Transient One Dimensional Numerical Simulation of Hall Thrusters

by

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S.B. Massachusetts Institute of Technology (1990)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Science
in
Aeronautics and Astronautics
at the
Massachusetts Institute of Technology

September, 1993

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Submitted to the Department of
Aeronautics and Astronautics
in partial fulfillment of the
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A numerical code for the plasma properties in the acceleration channel of a Hall thruster was written and its results compared with experimental data. The model was constructed to help understand the acceleration process in the channel and to accurately predict the thruster performance defined by specific impulse and efficiency. The model developed simulates a quasi one dimensional model for a Hall thruster. The model includes a kinetic energy equation for the ion distribution function, equations for neutral density and velocity, electron temperature and velocity, and electric field. Central to this analysis is the development of an ion distribution function, momentum transfer between ions and neutrals and careful treatment of the electron energy equation. Results show good agreement with experimental data for the axial distributions of the plasma properties and good quantitative agreement with experiments for thruster performance.
Acknowledgments

Foremost, I would like to thank my advisor, Professor Manuel Martinez-Sanchez, who was always available whenever I needed advice and guidance. I also need to thank the doctoral students in the lab, especially Eli Niewood and Eric Sheppard, for their patience and help in answering my numerous questions. I would like to thank the other graduate students in the lab for their support and camaraderie that helped to make my work all the more enjoyable. Finally, I would thank my parents for their continuing support over the years.
## Contents

### Acknowledgments

2

1 **Introduction**

1.1 Electric Propulsion  
11

1.2 Hall Thrusters  
11

1.3 Status of Hall Thruster Research  
13

1.4 Overview of This Research  
14

2 **Governing Equations**  
15

2.1 Species Conservation Equations  
15

2.1.1 Generalized Moment Equation  
15

2.1.2 Mass Conservation  
16

2.1.3 Momentum Conservation  
16

2.1.4 Energy Conservation  
17

2.2 Hall Thruster Flow Model  
17

2.2.1 Neutral Continuity Equation  
19

2.2.2 Electron Momentum Equation  
19

2.2.3 Ion Density and Velocity Equations  
20

2.2.4 Neutral Velocity  
23

2.2.5 Electron Energy Equation  
24

2.2.6 Channel Current  
27

2.2.7 Plasma Equations  
27

2.3 Source Terms  
28

2.3.1 Ionization Rate  
28

2.3.2 Wall Losses  
30

2.3.3 Energy Sources  
30

2.4 Performance Characteristics  
32
2.5 Boundary Conditions

3 Numerical Method

3.1 Vector Formulation

3.2 Numerical Time Steps

3.3 MacCormack's Method

3.4 Ion Distribution Function Integration

3.5 Overall Scheme

4 Numerical Results

4.1 Geometric Model

4.2 Argon Results

4.2.1 Inputs and Boundary Values

4.2.2 Numerical/Experimental Comparison

4.2.3 Performance Comparison

4.2.4 Effects of Varying Anode Current

4.2.5 Effects of Varying Mass Flow

4.2.6 Effects of Varying Magnetic Field

4.2.7 Shape Profile Influence

4.3 Xenon Results

4.3.1 Inputs and Boundary Values

4.3.2 Sample Results

4.3.3 Performance Comparison

5 Conclusions

5.1 Plasma Properties Determination

5.2 Performance Prediction

5.3 Further Work

Bibliography
List of Figures

1.1 Hall Thruster Diagram 12
2.1 Quasi One Dimensional Hall Model 18
2.2 Hall Thruster Current Diagram 27
2.3 Wall Fluxes 31
3.1 Program Flow Path 39
4.1 Geometric Dimensions of Test Thruster 40
4.2 Radial Magnetic Field Strength 41
4.3 Shape Factor Profile 41
4.4 Ion Density, Argon \( I_a = 1.5 \) Amps 42
4.5 Ionization Rate, Argon \( I_a = 1.5 \) Amps 43
4.6 Heavy Particle Velocity, Argon \( I_a = 1.5 \) Amps 44
4.7 Potential, Argon \( I_a = 1.5 \) Amps 45
4.8 Electron Temperature, Argon \( I_a = 1.5 \) Amps 45
4.9 Performance Results - Argon 46
4.10 Ion Densities - Variable \( I_a \) 49
4.11 Ionization Rates - Variable \( I_a \) 49
4.12 Ion Velocities - Variable \( I_a \) 50
4.13 Neutral Velocities - Variable \( I_a \) 50
4.14 Potentials - Variable \( I_a \) 51
4.15 Electron Temperatures - Variable \( I_a \) 51
4.16 Ion Densities - Variable Mass Flow Rate 53
4.17 Ionization Rates - Variable Mass Flow Rate 53
4.18 Ion Velocities - Variable Mass Flow Rate 54
4.19 Neutral Velocities - Variable Mass Flow Rate 54
4.20 Potentials - Variable Mass Flow Rate 55
4.21 Electron Temperatures - Variable Mass Flow Rate 55
4.22 Ion Densities - Variable Magnetic Field
4.23 Ionization Rates - Variable Magnetic Field
4.24 Ion Velocities - Variable Magnetic Field
4.25 Neutral Velocities - Variable Magnetic Field
4.26 Potentials - Variable Magnetic Field
4.27 Electron Temperatures - Variable Magnetic Field
4.28 Ion Densities - Shape Factor Comparison
4.29 Ion Velocities - Shape Factor Comparison
4.30 Potentials - Shape Factor Comparison
4.31 Electron Temperatures - Shape Factor Comparison
4.32 Ion Density, Xenon Ia = 1.0 Amp
4.33 Ionization Rate, Xenon Ia = 1.0 Amp
4.34 Heavy Particle Velocity, Xenon Ia = 1.0 Amp
4.35 Potential, Xenon Ia = 1.0 Amp
4.36 Electron Temperature, Xenon Ia = 1.0 Amp
4.37 Performance Results - Xenon
List of Tables

1.1 Typical Hall Thruster Parameters 12
4.1 Performance Results - Argon, $\dot{m} = 1.5$ Aeq 47
4.2 Performance Results - Argon, $\dot{m} = 2.0$ Aeq 47
4.3 Performance Results - Argon, $\dot{m} = 2.5$ Aeq 48
4.4 Performance Results - Argon, $I_a = 1.5$ Amps, $\dot{m} = 2.0$ Aeq 59
4.5 Performance Results - Xenon, $\dot{m} = 0.5$ Aeq 66
4.6 Performance Results - Xenon, $\dot{m} = 1.0$ Aeq 67
List of Symbols

$A$  Channel cross sectional area

$a_o$  Radius of the first Bohr orbit for Hydrogen

$a_t$  Derivative of $a$ with respect to time

$a_z$  Derivative of $a$ with respect to position

$a_{z2}$  Second derivative of $a$ with respect to position

$\alpha$  Ionization fraction

$B$  Magnetic field strength

$c_s$  Random velocity of species $s$

$\bar{c}_s$  Thermal velocity of species $s$

$D$  Diffusion coefficient

$\delta$  Secondary electron emission from the wall

$\Delta t$  Time step

$\Delta z$  Spatial step in the axial direction

$E$  Electric field strength

$e$  Electric charge of proton

$E_i^H$  Ionization energy for Hydrogen

$E_i$  Ionization energy

$\eta$  Thrust efficiency

$\eta_a$  Acceleration efficiency

$\eta_e$  Nonuniformity factor

$\eta_u$  Propellant utilization efficiency

$f_s$  Distribution function for species $s$

$\bar{F}_s$  External force p.u. volume on species $s$

$\Gamma_s$  Spitzer logarithm for species $s$

$\gamma$  Ion production cost
\( I_a \) Anode current
\( I_{sp} \) Specific impulse
\( J_s \) Electric current for species \( s \)
\( j_s \) Electric current density for species \( s \)
\( k \) Boltzmann constant
\( K_e \) Heat transfer coefficient
\( L \) Channel length
\( \dot{m} \) Mass flow rate
\( \dot{m}_i \) Ion mass flow rate
\( \dot{m}_n \) Neutral mass flow rate
\( m_e \) Electron mass
\( m_i \) Ion mass
\( n_s \) Particle density for species \( s \)
\( \dot{n}_e \) Ionization rate
\( \dot{n}_w \) Wall loss rate
\( \nu_{sr} \) Collision frequency of species \( s \) with species \( r \)
\( \bar{P}_s \) Kinetic pressure dyad for species \( s \)
\( \phi \) Electric potential
\( \phi_a \) Anode potential
\( Qin \) Ion-neutral collision cross section
\( q_s \) Electric charge of species \( s \)
\( \bar{q}_s \) Thermal conduction vector for species \( s \)
\( \rho_s \) Mass density of species \( s \)
\( S_1 \) Source term for continuity equation
\( S_2 \) Source term for energy equation
\( \sigma_i \) Ionization cross section
\( T_{no} \) Neutral inlet temperature
\( T_s \)  Temperature of species \( s \)
\( v_b \)  Bohm velocity
\( v_s \)  Velocity for species \( s \)
\( \overline{v}_s \)  Mean mass velocity for species \( s \)
\( W \)  Channel width
\( \xi_k \)  Number of electrons in \( k \)th level
\( \langle \ldots \rangle \)  Average value
Chapter 1
Introduction

1.1 Electric Propulsion

Conventional chemical rockets are limited in their exhaust velocity by the chemical energy stored in the propellant. While thrust may be unlimited in chemical rockets, their burn times are short and mass flows high. Electric propulsion devices, such as the electrothermal resistojet, arcjets and Hall thrusters, overcome these difficulties by imparting electric power to the propellant, greatly increasing the exit velocity. High specific impulses, greater than 1000 seconds, can be achieved in electric propulsive devices. However, the added weight of the power generation and conditioning systems limits the size of these thrusters. Because of the low thrust produced from small-sized thrusters, electric thrusters are limited to station keeping, orbital maneuvering and planetary transfers where time is not critical.

1.2 Hall Thrusters

Hall thrusters are one type of electric propulsive devices that are presently being researched and utilized in space operations. These thrusters use an axial electric field to accelerate ions down the channel to high exit velocities creating the thrust. The electric field is generated between the anode, which also serves as the back plate through which the propellant enters, and a cathode to neutralize the plasma flow beyond the channel exit. Ionization takes place in the acceleration channel through neutral-electron collisions. A radial magnetic field is imposed causing the electrons to travel azimuthally in the $\mathbf{E} \times \mathbf{B}$ direction. This prevents the electrons emitted into the plasma stream at the cathode from traveling directly to the anode. Additionally, to maintain the current balance with the axially
moving ions, the electrons drift toward the anode. Because of their large mass, the ions have a large gyroradius and their trajectories are not substantially altered from the axial direction in the span of the acceleration channel. A typical Hall thruster is shown in Figure 1.1, and typical operational parameters are given in Table 1.1.

![Hall Thruster Diagram](image)

**Figure 1.1: Hall Thruster Diagram**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Impulse</td>
<td>800 - 2000 sec</td>
</tr>
<tr>
<td>Anode Current</td>
<td>1 - 5 A</td>
</tr>
<tr>
<td>Efficiency</td>
<td>15 - 60%</td>
</tr>
<tr>
<td>Anode Potential</td>
<td>120 - 500 V</td>
</tr>
<tr>
<td>Thrust</td>
<td>0.1 - 2 N</td>
</tr>
<tr>
<td>Magnetic Field</td>
<td>0.06 - 0.1 T</td>
</tr>
<tr>
<td>Mass Flow Rate</td>
<td>2 - 10 mg/s</td>
</tr>
<tr>
<td>Service Life</td>
<td>&gt; 3000 hours</td>
</tr>
</tbody>
</table>

**Table 1.1: Typical Hall Thruster Parameters**

Xenon is the preferred propellant for Hall thrusters because of its low ionization potential and large atomic mass. However, because Xenon is relatively expensive, other rare gasses such as Argon and Krypton are often substituted as propellants.
1.3 Status of Hall Thruster Research

The first experimental work performed in the United States took place in the middle 1960's by Brown and Pinsley [3] and later by Janes and Lowder [8]. These early experiments showed specific impulses in excess of 1000 seconds and efficiencies as high as 40%. However, the thruster operations were accompanied by a number of instabilities and oscillations in the plasma and further research was apparently abandoned until the 1980's.

In the early 1960's, under A. I. Morozov, the Soviets developed the first concept of a thruster with closed electron drift and an extended acceleration zone. Between 1964-1969, these thrusters were improved and evolved into the Hall thruster. Much experimental work has been performed in the Soviet Union by Morozov [12, 13, 14], Bugrova [4, 5], Smirnov [15], Bishaev [2], Esipchuk [7] and Zubkov [16] to list a few. A great deal has been learned from the data about thruster design, scaling, plasma properties in the acceleration channel, and in improving performance.

The first Hall thruster was launched in 1971 on board the Soviet Meteor satellite. In its first mission, the thruster worked successfully, exceeding its requirements. Since then, the Soviets have launched over fifty Hall thrusters for satellite station keeping. To date there have been no operational failures of these thrusters in space.

Very little work had been performed on numerically modeling these thrusters. Komurasaki et. al. [9], at the University of Tokyo, created simplified one and two dimensional models for the plasma. These models assume a constant ionization to loss ratio, use a simplified ionization model, and neglect thermal conduction in the electron energy equation.

The thrust of this research was to develop a more detailed and accurate model of the physical processes in the acceleration region. This model was then used to examine
thruster performance over various operational power levels using Argon and Xenon as propellants.

1.4 Overview of This Research

This research defines a quasi-one dimensional transient model for the plasma properties down the length of the channel. The geometry for the model is taken from an actual thruster for comparison of the numerical results with collected data. The magnetic field in the model is assumed to be imposed and taken as constant in time and the plasma is assumed to be quasi-neutral. Careful treatment is given to the electron energy equation, including heat conduction in its derivation. An ion distribution function as a function of the electric field, ionization rates and wall losses are defined and calculated. The Drawin theory for ionization cross sections is employed in determining ionization rates. Additionally, momentum transfer between ions and neutrals is considered.

The analytical model is described in Chapter 2. Chapter 3 details the numerical methods employed to solve the equations. Results for Argon and Xenon as propellants are given in Chapter 4, and when possible, compared with the results from Komurasaki, Hirakawa and Arakawa [9]. Conclusions and recommendations for further research are given in Chapter 5.
Chapter 2

Governing Equations

Hall thrusters involve the interaction of neutrals, electrons and ions in the presence
of electric and magnetic fields creating a complex problem to solve. The governing
equations for this flow are an ion distribution function, neutral conservation and velocity
equations, electron momentum equation, a current equation and an energy equation.

2.1 Species Conservation Equations

2.1.1 Generalized Moment Equation

The fluid equations for species $s$ can be derived by taking the moments of the
Boltzmann Equation.

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \nabla f_s + \frac{\vec{F}_s}{m_s} \cdot \nabla_v f_s = \left( \frac{\partial f_s}{\partial t} \right)_{\text{coll}}$$

(2.1)

Boltzmann's Equation describes the rate of change of the distribution function $f$ with
respect to velocity and position in six dimensional space ($d^3r \ d^3v$). $\vec{F}_s$ represents an
externally applied force and the term on the right side is the time rate of change of the
distribution function due to collisions.

The general moment equation is obtained by multiplying (2.1) by some function $\phi$
and integrating over all phase space. The generalized moment equation is

$$\frac{\partial}{\partial t} \left( n_s \langle \phi \rangle_s \right) - n_s \left( \langle \frac{\partial \phi}{\partial t} \rangle_s \right) + \nabla \cdot \left( n_s \langle \phi \vec{v} \rangle_s \right) - \langle \vec{v} \cdot \nabla \phi \rangle_s$$

$$-n_s \left( \frac{\vec{F}_s}{m_s} \cdot \nabla_v \phi \right)_s = \int \phi \left( \frac{\partial f_s}{\partial t} \right)_{\text{coll}} \ d^3 \nu$$

(2.2)

where $<>$ represents the average value of the term inside. The fluid equations can be
obtained by substituting in values for $\phi$ and solving Equation 2.2.
2.1.2 Mass Conservation

The number conservation equation can be found for species \( s \) by taking \( \phi = 1 \) in the generalized moment equation resulting in

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{u}_s) = S_s = \int \left( \frac{\partial f_s}{\partial t} \right)_{\text{coll}} d^3\nu
\]

(2.3)

The source term, \( S_s \), represents the rate at which particles of species \( s \) are created or lost in the unit volume.

2.1.3 Momentum Conservation

The momentum equation can be found by taking the moment of \( \phi = m_s \vec{u}_s \) in Boltzmann's equation. The species momentum equation is obtained as

\[
\rho_s \left[ \frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \nabla) \vec{u}_s \right] + \nabla \cdot \vec{P}_s - n_s \langle \vec{F} \rangle_s = \vec{A}_s - \vec{v}_s S_s
\]

(2.4)

where \( \vec{P}_s \) is the kinetic pressure dyad defining pressure and shear forces. The right side terms represent the change in momentum of species \( s \) due to collisions. \( \vec{A}_s \) denotes the collisional change in mean momentum, while \( S_s \) is the same source term from Equation 2.3. \( \vec{A}_s \) may be expressed as

\[
\vec{A}_s = \rho_s \sum_r n_r (\vec{v}_r - \vec{u}_s)
\]

(2.5)

and the term representing externally applied forces may be rewritten as

\[
\langle \vec{F} \rangle_s = q_s (\vec{E} + \vec{u}_s \times \vec{B})
\]

(2.6)

where \( q_s \) is the electric charge of species \( s \). Rewriting Equation 2.4 utilizing Equations 2.5 and 2.6 results in the momentum equation for species \( s \) as
\[
\frac{\partial}{\partial t}(n_s m_s \tilde{v}_s) + \nabla \cdot (\rho_s \tilde{v}_s \tilde{v}_s) + \tilde{p}_s : \nabla - n_s \left( \vec{E} + \tilde{v}_s \times \vec{B} \right) = \sum_r \tilde{M}_{rs}
\]  

(2.7)

The right hand side of Equation 2.4 is often represented as \( M_{rs} \). Using this and rewriting the first term on the left side of Equation 2.7 results in a momentum equation of the form

\[
\frac{\partial}{\partial t}(n_s m_s \tilde{v}_s) + \nabla \cdot (\rho_s \tilde{v}_s \tilde{v}_s) + \tilde{p}_s : \nabla + n_s q_s \left( \vec{E} + \tilde{v}_s \times \vec{B} \right) = \sum_r \tilde{M}_{rs}
\]

(2.8)

### 2.1.4 Energy Conservation

The equation for species energy conservation is obtained by taking \( \phi = \frac{1}{2} m_s c_s^2 \). \( c_s \)
is defined as the particle velocity minus the mean mass velocity of species \( s \) (\( = \tilde{w}_s - \tilde{v}_s \)).

After manipulation, the internal energy equation can be written as

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_s k T_s \right) + \nabla \cdot \left( \frac{3}{2} n_s \tilde{v}_s k T_s + \tilde{q}_s \right) + \tilde{p}_s : \nabla \tilde{v}_s = \sum_r \left( E_{rs} - \tilde{v}_s \cdot \tilde{M}_{rs} \right)
\]

(2.9)

Where the right hand side represents the net energy transfer and \( M_{rs} \) is the same as in Equation 2.8. This equation will be further manipulated to obtain a total energy equation for electrons in Section 2.2.5.

### 2.2 Hall Thruster Flow Model

The model for the thruster includes six major equations - neutral continuity and momentum equations, an electron momentum equation, a detailed kinetic energy equation for the ion distribution function from which ion density and mean velocity can be calculated, an electron temperature equation and a current equation for electron velocity. These equations comprise a closed set of relations from which the plasma properties may be calculated. For this study, a number of simplifying assumptions are made. The first assumption is that quasi one dimensional flow exists - that there are no variations in the y
direction and that the values are uniform across the channel in the x direction. Therefore, it is assumed that
\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \quad (2.10)
\]

Additionally, it is assumed that there is only one component to the electric and magnetic fields and current
\[
\vec{E} = E\hat{z} \quad (2.11)
\]
\[
\vec{B} = B\hat{x} \quad (2.12)
\]
\[
\vec{J} = J\hat{z} \quad (2.13)
\]

illustrated in Figure 2.1.

The plasma is assumed to be macroscopically neutral or quasi neutral in the acceleration channel, and finally, the loss of electron-ion pairs to the walls of the channel is assumed to be ambipolar. Electron wall conduction, due to gyrating electrons colliding with the wall, is neglected.

![Figure 2.1 Quasi One Dimensional Hall Model](image-url)
2.2.1 Neutral Continuity Equation

The neutral conservation equation is Equation 2.3 where the source term $S_1$ has been replaced by

$$S_1 = \dot{n}_w - \dot{n}_e$$  \hspace{1cm} (2.14)

where $\dot{n}_w$ is the rate at which neutrals are formed per unit volume due to electron-ion recombination at the wall and $\dot{n}_e$ is the rate per unit volume at which neutrals are lost to ionization. The neutral conservation equation in differential form is

$$\frac{\partial n_n}{\partial t} + \frac{\partial (n_n \bar{v}_n)}{\partial z} = \dot{n}_w - \dot{n}_e$$  \hspace{1cm} (2.15)

2.2.2 Electron Momentum Equation

The electron diffusion equation originates from the momentum Equation 2.8. In applying this equation to electrons, several terms may be neglected, greatly simplifying the equation. The first term on the left (the electron inertia term) and the second term on the left (acceleration of heated electrons) of Equation 2.8 are neglected. Additionally, in the second term on the right side, only scattering by fluctuating fields (effective collisions) are considered, and, since the electron velocity is much greater than the heavy particle velocity, the $\bar{v}_e$ term is neglected. The resulting equation is

$$\nabla \cdot \ddot{p}_e = -en_e \left(\bar{E} + \bar{v}_e \times \bar{B}\right) - \rho_e v_e \ddot{v}_e$$  \hspace{1cm} (2.16)

The kinetic pressure dyad, neglecting off-diagonal terms, may be written as

$$\nabla \cdot \ddot{p}_e = \nabla p_e = \nabla (n_e kT_e)$$  \hspace{1cm} (2.17)

Using the definition of Bohm diffusion, an effective electron collision frequency may be obtained for Equation 2.16.

$$D = \frac{kT_e}{16eB} = \frac{kT_e}{m_e v_e}$$  \hspace{1cm} (2.18)
\[ v_e = \frac{16eB}{m_e} \]  

(2.19)

Finally, \( \nabla(n_e kT_e) \) in Equation 2.17 may be expanded using the chain rule into 

\[ \nabla(n_e kT_e) = n_e k \frac{\partial T_e}{\partial z} + k T_e \frac{\partial n_e}{\partial z} \]  

(2.20)

Applying Equations 2.19 and 2.20, expanding the cross product \( \vec{v}_e \times \vec{B} \), and retaining only the terms in the z-direction from Equation 2.16 results in 

\[ n_e k \frac{\partial T_e}{\partial z} + k T_e \frac{\partial n_e}{\partial z} = -en_e \vec{E} - 16eB n_e v_e \]  

(2.21)

Rewriting the electric field as \( E = -\frac{\partial \phi}{\partial z} \) and rearranging Equation 2.21 produces the electric field equation 

\[ \frac{\partial \phi}{\partial z} = 16Bv_e + \frac{k}{e} \frac{T_e}{n_e} \frac{\partial n_e}{\partial z} + \frac{k}{e} \frac{\partial T_e}{\partial z} \]  

(2.22)

2.2.3 Ion Density and Velocity Equations

To determine the ion density and velocity, a distribution function for ions is introduced. The Boltzmann Equation 2.1 is used to produce 

\[ \frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial z} + \frac{eE}{m_i} \frac{\partial f_i}{\partial v_i} = n_e \delta(v_i - \vec{v}_n) - \frac{\dot{n}_w}{n_e} f_i \]  

(2.23)

where \( f_i \) is the ion distribution function and the Dirac delta function, \( \delta \), is employed so that ions are created at the neutral velocity. In equation 2.23, \( n_e \) is the ionization rate and \( \dot{n}_w \) is the ion loss rate to the channel walls. Equation 2.23 does not include a term for momentum transfer between ions and neutrals. Adding in a term to approximate the momentum transfer between heavy species results in
\[
\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial z} + \frac{eE}{m_i} \frac{\partial f_i}{\partial v_i} = \frac{n_i}{n_e} \delta(v_i - v_n) - \frac{n_w}{n_e} f_i + n_n Q_{in} \frac{1}{2} \left[ n_e (\overline{v}_i - v_n) \delta(v_i - v_n) - (v_i - v_n) f_i \right]
\]  
(2.24)

The last term in Equation 2.24, the momentum term, is derived in such a way that it ensures conservation of ions and gives the correct momentum exchange rate (see Section 2.2.4). The addition of the factor of 1/2 preceding the last term is necessary to ensure the correct momentum transfer. Additionally, a constant value for the ion-neutral collision cross-section, \( Q_{in} \), is assumed. The value selected for Argon is \( Q_{in} = 1.4 \times 10^{-18} \text{ m}^2 \) [10] and is \( Q_{xe} = 2.145 \times 10^{-17} \text{ m}^2 \) for Xenon. Rearranging Equation 2.24 slightly produces

\[
\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial z} + \frac{eE}{m_i} \frac{\partial f_i}{\partial v_i} = \left[ \frac{n_i}{n_e} + \frac{1}{2} n_e n_n Q_{in} (\overline{v}_i - v_n) \right] \delta(v_i - v_n) - \left[ \frac{n_w}{n_e} + \frac{1}{2} n_n Q_{in} (v_i - v_n) \right] f_i
\]  
(2.25)

Equation 2.25 may be solved using the method of characteristics. The first step is to find the characteristic trajectories and then to calculate the rate of change of \( f_i \) along them. The characteristic system is

\[
\frac{dt}{1} = \frac{dz}{v_i} = \frac{dv_i}{eE} = \frac{df_i}{m_i} \left( n_i + n_e \overline{v}_{in} \right) \delta(v_i - \overline{v}_n) - \left( \frac{n_w}{n_e} + v_{in} \right) f_i
\]  
(2.26)

where

\[
v_{in} = \frac{1}{2} n_n Q_{in} (v_i - v_n)
\]  
(2.27)

\[
\overline{v}_{in} = \frac{1}{2} n_n Q_{in} (\overline{v}_i - v_n)
\]  
(2.28)

Notice that the first equality in Equation 2.26 simply relates position and velocity. The second and third terms produce the energy equation for ions. However, the equality of interest is between the last two terms:

21
\[
\left[ n_e + n_e \bar{V}_n \right] \delta (v_i - v_n) = \left[ \frac{n_i}{n_e} + v_{in} \right] f_i \right] dv_i = \frac{eE}{m_i} df_i
\]  
(2.29)

This equation may be integrated to create a distribution function. The integration is over the interval from \( v_i = \bar{V}_n \), representing ions that were just created, to the maximum velocity the ions could achieve at their present location. This maximum velocity would equal the ion velocity if they were created at the anode and had no collisions,

\[
\nu_{z=0} = \sqrt{\frac{2e}{m_i} \left( \phi_a - \phi_z \right) + \bar{c}_n^2}
\]  
(2.30)

where \( \phi_a \) is the anode potential and \( \phi_z \) is the potential at the ions present axial location. \( \bar{c}_n \) is the neutral sonic speed at the anode, \( \bar{c}_n = \sqrt{\frac{5 k T_{ne}}{3 m_i}} \), so that \( \bar{c}_n = \bar{V}_{n0} \). The distribution function is solved for each particular location \( z \). So for each \( v_i \), there is a different \( z_o \), or location where the ion was created, such that

\[
\phi(z_o) = \phi(z) + \frac{m_i}{2e} \left( v_i^2 - \bar{V}_{n0}^2 \right)
\]  
(2.31)

The solution to Equation 2.29 is

\[
f_i(v_i) = \left[ \frac{m_i (n_e + n_e \bar{V}_n)}{eE} \right] \frac{m_i}{e} \frac{1}{2} \bar{c}_n \delta (v_i - v_{in}) \right] dv_i
\]  
(2.32)

In the integral appearing in Equation 2.32, the quantities \( E, \frac{n_i}{n_e} \) and \( v_{in} \) are evaluated at the time \( t' \) and location \( z' \) at which an ion with velocity \( v_i \) at \((z, t)\) went through velocity \( v_i' \). Thus, \( v_i' \) acts as the dummy variable of integration and as a parameter specifying the integrand. The time and location \((z_0, t_0)\) are simply the \((z', t')\) at which this particular ion was created and at which \( v_i' = \bar{V}_n \). Also, note that the neutral speed \( \bar{V}_n \) is to be evaluated at \((z', t')\) as well.
The ion distribution function, integrated over velocity, produces the ion or electron density

\[ n_e(z) = \frac{u_i}{v_{\text{max}}} \int f_i d\nu_i \]  

(2.33)

The mass average ion velocity can be found by multiplying the right side of Equation 2.32 by \( \nu_i \) and integrating

\[ n_e \bar{\nu}_i(z) = \frac{u_i}{v_{\text{max}}} \int \nu_i f_i d\nu_i \]  

(2.34)

2.2.4 Neutral Velocity

Due to ion-neutral collisions, momentum is transferred from the ions to the neutrals resulting in an increase in the neutral velocity and a decrease in the ion velocity. For simplicity, the neutral velocities are all lumped together into their mean velocity so that a neutral distribution function does not need to be created. The neutral energy conservation equation may be written as

\[ n_n \frac{d}{dz} \left( \frac{1}{2} m_i \nu_n^2 \right) = M_{in} \]  

(2.35)

\( M_{in} \) is the momentum change due to collisions and, assuming the same mass for both neutrals and ions, can be defined as

\[ M_{in} = \int_{0}^{\infty} \frac{m_i}{2} n_n (\nu_i - \nu_n) Q_{in} \left[ n_e (\bar{\nu}_i - \nu_n) \delta(\nu_i - \nu_n) - (\nu_i - \nu_n) f_i \right] d\nu_i \]  

(2.36)

As in the previous section, Equation 2.36 satisfies the zeroth moment of Boltzmann’s equation and produces the correct answer for the first (momentum) moment, but is invalid at higher moments. Simplifying and solving Equation 2.36 yields
\[ M_{in} = \frac{m_i}{2} n_n Q_{in} \int_0^\infty (v_i - v_n)^2 f_i dv_i \]  \hspace{1cm} (2.37)

and evaluating the integral

\[ M_{in} = \frac{m_i}{2} n_n n_e Q_{in} \langle (v_i - v_n)^2 \rangle \]  \hspace{1cm} (2.38)

The term within \(< >\) represents the averaged value of the quantity inside. Thus, Equation 2.35 may be written as

\[ \bar{v}_n(z) = \sqrt{\bar{v}_e^2 + \Delta_n(z)} \]  \hspace{1cm} (2.39)

where the change in momentum is given as

\[ \Delta_n = Q_{in} \int_0^z (v_i - v_n)^2 f_i dv_i \]  \hspace{1cm} (2.40)

### 2.2.5 Electron Energy Equation

The electron energy relation starts from the energy moment Equation 2.9

\[ \frac{\partial}{\partial t} \left( \frac{3}{2} n_e k T_e \right) + \nabla \cdot \left( \frac{3}{2} n_e \vec{v}_e k T_e \bar{E} + \bar{q}_e \right) + \vec{p}_e \cdot \nabla = \sum_r \left( E_{re} - \bar{v}_e \cdot \vec{M}_{re} \right) \]  \hspace{1cm} (2.41)

Assuming Maxwellian collisions, the term \( E_{re} \) may be written as

\[ E_{re} = m_e v_{ea} n_e \left[ \left( \vec{v}_r + \frac{m_e}{m_r} \vec{v}_e \right) \cdot \left( \vec{v}_r - \bar{v}_e \right) + \frac{3k}{m_r} (T_r - T_e) \right] \]  \hspace{1cm} (2.42)

Using the momentum equation, Equation 2.8

\[ \frac{\partial}{\partial t} (n_e m_e \bar{v}_e) + \nabla \cdot (\rho_e \vec{v}_e \bar{v}_e) + \bar{p}_e \cdot \nabla + n_e \left( \vec{E} + \bar{v}_e \times \vec{B} \right) = \sum_r \vec{M}_{re} \]  \hspace{1cm} (2.43)

a total energy relation may be obtained by taking the internal energy equation and adding the dot product of \( \bar{v}_e \) with the momentum equation: (Equation 2.41) + \( \bar{v}_e \cdot \) (Equation 2.43).

The resulting expression

\[ 24 \]
\[ \frac{\partial}{\partial t} \left[ n_e m_e \left( \frac{3}{2} \frac{k}{m_e} T_e + \frac{v_e^2}{2} \right) \right] + \nabla \cdot \left[ n_e m_e \tilde{v}_e \left( \frac{3}{2} \frac{k}{m_e} T_e + \frac{v_e^2}{2} \right) \right] + \nabla \cdot (n_e k T_e \tilde{v}_e) = S_2 \]  

(2.44)

represents the electron total energy. The term \( \tilde{q}_e \) in Equation 2.44 is the thermal conduction vector

\[ \tilde{q}_e = -K_e \tilde{V} T_e \]  

(2.45)

The right side source term includes energy lost to collisions, to the wall and the energy gained due to the electric field (\( \tilde{j} \cdot \tilde{E} \)). This equation may be simplified by introducing the variable, \( X \) representing the total energy density, defined as

\[ X = n_e m_e \left( \frac{3}{2} \frac{k}{m_e} T_e + \frac{v_e^2}{2} \right) = \frac{3}{2} n_e k T_e \]  

(2.46)

So that Equation 2.44 may be rewritten as

\[ \frac{\partial X}{\partial t} + \nabla \cdot \left[ \tilde{v}_e X + \tilde{q}_e \right] + \nabla \cdot (n_e k T_e \tilde{v}_e) = S_2 \]  

(2.47)

Rearranging and expanding equation 2.47, its differential form is obtained as

\[ \frac{\partial X}{\partial t} + \frac{\partial (\tilde{v}_e X)}{\partial z} + \left( k_n E_e - \frac{\partial K_e}{\partial z} \right) \frac{\partial T_e}{\partial z} - K_e \frac{\partial^2 T_e}{\partial z^2} = S_2 - k T_e \frac{\partial (n_e v_e)}{\partial z} \]  

(2.48)

Here the term \( k T_e \frac{\partial (n_e v_e)}{\partial z} \) has been moved to the right hand side as a source term since it contains no temperature term.

The term \( K_e \) is the heat conduction coefficient, and is described as the sum of two limiting models. The first is due to electron-ion collisions and is the "classical" description in the presence of a magnetic field [11].

\[ K_{ei} = 2.689 \times 10^{-11} \frac{T_e^5}{\beta^2 \ln \Gamma_e} \]  

(2.49)

where
\[ \Gamma_e = 1.24 \times 10^7 \sqrt{\frac{T_e^3}{n_e}} \]  
\[ (2.50) \]

\[ \beta = \frac{eB}{m_e v_e} \]  
\[ (2.51) \]

The alternative heat conduction coefficient is derived according to Bohm diffusion using:

\[ K_e \equiv k n_e \bar{c}_e l_e^M \]  
\[ (2.52) \]

\[ \bar{c}_e = \sqrt{\frac{16 k T_e}{2 \pi m_e}} \]  
\[ (2.53) \]

\[ v_e = \frac{16 e B}{m_e} = \frac{\bar{c}_e}{(l_e^M)_{\text{eff}}} \]  
\[ (2.54) \]

where \((l_e^M)_{\text{eff}}\) represents an effective length between collisions. Thus,

\[ l_e^M = \frac{\bar{c}_e}{v_e} \]  
\[ (2.55) \]

Combining Equations 2.52 through 2.55 produces the Bohm thermal heat transfer coefficient

\[ K_{e2} = \frac{1}{2 \pi} \frac{k^2 n_e T_e}{eB} \]  
\[ (2.56) \]

Summing these two thermal conductivities produces the effective heat transfer coefficient.

\[ K_e = K_{e1} + K_{e2} \]  
\[ (2.57) \]

In Equation 2.57, because the Hall parameter, \( \beta \), is large \( K_e = K_{e2} \) and Bohm diffusion is the driving model for heat conduction.
2.2.6 Channel Current

An additional relation for the plasma properties in the channel is one for current conservation where

\[
\frac{I_a}{A} = j_i + j_e = en_e \overline{v_i} - en_e \nu_e
\]  

(2.58)

\(I_a\) is the anode current and \(j_{ez}\) and \(j_{iz}\) are current densities. The equation could also be written in the form \(I_a = J_e(z) + J_i(z)\) to use currents. The ion current \(J_i\) at the exit, is sometimes referred to as the beam current \(I_b\). Figure 2.2 diagrams the currents in the thruster. At the cathode, a part \(I_b\) of the emitted electrons, are required to neutralize the current of \(I_b\) ions. At the anode, the remaining \(J_e\) from the cathode and the electrons produced by ionization, \(J_i\), equals the anode current \(I_a\).

Figure 2.2 Hall Thruster Current Diagram

2.2.7 Plasma Equations

There are seven plasma equations used, determined by the number of unknown variables. Because the plasma is quasi-neutral, only one equation is needed to define both the electron and ion densities. The electron velocity is determined from the current equation and constitutes the last equation. Listed together here, the plasma equations are

\[
\frac{\partial n_e}{\partial t} + \frac{\partial (n_e \overline{v_e})}{\partial z} = \dot{n}_w - \dot{n}_e
\]  

(2.59)
\[ \bar{v}_n(z) = \sqrt{\tau^2_n + \Delta_n(z)} \]  
(2.60)

\[ n_e(z) = \int_{v_n}^{v_{emax}} f_i dv_i \]  
(2.61)

\[ \bar{v}_i(z) = \frac{1}{n_e} \int_{v_n}^{v_{i max}} v_i f_i dv_i \]  
(2.62)

\[ \frac{\partial X}{\partial t} + \frac{\partial (v_e X)}{\partial z} + \left( k n_e v_e - \frac{\partial K_e}{\partial z} \right) \frac{\partial T_e}{\partial z} - K_e \frac{\partial^2 T_e}{\partial z^2} = S_z - k T_e \frac{\partial (v_e v_e)}{\partial z} \]  
(2.63)

\[ \frac{\partial \phi}{\partial z} = 16 B v_e + \frac{k T_e}{e n_e} \frac{\partial n_e}{\partial z} + \frac{k}{e} \frac{\partial T_e}{\partial z} \]  
(2.64)

\[ \frac{I_s}{A} = j_i + j_e = en_e \bar{v}_i - en_e v_e \]  
(2.65)

2.3 Source Terms

2.3.1 Ionization Rate

The ionization rate represents the rate at which electrons and ions are created or neutrals lost due to electron-neutral collisions. The production rate of ions or electrons, \( n_i \), is given as

\[ n_i = n_e \int_{0}^{c_e} f_e c_e^3 4\pi \sigma_i(c_e) dc_e \]  
(2.66)

where \( f_e \) is a Maxwellian electron distribution

\[ f_e = n_e \left( \frac{m_e}{2\pi kT_e} \right)^{\frac{3}{2}} e^{-\frac{m_e c_i^2}{2kT_e}} \]  
(2.67)
\( \sigma_i \) is the ionization cross section and is calculated according to the Drawin cross section theory [11]. According to Drawin, the nonelastic cross section for ionization by electron impact can be expressed as a function of energy

\[
\sigma_i = 2.66 \pi a_o^2 B \left( \frac{E_i^H}{E_i} \right) \xi_k g(u) \tag{2.68}
\]

Here \( E_i^H \) is the ionization energy of Hydrogen (=13.6 eV), \( E_i \) is the propellant ionization energy, \( \xi_k \) is the number of electrons in the outer shell (6 for both Xenon and Argon), \( a_o \) is the radius of the first Bohr orbit for Hydrogen (=0.529x10\(^{-10}\) m\(^2\)),

\[
u = \frac{E}{E_i} \tag{2.69}
\]

\[
g(u) = \frac{u-1}{u^2} \ln(1.25B_2u) \tag{2.70}
\]

and \( B_1 \) and \( B_2 \) are adjustable constants of order unity. Gathering all the constants together into a value \( \bar{\sigma}_i \), Equation 2.68 can be rewritten as

\[
\sigma_i = \bar{\sigma}_i B_1 g(u) \tag{2.71}
\]

where \( \bar{\sigma}_i=1.04x10^{-19} \) for Argon and \( 1.77x10^{-19} \) for Xenon.

Making a change of variables in Equation 2.66 from units of \( c_e \) to units of \( u \), using the Maxwellian distribution, and defining

\[
c_e = \sqrt{\frac{2E}{m_e}} \tag{2.72}
\]

\[
dc_e = \frac{1}{2} \sqrt{\frac{2}{m_eE}} dE \tag{2.73}
\]

\[
\theta = \frac{kT_e}{E_i} \tag{2.74}
\]

\[
\frac{u}{\theta} = \frac{E}{kT_e} \tag{2.75}
\]
results in the expressions

\[ n_i = Q n_i n_n B_i \frac{I(\theta)}{\theta^2} \]  

(2.76)

\[ I(\theta) = \int_{1}^{\infty} e^{-\frac{u}{\theta}} g(u) du = \int_{1}^{\infty} e^{-\frac{u}{\theta}} \frac{u-1}{u} \ln(1.25B_2u) du \]  

(2.77)

\[ Q = 4\sigma \sqrt{\frac{E_i}{2\pi m_e}} \]  

(2.78)

The integral in Equation 2.77 is evaluated over all energies greater than the ionization energy. Energies beyond approximately 10 times the ionization energy do not contribute significantly to the quantity \( I(\theta) \) due to the tailing off of the function \( g(u) \). The constant \( Q \) for Argon is evaluated as \( Q_{Ar} = 2.77 \times 10^{-13} \), and \( Q_{Xe} = 4.13 \times 10^{-13} \) for Xenon.

### 2.3.2 Wall Losses

The ion loss to the walls per unit volume per unit time is given simply as

\[ n_w = \frac{2n_e v_b}{W} S(z) \]  

(2.79)

where \( v_b \) is the Bohm velocity given as:

\[ v_b = 0.607 \sqrt{\frac{kT_e}{m_i}} \]  

(2.80)

In Equation 2.79, \( S(z) \) represents a shape factor to account for imperfect wall-plasma contact near the injector and has values ranging from zero near the anode to one part way down the acceleration channel and beyond.

### 2.3.3 Energy Sources

The source terms for the energy Equation 2.63 include energy lost to the wall, energy lost in ionization and energy imparted to the plasma by the electric field. The energy input from the electric field is the \( j_e \cdot E \) term and is given as
\[ \bar{j}_e \cdot \bar{E} = -en_e v_e E = en_e v_e \frac{\partial \phi}{\partial z} \]  \hspace{1cm} (2.81)

Energy lost to ionization is

\[ (1 + \gamma) e E \bar{n}_e \]  \hspace{1cm} (2.82)

Where the term \((1 + \gamma)\) represents the net energy cost for each ion produced. For these calculations, a value of \(\gamma = 3.5\) is selected based on a more detailed analysis [6].

![Figure 2.3 Wall fluxes](image)

The energy lost to the wall consists of two electron fluxes, one into and one away from the wall, as shown in Figure 2.3. The first term, Equation 2.83, represents the energy lost by the electron flux to the wall and the second is the energy flux carried by the secondary electrons emitted at the wall back into the plasma.

\[ \Gamma_{E_i} = -\bar{\Gamma}_e \left( 2eT_e' + e\phi_w \right) \]  \hspace{1cm} (2.83)

where

\[ \Gamma_{E_i} = \bar{\Gamma}_e e \delta \phi_w \]  \hspace{1cm} (2.84)

\[ \bar{\Gamma}_e = \frac{n_e \bar{c}_e}{4} e \frac{e\phi_w}{m_e} \]  \hspace{1cm} (2.85)

\[ \bar{c}_e = \left( \frac{8}{\pi} \frac{k T_e}{m_e} \right)^{\frac{1}{2}} \]  \hspace{1cm} (2.86)
and $\delta$ is the secondary emission coefficient. The total energy lost to the wall is the potential drop at the wall defined as

$$\sum_i \Gamma_{E_i} = -e\Gamma_e [2T_e' + (1 - \delta)\phi_w]$$  \hspace{1cm} (2.87)$$

The term $T_e'$ represents the electron temperature in electron volts, and $\phi_w$ is the potential drop at the wall, given by $\Gamma_i = (1 - \delta)\Gamma_e$ and $\Gamma_i = n_z v_b$, which is valid for $\phi_w > T_e'$ or $1 - \delta > \sqrt{\frac{2e}{m_i}}$. This gives

$$\phi_w = T_e' \ln \left( 1 - \delta \right) \frac{e^2}{4} \sqrt{\frac{8m_i}{\pi m_e}}$$  \hspace{1cm} (2.88)$$

Equation 2.87 is represented in terms of the electron flux to the wall, but using $\Gamma_i = (1 - \delta)\Gamma_e$, it may be rewritten in terms of the flux of ions to the wall or the ion wall loss rate defined in Equation 2.79 as

$$\sum_i \Gamma_{E_i} = -en_i \left[ \frac{2T_e'}{1 - \delta} + \phi_w \right]$$  \hspace{1cm} (2.89)$$

Summing all the energy sources gives the total energy source term for Equation 2.63

$$S_2 = en_e v_e \frac{\partial \phi}{\partial z} - e \left( (1 + \gamma)E \dot{n}_e + n_i \left[ \frac{2T_e'}{1 - \delta} + \phi_w \right] \right)$$  \hspace{1cm} (2.90)$$

2.4 Performance Characteristics

The thrust produced by the thruster can be calculated knowing the exit and inlet conditions. The thrust is derived from a closed volume analysis as

$$F = \dot{m}_e \left( \bar{v}_w - \bar{c}_n \right) + \dot{m}_i \left( \bar{v}_{n_{exit}} - \bar{c}_n \right) + k(n_e T_e)_{exit} A + \frac{\varepsilon_0}{2} E_{exit}^2 A$$  \hspace{1cm} (2.91)$$
The specific impulse and thrust efficiency are calculated respectively by

\[ I_p = \frac{F}{mg} \]  

(2.92)

\[ \eta = \frac{F^2}{2mI_a\phi_a} \]  

(2.93)

Here \( \dot{m} \) is the mass flow, \( g \) gravitational acceleration, and \( I_a \) and \( \phi_a \) the acceleration current and potential.

To further define the flow efficiencies, three internal efficiencies are introduced as given by Komurasaki et. al. [9].

\[ \eta_u = \frac{\dot{m}_i}{\dot{m}} \]  

(2.94)

\[ \eta_a = \frac{I_b}{I_a} \]  

(2.95)

\[ \eta_e = \frac{m_i \left( \frac{F}{\dot{m}_i} \right)^2}{2e} \]  

(2.96)

\( \eta_u \) is the propellant utilization efficiency, \( \eta_a \) the primary electron utilization and \( \eta_e \) a nonuniformity factor representing the distribution in exit velocities. \( \eta_e \) penalizes the acceleration of ions over only part of the potential. \( \dot{m}_i \) is the ion mass flow rate and \( I_b \) the beam current.

Subsequently, by their definition, the thrust efficiency may be expressed as the product of the internal efficiencies

\[ \eta = \eta_u \eta_e \eta_a \]  

(2.97)
2.5 Boundary Conditions

For seven equations, Equations 2.59-2.65, and one being second order, eight boundary conditions are required. At the inlet, the anode current, $I_A$, is set. Additionally, a pre-ionization factor $\alpha$, a fraction of ions to neutrals, is prescribed. This value, knowing the mass flow rate, then determines the ion and neutral density at the anode. The neutrals enter the acceleration channel at their thermal speed so that $\bar{v}_{n_0} = \bar{c}_n$. Also, since the ions are born at the neutral thermal speed, the mean mass velocity of the ions at the anode equals $\bar{c}_n$. Then, by Equation 2.64, the electron velocity at the anode is set. Finally, the slope of the electron temperature is taken to be zero at both the anode and the end of the cathode. When calculating the potential from the electric field, the cathode potential is set to zero and the anode potential is free and solved as part of the calculation.
Chapter 3
Numerical Method

3.1 Vector Formulation

The governing equations for neutral conservation and electron energy can be written in the form

\[
\frac{\partial V}{\partial t} + \frac{\partial h(V)}{\partial z} + k(V) \frac{\partial V_1}{\partial z} + l(V) \frac{\partial^2 V_1}{\partial z^2} + S = 0
\]  

(3.1)

where \( V_1 \) is a function of \( V \). Writing Equations 2.59 and 2.63 in vector form:

\[
V = \begin{bmatrix} n_n \\ X \end{bmatrix}, \quad h(V) = \begin{bmatrix} \bar{e}_n n_n \\ v_e X \end{bmatrix}, \quad k(V) = \begin{bmatrix} 0 \\ k n_e v_e - \frac{\partial K_e}{\partial z} \end{bmatrix}
\]

(3.2)

\[
I(V) = \begin{bmatrix} 0 \\ K_e \end{bmatrix}, \quad S = \begin{bmatrix} \dot{n}_w - \dot{n}_e \\ S_2 - k T_e \frac{\partial (n_e v_e)}{\partial z} \end{bmatrix}
\]

These equations are written in this form to facilitate programming the numerical method.

3.2 Numerical Time Steps

The two differential equations, written as equation 3.2, have their own time scale that limits the step size used in the integration. For the neutral continuity equation, a purely convective equation of the form

\[
V_t + (vV)_z = 0
\]

(3.3)

the time step can be calculated from the Courant-Fredrichs-Lowy condition

\[
\Delta t_c \leq \frac{\Delta z}{v_{\text{max}}}
\]

(3.4)
\( \nu_{\text{max}} \) represents the maximum velocity associated with \( V \) in the axial direction. A modification factor of 10 is used to shorten the time step for the convective equation so that the convective time step can be held constant throughout the convergence routine. The reason for this is to simplify the programming and will be further described in Section 3.4.

For a convective-diffusive equation

\[
V_t + k(V)V_z + l(V)V_{zz} = 0
\]

the maximum time step given by Anderson [1] is

\[
\Delta t_d = \frac{(\Delta z)^2}{|k(V)| + 2l(V)} \quad (3.6)
\]

The two time steps can be quite different. Defining the number of times the diffusive equation must be iterated for one step in the conductive equation as

\[
\frac{\Delta t_c}{\Delta t_d} = \frac{|k(V)| + 2l(V)}{\nu_{\text{max}} \Delta z} \quad (3.7)
\]

This ratio can vary greatly, but generally is on the order of thousands to tens of thousands.

### 3.3 MacCormack's Method

MacCormack's method [1] is a predictor-corrector integration scheme that is ideal for diffusive equations and can also be used for simpler convective equations by ignoring some of the terms. MacCormack's method consists of a predictor step

\[
\overline{V}_j^{n+1} = V_j^n - \frac{\Delta t}{\Delta z} \left( h_{j+1}^n - h_j^n \right) - k_j^n \frac{\Delta t}{\Delta z} \left( V_{j+1}^n - V_j^n \right) - l_j^n \frac{\Delta t}{\Delta z} \left( V_{j+1}^n - 2V_j^n + V_{j-1}^n \right) \quad (3.8)
\]

and a corrector step

\[
V_j^{n+1} = \frac{1}{2} \left[ V_j^n + \overline{V}_j^{n+1} - \frac{\Delta t}{\Delta z} \left( h_{j+1}^{n+1} - h_j^{n+1} \right) - k_j^n \frac{\Delta t}{\Delta z} \left( \overline{V}_j^{n+1} - V_j^{n+1} \right) - l_j^n \frac{\Delta t}{\Delta z} \left( V_{j+1}^{n+1} - 2\overline{V}_j^{n+1} + V_{j-1}^{n+1} \right) \right] \quad (3.9)
\]
The index \( j \) denotes the axial position or node, \( n \) the time, and a barred variable the result of the predictor step or the variable evaluated using the predicted value of \( V \). This method is second order accurate for both the spatial and time derivatives.

### 3.4 Ion Distribution Function Integration

At the beginning of the routine to evaluate the ion distribution function, two matrices are formed. The first is a square velocity matrix with indices \( i \) and \( j \) whose maximum equals the number of nodes used in the axial direction. The cells of the matrix include the velocity an ion would have arriving at \( j \) if it were born at \( i \).

\[
v_{i,j} = \sqrt{\frac{2e}{m_i} \left( \phi_i - \phi_j \right) + \overline{u}_{ni}^2}
\]  
(3.10)

For the index \( i=j \), the velocity is \( \overline{u}_{ni} \) and for \( j<i \), \( v_{i,j} = 0 \). The second matrix, also with the same indices \( i \) and \( j \), contains the number of time steps backward in time the ion at \( j \) was born (at \( i \)). Note that the upper half of this birth/location matrix contains zeros. Since the convective time step is held constant throughout the convergence of the neutral continuity and density equations, the values of the birth/location matrix remain constant.

Equation 2.32 is evaluated as a summation and rewritten here as

\[
n_e(z) = \sum_{v_i = \overline{u}_{ni}} \left[ \frac{m_i \left( n_e + n_e \overline{V}_{e} \right)}{eE} \right] e^{-\frac{m_i}{e} \sum_{v_i = \overline{u}_{ni}} \left[ \frac{1}{E(v_i)} \right]} \Delta u_i \cdot \Delta v_i
\]  
(3.11)

Since the density, ionization and wall loss rates over time are needed, the values of these variables are stored in an array indexed by time and located by the values stored in the birth/location matrix. Rewriting Equation 3.11 in terms of the axial location

\[
n_e(j) = \sum_{i=j}^{i=0} \left[ \frac{n_e(i) + n_e(j) \overline{V}_{in}(i)}{eE(i)} \right] e^{-\frac{m_i}{e} \sum_{k=0}^{k=0} \left[ \frac{1}{E(k)} \left[ n_e(k) + \nu_{in}(i) \right] \right]} \left( \hat{u}_{k,j} - \hat{v}_{k-1,j} \right) \left( \hat{u}_{i,j} - \hat{v}_{i-1,j} \right)
\]  
(3.12)
where \( \hat{v}_{i,j} \) refers to the value in the velocity matrix for an ion born at \( i \) when the ion reaches \( j \), and \( i, j \) and \( k \) are nodal indices. The axial position for the node \( j \) can be computed as

\[
    z = \frac{j}{T} L
\]

(3.13)

where \( T \) is the total number of nodes used and \( L \) is the channel axial length.

3.5 Overall Scheme

As indicated in section 2.3, the time steps for the diffusive and convective equations can vary greatly. Integrating all the equations at the smallest time step would result in performing many more iterations than are actually necessary. To get around this problem, for every time the convective equation is iterated, the diffusive equation is iterated \( N \) times as calculated by

\[
    N = \frac{\Delta t_c}{\Delta t_d}
\]

(3.14)

At each convective time step, the ion distribution function is evaluated and the electron velocity and axial currents are determined. The overall scheme is outlined as follows and illustrated in Figure 3.1

1. Evaluate the time steps using the values of the previous time step.
2. Evaluate all source terms using the values of the previous time step.
3. Determine the ion distribution function and find the ion density and velocity.
4. Integrate the neutral conservation and momentum equations.
5. Integrate the electron energy equation \( N \) times.
6. Test if the ion density has converged. If it has not, return to step 1.
7. Solve for the electric field.
8. Test if the electric field has converged. If it has not, return to step 1.
Figure 3.1 Program Flow Path
Chapter 4
Numerical Results

4.1 Geometric Model

The thruster selected for verification of the numerical model is a Japanese thruster utilized by Komurasaki et. al. [9] where data for the plasma properties was provided at one operating condition for Argon, and a performance comparison was given at several different mass flows for both Xenon and Argon. The geometric dimensions of the thruster chosen for the verification are shown in Figure 4.1. The thruster has an inner radius of 2 cm, an outer radius of 2.4 cm, and accelerator length of 8 mm. The flow analysis extends out into the plume, where 6 mm past the accelerator channel exit plane, the cathode is located. The accelerator channel has a constant area to the physical exit. Beyond the exit, ions are still lost to a fictitious wall but no wall recombination takes place.

![Figure 4.1: Geometric Dimensions of Test Thruster](image)

4.2 Argon Results

4.2.1 Inputs and Boundary Values

Argon as the working gas has an ionization energy of 15.8 eV and an ion mass of $6.63 \times 10^{-26}$ kg. For the runs presented in this section, a preionization factor of $\alpha=0.0073$
was selected to agree with the experimental data at the anode. The inlet neutral temperature was set at 300° K. The average radial magnetic field is 0.1 Tesla and its profile is shown in Figure 4.2. The loss shape factor, $S(z)$, is shown in Figure 4.3. The secondary electron emission from the wall, $\delta$, is set to 0.6 and a value of 3.5 is used for $\gamma$.

![Figure 4.2: Radial Magnetic Field Strength](image1)

![Figure 4.3: Shape Factor Profile](image2)
4.2.2 Numerical/Experimental Comparison

For this run, the anode current was set to 1.5 Amps and the mass flow was 2.0 Aeq (8.283x10^{-7} kg/s) to correspond to the operating conditions given by Komurasaki. The numerical results are compared, when the data was given, to the experimental data from Komurasaki where the two dimensional profiles have been averaged over the radial direction for several axial locations.

![Graph showing Ion Density vs Axial Position](image)

Figure 4.4: Ion Density, Argon Ia = 1.5 Amps

Figures 4.4 through 4.8 show the profiles of the plasma properties. In Figure 4.4, the ion density rises more quickly than the experimental values. This is most likely due to incorrect modeling of the shape profile near the anode, in that there are actually more losses to the wall than modeled. However, the density reaches the same maximum density as the data does. The numerical results peak and fall off earlier than the data, which indicates that either the ionization rate is too low in this region or the wall loss is too large. The rapid falling off is more likely due to the ionization rate, which is lower than expected in the exit region.
The ionization rate (Figure 4.5) reaches its maximum quickly and drops away smoothly after that. The peak is reached early in the channel where, even though the ion density is low, the electron temperature and neutral density are at or near their maxima.

Figure 4.6 shows the ion and neutral velocity. The neutral velocity increases fairly smoothly due to the ion-neutral collisions. The ion velocity shows a slope discontinuity at the channel exit (8 mm location). This is due to continued expansion and acceleration of the plasma without any wall losses. As Figure 4.7 shows, from the accelerator exit to the end of the computational grid, the potential drops less than half of the anode potential, while the ion velocity almost doubles. This is due to the majority of ions actually having a smaller velocity than their mean. These ions then benefit more in the remaining potential drop that those ions with velocities greater than their mean.
Figure 4.7 demonstrates that the numerical code accurately calculates the anode potential, but that the potential drops away slightly quicker than the data. This indicates that over the accelerating channel, the electric field calculated is higher than in the experiment, while it is smaller outside the channel.

The electron temperature (Figure 4.8) peaks near the anode at almost fifteen electron volts. The data points shown in this case represent the channel centerline values and not channel averaged values. The experimental temperature at the anode is slightly lower and this could be due to several factors. The first is that the computed electric field is too large in this area and the ionization rate and wall losses are too small. Another explanation is that the value selected for $\gamma$ is too small and does not accurately account for secondary ionization.
Figure 4.7: Potential, Argon Ia = 1.5 Amps

Figure 4.8: Electron Temperature, Argon Ia = 1.5 Amps
4.2.3 Performance Comparison

The performance comparison was conducted to check the numerical code's validity at predicting the performance characteristics of the thruster over a wide range of operating conditions. The data provided by Komurasaki included a $\text{Isp- } \eta$ plot for three different mass flows. Results for this study were obtained by running the code at successive increments of the anode current. Figure 4.9 shows the composite results of this study.

![Graph](image)

Figure 4.9: Performance Results - Argon

Figure 4.9 shows good quantitative agreement with the experimental data. The thrust efficiency is seen to increase with specific impulse in all three cases, with an increase in mass flow being accompanied by an increase in efficiency at the same specific impulse.
For the 1.5 Aeq mass flow, the anode current was varied from 0.5 to 1.75 amps while the efficiency increased from 3% to 10%. Over this same range, the accelerating voltage increased from 250 V to 265 V and the propellant utilization efficiency increased from 20% to 67%. Both the acceleration efficiency and nonuniformity factor were relatively constant over the range studied, with values of 59% and 25% respectively. Their insensitivity to the accelerating current agrees with the results and conclusions from Komurasaki's analysis. The values obtained from the performance analysis are summarized in Table 4.1.

<table>
<thead>
<tr>
<th>$I_a$ Amps</th>
<th>$\phi_a$ Volts</th>
<th>$I_{sp}$ sec</th>
<th>$\eta_a$</th>
<th>$\eta_e$</th>
<th>$\eta_u$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>250.0</td>
<td>352.1</td>
<td>0.593</td>
<td>0.254</td>
<td>0.198</td>
<td>0.030</td>
</tr>
<tr>
<td>0.75</td>
<td>235.6</td>
<td>525.4</td>
<td>0.605</td>
<td>0.255</td>
<td>0.303</td>
<td>0.047</td>
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<tr>
<td>1.00</td>
<td>238.4</td>
<td>697.8</td>
<td>0.601</td>
<td>0.253</td>
<td>0.401</td>
<td>0.061</td>
</tr>
<tr>
<td>1.25</td>
<td>241.7</td>
<td>865.3</td>
<td>0.594</td>
<td>0.252</td>
<td>0.495</td>
<td>0.074</td>
</tr>
<tr>
<td>1.50</td>
<td>254.1</td>
<td>1070.3</td>
<td>0.593</td>
<td>0.254</td>
<td>0.593</td>
<td>0.089</td>
</tr>
<tr>
<td>1.75</td>
<td>265.6</td>
<td>1238.7</td>
<td>0.579</td>
<td>0.253</td>
<td>0.674</td>
<td>0.099</td>
</tr>
</tbody>
</table>

Table 4.1: Performance Results - Argon, $\dot{m} = 1.5$ Aeq

<table>
<thead>
<tr>
<th>$I_a$ Amps</th>
<th>$\phi_a$ Volts</th>
<th>$I_{sp}$ sec</th>
<th>$\eta_e$</th>
<th>$\eta_e$</th>
<th>$\eta_u$</th>
<th>$\eta$</th>
</tr>
</thead>
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<tr>
<td>0.75</td>
<td>148.5</td>
<td>357.5</td>
<td>0.638</td>
<td>0.299</td>
<td>0.239</td>
<td>0.045</td>
</tr>
<tr>
<td>1.00</td>
<td>151.2</td>
<td>455.9</td>
<td>0.624</td>
<td>0.282</td>
<td>0.312</td>
<td>0.055</td>
</tr>
<tr>
<td>1.25</td>
<td>169.3</td>
<td>581.1</td>
<td>0.614</td>
<td>0.271</td>
<td>0.382</td>
<td>0.064</td>
</tr>
<tr>
<td>1.50</td>
<td>181.4</td>
<td>708.1</td>
<td>0.610</td>
<td>0.263</td>
<td>0.461</td>
<td>0.074</td>
</tr>
<tr>
<td>1.75</td>
<td>193.8</td>
<td>833.0</td>
<td>0.605</td>
<td>0.255</td>
<td>0.529</td>
<td>0.082</td>
</tr>
<tr>
<td>2.00</td>
<td>201.6</td>
<td>963.0</td>
<td>0.605</td>
<td>0.250</td>
<td>0.605</td>
<td>0.092</td>
</tr>
<tr>
<td>2.25</td>
<td>230.0</td>
<td>1125.1</td>
<td>0.578</td>
<td>0.265</td>
<td>0.651</td>
<td>0.100</td>
</tr>
<tr>
<td>2.50</td>
<td>238.2</td>
<td>1331.2</td>
<td>0.598</td>
<td>0.260</td>
<td>0.748</td>
<td>0.116</td>
</tr>
</tbody>
</table>

Table 4.2: Performance Results - Argon, $\dot{m} = 2.0$ Aeq

For the 2.0 Aeq mass flow case, Table 4.2, the anode current was increased from 0.75 Amps to 2.5 Amps, with accompanying increases in voltage from 148 V to 238 V and
propellant utilization from 24% to 75%. The accelerating efficiency is again constant at about 60% and the nonuniformity factor has a value of 27%. The 2.5 Aeq case, Table 4.3, varied the anode current from 1 Amps to 3.5 Amps, with an increase in the accelerating voltage from 130 V to 175 V, and propellant utilization from 26% to 87%. The remaining two internal efficiencies had the same values as the 2.0 Aeq mass flow case.

<table>
<thead>
<tr>
<th>I_a Amps</th>
<th>( \phi_a ) Volts</th>
<th>I_sp sec</th>
<th>( \eta_s )</th>
<th>( \eta_e )</th>
<th>( \eta_u )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>130.6</td>
<td>364.7</td>
<td>0.642</td>
<td>0.307</td>
<td>0.257</td>
<td>0.051</td>
</tr>
<tr>
<td>1.50</td>
<td>137.7</td>
<td>533.1</td>
<td>0.636</td>
<td>0.283</td>
<td>0.382</td>
<td>0.069</td>
</tr>
<tr>
<td>2.00</td>
<td>144.1</td>
<td>705.3</td>
<td>0.633</td>
<td>0.268</td>
<td>0.506</td>
<td>0.086</td>
</tr>
<tr>
<td>2.50</td>
<td>155.2</td>
<td>891.3</td>
<td>0.629</td>
<td>0.258</td>
<td>0.629</td>
<td>0.102</td>
</tr>
<tr>
<td>3.00</td>
<td>164.9</td>
<td>1081.7</td>
<td>0.626</td>
<td>0.251</td>
<td>0.751</td>
<td>0.118</td>
</tr>
<tr>
<td>3.50</td>
<td>174.9</td>
<td>1280.8</td>
<td>0.623</td>
<td>0.246</td>
<td>0.873</td>
<td>0.134</td>
</tr>
</tbody>
</table>

Table 4.3: Performance Results - Argon, \( \dot{m} = 2.5 \) Aeq

With the exception of the 1.5 Aeq mass flow case at very low current, all the accelerating potentials increase with increasing anode current. The general increase in performance with an increase in anode current will be examined more closely in the following section.

### 4.2.4 Effects of Varying Anode Current

For the results presented in this section, the mass flow was 2.0 Aeq for all runs and all other parameters are as described in Section 4.2.1. Three different anode currents were run with the code - 1.0, 1.5 and 2.0 Amps - and their plasma distributions compared.

Figures 4.10 through 4.15 illustrate the effects of the anode current on the plasma. The immediate effect of increasing the anode current is to increase the electron velocity toward the anode. This in turn increases the electron temperature (Figure 4.15) and leads to an increase in the ionization rate and ion density and velocity (Figures 4.10 - 4.12).
However, a balance is achieved because ionization rate is a loss in the electron temperature equation. In the end, the electron temperature does not vary greatly between runs except at

![Graph of Ion Densities - Variable Ia (Amp)](image)

Figure 4.10: Ion Densities - Variable Ia (Amp)

![Graph of Ionization Rates - Variable Ia (Amp)](image)

4.11: Ionization Rates - Variable Ia (Amp)
4.12: Ion Velocities - Variable Ia (Amp)

4.13: Neutral Velocities - Variable Ia (Amp)
the anode, while the ionization rate and density both increase substantially.

The ion velocity (Figure 4.12) also remains relatively insensitive to the anode current, although a slight increase is observed with an increase in anode current. Because
of the increase in ion density and the slight increase in ion velocity, the neutral velocity (Figure 4.13) increases as the anode current rises. Finally, the anode potential also rises with the increase in anode current. With each current increase, the slope of the potential, or electric field, also increases near the anode, giving rise to an increase in the ion density and velocity.

The effects on performance of an increase in anode current are to increase the thrust efficiency and specific impulse. Although both the anode current and potential increase, the increase in ion density and velocity raise the thrust more substantially than the increase in power. For the cases in this section, the thrust efficiency was 5.5%, 7.4% and 9.2% for an anode current of 1.0, 1.5 and 2.0 Amps respectively. The specific impulse increased as 456, 708 and 963 for the anode current increase. With the increase in ion density and velocity, the propellant utilization nearly doubles along the anode current increase - 31% at Ia=1.0 Amps, 46% at Ia=1.5 Amps and 60% at Ia=2.0 Amps.

4.2.5 Effects of Varying Mass Flow

This section looks at the results for three different mass flows at the same anode current of 1.5 Amps. Figures 4.16 through 4.21 show the results of this study.

The ion density (Figure 4.16) shows a nearly proportional increase with increasing mass flow. Even with the increase in ion density and greater mass flow, the ionization rate (Figure 4.17) for the increased mass flow is greatest only near the anode. The reason for this is that the electron temperature (Figure 4.21) is lowest for the greatest mass flow. Electron temperature is lowest for the greater mass flow (and ion density) because it has the lowest electron velocity.

The lower the mass flow, the greater is the anode potential (Figure 4.20). With a larger potential, the lower mass flow has the greatest ion velocity (Figure 4.18). A greater ion velocity is also due to fewer ion-neutral collisions - there being fewer neutrals to collide with. The neutral velocity (Figure 4.19) is relatively insensitive to the mass flow. For the
lowest mass flow rate, even though the ion velocity is high, there are fewer ions and neutrals to collide. The converse is true for the greatest mass flow rate. The two effects act to cancel one another out and produce about the same neutral velocity for all mass flows.

![Figure 4.16: Ion Densities - Variable Mass Flow Rate](image)

![Figure 4.17: Ionization Rates - Variable Mass Flow Rate](image)
Figure 4.18: Ion Velocities - Variable Mass Flow Rate

Figure 4.19: Neutral Velocities - Variable Mass Flow Rate
In terms of performance, efficiency decreases with increasing mass flow at a constant anode current. The efficiency is 6.9% at 2.5 Aeq, 7.4% at 2.0 Aeq and 8.9% at
1.5 Aeq. Both specific impulse and propellant utilization also decrease with increasing mass flow.

4.2.6 Effects of Varying Magnetic Field

The results of three different average magnetic fields are compared in this section. The mass flow rate is set to 2.0 Aeq and the anode current is 1.5 Amps. Figures 4.22 through 4.27 show the results from these cases.

The ion density (Figure 4.22) shows a slight increase over the length of the acceleration channel with increasing magnetic field. The increase in ion density is accompanied by an increase in the ionization rate (Figure 4.23). Both the ion velocity and neutral velocity (Figures 4.24 and 4.25) show a slight increase with increasing magnetic field. These increases are all due to an increase the electric field as the magnetic field increases. The increase in electric field is apparent in the rise of the potential at the anode (Figure 4.26), and is due to the higher impedance of the plasma to the upstream drifting electrons.

![Figure 4.22: Ion Densities - Variable Magnetic Field](image)
As the electric field increases, the ion distribution function increases, increasing the ion density and velocity. The increase in ion density is accompanied by the increase in ionization rate. Since both ion density and velocity rise, more momentum is transferred to the neutrals by collisions, accounting for the rise in neutral velocity.

4.23: Ionization Rates - Variable Magnetic Field

4.24: Ion Velocities - Variable Magnetic Field
4.25: Neutral Velocities - Variable Magnetic Field

4.26: Potentials - Variable Magnetic Field
The electron temperature (Figure 4.27) shows a general decrease with increasing magnetic field. This is due to the rise in electric field being a loss in the source term for the electron temperature equation.

The efficiency rises as the magnetic field increases (Table 4.4). The increase in efficiency is due to an increase in the acceleration efficiency as well as in propellant utilization. Additionally, the specific impulse rises due to the increases in ion density and heavy particle velocities, increasing the thrust.

<table>
<thead>
<tr>
<th>B₀ Tesla</th>
<th>φa Volts</th>
<th>Iₛₚ sec</th>
<th>ηₐ</th>
<th>ηₑ</th>
<th>ηᵤ</th>
<th>η</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100</td>
<td>181.4</td>
<td>708.1</td>
<td>0.610</td>
<td>0.263</td>
<td>0.461</td>
<td>0.074</td>
</tr>
<tr>
<td>0.125</td>
<td>199.3</td>
<td>826.2</td>
<td>0.666</td>
<td>0.273</td>
<td>0.499</td>
<td>0.091</td>
</tr>
<tr>
<td>0.150</td>
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<td>893.9</td>
<td>0.703</td>
<td>0.269</td>
<td>0.527</td>
<td>0.100</td>
</tr>
</tbody>
</table>

Table 4.4: Performance Results - Argon, Iₐ = 1.5 Amps, m = 2.0 Aeq
4.2.7 Shape Profile Influence

This section looks at the impact the shape factor profile, \( S(z) \), has on the plasma. All the flow inputs are as described as in Section 4.2.1, except that the shape profile is given a constant value of one, \( S(z) = 1 \), for the entire accelerator.

A dramatic drop in the ion density (Figure 4.28) results with the constant shape factor. This stems directly from the fact that more ions are lost to the wall along the length of the accelerator channel. However, the ion velocity (Figure 4.29) increases slightly. This is due to the large jump in accelerating potential (Figure 4.30). With more ion wall loss, more ionization is required to give the same anode current. This requires a greater potential drop since most of the \( \Delta \phi \) goes into ionization work. Finally, the electron temperature (Figure 4.31) is greater over the acceleration channel. Even though more ions are lost to the wall and the source term tries to drag the electron temperature lower, the fewer ions (or electrons) result in a greater electron velocity and these act to increase the temperature. Both the ionization rate and neutral velocities are nearly identical to those shown in Figures 4.5 and 4.6 respectively.

![Figure 4.28: Ion Densities - Shape Factor Comparison](image)

---

60
Figure 4.29: Ion Velocities - Shape Factor Comparison

Figure 4.30: Potentials - Shape Factor Comparison
With the constant shape factor, the specific impulse increases to 766.8 seconds while the efficiency drops to 0.066. (Compare with 1.50 Amp case from Table 4.2.) Both the acceleration efficiency and utilization efficiency drop to 0.575 and 0.431 respectively, but the nonuniformity factor increases slightly to 0.266.

4.3 Xenon

4.3.1 Inputs and Boundary Values

Xenon has an ionization energy of 12.1 eV, and ion mass of 2.18x10^{-25} kg. The ion-neutral collision cross section used was $Q_{in}=2.145\times10^{-17}$ m$^2$ determined by a hard sphere collision cross section [11], with a scaling factor incorporated equal to $Q_{in}$ for Argon divided by the hard sphere collision cross section for Argon. A preionization factor of 0.02 was used and all other values are the same as detailed in Section 4.2.1.
4.3.2 Sample Results

For this run, the anode current was 1.0 Amp and the mass flow was 0.5 Aeq (6.807x10^{-7} kg/s). No data was given for the plasma distributions with Xenon as the working gas by Komurasaki.

![Graph of Ion Density vs Axial Position](image)

Figure 4.32: Ion Density, Xenon Ia = 1.0 Amp

Figures 4.32 through 4.36 show the profiles of the plasma properties for Xenon. The ion density (Figure 4.32) is greater than the ion density for Argon that ran at four times the mass flow rate and twice the anode current. This is due to the fact that Xenon ionizes far more easily than Argon (ionization energies of 12.1 eV vs. 15.8 eV). The ionization rate (Figure 4.33) is similar in shape to the Argon profiles, but has a lower peak due to the low mass flow rate.

Figure 4.34 shows the particle velocity. Because Xenon is heavier than Argon, the ion velocity is not as great as the Argon cases. Because the ion velocity is lower than the Argon cases, and also because there is a low mass flow rate, not as much momentum is imparted to the neutrals and the Xenon neutral momentum is one-half to two-thirds of the Argon neutral velocity.
The potential (Figure 4.35) at the anode is 127 Volts and drops nearly linearly to the cathode indicating an almost constant electric field down the length of the accelerator.
Finally, the electron temperature (Figure 4.36) does not peak near the anode as in Argon cases, but is nearly constant for the length of the accelerator channel.

Figure 4.35: Potential, Xenon $I_a = 1.0$ Amp

Figure 4.36: Electron Temperature, Xenon $I_a = 1.0$ Amp
For the Xenon case described here, the thruster had an efficiency of 14.0% and a specific impulse of 742 sec.

4.3.3 Performance Comparison

Cases at two mass flows, 0.5 and 1.0 Aeq, were run and their performance compared with data from Komurasaki. Figure 4.37 shows the results of this study. Again, as with Argon, good correlation with the experimental data is observed, including the existence of a maximum efficiency at a certain specific impulse.

The efficiency increases with increasing anode current until the utilization efficiency nears one where the increase slows with increasing specific impulse. The efficiency then begins to drop off because too much power is being poured into the plasma without any more increase in ion density. For the cases studied, the anode current was varied from 0.5 Amps to 1.75 Amps in the 0.5 Aeq mass flow rate case and 0.75 to 3.5 Amps in the 1 Aeq mass flow rate case. In both cases, the propellant utilization efficiency rose from 40% to 100%. The acceleration efficiency had almost constant values of 50% in both cases while the nonuniformity factor fluctuated, but generally rose with increasing anode current. The results for the two mass flows are summarized in Tables 4.5 and 4.6.

<table>
<thead>
<tr>
<th>I_a Amps</th>
<th>Φ_a Volts</th>
<th>I_sp sec</th>
<th>η_a</th>
<th>η_e</th>
<th>η_u</th>
<th>η</th>
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</thead>
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<td>0.527</td>
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<td>126.6</td>
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<td>141.1</td>
<td>968.9</td>
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<td>0.175</td>
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<td>1164.6</td>
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<td>0.394</td>
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<td>168.5</td>
<td>1301.5</td>
<td>0.472</td>
<td>0.379</td>
<td>1.00</td>
<td>0.186</td>
</tr>
</tbody>
</table>

Table 4.5: Performance Results - Xenon, \(\dot{m} = 0.5\) Aeq

66
Figure 4.37: Performance Results - Xenon

<table>
<thead>
<tr>
<th>$I_a$ Amps</th>
<th>$\phi_a$ Volts</th>
<th>$I_{sp}$ sec</th>
<th>$\eta_a$</th>
<th>$\eta_e$</th>
<th>$\eta_u$</th>
<th>$\eta$</th>
</tr>
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<tr>
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<td>90.0</td>
<td>255.0</td>
<td>0.546</td>
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<td>0.410</td>
<td>0.063</td>
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<td>97.8</td>
<td>398.7</td>
<td>0.502</td>
<td>0.417</td>
<td>0.502</td>
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<td>1.50</td>
<td>100.5</td>
<td>609.9</td>
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<td>0.351</td>
<td>0.830</td>
<td>0.161</td>
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Table 4.6: Performance Results - Xenon, $\dot{m} = 1.0$ Aeq
Chapter 5
Conclusions

5.1 Plasma Properties Determination

The numerical code produces axial profiles of the plasma properties with relatively good accuracy. These profiles help to visualize the acceleration process in the thruster channel, and to see the interaction between variables and the effects of running at different operating conditions. Momentum transfer between ions and neutrals due to heavy particle collisions was included, and, in order to accurately model wall losses, a shape factor modifying the wall loss rate had to be included. Additionally, heat conduction and energy lost to the wall and imparted back into the plasma by secondary emissions was included into the electron energy equation. These additions to the model yield a relatively accurate one dimensional model for the Hall thruster. This accuracy is a dramatic improvement over the one dimensional model presented by Komurasaki which was, at best, only qualitative.

4.2 Performance Prediction

The goal of producing an accurate code for predicting thruster performance was achieved, as is evident by the close agreement between the experimental data and numerical results relating efficiency and specific impulse. Both increasing mass flow and increasing anode current are seen to have significant effects on the thruster performance. However increasing the mass flow, without increasing the anode current, will not increase the performance. For Xenon, the efficiency peaks at a certain specific impulse after full propellant utilization is achieved.

In general, most of the changes in efficiency can be traced to corresponding changes in utilization efficiency, the other factors being insensitive to changes in anode
current. Thus, efficiency increases as higher ionization is achieved. This occurs when the mass flow is raised, with higher currents, or with the use of a more easily ionizable propellant.

4.3 Further Work

The simulation presented produces good results for performance estimation and is a valuable tool in this sense, but its weakness is in providing only axial distributions for the plasma quantities. In order to observe exactly what is occurring down the length of the acceleration channel, a two dimensional code, in the axial and radial directions, needs to be produced. Two dimensions will additionally allow observations of the interaction of the wall with the plasma, removing the necessity for a shape factor in the ion loss wall term. Various operating conditions can then be compared to see what influences wall losses and how to minimize them.

Using this code, the possible effects of wall conductivity should be investigated. Finally, further studies with the code should be made to try and model a Soviet thruster of the SPT series. This study would test the code’s viability at predicting efficiencies near 50% as well as predicting the distributions in a thruster with an extended acceleration channel.
Bibliography


