ADMISSIBLE ORDINALS AND RECURSION THEORY

by

Stephen G. Simpson

B.A., M.S., Lehigh University
(1966)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
September 1971

Signature of Author:
Department of Mathematics, August 16, 1971

Certified by....

Thesis Supervisor

Accepted by.............
Chairman, Departmental Committee on Graduate Students

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Submitted to the Department of Mathematics in August, 1971 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Let $\alpha$ be an admissible ordinal. The basic notions of $\alpha$-recursion theory are defined in terms of the Kleene-Kripke equation calculus. These notions are then redefined in terms of first order definability over $M^\alpha$. The $\alpha$th level of Gödel's constructible hierarchy. The relation between $\alpha$-recursion theory on the one hand, and $M^\alpha$'s properties as a model of a weak set theory on the other, is stressed.

Section 1 treats the problem of generalizing Friedberg's maximal set theorem to $\alpha$-recursion theory. Six notions of maximal $\alpha$-r.e. set are defined and their inequivalence discussed briefly. A fairly general sufficient condition on $\alpha$ for the non-existence of maximal $\alpha$-r.e. sets is established. Call $\alpha$ $\Delta_2$-collapsible if there is a $\Delta_2$ function with domain $\omega$ and range $\alpha$. Theorem. If $\alpha$ is $\Delta_2$-collapsible then maximal $\alpha$-r.e. sets exist. Conjectures are made concerning the problem of when maximal $\alpha$-r.e. sets exist.

Sections 2 and 3 treat non-hyperregular $\alpha$-r.e. sets. If $B$ is $\alpha$-r.e. define $rcf(B)$ to be the least $\beta$ such that there is an unbounded function with domain $\beta$ weakly $\alpha$-recursive in $B$. Thus $rcf(B)$ measures the magnitude of $B$'s failure to be hyperregular. Theorem. $B$'s $\alpha$-degree contains a non-regular $\alpha$-r.e. set if and only if $\alpha^{M_\alpha} < \alpha$ and $rcf(B) < cf^{\alpha}(\alpha^{M_\alpha})$. Theorem. Let $\beta$ be a regular cardinal of $M_\alpha$, then $\beta$ is the rcf of some $\alpha$-r.e. set if and only if there is an $\alpha$-recursive function $f$ with range $(f) \subseteq \beta = \lim_{\sigma} f(\sigma)$. In particular, if $\alpha$ is $\Delta_2$-collapsible then every regular cardinal of $M_\alpha$ is the rcf of some $\alpha$-r.e. set.
The following theorem answers questions of Sacks and Owings. Theorem. Suppose $\alpha^*$ is a regular cardinal of $M_\alpha$. Then (a) there are incomparable non-hyperregular $\alpha$-r.e. $\alpha$-degrees; (b) every non-hyperregular $\alpha$-r.e. $\alpha$-degree contains a simple $\alpha$-r.e. subset of $\alpha^*$. Examples show that the hypothesis of regularity cannot be dropped for either conclusion.

In section 3 a Skolem hull argument is used to prove: Theorem. If $\alpha$ is the least admissible ordinal such that $\langle M_\alpha, \epsilon \rangle$ satisfies some $\Sigma_4$ sentence, then $\alpha$ is $\Delta^2_2$ collapsible.

Section 4 treats generalizations to $\alpha$-recursion theory of Friedberg's completeness criterion. It is observed that all reasonable notions of $\alpha$-jump coincide when restricted to regular, hyperregular $\alpha$-degrees. Theorem. Let $\alpha$ be a countable admissible ordinal, then there is a cone of $\alpha$-degrees which are $\alpha$-jumps of regular, hyperregular $\alpha$-degrees.

Thesis Supervisor: Gerald E. Sacks
Title: Professor of Mathematics
Acknowledgements

I wish to acknowledge the help of all who have encouraged my work in mathematical logic.

To those of my contemporaries, including Irene Chang and Richard Shore, who by their examples have stimulated me to greater effort, I offer my sincere gratitude.

Thanks go also to my teachers, especially L. Tharp and H. Rogers, Jr.

Extensive intellectual debts are owed to G. D'Innocenzo, R. Jensen, G. Kreisel, J. MacIntyre, H. Putnam, and J. Owings.

My chief thanks go to Gerald Sacks, who in many long conversations has taught me to appreciate the subtleties and delicate variations possible in forcing and Löwenheim-Skolem arguments. Sacks' endless patience and willingness to explain and encourage are well known to his students. My debt to Sacks, both intellectual and personal, is much greater than to all the others combined.
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Introduction

The advance of science is not comparable to the changes of a city, where old edifices are pitilessly torn down to give place to new, but to the continuous evolution of zoologic types which develop ceaselessly and end by becoming unrecognizable to the common sight, but where an expert eye finds always traces of the prior work of past centuries.

-- H. Poincaré.

This thesis contains various contributions to $\alpha$-recursion theory. Here $\alpha$ is an admissible ordinal.

By $\alpha$-recursion theory we mean the recursion theory developed by Kreisel and Sacks [13] for $\alpha = \omega_1^{ck}$ (Church-Kleene $\omega_1$) and generalized by Kripke [14] and Platek [22] to arbitrary admissible $\alpha$.

In $\alpha$-recursion theory the domain of individuals is the set of ordinals less than $\alpha$. There is a sense in which $\alpha$-recursion theory is the most natural recursion theory with this domain of individuals. At present we are unable to make this precise.

The reader will need no previous knowledge of $\alpha$-recursion theory save a nodding acquaintance with the basic notions as expounded e.g. on pages 324-326 of Kreisel-Sacks [13]. However, many parts of the work (e.g. the section on Preliminaries) will be unclear unless the reader knows Gödel's proof of the G.C.H. assuming $V = L[7]$. 
The present work is merely the latest in a long series of Ph.D. theses (Driscoll, Owings, MacIntyre, Machtley, Sukonick, Stillwell, Myers) devoted to various questions in $\alpha$-recursion theory. Our emphasis differs somewhat from that of previous authors. Let us list the differences, hoping thus to convey to the reader an idea of our outlook and purpose.

1. Most previous authors have dealt only with the special case $\alpha^* = \omega$. Now it so happens that some of our results (e.g. 2.5.5 and 4.4) are new even for $\alpha^* = \omega$; however, our main focus has been arbitrary admissible ordinals.

2. J. MacIntyre [18] concentrated on $\alpha$-recursion theory for countable admissible ordinals $\alpha$. His methods did not depend on the specific nature of $\alpha$ beyond its countability. In contrast, a lot of our work depends on very specific "internal" properties of $M_\alpha$, the $\alpha$th constructible level, regarded as a model of a weak set theory. Some of the relevant internal properties are these: $\alpha^* < \alpha$; $\alpha^*$ is a regular cardinal of $M_\alpha$; there is a $\Delta_2$ function from $\omega$ onto $\alpha$.

Sometimes a result is true for all $\alpha$'s (or at least a wide class of them) but its proof
8.

splits into cases depending on the nature of \( \alpha \). For example, we had to give two completely different proofs of Lemma 2.9, one for \( \alpha^* \) a regular cardinal of \( M_\alpha \), the other for \( \alpha^* \) a singular cardinal of \( M_\alpha \). We see no hope of stretching either proof to cover both cases. Moreover neither of the two proofs bears the slightest resemblance to the classical proof of the corresponding theorem in ordinary recursion theory (Dekker [3]).

Another example of this sort is the proof [27] of the Friedberg-Muchnik theorem for all admissible \( \alpha \). Here the main split is between \( \alpha^* < \alpha \) and \( \alpha^* = \alpha \).

3. Previous authors have been concerned chiefly with "lifting" theorems from ordinary recursion theory into \( \alpha \)-recursion theory. (Our sections 1 and 4 below are devoted to two questions of this type: Friedberg's maximal set and completeness theorems respectively.) In contrast, the results of our sections 2 and 3 are not generalizations of results from ordinary recursion theory and in fact have no counterparts there.
9.

We feel strongly that results of this type will play an ever more important role as our subject evolves. For instance, work is now in progress on a method whereby $\alpha$-recursion theoretic priority arguments can be used to facilitate various forcing constructions over uncountable $M_\alpha$'s. We hope that this method will have applications in model theory and set theory. We think it unlikely that such applications would make sense in the case $\alpha = \omega$.

4. Previous authors in $\alpha$-recursion theory have emphasized the Kleene-Kripke equation calculus. We have based our thinking rather on Gödel's constructible hierarchy [7], [8]. Thus for instance a partial function $\phi \subseteq \alpha \times \alpha$ is $\alpha$-recursive just if its graph is $\Sigma_1$ over $M_\alpha$.

This approach has proved fruitful in many ways. To mention a few:

(i) Throughout this thesis we have scattered many examples of pathological and not-so-pathological admissible ordinals. These examples arise naturally from Gödel's proof [7] of the G.C.H. assuming $V = L$. 
(ii) In at least one instance (the proof of Theorem 3.4) we have employed Gödel's Skolem hull-condensation method in an essential way.

(iii) The use of $\mathcal{M}_\alpha$'s opens the way for set- and model-theoretic methods in priority arguments (see [27]).

(iv) The use of $\mathcal{M}_\alpha$'s suggests that one might try to apply the methods of $\alpha$-recursion theory to problems in set theory and in the model theory of infinitary languages.

We end the introduction by pointing out that many of the problems and methods of this thesis were suggested by Sacks' fundamental paper [25].
§0. Preliminaries

The reader should not read this section through, but rather refer to it as needed.

We use von Neumann's definition of ordinal. Thus an ordinal is identified with the set of smaller ordinals.

Let $\alpha$ be a limit ordinal. We assume familiarity with the Kleene-Kripke [14], [13] equation calculus for $\alpha$. Thus $S_0^E$ is the set of all equations deducible from $E$, an initial set of equations, in fewer than $\tau$ steps. Equations in $S_0^E$ are constrained to mention only numerals appearing in $E$ or denoting ordinals less than min $\{\sigma, \alpha\}$.

We call $\alpha$ admissible if $S_{\alpha+1}^E = S_\alpha^E$ for every finite set of equations $E$ in the equation calculus for $\alpha$. From now on $\alpha$ denotes an admissible ordinal. $E_\alpha$ is the set of all equations in the equation calculus for $\alpha$.

A partial function $\phi \subseteq \alpha \times \alpha$ is $\alpha$-recursive if there is a finite $E \subseteq E_\alpha$ such that for all $\langle \gamma, \delta \rangle \in \alpha \times \alpha$

$$\phi(\gamma) = \delta \text{ iff } \exists \zeta \in S_\alpha^E \ z(\gamma) = \delta.$$
We may define an $\alpha$-recursive pairing function by

$$<\beta, \gamma> = \left( \sum_{\xi<\beta+\gamma} \xi + 1 \right) + \beta.$$ 

Thus $<$, $>$ puts $\alpha \times \alpha$ into $\alpha$-recursive one-one correspondence with $\alpha$. We also write $<\beta, \gamma, \delta> = <<\beta, \gamma>, \delta>$, etc.

A subset of $\alpha$ is $\beta$-bounded ($\beta \leq \alpha$) if it is a subset of some $\eta < \beta$. We sometimes say bounded instead of $\alpha$-bounded.

A subset of $\alpha$ is $\alpha$-recursive if its representing function is $\alpha$-recursive. It is $\alpha$-finite if it is $\alpha$-recursive and bounded.

0.1 If $\phi$ is an $\alpha$-recursive partial function and $K \subseteq \text{dom}(\phi)$ is $\alpha$-finite, then $\phi[K]$ is $\alpha$-finite.

0.2 There is an $\alpha$-recursive function $k(\gamma, \eta)$ such that

(i) if $k(\gamma, \eta) = 0$ then $\gamma < \eta$;

(ii) $\{ \gamma | k(\gamma, \eta) = 0 \}$ ranges over the $\alpha$-finite sets as $\eta$ ranges over $\alpha$;
(iii) the finite set of equations defining $k$ mentions no numerals except those denoting finite ordinals.

Putting

$$K_\eta = \{ \gamma | k(\gamma, \eta) = 0 \}$$

we obtain the canonical indexing of the $\alpha$-finite sets. Note that $K_\eta \subseteq \eta$ for all $\eta$.

The projectum of $\alpha$, denoted $\alpha^*$, is the least $\beta$ such that there is a one-one $\alpha$-recursive function into $\beta$. Note that $\alpha^* \leq \alpha$; we say $\alpha$ is projectible if $\alpha^* < \alpha$.

A subset of $\alpha$ is $\alpha$-recursively enumerable (abbreviated $\alpha$-r.e.) if it is the domain of an $\alpha$-recursive partial function.

0.3 If $\eta < \alpha^*$ then every $\alpha$-r.e. subset of $\eta$ is $\alpha$-finite.

0.4 There is an $\alpha$-recursive function $r(\sigma, \epsilon)$ such that

$$(1) \quad K_{r(\sigma, \epsilon)} \subseteq K_{r(\tau, \epsilon)} \subseteq \tau \text{ for } \sigma \leq \tau;$$
(ii) \( \bigcup \{ K_r(\sigma, \epsilon) \mid \sigma < \alpha \} \) ranges over the \( \alpha \)-r.e. sets as \( \epsilon \) ranges over \( \alpha \);

(iii) the finite set of equations defining \( r \) mentions no numerals except those denoting finite ordinals.

Putting \( R^G_\epsilon = K_r(\sigma, \epsilon) \) and \( R_\epsilon = \bigcup \{ K_r(\sigma, \epsilon) \mid \sigma < \alpha \} \) we obtain our standard indexing of the \( \alpha \)-r.e. sets. Note that \( R^G_\epsilon \subseteq R^T_\epsilon \subseteq \tau \) for \( \sigma \leq \tau \).

A partial function \( \phi \subseteq \alpha \times \alpha \) is weakly \( \alpha \)-recursive in a set \( B \subseteq \alpha \) (abbreviated \( \phi \leq_{\text{wa}} B \)) if there is \( \epsilon < \alpha \) such that for all \( \langle \gamma, \delta \rangle \in \alpha \times \alpha \),

\[
\phi(\gamma) = \delta \quad \text{iff}
\]

\[
(E_\xi)(En)[\langle \gamma, \delta, \xi, \eta \rangle \in R_\epsilon \land K_\xi \subseteq B \land K_\eta \subseteq cB].
\]

For \( B \subseteq \alpha \) we write

\[
\Delta_B = \{ g(\gamma) = 0 \mid \gamma \in B \} \cup \{ g(\gamma) = 1 \mid \gamma \in cB \}.
\]

If \( E \subseteq E_\alpha \) then \( S^E, B \) is the set of all equations in \( E_\alpha \) deducible in any number of steps from the initial set of equations \( E \cup \Delta_B \). A partial function
\( \phi \subset \alpha \times \alpha \) is \( \alpha \)-calculable from \( B \) (abbreviated \( \phi \leq_{ca} B \)) if there is a finite set of equations \( E \subset \mathcal{E}_\alpha \) such that for all \( \langle \gamma, \delta \rangle \in \alpha \times \alpha \),

\[
\phi(\gamma) = \delta \text{ iff } \gamma(\gamma) = \delta \in S^{E,B}.
\]

It seems difficult to work directly with the notion of \( \alpha \)-calculability. This is because the deduction tree putting an equation into \( S^{E,B} \) need not be \( \alpha \)-finite (even though it is constrained to mention only numerals for ordinals less than \( \alpha \)). We therefore introduce some auxiliary notions:

We denote by \( R^{E,B} \) the set of all equations in \( S^{E,B} \) having \( \alpha \)-finite deduction trees. Call a partial function \( \phi \subset \alpha \times \alpha \) \( \alpha \)-finitely calculable from \( B \) if there is a finite set of equations \( E \subset \mathcal{E}_\alpha \) such that for all \( \langle \gamma, \delta \rangle \in \alpha \times \alpha \),

\[
\phi(\gamma) = \delta \text{ iff } \gamma(\gamma) = \delta \in R^{E,B}.
\]

0.5 ([26]) \( \phi \) is \( \alpha \)-finitely calculable from \( B \) if and only if \( \phi \leq_{\omega \alpha} B \).

Let \( B \subset \alpha \) be given. \( B \) is regular if \( B \cap \gamma \) is \( \alpha \)-finite for every \( \gamma < \alpha \). \( B \) is hyperregular if \( \phi[\gamma] \)
is bounded for every $\phi \leq_{w\alpha} B$, $\gamma < \alpha$, $\gamma \subseteq \text{dom} \ (\phi)$.

0.6 ([26]) The following properties of $B \subseteq \alpha$ are equivalent.

(i) $B$ is regular and hyperregular;

(ii) every deduction from $B$ is $\alpha$-finite, i.e. $SE, B = RE, B$ for every finite $E \subseteq \mathcal{E}_\alpha$.

Thus $\phi \leq_{w\alpha} B$ and $\phi \leq_{c\alpha} B$ are equivalent provided $B$ is regular and hyperregular.

0.7 If $B$ is $\alpha$-r.e. and hyperregular, then $B$ is regular.

For $A, B \subseteq \alpha$ we write $A \leq_{w\alpha} B$ (resp. $A \leq_{c\alpha} B$) if the representing function of $A$ is $\leq_{w\alpha}$ (resp. $\leq_{c\alpha}$) $B$. It turns out that $\leq_{w\alpha}$ is not usually transitive. We therefore consider a closely related notion: $A$ is $\alpha$-recursive in $B$ (abbreviated $A \leq_{\alpha} B$) if

$\{2n | K_n \subseteq A\} \cup \{2n + 1 | K_n \subseteq cA\}$ is weakly $\alpha$-recursive in $B$. We take $\leq_{\alpha}$ and $\leq_{c\alpha}$ as the basic reducibility notions in $\alpha$-recursion theory.
The equation calculus is not the only way to define the basic notions of \(\alpha\)-recursion theory and admissible ordinals. One particularly elegant approach, due to Platek ([22]; see also Jensen-Karp [9]), is to introduce primitive recursive functions of ordinals via schemata.

Unfortunately, neither the equation calculus nor the approach using schemata seems to suffice for proving the deeper results about admissible ordinals. A typical result of this sort is Kripke's theorem that \(\alpha^*\) is admissible. It does not seem possible to give a proof of Kripke's theorem purely in terms of equations or schemata.

These deeper results seem to depend on the methods introduced by Gödel [7] for proving the G.C.H. assuming \(V = L\). We therefore review Gödel's hierarchy of constructible sets. We shall need this material in §3.

The \textbf{ZF language} is just the first order language with \(=, \in\). Recall Lévy's [16] classification of formulas in the ZF language. A formula is \textbf{limited}
if it is built up from atomic formulae $x = y$, $x \in y$ using only propositional connectives $\&, \vee, \neg, \rightarrow, \leftrightarrow$ and limited quantifiers $(\text{Ex})_{x \in y}$ and $(x)_{x \in y}$. A formula is $\Sigma_0$ or $\Pi_0$ if it is limited. For $n \geq 1$, a formula is $\Sigma_n$ if it has the form

$(\text{Ex}_1)...(\text{Ex}_k)F$ where $F$ is $\Pi_{n-1}$; a formula is $\Pi_n$ if it has the form $(x_1)...(x_k)G$ where $G$ is $\Sigma_{n-1}$.

Let $t$ be a transitive set. A set $x \subseteq t$ is $\Sigma_n$ (over $t$) if it is first order definable over the structure $<t, \epsilon>$ by a $\Sigma_n$ formula with parameters from $t$; $x$ is $\Delta_n$ (over $t$) if it is both $\Sigma_n$ and $\Pi_n$ over $t$. Fodo ($t$) is the set of all subsets of $t$ which are $\Sigma_n$ over $t$ for some $n \geq 0$. The constructible hierarchy is defined by transfinite introduction thus: $M_0 = \emptyset$; $M_{\beta+1} = \text{fodo}(M_\beta)$;

$M_\lambda = \bigcup \{M_\beta | \beta < \lambda\}$ for limit ordinals $\lambda$. The constructible universe, $L$, is the union of all the $M_\beta$'s.

A formula of the ZF language is a sentence if it has no free variables.
0.8 THEOREM.

1. There is a $\Pi_2$ sentence $F$ such that for any transitive set $t$, we have $\langle t, \varepsilon \rangle \models F$ if and only if $t = M_\lambda$ for some limit ordinal $\lambda$.

2. There is a $\Pi_3$ sentence $G$ such that $\langle M_\beta, \varepsilon \rangle \models G$ if and only if $\beta$ is admissible.

The sentence $F$ of this theorem is sometimes referred to as $V = L$. This theorem will be needed in §3. Its proof is straightforward. For a detailed proof of $\aleph$-land refinements, consult Boolos [1].

Following Rogers [24] pp. 301-307 we can define an $\alpha$-arithmetical hierarchy. Thus a relation on $\alpha$ is $\Sigma_0$ if it is $\alpha$-recursive, $\Pi_n$ if its complement is $\Sigma_n$, and $\Sigma_{n+1}$ if it is the projection of a $\Pi_n$ relation. It turns out that, for $n \geq 1$, a relation is $\Sigma_n$ in this sense if and only if it is $\Sigma_n$ over $M_\alpha$.

Fact: a $\Sigma_n$ partial function with $\Delta_n$ domain is $\Delta_n$.

Warning: the bounded quantifier manipulations of Rogers [24] p. 311 do not lift to $\alpha$-recursion theory.
except in very special circumstances. However, Jensen [8] has proved the following remarkable theorem.

**0.9 Theorem (Jensen)** Every $\Sigma_n$ relation on $\alpha$ can be uniformized by a $\Pi_n$ partial function.

In this dissertation we shall have little to say about $\Sigma_n$, $n \geq 3$. But for $n = 1, 2$ the following trivialities should be borne in mind.

**0.10 Proposition.** A partial function $\phi \subset \alpha \times \alpha$ is

(i) $\Sigma_1$ iff it is $\alpha$-recursive;

(ii) $\Sigma_2$ iff there is an $\alpha$-recursive function $f$ such that always

$$\phi(\gamma) = \lim_{\sigma} f(\sigma, \gamma)$$

i.e.

$$\phi(\gamma) = \delta \text{ iff } (E\tau)(\sigma)_{\sigma \geq \tau} f(\sigma, \gamma) = \delta.$$ 

For $n \geq 1$, we say $\alpha$ is $\Sigma_n$-admissible if every $\Sigma_n$ function with domain $\gamma < \alpha$ has bounded range. Thus $\Sigma_1$-admissibility is equivalent to admissibility.
For $\beta \leq \alpha$ and $n \geq 1$, the $\Delta_n$-cofinality of $\beta$ (abbreviated $\Delta_n$-cf($\beta$)) is the least $\lambda$ such that there is a $\Sigma_n$ function with domain $\lambda$ and $\beta$-unbounded range. We say $\beta$ is $\Delta_n$-regular if $\Delta_n$-cf($\beta$) = $\beta$. Thus $\alpha$ is $\Sigma_n$-admissible if and only if it is $\Delta_n$-regular.

Recf($\alpha$), the recursively enumerable cofinality of $\alpha$, is the least $\lambda$ such that there is an unbounded $\alpha$-r.e. set of the form

$$\bigcup_{\xi<\lambda} B_\xi \times \{\xi\}$$

with each $B_\xi$ bounded. The following facts are easily verified:

0.10.1. Recf($\alpha$) = $\Delta_2$-cf($\alpha$).

0.10.2. If $\Delta_2$-cf($\beta$) < $\Delta_1$-cf($\beta$) then $\Delta_2$-cf($\beta$) = $\Delta_2$-cf($\alpha$).

In particular, $\alpha$ is $\Sigma_2$ admissible if and only if recf($\alpha$) = $\alpha$. The invariant recf($\alpha$) will play a major role in §§2,3.
There is a $\Delta_2$ function carrying an $\alpha^*$-unbounded subset of $\alpha^*$ order-preservingly onto an unbounded subset of $\alpha$. Hence; $\text{cf}(\alpha) = \text{cf}(\alpha^*)$, and $\Delta_n - \text{cf}(\alpha) = \Delta_n - \text{cf}(\alpha^*)$ for $n \geq 2$.

The structure $<M_\alpha, \in>$ can (and should) be thought of as a model of a weak set theory. In particular, all the usual definitions of ZF set theory apply within $M_\alpha$. In this connection it is important to keep in mind that $K \subseteq \alpha$ is $\alpha$-finite if and only if it is a member of $M_\alpha$.

Suppose for instance that $\beta < \alpha$. Then clearly $\Delta_1 - \text{cf}(\beta) = \text{cf}^{M_\alpha}(\beta)$ where $\text{cf}^{M_\alpha}(\beta)$ is the cofinality of $\beta$ as evaluated by a man living in $M_\alpha$, i.e. the least order type of an $\alpha$-finite, $\beta$-unbounded set. (The proof that $\Delta_1 - \text{cf}(\beta) = \text{cf}^{M_\alpha}(\beta)$ uses the fact that $\alpha$ is admissible.) Similarly, suppose $K$ is $\alpha$-finite; then we denote by $\text{card}^{M_\alpha}(K)$ the cardinality of $K$ as evaluated inside $M_\alpha$, i.e. the least $\beta$ such that there is an $\alpha$-finite function from $\beta$ onto $K$.

Let $\beta$ be an infinite ordinal less than $\alpha$. We say $\beta$ is an $\alpha$-cardinal if $\text{card}^{M_\alpha}(\beta) = \beta$. We say $\beta$
is a regular $\alpha$-cardinal if $\text{cf}^M\alpha(\beta) = \beta$. We say $\beta$ is a singular $\alpha$-cardinal if $\beta$ is an $\alpha$-cardinal which is not regular. If $K$ is an $\alpha$-finite set then the $\alpha$-cardinality of $K$ is just $\mathfrak{K}^M\alpha$.

0.11 A singular $\alpha$-cardinal is a limit of regular $\alpha$-cardinals.

0.12 If $\alpha^* < \alpha$, then $\alpha^*$ is the largest $\alpha$-cardinal.

The next two theorems are good to keep in mind when thinking about $\alpha$-cardinals.

0.13 THEOREM. Let $\langle I_\nu : \nu < \lambda \rangle$ be a sequence of $\alpha$-finite sets such that

$$\bigcup_{\nu < \lambda} I_\nu \times \{\nu\}$$

is $\alpha$-r.e. Suppose $\beta$ is an $\alpha$-cardinal such that $\lambda < \text{cf}^M\alpha(\beta)$ and each $I_\nu$ has $\alpha$-cardinality less than $\beta$. Then

$$\bigcup_{\nu < \lambda} I_\nu$$
is $\alpha$-finite and has $\alpha$-cardinality less than $\beta$.

PROOF. See Sacks-Simpson [27].

This Theorem has a corollary which is basic to the material of §2.

0.14 COROLLARY. Suppose $\alpha^*$ is less than $\alpha$. Let $C$ be an $\alpha$-r.e. subset of $\alpha^*$ such that $cc \cap \alpha^*$ is $\alpha^*$-unbounded. Then $cc \cap \alpha^*$ has order type $\geq \text{cf}\, \alpha^*(\alpha^*)$.

0.15 THEOREM (Gödel) Let $\beta$ be an $\alpha$-cardinal. Then every $\alpha$-finite, $\beta$-bounded subset of $\beta$ is $\beta$-finite.

PROOF. As in Gödel [7].

We shall sometimes employ the bounded $\mu$-operator. Thus $(\mu x)_{x < y} \ldots$ means "the least $x$ such that $x$ is less than $y$ and ... if such an $x$ exists, $y$ otherwise."

We denote by $\alpha^+$ the next admissible ordinal after $\alpha$. Apart from this, all our notation is self-explanatory.
§1. Existence of Maximal $\alpha$-r.e. sets

In this section we discuss generalizations to $\alpha$-recursion theory of the following theorem of Friedberg [6]: there is an r.e. set whose complement is infinite but cannot be split by an r.e. set into two infinite parts. Our treatment is far from complete. We aim only to present some results and conjectures which we hope will be useful when the time comes for a complete treatment.

In the first place, an $\alpha$-r.e. set with bounded complement can for most purposes be identified with an $\alpha$-r.e. subset of $\alpha^*$.  

---

1 Here we discount investigations of the lattice of $\alpha$-r.e. sets. (See Machtay [17] ) This thesis is mainly concerned with the combinatorial differences between various admissible ordinals and the effect of these differences on the associated recursion theories. The question "for which $\alpha$ do maximal $\alpha$-r.e. sets exist?" is far from answered. It therefore seems premature to start looking at lattice theoretic questions in this context.
In the second place, while several notions of generalized finiteness present themselves for consideration, some of these turn out to be irrelevant because of the specific nature of the theorem we are trying to generalize. The pertinent trivialities are these:

(i) If a bounded set is split into two parts, then both parts are bounded.

(ii) If an $\alpha$-finite set of $\alpha$-cardinality less than $\alpha^*$ is split by an $\alpha$-$r.e.$ set into two parts, then both parts are $\alpha$-finite.

(iii) An $\alpha$-finite set of $\alpha$-cardinality $\alpha^*$ can always be split by an $\alpha$-$r.e.$ set into two $\alpha$-infinite parts.

Thus for example, by (ii), the notion "$\alpha$-finite set of $\alpha$-cardinality less than $\alpha^*$" becomes
irrelevant.  

Bearing these things in mind, we see that there are only six reasonable notions of maximal set for $\alpha$-recursion theory.

1.1 DEFINITION. An $\alpha$-r.e. set is UU (resp. UA, UI)-maximal if its complement is unbounded and cannot be split by an $\alpha$-r.e. set into two unbounded (resp. $\alpha$-infinite, infinite) parts.

1.2 DEFINITION. An $\alpha$-r.e. subset of $\alpha^*$ is SA (resp. SU, SI)-maximal if its complement in $\alpha^*$ is $\alpha$-infinite and cannot be split by an $\alpha$-r.e. set into two $\alpha$-infinite (resp. $\alpha^*$-unbounded, infinite) parts.

---

2 Hence for example the following two notions of "maximal $\Pi^1_1$ set" coincide: a $\Pi^1_1$ set whose complement cannot be split by a $\Pi^1_1$ set into two infinite parts; a $\Pi^1_1$ set whose complement cannot be split by a $\Pi^1_1$ set into two non-hyperarithmetic parts.
1.3 REMARKS

1. The notation of 1.1 and 1.2 has been generated by writing $U$ for unbounded, $A$ for $\alpha$-infinite, $I$ for infinite, and $S$ for star.

2. $UI$ implies $UA$ which implies $UU$; $SI$ implies $SU$ which implies $SA$.

3. If $\alpha^* = \alpha$ then $UU = UA = SU = SA$ and $UI = SI$.

4. If $\alpha^* = \omega$ then $UA = UI$ and $SA = SU = SI$.

5. If $\alpha^*$ is a regular $\alpha$-cardinal then $SA = SU$.

6. The complement of a $UI$ or $SI$ set has order type $\omega$. If $\text{recf} (\alpha) > \omega$ then there are no $UI$ or $SI$ sets.

7. If $\alpha^* < \alpha$ and $\text{cf}^\alpha (\alpha^*) > \omega$ then there are no $SI$ sets.

Call an $\alpha$-r.e. set maximal (no prefix) if it is maximal in any of the six senses of Definitions 1.1 and 1.2. We give a condition on $\alpha$ sufficient for the non-existence of maximal sets.
1.4 THEOREM. Suppose \( \beta \) is less than \( \alpha^* \) and there are no \( \alpha \)-cardinals between \( \beta \) and \( \alpha^* \). Suppose further that \( \alpha^* \) is \( \Delta_3 \)-regular. Then there are no maximal \( \alpha \)-r.e. sets.

PROOF. It suffices to rule out the existence of UU- and SA-maximal sets.

Let \( M \) be a UU-maximal set. By hypothesis on \( \beta \) there is a one-one \( \alpha \)-recursive function \( H \) from \( \alpha \) into the \( \alpha \)-finite subsets of \( \beta \). For each \( \nu < \beta \) let

\[
S_\nu = \{ \sigma < \alpha | \nu \in H_\sigma \}.
\]

Then \( S_\nu \) is \( \alpha \)-recursive. Hence by UU-maximality either \( cM \cap S_\nu \) or \( cM - S_\nu \) is bounded by some \( \tau < \alpha \). The relation

\[
cM \cap S_\nu \subseteq \tau \lor cM - S_\nu \subseteq \tau
\]

is \( \Pi_2 \). Hence by Jensen's uniformization lemma 0.9, there is a \( \Delta_3 \) function \( t(\nu) \) which uniformizes this relation, i.e. \( cM \cap S_\nu \subseteq t(\nu) \) or \( cM - S_\nu \subseteq t(\nu) \).
By $\Delta_3$ regularity, $t[\beta]$ is $\alpha$-bounded. Let $\gamma, \delta$ be members of $cM$ such that $t[\beta] \subseteq \gamma < \delta < \alpha$. Then $H_\gamma = H_\delta$. This contradicts the fact that $H$ is one-one.

The proof that there are no $SA$-maximal sets is similar; just replace each occurrence of $\alpha$ by $\alpha^*$. Q.E.D.

1.5 COROLLARY. Suppose $\alpha^*$ is a successor cardinal in $L$, the constructible universe. Then there are no maximal $\alpha$-r.e. sets.

Sacks [25] first pointed out that there are no maximal $\omega_1^L$-r.e. sets.

Theorem 1.4 covers every known (to us) case of an $\alpha$ with no maximal $\alpha$-r.e. sets. We would not be a bit surprised if it turned out that the converse of 1.4 is true, i.e. that maximal $\alpha$-r.e. sets exist whenever the conditions of 1.4 do not obtain. Our best partial result in this direction is the following:

1.6 THEOREM

Hypothesis: There is a $\Delta_2$ function with domain $\omega$
and range \( \alpha \).

**Conclusion:** There is a UI-maximal \( \alpha \)-r.e. set.


**PROOF.** Let \( f(\sigma,n) \) be an \( \alpha \)-recursive function such that \( f(n) = \lim_{\sigma} f(\sigma,n) \) exists for each finite \( n \) and \( f \) has domain \( \omega \) and range \( \alpha \).

We shall define \( \alpha \)-recursive functions \( v(\sigma,e) (e < \omega) \) and \( M^\sigma \) by induction on \( \sigma \). \( \lambda \sigma M^\sigma \) will be an increasing sequence of \( \alpha \)-finite sets. At the end of the construction we shall put

\[
M = \bigcup \{ M^\sigma | \sigma < \alpha \}
\]

and prove that \( M \) is UI-maximal.

As a preliminary to stage \( \sigma \) of the construction we put

\[
M^{< \sigma} = \bigcup \{ M^\tau | \tau < \sigma \}.
\]

**Stage \( \sigma \):** For each \( n < \alpha \) we say \( n \) is in the \( j \)th \( e \)-state if \( n \notin M^{< \sigma} \) and

\[
j = \Sigma \{ 2^{e-1} | n \in R^\sigma_f(\sigma,i) \land i < e \}.
\]
We define $v(\sigma, e)$ ($e < \omega$) by induction on $e$.

Since $M^{< \sigma}$ is $\alpha$-finite, there is an $e$-state which contains an $n$ exceeding every member of

\[ \{v(\sigma, i) | i < e\} \cup \{f(\tau, i) | \tau < \sigma \land i \leq e\}. \]

Let $j(\sigma, e)$ be the highest such $e$-state. Let $v(\sigma, e)$ be the least $n$ which exceeds every member of

\[ \{v(\sigma, i) | i < e\} \cup \{f(\tau, i) | \tau < \sigma \land i \leq e\} \]

and is in the $j(\sigma, e)^{th}$ $e$-state.

We define $M^\sigma$ by

\[ M^\sigma = \{u | u < v(\sigma, 0) \lor (\forall n)(v(\sigma, n) < u < v(\sigma, n+1))\}. \]

This completes stage $\sigma$ of the construction.

Note that $M^{< \sigma} \subseteq M^\sigma$ and that $\{v(\sigma, n) | n < \omega\}$ are the first $\omega$ members of the complement of $M^\sigma$.

1.6.1 LEMMA. For each $e$, $v(e) = \lim_{\sigma} v(\sigma, e)$ exists, i.e. $(E\sigma)(\tau)_{\tau > \sigma} v(\tau, e) = v(\sigma, e)$.
PROOF. We argue by induction on \( e \). Suppose for each \( i < e \) that \( v(i) = \lim_{\sigma} v(\sigma,i) \) exists. Let \( \gamma \) be the least ordinal exceeding every member of

\[
\{v(i)|i < e\} \cup \{f(\sigma,i)|\sigma < \alpha & i \leq e\}.
\]

Let \( \sigma_0 \) be such that

\[
(\sigma)_{\sigma \geq \sigma_0} ((i)_{i < e} v(\sigma,i) = v(i) & (i)_{i \leq e} f(\sigma,i) = f(i)).
\]

Then for any \( \sigma \geq \sigma_0 \), \( j(\sigma,e) \) is the highest \( e \)-state containing an \( \eta \geq \gamma \). Let \( j \) be the largest member of \( \{j(\sigma,e)|\sigma \geq \sigma_0\} \). Let \( \beta \) be the least \( \eta \geq \gamma \) such that \( \eta \) is in the \( j \)-th \( e \)-state at some stage \( \sigma \geq \sigma_0 \).

Let \( \sigma_1 \geq \sigma_0 \) be such that \( \beta \) is in the \( j \)-th \( e \)-state at stage \( \sigma_1 \). Then \( j(\sigma_1,e) = j \) and \( v(\sigma_1,e) = \beta \).

Hence by induction \( j(\sigma,e) = j \) and \( v(\sigma,e) = \beta \) for all \( \sigma \geq \sigma_1 \). Q.E.D.

Recall that \( f(\sigma,e) < v(\sigma,e) < v(\sigma,e+1) \) for all \( \sigma \) and \( e \), and that the range of \( f \) is unbounded. It follows that \( \mathcal{M} = \{v(e)|e < \omega\} \) is unbounded and
has order type $\omega$.

1.6.2 LEMMA. $M$ is UI-maximal.

PROOF. Suppose not. Let $e$ be the least $n$ such that $R_e(n)$ splits $cM$ into two infinite parts. Then there are $c, d, i, j$ such that $e < c < d, i < j$, and $v(c)$ (resp. $v(d)$) is in the $i^{th}$ (resp. $j^{th}$) e-state at all sufficiently large stages $\sigma$. Let $\sigma_0$ be such that $\sigma_0 \geq \sigma \geq 0 (\sigma_{0}, i), i < d(v(\sigma, i) = v(i)$ & $f(\sigma, i) = f(i))$. Let $\sigma > \sigma_0$ be such that $v(c)$ (resp. $v(d)$) is in the $i^{th}$ (resp. $j^{th}$) e-state at stage $\sigma$. Let $i^*$ (resp. $j^*$) be the c-state of $v(c)$ (resp. $v(d)$) at stage $\sigma$. Then $v(d)$ exceeds every member of

$\{v(\sigma, i)\mid i < c\} \cup \{f(\tau, i)\mid \tau \leq \sigma \land i \leq c\}$

so $j(c, c) \geq j^* > i^*$. On the other hand $v(\sigma, c) = v(c)$ so $j(\sigma, c) = i^*$, a contradiction. Q.E.D.

The proof of 1.6 is complete.

1.7 COROLLARY.
Hypothesis: \(\omega\) is the largest \(\alpha\)-cardinal; \(\alpha\) is not \(\Sigma_2\) admissible.

Conclusion: There are UI- and SI-maximal \(\alpha\)-r.e. sets.

PROOF. For such an \(\alpha\) the conditions of Theorem 1.6 clearly obtain and so UI-maximal \(\alpha\)-r.e. sets exist. If \(\alpha^* = \alpha\) then any such set is also SI-maximal. If \(\alpha^* < \alpha\) then \(\alpha^* = \omega\), so the usual construction of a maximal \(\Pi_1^1\) set [13] gives an SI-maximal \(\alpha\)-r.e. set. Q.E.D.

We do not know whether \(\Sigma_2\) in this Corollary can be improved to \(\Sigma_3\) (we conjecture that it can). By 1.4 it cannot be improved to \(\Sigma_4\).

1.8 EXAMPLE. Let \(\alpha\) be the least admissible ordinal not projectible into \(\omega\). Then clearly \(\omega\) is the largest \(\alpha\)-cardinal and so \(\alpha^* = \alpha\). It follows that \(\alpha\) is the limit of the first \(\omega\) \(\alpha\)-stable ordinals. (See Sacks-Simpson [27] for a discussion of \(\alpha\)-stable ordinals.) Hence \(\alpha\) is not \(\Sigma_2\)-admissible. Hence by 1.7 there is a UI-maximal \(\alpha\)-r.e. set.
Kreisel [12] p. 159 asks whether the various notions of maximal set are extensionally inequivalent. Of the notions considered in Definitions 1.1 and 1.2, all pairs except \{UA, UI\} and \{SA, SU, SI\} are extensionally inequivalent in metarecursion theory\(^3\). Moreover \(UA \neq UI\) and \(SU \neq SI\) in \(\alpha\)-recursion theory for the \(\alpha\) of example 1.8. We do not know of an \(\alpha\) where \(SA \neq SU\).

Theorems 1.4 and 1.6 should of course be viewed as partial results on the following

1.9 PROBLEM For which admissible \(\alpha\) do maximal \(\alpha\)-r.e. sets exist?

There are really six problems here, corresponding to the different notions of maximal set. If \(Z \in \{UU, UA, UI, SA, SU, SI\}\) then we ask: for which \(\alpha\) do \(Z\)-maximal sets exist? All known (to us) results on

\(^3\) For \(UU \neq UA\) let \(M = \{\omega \oplus \beta | \beta \in B\}\) where \(B\) is a UI-maximal meta-r.e. set. Of course \(UA = UI\) and \(SA = SU = SI\) in metarecursion theory.
these six problems are covered by Remarks 1.3 and Theorems 1.4, 1.6, and 1.7 above.

In §3 we shall exhibit an (admittedly rather pathological) admissible \( \alpha \) such that UI-maximal \( \alpha \)-r.e. sets exist but SI-maximal \( \alpha \)-r.e. sets do not exist. Thus the six problems will not all have the same answer.

We conjecture that Z-maximal \( \alpha \)-r.e. sets always exist except when they are ruled out by Remarks 1.3 and Theorem 1.4.

To prove our conjecture one will have to abandon or radically alter Friedberg's notion of e-state. All known maximal set constructions use this notion and hence produce sets whose complements have order type \( \omega \).

Let us state some special cases of our conjecture which seem like sensible starting points.

1.10 CONJECTURE Let \( \alpha = \mathcal{H}^L_\omega \), the \( \omega \)th constructible cardinal. Then UI-maximal
α-r.e. sets exist

1.11 CONJECTURE. Let α be the first $\Sigma_2$-admissible ordinal greater than $\omega$. Then UA-maximal α-r.e. sets exist.

If one proved 1.11 then the method of proof combined with 1.4 would probably yield an exact determination of \{α|α is admissible & $\omega$ is the largest α-cardinal & maximal α-r.e. sets exist\}.

Closely related to 1.11 is the following conjecture for metarecursion theory.

1.12 CONJECTURE There is a hyperregular maximal meta-r.e. set.

---

4 J. Stillwell in his thesis (M.I.T. 1970) claimed to have proved this conjecture, but he has since withdrawn his claim.
§2. Non-hyperregular $\alpha$-r.e. sets and their $\alpha$-degrees

A. Consider a hyperregular $\alpha$-r.e. set. By 0.6 and 0.7 such a set is also regular and so every deduction from it is $\alpha$-finite. Hence arguments involving such a set are likely to have $\omega$-recursion theory counterparts and to resemble them closely. In short, hyperregular $\alpha$-r.e. sets are pleasant in every way.

Furthermore, Sacks has shown that for every $\alpha$ there is a hyperregular $\alpha$-r.e. set which is not $\alpha$-recursive$^5$.

All this has led some people, e.g. Sukonick [31] p. 72, to propose that hyperregular $\alpha$-r.e. sets rather than arbitrary $\alpha$-r.e. sets be taken as the correct generalization of r.e. sets to $\alpha$-recursion theory.

$^5$ The proof of this in [25] is correct in broad outline but contains several errors of detail. See our [28] or Sukonick [31] for a correct proof in the special case of metarecursion theory. A correct proof for $\alpha$-recursion theory is sketched in [27], section 6.
We see no really good reason for accepting this proposal. Moreover, the proposal would have a number of unpleasant consequences, e.g. it would mean rejecting all known maximal set constructions in \( \alpha \)-recursion theory for \( \alpha > \omega \). (This is because the sets yielded by these constructions are all non-hyperregular. We do not know whether there exist hyperregular maximal meta-r.e. sets. We conjecture that they do exist but that to prove this one will need an infinite injury argument.) In general, non-hyperregular meta-r.e. sets (e.g. \( \eta_1^1 \) sets) have played a significant role in metarecursion theory. This role would be lost if Sukonick's proposal were accepted.

A much better case could be made for another proposal which we label \( P : \text{accept all sets, but reject the notion of } \alpha \text{-degree and focus instead on } \alpha \text{-calculability degrees.} \) Kreisel [10], [11] has stressed the model-theoretic significance of \( \alpha \)-calculability in case \( \alpha \) is countable. We mention only that for \( \alpha \) countable and \( B \subseteq \alpha \), \( \alpha \)-calculability from \( B \) is equivalent to invariant implicit definability
from $B$ (in the sense of Kunen [15], over the transitive set $M_\alpha$).

For $\alpha$-r.e. sets the connection between $\alpha$-degrees and $\alpha$-calculability degrees is as follows:

2.1 PROPOSITION

1. Every $\alpha$-r.e. $\alpha$-degree containing a hyperregular set is an $\alpha$-calculability degree.

2. All non-hyperregular $\alpha$-r.e. sets fall into the same $\alpha$-calculability degree.\(^6\)

PROOF. 2.1.1 is an immediate consequence of 0.6 and 0.7. Let $B$ be a non-hyperregular set. We shall prove 2.1.2 by showing that every $\alpha$-r.e. set is $\alpha$-calculable from $B$. Let $\phi$ be a partial function weakly $\alpha$-recursive in $B$ such that $\text{dom}(\phi) = \lambda < \alpha$ and $\text{range}(\phi)$ is unbounded. Given an $\alpha$-r.e. set $A$,

\(^6\) This generalizes Spector's result that all non-hyperarithmetic $\Pi^1_1$ sets fall into the same hyperdegree.
let $A_\alpha(\sigma < \alpha)$ be an increasing $\alpha$-recursive sequence of $\alpha$-finite sets such that $A = \bigcup \{A_\alpha | \sigma < \alpha\}$. Then

(\#) $\gamma \in A$ iff $(E\delta)^{\delta < \lambda} \gamma \in A_{\phi}(\delta)$.

Since $A_\alpha(\sigma < \alpha)$ is $\alpha$-recursive and $\phi \leq B$, we can easily translate (\#) into a finite set of equations for calculating $A$ from $B$. Q.E.D.

Thus proposal $P$ would have the desirable effect of eliminating a lot of pathology from the upper semi-lattice of $\alpha$-r.e. $\alpha$-degrees. We could permit ourselves to concentrate instead on the hyperregular case. The arguments would tend to resemble those of $\omega$-recursion theory, hence presumably to shed more light on $\omega$-recursion theory.

A relevant technical observation here is this:

2.1.3 REMARK. Every theorem about $\omega$-degrees which has been lifted to metadegrees, also lifts to meta-calculability degrees. (E.g. Driscoll's proof of the density theorem for metadegrees [4] contains a proof of the same theorem for metacalculability degrees (see [4] Theorem 7, p. 410). Similarly
MacIntyre's minimal metadegree [18] and Sukonick's minimal pair of meta-r.e. metadegrees [31] are regular and hyperregular, hence also give the same results for metacalculability degrees. Another example is provided by our discussion of the Friedberg completeness theorem in metarecursion theory, §4 below.)

The notion of \( \alpha \)-calculability degree is thus seen to be important and interesting. Nevertheless, we reject proposal \( P \) at least for the time being.

Our reason is this: the reducibility notion \( \leq_{\omega \alpha} \) (weak \( \alpha \)-recursiveness) is an indispensable tool in all known arguments for lifting theorems about \( \omega \)-degrees to \( \alpha \)-calculability degrees. We have tried without success to avoid the use of metadegrees in the proof that there is a non-trivial incomplete meta-r.e. meta-calculability degree. (Of course a completely different proof may be required. We may be overlooking some model-theoretic trick for dealing directly with non-metafinite deductions.)

The proofs of the theorems mentioned in 2.1.3 all use metadegrees, even to get results about
metacalculability degrees only. (The general method of
attack here is to lift the theorem in question to
\( \alpha \)-degrees, but to mix in preservation requirements
which cause the sets constructed to be regular and
hyperregular. Thus by 0.6 and 0.5 all the relevent
\( \alpha \)-degrees turn out to be \( \alpha \)-calculability degrees.)

We think that so long as this state of affairs
prevails, the reducibility \( \leq_{\alpha} \), and hence the associated
notion of degree, should continue to be studied.

B. It is more difficult to explain our interest in
the topic of the present section, non-hyperregular
\( \alpha \)-r.e. sets and their \( \alpha \)-degrees. In view of 2.1.2,
the study of this topic cannot yield any information
about \( \alpha \)-calculability degrees.

Perhaps the best explanation we can offer is
as follows: our purpose in this thesis is to expose
the differences between various admissible ordinals.
Specifically we ask: how do these differences show up
in the associated recursion theories? We do not know
of any result saying that one admissible ordinal \( \alpha 
has a different \( \alpha \)-degree or \( \alpha \)-calculability degree
structure from another. We would not be a bit surprised if the upper semilattices of α-r.e. α-degrees and α-r.e. α-calculability degrees for all admissible α were elementarily equivalent. Hence, these upper semilattices would be a poor choice of topic for our purpose.

In contrast, the study of non-hyperregular α-r.e. α-degrees will be seen to reveal many interesting differences between various admissible ordinals. To mention just one example: the admissible ordinal \((\Omega^L_\omega)^+\) has exactly one non-hyperregular α-r.e. α-degree; while its "\(\Lambda_2\)-collapsible counterpart" (the α of Example 3.3 below) has many α-r.e. α-degrees of non-hyperregular sets.

The main conclusions of our investigation are these: (i) if \(\alpha^*\) is a regular α-cardinal, then the non-hyperregular α-r.e. α-degrees behave pretty much as they do for \(\omega^ck\); (ii) if \(\alpha^*\) is a singular α-cardinal, then the situation can be quite different.

C. We begin with a simple definition which is the obvious starting point for an investigation of
non-hyperregularity.

2.2 DEFINITION Let $B \subseteq \alpha$. Then we define $\text{rcf}(B) = \mu\beta[\text{there is a function } \phi \leq_{\omega\alpha} B \text{ with } \text{dom}(\phi) = \beta \text{ and range } (\phi) \text{ unbounded}].$

$\text{Rcf}(B)$ might be called the "recursive cofinality of } \alpha \text{ relative to } B." \text{ The following facts are easily verified.}

2.3 FACTS

1. $B$ is hyperregular $\iff \text{rcf}(B) = \alpha$.

2. $A \equiv \alpha B \Rightarrow \text{rcf}(A) = \text{rcf}(B)$. Thus rcf is an invariant of $\alpha$-degree.

3. Suppose $B$ is $\alpha$-r.e. Then rcf($B$) is a regular $\alpha$-cardinal.

4. $\text{Rcf}(\alpha) = \min \{\text{rcf}(B)|B \text{ is } \alpha\text{-r.e.}\}$

5. Suppose $B$ is $\alpha$-r.e. and non-regular. Then $\text{rcf}(B) \leq \text{cf}^\alpha(\alpha^\#)$.

2.4 DEFINITION. An $\alpha$-r.e. set $B$ is conscientious if $cB$ is unbounded and has order type rcf($B$).
2.5 THEOREM.

1. Every α-r.e. α-degree contains a conscientious α-r.e. set.

2. Every conscientious α-r.e. set is regular.

For the proof of 2.5.1 we shall need a lemma due to Sacks [25].

2.6 LEMMA (Sacks) Every α-r.e. α-degree contains a regular α-r.e. set.

PROOF. We're given an α-r.e. set B. We may assume without loss that B is non-regular. Hence $\alpha^* < \alpha$.

Let $\beta$ be such that $B \cap \beta$ is α-infinite. Let $g$ be a one-one α-recursuve function whose range is a subset of $\alpha^*$. Let $N = g[B \cap \beta]$. Thus $N$ is an α-infinite α-r.e. subset of $\alpha^*$ and $N \leq^* \alpha$.

Let $n$ be a one-one α-recursuve function whose range is $N$. Let

$$B^* = \{ n(\eta) | K_\eta \cap B \neq \emptyset \}.$$  

Then $B^*$ is an α-infinite α-r.e. subset of $\alpha^*$. 
Let $f$ be a one-one $\alpha$-recursive function whose range is $B^*$. Define the deficiency set of $f$ by

$$D_f = \{\gamma | (E\delta)_{\gamma < \delta} f(\delta) < f(\gamma)\}.$$ 

We claim: $D_f$ is $\alpha$-r.e. and regular, and $D_f \equiv_\alpha B$.

Obviously $D_f$ is $\alpha$-r.e. and $cD_f$ is unbounded.

$D_f$ is regular because: for each $\beta$ there is a $\tau$ such that

$$D_f \cap \beta = \{\gamma < \beta | (E\delta)_{\gamma < \delta < \tau} f(\delta) < f(\gamma)\}.$$ 

If $\tau$ did not exist, then in searching for $\tau$ one would develop a sequence $\tau_n (n < \omega)$ such that $f(\tau_{n+1}) < f(\tau_n)$ for all $n$.

Note that $f$ is increasing on $cD_f$. To show $D_f \subseteq_\alpha B$ we observe:

$$\gamma \in cD_f \iff f(\gamma) - f[\gamma] \subseteq cB^*.$$ 

Hence, for an $\alpha$-finite set $K$, 

\[ K \subseteq cD_f \iff K' \text{ is } \alpha^*\text{-finite} \land K' \subseteq cB^* \]
\[ \iff K'' \subseteq cB \]

where we write

\[ K' = \bigcup_{\gamma \in K} (f(\gamma) - f[\gamma]) \]

and

\[ K'' = \overline{\eta \in \text{en}^{-1}[K' \cap N]} \cdot K_\eta. \]

Thus, using \( N \preceq \alpha B \), it follows that \( D_f \preceq \alpha B \).

On the other hand, for each \( \nu < \alpha^* \) let us define

\[ p(\nu) = \nu \gamma[\nu < f(\gamma) \land \gamma \in cD_f]; \]

\( p(\nu) \) exists since \( cD_f \) is unbounded. Clearly \( p \) is weakly \( \alpha \)-recursive in \( D_f \), and

\[ \nu \in B^* \iff \nu \in f[p(\nu)]. \]

It follows at once that \( B \preceq \alpha D_f \). Q.E.D.
PROOF of 2.5.1 We're given an $\alpha$-r.e. $\alpha$-degree.

By 2.6 it contains a regular $\alpha$-r.e. set, B. If $rcf(B) = \alpha$ then B is hyperregular, hence conscientious.

Suppose now that $rcf(B) = \beta < \alpha$. Let $\phi$ be a function weakly $\alpha$-recursive in B with domain $\beta$ and unbounded range. Let $\rho$ be such that always

$$\phi(v) = \delta \leftrightarrow (\exists n)[\langle v, \delta, n \rangle \in R_{\rho} \land K_{\eta} \cap B = \phi].$$

Let $g$ be a one-one $\alpha$-recursive function whose range is B. Define

$$y(\sigma, v) = (\mu \tau)_{\tau < \sigma}(\exists n)[\langle v, \delta, n \rangle \in R_{\rho} \land K_{\eta} \cap g(\sigma) = \phi].$$

Note that for each $v < \beta$, $y(\sigma, v)$ is monotone increasing in $\sigma$; moreover $\lim_{\sigma} y(\sigma, v)$ exists since $\phi(v)$ is defined. By induction on $\sigma$ we define

$$f(<\sigma, v>) = \sup \{f(\tau, v) | \tau < \sigma\};$$

$$F(\sigma) = \sup \{f(<\sigma, v>) | v < \beta\};$$

and
51.

\[ f(\sigma, \nu) = \begin{cases} 
  f(<\sigma, \nu) & \text{if} \\
  \mu \leq \nu, y(\sigma + 1, \mu) = y(\sigma, \nu) \leq g(\sigma); \\
  F(\sigma) + 1 & \text{otherwise}. 
\end{cases} \]

For each \( \nu < \beta \), \( f(\nu) = \lim_{\sigma} f(\sigma, \nu) \) exists since \( B \)
is regular and \( \nu < \text{rcf}(B) \). Moreover \( f(\nu) \) is monotone increasing in \( \nu < \beta \) and weakly \( \alpha \)-recursive in \( B \). On the other hand \( f[\beta] \) is unbounded since \( \phi[\beta] \) is.

It follows that \( f[\beta] \) has order type \( \beta \).

Define \( M = \alpha - f[\beta] \). Note that \( M \) is regular and \( \alpha \)-r.e. Also \( M \leq B \); hence \( M \) is conscientious.

But \( f(\sigma, \nu) = f(\nu) \) implies \( B \cap y(\sigma, \nu) = g[\sigma + 1] \cap y(\sigma, \nu) \);
thus \( B \leq M \). The proof of 2.5.1 is complete.

For 2.5.2 let \( B \) be a non-regular \( \alpha \)-r.e. set.

Let \( \gamma \) be the least ordinal such that \( B \cap \gamma \) is not \( \alpha \)-finite. Let \( \{d(\nu) | \nu < \lambda\} \) be the members of \( cB \cap \gamma \) in increasing order. Obviously \( \lambda \) is a limit ordinal, and the sequence \( d(\nu) (\nu < \lambda) \) is weakly \( \alpha \)-recursive in \( B \). Let \( B^\sigma (\sigma < \alpha) \) be an increasing \( \alpha \)-recursive sequence of \( \alpha \)-finite sets whose union is \( B \). For each \( \nu < \lambda \) define \( \phi(\nu) \) to be the least \( \sigma \) such that
Then \( \phi \) is weakly \( \alpha \)-recursive in \( B \), and \( \phi[\lambda] \) is unbounded. Hence \( \text{rcf}(B) \leq \lambda \). Hence \( B \) is not conscientious. \( \Box \)

2.5.3 COROLLARY

\( \text{rcf}(\alpha) = \mu \beta \) [there is an \( \alpha \)-r.e. set \( B \) with \( cB \) unbounded of order type \( \beta \)]. In particular \( \text{rcf}(\alpha) = \alpha \) if and only if every \( \alpha \)-r.e. set is hyperregular.

This 2.5.3 answers a question of J. Stillwell.

By an \( \omega \)-set Owings [21] means a subset of \( \omega^c \) whose complement is unbounded and has order type \( \omega \).

2.5 COROLLARIES

4. Every meta-r.e. metadegree either is hyperregular or contains a meta-r.e. \( \omega \)-set.

5. Every meta-r.e. metadegree either is hyperregular or contains a \( \Pi^1_1 \) set.
PROOF. 2.5.4 is just 2.5.1 specialized to metarecursion theory. Then 2.5.5 follows from 2.5.4 and the results of [21]; see also Theorem 2.10 below.

REMARKS.

1. Corollary 2.5.5 answers Owings' questions Q1 and Q2 ([20] p. 87; see also [21] p. 201). In Owings' terminology, 2.5.5 says that every totally regular set is completely regular.

2. In what follows we shall investigate the extent to which 2.5.5 generalizes to α-recursion theory. It will turn out that (i) it does generalize provided α* is a regular α-cardinal; (ii) it does not generalize to all projectible α.

3. It is often useful to have at hand a theorem saying that we can pick "nice representatives" from every α-r.e. α-degree. The first theorem of this type was Sacks' lemma 2.6. Corollary 2.5.5 is a theorem of this type in metarecursion theory; for instance, Driscoll's proof of the density theorem [4] can be simplified considerably by the judicious use of 2.5.5
54.

and 2.12.1. Unfortunately 2.5.5 does not generalize to every $\alpha$ or even to every projectible $\alpha$. We therefore offer 2.5.1 as a general result of this type which holds for all $\alpha$.

4. Another way to use 2.5.1 is illustrated in Example 2.18 below.

D. Our next goal is to prove Theorem 2.10. This theorem characterizes the $\alpha$-degrees of non-regular $\alpha$-r.e. sets in terms of rcf.

We break the proof up into two lemmas. The first lemma is essentially due to J. Owings (in the metarecursive case).

Note that $\{k_\eta | \eta < \alpha\}$, our canonical indexing of the $\alpha$-finite sets, is uniform in the following sense: if $\beta < \alpha$ is admissible then $\{k_\eta | \eta < \beta\}$ is the canonical indexing of the $\beta$-finite sets.

An $\alpha$-complete set is one in which the complete $\alpha$-r.e. set is $\alpha$-recursive.
LEMMA (Owings) Suppose \( \alpha^* < \alpha \). Let \( B \) be an \( \alpha \)-r.e. set with

\[
M \text{ rcf}(B) \leq cf \alpha(\alpha^*).
\]

Then there is an \( \alpha \)-complete \( \alpha \)-r.e. subset of \( \alpha^*, C \), such that \( \{v < \alpha^* | K_v \cap C \neq \emptyset \} \) is weakly \( \alpha \)-recursive in \( B \).

2.8 REMARKS

1. The conclusion \( \{v < \alpha^* | K_v \cap C \neq \emptyset \} \leq \omega \alpha B \) in 2.7 is a little stronger than saying \( C \leq \omega \alpha B \). We shall need this extra strength in the proof of 2.10. Since \( C \) is \( \alpha \)-complete, we cannot expect to get \( C \leq \alpha B \).

2. Owings [21] proved the metarecursive case of 2.7 under the additional hypothesis that \( B \) be an \( \omega \)-set. Our 2.5.4 shows that this hypothesis was not necessary.

PROOF of 2.7 We are given an \( \alpha \)-r.e. set \( B \) with \( M \text{ rcf}(B) \leq cf \alpha(\alpha^*) \). By 2.5.1 we may assume that
B is conscientious. Let

\[ K_\alpha = \bigcup_{\rho < \alpha} P_\rho \times \{\rho\} \]

be the complete \( \alpha \)-r.e. set. Let \( g \) be a one-one \n\( \alpha \)-recursive function whose range is a subset of \( \alpha^\# \).

Define

\[ N = g[K_\alpha] \times \alpha^\# \]

Thus \( N \) is an \( \alpha \)-infinite \( \alpha \)-r.e. subset of \( \alpha^\# \) and is furthermore a cylinder.

Let \( n \) be a one-one \( \alpha \)-recursive function whose range is \( N \). Since \( cB \) is \( \alpha \)-unbounded and \( n \) is one-one, \( n(cB) \) is \( \alpha^\# \)-unbounded. For each \( \nu < \alpha^\# \) define

\[ p(\nu) = \nu \gamma[\gamma \in cB \& n(\gamma) > \nu] \]

Then let

\[ C = \{\nu < \alpha^\# | \nu \in n[p(\nu)]\} \]

By 2.5.2 \( B \) is regular. Hence \( C \) is \( \alpha \)-r.e. and

\[ \{\mu < \alpha^\# | K_\mu \subseteq cC\} \] is weakly \( \alpha \)-recursive in \( B \). It
remains to show that $C$ is $\alpha$-complete.

By definition $C = n[D]$ where

$$D = \{ \delta | (\gamma) \gamma < \delta [ \gamma \in B \land n(\gamma) < n(\delta) ] \}.$$ 

$D$ is regular since $B$ is. Let $\eta < \alpha$ be given. $B$ is conscientious so $cB \cap \eta$ has $\alpha$-cardinality less than $\text{cf}^M \alpha(\alpha^*)$. Hence $n[cB \cap \eta]$ is $\alpha^*$-bounded. But each member of $n[cD \cap \eta]$ is less than a member of $n[cB \cap \eta]$. Hence $n[cD \cap \eta]$ is $\alpha^*$-bounded. Hence $cD \cap \eta$ has $\alpha$-cardinality less than $\alpha^*$. Since this holds for all $\eta < \alpha$, we see that $C = n[D]$ must meet every set $\{\nu\} \times \alpha^* \ni \nu \in g[K_\alpha]$. Hence $K_\alpha \subseteq K_\alpha C$. It follows at once that $C$ is $\alpha$-complete. 

Q.E.D.

Before stating 2.9 let us make some convenient definitions.

Let $B, C$ be subsets of $\alpha^*$. We say $B$ is $\alpha^*$-finitely $\alpha$-calculable from $C$ (abbreviated $B \subseteq_{f\alpha} C$) if there is $\rho < \alpha$ such that for all $\nu < \alpha^*, K_\nu \subseteq B$ iff
(Eξ)ξ<α*(Em)η<α*[<2ν,ξ,η> ∈ Rρ & Kξ ≤ C & Kη ∩ C = ∅]

and Kν ∩ B = ∅ iff

(Eξ)ξ<α*(Em)η<α*[<2ν+1,ξ,η> ∈ Rρ & Kξ ≤ C & Kη ∩ C = ∅].

If B, C are in addition α-r.e., then the definition can be simplified to the following: there is ρ < α such that for all ν < α*

Kν ∩ B = ∅ iff (Em)η<α*[<ν,η> ∈ Rρ & Kη ∩ C = ∅].

The reducibility ≤_fα does not seem to deserve attention for its own sake. It enters merely as a useful, though not indispensable, auxiliary notion.

Suppose α* < α. An α-r.e. set D ⊆ α* is said to be simple if

(i) cD ∩ α* is α*-unbounded;

(ii) D ∩ H ≠ ∅ for every α-finite, α*-unbounded H ⊆ α*.

We call S ⊆ α closed if sup(α) ∈ S for every
α-bounded $K \subseteq S$. We call $D \subseteq \alpha$ open if $cD$ is closed.

2.9 LEMMA. Suppose $\alpha^* < \alpha$. Let $C$ be an α-infinite, α-r.e. subset of $\alpha^*$. Then there is a simple α-r.e. subset of $\alpha^*$, $D$, such that $C \equiv_f \alpha D$.

The method of Dekker [3] does not seem to work here. Instead we split into cases depending on whether $\alpha^*$ is regular or singular as an α-cardinal.

PROOF of 2.9. Let $\{H_\sigma | \sigma < \alpha\}$ be a canonical listing of the α-finite $\alpha^*$-unbounded subsets of $\alpha^*$. Let $g$ be a one-one α-recursive function whose range is $C$. There are two cases.

**Case I:** $\alpha^*$ is a regular α-cardinal. We shall have $D = \bigcup \{D^\sigma | \sigma < \alpha\}$ where $D^\sigma(\sigma < \alpha)$ is an increasing α-recursive sequence of σ-finite subsets of $\alpha^*$. The construction will be designed to make $D$ simple and $D \equiv_f \alpha C$. Our biggest headache will be to verify that $cD \cap \alpha^*$ is $\alpha^*$-unbounded.

We set $D^0 = \emptyset$ and $D^\lambda = \bigcup \{D^\sigma | \sigma < \lambda\}$ for limit ordinals $\lambda \leq \alpha$. Let $\{d(\sigma, v) | v < 0(\sigma)\}$ be the
members of $\mathcal{D}\sigma \cap \alpha^*$ in increasing order. If $0(\sigma) < \alpha^*$ then we set $\mathcal{D}\sigma+1 = \mathcal{D}\sigma$ and the construction grinds to a halt. Suppose $0(\sigma) = \alpha^*$. Let $\xi$ be the least member of $\mathcal{H}\sigma$ such that $d(\sigma, g(\sigma)) < \xi$. Then let $m(\sigma)$ be the least $\mu$ such that $\xi \leq d(\sigma, \mu)$. Finally define

$$\mathcal{D}\sigma+1 = \mathcal{D}\sigma \cup \{\delta | d(\sigma, g(\sigma)) < \delta \leq d(\sigma, m(\sigma))\}.$$ 

This completes the construction.

Clearly $\mathcal{D} = \mathcal{D}\alpha$ is $\alpha$-r.e. and $\mathcal{D}\sigma+1 \cap \mathcal{H}\sigma \neq \emptyset$ for all $\sigma < \alpha$. Thus $\mathcal{D}$ will be simple provided we can show $c\mathcal{D} \cap \alpha^*$ is $\alpha^*$-unbounded.

It is easy to see by induction on $\sigma$ that each $\mathcal{D}\sigma$ ($\sigma \leq \alpha$) is open. This will be the key fact in the proof of 2.9.1.

2.9.1 SUBLEMMA

$0(\sigma) = \alpha^*$ for all $\sigma \leq \alpha$.

PROOF. Suppose not. Let $\tau < \alpha$ be minimal with $0(\tau) < \alpha^*$. Clearly $0(\sigma+1) = 0(\sigma)$ for all $\sigma$, so $\tau$
must be a limit ordinal. Note that $d(\sigma, 0) = 0$ for all $\sigma$. Put

$$\eta_0 = \sup \{d(\tau, \nu) | \nu < 0(\tau)\}.$$  

Then $\eta_0 < \alpha^*$ by the regularity of $\alpha^*$. Since $cD^\tau$ is closed, $\eta_0 \in cD^\tau$. Hence $\eta_0 = d(\tau, \nu_0)$ where $0(\tau) = \nu_0 + 1$.

For each $\sigma < \tau$ we have $\eta_0 = d(\sigma, \nu)$ for some $\nu \geq \nu_0$; moreover this $\nu$ as a function of $\sigma$ is monotone decreasing. Hence there is a stage $\sigma_0 < \tau$ such that $(\sigma)_{\sigma_0 \leq \sigma < \tau} (d(\sigma, \nu_0) = \eta_0 < d(\sigma, \nu_0 + 1) < \alpha^*)$.

Here $d(\sigma, \nu_0 + 1)$ as a function of $\sigma$, $\sigma_0 \leq \sigma < \tau$, is continuous and monotone increasing. Put

$$\eta_1 = \sup \{d(\sigma, \nu_0 + 1) | \sigma < \tau\}.$$  

Obviously $\eta_0 < \eta_1$. By the construction

$$\{\sigma | d(\sigma, \nu_0 + 1) \neq d(\sigma + 1, \nu_0 + 1)\} \subset \{\sigma | g(\sigma) \leq \nu_0\}$$  

but since $g$ is one-one the latter set has $\alpha$-cardinality less than $\alpha^*$. Hence $\eta_1 < \alpha^*$ by the
regularity of \( \alpha^* \). Since each \( cD^\sigma \) is closed we must have \( \eta_1 \in cD^\sigma \) for each \( \sigma < \tau \). Hence 
\( \eta_1 \in cD^\tau \) since \( D^\tau = \bigcup \{ D^\sigma | \sigma < \tau \} \). Hence 
\( d(\tau, \nu_0 + 1) = \eta_1 \). Hence \( \nu_0 + 1 < \vartheta(\tau) \), a contradiction.
This proves 2.9.1.

It follows that \( cD \cap \alpha^* \) is \( \alpha^* \)-unbounded; hence \( D \) is simple. The construction is such that 
\( d(\sigma, \nu) \neq d(\sigma + 1, \nu) \) only when \( \nu > \varrho(\sigma) \). Thus \( D \leq_{f\alpha} C \).
Let \( \{ d(\nu) | \nu < \alpha^* \} \) be the members of \( cD \cap \alpha^* \) in increasing order. Then \( \nu \in C \) iff 
(\( E\sigma \))[\( d(\sigma, \nu + 1) \neq d(\nu + 1) \land \varrho(\sigma) = \nu \)]. Thus \( C \leq_{f\alpha} D \).

0.E.D.

The proof of 2.9 for Case I is complete.

Case II: \( \alpha^* \) is a limit of \( \alpha \)-cardinals. Our proof for this case is based on Post's [23] original construction of a simple set. We shall have 
\( D = \text{range} \ (h) \) where \( h(\sigma) \) will be an \( \alpha \)-recursive function defined by induction on \( \sigma \).

Put \( \beta_0 = 0 \) and let \( \{ \beta_\xi | 1 \leq \xi < \lambda \} \) be the \( \alpha \)-cardinals less than \( \alpha^* \) in increasing order.
Note that \( \{ \beta_{\xi} | 1 \leq \xi < \lambda \} \) is \( \alpha \)-finite.

Stage \( \sigma \): Let \( \xi < \lambda \) be such that 
\[ \beta_{\xi} \leq g(\sigma) < \beta_{\xi+1}. \]
If \( \beta_{\xi+2} \subseteq \beta_{\xi+1} \cup h[2\sigma] \) then 
\( h(2\sigma) \) is undefined and the construction grinds to a halt. Otherwise let \( h(2\sigma) \) be the least member of 
\[ \beta_{\xi+2} - (\beta_{\xi+1} \cup h[2\sigma]), \]
and let \( h(2\sigma+1) \) be the least member of \( H_{\sigma} - \beta_{\xi+1} \). This completes stage \( \sigma \).

Obviously \( \{ \tau | h(2\tau) < \beta_{\xi+2} \lor h(2\tau+1) < \beta_{\xi+2} \} \) \( \subseteq \{ \tau | g(\tau) < \beta_{\xi+1} \} \). But \( g \) is one-one so the latter set has \( \alpha \)-cardinality \( \leq \beta_{\xi+1} \). Hence 
\[ \beta_{\xi+2} - (\beta_{\xi+1} \cup h[2\sigma]) \]
is always nonempty and so \( h(2\sigma) \) is always defined.

Let \( D = \text{range } h \). For each \( \xi < \lambda \), 
\[ D \cap \beta_{\xi+2} \subseteq h[2\sigma] \text{ if and only if } C \cap \beta_{\xi+1} \subseteq g[\sigma]. \]
Hence \( D \subseteq f_{\alpha} C \). Of course \( h(2\sigma+1) \in H_{\sigma} \) so \( D \) is simple. \( \square \)

The proof of 2.9 is now complete.

2.10 THEOREM Suppose \( \alpha^# < \alpha \). Let \( B \) be an \( \alpha \)-r.e. set. Then the following are equivalent:
(i) B's $\alpha$-degree contains a non-regular $\alpha$-r.e. set.

(ii) B's $\alpha$-degree contains a simple $\alpha$-r.e. subset of $\alpha^*$. 

(iii) $\text{rcf}(B) \leq \text{cf}^M \alpha^*(\alpha^*)$.

PROOF. (i) $\Rightarrow$ (iii) is just Fact 2.3.5.
(ii) $\Rightarrow$ (i) is trivial. We shall prove
(iii) $\Rightarrow$ (ii).

Let B be an $\alpha$-r.e. set with $\text{rcf}(B) \leq \text{cf}^M \alpha^*(\alpha^*)$. Then Owings' lemma 2.7 provides an $\alpha$-infinite $\alpha$-r.e. set $C \subseteq \alpha^*$ with

$$\{ \nu < \alpha^* | K_\nu \cap C \neq \emptyset \} \leq_M B.$$ 

Let $g$ be a one-one $\alpha$-recursive function whose range is $C$. Define

$$B^* = \{ g(\eta) | K_\eta \cap B \neq \emptyset \}.$$ 

For $\nu < \alpha^*$ it is easy to see that $K_\nu \cap B^* = \emptyset$ if and only if
\[(E\mu)_{\mu<\alpha^*}(En)\{K_\nu = K_\mu \cup g[K_\eta] \& K_\mu \cap C = \emptyset \& \bigcup_{\sigma \in K_\eta} K_\sigma \cap B = \emptyset\}.\]

From this it follows that

\[
\{\nu < \alpha^* | K_\nu \cap B^* \neq \emptyset\} \subseteq_{\omega\alpha} B.
\]

Since \(B^*\) is an \(\alpha\)-infinite \(\alpha\)-r.e. subset of \(\alpha^*\),
2.9 provides a simple \(\alpha\)-r.e. set \(D \subseteq \alpha^*\) such that
\(D \equiv_{\alpha} B^*\). Using the simplicity of \(D\), it then follows
that \(D \equiv_{\alpha} B\).

\[\text{Q.E.D.}\]

2.11 COROLLARY

**Hypothesis:** \(\alpha^*\) is a regular \(\alpha\)-cardinal.

**Conclusion:** every \(\alpha\)-r.e. \(\alpha\)-degree either is
hyperregular or contains a simple \(\alpha\)-r.e. subset of \(\alpha^*\).

2.12 COROLLARY

1. Every non-zero \(\Pi_1^1\) metadegree contains a
simple \(\Pi_1^1\) set.

2. Every non-zero \(\Pi_1^1\) metadegree is the supremum
of two incomparable \(\Pi_1^1\) metadegrees.
We state the following result here for lack of a better place.

2.12.3 THEOREM. Every non-zero \( \Pi^1_1 \) metadegree contains a maximal \( \Pi^1_1 \) set.

The proof will appear elsewhere.

2.13 REMARKS

1. 2.11 is immediate from 2.10. 2.12.1 is just the implication (i) \( \Rightarrow \) (ii) of 2.10 specialized to metarecursion theory. 2.12.2 follows from 2.12.1 and Owings' splitting theorem for simple \( \Pi^1_1 \) sets ([20] p. 68).


3. One might ask whether the hypothesis "\( \alpha^* \) is regular" can be dropped from 2.11. In §3 we shall exhibit a pathological \( \alpha \) showing that some hypothesis is needed here. Namely, for this \( \alpha \) there will be an \( \alpha \)-r.e. set \( B \) with
Thus, although $B$ will be non-hyperregular, by 2.10 every $\alpha$-r.e. set in $B$'s $\alpha$-degree will be regular.

E. In [27] it is shown that for any admissible $\alpha$ there is an incomparable pair of hyperregular $\alpha$-r.e. $\alpha$-degrees. We now ask the corresponding question for non-hyperregular $\alpha$-r.e. $\alpha$-degrees.

2.14 PROBLEM. For which admissible $\alpha$ is there an incomparable pair of non-hyperregular $\alpha$-r.e. $\alpha$-degrees?

We are about to present some results and examples concerning this problem.

Historically, the first result on problem 2.14 was due to Sacks [25]. He showed that there is an incomparable pair of $\Pi^1_1$ metadegrees. We now generalize Sacks' result to all admissible $\alpha$ such that $\alpha^*$ is a regular $\alpha$-cardinal. We then present some examples showing that "regular" cannot be replaced by "singular". These examples answer in the negative Sacks' question Q2 [25] p. 21.
Sacks' proof in [25] is unnecessarily complicated, and we do not follow it. We take an alternate tack based on the idea of simplicity and Lemma 2.9. Note that if $A, B$ are simple $\alpha$-r.e. subsets of $\alpha^\#$, then $A \equiv_\alpha B$ if and only if $A \equiv_{f_\alpha} B$.

2.15. THEOREM

Hypothesis: $\alpha^\#$ is a regular $\alpha$-cardinal.

Conclusion: there are simple $\alpha$-r.e. subsets of $\alpha^\#$, $A$ and $B$, such that $A \equiv_{W_\alpha} B$ and $B \equiv_{W_\alpha} A$.

PROOF. Let $n$ be a one-one $\alpha$-recursive function from $\alpha$ into $\alpha^\#$. For each $\epsilon < \alpha^\#$ and $\sigma < \alpha$, define

$$p_{\epsilon}^{\sigma} = \begin{cases} R^{\sigma}_{n^{-1}(\epsilon)} \cap \alpha^\# & \text{if } \epsilon \in n[\sigma], \\ \phi & \text{otherwise.} \end{cases}$$

Define $P_{\epsilon} = \bigcup\{p_{\epsilon}^{\sigma} | \sigma < \alpha\}$. Then

$$P_{\epsilon} = \begin{cases} R_{n^{-1}(\epsilon)} \cap \alpha^\# & \text{if } \epsilon \in \text{range } (n), \\ \phi & \text{otherwise.} \end{cases}$$
Thus \( P_\varepsilon (\varepsilon < \alpha^*) \) is a standard simultaneous enumeration of the \( \alpha \)-\( r.e. \) subsets of \( \alpha^* \).

By Lemma 2,9 we are done if we can find \( \alpha \)-\( r.e. \) sets \( A, B, \subseteq \alpha^* \) and ordinals \( p(\varepsilon), q(\varepsilon) (\varepsilon < \alpha^*) \) such that

\[
p(\varepsilon) \in A \iff (En)[<p(\varepsilon), \eta> \in P_\varepsilon \& K_\eta \cap B = \emptyset],
\]

\[
q(\varepsilon) \in B \iff (En)[<q(\varepsilon), \eta> \in P_\varepsilon \& K_\eta \cap A = \emptyset].
\]

For suppose we have done this. Let \( A^*, B^* \) be simple \( \alpha \)-\( r.e. \) subsets of \( \alpha^* \) such that \( A^* \equiv_{f_\alpha} A \) and \( B^* \equiv_{f_\alpha} B \). Using the simplicity it is then easy to see that \( A^* \uparrow_{\omega_\alpha} B^* \) and \( B^* \uparrow_{\omega_\alpha} A^* \).

To get \( A \) and \( B \), one imitates the proof in Sacks - Simpson [27] Theorem 3.1, Case I, with the obvious modifications. The hypothesis "\( \alpha^* \) is regular" helps to insure that \( f(\varepsilon) = \lim_{\sigma} f(\sigma, \varepsilon) \) and \( g(\varepsilon) = \lim_{\sigma} g(\sigma, \varepsilon) \) are less than \( \alpha^* \). We shall not write out the details since they are almost a verbatim repetition of those in [27].
This completes the proof of 2.15.

2.16 COROLLARY

Hypothesis: \( \alpha^\# \) is a regular \( \alpha \)-cardinal.

Conclusion: the restriction of \( \leq_{W\alpha} \) to the \( \alpha \)-r.e. subsets of \( \alpha^\# \) is not transitive.

PROOF. By 2.15 there is an \( \alpha \)-infinite \( \alpha \)-r.e. set \( A \subseteq \alpha^\# \) such that \( A \) is not \( \alpha \)-complete. By 2.7 there is an \( \alpha \)-complete \( \alpha \)-r.e. set \( C \subseteq \alpha^\# \) such that \( C \leq_{W\alpha} A \).

If \( \leq_{W\alpha} \) were transitive on the \( \alpha \)-r.e. subsets of \( \alpha^\# \), then the complete \( \alpha \)-r.e. set would be \( \leq_{W\alpha} A \). Hence \( A \) would be \( \alpha \)-complete, a contradiction.

Corollary 2.16 for metarecursion theory is due to Driscoll [4] and Owings [21].

2.17 PROBLEM. For which admissible \( \alpha \) is the restriction of \( \leq_{W\alpha} \) to the \( \alpha \)-r.e. sets transitive?

By 2.5.3 this is the case if \( \text{recf}(\alpha) = \alpha \).

J. Stillwell claimed in his Ph.D. thesis that these were the only such \( \alpha \), i.e. that the restriction of
to the $\alpha$-r.e. sets is transitive if and only if 
$\text{recf}(\alpha) = \alpha$. The following examples refute 
Stillwell's claim.

These examples also show that the hypothesis 
"$\alpha^*$ is a regular $\alpha$-cardinal" cannot be dropped from 2.15.

2.18. EXAMPLE. $\mathcal{H}_\omega^L$ denotes, as usual, the 
$\omega$th constructible cardinal. Suppose $\alpha^* = \mathcal{H}_\omega^L$. Then 
we claim: there is exactly one non-hyperregular $\alpha$-r.e. 
$\alpha$-degree.

PROOF. If $\alpha = \mathcal{H}_\omega^L$ then $\{\mathcal{H}_n^L \mid n < \omega\}$ is the 
complement of an $\alpha$-r.e. set, and this set is clearly 
non-hyperregular. If $\mathcal{H}_\omega^L = \alpha^* < \alpha$ then there is a 
non-regular, hence non-hyperregular, $\alpha$-r.e. set.

Let $B$ be a non-hyperregular $\alpha$-r.e. set. We 
shall prove 2.18 by showing that $B$ is $\alpha$-complete.

First let us compute $\text{rcf}(B)$. The regular 
$\alpha$-cardinals are $\{\mathcal{H}_n^L \mid n < \omega\}$ and by 2.3.3 $\text{rcf}(B)$ is 
among these. But $\text{cf}(\alpha) = \text{cf}(\mathcal{H}_\omega^L) = \omega$ and by 2.5.1 we
may assume that \( B \) is conscientious. Hence \( \text{rcf}(B) \) is cofinal with \( \omega \). Hence \( \text{rcf}(B) = \omega \).

Let an \( \alpha \)-r.e. set \( A \) be given. We shall show \( A \sqsubseteq_\alpha B \). By 2.6 we may assume that \( A \) is regular. Let \( f \) be a one-one \( \alpha \)-recursive function whose range is \( A \). Let \( \{ \beta_n \mid n \in \omega \} \) be the members of \( cB \) in increasing order. For each \( n < \omega \) define

\[
j(n) = \mu j(A \cap \beta_n \subseteq f[\beta_j]).
\]

Since \( A \) is regular, \( j(n) \) exists. Now \( j : \omega \to \omega \) is constructible, hence \( \alpha \)-finite. Hence we may compute with \( j \) as a "parameter". Hence \( A \sqsubseteq_\alpha B \). Q.E.D.

N.B. A similar argument proves: if the regular \( \alpha \)-cardinals are \( \{ \nu_n^I \mid n < \omega \} \) then every non-hyperregular \( \alpha \)-r.e. set is \( \alpha \)-complete. In particular, for any such \( \alpha \), the restriction of \( \sqsubseteq_{\nu \alpha} \) to the \( \alpha \)-r.e. sets is transitive.

A related example is the following.

2.19 EXAMPLE Let \( \beta_0 = \nu_1^L \); \( \beta_{n+1} \) is the least stable ordinal greater than \( \beta_n \); and \( \alpha = \sup \beta_n \).
It is easy to see that

(i) $\alpha$ is stable, hence admissible;

(ii) the $\alpha$-cardinals are $\{\omega, H_\alpha^L\}$;

(iii) $\alpha^\# = \alpha$ and $\text{recf}(\alpha) = \omega$.

We claim: there is exactly one non-hyperregular $\alpha$-r.e. $\alpha$-degree.

The proof is similar to the one for Example 2.18.

The answers to Problems 2.14 and 2.17 may turn out to be quite complicated. We have been unable to frame a general conjecture. We are particularly curious about the exact connection between these two problems. Some relevant further examples appear in §3.
§3. $\Delta_2$-collapsible admissible ordinals

In this section we discuss certain rather pathological admissible ordinals. We show that these ordinals are counterexamples to a number of attractive conjectures in $\alpha$-recursion theory.

The main technical point of the section is this: it is possible for $\alpha$-cardinals to be "collapsed" by $\Delta_2$ functions; moreover, when this happens, it has important consequences for $\alpha$-recursion theory.

When we write of "$\Delta_2$-collapsible" admissible ordinals $\alpha$, we mean those with the property that there is a $\Delta_2$ function from $\omega$ onto $\alpha$.

3.1 EXAMPLE Let $\alpha$ be the least admissible ordinal such that there is an $\alpha$-cardinal greater than $\omega$. Then:

(i) $\omega < \alpha^* < \alpha = (\alpha^*)^+$;

(ii) $\alpha^*$ is the only $\alpha$-cardinal greater than $\omega$;

(iii) $\alpha^*$ is a regular $\alpha$-cardinal.
3.2 EXAMPLE. Let $\alpha$ be the least admissible ordinal such that there is no largest $\alpha$-cardinal. Then:

(i) $\alpha$ is the limit of the first $\omega$ $\alpha$-cardinals;
(ii) $\alpha^* = \alpha$.

3.3 EXAMPLE. Let $\alpha$ be the least admissible ordinal such that there is a singular $\alpha$-cardinal. Then:

(i) $\omega < \alpha^* < \alpha = (\alpha^*)^+$;
(ii) $\alpha^*$ is the limit of the first $\omega$ $\alpha$-cardinals.

The $\alpha$'s of examples 3.1, 3.2, and 3.3 may be thought of as the "$\Delta_2$-collapsible counterparts" of $(H_1)^+, H_\omega^L$, and $(H_\omega^L)^+$ respectively.

Our immediate goal is to show that each of these $\alpha$'s is $\Delta_2$-collapsible. For clarity's sake we shall prove this in some, although not the greatest possible, generality.

3.4 THEOREM

Hypothesis: $F$ is a $\Sigma_4$ sentence of the ZF language;
\( \alpha \) is the least admissible ordinal such that \( M_\alpha \models F \).

**Conclusion:** there is a \( \Lambda_2 \) function with domain \( \omega \) and range \( \alpha \).

**Proof.** Let \( G \) be a \( \Pi_3 \) sentence of the ZF language such that an ordinal \( \beta \) is admissible if and only if \( M_\beta \models G \). We may assume without loss that \( F \) is equivalent to \( G \) conjuncted with another \( \Sigma_4 \) sentence.

Let \( F \) be \((Ew)(x)(Ey)(z)P(w,x,y,z)\) where \( P \) is limited. Let \( w_0 \) be the \( \leq_L \)-least \( w \) such that \( M_\alpha \models (x)(Ey)(z)P(w,x,y,z) \). For each \( x \) in \( M_\alpha \) let \( \phi(x) \) be the \( \leq_L \)-least \( y \) such that \( M_\alpha \models (z)P(w_0,x,y,z) \). Let \( P_i(x_0,\ldots,x_{n_i},y) \) \((i \in \omega)\) be a recursive enumeration of the limited formulas having \( \geq 2 \) free variables. For all \( x_0,\ldots,x_{n_i} \) in \( M_\alpha \) let \( \phi_i(x_0,\ldots,x_{n_i}) \) be the \( \leq_L \)-least \( y \) in \( M_\alpha \) such that \( P_i(x_0,\ldots,x_{n_i},y) \) if such exists and 0 otherwise. Let \( Y \subseteq M_\alpha \) be the closure of \( \{w_0\} \) under the functions \( \phi, \phi_i \) \((i \in \omega)\).
3.4.1 LEMMA. There is a $\Delta_2$ function with
domain $\omega$ and range $Y$.

PROOF. Let $\theta_i (i \in \omega)$ be a recursive enumeration
of all formal compositions of the $\phi, \phi_i (i \in \omega)$ having
exactly one free variable. Thus for instance we might perhaps have

$$\theta_{25}(x) = \phi(\phi_2(x, x, \phi_5(x))).$$

Define $f(i) = \theta_i(w_0)$. Then $f$ is a function from $\omega$
onto $Y$.

To show that $f$ is $\Delta_2$ it suffices to show that
$\phi$ and the $\phi_i (i \in \omega)$ are uniformly $\Delta_2$. The
definition of $\phi$ can be expressed in a first order
way over $M_\alpha$ as follows: $\phi(x) = y$ iff

$$(z)P(w_0, x, y, z) \land (y')_{y' \in Ly}(Ey) \leadsto P(w_0, x, y', z).$$

Since $\alpha$ is admissible we can rewrite this as

$$(z)P(w_0, x, y, z) \land (Ey)(y')_{y' \in Ly}(Ey) \leadsto P(w_0, x, y', z).$$
This latter is clearly a $\Delta_2$ expression. Similarly $\phi_1$ can be defined over $M_\alpha$ by: $\phi_1(x_0, \ldots, x_{n_1}) = y$ iff

$$\left( P_1(x_0, \ldots, x_{n_1}, y) \& (y')_y \prec_{L_y} P_1(x_0, \ldots, x_{n_1}, y') \right) \& (y = 0 \& (y') \prec P_1(x_0, \ldots, x_{n_1}, y')).$$

These are again $\Delta_2$ expressions. Lemma 3.4.1 is proved.

3.4.2 Lemma. $Y = M_\alpha$

Proof. $Y$ is closed under the $\phi_1 (1 \in \omega)$. Hence $Y$ is a $\Sigma_1$ submodel of $M_\alpha$. In particular $Y$ satisfies the $\Pi_2$ sentence $V = L$. Hence $<Y, e>$ is isomorphic to some $<M_\beta, e>$, $\beta \leq \alpha$. On the other hand, $w_0 \in Y$ and $Y$ is closed under $\phi$. Hence $<Y, e> \models P$. Hence $M_\beta \models P$ so by leastness of $\alpha$, $\beta = \alpha$.

For each $y \in Y$ let $\bar{y}$ be the image of $y$ under the isomorphism $<Y, e> \cong <M_\alpha, e>$. Thus $\bar{y} \prec_{L} y$. We claim that $\bar{y} = y$. We shall prove this claim by induction on the length of a formal expression for $y$ in terms of
\( \phi, \phi_1 (1 \in \omega) \), and \( w_0 \),

First, \( w_0 \in Y \) and \( \langle Y, e \rangle \models (x)(E y)(z)P(w_0, x, y, z) \).

Hence by the isomorphism \( M_\alpha \models (x)(E y)(z)P(\bar{w}_0, x, y, z) \).

Hence by leastness of \( w_0 \), \( \bar{w}_0 = w_0 \). Next, suppose \( y = \phi(x), x \in Y \), where by the induction hypothesis \( \bar{x} = x \). Then \( M_\alpha \models (z)P(w_0, x, y, z) \).

Since \( Y \) is a \( \Sigma_1 \) submodel of \( M_\alpha \), \( \langle Y, e \rangle \models (z)P(w_0, x, y, z) \). But \( \bar{x} = x \), hence by the isomorphism \( M_\alpha \models (z)P(w_0, x, \bar{y}, z) \).

Now \( \phi(x) \) is defined as the least \( y \) such that \( (z)P(w_0, x, y, z) \). Hence \( \phi(x) \leq_L \bar{y} \). But \( y = \phi(x) \) and \( \bar{y} \leq_L y \). Hence \( \bar{y} = y \). Finally, suppose \( y = \phi_1(x_0, \ldots, x_{n_1}), x_0, \ldots, x_{n_1} \in Y \), where by the induction hypothesis \( \bar{x}_0 = x_0, \ldots, \bar{x}_{n_1} = x_{n_1} \).

If \( y = 0 \) then of course \( \bar{y} = y \). If \( y \neq 0 \) then \( M_\alpha \models P_1(x_0, \ldots, x_{n_1}, y) \) hence as before \( M_\alpha \models P_1(x_0, \ldots, x_{n_1}, \bar{y}) \) whence \( \bar{y} = y \).

We have just shown that \( \bar{y} = y \) for all \( y \in Y \).
Hence \( Y = M_\alpha \) and Lemma 3.4.2 is proved.

By Lemmas 3.4.1 and 3.4.2 there is a \( \Lambda_2 \) function from \( \omega \) onto \( M_\alpha \). Hence there is a \( \Lambda_2 \) function from \( \omega \) onto \( \alpha \). Q.E.D.

3.5 COROLLARY. Let \( \alpha \) be as in Example 3.1, 3.2, or 3.3. Then there is a \( \Lambda_2 \) function with domain \( \omega \) and range \( \alpha \).

PROOF. It is easy to see that each of these \( \alpha \)'s can be described as the least admissible ordinal satisfying some \( \Sigma_4 \) sentence.

N.B. Example 1.8, the least non-projectible admissible \( \alpha \) greater than \( \omega \), also falls under the purview of Theorem 3.4.

REMARKS. Theorem 3.4 is based on an idea of H. Putnam which he likes to express orally as follows: "to collapse \( M_\alpha \), show that you are dealing not with just any old \( \alpha \) but rather with the \( \alpha \), or the least \( \alpha \), enjoying some property." One should compare our 3.4 with the Main Technical Lemma of Boolos-Putnam [2].
R. B. Jensen [8] has greatly refined and generalized Putnam's ideas. Jensen's methods can be used to eliminate the hypothesis "α is admissible" from §3.4. We are firmly convinced that close interaction is possible between Jensen's methods and α-recursion theory.

We now use Examples 3.1, 3.2, and 3.3 to refute some attractive conjectures suggested by the material of §§1, 2. The following proposition is completely worthless except as a source of such counterexamples.

3.6 PROPOSITION

**Hypothesis:** there is a $\Delta_2$ function with domain $\omega$ and range $\alpha$.

**Conclusions:**

1. There is a UI-maximal $\alpha$-r.e. set.

2. There are $\alpha$-r.e. sets $A, B$ with $\text{rcf}(A) = \text{rcf}(B) = \omega$ and $A \not\leq_{\text{w}A} B, B \not\leq_{\text{w}A} A$.

3. Every regular $\alpha$-cardinal is the rcf of some $\alpha$-r.e. set.
3.6 has the following consequences:

3.7 PROPOSITION

1. There is an admissible $\alpha$ such that UI-maximal $\alpha$-r.e. sets exist but SI-maximal $\alpha$-r.e. sets do not exist.

2. There is an admissible $\alpha$ such that

   (i) $\alpha^*$ is a singular $\alpha$-cardinal;

   (ii) there are simple $\alpha$-r.e. subsets of $\alpha^*$, $A$ and $B$, such that $A \not\subseteq B$, $B \not\subseteq A$.

3. There is an admissible $\alpha$ and a non-hyperregular $\alpha$-r.e. set, $A$, such that

   (i) $\alpha^*$ is less than $\alpha$;

   (ii) every $\alpha$-r.e. set $\alpha$-recursive in $A$ is regular.

PROOF. For 3.7.1 let $\alpha$ be as in Example 3.1. That SI-maximal $\alpha$-r.e. sets do not exist follows from Remark 1.3.6.

For 3.7.2 and 3.7.3 let $\alpha$ be as in Example 3.3. The results follow from 3.6.2, 3.6.3, and 2.10.
REMARK. By 3.7.3 the hypothesis "regular" cannot be dropped from 2.11. The problem of whether it could be dropped was what originally motivated us to consider the examples of the present section.

It remains to prove 3.6. We've already proved 3.6.1 as Theorem 1.6. The proof of 3.6.2 is routine and we omit it.

PROOF of 3.6.3. We are given a $\Delta_2$-collapsible $\alpha$ and a regular $\alpha$-cardinal $\beta$. We seek an $\alpha$-r.e. set $B$ with $\text{rcf}(B) = \beta$. Since $\alpha$ is $\Delta_2$-collapsible, $\text{rcf}(\alpha) = \omega$ so by 2.3.4 there is an $\alpha$-r.e. set with $\text{rcf} \omega$. Thus we are done if $\beta = \omega$.

Suppose now that $\beta$ is a regular $\alpha$-cardinal greater than $\omega$. We shall show how to enumerate an $\alpha$-r.e. set $B$ with $\text{rcf}(B) = \beta$. Our $B$ will turn out to be conscientious (Definition 2.4). By 2.5.1 we lose nothing by aiming for such a $B$.

We shall arrange $\text{rcf}(B) \geq \beta$ by preserving certain computations so as to make $\phi[\gamma]$ bounded for each $\phi \leq_{\omega_\alpha} B$, dom $(\phi) = \gamma < \beta$. Simultaneously we shall make
cB unbounded of order type $\beta$. Thus $\text{ref}(B) = \beta$.

Since cB will be unbounded of order type $\beta$, any $\alpha$-finite subset of cB will have to have $\alpha$-cardinality less than $\beta$. Therefore, when we preserve computations, we shall be able to safely overlook those computations which involve $\alpha$-finite sets of $\alpha$-cardinality $\geq \beta$. This will be a big help in making the entire construction work.

Let us proceed to the details. By $\Delta^1_2$-collapsibility there is an $\alpha$-recursive function $f(\sigma, n)$ such that $f(n)$, defined by $f(n) = \lim_{\sigma} f(\sigma, n)$, has domain $\omega$ and range $\alpha$. For each $\sigma < \alpha$ define

$$g(\sigma) = \sup \{ f(\sigma, n) \mid n < \omega; f(\sigma, n) < \beta \}.$$ 

Thus $g$ is an $\alpha$-recursive function with range $(g) \subseteq \beta$, and $\beta = \lim_{\sigma} g(\sigma)$ in the weak sense that

$$(\beta')_{\beta' < \beta} (E^\tau)(\sigma)_{\sigma > \tau} \beta' \leq g(\sigma) < \beta.$$

The function $g$ will play a key role in the construction.
We shall define \( \alpha \)-recursive functions 
\( p(\sigma, \nu, e) (\nu < \beta; e < \omega) \), \( p(\sigma) \), and \( B^\sigma \) by induction 
on \( \sigma \). The \( B^\sigma (\sigma < \alpha) \) will be an increasing sequence 
of \( \alpha \)-finite sets. At the end of the construction we 
shall set \( B = \bigcup \{ B^\sigma | \sigma < \alpha \} \). Thus \( B \) will be \( \alpha \)-r.e. 
We shall then prove, among other things, that 
\( rcf(B) = \beta \).

As a preliminary to stage \( \sigma \) of the construction 
we let 
\[
B^{<\sigma} = \bigcup \{ B^\tau | \tau < \sigma \}
\]
and 
\[
\nu(\sigma) = \sup \{ \nu(\sigma, \xi) | \xi < \beta \}
\]
where \( \{ \nu(\sigma, \xi) | \xi < \beta \} \) are the first \( \beta \) members of 
the complement of \( B^{<\sigma} \) in increasing order.

Stage \( \sigma \): For each \( e < \omega \) and \( \nu < \beta \) define
\( y(\sigma, \nu, e) = \)
\[
\begin{cases}
(\mu \tau)_{\tau < \delta}(E \gamma)_{\gamma < \beta}(E \delta)(E \eta)[\langle \nu, \delta, \eta \rangle \in R_\tau(\sigma, e) \\
&\land K_\eta \subseteq \{v(\sigma, \xi) \mid \xi < \gamma\}]
\end{cases}
\]
if such a \( \tau \) exists;
\[
\sigma
\]
otherwise,

and
\( p(\sigma, \nu, e) = \)
\[
\begin{cases}
(\mu \gamma)_{\gamma < \delta}(E \delta)(E \eta)[\langle \nu, \delta, \eta \rangle \in R_\gamma(\sigma, \nu, e) \\
&\land K_\eta \subseteq \{v(\sigma, \xi) \mid \xi < \gamma\}]
\end{cases}
\]
if \( y(\sigma, \nu, e) < \sigma; \)
\[
0
\]
otherwise.

Then let
\[
p(\sigma) = \sup \{p(\sigma, \nu, e) \mid e < \omega; \nu < \pi(\sigma)\}.
\]

Note that \( p(\sigma) < \beta \) since \( \beta \) is a regular \( \alpha \)-cardinal.
Let $u = \max \{ p(\sigma), g(\sigma) \}$ and define

$$B^\sigma = B^{<\sigma} \cup \{ \gamma | v(\sigma, \mu) \leq \gamma < v(\sigma) + \sigma \}.$$ 

This completes stage $\sigma$ of the construction. Note that the first $\beta$ elements of the complement of $B^\sigma$ are just

$$\{ v(\sigma, \xi) | \xi < \mu \} \cup \{ v(\sigma) + \sigma + \xi | \xi < \beta \}.$$

3.6.4 Lemma. B is regular; cB is unbounded and has order type $\beta$.

Proof. Given $\beta' < \beta$ let $\sigma'$ be a stage such that $(\sigma)_{\sigma > \sigma'} \beta' \leq g(\sigma)$. Then by the construction

$$\{ v(\sigma', \xi) | \xi < \beta' \} \subseteq cB.$$ 
Hence cB has order type $\geq \beta'$. Since this holds for all $\beta' < \beta$, it follows that cB has order type $\geq \beta$.

On the other hand, it is immediate from the definition of $B^\sigma$ that $cB^\sigma \cap \sigma$ is $\alpha$-finite and has order type less than $\beta$, hence less than $\alpha'$. It follows that $cB \cap \sigma$ is $\alpha$-finite and has order type less than $\beta$. Since this holds for all $\sigma$, it follows that B is regular and cB is unbounded and has order type $\beta$. Q.E.D.
3.6.5 LEMMA. \text{Rcf}(B) = \beta.

PROOF. We begin with a trivial observation \( Q \):
for any \( K_\eta \subseteq cB \) there are \( \tau < \alpha, \gamma < \beta \) such that
\[
\{ v(\sigma, \xi) | \xi < \gamma \}.
\]

This observation \( Q \) is immediate from 3.6.4.

\text{Rcf}(B) \leq \beta \quad \text{in view of 3.6.4.}

Suppose \( \text{rcf}(B) < \beta \). Then there are \( \lambda < \beta \) and \( \phi \prec_{\omega\alpha} B, \text{dom}(\phi) = \lambda, \phi[\lambda] \) unbounded. Using
observation \( Q \) we shall show that \( \phi \) is \( \alpha \)-recursive, hence \( \phi[\lambda] \) is bounded.

Let \( \rho \) be such that always
\[
\phi(v) = \delta \iff (E\eta)[<v, \delta, \eta> \in R_\rho \land K_\eta \subseteq cB].
\]
Let \( e < \omega \) be such that \( f(e) = \rho \). Let \( \sigma_0 \) be a stage such that
\[
\{ f(\sigma, e) = \rho \land \lambda \leq g(\sigma) \}.
\]

We now describe an \( \alpha \)-recursive procedure for computing...
89.

\( \phi(v) \ (v < \lambda) \). Given \( v < \lambda \), find a stage \( \sigma \geq \sigma_0 \) such that \( (E\gamma')_{\gamma'} < \beta(E\delta')(E\eta')[v_0, \delta', \eta'] \in R^G_\rho \) & \( K_{\eta'} \subseteq \{v(\sigma, \xi) | \xi < \gamma'\} \). By \( Q \) such a \( \sigma \) must exist. Then let \( \delta, \eta \) be such that

\[ <v, \delta, \eta> \in R^v_\rho (\sigma, \nu, e) \quad \text{&} \quad K_{\eta} \subseteq \{v(\sigma, \xi) | \xi < p(\sigma, \nu, e)\}. \]

We claim that \( \phi(v) = \delta \). It suffices to show that \( \{v(\sigma, \xi) | \xi < p(\sigma, \nu, e)\} \subseteq cB \). Since \( \sigma \geq \sigma_0 \) we have

\[ p(\sigma, \nu, e) \leq p(\sigma) \] so at least \( \{v(\sigma, \xi) | \xi < p(\sigma, \nu, e)\} \subseteq cB^G \). But it is easy to see by induction on \( \tau \geq \sigma \) that \( p(\sigma, \nu, e) = p(\tau, \nu, e) \) whence again

\[ \{v(\sigma, \xi) | \xi < p(\sigma, \nu, e)\} \subseteq cB^T \]. Thus \( \phi(v) = \delta \). \( \Box \).

The proof of 3.6.3 is complete.

Define \( RS(\alpha) \), the rcf spectrum of \( \alpha \), to be the set of all \( \beta < \alpha \) such that \( \beta = rcf(B) \) for some \( \alpha \)-r.e. set \( B \). We give an example of an \( \alpha \) whose rcf spectrum has a gap i.e., there are regular \( \alpha \)-cardinals \( \beta_0 < \beta_1 < \beta_2 \) with \( RS(\alpha) = \{\beta_0, \beta_2\} \).

3.8 EXAMPLE. Let \( \alpha \) be the least admissible ordinal such that \( \alpha^* = \alpha \) and there is an \( \alpha \)-cardinal
greater than $\mathcal{H}_1^L$.

Define $\beta_0 = \omega$, $\beta_1 = \mathcal{H}_1^L$, and $\beta_2$ is the unique $\alpha$-cardinal greater than $\beta_1$. As in Example 1.8 $\text{recf}(\alpha) = \omega$; hence $\beta_0 \in \text{RS}(\alpha)$ and $\beta_1 \notin \text{RS}(\alpha)$. The proof of 3.4 shows: there is a $\Delta_2$ function with domain $\beta_1$ and range $\alpha$. An argument like that of 3.6.3 then shows: $\beta_2 \in \text{RS}(\alpha)$.

After writing the main text of this thesis, we obtained the following theorem.

3.9 THEOREM. Let $\beta$ be a regular $\alpha$-cardinal. Then the following are equivalent.

1. $\beta \in \text{RS}(\alpha)$.

2. There is a regular $\alpha$-r.e. set $B$ with $cB$ unbounded of order type $\beta$.

3. There is an $\alpha$-recursive function $g$ with range $(g) \subseteq \beta = \lim_{\sigma} g(\sigma)$ in the weak sense.

4. $\Delta_2 = \text{cf}(\beta) = \text{recf}(\alpha)$. 
PROOF (Sketch). The implications
\( 1 \Rightarrow 2 \iff 3 \iff 4 \) are straightforward. Also if
\( \beta = \text{recf}(\alpha) \) then \( 1 \) is obvious. Suppose now that
\( \Delta_2 - \text{cf}(\beta) < \beta \). Then, by the regularity of \( \beta \),
there is an \( \alpha \)-infinite \( \Sigma_2 \) subset of \( \beta \). Hence by
Jensen \([8]\) there is a \( \Sigma_2 \) function from a subset of
\( \beta \) onto \( \alpha \). Using this and the \( g \) of \( 3 \), imitate the
proof of \( 3.6.3 \) to get an \( \alpha \)-r.e. \( B \) with \( \text{rcf}(B) = \beta \).

N.B. We can refine the proof just given to show:
if \( \text{recf}(\alpha) \neq \beta \in \text{RS}(\alpha) \) then there is a tame
\( \Delta_2 \) function \( f \) from \( \beta \) onto \( \alpha \). By tame we mean:
there is an \( \alpha \)-recursive function \( h \) such that
\[
(\beta') \beta' < \beta (E\sigma')(\sigma) \sigma > \sigma'(v) \forall \beta' f(v) = h(\sigma, v).
\]

In particular, it is possible to use the ordinals less
than \( \beta \) to index requirements in \( \alpha \)-recursive priority
arguments.

The details will appear elsewhere \([29]\).
§4. Friedberg's completeness criterion

The jump of a set \( A \subseteq \omega \) is defined to be the recursive union of all sets recursively enumerable in \( A \). On trying to lift this definition to \( \alpha \)-recursion theory, we meet one difficulty: there are several notions of relative \( \alpha \)-recursive enumerability. The problem of choosing among notions of \( \alpha \)-jump is completely parallel to that of choosing among notions of relative \( \alpha \)-recursiveness.

We are concerned in this thesis with only two notions of relative \( \alpha \)-recursiveness: \( \preceq_\alpha \) and \( \preceq_{\alpha^*} \). (For countable \( \alpha \) these correspond to an emphasis on computability and definability, respectively. See Kreisel [12] pp. 157-8 for a discussion and references.) The associated notions of \( \alpha \)-jump are as follows.

Let \( A \subseteq \alpha \) be given.

4.1 DEFINITION \( A^{\omega_\alpha} \), the weak \( \alpha \)-jump of \( A \), is defined by
\[ A^w\alpha = \{ \varepsilon | (E\xi)(En)[<\xi,\eta> \in R_\varepsilon \& K_\varepsilon \subseteq A \& K_\eta \subseteq cA] \}. \]

4.2 DEFINITION. \( A^{\alpha} \), the \( \alpha \)-calculability jump of \( A \), is defined by

\[ A^{\alpha} = \{ \varepsilon | f(\varepsilon) = 0 \in S^{E,A} \text{ where } E \text{ is a finite set of equations in } E_{\alpha} \text{ with G\"{o}del number } \varepsilon \}. \]

This section deals with generalizations to \( \alpha \)-recursion theory of Friedberg's completeness theorem [5]. Our purpose is twofold:

(i) to exercise the notions of jump just introduced;

(ii) to study \( \alpha \)-recursion theoretic forcing arguments.

The point in (ii) is that most previous work\(^8\) in \( \alpha \)-recursion theory has dealt with \( \alpha \)-r.e. sets; hence priority methods have been the main focus. The Friedberg completeness theorem presents a different

\(^8\) An exception is MacIntyre's dissertation [18], q.v. for more information on metarecursion-theoretic forcing arguments.
sort of problem in that it concerns arbitrary sets. We are trying to control the jump of a set being constructed by successive approximations. Hence forcing is a natural tool. Even Friedberg's original argument [5] can be viewed as a forcing argument.

Our methods yield completeness theorems for both $\leq_\alpha$ and $\leq_{\text{co}}$. This is possible because of the following obvious fact.

4.3 Suppose $A \subseteq \alpha$ is regular and hyperregular. Then $A^{\text{co}}$, $A^{\text{co}}$ are $\alpha$-recursively isomorphic.

PROOF. Immediate by 0.6 and 0.5.

The sets which we construct will be regular and hyperregular. For such sets $A$, 4.3 justifies our writing $A'$ for either $A^{\text{co}}$ or $A^{\text{co}}$.

We insert the following result here for lack of a better place.

4.3.1 THEOREM. There is a hyperregular, non-$\alpha$-recursive, $\alpha$-r.e. set $A$ such that $A' \equiv_\alpha K_{\alpha}$. 
PROOF. Let $A$ be the hyperregular, non-$\alpha$-recursive, $\alpha$-r.e. set of Sacks ([25] p. 17 but see also our footnote 4, §2 above). Then it is easy to see that $A' \equiv_\alpha K_\alpha$.

In our discussion of Friedberg's completeness theorem we deal mainly with countable $\alpha$'s. On the one hand, Friedberg's argument lifts beautifully to $\Delta_2$-collapsible $\alpha$'s (e.g. $\alpha = \omega_1^{ck}$). On the other hand, every countable $\alpha$ turns out to be $\Delta_2$-collapsible relative to some regular, hyperregular subset of $\alpha$. Thus we get a Friedberg completeness theorem for every countable $\alpha$. We close the section by reporting what we know about extensions to uncountable $\alpha$'s.

We write $K_\alpha$ for the complete $\alpha$-r.e. set and $A \oplus B$ for the $\alpha$-recursive union of $A$ and $B$.

4.4 THEOREM$^9$

Hypothesis: there is a $\Delta_2$ function with domain $\omega$ and range $\alpha$.

$^9$ D. Myers [19] obtained a weak version of this theorem for the case $\alpha = \omega_1^{ck}$. 
Conclusion: let \( B \subseteq \alpha \) be given. Then there is a regular, hyperregular \( A \subseteq \alpha \) such that

\[
A' = A \oplus K \alpha = B \oplus K \alpha.
\]

Before giving the proof we must discuss \( \alpha \)-finite initial segments.

An \( \alpha \)-finite initial segment, sometimes called a condition, is an ordered pair \( P = <\xi, \eta> \) with

\[
K_\xi = \lambda - K_\eta; \text{ here } \lambda \text{ is the length of } P, \ \lambda h(P) = \lambda < \alpha.
\]

\( P, Q, R, ... \) are variables ranging over conditions.

We say \( P \subseteq Q \) (\( Q \) extends \( P \)) if \( P = <\xi, \eta>, Q = <\gamma, \delta>, \)

\( K_\xi \subseteq K_\gamma, K_\eta \subseteq K_\delta \). By \( P^\leftarrow Q \) (resp. \( P^\rightarrow \)) we mean a condition \( Q = <\gamma, \eta> \) with \( K_\gamma = K_\xi \cup \{\lambda h(P)\} \)

(resp. \( Q = <\xi, \delta> \) with \( K_\delta = K_\eta \cup \{\lambda h(P)\} \).

A condition should be thought of as an \( \alpha \)-finite sequence of 0's and 1's. If we are using a condition \( P = <\xi, \eta> \) to approximate a set \( A \subseteq \alpha \), then \( P \)

should be thought of as saying \( K_\xi \subseteq A \) & \( K_\eta \subseteq cA \).

We write \( PH[\varepsilon](\nu) = \delta \) (pronounced "\( P \) forces \( [\varepsilon](\nu) = \delta \)") if

\[
(E\tau)_t < \lambda h(P)(E\xi')(E\nu')(<\xi', \eta', \nu', \delta> \in R^t_\varepsilon
\]

\& \( K_{\xi'}, \subseteq K_\xi \) & \( K_{\eta'}, \subseteq K_\eta \).
We write $P \vdash [\varepsilon](v)$ (pronounced "P decides $[\varepsilon](v)$") if $P \vdash [\varepsilon](v) = 0$ for some 0. We say $P$ forces $[\varepsilon](v)$ to be undefined if there is no $Q \supseteq P$ which decides $[\varepsilon](v)$.

**PROOF of 4.4.** We shall define a sequence of conditions $P_n (n \in \omega)$, $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n \subseteq \cdots$, with $\sup \{ \text{lh}(P_n) | n < \omega \} = \alpha$, such that we get the desired conclusions by setting $A = \bigcup \{ P_n | n < \omega \}$.

Let $f$ be a $\Lambda_2$ function with domain $\omega$ and range $\alpha$. Define

$$B^* = \{ 2n | K_f(n) \subseteq B \} \cup \{ 2n+1 | K_f(n) \subseteq cB \}.$$  

Define $P_0 = \phi$, i.e. $P_0 = \langle \xi, n \rangle$ where $K_\xi = K_n = \phi$.

Suppose $P_n$ has been defined. Let

$$P_n^0 = \begin{cases} 
P_n & \text{if } n \in B^*, \\
0 & \text{otherwise}.
\end{cases}$$

Suppose $v < f(n)$, $P_n^v$ has been defined, and $P_n^v$ does not force $[f(n)](v)$ to be undefined. Then let $P_n^{v+1}$ be the least $Q \supseteq P_n^v$ such that $Q \vdash [f(n)](v)$. 


Suppose $\mu$ is a limit ordinal and all of $P_n^\nu (\nu < \mu)$ have been defined. Then let $P_n^\mu = \bigcup \{P_n^\nu | \nu < \mu\}$.

Define $\nu(n)$ to be the least $\nu$ such that $P_n^{\nu+1}$ is undefined. Note that $\nu(n) \leq f(n)$.

Define $P_{n+1} = P_n^{\nu(n)}$.

If $P_n$ is a condition then so is $P_{n+1}$; in fact $P_{n+1}$ is $\alpha$-recursive as a function of $P_n$, $E^\#(n)$, and $\nu(n)$. Clearly $\sup \{\text{th}(P_n) | n < \omega\} = \alpha$. Define $A = \bigcup \{P_n | n < \omega\}$.

We claim $A$ is regular and hyperregular. For $\varepsilon < \alpha$ let $[\varepsilon]^A$ be the $\varepsilon$th function weakly $\alpha$-recursive in $A$. Suppose $[\varepsilon]^A(\nu)$ is defined for all $\nu < \varepsilon$. Then the sequence $<[\varepsilon]^A(\nu) : \nu < \varepsilon>$ is completely determined by the $\alpha$-finite condition $P_{n+1}$ where $f(n) = \varepsilon$. Hence this sequence is $\alpha$-finite. Thus $A$ is regular and hyperregular.

Note that $f$ is weakly $\alpha$-recursive in $K_\alpha$.

Using this, the proof that $A' \equiv_\alpha A \oplus K_\alpha \equiv_\alpha B \oplus K_\alpha$.
is similar to Friedberg's [5].

The proof of 4.4 is complete.

The following Lemma allows us to extend 4.4 in a meaningful way to all countable admissible ordinals.

4.5 LEMMA

Hypothesis: \( \alpha \) is countable; \( f \) is a function with domain \( \omega \) and range \( \alpha \).

Conclusion: there is a regular, hyperregular \( D \subseteq \alpha \) such that

\[ f \preceq D' \models \alpha D \circ \kappa. \]

Thus \( \alpha \) is \( \Delta_2 \)-collapsible relative to \( D \).

PROOF. Similar to the proof of 4.4. The definition of the sequence of conditions \( P_n \) \((n < \omega)\) proceeds exactly as before, with one change: \( P_n^0 \) is defined to be \( P_n \) followed by a string of \( f(n) \) 0's followed by a 1. We omit the details.

4.6 THEOREM

Hypothesis: \( \alpha \) is countable.
Conclusion: there is an \( \alpha \)-degree \( \mathcal{D} \) such that every \( \alpha \)-degree \( \geq \mathcal{D} \) is the jump of a regular, hyperregular \( \alpha \)-degree.

PROOF. Let \( f \) and \( D \) be as in 4.5. The conclusion of 4.5 says among other things: \( \alpha \) is \( \Delta_2 \)-collapsible relative to \( D \). The argument of 4.4 can thus be relativized to \( D \). For every \( B \geq_{\alpha} D' \) we get a regular, hyperregular \( A \) such that

\[
A' \equiv_{\alpha} A \oplus K \equiv_{\alpha} B.
\]

We may therefore take \( \mathcal{D} \) to be the \( \alpha \)-degree of \( D' \).

An \( \alpha \)-degree is regular if it contains a regular set.

4.7 COROLLARY

Hypothesis: \( \alpha \) is countable.

Conclusion: there is an \( \alpha \)-degree \( \mathcal{D} \) such that every \( \alpha \)-degree \( \geq \mathcal{D} \) is regular.

PROOF. Let \( f \) and \( D \) be as in the proof of 4.5. Suppose \( D' \leq_{\alpha} B \). Then as in the proof of 4.6 we get
a regular, hyperregular $A$ with $B \equiv_\alpha A \# K_\alpha$. By Sacks' Lemma 2.6 the $\alpha$-degree of $K_\alpha$ is regular. Hence the $\alpha$-degree of $B$ is regular.

N.B. The proof of 4.7 shows: if $\alpha$ is $\Delta_2$-collapsible then we can take $\sim = \sim'$, the $\alpha$-degree of the complete $\alpha$-r.e. set.

4.8 COROLLARY. Every metadegree $\geq \omega'$ is regular.

We now discuss briefly the possibilities for extending 4.6 and 4.7 to uncountable admissible ordinals.

4.9 EXAMPLE. Assume $V = L$. Let $\alpha$ be of cardinality $\mathcal{H}_1$ but cofinal with $\omega$ (e.g. the $\alpha$ of Example 2.19). Then $\{A \leq_\alpha A \text{ is regular}\}$ has cardinality $\mathcal{H}_1$. But any final segment of $\alpha$-degrees has cardinality $2^{\mathcal{H}_1} = \mathcal{H}_2$. Hence, for such an $\alpha$, the conclusion of 4.7 fails.

4.10 CONJECTURE. Assume $V = L$. Let $\alpha$ be such that
cf(α) = cf(card(α)).

Then the conclusions of 4.6 and 4.7 hold.

The proof of this conjecture should be quite interesting since it will involve a forcing construction which proceeds in uncountably many steps. So far we can prove the conjecture in the following cases (still assuming V = L):

1. α is a cardinal.

2. card(α) = cf(α) = K and there are no α-cardinals between K and α.

The proofs of these special cases (and, hopefully, a proof of the full conjecture) will appear elsewhere [30].
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Biography

Stephen G. Simpson was born on September 8, 1945, in Allentown, Pennsylvania. He entered Lehigh University in 1962 without a high school diploma. He graduated from Lehigh in June 1966, summa cum laude, mathematics. From 1966 to the present he has attended M.I.T. as a graduate student. In the summers of 1967 and 1968 he worked in Daniel H. Wagner, Associates, Inc., Paoli, Pennsylvania, as an Operations Analyst. He spent most of the academic year 1969-1970 in the 9th floor lounge of Van Vleck Hall, the University of Wisconsin, Madison. From this lofty vantage point he viewed with disdain the political events unfolding on the campus below.