A 24-DIMENSIONAL SPIK MANIFOLD

by

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ABSTRACT

A 24-Dimensional Spin Manifold
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A brief review of the results of Anderson, Brown and Peterson on the structure of the spin cobordism ring shows that there is a 24-dimensional class for which no representative manifold was previously known. This thesis presents such a manifold.

The manifold is the "Grassmanification" of a certain vector bundle (the tangent bundle with a trivial line bundle split off and discarded) over an orientable 9-manifold X characterized by the non-vanishing of its Stiefel-Whitney number $w_9w_2w_2w_2w_2(X)$. Grassmanification of a vector bundle $E\to X$ is a generalization of projectification of a vector bundle, namely instead of considering the set of lines within E one considers the set of, say, m-planes. The resulting set which we denote $E^{m,n}$ (if E has dimension $m + n$) is a compact manifold, provided $X$ is.

Writing $\tau(M)$ for the tangent bundle of any manifold $M$, we compute $H^8(E^{m,n})$, a module over $H^8(X)$, and a basis of it over $H^8(X)$; the tangent bundle $\tau(E^{m,n})$, which equals the Whitney sum of $\tau(X)$ (pulled back to $E^{m,n}$) and the tensor product of the canonical $m$- and $n$-plane bundles on $E^{m,n}$; and thus, the Stiefel-Whitney class of $E^{m,n}$.

It is shown that in case the Stiefel-Whitney number of the orientable manifold $X$ above does not vanish, then for the 8-bundle $E$ indicated above, $E^{3,5}$ is a spinor manifold such that $w_6(E^{3,5}) \neq 0$, a condition which implies that $E^{3,5}$ is a representative of the 24-dimensional spin cobordism class.

Various results appear along the way, such as a method of computing E. Thomas' function $\psi_{m,n}$ which gives the Stiefel-Whitney class of the tensor product of bundles. The method involves a formula by which Milnor's symmetric polynomials $s_n$ may be calculated. Obtaining the Stiefel-Whitney class of a specific tensor product then becomes a straightforward, though tedious, calculation.

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1 Introduction

Thom [10] invented the study of manifolds by cobordism and determined the structure of the unoriented cobordism ring. Milnor [6], Wall [12] and others determined the oriented cobordism ring $\Omega^*_{SO}$ and Anderson, Brown and Peterson [2] described the additive structure of the spin cobordism ring $\Omega^*_{Spin}$, as well as most of its multiplicative structure. Manifolds representing many generating classes in $\Omega^*_{Spin}$ remain unknown, however. Every spin manifold of dimension $< 2^4$ is unoriented cobordant to the square of an orientable manifold; this is also true in dimensions $25, 26, 27, 28, 30,$ and $31$ (see [8]). There is however a $2^4$-dimensional spin cobordism class for which no representative manifold was previously known. This thesis presents such a manifold.

2 Spin Cobordism

2.1 KO Characteristic classes

To give an understanding of the place of the manifold in $\Omega^*_{Spin}$, we give here part of the description in [2] of $\Omega^*_{Spin}$: let $BO$ be the classifying space for the orthogonal group. Let $p: BO^{<n}> \to BO$ be the fibre space such that $\pi_1 BO^{<n>}$ is $0$ for $i < n$ and $p_*$: $\pi_1 BO^{<n>}$ is an isomorphism if $i \geq n$. Let $\alpha_n \in H^n (BO^{<n>})$ be the generator
when $n \equiv 0, 2 \pmod{8}$.

Let $\xi \in KO^0(X)$ be of filtration $n$, i.e. $\xi$ is to be trivial on the $(n-1)$-skeleton of $X$. Then there is a map $f : X \to BO \wedge^n$ such that $pf_\xi \simeq \xi : X \to BO$. We define $[\xi] \in H^n(X)$ by

$$[\xi] = \{f_\xi^* (a_0)\}$$

for all $f$ such that $pf_\xi \simeq \xi$.

Define $K$-theory Pontrjagin classes [1] as follows: let $T^m$ be the maximal torus in $SO(2m)$ (and $SO(2m + 1)$). Now $KO(BT^m) \cong \mathbb{Z}[[x_1, \ldots, x_m]]$ where each $x_i$ has dimension 1. Both $KO(BSO(2m))$ and $KO(BSO(2m + 1))$ are injected into $KO(BT^m)$ under the map $\chi$ which is the composition of the homomorphism "complexification of a bundle", with the $K$-theory homomorphism associated to $BT^m \to BSO(2m)$. Their common image is the invariants of the Weyl group of $T^m$ in $SO(2m)$ or $SO(2m + 1)$. Let $\bar{x}_i = -x_i/(1 - x_i)$. Then in $KO(BT^m)[t]$, $\Pi_{i=1}^m (1 + t(x_i + \bar{x}_i))$ is a polynomial in $t$ and we denote the coefficient of $t^l$ by $\pi^l \in KO(BT^m)$.

lies in the image of $\chi$; pulling back, one also writes $\pi^l \in KO(BSO(2m))$ or $KO(BSpin)$. If $J = (j_1, \ldots, j_k)$ is a sequence of integers such that $k > 0$ and each $j_i > 1$, let $\pi^J = \pi^{j_1} \pi^{j_2} \ldots \pi^{j_k} \in KO(BSpin)$, and let $n(J) = \Sigma j_i$.

2.2 Theorem

The filtration of $\pi^J$ in $KO(BSpin)$ is $4n(J)$ if $n(J)$
is even and is $4n(J) - 2$ if $n(J)$ is odd (see [2]).

Let $k$ be large and let $MSpin(8k)$ be the Thom space of the classifying bundle over $BSpin(8k)$. Let $\phi: H^*(BSpin(8k)) \to H^{8k}(MSpin(8k))$ and $\phi: KO^*(BSpin(8k)) \to KO^8(MSpin(8k))$ denote the Thom isomorphisms. $\phi$ raises filtration by precisely $8k$, and $\phi([\xi]) = [\phi(\xi)] \in H^{8k}(MSpin(8k))$ [5]. Let $\mathbb{MSpin}$ denote the spectrum associated to $MSpin(8k)$. We state the results of [3] in the language of spectra where the Thom isomorphism has degree 0. Let $BO<n>$ be the $\Omega$-spectrum whose $0^{th}$ term is $BO<n>$.

If $n(J)$ is even (respectively, odd), let $f^J: MSpin \to BO^{4n(J)}$ (respectively $BO^{4n(J)} - 2$) be a map corresponding to $\pi^J$. If $z \in H^*(MSpin)$, let $f^z: MSpin \to K(Z_2, \dim z)$ denote the corresponding map, where $K(Z_2, n)$ denotes the spectrum whose $0^{th}$ term is $K(Z_2, n)$.

2.3 Theorem

There is a collection of elements $z_i \in H^*(MSpin)$ such that the map

$$F: MSpin \to BO^{4n(J)} \times_{n(J) \text{ even}} BO^{4n(J)} - 2 \times_{n(J) \text{ odd}} \prod K(Z_2, \dim z_i)$$

given by $F = \Pi f^J \times \Pi f^z z_i$ induces an isomorphism on cohomology with $Z_2$ coefficients. Hence $F$ induces a $C_2$-isomorphism on homotopy groups, where $C_2$ is the class of finite groups of odd order.
Since $\pi_{n}(M_{\text{Spin}}) \neq \Omega^{n}_{\text{Spin}}$ has no $p$-torsion for odd primes $p$ [6], the above theorem allows one to compute the additive structure of $\Omega^{n}_{\text{Spin}}$. (In [2] is given a complicated counting procedure for the number of $z_{i}$'s in each dimension.) $\pi_{n}(BO<n>)$ is periodic of period 8 in dimensions $\geq n$, the sequence, starting in dimensions $\equiv 0$ (mod 8), being $\mathbb{Z}, \mathbb{Z}_{2} \mathbb{Z}_{2} \cdots \mathbb{Z}_{2} \mathbb{Z}_{2} \cdots \mathbb{Z}_{2} \mathbb{Z}$, $0_{\cdot}$, $0_{\cdot}$, $0_{\cdot}$, $0_{\cdot}$, $0_{\cdot}$, $(n \equiv n-1)$.  

2.4 Generators of $\Omega^{n}_{\text{Spin}}$ manifolds representing generators

The above shows that the classes $[\pi_{n}^{j} = \{f_{j}^{n}(a_{4n}(J))\}$ or $\{f_{j}^{n}(a_{4n}(J)) = 1\}$ in $\mathbb{H}(M_{\text{Spin}}) = \Omega^{n}_{\text{Spin}}$ are of interest as generators of $\Omega^{n}_{\text{Spin}}$. Manifolds $M_{j}$ representing $[\pi_{n}^{j}]$ are known in case all $j_{1}$ are even (the product of quaternionic projective spaces) or in case only one is odd [4].

If $M$ is an $n$-dimensional spin manifold, denote by $\pi^{J}(M) \in KO^{-n}(pt.)$ the characteristic number defined by $\pi^{J}[1]$. By 2.2, a representative $M_{j}$ of $[\pi^{J}]$ is a spin manifold of dimension $4n(J)$ (or $4n(J) = 2$ if $n(J)$ is odd) such that $\pi^{J}(M_{j}) \neq 0$. In [2] it is shown that $\pi^{J}(M) = P^{J}(M)$ where $P^{J} = p_{J}^{1} \cdots p_{J}^{k}$ and $p_{J} \in H^{J}(BSpin)$ is the Pontrjagin class. Since the reduction mod 2 of the Pontrjagin class $p_{1}$ of any bundle equals $w_{21}^{2}$ of that bundle, where $w_{21}$ is the Stiefel-Whitney class,

\begin{equation}
(1) \quad w_{21}^{2} \cdots w_{2j}^{2} \cdots w_{2k}^{2}(M_{j}) \neq 0
\end{equation}

will guarantee that $0 \neq P^{J}(M) = \pi^{J}(M)$.  

\[ w^{J}(M) = \pi^{J}(BO(n)) \]
3. Grassmanification of a vector bundle

Real projective space is a compact k-dimensional manifold which can be described as the set of lines (i.e., l-planes) through 0 in a real \((k + 1)\)-dimensional vector space. The Grassman manifold \(G_{m,n}\), a compact mn-dimensional manifold, is the set of \(m\)-planes (or \(n\)-planes, taking orthogonal complements) through 0 in an \((m + n)\)-dimensional space. Manifolds representing generators for the unoriented and oriented cobordism rings have been described which involve projectification of a vector bundle \([4]\), which is a special case of "Grassmanification" of a bundle.

If \(E+X\) is an \((m + n)\)-dimensional vector bundle, let \(E_{m,n}\) be the set of \(m\)-planes in \(E\), each within a fibre and through the 0-section. There is a fibration \(G_{m,n} \rightarrow E_{m,n} \rightarrow X\), and if \(X\) is a compact manifold of dimension \(k\), \(E_{m,n}\) is a compact manifold of dimension \(mn + k\), naturally.

Note that orthogonality is established in fibres of \(E\) by choosing a Riemannian metric; an \(m\)-plane in \(E\) then determines the orthogonal \(n\)-plane and conversely, so \(E_{m,n}\) may also be regarded as the set of \(n\)-planes in \(E\).

3.1 \(H^k(E_{m,n})\)

Let all cohomology in the sequel have \(\mathbb{Z}_2\) coefficients.

Theorem

\[ H^k(E_{m,n}) = H^k(X)[u,v]/(uv = w(E)) \]
where $u = 1 + u_1 + u_2 + \ldots + u_m$, $u_1 = w_i(\gamma_m)$,
$v = 1 + v_1 + v_2 + \ldots + v_m$, $v_1 = w_i(\gamma_n)$,

$\gamma_m$ is the canonical $m$-plane bundle over $E^m,n$ whose
fibre over an $m$-plane in $E$ (i.e. point of $E^m,n$) consists
of the points in that $m$-plane,

$\gamma_n$ is the analogous canonical $n$-plane bundle,

and by abuse of language we write $H^*(X)[u,v]$ for the poly-
nomial algebra $H^*(X)[u_1, \ldots, u_m; v_1, \ldots, v_n]$ (in the sequel
we often abbreviate this list of arguments by $u,v$).

Proof First suppose $X$ is a point. Then $E^m,n = G_{m,n}$ and
$w(E) = 1$; the result in this case is well-known. For
general $X$, map $H^*(X)[u,v]/(uv = w(E)) \to H^*(E^m,n)$ by send-
ing $u_i$ to $w_i(\gamma_m)$ and $v_j$ to $w_j(\gamma_n)$.

$f$ is well-defined since $w(\gamma_m)w(\gamma_n) = w(\gamma_m \cup \gamma_n)$, and an
easy argument shows $\gamma_m \cup \gamma_n = \pi^{-1}(E)$, $\pi:E^m,n \to X$ the pro-
jection.

$f$ is 1-1 since $G_{m,n} \to E^m,n$ gives $i^* w_i(\gamma_m) = w_i(\gamma_n)$ \neq 0 and
similarly for $\gamma_n$ (writing now $\overline{\gamma}_m$ and $\overline{\gamma}_n$ for the canonical bundles on $G_{m,n}$, and $\overline{u}, \overline{v}$ for their Stiefel-Whitney classes). In fact the only polynomials in $w_i(\gamma_m)$ and
$w_j(\gamma_n)$ carried to 0 by $i^*$ are those given by $i^*(uv) = 
\overline{u} \cdot \overline{v} = 1 = i^*w(E)$.

$f$ is onto because in the Serre spectral sequence for the
fibration $G_{m,n} \to E^m,n \times X$, $E_2 = H^*(X) \otimes H^*(G_{m,n})$, and it can be
shown that the rank of $E_2$ and that of $H^*(X)[u,v]/
(uv = w(E))$ are equal in each dimension. Since $f$ is
injective, $E_2, E_\infty$ and $H^*(\mathbb{P}^n)$ must be additively isomorphic, and $f$ must be onto.

3.2 A basis of $H^*(\mathbb{P}^n)$ over $H^*(\mathbb{X})$ is given by

$$1 = \sum_{i=0}^n \eta_i \cdot v_i.$$ 

We wish to find an additive basis for $H^*(\mathbb{P}^n)$ over $H^*(\mathbb{X})$ among the monomials in the $u_i$ and $v_j$. Write $w_i(E) = E_i$ and similarly for other bundles. Since $uv = w(E)$,

$$u_k + u_{k-1}v_1 + \ldots + v_k = E_k$$

for $k = 0, 1, \ldots, n$. We can write $v_1 = u_1 + E_1$ and inductively express $v_1, \ldots, v_n$ in terms of the $u_i$ (and $H^*(\mathbb{X})$).

Defining $u_i = 0$ for $i > m$ and $v_j = 0$ for $j > n$, one has (1) also for $n < k < m + n$. Then substitution for $v_1, \ldots, v_n$ gives relations among polynomials in the $u_i$. To express these relations we define polynomials $P_k$ and $P_k'$ by

$$(2) \quad P_0(u) = 1, \quad P_k(u) = \sum_{|J| = k} J(u)^J$$

where $J$ stands for a sequence of positive integers $(j_1, \ldots, j_r)$ for some $r$ and we use the notation for any sequence, $|J| = \Sigma j_i$; and

$$(3) \quad P_k'(u) = \sum_{i=0}^{k-1} P_i(u)(u_{k-i} + E_{k-i}).$$

It is easy to show that

$$(4) \quad P_k(u) = \sum_{i=0}^{k-1} P_i(u) u_{k-1}$$

and

$$(5) \quad v_k = P_k'(u), \quad k = 1, \ldots, n,$$

using (1). Substituting in (1) we then find

$$(6) \quad P_{n+j}(u) = \sum_{i=0}^{n+j-1} P_i(u) E_{n+j-i-1} = \sum_{i=0}^{n} P_i(u) \sum_{k=1}^{j} i_{n+k-1} P_{j-k}(E)$$
where $E$ stands for the $E_i$ and $P_k(E)$ is defined by a formula similar to (2).

Further if $j > m + n$ then $0 = P_j' (u) = \sum_{i=0}^j u_i v_{j-i}$ is already implied by $0 = v_{n+1} = \cdots = v_{n+m}$ according to (3), hence $P_j'(u) = 0$ yields no new relations among polynomials in the $u_i$ for $j > m + n$.

3.3 The tangent bundle of $E^m \times_n$

Write $\tau (M)$ for the tangent bundle of a manifold $M$.

Theorem

$$\tau (E^m \times_n) \cong \nu^{-1} \tau (X) \oplus (\gamma_m \oplus \gamma_n)$$

where $E^m \times_n X$ is the projection.

This can be deduced from the results of [9]. The idea is that the tangent bundle of the total space of a fibration of manifolds is the sum of the vectors along the fibres with an orthogonal subbundle. The latter is isomorphic to $\nu^{-1} \tau (X)$ and the former in our case can be identified with $\gamma_m \oplus \gamma_n$.

3.4 $\omega (\gamma_m \oplus \gamma_n)$

To compute the Stiefel-Whitney classes of $E^m \times_n$, we need a result of E. Thomas [11] which we state without proof.

Theorem

If $\xi$ is an $m$-plane bundle and $\eta$ is an $n$-plane bundle over $X$, 
\[ w(\gamma_1 \otimes \gamma_n) = \phi_{m,n}(w_1(\xi), \ldots, w_m(\xi), w_1(\eta), \ldots, w_n(\eta)), \]

where if \( \sigma_i \) is the \( i \)th elementary symmetric function in the \( s_k \) and \( \tau_j \) is the \( j \)th elementary symmetric function in the \( t_k \) in the ring \( \mathbb{Z}[s_1, \ldots, s_m, t_1, \ldots, t_n] \).

(7) \[ \phi_{m,n}(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n) = \prod_{1 \leq i \leq m} (1 + \sigma_i + \tau_j) \]

(see the Appendix for definition of \( \sigma_i \)). To compute \( \phi_{m,n} \) we express it in terms of Milnor's polynomials \( s_j \) (see Appendix). Let \( S \) denote the set of all sequences \( J = (j_1, \ldots, j_m) \) of \( m \) integers between 0 and \( n \). Write \( J \uparrow \) if \( j_1 \leq \ldots \leq j_m \).

Lemma

\[ \phi_{m,n}(\sigma_1, \ldots, \tau_n) = \sum_{J \uparrow \in S} s_J(\sigma_1, \ldots, \sigma_m), \]

\[ = \sum_{A \in S} \tau_{n-a_1} \cdots \tau_{n-a_m} \binom{a_1}{j_1} \cdots \binom{a_m}{j_m} \]

where \( \binom{a}{j} \) is the binomial coefficient.

The proof follows from (7) by expanding the product and collecting monomials in the \( s_1 \) into groups \( \sum s_1^{k_1} \cdots s_m^{k_m} = s_J(\sigma_1, \ldots, \sigma_m) \).

An aid to computation results from noticing that if \( (a_1, \ldots, a_m) \) is a permutation of \( (b_1, \ldots, b_m) \), then

\[ \tau_{n-a_1} \cdots \tau_{n-a_m} = \tau_{n-b_1} \cdots \tau_{n-b_m}. \]

We can collect identical monomials in \( \tau \) and add their coefficients together.
The manifold \( M(3,3) \)

4.1 When is \( E_{m,n} \) spinor?

Recall that if \( E \times X \) is an \((m+n)\)-bundle, \( H^*(E_{m,n}) \) is generated as a ring over \( H^*(X) \) by \( u_1, \ldots, u_m, v_1, \ldots, v_n \), subject only to \( uv = w(E) \). According to Thomas [11], the terms in \( \phi_{m,n}(u,v) \) of degree 0, 1, and 2 are

\[
1 + (mv_1 + nu_1) + \frac{(m)}{2}v_1^2 + \frac{(n)}{2}u_1^2 + mv_2 + nu_2 \\
+ (mn - 1)u_1v_1.
\]

By 3, if \( M = E_{m,n} \), using \( uv = w(E) \) we get

\[
w(M) = w(X)w(\gamma_m \otimes \gamma_n) = w(X)\phi_{m,n}(u,v) \\
= 1 + ((X_1 + mE_1) + (m + n)u_1) + ((X_2 + (m)E_1^2 + mX_1E_1) + u_1(X_1(m + n) + mE_1) + \\
(mn - 1)E_1) + u_2((\frac{m}{2} + \frac{n}{2}) + m + mn + 1) \\
+ u_2(m + n)) + \text{higher terms}
\]

where we write for a manifold \( X \), \( w_1(X) = X_1 \).

Thus if \( M \) is orientable, \( M_1 = 0 \), or

\[
\begin{align*}
(1) & \quad m + n \equiv 0 \pmod{2} \\
(2) & \quad X_1 + mE_1
\end{align*}
\]

For \( M \) to be spinor, \( M_2 = 0 \) as well, or

\[
\begin{align*}
(3) & \quad X_2 + \frac{(m)}{2}E_1^2 + mE_1^2 + mE_2 + m^2E_1^2 = 0 \\
(4) & \quad mE_1 + (m^2 - 1)E_1 = 0 \\
(5) & \quad \frac{(m)}{2} + \frac{(n)}{2} + m^2 + m + 1 \equiv 0 \pmod{2},
\end{align*}
\]

since \( u_1, u_1^2 \), and \( u_2 \) are independent in \( H^*(M) \) over \( H^*(X) \) if \( m, n > 1 \).

By (4), \( E_1 = m(m + 1)E_1 = 0 \), hence (2) shows \( X_1 = 0 \).
Then by (3) $X_2 = mE_2$. By (5), \((\frac{m}{2}) + (\frac{n}{2}) \equiv 1 \pmod{2}\).
This implies \((m,n) \equiv (0,2), (1,3), (2,0),\) or \((3,1) \pmod{4}\).
Collecting the above conditions, we see that \(M\) will be

spinor if and only if

\[
\begin{align*}
E_1 &= X_1 = 0 \\
X_2 &= mE_2 \\
m &\equiv n + 2 \pmod{4}
\end{align*}
\]

(6)

\[4.2 \] \(m\) or \(n = 1\) or \(2\) does not work.

I calculated \(M^E_6\) for some manifolds \(M\) involving

successive projectifications of vector bundles in up to

three stages, and found that no \(M(3,3)\) was among them.

For more than 3 stages the calculations seem lengthy and

rather than continue them I began looking among manifolds

\(E^m,n\) for \(1 < m < n\). By (6) the simplest case is \(m = 2,\)

\(n = 4\); but computation shows that no spin manifold \(E^2,4\) can

satisfy \(w_6^H(E^2,4) \neq 0\).

\[4.3 \text{ Relations in } H^8(E^3,5)\]

The next case is \(m = 3, n = 5\). Write \(u_{1j...k} = u_{ij...k}\) and \(u_{ij...k} = u_{ij}\) for brevity, and similarly for \(E\).

Using the results of §3 one can write the relations

\[P_k'(u) = 0, \quad k = 6,7,8,\]

as

\[
\begin{align*}
u_{33} &= u_{222} + u_{211} + u_{16} + P_6(u) \\
u_{322} &= u_{311} + u_{2221} + u_{215} + u_{1P6(u)} + P_7(u) \\
u_{18} &= u_{2P6(u)} + P_8(u)
\end{align*}
\]

(7)
where \( P_6, P_7, \) and \( P_8 \) are to be expanded using 3,2(6). In dimension 10 because \( u_{3222} = (u_{322})u_2 = (u_{33})u_{22} \) can be decomposed in two ways, there results a relation which can be written

\[
(8) \quad u_{25} = u_{2411} + (u_{31} + u_{14})P_6 + (u_{3} + u_{21})P_7 + (u_2 + u_{11})P_8
\]

(This can be further reduced using (7)). Choosing an additive basis of \( H^\cdot(E^3, 5) \) over \( H^\cdot(X) \) whose elements, monomials in \( u \), have no factor \( u_{33}, u_{322}, u_{18}, \) or \( u_{25} \) leads to \( u_{2411} \) as a basis in dimension 15. Let \( M = E^3, 5 \).

4.4 \( w_6(M) \)

Calculating \( w(\gamma_3 \circ \gamma_5) \) yields

\[
= 1 + (v_1 + u_1) + (v_2 + u_2 + v_{11}) + v_3 + v_{111} + u_1v_2 + u_1v_{11} + u_2v_1 + u_3 + v_{211} + v_{22} + v_4 + u_{11}v_2 + u_{1111}
\]

\[
+ (v_5 + v_{311} + v_{211} + u_1v_{22} + u_1v_{211} + u_1v_4
\]

\[
+ u_{11}v_3 + u_2v_3 + u_{111}v_2 + u_3v_2 + u_{1111}v_1 + u_{15}
\]

\[
+ (v_{33} + v_{411} + v_{222} + u_{11}v_{211} + u_2v_4 + u_2v_{22} + u_2v_{21} + u_3v_{21} + u_2v_3 + u_3v_3 + u_{14}v_{11} + u_2v_{11}v_2
\]

\[
+ u_{14} + \text{higher terms,}
\]

and applying relations 3.2(5) one has, assuming \( M \) is spinor.

(\( X_2 = E_2 \) implies \( X_3 = E_3 \) by the Wu relations),

\[
w_6(M) = w(X) w(\gamma_3 \circ \gamma_5)
\]

\[
= X_6 + X_4E_2 + E_42 + u_{1}E_{32} + u_{11}E_{22} + u_{11}E_4
\]

\[
+ u_2E_4 + u_3E_3 + u_{11}E_3 + u_{22}E_2 + u_{21}E_{22} + u_{22} + u_16
\]

\[
u_{221} + u_{214}.
\]
$H^u(X)$ is 0 above dimension 9 if $X$ is a 9-manifold, so
$(Ea_1)^u = Ea_1^u$, which holds in any $\mathbb{Z}_2$-module, shows that
any term in $w_6(M)$ involving $H^{u}(X)$ in dimension > 2 can
be neglected in calculating $w_6^u(M)$. Further, since
$E_1 = X_1 = 0$, $H^1(X)$ will never enter the calculation;
thus any term containing a factor in $H^8(X)$ must be 0
as well. This leaves

(1) \[ w_6^u(M) = u_{212} + u_{124} + u_{2816} + u_{12416} \]

To see if $w_6^u(M) = 0$ we then use 4.3(7) and express (1) in
terms of our additive basis for $H^{u}(M)$ over $H^{u}(X)$. Fully
expanded, 4.3(7) and (8) become:

\[
\begin{align*}
    u_{33} &= u_{222} + u_{21\bar{h}} + u_{1\bar{c}} + E_5u_1 + E_4(u_{11} + u_2) \\
    &+ E_3(u_{111} + u_3) + E_2(u_{1\bar{h}} + u_{211} + u_{22}) \\
    u_{322} &= u_{314} + u_{2221} + u_{215} + E_5u_2 + E_4(u_{31} + u_{211} + u_{22}) \\
    &+ E_2(u_{2111} + u_{311}) \\
    u_{1\bar{8}} &= E_5(u_{111} + u_{21} + u_3) + E_4(u_{1\bar{h}}) + E_3(u_{2111}) \\
    &+ u_{32} + u_{1\bar{5}} + u_{221} + u_{31\bar{1}} + E_2(u_{21}\bar{h} + u_{2211} + u_{222}) \\
    u_{25} &= u_{2\bar{4}11} + x \\
    x &= E_5u_1 + E_5u_{111} + E_4u_{22} + E_3(u_{1\bar{h}} + u_{211} + u_{22}) \\
    &+ E_5(u_{221} + u_{311}) + E_2(u_{21}\bar{h} + u_{2211} + u_{222}) \\
    &+ E_4(u_{21}\bar{h} + u_6) + E_3(u_{1\bar{7}} + u_{31\bar{h}}).
\end{align*}
\]
It is convenient to express $u^{2k}$ for $k > 5$ by using 4.3(8) repeatedly: $u^{2k} = u^{2^412k-8} + Q_k x^5 (k > 5)$, where $Q_0 = 1$, $Q_k = u^{12k} + u^{12k-2} + \ldots + u^{2k}$, $k > 0$. Thus $Q_{k+5} = u^{12k+10} + u^{12k+2} + (k+1)u^{2^412k+2} + R_k x$, where $R_{2k} = Q_k^2$ and $R_{2k+1} = u^{2}R_{2k}$.

Expanding the last 2 terms of (1) we find

$$w_6^4(M) = u^{124} + u^{2^4116} + u^{16}u^{2^418} + (u^{16} + u^{214} + u^{221} + u^{222})x + (u^{2^4116} + x(u^{114}) + u^{212} + u^{221} + u^{222} + u^{2416} + u^{2^416} + u^{2^416} + u^{2^416} + u^{22})x).$$

Now use 4.3(7):

$$= u^{124} + u^{2^4}(u^{6^2} + p^2 + p^2) + (u^{14} + u^{124} + u^{226} + 8x^{14} + 8x^{22}x^2.$$

$$= u^{124} + p^2(u^{6^2} + 2^414 + x(u^{6^2} + u^{22}) + u^{6^2}p^2$$

$$+ (u^{14} + u^{22})x^2.$$

It helps to calculate $p^2$, $p^2$, $p^2$, $u^{124}$, and $x^2$ separately, and finally combine them. This can take about 11 pages. The result is $w_6^4(M) = u^{6^2} + 2^417 + 3222$. Recall that we assumed that M is a spin manifold, 4.1(6).

4.5 The base manifold X

We must find a 9-manifold X and bundle $E+X$ satisfying
APPENDIX: SYMMETRIC POLYNOMIALS

(1) $E_1 = X_1 = 0$

(2) It is well-known [17] that if $E_2$ is the graded polynomial $E_2 = X_2$ ($= X$) $\neq 0$.

(3) In $n$ variables $3222$, $3222$ of dimension $3$.

For any such bundle, $M$ will be a suitable manifold $M$ of the $X_4$ form a subring $S$ which is a poly-

The tangent bundle of any orientable (by (1)) 9-

manifold $X$ splits off a trivial line bundle. The re-

maining $8$-bundle $E$ will then satisfy (1) and (2), so we

need only make sure that $X$ is $\neq 0$. This we do follow-

ing the construction of orientable manifolds in [4].

$X$ will be a product of complex projective space $CP^2$, of dimension $4$, and a manifold $Y$ ($Y_5$ or $M(3,2)$) in [4]: let $F + RP^2$ be the bundle $H \ominus 5T$ where $H$ is the canonical line bundle on $RP^2$ and $T$ is a trivial line bundle. Then we put $Y = F_{5, 5}$. It is easy to show that $X_{3222} = Y_{32}(CP^2)_{22}$ $\neq 0$ using §3, so $X$ satisfies (1), (2), and (3).
APPENDIX: SYMMETRIC POLYNOMIALS

It is well-known [12] that in the graded polynomial ring $R_m$ in $m$ variables $x_1, \ldots, x_m$, each of dimension 1, the symmetric polynomials (those invariant under permutations of the $x_i$) form a subring $S_m$ which is a polynomial ring on generators $\sigma_1, \ldots, \sigma_m$ of dimension $\sigma_1 = 1$, where

$$1 + \sigma_1 + \ldots + \sigma_m = \prod_{i=1}^{m} (1 + x_i). \tag{1}$$

Also well-known is their usefulness in the study of characteristic classes, where the Stiefel-Whitney classes of an $m$-bundle which splits into $m$ line bundles are the elementary symmetric functions of the first Stiefel-Whitney classes of the line bundles, by the Whitney product theorem and (1). Thomas' result in 3.4 uses them, too.

Milnor [7] defines the following additive basis for $S_m$: call two monomials equivalent if some permutation of the $x_i$ carries one into the other. If $J = (j_1, \ldots, j_m)$ define $s_J$ by the equation in $R_m$,

$$s_J(\sigma_1, \ldots, \sigma_m) = \sum x_1^{k_1} \cdots x_m^{k_m},$$

summing over all monomials equivalent to $x_1^{j_1} \cdots x_m^{j_m}$. $s_J$ is a polynomial (homogeneous of dimension $|J| = j_1 + \ldots + j_m$) since $S_m$ is the polynomial ring on the $\sigma_i$. If $J$ satisfies $0 \leq j_1 \leq \ldots \leq j_m$ write $J^+$. It is obvious that $\{s_J(\sigma) \mid J^+\}$ (abbreviating as usual $\sigma_1, \ldots, \sigma_m$ by $\sigma$)
forms an additive basis for $S_m$. Another additive basis consists of monomials $\sigma^J = \sigma^J_1 \cdots \sigma^J_m$ for all $J$ such that $J_i > 0$, $i = 1, \ldots, m$. The lemma of 3.4 shows knowledge of the polynomials $s_J$ is useful in computing $\phi_{m,n}$.

It seems to be easier to calculate $\sigma^K$ in terms of various $s_J(\sigma)$ and then invert the transformation, than to attack the problem directly. This can be done inductively: $\sigma^1 = s(\sigma)$, and once an expression for each $\sigma^J$ with $|J| < q$ is known, if $|J| = q$ we can write $\sigma^J = \sigma^{J'} \cdot \sigma_1$ with $|J'| < q$ for some $1$, and use the expression for $\sigma^{J'}$ to find that for $\sigma^J$, by the lemma we shall shortly state. The sequences we speak of below will all be ordered sequences of $m$ non-negative integers. If $N = (n_1, \ldots, n_m)$ denotes a sequence we write $N(j)$ for the number of $j$'s appearing in $N$, and $x^N$ for $x_1^{n_1} \cdots x_m^{n_m}$.

If $J$ is a sequence and $1 \geq 0$ an integer, we define a sequence $K$ to be a $(J,1)$-sequence, if one can obtain $K$ by increasing each of $1$ entries in $J$ by unity. If $K$ is a $(J,1)$-sequence, choose a suitable set $S$ of $i$ entries of $J$ to be thus increased. (S may contain several copies of any integer). Let $h(j)$ be the number of $j$'s in $S$. I claim $h(0), h(1), \ldots$ are all fixed by $J$ and $K$. For from the definitions we find that

$$K(j) = J(j) - h(j) + h(j-1), \ j > 0$$

since increasing a $j$ in $J$ takes away one $j$ from $K$ and increasing a $(j-1)$ in $J$ adds one. Transposing,
(1) \( h(j-1) = K(j) - J(j) + h(j), j > 0. \)

Since \( S \) is finite, there is a largest \( j = j_0 \) for which \( h(j_0) \neq 0 \). Thus using (1) for \( j = j_0 + 1, j_0, \ldots, 1 \) in turn gives a proof by decreasing induction that \( K \) and \( J \) determine the \( h(j) \).

Lemma

If \( J \) is a sequence and \( i > 0 \) an integer,

(2) \[ s_J(\sigma) \cdot \sigma_1 = \Sigma c_K s_K(\sigma) \]

summed over all \((J,i)\)-sequences \( K \) where \( c_K \) is the integer

\[ c_K = \prod_{j \geq 0} \frac{K(j)!}{J(j)! \cdot h(j)!} \]

Proof: Recall \( s_J(\sigma) = \Sigma x^T \) summed over monomials \( x^T \)
equivalent to \( x^J \), and \( \sigma_1 = \Sigma x^D \) summed over sequences \( D \)
containing \( i \) ones and \( m-i \) zeroes, so

(3) \[ s_J(\sigma) \cdot \sigma_1 = \Sigma x^T \Sigma x^D = \Sigma x^{T+D} \]

(adding sequences entrywise). It turns out that each \( T+D \)
is a \((T,i)\)-sequence, and hence \( x^{T+D} \) is equivalent to
some \( x^K \) where \( K \) is a \((J,i)\)-sequence. Since (3) is a
symmetric polynomial, each monomial occurs together with all
equivalent monomials, and there exists a formula (2) in
which only \((J,i)\)-sequences occur. If we know the number of
monomials \( x^{T+D} \) in the sum (3) which are equivalent to \( x^K \),
we can then find the coefficient \( c_K \) of \( s_K \) by dividing by
the number of monomials in \( s_K(\sigma) \). The latter is \( m! / \prod_{j \geq 0} J(j)! \).

Let \( K \) be a fixed \((J,i)\)-sequence. How many monomials in
(3) are equivalent to \( x^K \)? There are \( m! \prod_{j>0} J(j)! \) different monomials \( x^T \) equivalent to \( x^j \), and for each of them \( x^T \sigma_1 \) contain the same number of monomials equivalent to \( x^K \).

We might as well use \( x^J \) to compute this number.

(4) \[ x^J \sigma_1 = x^J + D \]

where \( D \) runs over sequences of \( i \) ones and \( (m-i) \) zeroes, and \( x^J + D \) is equivalent to \( x^K \) if and only if adding \( D \) to \( J \) increases exactly \( h(j) \) of the \( j \)'s in \( J \), for each \( j \). There are \( \binom{J(j)}{h(j)} \) ways to choose \( h(j) \) entries from \( J(j) \) candidates, and every possible selection of \( i \) unit increases occurs for some \( D \), hence (4) contains \( \prod_{j>0} J(j)! \) monomials equivalent to \( x^K \).

Combining the above we have

\[
\frac{m!}{\prod_{j>0} J(j)!} \prod_{j>0} J(j)! \]

which gives the formula of the lemma.


10 R. Thom, "Quelques Propriétés Globales des


12 Van Der Waerden, Algebra.

BIOGRAPHICAL NOTE

Carey Mann was born in October 1942, and educated in the public schools, in Lakewood, Ohio. He attended high school in Edgewood, Pa., and received a B.S. from M.I.T. in 1964.