REVENUE MANAGEMENT FOR OCEAN CARRIERS:
OPTIMAL CAPACITY ALLOCATION
WITH MULTIPLE NESTED FREIGHT RATE CLASSES

by

SPYRIDON A. MARAGOS
Diploma in Naval Architecture and Marine Engineering
National Technical University of Athens, 1987

Submitted to the Department of Ocean Engineering in partial fulfillment of the
requirements for the degrees of

Master of Science in Operations Research
and
Master of Science in Ocean Systems Management

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
May 1994

Author  Department of Ocean Engineering
March 1994

Certified by
Ernest G. Frankel
Professor of Marine Systems
Thesis Supervisor

Accepted by
Thomas L. Magnanti
G. Eastman Professor of Management Science
Co-Director, Operations Research Center

Accepted by
A. Douglas Carmichael
Professor of Power Engineering
Chairman, Ocean Eng. Departmental Graduate Committee

ARCHIVES
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
JUN 20 1994
© Massachusetts Institute of Technology 1994.
All rights reserved.
REVENUE MANAGEMENT FOR OCEAN CARRIERS:
OPTIMAL CAPACITY ALLOCATION
WITH MULTIPLE NESTED FREIGHT RATE CLASSES

by

SPYRIDON A. MARAGOS

Submitted to the Operations Research Center and the Department of Ocean Engineering
on April 1, 1994, in partial fulfillment of the requirements for the degrees of
Master of Science in Operations Research
and
Master of Science in Ocean Systems Management

Abstract

The success of the application of Yield Management to the seat inventory control of the
major Airlines, has forced several leading Liner Shipping companies to consider the use of
Yield Management tools for their capacity control and reservation systems. I have solved
the simple case of the two ports-two classes of goods problem. I have proved that the
solution of my formulation gives the same answer as the formula by Littlewood. I have
shown that the solution, is the capacity allocation that maximizes the expected revenues
of the vessel operator. I have expanded the model formulation to cover the two ports, M
classes of goods problem. I prove that any point that fulfills the necessary conditions for
unconstrained optimality, also fulfills the sufficient conditions for optimality. From this
observation we conclude that the necessary condition of optimality has a unique solution
that corresponds to the global maximum of the optimization problem. I show that the
necessary conditions for optimality derived here, are equivalent to the necessary conditions
of optimality derived by Brumelle and McGill. I further prove that the case of the two
ports M classes of goods problem, with stand-by cargo of low value, is a special case of
the above two ports M classes of goods problem.

Although, the model presented here is basically similar to the model by Curry, it is
notationally and computationally more compact. This compactness allows for the simplifi-
cation of the model and reduction of the integrals of multiplicity of up to M, to double
integrals. We simplify the model even further and we get a system of equations involving
only single integrals that represent the simplified necessary conditions for optimality.
Furthermore, I have developed a criterion for the acceptance or rejection of orders that
consist of more than one containers.

I also present an extension of the approximate model for the case of the N ports-M
classes of goods problem.

Thesis Supervisor: Ernest G. Frankel
Title: Professor of Marine Systems
Αφιερώνεται
Στούς Γονείς μου
Μαρία και Αποστόλη
Acknowledgments

I would like to thank my advisor Ernest G. Frankel for his guidance through out the project that resulted in this thesis.

Funding for this thesis, provided by the Neptune Orient Lines of Singapore, is gratefully acknowledged.

I would also like to thank Thanassis Tjavaras for the tutorials that he has offered me over the last few years on computers and related subjects.

Spyros Maragos
April 1, 1994
# Contents

1 Yield Management Problem  
   1.1 Introduction ................................................................. 13  
   1.2 Description of Yield Management Practice .............................. 18  

2 Literature Review  
   2.1 Airline Yield Management Review .......................................... 25  
   2.2 Hotel Yield Management Review ........................................... 38  

3 Nested Model  
   3.1 Two Ports, Two Classes of Goods Nested Problem ....................... 41  
      3.1.1 Model Formulation ...................................................... 43  
      3.1.2 First Order Optimality Condition .................................... 45  
      3.1.3 Second Order Optimality Condition .................................. 47  
   3.2 Two Ports, M Classes of Goods Nested Model ............................. 50  
      3.2.1 Model Formulation ...................................................... 53  
      3.2.2 First Order Optimality Condition .................................... 55  
      3.2.3 Second Order Optimality Condition .................................. 67  
      3.2.4 Conclusions .............................................................. 73  
   3.3 Nested Model with Stand-By Cargo .......................................... 74  
   3.4 Approximate Nested Models .................................................. 81  
      3.4.1 Approximate Nested Problem for Two Ports and M Classes of Goods 81
3.4.2 Approximate Nested Problem for N Legs and M Goods ....... 91

4 Nested Model: Results ........................................ 97
   4.1 Numeric Solution of the Nested Problem ................. 97
   4.2 Input and Output of the Computer Code .................. 99
       4.2.1 Input of the Computer Code ......................... 99
       4.2.2 Output of the Computer Code ....................... 100
       4.2.3 Revision of the Optimal Capacity Limits .......... 102
   4.3 Numerical Results ........................................ 104

5 Dynamic Programming Approach ................................ 111
   5.1 Two Ports, Two Classes of Goods Problem ............... 111
       5.1.1 Dynamic modeling formulation ....................... 113
   5.2 Conclusions and Recommendations ........................ 126

Bibliography .......................................................... 131
List of Figures

1-1 Vessel Itinerary example .................................................. 17

3-1 Illustration of the Meaning of $\varphi_i$ for the case $M = 5$ ................ 51
3-2 Marginal revenue and optimal capacity allocation .......................... 66
3-3 Input and output of the yield management optimization model .......... 89

4-1 Computer Code Input-Output Diagram .................................. 101
4-2 Criteria for the Running of the Program .................................. 103

5-1 Arrival Pattern for Low- ($z_1$) and High-Freight-Rate ($z_2$) Customers ..... 113
5-2 Definition of $W_t(i)$ for integer and non-integer values of $i$ .......... 119
Chapter 1

Yield Management Problem

1.1 Introduction

The Container cargo tariffs are invariably value based. As a result, the price paid by a Container of high value cargo is greater than the price paid by a similar container filled with low value cargo. The vessel operator accepts orders from the day the itinerary is announced until the day the vessel departs, unless the capacity of the ship has been reached before that day.

The objective of the ship operator is to maximize his profits. If we assume that the voyage costs are fixed, then the operator can maximize his profits by maximizing his revenue.

The agents of the operator are being compensated on a value basis. The agent maximizes his own profits when the vessel maximizes its revenues. Both the operator and the agents ask for the maximization of the ship revenues. As a result, the objectives of the agents and the operator are compatible.

The problem can be described as follows: The operator announces the itinerary adequate time before departure and he waits for the orders of the shippers. The shippers cover a wide range of cargo value. For example at the same itinerary we have containers loaded with computers, as well as containers loaded with small quantities of semi finished
materials and crude commodities.

As we have already mentioned, the tariff structure is given and for the time scope of our problem, it can be consider as constant. As a result, the operator does not have the option of negotiating the prices with the customers. Therefore, the operator does not have the ability to increase his revenues by changing the prices. Each customer pays a freight rate equal to a percentage of the value of the cargo he ships. This percentage is constant across the board.

The operator has limited capacity to offer, and this capacity can be easily exceeded by the demand for transportation. Therefore, he can increase his revenues by being selective with reference to the customers he accepts. An obvious choice for the operator is to offer capacity to the high value-high freight rate customers, and turn down other shippers with low value cargo. In this case, the shippers who are turned down, ask other vessel operators for capacity. The assumption is that they are lost revenues for our operator. Whether they are offered transportation capacity by other operators on not, they never return to our operator to ask again for capacity. That does not present a problem for the ship operator if he is able to fill his vessel to capacity with high value customers. Nevertheless, this is not always possible and the operator, has to weight the benefits of a success of his policy, when the vessel departs with full load, against an unsuccessful implementation of his policy when the low value cargo

has been turned down and the ship leaves port with cargo below its capacity. And that can happen when the operator does not get enough orders from high value customers that would enable him to fill his vessel.

An additional feature of the liner booking pattern is that the low value cargo asks for reservations enough time before departure, whereas the high value cargoes are last moment shoppers. From the description above, it seems that the problem at hand is similar to the yield management problem of the airlines. The major similarity is the pattern with with the customers arrive. The high fare customers start asking for transportation capacity after the low fare customers have decided on their transportation means and they do not
ask for transportation any more. Capacity that has not been booked does not create revenue for the operator.

Despite the similarities between the Containership and the Airline yield problems, the Containership yield problem can be more complicated. One source of complication is the number of the ports that the containership includes in its schedule. The number of ports a Containership visits is greater than the number of airports that an airplane includes at its itinerary.

A further complication arises from the time span that the trip of the containership covers relatively to the time it takes from the announcement of the trip to the departure of the vessel. An airline typically announces the flight two months before it takes place. The airplane visits the two or more airports of the itinerary within at most twenty-four to forty-eight hours from departure from the airport of origin. Consequently, we can argue that the customers who belong at the same customer class (i.e. high-value) ask for transportation the same period of time, whether the origin of their flight is the first airport or an intermediate one. On the contrary, when we consider the Containership problem we could have a difference of i.e. twenty days between the time the vessel sails from ports A and the time it sails from a following port B (see Figure 1-1). The week proceeding the departure of the vessel from port A we have offers of high-value cargo for transportation from port A to port D.

At the same time the operator accepts offers for transportation of goods from port B to port C. It is almost a month before departure from port C, and the offers that the operator gets for transportation from port C to port D, are offers for low-value cargo. The operator is offered at the same time high value cargo for transportation from port A to port D and low value cargo for transportation from port C to port D. These two classes of goods are competing for the same capacity constraint (namely the capacity constrain at the leg C-D). At the same time the operator has to reserve some space for the high-value cargo originating at port C with destination D. We, therefore, see that although the problem is nested with reference to some of its variables, it is not so, when it comes to
other variables.

An additional reason of complication is the lack of a clear origin and a final destination for the Liner. The liner visits the consecutive ports of the itinerary and every port of the itinerary could be considered to be the origin of the itinerary (rolling horizon problem). The capacity available thought, is reduced by the capacity that is already occupied from goods transported from the previous ports and the capacity that has been committed for transportation from the following ports. The horizon (i.e. the number of ports) over which we want to optimize depends on the confidence the operator has in the estimation of the demand at the following ports.

Additionally, the nature of the flow of the demand for the Shipping Liners is different than the flow of the demand for the Airlines. In the case of Airlines, the demand between two cities is balanced on average. The passengers who leave their city to visit an other have to return somehow to their home. There are some imbalances before and after major holidays (Thanksgiving etc.), but they are not characteristic of the year round nature of the business.

On the other hand the demand for capacity on the one of the legs of the trip of the Shipping Liner is usually stronger than the demand for the returning leg of the vessel. That imbalance of demand at the two legs of the trip of a vessel is directly proportional to the trade imbalance between the two region that are the hinterlands of the two ports. For instance the demand for capacity on the incoming leg from Japan is much stronger than the demand for capacity at the outgoing leg. This characteristic of the international trade that the Shipping Companies have to live with give a greater relative significance of the application of yield management at the case of the Shipping Liners.

In other words, when the shipping liners consider placing a vessel at a certain itinerary with imbalanced demand, they try to capture the demand of the “strong” leg. This demand is more profitable not only because it has a larger volume to offer but also because the unit value of the products transported in this direction is higher. As a result, the operators employ ample capacity that allows for them to capture the demand at the
One week before departure from Port A.

Twenty-seven days before departure from Port C.

Figure 1-1: Vessel Itinerary example
“strong” leg, whereas their capacity is underutilized at the other direction.

We have stated above that volume capacity is a constraint at the strong leg of the trip. As expected, the capacity is not a constraint at the weaker or returning leg of the trip. The nature of liner shipping is such that usually the cargo traffic at the weak leg consists of items that are of lower value, compared to the goods that are transported over the strong leg, and often times they are heavier. It is not unusual for a vessel on the returning leg of the trip, to have reached its dead weight limit before it has exhausted the available capacity. In cases like that the vessel is forced to decline extra cargo, exactly because they have reached the allowed carrying weight capacity.

We have therefore stated that, we always have two kinds of constraints. The one constraint, is the volume or the number of containerslots offered by the vessel. This is the constraint that we are more concerned about because the nature of the containership cargo is such that most of it is relatively light and occupying large volume. As a result, the volume constraint is the constraint that is reached more frequently and therefore of major concern. Next to it we have a weight constraint that is reached often enough at the returning leg of the trip. The control of the weight at the weak leg of the trip is a rather subtle form of yield management not really understood by many Shipping Liners.

Both areas of capacity and weight control should be taken into consideration during the booking process of the shipping companies. In addition they should do management of the available space for specialized containers and other equipment.

A final reason of potential complication is the large number of different fares that are the result of all the possible combinations of cargo class and origin destination pairs.

1.2 Description of Yield Management Practice

The experience with yield management tools is mainly concentrated in the area of the airline industry. Many shipping company executives wonder weather the nature of the shipping business is related closely enough to the nature of the airlines for the lessons
1.2. *Description of Yield Management Practice*

learnt in the practice of the airlines to be valid for shipping companies too. Furthermore, are the characteristics of the operations of the two different industries close enough for the transfer of experience and expertise between the two areas to be fruitful? Can the models of the airline yield management be useful to the vessel operators?

At that point we will describe briefly the operations of a typical airline and compare them against the operations of a typical shipping line.

**Airline Yield Management Problem**

A seller can maximize his revenues, and consequently his profits when he causes the market to segment, so as to be able to charge each segment of the market as much as this segment of the market can bear. A successful implementation of the segmentation of the market does not "allow" customers who belong in segments of the market that are able to pay a higher price for the service or product, to purchase the service or the product that is offered at a lower price at the lower end of the market.

The airline industry implements the above principle as follows: They start accepting reservations for a flight, as soon as the schedule is announced. They offer a variety of fares that go with certain constraints. The more constraints they impose on the fares, the cheaper the fares are. Some of the classes of the fares they offer are so low that they cover only the variable cost of the passenger, and in some they are unable to sustain the overhead costs that correspond to the resources they utilize. It is of no surprise that these fares are the most restrictive. The more usual form of constraint is the date of the purchase of the ticket. The further away from the departure day the purchase takes place, the lower the fare is. The mechanics of this strategy work as follows:

The higher end of the airliner market, namely the business travelers, usually travel on a short notice. As a result they do their reservations for a flight a few days before departure. Since the demand of the business market for airline transportation is relatively inelastic, the airlines are able to charge business travelers heavily. This segment of the market, that does the late reservations is ensuring the profitability of the airlines. Not
all of the capacity offered by the airlines can be covered by customers belonging at this segment of the market. That means that if the airlines want to increase their revenues they have to offer the remaining of their seats at prices affordable by the other segments of the market. The operator has to make sure that the prices at which he offers the airline seats to the low end of the market are high enough to cover at least the variable cost of these seats. The overhead cost has been covered by the fares the high end of the market has paid.

The travelers who belong to the lower end of the market, namely backpackers, students, seniors etc., do not have the need or the economic ability to travel frequently, and they are able to make their travel plans far in advance. As a result, they know their travel needs well ahead in time, and they can purchase their tickets long before departure.

Therefore, the pattern for airline reservations emerges as follows: The low end of the market has the ability and it is encouraged by the constraints of the low fares, to make their purchase well ahead of time. The passengers belonging at the upper end of the market know their travel plans only a short time ahead of departure. Since they are able to pay the increased fares charged to them, the operators are able to recover their costs and make their profits.

The operators face a problem that is a result of their booking policies. The lower end market is much more populous than the high end of the market and the demand for seats offered at low prices outstrips the capacity the operators can offer. If there were no further constraints imposed, the airplanes could end up being filled with low end customers who pay enough to cover only their variable costs. In this case the operator would be burdened with the deficit of the overhead costs. The operators have solved this problem by imposing restrictions on the number of the seats they offer at the discounted rates. The next problem, is to define the maximum number of the seats they offer to the low end customers so as to maximize their revenues.

The airline operators have segmented their market by imposing restrictions on the date of the purchase of the sale of the ticket. They have also imposed other restrictions with
1.2. Description of Yield Management Practice

reference to the length of stay, the days of the week the traveler can fly etc. As a result they have created the pattern where all the low end customers come enough time before departure and they are given seats, up to a certain number predefined by the operator and the remaining the seats are preserved for the high end customers.

Shipping Liner Yield Management Problem

The Liner Conferences discriminate with reference to the value of the cargo of the container and this distinction would be enough for the markets to be segmented. Nevertheless, it is true that the lower end cargo asks for transportation capacity earlier than the high end cargo. A possible reason that forces the low value customer to seek transportation capacity long time before departure might be routed into the fear that the limited capacity offered to the low end customers is not enough to satisfy all the demand for low value cargoes and therefore early action would be appropriate.

The fact, though, is that the timeliness of the purchase of the transportation capacity is not a factor affecting the price paid by the low value customers. A low value cargo shipper will pay the same price irrespectively of whether he asks for capacity two months or only a week before departure. Therefore, the pattern that we observed in the case of the airlines is not very strict in the case of the Liner Conferences. Nevertheless, the vessel operator is faced with a booking pattern that is similar to the booking pattern an airliner operator is faced with. The low fare customers in both cases, ask for transportation far in advance, whereas the high fare customers ask for it shortly before departure.

There are already a few Ocean Shipping Companies that use yield management techniques in areas of their operations. Usually, they apply yield management on their refrigerated cargo traffic. Nevertheless, there is a concern among Shipping executives that the Shipping Liner business is an environment where the assumptions of Airline yield management do not hold, and its practices can not be applied. In other words, the relative size of the average customer of an Airline is much smaller than the relative size of the average Shipping Liner customer. Therefore, the short term revenue maximization, that can very
well guarantee long term profitability for an Airliner, does not necessarily explain, it is argued, how the Shipping Company that is bound by contracts and has to deal with a few big customers that have negotiating power, can profitably apply yield management to their operations.

Furthermore, Shipping Companies value service contracts in key trades and they feel that the application of yield management does not address strategic concerns of the Company. In addition, the shippers who ask for the services of a Shipping Liner company in the form of service contracts, prefer low prices and freight rate stability. Customers like that, cannot relate to the capacity utilization problem of the operator, and they be alienated if they perceive the yield management practices of the operator for poor treatment.

An answer to the above questions and worries of the shipping companies is that yield management can be applied in two faces. The first face of yield management application allows for application of yield management and capacity and equipment control, for the part of the vessel capacity that the operator is not bound by contracts to have available to any of the few big shippers. This is the part of the vessel that is available to the smaller shippers who do not generate enough transportation volume to have contracts with the operator. The relation between these shippers and the operator is similar to the relation that an airline has with its customers. As a result, the application of yield management at this segment of the market could result in increased profitability.

The second face of yield management application can start when the shipping company, with the application of yield management tools (and their booking databases) estimates what is the contribution of the big customers to the bottom line of the company. These estimations can serve as an internal tool for the company to negotiate the new contracts with the big customers. In this second face, yield management can serve as an analytical tool to access individual transactions and contracts, and figure ways to make them more profitable for the Carrier. Yield management/capacity control technics are used by Shipping Liners. Sometimes, they are used under different names. Some other times they are used in a different form. Often times they are not very sophisticated. Often, Shipping
1.2. *Description of Yield Management Practice*

Companies that are members of the same Conference agree to withhold voluntarily freight carrying capacity from the market. This kind of capacity control is not a sophisticated but it is a form of yield management, nevertheless.

Airlines operate in a deregulated environment (at least in the U.S.), whereas the Shipping Liners operate in a regulated environment with many of the executive decision made at a Conference rather than Company level, with pricing being the most characteristic among them. Price inflexibility is an inherent characteristic of Shipping Conferences. Although every conference member has the right to independent action on any tariff item, Shipping Conferences try to work towards greater cohesion on pricing and not many of the operators look forward to start price wars with their fellow Conference members.

The above arguments, are essentially arguments in favor of the application of yield management in shipping. Airlines have the option of controlling their prices, and regulating their capacity availability through yield management. On the other hand, since Shipping Liners operate in an environment where capacity control through pricing is not always feasible, they have to use yield management technics as a straightforward way of controlling their capacity and their equipment allocation.
Chapter 2

Literature Review

The Yield Management Literature on Shipping Liners is virtually non existent. As a result, the following literature review will primarily be a review of the existing literature for Airline Yield Management. A brief review of the Hotel capacity control literature is also offered.

2.1 Airline Yield Management Review

Before 1970 most of the work done in the area of capacity control for Airlines had focused in the direction of the maximization of the aircraft load factors. By and large, maximization of the load factor would mean maximization of the revenues of the operator. As a result, the research of the operators was in the direction of overbooking.

Since then, changes in the structure of the airline fares and the introduction of multiple fares, the optimizing criterion has shifted from the load factor maximization to revenue maximization. This change had an obvious result. The scope of the research conducted on revenue management broadened in order to include seat allocation to the different fare classes of customers.

The research started with the simplest form of seat allocation, the one leg trip (non-stop flight between two cities). From then on the research included methods of Operation
Research techniques. Mathematical Programming and Network flow analysis were used in order to include all the possible passenger itineraries and the fare classes.

In 1972 Littlewood [1] introduced the model of a flight with one leg and two classes of customers. The aircraft was assumed to have remaining capacity $C$. The basic assumptions that Littlewood made, assumptions that are kept on in the later literature are:

- The low revenue customers ask for transportation capacity before the high revenue customers
- The demand for one class of customers is independent from the other
- The probability distribution of arrivals is a continuous distribution
- There are no cancellations of realized bookings

Littlewood, also introduced the concept of the expected marginal seat revenue, and he used is as his criterion for the rejection or the acceptance of a reservation from the class with the lower fare. When a lower fare customer asks for a reservation the revenue from this customer is equal to the fare $f_L$. This reservation decreases the available seats (capacity) by one unit. Alternatively this seat is not given to the customer from the lower paying class, but instead is reserved for the last customer from the high fare class (it is easier to think of the last rather than any other high fare customer). It is an implicit assumption that once we close our capacity to the low fare class we do not open it again. In other words, if we decline a booking from the low fare class, we will expect to accept only high fare customers until the end of the booking process. In this case, the seat that we have left empty now will be used only if we have the arrival of more or equal than $C$ higher paying customers. If we reserve the seat for the high fare customer, the expected revenue is

$$f_H \cdot P(x_H \geq C)$$

The above formula states that the revenue from a particular seat, if we reserve it for the higher paying is equal to the fare that the higher fare class pays multiplied by the
probability that this seat will be used. As a result, Littlewood concluded that the operator should accept the lower revenue class customer if the marginal seat revenue from this customer is greater (or equal) than the expected marginal revenue from a future high revenue customer.

\[ f_L \geq f_H \cdot P(x_H \geq C) \]  

(2.1)

In conclusion, Littlewood suggests that the revenue is maximized when the operator accepts low fare passengers up to the point where the probability of selling all the remaining capacity to high fare customers is equal to the ratio of the low fare to the high fare \( f_L / f_H \).

Several authors have worked on Littlewood’s model and have proposed several variations of the original model. Bhatia and Parekh [2] in 1973 proved that Littlewood’s criterion for capacity allocation is the condition that maximizes the revenues. This theoretical result was the main contribution of their work.

Richter [3] in 1982 introduced the concept of “differential revenue”. He defined it as the difference between the revenue realized from selling a seat to a low fare customer and the expected revenue from the same seat had it been kept for a high fare customer. It was Richter’s turn now to find that Littlewood’s allocation criterion is in fact the revenue maximization criterion for the two fare classes, single leg problem that Littlwood presented.

An observation that we can make here is that the seat allocation proposed by equation 2.1 is only a function of the demand for the high revenue customers. It is independent from the demand for the low fare customers. That means that weather the demand for the low fare class is unlimited or weak, the number of seats that we preserve for the higher paying customers is the same. The intuition behind this result can be found in the above analysis. If the low revenue customers ask for seats before any of the high fare customers, then by turning down a low fare customer, we implicitly mean that we do the same to all the low fare customers who ask for transportation between that moment and departure. The observation that we mentioned, is a direct result of the assumptions of the model.

The reservation process is a dynamic process, and the criterion of Littlewood, along
with the formulations of the problem by the above mentioned authors reveal that their formulation is a static one that ignores the dynamic aspect of the booking process. In practice, this theoretical shortcoming of the above models can be overcome if the operators update "constantly" their accessment of the expected demand for high fare customers (as well as the low fare customers), and recalculate the optimal capacity allocation.

Mayer [4] has tested the sensitivity of the main assumptions of the Littlewood model. He assumed that low fare customers book first in each of many periods before departure. His analysis suggested that the smaller the ratio \( \frac{f_c}{f_H} \) is, the more sensitive the expected revenue from the flight is to the non-optimal allocation of seats. Mayer showed that a heuristic application of Littlewood’s formula results in only a slight revenue decrease, relatively to more complex optimization methods.

Titze and Griesshaber (1983) [5] have used simulation for the same one leg problem with the basic assumptions that have been stated earlier. Their work shows that in practice the strict booking sequence can be relaxed somehow without any major reduction of the optimal revenues.

Buhr in 1982 [6] examined a multi-sector flight where the demand on certain flight segments competes with the demand on other flight segments for seats on the aircraft. This study considers only one fare class. Buhr proposes a criterion according to which the seat allocation should be done in such a way that the difference between expected revenues should be minimum. An iterative solution method was used to find the optimal values of the capacity allocation.

With reference to seat allocation between different fare classes on the same itinerary, Buhr suggested a two step approach. At the first step he finds the seat allocation for every itinerary. The second step is to allocate the capacity given to each itinerary, to the different fare classes that use this itinerary. Although, Buhr describes the process, he does not implement it. Furthermore, he offers no proof for his criterion nor does he generalize it for an itinerary with more than two legs.

Wang [7] (1983) extended Buhr’s model. His goal was to model the Yield Management
problem in the case of an itinerary with multiple legs and multiple fare classes. According to his model, a particular seat on a multi-leg itinerary, if necessary, can be allocated to several different origin-destination and fare combinations. He developed a method for the maximization of the expected marginal revenue of each seat across the multi-leg flight path. His suggestion was that the revenue is maximized when the seat is allocated to the origin-destination and fare combination that offers the highest expected marginal revenue. As a result, Wang suggests that we compare the expected marginal revenue of all the origin-destination and fare combinations, and reserve the seat for the combination with the highest marginal revenue. Although this method can be applied to relatively small networks, with few fare classes, it is not very practical to be used when we have a multi-leg multi-class real life application.

Hersh and Ladany [8] (1978), developed a sequential decision tool for an airline reservation system, for an aircraft flying on a two leg itinerary. In their model, the airline has the right to transport passengers over either the first or both legs of the trip. It cannot board passengers for only the second leg of the trip. This is a realistic situation often encountered in international flights, as a result of bilateral regulatory agreements. In their dynamic programming formulation, the authors have given consideration to the effects of waiting lists, stand-by customers and overbookings. They further incorporate a Bayesian reassessment of probabilities in the decision process. Despite the above realistic representation of the booking process, the authors restrict the fare classes to just one. As a result, the long haul and the short haul passengers do not compete for the same capacity, in the sense that the long haul customers are always given priority over the short haul customers. The Bayesian reassessment of probabilities, despite its theoretical intrigue, is doubtful if it can be useful (it needs an extended database to become meaningful), and it is questionable if the improved results are worth the added computational intensity. Furthermore, the proof of the inherent complicacy of their approach is the fact that the authors ran an example with aircraft capacity equal to just 6 customers. Nevertheless, we have to keep in mind, that the authors ran their examples back in 1978 with computers
of equivalent capabilities. Anyhow the work by Hersh and Ladany is valuable for the questions they raise, rather than the answers they give.

Glover et al. [9] (1982) formulated and developed the capacity control problem as a large network flow problem. The scope of the authors was to maximize the revenue of all the itineraries of an airline network. The task of the model was to find the optimal allocation to the different origin-destination and fare combinations over the network of a particular airline. The system built for Frontier Airlines, based on the above model, could handle a network of up to 600 flights and 30,000 passenger itineraries (origin-destination combinations) with up to five fare classes per passenger itinerary. The number of the side constraints of the model would be at the range 1,800-2,400. The side constraints were the aircraft capacity limits and the deterministic demand limits.

Wollmer [10] developed a large scale multi-leg multi-class network model into which he incorporated the probabilistic demands of the different classes. Each origin-destination and fare combination is represented as a binary variable. The objective of the model is the maximization of the expected revenue of the airline from the operation of the whole airline network. Wollmer's probabilistic formulation, combined with the binary representation of every variable of the model produce an extremely large problem, that can be unmanageable under non favorable conditions. Wollmer argues [11] in a subsequent paper, that despite this potential difficulty, only a small subset of the variables needs to be considered at any particular time. As a result, the solution to the general model can be found by solving a shortest path problem on a number of relatively small networks.

Dror et al. [12] present a formulation of the airline yield management problem that emphasizes passenger cancelations, multi-leg flights and a rolling planning horizon for the Airline. The authors present several network formulations in an order of increasingly realistic assumptions and therefore complication. The objective is to obtain flow (revenue) maximization. The first model they present is a simple network flow problem of a single flight with several intermediate stops. The basic formulation is expanded in order to include many flights. This second network model is a structured version of a maximal
flow, multi-commodity problem with sources and sinks at the nodes of the network.

The model can be implemented for relatively simple cases, when no switch-over passengers are allowed. Incorporation of the switch-over passengers into the model, transforms the network into a more complex multi-commodity flow problem with additional constraints. The authors do not address the following questions (i) the calculation of the parameters for this type of network, and (ii) presentation of the algorithms necessary for the solution of such network problems.

Belobaba [13] (1987) presented a heuristic approach to the one leg multi-class booking problem with ordered arrival of the customers according to the fare class where they belong. The heuristic is called the Expected Marginal Seat Revenue Method (EMSR) method. The objective is to calculate the number of the seats the operator must preserve for the higher fare classes, by protecting them from the lower fare classes. At each stage of his method, Belobaba calculates the number of seats he has to preserve for a fare-class over all the lower fare classes. According to the assumptions of the nested yield management problem lower fare classes ask for transportation capacity earlier than high fare classes.

Belobaba separates each stage into several steps. At every step he calculates how many seats he has to preserve for the higher fare class by protecting them over each one lower fare class. In order to find how many seats should be protected for a higher fare class over a lower fare class, the author treats the two classes in isolation, and he applies Littlewood’s methodology. By adding up the number of the seats that should be protected over all the lower fare classes, he finds the total number of seats that should be protected for the higher fare class. Belobaba repeats the same algorithm for all the classes of the flight, and he finds the protection levels for each one of them.

Littlewood gave the accurate solution to the one leg, two fare classes problem under a set of assumptions that became almost synonymous to the yield management problem for airlines. Other models have extended Littlewood’s model to include more than two nested fare classes. Under the assumption of the independence of demand for the nested classes of customers, the above generalization of Littlewood’s model has been solved independently
by Curry [14], Brumelle and McGill [15] and Wollmer [16].

In his work, Curry assumes continuous distributions of demand for the different fare classes and he has expressed the expected revenue from each fare class in the form of a series of recursive integral equations. These recursive equations suggest convolved multiple integrals of order up to the number of the fare classes of the model.

Curry formulates the problem as an optimization problem with non-linear objective function, and linear constraints. The non-linear objective function is the sum of the expected revenues from every fare class. Each one of the expected revenue expressions is formulated like the above mentioned convolved multiple integrals. The value of each one of the integrals is a function of the protection levels of all the lower fare classes and the class itself. The linear constraints of the problem are the capacity constraints of the vessel.

Curry's model is an accurate formulation for the multi-class, one leg problem, where the nesting of the classes is clear. Curry extends the model for the multi-class, multi-leg problem. That problem does not fulfill the basic assumption of the nested ordered fare classes. In other words, a businessman who books a seat shortly before departure in order to fly from San Francisco via Minneapolis to Boston, and a businessman who books a seat shortly before departure in order to fly from San Francisco to Minneapolis, on the same aircraft, book their seats at around the same time, compete for the same capacity and they do not belong to the same fare class.

The above example shows that the assumptions of Curry's (and Littlewood's) model are not met in the case of multi-leg itineraries. Nevertheless, Curry mentions that his multi-leg model is accurate only when the different origin-destination pairs do not share the same seat inventory. A practical problem that we encounter if we attempt to find the optimal capacity allocation using the model by Curry, is that we will have to calculate the multiple integrals (or their derivatives) of the objective function, at each step of the iterative nonlinear optimization method.

Brumelle and McGill [15], work on the one leg multiple fare classes problem, with
the usual airline yield management assumptions. The formulation of the model is done through dynamic programming. The major contribution of the paper by Brumelle and McGill, is the proof that the fixed-limit booking policies are optimal within the class of all admissible policies that depend only on the observed number of current bookings. The proof shows that the policies of protection levels for the higher fare classes studied by Littlewood and the other authors are optimal. The formulation by Brumelle and McGill accepts either discrete or continuous demand distributions.

Through subdifferential optimization within a stochastic dynamic programming framework, the optimality conditions are reduced to a set of probability statements that become equivalent to the Littlewood optimality condition for the simple case of the two fare classes. Despite the simplicity of their formulation and their theoretical value, the usefulness of the optimality criteria is hindered by the same drawbacks that limit the work by Curry. The authors consider just the case of the one leg itinerary, but the optimality criteria involve calculation of multiple convoluted integrals (or multiple convoluted sums in the case of the discrete demand distributions). These integrals are the same as the integrals in the model by Curry. The reservations that were expressed before about the usefulness of the Curry model, are expressed here as well.

Wollmer [16], assumes that the demand is described by discrete rather than continuous functions and he reaches results that are similar to the results by Curry, and Brumelle and McGill. Wollmer presents the optimal booking limits as decreasing functions of the fare price and increasing functions of the available capacity. He also presents an algorithm for the calculation of both the optimal protection levels and the optimal expected revenue. Brumelle and McGill as well as Wollmer present simulations of both their methods and EMSR method by Belobaba. The comparison of EMSR against the respective methods shows that although EMSR is clearly a suboptimal method, the slight revenue improvement (in the range of 1%) obtained by the Brumelle and McGill or Wollmer methods, does not always justify the use of the later methods over EMSR.

Brumell et al. [17], examine the two fare classes-one leg flight problem, and they
relax the assumption of the statistical independence between the demands for the two fare classes. They allow the unconditional probability demand of the higher fare class to become conditional on the fact that the lower fare class bookings have exceeded a booking limit, and they generalize the optimal criterion derived by Littlewood. The authors remark that:” With small cabin capacities relatively to demand, the booking limits in the dependent case are the same as those in the independent case, as there is no revenue benefit from taking dependency into account”.

The success of Yield Management depends on the implementation of the following routine:

- Demand forecasting for all the Classes of Customers
- Optimization of the Booking limits for each and every Class of Customers
- Revision of forecasts and booking limits as departure approaches

We therefore see that the development of good statistical methods that can adequately describe the reservations demand distributions is a prerequisite for the optimization process to be successful.

The derivation of the Bivariate statistics for the two classes of customers case is more challenging than the derivation of two independent distributions, one for each of the two classes of goods. That means that we need more data in order to derive the bivariate demand distribution for the two fare classes.

The model proposed by Brumell et al. tries to incorporate dynamic elements with the introduction of the correlation between the two classes of customers. It is, nevertheless static in its nature. The user of the model would always have to update periodically the estimation of the demand and the booking limits for the model. It is doubtful whether the application of the method suggested by Brumell et al. can be effective in any aspect except for the decreased frequency with which the user would have to update the model results.
2.1. **Airline Yield Management Review**

Robinson [18], relaxes the assumption of the monotonic increase of the fare classes arrival. His model allows for the fare classes to arrive in an arbitrary order. Robinson uses the optimality conditions to show that under the optimal booking limits, the ratio of the “current” fare to the highest remaining fare, is equal to the probability of filling the aircraft. Using this observation, Robinson shows that one can find good approximations to the optimal booking limits by employing Monte Carlo integration.

Williamson [19], focuses on the model of an airline itinerary network. The author introduced a nesting method based on shadow prices. The shadow prices are the change in revenue if one additional unit of capacity (seat) is allocated to a given origin-destination and fare class combination. In the case of deterministic demand these shadow prices are the shadow prices for the demand constraints of the given origin-destination and fare class combination.

As a variation of the above method, Simpson [21], developed a similar approach with shadow prices that pertain to the capacity constraints, rather than the origin-destination and fare class combinations. In Simpson’s work it is the shadow prices that determine which origin-destination and fare class combinations should be accepted or not, and the level at which the booking limits of the different combinations should be set.

Pfeifer [22], studies the one leg two fare classes problem. In an interesting departure from the typical airline yield management assumptions, he assumes that both classes of customers come from the same pool. He assumes that after the low fare class exhausts the space allocated to it, the customers who wanted to purchase low fare seats, accept to buy a high fare seat with some probability \( p_1 \).

There is one weak point in Pfeifer’s analysis. It has to do with the fact that Pfeifer wants to find the probability that the \((q+1)\)th customer (where \(q\) is the maximum number of low fare customers that the operator accepts) is a customer who will agree to a sellup and buy a high fare ticket. In order to find that probability one would need a definition of the demand as a function of time, whereas Pfeifer treats the problem as a static one that is updated periodically.
Furthermore, Pfeifer gives a solution to the airline yield management problem as an analogy of the newsboy inventory problem. He does not elaborate on the similarities of the two problems and why the solution to the newsboy problem could be used with no major modifications, to find the solution to Pfeifer's version of the yield management problem.

Alstrup et al. [23] present an overbooking model for a fixed non-stop flight with two types of passengers. Their model includes reservations, cancelations, as well as "no shows" (passengers who fail to arrive for the flight without notice). The authors incorporated denied boardings and downgrading of passengers. The model presents the airline booking process as a Markovian non-homogeneous sequential decision process. The model is solved by a Stochastic Dynamic Programming formulation. The model presented in the paper by Alstrup et al. is very similar to the two-variable Dynamic Programming model developed by Hersh et al. ([8]) and described earlier in the current literature review. The only difference between the two models is the initial conditions in the model by Alstrup et al. which takes into consideration up-grading and down-grading of passengers. The objective of the authors was to minimize the difference between the maximum expected potential gain and the actual gain. Several adjustments with reference to the grouping of seat reservations and the reduction of the range of the state variables was able to make the running time of the model short enough so that it could be applicable in airline yield management practice.

Weatherford et al. [24], feel that the Yield Management problem should be renamed Perishable-Asset Revenue Management (PARM) in order to describe more accurately the nature as well as the goals of the problem. They present a taxonomy (classification scheme) and a research overview of the Yield Management problem. They analysed situations in which Yield Management has been practiced, and they concluded in some common characteristics for all Yield Management applications. Those common characteristics are:

- One date on which the product is available and after which it is either not available or it ages.
2.1. *Airline Yield Management Review*

- A fixed number of units.
- The possibility of segmenting price sensitive customers.

The authors discuss the different possible management objectives from the application of yield management, and they conclude that in most cases, a risk neutral management would use the criterion of maximizing expected profit.

An interesting classification of fourteen distinguishing elements is presented. Some of these elements pertain to the nature of the specific yield management problem or situation. Some characteristics depend on the decision maker, and others on the formulation of the model and the assumptions of the booking process.

A review of the published research on Yield management problems is presented. Each variation is described briefly and then the optimal rules, along with the literature for this problem are presented.

At an other paper, Weatherford et al. [25], present a model for the customer arrivals pattern, which is presented as a non-homogeneous Poisson process. The model allows evaluation of different decision rules for yield management situations. It is also used to derive the probability distributions that are necessary for the operational implementation of the optimal decision rules for yield management problems with diversion and two classes of customers.

The authors present and compare several heuristic approaches to the optimal acceptance rules. The improvement to the expected contribution is the main criterion for the usefulness of the approach. Finally the sensitivity of the different models to the changes of the model's input parameters is examined. The paper focuses heavily on the airline models, data and experience.

Lee et al. [26], develop a discrete time dynamic programming model for finding an optimal booking policy, that would ideally be reduced to a set of critical values. The formulation of the model is valid for any pattern of arrivals and not only the usual ordered arrival pattern of the airline models. Multiple seat bookings are also incorporated in the
model. The basic properties of the model are studied. An attempt for the derivation of a demand probability distribution as a function of time is presented.

The authors give some limits for the case of multiple bookings. We have many bookings when one ordering is placed for several bookings. The operator either accepts them all or none. The option to accept only a few of them is not possible.

The limits for multiple bookings given by the authors for the cases of Non-nested Seat-Allocation Approach or the Nested Booking-Limit Approach, are simple extensions of the case of single bookings. The rules offered are the best that can be offered when the criteria for single bookings have to be extended for the case of many bookings, but they do not necessarily reflect the theory from which the criteria have been developed.

In order for a theory about multiple bookings to be developed, we have need of statistics that pertain to the probabilities of an ordering to be an ordering of multiple bookings. Given that we have a multiple bookings order, we need to define the probability distribution of the number of the bookings. It is rather difficult to find statistics that satisfy the above demands. On the other hand it would be rather costly for companies to keep records of offered bookings and sort them according to the number of the offered bookings. It is doubtful that the increased complication of the model would benefit its user with substantially increased expected revenues, that would justify the increased number of computations.

2.2 Hotel Yield Management Review

Rothstein [27], presents a sequential decision model for the hotel overbooking problem and the determination of the booking policies. He assumes that each booking is for only one day. He treats a booking for $S$ days as a series of $S$ independent bookings. He formulates the reservation system as a Markov process with transition probabilities that depend on the booking policy. The model presented by Rothstein has the added notion of economic rewards for transition from state to state. A dynamic programming algorithm is developed
for the computation of the optimal booking policies. According to the author, this paper has been mainly written in order to demonstrate the nature of the hotel booking problem, demonstrate one solution approach and indicate its shortcomings rather than provide a model for decision making.

Ladany [28] presents a Dynamic programming model for Hotel rooms reservations and capacity control. He treats bookings as a sequential dynamic process. The objective of his model is the maximization of expected profit contribution per rental day. The model includes the effects of cancellations and overbookings. It accepts Poisson or Normal distribution demand, but a non-fitted empirical distribution could be used without complicating the utilization of the model. The fact that the decision tool relies exclusively on the Dynamic programming model, confines the use of the model to the management of a small number of rooms.

Liberman et al. [29] address hotel overbooking, which is a form of the capacity control problem. The modeling of the process is as follows: M hotel rooms are available at a date n periods from now. Reservations are made by the customers for only one day. Reservations for more than one day are treated as independent events. Customers may cancel their previously confirmed reservations at any time prior to their arrival, with no penalty. On the other hand, new requests for rooms for the target day are generated randomly.

The authors want to find an optimal booking policy that will maximize the expected net profit (or the discounted net profit) realized over a time period. The problem is formulated as a Markovian Sequential Decision process, and they give the structure of an optimal booking policy. It is shown that the optimal strategy is a 3-region policy. Depending on the region, overbooking is either encouraged (unconditionally or up to some level) or discouraged.
Chapter 3

Nested Model

3.1 Two Ports, Two Classes of Goods Nested Problem

The Nested Model for the case of two different classes of goods, asking for transportation between two ports has been addressed by Littlewood [1]. We treat the same problem as Littlewood did and we repeat the assumptions that both Littlewood and several other authors have used in their research. These are the assumptions that we will employ in our research.

1. Lower valued freight rate customers book before higher valued freight rate customers.

2. If we refuse an order from a customer who belong to a lower freight rate class, we do not accept any customer from this class any more. This assumption is true for the purpose of describing the dynamic process of booking with a static model. In order to accommodate for the deficiencies of the static model, we have to run it at set time intervals through out the time that the vessel is booking. If the newer run of the model dictates the acceptance of customers from lowered valued classes that we had rejected in the past, we accept them.

3. There are no cancelations of bookings.
4. Demand among freight rate classes is independent. This is an approximate assumption in the airline industry, where sell ups are possible. In shipping, where we do not have the possibility of sellups, the assumption of freight rate class independence is accurate.

5. A denied request for a container slot is revenue lost for the liner company, i.e. we do not consider that a container that is denied a request will buy a container slot later. This assumption is especially true for high value containers. In the case of lower valued containers, the assumption does not describe accurately the booking process. actually, for low value containers we can also have stand by cargo that waits for the shipping liner to accept it for transportation.

Littlewood, presents the following treatment of the booking process. Let as assume that we are \( T \) days before departure and we have already allocated \( L \) seats to the low value customers. If the total capacity is \( K \), the remaining \( K - L \) seats have not been allocated yet. When a customer who belongs to the low value class asks for transportation capacity and we give the space to him, we get the fare from this customer, \( f_1 \). At the same time though we lose the fare \( f_2 \) of a higher class customer, in case more than \( K - L \) customers belonging to the higher class ask for transportation. This event happens with probability \( P(X_2 \geq K - L) \). We offer the container slot to the low end customer, in case, we consider that the revenues from the low value customer exceeds the expected revenues from a high value customer.

\[
f_1 \geq f_2 \cdot P(X_2 \geq K - L)
\]

From the special case of equality, in the above inequality, we reach the optimality criterion when we are indifferent between accepting customers from class 1, or waiting for the customers belonging to class 2. Therefore if the total capacity available is \( K \), we can accept up to \( L \) customers from class 1.
3.1. Two Ports, Two Classes of Goods Nested Problem

In the following we will employ a more rigorous approach to the two-ports two-goods Nested problem. We will show that the results of our formulation coincide with the results of Littlewood, and we will develop the theory for the more general case of two ports and \( M \) classes of goods problem.

3.1.1 Model Formulation

Our objective is to maximize the revenues from the particular trip from port A to port B. The number of customers from the low value class, that will ask for transportation is \( x_1 \), and the number of the high value customers will be \( x_2 \). Both \( x_1 \) and \( x_2 \) are probabilistically distributed, with probability distributions \( P_1(x_1) \) and \( P_2(x_2) \). The assumption of the model is that all the customers of the low value class who will ask for transportation, will do so before any of the higher class customers asks for capacity. So, if we put a limit \( L \) on the number of customers belonging to class 1, that we will accept, we will have \( K - L \) remaining seats for the high value customers, even if more than \( L_1 \) class 1 customers ask for transportation.

If fewer than \( K - L \) customers from class 2 ask for transportation, we will not be able to fill the rest of our capacity with class 1 customers. On the other hand if \( S \) where \( S < L \) customers from class 1 ask for transportation, we will be able to offer \( K - S \) seats to the high value customers. The fact that the number of seats for class 1 is nested in the seats for the high value cargo, has given its name to the problem.

Our maximization problem, has as follows:

\[
\max F(L) = f_1 \cdot E(x_1) + f_2 \cdot E(x_2) \tag{3.1}
\]

subject to:

\[
L \leq K,
\]

The expected number of customers of class 1, that we accept is the number of customers that ask for transportation up to the imposed limit \( L \). If more than \( L \) low freight rate
customers arrive, we do not accept them.

\[ E(X_1) = \int_{x_1=0}^{\infty} \varphi_1 \cdot p_1(x_1) \, dz_1 \]  \hspace{1cm} (3.2)

where,

\[ \varphi_1 = \min(x_1, L) \]

The expected number of customers from class 2 that we accept is conditional on the number of customers of class 1 that we have accepted. If we have accepted \( \varphi_1 = \min(x_1, L) \) low value customers, then the available space on the vessel is \( K - \varphi_1 \), (where \( K \) is the original capacity of the vessel). We accept all the high value customers up to the limit \( K - \varphi_1 \). If more high value customers show up we do not have any extra capacity to offer them. We do not know a priori the number of low value customers who will show up. Therefore, the expected number of the high value customers that we will accept for transportation is a function of the low value customers that we will accept and as a result is a function of the a priori distribution of the low freight rate class arrivals.

\[ E(X_2) = \int_{x_1=0}^{\infty} p_1(x_1) \cdot \left\{ \int_{x_2=0}^{\infty} \varphi_2 \cdot p_2(x_2) \, dx_2 \right\} \, dx_1 \]  \hspace{1cm} (3.3)

where,

\[ \varphi_2 = \min(x_2, K - \varphi_1) \]  \hspace{1cm} (3.4)

and,

\[ \varphi_1 = \min(x_1, L) \]  \hspace{1cm} (3.5)
3.1.2 First Order Optimality Condition

The necessary condition for a point $z^*$ to be the unconstrained extremum of a function, is that it satisfies the condition ([30]):

$$\nabla f(z^*) = 0$$

(3.6)

In our case, the function is univariate, and the necessary condition 3.6 becomes:

$$\frac{\partial F(L)}{\partial L} = 0$$

(3.7)

If we substitute equation 3.1 to equation 3.7, the necessary condition for optimality becomes:

$$\frac{\partial F(L)}{\partial L} = \frac{\partial E(z_1)}{\partial L} + \frac{\partial E(z_2)}{\partial L} = 0$$

(3.8)

The derivatives of the expected revenue of the two classes of customers can be found from equations 3.2 and 3.3

$$\frac{\partial E(z_1)}{\partial L} = \int_{z_1=0}^{\infty} \frac{\partial \varphi_1}{\partial L} \cdot p_1(z_1)dz_1$$

(3.9)

and,

$$\frac{\partial E(z_2)}{\partial L} = \int_{z_1=0}^{\infty} p_1(z_1) \cdot \left( \int_{z_2=0}^{\infty} \frac{\partial \varphi_2}{\partial L} \cdot p_2(z_2)dz_2 \right)dz_1$$

(3.10)

From equations 3.4 and 3.5, we see that the derivatives of $\varphi_1$ and $\varphi_2$ become:

$$\frac{\partial \varphi_1}{\partial L} = \begin{cases} 0 & \text{for } z_1 < L \\ 1 & \text{for } z_1 \geq L \end{cases}$$

(3.11)

and,

$$\frac{\partial \varphi_2}{\partial L} = \begin{cases} -1 & \text{for } z_2 \geq K - L, z_1 \geq L \\ 0 & \text{otherwise} \end{cases}$$

(3.12)
and the derivatives of the expected revenues from each one of the two classes of customers, becomes:

\[
\frac{\partial E(x_1)}{\partial L} = \int_{x_1=L}^{\infty} p_1(x_1)dx_1
\]

and,

\[
\frac{\partial E(x_2)}{\partial L} = - \int_{x_1=L}^{\infty} p_1(x_1) \cdot \left\{ \int_{x_2=K-L}^{\infty} p_2(x_2)dx_2 \right\} dx_1
\]

Closer observation of equation 3.14 shows that the inner integral of the right hand side, is independent of \(z_1\). As a result, the right hand side of equation 3.14, can be presented as the product of two single integrals. Therefore:

\[
\frac{\partial E(x_1)}{\partial L} = \int_{x_1=L}^{\infty} p_1(x_1)dx_1
\]

and,

\[
\frac{\partial E(x_2)}{\partial L} = - \int_{x_1=L}^{\infty} p_1(x_1)dx_1 \cdot \int_{x_2=K-L}^{\infty} p_2(x_2)dx_2
\]

If we replace the equations 3.15, and 3.16, into equations 3.8, the necessary condition for optimality becomes:

\[
f_1 \cdot \int_{x_1=L}^{\infty} p_1(x_1)dx_1 - f_2 \cdot \int_{x_1=L}^{\infty} p_1(x_1)dx_1 \cdot \int_{x_2=K-L}^{\infty} p_2(x_2)dx_2 = 0
\]

After some trivial manipulations, and change in the notation the above equation 3.17 becomes:

\[
f_1 \cdot P(x_1 \geq L) = f_2 \cdot P(x_1 \geq L) \cdot P(x_2 \geq K-L)
\]

If we exclude the trivial case \(P(x_1 \geq L) = 0\), the necessary condition for maximization of the revenues, becomes:

\[
f_1 = f_2 \cdot P(x_2 \geq K-L)
\]
or
\[
f_1 = f_2 \cdot \int_{z_2 = K - L}^{\infty} p_2(z_2)dz_2 \tag{3.20}
\]

3.1.3 Second Order Optimality Condition

We have proven that the necessary condition for a booking limit \( L \) to be the decision rule that maximizes the revenues, is the criterion 3.19. In order to prove that the above booking limit is indeed the maximizing criterion, we should do some further analysis.

A point \( z^* \) that satisfies the necessary condition of optimality and also satisfies the second order sufficient optimality condition:

\[
\nabla^2 f(z^*) : \text{negative definite} \tag{3.21}
\]

is a strict local maximum of the twice continuously differentiable function \( f(z) \) ([30]). In the case of our twice continuously differentiable function of one variable, the condition becomes:

\[
\frac{\partial^2 F(L)}{\partial L^2} < 0 \tag{3.22}
\]

From the objective function of the optimization problem 3.1, we get:

\[
\frac{\partial^2 F(L)}{\partial L^2} = f_1 \cdot \frac{\partial^2 E(z_1)}{\partial L^2} + f_2 \cdot \frac{\partial^2 E(z_2)}{\partial L^2} \tag{3.23}
\]

I define:

\[
A(L) = \int_{z_1 = L}^{\infty} p_1(z_1)dz_1 \tag{3.24}
\]

and,

\[
B(L) = \int_{z_2 = K - L}^{\infty} p_2(z_2)dz_2 \tag{3.25}
\]

In light of the above definitions, the derivatives of the expected revenues (equations 3.15
and 3.16 become:

\[
\frac{\partial E(x_1)}{\partial L} = A(L) \tag{3.26}
\]
and,

\[
\frac{\partial E(x_2)}{\partial L} = -A(L) \cdot B(L) \tag{3.27}
\]

Consequently:

\[
\frac{\partial^2 E(x_1)}{\partial L^2} = \frac{\partial A(L)}{\partial L} \tag{3.28}
\]
and,

\[
\frac{\partial^2 E(x_2)}{\partial L^2} = -\left\{A(L) \cdot \frac{\partial B(L)}{\partial L} + \frac{\partial A(L)}{\partial L} \cdot B'(L)\right\} \tag{3.29}
\]

From equations 3.24, and 3.25, we derive:

\[
\frac{\partial A(L)}{\partial L} = -p_1(L) \tag{3.30}
\]
and,

\[
\frac{\partial B(L)}{\partial L} = p_2(K - L) \tag{3.31}
\]

If we take into consideration the equations 3.30, and 3.31, the second derivatives of the expected revenues per class of goods, become:

\[
\frac{\partial^2 E(x_1)}{\partial L^2} = -p_1(L) \tag{3.32}
\]
and,

\[
\frac{\partial^2 E(x_2)}{\partial L^2} = -p_2(K - L) \cdot \int_{x_1=L}^{\infty} p_1(x_1)dx_1 + p_1(L) \cdot \int_{x_2=K-L}^{\infty} p_2(x_2)dx_2 \tag{3.33}
\]

We have to keep in mind that we examine for concavity at the point that satisfies the necessary criterion of first order. From equation 3.33, with the help of equation 3.20, we
3.1. Two Ports, Two Classes of Goods Nested Problem

get:

\[
\frac{\partial^2 E(x_2)}{\partial L^2} = -p_2(K - L) \cdot \int_{z_1=L}^{\infty} p_1(z_1)dz_1 + \frac{f_1}{f_2} \cdot p_1(L)
\]  

(3.34)

From the combination of equations 3.23, 3.32, and 3.34, we get after a few simple manipulations:

\[
\frac{\partial^2 F(L)}{\partial L^2} = -f_2 \cdot p_2(K - L) \cdot \int_{z_1=L}^{\infty} p_1(z_1)dz_1 < 0
\]  

(3.35)

As a result, the \( L \) that satisfies the first order criterion is the \( L \) that corresponds to a local maximum of the revenue function. As a matter of fact it is the only unconstrained maximum at the domain \((0, K)\). It is easy to prove that the values of the expected revenue function for \( L = 0 \) or \( L = K \) are smaller than the value of the expected revenues for the \( L \) given from the first order necessary condition. As a result, the \( L \) given as a root of the equation 3.19, is the global maximum.

We see that the above approach, results in the same optimization criterion that Littlewood [1] concluded. We have additionally proven, by using optimization that the criterion used by Littlewood is actually the criterion that maximizes the revenues of the airline/vessel operator.
3.2 Two Ports, M Classes of Goods Nested Model

We assume that a vessel operator does a round trip among two ports and there are $M$ classes of products that he can transport, from port 1 to port 2. The formulation of the problem has as follows: All the customers who belong to a certain fare class, and who would like to have a container transported with the vessel, ask for container slots after all the shippers who belong to the immediately lower value containers and who would like to get a container slot. They also ask for transportation before any shipper who belongs to the immediately more valuable category. The vessel operator can either accept or refuse a cargo offer. The customers who are being refused container slots are being serviced by other operators and they do not return to the same operator. The operator wants to define the maximum number of containers he will accept from each class, so as to maximize his revenue/profit.

We can alternatively understand the problem as follows: We have $M$ classes of goods and we conduct $M$ experiments, each of which is associated with a lottery that gives a number of containers that belong to a particular class and demand transportation capacity. Some of the containers we accept, and the rest we have to reject. The rejected containers go to other operators and they do not return to ask for service. The number of containers for each class has a probability distribution

$$p_i(z_i), \text{ for } i = 1, \ldots, N$$

First, we get the outcome of lottery 1, and this outcome corresponds to the number of containers of class 1 (the lowest valued class of containers), that demand transportation capacity from the vessel operator. Class 1 is the the lowest valued class of containers and consequently the class with the lowest fare. The operator has placed a priori, a cap $L_1$ on the number of class 1 containers that he will accept. He rejects all the containers from class 1 in excess of the number $L_1$. These excessive containers go to other operators, and none of them returns to ask for capacity with our vessel operator.
Next, the operator gets the outcome of the lottery 2, that defines the number of class 2 containers that would want space with the vessel. Class 2 is the class which has the containers that are immediately more expensive than the containers in class 1, but less expensive than any other class of containers. The operator puts (a priori) a cap $L_2$ on the combined number of containers from class 1 and class 2 that he accepts. He rejects all the containers from class 2 that would make him exceed the a priori set limit $L_2$.

The experiments continue with with classes that represent containers of always increasing value. As a result, the experiment with the lottery of class $i$ is the $i^{th}$ experiment that takes place. The containers in class $i$ have greater value than the containers in classes $1, \ldots, i-1$. Initially the operator puts a cap $L_i$ on the total number of containers belonging to classes $1, \ldots, i$ and he rejects the containers from class $i$ that violate this cap. (See Figure 3-1).

![Diagram](image)

Figure 3-1: Illustration of the Meaning of $\varphi_i$ for the case $M = 5$
Chapter 3. Nested Model

The problem of the operator is to find the $L_i$'s for which his expected revenue becomes maximum. If the optimal allocation is $L_1, \ldots, L_M = K$, at the first round of the experiment he accepts

$$\varphi_1 = \min(x_1, L_1)$$

number of containers from class 1.

We have to mention here that the solution that we have suggested for the two ports, two classes of goods problem, as well as the solution that we will suggest here for the two ports, $M$ classes of goods problem, are static and a priori, whereas the nature of the booking process is dynamic. In order to accommodate the discrepancy between the process and the modeling, we introduce the periodic updating of the model.

The capacity limits $L_1, \ldots, L_M$ ($L_M$ is the capacity constraint of the vessel) are the initial optimal constraints imposed before we start the experiment.

At the first iteration of the problem, the operator uses only the limit $L_1$ for the maximum number of containers he will accept from class 1. After he receives the offers for class 1, he accepts $\varphi_1 = \min(x_1, L_1)$ and the remaining capacity of the vessel is $K - \varphi_1$. At this remaining capacity he has to accommodate, the classes $2, \ldots, M$. He updates his estimations for the probability distribution of the classes $2, \ldots, M$ and he solves again the problem for the remaining vessel capacity $K - \varphi_1$.

As the experiment progresses, when for instance, the experiments with the first $i - 1$ lotteries have taken place, the operator has accepted

$$\varphi_j, \quad j = 1, \ldots, i - 1$$

containers from each class $j = 1, \ldots, i - 1$ and the remaining capacity of the vessel is

$$C(i - 1) = K - \sum_{j=1}^{i-1} \varphi_j$$
with,

\[ C(0) = K \]

At this stage, the variables we want to optimize, in order to maximize the revenue of the operator are \( L_i, \ldots, L_M = C(i - 1) \). The number \( L_i \) represents the maximum number of containers from class \( i \) that the operator will accept. \( L_{i+1} \) is the cumulative number of class \( i \) and class \( i+1 \) containers the operator will accept. In the same spirit \( L_M \) is the total number of containers the operator is willing to accept from the classes \( L_i, \ldots, L_{M-1}, L_M = C(i - 1) \)

In conclusion, the program at the stage \( i \), calculates the numbers \( L_i, \ldots, L_j, \ldots, L_{M-1}, L_M = C(i - 1) \). The number \( L_j \) is the maximum cumulative number of containers belonging to classes \( i, \ldots, j \) that the operator will accept. Nevertheless, \( L_i \) is the only limit that will used at this stage. Once the operator has accepted \( \varphi_i = \min(z_i, L_i) \), the capacity limits are calculated again for the remaining capacity and the remaining classes of goods.

### 3.2.1 Model Formulation

The operator wants to maximize his expected revenues over all the classes of goods he accepts. The constraint for the problem is the capacity the operator can offer his shippers. Class 1 is the lowest fare class, whereas class \( M \) is the highest fare class.

The mathematical formulation of the above problem has as follows:

\[
\max \quad F(L_1, \ldots, L_M) = \sum_{i=1}^{M} f_i \cdot E_i(L_1, \ldots, L_i) \tag{3.36}
\]

subject to:

\[
L_{i-1} \leq L_i, \quad i = 2, \ldots, M
\]

\[
L_M = K,
\]

\[
L_i \geq 0, \quad i = 1, \ldots, M
\]
where:

$$E_i(L_1, \ldots, L_i) = \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \cdots \int_{x_i=0}^{\infty} \varphi_i \cdot p_i(x_i) dx_i \cdots dx_2 dx_1$$  (3.37)

$$\varphi_i = \min(x_i, L_i - \sum_{j=1}^{i-1} \varphi_j), \quad i = 2, \ldots, M$$  (3.38)

$$\varphi_1 = \min(x_1, L_1)$$

and:

$$F(L_1, \ldots, L_M) = \text{the total expected revenue from the trip}$$

$$M = \text{the number of classes of goods that the Liner accepts for transportation between the two ports}$$

$$i = \text{the class of the particular good}$$

$$f_i = \text{the fare paid by a container of good belonging to class } i \text{ for transportation from port 1 to port 2}$$

$$x_i = \text{the number of the containers from class } i, \text{ that will demand transportation capacity from port 1 to port 2}$$

$$L_i = \text{the maximum vessel capacity that the vessel operator allocates cumulatively to classes } 1 \text{ up to } i \text{ for transportation from port 1 to port 2. } (\sum_{j=1}^{i} \varphi_j \leq L_i)$$

$$\varphi_i = \text{the number of class } i \text{ containers that the operator accepts for transportation}$$

$$E_i(L_1, \ldots, L_i) = \text{the expected number of containers of class } i \text{ that will be transported from port 1 to port 2.}$$

$$p_i(x) = \text{the probability distribution of transportation demand for goods belonging to class } i.$$ 

$$K = \text{the vessel capacity available at the leg of the trip from port 1 to port 2.}$$
3.2. Two Ports, M Classes of Goods Nested Model

In the above equation, we can make the following comments

2. A major assumption that should be underlined, is that there is a time pattern in the way that the low and high ends of the market ask for transportation capacity. The assumption is that the low freight rate customers ask for transportation capacity before any of the the high freight rate customers ask for transportation capacity. All low end customers that will not be accepted for transportation will balk to other operators.

3. When we solve the optimization problem 3.36, we have T remaining time units, until departure from port 1. The probability distribution \( p_m(z) \) is our best estimate of the number of goods belonging to class \( m \) that will ask for transportation from port 1 to port 2 in the remaining \( T \) time until the departure of the vessel from port 1.

4. The main constraint of the optimization problem is the capacity constraints of the vessel. The remaining \( M - 1 \) constraints, simply state that the maximum capacity allocated to the first \( i \) classes of goods is not larger than the capacity allocated to the first \( i + 1 \) classes of goods.

5. The available capacity is the capacity of the vessel that has not been committed to any shippers yet.

3.2.2 First Order Optimality Condition

The necessary condition for a point \( z^* \) to be the local maximum of a function \( f(z) \) over a convex set \( X \), is that it satisfies the condition ([30]):

\[
\nabla f(z^*)' \cdot (z - z^*) \leq 0
\]

The possible active constraints that we might encounter at our problem are the booking limits for several classes of goods, that will be equal to zero, when we do not accept containers from these classes of goods. (i.e. \( L_1, \ldots, L_j \) for some \( j \) are equal to zero).
The necessary condition for a point $z^*$ to be an unconstrained local extremum of a function $f(z)$, is that it satisfies the condition ([30]):

$$\nabla f(z^*) = 0$$ \hspace{1cm} (3.40)

We see from both equations 3.39 and 3.40 that we have to calculate the derivatives of the function that we want to optimize, in both the case of the unconstrained and the constrained optimization. We prefer to deal with the unconstrained optimization rather than the case of the constrained optimization for several reasons.

The first reason is that at our problem and for the case of the constrained optimization where some of the constraints are active, the equivalent elements of $L^*$ are equal to zero, whereas the partial derivatives that correspond to the inactive constraints are equal to zero. That reduces the necessary conditions for optimality to the following

$$\frac{\partial f(x^*)}{\partial x_j} \leq 0, \quad j = 1, \ldots, M - 1$$ \hspace{1cm} (3.41)

which does not give much more insight than the case of the unconstrained optimization that requires all the partial derivatives to be equal to zero.

The second reason is that it will be easier to show that a point that satisfies the necessary conditions for unconstrained optimization is the only global maximum, of the revenue function. We therefore proceed with the derivation of the necessary conditions for unconstrained optimization.

The constraint $L_M = K$ is an active constraint. Nevertheless, it is not an essential constraint. We could have given the definitions of $E_M(L_1, \ldots, L_M)$ and $\varphi_M$, as functions of $K$ instead of $L_M$. Equation $L_M = K$ makes it easy for $E_M(L_1, \ldots, L_M)$ to be described by equation 3.37, rather than by a separate equation. We therefore conclude that our optimization problem is a unconstrained optimization problem. From now on, we will continue using the previously established convention that the revenue is a function of all the $L_i$'s, the $L_M$ included. At the same time, we have to remember that $L_M$ is a parameter.
of the problem. \( L_M \) is the total capacity available collectively to all classes \( 1, \ldots, M \) of containers, and not a variable of the problem that can be optimized.

The necessary condition for optimality described by equation 3.40 becomes:

\[
\frac{\partial F(L_1, \ldots, L_{M-1})}{\partial L_j} = 0, \ j = 1, \ldots, M - 1
\]  

(3.42)

**Derivation of the Objective Function Gradient**

From the objective function of the optimization problem 3.36, the first order derivatives of the expected revenues become:

\[
\frac{\partial F(L_1, L_2, \ldots, L_M)}{\partial L_j} = \sum_{i=1}^{M} f_i \cdot \frac{\partial E_i(L_1, L_2, \ldots, L_i)}{\partial L_j}
\]  

(3.43)

For \( j > i \), we have:

\[
\frac{\partial E_i(L_1, L_2, \ldots, L_i)}{\partial L_j} = 0
\]  

(3.44)

From equations 3.43 and 3.44, we get:

\[
\frac{\partial F(L_1, L_2, \ldots, L_M)}{\partial L_j} = \sum_{i=j}^{M} f_i \cdot \frac{\partial E_i(L_1, L_2, \ldots, L_i)}{\partial L_j} = 0
\]  

(3.45)

For our convenience, we introduce the notation, \( E_i = E_i(L_1, L_2, \ldots, L_i) \). For \( i > j \) we have:

\[
\frac{\partial E_i}{\partial L_j} = \int_{x_{i-1}}^{\infty} p_1(z_1) \int_{x_{i-2}}^{\infty} p_2(z_2) \ldots \int_{x_{i-1}}^{\infty} p_{i-1}(z_{i-1}) \int_{x_i}^{\infty} \frac{\partial \varphi_i}{\partial L_j} \cdot p_i(z_i) \, dz_i \ldots dx_2 dx_1
\]  

(3.46)

From equation 3.38 and for \( i > j \)

\[
\frac{\partial \varphi_i}{\partial L_j} = \begin{cases} 
0 & \text{for } x_i < L_i - \sum_{k=1}^{i-1} \varphi_k \\
- \sum_{k=1}^{i-1} \frac{\partial \varphi_k}{\partial L_j} & \text{for } x_i \geq L_i - \sum_{k=1}^{i-1} \varphi_k
\end{cases}
\]  

(3.47)
At equations 3.38, we see that $\varphi_j$ is a function of $L_j$, $x_i$ and $\varphi_i$, $i = 1, \ldots, j$. We therefore conclude that:

$$\varphi_i = f(L_1, L_2, \ldots, L_i, x_1, x_2, \ldots, x_i)$$  \hspace{1cm} (3.48)

From the above equation 3.48, and for $j > i$ we get:

$$\frac{\partial \varphi_i}{\partial L_j} = 0$$  \hspace{1cm} (3.49)

From equations 3.47 and 3.48, we conclude that for $i > j$

$$\frac{\partial \varphi_i}{\partial L_j} = \begin{cases} 0 & \text{for } x_i < L_i - \sum_{k=1}^{i-1} \varphi_k \\ -\sum_{k=j}^{i-1} \frac{\partial \varphi_k}{\partial L_j} & \text{for } x_i \geq L_i - \sum_{k=1}^{i-1} \varphi_k \end{cases}$$  \hspace{1cm} (3.50)

In the following we will prove that for $i > j$:

$$\frac{\partial \varphi_i}{\partial L_j} = \begin{cases} 0 & \text{for } x_i < L_i - \sum_{k=1}^{i-1} \varphi_k \\ -1 \text{ or } 0 & \text{for } x_i \geq L_i - \sum_{k=1}^{i-1} \varphi_k \end{cases}$$  \hspace{1cm} (3.51)

We introduce the following notation:

$$\frac{\partial \varphi_i}{\partial L_j} = \delta_{i,j}$$  \hspace{1cm} (3.52)

and we define $\xi_i$ such that:

$$x_i < L_i - \sum_{j=1}^{i-1} \varphi_j \iff \xi_i = 0$$  \hspace{1cm} (3.53)

$$x_i \geq L_i - \sum_{j=1}^{i-1} \varphi_j \iff \xi_i = 1$$
If we incorporate the changes in notation from definitions 3.52 and 3.53 in equation 3.50, it becomes:

$$\delta_{i,j} = \begin{cases} 
0 & \text{for } \xi_i = 0 \\
-\sum_{k=j}^{i-1} \delta_{k,j} & \text{for } \xi_i = 1 
\end{cases} \quad (3.54)$$

We can easily observe that:

$$\delta_{i,i} = \begin{cases} 
0 & \text{for } \xi_i = 0 \\
1 & \text{for } \xi_i = 1 
\end{cases} \quad (3.55)$$

**Proposition 1** Let us assume that $\delta_{j,j} = 0$ and $i > j$. Then we have that $\delta_{i,j} = 0$.

**Proof:** The proof will be done by induction.

We have that $\delta_{j,j} = 0$. We assume that $\delta_{j+1,j} = 0, \delta_{j+2,j} = 0, \ldots, \delta_{i-1,j} = 0$. If $\xi_i = 0 \implies \delta_{i,i} = 0$. If $\xi_i = 1 \implies \delta_{i,i} = -\sum_{k=j}^{i-1} \delta_{k,j} = 0$ as a consequence of the above assumption.

Therefore, we have that $\delta_{i,j} = 0$ Q.E.D.

**Proposition 2** Let $\delta_{j,j} = 1$ and $A = \{t_1, t_2, \ldots, t_s, \ldots\}$ such that for every $t_i \in A$ we have that $\xi_{t_i} = 1$ and $t_i > j$, whereas for every $s > j$ and $s \not\in A$, $\xi_s = 0$. We also define $k = \min(t_1, t_2, \ldots, t_s, \ldots)$. Then we have:

1. $\delta_{k,j} = -1$

2. For $t \in A$ and $t > k \implies \delta_{t,j} = 0$

**Proof:** 1) We have that $\delta_{j,j} = 1$. We also know that $\delta_{p,j} = 0$ ($j < p < k$)

$$\implies \delta_{k,j} = -\sum_{z=j}^{k-1} = -1 - 0 - \ldots - 0 = -1, \implies \delta_{k,j} = -1$$

2) The proof will be done by induction. Let $t_1 \leq t_2 \leq \ldots \leq t_s \implies k = t_1$ We will prove that $\delta_{t_2,2} = 0$

$$\delta_{t_2,j} = -\delta_{j,j} - \delta_{j+1,j} - \ldots - \delta_{k-1,j} - \delta_{k,j} - \delta_{k+1,j} - \delta_{k+2,j} - \ldots - \delta_{t_2-1,j}$$

But $\xi_{k+1} = 0 \implies \delta_{k+1,j} = 0$
In a similar way, we have:
\[ \xi_{k+2} = 0 \implies \delta_{k+2,j} = 0 \]
\[ \ldots \]
\[ \xi_{t_2-1} = 0 \implies \delta_{t_2-1,j} = 0 \]
\[ \implies \delta_{t_2,j} = -1 - 0 - \ldots - 0 - (-1) - 0 - \ldots - 0 = 0 \]
\[ \implies \delta_{t_2,j} = 0 \]

We assume that \( \delta_{t_3,j} = \delta_{t_4,j} = \delta_{t_5,j} = 0 \). We will prove that \( \delta_{t_{s+1},j} = 0 \).
\[ \delta_{t_{s+1},j} = -\delta_{j,j} - \delta_{j+1,j} - \ldots - \delta_{k-1,j} - \delta_{k-1,j} - \delta_{k-1,j} - \ldots - \delta_{t_2-1,j} - \delta_{t_2,j} - \ldots - \delta_{t_3,j} - \ldots - \delta_{t_{s},j} \]
\[ \implies \delta_{t_{s+1},j} = -1 - 0 - \ldots - 0 - (1) - 0 - \ldots - 0 - 0 - \ldots - 0 = 0 \]
\[ \implies \delta_{t_{s+1},j} = 0 \text{ Q.E.D.} \]

From the above propositions, we see that when \( \delta_{j,j} = 0 \), we have \( \delta_{i,j} = 0 \), for every \( i > j \). When \( \delta_{j,j} = 1 \), we see that for the smallest \( k \) for which \( \xi_k = 1 \), we have that \( \delta_{k,j} = -1 \). All other \( \delta_{p,j} = 0 \), where \( j < p \leq i, p \neq k \).

As a result, for \( i > j \), \( \delta_{i,j} = -1 \) only when \( \delta_{j,j} = 1 \) and \( i \) is the smallest index for which both \( i > j \) and \( \xi_i = 1 \). Otherwise \( \delta_{i,j} = 0 \).

Combining the above result with the result from Equation \( 3.46 \), and by setting \( j = 1 \) we get:

\[ \frac{\partial E_i}{\partial L_1} = \int_{z_1=L_1}^{\infty} p_1(x_1) \int_{z_2=0}^{L_2-\varphi_1} p_2(x_2) \ldots \]
\[ \ldots \int_{z_{i-1}=0}^{L_{i-1}-\varphi_{i-2}-\ldots-\varphi_1} p_{i-1}(x_{i-1}) \int_{z_i=L_i-\varphi_{i-1}-\ldots-\varphi_1}^{\infty} (-1) \cdot p_i(x_i) dz_i \ldots dz_2 dz_1 \]
\[ (3.56) \]

At the above integral, we observe that:

\[ x_1 \geq L_1 \implies \varphi_1 = L_1 \]
\[ x_j < L_j - \sum_{r=1}^{j-1} \varphi_r \implies \varphi_j = x_j , j = 2, \ldots , i - 1 \]
\[ (3.57) \]
3.2. Two Ports, M Classes of Goods Nested Model

\[ z_i \geq L_i - \sum_{r=1}^{i-1} \varphi_r \implies \varphi_{i-1} = L_i - \sum_{r=1}^{i-1} x_r \]

From equations 3.56 and 3.57, we get:

\[
\frac{\partial E_i}{\partial L_1} = \int_{z_1 = L_1}^{\infty} \int_{z_2 = 0}^{L_2 - L_1} \int_{z_3 = 0}^{L_3 - z_2 - L_1} \ldots \int_{z_{i-1} = 0}^{L_{i-1} - z_{i-2} - \ldots - z_2 - L_1} p_i(x_i) \int_{x_i = L_i - \varphi_{i-1} - \ldots - \varphi_1}^{\infty} (1) \cdot p_i(x_i) dx_i \ldots dx_2 dx_1
\]

(3.58)

In general, we have that equation 3.46, i.e. the derivative of \( E_i \) with reference to \( L_j \) becomes:

\[
\frac{\partial E_i}{\partial L_j} = \int_{z_1 = 0}^{\infty} p_1(x_1) \int_{z_2 = 0}^{\infty} p_2(x_2) \ldots \int_{z_{j-1} = 0}^{\infty} p_{j-1}(x_{j-1}) \int_{z_{j} = L_{j} - \varphi_{j-1} - \ldots - \varphi_1}^{\infty} p_j(x_j) \int_{x_{j+1} = 0}^{L_{j+1} - \varphi_j - \ldots - \varphi_1} \ldots \int_{z_{i-1} = 0}^{L_{i-1} - \varphi_{i-2} - \ldots - \varphi_1} p_{i-1}(x_{i-1}) \int_{x_i = L_i - \varphi_{i-1} - \ldots - \varphi_1}^{\infty} (1) \cdot p_i(x_i) dx_i \ldots dx_2 dx_1
\]

(3.59)

The variables \( x_1, x_2, \ldots, x_{j-1} \), can take any value between 0 and \( \infty \). At the integral of the previous equation 3.59:

\[
\text{int} = \int_{x_i = L_i - \varphi_{i-1} - \ldots - \varphi_1}^{\infty} p_j(x_j)(\ldots) dx_j
\]

(3.60)

we have that: \( x_j \geq L_j - \sum_{i=1}^{j-1} \varphi_i \) (the domain of the innermost integral).

We remind here that \( \varphi_j = \min(x_j, L_j - \sum_{i=1}^{j-1} \varphi_i) \). Therefore, \( \varphi_j = L_j - \sum_{i=1}^{j-1} \varphi_i \) and
\( L_j = \sum_{i=1}^{j} \varphi_i \). As a result, the integral:

\[
\int_{x_{j+1}=0}^{L_{j+1}-\varphi_j-\ldots-\varphi_1} p_{j+1}(x_{j+1})(\ldots)dx_{j+1}
\]

becomes:

\[
\int_{x_{j+1}=0}^{L_{j+1}-L_j} p_{j+1}(x_{j+1})(\ldots)dx_{j+1}
\]

We make the same substitution at all the upper and lower limits of the multiple integral \( \text{int} \) (equation 3.60), and we get the following results.

\[
\begin{align*}
\varphi_{j+2} &= L_{j+2} - \varphi_{j+1} - L_j \\
\vdots & \\
\varphi_{i-1} &= L_{i-1} - \varphi_{i-2} - \ldots - \varphi_{j+1} - L_j \\
\varphi_i &= L_i - \varphi_{i-1} - \ldots - \varphi_{j+1} - L_j
\end{align*}
\]  

Consequently, the integral 3.59, becomes:

\[
\frac{\partial E_i}{\partial L_j} = \\
= \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \ldots \int_{x_{j-1}=0}^{\infty} p_{j-1}(x_{j-1}) \int_{x_j=L_j-\varphi_{j-1}-\ldots-\varphi_1}^{\infty} p_j(x_j) \\
\left[ \int_{x_{j+1}=0}^{L_{j+1}-L_j} p_{j+1}(x_{j+1})(\ldots)dx_{j+1} \right] \\
\left[ \int_{x_{j+1}=0}^{L_{j+1}-L_j} p_{j+1}(x_{j+1})(\ldots)dx_{j+1} \right] \\
\left[ \int_{x_{i}=L_i-\varphi_{i-1}-\ldots-\varphi_{j+1}-L_j}^{\infty} (-1) \cdot p_i(x_i)dx_i \ldots dx_2 dx_1 \right]
\]

For \( z_s \), such that \((j+1 \leq s < i)\) we have: \( z_s \leq L_s - \varphi_{s-1} - \ldots - \varphi_1 \implies \varphi_s = z_s \). Therefore:
\[ \frac{\partial E_i}{\partial L_j} = \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \ldots \int_{x_{j-1}=0}^{\infty} p_{j-1}(x_{j-1}) \int_{x_j=0}^{L_{j+1} - L_j} p_{j+1}(x_{j+1}) \int_{x_{j+2}=0}^{L_{j+2} - x_{j+1} - L_j} p_{j+2}(x_{j+2}) \ldots \int_{x_{i-1}=0}^{L_{i-1} - x_{i-2} - \ldots - x_{j+1} - L_j} p_{i-1}(x_{i-1}) \]

\[ \int_{x_1=L_i - x_{i-1} - \ldots - x_{j+1} - L_j}^{\infty} (-1) \cdot p_i(x_i) dx_i \ldots dx_2 dx_1 \quad (3.65) \]

The integral of the above equation 3.65, can be decoupled to two integrals. We can, therefore present derivative of the expected revenue from class \( i \) as the product of two integrals:

\[ \frac{\partial E_i}{\partial L_j} = A(L_1, \ldots, L_j) \cdot B(L_j, \ldots, L_i) \quad (3.66) \]

where:

\[ A(L_1, \ldots, L_j) = \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \ldots \int_{x_{j-1}=0}^{\infty} p_{j-1}(x_{j-1}) \int_{x_j=0}^{L_{j+1} - L_j} p_{j+1}(x_{j+1}) \int_{x_{j+2}=0}^{L_{j+2} - x_{j+1} - L_j} p_{j+2}(x_{j+2}) \ldots \int_{x_{i-1}=0}^{L_{i-1} - x_{i-2} - \ldots - x_{j+1} - L_j} p_{i-1}(x_{i-1}) \int_{x_1=L_i - x_{i-1} - \ldots - x_{j+1} - L_j}^{\infty} (-1) \cdot p_i(x_i) dx_i \ldots dx_2 dx_1 \quad (3.67) \]

and

\[ B(L_j, \ldots, L_i) = \int_{x_j=0}^{L_{j+1} - L_j} p_{j+1}(x_{j+1}) \int_{x_{j+2}=0}^{L_{j+2} - x_{j+1} - L_j} p_{j+2}(x_{j+2}) \ldots \int_{x_{i-1}=0}^{L_{i-1} - x_{i-2} - \ldots - x_{j+1} - L_j} p_{i-1}(x_{i-1}) \int_{x_1=L_i - x_{i-1} - \ldots - x_{j+1} - L_j}^{\infty} (-1) \cdot p_i(x_i) dx_i \ldots dx_2 dx_1 \quad (3.68) \]

We remind that the above results are valid for \( i > j \). We also have some special cases
that are mentioned below,

\[ \text{For } i = j + 1 \quad B(L_j, L_{j+1}) = \int_{x_i=L_{i+1}}^{\infty} (-1) \cdot p_i(x_i) dx_i \]

\[ \text{For } i = j \quad B(L_j) = 1.0 \]

\[ \text{For } j = 1 \quad A(L_1) = \int_{x_1=L_1}^{\infty} p_1(x_1) dx_1 \]

With the help of equation 3.66, the necessary condition for optimality, expressed by equation 3.45, becomes:

\[ \frac{\partial F(L_1, L_2, \ldots, L_M)}{\partial L_j} = \sum_{i=j}^{M} f_i \cdot A(L_1, \ldots, L_j) \cdot B(L_j, \ldots, L_i) = 0 \]

\[ \Rightarrow \frac{\partial F(L_1, L_2, \ldots, L_M)}{\partial L_j} = A(L_1, \ldots, L_j) \cdot \sum_{i=j}^{M} f_i \cdot B(L_j, \ldots, L_i) = 0 \]  \hspace{1cm} (3.69)

As a result, the necessary condition for optimality is expressed as the following system of equations.

\[ \sum_{i=j}^{M} f_i \cdot B(L_j, \ldots, L_i) = 0 \ , \ j = 1, \ldots, M - 1 \]  \hspace{1cm} (3.70)

where \( B(L_j, \ldots, L_i) \) is given by equation 3.68.

If we take into account the fact that \( B(L_j) = 1.0 \), the system of the necessary conditions for optimality becomes:

\[ f_j = - \sum_{i=j+1}^{M} f_i \cdot B(L_{j+1}, \ldots, L_i) = 0 \ , \ j = 1, \ldots, M - 1 \]  \hspace{1cm} (3.71)

At the above system 3.71 of necessary equations for optimality, we consider the \( M - 1 \)th
3.2. Two Ports, M Classes of Goods Nested Model

equation,

\[ f_{M-1} = -f_M \cdot B(L_{M-1}, L_M) \]  \hspace{1cm} (3.72)

If we keep in mind that \( L_M = K \), where \( K \) is the capacity of the vessel, we see that the booking limit \( L_{M-1} \), and consequently the protection level \( L_M - L_{M-1} \), is a function of \( f_{M-1} \) and \( f_M \) only. If we subsequently, examine all the equations, from the \( j - 2 \) to the 1st equation, we see that all the booking limits \( L_s \) and the protection levels \( L_{s+1} - L_s \), are functions only of the booking limits for the higher classes \( L_t \), \( t = s + 1, \ldots, M \) and the respective protection levels \( L_{t+1} - L_t \), \( t = s + 1, \ldots, M \).

The figure 3-2 shows graphically an example of a booking problem with three booking classes. The necessary equations of 3.71 describe the points where the marginal revenue of one class of containers, becomes equal to the freight rate of the immediately lower class of containers. If our capacity is less than \( L_3 - L_2 \), then we keep all the capacity for class 3 containers. If our remaining capacity is greater than \( L_3 - L_2 \), then we accept containers from class 2 as well. We accept class 1 containers, only if the remaining capacity is more than \( L_3 - L_1 \).

If we inspect closely \( B(L_j, \ldots, L_i) \) as given by equation 3.68, we see that:

\[
B(L_j, \ldots, L_i) = -P([x_{j+1} \leq L_{j+1} - L_j] \cap [x_{j+1} + x_{j+2} \leq L_{j+2} - L_j] \cap \ldots \cap [x_{j+1} + \ldots + x_{i-1} \leq L_{i-1} - L_j] \cap [x_{j+1} + \ldots + x_i \geq L_i - L_j])
\]  \hspace{1cm} (3.73)

The event \( x_{j+1} \leq L_{j+1} - L_j \) is the event that the arrivals of containers \( x_{j+1} \) belonging to class \( j + 1 \) are fewer than the protection level \( L_{j+1} - L_j \), that has been set for this class.

The event \( x_{j+1} + x_{j+2} \leq L_{j+2} - L_j \) is the event that the total arrivals \( x_{j+1} + x_{j+2} \) of containers belonging to classes \( j + 1 \) and \( j + 2 \) are fewer than the protection level \( L_{j+2} - L_j \) that has been set for the two classes \( j + 1 \) and \( j + 2 \), and so on.

The event \( x_{j+1} + \ldots + x_{i-1} \leq L_{i-1} - L_j \) is the event that the total arrivals from the classes \( j + 1, \ldots, i - 1 \) are fewer than the cumulative protection level \( L_{i-1} - L_j \) for all
Figure 3-2: Marginal revenue and optimal capacity allocation
3.2. Two Ports, M Classes of Goods Nested Model

these classes.

The event \( x_{j+1} + \ldots + x_i \geq L_i - L_j \) is the event that the total arrivals from the classes \( j+1, \ldots, i \) exceed the cumulative protection level \( L_i - L_j \) that has been set for all the classes \( j+1, \ldots, i \).

The probability distribution of the right hand side of equation 3.73 is the probability, that all of the above events occur during the booking process. In other words, it is the probability that if we start booking from the \( j_{th} \) class of containers, no upper booking limit is reached except for the booking limit of the booking limit of the \( j_{th} \) class of containers.

In the light of the last observation, we consider an example with three classes of containers, and the necessary conditions for optimality become:

\[
\begin{align*}
    f_1 &= f_2 \cdot P(x_2 \geq L_2 - L_1) + f_3 \cdot P([x_2 \leq L_2 - L_1] \cap [x_2 + x_3 \leq L_3 - L_1]) \\
    f_2 &= f_3 \cdot P(x_3 \geq L_3 - L_2)
\end{align*}
\]

(3.74)

If we substitute equation 3.73 at the system of the necessary equations for optimality (equations 3.68), we get a similar form of the general system of the necessary equations. With a few more manipulations at the new form of the system of necessary equations, we find the conditions for optimality as they are presented by Brumelle and McGill ([15]).

In the following subsection, we will try to prove that the solution of the necessary condition of optimality, fulfills the sufficient conditions for maximization. In other words, we will prove that the solution of the necessary conditions of optimality, is the capacity allocation that maximizes the revenues of the vessel.

3.2.3 Second Order Optimality Condition

First we will examine whether the points that satisfy the necessary conditions for optimality of the unconstrained problem, are a maximum of the revenue function. At the end of the section, we will talk briefly about what happens when some of the constraints of our problem are active.
A point \( L^* \) that satisfies the necessary condition of optimality and also satisfies the second order sufficient optimality condition:

\[
\nabla^2 F(L^*) : \text{negative definite}
\]

is a strict local maximum of the twice continuously differentiable function \( F(L) \) ([30]).

From equation 3.43, we get that the second derivatives of the expected revenue are:

\[
\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j \partial L_k} = \sum_{i=j}^M f_i \cdot \frac{\partial^2 E_i}{\partial L_j \partial L_k} \tag{3.76}
\]

For \( i < k \) and \( i < j \) we have:

\[
\frac{\partial^2 E_i}{\partial L_j \partial L_k} = 0 \tag{3.77}
\]

Therefore,

\[
\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j \partial L_k} = \sum_{i=\text{max}(j,k)}^M f_i \cdot \frac{\partial^2 E_i}{\partial L_j \partial L_k} \tag{3.78}
\]

We observe that for \( k < j < i \) we have that:

\[
\frac{\partial^2 E_i}{\partial L_j \partial L_k} = \frac{\partial A(L_1, L_2, \ldots, L_j)}{\partial L_k} \cdot B(L_j, \ldots, L_i) \tag{3.79}
\]

For \( j < k < i \), we get that:

\[
\frac{\partial^2 E_i}{\partial L_j \partial L_k} = A(L_1, L_2, \ldots, L_j) \cdot \frac{\partial B(L_j, \ldots, L_i)}{\partial L_k} \tag{3.80}
\]

We first examine equation 3.79.
\[ \frac{\partial A(L_1, \ldots, L_j)}{\partial L_k} = \]
\[ = \int_{z_1=0}^{\infty} p_1(z_1) \int_{z_2=0}^{\infty} p_2(z_2) \cdots \int_{z_{j-1}=0}^{\infty} p_{j-1}(z_{j-1}) \left\{ \frac{\partial}{\partial L_k} \cdot \int_{z_j=L_j-\sum_{r=1}^{j-1} \varphi_r}^{\infty} p_j(z_j) dz_j \right\} \cdots dz_1 \]
\[ (3.81) \]

After a few manipulations, we get that:
\[ \frac{\partial}{\partial L_k} \cdot \int_{z_j=L_j-\sum_{r=1}^{j-1} \varphi_r}^{\infty} p_j(z_j) dz_j = \left( \sum_{r=1}^{j-1} \frac{\partial \varphi_r}{\partial L_k} \right) \cdot p_j(L_j - \sum_{r=1}^{j-1} \varphi_r) \]
\[ = \left( \sum_{r=k}^{j-1} \frac{\partial \varphi_r}{\partial L_k} \right) \cdot p_j(L_j - \sum_{r=1}^{j-1} \varphi_r) \]
\[ (3.82) \]

From proposition 1 we know that if \( \frac{\partial \varphi_k}{\partial L_k} = 0 \) then all \( \frac{\partial \varphi_r}{\partial L_k} = 0 \), \( \forall \ r > k \). Therefore:
\[ \sum_{r=k}^{j-1} \frac{\partial \varphi_r}{\partial L_k} = 0 \]
\[ (3.83) \]

From proposition 2 we know that if \( \frac{\partial \varphi_k}{\partial L_k} = 0 \), then \( \exists \) one and only one \( s > k \) such that \( \frac{\partial \varphi_r}{\partial L_k} = -1 \), with all \( \frac{\partial \varphi_r}{\partial L_k} = 0 \), \( \forall \ r > k \), \( r \neq s \). As a result,
\[ \sum_{r=k}^{j-1} \frac{\partial \varphi_r}{\partial L_k} = \frac{\partial \varphi_k}{\partial L_k} + \frac{\partial \varphi_s}{\partial L_k} = 1 - 1 = 0 \]
\[ (3.84) \]

We have just proven that irrespectively of weather \( \frac{\partial \varphi_k}{\partial L_k} \) is equal to 0 or 1, the summation \( \sum_{r=k}^{j-1} \frac{\partial \varphi_r}{\partial L_k} = 0 \). Therefore,
\[ \frac{\partial}{\partial L_k} \cdot \int_{z_j=L_j-\sum_{r=1}^{j-1} \varphi_r}^{\infty} p_j(z_j) dz_j = 0 \implies \frac{\partial A(L_1, \ldots, L_j)}{\partial L_k} = 0 \]
\[ (3.85) \]
From equations 3.79 and 3.85, we conclude that for $k < j < i$ we get:

$$\frac{\partial^2 E_i}{\partial L_j \partial L_k} = 0 \quad (3.86)$$

In a similar way, we can prove that for $j < k < i$ we can also get:

$$\frac{\partial^2 E_i}{\partial L_j \partial L_k} = 0 \quad (3.87)$$

The combination of equations 3.78, 3.86 and 3.87 shows that for $j \neq k$,

$$\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j \partial L_k} = 0 \quad (3.88)$$

The above equation shows that the square matrix $\nabla^2 F(L)$ has all the non-diagonal elements equal to zero. We will now show that all the diagonal elements of $\nabla^2 F(L)$, are negative, making $\nabla^2 F(L)$, a negative definite matrix.

From equation 3.69, we have that the second derivatives of the expected revenues $F(L)$, with reference to $L_j$ become:

$$\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j^2} =$$

$$\frac{\partial A(L_1, L_2, \ldots, L_j)}{\partial L_j} \left\{ \sum_{i=j}^{M} f_i \cdot B(L_j, \ldots, L_i) \right\} + A(L_1, \ldots, L_j) \left\{ \sum_{i=j}^{M} f_i \cdot \frac{\partial B(L_j, \ldots, L_i)}{\partial L_j} \right\} \quad (3.89)$$

with

$$\frac{\partial A(L_1, \ldots, L_j)}{\partial L_j} =$$

$$- \int_{z_1=0}^{\infty} p_1(z_1) \int_{z_2=0}^{\infty} p_2(z_2) \cdots \int_{z_{j-1}=0}^{\infty} p_{j-1}(x_{j-1}) \cdot p_j(L_j - \sum_{r=1}^{j-1} \varphi_r)dz_{j-1} \ldots dz_1 \quad (3.90)$$
3.2. Two Ports, M Classes of Goods Nested Model

If we plug the equation 3.70, into the above equation 3.89, we get that:

\[
\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j^2} = A(L_1, \ldots, L_j) \cdot \left\{ \sum_{i=j}^{M} f_i \cdot \frac{\partial B(L_j, \ldots, L_i)}{\partial L_j} \right\} \tag{3.91}
\]

At this point, I rename \( B(L_j, L_{j+1}, \ldots, L_i) \), to \( B_{j+1}(L_j, L_{j+1}, \ldots, L_i) \), and I get the recursive equation:

\[
\frac{\partial B_{r+1}(L_j, L_{r+1}, \ldots, L_i)}{\partial L_j} = -p_{r+1}(L_{r+1} - z_r - \ldots - z_{j+1} - L_j) \cdot B_{r+2}(L_j, L_{r+2}, \ldots, L_i) + \int_{z_{r+1}=0}^{L_{r+1} - z_r - \ldots - z_{j+1} - L_j} p_{r+1}(z_{r+1}) \cdot \frac{\partial B_{r+2}(L_j, L_{r+2}, \ldots, L_i)}{\partial L_j} dz_{r+1} \tag{3.92}
\]

We get this recursive equation for all the indices \( r = j, \ldots, i - 1 \). We define \( B_{r+2}(L_j, L_{r+2}, \ldots, L_i) \) in the following equation:

\[
B_{r+2}(L_j, L_{r+2}, \ldots, L_i) = \int_{z_{r+2}=0}^{L_{r+2} - z_{r+1} - \ldots - z_{j+1} - L_j} p_r(z_{r+2}) \cdots \int_{z_{i-1}=0}^{L_{i-1} - z_{i-2} - \ldots - z_{j+1} - L_j} p_{i-1}(z_{i-1})
\]

\[
\int_{z_{i}=L_i - z_{i-1} - \ldots - z_{j+1} - L_j}^{\infty} (-1) \cdot p_i(z_i) dz_i \, dx_{i-1} \ldots \, dx_{r+2} \tag{3.93}
\]

For \( r = i \) we get that:

\[
B_i(L_j, L_i) = \int_{z_{i}=L_i - z_{i-1} - \ldots - z_{j+1} - L_j}^{\infty} (-1) \cdot p_i(z_i) dx_i \tag{3.94}
\]

and its derivative becomes:

\[
\frac{\partial B_i(L_j, L_i)}{\partial L_j} = -p_i(L_i - z_{i-1} - \ldots - z_{j+1} - L_j) < 0 \tag{3.95}
\]
We can prove by induction that

\[
\frac{\partial B_{j+1}(L_j, L_{j+1}, \ldots, L_i)}{\partial L_j} = \frac{\partial B(L_j, L_{j+1}, \ldots, L_i)}{\partial L_j} < 0 \tag{3.96}
\]

From equation 3.91 and the above equation 3.96 we get

\[
\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j^2} < 0 \tag{3.97}
\]

That concludes the proof that \( \nabla^2 F(L^*) \) is a negative definite matrix. As a result, every point that satisfies the necessary conditions for optimality, also satisfies the sufficient conditions for that point to be a local maximum.

The above proof shows that within the domain of the variables of our optimization problem, there is no local minimum. We can also show that there is only one local maximum, which consequently becomes the global maximum. It is relatively easy to show that if we had at least two local maxima, we would have at least one local minimum, But we have no local minimum, since \( \nabla^2 F(L^*) \) is a negative definite matrix, at every point \( L^* \), where \( \nabla F(L^*) = 0 \). Since we can not have a local minimum, we can not have more than one local maximum. That is, the solution of the equation \( \nabla F(L^*) = 0 \) is unique, and the point \( L^* \) is a global maximum.

**Second order sufficiency conditions for the constrained problem**

If a point satisfies equation 3.39 for the necessary conditions for constrained maximization, and

\[
(x - x^*)' \cdot \nabla^2 f(x^*) \cdot (x - x^*) < 0 \tag{3.98}
\]

then this point is the maximum of the function. The non diagonal elements of \( \nabla^2 f(x^*) \) are equal to zero for the case of constrained optimization too. When

\[
\frac{\partial A(L_1, L_2, \ldots, L_j)}{\partial L_j} \cdot \left\{ \sum_{i=j}^{M} f_i \cdot B(L_j, \ldots, L_i) \right\} > 0, j = 1, \ldots, M - 1 \tag{3.99}
\]
there is always room for all $\frac{\partial^2 F(L_1, L_2, \ldots, L_M)}{\partial L_j^2}, \forall j = 1, \ldots, M - 1$ to be negative. Then the condition 3.98 is valid, and the point that satisfies the necessary conditions for optimality is the maximum of the revenue function. Even when the matrix $\nabla^2 F(L^*)$ is not negative definite, the solution of equation 3.39, can be shown that it gives the maximum of the revenue function. In order to prove that, someone might be able to prove that equation 3.98 holds. Even if equation 3.98 is disproved, we could try to prove that the solution we have found is the maximum by employing duality. We could take the dual of our optimization problem and show that the primal and the dual give the same value for the objective function, in which case the optimality of the above suggested solution will have been proven.

3.2.4 Conclusions

The equation $\nabla F(L^*) = 0$ is the necessary condition that has to be satisfied by an optimal allocation of the vessel capacity. We have proven that every point that satisfies the necessary conditions of optimality, also satisfies the sufficient condition for $F(L^*)$ to be a local maximum. In light of that property, we have argued that the function $F(L)$, has no local minimum, and that helped as establish that the function $F(L)$ does not have more than one local maximum either. Based on that latter observation, our only local maximum, is elevated to global maximum. As we said before, the $L^*$ of this global maximum $F(L^*)$, is the only solution of the equation $\nabla F(L) = 0$. 
3.3 Nested Model with Stand-By Cargo

So far, we have examined the two ports problem with $M$ classes goods that arrive in a sequence of increasing value, and therefore, of increasing freight rate. What happens in the practice of liner shipping is different sometimes. There is a whole class of low unit value cargo, that is time insensitive, and buys space on container ships only when this space has not been bought by other higher valued classes of containers.

In other words, the owners of this stand-by cargo offer it to the vessel operator with the understanding that their cargo will be loaded to the ship only if there is adequate space, and if there is available space, that might be enough for only part of the stand-by cargo. The stand-by cargo that is not offered capacity space, will continue being stand-by cargo, usually for the next sailing of the same shipping company to the desired destination.

For non stand-by cargo, when a shippers offers some cargo for transportation to the shipping company, the company decides in a relatively short time weather they should accept that cargo or not. In case they decline to allocate transportation capacity to the cargo immediately, the offer is withdrawn and the shipper takes his containers to other vessel operators.

The difference between the typical container and the stand-by cargo is that the offer for the transportation of stand-by cargo is continuous and the verification of the transaction between the shipper and the carrier is pending for a long period of time, spanning the time from the offer of the cargo to the sailing of the vessel. The vessel operator decides whether he will accept any stand-by, shortly before departure and only when he is sure that no other offers for higher value cargo are going to be made.

The stand-by cargo is typically cargo of low value, like raw material, semi-finished goods etc. The nature of stand-by cargo is such that it can use the services of either liner or tramp shipping. The utility that the stand-by cargo owners receive from either of the two alternative, and for the nature of their cargo competing, forms of shipping is about the same. As a result, shipping companies charges low prices for stand-by cargo, in order
3.3. Nested Model with Stand-By Cargo

To be competitive with tramp shipping.

In our model we make the assumption that at all times we have ample quantities of stand-by cargo. For computational simplicity, that does not change the results of our study by much, we assume that the available quantities of stand-by cargo are always more than the capacity of the vessel. That simplifies our calculations without changing our results significantly.

We assume that the transportation demand for stand-by cargo is probabilistic. Nevertheless, the distribution is such that it is zero for quantities smaller than the capacity of the carrying capacity of the vessel.

The Model for the case of stand-by cargo is:

$$\max \ F(L_1, \ldots, L_M, L_{sb}) = \sum_{i=1}^{M} f_i \cdot E_i(L_1, \ldots, L_i) + f_{sb} \cdot E_{sb}(L_1, \ldots, L_{sb})$$

(3.100)

subject to:

$$L_{i-1} \leq L_i, \ i = 2, \ldots, M$$
$$L_M \leq L_{sb},$$
$$L_{sb} \leq K,$$
$$L_i \geq 0, \ i = 1, \ldots, M$$

(3.101)

where:

$$E_i(L_1, \ldots, L_i) = \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \ldots \int_{x_i=0}^{\infty} \varphi_i \cdot p_i(x_i) dx_i \ldots dx_2 dx_1$$

(3.102)

$$E_{sb}(L_1, \ldots, L_M, L_{sb}) = \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \ldots$$

$$\ldots \int_{x_M=0}^{\infty} p_M(x_M) \int_{x_{sb}=0}^{\infty} \varphi_{sb} \cdot p_{sb}(x_{sb}) dx_{sb} dx_M \ldots dx_2 dx_1$$
\begin{equation}
\varphi_{ab} = \min(x_{ab}, L_{ab} - \sum_{j=1}^{M} \varphi_{j})
\end{equation}
\begin{equation}
\varphi_{i} = \min(x_{i}, L_{i} - \sum_{j=1}^{i-1} \varphi_{j}), \quad i = 2, \ldots, M
\end{equation}
\begin{equation}
\varphi_{1} = \min(x_{1}, L_{1})
\end{equation}

Most of the symbols in the above optimization problem have been explained in section 3.2. Additionally:

\( f_{ab} = \) the fare paid by a container with stand-by cargo for transportation from port 1 to port 2

\( x_{ab} = \) the number of stand-by containers, that will ask for transportation from port 1 to port 2

\( p_{ab}(x_{ab}) = \) the probability distribution of standby cargo asking for transportation from port 1 to port 2. Let \( \alpha > K \), where \( K \) is the capacity of the vessel. Then we have:
\begin{equation}
p_{ab}(x) = \begin{cases} 
0 & \text{for } x < \alpha \\
g(x) & \text{for } x \geq \alpha 
\end{cases}
\end{equation}

\( L_{ab} = \) the maximum space that the vessel operator offers cumulatively to all classes of goods and the stand-by cargo, for transportation from port 1 to port 2. \( (\varphi_{1} + \ldots + \varphi_{M} + \varphi_{ab} \leq L_{ab}) \). Obviously \( L_{ab} = K \), where \( K \) is the vessel capacity

\( E_{X_{ab}} = \) the expected number of containers of stand-by cargo that will be transported from port 1 to port 2.
First we will find a simpler expression for the innermost integral of the right hand side of equation 3.103.

\[
\int_{z_{ab}=0}^{\infty} \varphi_{ab} \cdot p_{ab}(x_{ab}) \, dz_{ab} = \int_{z_{ab}=0}^{\infty} \varphi_{ab} \cdot g(x_{ab}) \, dz_{ab} \quad (3.106)
\]

\[= \int_{z_{ab}=0}^{\infty} (L_{ab} - \sum_{j=1}^{M} \varphi_j) \cdot g(x_{ab}) \, dz_{ab} \quad (3.107)\]

\[= L_{ab} - \sum_{j=1}^{M} \varphi_j \quad (3.108)\]

\[\Rightarrow \int_{z_{ab}=0}^{\infty} \varphi_{ab} \cdot p_{ab}(x_{ab}) \, dz_{ab} = L_{ab} - \sum_{j=1}^{M} \varphi_j \quad (3.109)\]

The transition from the left to the right hand side of equation 3.106, is a direct consequence of equation 3.105. If we substitute \( \varphi_{ab} \) from equation 3.104 into equation 3.106, we get equation 3.107. \( L_{ab} \) and \( \varphi_j \), \( j = 1, \ldots, M \), are independent of \( z_{ab} \). This observation along with the fact that \( \int_{z_{ab}=0}^{\infty} g(x_{ab}) \, dz_{ab} = 1 \), result in equation 3.108.

From the combination of equations 3.103 and 3.109, we get:

\[E_{ab}(L_1, \ldots, L_M) =\]

\[= \int_{z_1=0}^{\infty} p_1(z_1) \int_{z_2=0}^{\infty} p_2(z_2) \cdots \int_{z_M=0}^{\infty} (L_{ab} - \sum_{j=1}^{M} \varphi_j) \cdot p_M(x_M) \, dz_M \cdots dz_2 \, dz_1 \quad (3.110)\]

\[= L_{ab} - \sum_{j=1}^{M} \left\{ \int_{z_1=0}^{\infty} p_1(z_1) \int_{z_2=0}^{\infty} p_2(z_2) \cdots \int_{z_M=0}^{\infty} \varphi_j \cdot p_M(x_M) \, dz_M \cdots dz_2 \, dz_1 \right\} \quad (3.111)\]

\[= L_{ab} - \sum_{j=1}^{M} \left\{ \int_{z_1=0}^{\infty} p_1(z_1) \int_{z_2=0}^{\infty} p_2(z_2) \cdots \int_{z_j=0}^{\infty} \varphi_j \cdot p_j(x_j) \, dz_j \cdots dz_2 \, dz_1 \right\} \quad (3.112)\]

\[= L_{ab} - \sum_{j=1}^{M} E_j(L_1, \ldots, L_j) \quad (3.113)\]

The observation that \( L_{ab} \) is independent from \( x_1, \ldots, x_M \), justifies the transition from
equation 3.110, to equation 3.111. The transition to equation 3.112 is obtained with the observation that \( \varphi_j \), is independent from \( x_{j+1}, \ldots, x_M \). The expression into the parentheses of equation 3.112, is the definition of the expected number of reservations from equation 3.102. Therefore we have proven that:

\[
E_{ab}(L_1, \ldots, L_M, L_{ab}) = L_{ab} - \sum_{j=1}^{M} E_j(L_1, \ldots, L_j)
\]  \hspace{1cm} (3.114)

We accept all the stand by cargo up to the capacity limit of the vessel. Therefore, we substitute \( L_{ab} = K \), and equation 3.114, becomes:

\[
E_{ab}(L_1, \ldots, L_M, L_{ab}) = K - \sum_{j=1}^{M} E_j(L_1, \ldots, L_j)
\]  \hspace{1cm} (3.115)

Equation 3.115, suggests that the expected number of stand-by containers that the vessel will accept for transportation is equal to the vessel capacity decreased by the expected capacity of all the other classes of cargo that book with the shipping liner. Equation 3.115, results from the assumption that at all times we have enough stand by cargo to fill the vessel. Any capacity that is not occupied by the containers of the other freight classes will be filled with stand-by containers.

If we substitute the above equation at the objective function of the Nested Model with stand-by cargo, we get:
\[ \max F(L_1, \ldots, L_M) = f_{\text{sb}} \cdot K + \sum_{i=1}^{M} (f_i - f_{\text{sb}}) \cdot E_i(L_1, \ldots, L_i) \]

(subject to:

\[ L_{i-1} \leq L_i, \quad i = 2, \ldots, M \]

\[ L_M \leq K, \]

\[ L_i \geq 0, \quad i = 1, \ldots, M \]

where:

\[ E_i(L_1, \ldots, L_i) = \int_{x_1=0}^{\infty} p_1(x_1) \int_{x_2=0}^{\infty} p_2(x_2) \cdots \int_{x_i=0}^{\infty} \varphi_i \cdot p_i(z_i) dx_i \cdots dx_2 dx_1 \]

(3.117)

\[ \varphi_i = \min(z_i, L_i - \sum_{j=1}^{i-1} \varphi_j), \quad i = 2, \ldots, M \]

(3.118)

\[ \varphi_1 = \min(z_1, L_1) \]

In this section we have shown how we can get from the original formulation of equation 3.100, of the booking problem with standby cargo, to the final formulation of equation 3.116.

If we go back to our assumptions we see that the formulation of equation 3.116 make intuitive sense. We have assumed that we have "ample" stand-by cargo, i.e. that the available stand-by cargo is at least enough to fill the whole vessel.

The intuition behind the formulation is the following. If we have "ample" stand-by cargo to fill the vessel, we do so. In this case the revenues of the vessel operator is \( f_{\text{sb}} \cdot K \). Every time a container that belongs to one of the \( M \) classes of goods arrives, i.e. a
container of class $i$, and we decide to accept it, we unload a container of stand-by cargo, and we pay back its owner the freight rate $f_{sb}$. Then the container that belongs to class $i$ is loaded to the vessel, and the vessel operator earns the corresponding freight rate $f_i$. The net difference for the operator from this transaction is $f_i - f_{sb}$. This scenario gives the following interpretation to the stand-by cargo problem. The earning potential of the vessel is at least equal to the revenues from the stand-by cargo, and from then on, each container from every class of goods has an adjusted freight rate of $f_i - f_{sb}$. 
3.4 Approximate Nested Models

3.4.1 Approximate Nested Problem for Two Ports and M Classes of Goods

In the previous section, we have solved the two ports problem for $M$ Classes of goods. We have also presented a variation of the model that included a modification for stand-by cargo. At the above mentioned models we see that the objective function includes integrals up to order $M$, where $M$ is equal to the number of classes of containers that we accept for transportation.

In order to solve the optimization problem for the two ports and $M$ classes of goods, represented by the formulation 3.36 or the stand-by problem described at the optimization model 3.116, we will employ a gradient method. This gradient method can range from the steepest descent method, to a variation of the Newton method. No matter which method we choose for the solution of the optimization problem, we have to calculate the derivative of the objective function of the optimization problem. The derivative is the sum of some integrals that can be of order up to $M$. The calculation of these integrals, which has to be repeated at every iteration of the optimization algorithm, presents two potential problems. The first problem is, that even for a modest number of classes of goods, the order of multiplicity of the integrals becomes so big that the calculation of the integrals becomes costly and prevents the solution of problems that can have practical meaning and use. The second problem is associated with the first. The computational intensity necessary for the calculation of the multiple integrals makes the accuracy of the results of the computation to be questionable.
We will try to develop an approximate model which will not encounter the bottleneck of the calculation of multiple integrals of multiplicity that we have mentioned above. We formulate the problem as follows:

$$\max \quad F(L_1, \ldots, L_M) = \sum_{i=1}^{M} f_i \cdot E_i(L_{i-1}, L_i) \quad (3.119)$$

subject to:

$$L_{i-1} \leq L_i, \quad i = 2, \ldots, M$$
$$L_M \leq K,$$
$$L_i \geq 0, \quad i = 1, \ldots, M$$

where:

$$E_1(L_1) = \int_{x_1=0}^{\infty} \varphi_1 \cdot p_1(x_1)dx_1 \quad (3.120)$$

$$E_i(L_{i-1}, L_i) = \int_{y_{i-1}=0}^{\infty} g_{i-1}(y_{i-1}) \int_{x_1=0}^{\infty} \varphi_i \cdot p_i(x_i)dx_i dy_{i-1} \quad (3.121)$$

$$\varphi_1 = \min(x_1, L_1)$$
$$\varphi_i = \min(x_i, L_i - \nu_{i-1}), \quad i = 2, \ldots, M$$
$$\nu_i = \min(y_i, L_i), \quad i = 1, \ldots, M - 1 \quad (3.122)$$

where,

$$p_i(x_i) = \text{the probability distribution of goods that belong to class } i \text{ and ask for transportation from port 1 to port 2. We assume that the distribution } p_i(x_i) \text{ is Normal with mean } \mu_i \text{ and variance } \sigma_i^2$$

$$y_i = \text{the total number of the containers from class 1 up to class } i, \text{ that will ask for transportation from port 1 to port 2. } y_i = \sum_{j=1}^{i} x_j$$
3.4. Approximate Nested Models

$g_i(y_i) = \text{the probability distribution of the sum of goods of class } i \text{ up to } i \text{ asking for transportation from port 1 to port 2. This distribution is also Normal with mean equal to } \sum_{j=1}^{i} \mu_j \text{ and variance equal to } \sum_{j=1}^{i} \sigma_j^2$

$L_i = \text{the maximum space that the vessel operator assigns cumulatively to classes 1 up to } i, \text{ for transportation from port 1 to port 2.}$

$(\sum_{j=1}^{i} \varphi_j \leq L_i)$

$E_i(L_{i-1}, L_i) = \text{the expected number of containers of class } i \text{ that will be transported from port 1 to port 2.}$

All the other symbols have been explained in previous versions of our model.

**First Order Optimality Condition**

The necessary condition for a point $x^*$ to be the unconstrained extremum of a function, is that it satisfies the condition ([30]):

$$\nabla f(x^*) = 0 \quad (3.123)$$

In our case, the necessary condition 3.123 becomes:

$$\frac{\partial F(L_1, \ldots, L_M)}{\partial L_j} = 0, \; j = 1, \ldots, M \quad (3.124)$$

From the objective function of the formulation 3.119 we get:

$$\frac{\partial F(L_1, L_2, \ldots, L_M)}{\partial L_j} = \sum_{i=1}^{M} f_i \cdot \frac{\partial E_i(L_{i-1}, L_i)}{\partial L_j} \quad (3.125)$$

From equation 3.125, we see that for $j \neq i - 1$ or $j \neq i$,

$$\frac{\partial E_i(L_{i-1}, L_i)}{\partial L_j} = 0 \quad (3.126)$$
From the combination of equations 3.121 and 3.125, we get that:

\[
\frac{\partial F(L_1, L_2, \ldots, L_M)}{\partial L_j} = f_j \cdot \frac{\partial E_j(L_{j-1}, L_j)}{\partial L_j} + f_{j+1} \cdot \frac{\partial E_{j+1}(L_j, L_{j+1})}{\partial L_j} = 0 \quad (3.127)
\]

First, we will calculate the derivative \( \frac{\partial E_j}{\partial L_j} \). For \( j = 1 \), and from equation 3.120, we get:

\[
\frac{\partial E_1(L_1)}{\partial L_1} = \int_{x_1=0}^{\infty} \frac{\partial \varphi_1}{\partial L_1} \cdot p_1(x_1)dx_1 = \int_{x_1=L_1}^{\infty} p_1(x_1)dx_1 \quad (3.128)
\]

For \( j > 1 \), from equation 3.121 we have:

\[
\frac{\partial E_j(L_{j-1}, L_j)}{\partial L_j} = \int_{y_{j-1}=0}^{\infty} g_{j-1}(y_{j-1}) \int_{x_j=0}^{\infty} \frac{\partial \varphi_j}{\partial L_j} \cdot p_j(x_j) dx_j dy_{j-1} \quad (3.129)
\]

From equations 3.122, we get that,

\[
\frac{\partial \varphi_j}{\partial L_j} = \begin{cases} 
1 & \text{for } x_j \geq L_j - \nu_{j-1} \\
0 & \text{otherwise}
\end{cases} \quad (3.130)
\]

Equations 3.129 and 3.130 give the following result:

\[
\frac{\partial E_j(L_{j-1}, L_j)}{\partial L_j} = \int_{y_{j-1}=0}^{\infty} g_{j-1}(y_{j-1}) \int_{x_j=L_j-\nu_{j-1}}^{\infty} p_j(x_j) dx_j dy_{j-1} \quad (3.131)
\]

Next, we want to calculate the derivative \( \frac{\partial E_{j+1}}{\partial L_j} \). From equation 3.121, we get:

\[
\frac{\partial E_{j+1}(L_j, L_{j+1})}{\partial L_j} = \int_{y_j=0}^{\infty} g_j(y_j) \int_{x_{j+1}=0}^{\infty} \frac{\partial \varphi_{j+1}}{\partial L_j} \cdot p_{j+1}(x_{j+1}) dx_{j+1} dy_j \quad (3.132)
\]

and

\[
\frac{\partial \varphi_{j+1}}{\partial L_j} = \begin{cases} 
-1 & \text{for } x_{j+1} \geq L_{j+1} - \nu_j \text{ and } y_j \geq L_j \\
0 & \text{otherwise}
\end{cases} \quad (3.133)
\]
Equations 3.132, and 3.133 give

$$\frac{\partial E_{j+1}(L_j, L_{j+1})}{\partial L_j} = - \int_{y_j=L_j}^{\infty} g_j(y_j) \int_{x_{j+1}=L_{j+1} - \nu_j}^{\infty} p_{j+1}(x_{j+1}) dx_{j+1} dy_j \quad (3.134)$$

We can simplify equation 3.134 further, if we notice that the lower limit of the inner integral, is a function of $y_j$, since $\nu_j = \min(y_j, L_j)$. We also notice that the lower limit of the outer integral, forces the $y_j$ to be greater or equal to $L_j$. We therefore conclude that $\nu_j = L_j$. This substitution shows that the inner integral is independent of $y_j$. Therefore the double integral of equation 3.134 can be reduced to the product of two single integrals, shown in the following equation.

$$\frac{\partial E_{j+1}(L_j, L_{j+1})}{\partial L_j} = - \int_{y_j=L_j}^{\infty} g_j(y_j) dy_j \cdot \int_{x_{j+1}=L_{j+1} - L_j}^{\infty} p_{j+1}(x_{j+1}) dx_{j+1} \quad (3.135)$$

If we substitute equations 3.131 and 3.135, into the optimality equation 3.127, we get after a few manipulations:

$$f_1 = f_2 \cdot P(x_2 \geq L_2 - L_1) \quad (3.136)$$

$$f_j \cdot \{P(y_{j-1} < L_{j-1} \text{ and } y_j \geq L_j) + P(y_{j-1} \geq L_{j-1}) \cdot P(x_j \geq L_j - L_{j-1})\} = f_{j+1} \cdot P(y_j \geq L_j) \cdot P(x_{j+1} \geq L_{j+1} - L_j), \text{ for } j = 2, \ldots, M - 1 \quad (3.137)$$

We have to keep in mind that the events $y_{j-1} < L_{j-1}$ and $y_j \geq L_j$, are not independent, since $y_j = y_{j-1} + x_j$. As a result, the left hand side of equation 3.137 includes a double integral.

We can make the following comments on the system of equations 3.136 and 3.137.

As we have mentioned in section 3.2, we are mainly interested in finding the allocation limit $L_1$ for the class of containers 1, which is the current or the first class of containers
to reserve transportation capacity. When we have stopped accepting the 1st class of containers and we continue with the 2nd up to the $M_{th}$ class of containers, we can simply rename them to $1_{st}, \ldots, M - 1_{th}$. Equation 3.136 gives the optimal limit for $L_1$ as a function of $L_2$, which is unknown. If we employ equation 3.137, for $j = 2$, we see that this equation gives the relation between $L_1, L_2$, and $L_3$. We have to use equation 3.137 for $j = 2, \ldots, M - 1$, in order to have $M - 1$ equations for $M$ unknowns. The $M_{th}$ equation is given by the equation $L_M = K$. Each equation of the above system of equations have at most one double integral, that we have to calculate. This is a considerable improvement over the multiple integrals of $M_{th}$ order that we have to calculate at the original model that we developed.

The only difference between the optimal problem formulation that we presented in section 3.2 and here, is that in section 3.2 we presented $\varphi_i$ to be a function of the following:

$$\varphi_i = f(L_1, L_2, \ldots, L_i, z_1, z_2, \ldots, z_i)$$ (3.138)

whereas in the current formulation, we considered $\varphi_i$ to be a function of the following variables:

$$\varphi_i = f(L_{i-1}, L_i, y_i = \sum_{j=1}^{i} x_j)$$ (3.139)

The above reformulation does not increase the calculations in any way, if we use the Normal distribution (which is the distribution assumed by most of the researchers in the area of yield management). The distribution of $y_i$ is also a Normal distribution with mean the sum of the means of the distributions we add up to create $y_i$, and variance the sum of the variances of the same distributions.

If we go back to section 3.2, we will see that the first of the system of equations that we have to solve in order to find the optimal booking limits, is indeed equation 3.136. We solve the whole system of equations in order to find a solution for $L_1$. I am certain that the uncertainty, introduced at the original system of equations through the assumption of the demand distribution, and the previously mentioned numerical errors from the calculation
of the multiple integrals, is similar or greater than the uncertainty introduced by the downgrading from the full model to the approximate.

**Marginal Value Method**

In this section we developed the approximate model, which is a modeling relaxation of the Nested model that we developed before. The booking limits are the solutions of the system of equations 3.127 for all \( j = 1, \ldots, M - 1 \). (The derivatives of the expected value of the revenues are given by equations 3.131 and 3.135). The derivative \( \frac{\partial E_{i+1}(L_{i}, L_{i+1})}{\partial L_{i}} \) involves the calculation of two single integrals. On the other hand the calculation of \( \frac{\partial E_{i}(L_{i-1}, L_{i})}{\partial L_{i}} \), involves the calculation of a double integral. If we want to simplify our model further, we could make a further simplifying assumption. We could decouple the double integral, and consider it equivalent to the product of two single integrals. In other words, we suggest the following assumption:

\[
\frac{\partial E_{j}(L_{j-1}, L_{j})}{\partial L_{j}} \approx \int_{y_{j-1}=0}^{\infty} g_{j-1}(y_{j-1}) dy_{j-1} \cdot \int_{z_{j}=L_{j} - \bar{v}_{j-1}}^{\infty} p_{j}(x_{j}) dx_{j}
\]

or

\[
\frac{\partial E_{j}(L_{j-1}, L_{j})}{\partial L_{j}} \approx \int_{z_{j}=L_{j} - \bar{v}_{j-1}}^{\infty} p_{j}(x_{j}) dx_{j}
\]  

where

\[
\bar{v}_{j} = \int_{y_{j}=0}^{\infty} \nu_{j} \cdot g_{j}(y_{j}) dy_{j}, \ j = 1, \ldots, M - 1,
\]

\[
\bar{v}_{0} = 0
\]  

(3.140)
If we substitute the above simplifications (equations 3.140 and 3.135), at the equation of the necessary condition of optimality (equation 3.127), we get:

\[
\begin{align*}
  f_j \cdot \int_{x_j = L_j - \bar{\nu}_{j-1}}^{\infty} p_j(x_j) dx_j &= f_{j+1} \cdot \int_{y_j = L_j}^{\infty} g_j(y_j) dy_j \cdot \int_{x_{j+1} = L_{j+1} - \bar{\nu}_{j-1}}^{\infty} p_{j+1}(x_{j+1}) dx_{j+1} , \\
  j &= 1, \ldots, M - 1
\end{align*}
\]  

(3.142)

By changing the notation, we get:

\[
\begin{align*}
  f_j \cdot P(x_j \geq L_j - \bar{\nu}_{j-1}) &= f_{j+1} \cdot P(y_j \geq L_j) \cdot P(x_{j+1} \geq L_{j+1} - L_j) , \\
  j &= 1, \ldots, M - 1
\end{align*}
\]  

(3.143)

At the system of equations 3.143, we have \( M \) unknowns and \( M - 1 \) equations. The \( M \)th equation is

\[
L_M = K
\]  

(3.144)

where \( K \) is the capacity of the vessel. Therefore, we can solve the system of equations 3.143 and 3.144. The \( j \)th among the equations 3.143, is a function of \( L_{j-1}, L_j, L_{j+1} \) (we remind that \( \bar{\nu}_{j-1} \) is a function of \( L_{j-1} \)). The first among them, is a function of only the \( L_1 \) and \( L_2 \). If we make the notation a little more compact, our system of equations becomes:

\[
\begin{align*}
  f_j \cdot MVL(L_{j-1}, L_j) &= f_{j+1} \cdot MVR(L_j, L_{j+1}) , \\
  j &= 1, \ldots, M - 1
\end{align*}
\]

\[
L_M = K
\]  

(3.145)

If we solve the system of equations 3.145, we can get the booking limits, \( L_1, \ldots, L_{M-1}, L_M = K \). The solution of this system of equations is easy, because each equation \( j \) is a function of three single integrals that are easy to calculate.

We call this simplified method for the derivation of the booking limits, for the one leg, \( M \) classes of goods problem, the Marginal Value Method.
3.4. Approximate Nested Models

Figure 3-3: Input and output of the yield management optimization model

Customers with Many Containers

A major assumption of all the models on which we have worked, is that each arriving customer asks for transportation for only one container. Many times customers arrive with more than one container. There is a possibility that a customer might want transportation for several containers that belong to different classes of goods. The difference of the scenario mentioned above from the standard approach that we followed so far, is that when a customer gives an order for several containers, we either accept all the containers or we decline them all. We do not have the ability to accept some of the containers up to the booking limit of the class to which they belong, and decline to offer transportation capacity to the other containers, that would have to go to other vessel operators.

The approach that we propose is the following. The optimization model we have developed in section 3.2, gives the expected revenues if we impose optimal booking allocation.

The input of the optimization model is the probability distributions for the demand of the different classes of customers, and the remaining capacity of the vessel. The output of the model, is the optimal upper booking limits and the expected revenues under optimal booking policy. (See Figure 3-3)

When we treat the one customer/one container problem, we only have to use the
optimal upper booking limits from the model. When we consider the one customer/many containers problem, we will use the other output of the model, that is the expected optimal revenues. We therefore conclude that:

$$\text{Maximum Expected Revenues} = ER(K, DF)$$

(3.146)

where $K$ is the remaining vessel capacity, and $DF$ is the demand forecast.

When a customer with several containers wants to have them transported with the vessel of the shipping liner, the operator has two options. He can either accept the order of the customer or refuse it. The operator will accept a multiple order from a customer, if the revenue from this particular order, plus the expected optimal revenue from the remaining of the capacity of the vessel, is greater than the expected maximum revenue from the remaining capacity, if we do not accept the order. In other words, if a multiple order consists of $x_i$ containers, from each class of containers $i$, the vessel operator will accept this multiple order if:

$$\sum_{i=1}^{M} f_i \cdot x_i + ER(K - \sum_{i=1}^{M} x_i, DF) \geq ER(K, DF)$$

(3.147)

The demand forecast does not depend on our decisions, for the additional reason that the model assumes that any customers that were turned away, will find other shipping liners to transport their containers.

When all the containers offered by the customer belong to classes that we currently accept for transportation, and the number of containers per class, are fewer than the current upper limits, then there is no need to apply the method we developed in this paragraph. If any of the two conditions are violated, then we have to apply our new methodology.
3.4. Approximate Nested Models

3.4.2 Approximate Nested Problem for N Legs and M Goods

We have confined our analysis so far to the two ports problem. The assumption of the ordering of the classes of containers, is a quite reasonable assumption for the two ports problem. If we try to extend our model and the experience to the $M$ ports problem, we see that several problems arise. The first problem, is a modeling problem. We have extensively talked in section 1.1 about the weakness of the ordering assumption in the case of liner shipping, and we have shown it at figure 1-1. Several authors who have tried to extend the one leg model to cover the multi leg yield management problem for the airline industry, have agreed that the fidelity of the model to the process that it describes becomes weaker as we add legs to the itinerary. As we have already mentioned, the nature of the shipping industry is such that the multi leg extension of the ocean yield management model, becomes much less credible than the equivalent models of the airline industry.

In the absence of better formulation, we will follow the example of the airline yield management researchers, and we will extend the model, for the multi leg problem, with the additional assumption that there is no competition between different itineraries for the same capacity.

In other words, the model that we will develop, will be nested only for the classes of goods that have the same port of origin and the same port of destination. Our assumption of ordered arrival of the customers, according to the reverse order of value of their containers, is still valid for the goods with the same origin and destination.

The model that we develop here, is a network model for the different itineraries, and it becomes a nested model only for the classes of containers that have a common origin and destination.

A second problem we face when we extend the one leg model to cover the multi-leg problem, is the computational intensity needed for the calculation of the multiple integrals of the one leg problem. The multiplicity of the integrals does not become greater to the multi-leg problem, especially since we have assumed that we will treat the problem as nested only on every itinerary. What is the problem is that instead of only one, we have
different itineraries at the \( N \) leg model. The computational intensity of the multi-leg model would increase proportionally. In order to bypass this computational barrier of a model that would not give accurate results in the first place, instead of the "accurate" two ports, \( M \) goods model, we extend the approximate two ports, \( M \) goods model to describe the multi-leg multi-product ocean yield management problem.

In the following, we will develop the optimization model for the \( N \) legs (\( N + 1 \) ports) and \( M \) goods problem. The method for the derivation of the optimality conditions is similar to the method that we followed for the derivation of the one leg \( M \) classes of goods problem, and we will skip steps that are not necessary for the understanding of the process.

The mathematical formulation of the Approximate Nested Problem for \( N \) legs (\( N + 1 \) ports) and \( M \) goods, has as follows:

\[
\begin{align*}
\max \quad & F(LOD_{j,1}, \ldots, LOD_{j,M}, \forall j) = \sum_{\text{All itineraries}} \sum_{i=1}^{M} f_{OD_{j,i}} \cdot E_{OD_{j,i}}(LOD_{j,i-1}, LOD_{j,i}) \\
\text{subject to:} \quad & \sum_{\text{All itineraries}} \sum_{\text{OD}_{j,\text{that use legs},i+1}} LOD_{j,M} \leq K_{i,i+1}, \quad i = 1, \ldots, N \\
& LOD_{j,i-1} \leq LOD_{j,i}, \quad \forall OD_{j}, \quad i = 2, \ldots, M \\
& LOD_{j,i} \geq 0, \quad \forall OD_{j}, \quad i = 1, \ldots, M
\end{align*}
\]  

(3.148)  

(3.149)  

(3.150)  

(3.151)

where,

\[
E_{OD_{j,1}}(LOD_{j,1}) = \int_{x_{OD_{j,1}}=0}^{\infty} \varphi_{OD_{j,1}} \cdot p_{OD_{j,1}}(x_{OD_{j,1}}) \, dx_{OD_{j,1}}
\]  

(3.152)
3.4. Approximate Nested Models

\[ E_{OD,j,i}(L_{OD,j,i-1}, L_{OD,j,i}) = \]
\[ = \int_{y_{OD,j,i-1}}^{\infty} g_{OD,j,i-1}(y_{OD,j,i-1}) \int_{x_{OD,j,i}}^{\infty} \phi_{OD,j,i} \cdot p_{OD,j,i}(x_{OD,j,i}) dx_{OD,j,i} dy_{OD,j,i-1} \]  
(3.153)

\[ \phi_{OD,j,1} = \min(x_{OD,j,1}, L_{OD,j,1}) \]
\[ \phi_{OD,j,i} = \min(x_{OD,j,i}, L_{OD,j,i} - \nu_{OD,j,i-1}), \ i = 2, \ldots, M \]  
(3.154)
\[ \nu_{OD,j,i} = \min(y_{OD,j,i}, L_{OD,j,i}), \quad i = 1, \ldots, M - 1 \]

and,

\[ p_{OD,j,i}(z_{OD,j,i}) = \text{the probability distribution of goods that belong to class } i \text{ and ask for transportation over the itinerary } OD_j. \]

We assume that the distribution \( p_{OD,j,i}(z_{OD,j,i}) \) is Normal with mean \( \mu_{OD,j,i} \) and variance \( \sigma^2_{OD,j,i} \).

\[ y_{OD,j,i} = \text{the total number of the containers from class 1 up to class } i, \text{ that will ask for transportation over the itinerary } OD_j. \]

\[ y_{OD,j,i} = \sum_{j=1}^{i} x_{OD,j,j} \]

\[ g_{OD,j,i}(y_{OD,j,i}) = \text{the probability distribution of the sum of goods of class 1 up to } i \text{ asking for transportation over the itinerary } OD_j. \]

This distribution is also Normal with mean equal to \( \sum_{j=1}^{i} \mu_{OD,j,j} \) and variance equal to \( \sum_{j=1}^{i} \sigma^2_{OD,j,j} \).

\[ L_{OD,j,i} = \text{the maximum space that the vessel operator assigns cumulatively to classes 1 up to } i, \text{ for transportation over the itinerary } OD_j. \]

\[ (\sum_{j=1}^{i} \phi_{OD,j,j} \leq L_{OD,j,i}) \]
\[ E_{OD,j,i}(L_{OD,j,i-1}, L_{OD,j,i}) = \text{the expected number of containers of class } i \text{ that will be transported over the itinerary } OD_j. \]

whereas the other symbols have already been explained elsewhere.
The necessary condition for optimality is:

\[
\sum_{\text{All itineraries } OD_j} \sum_{i=1}^{M} \left\{ \frac{\partial F}{\partial LOD_{j,i}} \bigg|_{LOD_{j,i}} \cdot (LOD_{j,i} - LOD_{j,i}) \right\} \geq 0 \tag{3.155}
\]

where \( LOD_{j,i} \) is the optimal value of \( LOD_{j,i} \). The derivative \( \frac{\partial F}{\partial L} \) is:

\[
\frac{\partial F}{\partial LOD_{j,i}} = f_{OD_{j,i}} \cdot \frac{\partial E_{OD_{j,i}}(LOD_{j,i-1}, LOD_{j,i})}{\partial LOD_{j,i}} + f_{OD_{j,i+1}} \cdot \frac{\partial E_{OD_{j,i+1}}(LOD_{j,i}, LOD_{j,i+1})}{\partial LOD_{j,i}} \tag{3.156}
\]

First, we will calculate the derivative \( \frac{\partial E_{OD_{j,i}}}{\partial LOD_{j,i}} \). For \( i^* = 1 \), we have:

\[
\frac{\partial E_{OD_{j,1}}(LOD_{j,1})}{\partial LOD_{j,1}} = \int_{x=LOD_{j,1}}^{\infty} p_{OD_{j,1}}(x) dx \tag{3.157}
\]

For \( i > 1 \), we have:

\[
\frac{\partial E_{OD_{j,i}}(LOD_{j,i-1}, LOD_{j,i})}{\partial LOD_{j,i}} = \int_{y=0}^{\infty} g_{OD_{j,i-1}}(y) \int_{x=LOD_{j,i-1}, y}^{\infty} p_{OD_{j,i-1}}(x) dx dy \tag{3.158}
\]

and the derivative \( \frac{\partial E_{j,i+1}}{\partial LOD_{j,i}} \), becomes:

\[
\frac{\partial E_{OD_{j,i+1}}(LOD_{j,i}, LOD_{j,i+1})}{\partial LOD_{j,i}} = - \int_{y=LOD_{j,i}}^{\infty} g_{OD_{j,i}}(y) dy \cdot \int_{x=LOD_{j,i+1}, y}^{\infty} p_{OD_{j,i+1}}(x) dx \tag{3.159}
\]

We will not derive equations equivalent to the equations 3.136 and 3.137, because the equivalent system of equations does not necessarily correspond to the optimality criterion for our original yield management model. The more general optimality criterion, is given
by equation 3.155. The solution of the original optimization problem 3.148 will be done with one of the gradient methods.

It is relatively easy to extend the Marginal Value Method, originally developed for the one leg problem, for the current case of the multi leg, multi class problem.

Weather we use the approximate model that we have developed here, or the Marginal Value Method modified for the multi leg problem, our iterative solution has two steps at each iteration.

1. We solve a linear program to find the space allocation to the different itineraries.
   The space allocated to each itinerary, is to be used exclusively by customers who want to use this particular itinerary.

2. When we have allocated some space to an itinerary, we have reduced our problem to the solution of the two ports, $M$ classes of containers problem. We compute the optimal booking limits for each one of the container classes. The vessel capacity for the reduced problem, is the capacity allocated to this itinerary, at step 1.
Chapter 4

Nested Model: Results

4.1 Numeric Solution of the Nested Problem

The optimization problem at hand is a problem that has a non linear objective function with linear constraints. We will solve this problem with the iterative Feasible Directions Algorithm.

\[
F = \max f(x)
\]
\[
s.t. \quad Az \leq b
\]
\[
z \geq 0
\]

\(f(x)\) has continuous second partial derivatives. We choose an initial \(x^0\), that satisfies the constraints of the model. We solve the linear program:

\[
\max \quad \nabla f(x^k) \cdot z
\]
\[
s.t. \quad Az \leq b
\]
\[
z \geq 0
\]
The optimal solution of the above linear program is $z^k$. We create the difference,

$$r^k = z^k - z^k$$

and we calculate,

$$t_k' = -\frac{\nabla f(z^k) \cdot r^k}{D}$$

where

$$|(y - z)H_f(w)(y - z)| \leq D$$

$$\forall \ x, y, w \in G = \{x/Ax \leq b, x \geq 0\}$$

where, $H_f(w)$ is the Hessian matrix of the function $f(z)$.

We define

$$t_k = \min\{t_k', 1\}$$

and we find the new

$$z^{k+1} = z^k + t_k r^k$$

Each $z^k \in G$ and $r^k$ is a feasible direction of ascent at $z^k$. As a result, $z^{k+1} = z^k + t_k r^k$ is feasible for $t_k \geq 0$ and sufficiently small. Every $f(z^k)$ satisfies the following inequalities:

$$f(z^{k+1}) \geq f(z^k)$$

and

$$f(z^k) \leq F \leq f(z^k) + \nabla f(z^k) \cdot r^k$$

The above shows that a function that fulfills the assumptions stated here, converges, and a convenient criterion for the termination of the iterative process, is $|\nabla f(z^k) \cdot r^k|$. We stop the iterations when the following inequality is satisfied.

$$|\nabla f(z^k) \cdot r^k| \leq err$$
err, stands for the maximum error that we are willing to accept.

We could have also employed Newton's Method.

\[ z^{k+1} = z^k - \alpha_k \cdot D^k \cdot r^k \]  

(4.1)

where

\[ D^k = (\nabla^2 f(z^k))^{-1} \]  

(4.2)

provided that \( \nabla^2 f(z^k) \) is negative definite. In the area where that does not happen, we can modify \( D^k \), so as to make it negative definite. In our case we have found that \( \nabla^2 f(z^k) \) is a diagonal matrix. Therefore, the inverse of \( \nabla^2 f(z^k) \) is just the diagonal matrix with nonzero entries the inverse of the entries in \( \nabla^2 f(z^k) \). For more details see [30].

4.2 Input and Output of the Computer Code

4.2.1 Input of the Computer Code

Horizon of the Optimization The horizon of the optimization adapts to the believes, confidence and the experience of the vessel operator. The longer the time horizon is, the better the results of the operations are (improved revenues or profits). On the other hand, a long optimization horizon is more demanding, both in prediction accuracy, and computational intensity. Inaccuracies of the demand predictions can lead to suboptimization. A longer optimization horizon leads to larger models, whose solutions need increased computational intensity. The optimization horizon used at the present computer code, is equal to the number of the ports that the ship calls. In other words, the optimization horizon is equal to the round trip of the vessel.

Probability Distributions It is assumed that the distribution of the demand of the different classes of goods that ask for transportation between the ports that the vessel calls, are normally distributed. If the optimization horizon is \( k \) ports, the probabilistic demand of the different classes of goods that ask for transportation
at the next $k$ ports, are input of the code. That means that if the next port the vessel will call is the $i$ port, we have to input into the program the probabilities $p_{s,i,m}(x_{s,i,m})$, where $s$ is the port of origin of the cargo, $t$ is the port of destination and $m$ is the class where the particular container belongs to.

**Capacity Constraints** The capacity constraints control the capacity availability at the different legs of the trip. These constraints can be either weight or space constraints. Our computer program considers only space availability constraints. The number of the capacity constraints is equal to the horizon of optimization. If we want to optimize the financial performance for the next $k$ ports, we have to consider the capacity constraints at these and only these $k$ ports.

**Other Constraints** These are mainly cabotage constraints. Cabotage constraints do not allow the vessel operator to transport from every port that the vessel visits, the mix of cargo that the operator would prefer.

### 4.2.2 Output of the Computer Code

**Allocation Upper Limits** on the number of goods from every class and port combination we would accept for transportation, while the probability distribution and the capacity constraints remain the same.

**Expected Revenues** The expected revenues from the model is an output that does not have many practical uses, except for the case where the operator considers the acceptance of customers with many containers.

Figure 4-1, is a graph of inputs and the output of the computer code.
Figure 4-1: Computer Code Input-Output Diagram
4.2.3 Revision of the Optimal Capacity Limits

The capacity constraints and the estimates of the probability distributions change over time. The solution of the model has to be updated every time some of the parameters of the model change. We include a graph that gives the criteria for the running of the program (see Figure 4-2)
CRITERIA FOR THE RUNNING OF THE PROGRAM

Has the ship called at the next port of the itinerary since the program ran for the last time?

Yes

Has the operator accepted any new bookings for cargo transportation since the program ran for the last time?

Yes

The estimates of the probability distributions of the bookings for cargo transportation from the ports to be visited change?

Yes

New Input

Run the program

Output: New Upper Limits per cargo class and itinerary. We accept cargo up to these limits

Figure 4-2: Criteria for the Running of the Program
4.3 Numerical Results

In the following pages we will compare a Nested against a Non-Nested model. We will show the superiority of the operational results that we can get from a Nested Model, relatively to the results that we get from a Non-Nested Model. The Non-Nested Model is the model that assumes that the capacity allocated to a particular class is exclusively allocated to this class of goods. On the other hand, the Nested Model assumes that the capacity that is not used by the class of containers to which it has been assigned, can be used by another class of containers that books later. Brumelle and McGill [15], have shown that in the case of the one leg multiple fare classes problem, the fixed-limit booking policies are optimal within the class of all admissible policies that depend only on the number of the current bookings. In other words they have proved that the Nesting Policies of the Nested Model are the optimal Policies among all static policies. We show, through an example, that the same happens in the case of the multi leg, problem, with Nested itineraries.

Input for a Three Ports-Two Classes of Goods Problem

We run the P ports M classes of goods model for the case of a three ports-two goods example. We examine a round trip from the port of origin, and back to the same port. If we put all these ports in a line we have five ports, to consider. We accept cargo from any port to the following one or two ports. At port 4 we accept cargo only for port 5, and at port 5 we do not accept any cargo.

The input data is the same for both models. The revenue from the Nested model is \( \text{EXPECTED REVENUE} = \$309.47 \) and the revenue from the Non-Nested model is \( \text{EXPECTED REVENUE} = \$294.87 \). We see that the Non-Nested problem gives revenues that are about 5\% lower than the revenue from the Nested Problem. The input data of the computer code is:
4.3. Numerical Results

capacity = 200

<table>
<thead>
<tr>
<th>m, s, t</th>
<th>mean (20.0)</th>
<th>var (100.0)</th>
<th>freight (1.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1, 2</td>
<td>20.0</td>
<td>100.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1, 1, 3</td>
<td>20.0</td>
<td>100.0</td>
<td>2.0</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>20.0</td>
<td>100.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1, 2, 4</td>
<td>20.0</td>
<td>100.0</td>
<td>2.0</td>
</tr>
<tr>
<td>1, 3, 4</td>
<td>20.0</td>
<td>100.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1, 3, 5</td>
<td>20.0</td>
<td>100.0</td>
<td>2.0</td>
</tr>
<tr>
<td>2, 1, 2</td>
<td>20.0</td>
<td>100.0</td>
<td>1.1</td>
</tr>
<tr>
<td>2, 1, 3</td>
<td>20.0</td>
<td>81.0</td>
<td>2.2</td>
</tr>
<tr>
<td>2, 2, 3</td>
<td>20.0</td>
<td>64.0</td>
<td>1.1</td>
</tr>
<tr>
<td>2, 2, 4</td>
<td>20.0</td>
<td>49.0</td>
<td>2.2</td>
</tr>
<tr>
<td>2, 3, 4</td>
<td>20.0</td>
<td>36.0</td>
<td>1.1</td>
</tr>
<tr>
<td>2, 3, 5</td>
<td>20.0</td>
<td>25.0</td>
<td>2.2</td>
</tr>
</tbody>
</table>

The data show that the class m=2 of containers that goes from port s=1, to port t=3, has a normally distributed demand with a mean of mean=20 containers and a variance of var= 81.0. The freight rate for this origin-destination and class combination is freight= 2.2

Output from the Three Ports-Two Classes of Goods Problem

Results from the Nested Model  A printout of the results of my model for the case of the model of three ports-two goods follow. The multi leg model is nested for goods that travel on the same itinerary. The different itineraries compete for the same capacity. There is no nesting as far as different itineraries are concerned. SPACE(1,2,1) is the maximum space that we allocate to goods that go from port 1 to port 2 and belong to class 1 (i.e. low value class). SPACE(1,2,2) is the maximum space that the class of goods that go from port 1 to port 2 and belong to class 2 can occupy. SPACE(1,2,2) is the maximum space that class 1 and class 2 goods with origin Port 1 and destination Port 2
can cumulatively occupy. EXPVALUE is the expected number of containers for each origin-destination port and class combination that we can have (EXPVALUE is the $E_{s,t,i}$ of our yield management models).

In the printout, we include the results from some early iterations of the model. These results give an idea of the convergence speed of the capacity allocation SPACE, and the sensitivity of the objective function EXPECTED REVENUE to the changes of the capacity allocation. ERROR is the upper bound of the difference between the current and the optimal EXPECTED REVENUE.
### 4.3. Numerical Results

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Error</th>
<th>Expected Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>94.893</td>
<td></td>
</tr>
<tr>
<td>space(1,2,1)= 39.22</td>
<td>space(1,2,2)= 47.96</td>
<td></td>
</tr>
<tr>
<td>space(1,3,1)= 49.55</td>
<td>space(1,3,2)= 50.38</td>
<td></td>
</tr>
<tr>
<td>space(2,3,1)= 15.22</td>
<td>space(2,3,2)= 24.78</td>
<td></td>
</tr>
<tr>
<td>space(2,4,1)= 13.39</td>
<td>space(2,4,2)= 24.84</td>
<td></td>
</tr>
<tr>
<td>space(3,4,1)= 12.25</td>
<td>space(3,4,2)= 24.55</td>
<td></td>
</tr>
<tr>
<td>space(3,5,1)= 49.78</td>
<td>space(3,5,2)= 50.61</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>3.7651</td>
<td></td>
</tr>
<tr>
<td>space(1,2,1)= 48.42</td>
<td>space(1,2,2)= 56.39</td>
<td></td>
</tr>
<tr>
<td>space(1,3,1)= 41.52</td>
<td>space(1,3,2)= 42.21</td>
<td></td>
</tr>
<tr>
<td>space(2,3,1)= 19.27</td>
<td>space(2,3,2)= 28.86</td>
<td></td>
</tr>
<tr>
<td>space(2,4,1)= 17.80</td>
<td>space(2,4,2)= 28.93</td>
<td></td>
</tr>
<tr>
<td>space(3,4,1)= 16.05</td>
<td>space(3,4,2)= 28.17</td>
<td></td>
</tr>
<tr>
<td>space(3,5,1)= 41.71</td>
<td>space(3,5,2)= 42.90</td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>2.7911</td>
<td>309.47</td>
</tr>
<tr>
<td>SPACE(1,2,1) = 48.680</td>
<td>EXPVALUE= 20.079</td>
<td>FREIGHT= 1.000</td>
</tr>
<tr>
<td>SPACE(1,2,2) = 56.613</td>
<td>EXPVALUE= 19.176</td>
<td>FREIGHT= 1.100</td>
</tr>
<tr>
<td>SPACE(1,3,1) = 41.308</td>
<td>EXPVALUE= 20.023</td>
<td>FREIGHT= 2.000</td>
</tr>
<tr>
<td>SPACE(1,3,2) = 42.001</td>
<td>EXPVALUE= 15.888</td>
<td>FREIGHT= 2.200</td>
</tr>
<tr>
<td>SPACE(2,3,1) = 19.175</td>
<td>EXPVALUE= 15.681</td>
<td>FREIGHT= 1.000</td>
</tr>
<tr>
<td>SPACE(2,3,2) = 29.214</td>
<td>EXPVALUE= 11.650</td>
<td>FREIGHT= 1.100</td>
</tr>
<tr>
<td>SPACE(2,4,1) = 17.710</td>
<td>EXPVALUE= 15.024</td>
<td>FREIGHT= 2.000</td>
</tr>
<tr>
<td>SPACE(2,4,2) = 28.785</td>
<td>EXPVALUE= 12.078</td>
<td>FREIGHT= 2.200</td>
</tr>
<tr>
<td>SPACE(3,4,1) = 16.473</td>
<td>EXPVALUE= 14.124</td>
<td>FREIGHT= 1.000</td>
</tr>
<tr>
<td>SPACE(3,4,2) = 28.525</td>
<td>EXPVALUE= 12.592</td>
<td>FREIGHT= 1.100</td>
</tr>
<tr>
<td>SPACE(3,5,1) = 41.501</td>
<td>EXPVALUE= 20.026</td>
<td>FREIGHT= 2.000</td>
</tr>
</tbody>
</table>
SPACE(3,5,2) = 42.690  EXPVALUE= 18.251  FREIGHT= 2.200
Verification of the Model Robustness  We will employ the Littlewood formula in order to check the robustness of the results that we have shown in the previous section. We will check the relative robustness of the results:

\[
\text{SPACE}(1,2,1) = 48.680 \quad \text{XPVALUE} = 20.079 \quad \text{FREIGHT} = 1.000 \\
\text{SPACE}(1,2,2) = 56.613 \quad \text{XPVALUE} = 19.176 \quad \text{FREIGHT} = 1.100
\]

\text{SPACE}(1,2,1) \text{ is the maximum capacity allocated to class 1 goods traveling from port 1 to port 2 and SPACE}(1,2,2) \text{ is the maximum capacity allocated collectively to classes of goods 1 and 2 that travel from port 1 to port 2. The results of our model are robust if}\text{SPACE}(1,2,1) \text{ and SPACE}(1,2,2) \text{ satisfy the Littlewood formula:}

\[
\text{FREIGHT}_{1,2,1} = \text{FREIGHT}_{1,2,2} \cdot P[x_{1,2,2} \geq (\text{SPACE}_{1,2,2} - \text{SPACE}_{1,2,1})]
\]

where: \( P(x_{1,2,2} \geq a) \) is the cumulative probability that \( x_{1,2,2} \geq a \). The freight rate, the mean and the variance of the distributions of \( x_{1,2,1} \) and \( x_{1,2,2} \) is the following:

\[
m=1, s=1, t=2 \quad \text{**mean***} \quad 20.0 \quad \text{***var***} \quad 100.0 \quad \text{freight***} \quad 1.0 \\
m=2, s=1, t=2 \quad \text{**mean***} \quad 20.0 \quad \text{***var***} \quad 100.0 \quad \text{freight***} \quad 1.1
\]

Littlewood's formula becomes:

\[
1.0 = 1.1 \cdot P[x_{1,2,2} \geq 7.933] \Rightarrow \\
1.0 = 1.1 \cdot \left( 1.0 - \Phi \left[ \frac{7.933 - \left( \mu_{1,2,2} = 20.0 \right)}{\sigma_{1,2,2} = 10.0} \right] \right) \Rightarrow \\
1.0 = 1.1 \cdot \Phi[0.131] \Rightarrow \\
1.0 = 0.9935
\]

which is true within the desired accuracy. If we repeat the above process with all the different pairs of classes of goods that have the same origin and destination we will find that the results from our model, and the results from the formula by Littlewood's are compatible. We conclude that our results are robust.
Results from the Non-Nested Model

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Error</th>
<th>Expected Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>2.8161</td>
<td>294.87</td>
</tr>
</tbody>
</table>

SPACE(1,2,1) = 28.338  EVALUE = 18.871  FREIGHT = 1.000
SPACE(1,2,2) = 29.233  EVALUE = 19.040  FREIGHT = 1.100
SPACE(1,3,1) = 20.528  EVALUE = 16.269  FREIGHT = 2.000
SPACE(1,3,2) = 21.257  EVALUE = 17.003  FREIGHT = 2.200
SPACE(2,3,1) = 12.622  EVALUE = 11.285  FREIGHT = 1.000
SPACE(2,3,2) = 17.717  EVALUE = 15.538  FREIGHT = 1.100
SPACE(2,4,1) = 11.948  EVALUE = 10.758  FREIGHT = 2.000
SPACE(2,4,2) = 15.928  EVALUE = 14.713  FREIGHT = 2.200
SPACE(3,4,1) = 11.555  EVALUE = 10.447  FREIGHT = 1.000
SPACE(3,4,2) = 16.299  EVALUE = 15.315  FREIGHT = 1.100
SPACE(3,5,1) = 22.544  EVALUE = 17.154  FREIGHT = 2.000
SPACE(3,5,2) = 21.726  EVALUE = 18.751  FREIGHT = 2.200
Chapter 5

Dynamic Programming Approach

5.1 Two Ports, Two Classes of Goods Problem

In the Nested model on which I have worked so far, I have assumed that there is a particular order in which customers who belong to different freight-rate classes arrive. Namely, I was considering that the customers who belong to the lowest paying class arrive first, whereas the customers who belong to higher freight-rate classes arrive later.

A further assumption of the model was, that, once we have started accepting high freight-rate customers, we stop accepting customers from the low end of the market. This last assumption was a direct reference to the Airline Problem. At the Airline problem the price discrimination is based on the timing of the purchase of the ticket.

As a result it is impossible to sell both low and high price tickets at the same time, and the above mentioned assumption is a plausible one. On the contrary, Liner Shipping Conferences discriminate among different commodities on the basis of the value of the goods transported rather than on the timing of the transportation capacity request. Therefore, it is possible to sell transportation capacity to both high and low end customers at the same time, or in any order.

In order to accommodate the needs of the conference problem, we relax the restrictions of the Airline Problem. Because of the sequential nature of the new model we employ the
Dynamic Programming Approach.

Modeling Assumptions

We make several simplifying assumptions:

1. The itinerary of the liner, takes place between two ports. We have only two classes of commodities (Low- and High- freight-rate) that ask for transportation between the two ports. \( f_1 \) is the freight-rate per container for the low freight-rate (Class 1) containers and \( f_2 \) the freight-rate of the high value (Class 2) containers. Obviously \( f_1 < f_2 \).

2. The announcement of the itinerary takes place \( T \) periods before the departure of the vessel from Port A to Port B.

3. We divide the \( T \) remaining time periods until departure, into \( T \) equal time segments. Offers of goods that belong to class 1 (Low revenue customers with freight-rate \( f_1 \)) can arrive only at the beginning of the time segment \( t \) (i.e. they can arrive only exactly at time \( T, T - 1, \ldots, t, \ldots, 2, 1 \)). We have no potential class 1 arrivals at time \( t = 0 \). The probability with which we have an \( x_1 \) (i.e. class 1) arrival at time \( t \) is equal to \( \lambda \). The probability of zero \( x_1 \) arrivals is \( (1 - \lambda) \). The probability to have more than one arrivals on time \( t \) is zero. For the moment being we consider \( \lambda \) to be a constant, independent of time \( t \).

4. The high revenue customers (class 2 with freight-rate \( f_2 \)) can arrive anytime in the time interval \( (t, t - 1), \forall t \in \{T, T - 1, \ldots, 2, 1\} \).

The probability that we have \( x_2 \) (Class 2) arrivals in the time interval \( (t, t - 1), \forall t \in \{T, T - 1, \ldots, 2, 1\} \) is \( P_2(\tau = 1, x_2 = c) \). For the moment being we consider \( P_2(\tau = 1, x_2 = c) \) to be independent of time \( t \).

Figure 5-1 is an illustration of the above assumptions.
5.1. Two Ports, Two Classes of Goods Problem

5.1.1 Dynamic modeling formulation

Every time we have a Class 2 (High Revenue) arrival we accept the offer (given that we have available space). Nevertheless, when we have a Class 1 arrival, and before we accept or reject it, we have to consider whether the certain revenue $f_1$ that we get in case we accept the offer, outweighs the potential revenue losses from the unavailability of the one slot that we give to that container.

The stage of our cost functional is the remaining time units ($t$) until departure. The state is the number of the remaining container slots ($n$) that are still available, and have not been committed to any shipper yet. The cost functional is the following:

$$V_t(n) = \max_{I_t(n) \in \{0,1\}} [I_t(n) \cdot f_1 + W_t(n - I_t(n))] \quad (5.1)$$

$$W_t(n) = \sum_{c=0}^{\infty} [\gamma_n \cdot f_2 + \lambda \cdot V_{t-1}(n - \gamma_n) + (1 - \lambda) \cdot W_{t-1}(n - \gamma_n)] \cdot P_2(\tau = 1, x_2 = c) \quad (5.2)$$

where: $\gamma_n = \min[c,n]$.

$V_t(n)$ is the maximum Expected Revenue at time $t$ with $n$ container slots still available,
given that we have an arrival of Low Revenue Shipper who asks for transportation capacity.

$W_t(n)$ is the maximum Expected Revenue at time $t$ with $n$ container slots still available, given that we do not have an arrival of Low Revenue Shipper who asks for transportation capacity.

$I_t(n)$ is the control variable at time $t$ with $n$ container slots still available, when we have an arrival of Low Revenue Shipper who asks for transportation capacity.

$\gamma_n$ is the number of High Revenue customers that we accept in the time interval $(t, t - 1)$. The number of the customers we accept can not be greater than the remaining vessel capacity.

The Boundary Conditions are:

\begin{align*}
V_t(0) &= 0.0 \\
W_t(0) &= 0.0 \quad \forall \; t \in \{0, T\} \quad (5.3)
\end{align*}

\begin{align*}
V_0(n) &= 0.0 \\
W_0(n) &= 0.0 \quad \forall \; n \in \{0, N\} \quad (5.4)
\end{align*}

\begin{align*}
W_t(n) &= \sum_{c=0}^{\infty} [\lambda \cdot V_{t-1}(n - \gamma_n) + (1 - \lambda) \cdot W_{t-1}(n - \gamma_n)] \cdot P_2(\tau = 1, z_2 = c) + \\
&\quad + \sum_{c=0}^{\infty} \gamma_n \cdot f_2 \cdot P_2(\tau = 1, z_2 = c) \quad (5.5)
\end{align*}

\begin{align*}
W_t(n) &= \sum_{c=0}^{n} [\lambda \cdot V_{t-1}(n - \gamma_n) + (1 - \lambda) \cdot W_{t-1}(n - \gamma_n)] \cdot P_2(\tau = 1, z_2 = c) + \\
&\quad + \sum_{c=n}^{\infty} \gamma_n \cdot f_2 \cdot P_2(\tau = 1, z_2 = c) \quad (5.6)
\end{align*}
5.1. Two Ports, Two Classes of Goods Problem

\[ + \sum_{c=n+1}^{\infty} \left[ \lambda \cdot V_{t-1}(n - \gamma_n) + (1 - \lambda) \cdot W_{t-1}(n - \gamma_n) \right] \cdot P_2(\tau = 1, x_2 = c) + \]

\[ + \sum_{c=0}^{\gamma_n} f_2 \cdot P_2(\tau = 1, x_2 = c) \]  

Equation 5.3 gives \( V_t(0) = 0.0 \) and \( W_t(0) = 0.0 \) \( \forall \ t \in \{0, T\} \). Therefore, equation 5.6 becomes:

\[ W_t(n) = \sum_{c=0}^{n} \left[ \lambda \cdot V_{t-1}(n - \gamma_n) + (1 - \lambda) \cdot W_{t-1}(n - \gamma_n) \right] \cdot P_2(\tau = 1, x_2 = c) + \]

\[ + \sum_{c=0}^{\gamma_n} f_2 \cdot P_2(\tau = 1, x_2 = c) \]  

\[ W_t(n) = \sum_{c=0}^{n} \left[ \lambda \cdot V_{t-1}(n - c) + (1 - \lambda) \cdot W_{t-1}(n - c) \right] \cdot P_2(\tau = 1, x_2 = c) + \]

\[ + \sum_{c=0}^{\gamma_n} f_2 \cdot P_2(\tau = 1, x_2 = c) \]  

From [31, p. 49] we get the definition of concavity for a function \( g(x) \) whose domain is the set of non-negative integers. The first forward difference \( \Delta g(x) \) is defined as: \( \Delta g(x) = g(x + 1) - g(x) \), \( x = 0, 1, 2, \ldots \) The function \( g(x) \) is called concave, if its first forward difference is nonincreasing. That is: \( \Delta g(x) \geq \Delta g(x + 1) \), \( x = 0, 1, 2, \ldots \) Therefore,

We know the definition of concavity for a function \( g(x) \) whose domain is the set of non-negative integers. The first forward difference \( \Delta g(x) \) is defined as: \( \Delta g(x) = g(x + 1) - g(x) \), \( x = 0, 1, 2, \ldots \) The function \( g(x) \) is called concave, if its first forward difference is nonincreasing. That is: \( \Delta g(x) \geq \Delta g(x + 1) \), \( x = 0, 1, 2, \ldots \) Therefore,

\[ g(x) : \text{concave} \iff g(x + 1) \geq \frac{1}{2} \cdot g(x + 2) + \frac{1}{2} \cdot g(x), \ x \in \mathbb{N} \]  

(5.9)
Lemma 1 $V_t(n)$ and $W_t(n)$ are concave functions of $n$.

Proof: (We follow the outline of the proof for the Sequential Allocation Model given by [32, p. 17]). In order to prove that $V_t(n)$ is concave, we have to prove simultaneously, the concavity of $W_t(n)$.

1. From 5.4 we have that for $t = 0$ both $V_t(n)$ and $W_t(n)$ are concave.

2. We assume that $W_t(n)$ and $V_t(n)$, $i = 1, \ldots, t - 1$ are concave.

3. In order to prove that $W_t(n)$ and $V_t(n)$ are concave, we have to prove that

\[ W_t(n + 1) \geq \frac{1}{2} \cdot W_t(n + 2) + \frac{1}{2} \cdot W_t(n) \] and that

\[ V_t(n + 1) \geq \frac{1}{2} \cdot V_t(n + 2) + \frac{1}{2} \cdot V_t(n). \]

In order to prove the concavity of $W_t(n)$ we have to prove that:

\[
\sum_{c=0}^{n+1} \left[ \lambda \cdot V_{t-1}(n + 1 - c) + (1 - \lambda) \cdot W_{t-1}(n + 1 - c) \right] \cdot P_2(\tau = 1, z_2 = c) \]

\[ \geq \sum_{c=0}^{\infty} \gamma_{n+1} \cdot f_2 \cdot P_2(\tau = 1, z_2 = c) \]

\[
\frac{1}{2} \cdot \sum_{c=0}^{n+2} \left[ \lambda \cdot V_{t-1}(n + 2 - c) + (1 - \lambda) \cdot W_{t-1}(n + 2 - c) \right] \cdot P_2(\tau = 1, z_2 = c) \]

\[ \geq \frac{1}{2} \cdot \sum_{c=0}^{\infty} \gamma_{n+2} \cdot f_2 \cdot P_2(\tau = 1, z_2 = c) \]

\[
\frac{1}{2} \cdot \sum_{c=0}^{n} \left[ \lambda \cdot V_{t-1}(n - c) + (1 - \lambda) \cdot W_{t-1}(n - c) \right] \cdot P_2(\tau = 1, z_2 = c) \]

\[ \geq \frac{1}{2} \cdot \sum_{c=0}^{\infty} \gamma_{n} \cdot f_2 \cdot P_2(\tau = 1, z_2 = c) \]
In order to prove the validity of the above inequality, it is enough to prove that \( a_1 \geq \frac{1}{2} \cdot a_2 + \frac{1}{2} \cdot a_0 \) and that \( b_1 \geq \frac{1}{2} \cdot b_2 + \frac{1}{2} \cdot b_0 \). First,

\[
a_1 \geq \frac{1}{2} \cdot a_2 + \frac{1}{2} \cdot a_0 \Leftrightarrow
\]

\[
\Leftrightarrow \sum_{c=0}^{n} c \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + (n + 1) \cdot \sum_{c=n+1}^{\infty} f_2 \cdot P_2(\tau = 1, x_2 = c) \geq \
\]

\[
\frac{1}{2} \cdot \sum_{c=0}^{n} c \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + \frac{1}{2} \cdot (n + 1) \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) \
\]

\[
+ \frac{1}{2} \cdot (n + 2) \cdot \sum_{c=n+2}^{\infty} f_2 \cdot P_2(\tau = 1, x_2 = c) \
\]

\[
+ \frac{1}{2} \cdot \sum_{c=0}^{n} c \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + \frac{1}{2} \cdot n \cdot \sum_{c=n+1}^{\infty} f_2 \cdot P_2(\tau = 1, x_2 = c) \Leftrightarrow
\]

\[
\Leftrightarrow (n + 1) \cdot P_2(\tau = 1, x_2 = n + 1) + (n + 1) \cdot \sum_{c=n+2}^{\infty} P_2(\tau = 1, x_2 = c) \geq \
\]

\[
\frac{1}{2} (n + 1) \cdot P_2(\tau = 1, x_2 = n + 1) + \frac{1}{2} (n + 2) \sum_{c=n+2}^{\infty} P_2(\tau = 1, x_2 = c) \
\]

\[
+ \frac{1}{2} \cdot n \cdot P_2(\tau = 1, x_2 = n + 1) + \frac{1}{2} \cdot n \cdot \sum_{c=n+2}^{\infty} P_2(\tau = 1, x_2 = c) \Leftrightarrow
\]

\[
\Leftrightarrow (n + 1) \cdot P_2(\tau = 1, x_2 = n + 1) \geq \
\]

\[
\frac{1}{2} (n + 1) \cdot P_2(\tau = 1, x_2 = n + 1) \
\]

\[
+ \frac{1}{2} \cdot n \cdot P_2(\tau = 1, x_2 = n + 1)
\]

which is obviously true.

Now we have to prove the following:

\[
b_1 \geq \frac{1}{2} \cdot b_2 + \frac{1}{2} \cdot b_0 \Leftrightarrow
\]
\[ \Leftrightarrow \sum_{c=0}^{n+2} \left[ \lambda \cdot (V_{t-1}(n + 1 - \gamma_{n+1}) - \frac{1}{2} \cdot V_{t-1}(n + 2 - \gamma_{n+2}) - \frac{1}{2} \cdot V_{t-1}(n - \gamma_n)) \right] = \alpha \]

\[ + (1 - \lambda) \cdot (W_{t-1}(n + 1 - \gamma_{n+1}) - \frac{1}{2} \cdot W_{t-1}(n + 2 - \gamma_{n+2}) - \frac{1}{2} \cdot W_{t-1}(n - \gamma_n)) \right] = \beta \]

\[ \cdot P_2(\tau = 1, x_2 = c) \geq 0 \]

\( \alpha \geq 0 \) and \( \beta \geq 0 \) because, from our assumption, \( V_{t-1}(n) \) and \( W_{t-1}(n) \) are concave.

As a result, the above equation is equivalent to:

\[ \Leftrightarrow \sum_{c=0}^{n+2} [\lambda \cdot \alpha + (1 - \lambda)\beta] \cdot P_2(\tau = 1, x_2 = c) \geq 0 \]

which holds. Therefore, we have proven that \( W_t(n) \) is a convex function. In other words we have proven that:

\[ W_t(n + 1) \geq \frac{1}{2} \cdot W_t(n + 2) + \frac{1}{2} \cdot W_t(n) \quad (5.10) \]

I define \( W_t(x) \) in the interval \( (n, n + 1) \) as the linear interpolation of the values \( W_t(n) \) and \( W_t(n + 1) \). (See Figure 5-2). The function continues being concave.

We have:

\[ W_t(n + 1 - i) = [i \cdot W_t(n) + (1 - i) \cdot W_t(n + 1)], \quad i \in [0, 1] \quad (5.11) \]
In the following we will prove that if we relax the constraint for $I_i(n) \in \{0, 1\}$ and we allow $i = I_i(n) \in [0, 1]$, the value of $i$ will be either 0 or 1. Let:

$$i_j^* = \arg \max_{i_j \in [0, 1]} V_i(n + j) = \arg \max_{i_j \in [0, 1]} [i_j \cdot f_1 + W_i(n + j - i_j)]$$  \hspace{1cm} (5.12)

From equation 5.1 and 5.11, we get:

$$V_i(n + 1) = \max_{i_1 \in [0, 1]} [i_1 \cdot f_1 + W_i(n + 1 - i_1)]$$

$$= \max_{i_1 \in [0, 1]} [i_1 \cdot f_1 + i_1 \cdot W_i(n) + (1 - i_1) \cdot W_i(n + 1)]$$

$$= \max_{i_1 \in [0, 1]} [f_1 + W_i(n)) + (1 - i_1) \cdot W_i(n + 1)]$$  \hspace{1cm} (5.13)

At equation 5.13 we see that:

$$f_1 + W_i(n) \geq W_i(n + 1) \Rightarrow i_1^* = 1$$
\[ f_1 + W_t(n) < W_t(n+1) \Rightarrow i^*_1 = 0 \]  \hspace{1cm} (5.14)

In a similar way we can show:

\[ f_1 + W_t(n + j - 1) \geq W_t(n + j) \Rightarrow i^*_j = 1 \]
\[ f_1 + W_t(n + j - 1) < W_t(n + j) \Rightarrow i^*_j = 0 \]  \hspace{1cm} (5.15)

Therefore, the optimal value \( i^* \) that maximizes 5.13 is either 0 or 1. In short, what we have proven is, that if we relax the control variable constraint and allow \( I_t(n) = i \) to get values not only from \{0, 1\} but also from \( (0, 1) \), then the optimal values of \( i \) will again be either 0 or 1. Therefore, even if we do not restrict \( I_t(n) \) to get integer values it will do so anyway. What we want to prove now, is the concavity of \( V_t(n) \). We have:

\[
V_t(n+1) = i^*_1 + W_t(n+1 - i^*_1) \\
\geq \frac{1}{2} \cdot [i^*_0 + i^*_2] + W_t(n + 1 - \frac{1}{2}[i^*_0 + i^*_2]) \\
= \frac{1}{2} \cdot [i^*_0 + i^*_2] + W_t(\frac{1}{2}[n - i^*_0] + \frac{1}{2}[n + 2 - i^*_2]) \\
\geq \frac{1}{2} \cdot [i^*_0 + i^*_2] + \frac{1}{2} \cdot W_t(n - i^*_0) + \frac{1}{2} \cdot W_t(n + 2 - i^*_2) \\
= \frac{1}{2} \cdot [i^*_0 + W_t(n - i^*_0)] + \frac{1}{2} \cdot [i^*_2 + W_t(n + 2 - i^*_2)] \\
= \frac{1}{2} \cdot V_t(n) + \frac{1}{2} \cdot V_t(n + 2) \]  \hspace{1cm} (5.16)

The first inequality of 5.16 comes from 5.15, which shows that the maximum over \{0, 1\}, is equal to the maximum over \([0, 1]\). The second inequality of 5.16 comes from the concavity of \( W_t(n) \) with reference to \( n \), (equation 5.10). Therefore, we have concluded that:

\[ V_t(n+1) \geq \frac{1}{2} V_t(n) + \frac{1}{2} V_t(n + 2) \]  \hspace{1cm} (5.17)

From equation 5.17 we conclude that \( V_t(n) \) is a concave function of \( n \). That concludes
the proof of Lemma 1. \textbf{Q.E.D.}

From equation 5.1 we have:

\[ I_t^*(n) = \arg \max_{I_t(n) \in \{0,1\}} [I_t(n) \cdot f_1 + W_t(n - I_t(n))] \]  

In other words, \( I_t^*(n) \) is the optimal decision variable, when there are \( t \) remaining units of time till departure and \( n \) remaining seats. In the following we will show that \( I_t(n) \) is a non-decreasing function of \( n \), and a non-increasing function of \( t \).

\textbf{Theorem 2} \( I_t^*(n) \) is a non-decreasing function of \( n \).

\textbf{Proof:} Let us start by reminding that \( I_t^*(n) \) is a control variable that determines whether we should accept a low freight-rate customer or not, given that this customer asks for transportation. Furthermore, \( I_t^*(n) \in \{0,1\} \). We have to prove that \( I_t^*(i) \leq I_t^*(j) \), \( \forall \ i < j \). If \( I_t^*(i) = 0 \), it is trivial to prove that \( I_t^*(i) \leq I_t^*(j) \), \( \forall \ i < j \). If now \( I_t^*(i) = 1 \), then we have to prove that \( 1 = I_t^*(i) \leq I_t^*(j) \), \( \forall \ i < j \). In other words, we have to prove that when \( I_t^*(i) = 1 \) then \( I_t^*(j) = 1 \), \( \forall \ j > i \). By substituting \( I_t^*(i) = 1 \) in equation 5.1 we get that \( f_1 + W_t(i - 1) > W_t(i) \Rightarrow \)

\[ \Rightarrow f_1 > W_t(i) - W_t(i - 1) \]  

(5.19)

We will prove Theorem 2 by contradiction. We assume that \( I_t^*(j) = 0 \), \( \forall \ j > i \), in which case from equation 5.1 we will get: \( f_1 + W_t(j - 1) < W_t(j) \Rightarrow \)

\[ \Rightarrow f_1 < W_t(j) - W_t(j - 1) \]  

(5.20)

From equations 5.19 and 5.20 we have

\[ W_t(j) - W_t(j - 1) > W_t(i) - W_t(i - 1), \ \forall \ j > i \]  

(5.21)

From Lemma 1 we know that \( W_t(n) \) is a concave function of \( n \). From the concavity
we get that:

\[ W_t(j) - W_t(j - 1) < W_t(i) - W_t(i - 1), \quad \forall j > i \]  

(5.22)

The contradiction between equations 5.21 and 5.22 shows that the assumption \( I_t^*(j) = 0 \) is false. Therefore, we conclude that if \( I_t^*(i) = 1 \) then \( I_t^*(j) = 1, \quad \forall j > i \).

**Lemma 3** If \( I_t(n) = 0 \) \( \Rightarrow f_t + V_t(n - 1) \leq V_t(n) \)

**Proof:** From equation 5.1 we have:

\[ I_t^*(n - 1) = \arg \max_{I_t(n-1) \in \{0,1\}} [I_t(n-1) \cdot f_t + W_t(n-1 - I_t(n-1))] \]

\[ = \arg \max_{I_t(n-1) \in \{0,1\}} V_t(n-1) \]  

(5.23)

As a result,

\[ f_t + V_t(n - 1) = f_t + I_t^*(n - 1) + W_t(n - 1 - I_t^*(n - 1)) \]

\[ = f_t + W_t(n - 1) \]

\[ \leq V_t(n) \]

\[ \Rightarrow f_t + V_t(n - 1) \leq V_t(n) \]  

(5.24)

The second equality of equation 5.24 comes from the fact that \( I_t^*(n - 1) = 0 \) when \( I_t(n) = 0 \) (from Theorem 2). The inequality comes from equation 5.1. Therefore when \( I_t(n) = 0 \) \( \Rightarrow f_t + V_t(n - 1) \leq V_t(n) \). Q.E.D.

**Theorem 4** \( I_t^*(n) \) is a non-increasing function of \( t \).

**Proof:** We have to prove that \( I_{t+1}^*(n) \leq I_t^*(n) \).

If \( I_t^*(n) = 1 \), it is trivial to prove that \( I_{t+1}^*(n) \leq I_t^*(n) \).

If, now, \( I_t^*(n) = 0 \), then we have to prove that \( I_{t+1}^*(n) \leq I_t^*(n) = 0 \). In other words, it is enough to prove that when \( I_t^*(n) = 0 \) then \( I_{t+1}^*(n) = 0 \).

We have:

\[ f_t + W_t(n - 1) - f_t \cdot I_t^*(n) - W_t(n - I_t^*(n)) = \]
5.1. Two Ports, Two Classes of Goods Problem

\[ f_1 + W_t(n - 1) - W_t(n) \leq 0 \]

We will prove that

\[ f_1 + W_{t+1}(n - 1) \leq W_{t+1}(n) \]

which is equivalent to proving that \( I^*_t(n) = 0 \). From equation 5.8 we get:

\[
\begin{align*}
    f_1 + W_{t+1}(n - 1) &= f_1 + \sum_{c=0}^{\infty} \gamma_{n-1} \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + \\
    & \quad \lambda \cdot \sum_{c=0}^{n-1} V_t(n - 1 - c) \cdot P_2(\tau = 1, x_2 = c) + \\
    & \quad (1 - \lambda) \cdot \sum_{c=0}^{n-1} W_t(n - 1 - c) \cdot P_2(\tau = 1, x_2 = c) \\
    &= f_1 \cdot \sum_{c=0}^{n-1} P_2(\tau = 1, x_2 = c) + f_1 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) + \\
    & \quad \sum_{c=0}^{\infty} \gamma_{n-1} \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + \\
    & \quad \lambda \cdot \sum_{c=0}^{n-1} V_t(n - 1 - c) \cdot P_2(\tau = 1, x_2 = c) + \\
    & \quad (1 - \lambda) \cdot \sum_{c=0}^{n-1} W_t(n - 1 - c) \cdot P_2(\tau = 1, x_2 = c) \\
    &= f_1 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) + \\
    & \quad \sum_{c=0}^{\infty} \gamma_{n-1} \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + \\
    & \quad \lambda \cdot \sum_{c=0}^{n-1} [f_1 + V_t(n - 1 - c)] \cdot P_2(\tau = 1, x_2 = c) + \\
    & \quad (1 - \lambda) \cdot \sum_{c=0}^{n-1} [f_1 + W_t(n - 1 - c)] \cdot P_2(\tau = 1, x_2 = c)
\end{align*}
\]

(5.26)
I will prove now that:

\[
\sum_{c=0}^{n-1} \left[ f_1 + V_t(n-1-c) \right] \cdot P_2(\tau = 1, x_2 = c) \leq \sum_{c=0}^{n-1} V_t(n-c) \cdot P_2(\tau = 1, x_2 = c) = \sum_{c=0}^{n} V_t(n-c) \cdot P_2(\tau = 1, x_2 = c)
\]

(5.27)

For the first inequality of 5.27 to be true, it is enough to have \( f_1 + V_t(i-1) \leq V_t(i), \forall i \in \{0,1,\ldots,n-1\} \). For that to happen, is enough to show that \( I_t(i) = 0, \forall i \in \{0,1,\ldots,n-1\} \) (from Lemma 3). But we have that \( I_t(n) = 0 \) and since \( I_t(i) \) is a non-decreasing function of \( i \) (Theorem 2), we have that \( I_t(i) = 0, \forall i \in \{0,1,\ldots,n-1\} \). Therefore, it is true that \( f_1 + V_t(i-1) \leq V_t(i), \forall i \in \{0,1,\ldots,n-1\} \) (from Lemma 3). The equation of 5.27 is a trivial consequence of the Boundary Conditions for \( V_t(n) \) (equation 5.3). Therefore, we have proven the validity of 5.27. In a similar way we can prove that:

\[
\sum_{c=0}^{n-1} \left[ f_1 + W_t(n-1-c) \right] \cdot P_2(\tau = 1, x_2 = c) \leq \sum_{c=0}^{n} W_t(n-c) \cdot P_2(\tau = 1, x_2 = c)
\]

(5.28)

and we get:

\[
f_1 + W_{t+1}(n-1) \leq f_1 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) + \sum_{c=0}^{\infty} \gamma_{n-1} \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) + \lambda \cdot \sum_{c=0}^{n} V_t(n-c) \cdot P_2(\tau = 1, x_2 = c) + (1 - \lambda) \cdot \sum_{c=0}^{n} W_t(n-c) \cdot P_2(\tau = 1, x_2 = c)
\]

(5.29)
5.1. Two Ports, Two Classes of Goods Problem

I want to prove now the validity of:

\[
f_1 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) + \sum_{c=0}^{\infty} \gamma_{n-1} \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) \leq \sum_{c=0}^{\infty} \gamma_n \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) \quad (5.30)
\]

It is enough to prove that:

\[
f_1 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) \leq \sum_{c=0}^{\infty} (\gamma_n - \gamma_{n-1}) \cdot f_2 \cdot P_2(\tau = 1, x_2 = c) = f_2 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) \quad (5.31)
\]

But:

\[
f_1 \cdot \sum_{c=0}^{\infty} P_2(\tau = 1, x_2 = c) \leq f_2 \cdot \sum_{c=n}^{\infty} P_2(\tau = 1, x_2 = c) \quad (5.32)
\]

because \( f_1 < f_2 \). Therefore, the validity of 5.30 has been proven. We have proven that when:

\[
f_1 + W_t(n - 1) \leq W_t(n), \quad (i.e. \ I_t(n) = 0) \Rightarrow \]

\[
\Rightarrow f_1 + W_{t+1}(n - 1) \leq W_{t+1}(n), \quad (i.e. \ I_{t+1}(n) = 0)
\]

Therefore, if \( I_t^*(n) = 0 \Rightarrow I_{t+1}^*(n) = 0 \).

\( I_t(n) \) can take the two values 0 or 1. The above equation proves that \( I_t(n) \) is a non-increasing function of \( t \).

The above results are compatible with the intuitive understanding that:

The more slots available there are, the easier would be for a low freight rate customer to be accepted.

As the time of the departure approaches, and the easier it becomes to accept low freight rate customers.
5.2 Conclusions and Recommendations

The model developed in the previous sections, is a tool for the description and solution of the containership yield management problem. It is a generalization of the simple formula developed by Littlewood, used for the formulation of the airline yield management problem. Our model is a model with not many restrictive assumptions. That enables us to consider several versions of the yield management problem.

First, I have solved the simple, yet fundamental case of the two ports-two classes of goods problem. I have proved that the solution of my formulation gives the same answer as the formula by Littlewood. I have shown that the solution suggested by Littlewood, is the capacity allocation that maximizes the expected revenues of the vessel operator.

Second, I have expanded to the case of the two ports-M classes of goods problem. I give the necessary conditions for the optimal allocation. I prove that any point that fulfills the necessary conditions for optimality, also fulfills the sufficient conditions for optimality. That means that all points that fulfill the necessary conditions for optimality are at least local optima. I further prove that the solution found by the necessary conditions for optimality is unique. That suggests that the solution of the necessary conditions for optimality is the global maximum of the expected revenues.

Third, I have solved for the case of the two ports-M classes of goods problem, when we have low value stand-by cargo that we can accept shortly before departure. This problem reduces to a form of the two ports-M classes of goods problem where the objective function has modified coefficients.

The obvious bottleneck to the numerical solution of the two ports-M classes of goods problem, is the necessity to calculate at each iteration several integrals of multiplicity of up to M. In order to avoid these multiple integrals that create an obvious difficulty for the application of the model, we have developed an approximate model for the case of the two ports-M classes of goods problem that is much easier to solve numerically. The integrals of this approximate model are at most double.
We simplify the model even further and we get a system of equations that represent the simplified necessary conditions for optimality. Each of these equations involves three integrals, each one of them being single integrals. I call this method the Marginal Value method.

Furthermore, I have developed a criterion for the acceptance or rejection of orders that consist of more than one containers.

I have also presented an extension of the approximate model for the case of the \( N \) ports-\( M \) classes of goods problem.

The Nested modeling formulation that we have developed at the first part of this thesis, is a modeling formulation that had originally been developed for the airline industry. The major assumption of the nested model is that we assume an ordered arrival of the different classes of containers.

The ordered arrival of the different classes of customers, is a marketing strategy and a necessity for the segmentation of the markets in the airline industry. As a result, the modeling assumption that incorporates this marketing strategy into the yield management model, is an assumption that works very well for the airline industry.

In shipping, the pricing of the containers is done on the basis of the value of the cargo of each container. If we incorporate the assumption of the ordered arrival of the different classes of containers, into the yield management model for the shipping industry, then the results of the model are accurate only to the extend that the customers exhibit the desired arrival behavior.

On the contrary, if we do not impose any strict conditions on the arrival pattern of the customers, but we allow similar arrival patterns for all classes of customers, then we are obliged to use Dynamic programming formulation.

In conclusion, the differences between the airline and shipping industry can be summarized as follows:

1. The number of the ports that the containership includes in its schedule is larger than the number of airports that an aircraft includes in its schedule. That, combined with
the large number of different classes of goods, each of which has its own freight-rate, give us a fairly large number of possible combinations of cargo classes and origin-destination pairs. The increased number of the different origin-destination pairs that have to be considered separately, decrease the accuracy of the "nested modeling method". On the other hand, the shipping company network is smaller than the network of a typical airline, and it is covered with fewer vessels.

3. Shipping Liners do price discrimination on the basis of the value of the cargo. The freight-rate is a percentage of the value of the cargo, and not a function of the timing of the arrival. The assumption of the ordered arrival can be a weak assumption in the case of ocean shipping. Therefore, if we treated the problem as a sequential problem, we would get more accurate results.

4. A further complication arises from the ratio between the time length of a round trip of the containership, and the time it takes from the announcement of the trip, to the departure of the vessel. We could have a substantial time difference of i.e. twenty days between the time the vessel sails from ports A and the time it sails from a following port B. The equivalent lag for an aircraft is always less than 24 hours. Therefore, a static analysis that could be accepted for an airline operator, can not be considered optimal for the liner shipping operations. The booking problem in shipping is a dynamic problem, the nature of which is sequential.

5. There is a lack of a clear origin and a clear final destination for the liner. The liner visits the consecutive ports of the itinerary and at no port the vessel becomes totally empty. Therefore, every port of the itinerary could be considered as the origin of the itinerary, since all ports are in a circle (rolling horizon problem). The decisions that we make for some leg of the itinerary, can and do influence all subsequent decisions. The consequences of a decision that we make now, can influence our revenue potential for more than one round trip.

With the above arguments, we do not want to diminish the potential contribution of
the static models to the yield management for ocean carriers. What we want to stress is that an experimentation with dynamic models (i.e. dynamic programming) could be more fruitful than it might seem originally. What we have tried to do in the last chapter of this dissertation is to investigate the nature and the potential complications of the Dynamic programming formulation for the two ports, two classes of containers problem. For the dynamic programming model to become more realistic, we have to include more classes of goods traveling between more than two ports.
Bibliography


Bibliography


