SOUND SCATTERING FROM FLUID CYLINDERS

by

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ABSTRACT

The rigorous series solution to the scattering of plane waves from liquid cylinders in given for the general case of complex propagation constants. The solution is given in terms of functions already tabulated, and makes use of the fact that the absorption wave number is usually small in physically realizable liquids. The physical significance of the solution at low frequencies is pointed out.

The series solution is supplanted at high frequencies by approximate solutions developed by integral equations. The approximations can be easily interpreted physically at the high frequencies, and essentially reduce to the rigorous solution at low frequencies. The range of validity of the approximations are found, and one approximation is compared with experimental results.

The effect of large absorption changes in the liquid cylinders is shown to have a negligible effect upon the scattered radiation compared to relatively small changes in the velocity or the density.
ACKNOWLEDGEMENT

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\rho_I, \rho_{II}$</td>
<td>Density of the fluids in region I and region II in grams/cm$^3$.</td>
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<tr>
<td>$C_I, C_{II}$</td>
<td>Velocity of propagation of sound in region I and region II in cm/sec.</td>
</tr>
<tr>
<td>$\alpha_I, \alpha_{II}$</td>
<td>Attenuation parameter in region I and region II. For plane waves, it is the attenuation of the pressure in nepers per cm.</td>
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<tr>
<td>$\omega$</td>
<td>Angular frequency of sound waves</td>
</tr>
<tr>
<td>$\psi_I, \psi_{II}$</td>
<td>Velocity potential in region I and region II. The pressure is $-i\omega \psi$ and the velocity is $-\text{grad} \psi$.</td>
</tr>
<tr>
<td>$J_m(z)$</td>
<td>Bessel function of the first kind of integral order $m$.</td>
</tr>
<tr>
<td>$N_m(z)$</td>
<td>Bessel function of the second kind of integral order $m$.</td>
</tr>
<tr>
<td>$H_m^{(1)}(z)$</td>
<td>Hankel function of the first kind of integral order $m$.</td>
</tr>
<tr>
<td>$-i\omega t$</td>
<td>Time dependent factor implicitly contained in all expressions involving $\psi$.</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td></td>
</tr>
<tr>
<td>$\text{Re}(z)$</td>
<td>Real part of $z$.</td>
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<tr>
<td>$\text{Im}(z)$</td>
<td>Imaginary part of $z$.</td>
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<tr>
<td>$\nabla^2$</td>
<td>Laplacian operator.</td>
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I. INTRODUCTION

The scattering of waves from obstacles has been a subject of considerable experimental and theoretical interest, especially since the time of Lord Rayleigh. The investigation and understanding of scattering phenomena is important in both basic and applied physics. A great deal of fundamental knowledge has been gained, for example, on the nature of nuclear forces by nuclear scattering experiments, and a great many applications have been made, such as echo interpretation on sonar devices by underwater scattering calculations.

This study is concerned with the scattering of an acoustic plane wave from an infinitely-long, cylindrically shaped fluid. The acoustic properties of the scattering fluid considered are comparable to those of the fluid in the external medium. The properties of the fluid will be specified only by longitudinal wave velocities, by densities, and by absorption parameters; shear waves will not be considered. This problem differs from most acoustic scattering problems in that the effect of wave motion inside the cylinder must be considered. It is usual, for example, to assume that for rigid cylinders (very large stiffness and density), for soft cylinders
(very small stiffness and density), and for acoustically treated cylinders (coated by materials with a normal impedance), the scattering can be characterized completely by the surface properties.

Rayleigh first solved the corresponding problem for liquid spheres in the limit of wavelength much longer than the radius, and in the limit of zero absorption.\(^1\) More recently, Lowan, Morse, Feshback, and Lax,\(^2\) and Lax and Feshback,\(^3\) have published series methods and tables which facilitate calculation of scattering from fluid spheres and fluid circular cylinders. Their methods are useful for wavelengths small or comparable to the radius. Tamarkin has reported experiments on the scattering from fluid cylinders,\(^4\) and Anderson\(^5\) and Hart\(^6\) have presented computations on the scattering from fluid spheres.

The problem is mathematically analogous to the scattering of electromagnetic waves from a cylinder of differing permittivity, permeability, and conductivity, provided that either the electric or magnetic vector is polarized parallel to the axis of the cylinder.

Except in the long wavelength limit, the solutions of Morse, Feshback, et. al., require considerable numerical computation.
It is the purpose of this thesis to present approximate closed form solutions to the scattering which will circumvent some of the detailed computations, and which will not obscure the physical meaning of the scattering process.

In Section II, the solution to the scattering from circular cylinders will be found utilizing the series techniques of Morse, Feshback, et, al. It will be shown that their techniques can be specialized for this study to take account of absorption. The solution will be discussed at long wavelengths (low frequencies) for several limiting cases of fluid parameters.

Two approximate solutions will be given in Section III which agree with the rigorous solution of Section II at long wavelengths (low frequencies). The approximations are based upon a corresponding approximation devised by Born, which has been used to describe certain nuclear collision processes.

In Section IV, it will be shown that the approximate solution given in Section III is in agreement with the experimental results obtained on liquid cylinders for the relatively high frequency case corresponding to a wavelength of about 1/4 of the radius.
II. SOLUTION BY SERIES

The scattering of plane waves from a liquid cylinder of circular cross-section may be found rigorously by summing the scattering from each mode. Eigenfunction expansions valid separately in the external medium and in the scatterer can be adjusted term by term according to continuity conditions at the mutual surface, and the total scattering can be interpreted as a superposition of scattering from modes corresponding to each term. This method leads to results which are useful principally at low frequencies. At high frequencies, a large number of modes must be considered and the evaluation of the series becomes cumbersome. It is possible in some specialized cases to overcome some of the difficulty by summing the modes approximately with Fourier integrals, but these techniques will not be considered here.

In this section the general series solution will be presented for complex values of the propagation constants. It will be shown that for most physically realizable liquid scatterers, the solution can be related to functions which have already been tabulated. The general solution will be specialized for the low frequency case where only the first two modes are important. Finally, the limiting cases of cylinders
with zero absorption, equal density, and equal velocity will be given, and their physical significance will be indicated.

A. GENERAL SERIES SOLUTION

Solutions of the time independent scalar wave equation

$$(\nabla^2 + k^2)\psi = 0$$

for the coordinate system shown in Fig. 1 will be given for regions I and II. In this case, region II is an infinitely-long circular cylinder of radius, $a$.

1. Waves in Region I, the External Medium.

The direction of the incident plane wave is normal to the cylinder axis, and is incident from the $\varphi = \pi$ direction. The amplitude of the wave can be taken as unity at the origin of the coordinate system without loss of generality. The well-known expansion of the plane wave velocity potential, $\psi_{in}$, in terms of cylindrical wave function is:

$$\psi_{in} = e^{ik_{\perp}r} \cos \varphi = \sum_{m=0}^{\infty} e^{im} J_m(k_{\perp}r) \cos(m\varphi)$$

(2.1)
where

\[ e_m = \begin{cases} 
1 & m = 0 \\
2 & m > 0 
\end{cases} , \]

\( k_I \) is the propagation constant in I. \( k_I \) may be complex.

The scattered velocity potential, \( \Psi_s \), must be an outgoing cylindrical wave symmetrical about \( \varphi = 0 \) and can be written as:

\[
\Psi_s(r, \varphi) = \sum_{m=0}^{\infty} A_m H_m^{(1)}(k_I r) \cos(m \varphi), \quad r \geq a .
\]

(2.2)

The asymptotic form of \( H_m^{(1)}(k_I r) \) is proportional to \( e^{i k_I r} / \sqrt{r} \), so that the scattered wave satisfies the cylindrical wave condition.

2. Waves in Region II, the Cylinder.

A suitable solution of the time independent wave equation inside the cylinder is;

\[
\Psi_{II}(r, \varphi) = \sum_{m=0}^{\infty} B_m J_m(k_{II} r) \cos(m \varphi), \quad r \leq a ,
\]

(2.3)
where

\[ \psi_\text{II} \]
denotes the velocity potential in the cylinder

\[ k_\text{II} \]
is the propagation constant in the cylinder. \( k_\text{II} \)
may be complex.

This expression is finite at \( r = 0 \).


In order to evaluate the unknown coefficients \( A_m \) and \( B_m \),
two relations between \( \psi_{\text{in}}, \psi_s, \) and \( \psi_\text{II} \) are required. These
relations are the continuity conditions;

\[
\left. \rho \psi_\text{I} \right|_{r=a} = \left. \rho \psi_\text{II} \right|_{r=a}
\]

and

\[
\left. \frac{\partial \psi_\text{I}}{\partial r} \right|_{r=a} = \left. \frac{\partial \psi_\text{II}}{\partial r} \right|_{r=a}
\]

which demand continuity of pressure and continuity of normal
velocity across the surface of the circular cylinder. Note
that

\[ \psi_\text{I} = \psi_{\text{in}} + \psi_s \] .

The application of these boundary conditions provides the
joining relations between $\psi_1$ and $\psi_{\text{II}}$; the $m'$th conditions corresponds to the $m'$th scattering mode.

The boundary conditions at infinity are that there be a source distribution corresponding to the incident plane wave, $\psi_{\text{in}}$, and that the scattered wave, $\psi_s$, must vanish. These conditions have been implicitly satisfied by the choice of the form of the solutions in region I.

4. The Solution.

It is convenient to introduce the phase-shift technique for the evaluation of the coefficients.\(^{(2),(3),(8),(9)}\). The notation used by Lowan, Morse, et al.\(^{(2)}\) is used here. Some straight-forward manipulation yields for the coefficient of the scattered wave:\(^{(3)}\)

$$A_m = -e^{\phi m} \sin \eta_m e^{-i \eta_m}$$  \hspace{1cm} (2.5)

where the $m'$th phase shift angle, $\eta_m$, is given by:

$$\tan \eta_m = \tan \phi_m (k_{\text{II}} a) \frac{\tan \bar{\phi}_m (k_{\text{II}} a) + \tan \alpha_m (k_{\text{I}} a)}{\tan \bar{\phi}_m (k_{\text{II}} a) + \tan \beta_m (k_{\text{I}} a)}$$  \hspace{1cm} (2.6)

and where the intermediate phase angles are defined as:
\[
\tan \delta_m(k_\perp a) = -\frac{J_m(k_\perp a)}{N_m(k_\perp a)}
\]

\[
\tan \delta_m(k_{\|} a) = -\frac{\rho_\perp}{\rho_{\|}} \tan \alpha_m(k_{\|} a)
\]  
(2.7)

\[
\tan \alpha_m(k_\perp a) = -\frac{k_\perp a}{J_m(k_\perp a)} \frac{d}{d(k_\perp a)} \left[ J_m(k_\perp a) \right]
\]

\[
\tan \beta_m(k_\perp a) = -\frac{k_\perp a}{N_m(k_\perp a)} \frac{d}{d(k_\perp a)} \left[ N_m(k_\perp a) \right].
\]

The phase angles \( \delta_m \), \( \alpha_m \), and \( \beta_m \) are tabulated in "Scattering and Radiation from Circular Cylinders and Spheres" (2) for real values of \( k_\perp a \) between 0 and 10.

A slightly different, but equivalent form for \( \eta_m \) is given by Lax and Feshbach (3). They relate \( \eta_m \) to two other functions which they have computed and tabulated for real values of \( k_\perp a \) from 0 to 10.

In the general case in which both the scatterer and the external medium have absorption, \( k_\perp \) and \( k_{\|} \) are complex, and tables of Bessel functions for complex argument are necessary. Recently tables of the first and second kind Bessel functions
of complex argument for \( m \) equal to 0 and 1 have been published. *(10)

In the physical case, however, the imaginary part of \( k_\perp \) or \( k_{\|} \) is usually small compared to the real part (See Appendix I), and it is possible to relate approximately the complex phase angles to functions of the real phase angles already tabulated. The advantage of this procedure is that use can be made of the phase shift technique, which presents the solution in a more elegant fashion, and for which tables are available for \( m = 0 \) to 20. For the present, let

\[
k_\perp a = z = x + iy, \quad k_{\|} a = z' = x' + iy'
\]

\[
x = \frac{\omega}{c_\perp} a, \quad y = \alpha_\perp a, \quad x' = \frac{\omega}{c_{\|}} a, \quad y' = \alpha_{\|} a.
\]

The Taylor series expansion for \( J_m(x + iy) \) about \( x \) is

\[
J_m(x+iy) = J_m(x) + \frac{iy}{1!} \frac{dJ_m(x)}{dx} + \frac{(iy)^2}{2} \frac{d^2J_m(x)}{dx^2} + \frac{(iy)^3}{3!} \frac{d^3J_m(x)}{dx^3} + \ldots
\]

and converges for all finite values of \( z \). Similarly, the expansion for \( N_m(x + iy) \) about \( x \) is:

\[
* \quad \text{The higher order functions may be determined by the recursion relation valid for either } J_m \text{ or } N_m:
\]

\[
J_{m+1}(z) = \frac{2m}{z} J_m(z) - J_{m-1}(z).
\]
\[ N_m(x + iy) = N_m(x) + \frac{iy}{1!} \frac{dN_m(x)}{dx} + \frac{(iy)^2}{2!} \frac{d^2N_m(x)}{dx^2} + \]

and converges for all finite values of \(|z| > 0\), provided

\[ \frac{\pi}{4} > \arg z > -\frac{\pi}{4}. \]

In Appendix I, it is shown that

\[ \frac{y^2}{2} = \left( \frac{\alpha_T a}{2} \right)^2 \text{ and } \frac{y^2}{x^2} = \frac{(\alpha_T c_T)}{\omega^2} \]

are small compared to 1. Hence \( J_m(z) \) and \( N_m(z) \) can be given approximately by:

\[ J_m(z) \simeq J_m(x) + iyJ'_m(x) \]

\[ N_m(z) \simeq N_m(x) + iyN'_m(x) \quad (2.8) \]

* This requirement is necessary because the radius of convergence is equal to the distance from \( x \) to the singularity at the origin. The physics of the problem assures \( y < x \). Hence this condition is always satisfied.

** The notation \( J'_m(x) \) means \( \frac{d}{dx} [J_m(x)] \).
The approximation in (2.8) is possible because the third, fifth, etc. terms and the forth, sixth, etc. terms are an order of magnitude less than the first and second terms respectively. (See Appendix II). The phase angle \( \delta_m(z) \) can then be written as:

\[
\tan \delta_m(z) = -\frac{J_m(z)}{N_m(z)} \approx -\frac{J_m(x)}{N_m(x)} + iy \frac{J_m'(x)}{N_m(x)}
\]

and with the aid of relations (2.7):

\[
\tan \delta_m(z) \approx \tan \delta_m(x) \frac{1 - i \frac{y}{x} \tan \alpha_m(x)}{1 - i \frac{y}{x} \tan \beta_m(x)}
\]

or

\[
\tan \delta_m(z) \approx \tan \delta_m(x) \frac{1 + \frac{y^2}{x^2} \tan \alpha_m(x) \tan \beta_m(x) + i \frac{y}{x} \left[ \tan \beta_m - \tan \alpha_m \right]}{1 + \frac{y^2}{x^2} \tan^2 \beta_m(x)}
\]

(2.9)

Similarly, approximate relations for the other phase angles, \( \alpha_m \) and \( \beta_m \), can be found. The angle \( \alpha_m(z) \) is approximately
given by:

\[
\tan \alpha_m(z) = -z \frac{J'_m(x)}{J_m(x)} \alpha - z \frac{J''_m(x)}{J_m(x)} + iy \frac{J'_m(x)}{J_m(x)}
\]

The ratio \(J''_m(x)/J'_m(x)\) may be evaluated from the differential equation satisfied by \(J_m\):

\[
J''_m(x) + \frac{1}{x} J'_m(x) + \left[ 1 - \frac{m^2}{x^2} \right] J_m(x) = 0
\]

or

\[
\frac{J''_m(x)}{J'_m(x)} = \left[ \frac{m^2}{x^2} - 1 \right] + \frac{1}{x^2} \tan \alpha_m(x).
\]

Hence

\[
\tan \alpha_m(x) \left[ 1 + \frac{y^2}{x^2} \left( m^2 - x^2 \right) \right] + \frac{y^2}{x^2} (m^2 - x^2)
\]

\[
\tan \alpha_m(z) = \frac{1 + \frac{y^2}{x^2} \tan^2 \alpha_m(x)}{1 + \frac{y^2}{x^2} \tan^2 \alpha_m(x)} + \frac{iy}{x} \tan \alpha_m(x) \left[ \tan \alpha_m(x) + \frac{y^2}{x^2} (m^2 - x^2) \right] - (m^2 - x^2)
\]

\[
1 + \frac{y^2}{x^2} \tan^2 \alpha_m(x)
\]

(2.10)
The expression for $\tan \beta_m(z)$ is the same form as (2.10) and is:

$$\tan \beta_m(z) \approx \frac{\tan \beta_m(x) \left[ 1 + \frac{y^2}{x^2} (m^2-x^2) \right] + \frac{y^2}{x^2} (m^2-x^2)}{1 + \frac{y^2}{x^2} \tan^2 \beta_m(x)}$$

$$+ \frac{iy}{x} \cdot \frac{\tan \beta_m(x) \left[ \tan \beta_m(x) + \frac{y^2}{x^2} (m^2-x^2) \right] - (m^2-x^2)}{1 + \frac{y^2}{x^2} \tan^2 \beta_m(x)}.$$

(2.11)

Equations (2.9), (2.10), and (2.11) for $\delta_m$, $\alpha_m$, and $\beta_m$ permit the determination of the phase shift, $\eta_m$, for complex values according to the prescription in (2.6). These expressions have been determined using the fact that the imaginary part of the propagation constant in liquids in which wave motion is considered is small compared to the real part.

Lax and Feshbach(3) have outlined the general procedure
for determining the real and imaginary parts of \( \eta_m \) in terms of the specific acoustic admittance of the scatterer. Their methods are well suited to scatterers coated with materials of specified normal impedance and in which wave motion is not considered. The problem of a liquid scatterer is also included in their theory because an admittance can be associated with each scattering mode. In addition, they make no restriction on the relative values of the real and imaginary part of the admittance. In the formulation above, however, specific use is made of the fact that the absorption wave number is small, thus leading to the expressions (2.9), (2.10), and (2.11) which are more useful for this particular study.

B. FAR-FIELD SOLUTION

Often the field at large distances from the scatterer is desired. In that case:

* This asymptotic form is valid for \( |\arg k \| \pi/2 \).

The physics of this problem assures \( 0 \leq \arg k \| \pi/4 \), so that this condition is always satisfied.
\[ H^{(1)}_{m}(k_{r}r) \rightarrow \frac{2}{\sqrt{\pi k_{r}}} \frac{1}{\sqrt{r}} e^{\frac{i k_{r}}{r}} e^{-i \pi \left( \frac{2m+1}{4} \right)} \]

\[ |k_{r}r| > m^2. \]

Hence, using (2.5), \( \Psi_s \) becomes:

\[
\psi_s(r, \phi) \rightarrow -\frac{2}{\sqrt{\pi k_{r}}} \frac{1}{\sqrt{r}} \sum_{m=0}^{\infty} e^{-i \pi \left[ \frac{2m+1}{4} + \eta_m \right]} \frac{m+1}{2} \cos \eta_m \cos \phi.
\]

(2.12)

The scattered intensity per unit incident intensity at large distances from the scatterer (3) is:

\[
I_s(r, \phi) \propto \frac{\varepsilon^{-2a_{r}r}}{2\pi \varepsilon_{r}^{2} C_{r}} \sum_{m=0}^{\infty} e_n \cos(m\phi) \cos(n\phi) F_{mn},
\]

(2.13)

where

\[
F_{mn} = F_{nm} = 1 + \varepsilon \cos \left[ 2 \text{Re}(\eta_m - \eta_n) \right] + \varepsilon^{2} \cos \left[ 2 \text{Re}(\eta_m) \right] - \varepsilon^{2} \cos \left[ 2 \text{Re}(\eta_n) \right].
\]
In most acoustic scattering experiments, the quantity measured is the rms pressure, which in turn is proportional to the square root of \( I_s \) (provided the incident plane wave can be suitably collimated and restricted from the measurement region).

The scattering cross section, \( Q_s \), is defined as the ratio of the mean power scattered per unit length of the cylinder to the mean power incident per unit area. Similarly the absorption cross section, \( Q_a \), is the ratio of the mean power absorbed per unit length to the mean power incident per unit area. For zero absorption external to the cylinder, they are determined by the relations (3):

\[
Q_s = \frac{a}{\lambda} \sum_m e_m F_{mm}
\]

\[
Q_a = \frac{a}{\lambda} \sum_m e_m \left[ 1 - \epsilon - 4 \text{Im}(\eta_m) \right].
\]

(C. LOW FREQUENCY SOLUTION.

At low frequencies \((k_1a \ll 1)\) only the first two terms of (2.12) are important. The real phase angles become (2)

(neglecting \( x^2 \) compared to unity):

* Strictly speaking, the pressure is proportional to \( I_s^{1/2} \) only for plane waves. However at large distances from the scatterer the wave fronts are very nearly plane.
\[ \tan \alpha_m(x) \rightarrow -m \quad , \quad m > 0 \]

\[ \tan \alpha_0(x) \rightarrow \frac{x^2}{2} \]

\[ \tan \beta_m(x) \rightarrow m \quad , \quad m > 0 \]

\[ \tan \beta_0(x) \rightarrow -\frac{1}{\log x} \]

\[ \tan \delta_m(x) \rightarrow \frac{\pi m}{(m!)^2} \frac{x^{2m}}{2} \quad , \quad m > 0 \]

\[ \tan \delta_0(x) \rightarrow -\frac{\pi}{2} \frac{1}{\log x} \]

Then using (2.9), (2.10), and (2.11), the complex phase angles for the first two modes are:

\[ \tan \delta_0(x+iy) \rightarrow -\frac{\pi}{2} \frac{1}{\log x} \left[ 1 - i \frac{y}{x} \frac{1}{\log x} \right] \]

\[ \tan \alpha_0(x+iy) \rightarrow \frac{x^2}{2} \left[ 1 + 2i \frac{y}{x} \right] \]

\[ \tan \beta_0(x + iy) \rightarrow -\frac{1}{\log x} \left[ 1 - i \frac{y}{x} \frac{1}{\log x} \right] \quad (2.15) \]

and

\[ \tan \delta_1(x + iy) \rightarrow \pi \left( \frac{x}{2} \right)^2 \left[ 1 + 2i \frac{y}{x} \right] \]

\[ \tan \alpha_1(x + iy) \rightarrow -1 \]

\[ \tan \beta_1(x + iy) \rightarrow 1 \]
where the inequalities $x^2 \ll 1$ and $\frac{y}{x^2} \ll 1$ have been applied. Hence the phase shift for the zero order scattering mode, $\eta_0$, is given by:

$$\tan \eta_0 = \eta_0 = \frac{\pi}{2} \left( \frac{1}{\log x} \right) \left[ 1 - \frac{1}{x} \frac{1}{\log x} \right].$$  \tag{2.16}$$

$$\left[ 1 - \frac{\rho_I}{\rho_{II}} (C_{I})^2 \right] + 21 \frac{y}{x} \left[ 1 - \frac{\rho_{II}^a (C_{II})^2}{\rho_{II}^a (C_{II})^2} \right]$$

$$\left[ \frac{2}{x^2 \log x} + \frac{\rho_I}{\rho_{II}} (C_{II})^2 \right] - 21 \frac{y}{x} \left[ \frac{1}{x^2 \log x} - \frac{a_{II} (C_{II})^2}{a_I (C_{I})^2} \right]$$

For low frequencies, $\eta_0$ depends upon all six scattering constants: $\rho_I$, $C_I$, $\alpha_I$, and $\rho_{II}$, $C_{II}$, $\alpha_{II}$.

Similarly the phase shift for the first order scattering mode, $\eta_1$ is given by:

$$\tan \eta_1 = \eta_1 = \pi \left( \frac{x}{2} \right)^2 \left[ 1 + 21 \frac{y}{x} \right] \left( \frac{\rho_I - \rho_{II}}{\rho_I + \rho_{II}} \right)$$  \tag{2.17}$$

To this order of approximation, $\eta_1$ does not depend upon $C_{II}$ and $\alpha_{II}$.
D. LIMITING CASES

The general expressions for $\eta_m$ are somewhat cumbersome, but they can be better understood by considering a few special cases. In order to point out the physical importance of the phase shifts, the low frequency solutions will be considered, where, at most, two modes are required.

1. Attenuation Zero in I and II.

In this case, the phase shifts (2.16) and (2.17) are real and become (for low frequencies):

$$\tan \eta_0 \rightarrow \pi \left( \frac{x}{2} \right)^2 \frac{E_I}{E_{II}} - 1$$

$$\tan \eta_1 \rightarrow \pi \left( \frac{x}{2} \right)^2 \frac{\rho_I - \rho_{II}}{\rho_I + \rho_{II}}$$

(2.18)

where $E_I$ and $E_{II}$ are the bulk moduli (inverse of the compressibility) in regions I and II, and are given by $E = \rho c^2$. In (2.18) $E_I/E_{II}$ was assumed to be less than 20.*

* The largest ratio listed in the Handbook of Chemistry and Physics, Chemical Rubber Publishing Co., 30 Ed., 1947, is about 85 for mercury and nitric acid. However, all other liquids listed have a ratio less than about 5 (for approximately standard temperature and pressure).
Hence changes in compressibility lead to "monopole" scattering (independent of $\varphi$), and changes in density lead to dipole scattering ($\cos\varphi$ dependence). This result is in accord with the corresponding problem of scattering from a liquid sphere solved by Rayleigh.\(^{(1)}\)

The first order approximation for the scattered intensity per unit incident intensity at low frequencies is (from 2.13)

$$I_s(r, \varphi) \approx \frac{1}{2\pi \frac{\omega}{c_I} r} \left[ F_{00} + 4 \cos \varphi F_{10} + 4 \cos^2 \varphi F_{11} \right]$$

where

$$F_{00} = \left[ 2\pi \left( \frac{x}{2} \right)^2 \right]^2 \left[ \frac{E_I}{E_{II}} - 1 \right]^2$$

$$F_{10} = -\left[ 2\pi \left( \frac{x}{2} \right)^2 \right]^2 \left[ \frac{E_I}{E_{II}} - 1 \right] \left[ \frac{\rho_I - \rho_{II}}{\rho_I + \rho_{II}} \right]$$

$$F_{11} = \left[ 2\pi \left( \frac{x}{2} \right)^2 \right]^2 \left[ \frac{\rho_I - \rho_{II}}{\rho_I + \rho_{II}} \right]^2$$

Hence

$$I_s(r, \varphi) \approx \frac{\pi}{3} \frac{x^3 a}{r} \left[ \left( \frac{E_I}{E_{II}} - 1 \right) - 2 \frac{\rho_I - \rho_{II}}{\rho_I + \rho_{II}} \cos \varphi \right]^2$$

(2.19)

as $x \to 0$, the distribution in angle remains constant;
only the magnitude of the scattered intensity changes.

The scattering cross-section can also be obtained simply for this case and is:

\[ Q_s \approx \frac{\pi^2}{4} a x^3 \left[ \left( \frac{E_I}{E_{II}} - 1 \right)^2 + 2 \left( \frac{\rho_I - \rho_{II}}{\rho_I + \rho_{II}} \right) \right]. \]

The absorption cross-section is zero.

If \( E_{II} \) and \( \rho_{II} \) become very large, the scatterer approaches a rigid cylinder. Hence

\[ I_s \to \frac{\pi}{8} \frac{a x^3}{r} \left[ 1 - 2 \cos \phi \right]^2 \]

\[ Q_s \to \frac{3\pi^2}{4} a x^3 \quad \text{as} \quad E_{II}, \rho_{II} \to \infty \]

These results are in agreement with the scattered intensity and the scattering cross-section obtained by Morse\(^{(11)}\) for a rigid cylinder.

2. Equal Compressibilities and Densities in I and II.

In this case \( \eta_0 \) and \( \eta_1 \) become:
\[ \tan \eta_0 = -\frac{\pi}{2} \frac{1}{\log x} \left[ 1 - i \frac{\sqrt{x}}{x \log x} \right] \frac{21 \frac{\sqrt{x}}{x} \left[ \frac{a_{II}}{a_I} - 1 \right]}{2 + 21 \frac{\sqrt{x}}{x} \left[ \frac{a_{II}}{a_I} - \frac{1}{x^2 \log^2 x} \right]} \]  

(2.21)

\[ \tan \eta_1 = 0. \]

Hence, to this approximation, changes in the attenuation alone lead to monopole scattering but not to dipole scattering.

3. Equal Compressibilities and Densities, and Zero Attenuation in I.

Further simplification of (2.21) results if the attenuation in the external medium is zero \((a_I = 0)\). Then

\[ \tan \eta_0 = -\pi \left[ x^2 \frac{(a_{II}a)^2}{2} \log x + i \frac{a_{II}a}{2} x \right] \]  

(2.22)

\[ \tan \eta_1 = 0. \]

For this case the scattering cross section, \(Q_s\), is, from (2.14):

\[ Q_s = \frac{\pi^2}{4} a_{II}^2 a^3 x \]  

(2.23)
and the absorption cross section, \( Q_a \), is:

\[
Q_a = 2\pi \alpha_{II} a^2.
\]  

(2.24)

It is interesting to note that \( Q_s/Q_a \to 0 \) as \( x \to 0 \), indicating that at low frequencies the absorption in the cylinder accounts for a greater loss of energy than does scattering.

The scattering cross section given in Equation (2.20) for density and compressibility changes, and zero absorption, depends upon \( x^3 \) in the low frequency limit. Hence on the basis of (2.24)* it may be concluded that as \( x \to 0 \), the cross section for an absorbing cylinder is greater than for a cylinder of differing density and compressibility.

* The attenuation parameter \( \alpha_{II} \) depends upon the square of frequency for most liquids. Hence \( Q_s \) is proportional to \( x^5 \), and \( Q_a \) is proportional to \( x^2 \).
The series method presented in Section III suffers from two difficulties. One difficulty is that only cylinders of relatively simple cross section can be treated. The cross sections must belong to a separable coordinate system for which tabulated eigenfunctions are available. The other difficulty is that the solution is not easily modified for use at high frequencies. For $|k_Ia| > 1$, approximately $m$ modes must be considered, where $m > |k_Ia|$, before the series converges satisfactorily. Certain approximations of considerable mathematical intricacy have been employed in some cases to facilitate summing of the series.\(^{(6),(12),(13)}\). However, the integral solution is a more direct approach to approximate solution which removes, in part, some of the difficulties.

The wave equations valid separately in the external medium and in the scatterer can be recast into integral equations which implicitly contain the boundary conditions. These integral equations can be constructed for arbitrarily shaped scatterers. However, they are not, in principle, any easier to solve rigorously than the differential equations, but they do lend themselves more readily to
approximate solution and to numerical iteration.

In this section, the integral equations valid in the two regions will be formulated. It will be shown that the scattered radiation can be related to the velocity potential inside the scatterer and to the normal velocity on the surface of the scatterer. This representation emphasizes the physical dependence of the scattering on the geometry and on the properties of the scatterer, and points the way to physically intuitive approximations. With the utilization of some relatively simple trial functions, two solutions to the scattering at large distances from a circular cylinder will be found. One solution is useful for $|k_r a| < 10$, and the other solution is useful for higher frequencies.

A. FORMULATION OF THE INTEGRAL EQUATION.

Consider an infinitely-long cylinder of arbitrary cross section. The coordinate system is shown in Fig. 2. In general the density, the velocity, and the absorption, inside the cylinder are not equal to the density, velocity and absorption outside the cylinder. The time independent
scalar wave equation for the velocity potential, $\psi(\vec{r})$, is:

$$(\nabla^2 + k_\text{II}^2 + w^2)\psi(\vec{r}) = 0 \ , \quad (3.1)$$

where

$$w^2(\vec{r}) = \begin{cases} k_\text{II}^2 - k_\text{I}^2 & \text{inside the cylinder, Region II.} \\ 0 & \text{outside the cylinder, Region I.} \end{cases}$$

$\vec{r}$ is the two dimensional vector from the origin to the measurement point.

The boundary conditions on $\psi$ between regions I and II are

$$\rho_\text{I}\psi_\text{I} \bigg|_s = \rho_\text{II}\psi_\text{II} \bigg|_s \ , \quad (3.2)$$

and

$$\frac{\partial\psi_\text{II}}{\partial n_\text{II}} \bigg|_s = -\frac{\partial\psi_\text{I}}{\partial n_\text{I}} \bigg|_s \quad , \quad (3.3)$$

where $n_\text{I}$ and $n_\text{II}$ are coordinates in the direction of the normal unit vectors to $S$. $\vec{n}_\text{I}$ is the outward pointing normal from region II. The boundary conditions on the surface at
infinity are the same as those given in Section II. \( \mathbf{k}_0 \) and \( \mathbf{k}_1 \) are vectors which both have magnitude \( k_1 \), and which have the direction of the incident radiation and of the measurement direction respectively.

The inhomogeneous scalar wave equation for the two dimensional Green's function, \( G(\mathbf{r},\mathbf{r}') \), will be associated with (3.1) and is:

\[
(\nabla^2 + k_1^2) G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{in (3.4)}
\]

\( G(\mathbf{r},\mathbf{r}') \) represents a wave at \( \mathbf{r}' \) due to a source at \( \mathbf{r} \) described by \( \delta(\mathbf{r}-\mathbf{r}') \) where \( \delta(\mathbf{r}-\mathbf{r}') \) is the two dimensional Dirac delta function. The boundary condition on \( G(\mathbf{r},\mathbf{r}') \) is that it be an outgoing cylindrical wave at infinity. The appropriate solution is (14):

\[
G(\mathbf{r},\mathbf{r}') = \frac{1}{4} H_0^{(1)} \left\{ k_1 \left[ (r\cos\varphi - r'\cos\varphi')^2 + (r\sin\varphi - r'\sin\varphi')^2 \right]^{\frac{1}{2}} \right\}.
\]

The volume perturbation term, \( w^2 \psi \), may be considered the source distribution for the scattered radiation. Then, using Helmholtz's theorem (5), (6) in two dimensions for the two regions, the following set of integral equations result:
Region I, External Medium

\[ \mathbf{1} \mathbf{k}_0 \cdot \mathbf{r} + \int_{\Sigma} \left[ \frac{\partial \Psi_I(\mathbf{r}')}{\partial n^I} G(\mathbf{r}, \mathbf{r}') - \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n^I} \Psi_I(\mathbf{r}') \right] dS' \]

\[ = \begin{cases} 
\Psi_I(\mathbf{r}) , & \mathbf{r} \text{ in } I . \\
0 , & \mathbf{r} \text{ in } II .
\end{cases} \quad (3.5) \]

Region II, The Scatterer

\[ \int_{\Sigma} \left[ \frac{\partial \Psi_{II}(\mathbf{r}')}{\partial n^{II}} G(\mathbf{r}, \mathbf{r}') - \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n^{II}} \Psi_{II}(\mathbf{r}') \right] dS' + \]

\[ \int_{\Omega} \mathbf{w}(\mathbf{r}') \Psi_{II}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d\Omega' = \begin{cases} 
0 , & \mathbf{r} \text{ in } I . \\
\Psi_{II}(\mathbf{r}) , & \mathbf{r} \text{ in } II .
\end{cases} \quad (3.6) \]

The term \( \frac{\mathbf{k}_0}{\mathbf{r}} \) results from evaluating the surface integral at infinity, and is the incident plane wave of unit amplitude. Hence, from (3.5), \( \Psi(\mathbf{r}) \), for \( \mathbf{r} \) external to the scatterer, is determined completely and uniquely by the values of \( \Psi_I \) and \( \frac{\partial \Psi_I}{\partial n^I} \).
on the surface of the scatterer. However it is often more 
convenient to express $\psi_I$ in terms of integrals involving 
$\psi_{II}$. This can be done by suitably combining (3.5) and (3.6).

In order to join the two integral equations, the continuity 
conditions (3.2) and (3.3) are used. Consider $\vec{r}$ outside 
the scatterer (in region I). Then, from (3.5):

$$
\int_{s} \psi_{II}(\vec{r}') \cdot \frac{\partial G(\vec{r}, \vec{r}')}{\partial n_{II}} \, dS' = \int_{v} w^2(\vec{r}) \psi_{II}(\vec{r}') \, G(\vec{r}, \vec{r}') \, dv'
$$

$$
+ \int_{s} \frac{\partial \psi_{II}(\vec{r}')}{\partial n_{II}} \, G(\vec{r}, \vec{r}') \, dS', \quad \vec{r} \text{ in I.} \quad (3.7)
$$

The continuity equation (3.2) for the pressure can be applied 
to $\psi_{II}$ in the surface integral, and (3.7) becomes

$$
- \int_{s} \psi_I(\vec{r}') \cdot \frac{\partial G(\vec{r}, \vec{r}')}{\partial n_I} \, dS' = \frac{\rho_{II}}{\rho_I} \int_{v} w^2(\vec{r}') \psi_{II}(\vec{r}') \, G(\vec{r}, \vec{r}') \, dv'
$$

$$
+ \frac{\rho_{II}}{\rho_I} \int_{s} \frac{\partial \psi_{II}(\vec{r}')}{\partial n_{II}} \, G(\vec{r}, \vec{r}') \, dS', \quad \vec{r} \text{ in I,} \quad (3.8)
$$

where use has been made of the relation
\[
\frac{\partial G(\vec{r}, \vec{r}')}{\partial n_{II}} \bigg|_s = - \frac{\partial G(\vec{r}, \vec{r}')}{\partial n_I} \bigg|_s ,
\]

Equations (3.8) and (3.5) can be added to obtain:

\[
\psi_I(\vec{r}) = \epsilon^{ik_0 \cdot \vec{r}} + \frac{\rho_{II}}{\rho_I} \int_{\Omega} W^2(\vec{r}') \psi_{II}(\vec{r}') G(\vec{r}, \vec{r}') \, d\Omega'
\]

\[
+ \int_s \left[ \frac{\rho_{II}}{\rho_I} \frac{\partial \psi_{II}(\vec{r}')}{\partial n_{II}} + \frac{\partial \psi_I(\vec{r}')}{\partial n_I} \right] G(\vec{r}, \vec{r}') \, ds' , \quad \vec{r} \text{ in } I.
\]

(3.9)

Finally, the continuity equation (3.3) for the normal velocity can be used for the second term in the surface integral and equation (3.9) becomes:

\[
\psi_I(\vec{r}) = \epsilon^{ik_0 \cdot \vec{r}} + \frac{\rho_{II}}{\rho_I} \int_{\Omega} W^2(\vec{r}') \psi_{II}(\vec{r}') G(\vec{r}, \vec{r}') \, d\Omega'
\]

\[
+ \left[ \frac{\rho_{II}}{\rho_I} - 1 \right] \int_s \frac{\partial \psi_{II}(\vec{r}')}{\partial n_{II}} G(\vec{r}, \vec{r}') \, ds' , \quad \vec{r} \text{ in } I.
\]

(3.10)
Hence $\psi(\mathbf{r})$, for $\mathbf{r}$ external to the scatterer, can be determined also from knowledge of the potential, $\psi_{II}$, inside the cylinder, and the normal velocity, $\frac{\partial \psi_{II}}{\partial n_{II}}$, on the surface. $\psi_{II}$ is sometimes called the "internal wave function", or the "inside wave function."

Obviously, equation (3.5) or (3.10) does not provide the solution to the problem. The values of $\psi_I$ and $\frac{\partial \psi_{II}}{\partial n_{II}}$ on the scatterer surface, or the values of $\psi_{II}$ in the scatterer and $\frac{\partial \psi_{II}}{\partial n_{II}}$ on the surface, must still be determined. However it is possible to approximate these functions, and so obtain a solution for $\psi_I(\mathbf{r})$.

For example, for cylinders of very different properties than the external medium, $\psi_I$ and $\frac{\partial \psi_{II}}{\partial n_{II}}$ at the surface may be associated approximately at high frequencies with specularly reflected rays. Or, for cylinders of very similar properties, the inside wave function may be taken as the unperturbed incident wave. Or, further, approximations could be iterated in the following manner: In the case of integral equation (3.5), the value of $\psi_I(\mathbf{r})$ calculated on the basis of first trials $\psi_I$ and $\frac{\partial \psi_{II}}{\partial n_{II}}$ on the surface, can be re-entered into (3.5). In the case of integral equation (3.10), an analogous equation can be
obtained for $\psi_{II}(\mathbf{r})$ in terms of $\psi_I$ and $\partial \psi_I / \partial n_I$ (which can be evaluated from (3.10) by a trial $\psi_{II}$). The equation obtained for $\psi_{II}$ can be used to obtain corrections to the trial. In both cases, the success of iterative procedures depends upon the rapidity of convergence. In practice, however, one usually attempts to arrive at the best tractable first approximation; successive integrations are usually difficult. Hence a priori knowledge of the character of the inside field is desirable; the better the approximation to the surface or internal values, the better the solution will be.

The first integral equation (3.5) is most useful when wave motion is not considered inside the cylinder. In those cases, $\psi_I$ or $\partial \psi_I / \partial n_I$ approach zero on the surface, corresponding to a soft cylinder ($E_{II} \rightarrow 0$, $\rho_{II} \rightarrow 0$) or to a rigid cylinder ($E_{II} \rightarrow \infty$, $\rho_{II} \rightarrow \infty$). When wave motion is considered inside the cylinder, equation (3.10) is often useful and will be considered principally in this study. In parts C and D of this section, relatively simple functions will be taken for the inside wave function of integral equation (3.10), and the first order approximation to the scattering from circular cylinders will be found.
B. FAR-FIELD SOLUTION

Often the field at large distances \(|k_r r| > 10\) from the cylinder is desired. In that case the asymptotic form of \(G(\vec{r}, \vec{r}')\) can be used and its argument approximated:

\[
G(\vec{r}, \vec{r}') = \frac{1}{4} H_0^{(1)}(k_I R) \to \frac{1}{4} \frac{2^{1/2}}{\pi k_I} \frac{e^{ik_I r}}{r} \frac{\kappa}{\sqrt{r}} e^{-ik_1 \cdot \vec{r}'}
\]

where

\[ R = |\vec{r} - \vec{r}'| \]

(3.11)

\(\vec{k}_1\) is a vector of magnitude \(k_I\) and direction in the measurement direction; \(\vec{r}'\) is the two dimensional vector to the integration element. Then (3.10) becomes

\[
\psi_I(\vec{r}') \approx \kappa_o e^{ik_0 \cdot \vec{r}} + \frac{1}{4} \frac{2^{1/2}}{\pi k_I} \frac{e^{ik_I r}}{\sqrt{r}} A(\vec{k}_1, \kappa_o),
\]

(3.12)

\*

The argument \(k_I R \approx k_I (r-r' \cos \varphi') = k_I r - \vec{k}_1 \cdot \vec{r}'\), for \(R\) much greater than the dimensions of the scatterer. This approximation is used in the phase of \(G(\vec{r}, \vec{r}')\); for the amplitude of \(G(\vec{r}, \vec{r}')\), \(k_I R \approx k_I r\).
where

\[ A(\vec{k}_1, \vec{k}_0) = \frac{\rho_{II}}{\rho_I} \int_V \psi_{II}(\vec{r}') e^{i\vec{k}_1 \cdot \vec{r}'} dV' \]

\[ + \left[ \frac{\rho_{II}}{\rho_I} - 1 \right] \int_S \frac{\partial \psi_{II}(\vec{r}')}{\partial n_{II}} e^{-i\vec{k}_1 \cdot \vec{r}'} dS'. \]

(3.13)

The second term on the right of equation (3.12) is recognized as the scattered radiation, \( \psi_s \). \( A(\vec{k}_1, \vec{k}_0) \) is the amplitude of the scattered cylindrical wave measured in a direction \( \vec{k}_1 \) due to a wave incident in a direction \( \vec{k}_0 \). Hence \( A(\vec{k}_1, \vec{k}_0) = A(\phi, 0) \).

In general, equation (3.12) for the far field is easier to integrate than equation (3.10), but (3.12) does not lend itself readily to iteration. The reason for this is that only the asymptotic value of \( \psi_I \) is determined by (3.12) and values of \( \psi_I \) on and near the scatterer are useful for iteration. Nevertheless, it is still possible to obtain corrections to the trial inside function by application of integral equation (3.6). In any practical case, however, only the far field is of interest, and it is often possible
to obtain useful first approximations without iteration.

The scattered intensity per unit incident intensity, $I_s$, is given by: 

$$I_s(r, \varphi) = \frac{cI}{8\pi\omega} \frac{\varepsilon_{2}^{II} r}{r} \left| A(\varphi, 0) \right|^2,$$

(3.14)

and the scattering cross section, $Q_s$, for zero absorption in region I, is given by: 

$$Q_s = \frac{cI}{8\pi\omega} \frac{2\pi}{0} \left| A(\varphi, 0) \right|^2 d\varphi.$$

(3.15)

In order to determine the absorption cross section, a theorem relating the total cross section, $Q_t$, and $A(k_1, k_0)$ is proved in Appendix III. The theorem states:

$$Q_t = Q_s + Q_a = \frac{cI}{\omega} \text{Im} \left[ A(k_0^*, k_0) \right].$$

(3.16)

Hence, if the absorption cross section is zero, equation (3.16) provides an alternative method for the calculation of $Q_s$. If it is not zero, the absorption cross section is
obviously given by:

\[ Q_a = -\frac{c}{\omega} \text{Im} \left[ A(\vec{k}_0, \vec{k}_0) \right] + \frac{c}{2\pi\omega} \int_0^{2\pi} A(\vec{k}_1, \vec{k}_0)^2 \, d\varphi. \]

(3.17)

C. THE BORN APPROXIMATION FOR A CIRCULAR CYLINDER

One of the simplest inside trial functions is a plane wave equal to the incident plane wave. This trial function leads to the first "Born approximation."

* From an intuitive point of view, the closer the properties of the scatterer are to the external region, and the lower the frequency, the better the approximation will be.

For the specialized case of a circular cylinder of radius a, the volume and surface integrals for the scattering amplitude, \( A(\vec{k}_1, \vec{k}_0) \) are taken over the surface and circumference of the circle.

* The Born approximation has been used for the calculation of spherical nuclear interaction processes. See, for example, Schiff, Quantum Mechanics, reference (9).
The inside function is chosen to be a plane wave of propagation constant, \( \vec{k}_{II} \), which is equal to the propagation constant in the external medium. However it is important that the continuity equations be satisfied as nearly as possible. Hence, \( \psi_{II} \) and \( \frac{\partial \psi_{II}}{\partial n_{II}} \) are taken as

\[
\psi_{II} = \frac{\rho_{II}}{\rho_{II}} \in \frac{i\vec{k}_0 \cdot \vec{r}}{r},
\]

\[
\frac{\partial \psi_{II}}{\partial n_{II}} = i k_I \cos \varphi \in \frac{i\vec{k}_0 \cdot \vec{r}}{r}. \tag{3.18}
\]

The assumptions are somewhat inconsistent, inasmuch as \( \frac{\partial \psi_{II}}{\partial n_{II}} \) is not obtainable from \( \psi_{II} \), but they both satisfy the continuity conditions separately. Using (3.18), the scattering amplitude becomes:

\[
A(\vec{k}_1, \vec{k}_0) = \bar{w}^2 \int \epsilon \in \frac{1}{v} \frac{i(\vec{k}_0 - \vec{k}_1) \cdot \vec{r}'}{\vec{r}'} dV'.
\]

\[
+ \left[ \frac{\rho_{II}}{\rho_{II}} - 1 \right] i k_I \int \epsilon \in \frac{i(\vec{k}_0 - \vec{k}_1) \cdot \vec{r}'}{\vec{r}'} \cos \varphi dS'. \tag{3.19}
\]

The volume integral may be evaluated as follows: Consider a vector \( \vec{K} \) such that
\[ \vec{K} = \vec{k}_0 - \vec{k}_1. \]

Hence

\[ |\vec{K}| = 2k_1 \sin \frac{\varphi}{2}, \]

\[ \vec{K} \cdot \vec{r'} = 2k_1 r' \sin \frac{\varphi}{2} \cos(\varphi', \vec{K}). \]

This can be seen from inspection of the diagram below.

If a change of variable is made to the angle coordinate \( \theta \), where \( \theta = \varphi' + \frac{1}{2} (\pi - \varphi) \) then \( \cos(\varphi', \vec{K}) = \cos \theta \), and the volume integral becomes:

\[ w^2 \int_0^a r' \mathrm{d}r' \int_{\frac{\pi}{2} - \frac{\varphi}{2}}^{\frac{5\pi}{2} - \frac{\varphi}{2}} \mathrm{d}\theta \in 12k_1 r' \sin \frac{\varphi}{2} \cos \theta. \]

The angle integration is independent of the limits, provided they describe a single closed circle. It is
an integral representation of \( J_0 \), and the radial integration can also be simply determined. Hence the volume integral is:

\[
\pi w^2 a^2 \left[ \frac{J_1(2k_I a \sin \phi/2)}{k_I a \sin \phi/2} \right].
\]

(3.21)

The surface integral of (3.18) is evaluated in a similar fashion. Using the same change of angle variable, the integral becomes:

\[
\left[ \frac{\rho_{II}}{\rho_I} - 1 \right] ik_I a \left\{ \sin \frac{\phi}{2} \int_{\pi/2 - \phi/2}^{5\pi/2 - \phi} \frac{5\pi - \phi}{2} \epsilon i2k_I a \sin \frac{\phi}{2} \cos \theta \cos \theta \, d\theta \right.
\]

\[+ \cos \frac{\phi}{2} \int_{\pi/2 - \phi/2}^{5\pi/2 - \phi} \epsilon i2k_I a \sin \frac{\phi}{2} \cos \theta \sin \theta \, d\theta \right\}.
\]

(3.22)

The first integral in (3.22) is an integral representation of \( J_1 \), (18) and the second integral is zero. Hence the surface integral is:
\[-2\pi \left[ \frac{\rho_{\Pi}}{\rho_I} - 1 \right] k_I a \sin^2 \frac{\varphi}{2} J_1 \left( 2k_I a \sin \frac{\varphi}{2} \right). \quad (3.23)\]

The scattering amplitude can then be written:

\[ A(k_1, k_0) = \pi (k_I a)^2 \left[ \frac{J_1(2k_I a \sin \frac{\varphi}{2})}{k_I a \sin \frac{\varphi}{2}} \right] \left[ \frac{w^2}{k_I^2} - 2 \left( \frac{\rho_{\Pi}}{\rho_I} - 1 \right) \sin^2 \frac{\varphi}{2} \right]. \quad (3.24) \]

In the limit of zero absorption in both media, and at low frequencies, the scattered intensity per unit incident intensity is, (from (3.14) and (3.24)):

\[ I_s(r, \varphi) = \frac{\pi}{8} \frac{(k_I a)^3 a}{r} \left[ \left( \frac{E_I}{E_{\Pi}} - 1 \right) \frac{\rho_{\Pi}}{\rho_I} - \left( 1 - \frac{\rho_{\Pi}}{\rho_I} \right) \cos \varphi \right]^2. \quad (3.25) \]

This equation agrees excellently with the rigorous expression for \( I_s \) given in Section II, equation (2.19), particularly as \( \rho_{\Pi} \to \rho_I \).

The expression (3.16) cannot be used to calculate \( Q_s \) inasmuch as (3.24) is calculated only to first order in \( \left( \frac{E_I}{E_{\Pi}} - 1 \right) \) and \( \left( 1 - \frac{\rho_{\Pi}}{\rho_I} \right) \); the expression (3.15) must be used. However
it can be seen by the close correlation between (3.25) and (2.19) that \( Q_s \) will also be predicted adequately.

The scattering amplitude calculated by the Born approximation (3.24) may be interpreted as a product of two factors. The first factor is the solution valid at low frequencies which leads to (3.25), and the second factor is the high frequency directivity factor

\[
\frac{J_1(2k_\perp a \sin \frac{\varphi}{2})}{k_\perp a \sin \frac{\varphi}{2}}
\]

which "modulates" the first. As expected, the directivity factor has a peak in the forward direction; the angular width of the peak can be determined by

\[
\varphi \approx 2 \sin^{-1} \left[ \frac{1.2 \pi}{2k_\perp a} \right].
\]

The range of validity of the Born approximation will depend both on the frequency and on the relative density and compressibility changes. In order to investigate this quantitatively, consider the integral equation
\[ \psi_{\text{II}}(\vec{r}) = \frac{\rho_{\text{I}}}{\rho_{\text{II}}} \epsilon \hat{k}_o \cdot \vec{r} + \left[ 1 - \frac{\rho_{\text{I}}}{\rho_{\text{II}}} \right] \int_s \frac{\partial \psi_{\text{II}}(\vec{r}')}{\partial n_{\text{II}}} \mathcal{G}(\vec{r}, \vec{r}') \, ds' \]

\[ + \int_V w^2(\vec{r}') \psi_{\text{II}}(\vec{r}') \mathcal{G}(\vec{r}, \vec{r}') \, dv' , \quad \vec{r} \text{ in II}, \]

(3.26)

which can be derived from (3.5) and (3.6) in a manner analogous to the derivation of (3.10). The term,

\[ \frac{\rho_{\text{I}}}{\rho_{\text{II}}} \epsilon \hat{k}_o \cdot \vec{r} \]

is the inside wave function assumed for the Born approximation. Hence the value of the remaining terms of (3.26) indicates the error made, and can then be used to establish the range of validity in terms of the frequency and the compressibility and density changes.

The Green's function is difficult to integrate for arbitrary \( \vec{r} \) inside the cylinder, but a useful criterion, analogous to a criterion given by Schiff for the spherical case, can
be established by finding the error at \( \bar{r} = 0 \). From (3.26), the inside function at \( \bar{r} = 0 \) is:

\[
\psi_{II}(0) = \frac{\rho_I}{\rho_{II}} + \left[1 - \frac{\rho_I}{\rho_{II}}\right] \int s \frac{\partial \psi_{II}(\bar{r}')}{\partial n_{II}} G(0, \bar{r}') \, ds'
\]

\[
\quad + \int v \omega^2(\bar{r}') \psi_{II}(\bar{r}') G(0, \bar{r}') \, dv'
\]

(3.27)

The trial functions (3.18) will be used in (3.27) and the ratios of the volume and surface integrals to the plane wave term will be formed:

\[
R_1 = \frac{\rho_{II}}{\rho_I} \int v \omega^2(\bar{r}') \psi_{II}(\bar{r}') G(0, \bar{r}') \, dv'
\]

(3.28)

\[
R_2 = \left[\frac{\rho_{II}}{\rho_I} - 1\right] \int s \frac{\partial \psi_{II}(\bar{r}')}{\partial n_{II}} G(0, \bar{r}') \, ds'
\]

(3.29)

They will be required to be much less than 1 to establish the range of validity. \( G(0, \bar{r}') \) is given by (see Part A of this section):
\[ g(0, \vec{r}') = \frac{1}{4} H_0^{(1)}(k_{I} r'). \]

Hence
\[ R_1 = \frac{1}{4} w^2 \int_0^{2\pi} d\varphi' \int_0^a r'dr' \frac{ik_{I} r'}{\varepsilon_{I} r'} \cos \varphi' H_0^{(1)}(k_{I} r') \]
\[ = \frac{\pi}{2} \frac{w^2}{k_{I}} \int_0^{k_{I} a} \frac{k_{I} a}{x} J_0(x) H_0^{(1)}(x) x \, dx, \]

(3.30)

and
\[ R_2 = -\frac{1}{4} \left[ \frac{\rho_{II}}{\rho_{I}} - 1 \right] k_{I} a \ H_0^{(1)}(k_{I} a) \int_0^{2\pi} \frac{1}{x} k_{I} a \cos \varphi \cos \varphi' d\varphi' \]
\[ = \frac{\pi}{2} \left[ \frac{\rho_{I} - \rho_{II}}{\rho_{I}} \right] k_{I} a \ H_0^{(1)}(k_{I} a) J_1(k_{I} a). \]

(3.31)

The quantities \[ R_1 \frac{k_{I}^2}{w^2} \] and \[ R_2 \frac{\rho_{I}}{\rho_{I} - \rho_{II}} \] are plotted in Fig. 3 as a function of \( k_{I} a \) (for purposes of this plot, \( k_{I} \) is considered real).

For given values of the relative compressibility, absorption, and density changes, expressed by the ratios
\[
\frac{w^2}{k_I^2} \quad \text{and} \quad \frac{\rho_{II} - \rho_I}{\rho_I}
\]

the upper limit of \(k_Ia\) can be found. For example, if

\[
\rho_{II} = \rho_I \quad \text{and} \quad \frac{w^2}{k_I^2} = .02
\]

corresponding to a 1\% change in the propagation constants (a 2\% change in the compressibility), then the range of validity of the Born approximation is about \(k_Ia < 10\).

D. MODIFIED BORN APPROXIMATION FOR A CIRCULAR CYLINDER

If the propagation constant of the trial inside wave function is chosen equal to that of the medium of the scatterer, a "modified" Born approximation to the scattering is obtained. For given changes in the scatterer parameters, this approximation should be valid for higher frequencies than the ordinary Born approximation. It is expected intuitively that as the radius of the scatterer becomes greater than a wavelength, the inside wave function should be more nearly characterized by the propagation constant of the scatterer rather than by the propagation constant of the
exciting wave.

In this case, the inside wave functions are assumed to be

$$
\psi_{\text{III}} = \frac{\rho_{\text{III}}}{\rho_{\text{III}}}, \quad \dot{e}^{i \vec{k}_{\text{III}} \cdot \vec{r}(\vec{k}_{\text{III}})}
$$

$$
\partial_{n_{\text{III}}} = \frac{\vec{k}_{\text{III}}}{k_{\text{III}}} \partial_{n_{\text{III}}} \dot{e}^{i \vec{k}_{\text{III}} \cdot \vec{r}(\vec{k}_{\text{III}})} = i k_{\text{III}} \cos \varphi \dot{e}^{i \vec{k}_{\text{III}} \cdot \vec{r}(\vec{k}_{\text{III}})}
$$

(3.32)

Here again the amplitudes are chosen to satisfy the continuity conditions (in the sense of absolute value) at the scatterer surface. The scattering amplitude is:

$$
A(\vec{k}_{\text{III}}, \vec{k}_{\text{III}}) = w^2 \int_V \epsilon \dot{e}^{i \vec{k}_{\text{III}} \cdot \vec{r}'} d\vec{r}'
$$

$$
+ i k_{\text{III}} \left[ \frac{\rho_{\text{III}}}{\rho_{\text{III}}} - 1 \right] \int_S \epsilon \dot{e}^{i \vec{k}_{\text{III}} \cdot \vec{r}'} \cos \varphi' dS'
$$

(3.33)

where $\vec{K}$ in this case is
\( \vec{K} = \vec{k}_0 \left( \frac{k_{II}}{k_I} \right) - \vec{k}_I \)

\[ |\vec{K}|^2 = k_I^2 + k_{II}^2 - 2k_I k_{II} \cos \phi \]

The volume integral is evaluated exactly the same as in Part C, and is obviously:

\[ \pi w^2 a^2 \left[ \frac{2J_1(\frac{|\vec{K}|a}{|\vec{k}|a})}{|\vec{K}|a} \right] \]

Similarly, after some straightforward manipulation, the surface integral is:

\[ -\pi \left( \frac{\rho_{II}}{\rho_I} - 1 \right) k_I^2 a^2 \left[ \frac{k_{II}}{k_I} - \cos \phi \right] \left[ \frac{2J_1(\frac{|\vec{K}|a}{|\vec{k}|a})}{|\vec{K}|a} \right] \]

Hence the scattering amplitude can be written as:

\[ A(\vec{k}_1, \vec{k}_0) = \pi (k_I a)^2 \left[ \frac{2J_1(\frac{|\vec{K}|a}{|\vec{k}|a})}{|\vec{K}|a} \right] \left[ \frac{w^2}{k_I} - \left( \frac{\rho_{II}}{\rho_I} - 1 \right) \left( \frac{k_{II}}{k_I} - \cos \phi \right) \right] \]

(3.36)
This expression exhibits the same general characteristics as the corresponding expression (3.24) for the ordinary Born approximation, but the important difference is that the high frequency directivity factor is now a function of both \( k_I \) and \( k_{II} \).

It is not convenient in this case to apply the integral equation test for the range of validity. The reason for this is that the inside wave is not

\[
\frac{\rho_I}{\rho_{II}} \in \ii k_0 \cdot \hat{r},
\]

and, hence, the integrals in (3.26) can not be interpreted simply as error terms. In order to establish the usefulness of the solution, however, it is possible to compare (3.36) directly with a rigorous solution computed by series.

Recently, Anderson\(^{(5)}\) has computed and published series solutions to sound scattering from fluid spheres. It is easier to construct a solution for fluid spheres analogous to (3.36) than to compute the series solution for the cylindrical case, particularly for \( k_I a > 1 \). Hence, the modified Born solution for the spherical case is given
in Appendix IV, and in Fig. 4, it is compared with Anderson's result for a sphere of $k_1a = 4.0$, and $k_1/k_{II} = 0.8$.

Even for this relatively large velocity change, the approximate solution predicts the distribution in angle of the pressure level, $p$, to within about 5 decibels except near the minima. Nevertheless, the angle of the minima are predicated fairly well. Further, if the ordinary Born solution for a sphere is compared with Anderson's curve, the agreement is within about 5 decibels for $0^\circ \leq \phi \leq 120^\circ$, and within about 12 decibels for $120^\circ \leq \phi \leq 180^\circ$, but the minima are not predicted very well. Hence it may be concluded that the modified Born approximation represents a considerable improvement over the ordinary Born approximation, and it is reasonable to expect that the modified Born solution would predict fairly adequately the scattered radiation from a circular cylinder under similar conditions.

Recently Hart\(^{(6)}\) has published a rather intricate approximate series solution to scattering from fluid spheres which also compares well with Anderson's exact curve given in Fig. 4. In the author's opinion, however, the methods of solution given in this section (and in Appendix IV) are more satisfactory
because they are easier to handle and because they stem from approximations which do not obscure the physical significance of the scattering mechanism.

In Section IV, the modified Born solution (3.36) will also be compared with experimental results on scattering from liquid circular cylinders for \( k_I a = 23.5 \).

Equations (3.36) and (3.24) provide a convenient basis upon which to discuss the relative importance of absorption changes relative to velocity changes. The rms pressure is proportional to the absolute magnitude of \( A(k_I, \vec{k}_0) \). Hence the quantity

\[
\frac{\text{Re} \left( \frac{W^2}{k_I} \right)}{|\frac{W^2}{k_I^2}|}
\]

is a measure of the relative magnitude of the contribution to the scattering from velocity changes, compared to both velocity and absorption changes. It is plotted as a function of \( \frac{c}{\Delta c} \)( \( \frac{\omega}{c} \)) in Fig. 5 for three values of \( (\alpha_{II} - \alpha_I) \), given by (in nepers/cm):

\[
\alpha_{II} - \alpha_I = a \omega^2
\]

\[
\alpha_{II} - \alpha_I = a 10^{-1} \omega^2
\]
and

\[ \alpha_{\text{II}} - \alpha_{\text{I}} = \alpha \, 10^{-2} \, \omega^2, \]

where

\[ \alpha = 1.93 \, 10^{-15} \]

is the absorption for carbon disulfide and is the maximum absorption measured to date for liquids.\(^{(19),(20)}\). For example, if \( \Delta c/c = 1/10 \), \( \omega/c \) can be as high as 300 per cm before absorption has an appreciable effect on the scattering amplitude, even for liquids of the highest absorption.

E. OTHER APPROXIMATIONS

Thus far only plane waves have been chosen for the inside wave functions. At high frequencies, however, it is expected that the waves will be refracted as they enter and leave the scattering medium. In addition, it is expected that some type of partial standing wave structure will be set up inside the scatterer, due to the reflections at the boundaries. A wave traveling in the negative \( \vec{k}_0 \) direction, for example, would make the value of the scattered rms pressure non zero at the minimums and would increase the value of the back scattering. Hence better agreement
between exact and approximate theories could be expected.

If the wave at high frequencies is assumed to be a ray, it can readily be shown that the inside function for a circular cylinder can be taken as (with neglect of standing wave effects):

\[ \psi_{\Pi} = \frac{2}{\frac{k_{\Pi}}{k_I} g(\phi) + \frac{\rho_{\Pi}}{\rho_I}} \epsilon^{1} k_{\Pi} r g(\phi) \]

(3.37)

where

\[ g(\phi) = \sqrt{1 - \left(\frac{k_I}{k_{\Pi}}\right)^2 \sin^2 \phi} \]

The complicated dependence on \( \phi \) enters because each ray strikes the surface at different angles, and makes the expression difficult to integrate. Further, the evaluation of an inside function which expresses the partial standing wave in the circular cylinder, appears even more remote. However, if the cross section of the cylinder is taken as a rectangle (see Fig. 6), the complicated dependence on \( \phi \) disappears, and the partial standing wave can readily be calculated. This specialized case is treated in Appendix V.
Recently Levine and Schwinger,\(^{(21),(22),(23)}\). Papas\(^{(24)}\)
and Kohn\(^{(25)}\) have applied variational techniques to the
handling of approximations to the inside function.
Levine and Schwinger have treated the diffraction of
waves through an aperture, Papas has treated the scattering
of electromagnetic waves from a conducting cylinder, and
Kohn has treated certain nuclear collision processes. It
is also possible to develop analogous variational principles
for the scattering of sound from fluid cylinders, and it
has been carried out by the author. However the manipulation
becomes quite involved (there are both surface and volume
integrals to consider where as only one or the other appears
in the applications mentioned above) and the results will
not be given here.
IV. EXPERIMENTAL RESULTS

The principal objective of the experimental work is to determine the usefulness of the approximate scattering solution given by (3.36):

\[ A(\mathbf{k}_1, \mathbf{k}_0) = \pi (k_1 a)^2 \left[ \frac{2 J_1(\sqrt{a} k_1)}{k_1 a} \right] \left\{ \frac{w^2}{k_1^2} \left[ \frac{\rho_{II}}{\rho_I} - 1 \right] \frac{\mathbf{k}_{II}}{k_1} - \cos \varphi \right\} \]

for \( k_1 a \) greater than 1. The experiments reported by Tamarkin on the scattering by liquid cylinders were made for \( k_1 a \approx 30 \). However his measurements were made principally in the forward direction, where considerable interaction between the incident and scattered waves occur. Hence they cannot be conveniently compared with (3.36).

A. EXPERIMENTAL EQUIPMENT

The experiment was conducted on equipment used previously by Fay and Fortier,\(^{(26)}\) and by Pietrasanta,\(^{(27)}\) for the measurement of the transmission of sound through steel and glass plates immersed in water.
The transmitting circuit consisted of a pulse modulated radio frequency transmitter operating at 1.5 megacycles per second, and an x-cut quartz transducer. The receiving circuit consisted of another x-cut quartz transducer, a calibrated attenuator, a tuned amplifier, and an A-R cathode ray oscilloscope. The pulse length in water was greater than 4 cm, and the pulse contained more than 40 cycles, so that essentially steady-state conditions prevailed at the center of the pulse.

Both transducers were $1\frac{1}{4}$ cm in diameter and were $\frac{1}{2}$ wavelength thick. The sound was beamed at the liquid cylinder which was suspended about 30 cm from the transmitter, and the scattered sound was measured at a radius of 24 cm from the cylinder center, as a function of the angle $\varphi$.

The distribution in angle of the incident sound measured as a function of $\varphi$ on the 24 cm measurement circle is presented in Fig. 7. The radius of the cylinder was 3/8 cm; hence the angle subtended by the cylinder at the source was 0.72 degrees, corresponding to $\varphi \approx 1.5$ degrees. According to Fig. 7, the amplitude of the incident wave did not change measurably between $\varphi = 0$ and $\varphi = 1.5$ degrees. The criterion for beginning of the far-field
radiation of a piston is $a^2/wavelength$, and is exceeded in this case by a factor of 13. Hence it may be concluded that the incident field at the cylinder is essentially a plane wave over the cross-sectional dimensions of the cylinder.

The cylinder shell was constructed of polyethylene sheet, 0.0038 cm thick. At 1.5 megacycles the wavelength in polyethylene is the order of 0.2 cm. Hence the shell will not have an appreciable effect upon the scattering. This was verified by placing water inside the cylinder and measuring the total field; the total field was within 1 decibel of the incident field for all measurement angles. The length of the cylinder was about 15 cm, or about three times the pulse length in water. Hence the cylinder may be considered essentially infinite in length.

The reading at $\varphi = 0$, with the cylinder removed, established a reference incident pressure level prior to each test. The origin of the angle coordinate, $\varphi$, was determined by noting the center of symmetry of the scattered field. In order to determine the pressure level of the scattered field for each measurement angle, the signal on the A-R oscilloscope was brought to a standard height by means of
the calibrated attenuator.

The total measurement error is estimated to be \( \pm 1 \) decibel in pressure level, and \( \pm \frac{1}{2} \) degree in angle for any one test. The error in the pressure level is caused principally by the output variations of the transmitter, and the error in the angle is due principally to the difficulty in determining the center of symmetry.

B. OBSERVED RESULTS

Most of the measurements were made on cylinders immersed in water containing mixtures of tertiary butyl alcohol and water. Recently Burton\(^{28}\) and Langmuir\(^{29}\) have measured independently the absorption and velocity of tertiary butyl alcohol (tertiary butanol) water mixtures. Their results are in good agreement, and are presented in Fig. 8 as ratios of the absorption and the velocity of the mixture, to that of the water. The peak in the absorption curve corresponds to

\[
\alpha_{II} = 0.96 \times 10^{-15} \omega^2
\]

which is one-half of the maximum absorption measured in a liquid
to date (see Section III, Part D). The maximum velocity change is the order of 25 percent (for pure tertiary butanol). The ratio of the densities, $\rho_{II}/\rho_I$, is linearly related to the percentage of the mixture, and is 1.00 at 0 percent tertiary butanol and is 0.79 at 100 percent tertiary butanol. Hence the choice of tertiary butanol-water mixture as the liquid scatterer affords a wide range of easily adjusted variations in $C_{II}$, $\rho_{II}$, and $\alpha_{II}$.

One experiment was performed using mercury as the cylinder fluid. For mercury, $C_{II}/C_I = 0.968$, $\rho_{II}/\rho_I = 13.59$, and the absorption may be neglected.

Measurements were made for tertiary butanol-water mixtures of 100, 75, 50, 40, 35, 25, 20, and 0 percent. The observed results for the 100, 75, 40, and 35 percent mixtures are presented in Figs. 9, 10, 11, and 12. The curves for 50, 25, and 20 percent are very similar to the curves for 40 and 35 percent, and are not presented. The curve for 0 percent is the curve of the incident radiation to within 1 decibel, and is also not presented. The measurements on mercury are shown in Figure 13. The total field (scattered plus incident field) is plotted as pressure level in decibels; the reference level was taken as the incident pressure level.
at $\varphi = 0^\circ$

The $k_1$ for these measurements is equal to 23.5. It is to be expected that at this relatively high frequency, most of the scattering will occur at relatively small angles. Readings were taken, however, throughout the entire angular scale, but only the reading up to $\varphi = 50^\circ$ degrees are presented. For almost all cases, the readings for $\varphi$ greater than $50^\circ$ were obscured by interaction with side lobes of the incident radiation. (The value of the side lobes were the same order of magnitude as the scattered field). In addition, the total field from the "weak" scatterers were also predominantly incident radiation, even in the range $0^\circ \leq \varphi \leq 50^\circ$. The total field for $15^\circ \leq \varphi \leq 50^\circ$ may be taken as the scattered field only if the total field is 10 decibels greater than the incident field given in Fig. 7.

For example, the pressure level measured for 100 and 75 percent tertiary butanol (given in Figs. 9 and 10) represent the scattered field for $\varphi$ greater than about $13^\circ$, because the total field is the order of 10 decibels greater than the incident field. Hence these curves should lend themselves to comparison with theory. On the other hand, for 40 and 35 percent
tertiary butanol, and for mercury, (Figs. 11, 12, 13) considerable interaction occurred between the incident and scattered fields (the incident field is shown again with a dash line in Fig. 11), and interpretation of these curves in the light of the theory is difficult.

C. COMPARISON OF THE MODIFIED BORN THEORY WITH EXPERIMENTAL RESULTS

The modified Born theory may be compared with experiment either by comparing the distribution in angle of the scattered sound, or by comparing the total cross-section, or by both.

It was impossible to obtain the total cross section because the measurements in the forward direction were obscured by the incident radiation. (At high frequencies, the forward scattering contributes considerably to the cross section). The distribution in angle however, is a more delicate method of comparison, but here too, the forward direction cannot be considered. Therefore, the comparison will be made between the measured and predicted distribution in angle, over a restricted range of angle, $13^\circ \leq \varphi \leq 50^\circ$. 
1. Comparison of the Angles of the Minima

The zeros of the directivity factor,

\[ \frac{2 J_1(|K|a)}{|K|a} \]  \hspace{1cm} (4.1)

for cylinders of 100 and 75 percent tertiary butanol are indicated by arrows on Fig. 9 and Fig. 10. The theory agrees with the measured minima with an error of less than 1°. The number associated with each arrow indicates the \( n \)th zero of (4.1); the angle, \( \varphi_n \), of the \( n \)th zero is determined by:

\[ k_{II} a \left[ 1 + \left( \frac{k_{II}}{k_I} \right)^2 - 2 \left( \frac{k_{II}}{k_I} \right)^2 \cos \varphi_n \right]^{\frac{1}{2}} \leq \pi \left( n + \frac{1}{4} - \frac{0.15}{4n+1} \right) \]  \hspace{1cm} (4.2)

It is interesting to note that the first solution for (4.2) may not be \( n = 1 \), but rather some higher value of \( n \). In order to see this, (4.2) can be rewritten as:

\[ \cos \varphi_n = \frac{1 + \left( \frac{k_{II}}{k_I} \right)^2 - \left( \frac{\pi}{k_I a} \right)^2 \left[ n + \frac{1}{4} - \frac{0.15}{4n+1} \right]^2}{2 \frac{k_{II}}{k_I}} \]  \hspace{1cm} (4.3)
For a small enough value of \( n \), it may be possible that the right hand side of (4.3) is greater than 1, since,

\[
1 + \left( \frac{k_{II}}{k_I} \right)^2 > 2 \frac{k_{II}}{k_I}
\]

for all values of \( k_{II}/k_I \) (not equal to 1). The value of \( n \) for the first zero may be determined from (4.3) by finding the lowest integral value of \( n \) necessary to make the right hand side less or equal to 1. For example, the first zero of (4.1) for 100 percent tertiary butanol \( (k_{II}/k_I = 1.34) \) occurs for \( n = 3 \). Similarly, the first zero for 75 percent tertiary butanol \( (k_{II}/k_I = 1.17) \) occurs for \( n = 2 \).

In addition, the magnitude of (4.1) is always less than 1, because \(|k| \) cannot be zero (except for \( k_{II}/k_I = 1 \)).

2. Comparison of Magnitude

In order to compare measurement and theory, it is necessary that the theoretical pressure level curve be computed using the same reference level as was used for the measured levels. The theoretical pressure level is given by:
\[ 20 \log_{10} \left| \frac{A(\mathbf{r}_1, \mathbf{r}_0)}{A(0, \mathbf{0})} \right| \] - 40.7 decibels \hspace{1cm} (4.4)

The 40.7 decibel correction converts the reference of unit amplitude incident velocity potential at the cylinder to the reference of incident pressure determined at \( \varphi = 0 \) and \( r = 24 \text{ cm} \) for the measurements.

Equation (4.4) has been computed for the case of the cylinder containing 75 percent tertiary butanol, and the measured and theoretical curves are shown together in Fig. 14. The qualitative agreement is very good in the angular region of \( 13^\circ \leq \varphi \leq 50^\circ \). The measured values fall below the predicted values at the maxima, and above the predicted values at the minima. In view of the crude approximation involved, however, the agreement is remarkable.

The discrepancies may be understood by considering two factors. (a), the finite size of the measuring probe, and (b), the type of approximation involved.

(a) The finite measuring probe subtends a double angle of \( 3^\circ \) at the cylinder. Hence, to a first approximation, the measured maxima are reduced relative to the actual values and the measured minima are increased relative to the actual values, because of averaging over the probe dimensions.
(b) The approximation made in obtaining (3.36) neglected standing wave effects and refraction in the cylinder. The inclusion of a wave traveling in the negative $\vec{k}_0$ direction, of reduced amplitude, will make the value of the potential non zero at the minima. The vector $\vec{K}'$ for this type wave is given by:

$$\vec{K}' = -\vec{k}_0(k_{II}/k_I) - \vec{k}_I$$

$$|\vec{K}'|^2 = k_1^2 \left[ 1 + \left( \frac{k_{II}}{k_I} \right)^2 + \frac{k_{II}}{k_I} \cos \varphi \right]$$

(4.5)

Therefore, by comparing (4.5) with (4.2), it is obvious that the zeros of $2 J_1(|\vec{k}|a) / |\vec{k}| a$ and $2 J_1(|\vec{k}'|a) / |\vec{k}'|a$ will not, in general, be coincident; at a zero of one, the other will have a value. The neglect of the effect of refraction does not appear to have any simple interpretation.

3. Relative Importance of Velocity, Density and Absorption

Changes Upon Scattering

A 75 percent mixture of tertiary butanol has a velocity ratio, $C_{II}/C_I$, equal to 0.86, a negligible absorption ratio, and a density ratio, $\rho_{II}/\rho_I$, equal to 0.84. Hence the monopole and dipole coefficients are:

$$\frac{W^2}{k_I^2} = 0.37$$

$$\frac{\rho_{II}}{\rho_I} - 1 = 0.16$$

(4.6)
On the other hand, for the 40 percent mixture:

\[
\frac{c_{\text{II}}}{c_1} = 1
\]

\[
\frac{a_{\text{II}}}{a_1} = 114
\]

\[
\frac{\rho_{\text{II}}}{\rho_1} = 0.92
\]

The corresponding quantities to (4.6) are:

\[
\frac{W^2}{k_1} - 2i \frac{c_1 a_{\text{II}}}{\omega} = 1.71 \times 10^{-3}
\]

(4.7)

\[
\frac{\rho_{\text{II}}}{\rho_1} - 1 = 8 \times 10^{-2}
\]

On the basis of (4.6) and (4.7), it may be predicted that the scattering amplitude for a 40 percent mixture is at least an order of magnitude, (20 decibels), less than for a 75 percent mixture. The measurement conducted on a 40 percent tertiary butanol mixture (see Fig. 11) verifies this conclusion. A similar conclusion can be reached for the 35 percent mixture, even though the absorption ratio is at
its peak. Hence it may be concluded, on the basis of both experiment and theory, that at this frequency the scattering from a large change in absorption is several orders of magnitude less than scattering from relatively small changes in velocity or density.

Finally, for the case of mercury:

\[
\frac{W^2}{k_1^2} = 0.07
\]

\[\frac{\rho_{\text{II}}}{\rho_{\text{I}}} - 1 = 12.59\]  

(4.8)

On the basis of (4.8), the modified Born theory predicts a large dipole scattering amplitude. This has not been verified by experiment (see Fig. 13). The disagreement is not surprising; the mercury cylinder represents a substantial discontinuity in the water medium. It may be concluded that the modified Born theory does not adequately describe scattering from liquid cylinders with changes as large as this.
V. CONCLUSION

The scattering of plane waves of sound from liquid cylinders has been treated from both a theoretical and experimental viewpoint.

A rigorous series solution has been obtained for the general case of complex propagation constants. The solution depended upon the fact that for most cases of practical interest, the absorption wave number can be considered small. At low frequencies, the physical significance of the monopole and dipole terms were discussed, and the relative importance of changes in compressibility, density, and absorption were treated in limiting cases. It was found that for zero absorption in both media, the monopole term depended only on compressibility changes, and the dipole term depended only on density changes.

The rigorous series solution was supplanted by two approximate solutions, which agree essentially with the series solution at low frequencies, but which may be used at higher frequencies without obscuring the physical significance of the scattering expression. The approximations followed from an integral equation formulation which has general
usefulness in other coordinate systems. At the higher frequencies, each of the two approximations may be interpreted as the low frequency solution multiplied by a directivity factor. The range of validity of one the approximations, the Born solution, has been found in terms of the relative media changes in the cylinder. The usefulness of the other approximation, the modified Born solution, has been established by comparison with a rigorous solution for the corresponding problem of a fluid sphere, and by comparison with experiment. The modified Born solution has been shown to describe the minima of the scattered radiation very closely, and the magnitude qualitatively, in a restricted range of observation for relatively high frequencies.

It has been determined that relatively large absorption changes are not important compared to relatively small velocity or density changes, and that for all practical purposes, the absorption may be neglected in describing the scattered field.
APPENDIX I

Absorption may be introduced into the time independent scalar wave equation by considering the dissipationless propagation constant, $\omega/c$, to be modified by the addition of an imaginary term, $i\alpha$. For most liquids in which wave motion is to be considered, the imaginary part of the propagation constant can be considered either small compared to the real part, or small compared to $1/a$, (where $a$ is the radius of the cylinder). To show this, consider the limiting case of carbon disulfide, which has the greatest absorption measured in a liquid to date:\textsuperscript{(19),(20)}.

$$\alpha = 1.93 \times 10^{-15} \omega^2 \text{ nepers/cm}$$

$$c = 1.449 \times 10^5 \text{ cm/sec} \quad (1.1)$$

The ratio of the imaginary part, $\alpha$, to the real part $\omega/c$, is

$$\frac{\alpha c}{\omega} = 1.93 \times 10^{-15} \omega c \quad (1.2)$$

The product of $\alpha$ and the cylinder radius is:

$$\alpha a = 1.93 \times 10^{-15} \omega c(ka) \quad (1.3)$$
Hence, if the frequency is less than about $10^7$ cps, and if $ka$ is not larger than about 10, the squares of (1.2) and (1.3) are both small compared to unity, and the Taylor series approximation given in Section II is valid. However, many liquids have absorption orders of magnitude less than carbon disulfide, so that the maximum frequency for which the Taylor series approximation is useful is usually much greater. On the other hand, the method of series solution is not readily applied for $ka$ greater than 10, and a practical lower limit of the cylinder radius $\frac{1}{2}$ mm corresponds to $10^7$ cps. Hence it may be concluded that the Taylor series approximation is valid for virtually all cases of practical interest.
APPENDIX II

To show that \( \frac{y^2}{2} \frac{J''_m(x)}{J_m(x)} \) is much smaller than 1,

consider three cases: (1) \( x \to 0 \), (2) \( x \sim m \), (3) \( x \to \infty \).

The differential equation satisfied by \( J_m(x) \) is:

\[
J''_m(x) + \frac{1}{x} J'_m(x) + \left[ 1 - \frac{m^2}{x^2} \right] J_m(x) = 0.
\]

Hence

\[
\frac{J''_m(x)}{J_m(x)} = \left[ \frac{m^2}{x^2} - 1 \right] + \frac{1}{x^2} \tan \alpha_m(x).
\]

In the first case \( x \to 0 \):

\[
\tan \alpha_m \to -m \quad m > 0
\]

\[
\tan \alpha_0 \to \frac{x^2}{2}.
\]

Hence

\[
\frac{y^2}{2} \frac{J''_m}{J_m} = \frac{y^2}{x^2} \frac{m(m-1)}{2}, \quad m > 0,
\]

\[
\frac{y^2}{2} \frac{J''_0}{J_0} = -\frac{y^2}{4}.
\]
In the second case ($x \sim m$):

$$\tan a_m \approx \left[ .2 - .92 m^{2/3} \right].$$

Hence:

$$\frac{\frac{y^2}{2}}{\frac{J_m''}{J_m}} \approx \frac{\frac{y^2}{x^2}}{\frac{.1 - .46 m^{2/3}}{x^2}}.$$ 

In the third case ($x \to \infty$):

$$\frac{\frac{y^2}{2}}{\frac{J_m''}{J_m}} \approx \frac{\frac{y^2}{2}}{2}.$$ 

In all three cases, the ratio \( \left| \frac{\frac{y^2}{2} - \frac{J_m''}{J_m}}{\frac{y^2}{2}} \right| \) is much less than 1 because

$$\frac{\frac{y^2}{x^2}}{\frac{y^2}{x^2}} \ll 1$$

and

$$\frac{y^2}{y^2} \ll 1.$$ 

Similar results are easily obtained for the ratio of the third term in the Taylor expansion to the first term, and for \( N_m(x + iy) \), and hence the Taylor series approximation is a valid representation.
APPENDIX III

The scattered intensity can be written as:

\[ I_s = \frac{\omega \rho_I}{2} \text{Im} \left( \psi_s^* \frac{\partial \psi_s}{\partial n} \right) , \]  

(3.1)

where \( \psi_s^* \) is the complex conjugate of \( \psi_s \).

Hence the total scattered power passing through a closed surface, \( S \), is:

\[ P_s = \frac{\omega \rho_I}{2} \text{Im} \int_S \psi_s^* \frac{\partial \psi_s}{\partial n} \, dS. \]  

(3.2)

The derivative is taken as the outward normal to \( S \).

The incident plane wave intensity is

\[ I_{in} = \frac{k_I \rho_{I\omega}}{2} . \]

Hence the scattering cross section, \( Q_s \), is:

\[ Q_s = \frac{1}{k_I} \frac{1}{2} \text{Im} \int_S \psi_s^* \frac{\partial \psi_s}{\partial n} \, dS . \]  

(3.3)
The scattered field, $\psi_s$, can be written as the difference between the total and incident fields:

$$ \psi_s = \psi_I - e^{i k_o \cdot \hat{r}}. $$

Then (3.3) becomes:

$$ Q_s = \frac{1}{k_I} \text{Im} \left\{ \int_s \psi_I^* \frac{\partial \psi_I}{\partial n} \, dS - \int_s e^{-i k_o \cdot \hat{r}} \frac{\partial \psi_I}{\partial n} \, dS \right. $$

$$ - \left. \int_s \psi_I^* \frac{\partial \varepsilon}{\partial n} e^{i k_o \cdot \hat{r}} \, dS \right\}. $$

If there is no absorption in the external medium, the power leaving the scatterer surface is equal to the power passing through $S$. Hence $S$ can be taken as the scatterer surface, and the continuity relations applied:

$$ Q_s = \frac{1}{k_I} \text{Im} \left\{ \int_s \psi_I^* \frac{\partial \psi_I}{\partial n_{II}} \, dS - \int_s e^{-i k_o \cdot \hat{r}} \frac{\partial \psi_{II}}{\partial n_{II}} \, dS \right. $$

$$ - \left. \frac{\rho_{II}}{\rho_I} \int_s \psi_{II}^* \frac{\partial \varepsilon}{\partial n_{II}} e^{i k_o \cdot \hat{r}} \, dS \right\} \quad (3.4) $$
The scattering amplitude in the forward direction is:

\[
A(\vec{k}_o, \vec{k}_o) = \frac{\rho_{II}}{\rho_I} \int_V w^2 \psi_{II} \epsilon^{-i\vec{k}_o \cdot \vec{r}} \, dv
\]

\[+ \left( \frac{\rho_{II}}{\rho_I} - 1 \right) \int_s \frac{\partial \psi_{II}}{\partial n_{II}} \epsilon^{-i\vec{k}_o \cdot \vec{r}} \, ds. \quad (3.5)\]

If the integral equation for \(\psi_{II}\) valid for \(\vec{r}\) in I, is specialized by taking the measurement direction as \(\vec{k}_o\), and is then combined with (3.5), the result is:

\[
A(\vec{k}_o, \vec{k}_o) = \frac{\rho_{II}}{\rho_I} \int_s \frac{\partial \epsilon^{-i\vec{k}_o \cdot \vec{r}}}{\partial n_{II}} \psi_{II} \, ds
\]

\[-\int_s \frac{\partial \psi_{II}}{\partial n_{II}} \epsilon^{-i\vec{k}_o \cdot \vec{r}} \, ds. \quad (3.6)\]

Hence (3.4) becomes:

\[
Q_s = \frac{1}{k_I} \text{Im} \left[ \int_s \psi_{II}^* \frac{\partial \psi_{II}}{\partial n_{II}} \, ds + A(\vec{k}_o, \vec{k}_o) \right] \quad (3.7)
\]
where use has been made of the fact that:

\[ \text{Im} \int_{s} \psi_{II}^{*} \frac{\partial}{\partial n_{II}} \left( \hat{k}_{o} \cdot \hat{r} \right) \, ds = - \text{Im} \int_{s} \psi_{II} \frac{\partial}{\partial n_{II}} \left( -\hat{k}_{o} \cdot \hat{r} \right) \, ds. \]

The power lost to the cylinder by absorption is:

\[ - \frac{\omega P_{I}}{2} \text{Im} \int_{s} \psi_{I}^{*} \frac{\partial \psi_{I}}{\partial n_{II}} \, ds. \]

Hence the absorption cross section is:

\[ Q_{a} = - \frac{1}{k_{I}} \frac{1}{\omega} \text{Im} \int_{s} \psi_{I}^{*} \frac{\partial \psi_{I}}{\partial n_{II}} \, ds. \]

The total cross section, \( Q_{t} \), is (from 3.7):

\[ Q_{t} = Q_{s} + Q_{a} = \frac{1}{k_{I}} \frac{1}{\omega} \text{Im} \left[ A(k_{o}^{2}, k_{o}^{2}) \right]. \]
APPENDIX IV

In three dimensions \((r, \theta, \varphi\), coordinates\), the total wave is:

\[
\psi_i = \epsilon \frac{\widehat{1}_0 \cdot \widehat{r}}{k_0 r} + \epsilon \frac{\widehat{1}_I \cdot \widehat{r}}{4\pi r} B(\widehat{k}_1, \widehat{k}_0). \tag{4.1}
\]

If the densities are equal, the scattering amplitude, \(B(\widehat{k}_1, \widehat{k}_0)\) for the modified Born inside function, is:

\[
B(\widehat{k}_1, \widehat{k}_0) = w^2 \int \nu \epsilon \frac{k_{II}}{k_I} \left( \frac{\widehat{k}_1}{k_0} - \frac{\widehat{k}_1}{k_I} \right) \cdot \widehat{r} \, d\nu, \tag{4.2}
\]

(The extension to the more general case of unequal density follows closely the development in Section III). Let

\[
\overrightarrow{K} = k_0 \frac{k_{II}}{k_I} - \frac{\widehat{k}_1}{k_1}.
\]

Then, if the direction of \(K\) is taken as the polar axis, the integral over the volume of the sphere is:

\[
B(k_1, k_0) = w^2 \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^a r^2 \sin \varphi \, dr \epsilon \frac{\overrightarrow{K} \cdot \widehat{r} \cos \varphi}{r}.\]
The integral over $\Theta$ is $2\pi$. The integral over $\varphi$ is also easy and is:

$$
\int_0^\pi \left[ \frac{1}{r} \frac{\cos \varphi}{\sin \varphi} \right] \sin \varphi \, d\varphi = \frac{2 \sin (|\vec{k}|r)}{|\vec{k}|(r)}.
$$

The $r$ integration follows simply and yields:

$$
B(\vec{k}_1, \vec{k}_0) = \frac{4\pi a^3 w^2}{|\vec{k}|^2} \left[ \frac{\sin |\vec{k}|a}{|\vec{k}|a} - \cos |\vec{k}|a \right], \quad (4.3)
$$

or equivalently by:

$$
B(\vec{k}_1, \vec{k}_0) = (2\pi a^2)^{3/2} w^2 \frac{J_{3/2} (|\vec{k}|a)}{(|\vec{k}|a)^{3/2}}, \quad (4.4)
$$

The analogy between this expression and the corresponding expression for the cylinder is obvious. (4.4) was used to calculate the scattered field shown in Fig. 4, for $k_1a = 4$.  

APPENDIX V

The coordinate system for the rectangular scatterer is shown in Fig. 6. For $k_0$ large, it is expected that the inside function can be given as:

$$\varphi_{II} = A_o e^{ik_{II}^r \cos \varphi} + B_o e^{-ik_{II}^r \cos \varphi}$$

(5.1)

$A_o$ represents the amplitude of the wave traveling in the $\vec{k}_o$ direction, $B_o$ represents the amplitude of the wave traveling in the negative $\vec{k}_o$ direction. They may be found by standard transmission line techniques. For simplicity, take $\rho_I = \rho_{II}$; the essential feature of the solution will be evident in spite of this specialization. The scattering amplitude is:

$$A(\vec{r}_1, \vec{r}_0) = W^2 \int \left[ A_o e^{ik_{II}^x} + B_o e^{-ik_{II}^x} \right] e^{ik_{II}^x \cos \varphi}$$

$$\cdot e^{-ik_0 y \sin \varphi} \, dV$$

(5.2)

where

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$
The integrals involved in (5.2) are elementary, and the scattering amplitude becomes:

$$A(\vec{k}_1, \vec{k}_0) = 4 \text{abw}^2 \sin \gamma \left[ A_0 \frac{\sin \tau}{\tau} + B_0 \frac{\sin \beta}{\beta} \right]$$

(5.3)

where

$$\gamma = k_1 a \sin \varphi$$

$$\tau = k_1 b \left( \frac{k_{\text{II}}}{k_1} - \cos \varphi \right)$$

$$\beta = k_1 b \left( \frac{k_{\text{II}}}{k_1} + \cos \varphi \right).$$

In (5.3), it is easy to see the importance of the negative traveling wave in the inside function. $\tau$ is important near $\varphi$ equal to zero, and is associated with the positive traveling wave. However $\beta$ is important near $\varphi$ equal to $\pi$ and is associated with the negative traveling wave. Hence, omission of the negative traveling wave can lead to error in the back scattering. It is therefore reasonable to expect that neglect of the negative traveling wave in the case of the circular cylinder will also lead to error in the back scattering.
Figure 1
Figure 2
Figure 5
Figure 6


(10) "Tables of the Bessel Functions J_0(z) and J_1(z) for Complex Arguments", 2nd Ed. (1949) and "Tables of the Bessel Functions Y_0(z) and Y_1(z) for Complex Arguments" (1950), New York, Columbia University Press.

(11) P. Morse, op. cit., p. 350.


(16) P. Morse and H. Feshbach, op. cit., p. 151.

(17) P. Morse and H. Feshbach, op. cit., p. 349.


