the limiter will be accommodated over the whole extent of its instantaneous frequency excursions within an essentially flat portion of the amplitude characteristic, and an essentially linear portion of the phase characteristic. This is evident from the fact that the stipulation

$$K_{B_{\lim}} > 5/2$$

implies the restriction $x_{\text{max}} < 2/5$. Consequently, the amplitude characteristic will drop from its maximum center frequency value by less than 8% at the maximum frequency deviation, and the phase characteristic will deviate from linearity by less than about 5%. Of course, the filter will be capable of following the excitation through quasi-stationary states regardless of whether the instantaneous frequency variations sweep the peak or the sides of the amplitude characteristic. Also, the validity of the quasi-stationary solution will meet its stiffest test when the instantaneous frequency of the excitation becomes equal to the resonant frequency of the filter at an instant in the modulation cycle when $\Theta'(t)$ assumes its maximum value.

For a critically-coupled pair, the amplitude response characteristic may be normalized to the form

$$|Z(y)| = (1 + y^4)^{-1/2}$$  \hspace{1cm} (60)

and the phase characteristic may be expressed as

$$\phi(y) = \tan^{-1} \frac{\sqrt{2}y}{1 - y^2}$$  \hspace{1cm} (61)

to within an additive $\pi/2$. Here, the variable $y$ is the frequency deviation from the center frequency of the filter, measured in units of one-half the overall bandwidth of the filter. At $y = x_{\text{max}}$, $|Z(y)|$ is below its center-frequency
value by less than 0.2% if the stipulation $KB_{\text{lim}} > 4$ is imposed to insure an acceptable approximation to a quasi-static, response; $\Phi(y)$ deviates from linearity by less than about 2%. Again, the filter will respond in a quasi-static manner regardless of which portion of the response characteristic is swept over by the variable instantaneous frequency of the excitation.

If the instantaneous frequency sweeps of the excitation are confined within the limits of $\pm \Omega_{\text{max}}$ off center frequency of the filter, then Eq. 58 shows clearly that the response will exhibit an amplitude modulation which grows increasingly insignificant as the condition for quasi-stationary response is approached more and more closely. On the other hand, if the operation is centered about a frequency falling on a sloping portion of the amplitude characteristic, then FM-to-AM conversion (or discriminator action) results. In either case, to the extent to which the second term in the expansion $\mathcal{B}$ is negligible compared with the first term, the response of the filter to the excitation

$$i(t) = e^{j[pt + \Theta(t)]}$$

will be approximated by

$$e(t) = Z(j\omega_1) e^{j[pt + \Theta(t)]}$$

$$= |Z(j\omega_1)| e^{j[pt + \Theta(t) + \Phi(\omega_1)]}$$  \hfill (\ref{52})$$

where $\Phi(\omega_1)$ denotes the phase shift associated with $Z(j\omega_1)$. The instantaneous frequency of the response will therefore be given by

$$\omega(t) = p + \frac{d\Theta}{dt} + \Phi'(\omega_1) \frac{d^2\Theta}{dt^2}$$  \hfill (\ref{53})$$

where, if desired, $\Phi'(\omega_1)$ may be termed the instantaneous time delay. Clearly, even under conditions of quasi-static
response, the filter will introduce some distortion into
the instantaneous frequency waveform. This distortion is
seen to be a function of the slope of the phase characteristic
of the filter and the rate at which the instantaneous
frequency of the excitation is varied only. As shown by
van der Pol, this distortion amounts to a time delay only
if the phase characteristic is a linear function of frequency
over the range of instantaneous frequency variation. This
follows from the fact that the instantaneous deviation
from the frequency \( p \), in Eq. 63, may be expressed in the form

\[
D(t) = \Theta'(t) + \Phi'(p + \Theta')\Theta''(t) \\
= \Theta'(t) + \Theta''(t) \left[ \Phi'(p) + \Phi''(p)\Theta'(t) + \ldots \right] \\
= \Theta'[t + \Phi'(p)] + \Theta''(t) \left[ \Phi''(p)\Theta'(t) + \ldots \right] \\
- \left[ \frac{1}{2} \Theta''(t) \Phi'(2\omega_o) + \ldots \right].
\] (64)

The first term is a delayed replica of \( \Theta'(t) \), the remainder
is distortion. Application to a parallel resonant circuit,
as described by Eq. 37 yields for the delay

\[
\Phi'(p) = \left( \frac{2}{BW} \right)_{\lim} \frac{1}{1 + \frac{4}{(BW)^2_{\lim}} \left( p - \omega_o \right)^2} .
\] (65)

The delay is greatest when \( p = \omega_o \), and is then given by

\[
\Phi'(\omega_o) = 2/(BW)_{\lim}.
\] (66)

Since

\[
\Phi(\Omega) = \tan^{-1} \frac{2\Omega}{(BW)_{\lim}},
\]

it is clear that the larger \((BW)_{\lim}\) gets, relative to \( \Omega \), the
better the approximation in Eq. 62 gets, and

\[
\Phi(\Omega) \text{ approaches } \frac{2\Omega}{(BW)_{\lim}} \ll 1
\]

more closely, making all the terms except the first, in Eq. 64,
as well as the time delay, more negligible.

With reference to Eq. 63, it is significant to note that at $\tau t = \pi$, $d^2\phi/dt^2 = 0$, and the magnitude of the instantaneous frequency spike of the response is identical with that of the excitation. Therefore, when the filter is able to follow the excitation closely, the maximum deviation from the average frequency is not affected by the filter.

From Eq. 62, it is evident that under conditions of quasi-static response, the amplitude characteristic of the filter describes the envelope of the response. If $|Z(j\omega_1)|$ varies with $\omega_1$ over the range of instantaneous frequency variations, then discriminator action, or FM-to-AM conversion, results. For proper undistorted conversion, the time variations in the resulting envelope should be linearly related to the time variations of the instantaneous frequency. This requirement can, evidently, be met only if $|Z(j\omega_1)|$ is a linear function of $\omega_1$ over the entire range of desired undistorted conversion.

Before terminating this section we would like to consider three additional forms for the phase-modulating function $\phi(t)$. The choice of these forms will be based upon the properties of the function

$$f(\phi) = \frac{\sin A\phi}{\sin \phi}$$

(67)

where $A$ is an integer. Many interesting properties of this function as it relates to the Gibbs phenomenon have been presented by Guillemin in Ref. 7. For our contemplated use of the function, we first note that for infinitesimally small $\phi$, the numerator and the denominator are closely approximated by their arguments. Therefore, $f(0) = A$.

A similar argument in terms of $\phi = \pi - \delta$, where $\delta$ is an infinitesimal quantity, shows that $f(\pi) = -A \cos A\pi$.

Therefore, if $A$ is even $f(\pi) = -A$, and if odd, $f(\pi) = A$.

A sketch of $f(\phi)$ for $A$ even is shown in Fig. 3. The humps
$$f(\varphi) = \frac{\sin A\varphi}{\sin \varphi}, \quad A = \text{even integer}$$

(a)

(b)

(c)

FIG. 3
at $\Phi = 0$ and $\Phi = \pi$ are integral units high, and $2\pi/A$ radians wide at the base. We immediately recognize that $f(\Phi)$ with $A$ even is quite valuable because its integral, as shown in Fig. 3, is a reasonable analytical approximate to a square wave, and $f(\Phi)$ is just as reasonable as an analytical representation of a series of recurring pulses of alternating polarities. The parameter $A$ is seen to control the height and the width of each pulse keeping the area under the hump approximately constant, especially for large values of $A$. For the integral of $f(\Phi)$, the parameter $A$ is equal to the maximum slope, and is inversely proportional to the rise time, of the step. The properties of $f(\Phi)$ for $A$ odd are very much like those for $A$ even with obvious differences. For $A$ even, the integral of $f(\Phi)$ is a rising staircase, and $f(\Phi)$ itself is a series of recurring pulses of the same polarity.

Let us now set $\Phi = rt$, and consider the case in which

$$
\Theta(t) = \int_{0}^{t} \frac{\sin A\Phi}{\sin \Phi} d\Phi,
$$

where $A$ is any integer, even or odd. This case represents either square wave, or staircase, phase modulation, and therefore, the instantaneous-frequency modulating function is a train of recurrent pulse-like humps of the same or alternately different polarity. Aside from the obvious theoretical as well as practical aspects of the interest in this type of modulation, we would like to point out that the results of the present computation are of value in the study of the transmission of impulsive noise of the more serious "pop" variety through the limiter filters and the conversion of the pops to AM noise by the discriminator network.

The maximum rate at which the modulation of Eq. 68 will vary the instantaneous frequency of the carrier is given by

$$
\Theta''(t)_{\text{max}} = r^2 | \frac{d}{d\Phi} \left( \frac{\sin A\Phi}{\sin \Phi} \right) |_{\max}
$$
The determination of the maximum on the right involves the solution of a cumbersome transcendental equation. The problem is greatly simplified if it is noted that for large \( A (> 10, \text{ say}) \), and in the region covering the large hump and a few neighboring undulations,  

\[
\frac{\sin A\Phi}{\sin \Phi} \approx A \frac{\sin x}{x}
\]

where \( x = A\Phi \). The maximum slope of \( \sin x/x \) occurs at (very closely)  

\[
x = \pm \frac{2}{3} \pi
\]

Therefore, for large \( A \),  

\[
|\theta(t)|_{\text{max}} = 0.436 A^2 r^2.
\]

(70)

For \( A = 10 \), the maximum value is given by \( 0.41745 A^2 r^2 \), so that the value given by Eq. 70 is the upper bound on the value for an arbitrary odd \( A \). Substitution from Eqs. 55 and 70 into condition 32 shows that for a filter, whose \( |Z''/Z| \) has a maximum given by Eq. 55, to follow the instantaneous frequency pulses produced by the phase modulating function of Eq. 68, the condition  

\[
k \frac{Ar/2}{(BW)}^2 \ll 1
\]

(71)

must be satisfied. If we note that, from Eq. 68,  

\[
\frac{d\theta}{dt} = r \frac{d\Phi}{d\Phi} = r \frac{\sin A\Phi}{\sin \Phi}
\]

(72)

the condition 71 may be interpreted as stating that the product of the index of the filter, \( k \), and the square of one-half the instantaneous frequency pulse amplitude, \( Ar \), measured in units of one filter bandwidth between the half-power points, must be negligible compared with unity.
in order for the filter to be capable of following the excitation through quasi-stationary states.

As a second example of both theoretical and practical interest, let

\[ \Theta(t) = \frac{\sin A\Phi}{\sin\Phi} \]

where again \( \Phi = rt \). As before, a factor of unity has been arbitrarily chosen as a multiplier, although for purposes of proper scaling any other constant multiplier may be used as deemed desirable. This form of the phase-modulating function, where \( A \) is an integer that may be even or odd, represents a pulse-type phase modulation. We might add that this representation of the phase modulation interests us because of its applicability to the second, but milder type of impulsive interference known as the "click" noise, just as the form we assumed in Eq. 68 was applicable to the more troublesome "pop". The convenience and value of this formulation of the impulsive interference problem cannot be overemphasized.

The maximum rate at which the modulation of Eq. 73 will vary the instantaneous frequency is readily shown to be

\[ |\Theta(t)|_{\text{max}} = \frac{r^2 A^3}{3} \left( 1 - \frac{1}{A^2} \right) \]

\[ = \frac{r^2 A^3}{3} \quad \text{for} \ A > 3 \quad . \] (74)

The maximum phase deviation is given by \( A \), to within a normalization factor implied in Eq. 73, the maximum frequency deviation is given by \( f_{\text{max}} = 0.436 A^2 r \), for large \( A \). The condition for quasi-stationary response, and hence for no significant distortion of the frequency modulation, is given by

\[ \frac{k}{cA} \left( \frac{\Delta f}{(BW)/2} \right)^2 \ll 1 \] (75)
where \( k \) is again the index of the filter. In terms of the phase modulation pulse height, \( A \), and the pulse repetition rate, \( r \), the condition is

\[
\frac{k}{6} A^3 \left[ \frac{r}{(BW)} \right]^2 \ll 1.
\]  

(76)

Finally, consider the case in which

\[
\frac{d\Theta}{dt} = \Delta\omega \cdot \int_0^\phi \frac{\sin A\phi}{\sin \phi} \, d\phi, \quad \phi = rt
\]  

(77)

where we shall restrict \( A \) to even integer values only.

From Fig. 3 it is clear that the present type of frequency-modulation is a convenient analytical formulation of the problem of square-wave modulation. The instantaneous frequency shifts during a short interval of time from about \( \pm \pi A/2 \) to \( \mp \pi A/2 \) with \( \Delta\omega = 1 \). The magnitude of the steepest rise or fall of the square wave is given by

\[
|\dot{\Theta}(t)|_{\text{max}} = \Delta\omega \cdot A r.
\]  

(78)

Consequently, in order for a filter of index \( k \) to follow the excitation, and reproduce the instantaneous-frequency steps without significant distortion, the bandwidth of the filter between half-power points must be so chosen that (with \( S = \pi A\Delta\omega = \) step size)

\[
\frac{1}{2\pi} \cdot \frac{k}{(BW)} \cdot \frac{r}{(BW)} \ll 1.
\]  

(79)

This condition states that the product of the index of the filter, the magnitude of the frequency jump of the square wave measured in units of the filter bandwidth, and the fundamental frequency of the square wave, also measured in units of the filter bandwidth, should be negligible.
III.3 THE CAPTURE RATIO: CONCEPT AND MEASUREMENT

A. Definition and Criteria for Measurement

The concept of capture ratio is a convenient as well as useful way of describing and gauging the interference rejection ability of an FM receiver. A precise statement of what it actually signifies and measures is of great value to our investigations, and particularly to the discussion of the next section. The use of the term seems to have evolved into prominence as a result of investigations directed by Arguimbau, aimed at determining the design conditions that must be met in FM receivers in order to enhance their ability to handle high-level two-signal interference and exhibit the predictable capture qualities of a wide-band FM system. The common understanding has been that, under conditions of two-signal interference, an FM receiver will generally capture the stronger of the two signals and effectively suppress the disturbance of the weaker signal. Such performance constitutes effective separation of the two signals by what is equivalent to a great increase in the predominance of the stronger of the two signals in the process of amplitude limiting and frequency detecting followed by low-pass filtering. The capture ratio of an FM receiver thus signifies the highest value of weaker-to-stronger signal amplitude ratio that the receiving circuits can handle in this manner under the most adverse condition of interference, namely, when the two signals are furthest apart (by one i-f bandwidth) in frequency. In other words, it is a measure of how closely the weaker signal could approach the stronger one in amplitude and still remain separable, or suppressable, under the most troublesome interference conditions possible. The capture ratio is a measure of the smallest amplitude difference which a given receiving circuit arrangement can recognize under any prescribed interference conditions.

Conceptually, the above definition of capture ratio
is acceptable without any further elaborations. Practically, it remains vague and intangible until a satisfactory criterion for what constitutes capture in the laboratory has been put forward. A careful study of this matter has led us to a criterion which we shall discuss presently. Arguimbau and Granlund used a figure of 30 db suppression of the interference as a criterion for capture with no careful justification of this attitude. They tacitly assumed the existence of an unjustifiable degree of correlation between this figure and what theory would indicate to constitute capture in the definition of the capture ratio, as we shall presently clarify. This issue is not only important for standardization purposes, but all the more so in the evaluation of the significance of experimental data and the correlation of conclusions reached on this basis with theoretical predictions. Proper evaluation and correlation of experimental evidence with a proposed theory pre-supposes, of course, that the yardstick used in the experiments conforms to what the theory assumes and demands, within acceptable tolerances.

Theoretically, the stronger of two carriers is in capture if the value of the direct voltage level dictated by the frequency of this carrier at the output of the discriminator, is equal to the average (over one cycle of the frequency difference between the two carriers) of the detected voltage appearing across the output terminals of the discriminator, before de-emphasis or audio filtering. When the two signals are furthest apart in frequency within a flat i-f passband with sharp cutoffs, the condition of capture of the stronger carrier will be maintained for all values of the weaker-to-stronger carrier amplitude ratio, from zero up to and including the value nominally quoted as the value of the capture ratio. If we denote the value of the capture ratio by \( r \), and the ratio of weaker-to-stronger signal amplitude by \( a \), then capture of the stronger signal
under the most adverse condition of interference will be maintained for all $0 \leq a \leq \rho$. If the amplitude of the stronger signal is denoted by $E_s$, the limiting condition of capture will prevail when the value of the difference frequency, $r$, is one (BW) if the amplitude of the weaker signal is given by $E_s$. If we now let $a$ increase beyond $\rho$, then a deterioration in the reception sets in because for one reason or another the receiver becomes unable to keep the average value of the detected voltage at the discriminator output equal to the value dictated by the frequency of the stronger signal. Therefore, in a practical measurement, we may consider the capture condition to be the border-line condition for which $a = \rho$, if as $a$ is increased slightly beyond $\rho$ a sudden deterioration in the reception (as monitored on a sufficiently sensitive and appropriate measuring instrument) begins to set in. If $a$ is varied by varying the amplitude of the weaker signal, then a measurement of this amplitude at its limiting value of $E_1 = \rho E_s$ may be taken for later reference.

Suppose, next, that after measuring the value $E_1$, the amplitude of the weaker signal, $aE_s$, is increased beyond $E_1$ and beyond $E_s$, making $a$ greater than unity by an amount sufficient to make this signal take over control of the reception. Another limiting condition of capture therefore prevails when the amplitude of the originally weaker signal becomes greater than $E_s$ and is given by $E_2 = E_s/\rho$, because by hypothesis, the capture ratio of the receiver is $\rho$. If this second limiting capture condition is also determined as being the condition of capture that just precedes the setting in of observed deterioration, then a measurement of $E_2$ is in order. The capture ratio of the receiver may now be computed from the readings $E_1$ and $E_2$ taken on the same r-f attenuator dial, since

$$\frac{E_1}{E_2} = \frac{\rho E_s}{E_s/\rho} = \rho^2$$
and
\[ \rho = \sqrt{E_1/E_2} \]  

The critical decisions in this measurement concern the question of when to take the readings \( E_1 \) and \( E_2 \). These decisions are obviously subject to the details of the measurement procedure, especially those pertaining to the character of the signals used and the way in which the observations are made. In a practical situation, the measurement may be carried out under either of two conditions. Either two unmodulated carriers which are apart in frequency by one i-f bandwidth may be used; or, one unmodulated carrier located above or below the center frequency of the i-f amplifier by \( (3W)_{IF}/2 \), and one carrier having a slow sinusoidal modulation which deviates its frequency by \( t(3W)_{IF}/2 \) about the center frequency of the i-f amplifier. Under the first of these two conditions, the most troublesome interference condition is maintained at every instant throughout the measurements, and reasonable visual indications of the condition of capture at the output of the discriminator (and before audio and de-emphasis filtering) may be made by displaying the detected spike pattern on an oscilloscope screen. The pattern will appear much like the plots shown in Fig. 2 of Chapter One. The spikes will point upward or downward depending upon which signal is captured, and a reasonable visual estimate of where either of the two limiting capture conditions lies seems possible from a study of the way in which these spikes are affected by variations in \( \rho \).

On the other hand, the second condition in which one of the two signals is sinusoidally modulated and the other is unmodulated, offers other possibilities. This choice of signals does meet the requirement stipulated in the definition of the capture ratio concerning measurements under the most adverse interference condition. For if the modulation index of the sinusoidally-modulated wave is sufficiently high, then in the neighborhood of the peak swing away from the
unmodulated signal, the condition of most troublesome interference will maintain for several cycles of the peak frequency differency of one i-f bandwidth between the two signals. Observations made on the filtered output of the discriminator by means of an audio voltmeter offer a greater degree of objectivity and consistency in timing the readings on the r-f attenuator dial. However, the audio voltmeter will be helpful only when the signal with the sinusoidal modulation is coming through, and therefore, only one of the two necessary readings may be taken with reliability of attitude at the appropriate time; namely, the reading marking the limiting condition for the capture of the sinusoidally-modulated signal. Denote this reading by $E_1$, and let the audio voltmeter indication be $V_m$. When it comes to the second reading, $E_2$, which corresponds to the limiting condition of capture of the unmodulated signal, it does not seem clear at the outset how one would go about making the decision to take or not to take the reading. For under the present condition, any sinusoidal modulation coming through constitutes disturbance. On the other hand, it does not seem advisable to interchange modulation and lack of it between the two signals in midstream because of the danger of introducing unpredictable departures from the reliability which results from taking a ratio of two readings on the same attenuator dial under identical operating conditions for the signal generator supplying the signal whose amplitude is varied in the measurement. The question, therefore, becomes: how far below $V_m$ in db, should one see the audio voltmeter indication dip before deeming it appropriate to take a reading on the radio-frequency attenuator dial of the signal generator that would correspond to $E_2$? Arguimbau and Granlund chose 50 db.

We have carried out a detailed study of this question through a series of repeated measurements, first using a laboratory receiver used by Arguimbau and Granlund in a series of transatlantic FM transmission tests, and later using a
much simpler design which corresponds to a standard low-quality receiver. These observations were also checked informally at several occasions using other receivers. Briefly, the technique involved making a series of repeated capture ratio measurements, first using two unmodulated signals and deciding from visual observation of the detected spike trains when to take readings; and second, using one sinusoidally modulated signal and one unmodulated signal with criterion for the suppression of the sinusoidal signal at the output ranging between 15 and 30 db, and in each case keeping vigil on the behavior of the detected spike pattern under the conditions of the measurements. The results indicated a striking consistency between results obtained by the first method (which clearly purports to attach to the measured its theoretical significance) and results obtained by the second method using a criterion of 19 to 30 db suppression of the sine wave at the output. Consequently, we have adopted a figure of 20 db suppression below the level of $V_m$ on the audio voltmeter, as a criterion for measuring the capture ratio by this method. We have observed that the behavior of the spike patterns under conditions of the first type of measurement that correspond to a dip of the output voltmeter indicator by more than 20 db below $V_m$, corresponds rather well to their expected behavior when $\alpha$ is decreased below 1. Under conditions corresponding to a dip of less than 20 db, telltale distortion of the observed spike patterns has been observed to set in. Thus, 20 db suppression of a detected sine wave modulation below its capture level seems to correspond rather closely to the condition of proper turning over of the spikes in the manner demanded by the theory for the establishment of the capture of the unmodulated signal used in the measurement.

3. A Generalization of the Concept

An extended meaning of the term "capture ratio" beyond the significance attached to it in the preceding paragraphs is possible, and often helpful. For, frequently, under
conditions of two-signal interference, it is convenient to specify the capture conditions at any point within the high-frequency section of the receiver by the ratio of weaker-to-stronger signal amplitude at that point under the most adverse condition of interference. This, of course, assumes that the most troublesome resultant signal at that point is still specifiable in terms of only two carriers corresponding to the carriers accommodated within the i-f passband. Such is the situation, for instance, when an idealized filter having one i-f bandwidth is assumed to follow a limiter and the input signals lie at the opposite edges of the band.

Under this condition of interference, and when a chain of such idealized narrow-band (one i-f bandwidth) limiters is employed, it is convenient to say that the "capture ratio" is decreased as the resultant signal travels down the chain, meaning the ratio of the weaker-to-stronger spectral component amplitude at the output of each limiter when the corresponding weaker-to-stronger signal ratio within the i-f passband is the capture ratio of the receiver. To avoid any confusion with the meaning of the term as it applies to a measurement at the input to the receiver, the name "interference ratio" will, henceforth, be used.

It often happens, however, that at the input of the second, or later, limiter, or at the input of a limiter with feedback (as explained in Chapter Four), the spectrum is changed by a combination of amplitude limiting and narrow-band filtering into a form that has several components instead of the two that represented the original signals. Under such conditions, the seriousness of the interference does not appear to be describable with the same simplicity that marked the use of the usual capture ratio concept. If the capture conditions at these points are investigated in order to determine the effect of any constituent stage (or stages) of the receiver upon these conditions, then it would seem that an acceptable, and logical method of specification is called for.
We find that these requirements on the specification are adequately met by an extension of the meaning of interference ratio in the following way.

With any given configuration of sideband components owing its origin to the simultaneous presence of two carriers within the i-f passband, the most serious aspects of the disturbance relate to the properties of the instantaneous frequency deviations from an average value that corresponds to the frequency of the stronger of the two input signals. Fundamentally, the most serious characteristics of this so-called instantaneous frequency spike train, relate to the frequency of recurrence of the spikes, their greatest rate of change on the steep sides of the spikes, and the magnitude of the frequency spike at $\pi t = \pi$. Additional details concerning the instantaneous variation with time are of importance only to the evaluation of the relative amplitudes of the Fourier components in the structure of the detected spike trains, a matter of minor significance in the overall picture. One may, therefore, replace the actual spike pattern by a spike pattern having the same fundamental repetition frequency and spike magnitude, but otherwise varying with time as if it were the result of the superposition of two carriers occupying the same frequency positions as the two carriers delivered by the i-f amplifier with the ratio of the weaker-to-stronger carrier amplitude having the value necessary to produce the required spike magnitude. With reference to Fig. 4, if the given configuration of sideband components arises with a value of difference frequency, $r$, and in turn gives rise to an instantaneous-frequency spike of normalized magnitude

$$\lambda = \left[ \frac{\omega}{\pi} \right] / r,$$

then an equivalent frequency spike magnitude, recurring every $2\pi / r$ seconds, will also be produced when all the sideband components that are considered are replaced by one component at $p + r$ rad/sec whose amplitude is multiplied by the
amplitude of the component at $p$ rad/sec, where

$$\gamma' = \frac{\Lambda}{1 + \Lambda}.$$  \hspace{1cm} (83)

In terms of $\gamma'$, which we may term the equivalent interference ratio, the capture conditions at the output of a limiter with an arbitrary bandwidth may be described and compared with conditions at some other point in the receiver, regardless of the particular configurations of Fourier components making up the most troublesome interference condition at the various points. The comparison is then made on the basis of those features that count the most in the final analysis.

In the preceding section, we showed that bandwidths of two-and-a-half, or greater, times the bandwidth predictable on the basis of the spike magnitude are necessary in order for a filter to follow and reproduce the variations in the instantaneous frequency of the resultant of two carriers. This conclusion agreed with estimates made in Chapter One, on the spectrum and ideal filter basis. Consequently, filters having bandwidths amounting to a small fraction of this value (but meeting the minimum requirements computed in Chapter One) will be too sluggish to follow those instantaneous frequency variations and will, as a result, exhibit a response whose instantaneous frequency variations will, as shown in Chapters One and Two, be a distorted version of the original pattern, with the spikes materially diminished in size and blunted, but recurring at the same rate as before. This effect of the filter sluggishness then amounts to a decrease in the equivalent interference ratio at the output, below its value at the input, a decided improvement in the capture conditions. In terms of this effect that a sluggish filter following an amplitude limiter has upon the interference ratio, one may, at least qualitatively, fortell the various by-products of narrow-band limiting examined under idealized conditions, and on a spectrum basis, in Chapter Two. In effect, by being
unable to follow the instantaneous frequency variations of
the excitation, the sluggish filter acts as a "frequency
limiter", in the sense of setting a limit to the size, rate
of change, and repetition rate of the instantaneous frequency
variations that it will reproduce without distortion, or that
its own response can display. These effects are inherent to
the energy storage character and consequent inertia of the
filter.

The results of Section I.6 are readily and conveniently
expressible in terms of the equivalent interference ratio
concept. Thus, combined use of Eq. 83 of this section and
Tables I and IV of Section I.6 has resulted in the plotted
values of $\gamma'$, the equivalent interference ratio at the output
of an idealized limiter followed by an idealized filter
having a bandwidth $X$ times the i-f bandwidth, as a function
of $X$, with $\alpha$, the ratio of weaker-to-stronger signal amplitude
at the input of the limiter, taken as a parameter, in Fig. 5.
The value of $\gamma'$, which of course is an index of the most
adverse capture condition at the output of the limiter filter,
is seen to be always smaller than the corresponding $\alpha$, and
approaches the corresponding $\alpha$ value only as $X$ grows very
large. A formula relating $\gamma'$ to $\alpha$ and $X$ is readily obtainable
by substitution from Eq. 51 of Chapter One into Eq. 83 of
the present section.

The following theorems are more or less restatements,
in terms of the equivalent interference ratio designation of
the interference or capture conditions at the output of
a limiter filter, of previously demonstrated facts. The term
"primary spectrum" is used in reference to that portion of the
total spectrum produced by amplitude-limiting the resultant
of two sinusoids which is centered about the frequency of
the stronger sinusoid, assuming the two sinusoids to differ
in frequency by an amount which is negligible compared with
the frequency of either.
EQUIVALENT INTERFERENCE RATIO
AT THE OUTPUT OF ONE IDEAL LIMITER

\[ q = 0.95 \]
\[ q = 0.90 \]
\[ q = 0.85 \]
\[ q = 0.80 \]
\[ q = 0.70 \]

IDEAL LIMITER BANDWIDTH
IN UNITS OF ONE \( (BW)_{IF} \)

FIG. 5
Theorem I:

If the bandpass filter following an idealized limiter is capable of following the instantaneous frequency variations of the resultant of two carriers (or of spectral components resulting from limiting the resultant of these carriers) which is impressed at the input to the limiter, through quasi-stationary states — or, equivalently, if the filter passband is sufficiently wide to accommodate the entire significant primary spectrum caused by the ideal limiting action — then the equivalent interference ratio of the composite signal delivered to the limiter will not be affected by the bandpass limiting action. Such a bandpass limiter will, however, deliver a constant-amplitude signal at its output.

Theorem II:

The equivalent interference ratio of a composite signal will be diminished by a bandpass filter only if the significant spectrum of the resultant impressed signal is spread out beyond the extent of the filter passband. Under these conditions, a process of instantaneous frequency limiting is exhibited by the response of the narrow-band filter because this filter is unable to follow closely the fast rates of instantaneous frequency variations of the excitation.

Theorem III:

When the ratio of weaker-to-stronger signal amplitude delivered by the intermediate-frequency (i-f) amplifier is less than 0.84, the greatest reduction in the equivalent interference ratio resulting from passing a composite signal through a stage of narrow-band limiting is achieved when the limiter bandwidth used has the smallest permissible value of one i-f bandwidth.

Corollary:

When the ratio of weaker-to-stronger signal amplitude delivered by the i-f amplifier is less than 0.84, the number of cascaded stages of narrow-band limiting necessary to achieve a prescribed decrease in the equivalent interference

(114)
ratio delivered to an amplitude-insensitive discriminator is a minimum when the least permissible bandwidth of one i-f bandwidth is used in each stage of limiting.

Theorem IV:

Under conditions of two-signal interference, in which the weaker-to-stronger signal amplitude ratio is less than unity, the equivalent interference ratio may be reduced by any desired amount by passing the composite signal through an appropriate number of cascaded narrow-band limiters having appropriately chosen bandwidths.
III.4 CRITIQUE AND REEXAMINATION OF THE "WIDEBANDING"
THEORY FOR IMPROVED CAPTURE PERFORMANCE

A fortiori of their experimental findings, Arguimbau and Granlund (Refs. 8, 9, and 10) were led into proposing an account of the performance of their experimental receivers which will henceforth be referred to as the "widenaming" theory. A brief statement of the receiver design criteria prescribed by this theory for the suppression of high-level interference was presented at the beginning of Chapter One. In what follows, we would like to re-examine the foundations of this theory and its various conclusions, and present a new account and justification of the performance of the Arguimbau–Granlund receivers where the old theory fails to meet the test.

At the basis of the "widenaming" theory lies the argument that if the instantaneous frequency variations caused by the interference can be preserved undistorted in going through the non-linear sections (limiter and discriminator), then it can be guaranteed that their average over one cycle of the difference frequency between the two carriers will always be zero, and hence their final effect upon the output message will either be completely filterable (as when the difference frequency is beyond the range of audibility), or be capable of being minimized, otherwise. Our demonstration in Chapters One and Two of the fact that it is possible to distort this instantaneous frequency pattern in various ways and into many forms without in any way affecting the desired zero average deviation from the desired frequency proves that although this first fundamental assumption of the "widenaming" theory is a sufficient condition, it certainly is not necessary. Indeed, we have shown in Chapters One and Two that it is always advantageous to disregard this condition altogether.

Next, in attempting to provide limiter bandwidth design criteria that would insure the absence of harmful distortion
of the instantaneous frequency pattern, the "widebanding" theory makes the (unjustified) assumption that if this filter bandwidth is designed on the basis of the extent of the frequency spikes, then a quasi-stationary analysis is permissible, and it can safely be said that the instantaneous frequency spike pattern of the filter response will be almost identical with that of the excitation. This turns out to be the weakest link in the whole argument and the most serious difficulty with the "widebanding" theory. No clear-cut justification of this assumption was provided and an effort in that direction, based on a study of the spectrum after limiting, was attempted (Ref. 8, p. 39), but lead to no conclusions. We have shown this assumption to be invalid: first on the spectrum and ideal filter basis, as is clear from Figs. 18 and 25 of Chapter One, and second by examining the foundations for the validity of a quasi-stationary approach using an argument which provides an alternative check on the results of Chapter One, since it does not involve the use of the spectrum and the idealized filter. It is clear, that in case of any lack of attention for the ideal filter, it is merely a matter of laborious computation, using any other type of filter along with the Fourier approach, to show with numbers that the "widebanding" theory is not justified in using a quasi-stationary approach assuming only the bandwidths it prescribes. A study of the spread of the spectral amplitudes will provide a quick qualitative argument, for it will be immediately appreciated that many sideband components of significance in the structure of the amplitude-limited resultant of the two input carriers will fall out on the sloping sides of the filter amplitude response characteristic, as well as on nonlinear portions of the phase characteristic, with the result that a combination of sideband clipping and phase distortion will invalidate the results of a quasi-stationary approach.

Both the Fourier approach of Chapter One, and the necessary requirements for the validity of the instantaneous frequency
(quasi-stationary) approach explored in Section III.2, reveal that a bandwidth of the order of more than three times the bandwidth prescribed by the "widebanding" theory would be required in order for the filter to follow the prescribed excitation through quasi-stationary states, and thus justify the assumption that the system function changes instantaneously with the instantaneous frequency of the excitation.

We are therefore forced to conclude that the bandwidths prescribed by the "widebanding" theory are not sufficiently wide: that is to say, the "wideband" of the "widebanding" theory is only one third, or less, as wideband as the "widebanding" theory fundamentally intended it to be. In line with the definitions made of these terms at the beginning of Section III.2, the bandwidths prescribed and used by Arguimbau and Granlund, wide as they were believed to be, still fall safely within the classification of narrow-band filters.

Contrary to the assumption of the "widebanding" theory, it is not just the extent of the maximum instantaneous frequency deviation that decides the necessary bandwidth for the validity of an instantaneous frequency approach. Rather, it is a combination of the maximum deviation, the fundamental rate of recurrence, and the rate of change of frequency with time. All of these fundamental factors can be well taken care of, as shown in Section III.2, simply by satisfying the requirement involving the maximum rate of change that the instantaneous frequency of the excitation will experience.

Collaterally with this topic, we would like to point out that it would be incorrect to suppose that because the instantaneous frequency of the resultant of two sinusoids varies slowly during a part of the difference frequency cycle, that over this part of the cycle, a filter, whose bandwidth does not satisfy the fundamental requirement for the validity of a quasi-stationary approach, will be able to follow the instantaneous frequency variations of the excitation closely.
Indeed the whole instantaneous frequency pattern will be distorted if the filter cannot follow the fast changes that give rise to the instantaneous frequency spike. The appearance of the instantaneous frequency pattern will be much like the patterns illustrated by the instantaneous frequency plots shown in Chapters One and Two. Similar plots could readily be made to describe comparable conditions in which a parallel-resonant circuit, or any other bandpass filter, is used instead of the idealized filter to demonstrate the narrow-banding effect. This task, however, is not deemed necessary. Only the repetition frequency of the spikes will be preserved in the response. Otherwise, not only will the spikes be compressed and blunted, but undulations will appear in the pattern of the response where in the pattern of the excitation the instantaneous frequency was varying slowly. The undulations may be attributed to undying FM transients that recur every time the excitation goes through the faster portions of its instantaneous frequency excursions. The instantaneous frequency of the filter response is unable to settle down to a quasi-stationary mode before the next occurrence of the sharp changes.

The "widebanding" theory is therefore not correct in assuming that a bandwidth given by

\[(3W) = \frac{1 + a}{1 - a} \]  

is sufficient for the validity of a quasi-stationary argument. Consequently, it is also incorrect to say that if the limiter bandwidth is given by this formula, the linearity of the FM-to-AM conversion characteristic must extend over the same range. Before we present a new theory that meets these objections, we would like to make it clear that other important aspects of the "widebanding" theory relating to its stress on careful i-f amplifier design, the use of limiters and discriminators with low, or no, time constants, and last, but not least, its pointing out that the considerations that enter into the
design of the discriminator circuit bandwidth do not relate merely to the fundamental requirements of a message signal normally swept in frequency within the limits of one i-f bandwidth — all of these are fundamentally sound and necessary considerations. But the original work left many questions which concern these considerations unanswered. The theory did not point out quantitatively how rapid acting was "rapid-acting", and how short were the short time constants that could not be exceeded in situations where other fundamental considerations made it imperative that a non-zero time constant be used. The discussions of Section II.2 of this thesis purport to fulfill this need for the appropriate numbers and method of attack. Moreover, by stressing a need for fast action in the limiter filter as well (to insure the validity of arguing on the instantaneous frequency basis) without clarifying the necessity for this attitude, or the extent of the implied penalty for not meeting this fast action requirement, the "widebanding" theory both overlooked and left in doubt the role of the bandwidth of the limiter stages in the overall interference rejection picture.

The most important conclusion that the "widebanding" theory draws, basing the argument on a quasi-stationary approach to the limiter filter response problem, is that the FM-to-AM conversion characteristic of the discriminator circuit must be linear over a range prescribed by

$$\left(3W\right)_{\text{disc}} = \frac{1 + a}{1 - a} \left(3W\right)_{\text{if}}$$

(8) if the worst situation in which \( r = (3W)_{\text{if}} \) is to be successfully handled. On the other hand, the results of Section I.3, which take into account the fact that a limiter filter whose bandwidth is given by this same formula is still too narrow to reproduce the two-carrier frequency spike pattern without significant distortion — or equivalently, too narrow to accommodate all of the sideband components of significance — indicate that the bandwidth requirement in the discriminator
can only be stated after the number of limiter stages used, together with the bandwidth of each, has been specified. If only one such stage with the above bandwidth is used, the discriminator characteristic linearity should only extend over a range which is a little over two-thirds of the figure specified by Eq. 85. Lower figures accompany the use of narrower limiter bandwidths, and of more than one narrow-band limiting stage. The importance of these lower figures is best appreciated by those who have to build and maintain a discriminator circuit with that bandwidth specification.

We now turn to a different, or new account of the reasons for the high degree of capture performance exhibited by the Argushima-Granlund receiver. This new theory is based on the various pertinent results and conclusions reached in this and in the preceding chapters, as well as a study of the Argushima-Granlund receiver. We shall start with a brief description of the design of this receiver, high-lighting only the prominent features that bear directly upon the ensuing argument. These and other features and details of this design, as well as an account of the "widebanding" theory, have been presented by Argushima and Granlund in Refs. 8, 9, and 10. Henceforth, this receiver is referred to as the experimental receiver.

The experimental receiver utilizes a 7-pole Butterworth-type 1-f amplifier which, with proper alignment, has a bandwidth between half-power points of 200 kc, and maintains an essentially flat response over a 150 kc band with a drop of less than one percent at the extremes of this band. The response is about 25 db down at 150 kc off center frequency. The 1-f amplifier drives a four-stage pre-limiter which in turn is followed by a four-stage limiter. The pre-limiter was intended as a substitute for a system of automatic gain control in order to avert the detuning effects of the variation of input capacitance with variable grid bias. Each pre-limiter stage makes use of a pair of unbiased crystals so arranged that
one of them is conducting while the other is off, each conducting once every radio-frequency cycle. At the low signal levels the drive may not be sufficient to drive the crystal conduction into regions of reasonable limiting. Each stage experiencing such low-drive conditions will behave as an approximately linear amplifier, with the crystals maintaining a sufficiently low damping across the plate-circuit tank to make the bandwidth of the stage about 3 mc wide. This bandwidth is calculated to maintain the same degree of flatness as is exhibited by the i-f response over the usable bandwidth, so that no deterioration in the overall i-f response would result from any of the pre-limiters operating as an amplifier. At these low signal levels, the pre-limiters make up for deficiency in signal strength coming in from the i-f amplifier and thus maintain the drive on the biased crystals in the limiting stages sufficiently hard to keep these crystals conducting, and therefore the limiters limiting. As the signal level delivered by the i-f amplifier is increased, the conduction of the unbiased pre-limiter crystals is boosted into regions of high crystal current and consequent significant limiting. The pre-limiters, therefore, operate as "shock-absorbers", amplifying at the lower levels and limiting at the higher levels, but always maintaining a sufficiently hard drive on the biased crystals of the succeeding limiter stages to keep them in "full-time" limiting operation. In the heavy conduction (and limiting) region of the crystals in the pre-limiters, the damping offered by the crystals in the plate tanks widens the bandwidth of each stage to something of the order of 3 mc. This bandwidth is designed in accordance with the prescriptions of the "widebanding" theory, since the receiver was intended to handle weaker-to-stronger signal amplitude ratios up to $\alpha = 0.95$. The 3-mc figure is 40 times a usable i-f bandwidth of 150 kc.
The limiter stages employ biased crystals for improved limiting. Each stage is designed in accordance with formula 84 to have a bandwidth of 6 mc. The drive on these stages is sufficiently hard to keep them limiting even at the random noise level coming through the i-f amplifier and pre-limiters. It is important to observe that the pre-limiters and limiters do not depend for their operation on the charging and discharging of any capacitors, and therefore, they have virtually no time constants.

In the discriminator circuit design, the desire to utilize a circuit in which there was virtually no danger of any distortion by diagonal clipping in the output low-frequency circuit, led Granlund to the invention of an extremely interesting discriminator. Diagonal clipping in the load circuit of a peak detector results from the charging and discharging of the by-pass condenser through resistances of widely different orders of magnitude. If in the discriminator circuit the two capacitors (usually associated with the two diode detectors) are lumped into one and this one is arranged to be charged and discharged by the same low-impedance radio-frequency source, or by two such sources of comparable impedances, then, it was reasoned, the diagonal clipping trouble would be virtually absent. Actually, other receiver design features being appropriately met, the trouble from diagonal clipping will continue to pose a limitation unless the effective charging and discharging resistance seen by the output capacitor is zero. With the Granlund circuit, the output capacitor sees a dynamically-measured source resistance of about 150 ohms at the center frequency of operation. With a 750-µuf output capacitor, the time constant of the output circuit is about 0.11 µsec. In the notation of Section II.2, and with a usable Wif of 150 kc, this represents a $\tau_d = 0.017$. Consequently, with more than one narrow-band limiting stages preceding the discriminator circuit, the discriminator output-circuit time constant of the Granlund circuit presents no practical limitation on the capture performance of an FM receiver.
In compliance with the dictates of the "widebanding" theory, the two tank circuits constituting the r-f portion of the discriminator were designed to resonate at frequencies 6 mc apart.

We have had occasion to align this experimental receiver, demonstrate and measure its capture performance, and study the effect upon this performance of introducing, in the limiter section, filters having bandwidths several times narrower than the bandwidth originally used in that section. The results of our observations will now be detailed.

In aligning the i-f amplifier for the measurements, we were satisfied with a response that was flat over a 150 kc band to about 3 measured ripple. The pre-limiters and limiters were checked and no alignment was found necessary. The limiter bandwidths were of the order 6 mc. The alignment of the discriminator was quite troublesome, because all attempts at securing a linear FM-to-AM conversion characteristic within the 6-mc range proved of no avail. As illustrated in the sketch of Fig. 6, which is based on a measurement of the characteristic under the conditions of the measurements that will be quoted presently, the tell-tale curvature on the right seemed inherent to the performance of the discriminator circuit used. Our desire to refrain from alterations that changed materially the character of any of the key constituent stages.
of the receiver from what they were at the time of the tests reported by Arguimbau and Granlund, prevented us from attempting an amelioration of the defect by redesign. The effect of this curvature in the discriminator characteristic was indeed noticeable on spike trains that swept over it. Such spikes, displayed on a scope, seemed naturally unsymmetrically disposed with respect to those pointing the other way under conditions of interchanged signal strengths.

In order to test the effect of inserting a filter whose bandwidth was much narrower than 5 mc upon the capture performance of the receiver, we designed several filters each of which was made up of two parallel-resonant critically-coupled circuits isolated at the input and output by vacuum tubes so as to minimize interactions between the coupled pair and their surroundings where inserted. In this way the filter would truly act as an independent box inserted after a limiter with a much wider bandwidth, and almost unaffected by what goes on at the grid circuit of the next limiter stage. At the output end, a cathode follower (with its high frequency shortcomings carefully watched for and minimized) served as a coupling stage to the grid of the next limiter. Careful provisions were arranged for quickly inserting and removing the narrow-band filter without upsetting the operating conditions of the limiter or of the filter circuit. Two such filters were extensively used in these tests. The first had a 210-kc bandwidth between half-power points, the response dipping about 8% at ±75 kc from center frequency. The second filter had a bandwidth of 480 kc between half-power points, with a response that dipped by less than 0.75% at ±100 kc from center frequency, and by 8% at the edges of a 325 kc band. Each filter arrangement had a measured gain of a little over unity between input and output.

† In a private conversation with J. Granlund, at the time of the first measurements, the author learned that this curvature of the characteristic had also been noticed previously, but attempts at eliminating it had failed.
The capture ratio measurements were carried out in accordance with the discussion of the preceding section. The filters were inserted separately between the first and second limiter stages immediately following the pre-limiters.

First of all, the capture ratio of the receiver without any extraneous filters inserted was measured using a 20 db criterion when one of the signals was modulated, and the spike turn-over criterion when both signals were unmodulated. Results following both criteria checked consistently, and values of capture ratio in excess of 0.95, and close to 0.96 were observed. With 30 db suppression, the figure was a little less than 0.94. The overall alignment was not as good as it could have been. The i-f ripple over a 150-kc band exceeded 2% slightly and the discriminator characteristic appeared as sketched in Fig. 8. On the other hand, we became quickly convinced that the figure of 0.95 with 30 db suppression, reported by Granlund (Ref. 8), and measured under conditions of superior overall alignment, must have been indicative of a capture ratio (with 20 db suppression, or spike turn-over) of about 0.96. This guess would seem to indicate that the limitation on the degree of capture did not lie so much with the design of the limiters (including the pre-limiters, particularly at the higher signal levels) or the discriminator circuits, as with the i-f ripple, or the background noise.

At a later time, the effect of inserting the narrow-band filters described above was studied. During these measurements, the i-f ripple over a 150-kc band was about 3%, and the discriminator characteristic was in poor alignment condition, with point A (marking the start of curvature) in Fig. 5 a little over 400 kc to the right of 0, and point 3 lying about 1 mc to the left of 0. With no extraneous filters inserted, the capture ratio was measured at a little over 0.9. With the 480-kc filter inserted between the first and second limiters (following the pre-limiters), several
measurements, consisting of repeated trials, yielded average figures between 0.88 and 0.9, with an overall average of 0.89. The introduction of this filter did not, therefore, materially affect the performance. However, when the 200-kc filter was introduced, the capture ratio dropped to an overall average of about 0.76, and no values less than 0.75 were observed. Still another measurement, with the overall alignment so poor that the capture ratio before insertion of filter was 0.85. With the 480 kc filter inserted, no noticeable effects were observed. The capture ratio measurement indicated 0.85 also. The significant deterioration with the 200-kc filter was to be expected on account of the fact that its response dropped by 8% at 75 kc off center frequency.

The above results are in direct qualitative agreement with our theoretical predictions, and contradict predictions on the basis of the "widebanding" theory. For according to the "widebanding" theory, a limiter bandwidth of $W_{lim}$, in cycles per second between half-power points, would limit the capture ratio to a value of

$$a \leq \frac{W_{lim} - W_{lf}}{W_{lim} + W_{lf}}$$

if $W_{lf}$ is the (usable) i-f bandwidth in cycles per second. Consequently, with $W_{lf} = 150$ kc, the limitation on $a$ with $W_{lim} = 480$ kc is $a \leq 0.524$

while with $W_{lim} = 210$ kc, it is $a \leq 1/5$.

Even with the (unjustified) 50 db suppression criterion, the measured values averaged $a = 0.81$ with $W_{lim} = 480$ kc. On the other hand, considering the departures from the idealizations involved in reaching the specific results in Chapters One and Two, these measured values check well with our
theoretical predictions. We might mention, in passing, that it has been of reassurance to us to hear of the results of recent more elaborate field tests, carried out by other investigators to whom our results pointed out the reasons for the superior capture performance they were able to achieve by cascading several stages of narrow-band limiting, while the "widebanding" theory had set a severe limitation on their expectations because of practical restrictions on achievable discriminator bandwidths.

The inevitable explanation is that the high capture performance achieved by Arguinbau and Granlund was by and large due to the chain of good "narrow-band" limiters incorporated in their laboratory receiver. In an effort to secure close to perfect limiting by cascading four good pre-limiters followed by four good limiters, each possessing the decidedly narrow bandwidth of a little over 3 mc, Arguinbau and Granlund constructed the first effective interference suppressor: the chain of cascaded narrow-band limiters. Of course, other incorporated design aspects were quite helpful if not necessary. Such aspects included the close-to-ideal i-f filtering, the use of limiters with virtually no time constants, and the development of the Granlund discriminator circuit (which relieved the pressure of the often severe requirements on the discriminator output circuit time constant); and the specification, for the first time in the FM art, of the necessity for perfectly reproducing the spike trains for the preservation of the zero average deviation from the desired frequency. But it should be clear that the "widebanding" theory does not provide an adequate explanation of the results. The real explanation, other fundamental requirements being met, is found in the chain of cascaded narrow-band limiters.

Before bringing this discussion to an end, we would like to mention two more or less obvious advantages that might result from using wider bandwidths than the minimum requirements
would demand in the limiter stages, and in the discriminator. At the cost of a need for wider discriminator bandwidths, one may be willing to choose the wider limiter bandwidths in order to secure the greater freedom from amplitude variations that accompanies the use of the wider limiter bandwidths under the most adverse conditions of interference. Limiters and discriminators with bandwidths exceeding the minimum requirements, and are a few times the i-f bandwidth, are also helpful in minimizing the effects of frequency drifts. Thus AFC systems are quite dispensable in receivers in which the limiter and discriminator bandwidths are a few times the i-f bandwidth, provided the limiter section has a low value of limiting threshold. The requirement of a low limiting threshold is dictated by the possibility of a frequency drift out of the i-f passband. Such a drift would not only decrease the available drive on the limiter rather dangerously, but it would also produce FM-to-AM conversion on the sloping sides of the i-f characteristic. A sufficiently low limiting threshold would handle these often serious effects successfully, and the reception would not be noticeably impaired unless some other signal, of whatever origin, happens to fall on a more favorable position on the i-f characteristic, or for any reason be stronger than the desired signal—often a likely possibility.
REFERENCES


CHAPTER FOUR

THE THEORY OF FEEDBACK AROUND THE LIMITER
INTRODUCTION

In the preceding chapters, the interest centered on the effect of a process of ideal amplitude limiting followed by ideal filtering (or frequency limiting) upon the instantaneous frequency of the resultant of two carriers differing in frequency as well as amplitude. The main objective was to determine the changes in the basic design requirements to secure proper operation under the most adverse conditions to be met, with the expressed purpose of enhancing the capture of the stronger signal. Our philosophy has not been guided merely by a desire to specify the requirements leading to the realization of an ideal frequency demodulator which is both insensitive to amplitude changes and meets the requirements in bandwidth dictated by the extraneous instantaneous frequency variations caused by the presence of the interference. We have, in addition, recognized certain fundamental features in the nature of what usually constitutes serious interference, and have chosen to capitalize upon these features in preparing the receiving circuits, by design, to minimize, or completely eliminate, the disturbance.

Generally speaking, interference may be described as being any extraneous modifications of the instantaneous variations of the parameter of the carrier wave modulated by the message. The interference may be tackled, for its possible suppression, both in the radio-frequency as well as in the low-frequency sections of the receiver. Whatever the extraneous effects of the interference are to be minimized, or eliminated, it is important to realize that, at the basis of any interference suppressing scheme, lies what we might term the fundamental principle of interference rejection: The interference may be suppressed if its disturbance is in the form of modifications, in the instantaneous variations of the message-bearing parameter of the carrier wave, that differ in a fundamentally distinguishable manner from the type of variations that an expected message modulation would
influct upon this parameter. If the extraneous variations cannot be distinguished from the variations caused by the message modulation, then the interference cannot be suppressed. A successful scheme for interference suppression is one which is capable of discriminating against those characteristic features of the disturbance which are not normally expected in the proper message modulation, without, in any significant way, affecting the message modulation itself. Recognition and appreciation of these facts helps one in accounting for the pronounced capture possibilities of an FM system, and the inherent vulnerability of an AM system, particularly to cochannel disturbances.

The most important distinguishable feature of an FM disturbance is the highest rate at which the interference causes the instantaneous frequency of the resultant signal, delivered by the i-f amplifier, to vary. This rate of instantaneous frequency variation combines the highest frequency deviation, in one package. Under conditions of high-level interference, this rate is sufficiently higher than the highest rate of variation that can be expected in the message modulation to make it possible to insert filters, at appropriate places in the signal path, that would be too sluggish to follow the disturbance, without being capable of distorting the message modulation noticeably. The appropriate places for these filters in the high-frequency sections of the receiver are not in the linear stage, because in these stages the desired carrier and the interference combine linearly and their resultant spectrum is fully accommodated within the i-f passband. The concentration of the spectrum of this resultant signal within the reaches of the i-f passband makes impossible to separate the two signals, or improve the predominance of the stronger one, purely by linear filtering. On the other hand, a process of ideal amplitude limiting will spread out the significant spectrum, necessary for the reproduction of the original frequency disturbance.
of the interference, over several i-f bandwidths. The extent of this range increases with an increase in the gravity of the interference resulting from a decrease in the amplitude difference, and an increase in the frequency difference between the two signals. Since the instantaneous frequency of the desired signal will always place this signal within the extent of one i-f bandwidth, we immediately recognize that we can filter after the limiting process to exclude sizable portions of the interference spectrum, without affecting the message-bearing spectrum. One stage of limiting and filtering, however, will still retain at its output a spectrum which contains a significant amount of the interference in components that could not be rejected without impairing some important phase of the operation. Therefore, subjecting the resultant of this retained spectrum to a process of ideal limiting will spread out the spectrum of the retained disturbance, again, over a sufficiently wider frequency range to enable additional filtering to be effective. This cycle of spreading out of the interference spectrum, followed by rejecting the outer portions of the significant spread-out spectrum, may be repeated until so little is left of the disturbance that the significant spectrum of the amplitude-limited resultant signal at the end of the chain becomes essentially confined within the limits of the minimum permissible passband. At this point, the maximum rate at which the disturbance will vary the instantaneous frequency of the resultant signal becomes comparable, and almost indistinguishable, from the variations that may arise with the expected message modulation. In other words, the cascading scheme, discussed in Chapter Two, can so sap the energy in the spectrum of the disturbance until what is left of this energy gives rise to a spectrum which is not significantly distinguishable in its extent from the spectrum that can arise with the usual type of message modulation. Phrased differently, the cascading of limiters followed by sluggish filters will remain effective
in the abatement of the disturbance until the variations in the instantaneous frequency caused by this disturbance begin to resemble the variations that the message modulation may be expected to give rise to. Beyond this point, continuation of this scheme is not profitable.

Thus a properly designed narrow-band limiter in the path of the resultant of two signals differing in strength by an arbitrarily small amount, will modify the character of this resultant signal in such a way as to reduce the effective disturbance caused by the presence of the weaker signal, and enhance the capture of the stronger signal. The amount of improvement in the conditions favorable to the capture of the stronger signal, per stage of narrow-band limiting, is predictable in accordance with the techniques and results of Chapters One and Two. We shall now raise the question of whether the performance of each stage of narrow-band limiting in reducing the interference may be improved by some circuit modification that does not alter the character of the narrow-band limiter per se, but rather involves such a limiter as a building block in an effective scheme. An important possibility, along these lines, is the rise of feedback at a proper angle from the output of the first, second or any other limiter, or several of them, to the input of the first or any other intermediate stage in a chain. The purpose of this chapter is to explore this important possibility. A simple theory of the effect of feedback around the limiter will be the focus of interest in this exploration. The philosophy of the method of attack is amazingly simple and will be found convenient for a preliminary study of any scheme involving limiters and filters together with amplifiers and phase shifters.

The background of the proposed theory lies in the results and experience gained in the studies of the first three chapters. Simplifying assumptions will be made that will make it possible for us to concentrate on the very basic
aspects of the mechanisms of operation involved, without getting entangled in considerations of secondary importance that might at best obscure the issue with unnecessary details. The assumptions to be made will be consistent with whatever assumptions we have found useful in the preceding chapters, and this will consequently allow us to draw upon the store of results and experience reached there.
IV.1 BACKGROUND

The idea of using feedback around the limiter has first appeared in the literature in an article, by W. A. Wilmore, in the Proceedings of the IRE, June 1952 (Ref. 1). In this article, an attempt is made at explaining how such a scheme might help in improving the capture performance of an FM receiver, but the theory proves both incomplete and inadequate. In fact, the type of feedback suggested is inverse feedback which, according to our analysis and measurements, can only cause deterioration in the interference suppression ability of the receiver by decreasing the amplitude difference between the two signals competing for capture and increasing the limiting threshold of a practical limiter. Consequently, this first feedback proposal has led some investigators to discredit the feedback idea altogether (Ref. 6).

The basic difficulty with the approach in Ref. 1 may be ascribed to a heterogeneous mixture of quasi-quantitative and qualitative reasoning. Starting out with an incorrect and misinterpreted phasor diagram, the argument suddenly turns qualitative in reasoning and curve sketching, with no clear relation (to show the effect of feedback) between the results of the phasor diagram computation and the qualitatively sketched instantaneous phase and frequency curves. Indeed, the qualitative discussion leaves the reader with no clear idea as to how the inverse feedback part of the mechanism of the suggested scheme combines with the effect of the sluggish filter in the feedback branch, in bringing about the claimed performance. In conclusion, it is claimed that the scheme suggested has made it possible "... to separate adequately field signals which have a ratio of amplitude of less than 200 below [earlier in the article, "1 or 20"] closer ratios could probably have been used if flatter IF amplifiers had been available." No description or schematics of the circuits used are provided, but the strong implication is that conventional circuits were used in the tests, which
renders the quoted figures nothing short of extraordinary.

A new approach to the problem has led us to a more convincing theory which not only clearly predicts the inadequacy of the Wilmotte scheme, but also points out that improvement may be reaped only with positive feedback. Furthermore, this new theory points out various possibilities that are in themselves sufficiently interesting and desirable to be worth aiming at in any design, and may perhaps parallel the function of increasing the difference in relative amplitudes between the stronger and the weaker signals in importance to the interference suppression ability of the FM receiver.

As before, we make some simplifying assumptions about the character of each building block in our suggested schemes so as to sharpen the focus on those basic aspects of the behaviour of each block that are really essential to the overall performance of the scheme. This will also focus our attention on the basic mechanism of the performance and will spare us the distractions contributed by side-tracks. Since all of the idealizations made may be closely approached in practice, the value of the argument, as well as the numerical results and novel conclusions made, is not significantly affected by any practical deviations from the ideal that may be cited in any particular situation. Indeed, the fundamental aspects of the argument and conclusions have already stood the test of laboratory measurements carried out with extremely simple and highly conventional circuits (Ref.5). Thus, although the numerical results arrived at apply strictly under the assumptions and conditions of the analysis, practical deviations from these assumptions and conditions that cannot be classified as truly significant will not impair the value of these numbers.
IV.2 PRELIMINARY THEORY: WIDEBAND FEEDBACK LIMITERS

On the basis of feedback bandwidth, we recognize first, two types of feedback around the limiter: wideband feedback, and narrow-band (or band-limited) feedback. Since we shall be mainly concerned with the effect of feedback around the limiter upon the capture performance in the presence of two-signal interference, we define the feedback bandwidth with respect to the spectrum of the amplitude-limited resultant of two carriers differing in amplitude and in frequency. Of the total spectrum that a limiter of unlimited bandwidth would deliver, our interest will be confined to that portion that is centered about the frequency of the stronger signal, assuming, as in Chapters One and Two, that the two carriers differ in frequency by a negligible fraction of the frequency of either. This portion of the spectrum (which has received detailed treatment in Chapters One and Two) shall henceforth be referred to as the primary spectrum at the output of the limiter. When all of the spectral components of significant amplitudes in the structure of the primary spectrum are feedback from the output of the limiter to its input, the operation will be termed wideband feedback. Band-limited, or narrow-band, feedback will result if the limiter, or its associated feedback amplifier, has incorporated in it a filter of such bandwidth as to exclude significant portions of the primary spectrum from reaching the input of the limiter. The feedback bandwidth is, therefore, defined either by the bandwidth of the limiter from whose output the feedback voltage is taken, or by the bandwidth of the amplifier in the feedback path, or by both. This division on the basis of feedback bandwidth will be shown to simplify the analysis and enable us to have a better grasp of the mechanism involved.

In the present section, the interest will center on the effect of wideband feedback around the limiter upon the capture performance of an FM receiver.

Figure 1 illustrates the basic scheme to be used in our analysis. The resultant signal delivered by the i-f amplifier
is assumed to be given by

\[e_i(t) = E_s \cos pt + aE_s \cos (P + r)t, \quad (1)\]

where the fundamental assumptions made in Section 1.2 with respect to the component signals apply. The idealized wideband limiter will be assumed to be such that in response to \(e_i(t)\) as given by Eq. 1 (with the switch \(S\) closed), the output \(e_o(t)\) is given by

\[e_o(t) = k \cos (pt + \theta), \quad (2)\]

where \(\theta(t)\) is the instantaneous deviation from the phase angle of the stronger signal caused by the presence of the weaker signal within the i-f passband. The amplitude factor \(k\) is by definition a factor of the limiter, and represents the value of the constant amplitude of the voltage at the output of the limiter when the amplitude of the input signal exceeds a threshold value which, for the time being, will be assumed negligibly small.

\[\text{Fig. 1}\]
The idealized output-input characteristic of the limiter is shown in Fig. 1(b). The role of a nonzero threshold in the overall picture is taken up in a later discussion. The idealized limiter and its associated ideal wideband filter are assumed to contribute no phase shift to the resultant signal going through them, for the convenience of associating the total phase shift around the loop with the feedback amplifier. In going around the feedback loop, through the feedback amplifier, the resultant signal is assumed to undergo a total phase shift given by \( \phi_{fb} \). Also, associated with this amplifier will be an ideal wideband filter that will pass essentially the whole primary spectrum with a passband gain of \( A \). Finally, let us assume that the ratio of feedback voltage seen across the terminals of switch \( S \) to the voltage \( e_o(t) \) at the output of the limiter is \( G \), where \( G \) lumps any voltage division ratio, \( d_o \), seen between the output of the limiter and the input to the amplifier, and \( d_i \) between the output of the amplifier and the input to the limiter. Thus, we may write \( G = d_o d_i \).

We shall start with a discussion of the two important special cases of positive \( (\phi_{fb} = 2\pi m, m = 0, 1, 2, \ldots) \) and inverse \( (\phi_{fb} = \pi m, m = 1, 3, 5, \ldots) \) wideband feedback.

A. **POSITIVE FEEDBACK:**

Starting with positive feedback, we first note that, with the switch \( S \) closed, the signal \( e_o(t) \), as given by Eq. 2, will appear at the output of the limiter, when the input to the limiter is given by \( e_i(t) \) of Eq. 1. However, assuming the time delay around the loop to be much less than \( 2\pi/r \) secs, for simplicity, an initial feedback voltage

\[
e_{fb}(t) = kG \cos \left[ \pi t + \Theta(t) \right]
\]

will appear across the terminals of the switch \( S \), just after the opening of this switch. The assumption of negligible time delay around the loop will be examined later on. The initial feedback voltage adds to the voltage
\[ e_1(t) = E_s (1 + 2a \cos r t + a^2)^{1/2} \cos (pt + \theta) \]
in phase, and the resultant input to the limiter becomes
\[ e_{R+}(t) = E_s \left[ (1 + 2a \cos r t + a^2)^{1/2} + k_s \right] \cos (pt + \theta) \]
where, by definition,
\[ k_s = kG/E_s. \]  \hspace{1cm} (4)
\[ k_s = kG/E_s. \]  \hspace{1cm} (4a)
This initial feedback addition is, therefore, seen to affect only the instantaneous amplitude of the resultant signal at the input to the limiter, but otherwise it leaves the instantaneous phase variations unaltered. This, of course, is a consequence of the assumption that the time it takes \( pt + \theta(t) \) to change by a noticeable amount is much longer than the delay time in the transmission around the loop. In fact, as will be explained shortly, only \( \theta(t) \) should remain essentially constant during the transmission time because phase changes arising from \( pt \) may be readily lumped into the feedback phase angle. This would call for a restoration of the proper positive feedback phase angle by means of a readjustment of the midband phase shift around the loop.

Neglecting the effect of time delay around the loop, enables us then to assume that the instantaneous phase angle of the feedback voltage will not differ noticeably from the instantaneous phase angle of the resultant input signal at any time. The effect of the positive (in-phase) feedback will, therefore, amount simply to an increase in the instantaneous amplitude of the input signal, and this in itself will leave the output of the limiter, as given by Eq. 2, unchanged.

The limiter, with its associated wideband filter, exerts a constraining effect in the signal path which clamps the signal amplitude at a specified level at its output and leaves the instantaneous phase of the output signal identical with that of the input signal. Since this limiter will deliver a voltage \( e_o(t) \), as given by Eq. 2, regardless of what the amplitude of the input signal is (but only so long as the instantaneous phase angle at the input and output remain
essentially the same), it is readily seen that a positive feedback steady-state will immediately be reached with the feedback voltage as given by Eq. 3, and therefore, with the net input to the limiter as given by Eq. 4. This result may be summarized by the following theorem.

Theorem I

Positive feedback of the primary spectrum from the output of the limiter to its input, without significant band limitation of the feedback spectrum, will not affect the instantaneous frequency variations of the resultant signal at the input to the limiter.

In general this type of feedback increases the amplitude of the resultant signal, thus improving the limiting conditions by decreasing the limiting threshold of a practical limiter, as measured by the smallest voltage that the i-f amplifier must supply for saturation of the limiter. It also smoothes out the instantaneous amplitude variations of the input signal, thus increasing the maximum permissible value of the time constant for limiters that depend upon the charging and discharging of capacitors for their operation.

Matters relating to the questions of stability and oscillations under positive feedback conditions will be taken up later in a separate section.

In Fig. 2, we present a phasor diagram to illustrate the effect of wideband positive feedback. The feedback is seen to effect a noticeable degree of smoothing that increases with the magnitude of the feedback signal relative to the input signals delivered by the i-f amplifier. The bearing of this smoothing in the amplitude upon the maximum permissible time constant at the input to a grid-bias limiter, for example, is readily determined by the methods of Section II.2. If the instantaneous amplitude is denoted by \( A(t) \), the grid-circuit time constant should obey the restriction

\[
\frac{C_T}{R_G} < \left[ -\frac{A(t)}{A(t)} \right]_{\text{min}}
\]
Fig. 2
which is condition 7 of Chapter Two. From Eq. 4, we observe that the positive feedback increases $A(t)$ by a constant value given by the amplitude of the feedback voltage, while $A'(t)$ is unaffected. We therefore conclude that the minimum value on the right in condition 5 is increased by the positive feedback. The larger the feedback factor $K_s$ the more nearly constant the resultant signal amplitude is (since it will then approximate the constant amplitude of the feedback voltage), and the less stringent the requirements on the limiter time constant become.

B. NEGATIVE FEEDBACK:

With inverse feedback, the resultant signal at the input to the limiter is given by

$$e_{R_\neg}(t) = E_s \left[ \left( 1 + 2a \cos rt + a^2 \right)^{1/2} - K_s \right] \cos (pt + \theta) \quad (6)$$

in the feedback steady state, if the time delay around the loop is again negligible. If the feedback factor $K_s$ is smaller than $(1 - a)$, then the resultant signal with feedback will have the same instantaneous frequency variations as in the absence of feedback, but its amplitude level will be lower. This constitutes a deterioration in the limiting conditions. In other words, the limiting threshold as gauged by the minimum necessary i-f signal that saturates the limiter will be increased by the inverse feedback. If $K_s$ equals $(1 - a)$, then the amplitude of the resultant signal with inverse feedback will vanish once during every difference frequency cycle, every time the two input carriers go through phase opposition. Although distortion will leak through the limiter in the neighborhood of this zero amplitude, the fact that the amplitude does not turn negative beyond this zero shows this is not technically an additional "zero crossing", and the instantaneous frequency is not ruffled by that zero. However, when the value of $K_s$ lies between $(1 - a)$ and $(1 + a)$, the instantaneous amplitude of the resultant signal will contribute to the
zero crossings of this resultant by going through zero (and changing sign in the process) twice in every difference-frequency cycle. This condition of feedback is also illustrated in Fig. 2. This figure shows the loci traced over one period of the difference frequency, \( r \), by the end point of the phasor that represents the resultant signal at the input to the limiter under conditions of no feedback (\( \text{OR}_0 \)), of positive wideband feedback (\( \text{OR}_+ \)), and of inverse wideband feedback (\( \text{OR}_- \)). The sign reversal of the instantaneous amplitude every time this envelope crosses through zero indicates that an impulse of instantaneous frequency, of area \( \pi \), is introduced at every zero crossing.

Finally, when \( K_s \) is much larger than \((1 + a)\), the resultant signal is given essentially by the feedback voltage, \(-K_s E_s \cos (pt + \theta)\), and is therefore of almost constant amplitude. The instantaneous phase variations are therefore identical with those of the resultant signal delivered by the i-f amplifier.

**Theorem II**

Inverse feedback of the primary spectrum from the output of the limiter to its input, without significant band limitation of the feedback spectrum, does not affect the instantaneous frequency variations of the resultant signal at the input to the limiter as long as the instantaneous amplitude does not contribute to the zero crossings of the resultant signal.

In general, this type of feedback decreases the amplitude of the resultant signal with the result that the limiting conditions deteriorate because of the increased limiting threshold of a practical limiter. Moreover, reference to Eq. 6 and condition 5 shows that the decrease in \( A(t) \) by a constant subtractive quantity leaves \( A'(t) \) unaffected, and therefore, decreases the maximum permissible value of the limiter time constant for a grid-bias limiter, unless the feedback voltage is large compared with the input signal level. Since the feedback factor \( K_s \) is the ratio of the constant feedback voltage amplitude to the generally varying amplitude \( E_s \) of the stronger component signal in \( e_1(t) \), it is
clear that $K_e$ will vary over a wide range with variations in the input signal level. In view of the discussion of the various possibilities that arise with various values of $K_e$, and of the severe distortion and bad limiting that will accompany any zero crossings contributed by the instantaneous amplitude, wideband inverse feedback proves quite undesirable, and must be avoided.

C. WIDEBAND FEEDBACK AT A GENERAL ANGLE $\Phi_{fb}$:

Going back to Fig. 1, we again observe that prior to opening the switch $S$, and with $e_1(t)$ of Eq. 1 at the input, $e_o(t)$ of Eq. 2 will appear at the output of the limiter. However, if we assume that the total phase shift around the loop at the center frequency, $\omega_0$, is given by $\Phi_0$, and let the slope of the net linear phase-shift characteristic around the loop be $t_d$, then when we open the switch $S$, the initial feedback voltage appearing at the input at time $t$ will be

$$e_{fb}(t) = KG \cos \left[ p(t + t_d) + \Theta(t + t_d) + \Phi_0 - \omega_0 t_d \right]$$

the substitution $t + t_d$ indicates that the instantaneous phase angle of the feedback voltage at time $t$ is equal to the value taken up by the instantaneous phase of the resultant at the input $t_d$ seconds earlier to within an additive constant. The time delay, $t_d$, is seen to be of significance only in so far as it affects the value of $\Theta(t)$, since the constant phase $p t_d$ could be easily lumped with $\Phi_0 - \omega_0 t_d$ to make up a new feedback phase angle $\Phi_{fb}$. The effect upon the instantaneous value of $\Theta(t)$ is best seen from

$$\tan \Theta(t + t_d) = \frac{a \sin r(t + t_d)}{1 + a \cos r(t + t_d)}$$

so that, if $t_d$ is a negligible fraction of one period of $2\pi/r$ sec, of the difference frequency, $r$, then we may assume the value of $\Theta(t)$ to be essentially unchanged during the interval of time $t_d$. The most severe test of such an assumption arises actually over a short interval of time of duration $\Delta t = 2(\cos^{-1}a)/r$ centered about $rt = \pi$, because $\Theta(t)$ varies most rapidly during this interval of time.
In order to appreciate the importance of $t_d$ to the present analysis, we recall some fundamental considerations relating to physically realizable ideal bandpass filters from network theory. For such filters, it may be shown (Ref. 2) that if the filter has zero attenuation over a bandwidth (BW) rad/sec, and an attenuation of $A$ nepers outside this band, then the phase characteristic associated with this attenuation characteristic will have a slope of $t_d = 4/\pi \cdot A/(BW)$ at the center frequency of the band. The assumption of discontinuities in the attenuation characteristic at the cut off frequencies makes the stated value of the slope at the center frequency most pessimistic. A filter which cuts off gradually at the edges of the band will display a phase characteristic having a lower slope at the center. In any case, we note appreciatively that the slope of the phase characteristic in the neighborhood of the center of the band varies inversely with the bandwidth of the filter and that

$$\frac{t_d}{\Delta t} = \frac{2A}{\pi \cos^{-1}a} \cdot \frac{r}{(BW)} \quad (7)$$

With reference to Chapter One, it is readily appreciated that the value of bandwidth necessary after the limiter and in the feedback amplifier in order to justify the description of the feedback voltage by Eq. 3, of the present chapter, may be considered to be sufficiently greater than $r$ to make $t_d$ negligibly smaller than $\Delta t$. This is particularly agreeable if it is recalled that an attenuation, $A$, of a few nepers outside the passband is sufficient for our purposes ($e^{-4.5} = 10^{-2}$) and $\cos^{-1}a = 0.2$ for $a = 0.98$, while $\cos^{-1}a = 0.1$ for $a = 0.995$.

In the light of the preceding examination of the importance of $t_d$ to the present discussion, we feel quite justified in neglecting the delay time $t_d$ around the loop, and this will do with relief in view of the simplicity that this adds
to the problem. We are in addition availing ourselves of
the further convenience of assuming the phase characteristic
of the ideal filter to be linear at least over that portion
of the passband which accommodates the significant components
of the primary spectrum, an assumption which in the light of
Ref. 2 is not unreasonable.

We might add, on the other hand, that in the light of
the results of Section III.2, we may just as well assume that
the filter involved in the present analysis is a simple
parallel-resonant circuit, or, perhaps, a two-pole Butterworth
type filter, which is sufficiently wideband to follow the
instantaneous frequency variations of the amplitude-limited
resultant of the two input sinusoids through stationary
states. For the parallel resonant circuit, Eq. 66 of Chapter
Three shows that \( t_d = \frac{2}{(3W)} \), so that

\[
\frac{t_d}{\Delta t} = \frac{1}{\cos^{-1} a} \cdot \frac{r}{(BW)} .
\]  

(8)

Again (BW) may be assumed to be larger than \( r \) by a sufficient
factor to justify the negligibility of \( t_d \). The use of Eq. 56
of Chapter Three enables us to write

\[
\frac{t_d}{\Delta t} = \frac{1}{1 + a} \cdot \frac{1 - a}{\cos^{-1} a} \cdot \frac{1}{KB_{lim}} .
\]

The ratio \((1 - a)/\cos^{-1} a\) approaches zero as \( a \) approaches unity.
For values of \( a \) between 0.5 and unity it is not difficult to
fix a value of \( K \) that will insure the negligibility of \( t_d \) in
comparison with \( \Delta t \).

Thus if we call the feedback angle \( \phi_{fb} \) we may write for
the initial feedback voltage after opening the switch

\[
e_{fb}(t) = KG \cos \left[ pt + \phi(t) + \phi_{fb} \right] .
\]  

(9)

This voltage is superimposed upon \( e_i(t) \) and the limiter next
FEEDBACK STEADY STATE PHASOR DIAGRAM

FIG. 3
sees the sum. The new resultant may be traced again around the loop to determine the succeeding modification of the signal at the input to the limiter. In the feedback steady state we should expect a resultant signal at the input to the limiter having an instantaneous phase deviation of $\Psi(t)$ from $Pt$. When this resultant is traced around the loop, it should return with a constant amplitude and an instantaneous phase deviated from $Pt$ by $\Psi(t) + \Phi_{fb}$, in turn to be added to the two signals delivered by the i-f, and to yield the same steady-state resultant signal at the input. The feedback steady-state phasor diagram is shown in Fig. 3. It is clear from this diagram that

$$\tan \Psi(t) = \frac{\text{asinrt} + K_s \sin(\Psi + \Phi_{fb})}{1 + \text{acosrt} + K_s \cos(\Psi + \Phi_{fb})}.$$  

Straightforward manipulations reduce this equation to the form

$$\tan \Psi(t) = \tan \Theta(t) + \frac{K_s \sin \Phi_{fb}}{\cos \Psi(1 + \text{acosrt})}.$$  

(10)

We observe that if $\Phi_{fb} = 0$ or $\pi$, then $\tan \Psi = \tan \Theta$, and $\Psi = \Theta$, or $\Theta + \pi$. In the case of $\Phi_{fb} = 0$ it is readily appreciated, from Fig. 2, that $\Psi(t) = \Theta(t)$ always. However, if $\Phi_{fb} = \pi$, then for certain feedback conditions ($K_s < 1 - a$), $\Psi$ equals $\Theta$, for others ($K_s > 1 + a$) it equals $\Theta + \pi$, while for $1 - a < K_s < 1 + a$, $\Psi$ will equal $\Theta$ over part of the difference frequency cycle, and $\Theta + \pi$ over the rest of the cycle.

If a well-known formula for the difference between two tangents is used, Eq. 10 becomes

$$\sin(\Psi - \Theta) = \frac{K_s \sin \Phi_{fb}}{\sqrt{1 - 2 \text{acosrt} + a^2}}.$$  

(11)

Equation 11 could have been written down by inspection of Fig. 5, or by using the law of sines. Equation 11 will lead to a physical $\Psi - \Theta$ only if $K_s \sin \Phi_{fb} \leq 1 - a$, which imposes a
severe restriction on the permissible value of $K_s$ when $\phi_{fb} \neq 0$ or $\pi$. Before pressing this matter any further at this point, let us pursue the discussion assuming for the moment that $K_s \sin \phi_{fb} \leq 1 - a$. A straightforward differentiation yields

$$
\frac{d\Psi}{dt} = \frac{de}{dt} + \frac{ar K_s \sin \phi_{fb} \sin rt}{(1 + 2a \cos rt + a^2) \sqrt{1 + 2a \cos rt + a^2 - K_s \sin^2 \phi_{fb}}} \tag{12}
$$

Recalling the assumption of $K_s \sin \phi_{fb} < 1 - a$, we observe that the quantity under the radical sign will always give a positive real (nonzero) number. Therefore, at $rt = \pi$, the second term in Eq. 12 is zero and

$$
\left. \frac{d\Psi}{dt} \right|_{rt=\pi} = \left. \frac{de}{dt} \right|_{rt=\pi}, \tag{13}
$$

indicating that the spike magnitude has not been affected by the wideband feedback. For all $\phi_{fb}$ in the range $0 < \phi_{fb} < \pi$, $\sin \phi_{fb} > 0$, and therefore, the second term in Eq. 12 adds to the value of $d\theta/dt$ over $0 < rt < \pi$, and subtracts from it over $\pi < rt < 2\pi$. This amounts to a distortion of the shape of the familiar spike pattern (without affecting the peak value of the spike) so as to render it unsymmetrical about $rt = 0$ or $rt = \pi$, through the superposition of an odd term upon the even function $d\theta/dt$.

The instantaneous amplitude of $\overline{OR}$ is, from Fig. 3,

$$
A(t) = E_s \left[ K_s \cos \phi_{fb} + \sqrt{1 + 2a \cos rt + a^2} \cos (\Psi - \Theta) \right] = E_s \left[ K_s \cos \phi_{fb} \sqrt{1 + 2a \cos rt + a^2 - K_s \sin^2 \phi_{fb}} \right], \tag{14}
$$

where the curious condition $K_s \sin \phi_{fb} < 1 - a$ must again be observed. This expression of course checks with the previous expressions for $\phi_{fb} = 0$ and $\pi$, but the nature of the restriction on $K_s \sin \phi_{fb}$ remains to be explored.
To appreciate the significance of the restriction $K_s \sin \phi_{fb} < 1 - a$, consider the steady state phasor diagram of Fig. 3 drawn for the instant of time $rt = \pi$, as in Fig. 4. By the law of sines

$$\sin \psi(\pi/r) = \frac{K_s \sin \phi_{fb}}{1 - a}. $$

If $K_s \sin \phi_{fb} = 1 - a$, then $\psi(\pi/r) = \pi/2$, while if $K_s \sin \phi_{fb} > 1 - a$ the triangle $OR_OR_f$ is hypothetical. For a given circuit and given minimum signal amplitude $(1 - a)E_s$ delivered by the 1-f, $\phi_{fb}$ and $K_s$ are fixed, and unless $K_s \sin \phi_{fb} \leq 1 - a$, no fixed steady-state value of $\sin \psi(\pi/r) \leq 1$ can be established. This means that no $\psi(\pi/r)$ value can be found corresponding to which a triangle $OR_OR_f$ can be drawn in such a way that its sides $R_OR_f$ and $OR$ make angles $\psi(\pi/r) + \phi_{fb}$ and $\psi(\pi/r)$ with $OR_O$. Since the existence of a triangle satisfying these specifications is necessary for the establishment of a feedback steady-state, we conclude that if $\phi_{fb} \neq 0$ or $\pi$, and if $K_s \sin \phi_{fb} > 1 - a$, then no feedback steady-state in the sense implied in Figs. 3 and 4 can be established.

What happens if $\phi_{fb} \neq 0$ or $\pi$, and $K_s \sin \phi_{fb} > 1 - a$? This is best answered by going back to Fig. 1 and chasing the signal around the loop over and over. Everytime the signal goes around the loop, a new feedback voltage $e_{fb}(t)$, portraying the instantaneous phase behaviour of the resultant signal.

\[\text{FIG. 4}\]
assumed at the input at the start of the round, appears back at the input to remodify the resultant signal seen by the limiter. If no instantaneous phase function \( \Psi(t) \) can be found which will be perpetuated (in the sense that when the resultant input signal has an instantaneous phase \( \Psi(t) \), then the feedback voltage \( e_{fb}(t) \) will have an instantaneous phase \( \Psi(t) + \Phi_{fb} \) and a magnitude compared to the signals delivered by the i-f such that when \( e_{fb}(t) \) is added to these signals the resultant signal assumed with phase \( \Psi(t) \) at the start of the round is again regained), then the modification and remodification procedure of the resultant signal at the input will go on and on. We may now state the following theorem.

**Theorem III**

When the resultant constant-amplitude signal appearing at the output of the wideband limiter, and having an instantaneous phase identical with that of the resultant input signal, is fed back (with a fixed amplitude of \( K_s \) times the stronger signal amplitude delivered by the i-f and) with its phase changed only by a constant value \( \Phi_{fb} \), then if \( K_s \sin \Phi_{fb} < 1 - a \), where \( a \) is the weaker-to-stronger signal amplitude ratio delivered by the i-f, a feedback steady-state will be reached in which the instantaneous frequency pattern of the resultant signal will be distorted in a manner (described by Eq. 8) which is not favorable for improvement in the interference rejection conditions. If \( K_s \sin \Phi_{fb} > 1 - a \), then no feedback steady state can be established and no improvement in the capture conditions may be expected.

A restatement of the above theorem may be put in the following form.

**Theorem IV**

Regardless of the feedback angle, wideband feedback around the limiter will not affect the instantaneous-frequency pattern produced by the presence of the weaker signal in a way that is favorable for improvement in the interference rejection conditions.
IV.3 POSITIVE VS INVERSE BAND-LIMITED FEEDBACK

As before, we shall start with a study of the two most important special cases of feedback, namely, positive and negative feedback. Although our first objective will be to develop a theory of the basic mechanism of the feedback from the output of the first limiter stage to the input of this stage under one or the other of these two types of feedback, the effect of feedback at an arbitrary angle will not be overlooked, particularly since a study of this effect will point out the importance of the feedback angle in greater perspective, and will show to what degree it would be necessary to keep to a specified value of this angle.

In the scheme shown in block diagram form, in Fig 5(a), the ideal amplitude limiter is characterized by the input-output voltage characteristic shown in Fig. 5(b). The nature of this characteristic in the neighborhood of the zero input voltage may, if desired be defined by a limiting process; or one may say that the ideal limiter is such that in response to any input voltage differing arbitrarily from zero, there is perfect saturation at a value of output voltage amplitude of k volts. In a later discussion of the possibility of operating the scheme with a feedback loop gain smaller than unity, we shall find it necessary to redefine the behavior of this characteristic in the neighborhood of the zero input voltage by allowing it to rise linearly to the value k and reach this value for a minimum $E_{in}$ corresponding to a finite nonzero threshold. It will be convenient, hereafter to refer to k as the ideal limiter factor. For convenience, it will be assumed that in going through the limiter, with its associated ideal filter, no changes in phase are sustained by a signal wave. The net phase change around the loop will be assumed lumped into the phase angle $\Phi_{fb}$ associated with the system function of the feedback amplifier, although in practice $\Phi_{fb}$ would not be so concentrated. The feedback amplifier will thus be assumed to behave like an ideal bandpass filter of bandwidth $(3N) f_{a}$, having a passband voltage gain A, a linear
FIG. 5

(a)
phase characteristic, and a midband phase shift of $\Phi_0$ radians. The intermediate-frequency (i-f) section of the receiver delivers a superposition of two carriers having an initial ratio of weaker-to-stronger signal amplitude, $a$, with frequencies $p$ and $p + \tau$ rad/sec, the former being the frequency of the stronger signal. With the switch, $S$, shorting out the output of the feedback amplifier, the conditions at the input of the ideal limiter could be described exactly as in Section I.2 of Chapter One, and the results of the analysis of the spectrum at the output of this ideal limiter, derived in that section, would be directly applicable. In view of the insight and experience gained in Chapter One, we would expect the situation in which the two signals were separated by one i-f bandwidth to represent, in general, the most unfavourable interference condition. As for the limiter filter bandwidth, we may, in the present example, get away without imposing any assumption beyond saying that it should be greater than or equal to one i-f bandwidth, provided we restrict the ideal filter of the feedback amplifier to one i-f bandwidth for the present analysis. This freedom
in the choice of the limiter filter band width will allow us, for the present, to meet the minimum requirements in limiter bandwidth for any value of \( a \) without any explanations or alterations in the analysis and results. We might add that as far as our present purposes are concerned, we may, just as well, restrict the bandwidth of the limiter to one i-f bandwidth and allow the feedback amplifier bandwidth to have any desired value equivalent \( \omega_0 \), or in excess of, that value. Under the latter conditions, it is easily realized (in the light of Chapter One), that the configuration of sidebands that will require the greatest value of minimum discriminator bandwidth (and therefore represent the most adverse interference condition) will be the one in which only the components \( A_0 \) and \( A_{-1} \) are passed by the limiter filter separated in frequency by the bandwidth of the i-f, for all values of \( a \) up to 0.84. Our interest in the value of one i-f bandwidth after the limiter or in the feedback amplifier is thus betrayed by the obvious convenience and simplification of the problem introduced by restricting the feedback signal to the minimum of two spectral components, and knowing at the same time that we are treating only the most adverse condition of interference over a wide range of values of \( a \).

In brief, let us consider that the signal amplitudes at the input are given by \( E_0 \) at \( p \), and \( aE_0 \) at \( p + r \) rad/sec, where \( a \) is less than unity, and \( E_0 \) is an arbitrary quantity that defines the signal level at the input; that the ideal limiter characteristic is such that for any \( E_{in} \leq 0 \), \( E_{out} \) equals a constant of \( k \) volts peak; that the ideal limiter filter has a bandwidth equal to \( (BW)_{if} \); that the amplifier in the feedback path has a voltage gain given by \( Ae^{j\phi_{fb}} \); that the coupling between the output of the limiter and input to this amplifier involves a voltage division ratio of \( d_0 \), without any phase shift; that the voltage division ratio between the output of the amplifier and input to the limiter is \( d_1 \); and that the net effect of the feedback branch on the voltage is described by \( Ge^{j\phi_{fb}} \). From Chapter One, we know that for \( a < 0.84 \), the
most unfavorable situation at the output of the limiter arises when only the spectral components located at \( p \) and \( p + r \) rad/sec are passed with \( r = (3W)_{1f} \). We also know that the most adverse condition of interference delivered by the i-f amplifier is the one in which the two signals are separated in frequency by one i-f bandwidth for a given value of weaker-to-stronger signal amplitude ratio. Therefore, it is logical to confine our present interest to this situation and gauge the effect of feedback by the way it influences this most adverse condition.

With the switch, \( S \), closed, we have, in the steady-state, the signals with amplitudes \( E_s \) and \( aE_s \) at the input, and the corresponding components \( kA_o(a) \) and \( kA_{-1}(a) \) at the output. When we first open the switch \( S \), the resultant signal at the input becomes composed of

\[
E_s + Ge^{j\Omega fb}kA_o(a) \quad \text{at \( p \) rad/sec}
\]

\[
aE_s + Ge^{j\Omega fb}kA_{-1}(a) \quad \text{at \( p + r \) rad/sec},
\]

and where \( G = \frac{d_1}{d_0}A \).

If we assume \( \Omega fb = 0 \), or an integral multiple of \( 2\pi \), we are then confining the interest to positive feedback for the time being, and we may say that the resultant signal at the input is composed of

\[
E'_s \quad \text{at \( p \) rad/sec}
\]

and

\[
a'E'_s \quad \text{at \( p + r \) rad/sec},
\]

where

\[
a' = \frac{aE_s + GkA_{-1}(a)}{E_s + GkA_o(a)} = \frac{aE_s + uA_{-1}(a)}{E_s + uA_o(a)}
\]

where \( u = kG \). We immediately note that since \( A_{-1}(a) < A_o(a) \), \( a' \) must be less than \( a \). With this new ratio of weaker-to-stronger signal at the input to the limiter, the spectral components at the output of the limiter filter become \( kA_o(a') \) and \( kA_{-1}(a') \),
and so, the feedback components become \( uA_0(a') \) and \( uA_{-1}(a') \). Since \( a' < a \), we note from the plots of \( A_0(a) \) and \( A_{-1}(a) \) shown in Fig. 3, that \( A_0(a') > A_0(a) \), and \( A_{-1}(a') < A_{-1}(a) \). The signals \( E_s \) and \( aE_s \) delivered by the i-f amplifier are unaffected by the feedback. However, the originally feedback components \( uA_0(a) \) and \( uA_{-1}(a) \) are now changed, in the second round, to \( uA_0(a') \) and \( uA_{-1}(a') \), to transform the limiter input into \( E_s + uA_0(a') \) at \( p \), and \( aE_s + uA_{-1}(a') \) at \( p + r \) rad/sec. The third round now starts with the spectral components at the limiter filter output becoming \( ka_0(a'') \) and \( ka_{-1}(a'') \), where

\[
a'' = \frac{aE_s + uA_{-1}(a')}{E_s + uA_0(a')} < a' < a,
\]

and at the end of this third cycle, the resultant weaker-to-stronger signal amplitude becomes

\[
a''' = \frac{aE_s + uA_{-1}(a'')}{E_s + uA_0(a'')} < a'' < a' < a,
\]

and so on. As the signal flow is traced around the feedback loop, the modified weaker-to-stronger signal amplitude ratio at the input to the limiter will continue to decrease. This ratio, however, will converge toward a finite non-zero value, as a steady-state condition is approached. That the limiting value of this ratio must remain non-zero, follows from the fact that it must obviously be larger than the lower bound

\[
a < b = \frac{aE_s}{E_s + u} 
\]

since \( A_{-1}(a) \rightarrow 0 \) and \( A_0(a) \rightarrow 1 \) only as \( a \rightarrow 0 \). Indeed, if it is remembered that a feedback steady-state will be reached only when the values of \( A_0 \) and \( A_{-1} \) at the output to the limiter filter correspond to those demanded by the resultant ratio of weaker-to-stronger signal at the limiter input, then it becomes obvious that the limiting value of this ratio is given by

\[
a \rightarrow b = \frac{aE_s + uA_{-1}(a)}{E_s + uA_0(a)} \quad (15a)
\]
If we set \( K_s = u/E_s \), Eq. 15(a) becomes

\[
s'_e = \frac{a + K_sA_{-1}(a'_e)}{1 + K_sA_0(a'_e)}
\]

(15)

It is interesting to observe that his result may be more quickly reached by "arguing backward" as follows.

With the i-f delivering \( e_1(t) = E_s \cos pt + aE_s \cos(p + r)t \) to the limiter input (where \( r = (3W)c_1 \)), and with the feedback applied, we assume that the voltage at the output of the narrow-band limiter (having one i-f bandwidth), after a feedback steady-state has been reached, is given by

\[
e_0(t) = k\left[A_0(a'_e) \cos pt + A_{-1}(a'_e) \cos(p + r)t\right].
\]

This voltage, multiplied by \( G = d_0a_0 A \), is feedback and superimposed upon the input signals to yield a net input to the limiter given by

\[
\left[E_s + uA_0(a'_e) \right] \cos pt + \left[aE_s + uA_{-1}(a'_e) \right] \cos(p + r)t
\]

with a ratio of weaker-to-stronger signal amplitude of

\[
a'_e = \frac{a + K_sA_{-1}(a'_e)}{1 + K_sA_0(a'_e)}
\]

which checks the previous result.

With inverse feedback, the phase angle \( \Phi_{fd} \) is chosen equal to some odd multiple of \( \pi \), but, otherwise the same reasoning applies. If the steps in the above argument are retraced with \( K_s \) replaced by \(-K_s\), then, corresponding to Eq. 15, we obtain

\[
a'_e = \frac{a - K_sA_{-1}(a'_e)}{1 - K_sA_0(a'_e)} > a
\]

(13)

Before we enter into a detailed discussion of the significance and implications of these results we would like to
derive another pair of useful equations. First, we recall that our choice of the signal amplitudes at the output of the 1-f as being $E_s$ and $aE_s$ was entirely arbitrary. Another possible and useful choice is $E_w/a$ for the signal at $p$ and $E_w$ for the weaker signal. The importance of each of these two choices is yet to be demonstrated, but, at the moment, we observe that with this second choice, an identical argument leads to

$$a\ell = \frac{E_w + uA_{-1}(a\ell)}{(E_w/a) + uA_0(a\ell)} \quad \frac{1 + K_wA_{-1}(a\ell)}{(1/a) + K_wA_0(a\ell)}$$  \hspace{1cm} (17)$$

for positive feedback, and

$$a\ell = \frac{1 - K_wA_{-1}(a\ell)}{(1/a) - K_wA_0(a\ell)}$$  \hspace{1cm} (18)$$

for negative feedback, where $K_w = u/E_w$. The feedback factors $K_s = u/E_s$ and $K_w = u/E_w$ are related by the formula $K_s = aK_w$. (19)

It is clear that if we allow the feedback factors $K_s$ and $K_w$ to take negative as well as positive values, then we may regard Eq. 15 and 17 as applying to positive feedback when $K_s$ and $K_w$ are positive, and to negative feedback when $K_s$ and $K_w$ are assigned negative values. With this agreement, we now turn to the plots of Figs. 7 and 8, where the information embodied in these equations is displayed graphically and in a most effective way. It is immediately observed that when the feedback is positive ($K_s$ and $K_w$ positive), the resultant ratio of weaker-to-stronger signal, $a\ell$, at the input to the limiter, is less than the original ratio, $a$, delivered by the 1-f, indicating an improvement in the conditions for the capture of the stronger signal through an increase in the amplitude difference between the two signals. This is readily understood if it is remembered that what we are doing is to add to the originally stronger signal at $p$ a component of the same phase and frequency which is larger than the corresponding component added to the weaker signal at $p + r$ (since $A_0(a) \geq A_{-1}(a)$,
the equality holding only for \( a = 1 \). In so doing we are of course increasing both signals in amplitude, but the originally stronger signal is being favored with a larger boost than is the weaker signal. As far as the resultant signal across the input terminals of the limiter in Fig. 5 is concerned, we are replacing the original two signals having relative amplitudes of 1 and \( a \) by two signals of the same corresponding frequencies, having relative amplitudes 1 and \( a' \), where \( a' < a \). The amount by which \( a' \) is less than \( a \) varies with the feedback factors \( K_s \) and \( K_w \). It is recalled that these factors combine the effect of the factor of the limiter, \( k \), the net feedback gain (as measured by the ratio of the feedback voltage actually appearing between the terminals of the switch, \( S \), in Fig. 5, to the voltage at the output of the limiter filter) and the signal level (\( E_s \) or \( E_w \)) delivered by the i-f amplifier. With reference to the discussion of wideband feedback, \( K_s \) measures the amplitude of the feedback voltage relative to the amplitude of the stronger of the two input signals, under conditions of wideband feedback. The constant \( K_w \) is the ratio of the feedback voltage to the weaker input signal amplitude, also under conditions of wideband feedback.

With inverse feedback, \( a' \) is always greater than \( a \) (in the range \( 0 < a < 1 \)) also by an amount which depends upon the same feedback factors, \( K_s \) and \( K_w \). This indicates a deterioration in the capture ratio with inverse feedback, which is contrary to Wilmotte's suggestion (Ref. 1). This conclusion is similarly appreciated by observing that what we are doing, basically, is to decrease the difference in relative amplitude between the two signals by subtracting a larger amount from the amplitude of the stronger signal than we are from the amplitude of the weaker signal. In so doing, we are of course decreasing both signals in amplitude, but the weaker signal amplitude sustains a smaller reduction in the process.

It is also important to remember, in anticipation of a detailed discussion, that by increasing the amplitude of the overall resultant signal at the input to the limiter with positive
feedback we are, in effect, helping a limiter that deviates from the ideal by having a finite nonzero limiting threshold, operate more safely beyond this threshold of full saturation with a lower value of signal amplitude from the i-f amplifier. The reverse is of course true of inverse feedback, where the resultant signal amplitude at the input to the limiter is being pushed down by the feedback.

We may now summarize the above conclusions with the following theorems.

Theorem V

If the feedback is taken from the output of one stage of ideal limiting to its input through a feedback bandwidth equal to one i-f bandwidth, then, under the most adverse interference condition passed by the i-f amplifier, positive feedback will improve the capture conditions at the input to the limiter by: (a) increasing the difference in relative amplitude between the stronger and the weaker signals; and (b) decreasing the limiting threshold of a practical limiter through an increase in the effective signal amplitude seen by this limiter.

Theorem VI

If the feedback is taken from the output of one stage of ideal limiting to its input through a feedback bandwidth equal to one i-f bandwidth, then, under the most adverse interference condition passed by the i-f amplifier, inverse feedback will cause the capture conditions to deteriorate at the input to the limiter by: (a) decreasing the difference in relative amplitude between the stronger and the weaker signals; and (b) increasing the limiting threshold of a practical limiter through a decrease in the effective signal amplitude seen by this limiter.

In Figs. 7 and 8, the lines marked $K_g = 0$ and $K_w = 0$ mark the relation between the amplitude ratio, $\alpha$, delivered by the i-f and the amplitude ratio, $\alpha'$, seen across the input terminals of the limiter in the absence of feedback. The two ratios are of course identical under this condition. Curves relating $\alpha'$ and $\alpha$ and lying to the right of the line
K = 0 (as with positive feedback) show that \( a \) is always less than \( a \), and, therefore, reflect improvement in the conditions for capture. Curves lying to the lift of \( K = 0 \) (as with inverse feedback) indicate deterioration in the conditions for capture. A curious and remarkable phenomenon is revealed by the positive feedback curves, which marks a welcome feature of this scheme. Under the conditions of the analysis, if at the start we have a signal \( a \) rad/sec which is stronger than the signal at \( + r \) rad/sec, then we have an original value of \( a \) lying in the range \( 0 < a < 1 \). For all such value of \( a \), we enter the curves between zero and unity, on the horizontal scale, and say that what the limiter sees is the resultant of two signals, one at \( p \) and a weaker one at \( p + r \), with a ratio of weaker-to-stronger signal amplitude of \( a_0 < a \).

However, suppose now that either of the two signals starts undergoing a change in amplitude due to a change in the propagation characteristics of its path, to its own build up or decay, or to any other physical reason. To be specific, suppose we start with original amplitudes of \( E_s \) and \( aE_s \) for the two signals, and let the amplitude of the weaker signal (or path) start increasing while that of the stronger signal remains constant. This state of affairs may be adequately described by saying that the value of the amplitude ratio, \( a \), undergoes an increase, starting from an original value between 0 and 1. As the weaker signal continues to increase in amplitude, the value of \( a \) may approach unity, attain this value, and even proceed to exceed it. On the assumption that the signal frequencies remain separated by more than one-half the i-f bandwidth while the weaker signal undergoes this increase in amplitude, the effect of this change in amplitude of the weaker signal upon the prevailing capture conditions is best followed on the plots of Fig. 7, since with \( E_s = \) constant, \( K_s \) will remain constant. From these plots, it is seen that \( a \) increases with \( a \) and despite the fact that \( a \) may attain a value of unity (signifying equality of the two signals) and may then exceed this value (indicating
that the originally weaker signal is now the stronger one),
the effective amplitude ratio, $a_\varepsilon$, at the input to the
limiter remains less than unity over a range that depends
upon the original value of $K_s$, and this indicates persisting
capture of the originally stronger (but now weaker) signal
at $p$ rad/sec. Thus, the fact that $a_\varepsilon$ will remain less than
one even when $a$ gets greater than one (due to a sudden increase
in the amplitude of the signal at $p + r$), indicates that the
reception of the message carried by the signal at $p$ will not
be interrupted by this change in the strength of the path
at $p + r$ (within limits to be defined). The in-phase feedback
process has thus added a measure of inertia in the reception
favorable to the capture of the originally stronger signal,
so that once this signal has captured, the receiver will
persist in letting it through, or remain locked to it, during
periods in which the originally weaker signal becomes larger
than the capturing signal by amounts whose bounds vary with
the feedback factor $K_s$. For any specified value of $K_s$, the
receiver will remain locked to the originally stronger signal
until $a$ reaches the critical value, $a_{\text{crit}}$, for which $da_\varepsilon/da$
becomes infinite (e.g., $a_{\text{crit}} = 1.85$ for $K_s = 5$). The course
of events beyond this point is clarified by the fact that the
value of $a$ is dictated by conditions beyond the control of
the limiter and its associated feedback circuit. Therefore,
since by assumption $K_s$ remains constant throughout (because
$E_s$ remains constant) we cannot predict the course of events
beyond this critical point by just following the curve as it
goes upward to the right with $a$ grows infinitely steep at the
critical point, and then turns backwards to the upper left,
since $a$ may keep right on increasing. The only tenable
conclusion under the assumed conditions is that this state
of "locking" to (or persistent and tenacious capture of) the
now weaker signal at $p$ will be broken as $a$ grows through the
critical point, and the now stronger signal at $p + r$ will
take over. The course of events resulting from a continued
increase in $a$ beyond the critical point may be pursued by now
defining a new \( a < 1 \) given by the reciprocal of the original \( \tilde{a} \), calling the old \( E_s, E_w \) (it being now the amplitude of the weaker signal), and accordingly renaming \( K_s, K_w \). We then plot the pertinent curve for \( K_w \) at the value of \( a \) (between 0 and 1) given by \( 1/a_{\text{crit}} \) and slide down along the curve of constant \( K_w \) to the lower left corner as the value of the newly defined \( a \) continues to decrease with continued increase in the amplitude of the signal at \( p + r \) rad/sec. Alternatively one may use the plots of Fig. 9, in which case one may retain the original \( a \) (i.e., not take its reciprocal as is done for Fig. 8), and read directly along the axis of abscissas in Fig. 9, which being taken in units of \( 1/a \), takes care of the reciprocality automatically. On the pertinent curve of constant \( K_w \) in Fig. 9, one would then slide down to the right as the amplitude of the originally weaker signal continues to increase past the point corresponding to \( a_{\text{crit}} \).

It is of interest to note what happens if, short of reaching the critical value indicated above, or upon attaining this value, the amplitude of the signal at \( p + r \) either starts to decline, with a consequent decrease of \( a \), or stops changing altogether with the result that \( a \) assumes a constant value below \( a_{\text{crit}} \). If the signal at \( p + r \) starts to decline in amplitude before \( a \) attains the value \( a_{\text{crit}} \), then it seems that the corresponding value of \( a \) should decrease in accord with the curve relating to the pertinent \( K_s \) value, by retracing that curve backward to the lower left-hand corner. If, however, \( a \) reaches a peak value corresponding to \( a_{\text{crit}} \) and then starts back on the wave, the character of the curves of constant \( K_s \) in the neighborhood of \( a = a_{\text{crit}} \) shows that either \( a \) should keep right on increasing, thus causing a rapid deterioration of the capture conditions of the now weaker signal at \( p \), by following the curve for the proper \( K_s \) and increasing \( a \), or the "looking" to the signal at \( p + r \) taking right control, the latter being perhaps more likely in view of inevitable fluctuations in amplitude that may push.
a beyond \( a_{\text{crit}} \), for various causes.

If, on the other hand, \( a \) reaches a value in the range \( 1 < a < a_{\text{crit}} \) and that value is maintained for any length of time without any further change, then provided the frequencies of the input signals remain separated by more than one i-f bandwidth, it seems that the only thing that can break the capture of the now weaker signal at \( p \) is nothing short of a severe interruption in the operation of the receiver, such as turning it off and back on again. Since \( a \) remains constant at that value, by assumption, the fact that the curve shows two possible values for \( a \) corresponding to each \( a \) in the range \( 1 < a < a_{\text{crit}} \) does not cause any ambiguity in the prevailing situation, since the higher value of \( a \) can only be reached if \( a \) first goes smoothly through the value \( a = a_{\text{crit}} \). The only point at which an uncertainty prevails is the point corresponding to \( a = a_{\text{crit}} \). As previously implied, if \( a \) tends to decline from a peak of \( a = a_{\text{crit}} \), then the odds are in favor of a continued rise in \( a \) to deteriorate the quality of the "illegitimate" capture, rather than a rolling back in the value of \( a \) into the region of safer and improved capture conditions. If \( a \) continues to increase with decreasing \( a \) (after the latter has passed through a peak value of \( a_{\text{crit}} \)), then as \( a \) approaches one from the right, \( a \) also approaches one, and the two signals become increasingly more difficult to separate in a small region on both sides of \( a = 1 \).

It is exceedingly important to note the bearing that the above modes of operation have upon the quality of FM reception under conditions of impulsive interference arising from arcing in car-ignition systems and in electric machinery. This important topic will be handled in due course, but is mentioned here in view of its pertinence to the discussion of the locking phenomenon.

Before terminating a discussion that hinges upon the curves of constant positive \( K_\alpha \) values, it is of interest to add a few additional observations about these curves. It
is clear from these curves (as well as from Eq. 15) that the positive feedback in effect decreases the dependence of the capture conditions at the input to the limiter upon the characteristics of the signal paths and other influences external to the receiver. An i-f amplifier amplitude vs frequency characteristic which displays a ripple within the passband will cause amplitude variations that are synchronous with the instantaneous frequency of the signals. Provided this ripple lies within reasonable bounds that will favor the desired (possibly consistently) stronger of the two paths with a sufficient initial margin to capture, the capture conditions at the input to the limiter with positive feedback will also prove less dependent upon the ripple in the i-f passband. The reason for this decreased dependence is best seen by referring to Eq. 15 and observing that the feedback terms added to the numerator and denominator provide a cushioning (or buffer) effect against fluctuations in the signal amplitudes delivered by the i-f amplifier. This buffer effect is greater the higher the value of \( K_s \), and is directly responsible for the "locking" phenomenon discussed above.

For purposes of plotting, it is convenient to rewrite Eq. 15 in the form

\[
a = a_x + K_s \left[ a_x A_0(a_x) - A_{-1}(a_x) \right].
\]

(20)

If the value of \( K_s \) for which the locking phenomenon will just fail to arise is desired, then we set \( a = 1 \) in Eq. 20 less than unity, and manipulate the inequality

\[
a_x + K_s \left[ a_x A_0(a_x) - A_{-1}(a_x) \right] < 1
\]

to get

\[
K_s < K'(a_x) = \frac{1 - a_x}{a_x A_0(a_x) - A_{-1}(a_x)}.
\]

(21)

Since \( K'(a_x) = 0 \) when \( a_x = 1 \), it is evident, from the inequality 21, that the locking phenomenon is, at least theoretically, possible so long as \( K_s > 0 \); in other words, all curves of
constant positive values of $K_s$ will cross and extend to the right of the line $a = 1$ for some nonzero range of $\varepsilon$ values. It is important to remember that a basic assumption in the derivation of Eq. 15 has been that $a$ be less than one. This assumption allowed us to state that the spectrum at the output of the limiter would be described by the results of Section I.2 of Chapter One. On that basis, we were able to trace the chain of events around the feedback and thus establish Eq. 15. The discussion of the various events relating to situations in which $a$ was allowed to undergo variations making it greater than one, has, of course, assumed for a starting point a feedback steady-state in which $a$ was less than one. Furthermore the detailed description of the events following a change in the amplitude of the signal at $p + r$ rad/sec has been carried out on the assumption that the difference frequency, $r$, remained greater than one-half the i-f bandwidth for the duration of those amplitude changes, in order to enable us to trace the variations of $\varepsilon$ with $a$ along one of the curves plotted in Fig. 7. On the other hand, should the difference frequency become smaller than, or just equal to one-half the i-f bandwidth, the feedback spectrum becomes different from that assumed in the analysis leading to the curves of Figs 7, 8 and 9, and consequently those curves would not adequately describe the variations of $\varepsilon$ with $a$. As will be shown in a later section of this chapter, the effect upon the capture conditions of feeding back a more complicated spectrum than the one assumed in the present section, may still be predicted by direct (though much more laborious) analysis. It will also be shown that the effect of narrow-bandening the feedback path will diminish and become increasingly less pronounced as the value of $r$ is decreased to allow more of the significant spectral lines to be fed back. When $r$ becomes sufficiently small to enable the filter defining the feedback bandwidth to follow the instantaneous frequency variations of the resultant of the two input sinusoids through quasi-stationary states, then the feedback falls into the
wideband category discussed in Section IV.2, and the capture improvement brought about by narrowbanding are absent. In the light of these facts, it may be concluded that when either, or both, of the signals is frequency modulated (slowly of course), the locking to the originally weaker of the two signals will break when the difference frequency decreases to a value for which the spectrum feedback does not result in a sufficiently pronounced inertia for the illegal capture to be sustained. The significance of these remarks will be better appreciated from the discussions of Sections IV.7 and IV.8.

Also of importance is the fact that the feedback factor $K_s$ is given by $u/E_s$ where $u$ is the amplitude of the feedback voltage across the terminals of switch $S$ in Fig. 5, under wideband feedback conditions, while $E_s$ is the amplitude of the initially stronger signal as delivered by the i-f amplifier. For any specific design, $u$ is a constant. However, the stronger signal amplitude $E_s$ delivered by the i-f may vary over a wide range. It is significant to note that $K_s$ varies inversely with $E_s$ and, therefore, assumes its largest values when $E_s$ assumes its smallest values, and vice versa. With reference to Fig. 7, we consequently conclude that since the positive feedback effects discussed above become more and more pronounced with increasing values of $K_s$, and since, other conditions being identical, $K_s$ is larger for smaller $E_s$, the phenomena associated with positive feedback discussed above, namely the improvement of the capture effect at the input to the limiter and the locking to the originally stronger signal when the weaker signal amplitude changes so as to make it the stronger one, will be increasingly more pronounced at the lower i-f signal levels. This fact will also be even better appreciated if it is remembered that, basically, the positive feedback effects depend upon the strength of the input signals relative to the magnitudes of the feedback components. Since, under the conditions of the present analysis, the feedback components are given by $uA_0(a)$ at $p$ rad/sec and
\( u_{A-1}(a) \) at \( p + r \) rad/sec, we observe that these components depend only upon the ratio of weaker-to-stronger signal amplitude, \( a \), delivered by the i-f and not upon the absolute values of these signals. Hence, if \( a \) is fixed, the feedback components are of fixed magnitudes, and therefore the smaller the absolute value of the initially stronger signal, \( E_s \), the greater the relative importance (and therefore the cushioning or buffer effect) of the feedback components. Indeed, the more predominant the feedback components are over the components delivered by the i-f (i.e. the larger the value of \( K_s \)) the more pronounced is the effect of the feedback upon the capture conditions at the input to the limiter, and the less dependent will these conditions be upon variations in the amplitudes of the signals delivered by the i-f amplifier. The degree of improvement in the capture conditions may be made fairly independent of the input signal level, and hence, left more reliably to the control and discretion of the designer, by the use of buffer schemes of the type discussed in Section IV.5.

Consider next the situation in which the signal at \( p + r \) rad/sec maintains a constant amplitude, \( E_w \), whereas the originally stronger signal \( E_w/a \) at \( p \) rad/sec starts decreasing in amplitude after a feedback steady-state has been established in which the signal at \( p \) is capturing. Here the variation of \( \alpha E \) (the modified ratio of weaker-to-stronger signal amplitude seen by the limiter) with \( a \), can be traced on the pertinent curve of constant \( K_w \) in Fig. 8. The fading of the amplitude \( E_w/a \) is equivalent to an increase in \( a \), since \( E_w \) is assumed constant. It may also be interpreted as a decrease in \( 1/a \) for the purpose of tracing the variation of \( \alpha E \) with \( a \) on the plots of Fig. 9. Figures 8 and 9 are in a sense complementary, the former portraying the variations of \( \alpha E \) in the region of low \( a \) values, the latter presenting the variations in the region of very large \( a \) values. The need for the plots of Fig. 9 is suggested by the curious behaviour of the curves of constant \( K_w \) for values of \( K_w \gtrsim 1/0.329 = 3.067 \). The curve for \( K_w = 4.357 \) (not shown) in Fig. 4 rises slowly with large values of \( a \),
approaching a horizontal asymptote at a value of \( \varepsilon \) of approximately 0.657. Above this asymptote, another branch of the curve starts from \( a = 1 \), \( \varepsilon = 1 \) and descends toward the same horizontal asymptote, touching it at \( a = \infty \). If plotted against \( 1/a \), instead of against \( a \), this curve would be tangent to the \( a \) - axis \( (1/a = 0) \) at \( \varepsilon = 0.657 \). The curves for \( K_w > 4.367 \) all have three pieces each, and have infinite abscissae for two values of \( \varepsilon \). In the plots of Fig. 9, these curves cross the \( \varepsilon \) - axis at two points. This behaviour is to be expected of these curves, since Eq. 17 may be rewritten in the form

\[
a = \frac{\varepsilon}{1 - K_w \left[ \varepsilon A_0(\varepsilon) - A_{-1}(\varepsilon) \right]}
\]

which is more convenient for plotting.

It is curious to observe that, in this instance, the originally stronger signal may fade out to zero (as \( 1/a \) goes to zero, or \( a \) becomes infinite) for \( K_w > 4.367 \), apparently without upsetting the capture. Of course, it would be hard to conceive of the signal at \( \beta \) rad/sec maintaining a zero value for any length of time, and for the receiver still to know what the message carried by this signal is. Indeed, we shall see later on that the capture of the vanishing signal will be broken, and this break may even happen before the amplitude of this signal goes to zero. The crossover of curves of \( K_w > 4.367 \) into regions of negative \( 1/a \) may perhaps be of mathematical interest only, since the significance of a continued variation of \( 1/a \) into the negative region indicates that the stronger signal decreases in amplitude to zero, and builds up again with its initial phase (relative to that of the signal at \( \beta + \gamma \)) reversed.

That the "locking" phenomenon will be displayed, at least in theory, in some finite nonzero range of \( a \) values for all \( K_w > 0 \), is readily shown as was done with the \( K_s \) -curves. As before, we seek the maximum value of \( K_w \) for which the corresponding \( \varepsilon \) vs \( a \) curve will just fail to cross the \( a = 1 \)
line for any value of a between 0 and 1. The result is a condition on $K_w$ exactly the same as that on $K_s$ in Eq. 21, whence, a cross-over to the right of $a = 1$ with $a < 1 \text{ will arise with all } K_w > 0$.

The various remarks made in connection with the curves of constant $K_s$ are applicable to curves of constant $K_w$, with minor additions that may be prompted by the detailed differences in behaviour between the two families.

Finally, the desirability of defining the signal amplitudes in the two ways indicated at the beginning of this section is now quite evident. For, if because of any circumstances befalling either of the original two signals at the input to the receiver and delivered by the i-f amplifier to the limiter, the value of $a$ should undergo significant variations during an interval of time that is either a small fraction of a message modulation period, or whose duration is smaller than the time required for either or both frequencies $p$ and $p + r$ to change by a sufficient amount to alter the feedback spectral configuration ($a$ always reserving the interpretation of amplitude ratio of signal at $p$ to that at $p + r$), then the effect of the variations in $a$ on the capture conditions at the input to the limiter (indicated by the effective ratio $a$ seen by the limiter), may be conveniently traced from: (a) Fig. 7, if $E_s$ (the amplitude of the originally stronger signal at $p \text{ rad/sec}$) remains constant, but $aE_s$ (the originally weaker signal at $p + r \text{ rad/sec}$) undergoes the variation; (b) Fig. 8 or 9, if $E_w$ (the originally weaker signal at $p + r \text{ rad/sec}$) remains constant, but $E_w/a$ (the originally stronger signal at $p \text{ rad/sec}$) undergoes the variation.

Later discussions of the effect of feeding back other spectral configurations will throw additional light upon the "locking" phenomenon, and related topics.

Figures 7, 8 and 9 include plots relating to the effect of inverse feedback upon the capture conditions at the input to the limiter. From these curves it is at once evident that
inverse feedback aggravates the capture problem by bringing the signals closer together in relative amplitude, as indicated by the fact that $\alpha_x$ is always greater than $a$ in the range $0 < a < 1$. The capture of the stronger signal is, consequently, obstructed rather than helped, and the deterioration increases with increase of the amplitude of the voltage fed back relative to that of the input signal. Since for any specific design, the feedback voltage maintains a constant amplitude (under the conditions of the analysis) the deterioration in the capture ratio of the receiver increases with a decrease in the signal level ($E_s$ for $K_s$, or $E_w$ for $K_w$) delivered by the i-f amplifier.

In view of the fact that significant portions of the curves of constant $-K_s$, for $K_s$ of the order of 3 or greater, extend over negative values of $\alpha$, and particularly over $a$ values in the range $-1 < a < 0$, the question may be raised as to what an initial negative value of $a$ really signifies in this analysis, and whether conditions starting with such values of $a$ should be traced on the $K_s$-curves by starting in the range $-1 < a < 0$ on the curves shown in Fig. 7. In this connection, one may well recall that in the analysis of the spectrum at the output of the limiter, in Section I.2 of Chapter One, the two signals were assumed to be in phase at $t = 0$, and so, one may associate positive values of $a$ with this choice of time reference. Consequently, we may agree to interpret negative values of $a$ as meaning that the signals are $180^\circ$ out of phase at $t = 0$. If, for convenience, we choose the originally stronger signal at $p$ rad/sec (whose amplitude is denoted by $E_s$ and $E_w/a$ in the derivation of Eqs. 20 and 22, respectively) for phase reference at $t = 0$, then it is helpful in avoiding algebraic pitfalls to note that this choice of reference results in $E_s = E_w/a$ (at $p$ rad/sec) as well as $aE_s = E_w$ (at $p + r$ rad/sec) positive at $t = 0$ when $\alpha$ is positive, and $E_s = E_w/a$ positive while $aE_s = E_w$ negative when $\alpha$ is negative. With this in mind,
we then observe that, for algebraic consistency, when we replace \( a \) by \(-a\) in Eqs. 20 and 22, then \( K_s \), by agreement, maintains the same positive sign in the algebraic manipulations, whereas, \( K_w = (1/a) K_s \) must be given a negative sign. Therefore, if we write

\[
\alpha_+ (a \ell) = a + \frac{a \ell}{1 \pm K_w \left[ a \ell A_0 (a \ell) - A_{-1} (a \ell) \right]} \quad (20')
\]

and

\[
\alpha_- (a \ell) = \frac{a \ell}{1 \pm K_w \left[ a \ell A_0 (a \ell) - A_{-1} (a \ell) \right]} \quad (22')
\]

then, if we multiply both sides of each equation by \(-1\) and remember that \( A_0 (-a \ell) = A_0 (a \ell) \) while \( A_{-1} (-a \ell) = -A_{-1} (a \ell) \), we obtain the relations

\[
-a_+ (a \ell) = -a + \frac{-a \ell}{1 \pm K_s \left[ -a \ell A_0 (-a \ell) - A_{-1} (-a \ell) \right]}
\]

\[
= -a \ell \mp K_s \left[ -a \ell A_0 (-a \ell) - A_{-1} (-a \ell) \right]
\]

\[
= a_+ (-a \ell), \quad (23)
\]

and (remembering that \( K_w \to -K_w \) when \( a \to -a \))

\[
-a_- (a \ell) = \frac{-a \ell}{1 \mp (-K_w) \left[ a \ell A_0 (a \ell) - A_{-1} (a \ell) \right]}
\]

\[
= a_- (-a \ell) \quad (24)
\]

Equations 23 and 24 reveal that the \( K_s \) and \( K_w \) curves in Figs. 7, 8, and 9 are odd-symmetrical with respect to both \( a \) and \( a \ell \), and so, conditions starting with negative values of \( a \) may be traced directly on either set of curves, after the signs on the \( a \) and \( a \ell \) scales have been changed from plus to minus, by entering on the new \(-a\) scale (now extending to the right in each family) between \( 0 \) and \(-1\) corresponding to the range \( 0 < a < 1 \) on the curves as presented in Figs. 7, 8 and 9). This procedure confirms expectations accruing from arguments
on a purely physical basis, and, here, completes the resolution of the questions that ushered in this topic.

Finally, the important conclusion is obvious that inverse feedback around the limiter is not desirable as far as generally improving the capture conditions and limiter operation (in practice) is concerned, and is therefore to be avoided.
IV.4 BAND-LIMITED FEEDBACK FROM OUTPUT OF \( n \)TH LIMITER TO INPUT OF FIRST

We shall now consider the properties of positive and negative feedback from the output of the second, third, or later, limiter to the input of the first. From our earlier studies, we may reason that if, for the time being we associate one i-f bandwidth with each limiter stage, then we may concentrate the initial interest on the capture conditions prevailing when the most adverse condition (and simplest to handle) arises, in which the two signals are separated in frequency by one i-f bandwidth. Under this condition, the second limiter sees only \( kA_0(a) \) and \( kA_{-1}(a) \) at its input, where \( a \) is the ratio of weaker-to-stronger signal amplitude at the input to the first limiter. It is recalled that this configuration of components will also be passed on to the second limiter whenever the two carriers delivered by the i-f amplifier differ in frequency by more than one-half the i-f bandwidth, but it will represent the most troublesome resultant signal delivered to the second limiter only for \( a < 0.84 \).

In the structure of the resultant signal at the input to the second limiter, under the above conditions, the ratio of weaker-to-stronger component amplitude will, therefore, be less than \( a \). It follows that the spectral components with frequencies \( p \) and \( p + r \) rad/sec at the output of the second stage will also differ in amplitude by more than they did at the output of the first stage. As a consequence, when these components are fed back to the input of the first limiter and superimposed upon the two corresponding sinusoids present there, the resultant sinusoids at \( p \) and \( p + r \) rad/sec will be further apart in relative amplitudes with positive feedback than they would otherwise be, with the feedback from the output of the first limiter itself. Thus, we are justified in expecting the feedback effects on the capture conditions to be more pronounced when the feedback is taken from the output of the second (or later) stage.
than from the output of the first. The effect of time delay around the loop, so far ignored, will be equivalent to an almost constant shift in the phase of the components fed back, relative to the phase of the input components, assuming the message modulation of each carrier to be sufficiently slow to make the time delay around the loop a negligible fraction of an audio cycle. The latter assumption is quite reasonable, particularly if it is recalled that a modulation at an audio rate of 15000cps has a period of 66.7 microseconds. The effect of this phase shift will be to shift the components fed back with respect to the input components from the positions of direct in-phase or out-of-phase superposition. Such shifts can, of course, be anticipated and approximately counter-balanced. The effect of deviations from the in-phase or out-of-phase positions is taken up in a later section.

Of special interest is the possibility that after the resultant of the two signals delivered by the i-f has been sent through a sufficiently long chain of limiters to decrease the amplitude of the originally weaker signal at p + r to a negligible value compared to the amplitude of the signal at p, a feedback of the signals at the output of the last limiter in the chain in phase opposition to the signals at the input to the first limiter will result in the amplitude of the signal at p + r essentially unchanged while that at p sustains a sufficient decrease in amplitude to make it the weaker signal. Of course, it is the conditions prevailing in a feedback steady-state that will show the final effect upon the relative amplitudes of the two signals at the input to the first limiter and thus determine whether such a scheme can pave the way for the capture of the weaker signal. Indeed, it is not difficult to realize, at this point, that the scheme will not work, for by tracing events around the loop over several rounds we can easily appreciate that no significant amplitude differences can be tolerated at the input to the first limiter. That is to say, this scheme will only suppress amplitude differences between the input
signals rather than favor either of them, and the signals
will be closer in amplitude the longer the chain of bandpass
limiters in the feedback loop. Thus, any contemplated use
of inverse feedback in any schemes aimed at creating conditions
favorable to the capture of either of the two signals should
indeed be quickly dispelled. The preceding thoughts will
now be pursued quantitatively.

Consider the scheme shown in Fig. 10. The first ideal
limiter bandwidth is assumed to be equal to that of the
1-f, as is the feedback bandwidth. If the feedback bandwidth
is defined by the idealized filter in the feedback amplifier,
the bandwidth of the second limiter may have any desired
value starting from one 1-f bandwidth. With the switch S
closed,

\[ e_{1f} = e_{1i} = E_s \left[ \cos pt + \cos (p + r)t \right] \]

\[ = E_w \left[ (1/\alpha) \cos pt + \cos (p + r)t \right] \]

where \( E_s = aE_w \), and \( a < 1 \). Also, under the most adverse
condition \( r = (3\omega)_{1f} \), for \( a < 0.84 \),

\[ e_{12} = kA_o(a) \left[ \cos pt + a_1 \cos (p + r)t \right] \]

where \( a_1 = A_{-1}(a)/A_o(a) \)

and

\[ e_o = k \left[ A_o(a_1) \cos pt + A_{-1}(a_1) \cos (p + r)t \right]. \]

FIG. 10
When the switch is opened, assuming the feedback is positive, the input signal amplitudes change to

\[ E_s + uA_o(a_1) \text{ at } p \text{ rad/sec} \]

and

\[ aE_s + uA_{-1}(a_1) \text{ at } p + r \text{ rad/sec}, \]

with

\[ u = kd_o A = kG, \text{ as before, and} \]

\[ a' \equiv \frac{aE_s + uA_{-1}(a_1)}{E_s + uA_o(a_1)} . \]

The input to the second limiter becomes

\[ kA_o(a') \text{ at } p, \text{ and } kA_{-1}(a') \text{ at } p + r \text{ with a ratio} \]

\[ a_2 = A_{-1}(a')/A_o(a'). \] This changes the amplitude of the components fed back to \( uA_o(a_2) \) and \( uA_{-1}(a_2) \), which completes the first round. In the steady state, the input signals settle to the amplitudes

\[ E_s (\text{or } E_w/a) + uA_o(a_1) \text{ at } p \]

and

\[ aE_s (\text{or } E_w) + uA_{-1}(a_1) \text{ at } p + r \]

with a ratio of weaker-to-stronger signal amplitude of

\[ a_2 = \frac{a + K_{s}A_{-1}(a_1)}{1 + K_{s}A_o(a_1)}, \quad K_s = u/E_s \]

\[ \frac{1 + K_{w}A_{-1}(a_1)}{(1/a) + K_{w}A_o(a_1)}, \quad K_w = u/E_w \]

(25)

(26)

where \( a_{-1} = A_{-1}(a')/A_o(a') \), and \( u = kG \), \( k \) being the constant (27) of the second limiter as defined in Fig. 1(b). Evidently, only the constant of the second limiter will enter into the definitions of \( K_s \) and \( K_w \) in the present scheme. Curves of \( a_2 \) vs \( a \) for constant values of \( K_s \) and \( K_w \), positive for positive feedback, and negative for inverse feedback, are presented in Figs. 11, 12 and 13. A comparison of these curves with the curves of Figs. 7, 8, and 9, of the preceding section, shows the difference between the two sets to be mainly in the
fact that the present curves are appreciably more spread (or stretched) horizontally than the corresponding curves of Figs 7, 8, and 9.

With positive feedback, the curves show the improvement in the capture conditions to be considerably more pronounced than in the case of feedback from the output of the first limiter; that is to say, for the same values of $a$ and $K_w$, the corresponding $aE$ values are significantly lower in the present situation than they are for feedback from the output of the first limiter.

The curves also show clearly that the capture condition at the input of the first limiter sustain a significantly greater deterioration when the feedback is from the output of the second stage to the input of the first, since the larger values of $aE$ show that the signals there are now closer together in amplitude and hence, harder to separate.

The above results may readily be extended to the case of feedback from the output of the third, fourth, or nth limiter to the input of the first. Of all the assumptions that may be carried over to the general case unaltered, we must emphasize that the total time delay around the loop must amount to a negligible fraction of the period of the highest modulation frequency carried by either of the two signals delivered by the i-f. The experience acquired so far enables us to write down the pertinent relations under feedback steady state conditions, assuming the components fed back appear in phase (for positive feedback) or out of phase (for inverse feedback) with the component signals delivered by the i-f amplifier. Thus, for feedback from output of the third limiter to the input of the first, we may write

$$aE = \frac{a + K_s A_1 (a E_0)}{1 + K_s A_0 (a E_0)}$$

$$K_s = \frac{K G}{E_s}$$

$$= \frac{1 + K_w A_1 (a E_0)}{(1/a) + K_w A_0 (a E_0)}$$

$$K_w = \frac{K G}{E_w}$$

where $K$ in the definitions of $K_s$ and $K_w$, is the constant the third limiter, and
\[ a_o = \frac{A^{-1}(a_o)}{A_o(a_o)}, \quad a_{-1} = \frac{A^{-1}(a_{-1})}{A_0(a_{-1})}, \quad (30) \]

these being the ratios of weaker-to-stronger signal amplitude seen by the third and second limiters, respectively, under the conditions of the analysis. From the fact that \( a_{-2} < a_{-1} \) (due to the action of the second narrow-band limiter) and, therefore, \( A^{-1}(a_{-2}) < A^{-1}(a_{-1}) \) while \( A_0(a_{-2}) > A_0(a_{-1}) \), it is readily appreciated that with positive (respectively negative) feedback, the present scheme adds (respectively subtracts) more to (respectively from) the amplitude of the signal at \( p_1 \) relative to that at \( p + r \), than does the scheme of feedback from the output of the second limiter to the input of the first under otherwise identical conditions. Therefore, feedback from the output of the third limiter will cause the difference in amplitudes to be greater with positive feedback, and smaller with negative feedback, than those produced by feedback from the output of the second or the first limiter. Indeed, the further down a chain of limiters we go to tap off a voltage for feedback to the input of the first limiter, the smaller the resultant ratio of weaker-to-stronger signal, \( a_{-2} \), gets with positive feedback, and the closer to unity a \( a_{-2} \) gets with negative feedback. We may, therefore, conclude that a scheme using several cascaded bandpass limiters fitting into the above specifications, with feedback taken from the output of the \( n \)th stage to the input of the first stage will result in a \( a_{-2} \) closer to zero with positive feedback, and closer to one with negative feedback, the larger the value of \( n \) is. With positive feedback, if we recall that \( a_{-2} \) is, in addition, quickly decreased by the action of the chain of bandpass limiters before the final resultant signal is fed to the discriminator, it is immediately appreciated that we have here a tremendously effective scheme for suppressing almost completely (at least in theory) the interference due to the presence of the weaker signal, and with that, eliminating any added requirements in limiter (for the later stages) and discriminator circuit design imposed by the desire to
handle the disturbance of the weaker signal successfully. Equations 28 and 29 may be generalized for the case of feedback from the output of the \( n \)th limiter (each of the limiters having one i-f bandwidth) to the input of the first, to yield

\[
a(\tau) = \frac{a + K_sA_{-1}(a_{n-1})}{1 + K_sA_0(a_{n-1})} = \frac{1 + K_sA_{-1}(a_{n-1})}{(1/a) + K_sA_0(a_{n-1})}
\]

where the \( k \) in the definition of \( K_s \) and \( K_w \) is that of the \( n \)th limiter, and

\[
a_{n-1} = A_{-1}(a_{n-2})/A_0(a_{n-2}), \ldots, a_1 = A_{-1}(a_0)/A_0(a_0)
\]

With reference to Fig. 14, we observe that under the conditions of the analysis, the relationship between \( a \) and \( a(\tau) \), in the feedback steady state, may be written down at a glance after the ratio of weaker-to-stronger signal at the input to each narrow-band limiter stage has been assigned a convenient designation. One may say that the scheme transforms the original ratio \( a \) into a new ratio \( a(\tau) = F(a, a) \), the nature of the dependence of \( a(\tau) \) upon itself being traceable through functional relationships, dictated by the individual narrow-band limiters, of the form \( a_1 = f(a_0), a_2 = f(a_1), \ldots \) assuming the narrow-band limiters as well as the spectral configurations (but not the magnitudes of the spectra) constituting the signals at the input of each to be identical.
If each of the ideal limiters has one i-f bandwidth and the performance of the scheme is studied under the interference conditions in which the difference frequency r lies in the range \(1/2(3W)_{1f} < r \leq (3W)_{1f}\), then the spectral distribution at the input of each limiter will be such that

\[ f(a) = A_{-1}(a)/A_0(a) \]

\[ = (1/a) \left[ 1 - (1 - a^2)K(a)/E(a) \right] \]

(33)

where \(K(a)\) and \(E(a)\) are the complete elliptic integrals of the first and second kind. The expressions for \(A_{-1}(a)\) and \(A_0(a)\), readily available from Section 1.2 of Chapter One, have been used. Equation 33 is useful in computations relating to Eqs. 31 and 32 for any specific case, and is plotted in Fig. 20 of Chapter One.

In the more general problem involving limiters with bandwidth values greater than the bandwidth of the i-f, the functional relationship \(f(a)\) for such limiters is not nearly as easily and neatly expressed as was done in Eq. 33 for the above problem. Indeed, the concept of "ratio of weaker-to-stronger signal amplitude" at the output of a limiter filter that is wider than the i-f loses its logical meaning, since such a limiter filter will (if sufficiently wider than one i-f bandwidth) always accommodate more components than just \(A_0\) and \(A_{-1}\) for \(r\) in the range \((1/2)(3W)_{1f} < r \leq (3W)_{1f}\). In Section IV.3, problems falling in this category will be treated, and the use of a generalized concept of capture ratio, defined on the basis of equivalent instantaneous frequency spike magnitudes, will be explored for this purpose. The following theorem summarizes the above conclusions.

Theorem VII

If the feedback is taken from the output of the second, third, or \(n\)th stage of ideal limiting to the input of the first (each stage, except possibly the (n-1)st, having one i-f bandwidth) and the feedback bandwidth is equal in extent to one i-f bandwidth, then, under the most adverse
interference condition delivered by the i-f amplifier, the effect upon the capture performance caused by an increase in the amplitude difference between the two signals with positive feedback, and a decrease in the amplitude difference with inverse feedback, becomes increasingly more pronounced with higher values of \( n \).

**Corollary**

The longer the chain of narrow-band limiters that is spanned by the feedback branch, the more insignificant the amplitude of the weaker signal relative to the stronger signal becomes with positive feedback; and the closer together in amplitude the two signals will be with inverse feedback, both signals being observed at the input to the first limiter in the chain.

Let us next consider the situation in which the signal is fed back from the output of the second limiter through an amplifier of one i-f bandwidth (or if preferred this bandwidth could be attributed to the second limiter itself) while the band of the first limiter is \( 3(3\times 1) \) if. From Chapter One, the resultant signal that is most unfavorable for capture, and which the first limiter can deliver to the second, arises here, also, with \( r = (3\times 1) \), and is specified by \( M = 1, N = 2 \). The spectrum at the output of the second limiter which results from the presence of this configuration at the input to this limiter appears in Table IIIA of Chapter Two for a few values of \( a \). The bandwidth of the second limiter need not exceed one i-f bandwidth, but this should not be of concern to the present discussion so long as the feedback amplifier bandwidth is restricted to one i-f bandwidth.

Thus, in the feedback steady state (with \( (2/\delta) \leq (r/(3\times 1)) \leq 1 \)),

\[
a = \frac{a - K_s B_{-1}(a \epsilon)}{1 + K_s B_0(a \epsilon)} = \frac{1 + K_n B_{-1}(a \epsilon)}{(1/\delta) + K_s B_0(a \epsilon)},
\]

(34)

where the \( k \)'s have the same significance as before, and are positive for positive feedback, and negative for inverse feedback. The \( B \)'s are the amplitudes of the spectral lines.
scheme in which the first limiter has only one i-f bandwidth, but superior to that of the scheme in which the feedback is taken from the output of the first limiter. This is equivalent to stating that (for both positive and negative feedback) the capture performance of the present scheme, as indicated by Eq. 34, is intermediate between that indicated in the plots of Figs. 7, 8, and 9, on the one hand, and that indicated in the plots of Figs. 11, 12, and 13 on the other.

In a similar fashion, let us next consider a scheme in which the feedback is taken from the output of the second limiter (with either the bandwidth of this limiter or the bandwidth of the feedback amplifier equal to \((BW)_{1f}\)) to the input of the first limiter, the latter having a bandwidth of \(7(BW)_{1f}\). For this scheme,

\[
\frac{a_l}{1 + K_w C_{-1}(a_l)} = \frac{1 + K_w C_{-1}(a_l)}{1/a + K_s C_0(a_l)}
\]

(35)

where the \(C\)'s are the amplitudes of the spectral lines appearing at the output of the second limiter when the \(7(BW)_{1f}\)-filter of the first limiter delivers its most troublesome configuration of \(M = 3, N = 4\), with \(r = (BW)_{1f}\). The values of the \(C_n\)'s have been presented in Table II B of Chapter Two. By direct comparison of the various quantities, we observe that

\[C_0(a_l) > A_0(a_l) \text{ while } C_{-1}(a_l) < A_{-1}(a_l),\]

and

\[C_0(a_l) < B_0(a_l) < A_0(a_{\varphi 1}),\]

while

\[C_{-1}(a_l) > B_{-1}(a_l) > A_{-1}(a_{\varphi 1}),\]

for all values of \(a_l\). Consequently, we conclude that (with positive feedback) \(a\) as given by Eq. 35 is always larger than the \(a_l\) given by Eq. 34 or Eqs. 25 and 26, but always smaller than the \(a\) of the scheme in which the positive feedback is taken from the output of the first limiter.
With reference to the curves of constant $K_s$ and $K_w$, this means that (for both positive and negative feedback) the curves pertinent to the present scheme would lie between the corresponding $K_s$ and $K_w$ curves shown in Figs. 7, 8, and 9, and those shown in Figs. 11, 12, and 13, just as the curves as plotted from Eq. 34 would. However, the curves as plotted from Eq. 35 would lie closer to the curves of Fig. 7, 8, and 9 than would those plotted from Eq. 35.

It is now clear from the above discussion that when positive feedback is taken from the output of the second limiter to the input of the first, the capture performance is best (i.e. $a_l$ is smallest, under otherwise identical conditions) when the bandwidth of the first limiter is smallest; i.e. equal to $(3W)_f$. The achieved improvement in capture performance (with positive feedback) decreases as the bandwidth of the first limiter is increased from its minimum value of one $1/f$ bandwidth, but this performance remains better than that available with positive feedback from the output of the first limiter. In fact, it is readily appreciated that in the limit as the bandwidth of the first limiter grows very large, the resultant signal delivered by this limiter to the second limiter approaches the resultant of essentially all of the primary spectrum (that centered about the frequency $p$) as given by Eq. 2, and therefore, the capture performance of this scheme approaches that of the scheme in which the feedback is taken directly from the output of the first limiter, more and more closely.

The conclusions just reached in connection with feedback from the output of the second limiter when the bandwidth of the first limiter is increased from its minimum value of one $1/f$ bandwidth, may be generalized to the case in which the feedback is taken from the output of the $n$th limiter in a chain, to the input of the first, through a feedback bandwidth of one $(3W)_f$. Thus, we may readily show that under positive feedback conditions, such a scheme exhibits its best possible capture performance (that is, the associated $a_l$ is smallest under otherwise identical feedback conditions)
when the first \( n - 1 \) limiters have their minimum permissible bandwidth value, \( (BW)_{if} \). This performance will be predictable from Eq. 31 and 32 under the most adverse interference conditions assumed in deriving those equations. The effect of increasing the bandwidth of any one of the intermediate limiters above its minimum permissible value, while all the other stages are left with one \( (BW)_{if} \) each and the feedback conditions are left unchanged, will be to make the overall performance intermediate between that of the scheme with feedback from the output of the \( n^{th} \) stage and that of the same scheme but with the feedback taken from the output of the \((n - 1)^{st}\) limiter, instead. In the limit as the varied bandwidth becomes very large, the performance of the overall scheme approaches the performance of an otherwise identical scheme in which the feedback is taken from the output of the \((n-1)^{st}\), instead of the \( n^{th} \) limiter. If the bandwidth of more than just one of the intermediate limiters is changed the same trend in the quality of the performance will be in evidence. Indeed, if all of the first \( n-1 \) limiters have their bandwidths increased to the extent that each of them is able to pass all of the significant fundamental spectrum more and more closely, then the performance of the overall scheme will approach that of the scheme in which the feedback is taken from the output of only the first ideal limiter more and more closely.

A fortiori of the above study and conclusions, the following theorem may be stated.

**Theorem VIII**

When the feedback is taken through a feedback bandwidth of one i-f bandwidth, from the output of the \( n^{th} \) limiter in a chain to the input of the first limiter, then the effect of the feedback upon the ratio of weaker-to-stronger signal amplitude at the input to the first limiter is greatest when each limiter, up to and including the \((n-1)^{st}\), possesses its lowest permissible bandwidth value of one i-f bandwidth.
Finally, the truth of the statement of the following theorem may now be considered more or less obvious.

**Theorem IX**

When the feedback bandwidth is the only parameter subject to variation, then the effect of the feedback is greatest when the feedback bandwidth takes on its lowest permissible value of one $1-f$ bandwidth.
IV.5 CASCADING NARROW-BAND FEEDBACK LIMITERS

The preceding discussions emphasized the importance of incorporating positive narrow-band feedback schemes as an expedient for processing the resultant signal delivered by the i-f amplifier and preparing it for eventual demodulation by introducing modifications in its character which are conducive to the minimization of the disturbance and the consequent enhancement of the capture ability of the FM receiver. A vast number of possible combinations and schemes immediately suggest themselves, and the process of assessing their potentialities may follow much of the same type of argument we have introduced in the preceding sections. Thus, for example, in addition to introducing a feedback branch connecting the output of the \( n \)th limiter to the input of the first, one may contemplate the introduction of other feedback branches connecting the output of some intermediate stage with the input of another stage that precedes it, or carry out various feasible variations on that theme.

Of the many interesting possibilities, we would like to single out for discussion, one scheme which offers properties of intense interest. This scheme may be considered as a special case of a cascade of black boxes each of which involves one or more ideal limiters with feedback from the output of the box to its input. From the conclusions of the preceding section, we would expect the improvement in capture performance to be greatest when the feedback bandwidth, as well as the bandwidth of each limiter (with the possible exception of the last in the chain) is just one i-f bandwidth. Again, the scheme will be discussed in relation to its effect upon the most adverse condition of interference that can arise at the input terminals of the scheme.

The scheme of interest is illustrated in Fig. 15. Under the stated conditions, the band-limited positive feedback transforms the original interference ratio \( a_1 = a \), as delivered by the i-f amplifier, into a smaller ratio \( a_0 \) at the input to the first limiter. Narrow-band limiter action, alone,
in the first stage, transforms $a_L^1$ into a still smaller ratio, $a_2$, at the output of the first limiter filter.
Positive narrow-band feedback around the second stage reduces $a_2$, further, into $a_L^2$, which in turn is reduced by the second narrow-band limiter into the ratio $a_3$, and so forth. The various ratios, starting with $a_1 = a$, are related by the now familiar relations

$$a_L^n = \frac{a_n + K_{sn} A_{-1}(a_L^n)}{1 + K_{sn} A_o(a_L^n)}, \quad n = 1, 2, 3 \ldots$$  \hspace{1cm} (36)$$

and

$$a_{n+1} = A_{-1}(a_L^n)/A_o(a_L^n)$$  \hspace{1cm} (37)$$

If the $n^{th}$ limiter has a constant, $k_n$, defining the amplitude of the resultant signal it delivers to its associated filter, and the corresponding feedback branch transforms this amplitude to a $k_n G_{fb}$ product denoted by $u_n$, then

$$K_{s1} = u_1/E_s$$

and

$$K_{sn} = \frac{u_n}{k_{n-1} A_o(a_L^{n-1})}, \quad n = 2, 3, \ldots$$  \hspace{1cm} (38)$$

An important feature of this scheme is immediately evident from Eq. 38. Since $A_o(a)$ varies slowly between $A_o(0) = 1$ and $A_o(1) = 2/e = 0.637$ as $a$ varies between zero to one, the ratio $K_{sn}$ of $u_n$ (which approximates the mean amplitude of the feedback voltage at the input to the $n^{th}$ stage) to the amplitude of the stronger component delivered by the $(n-1)^{th}$ stage, will vary over a sufficiently small range (with the small variations of $a_L^{n-1}$ of the preceding stage) to be essentially a constant. In fact, if the limiters used are all identical, the feedback factor, $K_{sn}$, of the $n^{th}$ stage is almost exactly given by the $G_{fb}$ of that stage, for all stages except the first, and the close approximation improves as one goes down the chain. This indicates that the
FIG. 15

CASCADING SCHEME

IDEAL LIMITER

(\text{BW}) = (8W)_{72}^{1/2}

AMP.

a_2

a

a_3

ACE

IDEAL LIMITER

(\text{BW}) = (8W)_{72}^{1/2}

AMP.
degree of improvement in the capture performance achieved by each of the stages following the first, is quite independent of the amplitude of the stronger signal at the input to the first stage, while its dependence upon the ratio of weaker-to-stronger signal amplitude coming through the i-f amplifier is so weak as to make it essentially a constant. Only the performance of the first stage will depend markedly upon the signal amplitude delivered by the i-f amplifier. Since the feedback around the first stage will compress the variations of the interference ratio significantly, this compression of the range, together with the slow variation of $A_0(a)$, helps make the $K_s$ factor of the second and later stages almost completely independent of the signal level, and the ratio of amplitudes at the output of the i-f amplifier. The significance and importance of these observations cannot be overemphasized, since a $K_s$ value that is heavily dependent upon the signal level delivered by the i-f (as is $K_s$, in the present scheme) will, as is evident from the preceding two sections, imply a degree of improvement in capture performance that varies markedly from one input signal level to another. Uniformity in the degree of capture improvement, and relative independence of the degree of overall improvement from the signal level delivered by the i-f amplifier, may thus be secured by cushioning off the effect of variations in the signal amplitudes by means of one or two stages of ideal limiting each having its own positive feedback arrangement. The $K_s$ values of the succeeding stages then become design factors controlled chiefly by the designer to meet prescribed capture performance patterns following specifically chosen $K_s$ curves for the various stages. Evidently, after the first stage or two have been thus designed, mainly to supply the desired buffer action, the remaining stages preceding the discriminator could be designed with any desired variations in the feedback
schemes.

As for the $K_w$ factors, we note that

$$K_{w1} = u_1/E_w$$

and

$$K_{wn} = \frac{u_n}{k_{n-1}A_{-1}(aE_{n-1})}, \quad n = 2, 3, \ldots$$

(39)

Since the interference ratio decreases rapidly, as does its range of possible variations, as one goes down the chain, the $K_w$ factors are also seen to be relatively independent of the conditions at the input to the chain. Since $A_{-1}(a)$ is approximately equal to $(1/2)a$ for small $a$, the dependence of $K_{wn}$ upon $aE_{n-1}$ is quite evident.

It is perhaps of interest to compare the performance of a number of limiters in cascade, each having its own feedback loop, with that of a chain of the same number of limiters for which the feedback is taken from the output of the last limiter to the input of the first. The comparison will be made, here, on the basis of the value of interference ratio delivered by each of the two schemes to a discriminator driven by the chain, starting with a ratio $a$ at the input to the chain. Other considerations (such as the relative independence of the overall performance from the input signal conditions) may favor one scheme or the other, but we are only interested, here, in the basis of comparison just stated.

For this purpose, the plots of Fig. 16 show the variation of $a_{out}$ versus $a_{in} = a$ for two such schemes, each using only two limiters. The solid and the dashed curves of this figure describe the relation of $a_{out}$ to $a$ for two cascaded limiters having separate feedback branches, for the choices of $K_{s1} = K_s$ and $K_{s2}$ indicated on the plots. The dotted curves relate to two narrow-band limiters in cascade with feedback taken from the output of the second to the input of the first. The comparison is obvious from the graphs and requires no special commentary.

Let us next illustrate how a slightly involved com-
\[ a_{out} = \frac{A_i(a_e)}{A_o(a_e)} \]

\[ a_{e_i} = \frac{a_2 + k_3 A_i(a_e)}{1 + k_3 A_o(a_e)} \]

\[ a_{e_2} = \frac{A_i(a_2)}{A_o(a_2)} \]

\[ a_{e_2} = \frac{a + k_3 A_i(a_e)}{1 + k_3 A_o(a_e)} \]

**FIG. 16**
bination with feedback loops may have its capture characteristics quickly described mathematically under the idealizations and assumptions outlined previously, to set the pattern for a possible approach to any other contemplated schemes. For this purpose consider the scheme shown in Fig. 17. The rectangular boxes represent ideal limiters with constants, k, and bandwidths as indicated. The gain factors, \( G \), indicated for the feedback amplifiers include any voltage division ratios that may be present at the input and output connections of these amplifiers as well as the midband gains of the amplifiers. Only positive feedback is considered of interest. The ratios of weaker-to-stronger signal amplitudes at the various points, when a steady state has been reached are also indicated. These ratios are related together by the formulas

\[
a_2 = \frac{a + K_s A_{-1}(a_2)}{1 + K_s A_0(a_2)}, \quad K_s = k_3 G_1 / E_s
\]

where \( a_2 = A_{-1}(a_2)/A_0(a_2) \), \( a_{out} = A_{-1}(a_2)/A_0(a_2) \).
and  \[ a_{L2} = \frac{k_1 A_{-1}(a_L) + k_2 G_{2} A_{-1}(a_{L2})}{k_1 A_0(a_L) + k_2 G_{2} A_0(a_{L2})}. \]

If the ideal limiters have identical characteristics, the \( k \)'s drop out of the last formula. If a feedback factor \( K_{s2} = G_2/A_0(a_L) \) is defined for the second stage and \( a_{V1} = A_{-1}(a_L)/A_0(a_L) \) is used, the expression for \( a_{L2} \) may be put in a form similar to that for \( a_L \). Note that the constant \( K_{s2} \) is not strongly dependent upon \( K_g \), and hence upon the signal level delivered by the i-f, and therefore can be controlled by the designer within limits, independently of the input signal conditions. For a graphical display of \( a_L \) versus \( a \), or even \( a_{out} \) versus \( a \), if desired, the computation would best start with assumed values of \( a_{L2} < 1 \).
IV.6 EFFECT OF PHASE SHIFT OFF CENTER FREQUENCY

In the analysis of the preceding sections, it was assumed that when the spectral components were fed back from the output of the limiter to its input, those components arrived either directly in phase or directly out of phase with the corresponding sinusoids at the input to the limiter. If this phase relationship could be maintained over all the frequencies within the feedback passband, the analysis of the preceding sections would be complete, and additional comment would be hardly justified, except in so far as determining how critical the feedback phase angle adjustment should be in practice. However, practical filters usually exhibit phase-shift characteristics that can vary significantly from one end of the band to the other. The effect of such variations in the phase shift upon the quality of the performance, particularly with positive feedback at the center frequency, will now be investigated.

As far as the present study is concerned, the exact frequency dependence of the phase shift within the passband is immaterial as long as this characteristic exhibits the usual odd symmetry about the center frequency. Our first task will be to determine the conditions prevailing at the input to the limiter when the spectral components fed back undergo phase shifts different from zero or $\pi$ in going from the output of the limiter proper to its input. To this end, we go back to Fig. 5, assume that the difference frequency, $r$, between the two signals, is greater than one-half the i-f bandwidth, consider what happens following the opening of the switch 5, and determine the feedback steady state relations. Let us also assume, for the sake of argument, that the overall phase characteristic around the feedback loop is given by

$$\phi(\omega) = \Phi_0 - (\omega - \omega_0)t_d$$

(40)

where $\omega_0$ is the center frequency, and $\Phi_0 = 0$, or an even multiple of $\pi$, for positive feedback, $\Phi_0 = \pi$, or an odd multiple of $\pi$, for negative feedback. With the feedback bandwidth assumed sharply defined by a value of one i-f bandwidth centered about the frequency $\omega_0$, and with the
difference frequency, \( r \), greater than one-half this bandwidth, we have, following the opening of the switch \( S \) in Fig. 5, a feedback voltage given by

\[
e_{fb}(t)/u = A_0(a)\cos \left[ p(t + t_d) + \Phi_o - \omega_o t_d \right] + A_{-1}(a)\cos \left[ (p + r)(t + t_d) + \Phi_o - \omega_o t_d \right]
\]

where \( a \) is the ratio of weaker-to-stronger signal amplitude delivered by the i-f amplifier. We now recall that in the Fourier analysis of the spectrum of the amplitude-limited resultant of the two sinusoids at \( p \) and \( p + r \) rad/sec, leading to the \( A_n(a) \) of the present analysis, we stipulated that if there was any message modulations of the frequencies \( p \) and \( p + r \), the periods of those modulations were much larger than a period of \( 2\pi/r \) sec. Therefore, as far as the time delay, \( t_d \), is concerned, \( p \) and \( p + r \) may be considered essentially constant for several periods of the difference frequency, \( r \). As noted in the preceding section, \( t_d \) would have to be negligible compared with the 67-microsecond period of a 15-kc audio modulation for the validity of the assumption that \( p \) and \( p + r \) remain constant during the time of travel around the feedback loop. Consequently, if we set

\[
\Psi_{fb} = -(p - \omega_o)t_d
\]

and

\[
-\Phi_{fb} + 2\pi(t_d/T_r) = (p + r - \omega_o)t_d
\]

where \( T_r = 2\pi/r \), then, with \( \delta = t_d/T_r \), we have for the initial feedback voltage

\[
e_{fb}(t)/u = A_0(a)\cos(pt - \Phi_{fb} + \Phi_o) + A_{-1}(a)\cos[(p + r)t - \Phi_{fb} + \Phi_o - 2\pi\delta]
\]

Since the value of \( r \) is now comparable with the feedback bandwidth, \( \delta \) cannot be considered negligible compared with unity. If we now restrict the interest to the most adverse condition of interference at the input to the limiter, which
arises with \( r = (3W)_{ir} \), then the component at the frequency
\( p = \omega_0 - \frac{1}{2}(3W)_{ir} \) is shifted in phase by \( \Phi_o - \frac{1}{2}(3W)_{ir}t \),
while that at \( p + r = \omega_0 + \frac{1}{2}(3W)_{ir} \) is shifted by \( \Phi_o + \frac{1}{2}(3W)_{ir}t \). Under this condition,
\[
\Phi_{fb} = \frac{1}{2}(3W)_{ir}t \tag{43}
\]
and the initial feedback voltage may be expressed as
\[
e_{fb}(t)/u = A_o(a)\cos(pt + \Phi_o - \Phi_{fb}) + A_{-1}(a)\cos[(p + r)t + \Phi_o + \Phi_{fb}] \tag{44}
\]
It is clear that with \( \Phi_{fb} \) properly interpreted, the overall phase characteristic need not be linear, but could be given any other possible odd-symmetrical representation. The superposition of this initial feedback voltage upon the carriers delivered by the i-f amplifier, results in a new ratio of weaker-to-stronger signal at the input to the limiter, each resultant signal having its initial phase angle changed also. This causes the amplitudes, as well as the phases of the components fed back to be altered. In the feedback steady state, a closed and stable vector triangle may be constructed for each signal as illustrated in Fig. 18. For the signal at \( p + r \), for example, the steady state resultant \( CR_{p+r} \) is so oriented with respect to the component delivered by the i-f that when the spectral component corresponding to it at the output of the limiter is advanced in phase by \( \Phi_{fb} \) and then superimposed upon the i-f component the result is again \( CR_{p+r} \). From Fig. 18(a), the feedback steady state amplitude of the component at \( p \) rad/sec is
\[
A_p = \left[1 - k_sA_o^2(s_e)\sin^2\Phi_{fb}\right]^{-1/2} + k_sA_o(a_e)\cos\Phi_{fb} \tag{45}
\]
while that at \( p + r \) rad/sec is
\[
A_{p+r} = \left[1 - k_sA_{-1}^2(s_e)\sin^2\Phi_{fb}\right]^{-1/2} + k_sA_{-1}(a_e)\cos\Phi_{fb} \tag{46}
\]
where the resultant ratio of weaker-to-stronger signal amplitude at the input to the limiter is given by
STEADY STATE PHASOR DIAGRAMS

FIG. 18
\[ a_L = \frac{A_{p+r}}{A_p} \]  

(47)

It is interesting to observe that the validity of the above expressions for \( A_p \) and \( A_{p+r} \), and hence for the assumption that a feedback steady state with stable closed triangles as in Fig. 18 is possible, implies restrictions that insure that the radical in each of the expressions be real. From Fig. 18(a), we note that the existence of a physical angle \( \Psi_p \) that will make it possible for three necessary magnitudes \( 1, K_sA_o(a_L) \) and \( A_p \) to form the sides of a closed triangle in which the angle between \( A_p \) and \( K_sA_o(a_L) \) is given by \( \Phi_{rb} \), is possible only if

\[ \sin \Psi_p = K_sA_o(a_L) \sin \Phi_{rb} \leq 1, \]  

(48)

that is, only if

\[ \sin \Phi_{rb} \frac{1}{K_sA_o(a_L)} = \alpha_{49} \]  

(49)

With reference to Eq. 45, condition 49 is seen to insure the realness of the square-root term in the expression for \( A_p \). Similarly, from Fig. 18(b), the condition for the existence of a physical \( \Psi_{p+r} \) is

\[ \sin \Psi_{p+r} = K_s \frac{A_{-1}(a_L)}{a} \sin \Phi_{rb} \leq 1 \]  

(50)

which is insured by

\[ \sin \Phi_{rb} \frac{1}{K_s} \frac{a}{A_{-1}(a_L)} = \alpha_{51} \]  

(51)

which, in turn, also insures the realness of the square-root term in the expression for \( A_{p+r} \). Evidently, the more severe of the two conditions 49 and 51 imposes the necessary restriction upon \( \Phi_{rb} \). To determine which of the two conditions is the more restrictive, we note that

\[ \frac{\alpha_{49}}{\alpha_{51}} = \frac{A_{-1}(a_L)}{A_o(a_L)} \]  

\[ \frac{a}{a} < \frac{a}{a} \]  

(52)

Since \( a \) may be expected to be smaller than \( a \) for all values of \( \Phi_{rb} \) between zero and some value near \( \pi/2 \) when \( \Phi_o = 0 \), it is
evident that condition 49 will be more restrictive for all values of $\Phi_{fb}$. However, if $\Phi_0 = \pi$, condition 51 may become more restrictive for $\Phi_{fb}$ in the neighborhood of zero. These conditions may also be interpreted as being restrictions on $K_s$ for possible $\Phi_{fb}$, $A$ and $a$ values.

If neither of the above two restrictions is satisfied, then the fact that neither of the two triangles in Fig 18 may be perpetuated after any length of time, signifies that no feedback steady state in the usual sense may be reached. However, for certain values of $\Phi_{fb}$, $K_s$ and $a$, it is conceivable that the modifications of each of the two triangles that result from the various signal flow rounds around the feedback loop may fall into some kind of a limit cycle of modifications from one triangle into the next in the cycle and back into some initial pattern which recurs with a definite periodicity. The effect upon the capture conditions is equivalent to that of a periodic modulation of the resultant ratio, as seen by the limiter. If the superposition of the components feedback upon the components delivered by the i-f is such as to always increase the amplitudes of the two signals, then the result is an improvement in the capture conditions of the type encountered with positive feedback. On the other hand, if the superposition decreases the amplitudes of the signals, then deterioration of the type encountered with inverse feedback is the result. In either case, the quality of the improvement or the deterioration varies at the period of the limit cycle.

If the feedback at the center frequency is positive, then as stated above, condition 49 will be more restrictive, possibly for all values of $\Phi_{fb}$ smaller in magnitude than $\pi/2$. It is possible, therefore, that condition 51 may be satisfied while condition 49 is not. Remarks similar to those in the preceding paragraph could, also, be made, since the failure of either condition signifies that neither of the two triangles will be perpetuated after any length of time. Under these conditions, we are of course unable to
determine the capture performance from the usual type of steady state analysis, because no such state, apparently, can be reached. The significance of a feedback steady state, or lack of it, will be explored in the light of the discussion of the next section also.

Let us now turn to the relationship between $a$ and $\bar{a}$, as a function of $\Phi_{fb}$, when conditions 49 and 51 are both satisfied. Clearly, only a study of this relationship under positive feedback conditions ($\Phi_o = 0$), at the center frequency, will be of value to us since the results of the preceding sections rule out any desire to use inverse feedback around the limiter, and since the effect of inverse feedback can be included in the discussion by letting $\Phi_{fb}$ take up values near $\pi$. Under these conditions, therefore, the pertinent restrictive condition, wherever $a/\bar{a}$ exceeds or equals unity, is

$$\sin \Phi_{fb} \leq \frac{1}{K_s A_0(a)}$$

(49)

in order for a feedback steady-state to be reached. If Eqs. 5, 7, and 8 are combined and solved for $a$, the result may be put in the form

$$a^2 = K_s A_0^2(\bar{a}) \sin^2 \Phi_{fb} + \left\{ \bar{a} \sqrt{1 - K_s A_0^2(\bar{a}) \sin^2 \Phi_{fb}} + K_s \cos \Phi_{fb} \left[ \bar{a} A_0(\bar{a}) - A_{-1}(\bar{a}) \right] \right\}^2$$

(52)

Condition 49 is seen to enter explicitly into the evaluation of this expression for $a$. This condition is seen to be restrictive only when

$$K_s > 1/A_0(\bar{a}) > 1.$$

Let us now determine the maximum deviations from the desired in-phase condition of the narrow-band feedback, that can be tolerated without seriously impairing the degree of capture improvement that can be brought about by this feedback. The phase sensitivity of this scheme is best illustrated by the plots of Fig. 13. In this figure values of $a/\bar{a}$, related to $K_s$ and $\Phi_{fb}$ by means of Eq. 52, are plotted for various choices of $\bar{a}$ and $K_s$ values, against $\Phi_{fb}$. 
These plots bring out the fact that the phase condition gets increasingly more severe as the value of $K_s$ is increased, which is also implied by condition 49 for positive feedback, and condition 51 for inverse feedback.

For values of $K_s$ of about unity or less, a feedback steady state exists for all values of $\phi_{fb}$, and the effect upon the capture performance of varying $\phi_{fb}$ from zero to $\pi$ is obvious from Fig. 19. For an understanding of the behavior of these curves, the vectorial representation of Fig. 20 is quite helpful. In this figure, which corresponds to Fig. 18(a), the amplitude, $OR_p$ (= $A_p$), of the stronger signal at $p$ rad/sec, as seen across the input terminals of the limiter in the feedback steady state, is held constant, as is the amplitude, $kG_{fb}A_o(a)$, of the corresponding spectral component fed back from the output of the limiter to its input. Evidently, this implies that the interference ratio, $a$, and, hence, the resultant amplitude, $A_{p+r}$, of the weaker signal at $p + r$ rad/sec, is held constant. The feedback angle, $\phi_{fb}$, is varied, and with it the amplitude of the component signal, $E_s$, as it is delivered by the i-f amplifier, must consequently vary, the tip of its representative phasor tracing out the circular locus shown, as $\phi_{fb}$ is varied from zero to $2\pi$. It is now recalled that the advantages of the positive feedback are fundamentally the result of the boost in amplitude undergone by each of the component signals, as seen by the limiter, the stronger signal being favored by the larger increase. The effect of the feedback simulates that of direct in-phase feedback if both components are symmetrically disposed with respect to the center frequency, $\nu_0$, and, hence, fed back at angles having the same magnitude $\phi_{fb}$, but different signs, and $\phi_{fb}$ is such that the resultant amplitude at each of the two frequencies exceeds the corresponding input amplitude. From Fig. 20, it is seen that the increase in amplitude is most pronounced for small values of $\phi_{fb}$ because of the direct in-phase superposition of $kG_{fb}A_o(a)$ upon $CE_{20}$ at zero $\phi_{fb}$. The deterioration in the amount of increase in the amplitude
sets in slowly as $\Phi_{fb}$ increases from zero, and speeds up near a value of $\Phi_{fb} = \Phi_{fb1}$, where $OE_s$ and $GR_p$ approach equality. The feedback effect becomes nil, except for a shift in the initial phase of each resultant signal from the phase of the corresponding input component, when the value of $\Phi_{fb}$ is the proper one for $OE_s = GR_p$. Obviously, this equality must occur for a value of $\Phi_{fb}$ smaller than $\pi/2$. Beyond this value of $\Phi_{fb}$, the amplitude of the resultant signal at the input to the limiter, at each component frequency, becomes smaller than the corresponding value delivered by the i-f amplifier. At such values of $\Phi_{fb}$, the capture conditions deteriorate with the application of the feedback, the deterioration reaching a peak at $\Phi_{fb} = \pi$. It is clear, from Fig. 20, that $OE_s$ and $GR_p$ differ in amplitude by an amount which varies slowly over a larger range of $\Phi_{fb}$ values near $\pi$ than for values near zero. The speed of the variation near zero, and its slowness near $\pi$, get more and more pronounced as the magnitudes of the feedback component and the component coming through the i-f amplifier approach equality.

With the frequencies $p$ and $p + r$ assumed symmetrically disposed with respect to the center frequency $\omega_o$, as assumed in the derivation of Eq. 52, the effect of the above dependence of the relation between $A_p$ and $E_s$, or $A_{p+r}$ and $aE_s$, upon $\Phi_{fb}$, reflects directly upon the value of $a$ that corresponds to the assumed value of $a\omega$. The reason is clear from the fact that where the amplitudes of the input components vary slowly (but at different rates) with $\Phi_{fb}$, the ratio of these amplitudes will also vary slowly with $\Phi_{fb}$.

Evidently, although the situation in which the two frequencies, $p$ and $p + r$, are symmetrically disposed with respect to $\omega_o$, is the most convenient to discuss, it is not the most general feedback condition. With the feedback angle, within the desired passband, kept within the limits for good improvement, brought out by a study based upon the symmetrical case, the situation in which the feedback angle
at the frequency of the weaker of the two impressed signals is smaller than the phase shift at the frequency of the stronger signal, will only be slightly more unfavorable than the symmetrical case. At any rate, the way to handle the more general case is well illustrated by the above discussion of the symmetrical case.
IV.7 CONSIDERATIONS OF STABILITY AND OSCILLATION

The phase relationship at the center frequency around the loop with positive feedback, in the previous analysis, is just the right one for instability and oscillation. It is important, therefore, to examine the fundamental limitations of an implicit assumption that has predominated our theory so far, namely, the complete feedback loop, limiter and all, has no self-generated oscillation present at the center frequency, that requires to be taken into account in the analysis of the effect of the scheme upon the capture conditions at the input and output of the limiter. These fundamental limitations may either have to do with restrictions that must be imposed upon the loop gain to insure the failure of this gain to satisfy the requirements for a stable oscillation, or, if this oscillation is unavoidable under the desired conditions of operation, those limitations may decide the locking range of the oscillating limiter.

The presence of the limiter in the feedback loop might at first suggest complications and unfamiliarity. But a little thought will remind us of the fact that barely any other circuit could look more like an oscillator than this one, particularly if the limiter used is of the conventional grid-bias variety. With a grid-bias limiter, the fundamental mechanism whereby the limiter manages to eliminate amplitude variations in the input signal and stabilize the value of amplitude it delivers at its output, is indistinguishable from the mechanism whereby the self-bias in the grid circuit of a stable oscillator manages, through its influence upon the conduction angle and height of the plate current pulses, to regulate the amplitude of the stable oscillation.

Let us first recall that our fundamental assumptions regarding the limiter include the practical matter of associating with this limiter sufficient selectivity to pass only the component having the fundamental carrier frequency and its associated sidebands, and this bandwidth is considered essentially infinite for the purpose of our
analysis of the interference problem when (for any prescribed
close value of \( a < 1 \)) it is sufficiently wider than one i-f band-
width to accommodate essentially all of the sideband components
about the frequency of the stronger signal that will influence
the instantaneous amplitude and phase of the resultant signal
noticeably. Under these conditions a sinusoidal voltage
whose frequency is in the neighborhood of the center frequency
will result in a sinusoid of the same frequency at the output
of the limiter. Under these conditions an output-amplitude
versus input-amplitude characteristic may be plotted which,
for an idealized practical limiter, will appear as shown
in Fig. 21(a). A corresponding voltage amplification
characteristic may then be easily determined, which is
illustrated in Fig. 21(b). The limiting threshold is denoted
by \( E_{\text{th}} \), and it is clear that for \( E_{\text{in}} < E_{\text{th}} \) the idealized
limiter behaves like a linear amplifier with a gain of
\( G_{\text{limmax}} = k/E_{\text{th}} \). In our previous analysis, the limiter was
idealized to the point of making \( E_{\text{th}} = 0 \), or infinitesimally
less than zero, otherwise we would have had to add the
stipulation that \((1 - a)E_{\text{s}} \gg E_{\text{th}}\) at every point.

When the feedback is such that the feedback voltage
arrives in phase with the input voltage, the loop gain is
given by the product \( G_{\text{fb}} \cdot G_{\text{L}} \), of the transfer voltage ratio,
\( G_{\text{fb}} \), of the feedback network and the transfer voltage ratio
of the limiter itself. With reference to Fig. 22, when the
switch \( S \) is closed, the circuit will be prone to build up
and sustain a stable oscillation if the closed loop gain
\( G_{\text{loop}} = G_{\text{fb}} \cdot G_{\text{L}} \gg 1 \). Since \( G_{\text{L}} = k/E_{\text{in}} \), for \( E_{\text{in}} \gg E_{\text{th}} \),
the voltage transfer ratio of the limiter will vary with
\( E_{\text{in}} \) in the indicated range, and will subside to the value
\( L_{\text{0}} = 1/G_{\text{fb}} \), when the condition of stable oscillation is
reached. The frequency of the oscillation will clearly be
the center frequency of the band since the phase condition
is by assumption satisfied at this frequency, and \( E_{\text{in}} \) will
then be given by \( E_{\text{osc}} = kG_{\text{fb}} \).  
\[ \text{(53)} \]
\[ \text{(54)} \]
(a) IDEALIZED LIMITER CHARACTERISTIC

$$G_l = \frac{E_{out}}{E_{in}}$$

$$\frac{k}{E_{th}}$$

(b) AMPLIFICATION CHARACTERISTIC

FIG. 21

FIG. 22
For stability, and hence no self-sustained oscillation, the feedback voltage transfer ratio must satisfy the condition
\[ G_{fb} < \frac{1}{K} E_{th} \]  \hspace{1cm} (55)
which is imposed by the maximum value that the limiter voltage transfer ratio can attain. In view of this condition, the ideal limiters of the preceding sections must be assumed to have a nonzero limiting threshold, \( E_{th} \), in order to allow for the possible existence of a condition of no self-sustained oscillation with nonzero values of \( G_{fb} \). It is also clear, in view of Eq. 54, that if condition 55 is not satisfied, but \( kG_{fb} \ll aE_s = E_w \), the presence of a locally generated oscillation may still be ignored. The significance of the condition for no self-sustained oscillation will now be explored. We shall also discuss the operation in which the feedback arrangement gives rise to a stable self-sustained oscillation.

A. OSCILLATION-FREE OPERATION

If no oscillation is desired, it is readily appreciated that the condition 55 is rather severe as far as the attainment of large values of \( K_s = kG_{fb}/E_s \) is concerned. With condition 55 satisfied, \( K_s \) will always be less than \( E_{th}/E_s \), which makes the desirability of low limiter threshold and pronounced improvement in capture conditions with feedback incompatible. If it is recalled that \( E_{th} \) itself must be smaller than \((1-a)E_s\), it becomes apparent that the condition for no oscillation implies that \( K_s \ll E_{th}/E_s < 1 \) - a which brings the severity of this condition into even sharper focus. It may, therefore, appear that the positive feedback scheme is severely hampered by the requirement of stable oscillation-free operation.

This difficulty may perhaps be somewhat relieved by the following simple modifications in the basic scheme.

First, we recognize that the importance of \( K_s \) as an index of the degree of improvement in the capture conditions with feedback stems from the fact that \( K_s \) is a measure of the amplitude of the feedback voltage relative to the amplitude of the stronger signal delivered by the i-f, both being
FIG. 23

FIG. 24
compared where they are superimposed. That is to say, the
importance of feedback as far as the degree of improvement
in the capture conditions is concerned, rests upon the
importance of the amplitude of the feedback voltage in com-
parison with the amplitude of the signals upon which the
feedback voltage is superimposed. Whereupon, two remedies
immediately suggest themselves: namely, either superimpose
a high-level feedback voltage upon the input signals and
apply the resultant to the limiter through a voltage divider
which introduces a compensating gain loss to keep the loop
gain below unity (Fig. 23), or introduce the low-level
feedback required for stability upon the input signals
while the latter are still at a comparably low level themselves
(Fig. 24).

In the scheme of Fig. 23, $a_x$ and $a$ are related by the
very curves of Figs. 7, 8 and 9, assuming the threshold
of the limiter to be sufficiently low for the drop in signal
level through the voltage divider to leave the operating
point of the limiter within the saturation region. As for
the loop gain, $G_{\text{loop}}$, we have, under the maximum limiter
gain condition, Max. loop gain $= G_{\text{Lmax}} \cdot G_{\text{loss}} \cdot G_{fb}$, whence,
for stability,

$$
G_{fb} \cdot G_{\text{loss}} < \frac{1}{G_{\text{Lmax}}} = \frac{E_{\text{th}}}{K}.
$$

(57)

The restriction that this imposes upon $K_s$ is indicated
by

$$
K_s < \frac{E_{\text{th}}}{G_{\text{loss}} E_s} < \frac{1 - a}{G_{\text{loss}}},
$$

(58)

which appears to be a little better than the restriction 56,
but is still limited by the amount of loss in signal level
that can be tolerated at the input to the limiter without
upsetting the operation of this limiter.

As for the scheme of Fig. 24, those aspects of the
scheme having to do with improvement in the capture conditions
are readily seen from the feedback steady state relationship
\[
\frac{a_{\text{out}}}{E_s} = \frac{a + K' A_{\text{in}}(a_{\text{out}})}{1 + K' A_o(a_{\text{out}})}
\]

where

\[
K' = \frac{kG_{\text{loss}}}{E_s}
\]

Formally, relationship 59 is identical with Eq. 15. Therefore, the capture performance of this scheme is also portrayed by the plots of Figs. 7, 8, and 9. Since it is assumed that the feedback is introduced at a point of low input signal level, \( E_s \) is small, and the voltage divider is introduced to bring down the limiter output voltage to a level comparable with \( E_s \) and offer some gain margin for the stability requirements. The amplifier in the forward path of the signal introduces a voltage gain that brings the signal level up to a value that meets the requirement \((1 - a)E_s G_{\text{amp}} > E_{\text{th}}\).

The stability requirements impose the condition

\[
G_{\text{amp}} G_{\text{loss}} < \frac{E_{\text{th}}}{E_s} < (1 - a)E_s G_{\text{amp}}/k,
\]

whence

\[
K' = \frac{kG_{\text{loss}}}{E_s} \ll (1 - a).
\]

This result obviously strips the scheme under consideration of any advantage over the scheme of Fig. 21.

The real promise for reasonable capture performance without instability is, therefore, seen to lie with the approach illustrated in Fig. 23. The shortcoming introduced by the practical limitation on the value of \( G_{\text{loss}} \) is, however, the price that one must pay.

Consider next the scheme of Fig. 25. The capture performance of this scheme is portrayed by the plots of Figs. 11, 12, and 13. The parameter \( K_s \) is here given by

\[
K_s = kG_{\text{fb}}/E_s.
\]

At the input to the first limiter, we require that \((1 - a)E_s < E_{\text{th}}\).
If the first limiter is assumed to have one i-f bandwidth, then for proper limiter action in the second stage, we require that

\[ k_1 \left[ A_0(a) - A_{-1}(a) \right] G_{\text{loss}} > E_{\text{th}_2}. \]  

(62)

This represents the very minimum amplitude value that may be attained, under the conditions of the analysis, while a switch shorting out the feedback voltage is closed, and which may prove somewhat pessimistic. In the feedback steady state, this condition may be replaced by

\[ k_1 \left[ A_0(a^a) - A_{-1}(a^a) \right] G_{\text{loss}} > E_{\text{th}'}. \]  

(63)

which can be considerably less severe than the above condition which applies before the feedback steady state is established. The stability requirement is expressed by

\[ \frac{k_1}{E_{\text{th}_1}} \cdot G_{\text{loss}} \cdot \frac{k_2}{E_{\text{th}_2}} \cdot G_{\text{fb}} < 1 \]  

(64)

If condition 62 is applied, condition 64 leads to

\[ K_s < (1 - a) \left[ A_0(a) - A_{-1}(a) \right] < 1 - a. \]  

(65)

The introduction of the voltage divider between the two limiters does not affect the restriction on \( K_s \), and, therefore, serves no useful purpose. This conclusion is also evident from the fact that if the second limiter is operating in the saturation region at the minimum value of signal amplitude, the amplitude of the output signal is \( k_2 \) independently of the input signal amplitude. Condition 65 is the condition on \( K_s \) when the feedback is taken from the output of the second limiter to the input of the first. This condition is seen to be even more severe than condition 56, but it must be remembered that the effect of feedback from the output of the second limiter may be sufficiently more pronounced than the effect of feedback from the output of the first limiter to offset the apparently
increased limitation.

When the feedback is taken from the output of the n-th limiter to the input of the first, through a feedback bandwidth of one i-f bandwidth, each limiter (with the possible exception of the n-th one) having one i-f bandwidth, the condition for no oscillation requires that

\[ G_{fb} \cdot \prod_{i=1}^{n} \frac{k_i}{E_{th_i}} \leq 1. \]  

(36)

The feedback factor is given by \( K_s = k_n G_{fb}/E_s \). The following conditions must also hold for proper limiting, while the feedback voltage is shorted out:

\[(1 - a)E_s > E_{th_i} \]

and

\[ k_i \left[ A_0(a_i) - A_{-1}(a_i) \right] > E_{th_{i+1}} \]

where \( a_1 = a, a_2 = A_{-1}(a_1)/A_0(a_1) \)

and \( a_{i+1} = A_{-1}(a_i)/A_0(a_i) \).

Substitution into condition 66 leads to

\[ K_s \leq (1 - a) \prod_{i=1}^{n-1} \left[ A_0(a_i) - A_{-1}(a_i) \right]. \]  

(37)

3. THE "OSCILLATING LIMITER"

Let us now turn to the situation in which the feedback loop gain is sufficiently high for the establishment and maintenance of an oscillation. The limiting threshold may now be assumed arbitrarily close to zero, and even equal to zero, depending upon the needs of the ensuing discussion. As noted earlier, the frequency of the self-oscillation will be the frequency at which the net phase shift around the loop is zero or an even integral multiple of \( \pi \). The amplitude of the steady oscillation at the limiter input will be as given by Eq. 54.

Suppose now, that after a steady oscillation has been
established having an amplitude given by $kG_{rb}$, the switch $S$
in Fig. 22 is opened to introduce a signal source having an
internal impedance of $A_s$. Also, let the input impedance of
the limiter be $Z_{in}$. Then the feedback voltage, initially
of amplitude $kG_{rb}$ across $Z_{in}$, will drop to

$$E_{in} = \frac{Z_{in}}{Z_{in} + Z_s} \times kG_{rb},$$

and at this value of $E_{in}$

$$G_L = \frac{k}{E_{in}} = \frac{Z_{in} + Z_s}{Z_{in}G_{rb}}$$

and

$$G_{rb} \cdot G_L = \frac{Z_{in} + Z_s}{Z_{in}} > 1.$$ 

The oscillation will be maintained also with this new value
of the product of feedback amplifier and limiter gain. This
product may also be altered by introducing a signal from the
source which will alter the amplitude of the resultant signal
at the input to the limiter. The introduction of such a
signal will have the effect of shifting the operating point
on the output-vs-input amplitude characteristic in Fig. 21(a),
and, hence, altering the value of $G_L$, the amplification of
the limiter. If the frequency of the introduced signal
coincides with the frequency of the self-oscillation, then
regardless of the initial phase relationship, the phase of
the resultant signal circulating around the feedback loop
will always settle to the phase of the impressed signal,
and the result will be a net increase in the amplitude of
the signal at the input to the limiter. Since the result of
increasing $E_{in}$ beyond $E_{osc}$ is to decrease the amplification
of the limiter below the value necessary for an overall
feedback loop gain of unity, it would be quite correct to
overlook the fact that the scheme had a self-oscillation
before the introduction of the source signal, in computing
the feedback steady-state value of the resultant input.
amplitude. For if the amplitude of the input signal is given by $E_s$ while the amplitude of the self-oscillation is $kG_{fb}$ (which incidentally is also the amplitude of the feedback voltage in the steady state), then the amplitude of the resultant signal at the input to the limiter will be given by $E_s + E_{osc} = E_s(1 + K_s)$, whether we view it as being the result of a positive feedback of the input signal, or the superposition of a voltage of amplitude $E_s$ upon $E_{osc}$ at the input to the limiter. Since the initial phases may be different, while in the steady-state the phases will be identical, the result may also be interpreted as a locking of the self-oscillation to the phase of the input signal. It is clear that the oscillation will lock to the input signal regardless of the value of

$$K_s = \frac{E_{osc}}{E_s} = \frac{kG_{fb}}{E_s}$$

relative to unity.

If the instantaneous phase of the impressed signal is now varied slowly, it is not difficult to see how the locking of the self-oscillation to the input signal will be maintained, while the instantaneous phase variation is made to drift the frequency of the input signal by a small fraction of the center frequency. If the instantaneous frequency variation of the input signal is now speeded up a little, and the selective filters around the feedback loop are able to follow these variations through stationary states, then the result may be interpreted either as a locking of the instantaneous frequency of the self-oscillation to the instantaneous frequency of the input signal with a resultant amplitude at the input to the limiter given by $E_s(1 + K_s)$, or as a positive feedback effect resulting from feeding back a component at the same frequency as the input signal. Evidently, the negligibility of the time delay around the loop is implicit, and this attitude has already been justified.
Let us now consider that the impressed signal has an amplitude $E_s$ and a frequency of $p$ rad/sec, and let the oscillation frequency be denoted by $p + r$, where $r$ is a negligible fraction of $p$, and the oscillation amplitude be given by $kG_{fb} = K_s E_s$. Assume, to start with, that $K_s < 1$. At the initial instant of opening the switch $S$, in Fig. 22, the resultant signal at the input to the limiter is given by

$$E_s \cos pt + K_s E_s \cos(p + r)t.$$

If the feedback bandwidth is sharply defined by a value of $2r$ rad/sec, and the frequency $p$ is assumed to be just within the lower cut off frequency, then in response to this initial resultant signal, the feedback voltage will be given by

$$kG_{fb}A_0(K_s)\cos(pt - \Phi_{fb}) + kG_{fb}A_{-1}(K_s)\cos(p + r)t - kG_{fb}A_{-2}(K_s)\cos[(p + 2r)t + \Phi_{fb}],$$

where $\Phi_{fb}$ is the phase shift at $r$ rad/sec off the center frequency. For simplicity, let us first assume that $\Phi_{fb}$ is small, and that $K_s A_{-2}(K_s)$ is negligible compared with $1 + K_s A_0(K_s)$. The latter is a good assumption for all $K_s < 1$. For $K_s = 0.38$, for instance, $K_s A_{-2}(K_s)$ is 1/9 times $1 + K_s A_0(K_s)$.

If desired, the assumption may be made that the feedback bandwidth is a little smaller than $2r$, so that, with $p$ just to the right of the lower cutoff frequency, and the oscillation frequency lying sufficiently to the right of the center frequency of the feedback passband, the component at $p + 2r$ falls in the heavy attenuation region. Neglecting the relatively small component at $p + 2r$, even if it is passed, simplifies the present task tremendously, without affecting the validity of the results.

Thus, at the beginning of the second round, we have, neglecting $\Phi_{fb}$ for the time being, a resultant voltage at the input to the limiter with component amplitudes of
\[ E_S + kG_{fb}A_o(K_s) \text{ at } p \]

and

\[ kG_{fb}A_{-1}(K_s) \text{ at } p + r. \]

Compared to the amplitude of the component at the frequency of the impressed signal, the amplitude of the component at the oscillation frequency is now given by

\[ K_s \frac{A_{-1}(K_s)}{1 + K_sA_o(K_s)} = K_s f(K_s). \]

As \( K_s \) approaches unity, \( A_{-1}(K_s) \) approaches \( 2/n \) from below, and \( A_o(K_s) \) approaches \( 2/n \) from above. Therefore, the maximum value of \( f(K_s) \), which occurs at \( K_s = 1 \), is approximately 0.39. Thus, the ratio of the self-oscillation amplitude to the amplitude of the impressed signal undergoes a sizable reduction, in the first round, from \( K_s \) to something less than 0.4\( K_s \). If the ratio at the beginning of the second round is denoted by \( K_2 \), then at the beginning of the third round \( K_2 \) is reduced by a factor of \( f(K_2) \), which is considerably less than unity.

As the signal flow is traced around the loop, it is immediately realized that the amplitude of the self-oscillation quickly goes to zero while the amplitude of the impressed signal quickly builds up to \( E_S(1 + K_s) \), the value that may be predicted by reasoning purely in terms of feedback, ignoring the self-oscillation. This quick decay of the self-oscillation and the accompanying transfer of the energy, initially residing in it, to the frequency of the impressed forcing excitation, is a manifestation of the familiar oscillator "locking" phenomenon. The introduction of the signal at \( p \), whose amplitude was assumed greater than the amplitude of the self-oscillation, will quickly pull the oscillation from the frequency \( p + r \) to the frequency \( p \), and literally superimpose the amplitude of the oscillation upon the amplitude of the impressed signal at the frequency \( p \).
If the feedback angle $\Phi_{fb}$ at the frequency $p$ of the impressed signal cannot be neglected, then at the end of the first round around the loop, the feedback component having the frequency $p$ of the impressed signal will have an amplitude given by $K_s A_o(K_s)$ but its phase will be deviated from that of the impressed signal by $\Phi_{fb}$. The component at the frequency of the self-oscillation will of course sustain no phase shift around the loop. The ratio of amplitudes at the beginning of the second round will, therefore, be

$$K_s \frac{A_{-1}(K_s)}{\sqrt{1 + 2 K_s A_o(K_s) \cos \Phi_{fb} + K_s^2 A_o^2(K_s)}} = K_s f'(K_s)$$

Evidently, the value of the denominator will exceed unity for all $\cos \Phi_{fb} > -(1/2)K_s A_o(K_s)$. Since the advantages of positive feedback relate fundamentally to the fact that it increases the amplitudes at the input rather than subtracts from them, it is logical to desire that the angle $\Phi_{fb}$ be smaller in magnitude than $\pi/2$ throughout that range of the feedback passband which will be occupied by the expected signal carrier. For values of $\Phi_{fb}$ satisfying these restrictions, the maximum value of $f'(K_s)$ will always be bounded by 0.54, which is the value for $K_s = 1$ and $\Phi_{fb} = \pi/2$. This again indicates a considerable decrease in the amplitude of the self-oscillation in the first round. A steady state will, therefore, be reached quickly in which the energy in the oscillation frequency is shifted to the frequency of the impressed signal. The pertinent phase relationships in this steady state are portrayed in Fig. 26. The phasor $CR$ represents the resultant amplitude of the source-frequency signal signal appearing across the input terminals of the limiter. The energy originally oscillating at the zero phase-shift frequency of the feedback passband, now resides in a sinusoid at the frequency $p$ of the impressed signal, and shifted in phase relative to this signal by $\Phi_{fb} + \theta$. The angle $\theta$ is given by
\[
\sin \psi = k_s \sin \phi_{fb},
\]

and, hence, \(\psi\) is a real angle for all \(k_s \sin \phi_{fb} \leq 1\), which is insured by the stipulated assumptions.

Let us now consider the situation in which the amplitude of the impressed signal at \(p\) is smaller than the amplitude of the self-oscillation at \(p + r\). Thus, if the amplitude of the impressed signal is given by \(E_s\), let the amplitude of the self-oscillation be given by

\[
k \frac{G_{fb}}{\phi} = \frac{E_s}{\alpha},
\]

where \(\alpha = 1\). Again we start by assuming the angle \(\phi_{fb}\) to be negligible. At the end of the first round, after opening the switch \(S\) in Fig. 22, the resultant signal at the input to the limiter is made up of the components

\[
E_s \left[ 1 + \frac{1}{\alpha} A_2 (\alpha) \right] \quad \text{at } p \text{ rad/sec}
\]

and

\[
E_s A_0 (\alpha) / \alpha \quad \text{at } p + r \text{ rad/sec},
\]

assuming the component at \(p + 2r\) falls in a region of heavy attenuation, to simplify the analysis. The amplitude of the rejected component at \(p + 2r\) would be only a few times smaller than the amplitude of the component at \(p\), if passed, at the end of the first round. Its inclusion in the computation has the effect of slowing down the transfer of the
energy of the self-oscillation into the frequency of the impressed signal during the second and following few rounds. This effect can be determined from an extremely involved computation which does not contribute to the discussion in proportion to the amount of labor it requires. Therefore, the assumption of the oscillation frequency as being off the center of the sharply defined feedback passband, on the side away from the frequency p of the impressed excitation, allows us to reject the component at p + 2r without impairing the portrayal of the fundamental frequency-pulling mechanism brought out by the analysis, or affecting the final result.

At the end of the first round around the loop, the ratio of the amplitude of the signal at the frequency of the source to the amplitude of the self-oscillation, at the input to the limiter, is given by

\[ \alpha_2 = \frac{\alpha}{A_0(\alpha)} + \frac{A_{-1}(\alpha)}{A_0(\alpha)} \]

which is greater than \( \alpha \), first because \( A_0(\alpha) \) is less than unity, and second because the ratio \( A_{-1}(\alpha)/A_0(\alpha) \) is always greater than or equal to one-half of \( \alpha \), and approaches unity. Thus, we may write

\[ \alpha_2 > (\frac{3}{2})\alpha, \]

which indicates a significant boost in the amplitude of the impressed signal at \( p \), at the expense of a decrease in the amplitude of the self-oscillation.

If the signal flow around the feedback loop is traced \( n \) times, let us say, the ratio of the amplitude of the component at \( p \) to the component of self oscillation at \( p + r \), as seen by the limiter, becomes, at the end of the \( n \)th round,

\[ \alpha_{n+1} = \frac{\alpha_n + A_{-1}(\alpha_n)}{A_0(\alpha_n)} \]

The value of \( \alpha_n \), for which \( \alpha_{n+1} \) equals unity, exceeds 0.39 slightly. Beyond this value of \( \alpha_n \), the amplitude of the
is not satisfied, the phase of the sinusoid carrying the self-oscillation energy will not be fixed relative to the impressed sinusoid, but will keep clipping. The method of analysis whereby the signal flow around the loop is traced around and around will never converge to a repeatable pattern. We may, therefore, conclude that no true locking exists if the above condition on \( K_s \) and \( \Phi_{fb} \) is not satisfied.

Finally, an important situation remains to be explored, which bears directly upon the approach followed in handling the positive feedback problem in the earlier sections of this chapter. Suppose two signals are impressed, instead on one, across the terminals of the switch \( S \) in Fig. 22. In line with the assumptions of Sections IV.3 and IV.4, we shall consider the two signals to differ in frequency by more than one-half the feedback bandwidth, which in turn is equal to one i-f bandwidth. The problem is to determine the components of the resultant signal across the input terminals of the limiter, after a steady state has been established, starting with the switch \( S \) shorting out the source rather than the feedback voltage. Here, following the opening of the switch \( S \), three signals will be present across the input to the limiter: the source signals having amplitudes \( E_s \) and \( aE_s \) at \( p \) and \( p + r \) rad/sec, and the self-oscillation having the amplitude \( KG_{fb} \) and the frequency \( \omega_0 \) at which the phase shift around the loop is exactly zero. The problem could, of course, be handled in terms of all three sinusoids by tracing several rounds of signal flow around the loop, but the analysis and computation are greatly simplified, if, instead of applying the two source signals simultaneously at the start, we switch on one of them first, allow it to pull the oscillation to its frequency, and then switch on the other signal.

Suppose we switch on the signal of amplitude \( E_s \) at \( p \) rad/sec, first. This signal, regardless of its relative strength with respect to the self-oscillation (provided \( \Phi_{fb} \) is considered negligible), will pull this oscillation from the frequency \( \omega_0 \) to the frequency \( p \), and in the steady state, the net
amplitude at $p$ rad/sec will be given by $E_s + kG_{fb} = E_s(1 + K_s)$. After this locking of the oscillation has been achieved, let us switch on the signal at $p + r$ rad/sec whose amplitude is given by $aE_s$, $a$ being less than one. The problem is thus reduced to a two-signal problem from the start. The initial ratio of weaker-to-stronger signal seen by the limiter is given by

$$\alpha_1 = \frac{a}{1 + K_s}.$$ 

At the end of the first round around the loop, the amplitudes of the signals as seen by the limiter will therefore become

$$E_s\left[1 + K_sA_0(\alpha_1)\right]$$

at $p$ rad/sec

and

$$E_s\left[a + K_sA_{-1}(\alpha_1)\right]$$

at $p + r$ rad/sec,

and the amplitude ratio is now given by

$$\alpha_2 = \frac{a + K_sA_{-1}(\alpha_1)}{1 + K_sA_0(\alpha_1)}.$$ 

The second and third rounds may also be similarly traced, and so on. Evidently, the steady state will be reached when the resultant ratio of weaker-to-stronger-signal amplitude becomes

$$\alpha = \frac{a + K_sA_{-1}(\alpha)}{1 + K_sA_0(\alpha)},$$

which checks the result reached in Section IV.5 on a different basis. The weaker signal is seen to gain in amplitude at the expense of the stronger by taking away some of the oscillation energy originally captured by this stronger signal. If the balance of oscillation energy is now added up, a portion of it will appear unaccounted for. It has been split up among components whose frequencies were assumed to fall outside the feedback passband.
Let us next start by switching on the weaker signal at \( p + r \), and allow it to build up its amplitude, at the limiter input, to the value \( E_s(a + K_s) \) by pulling the self-oscillation into the frequency \( p + r \). If the stronger signal at \( p \) is next switched on, then two initial possibilities exist. The initial amplitude ratio is given by \( a + K_s \). This ratio may either be less than one or greater than one. The case in which this ratio equals one may be treated as a special case of either of the stated possibilities.

Starting with \( a + K_s = \alpha_1 \) less than unity, the amplitude ratio at the end of the first round will be

\[
\alpha_2 = \frac{a + K_s A_{-1}(\alpha_1)}{1 + K_s A_{0}(\alpha_1)},
\]

and it is easily seen that this ratio will settle to the value given by Eq. 68, in the steady state. This conclusion is also readily seen to apply for \( \alpha_1 = 1 \).

If \( a + K_s \) is greater than one, let

\[
\alpha_1 = \frac{1}{a + K_s}.
\]

At the end of the first round, the ratio of signal amplitude at \( p \) to that at \( p + r \) is

\[
\alpha_2 = \frac{1 + K_s A_{-1}(\alpha_1)}{a + K_s A_{0}(\alpha_1)},
\]

which is obviously larger than \( \alpha_1 \). The second round will, therefore, bring back a larger addition to the amplitude at \( p \), and a smaller one to that at \( p + r \). This follows from the fact that \( A_{-1}(\alpha) \) increases, while \( A_{0}(\alpha) \) decreases with increasing \( \alpha \). The ratio at the end of the second round will, therefore, be larger than that at the end of the first round. This ratio will, therefore, build up as we trace the signal.
flow over successive rounds. At the end of the \( n \)th round,

\[
\alpha_{n+1} = \frac{1 + K_s A_{-1}(\alpha_n)}{a + K_s A_0(\alpha_n)}
\]

and \( \alpha_{n+1} \) will be larger than \( \alpha_n \). This build up will continue until the signal at \( p \), as seen by the limiter becomes the stronger of the two. If \( \alpha_{n+1} \) is greater than unity, while \( \alpha_n \) is less than or equal to unity, the argument continues in the \( (n + 1) \)th round with \( A_0(1/\alpha_{n+1}) \) at \( p \) rad/sec, and \( A_{-1}(1/\alpha_{n+1}) \) at \( p + r \). In the steady state, the ratio of signal amplitude at \( p + r \) to that at \( p \) becomes that given by Eq. 16, above.

The preceding discussion features an interesting approach to the phenomenon of locking and pulling of oscillations. The fact that the results of this approach check with the results of the reasoning introduced in Section IV.3, suggests that we now have at least two novel ways of studying the locking and pulling of oscillators by one or more signals whose frequencies lie in the neighborhood of the self-oscillation frequency.

At this point, we would like to comment on the phenomenon of locking to the initially stronger of two signals which was revealed in the plots of Figs. 7, 8, 9, 11, 12, and 13, in the light of the present discussion. In this connection, we would like to emphasize that, fundamentally, the establishment of a steady-state in the present analysis, as well as in the analyses of Sections IV.3 and IV.4, is a matter of reaching a condition in which the limiter sees a ratio of signal amplitudes given by \( \epsilon \), and delivers spectral components whose amplitudes depend upon \( \epsilon \) according to \( A_0(\epsilon) \) and \( A_{-1}(\epsilon) \), which when fed back to the input, do not result in a change of the amplitude ratio \( \epsilon \). Whether the mechanism of operation is viewed as a splitting up of the energy of the self-oscillation among the two impressed signals, or as a regenerative feedback process which ignores the possibility
of self-oscillation in the absence of external excitation, the conclusions reached in Section 3 concerning the persistence of the locking to the stronger signal even though this signal may become the weaker of the two at the source, still hold. However, we may now speak of the energy of self-oscillation, originally pulled by the initially stronger of the two source signals, as providing for the continued predominance of this signal over the other signal at the input to the limiter through a fly-wheel effect even though this other signal may now be the stronger one at the source. This outlook justifies our reference to the phenomenon as one of persistent "locking" to the originally stronger signal. Moreover, the present outlook helps clarify what happens when the value of \( a \) reaches to, or exceeds the value denoted in Section IV.3 by \( a_{\text{crit}} \). In the first place, the division of the oscillation energy shared by the two signals is dictated by the value of \( a_{\text{c}} \), which corresponds to given \( a \) and \( K_{a} \) values, according to the amplitude laws \( A_{0}(a_{\text{c}}) \) and \( A_{-1}(a_{\text{c}}) \), \( A_{0}(a_{\text{c}}) \), the predominantly larger of the two, going to the originally stronger signal. But, for \( a \) values exceeding \( a_{\text{crit}} \), there is no value of \( a_{\text{c}} \) that will allow this lop-sided division to persist, or, equivalently, to allow a continuation of the original steady-state conditions. The locking of the predominant share of the oscillation energy to the originally stronger signal will, therefore, be broken, and in a few signal flow rounds around the loop, the better part of this energy will be pulled over to the frequency of the now stronger signal to boost its capture. At \( a = a_{\text{crit}} \), the equilibrium of the division of the self-oscillation energy is unstable, and the balance will tip in favor of the now stronger of the two source signals.

If the phase shift off the center frequency is not negligible, an argument that has so far been well illustrated will lead to conclusions identical with those of the preceding section. In particular, Fig. 18 of that section applies in the steady state here, also. Condition 49, also, imposes
restrictions on $K_s$ as well as the maximum permissible phase shift, $\Phi_{fb}$, for a true state of locking of self-oscillation energy to the frequencies of the impressed signals to exist. Incidentally, this condition is seen to involve the ratio of the amplitudes of the impressed signals through the very slowly varying function, $A_o(a)$.

Before concluding the present discussion, we would like to point out an interesting property of the "oscillating limiter" scheme. In the absence of an input signal, the self-oscillation at the center frequency of the band, having an amplitude given by $kG_{fb}$, is usually sufficiently stronger than the background random noise level to give rise to a considerable degree of squelch, or noise quieting. The degree of quieting is predictable on the basis of the familiar analysis of low-level FM noise applying when a strong carrier is imbedded in a background of low-level thermal noise. The noise delivered by the i-f amplifier modulates the strong self-oscillation, in the familiar manner, to produce a triangular FM noise spectrum at the output. The result is that, after detection and low-pass filtering, an appreciable degree of background, or interstation, noise quieting may be expected in the absence of an input signal, or when the input signal is unable to lock the self-oscillation because it is sufficiently weak to be swamped in the ambient background noise. This phenomenon has been confirmed experimentally, and over 20db of quieting has been observed below the limiting threshold, using a conventional low-quality circuit design(Ref. 3).

Additional comments relating to the effect of feedback upon the appearance of the amplitude limiting characteristic of a practical circuit arrangement are also in order, at this point.
If the amplitude of the self-oscillation exceeds the limiting threshold, then the operating point of the scheme on the output amplitude vs input-amplitude characteristic, Fig. 21(a), will always lie in the saturation region, even in the absence of an input signal, as is evident from the above discussions. On the other hand, if the value of the limiter gain, $G_L$, necessary for the satisfaction of the condition of unity gain around the loop is given by $k/E_{th}$, then for the idealized limiter characteristic shown in Fig. 21(a), any value of $E_{in} = E_{osc}$ which puts the operating point on the linear portion of the characteristic, below the limiting threshold is a likely value for the amplitude of the oscillation. In practice, with the curve rounded off at the knee, and near the origin, a more or less well-defined point on the characteristic will qualify, and the value of $E_{osc}$ will take up the value of $E_{in}$ at that point. Therefore, if we plot an input-output amplitude characteristic for the limiter with feedback, by injecting an input signal of known amplitude at the center frequency at the terminals across which the i-f output would normally be impressed, and measuring the amplitude of the output voltage across the
limiter filter, we may expect the effect of feedback upon the limiting characteristic to be somewhat as sketched in Fig. 27. The positive feedback, even in the absence of oscillation, will improve the limiting threshold by boosting the signal level at the input to the limiter, with the familiar increase in the gain of the stage, in the region below the limiting threshold without feedback, thus enabling an earlier realization of complete saturation. The prediction of the boost in the driving signal in the region below the threshold, on the basis of the positive feedback argument even if there is a self-oscillation for a gain value at a point in this region, is justified, since it may be argued that the introduction of the source signal will shift the operating point into regions of lower limiter gain, and hence, upset the gain condition for self-oscillation. Fig. 27 also illustrates the effect of inverse feedback upon the limiting characteristic.

The preceding observations have also stood the test of laboratory measurement (Ref. 3).
IV.8 FEEDBACK OF MORE COMPLICATED SPECTRAL CONFIGURATIONS
AND IMPULSE NOISE CONSIDERATIONS

It is unquestionable that the analysis of the previous
sections has been greatly simplified by our restricting that
study to the situations governed by the most adverse inter-
ference conditions possible. Indeed, it is against these
very conditions that the success and performance of any
counter-interference schemes must be put to test. In that
respect, it is truly fortunate that the most adverse inter-
ference conditions should be the easiest and most convenient
to handle analytically.

The question of what happens under conditions which,
interference-wise, might be classified as being of a milder
nature than the conditions of the preceding study, but
analytically may be of greater complexity to handle (while
practically they are of a significantly greater frequency of
occurrence) remains of great interest and value. We become
particularly appreciative of this when we recall that some
questions remain to be clarified in the light of happenings
with feedback of more complicated spectral configurations,
relating to:

(a) the effect, upon the locking to the originally
stronger signal, of a change in the spectrum that is fed
back to the input, which in turn arises from a change in the
difference frequency when either or both signals are slowly
modulated.

(b) the detailed disturbance-suppressing qualities of
the feedback scheme in the presence of impulsive interference.
This type of interference arises from the ringing of the
tank circuits of a receiver in response to an impulse-like
excitation, and may be closely represented by a carrier at
the center frequency of the intermediate-frequency amplifier,
having an envelope variation largely dictated by the impulse
response of the i-f amplifier.
It is, therefore, appropriate to consider situations in which the feedback spectrum is more complicated than the spectrum used in the preceding analyses, even though we may anticipate, in advance, that the greater the number of significant sideband components feedback, the less pronounced are the feedback effects explored for the limiting situation of the preceding sections. The spectral configuration that we shall consider is one that is perhaps next in the order of seriousness, interference-wise, to the configuration considered in the preceding sections. This is the configuration made up of the desired component, $A_o(a)$, and the upper sideband components $A_{-1}(a)$ and $A_{-2}(a)$, located at $p, p + r,$ and $p + 2r$ rad/sec, respectively. If the weaker of the two input sinusoids is assumed to be at the center frequency of the i-f passband, and hence of the feedback passband, then the present configuration may arise with all values of $r$ ranging from a little over $(1/4)(3W)_{if}$ to $(1/2)(3W)_{if}$, assuming a sharply defined feedback passband. We shall consider the effect of phase shift off the center frequency to be negligible, for simplicity.

The initial signal seen by the limiter, following the opening of the switch, $S$, in Fig. 3, is given by

$$\left(e_1 + e_2\right)/E = \left[1 + K_s A_o(a)\right] \cos pt + \left[a + K_s A_{-1}(a)\right] \cos(p + r)t - K_s |A_{-2}(a)| \cos(p + 2r)t. \quad (52)$$

This will change the spectral amplitudes present at the output of the limiter and, consequently, remodify the feedback components. In the feedback steady state, assume that the amplitudes of the signals at the input to the limiter are in the ratios of $a_\alpha$ to $a_\beta$ at $p, p + r,$ and $p + 2r$ rad/sec, respectively. These amplitude ratios will obviously be given by

$$a_\alpha = \frac{a + K_s A_{-1}(a_\alpha, b_\alpha)}{1 + K_s A_o(a_\alpha, b_\alpha)}.$$
and

\[ b_L = \frac{K_s |A_{-2}(a_L, b_L)|}{1 + K_s A_0(a_L, b_L)}. \]

The evaluation of the indicated spectral amplitudes leads to an extremely difficult elliptic integral, and attempts at numerical analysis are quickly discouraged by the labor that looms ahead. To get around this difficulty, two possibilities may be resorted to, each of which involves an approximation. The first of these recognizes that the ratio of the amplitude of the component at \( p + 2r \) to that at \( p \) is initially given by

\[ \frac{K_s A_{-2}(a)}{1 + K_s A_0(a)} \]

which is less than \( 1/10 \) for all \( K_s \) equal to \( \delta \), or less, and for all \( a \) less than \( 0.8 \), and this ratio is less than \( 1/10 \) times the corresponding amplitude ratio of the component at \( p + r \) over the same range of \( K_s \) and \( a \) values. Therefore, for such values of \( a \) and \( K_s \), it is reasonable to ignore the presence of the component at \( p + 2r \) without anticipating serious error. If this is done, the problem reduces to the one solved in Section IV.3 for which the plots of Figs. 7, 8, and 9 apply. Therefore, we may conclude that, for \( K_s \) less than or equal to \( \delta \) and for all \( a \) of about \( 0.8 \), or less, the curves of Fig. 7 apply closely for all values of \( r \) greater than \((1/4)(3\pi)_{1-f}\), if the originally weaker signal is nearer or at the center of the passband for \( r \leq (1/2)(3\pi) \). The curves for the present problem will still cross to the right of \( a = 1 \) at reasonably low values of \( a_L \) for values of \( K_s \) that are only a little higher than the corresponding values of the situation described by Fig. 7. This conclusion is of great importance with regard to disturbances arising from impulsive interferences. It indicates that with values of \( K_s \) for which there is significant crossover to the right of \( a = 1 \), the locking phenomenon would enable the reception to persist uninterrupted even in the presence of impulsive ringing of the 1-f tank circuits,
the height of whose envelope may rise to a few times the desired signal amplitude under the more severe interference conditions. The higher impulses may, however, still interrupt the reception. Although this interruption may be delayed on the leading edge of the pulse, the recapture of the desired signal may be delayed even more, especially with large $K_w$, on account of the difference in the appearance between the curves of constant $K_w$ and constant $K_s$ for very large values of these constants. This discussion, of course, assumes that the build up and decay of the ringing pulse is slow compared to the time delay around the feedback loop. Whether or not this is a good assumption is a function of the specific circuits involved.

The second approximate treatment of the problem involves the use of the concept of the equivalent interference ratio, defined on the basis of the frequency spike magnitude and the repetition rate of the spike pattern associated with the resultant of a given spectral configuration, in Section III.3. The use of this concept is involved in the argument in the following manner.

Following the opening of the switch $S$ in Fig. 5, we have, in the present problem, an initial resultant voltage at the input to the limiter given by Eq. 69. According to Section III.3, the initial feedback spectrum may be replaced by two components described by the amplitudes

$$uA_o(a)$$ at $p$ rad/sec

and

$$u'A_o(a)$$ at $p + r$ rad/sec,

where, with reference to Eqs. 32 and 33 of Chapter Three,

$$A'_f = \frac{A_{-1}(a) + 2|A_{-2}(a)|}{A_o(a) + 4|A_{-2}(a)|} > \frac{A_{-1}(a)}{A_o(a)}.$$

Accordingly, the resultant signal, as seen by the limiter, may be replaced by
\(E_s + uA_o(a) \text{ at } p\)

and

\(aE_s + a'pA_o(a) \text{ at } p + r,\)

with a new ratio of weaker-to-stronger signal amplitudes given by

\[ a' = \frac{a + K_s a' A_o(a)}{1 + K_s A_o(a)} . \]

In the feedback steady state, the feedback voltage is, accordingly, replaced by

\(uA_o(a_{le}) \text{ at } p\)

and

\(ua_r A_o(a_{le}) \text{ at } p + r,\)

where

\[ a_r = \frac{A_{-1}(a_{le}) + 2|A_{-2}(a_{le})|}{A_o(a_{le}) + |A_{-2}(a_{le})|} \quad (70) \]

and

\[ a_{le} = \frac{a + K_s a_r A_o(a_{le})}{1 + K_s A_o(a_{le})} . \quad (71) \]

When solved for \(a\), this equation yields

\[ a = a_{le} + K_s A_o(a_{le}) \left[ a_{le} - a_r \right] . \quad (72) \]

A similar relationship is readily obtainable in terms of \(K_w\).

Equation 72 and the corresponding relationship in terms of \(K_w\) are plotted in Figs. 28 and 29 for illustration. The curious convergence of the curves at the point

\[ a = a_{le} = 0.897 \]

is due to the fact for this value of \(a_{le}\), the ratio on the right, in Eq. 70, is equal to \(a_{le}\). The curves of Figs. 28 and 29 are only of mathematical interest for \(a_{le}\) values exceeding about 0.8. Their accuracy goes from fair (on the pessimistic side) for \(a_{le} = 0.4\), to good below 0.2. It is worth noting that for values of \(a_{le}\) below 0.8, Eq. 72 is closely approximated by
\[ \alpha = q_x + K_3 A_0(a_x)[a_x - a_f] \]

\[ q_f = \frac{A_1(a_x) + 2|A_2(a_x)|}{A_0(a_x) + |A_2(a_x)|} \]
\[ a = \frac{a_0}{1 - K \cdot A_0(a_0)(q_i - q_f)} \]

\[ a_f = \frac{A_0(a_f) + 2|A_2(a_f)|}{A_0(a_f) + |A_2(a_f)|} \]

Fig. 29
\[ a = a_e + K_s \left[ a_e A_0(a_e) - A_1(a_e) \right] - 2K_s |A_2(a_e)|. \]  

Equation 73 should be compared with Eq. 20 for a better appreciation of its significance.

An alternative approximate approach to the problem of feeding back the more complicated spectral configurations, also making use of the equivalent interference ratio concept, will now be illustrated. This method of reasoning is believed more appropriate than the one just presented, for reasons that will be clarified presently. Instead of replacing the feedback spectrum itself by a two-component equivalent; as in the preceding argument, the present approach replaces the total spectrum seen by the limiter by a two-component equivalent, as far as spike magnitude and repetition rate are concerned. The latter attitude makes use of the equivalence at the more appropriate point, because the spike magnitude as seen by the limiter circuit differs from that computed on the basis of the former approach, as an easy computation readily shows. The difference is insignificant for values of \( a_e \) below 0.2.

Thus, we may say that, in the feedback steady state, the equivalent weaker-to-stronger signal amplitude ratio is given by

\[ a_e = \frac{a + K_s \left[ A_1(a_e) + 2|A_2(a_e)| \right]}{1 + K_s \left[ A_0(a_e) + |A_2(a_e)| \right]} . \]  

When solved for \( a \), this equation yields

\[ a = a_e + K_s \left[ a_e A_0(a_e) - A_1(a_e) \right] - K_s |A_2(a_e)| \left( 2 - a_e \right) . \]  

This formula is seen to agree with Eq. 73 for \( a_e \) values of 0.2 or less. Incidentally, the third term on the right is a measure of the deviation of the curves relating \( a \) and \( a_e \)
according to Eq. 75, from the curves of Figs. 7, 8, and 9 applying for the two-component situation. The negativenss of this third term indicates that the capture improvements under the present condition of positive feedback, are inferior to those under the most adverse condition of interference treated in Sections IV.3 and IV.4. The values of $K_s$ and $a_L$ for which this third term may be considered negligible are easily determined from Eq. 75.

The preceding application of the concept of the equivalent interference ratio is of great value in leading quickly to a reasonably close result when the number of significant spectral components fed back is greater than two, because it reduces such problems to an approximate two-signal problem.

As the difference frequency between the two input signals is decreased below one-half of the i-f bandwidth, the number of significant spectral components fed back to the input, increases, and the effect of the feedback grows less pronounced. This implies, among other things, that in the plots of the equivalent interference ratio at the input to the limiter against the ratio of weaker-to-stronger signal amplitude delivered by the i-f amplifier, the range of $a$ values for which the constant $K$ curves cross into the region of $a$ greater than unity is decreased below the indications of the plots in Sections IV.3 and IV.4. These curves would appear generally shrunk along the $a$-axis, and so, corresponding to a given value of $a$, below unity, the values of $a_L$ for the various $K_s$ and $K_w$ curves would be higher than the values applying under the most adverse interference condition. In terms of the self-oscillation energy, the portion pulled into the frequency of the stronger signal under the most adverse interference conditions at the input, gets frittered away among the increasing number of feedback components, as $r$ decreases. Equivalently, as the feedback bandwidth is increased, until when the value of $r$, relative to the feedback bandwidth, gets sufficiently small for all the signi-
significant sideband components in the spectrum of the amplitude-limited resultant of the two source signals to be accommodated within the feedback passband, not enough of that original portion of self-oscillation energy is left at the frequency, \( \gamma \), to enable the stronger signal to experience any boost in its predominance over the weaker signal. Concerning the locking to the originally stronger of the two signals, discussed in Section IV.3, these observations indicate that, with a decrease in the value of \( r \) relative to the feedback bandwidth, the fact that less self-oscillation energy remains locked to the frequency of the originally stronger signal indicates that the fly-wheel effect that enabled its illegal capture to be sustained under the interference conditions, discussed in Sections IV.3 and IV.4, will now be decreased. When \( r \) becomes sufficiently small relative to the feedback bandwidth to enable the filter defining this bandwidth to follow the impressed instantaneous frequency variations through stationary states, then the feedback becomes of the wideband variety which was shown, in Section IV.2, not to affect the capture conditions at the input to the limiter. One may, therefore, say that the locking to the originally stronger of the two signals will be maintained under conditions of slow frequency modulation of either or both of the two input signals, until the frequency difference falls to a value at which the curve of equivalent interference ratio versus \( \alpha \), for a given constant \( K_s \) or \( K_w \), has an \( \alpha_{\text{crit}} \) value (see Section IV.3) which is less than or equal to the \( \alpha \) (greater than unity) value at which the locking was initially maintained.

Assuming a feedback bandwidth of one i-f bandwidth, the value of \( r = r_{\text{min}} \) below which no noticeable improvement in the capture conditions may be expected, may be determined for any given filter which defines the feedback bandwidth, on the basis of the argument of Section III.2. The result expressed in Eq. 38 may be adapted, for this purpose to give
\[ r_{\text{min}} = \frac{l}{K_{3\text{lim}}} \cdot \frac{l - a}{1 + a} \cdot (3\omega)^{1r}. \] (78)

For values of \( a \) between 0.5 and unity, \( r_{\text{min}} \) is closely approximated by replacing \( K_{3\text{lim}} \) by \( 5/2 \) if a single parallel-resonant circuit is used, and by 4 if a two-pole Butterworth-type filter is used.
REFERENCES


BIOGRAPHICAL NOTE
Biographical Note

Elie J. Baghdady was born in Zahleh, Lebanon, on November 10, 1930. Between 1942 and 1947, he attended Gerard Institute, an American missionary boarding high school in Sidon, Lebanon, leading a class of near 30 students, academically, until his graduation as the valedictorian of his class in June 1947. Between 1947 and 1951, he attended the American University of Beirut, in Beirut, Lebanon where he lead the entire Junior class, academically, to receive the university first-prize scholarship in 1950, and graduate at the head of the class with a Bachelor of Arts with High Distinction, in physics, in 1951. While at A.U.B., he was a British Council Scholar between 1949 and 1951.

In the summer of 1951, Mr. Baghdady came to the United States on a Georgia Institute of Technology World Student Fund fellowship and enrolled in the Institute's graduate division of electrical engineering. In June of 1952, he joined the staff of the Massachusetts Institute of Technology as a research assistant and graduate student. After a summer in the microwave tube laboratory at the Research Laboratory of Electronics, he joined the multipath transmission group where he got engaged in research on frequency-modulation receiver design for rejecting interference. He received his Master of Science degree in electrical engineering from M.I.T. in February of 1954. His master's thesis was entitled "Frequency-Modulation Interference Rejection with Narrow-Band Limiters".

In July of 1954, Mr. Baghdady married Miss Virginia R. Benim of Stratford, Conn. His achievements as a doctoral candidate included a son, Elie Joseph II, born on April 28, 1965.

In July of 1955, Mr. Baghdady was promoted to Instructor in electrical engineering. He has taught courses in this department starting with 3.111, Electronic Circuits for Communications, a senior elective, in the fall term of 1954, 3.325, Principles of Frequency Modulation, a graduate course and senior elective, in the spring term of 1955, 3.52, Electronic Circuits, an elementary course, in the summer of 1955, 3.56,

To date, Mr. Baghdady's professional publications include:

To the Editor, Proc. I.E.E., on Reciprocal Inductance," vol. 43, p. 1116;