Combined $\mathcal{H}_\infty$/LQG Control via the Optimal Projection Equations: On Minimizing the LQG Cost Bound

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Abstract

The Optimal Projection equations for combined $\mathcal{H}_\infty$/LQG control are considered. Positive semidefiniteness of the associated Lagrange multiplier is shown to be necessary for the LQG cost bound to be minimal. It follows that all four Optimal Projection equations have a role to play, even in the full-order case.

1 Introduction

The design of feedback controllers which satisfy both $\mathcal{H}_\infty$ and LQG criteria has been the focus of much attention recently (see for example [1, 3, 10, 12]). Such controllers are interesting because they offer both robust stability (via an $\mathcal{H}_\infty$-norm bound) and nominal performance (via an LQG cost bound).

Our concern in this paper is with the 'equalized $\mathcal{H}_\infty$/LQG weights' case of the $\mathcal{H}_\infty$/LQG control problem considered in [1]. Thus a full or reduced-order stabilizing controller is sought to minimize the auxiliary cost (i.e., a certain LQG cost bound) of a specified closed-loop system subject to an $\mathcal{H}_\infty$-norm bound on the same closed-loop.

In [1], a Lagrange multiplier approach is used to tackle the combined $\mathcal{H}_\infty$/LQG problem. From the stationarity conditions a set of four coupled modified algebraic

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Riccati equations (the Optimal Projection equations) are derived which characterize the controller gains. However it is not apparent if the auxiliary cost has actually been minimized, since the constraint equation used to define the auxiliary cost has a nonunique solution. In this paper we consider this issue. Non-negativity of the Lagrange multiplier turns out to play a key role, and all four Optimal Projection equations are needed to ensure this.

The present paper is an expanded version of [7] where a special case was treated. In Section 2 a concise description of the combined $\mathcal{H}_\infty$/LQG problem of interest to us is given—mostly standard material for those familiar with [1, 4]. Section 3 contains the main results. A key lemma is introduced which relates non-negativity of the Lagrange multiplier to minimality of the auxiliary cost. The lemma is then applied in the reduced-order and then full-order cases, to bring out the role of the four Optimal Projection equations with regard to minimality of the auxiliary cost, strictness of the $\mathcal{H}_\infty$-norm bound and minimality of the controller realization. Some proofs are relegated to the Appendix for clarity.

2 Combined $\mathcal{H}_\infty$/LQG Control

Consider an $n$-state plant $P$ with state-space realization

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{12}u \\
y &= C_2x + D_{21}w
\end{align*}
\]

mapping disturbance signals $w \in \mathbb{R}^{m_1}$ and control signals $u \in \mathbb{R}^{m_2}$ to regulated signals $z \in \mathbb{R}^{p_1}$ and measured signals $y \in \mathbb{R}^{p_2}$. The $n_c$-state feedback controller $K$ with state-space realization

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y \\
u &= C_c x_c
\end{align*}
\]

is to be designed.

The closed-loop transfer function $H = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ from $w$ to $z$ is given by

\[
\begin{align*}
\tilde{A} &= \begin{bmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_cD_{21} \end{bmatrix} \quad \text{and} \quad \tilde{C} = \begin{bmatrix} C_1 & D_{12}C_c \end{bmatrix}.
\end{align*}
\]

Internal stability of the closed-loop $H$ corresponds to asymptotic stability of $\tilde{A}$. It is standard that $H$ is a linear fractional map $F(P,K)$ of $P$ (appropriately partitioned) and $K$:

\[
H = F(P,K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.
\]

2.1 Assumptions

It is assumed that
• \((A, B_2)\) is stabilizable and \((C_2, A)\) is detectable

which is a necessary and sufficient condition for the existence of stabilizing controllers.

In common with [1], it is also assumed that

• \(D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & R_2 \end{bmatrix}\) where \(R_2 > 0\)

• \(\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 & V_2 \end{bmatrix}\) where \(V_2 > 0\)

which are the standard assumptions of a non-singular control problem without cross-weightings between the state and control input, and plant disturbance and sensor noise.

As explained in Section III-A of [4], without loss of generality suitable scaling of \(u\) and \(w\) can then be applied to make \(R_2 = I\) and \(V_2 = I\), and we assume that this has been done. Note that, as pointed out on page 2482 of [13], no stabilizability/detectability assumptions are needed on \((A, B_1, C_1)\).

2.2 \(\mathcal{H}_\infty\)-norm and LQG cost

The concepts of \(\mathcal{H}_\infty\)-norm and LQG cost are well-known. We will therefore omit detailed discussion and content ourselves with stating the following definitions for reference. Given a system \(H = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}\) where \(\tilde{A}\) is asymptotically stable, the \(\mathcal{H}_\infty\)-norm of \(H\) is

\[
\|H\|_\infty := \sup_{\omega} \lambda_{max}^{1/2} \{H^*(j\omega)H(j\omega)\},
\]

where \(H^*(s) := HT(-s)\), and the LQG cost of \(H\) is

\[
C(H) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[H^*(j\omega)H(j\omega)]d\omega.
\]

It is a well-known fact that

\[
C(H) = \text{trace}[\tilde{Q}\tilde{C}^T\tilde{C}]
\]

where \(\tilde{Q} \geq 0\) satisfies

\[
0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}\tilde{B}^T.
\]

An 'exact' \(\mathcal{H}_\infty/LQG\) control problem would be to minimize the LQG cost \(C(H)\) over all stabilized closed-loops \(H\) satisfying \(\|H\|_\infty < \gamma\). As far as this author is aware, this remains an interesting unsolved problem. Instead, the combined \(\mathcal{H}_\infty/LQG\) problem considered in [1] and in the present paper minimizes LQG cost overbound subject to an \(\mathcal{H}_\infty\)-norm bound. We turn now to the definition of this LQG cost bound.

2.3 Auxiliary Cost

The following lemma, basically Lemma 2.1 of [1], is the key to defining the LQG cost bound used in the combined \(\mathcal{H}_\infty/LQG\) problem.
Lemma 2.1 Suppose $\tilde{A}$ is asymptotically stable and $\gamma \in \mathbb{R}^+$. If there exists $Q \geq 0$ satisfying
\[ 0 = \tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}Q\tilde{C}^T\tilde{C}Q + \tilde{B}\tilde{B}^T \] (3)
then the following hold true:
(i) $\|\tilde{C}(sI - \tilde{A})^{-1}\tilde{B}\|_{\infty} \leq \gamma$;
(ii) $Q \geq \tilde{Q}$ where $\tilde{Q} \geq 0$ satisfies (2).

Given a system $H = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ where $\tilde{A}$ is asymptotically stable, and $\gamma \in \mathbb{R}^+$ such that there exists $Q \geq 0$ satisfying (3), the auxiliary cost [1] is then defined by
\[ J(H,\gamma) := \text{trace}[QC^T\tilde{C}] \] (4)

By Lemma 2.1, and comparing (4) with (1), it is immediate that
- $\|H\|_\infty \leq \gamma$;
- $C(H) \leq J(H,\gamma)$.

Thus the single equation (3) leads to an $H_\infty$-norm bound $\gamma$ and an LQG cost overbound $J(H,\gamma)$. The combined $H_\infty$/LQG control problem considered in [1] and in the present paper is to minimize $J(H,\gamma)$ over all stabilized closed-loops $H$ satisfying $\|H\|_\infty \leq \gamma$.

The solution to (3) is, however, nonunique. Out of the set of $Q \geq 0$ satisfying (3) there exists a minimal solution $Q \geq 0$ which satisfies $Q \leq Q$ where $Q$ is any $Q \geq 0$ satisfying (3). Clearly to obtain the lowest auxiliary cost $J(H,\gamma)$ we need to use $Q$:
\[ J(H,\gamma) := \text{trace}[QC^T\tilde{C}] \leq J(H,\gamma) \] (5)

This was pointed out in [8]. Our main results regard the solution of the combined $H_\infty$/LQG problem as given in [1], with particular concern for ensuring that the minimal solution to (3) (hence the smallest auxiliary cost) is indeed taken.

2.4 Entropy

The auxiliary cost of a system satisfying a strict $H_\infty$-norm bound was shown in [8] to equal the entropy of the system. (See [10] for a comprehensive treatment of entropy in the context of $H_\infty$-control). Given a system $H = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ where $\tilde{A}$ is asymptotically stable, and $\gamma \in \mathbb{R}^+$ such that $\|H\|_\infty < \gamma$, the entropy is defined by
\[ I(H,\gamma) := -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln|\det(I - \gamma^{-2}H^*(j\omega)H(j\omega))|d\omega. \]

The entropy is an upper bound on the LQG cost [9, 10] so
- $\|H\|_\infty < \gamma$;
- $C(H) \leq I(H,\gamma)$. 

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The Strict Bounded Real Lemma (see e.g., [11] for proof) which we now state, gives a useful characterization of stable, strictly proper systems satisfying a strict $\mathcal{H}_\infty$-norm bound. The entropy is well-defined and finite for such systems.

**Lemma 2.2** The following are equivalent

(i) $\hat{A}$ is asymptotically stable and $\|\hat{C}(sI - \hat{A})^{-1}\hat{B}\|_\infty < \gamma$.

(ii) There exists a stabilizing solution $Q > 0$ to (3).

By the stabilizing solution $Q > 0$ to (3) we mean the unique $Q > 0$ satisfying (3) such that $\hat{A} + \gamma^{-2}Q\hat{C}^T\hat{C}$ is asymptotically stable. The above lemma allows a state-space evaluation of the entropy (Lemma 5.3.2 of [10]):

$$I(H, \gamma) = \text{trace}[QC^T]$$

where $\hat{Q}, > 0$ is the stabilizing solution to (3).

Now compare the definitions of the entropy $I(H, \gamma)$ in (6) and of the auxiliary cost $J(H, \gamma)$ in (5). The entropy is $\text{trace}[Q\hat{C}^T\hat{C}]$ for a system satisfying $\|H\|_\infty < \gamma$; the auxiliary cost is $\text{trace}[Q\hat{C}^T\hat{C}]$ for a system satisfying $\|H\|_\infty \leq \gamma$. We will see these differences disappear when the solution to the combined $\mathcal{H}_\infty$/LQG problem is analysed in the next section.

## 3 On Minimizing the LQG Cost Bound

### 3.1 Stationarity Conditions

The combined $\mathcal{H}_\infty$/LQG problem reduces to finding controller matrices $A_c, B_c, C_c$ which make $\hat{A}$ asymptotically stable and minimize $J(H, \gamma) = \text{trace}[Q\hat{C}^T\hat{C}]$ subject to $Q > 0$ satisfying (3). Define the Lagrangian

$$L(A_c, B_c, C_c, P, Q) := \text{trace}[Q\hat{C}^T\hat{C} + \{\hat{A}Q + Q\hat{A}^T + \gamma^{-2}Q\hat{C}^T\hat{C}Q + \hat{B}\hat{B}^T\}P]$$

where $P \in \mathbb{R}^{(n + n_c) \times (n + n_c)}$ is a Lagrange multiplier. The stationary points are then characterized by

$$\frac{\partial L}{\partial A_c} = 0, \quad \frac{\partial L}{\partial B_c} = 0, \quad \frac{\partial L}{\partial C_c} = 0, \quad \frac{\partial L}{\partial P} = 0, \quad \text{and} \quad \frac{\partial L}{\partial Q} = 0. \quad (7)$$

Expansion of these, followed by some algebraic manipulations, is carried out in [1] to obtain four coupled algebraic Riccati equations. The controller gains are defined in terms to the solution to these 'Optimal Projection equations.' Before stating them we will find it beneficial to do some further analysis of the stationarity conditions. The point is that we seek the minimum of $J(H, \gamma)$ not just of $J(H, \gamma)$. 

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3.2 The Lagrange Multiplier

Consider the last of the stationarity conditions in (7). One obtains the following Lyapunov equation for the Lagrange multiplier $\mathcal{P}$:

$$0 = \mathcal{P} (\dot{A} + \gamma^{-2} Q \ddot{C}^T \ddot{C}) + (\dot{A} + \gamma^{-2} Q \ddot{C}^T \ddot{C})^T \mathcal{P} + \ddot{C}^T \dot{C}.$$  

(8)

As the next lemma shows, the existence of a positive semidefinite solution for the Lagrange multiplier is intimately related to minimality of $Q$ and strictness of the $\mathcal{H}_\infty$-norm bound.

Lemma 3.1 Suppose $\dot{A}$ is asymptotically stable. Then $\|\dot{\mathcal{C}}(sI - \dot{A})^{-1} \dot{\mathcal{B}}\|_\infty < \gamma$ if and only if there exists $Q \geq 0$ satisfying (3) such that there exists $\mathcal{P} \geq 0$ satisfying (8). If so, then the following statements are equivalent:

(i) $Q \geq 0$ is the minimal solution to (3).

(ii) $Q \geq 0$ is the stabilizing solution to (3).

(iii) $Q \geq 0$ satisfies (3) such that there exists $\mathcal{P} \geq 0$ satisfying (8).

Proof Appendix A. 

The equivalence of (i) and (ii) has also been obtained (independently) in Theorems 2.2 and 2.3 of [15]. For our purposes however, the equivalence of (i) and (ii) with item (iii) is the most important: the key point being that $Q$ is minimal if and only if the Lagrange multiplier $\mathcal{P}$ associated with $Q$ is positive semidefinite. Then we know that

- $J(H, \gamma) = J(H, \gamma)$, rather than just $J(H, \gamma) \geq J(H, \gamma)$.
- $\|H\|_\infty < \gamma$, rather than just $\|H\|_\infty \leq \gamma$.
- $J(H, \gamma) = \mathcal{J}(H, \gamma) = \text{trace}[Q \dot{C}^T \dot{C}] = \text{trace}[Q \dot{C}^T \dot{C}]$.

Now we can apply the above lemma to the solution of the combined $\mathcal{H}_\infty$/LQG control problem.

3.3 Reduced-Order Case

The next theorem, a quotation of Theorem 6.1 and Appendix (A.21)-(A.23) of [1], characterizes the combined $\mathcal{H}_\infty$/LQG controller gains in the reduced-order case ($n_c \leq n$). We will then apply Lemma 3.1.

Theorem 3.2 ([1]) Suppose there exists $Q \geq 0$, $P \geq 0$, $\dot{Q} \geq 0$ and $\dot{P} \geq 0$ satisfying

$$0 = AQ + QA^T + B_1 B_1^T + Q(\gamma^{-2} C_1^T C_1 - C_2^T C_2)Q + \tau_1 Q C_1^T C_2 Q \tau_1^T$$

$$0 = (A + \gamma^{-2} [Q + \dot{Q}] C_1^T C_1)^T P + P(A + \gamma^{-2} [Q + \dot{Q}] C_1^T C_1) + C_1^T C_1$$

$$-S^T P B_2 B_2^T P S + \tau_1^T S^T P B_2 B_2^T P S \tau_1$$

(9) (10)
\[ 0 = (A - B_2B_2^T PS + \gamma^{-2}Q C_1^T C_1) \dot{Q} + \dot{Q}(A - B_2B_2^T PS + \gamma^{-2}Q C_1^T C_1)^T + \gamma^{-2}\dot{Q}(C_1^T C_1 + S^TP B_2B_2^T PS)\dot{Q} + QC_2^T C_2 Q - \tau_1 Q C_2^T C_2 \tau_1^T \] (11)

\[ 0 = (A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2)) \dot{\hat{P}} + \dot{\hat{P}}(A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2)) + S^TP B_2B_2^T PS - \tau_1 S^TP B_2B_2^T P S \tau_1 \] (12)

where \( S := (I + \gamma^{-2}\dot{Q} P)^{-1} \) and

\[ \text{rank}(\dot{P}) = \text{rank}(\dot{Q}) = \text{rank}(\dot{\hat{P}}) = n_c. \]

The projection \( \tau_1 \) is defined via the factorization \( \dot{Q} \dot{P} = G^T M \Gamma \) by \( \tau_1 := I - G^T \Gamma \), where \( G \) and \( \Gamma \) are \( n_c \times n \) and satisfy \( \Gamma G^T = I \) and \( M \) is \( n \times n_c \) and invertible. Let

\[ A_c = \Gamma(A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2) - B_2B_2^T PS)G^T \]
\[ B_c = \Gamma QC_2^T \]
\[ C_c = -B_2^T P S G^T. \]

Then \((\hat{A}, \hat{B})\) is stabilizable if and only if \( \hat{A} \) is asymptotically stable. In this case

(i) \( K = (A_c, B_c, C_c) \) stabilizes \( P \);

(ii) \( \|F(P, K)\|_\infty \leq \gamma \);

(iii) \( J(F(P, K), \gamma) = \text{trace}[(Q + \dot{\hat{P}})C_1^T C_1 + \dot{\hat{P}} S^TP B_2B_2^T PS]; \)

(iv) \( Q = \begin{bmatrix} Q + \dot{\hat{P}} & \dot{\hat{P}} G^T \\ \Gamma \dot{Q} & \Gamma \dot{Q} \Gamma^T \end{bmatrix} \) and \( P = \begin{bmatrix} P + \dot{\hat{P}} G^T & -G \dot{\hat{P}} \\ -G \dot{\hat{P}} & G \dot{\hat{P}} G^T \end{bmatrix} \).

On applying Lemma 3.1, the following corollary is obtained.

**Corollary 3.3** Suppose all the conditions of Theorem 3.2 are satisfied. Then the following hold true:

(i) \( \|F(P, K)\|_\infty < \gamma \);

(ii) \( J(F(P, K), \gamma) = J(F(P, K), \gamma); \)

(iii) \( J(F(P, K), \gamma) = I(F(P, K), \gamma). \)

**Proof** From Theorem 3.2(iv),

\[ Q = \begin{bmatrix} I & \dot{\hat{P}} \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix}^T + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \] and \( P = \begin{bmatrix} I & \dot{\hat{P}} \end{bmatrix} \begin{bmatrix} I \\ -G \end{bmatrix}^T + \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \).

Hence \( Q \geq 0 \) and \( P \geq 0 \) since \( Q \geq 0, P \geq 0, \dot{Q} \geq 0 \) and \( \dot{\hat{P}} \geq 0 \). Lemma 3.1 therefore applies (which gives (i) immediately) implying that \( Q \) is minimal (which gives (ii)) and stabilizing (which gives (iii) by (5) and (6)). \( \square \)
3.4 Full-Order Case

To specialize Theorem 3.2 to the full-order case \( n_c = n \) simply set \( G = \Gamma = I \) and \( \tau_1 = 0 \).

**Theorem 3.4** Suppose there exists \( Q \geq 0, P \geq 0, \dot{Q} \geq 0 \) and \( \dot{P} \geq 0 \) satisfying

\[
0 = AQ + QA^T + B_1B_1^T + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2)Q
\]

(13)

\[
0 = (A + \gamma^{-2}[Q + \dot{Q}]C_1^T C_1)^T P + P(A + \gamma^{-2}[Q + \dot{Q}]C_1^T C_1)
+ C_1^T C_1 - S^T PB_2 B_2^T PS
\]

(14)

\[
0 = (A - B_2 B_2^T PS + \gamma^{-2}Q C_1^T C_1)\dot{Q} + \dot{Q}(A - B_2 B_2^T PS + \gamma^{-2}Q C_1^T C_1)^T
+ \gamma^{-2}\dot{Q}(C_1^T C_1 + S^T PB_2 B_2^T PS)\dot{Q} + QC_2^T C_2 Q
\]

(15)

\[
0 = (A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2))^T \dot{P}
+ \dot{P}(A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2)) + S^T PB_2 B_2^T PS
\]

(16)

where \( S := (I + \gamma^{-2}\dot{Q}P)^{-1} \). Let

\[
A_c = A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2) - B_2 B_2^T PS
\]

(17)

\[
B_c = QC_2^T
\]

(18)

\[
C_c = -B_2^T PS.
\]

(19)

Then \((\hat{A}, \hat{B})\) is stabilizable if and only if \( \hat{A} \) is asymptotically stable. In this case

(i) \( K = (A_c, B_c, C_c) \) stabilizes \( P \);

(ii) \( \|F(P, K)\|_\infty \leq \gamma \);

(iii) \( J(F(P, K), \gamma) = \text{trace}[(Q + \dot{Q})C_1^T C_1 + \dot{Q}S^T PB_2 B_2^T PS] \);

(iv) \( Q = \begin{bmatrix} Q + \dot{Q} & \dot{Q} \\ \dot{Q} & \dot{Q} \end{bmatrix} \) and \( P = \begin{bmatrix} P + \dot{P} & -\dot{P} \\ -\dot{P} & \dot{P} \end{bmatrix} \).

Note that the equations for \( Q, P \) and \( \dot{Q} \) do not depend on \( \dot{P} \), and neither does the controller nor the cost. There is thus a great temptation to discard the \( \dot{P} \) equation as superfluous—as is done in Remark 6.1 and Theorem 4.1 of [1]. Our analysis will show that \( \dot{P} \) really does have a role to play. To see this role emerge, we need only apply Lemma 3.1 to Theorem 3.4 to obtain the following corollary.

**Corollary 3.5** Suppose all the conditions of Theorem 3.4 are satisfied. Then the following hold true:

(i) \( \|F(P, K)\|_\infty < \gamma \);

(ii) \( J(F(P, K), \gamma) = J(F(P, K), \gamma) \);

(iii) \( J(F(P, K), \gamma) = I(F(P, K), \gamma) \).
Proof From Theorem 3.4(iv),

\[
Q = \begin{bmatrix} I & I \\ I & I \end{bmatrix} \dot{Q} \begin{bmatrix} I \\ I \end{bmatrix}^T + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} 
\quad \text{and} \quad
P = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \dot{P} \begin{bmatrix} I \\ -I \end{bmatrix}^T + \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}.
\]

Hence \(Q \geq 0\) and \(P \geq 0\) since \(Q \geq 0\), \(P \geq 0\), \(\dot{Q} \geq 0\) and \(\dot{P} \geq 0\). Lemma 3.1 therefore applies (which gives (i) immediately) implying that \(Q\) is minimal (which gives (ii)) and stabilizing (which gives (iii) by (5) and (6)).

It is now clear that the existence of \(\dot{P} \geq 0\) as well as existence of \(Q \geq 0\), \(P \geq 0\) and \(\dot{Q} \geq 0\) is needed to ensure positive semidefiniteness of the Lagrange multiplier which is necessary for minimality of the auxiliary cost. All four Optimal Projection Equations have a role to play. But whilst we do need to calculate \(Q\), \(P\) and \(\dot{Q}\) to write down the controller and its cost, knowing that \(\dot{P} \geq 0\) exists is enough. The following lemma gives a convenient way of ensuring this.

Lemma 3.6 \(\dot{P} \geq 0\) exists satisfying (16) if \(Q \geq 0\) is the stabilizing solution of (13).

Proof If \(Q \geq 0\) is the stabilizing solution to (13) then \(A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2)\) is asymptotically stable. Then by Lemma 12.1 of [14] there exists a unique \(\dot{P}\) satisfying (16), and \(\dot{P} \geq 0\).

So either we solve for \(Q \geq 0\), \(P \geq 0\), \(\dot{Q} \geq 0\) and \(\dot{P} \geq 0\), or we solve for the stabilizing solution \(\dot{Q} \geq 0\) (knowing that existence of \(\dot{P} \geq 0\) is then ensured), and for \(P \geq 0\) and \(\dot{Q} \geq 0\).

3.5 Controller Minimality

One of the themes of this paper is that even in the full-order case, the fourth Optimal Projection equation is needed. In this section we will give a further use for this \(\dot{P}\) equation, with regard to minimality of the full-order controller realization (17)-(19).

Firstly consider the pure LQG problem i.e., \(\gamma \to \infty\) so that [9, 1] \(I(H, \infty) = J(H, \infty) = C(H)\) and the LQG cost is minimized over all stabilized closed-loops. It is well-known that the LQG controller may or may not be stable and may or may not be minimal. To remove non-minimal states regardless of stability or not of the LQG controller, [16] proposed a balanced reduction method, called the BCRAM (Balanced Controller Reduction Algorithm, Modified). In fact the same idea can be applied to the \(H_\infty\) case, when we will refer to the method as \(H_\infty\)-BCRAM. It enables non-minimal states to be removed from the full-order combined \(H_\infty/LQG\) controller given in (17)-(19), even if the controller is unstable. The following lemma underlies the method.

Lemma 3.7 Definitions as in Theorem 3.4. Then the following hold:

(i) Suppose \(A_c + B_c C_2\) is asymptotically stable. Then \((A_c, B_c)\) is controllable if and only if there exists \(\bar{M} \geq 0\) satisfying

\[
0 = (A_c + B_c C_2) \bar{M} + \bar{M}(A_c + B_c C_2)^T + B_c B_c^T.
\]
(ii) Suppose \( A_c - B_2C_c \) is asymptotically stable. Then \((C_c, A_c)\) is observable if and only if there exists \( \hat{N} > 0 \) satisfying

\[
0 = (A_c - B_2C_c)^T \hat{N} + \hat{N}(A_c - B_2C_c) + C_c^T C_c.
\]

(21)

**Proof** Appendix B.

With reference to part (ii), note that since \( A_c - B_2C_c = A + Q(\gamma^{-2}C_1^T C_1 - C_2^T C_2) \), \( A_c - B_2C_c \) is asymptotically stable if and only if \( Q \) is the stabilizing solution to (13). This is the solution that Lemma 3.6 needs.

\( \mathcal{H}_\infty \)-BCRAM simply involves applying a similarity transformation to the plant \( P \) to make \( M \) and \( N \) equal and diagonal. Non-minimal states in the controller then show up as zeros on the leading diagonal of \( M \) and \( N \); such states may be removed from the controller realization to leave a minimal realization. It is easy to see that equation (16) is identical to equation (21), so \( \hat{P} = \hat{N} \), and we see \( \hat{P} \) playing a role again. In [2], a similar observation has been made on the role of \( \hat{P} \) in the LQG case of BCRAM.

Non-minimal states in the full-order combined \( \mathcal{H}_\infty \)/LQG controller can therefore be removed using \( \mathcal{H}_\infty \)-BCRAM by balancing \( M \) and \( \hat{P} \), and deleting balanced controller states corresponding to the zero eigenvalues of \( M \hat{P} \). Carrying this further, reduced-order controllers could be obtained by removing (observable and controllable) balanced controller states corresponding to the smallest non-zero eigenvalues of \( M \hat{P} \). However no \textit{a priori} guarantees are known, so such a controller might not even be stabilizing. A closed-loop balance-and-truncate method such as \( \mathcal{H}_\infty \)-balanced truncation [10], for which \textit{a priori} guarantees are available, is a more attractive method in this respect. Alternatively, if the extra computational complexity is not an issue, the reduced-order Optimal Projection equations of Section 3.3 could be solved.

4 Conclusions

By analysing the stationarity conditions associated with the combined \( \mathcal{H}_\infty \)/LQG control problem, we have obtained some insights into the Optimal Projection equations which characterize combined \( \mathcal{H}_\infty \)/LQG controllers. Existence of a positive semidefinite Lagrange multiplier was shown to be necessary for a minimized auxiliary cost and implies a strict \( \mathcal{H}_\infty \)-norm bound. Fortunately, existence of positive semidefinite solutions to the four Optimal Projection equations was shown to imply existence of a positive semidefinite Lagrange multiplier. All four Optimal Projection equations have a role to play here, even in the full-order case when one equation otherwise appears to be redundant. That equation was shown to be closely involved in the issue of controller minimality too.

The results of this paper are concerned with the 'equalized' \( \mathcal{H}_\infty \)/LQG weights case of combined \( \mathcal{H}_\infty \)/LQG control. That is, when the \( \mathcal{H}_\infty \)-norm bound and the LQG cost bound apply to the same closed-loop transfer function. Extension of the results of this paper to the non-equalized case, where the \( \mathcal{H}_\infty \)-norm bound is applied to a different closed-loop transfer function to that for the LQG cost bound, is an interesting open problem.
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Appendix

A Proof of Lemma 3.1

This proof, taken from [7], is included for completeness. To prove the first part of Lemma 3.1, suppose $\tilde{A}$ is asymptotically stable. If $\|\tilde{C}(sI - \tilde{A})^{-1}\tilde{B}\|_\infty < \gamma$ then by Lemma 2.2 there exists a stabilizing solution $Q \geq 0$ to (3). That is, $\tilde{A} + \gamma^{-2}Q\tilde{C}^T\tilde{C}$ is asymptotically stable. Then by Lemma 12.1 of [14] there exists a unique symmetric solution $P$ to (8), and this solution is positive semidefinite. Conversely, let $Q \geq 0$ satisfy (3) such that there exists $P \geq 0$ satisfying (8). Since $\tilde{A}$ is asymptotically stable, $(\tilde{C}, \tilde{A})$ is certainly detectable, and hence so is $(\tilde{C}, \tilde{A} + \gamma^{-2}Q\tilde{C}^T\tilde{C})$. Then Lemma 12.2 of [14] gives that $\tilde{A} + \gamma^{-2}Q\tilde{C}^T\tilde{C}$ is asymptotically stable. That is, such a $Q$ is stabilizing, hence by Lemma 2.2, $\|\tilde{C}(sI - \tilde{A})^{-1}\tilde{B}\|_\infty < \gamma$.

We may now proceed with the proof of the equivalence of parts (i), (ii) and (iii) of the lemma.

(iii)\(\Rightarrow\) (ii) Since $\tilde{A}$ is asymptotically stable, $(\tilde{C}, \tilde{A})$ is certainly detectable, and hence so is $(\tilde{C}, \tilde{A} + \gamma^{-2}Q\tilde{C}^T\tilde{C})$, where $Q \geq 0$ satisfies (3). Then Lemma 12.2 of [14] states that existence of $P \geq 0$ satisfying (8) implies that $\tilde{A} + \gamma^{-2}Q\tilde{C}^T\tilde{C}$ is asymptotically stable.

(ii)\(\Rightarrow\) (iii) Since $Q$ is stabilizing, $\tilde{A} + \gamma^{-2}Q\tilde{C}^T\tilde{C}$ is asymptotically stable. By Lemma 12.1 of [14] there exists a unique symmetric solution $P$ to (8), and this solution is positive semidefinite.

(ii)\(\Rightarrow\) (i) Let $Q_*$ be the stabilizing solution and let $Q$ be any other solution to (3). Subtracting their respective equations and rearranging we obtain

\[
0 = (Q - Q_*)(\tilde{A} + \gamma^{-2}Q_*\tilde{C}^T\tilde{C})^T + (\tilde{A} + \gamma^{-2}Q_*\tilde{C}^T\tilde{C})(Q - Q_*) \\
+ \gamma^{-2}(Q - Q_*)\tilde{C}^T\tilde{C}(Q - Q_*) .
\]

Since $Q_*$ is stabilizing, $\tilde{A} + \gamma^{-2}Q_*\tilde{C}^T\tilde{C}$ is asymptotically stable, hence Lemma 12.1 of [14] gives that $Q - Q_* \geq 0$. That is, $Q_*$ is minimal.

(i)\(\Rightarrow\) (ii) Let $Q$ be the minimal solution to (3) and let $Q_*$ be the stabilizing solution. Then by definition $Q_* - Q \geq 0$. We have already proved that (ii)\(\Rightarrow\) (i), so $Q_*$ is minimal too: $Q - Q_* \geq 0$. Thus $Q = Q_*$. \(\square\)
B Proof of Lemma 3.7

We only need to prove part (i); part (ii) follows by duality. Firstly note the well-known fact that \((A_c, B_c)\) is controllable if and only if \((A_c + B_cC_2, B_c)\) is controllable.

‘If’ Let \(\dot{M} > 0\) satisfy (20). Suppose that \((A_c + B_cC_2, B_c)\) is not completely controllable. Then by the PBH test (see Theorem 2.4-8 of [6]), there exists \(\lambda\) and a non-zero vector \(y\) such that

\[ y^*(A_c + B_cC_2) = \lambda y^* \quad \text{and} \quad y^*B_c = 0. \]

But then \(y^*(20)y\) implies \((\lambda + \bar{\lambda})y^*\dot{M}y = 0\), which implies \(\text{Re}(\lambda) = 0\) since \(\dot{M} > 0\). But this contradicts the asymptotic stability of \(A_c + B_cC_2\). Hence \((A_c + B_cC_2, B_c)\) is controllable.

‘Only If’ Theorem 3.3(4) of [5], tells us that controllability of \((A_c + B_cC_2, B_c)\) implies that there exists \(\dot{M} > 0\) satisfying (20) if and only if \(A_c + B_cC_2\) is asymptotically stable, which it is by assumption. \(\square\)

References


