ELECTROMAGNETIC WAVE PROPAGATION

ON HELICAL CONDUCTORS

by

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Electromagnetic Wave Propagation on Helical Conductors

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Abstract

The results of a theoretical investigation of the properties of the natural waves or "free modes" which propagate along infinite helical conductors are reported here. After a brief review of the background of the problem, the sheath model which replaces the helix by an anisotropic cylindrical sheet is considered. The higher modes, in particular, are investigated, and it is found that several waves per mode can exist. The significance of these waves in terms of inward and outward traveling waves from a source is shown. The manner in which the characteristics of these waves change as the parameters of the system are altered is discussed in detail.

A more physically realistic model of the helix than the sheath model is analyzed. The better model consists of a helix wound of a tape of infinitesimal thickness. In this case an exact formal solution can be obtained through the use of characteristic function expansions, but it is impractical or impossible to use this solution. However, the exact formulation shows the existence of bands where "free mode" waves are not permitted. In order to obtain useful results the extreme cases of a single wire helix wound with a very narrow tape and a very wide tape for which reasonably valid approximations can be made are analyzed, and solutions for these are obtained. The usual low frequency behavior predicted by the sheath model and the anomalous behavior of the propagation constant in the region where the circumference of the helix is approximately equal to a wavelength result from these solutions. Other properties of the helix are also derived. The problem of multiwire helices is considered, and the manner in which the sheath helix model is obtained as the number of wires becomes infinite is shown.

The integral formulation of the small wire helix problem is shown to yield results essentially identical to those obtained by means of the methods and approximations indicated above. This formulation allows a solution for the infinite driven helix to be obtained, and although this is not completely evaluated, the "free mode" portion is extracted.
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CHAPTER I

INTRODUCTION

After a brief description of various aspects of the problem, the theoretical and experimental results which have been presented elsewhere are reviewed. (Sections I.1, I.2, I.3, I.4) The purpose and results of the present investigation are then discussed, and the manner in which these results are related to those presented elsewhere is pointed out. (Section I.5)

Review of Previous Theoretical Results

I.1 The Helix Problem

Although helical structures have long been used in various ways as electromagnetic devices, their recent use in traveling-wave tubes and in antenna structures has stimulated a renewed interest in their properties, particularly at frequencies higher than those normally used in the past. As a result of this interest, numerous papers and reports have appeared describing the various efforts made to analyze and to measure the electromagnetic properties of helical structures. Before considering the relevant history of the problem, it is advisable to describe its nature.

As in the case of other structures which can support electromagnetic waves, one desires to know the characteristics of those solutions of Maxwell's equations which match the boundary conditions prescribed by the helix. In particular, because of the cylindrical nature
of the helix, the waves which propagate with only exponential dependence on the axial coordinate are of considerable interest. These waves with exponential dependence, or natural waves, or "free modes" as they are called throughout this report, are of fundamental concern since, if the axial dependence is like $e^{-jhz}$ with $z$ the axial coordinate and $h$ the propagation constant, for real values of the propagation constant such waves persist at arbitrarily large distances from any source. The steady state solutions are the only ones considered here with harmonic time dependence $e^{j\omega t}$. The values of $h$ as a function of the frequency and the various electrical and physical properties of the system are of primary interest. The term "free modes" is applied only to those solutions of the field equations which correspond to physically realizable situations, that is, to those waves which are associated with finite total power.

The helical system can take many forms. The one towards which the most interest has been confined in the past is the one consisting of a helix of infinite extent assumed to exist in an infinite unbounded uniform medium. The problem then becomes one of finding those solutions which satisfy the homogeneous field equations and the boundary conditions in the absence of sources in any finite portion of space, the sources being assumed to be located at infinity. Related forms which have been considered in the past are multiwire helices, helices surrounded by cylindrical shields or mediums of different electrical properties, and others.

The problem of the driven helix, both finite and infinite, has received much less theoretical treatment than has been directed towards the source-free helix problem. As in other transmission systems, the solution of the driven or source-present problem is more difficult than that of the source-free problem. The solution of the driven helix problem can, however, yield important information concerning the impedance
and radiation properties of the helix. In the case of the driven helix where the helix is of infinite length, the "free mode" portion of the solution may be of most interest, although the manner in which the helix modifies the radiation field of the source is also of some concern. In the case of the finite driven helix, the radiation and impedance properties are of primary interest, although an analysis of the role played by the "free modes" in determining these properties may also be important.

1.2 Early Work

The various analyses of the helix which have been presented can be classified, in general, into three separate categories. These are the thin wire approximation, the exact approach, and the sheath model approximation. In the thin wire approximation it is assumed that the current and electric field in the direction of the wire are of dominant importance, and the approximations are carried out accordingly. In the exact approach Maxwell's equations are written in terms of a coordinate system describing the surface of the helix. Although this is certainly an exact procedure, the intractability of the resulting set of partial differential equations has so far required that approximations be made to obtain solutions. Finally, the sheath model approximation replaces the wire or wires of which the helix is wound by an anisotropic current conducting sheet or sheath. Although this allows the Maxwell equations to be solved, the validity of the model is subject to question.

The earliest work on the helix appears to have been that done by Pocklington (6). In a paper, which must be considered a model of succinctness, the thin wire approximation was used, and an integral equation for the "free modes" was derived. The helix wire was assumed to be a perfect conductor. Approximate solutions were obtained which predicted a traveling wave whose axial phase velocity is near the
velocity of light $c$, assuming the helix is immersed in free space, for low frequencies and whose axial phase velocity is reduced to $c \sin \psi$, where $\psi$ is the pitch angle of the helix, for high frequencies. This latter is equivalent to a wave with a phase velocity $c$ traveling along the wire. Although these solutions are more or less correct in view of present knowledge, the fine structure of the solutions was completely missed, and the approximate methods used for the evaluation of the integral must be considered not entirely satisfactory.

The next pertinent work seems to have been that of Nicholson (20) who used, what has been called above, the exact approach. Employing a set of coordinates which describes the surface of a helix wound of a round wire, he derived the form of Maxwell's equations in these coordinates. Since these are not solvable by known methods, he was forced to approximate and only obtained what might be called a low frequency solution. His results are of doubtful validity, however, since he assumed the coordinate system to be orthogonal, which it is not as was shown by Sollfrey (16) and Bagby (31). Further, his approach was such that the phase velocity was assumed rather than determined.

The first investigation using the sheath model approximation was apparently undertaken by Ollendorf (7). He obtained the solution for what is called the zeroth mode, or a solution in which the fields have no angular variation. Although his results are not quite in the convenient form given by most recent investigators, they are essentially equivalent to these as Kornhauser (13) has shown.

I.3 Recent Work

In order to simplify the presentation the recent work will be reviewed in the classifications given in the previous section although not in the chronological order noted there.
(a) The Sheath Model Approximation

Rudenberg (14) gave an analysis using a sheath model in which both the electric and magnetic fields along the "windings" were assumed to be zero. He took a plane sheet model, that is, a developed sheath helix, rather than one in cylindrical form and obtained a solution essentially identical to the one given by the Ollendorf sheath model, at least at high frequencies. Brillouin (12) extended this work and showed that the assumption of these boundary conditions leads to a low frequency cut-off property as well as to a complete isolation of the fields inside and outside the cylindrical sheath. Phillips (18) has also investigated this model in great detail. However, Kornhauser (13) has shown that the sheath model with these boundary conditions is unacceptable. Among its shortcomings is a very considerable disagreement with experimental data. Incidentally, Kornhauser (13) has given an excellent summary of much of the work on the helix which had been published as of the early part of 1949.

In a paper by Pierce (8) the results of some analysis made by Schelkunoff on a cylindrical sheath model were revealed. This analysis used the boundary condition that only the tangential electric field was required to be zero in the direction of the "windings". Only the zeroth mode was considered, and the results were put in a form convenient for application to traveling-wave tube design. Similar data in slightly different form have been given by Chu and Jackson (10) and Schulman and Heagy (11), to mention only a few.

Kornhauser (13) analyzed a sheath model made up of two plane parallel infinite anisotropic current carrying sheets. Using boundary conditions equivalent to those of the Schelkunoff (8) model, he obtained similar results. He also showed how such a model is the limit of a somewhat more physically satisfactory configuration and determined
approximately the effect of wire size on the propagation constant. Pierce (9) also analyzed a plane sheath model but used only one sheet. Although not properly part of a discussion on sheath models, it seems well to mention that Pierce (9), using a method attributed to R. S. Julian, has determined approximately the effect of finite wire size on the fields adjacent to a helix.

Other work which has been done which has made use of the sheath helix model operating in the zeroth mode, but with the helix surrounded by either different mediums, by a metallic shield, or by other helices should be noted. Harris and others (23) have considered the problem of a sheath helix in which the permittivity of the medium for $r < a$ is different from that of the medium for $r > a$, where $a$ is the radius of the cylindrical sheath. From their data it appears that the only effect of this change is to alter the magnitude of the axial phase velocity and the frequency range over which it is essentially constant. Jones (63) has analyzed the properties of a system in which the permittivity of the medium for $r > a$ is stratified. This arrangement results in a structure with negative dispersion, but except for this change the results are not too different from those for a sheath helix immersed in an unbounded uniform medium. Haus (62) has briefly considered the problem of concentric sheath helices analyzed from a field viewpoint, but no results are given. Pierce (9) analyzed such a structure from a circuit viewpoint and indicated that the results obtained in this manner are in agreement with those obtained through a field analysis. Lund (64) has described some results obtained for a sheath helix surrounded by a cylindrical conducting shield and quotes other references on this structure. The general character of the solution for the phase velocity in this case is very much like that which results for the unshielded sheath helix.
I.3(b)

The higher mode solutions of the sheath helix — those in which the field components vary in angle — have been examined only by a few investigators. The most complete analysis so far is that given by Phillips and Malin (19, 65). The presence of several waves per mode and cut-off characteristics were determined, and the manner in which these change with the different parameters was partially investigated. Schulman and Heagy (11) have also reported some analysis of the higher modes.

(b) The **Exact Approach**

Using a coordinate system identical to that used by Nicholson (20), Sollfrey (16) obtained the correct form of the Maxwell equations in this system and showed the possibility of exponential solutions. However, in solving the resulting set of partial differential equations by an approximation procedure, he obtained only the solution corresponding to a wave traveling along the helix wire itself with the velocity of light. Using a helical coordinate system, Parzen (15) has considered the helix problem, and, as reported by Kornhauser (13), he obtained results similar to those given for the sheath model.

The most comprehensive investigation of the possibilities of obtaining an exact solution for the helix problem has been that which was made by Bagby (31). He obtained the Maxwell equations in several different coordinate systems which might be used to describe a helix or a sufficiently close approximation to one. Choosing one of these, he derived an approximate solution which gave results which agreed in a fair manner with some experimental results obtained by Kraus and Williamson (27). However, the lack of full knowledge of the character of the solution plus the apparent need to examine different frequency ranges in a different manner lead to some uncertainty as to the validity of the procedures used.
(c) The Thin Wire Approximation

Phillips (17) obtained the fields surrounding a line helix carrying an unattenuated traveling wave of current by evaluating the vector potential in integral form. His work appears to have been the first published in this country which shows the influence of the periodic nature of the helix. From the form of his series expansion representations he deduced that certain frequency bands exist where the character of the solutions is different. However, he seems not to have realized the full significance of this difference. Further, he approximated the boundary conditions in only a limiting fashion and thereby obtained merely the usual solution corresponding to a wave traveling along the helix wire itself with the velocity of light. Phillips (17) did point out the possibility of obtaining solutions for a tape helix using expansions similar to those he gave. Sollfrey (16) derived field representations identical to those given by Phillips (17), but in a somewhat simpler fashion. The former, however, did not attempt to use these field expressions to determine the propagation constant required to meet the boundary conditions. Hsu (52) also used a thin wire approximation and considered the problem of the driven helix.

The most satisfactory solution for the helix problem published up to this time appears to be that given by Kogan (51) whose work was only very recently brought to the attention of the English speaking world. Using an integral equation method related to the approach of Pocklington (6), Phillips (17), and Sollfrey (16), he obtained results which agree very well over a wide range of frequencies with known experimental data. Kogan (51) also appears to have realized the limitations resulting from the periodic nature of the helix. However, he seems not to have realized the nature of the several exponential waves he obtained. Further, he did not extend his solution over the full frequency range and did not put it
in a convenient form for generalization. However, it would seem that Kogan (51), although he was apparently unaware of Pocklington's (6) work, is the first to have obtained a numerical solution of the integral equation for the helix.

(d) Related Work

Since in the limit of $\psi = 90^\circ$, where $\psi$ is the pitch angle, the helix becomes a straight wire, it would be expected that the theory of electromagnetic waves on straight wires would be of interest. This problem has had a long, and occasionally bitter, history, and the following references must be considered as only a sample of the available literature. Stratton (1, page 527) has given results first derived by Sommerfeld. Schelkunoff (2, page 117; 66), also, has considered the problem, as has Hallén (67). Sunde (58) devoted some parts of his book to this and related problems. Goubau (59) has recently examined the problem in some detail and has demonstrated the role played by loss and surface or space perturbations on the propagation characteristics of an infinitely long straight conductor. If the conductor is considered to be finite, the problem is related to antenna theory; and this is certainly not the place to relate the history of the antenna problem. However, the works of Schelkunoff (66) and Hallén (67) are of interest in this connection since they have examined the relationships which exist between the wave solutions on straight wires of infinite and finite length.

The helix is obviously a periodic structure, and it might be anticipated that the voluminous theory concerning such systems might prove pertinent. In this connection the papers of Slater (37, 38) and Walkinshaw (55) may be considered as merely representative. Brillouin (5) has given a rather full discussion of periodic structures and indicated in some detail the band properties of the propagation constants which may be anticipated for such systems. Since the helix is an "open"
system as well as a periodic one, the work of Rotman (61) is also of
interest.

"Open" structures which are not periodic have been examined by
several investigators, and, of course, the simple straight wire mentioned
above may be put in this class. Other types of "open" structures which
are capable of supporting transmission modes are the dielectric rod and
slab. These have been investigated by Schelkunoff (2, page 125), Adler
(39), Whitmer (48), Elsasser (49), and Roe (57), to mention only a few.

Review of Experimental Results

I.4  "Free Modes" on a Helix

Hertz was evidently the first to have performed an experiment to
determine the phase velocity of waves traveling along the helix. As
shown by Kornhauser (13), Hertz's result (really only one point at a
relatively low frequency) agrees quite well with the solution given by
the Schelkunoff (8) sheath model.

In recent years the experiments have been extended to cover a wider
frequency range. Those interested in traveling-wave tubes have confined
their attention to low frequencies, in terms of the helix dimensions,
whereas those interested in helical antennae have covered a somewhat
broader band. Cutler (22) reported measurements of phase velocity which
agreed very closely with predictions made by the acceptable sheath model
for relatively low frequencies, and Harris and others (23) have obtained
similar results. Cutler (22) observed only one wave with a reduced axial
velocity corresponding to the zeroth sheath mode, whereas Harris and
others (23) noted this and a "fast" wave, the presence of the latter
being dependent on the manner of excitation. Marston (24) has reported
some measurements which indicate the presence of the sheath modes but
only over restricted frequency bands.
A quite comprehensive series of measurements on the helix have been performed by Kraus and his co-workers, and the results of these have been summarized by him (25, Chapter 7). Others associated in this work have been Williamson (27), Glasser (28), Marsh (29), Aronoff (33), and Tice (34). These investigators have found that for low frequencies the phase velocity of the only observable wave on a helix is essentially that predicted by the sheath model, but that in the region where the circumference of the cylinder on which the helix is wound becomes equal to about one wavelength, an anomalous change in the propagation constant occurs. Further, this action occurs for a relatively wide range of pitch angles and appears to be essentially independent of conductor dimensions. Quite important, also, has been the observation that in the anomalous propagation region several waves with different propagation constants appear to exist. Kraus (25, Chapter 7) has also given considerable data on the radiation and impedance properties of finite helices.

*     *     *

Even for the very particular problem which is considered here, it is impossible to review or even to mention all the pertinent literature. For those interested in a further examination of the history of the problem the references quoted will serve as a guide to still other work. To those whose work has not been mentioned and to those whose results may not have been properly interpreted in the above review, the writer can only offer his assurances that such omissions or distortions were not intentional.
The Present Work

1.5 Purpose of the Investigation; Results and Relationship to Previous Work

(a) Purpose

When the work reported in the following chapters was started, the theory of electromagnetic wave propagation on helical conductors appeared to be in an unsatisfactory state. Although the sheath model gave useful results for low frequencies for the zeroth mode, it ignored the periodic nature of the structure and the dimensions of the wire. It was desired, therefore, to obtain a better solution which might indicate the range of usefulness of the sheath model. It was also hoped that the attainment of a more exact solution would explain in a consistent fashion the experimental results observed by Marston (24) and Kraus (25).

(b) Results and Relationship to Previous Work

The investigation was divided into three parts, and the results are correspondingly presented in Chapters II, III, and IV. In Chapter II the sheath helix model is analyzed, and the higher modes, in particular, are discussed in detail. The outstanding result here is that several "free mode" waves with differing phase and group velocity properties as a function of frequency and pitch angle can exist for all modes except the zeroth. The manner in which these velocities change as the parameters of the system are changed is carefully examined. The source-present as well as the source-free case is analyzed, and, indeed, a consideration of the former problem leads to a more complete understanding of the characteristics of the "free modes". It should be noted that the work of Chapter II parallels to a considerable degree the work of Phillips and Malin (19, 65), although most of the results given in Chapter II were obtained before the writer became aware of their work. Also, it is felt that although the procedures used here are, perhaps,
not as rigorous as the ones used by those authors, the more complete consideration given here to the properties of the modes as a function of the various parameters should be of interest.

In Chapter III the theory of the tape helix, a helix wound of a tape of finite axial but zero radial extent, is developed by using characteristic function expansions. Although an exact formal solution is possible, most of the chapter is devoted to obtaining useful results by employing approximations which are valid when the tape is either very narrow or very wide. It is shown why the sheath helix model gives such good results at low frequencies, and the modifications which result because of the periodic character of the helix are made evident. The anomalous behavior of the propagation constant and the existence of several waves as noted by Kraus and his co-workers (25) are predicted by the theory, and excellent agreement with experimental data is obtained. Other useful results concerning power flow, axial electric field, and power loss are presented. Finally, the theory of multiwire helices is given, and the manner in which the sheath model results when the number of wires becomes infinite is demonstrated.

Many of the results of Chapter III had been obtained when the work of Sollfrey (16) and Phillips (17) became available to the writer. Although the approach used by these investigators is different from the one used in Chapter III, essentially equivalent formulae may be obtained as is pointed out in Chapter IV. Indeed, unknown to the writer until very recently, Kogan (51) had used the integral equation approach and obtained some results similar to those presented in Chapter III. Perhaps the most interesting result in Chapter IV is the expression for the current at any point on an infinite driven helix. Although an exact solution of this appears difficult or impossible, the "free mode" portion is readily obtained, and, as a consequence, the significance of the several waves which exist on a helix is made clear.
CHAPTER II

THE SHEATH HELIX

Even though, as noted in the preceding chapter, several investigators have previously presented many analyses of an approximate representation of the helix which is called the sheath helix here, there are still many more properties of this model which should be discussed and clarified. Further, this work serves as an introduction to the more exact representation considered in Chapter III and allows useful comparisons to be made.

The various parameters and conventions used in the analysis are defined, and the boundary conditions are discussed first. (Sections II.1, II.2) The source-free problem is then considered. After a determination of appropriate forms for the solutions (Section II.3), the boundary conditions are applied, and the determinantal equation and explicit forms for all the field components are obtained. (Section II.4) The determinantal equation is solved, and the behavior of the various "free mode" waves as a function of $ka$, $\psi$, and $n$ is determined. (Section II.5, Appendix B) Several graphs showing how the propagation constants vary with the parameters are given. (Figs. II-4 through II-6, II-8 through II-11, II-13, II-14) The presence of several waves per mode for $\psi < 90^\circ$ is shown. The meaning of these various waves is understood best by considering what happens if they are excited by a source, and they are explained on that basis, although the full treatment of this latter problem is reserved for a later section. The special limiting
cases of the sheath ring, $\psi = 0^\circ$, and the sheath tube, $\psi = 90^\circ$, are of considerable interest and are given special treatment. (Section II.5 (b), (c))

The power flow associated with the different modes is calculated, and the usual relationship between group velocity and average power flow is proved. (Section II.6)

The source-present or driven helix problem is then solved for the particular case of a gap source. (Section II.7) The modified radiation field portion of the solution is not determined but is shown to be small at large distances from the source. However, the "free mode" portions of the solution are examined, and complete solutions for these for all values of $\psi$ and $n$ are determined. The disposition of the "free mode" roots of the determinantal equation in the complex propagation constant plane is considered in detail. (Section II.7, Figs. II-21 through II-26) The chapter is concluded with a brief discussion of other types of sources.

Formulation of the Problem

II.1 Definitions

The circular cylindrical coordinate system is used and is defined in the usual manner as shown in Fig. II-1. $\bar{a}_r$, $\bar{a}_\theta$, and $\bar{a}_z$ are unit vectors in the $r$, $\theta$, and $z$ directions, respectively. If on a cylinder of radius $a$, coaxial with the $z$ axis, a helix of pitch $p$ is wound, the configuration appears as in Fig. II-2. The helix is assumed to extend to infinity in both directions along the $z$ axis, and the medium is unbounded. The physical helix would, of course, be wound of wire of finite diameter, usually of circular cross section. If the cylinder on which the helix is wound is now cut by a plane of constant $\theta$ and unrolled, the resulting development appears, when viewed from the "inside", as in Fig. II-3.
Circular Cylindrical Coordinates

FIG. II-1

Helix

FIG. II-2

Developed Helix

FIG. II-3
II.1

The pitch angle $\psi$ is given by

$$\psi = \cot^{-1} \frac{2na}{p} .$$  \hspace{1cm} (1)

The unit vectors $\vec{a}_r$, $\vec{a}_\theta$, and $\vec{a}_z$ are parallel and perpendicular to the helix wire, and are useful and are shown in Fig. II-3. These are related to $\vec{a}_r$, $\vec{a}_\theta$, and $\vec{a}_z$ by

$$\vec{a}_r \times \vec{a}_n = \vec{a}_ \perp ,$$  \hspace{1cm} (2)

$$\vec{a}_n = \vec{a}_z \sin \psi + \vec{a}_\theta \cos \psi ,$$  \hspace{1cm} (3)

$$\vec{a}_\perp = \vec{a}_z \cos \psi - \vec{a}_\theta \sin \psi .$$  \hspace{1cm} (4)

Similarly, $\vec{a}_z$ and $\vec{a}_\theta$ can be expressed in terms of $\vec{a}_n$ and $\vec{a}_\perp$.

Since the time variation of the fields is taken to be harmonic, complex field vectors are used exclusively. Thus, for example, if $E_r$ is the complex component representing the radial electric field, the real component equal to the radial electric field is the real or imaginary part of $E_r e^{j\omega t}$. Similarly, the total real electric field vector is the real or imaginary part of $\vec{E} e^{j\omega t} = (\vec{a}_r E_r + \vec{a}_\theta E_\theta + \vec{a}_z E_z) e^{j\omega t}$.

The rationalized MKS system is used throughout this report. The medium in which the helix is immersed is considered to be homogeneous, isotropic, and linear and is characterized by $\varepsilon$, $\mu$, and $\sigma$, the permittivity permeability, and conductivity, respectively, of the medium. In this case

$$k^2 = (-j\omega \mu)(j\omega \varepsilon) = (-j\omega \mu)(\sigma + j\omega \varepsilon) = \omega^2 \varepsilon \mu - j\omega \mu \sigma ,$$  \hspace{1cm} (5)

where $\omega$ is the radian frequency. For a lossless medium, $\sigma = 0$,

$$k^2 = \omega^2 \varepsilon \mu = \left(\frac{2\pi}{\lambda}\right)^2 ,$$  \hspace{1cm} (6)

where $\lambda$ is the wavelength of a uniform plane wave in the medium. Other notation and additional definitions are introduced as they are required.
II.2 Boundary Conditions

The solution of an electromagnetic wave boundary value problem is effected by obtaining appropriate solutions of the Maxwell equations which also satisfy the boundary conditions. By appropriate solutions it is meant that in the absence of singular sources, the field representations must be finite and single-valued. The boundary conditions are that the tangential components of the electric and magnetic fields must, in general, be continuous through any surface. If metallic conductors which are assumed perfect, that is, have infinite conductivity, are present, then the tangential electric field, and consequently the normal magnetic field, must be zero on such surfaces. Also, if any perfectly conducting surfaces are assumed to be of infinitesimal thickness so that any current which flows may be considered to flow in an infinitesimally thick layer, then a discontinuity in the tangential magnetic field perpendicular to and equal to the total surface current density must exist at these surfaces. In addition, the radiation condition, or condition at infinity, must be satisfied by the physically appropriate solutions. Simply put, this requires that only outgoing waves be generated by any sources, and that the fields cannot grow increasingly large in the direction of energy propagation in a passive system which contains finite or, as a limiting case, zero losses.

Consider a helix wound of a wire of uniform cross section, say circular, and of infinite conductivity. If Maxwell's equations could be solved in an appropriate coordinate system in which the surface of the wire is a surface described by keeping one of the coordinates constant, then the problem could be solved by a procedure identical to that used in simpler waveguide problems. Specifically, the electric field tangential to the wire surface would be expressed in terms of the appropriate coordinate functions and the helix parameters. By requiring this to be
II.2

zero, the only necessary boundary condition aside from finiteness of the fields if the conductor is perfect, a determinantal equation would be obtained which would then be solved for the unknown propagation constant. This, inserted in the appropriate field expressions, would yield a solution to the Maxwell equations which would satisfy the boundary conditions and which would be, therefore, a unique solution. Unfortunately, as noted in Chapter I, although it is possible to define coordinate systems which describe the helix in the required manner and to write down the field equations for such systems, it is not possible, or has so far proved impossible, to solve the resulting equations. Consequently, one approach has been to replace the physical helix with a model which seems to retain many of its characteristics and which allows a solution to be obtained.

Considering Fig. II-2, assume that another wire helix is wound on the cylinder of radius \( a \), but displaced slightly in the \( z \) direction from the first. A third wire is now placed alongside the second, and so on until the entire pitch distance is filled up. In this manner a multi-wire helix is obtained. Now assume that the wires are allowed to become of infinitesimal radius so that current can be conducted only in the wire direction. Further, assume that the spacing between the wires becomes infinitesimal and that the number of wires becomes infinitely large. In the limit, the "wires" may be replaced by a sheath or sheet which can conduct current only in the "wire" direction; this is the sheath helix. It can be considered to consist of an anisotropic conducting sheet wound on a circular cylinder of radius \( a \). The problem is now to find solutions of Maxwell's equations for the two regions \( a \gg r \gg 0 \) and \( \infty \gg r \gg a \) which are connected by the appropriate continuity conditions at \( r = a \).

From the limit method by which the above model is derived these are

\[
E_i^i = E_\theta^\theta = 0 ,
\]  
(1)
\[ E_i = E_e, \quad (2) \]
\[ H_i = H_e, \quad (3) \]

for \( r = a, \ \theta \geq 0, \) and \(+ \infty > z > -\infty .\) The superscripts \( i \) and \( e \) are used to distinguish the expressions for the internal fields, \( a \geq r \geq 0, \) from those for the external fields, \( \infty \geq r \geq a.\) The subscripts \( i \) and \( e \) refer to the particular components of the field, parallel and perpendicular to the "wires" in that order. These equations are essentially expressions of the assumptions that the "wires" are taken to be perfect conductors and to conduct only in the direction of the "windings". In terms of the \( r, \theta, \) and \( z \) components, using (II.1-3) and (II.1-4), the continuity conditions become

\[ E_i^z = E_e^z, \quad (4) \]
\[ E_i^\theta = E_e^\theta, \quad (5) \]
\[ E_i^z e = -E_i^\theta e \cot \psi, \quad (6) \]

and

\[ H_i^z + H_i^\theta \cot \psi = H_e^z + H_e^\theta \cot \psi, \quad (7) \]

for \( r = a, \ \theta \geq 0, \) and \(+ \infty > z > -\infty .\) In addition to the above, for simplicity, it is assumed that the medium is the same and lossless for \( r < a \) as for \( r > a.\)

Some of the shortcomings of the sheath helix representation are clear immediately. The most serious deficiency is that the periodic structure of the physical helix is completely ignored. As will be seen in Chapter III of this report, it is this periodic structure which gives the actual helix some interesting and unusual properties. Further, the effect of finite wire size is nowhere considered in the sheath model. Despite these shortcomings, the assumption of the sheath helix with its uniform boundary conditions allows a solution which has considerable utility to be obtained in a fairly straightforward manner.

Before proceeding with the solution for the sheath helix, it is useful to consider what some of the results may be. Since the
configuration being examined has cylindrical symmetry, it is to be expected that solutions which have exponential dependence on $z$ exist. These correspond to waves guided by the helix. Because of the uniformity of the boundary conditions, it is also to be expected that an infinite set of modes characterized by different angular variations exists, although these modes undoubtedly do not constitute a complete set.\footnote{39} Further, it can be anticipated, in view of the peculiar nature of the boundary conditions, that the solutions consist of a mixture of TE (transverse electric) and TM (transverse magnetic) waves.\footnote{39}

In solving a problem of the type being considered here, there are two related approaches. In one, the homogeneous or source-free field equations are solved subject only to all the boundary conditions. In this, the manner in which the fields are generated is considered immaterial, and the sources are taken to be located at an infinite distance from the point at which a solution is desired. By this procedure, the so-called "free modes" or "natural waves" and their dependence solely on the geometry and other parameters of the system are determined. In the other approach, the inhomogeneous or source-present field equations are solved subject, of course, to the boundary conditions also. This procedure is perhaps more difficult than the first, but it can yield more information. Specifically, using the latter approach, one can in general determine which of the "free modes" are excited by a particular configuration of driving source, and the amplitudes of the "free modes" can be related to the strength of the source. Both approaches are used here with the source-free problem being considered first.
II.3 The Source-Free Problem

The fundamental equations which must be solved subject to the boundary conditions are, of course, the homogeneous or source-free Maxwell equations which are written as

\[ \nabla \times \vec{E} = -j \omega \mu \vec{H}, \]  
\[ \nabla \times \vec{H} = j \omega \varepsilon \vec{E}. \]  

(1)  
(2)

As is well-known\(^1\), solutions to these can be obtained in terms of electric and magnetic Hertzian vector potentials, \( \vec{\Pi} \) and \( \vec{\Pi}^* \), respectively, by

\[ \vec{E} = \nabla \times \nabla \times \vec{\Pi} = -j \omega \mu \nabla \times \vec{\Pi}^* , \]  
\[ \vec{H} = j \omega \varepsilon \nabla \times \vec{\Pi} + \nabla \times \nabla \times \vec{\Pi}^* , \]  

(3)  
(4)

where \( \vec{\Pi} \) and \( \vec{\Pi}^* \) both satisfy an identical vector wave equation

\[ \nabla \times \nabla \times \vec{\Pi} - \nabla \nabla \cdot \vec{\Pi} - k^2 \vec{\Pi} = 0, \]  
\[ \nabla \times \nabla \times \vec{\Pi}^* - \nabla \nabla \cdot \vec{\Pi}^* - k^2 \vec{\Pi}^* = 0. \]  

(5)  
(6)

If \( \vec{\Pi} \) and \( \vec{\Pi}^* \) are taken to have only \( z \) components, \( \Pi_z \) and \( \Pi_z^* \), respectively, then the electric and magnetic field components in circular cylindrical coordinates can be obtained as follows:

\[ E_r = \frac{\partial^2 \Pi_z}{\partial z \partial r} - j \frac{\omega \mu}{r} \frac{\partial \Pi_z^*}{\partial \theta}, \]  
\[ E_\theta = \frac{1}{r} \frac{\partial^2 \Pi_z}{\partial z \partial \theta} + j \frac{\omega \mu}{r} \frac{\partial \Pi_z^*}{\partial r}, \]  
\[ E_z = \frac{\partial^2 \Pi_z}{\partial z^2} + k^2 \Pi_z, \]  

(7)  
(8)  
(9)

\[ H_r = j \omega \varepsilon \frac{1}{r} \frac{\partial \Pi_z}{\partial \theta} + \frac{\partial^2 \Pi_z^*}{\partial z \partial r}, \]  
\[ H_\theta = -j \omega \varepsilon \frac{\partial \Pi_z}{\partial r} + \frac{1}{r} \frac{\partial^2 \Pi_z^*}{\partial z \partial \theta}, \]  

(10)  
(11)
\[ H_z = \frac{\partial^2 \Pi^*}{\partial z^2} + k^2 \Pi^* \]  
(12)

\( \Pi_z \) and \( \Pi^*_z \), being rectangular components, satisfy the same scalar wave equation

\[ \nabla^2 \Pi_z + k^2 \Pi_z = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Pi_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Pi_z}{\partial \theta^2} + \frac{\partial^2 \Pi_z}{\partial z^2} + k^2 \Pi_z = 0, \]  
(13)

and

\[ \nabla^2 \Pi^*_z + k^2 \Pi^*_z = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Pi^*_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Pi^*_z}{\partial \theta^2} + \frac{\partial^2 \Pi^*_z}{\partial z^2} + k^2 \Pi^*_z = 0. \]  
(14)

(13) and (14) are used to simplify the expressions for \( E_z \) and \( H_z \) obtained from (3) and (4). If \( \Pi^*_z \) vanishes but \( \Pi_z \) does not, \( H_z = 0 \), and a TM wave results; if \( \Pi_z \) vanishes but \( \Pi^*_z \) does not, \( E_z = 0 \), and a TE wave results. The total electromagnetic field obtained by including both \( \Pi_z \) and \( \Pi^*_z \) is of such generality that a given set of boundary conditions for some constant \( r \) can be satisfied.

Applying the usual separation of variables procedure in (13) and (14), it is found that the \( z \) and \( \theta \) solutions are expressible in exponential form. Thus, by using in (13) and (14) the form

\[ \Pi_z = f(r) e^{-jhz} e^{-jn\theta}, \]  
\[ \Pi^*_z \]  
(15)

where \( n \) must be integer since the potential and the fields must be single-valued, there results that \( f(r) \) must be a solution of

\[ r \frac{d}{dr} \left( r \frac{df}{dr} \right) - \left[ (h^2 - k^2)r^2 + n^2 \right] f = 0. \]  
(16)

\( h \) is the propagation constant whose value must be determined. As can be seen from Section A.1, the independent solutions of (16) are the modified Bessel functions of order \( n \) and argument \( \xi r \) where \( \xi \) is given
by
\[ \zeta = \sqrt{h^2 - k^2}. \] (17)

Since the fields must be finite, the \( I_n \) function is chosen for the solution for \( a > r > 0 \) and the \( K_n \) function for \( \infty > r > a \). Consequently, for the fundamental solutions there results

\[ \Pi^i_z = A_n^i I_n(\zeta r) e^{-jhz} e^{-jn\theta}, \quad a > r > 0, \] (18)

\[ \Pi^e_z = A_n^e K_n(\zeta r) e^{-jhz} e^{-jn\theta}, \quad \infty > r > a, \] (19)

\[ \Pi^{*i}_z = B_n^i I_n(\zeta r) e^{-jhz} e^{-jn\theta}, \quad a > r > 0, \] (20)

\[ \Pi^{*e}_z = B_n^e K_n(\zeta r) e^{-jhz} e^{-jn\theta}, \quad \infty > r > a. \] (21)

\( A_n^i, e \) and \( B_n^i, e \) are coefficients which are related by the boundary conditions and are functions of \( n, \omega \), and the character of the exciting sources, but not of the coordinates. Incidentally, it is clear from the above, as is well known otherwise, that \( E_z \) and \( H_z \) rather than \( \Pi_z \) and \( \Pi^*_z \) can be used as the scalar quantities from which the other field components can be derived.

II.4 Application of the Boundary Conditions; the Determinantal Equation; and the Sheath Helix Fields

Generally, in constructing a complete solution from the fundamental solutions, (II.3-18) through (II.3-21) above, one has to sum over all values of \( n \). However, since the boundary conditions for the sheath helix are the same for all \( \theta \) and \( z \), the orthogonality in \( \theta \) of the fundamental solutions allows the possibility of the boundary conditions being satisfied for each \( n \) separately.

Using the expressions for the Hertzian potentials, (II.3-18) through (II.3-21), in (II.3-8), (II.3-9), (II.3-11), and (II.3-12) to determine
the tangential fields, and then using these in the boundary condition equations, (II.2-4) through (II.2-7), one obtains the following:

from (II.2-4)

$$ I_n A_n^i - K_n A_n^e = 0, \quad (1) $$

from (II.2-5)

$$ - \frac{h_n}{a} I_n A_n^i + \frac{h_n}{a} K_n A_n^e + j\omega \xi I_n E_n^i = j\omega \xi K_n E_n^e = 0, \quad (2) $$

from (II.2-6)

$$ (\xi^2 + \frac{h_n}{a} \cot \psi) I_n A_n^i + j\omega \xi \cot \psi I_n E_n^i = 0, \quad (3) $$

and from (II.2-7)

$$ -j\omega \xi \cot \psi I_n A_n^i + j\omega \xi \cot \psi K_n A_n^e = (\xi^2 - \frac{h_n}{a} \cot \psi) I_n B_n^i + (\xi^2 + \frac{h_n}{a} \cot \psi) K_n E_n^e = 0. \quad (4) $$

Here, the argument of all the modified Bessel functions is $\xi a$, and the prime means differentiation with respect to the argument. Equations (1) through (4) are a homogeneous system of linear equations relating $A_n^i, A_n^e$ and $B_n^i, E_n^e$. A nontrivial solution exists only if the determinant of the system is zero, and the propagation constant $h$ is determined from the equation resulting from the requirement that the determinant vanish.

Solving directly, it is found that

$$ A_n^e = \frac{I_n}{K_n} A_n^i, \quad (5) $$
$$ B_n^i = \frac{(\xi^2 + \frac{h_n}{a} \cot \psi)}{j\omega \xi \cot \psi} I_n A_n^i, \quad (6) $$

$$ B_n^e = \frac{K_n}{K_n} B_n^i = \frac{(\xi^2 + \frac{h_n}{a} \cot \psi)}{j\omega \xi \cot \psi} \frac{I_n}{K_n} A_n^i, \quad (7) $$

and, finally,

$$ \frac{I_n'}{(\xi a) K_n'} (\xi a) \frac{I_n (\xi a) K_n (\xi a)}{I_n} = - \frac{(\xi^2 + \frac{h_n}{a} \cot \psi)^2}{k^2 \alpha^2 \beta^2 \cot^2 \psi}, \quad (8) $$
as the determinantal equation. In obtaining (8) from (5), (6), and (7) it is required that the Wronskian \( \mathcal{W}(I_m, K_m) \), \( (A.1-8) \), be nonzero.

Using \( (II.3-18) \) through \( (II.3-21) \) in \( (II.3-7) \) through \( (II.3-12) \), and then using (5) through (7) above, it is possible to express all the field components in terms of \( A_n^i \) alone. It is more convenient, however, to express the fields in terms of the surface current density. This is readily done since the surface current density vector \( \mathbf{K} \), where

\[
\mathbf{K} = a_\theta K_\theta + a_z K_z ,
\]

is related to the discontinuity in the tangential magnetic field by

\[
\begin{align*}
\bar{a}_r \times (\mathbf{H}^e - \mathbf{H}^i)_{r=a} &= \mathbf{K} .
\end{align*}
\]

Expanding (10) results in

\[
K_\theta = (H_z^i - H_z^e)_{r=a} \quad (11), \quad K_z = (H_\theta^e - H_\theta^i)_{r=a} . \quad (12)
\]

Further, since

\[
K_\theta = K_z \sin \psi + K_\theta \cos \psi , \quad (13) \quad \text{and} \quad K_z = K_\theta \cos \psi - K_\theta \sin \psi , \quad (14)
\]

--see \( (II.1-3) \) and \( (II.1-4) \) -- it is found that \( K_z \) vanishes, as would be expected from the boundary conditions, while (13) yields a relationship between \( A_n^i \) and \( K_n^i \). In the course of carrying through these calculations, it is necessary to use the Wronskian identity and the determinantal equation (8). There finally results for the field expressions

\[
E_\theta^e = \mu_0 \omega \left[ n h a \frac{\xi^2 a^2}{k^2 a^2} \beta_n \cot \psi \right] I_n(\xi a) K_n(\xi r)
+ \cot \psi I_n(\xi a) K_n'(\xi r) \sin \psi |K_n| e^{-j Hz} e^{-jn\theta} , \quad (15)
\]

\[
E_r^e = - \frac{\omega a}{\xi a} \left[ n a \frac{\xi^2 a^2}{k^2 a^2} \beta_n \cot \psi \right] I_n(\xi a) K_n'(\xi r)
+ n \frac{a}{r} \cot \psi I_n(\xi a) K_n(\xi r) \sin \psi |K_n| e^{-j Hz} e^{-jn\theta} , \quad (16)
\]
\[ E_z^e = j\omega a \left( \frac{\xi^2 a^2 + \text{nha cot}\psi}{k^2 a^2} \right) I_n(\xi a) K_n'(\xi r) \sin\psi |K_n| e^{-jhz e^{-jn\theta}}, \quad (17) \]

\[ H_z^e = -\frac{1}{\xi} \left[ (\xi^2 a^2 + \text{nha cot}\psi) I_n(\xi a) K_n'(\xi r) \right. \\
\left. \quad + \text{nha } a \frac{\xi^2 a^2}{r} \cot\psi I_n'(\xi a) K_n(\xi r) \sin\psi |K_n| e^{-jhz e^{-jn\theta}} \right], \quad (18) \]

\[ H_r^e = -j \left[ n \frac{\xi^2 a^2 + \text{nha cot}\psi}{\xi^2 a^2} I_n(\xi a) K_n(\xi r) \right. \\
\left. \quad + \text{ha cot}\psi I_n'(\xi a) K_n'(\xi r) \sin\psi |K_n| e^{-jhz e^{-jn\theta}} \right], \quad (19) \]

\[ H_z^e = -\xi a I_n'(\xi a) K_n(\xi r) \cos\psi |K_n| e^{-jhz e^{-jn\theta}}, \quad (20) \]

where \( |K_n| \) is the magnitude of the surface current density in the direction of the "wires". Only the external field expressions are given since the internal field expressions are identical, except that the \( I_n \) and \( K_n \) functions are interchanged everywhere. \( |K_n| \) is the usual undetermined constant which remains in "free mode" solutions and can only be evaluated if the power flow associated with the mode or the character of the exciting sources is known.

It is worthy of note that the continuity condition

\[ \nabla_s \cdot \bar{F} + j\omega q_s = \nabla_s \cdot \bar{F} + j\omega \varepsilon (E_r^e - E_r^i) = 0, \quad r = a, \quad (21) \]

where \( \nabla_s \cdot \bar{F} \) is the surface divergence of the surface current density and \( q_s \) is the surface charge density, is satisfied for all \( n \) as well as for \( n = 0 \). The fulfillment of the continuity condition for \( n = 0 \) alone for the sheath model considered here had previously been noted.\(^\text{13}\)

The expressions (15) through (20) constitute the proper fields for the "free modes" on a sheath helix if values of \( h \) and \( \xi \) are used which
satisfy the determinantal equation (8). In order to complete the solution these values must be determined. This is considered in detail in the following section and Appendix B.

II.5 Solutions of the Determinantal Equation

(a) Introduction

For simplicity, it is preferable to relegate to Appendix B a relatively complete analysis of the determinantal equation and its roots. Consequently, in this section only the results which are derivable for the most part from Appendix B are given. It should be emphasized that, in general, it has been the intent to determine the existence and general character of the solutions rather than specific numerical results. However, some of the latter is available and is noted in the following material where it seems of interest.

As shown in Section B.1, the propagation constant $\kappa$ is required in general to be real and in magnitude larger than $k$ if "free mode" solutions are to exist when the medium is lossless. This limitation has also been noted in other open-boundary problems and immediately reduces the magnitude of the task of looking for solutions of the determinantal equation, since one need only investigate this equation for $\zeta$ real and $\zeta > 0$.\textsuperscript{39}

In addition to the general case of the sheath helix for $\psi \neq 0^\circ$ or $90^\circ$, there are the two special cases of $\psi = 0^\circ$, called the sheath ring, and $\psi = 90^\circ$, called the sheath tube. For $\psi = 0^\circ$ there are solutions for $|\kappa| > k$, and these turn out to have considerable interest not only for themselves but also for their rather close connection with the solutions for $\psi \neq 0^\circ$. Further, a knowledge of the solutions for $\psi = 0^\circ$ is exceedingly useful in investigating the cases for $\psi \neq 0^\circ$, as may be seen in Appendix B and the following. For $\psi = 90^\circ$, "free mode" solutions exist for
values of $|n| \gg 1$ and $h \approx k$. It turns out that these solutions are the
limiting ones for a set of the sheath helix modes and are therefore of
considerable interest also. Ordinary waveguide modes also exist for $\Psi = 0^\circ$ and $90^\circ$.

(b) \underline{The Sheath Ring, $\Psi = 0^\circ$}

For $\Psi = 0^\circ$ the determinantal equation is

$$\frac{I_n'(\xi a)K_n'(\xi a)}{I_n(\xi a)K_n(\xi a)} = -\frac{2\xi^2 a^2}{k^2 a^2 \xi a^2}.$$  \hspace{1cm} (1)

This results if the Maxwell equations are solved subject to the boundary
conditions

$$E^i_n = E^e_n = E^i_\theta = E^e_\theta = 0, \hspace{1cm} (2) \hspace{1cm} E^i_z = E^e_z,$$

$$H^i_n = H^e_n = H^i_\theta = H^e_\theta.$$  \hspace{1cm} (3)

for $r = a, 2\pi \gg \theta \gg 0$, and $+ \infty \gg z \gg - \infty$. These correspond to (II.2-1)
through (II.2-7) for the special case of $\Psi = 0^\circ$. The sheath ring system
may be considered as the limiting one which results from an infinite
series of equally spaced perfectly conducting circular rings coaxial with
the $z$ axis as the wire of which the rings are made becomes infinitesimally
small and the spacing between the rings becomes likewise small. (1) also
results if in (II.4-8) $\Psi$ is allowed to approach zero and the limit
taken in the usual manner. The field expressions in this case may like-
wise be obtained from (II.4-15) through (II.4-20) if, there, $\Psi$ is
allowed to approach and finally equal zero. Because of this these are
not written out explicitly. As might be expected from the symmetry of
the boundary conditions, for this case the determinantal equation (1) is
an even function of $h$ and $n$. Consequently, there exist field solutions
whose angular dependence is $\sin n\theta$ or $\cos n\theta$ as well as $e^{-jn\theta}$, and whose
$z$ dependence is $\sin hz$ and $\cos hz$ as well as $e^{-jhz}$. Such solutions are
obtained by a proper linear combination of the expressions given by (II.4-15) through (II.4-20) with \( \psi = 0^0 \), of course, and correspond to solutions which are standing waves in \( z \) or \( \theta \) or both. Such solutions would fit the boundary conditions imposed if perfectly conducting infinite planes were placed perpendicular to the \( z \) axis or along planes of constant \( \theta \) -- at intervals of \( \frac{\pi}{|n|} \) -- or both. In the following only the unbounded system is considered.

In solving (1) it is most convenient to find those values of \( \xi a \), for a given \( n \) as \( ka \) varies, for which (1) is satisfied. This is done in Section 8.2. However, it is the propagation constant \( h \) and the ratio \( \frac{ka}{ha} \) which are of greatest interest, and it is these quantities which are considered below. \( h, \xi, \) and \( k \) are, of course, related by \( h^2 = \xi^2 + k^2 \).

For \( \psi = 0^0 \) for \( n = 0 \) there are no "free mode" solutions of the type which are of most interest here, whereas for \( |n| \geq 1 \) the solutions can be shown as in Fig. II-4. In \( h_{n,m} \) the \( n \) subscript refers to the mode number, and the \( m \) subscript refers to the branch of the solution. The reason for distinguishing between the different parts of a solution for a given \( n \) by using \( m \) and by showing the different portions of the curves as solid and dotted lines is considered below. Quite similar curves occur for \( -h_{n,m} \), which can be obtained by reflection about the \( ka \) axis in Fig. II-4 if care is taken to account for a complication which is also discussed below. Another form of presentation is shown in Fig. II-5, where the corresponding parts of the curves are related to those of Fig. II-4 in an obvious manner. Since \( ka = \frac{2\pi}{\lambda} a \), increasing \( ka \) corresponds to decreasing wavelength or increasing frequency. Because (1) is an even function of \( n \) and \( h \), it is evident that \( |h_{n,m}| = |h_{-n,m}|. \) The \( n \)th mode exists only in the range \( |n| + \Delta_n \geq ka > \sqrt{n^2 - 1} \), where \( \Delta_n \) is a very small positive number. \( \Delta_1 \approx 0.015, \Delta_2 \approx 0.01, \) and \( \Delta_n \) becomes increasingly small as \( |n| \) increases. The curves of Figs. II-4 and II-5
<table>
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<th>$l_i$</th>
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$\psi = 0^\circ$

The Sheath Ring

$|n| \geq 1$

$|h_{1,1}|a$

$|h_{2,1}|a$

$|h_{3,1}|a$

$|h_{1,2}|a$

$|h_{2,2}|a$

$|h_{3,2}|a$

Curves continue in a similar manner for $|n| \geq 5$

| $k_a/|h_{n,m}|a$ | $\sqrt{3}$ | $6$ | $n^2 - 1$ |
|------------------|-----------|------|-----------|
| $|n| = 1$ | $|n| = 2$ | $|n| = 3$ | $|n|$ |
| $ka$ | $ka$ vs. $ka$ |

| $ka$ vs. $|h_{n,m}|a$ |
|------------------------|
| $|n| \geq 1$ |

$\psi = 0^\circ$

The Sheath Ring

$ka = |h_{n,m}|a$
are only approximately to scale, and the effect of $\Delta_n$ is exaggerated very much.

Since the sheath ring may be considered as the limit of an infinite series of circular rings when the rings have become infinitesimally wide and similarly spaced, one might expect from an approximate lumped circuit analogy for such a transmission system that it would support a "free mode" at or near the frequency for which each individual element -- that is, each ring -- is separately resonant. This would occur when the circumference of the rings is an integral number of wavelengths, or for $\frac{2\pi a}{\lambda} = ka = |n|$. This is essentially the performance of the sheath ring, and the spread in frequency over which each "free mode" solution exists can be considered to be the result of the coupling between the elements -- the "rings" -- of this system. A physical explanation for the absence of the $n = 0$ mode of the type considered here is simply that no "ring" resonance can exist which produces such a mode. The persistence of the $|n| = 1$ mode down to low frequencies distinguishes it somewhat from the others. This is a property which the sheath ring has in common with the open dielectric rod.\textsuperscript{2,39,49} An explanation for this action would seem to be that for $ka$ small, and thus $\xi$ exceedingly small, the solution appears as essentially a perturbed uniform plane wave which is linearly or circularly polarized depending on whether the trigonometric or exponential dependence on $\theta$ is chosen. This may be shown from the field expressions by taking $ka$ very small so that $|h_{\pm 1, 1}|a$ is near $ka$ with $\xi$ exceedingly small, and by considering the region where $\xi r$ is small but $r$ is not necessarily small. In this case the $E_z^{i,e}$ and $H_z^{i,e}$ components are substantially $ka$ times as large as the $E_r^{i,e}$ and $H_r^{i,e}$ components, the latter are essentially independent of $r$, and the ratios $\frac{E_{\theta}^{i,e}}{H_{\theta}^{i,e}}$ and $\frac{E_r^{i,e}}{H_r^{i,e}}$ are given by $\pm j \sqrt{\frac{\mu}{\epsilon}}$.

In Figs. II-4 and II-5 portions of the curves are shown dotted. The reason for this is that the solutions corresponding to the dotted and
solid lines are really different branches of the curves, so that for
$|n| + \Delta_n > \kappa a > |n|$ there are two waves per mode. Assuming $z > 0$ for the
moment, one of these has $z$ dependence like $e^{-j|n|_1 a} z$; whereas the other
has $z$ dependence like $e^{+j|n|_2 a} z$. The absolute bars are used to avoid
any confusion of sign, the sign of the exponential being given explicitly.
The procedure whereby the above is deduced is explained best by consider-
ing the inhomogeneous or source-present problem. This is done in Section
II.7. Here, only the results and their physical significance are
discussed.

The phenomenon can be interpreted by considering a curve for both
positive and negative $|h_{n,m}| a$ for a particular $n$ as shown in Fig. II-6.
Assume that a finite source is placed at $z = 0$ which can excite only the
nth mode. The $z$ axis is, of course, taken to coincide with the axis of
the sheath ring, and the radius of the sheath ring, $a$, is assumed to
remain constant while the frequency is varied in the following. Now
assume that one stations oneself on the positive $z$ axis so that the
source is very far away. The sheath ring system continues on towards
$z = + \infty$. As the frequency of the source or $\kappa a$ is increased, no field
is observed --- it is assumed that the source is sufficiently far away
so that the radiation field is negligible --- until $\kappa a = \sqrt{n^2 - 1}$ after
which a "free mode" wave traveling towards $z = + \infty$ appears with a
propagation constant $|h_{n,1}|$. Here, traveling refers to the axial phase
velocity and its direction or sign. As $\kappa a$ increases, the propagation
constant of this wave increases, varying along the solid $|h_{n,1}| a$
curve in Fig. II-6 in the direction indicated there. For $\kappa a$ larger than
$|n|$ by any small amount, another "free mode" wave appears which travels
towards the source with a propagation constant of magnitude $|h_{n,2}|$.
This varies with increasing $\kappa a$ along the solid $-|h_{n,2}| a$ curve in
Fig. II-6 in the manner shown there. Both waves continue to exist
until $ka = |n| + \Delta_n$. At this point they become waves traveling in both directions with propagation constants of equal magnitude and, it turns out, field components of equal amplitude, so that a pure standing wave exists on the system. For $ka > |n| + \Delta_n$, nothing is observed again, the "free mode" waves disappearing. If one stations oneself on the negative $z$ axis so that the source is very far away, the system continuing on towards $z = -\infty$, a situation similar to the above results, except that the dotted curves in Fig. II-6 are traced. It should be remembered that solutions whose $z$ and $t$ behavior is like $e^{-jhz} e^{j\omega t}$ are being used.

A physical interpretation of the inward traveling wave, the wave traveling towards the source, is that it is the total result of the backward scattering which occurs when the frequency is such that the "rings" are slightly beyond resonance, or for $ka > |n|$. It should be emphasized that for $|n| + \Delta_n > ka > |n|$ the total "free mode" consists of the inseparable sum of both inward and outward traveling waves. The inward traveling wave does not come from any source at infinity, but again, only from the scattering resulting from "ring" resonance. This wave does not violate the condition at infinity; and, in fact, if the medium is taken to be slightly lossy, it has a $z$ dependence for $z > 0$ like

$$e^{-j|\alpha_{n,2}|z}$$

the outward traveling wave having a $z$ dependence like

$$e^{-j|\alpha_{n,1}|z}$$

For a slightly lossy medium $|\alpha_{n,1}|$ and $|\alpha_{n,2}|$ are very small numbers, but the important point is that both waves are exponentially damped at large distances from the source. When the inward traveling wave first appears, the fields associated with it are appreciable only for $r$ very near $a$ since $|h_{n,2}|$ is very large. However, as the frequency increases, $|h_{n,2}|$ rapidly becomes smaller. The radial extent of the fields of this wave increases so that at $ka = |n| + \Delta_n$, where $|h_{n,1}| = |h_{n,2}|$, the field components of the outward and inward traveling waves vary radially in an.
identical manner. The angular dependence of the waves is the same under all circumstances, and their amplitudes are intimately related, as might be expected from the physical interpretation given above. The action here is somewhat reminiscent of the performance of structures which are periodic in the \( z \) direction.\textsuperscript{5,37} There, however, a multiplicity of pass and attenuation bands usually exists even for a particular mode as characterized by the transverse field structure, and the inward traveling wave which occurs is interpreted as the total result of scattering from the periodically located elements. Here, only one pass band exists for a particular mode — the mode does not even exist outside this band —, the system is continuous in \( z \), and the inward traveling wave occurs only when the elements of the system themselves, or the "rings", are slightly beyond resonance.

The ratio of the phase velocity of a "free mode" wave along the axis of a transmission system, \( v_p \), to the phase velocity of a uniform plane wave in the medium, \( v_o \), is given by

\[
\frac{v_p}{v_o} = \frac{ka}{ha}.
\]

If the medium is free space, \( v_o = c \) where \( c \) is the velocity of light in free space. Also,

\[
\frac{\lambda_g}{\lambda} = \left| \frac{v_p}{v_o} \right| = \left| \frac{ka}{ha} \right|,
\]

where \( \lambda_g \) is the wavelength of the "free mode" waves on the system. Phase velocity is the velocity with which a constant phase front or point of a steady state wave may be considered to travel. In the previous discussion the inward and outward traveling waves were distinguished by the direction of phase change in the axial direction. With \( e^{j\omega t} \) as the time variable, outward traveling waves are those with \( z \) variation like \( e^{-j|h|z}, z > 0 \), or \( e^{j|h|z}, z < 0 \), that is, with increasing phase retardation away from the
source. On the other hand, inward traveling waves are those with \( z \) variation like \( e^{jhwz}, z > 0 \), or \( e^{-jhwz}, z < 0 \), that is, with increasing phase advance away from the source. Group velocity is generally considered as the envelope velocity of a narrow spectrum of waves, or as the velocity with which energy is propagated along the system.\(^{1,5,39}\) In either case, with \( v_g \) as the group velocity and \( v_o \) as before,

\[
\frac{v_g}{v_o} = \frac{d(ka)}{d(ka)}.
\]

(See also Section II.6.)

In Figs. II-4 and II-6 the slope of a straight line from the origin to some point on a curve equals \( \frac{v_p}{v_o} \), whereas the slope of a tangent line at the point is the value of \( \frac{v_g}{v_o} \), both for a particular \( ka \) and \(|n|\). It is evident that \( \frac{v_p}{v_o} \) and \( \frac{v_g}{v_o} \) may be positive or negative. Further, it should be noted that it is the phase and group velocities in the \( z \) direction which are defined above. Only these are considered throughout this chapter.

For the purposes of the following discussion, which is quite similar to the previous one except that the remarks refer to the phase and group velocities, it is convenient to consider \( z \) very large and positive and to confine attention to the solid curves in Fig. II-6. However, similar remarks can be made if one takes \( z < 0 \) and regards only the dotted curves in Fig. II-6. For \(|n| > ka > \sqrt{n^2 - 1}\) only an outward traveling wave exists. For all modes the initial value of \( \frac{v_p}{v_o} \) for this wave is unity and decreases as \( ka \) increases. The initial value of \( \frac{v_g}{v_o} \) is unity only for \(|n| = 1, 2\) and can be shown to be given by

\[
\frac{v_g}{v_o} = \frac{1}{1 + \frac{2(n^2 - 1)n^2}{5n^2 - 2}},
\]

for \(|n| > 3\). These values are limiting values which result if \( ka \) is allowed to approach \( \sqrt{n^2 - 1} \) from above. Note that the group velocity of the outward traveling wave is positive as would be expected since the total average
flow of power must be outward. For \(|n| + \Delta_n n > |n|\) outward and inward traveling waves exist, and the phase velocity of the latter is, of course, negative for \(z > 0\). The magnitude of the phase velocity of the inward traveling wave is at first zero and increases rapidly as \(ka\) increases, so that at \(ka = |n| + \Delta_n\) it equals the magnitude of the phase velocity of the outward traveling wave. It can be seen that the group velocity of the inward traveling wave is at first zero, increases to a small positive value, and then decreases to zero again for \(ka = |n| + \Delta_n\). Even though the phase velocity of the inward traveling wave is negative, its group velocity is positive or zero. Since both waves have positive group velocities, the total average flow of power is outward for \(|n| + \Delta_n > ka > |n|\), as well as for \(|n| > ka > \sqrt{n^2 - 1}\) where only an outward traveling wave exists. As already noted, for \(ka = |n| + \Delta_n\) the magnitude of the phase velocity of both the outward and inward traveling waves is the same, and the amplitudes of their field components are likewise equal. Consequently, a pure standing wave in \(z\) exists, and there is no net average power flow in either direction. This is indicated by the zero slope of the \(ka\) versus \(|h_{n,m}|\) curves in Figs. II-4 and II-6 for \(ka = |n| + \Delta_n\). The problem of power flow is considered in greater detail in Section II.6.

(c) The Sheath Tube, \(\Psi = 90^\circ\)

For \(\Psi = 90^\circ\) the boundary conditions (II.2-1) through (II.2-7) become

\[ E_i = E_e = E_z = E_\theta = 0, \quad (9) \quad E_i = E_e = E_i = E_\theta, \quad (10) \]

\[ H_i = H_e = H_z = H_\theta, \quad (11) \]

for \(r = a, 2\pi > \theta > 0, \) and \(\infty > z > -\infty\). The sheath tube system may be considered as the limiting one which results from several perfectly conducting infinitely long \(z\) directed wires spaced uniformly on the
circumference of a circle of radius $a$, when the diameter of the wires and 
their spacing become infinitesimally small, while the number of wires be-
comes infinitely large. It is shown in Section B.3 that for $h = \pm k$ "free 
mode" solutions can exist, and in this case the solutions are TEM waves, 
$E_z = H_z = 0$. The field expressions may therefore be readily derived from 
static considerations since, as is well-known, for TEM waves the distribu-
tion of the fields in the transverse plane, the $xy$ plane here, is a static 
one. Although this section is concerned primarily with a discussion of 
the determinantal equation, it seems worthwhile to digress briefly and 
derive the fields for $\Psi = 90^\circ$ in a simple manner.

For $n = 0$, although a solution to the Maxwell equations exists for 
$r > a$ for these boundary conditions, it is not a "free mode" in the sense 
used here since it can never be excited by a finite source. For $|n| \gg 1$, 
however, "free mode" fields exist which can be simply derived in the 
following manner. Assume that current flows only in the $z$ direction on 
the surface of a cylinder of radius $a$, and that the current density dis-
tribution $K_z$ is given by

$$K_z = |K_u| \cos n \theta e^{-j k z},$$

with $|n| \gg 1$. A cross sectional view of such an arrangement is shown in 
Fig. II-7 and is, of course, the sheath tube. The mediums for $r > a$ and

![The Sheath Tube](image)
II.5(c)

$r \lessgtr a$ are assumed, as before, to be the same and lossless. The continuity condition (II.3-21) gives for the surface charge density

$$q_s = -\frac{\nabla S\cdot \mathbf{E}}{j\omega} = \frac{K}{\omega} |K_n| \cos m\theta \cos e^{-jkr}.$$  

(13)

The potential $V$ at point $P$ resulting from this surface charge is given by

$$V = \frac{a}{2\pi \varepsilon} \int_0^{2\pi} q_s \text{ln} \frac{1}{R} \, d\theta' + C,$$  

(14)

where $R$ and $\theta'$ are defined as shown in Fig. II-7, and $C$ is some arbitrary constant. By using the expansions

$$\text{ln} \frac{1}{R} = \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{2}{a}\right)^m \cos m\theta' \cos m\theta + \sin m\theta' \sin m\theta - \text{ln} r, \ r > a,$$  

(15a)

and

$$\text{ln} \frac{1}{R} = \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{2}{a}\right)^m \cos m\theta' \cos m\theta + \sin m\theta' \sin m\theta - \text{ln} a, \ r < a,$$  

(15b)

(see reference 50, page 252) in (14), there results

$$V^e = \frac{a}{2\sqrt{\varepsilon}} |K_n| \cos m\theta \text{ln} \frac{1}{m} \left(\frac{2}{a}\right) |m| e^{-jkr} + C, \ r > a,$$  

(16a)

and

$$V^i = \frac{a}{2\sqrt{\varepsilon}} |K_n| \cos m\theta \text{ln} \frac{1}{m} \left(\frac{2}{a}\right) |m| e^{-jkr} + C, \ r < a.$$  

(16b)

The electric field components are given by

$$E_r = -\frac{\partial V}{\partial r}, \quad (17) \quad E_{\theta} = -\frac{1}{r} \frac{\partial V}{\partial \theta}, \quad (18)$$

with $E_z = 0$, and the magnetic field components can be obtained from (17) and (18) by using the Maxwell equation (II.3-1). Proceeding in this manner, one finds the field components to be

$$E_r^e = -\frac{1}{2} \sqrt{\varepsilon} |K_n| \cos m\theta \left(\frac{2}{a}\right) |m| -1 e^{-jkr},$$  

(19a)

$$E_r^i = \frac{1}{2} \sqrt{\varepsilon} |K_n| \cos m\theta \left(\frac{2}{a}\right) |m| +1 e^{-jkr},$$  

(19b)
\[ E^i_{\theta} = \frac{1}{2} \sqrt{\frac{\mu}{\varepsilon}} |K_{\|}| \sin m\theta \left(\frac{a}{r}\right)^{m+1} e^{-jkz} \quad , \tag{20a} \]
\[ E^o_{\theta} = \frac{1}{2} \sqrt{\frac{\mu}{\varepsilon}} |K_{\|}| \sin m\theta \left(\frac{a}{r}\right)^{m} e^{-jkz} \quad , \tag{20b} \]
and
\[ H^i_r = -\frac{1}{2} |K_{\|}| \sin m\theta \left(\frac{a}{r}\right)^{m-1} e^{-jkz} \quad , \tag{21a} \]
\[ H^o_r = -\frac{1}{2} |K_{\|}| \sin m\theta \left(\frac{a}{r}\right)^{m} e^{-jkz} \quad , \tag{21b} \]
\[ H^i_\theta = -\frac{1}{2} |K_{\|}| \cos m\theta \left(\frac{a}{r}\right)^{m-1} e^{-jkz} \quad , \tag{22a} \]
\[ H^o_\theta = \frac{1}{2} |K_{\|}| \cos m\theta \left(\frac{a}{r}\right)^{m} e^{-jkz} \quad . \tag{22b} \]

It can be shown, as should be evident perhaps from the derivation, that the above constitute permissible "free mode" solutions for \(|n| \geq 1\), since for each \(n\) they satisfy the Maxwell equations and the boundary conditions, and can be excited by a finite source.

If it is now assumed that a surface current density of the form

\[ K_z = j |K_{\|}| \sin m\theta e^{-jkz} \quad (23) \]

exists, the resulting field expressions derived by the above procedure are quite similar in form to (19) through (22), the component \(E^o_{\theta}\), for example, being given in this case by

\[ E^o_{\theta} = -j \frac{1}{2} \sqrt{\frac{\mu}{\varepsilon}} |K_{\|}| \cos m\theta \left(\frac{a}{r}\right)^{m} e^{-jkz} \quad . \tag{24} \]

Since the field equations are linear, (12) and (23) can be added to obtain a total current density like

\[ K_z = |K_{\|}| e^{j m\theta} e^{-jkz} \quad , \tag{25} \]

the resulting field components being obtained by adding (19) through (22) to their counterparts obtained from (23). Thus, for example, with a
current density like (25), the \( E_\theta^e \) component becomes

\[
E_\theta^e = - \frac{j}{2} \sqrt{\frac{\mu}{\varepsilon}} |K_u| \left( \frac{a}{r} \right)^{ln^1 + 1} e^{j \ln |\theta|} e^{-j k z} . \tag{26}
\]

As might be expected from physical considerations, a set of solutions for the sheath helix approaches the type of waves considered here as \( \Psi \) approaches \( 90^0 \). In fact, for these it is found that as \( \Psi \) approaches \( 90^0 \), some of the \( h \) approach \( k \) in magnitude and the corresponding \( I \) approach zero. Using this in the field expressions (II.4-15) through (II.4-20) and similar ones for the internal fields, it can be shown that in the limit of \( \Psi = 90^0 \) one obtains precisely the current density given by (25) and its associated field components. Of course, one may obtain solutions whose \( \theta \) dependence is like \( e^{-j \ln \theta} \) by finding the fields resulting from a current density distribution like

\[
K_z = - j |K_u| \sin \ln \theta e^{-j k z} , \tag{27}
\]

and adding these to the fields obtained using (12); or again, with some precautions these may be obtained from the limiting values of the sheath helix solution. The limit of the sheath helix solution always gives fields whose \( \theta \) dependence is like \( e^{-j n \theta} \), since, indeed, for \( \Psi \neq 0^o \) or \( 90^0 \) only this form of \( \theta \) dependence is allowed by the boundary conditions. However, as seen above and as would be expected from the symmetry of the boundary conditions, for \( \Psi = 90^0 \) linearly as well as circularly polarized fields are possible. It should be noted, and more is said of this in Part (a) of this section, that only those sheath helix modes for which the product \( nh \) is negative go over into the TEM waves discussed in the foregoing as \( \Psi \) approaches the limit of \( 90^0 \).

The solutions with \( e^{-j k z} \) dependence correspond to waves whose phase retardation increases in the positive \( z \) direction. Solutions whose phase retardation increases in the negative \( z \) direction, that is,
with $z$ dependence like $e^{jkz}$, can also exist on a sheath tube and can be readily found in the above manner. Thus, solutions with $z$ dependence like $\sin kz$ or $\cos kz$, as well as with $\theta$ dependence like $\sin|n|\theta$ and $\cos|n|\theta$ can occur; and, of course, these correspond to standing waves in $\theta$ or $z$ or both. Here, as for $\psi = 0^\circ$, such solutions would fit the boundary conditions imposed if perfectly conducting infinite planes were placed perpendicular to the $z$ axis or along planes of constant $\theta$ -- at intervals of $\frac{\pi}{|n|}$ -- or both. However, contrary to the sheath ring case, only outward traveling waves can occur on an unbounded sheath tube as the result of excitation by a single finite source. Further, the magnitude of the ratios $\frac{V_P}{V_O}$ and $\frac{V_E}{V_O}$ for these waves is always unity for all values of $ka$ as would be expected for TEM waves. The solutions discussed here correspond to the various symmetrical component waves or modes which can exist on a multiwire circular "cage" transmission line as the number of wires and, consequently, the number of component waves becomes infinitely large. The positive and negative sequence waves correspond to the $e^{\pm|n|\theta}$ solutions, and the zero sequence solution, $n = 0$, may be said not to be possible since there is no "ground return".

(d) The Sheath Helix, the General Case

For the sheath ring, $\psi = 0^\circ$, and the sheath tube, $\psi = 90^\circ$, solutions whose $\theta$ dependence is like $\sin|n|\theta$ or $\cos|n|\theta$ are possible. In the general sheath helix case, $\psi \neq 0^\circ$ or $90^\circ$, it can be shown that such solutions are not possible, and the $\theta$ dependence must be like $e^{-jn\theta}$ if the boundary conditions are to be satisfied. Of course, for $n = 0$ the sheath helix solutions are independent of $\theta$, but the effect of the boundary conditions is still evident as noted below. In Figs. II-2 and II-3 the single wire helix from which the sheath helix is developed by the multiple winding and limit process is shown as positively wound. Such a helix is one in which motion of a point along the wire axis, which
increases the \( z \) coordinate of the point positively, increases its \( \theta \) coordinate in a positive direction, and for this \( \cot \psi \) and \( \psi \) are positive. A negatively wound helix is one in which motion of a point along the wire axis, which increases its \( z \) coordinate positively, decreases \( \theta \), and for this \( \cot \psi \) and \( \psi \) are negative. The same definition applies in an obvious manner to the sheath helix. It is clear that either a positively or negatively wound helix may be considered without altering the really fundamental characteristics of the possible solutions. In other words, if field component solutions exist for a positively wound sheath helix with a functional form like \( f_n(r)e^{-jhz}e^{-jn\theta} \), then solutions of the form \( f_{-n}(r)e^{-jhz}e^{jn\theta} \) also exist for the negatively wound sheath helix, both for a given positive or negative \( n \) and both with identical characteristics. That this must be so can also be seen from the determinantal equation

\[
\frac{I_n'((\xi a)K_n((\xi a)))}{I_n((\xi a)K_n((\xi a)))} = -\frac{(\xi^2a^2 + nba \cot \psi)^2}{k^2a^2 \xi^2a^2 \cot^2 \psi}, \quad (II.4-8)
\]

where the product \( n \cot \psi \) occurs and where the other functions of \( \psi \) and \( n \) are even. The choice of positive or negative \( \psi \) fixes the boundary conditions, that is, the direction of "skewness" with respect to the coordinates, and once having taken either sign of \( \psi \), this must be retained throughout as the condition to which the solutions apply. Here, it is assumed that the helix is positively wound so that \( 90^\circ > \psi > 0^\circ \) or \( 0 < \cot \psi < \infty \).

Because of the "skewness" of the boundary conditions, it seems evident from physical considerations that if outward traveling wave solutions exist for the sheath helix with \( \theta \) dependence like \( e^{-j|n|\theta} \) and \( e^{+j|n|\theta} \) for a given \(|n|\), then these must have different propagation characteristics. Roughly speaking, one of these solutions represents waves traveling along the "wires", whereas the other represents waves traveling across the "wires". It would be expected that the properties
of these solutions must be different. A similar remark can clearly be made about inward traveling wave solutions. The determinantal equation indicates this since in the numerator of the right side of (II.4-5) the product $nh$ occurs. Consequently, with $h$ positive or negative, this numerator is different for $n$ positive than for $n$ negative, the other factors in the determinantal equation being unchanged since they are even functions of $n$ and $h$. Note, however, that if solutions are obtained for $h$ positive for both positive and negative $n$, then the same solutions exist for $h$ negative but with the $n$ numbering reversed. This is to be expected from the required $z$ and $\theta$ dependence of the field solutions, $e^{-jhz}e^{-j\theta}$, and from the physically evident requirement that if solutions exist for increasing or decreasing phase retardation in the positive $z$ direction with positive and negative angular phase retardation, then there must exist identical solutions with increasing or decreasing phase retardation in the negative $z$ direction with negative and positive angular phase retardation, respectively. It should be noted that in Section B.4 the determinantal equation is discussed on the assumption that $h$ is positive, but this leads to no loss of generality in view of the above remarks.

For $n = 0$ the sheath helix determinantal equation becomes

$$\frac{I_1(\xi a)K_1(\xi a)}{I_0(\xi a)K_0(\xi a)} = \frac{\xi}{k^2} \frac{2a^2}{2\cot^2 \psi}.$$  \hspace{1cm} (28)

Fig. II-8 shows some exact results obtained from (28) for particular values of $\psi$, where $\frac{ka}{h_0^m a}$ is plotted versus $ka$. The $m$ subscript on $h_{n,m}$ for $n = 0$ is unnecessary and is omitted. This sheath helix mode has been rather thoroughly considered by other investigators, since for $ka$ between about 0.1 and 0.5 its characteristics closely approximate those of the single wire helix which has been extensively used in traveling-wave
If ka is plotted versus $|h_o|a$ for this mode, for a given $\psi$ the result appears as in Fig. II-9. Since (28) is even in $h$, an identical curve exists for negative $|h_o|a$ which can be obtained by reflection about the ka axis in Fig. II-9. The initial magnitude of \[ \frac{ka}{|h_o|a} = \frac{V_p}{V_o} \] is unity and decreases as ka increases, approaching the asymptotic value of $\sin \psi$ from above. The magnitude of the initial slope of the ka versus $|h_o|a$ curve, $\left| \frac{V_p}{V_o} \right|$, is also unity and decreases as ka increases, also approaching the asymptotic value of $\sin \psi$. This asymptotic value is the result one might expect from simple considerations since, if a wave traveled along the sheath "wires" with the velocity $v_o$, it would be anticipated that the axial velocity, or velocity in the z direction, would be $v_o \sin \psi$. 9, 10
For a given $ka$, the magnitude of the phase velocity decreases as $\psi$ becomes smaller. For any small but finite $\psi$, a solution exists, although for $\psi$ exceedingly small, the phase and group velocities have correspondingly small magnitudes. For $\psi = 0^\circ$, that is, for the sheath ring, there is no "free mode" solution for $n = 0$ with finite fields for $r > a$ and $r < a$ together. As $\psi$ approaches $90^\circ$, the magnitudes of the phase and group velocities differ by only a small amount from $v_\psi$, and $\zeta a$ becomes small for all $ka$. In fact, for $\psi$ near but not equal to $90^\circ$, for the region where $\zeta r$ is small but $r$ is not small, it can be shown from the field expressions (II.4-15) through (II.4-20), with $n = 0$, that all the external field components except $E_r^e$ and $H_\theta^e$ are of small order, that these two are proportional to $\frac{1}{r}$, that the magnitude of their ratio is given by $\sqrt{\frac{1}{\varepsilon}}$, and that $\int rH_\theta^e d\theta$ equals the total enclosed conduction current. Thus, for $\psi$ close to $90^\circ$, the external fields of the $n = 0$ mode resemble those of the TEM wave surrounding a straight infinite perfectly conducting wire. However, as noted previously and in Section 8.3, for $r > a$ the $n = 0$ mode for the sheath tube, $\psi = 90^\circ$, is a nonphysical solution since it requires that an infinite amount of power be supplied to the system.

Since the determinantal equation for $n = 0$ is even in $h$, similar solutions are obtained for positive and negative $h$, that is, $h = \pm |h_0|$. These are interpreted in the usual manner as similar waves traveling in the positive and negative $z$ direction. However, despite this symmetry, it must not be thought that the wave traveling in the negative $z$ direction can be obtained from the wave traveling in the positive $z$ direction simply by reflection from an infinite perfectly conducting plane placed perpendicular to the $z$ axis. As indicated in reference 22, even for the $n = 0$ mode, "a wave reflected at a plane boundary tends to spiral in the wrong direction." Another way of viewing this is to note that from the field expressions (II.4-15) through (II.4-20) it is not possible to choose
positive and negative traveling waves which result in a pure standing
wave in \( z \) for which both \( E_x \) and \( E_\theta \) are simultaneously zero over a plane
of constant \( z \). Incidentally, some field plots for this mode are shown
in references 19 and 22.

For \( n \neq 0 \) the propagation characteristics of the sheath helix modes
are somewhat more complicated than for the various special cases discussed
so far. In the curves which are presented now for these propagation char-
acteristics, portions of the lines are shown solid and dotted in differ-
ent ways. As in the sheath ring case these different parts correspond to
different branches of the curves. The procedure whereby this is deduced
is explained best by considering the inhomogeneous or source-present
problem; this is done in Section II.7. Here, only the results and their
physical significance are discussed. The modes for which \( n h_n^1 m > 0 \) are
considered first. The single prime is used here to distinguish these
solutions for \( \psi \neq 0^\circ \) from another set for \( \psi \neq 0^\circ \) considered shortly, and
from those for \( \psi = 0^\circ \). The \( m \) values here are related to the ones used
for \( \psi = 0^\circ \), although now, for \( \psi \neq 0^\circ \), an additional solution may occur.
The following discussion should make these remarks clearer. Graphs of
\( \frac{ka}{|h_n^1|} \) versus \( ka \) for \( |n| = 1 \) and \( |n| = 2 \) are shown in Fig. II-10. For
\( |n| > 2 \) the characteristics of the modes are essentially identical. As
in the sheath ring case the \( |n| = 1 \) mode is somewhat exceptional in
that a solution persists to arbitrarily small values of \( ka \). Plots of
\( ka \) versus \( |h_n^1| \) are shown in Fig. II-11 for \( |n| = 1 \) and \( |n| = 2 \). In
Figs. II-10 and II-11 the \( \psi = 0^\circ \) curve is included for reference and to
indicate how the transition occurs. As \( \psi \) approaches and becomes equal
to \( 0^\circ \), \( |h_n^1| \) becomes \( |h_n^2| \), \( |h_n^3| \) becomes \( |h_n^2| \), and \( |h_n^3| \) vanishes.
Note that for \( ka \) slightly larger than \( |n| \) and \( \psi > 0^\circ \) three solutions or
waves occur. This happens only for exceedingly small values of \( \psi \) and
over a narrow range of \( ka \) for \( ka \) slightly larger than \( |n| \). For example,
by calculation using the determinantal equation (II.4-8), it has been
found that for \(|n| = 1\) the propagation characteristics are given by a
curve like the one labeled \(1 \Psi_1\) for \(0.191^\circ \leq \psi \leq 0^\circ\), whereas for
\(\psi > 0.286^\circ\) the curve becomes like the one labeled \(1 \Psi_2\). The transition
occurs for \(0.286^\circ > \psi > 0.191^\circ\) and for \(ka\) approximately equal to 1.026.
It has not been possible to find a simple expression which gives the
largest \(\psi\), say \(\Psi_{\text{max}}\), for which curves like those labeled \(1 \Psi_1\) for
\(|n| = 1\), \(2 \Psi_1\) for \(|n| = 2\), etc. just barely occur, that is, the \(|h_{n,2}'|\)
and \(|h_{n,3}'|\) branches just barely occur, although a straightforward but
lengthy calculative procedure for finding \(|n| \Psi_{\text{max}}\) is available. Since
the specific value of \(|n| \Psi_{\text{max}}\) seems to be of minor interest, rather
than describe this procedure fully, it should be sufficient to note that
\(|n| \Psi_{\text{max}}\) is less than 0.286\(^\circ\) for \(|n| = 1\), and that it becomes increasing-
ly smaller as \(|n|\) increases, as may be deduced from the discussion in
Section B.4. Further, as \(|n|\) increases, the transition occurs for values
of \(ka\) which approach \(|n|\) more closely. It can be realized from the above
that the transition effect has been very much exaggerated in Figs. II-10
and II-11 in order to show it in detail.

As in the \(n = 0\) case for \(ka\) becoming larger, for all \(|n|\) the value
of \(\frac{ka}{|h_{n,m}'| a}\) decreases from its initial value and approaches the asymptotic
value of \(\sin \psi\) from above for both the \(|h_{n,1}'|\) and \(|h_{n,3}'|\) branches.
However, the values for different \(|n|\) are still slightly different, the
magnitudes being such that

\[
\left[ \frac{ka}{|h_{0}'| a} \right]_{n=0} < \left[ \frac{ka}{|h_{n,m}'| a} \right]_{|n|=1} < \left[ \frac{ka}{|h_{n,m}'| a} \right]_{|n|=2} \quad \text{etc.} \quad (29)
\]

for a fixed value of \(\psi\) for large values of \(ka\). This result can be
deduced for large \(ka\) from Fig. B-5d and the attendant discussion in
Section B.4. Actually, (29) holds for any value of \(ka\), as well as for
\(ka\) large, for a fixed value of \(\psi\) where \(90^\circ > \psi > 0^\circ\), as can be seen from
an examination of Figs. II-8 and II-10.

In the sheath ring case the $|n|^{th}$ mode solutions occur only for $|n| + \Delta_n k_a \geq \sqrt{n^2 - 1}$. In other words, for $\Psi = 0^\circ$ there is both an upper and lower "cut-off" or "divergence" frequency, and outside the band defined by these the $|n|^{th}$ mode does not exist. For $\Psi \neq 0^\circ$ an upper frequency limit no longer exists, and for $|n| \geq 2$ the low frequency limit is modified. It is shown in Section B.4 that if $k_a < n k_d a$ for a given $\Psi$, where

$$n k_d a = \frac{(n^2 - 1) + \sqrt{(n^2 - 1)^2 + (n^2 - 1)n^2 \cot^2 \Psi}}{|n| \cot \Psi},$$

(30)

or if $\cot \Psi < \cot n \Psi_{min}$ for a given $ka$, where

$$\cot n \Psi_{min} = \frac{2ka}{|n|(\frac{k_a^2}{n^2 - 1} - 1)},$$

(31)

then there is no solution for the $|n|^{th}$ mode. It should be recalled that the modes for which $n h_n^l, m > 0$ are being considered. These relationships are presented graphically in Fig. II-12 for $|n| = 2, 3, 4, 5$, and the curves there are to be interpreted in the following manner.

If, for example, $\Psi = 20^\circ$, there are solutions for the $|n| = 2$ mode only for $ka > 2.36$, the $|n| = 3$ mode only for $ka > 3.96$, the $|n| = 4$ mode only for $ka > 5.47$, the $|n| = 5$ mode only for $ka > 6.95$, etc. Conversely, if $ka = 4.0$, there are no solutions for those modes for which $|n| \geq 5$, whereas there are solutions for the $|n| = 4$ mode but only for $\Psi < 1.9^\circ$; for the $|n| = 3$ mode but only for $\Psi < 20.5^\circ$; for the $|n| = 2$ mode but only for $\Psi < 47.3^\circ$; and for the $|n| = 1$ mode for $90^\circ > \Psi > 0^\circ$. Thus, as $\Psi$ becomes increasingly large, the minimum value of $ka$ for which the $|n| \geq 2$ modes can occur, $n k_d a$, also becomes increasingly large, such that in the limit of $\Psi = 90^\circ$ these modes vanish. For $\Psi$ very close to $90^\circ$ and for the $|n| = 1$ mode the solution appears as essentially a perturbed uniform
plane wave which is circularly polarized. This can be shown from the field expressions by taking \( \cot \psi \) very small so that \( |h_{\pm l, l}^i| a \) is close to \( k a \) with \( \xi a \) exceedingly small and by considering the region where \( \xi r \) is small but \( r \) is not necessarily small. In this case the \( E_{z e}^i \) and \( H_{z e}^i \) components are substantially \( \cot \psi \) times as large as the \( E_{r e}^i \) and \( H_{r e}^i \) components, the latter are essentially independent of \( r \), and the ratios \( \frac{E_{z e}^i}{H_{z e}^i} \) and \( \frac{E_{r e}^i}{H_{r e}^i} \) are given by \( -j \frac{H}{k} \). Since the magnitudes of both \( \frac{H_{r e}^i}{H_{z e}^i} \) and \( \frac{H_{r e}^i}{H_{r e}^i} \) are equal to \( |K_{ll}| \) for \( \cot \psi \) very small, if this mode existed in the limiting case of \( \psi = 90^\circ \) with finite \( |K_{ll}| \), it would transmit infinite energy. Consequently, such a "free mode" must be excited to only a very small amplitude by any finite source for \( \psi \), near \( 90^\circ \), and \( |K_{ll}| \) must approach and finally equal zero as \( \psi \) approaches and finally equals \( 90^\circ \). Therefore, for all \( |m| \) and \( nh_{n, m}^i > 0 \) for any \( k a \), with \( \psi \) increasing, the solutions vanish in the manner indicated above as the sheath helix becomes the sheath tube.
For \( nh_{n,m}^n < 0 \), graphs of \( \frac{ka}{h_{n,m}^n} \) versus \( ka \) for \( |n| = 1, 2, \) and 3 are shown in Fig. II-13. For identical values of \( |n| \), Fig. II-14 shows curves of \( ka \) versus \( -|h_{n,m}^n|a \). The double prime is used in these to distinguish them from the previous solutions for \( \psi \neq 0^\circ \) but where \( nh_{n,m}^1 > 0 \), and from the solutions for \( \psi = 0^\circ \). Here again, the \( m \) values correspond to those for \( \psi = 0^\circ \), although an additional solution occurs for \( \psi \neq 0^\circ \) also. As in many previous graphs of the propagation constants, the curves in Figs. II-13 and II-14 are essentially qualitative only and are considerably distorted for convenience in plotting; nevertheless, their relative positions for different \( \psi \) and the signs of their slopes are quite correct. The propagation characteristics of the various modes are similar, particularly for \( |n| > 3 \). The solutions are shown in Fig. II-14 for negative \( |h_{n,m}^n| \) for convenience in a later discussion. The \( \psi = 0^\circ \) curve is included in Figs. II-13 and II-14 for reference and to indicate how the transition occurs. As \( \psi \) approaches and becomes equal to \( 0^\circ \), \( |h_{n,1}^n| \) becomes \( |h_{n,1}'| \), \( |h_{n,2}^n| \) becomes \( |h_{n,2}'| \), and \( |h_{n,4}^n| \) vanishes. The fact that for \( \psi = 0^\circ \) the \( -|h_{n,1}^n|a \) and \( -|h_{n,2}^n|a \) branches as a function of \( -|h_{n,m}^n|a \) are shown as solid and dotted curves, respectively, in Fig. II-14 and in an opposite manner in Fig. II-6 is unimportant and should lead to no confusion. The prime on the various \( \psi_m \) in Figs. II-13 and II-14 is used merely to distinguish them from those in Figs. II-10 and II-11 since, in general, they need not be equal. Also, in both cases the \( n, m \) subscript numbering of \( \psi \) is used only to aid in labeling the curves and to indicate what happens as \( \psi \) increases and has no other significance.

For \( |n| = 1 \) with \( nh_{n,m}^n < 0 \) for \( 90^\circ > \psi > 0^\circ \), three solutions or waves occur for \( ka > 0 \) but not too large. As \( ka \) increases for a given \( \psi \), two of these, the \( |h_{n,1}^n| \) and \( |h_{n,2}^n| \) branches, come together, and for \( ka \) still larger only the \( |h_{n,4}^n| \) branch remains. Although not entirely evident from Fig. B-7a from which the solutions for the \( |h_{n,1}^n| \)
FIG. II-13a

$\psi = 0^\circ$

$\psi'_{2}$

$\psi'_{1}$

$\psi_{2}$

$\psi_{1}$

$|h'_{n,m}|a$

$|n| = 1$ and $nh'_{n,m} < 0$

The Sheath Helix

$\psi'_{2} > \psi'_{1} > 0^\circ$

$ka$ vs. $ka$

FIG. II-13b

$\psi = 0^\circ$

$\psi'_{2}$

$\psi'_{1}$

$\psi_{2}$

$\psi_{1}$

$|h'_{n,m}|a$

$|n| = 2$ and $nh'_{n,m} < 0$

The Sheath Helix

$\psi'_{2} > \psi'_{1} > 0^\circ$

$ka$ vs. $ka$

FIG. II-13c

$\psi = 0^\circ$

$\psi'_{2}$

$\psi'_{1}$

$\psi_{2}$

$\psi_{1}$

$|h'_{n,m}|a$

$|n| = 3$ and $nh'_{n,m} < 0$

(Similar for $|n| > 4$)

The Sheath Helix

$\psi'_{2} > \psi'_{1} > 0^\circ$

$ka$ vs. $ka$
and \(|h_{n,2}^{n}|\) branches are obtained for \(|n| = 1\), calculation shows that these branches exist for \(90^\circ > \psi > 0^\circ\). However, they vanish for smaller values of \(ka\) as \(\psi\) approaches \(90^\circ\). It is shown in Section B.4 that for \(|n| = 2\) with \(nh < 0\) for a given \(\psi\), if

\[
ka < \frac{(n^2 - 1)}{nh\cot\psi} \left( \sqrt{1 + \frac{n^2 \cot^2\psi}{n^2 - 1}} - 1 \right), \quad (32)
\]

then only two solutions, the \(|h_{n,2}^{n}|\) and \(|h_{n,4}^{n}|\) branches, occur. For \(ka\) larger than the limit given by (32) the \(|h_{n,1}^{n}|\) branch also occurs, but only over a small range of \(ka\). As \(ka\) increases, the \(|h_{n,1}^{n}|\) and \(|h_{n,2}^{n}|\) branches come together, and for \(ka\) still larger only the \(|h_{n,4}^{n}|\) branch remains. For \(|n| = 2\) as for \(|n| = 1\) for any \(\psi < 90^\circ\), the \(|h_{n,1}^{n}|\) and \(|h_{n,2}^{n}|\) branches always occur. However, for \(|n| \geq 3\), although the limitation given by (32) still exists, if \(\psi\) is larger than a minimum value for a particular \(n\), the \(|h_{n,2}^{n}|\) and \(|h_{n,4}^{n}|\) branches become the only solutions. In this case the propagation characteristic curve becomes like the one labeled \(\psi_2\) in Fig. II-11c. From the discussion and equations in Section B.4 associated with Fig. B-9a, it can be shown that this occurs for \(\psi > 16^\circ\) for \(|n| = 3\), for \(\psi > 9^\circ\) for \(|n| = 4\), etc. It has not been possible to find a simple expression which gives the value of \(ka\) for a given \(\psi\) and \(n\) for which the \(|h_{n,1}^{n}|\) and \(|h_{n,2}^{n}|\) branches come together when both such branches exist. It is clear that this occurs for \(ka\) greater than the value given by (32) satisfied with an equal sign, and it is believed to be within \(\Delta_n\) of this value.

As \(ka\) becomes larger for all \(|n|\), the value of \(\frac{ka}{|h_{\nu,1}|a}\) increases from its initial value and approaches the asymptotic value of \(\sin \psi\) from below. However, the values for different \(|n|\) are still slightly different, the magnitudes being such that

\[
\left[ \frac{ka}{|h_0|a} \right]_{n=0} > \left[ \frac{ka}{|h_{n,1}|a} \right]_{|n|=1} > \left[ \frac{ka}{|h_{n,1}|a} \right]_{|n|=2}, \quad \text{etc.} \quad (33)
\]
FIG. II-1a
\[-|h''_{n,m}|a\] ka vs. \[-|h''_{n,m}|a\]
\[|n|=1 \text{ and } nh_{n,m} < 0\]
The Sheath Helix
\[1\Psi_2' > 1\Psi_1' > 0^\circ\]

FIG. II-1b
\[-|h''_{n,m}|a\] \[\sqrt{ka} = |h''_{n,m}|a\]
\[|n|=2 \text{ and } nh_{n,m} < 0\]
The Sheath Helix
\[2\Psi_2' > 2\Psi_1' > 0^\circ\]

FIG. II-1c
\[-|h''_{n,m}|a\] \[ka = |h''_{n,m}|a\]
\[|n|=3 \text{ and } nh_{n,m} < 0\]
(Similar for \[|n| \geq 4\])
The Sheath Helix
\[3\Psi_2' > 3\Psi_1' > 0^\circ\]
for a fixed value of $\psi$ for large $ka$. This result is evident from Fig. B-6
and the accompanying discussion in Section B.4. It is of interest to note,
as can be seen in Fig. II-11; and as can be shown from (B.4-6), that for
$k_a$ approaching zero both $|h_{n,2}''|$ and $|h_{n,1}'| a$ approach $|n| \cot \psi$. It
should now be clear how the propagation characteristics of the modes for
which $n h_{n,m}'' < 0$ change as $\psi$ varies from $0^\circ$ to $90^\circ$. As $\psi$ approaches $90^\circ$,
the $|h_{n,1}'| branch — it it exists at all — and the $|h_{n,2}''|$ branch exist
over a smaller range of $ka$, whereas the $|h_{n,1}'| branch approaches its
asymptotic value more closely over a larger range of $ka$. Thus, in the
limit of $\psi = 90^\circ$ for all $|n|$, the $|h_{n,1}'|$ and $|h_{n,2}''|$ branches disappear
leaving only the $|h_{n,1}'| branch for which $\frac{ka}{|h_{n,1}'| a} = 1$ for all $ka$. These
are TEM waves and are, in fact, the symmetrical component waves of the
sheath tube discussed in Part (c) of this section.

The curves of Figs. II-11 and II-14 are best understood by consider-
ing what happens when the sheath helix is driven by a finite source placed
at $z = 0$ which can excite only the $n^{th}$ mode. With no loss of generality
as far as this description is concerned, it is assumed that the $\theta$
dependence of the source and the resulting waves are of the form $e^{-jn\theta}$ with
$n > 0$. The $z$ axis coincides with the axis of the sheath helix, and the
radius of the sheath helix, $a$, as well as the pitch angle, $\psi$, are
considered to remain constant while the frequency is varied in the
following. It is now assumed that one stations oneself on the positive
or negative $z$ axis at such large distances from the source that the
radiation field from the source is negligible. If one is on the posi-
tive $z$ axis, depending on the value of $\psi$, one observes waves whose pro-
pagation constants and corresponding phase and group velocities vary
along the $|h_{n,1}'|$, $|h_{n,3}'|$, and $-|h_{n,2}''|$ branches as $ka$ is varied; these
waves have a $z$ dependence like $e^{j h_{n,1}'|z}$, $e^{-j h_{n,3}'|z}$, and $e^{+j h_{n,2}''|z}$,
respectively. Conversely, if one is on the negative $z$ axis, again
depending on the value of $\psi$, one observes waves whose propagation constants and corresponding phase and group velocities vary along the 

$|h_{n,2}|$, $-|h_{n,1}|$, and $-|h_{n,4}|$ branches as $ka$ is varied; these waves have a $z$ dependence like $e^{-jn\theta}$, $e^{jn\theta}$, and $e^{jn\theta}$, respectively.

The $|h_{n,2}|$ and $|h_{n,3}|$ waves may not exist if $\psi$ is larger than a quite small maximum value, and also the $|h_{n,1}|$ wave might not occur for $|n| \geq 3$ for $\psi$ still relatively small. Note that all the waves which exist for positive $z$ have positive group velocities, whereas all the waves which exist for negative $z$ have negative group velocities. Thus, the total power flow is always outward from the source, irrespective of the number of waves which occur for a particular $\psi$ and $ka$ or their phase velocities.

Further, if the medium is considered slightly lossy, all the waves become exponentially attenuated in the direction of energy propagation, again irrespective of their phase velocities.

A few other points are of interest. It would appear that the waves for a given mode or $n$ are inseparable. If the sheath helix is excited by a finite source having a $\theta$ dependence like $e^{-jn\theta}$, and if $ka$ and $\psi$ are of such value that more than one wave can exist, then this multiplicity of waves seems to occur always. Although the source may possibly be arranged so that one wave is more strongly excited than another, it does not appear possible that any of the waves can be completely eliminated in general. In the discussion of Figs. II-11 and II-14 it was assumed that the source and the resulting waves have $\theta$ dependence like $e^{-jn\theta}$ with $n > 0$. It should be clear that if a source with $\theta$ dependence like $e^{-jn\theta}$ with $n < 0$ is assumed, the waves which now appear for $z > 0$ and $z < 0$ are exactly those which appeared for $z < 0$ and $z > 0$, respectively, with $n > 0$. In the case of $n < 0$, the propagation constants may be considered as being obtained by a reflection about the $ka$ axis of Figs. II-11 and II-14 but with a change in the sign of $n$. This point
may be made clearer by noting again that the determinantal equation is unchanged if the sign of \( nh \) is unchanged. Thus,

\[
|n|h_{nl}^{\prime} = -|n|h_{nl}^{\prime \prime}, \quad (3la) \quad h_{nl}^{\prime}, m = -h_{nl}^{\prime \prime}, m, \quad (3lb)
\]

and

\[
|h_{nl}^{\prime}, m| = |h_{nl}^{\prime \prime}, m|. \quad (3lc)
\]

Also,

\[
|n|h_{nl}^{\prime} = -|n|h_{nl}^{\prime \prime}, \quad (35a) \quad h_{nl}^{\prime}, m = -h_{nl}^{\prime \prime}, m, \quad (35b)
\]

and

\[
|h_{nl}^{\prime}, m| = |h_{nl}^{\prime \prime}, m|. \quad (35c)
\]

Note that

\[
|h_{nl}^{\prime}, m| = |h_{nl}^{\prime \prime}, m| \neq |h_{nl}^{\prime}, m| = |h_{nl}^{\prime \prime}, m|, \quad (36)
\]

except for \( \Psi = 0^\circ \), so that

\[
|h_{nl}^{\prime}, m|_{\Psi = 0^\circ} = |h_{nl}^{\prime \prime}, m|_{\Psi = 0^\circ} = |h_{nl}^{\prime \prime}, m|, \quad (37)
\]

with the \( \pm \) chosen in any order in (37). If the source has \( \Theta \) dependence like \( \sin n\Theta \) or \( \cos n\Theta \), an identical mixture of all the waves occurs for \( z > 0 \) and \( z < 0 \). The final point here is concerned with noting a somewhat anomalous situation which occurs for \( \Psi \) near and equal to \( 90^\circ \). As previously mentioned, for \( \Psi \) approaching \( 90^\circ \) the modes for which \( nh < 0 \), in particular the \( h_{nl}^{\prime}, m \) waves, approach the symmetrical component waves of the sheath tube, whereas the modes for which \( nh > 0 \) occur only for increasingly large values of \( k\alpha \) or become of increasingly small amplitude. Thus, if the limit of \( \Psi = 90^\circ \) is approached in this manner, only waves with \( \Theta \) and \( z \) dependence like \( e^{-jkz} e^{\pm j|n|\Theta} \) with \( z > 0 \) or \( e^{+jkz} e^{\pm j|n|\Theta} \) with \( z < 0 \) seem possible. However, it is evident from symmetry and the discussion of Part (c) of this section that for \( \Psi = 90^\circ \) the \( \Theta \) and \( z \) dependence of the waves may be of the form \( e^{-jkz} e^{\pm j|n|\Theta} \) with \( z > 0 \) and \( e^{+jkz} e^{\pm j|n|\Theta} \) with \( z < 0 \). Consequently, it is evident that waves whose coordinate
dependence is like \( e^{-jkz e^{-j|n|\theta}} \) with \( z > 0 \) and \( e^{+jkz e^{+j|n|\theta}} \) with \( z < 0 \) are not stable on the sheath system considered here. For \( \Psi \) any small amount less than 90° and \( ka \) finite, such waves vanish for \( |n| > 2 \), although for \( |n| = 1 \) a wave with exceedingly small amplitude can exist.

II.6 Power Flow

Since the field expressions of the "free modes" are available, the real power flow associated with these modes can be readily determined. In the following it is only the real power which is considered. The real part of the complex Poynting vector gives the average intensity of the energy flow, and integrating this quantity over the appropriate cross section yields the desired power.\(^1\) Thus, with

\[
\mathbf{S}^* = \frac{1}{2} (\mathbf{E} \times \mathbf{H}) , \tag{1}
\]

where \( \mathbf{S}^* \) is the complex Poynting vector and the tilde means complex conjugate, the total average flow in the \( z \) direction is

\[
P_z = \text{Re} \left[ \int_A \mathbf{S}_z^* \, d\mathbf{A} \right] = \text{Re} \left[ \frac{1}{2} \int_A \mathbf{a}_z \cdot (\mathbf{E} \times \mathbf{H}) \, d\mathbf{A} \right] = \text{Re} \left[ \frac{1}{2} \int_0^\infty \int_0^{2\pi} \mathbf{a}_z \cdot (\mathbf{E} \times \mathbf{H}) \, r \, dr \, d\theta \right]. \tag{2}
\]

Since the field expressions for \( r < a \) and \( r > a \) are different, the integration on \( r \) must be performed in two steps so that

\[
P_z^{i,e} = \text{Re} \left[ \frac{1}{2} \int_0^{2\pi} \int_{0,a}^{a,\infty} (E_{r}^{i,e} H_{\theta}^{i,e} - E_{\theta}^{i,e} H_{r}^{i,e}) r \, dr \, d\theta \right] , \tag{3}
\]

and the total average power \( P_z \) is obviously the sum of \( P_z^{i} \) and \( P_z^{e} \). It is necessary, of course, that the values of \( \zeta a \) and \( ha \) which satisfy the determinantal equation (II.6-8) for particular values of \( ka \), \( n \), and \( \Psi \) be used in the power expression. In the following calculation of (3) it is assumed that only one wave occurs on the system, but this imposes no undue restriction on the final results as noted later.
Using the appropriate field expressions of the set (II.4-15) through (II.4-20) in (3), one finds after some manipulation using the determinantal equation and the Wronskian relationship that

$$P_z^e = \pi \sin^2 \psi |K_n|^2 \omega a \left( \frac{\hbar a \cot^2 \psi}{\xi a} \right) \int_0^\infty \left[ K_n^2(\xi r) + \frac{n^2 a^2 \xi a^2}{r} K_n^2(\xi r) \right] r dr$$

$$+ 2n a \cot \psi \left( \frac{\xi^2 a^2 + n^2 a^2 \xi a}{\xi^2 a^2} \right) (1 + \frac{\hbar a^2}{k a^2}) K_n(\xi a) K_n(\xi a) \int_0^\infty K_n(\xi r) K_n^2(\xi r) dr \right\}.$$ (4)

In a similar manner by using the internal field expressions, there results

$$P_z^i = \pi \sin^2 \psi |K_n|^2 \omega a \left( \frac{\hbar a \cot^2 \psi}{\xi a} \right) \int_0^\infty \left[ I_n^2(\xi r) + \frac{n^2 a^2 \xi a^2}{r} I_n^2(\xi r) \right] r dr$$

$$+ 2n a \cot \psi \left( \frac{\xi^2 a^2 + n^2 a^2 \xi a}{\xi^2 a^2} \right) (1 + \frac{\hbar a^2}{k a^2}) K_n(\xi a) K_n(\xi a) \int_0^\infty I_n(\xi r) I_n(\xi r) dr \right\}.$$ (5)

Using (A.3-3) through (A.3-6), one can carry through the integrations in (4) and (5). Adding the resulting $P_z^e$ and $P_z^i$ to obtain $P_z$, combining terms, and using the determinantal equation and the Wronskian relationship, one obtains the following expression for $P_z$ after some simplification:

$$P_z = -\pi |K_n|^2 \omega a \left( \frac{\hbar a}{\xi a^2} \right) \cos^2 \psi \left\{ \frac{\hbar f_n^\circ(\xi a)}{\xi a^2} \right\} - n a \cot \psi \left( \frac{\xi^2 a^2 + n^2 a^2 \xi a}{\xi^2 a^2} \right) \right\}.$$ (6)

where $f_n^\circ(\xi a)$ is given by

$$f_n^\circ(\xi a) = \xi a \left[ \frac{I_n(\xi a)}{I_n(\xi a)} + \frac{K_n(\xi a)}{K_n(\xi a)} \right] - \frac{n^2 a^2 \xi a^2}{r} \left[ \frac{I_n(\xi a)}{K_n(\xi a)} - \frac{K_n(\xi a)}{I_n(\xi a)} \right] - \frac{n^2 a^2 \xi a^2}{r} \left[ \frac{I_n(\xi a)}{K_n(\xi a)} - \frac{K_n(\xi a)}{I_n(\xi a)} \right].$$ (7)

It can be readily shown that (6) agrees with the expression given in reference 9 for the power flow on the sheath helix for the $n = 0$ mode alone. Further, (6) is valid for $\psi = 0^\circ$ although in this case the
second term in the braces \[ \{ \} \] becomes \( - \frac{(k^2 a^2 + h^2 a^2)}{ha} \). In this case if the parallel current density, which is \( K_\| \) only for \( \Psi = 0^\circ \), is given by \( |K_\| | \sin n\theta e^{-jhz} \) or \( |K_\| | \cos n\theta e^{-jhz} \) instead of \( |K_\| | e^{-jn\theta} e^{-jhz} \), \( n \) on the right side of (6) must be replaced by \( \frac{n}{2} \). Using the series expansions and approximations of Section A.2, in particular (A.2-1) through (A.2-11), it can be shown that

\[
 f_0^*(\zeta a) \approx -2 + \frac{1}{\ln \frac{\zeta ae^\gamma}{2}}, \quad (8)
\]

\[
 f_1^*(\zeta a) \approx \frac{\zeta^2 a^2}{2} \left( 1 + 4 \ln \frac{\zeta ae^\gamma}{2} \right), \quad (9)
\]

\[
 f_n^*(\zeta a) \approx -\frac{2}{n^2 - 1} \frac{\zeta^2 a^2}{2}, \quad n \geq 2, \quad (10)
\]

for \( \zeta a \) small, where \( \gamma \) is given by (A.2-6); whereas

\[
 f_n^*(\zeta a) \approx -2 + \frac{\ln^2 - 1}{2 \zeta^2 a^2}, \quad (11)
\]

for \( \zeta a \) large for all \( n \). Note that \( f_n^*(\zeta a) \) is an even function of \( n \).

Fig. II-15 shows \( f_n^*(\zeta a) \) for \( 3 \leq n \geq 0 \) for the most useful range of \( \zeta a \).

The manner in which the values of \( f_n^*(\zeta a) \) are ordered for a given \( \zeta a \) as \( n \) varies and the asymptotic values of \( f_n^*(\zeta a) \) for \( \zeta a \) small and large are clear from (8) through (11) and Fig. II-15. For \( \zeta a \leq 0.1 \), (8), (9), and (10) give \( f_n^*(\zeta a) \) to within 1%, and (10) may be used for \( \zeta a \geq 10 \) to obtain \( f_n^*(\zeta a) \) to within better than 1% for \( 3 \geq n \geq 0 \). For larger values of \( n \), (10) is increasingly in error, and direct calculations from tables are required (see Table C-II); or additional terms in the asymptotic forms useful for large argument alone, or the asymptotic forms good for both large order and argument discussed in Section A.2 must be used.

It can be shown that (6) also gives the proper expression for the power flow for the sheath tube modes in the limit of \( \Psi = 90^\circ \). In
evaluating the limiting expression the fact that it is the sheath helix modes for which \( nh < 0 \), specifically the \( h_{n}^{m} \) waves, which go over into the TEM waves on the sheath tube must be used. The power can also be evaluated by using (2) and (3) and the field expressions given in Section II.5(c).

In either case there results for \( \Psi = 90^\circ \) the simple expression

\[
P_z = \frac{\pi}{2|n|} \sqrt{\frac{\mu}{\varepsilon}} |K_n|^2 a^2, \quad |n| \geq 1, \quad (12)
\]

if the parallel current density, which is \( K_z \) only for \( \Psi = 90^\circ \), is given by \( |K_n| e^{-jnh} e^{-jkz} \). If the current density is given by \( |K_n| \cos |n| \theta e^{-jkz} \) or \( |K_n| \sin |n| \theta e^{-jkz} \), \( \frac{\pi}{2} \) on the right side of (12) must be replaced by \( \frac{\pi}{4} \).

In calculating the total average power flow given by (6) it is assumed that only one wave is present for a particular mode or \( n \). If more than one wave is present, (6) is used for each wave separately, and the total average power flow is then given by the sum of the powers contributed by each wave alone. To prove this it is first necessary to show that the total average axial power flow is independent of \( z \) irrespective of the number of waves present. This in turn is readily proved by applying Poynting's theorem to the lossless source-free volume enclosed by two planes placed perpendicular to the \( z \) axis an arbitrary distance apart and a cylindrical surface at \( r = R \), where \( R \) approaches infinity.\(^1\) The average radial power flow across the cylindrical surface approaches zero since the fields become exponentially or otherwise sufficiently small as \( R \) approaches infinity. Actually, the average radial power flow is identically zero for all \( r \) as noted below. Consequently, the total average power flow across the two planes must be equal, and, further, since they are an arbitrary distance apart, it follows that under all circumstances the total average axial power flow for "free mode" propagation is not a function of \( z \). If \( P_z \) is calculated from (2) or (3) for the situation where several waves for a given \( n \) are present, some consideration shows that the resulting expression
is a sum of terms resulting from each wave separately, plus cross terms resulting from the $\mathbf{E} \times \mathbf{H}$ product of the different waves, these latter being functions of $z$. Since the total fields are proper solutions of the Maxwell equations, and Poynting's theorem certainly applies, it is clear from the above that these cross terms must vanish. In this manner the validity of the procedure whereby the entire total average power is obtained by adding the power in each wave alone is established. In order to complete the calculation for the power flow it is necessary to know the $|K_n|$ associated with each wave, and this can only be determined by solving the source-present problem as in the next section. It is evident that the power flow resulting from waves in different modes, that is, different $n$, can be calculated separately since in this case the field representations are orthogonal in $\theta$. It is worthy of note, as can be realized from the field expressions (II.4-15) through (II.4-20), that $S^\ast_r = \frac{1}{2}(E_\theta \tilde{H}_z - E_z \tilde{H}_\theta)$ is pure imaginary or zero for all $\Psi$, so that the average power flow in the radial direction is zero at all radii. This is true irrespective of the number of waves which occur on the system. It is also interesting to note that $S^\ast_\theta = \frac{1}{2}(E_z \tilde{H}_r - E_r \tilde{H}_z)$ has a finite real part which is independent of $\theta$ but which is a function of $z$ if more than one wave is present.

In the previous section it was stated without proof that the group velocity is the velocity of energy propagation, so that the direction or algebraic sign of total average power flow is the same as that of the group velocity. Although this property has been proved for transmission systems with various assumed boundary conditions, it appears not to have been proved for such sheath systems as are being discussed here.\textsuperscript{1,5,39} The proof is given below where the procedure used in Section 4.2 of reference 39 for a similar proof for different boundary conditions is followed rather closely. In the following it is necessary to split off the $z$ dependence of the field vectors. Since this is the same for all
components, one writes $\mathbf{E} = \mathbf{E} e^{-jhz}$ and $\mathbf{H} = \mathbf{H} e^{-jhz}$, where $\mathbf{E}$ and $\mathbf{H}$ are now complex field vectors independent of $z$. Note that the product of a vector by a complex conjugate vector is the same whether one uses $\mathbf{E}$, $\mathbf{H}$, or $\overline{\mathbf{E}}$, $\overline{\mathbf{H}}$. The basis for the proof is the energy theorem for lossless systems which states that

$$ \nabla \cdot (\mathbf{E} \times \frac{\partial \overline{\mathbf{H}}}{\partial \omega} + \frac{\partial \mathbf{E}}{\partial \omega} \times \overline{\mathbf{H}}) = -j (\varepsilon \mathbf{E} \overline{\mathbf{E}} + \mu \mathbf{H} \overline{\mathbf{H}}) . $$

(13)

This may be readily shown by expanding the left side of (13) and substituting from the Maxwell equations (II.3-1) and (II.3-2) for the various terms. Since each wave on the sheath system is separately a solution of the Maxwell equations for the required boundary conditions, (13) can be applied to the field solutions for each wave alone. Performance of the indicated differentiations, after splitting off the $z$ dependence as noted above, results in

$$ \nabla \cdot \left[ (\mathbf{E} \times \mathbf{H} + \mathbf{E} \times \overline{\mathbf{H}}) z \right] = 2 \nabla \cdot \left[ \text{Re}(\mathbf{E} \times \mathbf{H}) z \right] , \quad (14) $$

Since

$$ \nabla \cdot \left[ (\mathbf{E} \times \mathbf{H} + \mathbf{E} \times \overline{\mathbf{H}}) z \right] = 2 \nabla \cdot \left[ \text{Re}(\mathbf{E} \times \mathbf{H}) z \right] , \quad (15a) $$

or

$$ = 2 \text{Re} \left[ z \nabla \cdot (\mathbf{E} \times \overline{\mathbf{H}}) + a_z \cdot (\mathbf{E} \times \overline{\mathbf{H}}) \right] , \quad (15b) $$

and since the complex Poynting theorem, which is also directly obtainable from the Maxwell equations, states that for lossless systems

$$ \nabla \cdot \mathbf{S}^\mp = \nabla \cdot \left( \frac{1}{2} \mathbf{E} \times \overline{\mathbf{E}} \right) = j \frac{c}{2} (\varepsilon \mathbf{E} \cdot \overline{\mathbf{E}} - \mu \mathbf{H} \cdot \overline{\mathbf{H}}) , \quad (16) $$

it is evident, taking the real part of both sides of (16) and noting that this requires the first term on the right side of (15b) to vanish, that (13) becomes

$$ \nabla \cdot (\mathbf{E} \times \frac{\partial \overline{\mathbf{H}}}{\partial \omega} + \frac{\partial \mathbf{E}}{\partial \omega} \times \overline{\mathbf{H}}) - j 2 \frac{\partial}{\partial \omega} \text{Re} \left[ a_z \cdot (\mathbf{E} \times \overline{\mathbf{H}}) \right] = -3 (\varepsilon \mathbf{E} \cdot \overline{\mathbf{E}} + \mu \mathbf{H} \cdot \overline{\mathbf{H}}) . $$

(17)
Integrating (17) over the region \( a \gg r \geq 0 \), there results by using (3)

\[
\oint_{a \rightarrow r} \mathbf{E} \cdot \left( \frac{\partial \mathbf{H}}{\partial \omega} + \frac{\partial \mathbf{E}}{\partial \omega} \times \mathbf{H} \right) \, dl = j \ im \left( \frac{\partial \mathbf{H}}{\partial \omega} \right)_z \mathbf{P}_z
\]

\[= - j \int_{A^i} \left( \varepsilon \mathbf{E}^i \cdot \mathbf{E}^i + \mu \mathbf{H} \cdot \mathbf{H}^i \right) dA \quad (18)\]

Similarly, for \( \omega \gg r \gg a \), since the fields become exponentially or otherwise sufficiently small for \( r = \omega \),

\[
-\oint_{a \rightarrow r} \mathbf{E} \cdot \left( \frac{\partial \mathbf{H}}{\partial \omega} + \frac{\partial \mathbf{E}}{\partial \omega} \times \mathbf{H} \right) \, dl = j \ im \left( \frac{\partial \mathbf{H}}{\partial \omega} \right)_z \mathbf{P}_z
\]

\[= - j \int_{A^e} \left( \varepsilon \mathbf{E}^e \cdot \mathbf{E}^e + \mu \mathbf{H}^e \cdot \mathbf{H}^e \right) dA \quad (19)\]

Adding (18) and (19), dropping the superscripts where their absence should cause no confusion, noting that only the tangential components of the fields need be retained in the line integral and that \( \mathbf{E}^i_t = \mathbf{E}^e_t = \mathbf{E}^t_t \) for \( r = a \), one obtains

\[
\oint_{a \rightarrow r} \mathbf{E} \cdot \left[ \left( \frac{\partial \mathbf{H}}{\partial \omega} - \mathbf{H} \right)_t + \left( \frac{\partial \mathbf{E}}{\partial \omega} \times \mathbf{H} \right)_t \right] \, dl = j \ im \left( \frac{\partial \mathbf{H}}{\partial \omega} \right)_z \mathbf{P}_z
\]

\[= - j \int_{A} \left( \varepsilon \mathbf{E} \cdot \mathbf{E} + \mu \mathbf{H} \cdot \mathbf{H} \right) dA \quad (20)\]

If the tangential fields for \( r = a \) are resolved in directions parallel and perpendicular to the "windings" of the sheath system, it is clear from the assumed boundary conditions (II.2-1) through (II.2-3) that \( \mathbf{E}_t \) and \( \left( \mathbf{H}_t - \mathbf{H}^e \right)_t \) are both nonzero only in the perpendicular direction. Consequently, both vector cross products of the integrand of the line integral vanish, and with \( \frac{\partial \mathbf{h}}{\partial \omega} = \frac{1}{V_0} \frac{dh}{dk} = \frac{1}{V_0} \frac{1}{V} \) from (II.5-7), (20) becomes

\[
V = \frac{P_z}{\frac{1}{V} \int_{A} \left( \varepsilon \mathbf{E} \cdot \mathbf{E} + \mu \mathbf{H} \cdot \mathbf{H} \right) dA} \quad (21)
\]
Since the denominator of the right side of (21) is surely positive, the
total average axial power flow resulting from each wave and the group
velocity of the wave have the same direction or algebraic sign, and the
usual definition of \( v_g \) as the ratio of the total average real power flow
to the average energy stored per unit length of the system applies.

The Source-Present Problem

II.7 The Gap Source

(a) Introduction

In order to understand more completely the significance of
the many waves which occur on a sheath system, to understand the reason
for calling some of these inward and others outward traveling waves as
in Section II.5, and to illustrate other points, it is necessary to
solve the inhomogeneous or source-present problem. The procedure where-
by this is accomplished requires that the fields which result when the
sheath system is excited by an infinite line or tube source be deter-
mined first. By superimposing these cylindrical wave solutions, one
can determine the total field resulting from a finite essentially
spherical wave source.\(^1,2,39,48,50\)

(b) Derivation of General Results

Assume that the \( z \) component of the electric field has a dis-
continuity on the surface of the sheath helix which is zero everywhere
except for \( \frac{1}{2} > z > -\frac{1}{2} \) where it has a value \( E_n e^{-jn\theta} \) and where \( E_n \) is
independent of \( z \) and \( \theta \). By the methods of Fourier integral analysis,
this distribution can be represented by

\[
(z^e - z^i)_{r=a} = \frac{E_n e^{-jn\theta}}{\frac{\pi}{\hbar}} \int_{c} \frac{\sin \frac{\hbar z}{\hbar}}{\frac{\hbar z}{\hbar}} e^{-jnhz} \, dh = \frac{E_n e^{-jn\theta}}{0} \quad , \quad |z| < \frac{1}{2} \]

The contour \( c \) is merely the entire real axis of the \( h \) plane from
-∞ to +∞. Here, h is considered as a general complex variable and should not be confused with the axial propagation constants of the sheath helix modes, although these latter values are, of course, definite points on the real axis of the complex h plane for lossless systems. A similar remark holds for the variable ξ used below. It is assumed that the above discontinuity in the electric field is maintained by an impressed electric field which is the negative of (1). This can be considered to be the result of generators placed in the gap, |z|<\frac{a}{2} and r = a, as shown in Fig. II-16. The analysis in this section refers to what might be called an internally shielded gap source; that is, the sheath system and the generators are arranged for |z|<\frac{a}{2} so that E^i_z is zero for |z|<\frac{a}{2} at r = a without disturbing the sheath magnetic field boundary conditions. A less artificial choice for a gap source might have been an unshielded one in which the electric field discontinuity in the gap is in the direction of the "windings". However, the type of source used here appears to lead to somewhat simpler formulae and prevents the appearance of the ordinary waveguide modes at Ψ = 0° and 90°.

For E_n increasing large and l increasingly small, one can write with some caution

\[
(E^0_z - E^i_z)_{r=a} = V_n e^{-jnθ} u_o(z) = \frac{V_n e^{-jnθ}}{2π} \int_{C} e^{-jhz} dh,
\]

where u_o(z) is the unit impulse at z = 0 and V_n = E_n l is the voltage across the gap. The use of (2) in place of (1) is permissible since, although the integral in (2) is no longer convergent, the integrals to be derived throughout this discussion using (2) as the limiting form of (1) are in general convergent for |z| > 0. (2) represents the electric field distribution in the gap as a superposition of infinitesimal z-directed electric fields of the form \[\frac{V_n e^{-jnθ}}{2π} e^{-jhz} dh\] existing over
FIG. II-16
Gap Source Driven Sheath Helix

FIG. II-17
Contour $C_2$
Indentations around poles of integrand not shown.

FIG. II-18
Method for Determining
arg $\zeta$
$\varphi_1 = \arg (h+k)$
$\varphi_2 = \arg (h-k)$

FIG. II-19
Contour of Integration

FIG. II-20a
Modified Contour for $z > 0$

FIG. II-20b
Modified Contour for $z < 0$
the entire length of the sheath helix at \( r = a \). Since the system is linear, it is evident that if the fields which result when the \( z \) component of the electric field has a discontinuity of the form \( \frac{V}{2\pi} e^{-jn\theta} e^{-jhz} \) at \( r = a \) for all \( z \) can be determined, by superposing these fields one can obtain the final fields resulting from the finite gap source.

The regions for \( r > a \) and \( r < a \) are source-free so that the field components can still be obtained from Hertzian potentials like those given in (II.3-18) through (II.3-21) using (II.3-7) through (II.3-12). Also, the boundary conditions at \( r = a \) are again given by (II.2-5) through (II.2-7) using the \( i \) superscript in (II.2-6), but (II.2-4) must be modified to read

\[
E^e_z - E^i_z = \frac{V}{2\pi} e^{-jn\theta} e^{-jhz},
\]

for \( r = a, 2\pi \theta > 0, + \infty > z > -\infty \). Carrying through the calculations indicated above merely leads to a set of equations precisely like (II.4-1) through (II.4-4) except that on the right side of (II.4-1) \( \frac{V}{2\pi} \xi^2 \) replaces 0. Thus, an inhomogeneous system of equations results which can be solved for the coefficients \( A^i_n, e_{ng} \) and \( B^i_n, e_{ng} \). If these are distinguished from the ones used previously by the subscript \( g \), there results

\[
A^i_{ng} = \frac{V}{2\pi} \xi^2 \frac{\xi a \cot \Psi}{\delta_n} \frac{I^I_n}{I^I_n} \left[ nha (\xi^2 a^2 + nha \cot \Psi)K_n^2 - k^2 a^2 \xi^2 a^2 \cot \Psi K_n^2 \right],
\]

\[
A^e_{ng} = -\frac{V}{2\pi} \xi^2 \frac{\xi a \cot \Psi}{\delta_n} \left[ k^2 a^2 \xi^2 a^2 \cot^2 \Psi K_n^1 I_n^2 + (\xi^2 a^2 + nha \cot \Psi)(\xi a - nha \cot \Psi I_n K_n^1)I_n \right],
\]

\[
B^i_{ng} = \frac{(\xi^2 a^2 + nha \cot \Psi)}{\xi a \cot \Psi} \frac{I^I_n}{I^I_n} A^i_{ng},
\]

\[
B^e_{ng} = \frac{V}{2\pi} \xi^2 \frac{1}{\xi a \cot \Psi} \left[ nha \left[ (\xi^2 a^2 + nha \cot \Psi)I_n^2 - k^2 a^2 \xi^2 a^2 \cot^2 \Psi I_n^2 \right] K_n \right. \\
\left. + k^2 a^2 \xi a \cot \Psi K_n^1 I_n^1 \right],
\]
where
\[ D_n = D_n (\zeta a) = \left[ (\zeta^2 a^2 + n a \cot \Psi)^2 I_n^2 + k^2 a^2 \zeta^2 a^2 \cot^2 \Psi I_n^2 K_n^2 \right]. \tag{8} \]

The argument of all the modified Bessel functions in \((4)\) through \((8)\) above is \(\zeta a\) and has been omitted for convenience. The Hertzian potentials for the fields become

\[ \Pi_{ii} = e^{-jn\theta} \int_{C_2} A_{ng}^i I_n (\zeta r) e^{-jhz} dh, \tag{9} \]

\[ \Pi^{*i}_{ii} = e^{-jn\theta} \int_{C_2} B_{ng}^i I_n (\zeta r) e^{-jhz} dh, \tag{10} \]

\[ \Pi_{eg} = e^{-jn\theta} \int_{C_2} A_{ng}^e k_n (\zeta r) e^{-jhz} dh, \tag{11} \]

\[ \Pi^{*g}_{eg} = e^{-jn\theta} \int_{C_2} B_{ng}^e k_n (\zeta r) e^{-jhz} dh. \tag{12} \]

The fields can be obtained from \((9)\) through \((12)\) by using \((II.3-7)\) through \((II.3-12)\) since differentiation under the integral sign is permitted. Thus, for example,

\[ H_{eg} = -e^{-jn\theta} \int_{C_2} \zeta^2 B_{ng}^i I_n (\zeta r) e^{-jhz} dh. \tag{13} \]

The variable \(\zeta\) is related to \(h\) and \(k\) in the usual way since these solutions must satisfy the wave equation, that is,

\[ \zeta^2 = h^2 - k^2. \tag{11b} \]

The contour \(C_2\) is the real \(h\) axis from \(-\infty\) to \(+\infty\) indented around the branch points at \(\pm k\), as shown in Fig. \(II-17\), if the medium is lossless, with other proper indentations around any poles of the integrand on the real axis. These latter indentations are not shown in Fig. \(II-17\) but are discussed in detail below. If the fields obtained by using \((9)\)
through (12) are to be the proper ones obtained from the gap distribution (1) or (2) acting like a source, it is necessary that \( \frac{\pi}{2} \geq \arg \zeta > 0 \) on \( C_2 \).

Reference 2, page 415, shows an easily used procedure for following \( \arg \zeta \) as \( h \) varies, with \( \arg \zeta = 0 \) and \( \zeta \) positive real at \( h = -\infty \). This is illustrated in Fig. II-18 with

\[
|\zeta| = \frac{1}{2} \left| h - k \right|^{\frac{1}{2}} \left| h + k \right|^{\frac{1}{2}},
\]

\[
\arg \zeta = \frac{1}{2} \arg (h - k) + \frac{1}{2} \arg (h + k).
\]

For lossless systems, with \( \pm k \) on the real axis, \( \arg \zeta = 0 \) for \( -k > h > -\infty \), \( \arg \zeta = + \frac{\pi}{2} \) for \( k > h > -k \), and \( \arg \zeta = 0 \) for \( h > k \).

The expressions for the current density on the surface of the sheath system are given by (II.4-9) through (II.4-14) where now the fields produced by the gap source must be used. In this case one finds

\[
K_{tg} = i \omega a e^{-j\theta} \int_{C_2} \frac{V_n}{2\pi} \frac{1}{D_n} \left[ n a (K_n I_n' + I_n K_n') \cot \psi \csc \psi 
+ \zeta^3 a^3 K_n' I_n' \csc \psi \right] e^{-j\phi} d\phi,
\]

where again the modified Bessel functions all have the argument \( a \). The functional notation will often be omitted throughout the remainder of this section; in all cases the argument of the modified Bessel functions is \( a \) unless otherwise noted. \( K_{tg} \) is, of course, identically zero. If one should try to evaluate \( K_{tg} \) at \( z = 0 \), some consideration shows that the integral is no longer convergent, the integrand, in fact, going like \( 1/|h| \) for large \( h \). This is to be expected since the current density caused by a finite field discontinuity between an infinitesimally small gap would, indeed, be infinite. This difficulty can be avoided by assuming a finite gap and replacing \( V_n \) in (16) by \( V_n \frac{\sin \frac{hf}{2}}{hf} \) as is evident from
(1) and (2). This same procedure can be used, if necessary, in the other integral representations. The input admittance for the $n$th mode can now be defined as the sum of the admittances of all the infinitesimal "windings" making up the sheath helix, or

$$Y_n = - \int_0^{2\pi} \left[ \frac{K_n^g}{v_n e^{-j\omega t}} \right]_0^\infty \text{ad} = - j\omega a^2 \int_{\gamma_2} \frac{N_n(\zeta a)}{D_n(\zeta a)} \frac{\sin \frac{h_2}{2}}{\frac{h_2}{2}} e^{-j\frac{h_2}{2}} dh , \quad (17)$$

the minus sign occurring since the applied voltage is opposite to the induced voltage. $N_n(\zeta a)$ is given by the function in the brackets in (16), or

$$N_n(\zeta a) = nha \zeta a (K_n^I + I_n K_n^I) \cot \psi \sec \psi + \zeta^3 a^3 K_n^I \sec \psi . \quad (18)$$

The complete and exact evaluation of the integral representations given above for the various field components is a difficult or even impossible task. Fortunately, it is the "free modes" only which are of most interest, and these parts of the solution can be fairly readily determined. For $|z| > 0$, (16) converges as it stands, and it becomes merely

$$K_n^g = j\omega a \frac{v_n e^{-j\omega t}}{2\pi} \int_{\gamma_2} \frac{N_n(\zeta a)}{D_n(\zeta a)} e^{-j\omega t} dh . \quad (19)$$

The only singularities of the integrand in (19) are branch points at $\pm k$ and $\infty$, and poles wherever $D_n(\zeta a) = 0$. But $D_n(\zeta a) = 0$ is precisely the determinantal equation (II.13-8) for the "free modes" which is known to have roots only at various points on the real $h$ axis for a lossless medium, depending on the values of $n$, $ka$, and $\psi$. In carrying out the integration over the contour $\gamma_2$ it is necessary to indent the contour properly around these poles. If the medium is assumed to be slightly lossy, the roots of $D_n(\zeta a) = 0$ move off the real axis, the branch
points $-k$ and $+k$ move into the second and fourth quadrants, respectively, of the $h$ plane (see Fig. II-18), and the contour $C_2$ then becomes merely the entire real $h$ axis from $-\infty$ to $+\infty$ without indentations. The lossless system is now considered to be the limiting case of the slightly lossy system, and the contour $C_2$ is indented in an obvious fashion as the roots and the branch points approach the real axis. Thus, in order to decide how to indent $C_2$ around the roots of $D_n(\xi a)$, it is necessary to investigate $D_n(\xi a) = 0$ in the case where the medium is slightly lossy or to obtain the required information in some equivalent manner. Assume that this has been done and that in the limit of a lossless medium the contour $C_2$ appears as in Fig. II-19, where $-|h_a|$, $-|h_b|$, $|h_c|$, and $|h_d|$ are the various roots of $D_n(\xi a) = 0$. In general, the roots appear only for values of $|h| > k$, as already noted in detail in Sections B.1 and II.5 (although they can occur for $|h| = k$ in some limiting cases), and need not be symmetrically disposed about the origin. For $z > 0$, because of the $e^{-jhz}$ factor in (19), the integrand vanishes on a circle of infinite radius in the third and fourth quadrants, and the contour can be deformed as in Fig. II-20a. It becomes an integration around the poles of the integrand at $h = -|h_b|$ and $h = +|h_c|$, and an integration along a branch cut, which can be somewhat arbitrarily chosen. Here, it is taken along both sides of the negative imaginary axis from 0 to $-j\infty$, along both sides of the real axis from 0 to $(k - \delta)$, and around a circle of radius $\delta$ centered at $+k$ where $\delta$ approaches zero. The contour cannot be closed at $\infty$ because of the branch point there, and the integrations out to the poles and back cancel each other. In a similar way, for $z < 0$ the contour can be deformed into the upper half plane, and the result appears as in Fig. II-20b. It should now be clear that the $K_{ng}$ which results from the gap source is a sum of the "free mode" terms obtained from the contributions of the poles of the integrand, plus a term contributed by the branch
cut integration. This remark obviously applies to all the field components as well.

If the case for $z > 0$ is considered so that the contour shown in Fig. II-20a applies, $K_{h_g}$ can be divided into four parts, or

$$K_{h_g} = K_{h(f.m.)g} + K_{h_g}^1 + K_{h_g}^n + K_{h_g}^{n^*}.$$  \hspace{1cm} (20)

$K_{h(f.m.)g}$ is the result of the contributions from the poles of the integrand, $K_{h_g}^1$ results from the integration along both sides of the negative imaginary axis, $K_{h_g}^n$ results from the integration along both sides of the real axis, and $K_{h_g}^{n^*}$ results from the integration around $\delta$. Using the formulae in Section A.4, and Fig. II-18 to follow arg $\delta$, it can be shown that $K_{h_g}^1$ has the form

$$K_{h_g}^1 = \omega \varepsilon e^{-j\eta \theta} \frac{V_n}{2\pi} \int_0^\infty F(\alpha) e^{-\alpha z} d\alpha,$$ \hspace{1cm} (21)

where $F(\alpha)$ is, in general, a complex function of a real variable $\alpha$ and the parameters $k, \psi$, and $n$, and where no singularities occur in $F(\alpha)$ for $\infty > \alpha > 0$. $F(\alpha)$ is expressible in terms of the $J_n$ and $N_n$ functions of Section A.4, but since (21) has not been evaluated and since there is no need for the explicit form of $F(\alpha)$, it is not given here. The important point is that in view of the bounded character of $F(\alpha)$, $K_{h_g}^n$ becomes small like $\frac{1}{z}$ as $z$ becomes increasingly large. In a similar fashion it can be shown that

$$K_{h_g}^n = \omega \varepsilon e^{-j\eta \theta} \frac{V_n}{2\pi} \lim_{\delta \to 0} \int_0^{k-\delta} F(\beta) e^{-j\beta z} d\beta,$$  \hspace{1cm} (22)

where $F(\beta)$ is in all cases a real function of a real variable $\beta$ and the parameters $k, \psi$, and $n$, and where no singularities occur in $F(\beta)$ for $k - \delta \beta > 0$. As in the case of $F(\alpha)$, $F(\beta)$ is expressible in terms of the $J_n$ and $N_n$ functions, but for the discussion here it need not be
given explicitly. Again, the important point is that $K''_g$ becomes small like $\frac{1}{z}$ as $z$ becomes increasingly large. Using (15a) and (15b) to find $\zeta$ on the circle of radius $\delta$ around $+k$ in Fig. II-20a, and then using the formulae in Section A.2 to express $I_n(\zeta a)$ and $K_n(\zeta a)$ for small $\zeta a$, it can be shown that generally $K''_g$ approaches zero as $\delta$ approaches zero.

Exceptions to this rule may occur if $|h_c|$ coincides with $+k$, and in this case a finite value may result for $K''_g$. However, these cases are best considered as limiting ones for which $|h_c|$ approaches and finally equals $+k$; with this precaution $K''_g$ can be considered to be zero under all conditions. Since the $K''(f.m.)_g$ terms have a $z$ dependence like $e^{-j|h_b|z}$ and $e^{-j|h_c|z}$, it is clear that as $z$ becomes increasingly large, it is only these terms which are significant, the $K'_g$ and $K''_g$ terms vanishing like $\frac{1}{z}$. Thus, very far away from the source only the surface current density associated with the "free modes" need be considered. The physical significance of the $K'_g$ and $K''_g$ terms is that they correspond to the local induction and radiation effects of the source and system for $z$ not large. It is evident that for $z<0$ an argument similar to the above applies, and for $z<0$ only "free mode" fields with $z$ dependence like $e^{+j|h_a|z}$ and $e^{-j|h_d|z}$ persist. Using the usual methods of the theory of residues in complex integration, \cite{1,2,50} since the poles of the integrand are simple, one obtains

$$
K''(f.m.)_g(z>0) = \omega e a V_n e^{-j\theta} \sum_{h=-|h_b|,+|h_c|, \text{etc.}} \left\{ \frac{N_n(\zeta a)e^{-jhz}}{\frac{d}{dh} \left[ D_n(\zeta a) \right]} \right\}, \quad (23a)
$$

and

$$
K''(f.m.)_g(z<0) = \omega e a V_n e^{-j\theta} \sum_{h=-|h_a|,+|h_d|, \text{etc.}} \left\{ \frac{N_n(\zeta a)e^{-jhz}}{\frac{d}{dh} \left[ D_n(\zeta a) \right]} \right\}. \quad (23b)
$$
The quantity in the braces \( \{ \} \) can be evaluated, and after some manipulation one finds

\[
\left\{ \frac{N_n(\xi a)e^{-jhz}}{\frac{d}{dh} D_n(\xi a)} \right\} = \frac{\xi^2 a^2}{2 \xi a(\xi^2 a^2 + nha \cot \psi)(2ha + n \cot \psi)} e^{-jhz} \left[ \frac{I_n}{T_n} + \frac{K'_n}{K_n} \right]
\]

(24)

In obtaining (24) from (23), the relationship (14), the expressions for \( I_n \) and \( K_n \) available from the differential equation (A.1-1), and the fact that the evaluation occurs where \( D_n(\xi a) = 0 \) must be used. From (23) and (24) the amplitudes of the surface current density associated with the different waves of the \( n^{th} \) "free mode" can be obtained, and using these in (II.4-15) through (II.4-20), one can determine the various "free mode" field components. Also, using these current density amplitudes in (II.6-6), one can find the average axial power flow in the "free modes". The results given so far are quite general and applicable for any value of \( n \), although it has been assumed that the \( \theta \) dependence of the source is like \( e^{-j\theta} \) only. This is sufficiently general since, if the \( \theta \) and the \( z \) dependence of the source are like \( u_\theta(z) \sum_n V_n e^{-j\theta} \), the total solution can be obtained because of linearity and orthogonality in \( \theta \) by merely adding the solutions obtained above for each \( n^{th} \) harmonic term alone.

In order to complete the solution for the "free modes" it is necessary to decide how the contour \( C_2 \) must be indented around the poles of the integrand in (19). For the special cases of \( \psi = 0^\circ \) with \( |n| > 1 \)
and of \( n = 0 \) with \( \psi \neq 0^\circ \), the location of the \( h \) roots of the determinantal equation \( D_n(\xi a) = 0 \) or (II.4-8) for a slightly lossy medium can be determined in a rather tedious fashion from this equation alone. With the values for \( \psi = 0^\circ \) with \( |n| \gg 1 \) determined, it is possible to continue the solutions and thereby find the root locations for all \( \psi \). Although this has been done, the procedure is not presented here since the following much simpler analysis, which yields precisely the same results, avoids the necessity of the lengthy approach. Further, it allows information which is already available to be used. To explain this it is necessary to introduce some new notation momentarily. If the medium in which the sheath system is immersed is taken to be slightly lossy, then from (II.1-5) in place of \( k^2 = \omega^2 \mu \varepsilon \), one can write \( k^2 = \omega^2 \mu \varepsilon' = \omega^2 \mu \varepsilon - jk_0 \mu \sigma = k_0^2 - jk_0 v_0 \mu \sigma \). \( k_0^2 \) is taken to be the value which \( k^2 \) approaches as the conductivity of the medium approaches zero. \( v_0 \) is, as in Section II.5, the velocity of a uniform plane wave in a lossless medium of constants \( \mu \) and \( \varepsilon \). The nondimensional complex variable \( h a \) can be written in terms of its real and imaginary parts as

\[
ha = \beta a + j\alpha a \ .
\tag{26a}
\]

It is evident both from the functions involved and the physical requirements of the solution that, at least in any restricted region, \( h a \) can be considered an analytic function of \( n \), \( \cot \psi \), and \( ka \). Of course, an explicit form for \( h a \) cannot be written, but it can be represented as

\[
ha = f_n(\cot \psi, ka) \ .
\tag{26b}
\]

For \( ka = k_0 a \) the solutions of (26b) have already been considered. In this case the \( h a \) roots are pure real, and there can be several even for a given \( \cot \psi \) and \( n \). Calling any one of these \( h a = \beta_0 a \) results in

\[
\beta_0 a = f_n(\cot \psi, k_0 a) \ .
\tag{27}
\]

Since only positive frequencies need be considered, one can write, dropping
terms of order \(\sigma^2\),

\[
ka = \pm \sqrt{k_o^2 a^2 - j k_o v_o \mu_o a^2} \approx k_o a \left( 1 - \frac{j}{2} \frac{v_o \mu_o}{k_o} \right). \tag{28}
\]

With \(h_a\) an analytic function, an expansion can readily be written as

\[
h_a = \beta a + j \alpha a = f_n(\cot \psi, k_o a) + \frac{\partial f_n}{\partial (ka)} \bigg|_{ka=k_o a} (ka - k_o a) + \ldots, \tag{29}
\]

with \(n\) and \(\psi\) constant. From (27) and (28), (29) becomes

\[
\beta a + j \alpha a \approx \beta_o a + \frac{\partial f_n}{\partial (ka)} \bigg|_{ka=k_o a} (-j \frac{v_o \mu_o a}{2}), \tag{30}
\]

where terms of order \(\sigma^2\) are omitted. But from (II.5-7)

\[
\frac{\partial f_n}{\partial (ka)} \bigg|_{ka=k_o a} = \frac{d(h_a)}{d(k_o a)} = \frac{v_o}{v_g}, \tag{31}
\]

with \(v_g\) the group velocity. For a slightly lossy medium from (30) and (31)

\[
\beta a \approx \beta_o a, \tag{32a}
\]

\[
\alpha a \approx -\frac{v_o^2 \mu_o a}{2v_g}. \tag{32b}
\]

As expected, the real part of \(h_a\) is unaltered to this order. The imaginary part is dependent on \(\sigma\) and approaches zero as \(\sigma\) approaches zero. Since \(v_o^2 \mu_o a\) is surely positive, (32b) requires the product of \(\alpha a\) and \(v_g\) to be negative. \(\alpha\) is, of course, the displacement of the root perpendicular to the real \(h\) axis in the \(h\) plane. \(v_g\) is already known since the manner in which the \(h\) roots vary along the real axis as \(k_o\) varies is known. Consequently, the direction of displacement of the roots for a slightly lossy system or, conversely, the manner of indentation of \(C_2\) for a lossless system is readily obtained. Incidentally, the attenuation in the radial direction can be derived by substituting (32) in (14).
In the remainder of this section the solutions of the determinantal equation for the sheath helix are briefly discussed in view of the above remarks. In particular, the disposition of the roots of \( D_n(\zeta a) = 0 \) in the \( h \) plane for various values of \( \Psi \) and \( n \) as \( ka \) varies is shown. For convenience, these are shown on the assumption that the medium is slightly lossy, although it is to be understood that for the lossless case the roots and branch points appear on the real axis and \( C_2 \) is correspondingly indented. It should be clear now, in view of the way the contour is deformed for \( z > 0 \) and \( z < 0 \) (see Fig. II-20), why some waves are referred to as inward and others as outward traveling waves in Section II.5. Thus, for \( z > 0 \) a root in the fourth quadrant results in a \( z \) dependence like \( e^{-j|h_c|z} \), an outward traveling wave, whereas a root in the third quadrant results in a \( z \) dependence like \( e^{+j|h_d|z} \), an inward traveling wave.

Similarly, for \( z < 0 \) a root in the second quadrant results in a \( z \) dependence like \( e^{+j|h_a|z} \), an outward traveling wave, whereas a root in the first quadrant results in a \( z \) dependence like \( e^{-j|h_d|z} \), an inward traveling wave.

Note again, as mentioned in Section II.5, that for a lossy system all "free mode" waves are exponentially damped at large distances from the source.

(c) The Sheath Helix, \( n = 0 \)

Since the \( n = 0 \) mode is somewhat special, it is treated separately. In this case (2h) becomes

\[
\left\{ \frac{d}{dh} \left[ D_0(\zeta a) \right] \right\}_{D_0(\zeta a) = 0} = \frac{\zeta^2 a^2}{\csc \Psi} \left[ 1 - \frac{K_1}{K_0} \right] e^{-jhz} \]

It is evident from (33) that for \( \Psi = 0^\circ \), the sheath ring, no "free mode"
wave is excited by the type of source considered here. Indeed, (19) shows that \( K_{n_{h_{g}}} \) is identically zero for all \( z \) for this case. Further, some consideration of (9) through (12) shows that the field components resulting from the internally shielded gap source are unaffected by the presence of the sheath ring. Physically, it is evident that for \( n = 0 \) and \( \psi = 0^\circ \) there is no electric field in the direction in which current can flow. Consequently, the field components around this gap source are precisely what they would be if the sheath ring were not present. However, it should be noted that for \( \psi = 0^\circ \) the \( TE_{0m} \) modes, which can exist in the interior of a perfectly conducting hollow circular cylinder, can exist in the sheath ring. Such modes correspond to values of \( h \) such that \( h \) is real and less than \( k \) in magnitude, or \( h \) is pure imaginary. For \( \psi \) any small amount larger than \( 0^\circ \), the analysis in Section B.1 shows that such values of \( h \) cannot occur. Consequently, it can be concluded that these modes are not stable on the sheath system.

For \( 90^\circ > \psi > 0^\circ \) Fig. II-21 shows the position of the \( h \) roots, \( \pm |h_0| \). Here, as in subsequent plots, the arrows show the relative motion of the roots as \( k \) is varied with \( \psi \) constant. Also, the subscript and superscript notations on the \( h \) roots conform to those of Section II-5.

This view of the roots should be compared with that given by Figs. II-8 and II-9. For \( \psi = 90^\circ \), the sheath tube, there is no "free mode" wave for \( r > a \), as noted in Sections B.3 and II.5(c). \( \psi = 90^\circ \) with \( n = 0 \) corresponds in many respects to a perfectly conducting circular cylinder excited by a symmetrical gap source. It is worthy of note that for \( \psi = 90^\circ \) with \( n = 0 \) (17) becomes

\[
Y_0 = j \omega a \int_{c_2} \frac{K_1(\xi a)}{\xi K_0(\xi a)} \frac{\sin \frac{h l}{2}}{\frac{h l}{2}} e^{-j \frac{h l}{2}} dh.
\]  

(3h)
Except for a difference in notation this is precisely the equation given on page 418 of reference 2 for the input admittance of a perfectly conducting infinite circular cylindrical wire excited in the above manner. (24) gives only the external admittance for this case, and in relationship to this the remarks of Part (f) of this section are pertinent.

(d) **The Sheath Ring**, \( \psi = 0^\circ \) for \(|n| \geq 1 \)

For \( \psi = 0^\circ \) (24) becomes

\[
\left\{ \frac{\frac{d}{dn} \left( \frac{D_n(\xi a)}{D_n(\xi a) = 0} \right)}{D_n(\xi a) = 0} \right\} = \frac{\xi a^2}{2 \pi} \frac{n \left( \frac{I_n}{I_n} + \frac{K_n}{K_n} \right) e^{-jhz}}{2n^2 \xi a + \left( n^2 + \xi a^2 \right) k^2 a^2 + n^2 h^2 a^2} \left[ \frac{I_n}{I_n} + \frac{K_n}{K_n} \right].
\]

(35)

The position of the roots is shown for this case in Fig. II-22. This should be compared with Figs. II-4, II-5, and II-6. It is interesting to note that if the frequency of the source is reduced so that the "divergence" boundary \( ka = \sqrt{n^2 - 1} \) is approached from larger values of \( ka \), then with \(|h_n| \) approaching \( k \), and \( \xi \) approaching zero, (35) and (23) show that the amplitude of the current density of the "free mode" wave decreases like \( \xi^2 \).

(e) **The Sheath Helix**, \( 90^\circ > \psi > 0^\circ \) for \( n \neq 0 \)

In this case, since the roots are not symmetrically located about the origin, it is necessary to decide in advance whether \( n > 0 \) or \( n < 0 \) is to be considered. In the following \( n > 0 \) is chosen although this results in no loss of generality, as should be clear from the discussions in Sections II.5(d) and B.4. For \( n < 0 \) the root positions are those which result if the \( h \) plane with its roots for \( n > 0 \) is first reflected about the real \( h \) axis and then about the imaginary \( h \) axis, with the subscript and superscript notations on the roots being maintained except that \( n = +|n| \) becomes \( n = -|n| \). Note from Fig. II-22 that if this transformation
The Sheath Helix
$n=0, 90^\circ > \psi > 0^\circ$

No roots for $ka < \sqrt{n^2 - 1}$

The Sheath Ring, $\psi = 0^\circ$
$|n| \geq 1$
is performed on the h plane for $\psi = 0^\circ$, the resulting h plane with its roots is identical to the original h plane with its roots since in this case $D_n(\zeta a) = 0$ is a function of $n^2$ and $h^2$.

For $n = 1$ the position of the roots is shown in Figs. II-23 and II-24. These should be compared with Figs. II-11a and II-14a. Figs. II-10a and II-13a are also pertinent. $\psi_{\text{max}}$ is defined in Section II.5(d) as the largest $\psi$ for which $|h_{n,2}^1|$ and $|h_{n,3}^1|$ roots occur, and, as noted there, although the precise value of $\psi_{\text{max}}$ is not available, it is known that $0.286 \Psi_{\text{max}} > 0.191$. Incidentally, it can be shown that for $\psi$ approaching $90^\circ$ and $\zeta a$ approaching zero, the magnitude of (24) becomes small like $\zeta^2 a^2 (\ln \zeta a)^2$ for the $|h_{1,1}^1|$ root. This confirms the remarks made in Section II.5(d) concerning this solution.

Figs. II-25 and II-26 show the position of the roots for $n = 2$. These should be compared with Figs. II-11b and II-14b. Figs. II-10b and II-13b are also pertinent. The exact value of $\psi_{\text{max}}$ is not known, but it is less than $\psi_{\text{max}}$. For $n \geq 3$ the roots appear very much as they are shown in Figs. II-25 and II-26 for $n = 2$, although the $|h_{n,1}^n|$ root may not occur at all, as was shown in Section II.5(d).

Some consideration of the form of the integrand in (12) and the disposition of the poles of this integrand, the zeros of $D_n(\zeta a)$, indicates that the current density amplitudes of the "free mode" waves approach infinity as the loss approaches zero near the points where the group velocity becomes small. In all cases this corresponds to the values of $ka$ and $\cot \psi$ where the roots appear in juxtaposition across the real h axis as in Figs. II-22c, II-24b, II-25c, and others. For lossless systems this is the point where the roots coalesce, and after which they disappear. In the case of the sheath ring, $\psi = 0^\circ$, this occurs for $ka = |n| + \Delta n$, or very near where the individual "rings" are resonant. However, even for $\psi$ relatively large the group velocity can
II.7(e)

\[ \begin{align*}
\text{h plane} & \quad - |h''_n, l\rangle - |h''_n, 1\rangle - k \\
& \quad +k' + |h''_n, 1\rangle \\
& \quad - |h''_n, 2\rangle \\
& \quad 1 \gg ka > 0 \\
\text{ka slightly larger than 1} \\
\hline
\text{h plane} & \quad - |h''_n, l\rangle - |h''_n, 1\rangle - k \\
& \quad +k' + |h''_n, 1\rangle + |h'_n, 3\rangle \\
& \quad - |h''_n, 2\rangle \\
& \quad \text{ka larger than in FIG. II-23b} \\
\text{ka slightly larger than in FIG. II-23c; similar for ka increasing} \\
\hline
\text{The Sheath Helix, } n=1, n \psi_{\text{max}} > \psi > 0^\circ \\
\end{align*} \]

\[ \begin{align*}
\text{h plane} & \quad - |h''_n, l\rangle - |h''_n, 1\rangle - k \\
& \quad +k' + |h''_n, 1\rangle \\
& \quad - |h''_n, 2\rangle \\
& \quad 1 \gg ka > 0 \\
\text{ka approximately 1} \\
\text{for small } \psi \text{ but increasingly smaller for larger } \psi \\
\hline
\text{h plane} & \quad - |h''_n, l\rangle - |h''_n, 1\rangle - k \\
& \quad +k' + |h''_n, 1\rangle \\
& \quad - |h''_n, 2\rangle \\
& \quad \text{ka larger than in FIG. II-24b; similar for ka increasing} \\
\hline
\text{The Sheath Helix, } n=1, 90^\circ > \psi > n \psi_{\text{max}} \\
\end{align*} \]
become zero for values of \(ka\) much smaller than \(|n|\). It appears that in these cases each complete "turn" of the sheath helix is resonant, that is, a standing wave occurs on each "turn". From this viewpoint it is possible to understand how a zero group velocity in the \(z\) direction can occur even though the phase velocity is not zero. For if standing waves occur on the "turns", there is no power flow along the "turns" and, consequently, none in the axial direction; therefore, the group velocity is zero. But if the time at which the standing waves reach their peak amplitude varies in a progressive manner from "turn" to "turn", there is a resultant axial phase velocity which can be either positive or negative.

(f) The Sheath Tube, \(\psi = 90^\circ\)

For \(\psi\) approaching 90°, \(\xi\) approaching zero, and \(|h_n^\infty|\) approaching \(k\), it can be shown that (24) becomes

\[
\lim_{\psi \to 90^\circ} \left\{ \frac{N_n(\xi a)}{\frac{d}{dh} D_n(\xi a)} \right\} = \frac{|n|}{2ka^2} e^{+j|n|\theta}, \quad |n| \geq 1. \quad (36)
\]

For the limiting case of \(\psi = 90^\circ\) there can be waves with \(z\) and \(\theta\) dependence like \(e^{-j|n|\theta} e^{+j|n|\theta}\) with \(z > 0\) and \(e^{+j|n|\theta} e^{+j|n|\theta}\) with \(z < 0\), all with identical amplitudes for \(|n| \geq 1\). However, as noted at the close of Section II.5, the waves whose coordinate dependence is like \(e^{-j|n|\theta} e^{-j|n|\theta}\) with \(z > 0\) and \(e^{+j|n|\theta} e^{+j|n|\theta}\) with \(z < 0\) are not stable.

The sheath tube also supports the TM modes which normally can exist in the interior of a perfectly conducting hollow circular cylinder, and which have been discussed at length elsewhere.\(^1,2\) In this case only longitudinal currents flow so that the boundary conditions (II.5-9) through (II.5-11) are satisfied, with the additional requirements that \(H_z = 0\) everywhere and \(E_\theta = 0\) for \(r = a\). It can be shown that an unshielded or an externally shielded gap source excites the \(T_{n,m}\) modes.
The Sheath Helix, \( n = 2, n \psi_{\text{max}} > \psi > 0^\circ \)

The Sheath Helix, \( n = 2, 90^\circ > \psi > n \psi_{\text{max}} \)
Such modes correspond to values of h such that h is real and less than k in magnitude, or h is pure imaginary. For any small amount less than 90°, the analysis in Section B.1 shows that such values of h cannot occur. As a consequence, it can be concluded that these modes are not stable on the sheath tube system.

II.8 Other Types of Sources

Although the gap source is perhaps the most useful type of source to consider in view of the usual dimensions and wavelengths encountered, there are other types of sources which can excite the "free modes". In particular, electric or magnetic multipoles placed near the sheath helix can excite the various modes, depending on the geometry and character of the source. The procedure for determining the resultant fields is much like that used in the previous section. The spherical wave source is expressed as a superposition of cylindrical waves in the form of an integral representation. The fields produced by the sheath helix in the presence of the source are then also expressed as a superposition of cylindrical waves of known form but with unknown amplitudes. These latter terms contain no singularities in the regions where they are valid. Matching the boundary conditions for the total fields yields the values of the unknown amplitudes in terms of the amplitude of the source and leads to a complete formal solution of the problem. In all cases the resultant integrals are quite similar to those met in the previous section and can be resolved in an identical manner. The total fields are found to be those of a modified spherical wave corresponding to the branch-cut integration which becomes small at large distance from the source, plus "free mode" waves corresponding to the contributions from the poles of the integrand which persist at arbitrarily large distances from the source in a lossless system. Although several examples of the
above have been worked out, in view of the similarity of the results to
those of the previous section and because the procedures are quite stand-
ard and explained at length elsewhere,\textsuperscript{1,2,39,48,50} these are not shown
here.

\textbf{Summary}

From the source-free Maxwell equations and the boundary conditions
for a sheath helix, a determinantal equation for the various "free modes"
characterized by a given angular variation is obtained. Solving this
determinantal equation yields the allowed values of the propagation con-
stants for the different modes. It is found that several waves per mode
with different propagation characteristics exist in restricted frequency
regions. In order to determine the significance of these waves, the
source-present problem is considered, and the resolution of the waves into
inward and outward traveling waves is clarified. The incompleteness of
the set of "free modes" is also made evident from the source-present
solution since it turns out, as in other open boundary problems, that
it is not possible to express the total fields resulting from even a
particular source in terms of the "free modes" alone. In the course of
considering the sheath helix, the limiting cases of the sheath ring and
the sheath tube are considered, and it is shown how the solutions vary
as the sheath helix varies between these extreme limits. Although nu-
merical results are not the primary purpose here, many useful expressions
are derived and procedures are indicated so that such results can be
readily obtained.
CHAPTER III

THE TAPE HELIX

In this chapter a more physically realistic model of the helix than the sheath model is considered. In particular, by assuming the helix to be wound of a thin tape, exact expressions for the fields in terms of the surface current density are obtained. (Section III.2) An exact formal solution for the thin tape helix problem is then obtained, but the completion of this requires solving for the roots of a determinantal equation in the form of an infinite determinant. (Section III.3) Although it does not seem possible to obtain these, the exact approach reveals the existence of regions — the forbidden regions — where "free mode" solutions are not possible. (Section III.4, Figs. III-2,3)

In order to obtain numerical results a helix wound of a quite narrow tape is considered. Reasonable assumptions concerning the boundary conditions lead to an approximate determinantal equation. (Section III.5) A means for solving this equation is described (Section III.6, Appendix C), and the results for a particular case are given. (Section III.7) The anomalous behavior of the propagation constant is a natural consequence of the solution, and agreement with experimental results given elsewhere is very good. (Section III.7, Fig. III-8) A simple method for determining the propagation constants approximately without extensive calculation is described. (Section III.8) An expression for the power flow is derived, a method for evaluating this is described (Section III.9, Appendix C), and the results of calculations for a typical case are shown. (Fig. III-10)
These results are used to compare the relative axial electric field in the sheath and tape helix models for a typical case. (Fig. III-12) Agreement with experimental results given elsewhere appears to be good. Approximate expressions for the power loss are also given. (Section III.9)

By methods and approximations similar to those used in the narrow tape case, the wide tape or narrow gap helix is analyzed, and an approximate determinantal equation is obtained. (Section III.10) The solutions for this case are essentially identical to those for the narrow tape case except for small values of ka. Expressions for the power flow and the equivalent tape current in this case are derived, and means for determining these are described although no numerical calculations are carried through. (Section III.11)

The effect of different assumed distributions for the current density on the tape and the electric field in the gap for the narrow tape and narrow gap helix, respectively, is discussed and shown to be quite small. (Section III.12)

The problem of multiwire helices is then considered, and it is shown how solutions for these systems can be obtained by the approximation methods used for the single wire helix. (Section III.13) The manner in which the forbidden region restriction is modified in these cases is also described. The sheath helix determinantal equation and field expressions result when the number of wires in a multiwire helix becomes infinitely large, and the details of this limit process are shown. Finally, the tape ring system is solved by using approximations similar to those used previously, and it is shown that in the limit of zero spacing the sheath ring system is obtained. (Section III.14)
Formulation and Formal Solution of the Problem

III.1 Definitions

As in the previous chapter the cylindrical coordinate system is used, and essentially all of the data given in Section II.1 is applicable to the analysis given in the following. Here, however, the structure is a tape helix, that is, a helix wound of a wire which is assumed to be a perfect conductor with very small (actually, it is taken to be zero) radial extension but finite axial extension. Such a helix is shown in Fig. III-1a, and a developed view is shown in Fig. III-1b. \( p, a \), and \( \psi \) are defined as before, and \( \delta \) and \( \delta' \) are the tape width and gap width, respectively, in the axial direction. The helix, whose axis is taken to coincide with the \( z \) axis as in Fig. II-2, is assumed to be positively wound and to be infinite in extent. The medium is also taken to be infinite in extent and is assumed as before to be linear, isotropic, and homogeneous. Although the procedure to be used would be applicable even if the mediums were not the same, for simplicity these are again taken to be the same for \( r > a \) as for \( r < a \) and lossless, and to be characterized by \( \mu, \epsilon, \) and \( k \) as defined in Section II.1. The assumption of zero radial thickness for the tape is certainly nonphysical, but if the actual thickness is very small compared with the other dimensions and with the wavelength, this approximation should influence the results to a very small degree. One might expect that the major properties of a helix would be affected to only a minor degree by the cross sectional shape of the conductor, particularly if its dimensions are small. The validity of this supposition is shown by the results obtained in this and the following chapter.

III.2 The Field Expressions

It will be noted that the helix structure is a periodic one, and, consequently, an approach can be used which has been often employed in
Fig. III-1a

Fig. III-1b

Tape Helix
related problems. This is essentially an application of Floquet's theorem, which for the case here states that the fields are multiplied only by some complex constant if one moves down the helix a distance \( p \). This is clear since, if the helix is displaced along the \( z \) axis by a distance \( p \), it coincides with itself, and the new fields can differ from the previous ones by only a constant factor. There is also an additional characteristic of symmetry which is made use of shortly. From the above it should be evident that a simple form of \( z \) dependence which meets the requirements of the periodic character of the structure is

\[
e^{-jhz} e^{-jm\frac{2\pi z}{p}} = e^{-j\frac{m}{p}z}, \tag{1}
\]

where \( h_m \) is given by

\[
h_m = h + m\frac{2\pi}{p}, \tag{2}
\]

and where \( m \) can have any integer value including zero. Since the Hertzian potentials \( \Pi_z \) and \( \Pi^*_{-z} \) must satisfy the scalar wave equation (II.3-13), it is readily found that the \( \theta \) dependence can be expressed in the form \( e^{j\eta \theta} \), where \( n \) can be any integer including zero. The function \( f(r) \) which contains the \( r \) dependence of the elementary solutions must then be a solution of the differential equation

\[
r \frac{d}{dr} (r \frac{df}{dr}) - \left[ (h_m^2 - k^2)r^2 + n^2 \right] f = 0. \tag{3}
\]

As noted in Section A.1, the solutions to (3) are the modified Bessel functions of order \( n \) and argument \( \eta \), where \( \eta_m \) is given by

\[
\eta_m = \left[ (h_m^2 - k^2)a^2 \right]^{\frac{1}{2}} = \left[ m^2 \cot^2 \psi + 2m \hbar a \cot \psi + h_m^2 a^2 - k^2 a^2 \right]^{\frac{1}{2}}. \tag{4}
\]

Since the fields must be finite, the \( I_n \) function is chosen for the solution for \( a > r > 0 \) and the \( K_n \) function for \( \omega r > a \). Further, since the complete set of functions must be used to satisfy the boundary conditions.
at \( r = a \), and since the field equations are linear so that the elementary solutions can be added, the representation for \( \Pi_z \) becomes

\[
\Pi_z^{i,e} = e^{-jhz} \sum_{m,n} A_{m,n}^{i,e} I_n \left( \frac{r}{a} \right) e^{j\eta e} e^{-jm \frac{2\pi}{p} z},
\]

(5)

where the superscripts \( i \) and \( e \) refer to the internal, \( r \leq a \), and external, \( r \geq a \), regions as in Chapter II. The meaning of completeness is the usual one and refers to the ability of a series expansion to represent an arbitrary function within certain limitations. (See reference 50, page 179.) \( A_{m,n}^{i,e} \) are coefficients which are dependent on the boundary conditions, \( \sigma \), and the character of the exciting source, but independent of the coordinates.

As mentioned above, there is an additional characteristic of symmetry which simplifies the representations. It can be readily seen that if the helix is translated along its axis some distance less than \( p \), it may then be rotated so that it coincides with itself. It thus appears clear that Floquet's theorem can be applied to the angular coordinate as well as to the axial coordinate. The implications of this can be seen as follows.

Let \( z = z' + \gamma \) or \( z' = z - \gamma \), and let \( \theta = \theta' + \phi = \theta' + \frac{2\pi}{p} \gamma \) or \( \theta' = \theta - \frac{2\pi}{p} \gamma \); \( \phi = \frac{2\pi}{p} \gamma \) is the angle through which the helix must be rotated after a translation of \( \gamma \) to make it coincide with itself. This is equivalent to putting the \( x' \) axis through the helix at \( z = \gamma \) and measuring \( \theta' \) from \( x' \). After substitution of these values of \( \theta \) and \( z \), (5) becomes

\[
\Pi_z^{i,e} = e^{-j\gamma} e^{-j\gamma z'} \sum_{m,n} A_{m,n}^{i,e} I_n \left( \frac{r}{a} \right) e^{-j(m-n) \frac{2\pi}{p} \gamma} e^{j\eta e} e^{-jm \frac{2\pi}{p} z'}. \]

(6)

Since a rotation and a translation make the helix coincide with itself, the form of the solutions as a function \( \theta' \) and \( z' \) must be the same as the form expressed as a function of \( \theta \) and \( z \). It can be seen from (6) that if this is to be true for all \( \gamma \) as well as \( \gamma = p \), it is necessary
to take \( m = n \), and for \( m \neq n \) put \( A_{m,n}^{i,e} = 0 \). This constraint which results when uniform helices are considered requires that

\[
\Pi_{i,e}^z = e^{-jhz} \sum_{m} A_{m}^{i,e} \frac{I_{m}}{K_{m}} (\eta_{m} \Sigma_{a}) e^{-jm \frac{2\pi}{p} z - \phi},
\]

(7)

be taken as the representation for the electric Hertzian potential, and that an identical form

\[
\Pi_{i,e}^z = e^{-jhz} \sum_{m} B_{m}^{i,e} \frac{I_{m}}{K_{m}} (\eta_{m} \Sigma_{a}) e^{-jm \frac{2\pi}{p} z - \phi},
\]

(8)

be taken for the magnetic Hertzian potential. It should be pointed out that if the helix is not uniform, that is, if a developed turn of the helix is not a simple parallelogram as in Fig. III-1b, a summation over \( m \) and \( n \) as in (5) would be required. In other words, if an axial translation of \( p \) moves the helix into itself, but a translation of less than \( p \) and a rotation cannot move the helix into itself, (5) must be used. This is obviously a considerably more complicated situation than the uniform helix and is not considered any further.

Using (7) and (8) and (II.3-7) through (II.3-12), one can readily find the field components in terms of the \( A_{m}^{i,e} \) and \( B_{m}^{i,e} \) coefficients which are related by the continuity requirements of the field boundary conditions at \( r = a \). These are, of course, that at \( r = a \) the tangential electric field is continuous everywhere, and that the discontinuity in the tangential magnetic field is proportional to the total surface current density; that is,

\[
E_{z,\theta}^i = E_{z,\theta}^e',
\]

at \( r = a \)

\[
K_{\theta} = H_{z}^i - H_{z}^e',
\]

\[
K_{z} = H_{\theta}^i - H_{\theta}^e',
\]

(12)
(see II.4-11) and (II.4-12). Since the representations for the surface current density components must have the same form as the field components,

$$K_\theta = e^{-jhz} \sum_m \kappa_{\theta m} e^{-jm(\frac{2\pi z-\theta}{p})},$$  \hspace{1cm} (13)$$

and

$$K_z = e^{-jhz} \sum_m \kappa_{zm} e^{-jm(\frac{2\pi z-\theta}{p})},$$  \hspace{1cm} (14)$$

where \(\kappa_{\theta m}\) and \(\kappa_{zm}\) are the Fourier coefficients of the current density expansions. It should be emphasized that (13) and (14) are representations for the components of the total surface current density, that is, the sum of the current density on both "sides" of the infinitesimally thick perfectly conducting tape. Proceeding as pointed out above to find the field components in terms of \(A_m^i,e\) and \(B_m^i,e\), and using (12), (13), and (14) to find the values of \(A_m^i,e\) and \(B_m^i,e\) in terms of \(\kappa_{\theta m}\) and \(\kappa_{zm}\), where use is made of the orthogonality of the Fourier space harmonics for \(2\pi > \theta > 0\) and \(p > z > 0\), one obtains after considerable algebra

$$E_r^e = \frac{e^{-jhz}}{\omega \varepsilon a} \sum_m \left\{ - \frac{h_m a_m I_m (\eta_m) K_m (\eta_m \frac{r}{a}) \kappa_{zm}}{m \eta_m} ight. $$

$$+ \left. \frac{m}{\eta_m} \left[ \frac{h_m^2 \varepsilon_m (\eta_m) K_m (\eta_m \frac{r}{a}) + \frac{2a}{r} K_m (\eta_m \frac{r}{a}) K_m (\eta_m \frac{r}{a}) \right] \kappa_{\theta m} \right\} e^{-j(\frac{2\pi z-\theta}{p})},$$  \hspace{1cm} (15)$$

$$E_\theta^e = \frac{j e^{-jhz}}{\omega \varepsilon a} \sum_m \left\{ - \frac{a}{r} m h_m a I_m (\eta_m) K_m (\eta_m \frac{r}{a}) \kappa_{zm} ight. $$

$$+ \left. \frac{1}{\eta_m} \left[ \frac{a}{r} m^2 h_m^2 a^2 I_m (\eta_m) K_m (\eta_m \frac{r}{a}) + \eta_m^2 k^2 a^2 I_m (\eta_m) K_m (\eta_m \frac{r}{a}) \right] \kappa_{\theta m} \right\} e^{-j(\frac{2\pi z-\theta}{p})},$$  \hspace{1cm} (16)$$
\[
H_e = e^{-jhz} \sum_m \left\{ \eta_m I_m(\eta_m) K_m(\eta_m/\alpha) \mathcal{K}_{zm} \right\} e^{-jm(2\pi z/\rho - \theta)} e^{-jm(2\pi z/\rho - \theta)}, \tag{17}
\]

\[
H_r = j e^{-jhz} \sum_m \left\{ \eta_m I_m(\eta_m) K_m(\eta_m/\alpha) \mathcal{K}_{zm} \right\} e^{-jm(2\pi z/\rho - \theta)}, \tag{18}
\]

\[
H_\theta = e^{-jhz} \sum_m \left\{ -\eta_m I_m(\eta_m) K_m(\eta_m/\alpha) \mathcal{K}_{zm} \right\} e^{-jm(2\pi z/\rho - \theta)}, \tag{19}
\]

\[
H_z = -e^{-jhz} \sum_m \left\{ \eta_m I_m(\eta_m) K_m(\eta_m/\alpha) \mathcal{K}_{zm} \right\} e^{-jm(2\pi z/\rho - \theta)}. \tag{20}
\]

In obtaining the above it is necessary to make use of the Wronskian relationship \( \mathcal{W} \left[I_m(\eta_m), K_m(\eta_m)\right] = -\frac{1}{\eta_m} \). The prime on the \( I_m \) and \( K_m \) functions means differentiation with respect to the argument. Only the external field components are given above since the expressions for the internal components are identical except that the \( I_m \) and \( K_m \) functions are interchanged everywhere.

It is readily shown that (15) through (20) and their counterparts for the internal components satisfy the Maxwell equations as well as the continuity condition (II.4-21). In addition, they are bounded, single-valued, and have the proper form to represent the fields around a uniform helix. Finally, they are made up of a complete set of functions in \( \theta \) and \( z \) so that the boundary conditions at \( r = a \) can be satisfied. One is assured, therefore, that subject to these boundary conditions and the condition at infinity, these representations are the proper ones.
III.3 The Formal Solution

Having the proper representations of the field components, one must apply the boundary conditions in order to complete the solution and to find those values of $h$ which correspond to the "free modes". Since expressions are available which already satisfy the continuity requirements of the field boundary conditions at $r = a$, the only condition which remains is that the tangential electric field on the tape be zero. The application of this condition automatically insures that the tangential magnetic field will be continuous through the gap. The reasons for this are clear since if by making the tangential electric field zero on the tape one could solve for an $h$ which would lead to a discontinuity in the tangential magnetic field in the gap, a current density in the gap would thereby be implied. But this implies a zero tangential electric field there, which would be contrary to the original constraint. It is evident that instead of requiring the tangential electric field to be zero on the tape, one might take the determining boundary condition to be the continuity of the tangential magnetic field through the gap. It seems simpler to use the electric field condition, and it is used in the following. The necessity of this condition is obvious, and its sufficiency is at least indicated by the fact that a unique formal solution results.

The conditions that must be imposed are

$$E_{\theta}^{1,e} = 0,$$
$$E_{z}^{1,e} = 0,$$

for $r = a$, $\frac{D}{2\pi} \theta + \delta > z > \frac{D}{2\pi} \theta$, and $2\pi > \theta > 0$. In order to make the procedure somewhat clearer, it is assumed that Fourier expansions exist for the electric field components in the interval $\frac{D}{2\pi} \theta + \delta > z > \frac{D}{2\pi} \theta$ and $2\pi > \theta > 0$ for $r = a$ of the form
\[ E_\theta = j \frac{e^{-jhz}}{\omega a} \sum_n \nu_{\theta n} e^{-jn \frac{P}{5} \left( \frac{2\pi}{P} z - \theta \right)} \], \quad (2)

and

\[ E_z = j \frac{e^{-jhz}}{\omega a} \sum_n \nu_{zn} e^{-jn \frac{P}{5} \left( \frac{2\pi}{P} z - \theta \right)} \], \quad (3)

where \( \nu_{\theta n} \) and \( \nu_{zn} \) are the Fourier coefficients of the electric field expansions. Equating the corresponding components given in (III.2-16) and (III.2-17) to (2) and (3) above so that

\[
\sum_m \left\{ -\eta_m a_m K_m K_{zm} + \frac{1}{\eta m} \left[ \eta_m ^2 K_m ^2 K_{zm} ^2 + \eta_m ^2 K_m ^2 K_{zm} ^2 \right] \kappa_{\theta m} \right\} e^{-jm \left( \frac{2\pi}{P} z - \theta \right)} = \sum_n \nu_{\theta n} e^{-jn \frac{P}{5} \left( \frac{2\pi}{P} z - \theta \right)}, \quad (4)
\]

\[
\sum_m \left\{ \eta_m ^2 K_m ^2 K_{zm} - \eta_m a_m K_m K_{zm} \kappa_{\theta m} \right\} e^{-jm \left( \frac{2\pi}{P} z - \theta \right)} = \sum_n \nu_{zn} e^{-jn \frac{P}{5} \left( \frac{2\pi}{P} z - \theta \right)}, \quad (5)
\]

multiplying through by \( e^{jq \frac{P}{5} \left( \frac{2\pi}{P} z - \theta \right)} \), where \( q \) is integer, and integrating from \( z = \frac{P}{2\pi} \theta \) to \( \frac{P}{2\pi} \theta + 5 \) keeping \( \theta \) constant, one finds that only for \( q = n \) can the right side of (4) and (5) be nonzero, and for \( q = n \) these become, after dividing by \( 6 \) which is nonzero, \( \nu_{\theta n} \) and \( \nu_{zn} \), respectively. On the left side of (4) and (5) the functional form of the Bessel functions is omitted for convenience since the argument in every case is \( \eta_m \). The variables in \( \theta \) drop out, as might be expected from symmetry, and an integration over \( \theta \) is not required. Since \( E_\theta \) and \( E_z \) are zero for \( \frac{P}{2\pi} \theta + 5 > z > \frac{P}{2\pi} \theta \) and \( r = a \), it is clear from (2) and (3) that \( \nu_{\theta n} \) and \( \nu_{zn} \) must themselves be identically zero. Thus, since the integrated forms of (4) and (5) must be true for all \( n \), there results two sets of infinite simultaneous equations. These may be expressed in the following matrix form:
\[
\begin{align*}
\mathbf{P}_{nm} \times \mathbf{K}_{zm} &= \mathbf{Q}_{nm} \times \mathbf{K}_{\theta m}, \\
\mathbf{P}_{nm} \times \mathbf{K}_{zm} &= \mathbf{S}_{nm} \times \mathbf{K}_{\theta m},
\end{align*}
\]

where \( \mathbf{P}_{nm}, \mathbf{Q}_{nm}, \mathbf{R}_{nm}, \) and \( \mathbf{S}_{nm} \) are matrices with an infinite number of rows and columns, and \( \mathbf{K}_{zm} \) and \( \mathbf{K}_{\theta m} \) are infinite column matrices. The elements of \( \mathbf{P}_{nm}, \mathbf{Q}_{nm}, \mathbf{R}_{nm}, \) and \( \mathbf{S}_{nm} \) are given by

\[
\begin{align*}
P_{nm} &= m^2 \alpha_1 I_{m} \mathbf{M}_{nm}, \\
Q_{nm} &= \frac{1}{2} \left[ m^2 h^2 a^2 K_{m} I_{m} + \eta^2 m^2 h^2 a^2 K_{m} I_{m} \right] \mathbf{M}_{nm}, \\
R_{nm} &= \eta^2 K_{m} I_{m} \mathbf{M}_{nm}, \\
S_{nm} &= m^2 \alpha_1 K_{m} I_{m} \mathbf{M}_{nm} = P_{nm},
\end{align*}
\]

where

\[
M_{nm} = \frac{e^{j2\pi \left( \frac{n}{\delta} - \frac{m}{p} \right) \delta}}{j2\pi \left( \frac{n}{\delta} - \frac{m}{p} \right) \delta} - 1,
\]

with \( M_{nm} = 1 \) if \( \frac{n}{\delta} = \frac{m}{p} \). The elements of \( \mathbf{K}_{\theta m} \) and \( \mathbf{K}_{zm} \) are, of course, \( \mathbf{K}_{\theta m} \) and \( \mathbf{K}_{zm} \). With certain restrictions the usual matrix rules for \( n \) and \( m \) finite may be taken to be valid for \( n \) and \( m \) becoming infinitely large. Assuming these restrictions to be met and the non-singularity of \( \mathbf{Q}_{nm} \), from (6) one obtains

\[
\mathbf{Q}_{nm}^{-1} \times \mathbf{P}_{nm} \times \mathbf{K}_{zm} = \mathbf{K}_{\theta m},
\]

where \( \mathbf{Q}_{nm}^{-1} \) means the inverse of \( \mathbf{Q}_{nm} \) so that

\[
\mathbf{Q}_{nm}^{-1} \times \mathbf{Q}_{nm} = \mathbf{U}.
\]
\([U]\) is the infinite unit matrix with all diagonal elements equal to unity and all nondiagonal elements equal to zero. Substitution of (13) in (7) gives

\[
\left\{ \left[ R_{nm} \right] - \left[ S_{nm} \right] \times \left[ Q_{nm} \right]^{-1} \times \left[ P_{nm} \right] \right\} \times \left[ \kappa_{zm} \right] = 0 .
\] (15)

Since the \(\kappa_{zm}\) must be nonzero, (15) has no nontrivial solution unless the determinant of the matrix in the braces \(\{\}\) vanishes. Setting this determinant equal to zero provides the determinantal equation for the unknown propagation constant \(\xi\) since it is assumed that \(a, p, k,\) and \(\delta\) are known.

Before discussing (15) another approach will be pointed out which is somewhat related to the above. The expression

\[
\mathcal{E} = \frac{1}{2\pi a s} \int_{\text{tape}} |E|^2 \, dA
\] (16)

is considered, where \(\mathcal{E}\) is the mean square value of the electric field on the tape, and

\[
|E|^2 = |\tilde{E}\tilde{E}| = E_\theta \tilde{E}_\theta + E_z \tilde{E}_z ,
\] (17)

where the tilde \(\sim\) means the complex conjugate. It should be noted that if \(\tilde{E}\) has any nonzero value on the tape, \(\mathcal{E}\) will be finite and positive. \(\mathcal{E}\) may be considered as a measure of the error between an approximate representation of the fields which might be obtained in some manner, and a true representation which would give \(\mathcal{E} = 0\). It would be expected, therefore, that the current distribution which makes \(\mathcal{E}\) its smallest value would approximate most closely the true distribution. Proceeding in the manner indicated by the above, for \(r = a\) from (II.2-17) and (III.2-18) one obtains

\[
E_\theta = j \frac{e^{-jhz}}{\omega e a} \sum_m \left( a_m \kappa_{zm} + b_m \kappa_{\theta m} \right) e^{-jm(2\pi z - \theta)} ,
\] (18)
where \( a_m, b_m, c_m, \) and \( d_m \) are used to represent more complicated expressions momentarily. Forming (17) and then (16), where \( dA = ad\theta dz \) and the integration is on \( z \) from \( \frac{P}{2\pi} \) \( \theta \) to \( \frac{P}{2\pi} \) \( \theta + \delta \) first and then on \( \theta \) from 0 to \( 2\pi \), and where the \( \theta \) integration is simply performed because of the elimination of the \( \theta \) variables after \( z \) integration, one obtains

\[
E = \frac{1}{(\omega a)^2} \sum_{mn} \left[ \left( a_m \kappa_{zm} + b_m \kappa_{\theta m} \right) \left( \tilde{a}_n \kappa_{zn} + \tilde{b}_n \kappa_{\theta n} \right) \right. \\
+ \left. \left( c_m \kappa_{zm} + d_m \kappa_{\theta m} \right) \left( \tilde{c}_n \kappa_{zn} + \tilde{d}_n \kappa_{\theta n} \right) \right] N_{nm}.
\]  

Here,

\[
N_{nm} = \frac{e^{-j(m-n)\frac{2\pi}{P} \delta}}{\frac{2\pi}{P} \delta - 1},
\]

with \( N_{nm} = 1 \) for \( m = n \). For reasons which are considered shortly, it is assumed in writing (20) and (21) that only real values of \( h \) need be considered. For \( \frac{\partial E}{\partial \kappa_{zq}} \) and \( \frac{\partial E}{\partial \kappa_{\theta q}} \) one obtains

\[
\frac{\partial E}{\partial \kappa_{zq}} = \frac{1}{(\omega a)^2} \sum_m \left[ \tilde{a}_q \left( a_m \kappa_{zm} + b_m \kappa_{\theta m} \right) + \tilde{c}_q \left( c_m \kappa_{zm} + d_m \kappa_{\theta m} \right) \right] N_{qm},
\]

\[
\frac{\partial E}{\partial \kappa_{\theta q}} = \frac{1}{(\omega a)^2} \sum_m \left[ \tilde{b}_q \left( a_m \kappa_{zm} + b_m \kappa_{\theta m} \right) + \tilde{d}_q \left( c_m \kappa_{zm} + d_m \kappa_{\theta m} \right) \right] N_{qm},
\]

with q taking on all values of n. \( \frac{\partial E}{\partial \kappa_{zq}} \) and \( \frac{\partial E}{\partial \kappa_{\theta q}} \) may be obtained also, but the resulting equations yield no additional information as they are merely the conjugates of (22) and (23) since \( \bar{E} \) is real and \( \bar{N}_{nm} = N_{nm} \).

\( E \) will be an extremum and presumably a minimum if \( \frac{\partial E}{\partial \kappa_{zq}} \) and \( \frac{\partial E}{\partial \kappa_{\theta q}} \) are put equal to zero.
equal to zero. Doing this, one may express the results in matrix form

\[
\begin{bmatrix} T_{nm} \end{bmatrix} \times \begin{bmatrix} \kappa_{zm} \end{bmatrix} = \begin{bmatrix} U_{nm} \end{bmatrix} \times \begin{bmatrix} \kappa_{m} \end{bmatrix},
\]

\[
\begin{bmatrix} V_{nm} \end{bmatrix} \times \begin{bmatrix} \kappa_{zm} \end{bmatrix} = \begin{bmatrix} W_{nm} \end{bmatrix} \times \begin{bmatrix} \kappa_{m} \end{bmatrix},
\]

where \([T_{nm}], \ [U_{nm}], \ [V_{nm}], \ [W_{nm}]\) are matrices with an infinite number of rows and columns, and \([\kappa_{zm}], \ [\kappa_{m}]\) are as before. Since only real values of \(h\) need be considered, the coefficients \(a_m, b_m, c_m, \text{ and } d_m\) are purely real. Consequently, the elements of \([T_{nm}], \ [U_{nm}], \ [V_{nm}], \ [W_{nm}]\) are given by

\[
T_{nm} = (\frac{m^2 a_h^2}{\eta_m^2}) K_I K_I N_{nm},
\]

\[
U_{nm} = \left(\frac{(n h a_n a_h^2 n^2 + m h a_n a_h^2 n^2)}{\eta_m^2} K_I K_I N_{nm}\right) N_{nm},
\]

\[
V_{nm} = \left(\frac{(n h a_n a_h^2 n^2 + m h a_n a_h^2 n^2)}{\eta_m^2} K_I K_I N_{nm}\right) N_{nm},
\]

\[
W_{nm} = \left(\frac{1}{\eta_m^2} \left(\frac{n^2 a_h^2}{\eta_m^2} K_I K_I + \frac{m^2 a_h^2}{\eta_m^2} K_I K_I + \frac{m^2 a_h^2}{\eta_m^2} K_I K_I\right) + \frac{m n h a_n a_h^2}{\eta_m^2} K_I K_I N_{nm}\right) N_{nm},
\]

with \(N_{nm}\) as in (21). Note the symmetrical form of \([U_{nm}]\) and \([V_{nm}]\) so that \(U_{nm} = \tilde{V}_{nm}\). Proceeding as before, assuming that the necessary conditions for the process are met and that \([U_{nm}]\) is nonsingular, from (21h) one obtains

\[
[U_{nm}]^{-1} \times [T_{nm}] \times [\kappa_{zm}] = [\kappa_{zm}],
\]

and using this in (25) gives

\[
\left\{ \begin{bmatrix} V_{nm} \end{bmatrix} - \begin{bmatrix} W_{nm} \end{bmatrix} \times \begin{bmatrix} U_{nm} \end{bmatrix}^{-1} \times \begin{bmatrix} T_{nm} \end{bmatrix} \times \begin{bmatrix} \kappa_{zm} \end{bmatrix} = 0. \right.\]
Since the $\kappa_{zm}$ must be nonzero, (31) has no nontrivial solution unless the determinant of the matrix in the braces $\left\{ \right\}$ vanishes. Setting this determinant equal to zero again provides a determinantal equation for the unknown propagation constant $h$ since it is assumed that $a, p, k,$ and $\delta$ are known. In the above it has been assumed that $n$ and $m$ are infinite and that, therefore, the complete set of functions for representing the electric field at $r = a$ is used. In this case the Fourier expansions for the electric field components exactly represent the true electric field in the least square sense, the minimum of $\mathcal{E}$ is consequently zero, the boundary conditions are therefore satisfied, and, thus, a unique solution to the problem would result if the solutions to the determinantal equation could be obtained.

By a reexpansion procedure and by a minimization of mean square error procedure, two determinantal equations have been derived from either of which the proper values of the unknown propagation constant $h$ of the "free modes" can in theory be obtained. If these values of $h$ are known, they can be used in (15) or (31) to find the coefficients $\kappa_{zm}$, and these, in turn, can be used to find the coefficients $\kappa_{6m}$ from (13) or (30). Thus, there is a formal procedure for solving the problem, although practically, because of the infinite order of the determinants, the process cannot be carried out. It will be noted that the determinantal equation obtained from (15) appears different from the one which results from (31). Since both are solutions to the problem, this difference may be somewhat surprising. The significance of this difference can be explained in the following manner. The meaning of the process used to obtain (6) and (7) is that the weighted average of the electric field components on the tape, \[ \frac{1}{2\pi a\delta} \int_{\text{tape}} E_{0, z} e^{jn \frac{p}{p} (2\pi z - \delta)} \] ad$\delta$dz, where the weight factor is $e^{jn \frac{p}{p} (2\pi z - \delta)}$, is required to be zero for all $n$. On the
other hand, (24) and (25) are obtained by requiring the mean square deviation of the magnitude of the total electric field on the tape at \( r = a \) from its true value to be a minimum. Thus, the difference in the form of the determinantal equations may be said to be a result of the difference in the manner of approximating the electric field boundary condition for finite \( m \) and \( n \). For \( m \) and \( n \) becoming infinitely large the two methods should yield the same results. There are still other ways of approximating the electric field boundary condition for finite \( m \) and \( n \), and these would give still other forms for the determinantal equation, although all should yield the same result for \( m \) and \( n \) becoming infinitely large.

Although the above shows that a formal solution to the tape helix problem is possible, it is of no practical use for obtaining useful numerical results or for determining the detailed character of the solutions. The usual procedure for handling such infinite determinants is to assume that only the central rows and columns, that is, for small \( m \) and \( n \), are of any consequence and to expand the resulting finite determinant in the ordinary way. The significance of this procedure is that often only the lower order terms in the series expansions are important, and small error results if the higher order terms are ignored. The validity of the process may be checked by using more and more of the rows and columns, and if convergence is rapid, these operations need not be carried too far. Examples of this approach are shown on page 458 in reference 41, in reference 55, and in many other places in the recent literature on electromagnetic wave boundary value problems. A still more elegant method is shown in reference 38. The method shown there uses information concerning the physical characteristics of the field and the asymptotic forms of the functions involved to adjust the low order space harmonics to obtain a solution which quite closely approximates the requirements for all the higher order harmonics. Despite these possibilities for obtaining an
approximate solution to the helix problem, another method, which is discussed at length later in this chapter, has been preferred here.

In concluding this section it should be pointed out that the helix problem appears more complicated than the usual perturbed boundary problem which finally requires the solution of a single set of infinite homogeneous equations. This is seen from the previous development where, for example, from (24) and (25) it will be noted that two infinite sets of infinite linear equations must be solved simultaneously.

Or, if the final form (31) is considered, only one infinite set of infinite homogeneous equations must be solved; but this is complicated by the presence of the $[W_{nm}] \times [U_{nm}]^{-1} \times [T_{nm}]$ term. From a practical viewpoint the appearance of the double product and of the inverse matrix makes exceedingly onerous even some of the approximation procedures mentioned above. This complication is a direct result of the need for both TE and TM waves to describe the field and to satisfy the boundary conditions.

III.4 The Forbidden Regions

Although the formal solutions described in the previous section appear to have little immediate practical value, the field expressions of Section III.2 are of considerable use in later developments, and, further, some results which may be deduced from these expressions are of great interest.

In writing the representations for the external fields as a sum of terms whose radial dependence is of the form $K_m(\eta \frac{r}{a})$, it has been tacitly assumed that $\eta_m$ is real and positive. Without for the moment discussing the detailed implications of this, the necessary requirements for this condition to be met are considered next. It is assumed that $h$ is real, and for a lossless medium $k$ is also real. It should be recalled that the
tape is assumed to be a perfect conductor. From (III.2-4)

$$\eta_m = \left[ \left( h_m^2 - k^2 \right) a^2 \right]^{\frac{1}{2}} > 0$$  \hspace{1cm} (1)

requires

$$h_m^2 a^2 > k^2 a^2$$  \hspace{1cm} (2)

or

$$|h_m a| > k a$$  \hspace{1cm} (3)

If \( h \) is assumed to be positive, (3) is surely satisfied for \( m > 0 \) if \( |h| > k \). For \( m < 0 \), however, (3) is satisfied only within restricted regions. For \( m < 0 \), from (3) and (III.2-2), the condition becomes

$$|ha - |m| \cot \psi| > ka.$$  \hspace{1cm} (4)

Examination of (4) shows that this inequality can be expressed as

$$|m| \cot \psi + ka < |ha| < |m| \cot \psi - ka$$  \hspace{1cm} (5)

or

$$|m| + \frac{ka}{\cot \psi} < \frac{ha}{\cot \psi} < |m| - \frac{ka}{\cot \psi}, \quad |m| > l, \quad h > k.$$  \hspace{1cm} (6)

If \( h \) is negative and in magnitude larger than \( k \), (3) is surely satisfied for \( m < 0 \) but only for certain restricted regions for \( m > 0 \). Carrying through the argument as above gives results that can be expressed in a similar manner to (6) or, finally, for both positive and negative \( h \)

$$|m| + \frac{ka}{\cot \psi} < \frac{|ha|}{\cot \psi} < |m| - \frac{ka}{\cot \psi}, \quad |m| > l,$$  \hspace{1cm} (7)

$$|h| > k.$$  \hspace{1cm}

The significance of (7) is most easily realized from Fig. III-2 where

$$\frac{ka}{\cot \psi}$$ has been plotted versus \( \frac{|ha|}{\cot \psi} \). The shaded areas are the regions where the variables do not satisfy the inequality, with the boundaries being given by (7) satisfied with equal signs. The regions are labeled with the \( m \) to which they correspond. It should be noted that the entire region for \( \frac{ka}{\cot \psi} > \frac{1}{2} \) is shaded. The reason for this is most readily seen as follows. Since (7) must hold for all integer \( m \), it must
FIG. III-2
\( \frac{ka}{\cot \psi} \) vs. \( \frac{ha}{\cot \psi} \)

Chart of Forbidden Regions
also hold when \(|m| = |n|\), for example, is put on the left side, and
\(|m| = |n| + 1\) on the right side. If \(\frac{ka}{\cot \Psi} = \frac{1}{2} + \Delta\) is taken, where \(\Delta > 0\), (7) requires
\[|n| + \frac{1}{2} + \Delta < \frac{|h|a}{\cot \Psi} < |n| + \frac{1}{2} - \Delta;\]
this is clearly absurd and means that only for \(\frac{ka}{\cot \Psi} < \frac{1}{2}\) can (7) be satisfied. Since \(\cot \Psi = \frac{2\pi a}{p}\), this restriction immediately becomes
\[\frac{2\pi a}{\lambda} = ka < \frac{\cot \Psi}{2} = \frac{na}{p},\]
or
\[p < \frac{\lambda}{2}.\] (8)

The significance of the above restrictions can be seen in the following manner. Assume for the moment that a solution is possible at some \(ka < \frac{\cot \Psi}{2}\) with \(ha\) real and in the \(|n|\)th shaded region of Fig. III-2. In this case, in the external field representations all the terms of the series have radial dependence like \(K_m(\frac{r}{a})\) except one which is an \(H_n^{(2)}(\frac{r}{a})\) function. (See (A.4-14)). But for large \(r\) this would correspond to an outgoing radial wave which can only come from an infinitely long (in the \(z\) direction) line type source located at some finite \(r\). Such sources have not been postulated in the assumptions, and, in fact, the concern here is with the source-free problem. Thus, if a solution exists at all, it must exist with the restrictions given above and with only \(K_m(\frac{r}{a})\) radial dependence for all the terms of the series. Therefore, "free mode" solutions must exist only in the unshaded regions of Fig. III-2, and the shaded portions are called the forbidden regions. Another viewpoint is to recognize that the total average power flow must be the same and finite across any infinite plane perpendicular to the \(z\) axis for any "free modes" with \(h\) real. But a \(H_n^{(2)}(\frac{r}{a})\) term would result in an average radial power flow and, consequently, a violation of the Poynting...
theorem if no line sources are present. In the above it was assumed that only one term might have $H_n^{(2)}$ radial dependence, although clearly the same reasoning applies if it is assumed that more than one term of this nature could exist. The exclusion of "free mode" solutions from the forbidden regions does not insure that solutions will exist in the allowed regions since the boundary conditions must also be matched at $r = a$ to fulfill all the requirements for a unique solution.

In the above it has been assumed that $h$ is real. The possibility remains that solutions may exist for $h$ complex or pure imaginary. Although it would be expected that such solutions do not occur here as they do not for other infinite lossless open cylindrical structures, it has not been possible to prove in a rigorous fashion that such is the case.\textsuperscript{39} It should be noted from Section B.1 that the impossibility of the existence of complex or pure imaginary $h$ roots was proved for the sheath helix case by recourse to the determinantal equation. Attempts to use similarly the exact determinantal equation for the tape helix, (III.3-15), were not successful. Nor did the procedures of reference 39, which included use of symmetry and violation of Poynting's theorem as the key steps, and which supplied a satisfactory proof in a similar problem, lead to a definite proof here; although with further work these steps may supply the answer. The matter of a rigorous proof for the pure reality of the $h$ roots in the exact tape helix case must be considered still open. However, it is possible to prove from the approximate determinantal equations for the narrow tape or small wire helix and for the wide tape helix that only real $h$ roots are possible in those cases. The proof is given in Section IV.3(b). It seems reasonable to suppose that this character of the $h$ roots in maintained in the exact case also.

From the results of the previous analysis and discussion it is evident that one need only consider $h$ real with $|h| > k$ and that one can exclude
the forbidden regions in seeking "free mode" solutions. Further, it seems clear in view of the physical symmetry of the single wound tape helix and the manner in which $h$ and $m$ occur in the radial argument function $\eta_m$, as well as elsewhere in the field representations, that identical solutions exist for positive and negative $h$.

It is often useful to consider the ratio $\frac{ka}{ha}$ which corresponds to the ratio of the phase velocity of the zeroth order space harmonic to the velocity of a uniform plane wave in the medium. A plot of $\frac{ka}{|h|a}$ versus $\frac{ka}{\cot \psi}$ is shown in Fig. III-3 and represents another type of chart for the forbidden regions. From (7) these regions correspond to values such that

$$\left| \frac{1}{|m| \cot \psi} - 1 \right| > \frac{ka}{|h|a} > \left| \frac{1}{|m| \cot \psi} + 1 \right| , \ |m| > 1.$$  \hspace{1cm} (9)

As for the sheath helix model, since it is necessary that $|h| > k$, only slow waves, referred to the $z$-axis, can exist on the tape helix.

The limitation given in (8) shows that under no circumstances will "free modes" exist if $p > \frac{\lambda}{2}$. If each turn of the helix is considered as constituting an element of a radiating linear array, then the requirement that the spacing between elements, $p$, be less than $\frac{\lambda}{2}$ corresponds to the usual requirement for a single major lobe to exist only along the axis of the array. (See, for example, reference 56, page 275.) The forbidden region requirement can also be interpreted in this manner and suggests the somewhat crude picture of the helix as an infinite circular diffraction device. In this case a "free mode" results at those frequencies for which the waves diffracted at the gap have the proper phase and amplitude so as to interfere and to prevent radiation in a radial direction.

It should be emphasized that the results of this section concerning the forbidden regions are not dependent on any approximations and would apply to the exact solution for the lossless helix problem if it were
available. Further, it is evident from the remarks of Section III.2 that similar forbidden region restrictions exist for many types of helical structures and, actually, for all other open cylindrical periodic systems.

Some similarities between the results given here and those given elsewhere for periodic structures will, no doubt, be noted. However, in the present case there is an upper frequency limit beyond which no "free mode" solutions can exist, as well as certain regions where the same restriction applies. This is in contrast to many other problems concerned with periodic structures where, although wave propagation does not occur within restricted frequency bands, exponentially damped solutions which satisfy the boundary conditions do exist in such bands. This difference is a direct consequence of the "open" character of the helix and the resulting need to satisfy the proper boundary condition for larger r.

The Narrow Tape Approximation

III.5 Boundary Conditions; Derivation of Approximate Determinantal Equation

Since an exact solution to the tape helix problem seems impossible, as noted in Section III.3, and since even the approximation procedure in which only a few terms of the exact infinite determinantal equation is used appears excessively burdensome, another approach is used to obtain useful numerical results. In this the tape is taken to be quite narrow so that the current distribution may be assumed with fair validity to be essentially quasi-static. By approximating the electric field boundary condition, one can then obtain a determinantal equation. In a later section the case where the tape is quite wide is considered.

If the tape is taken to be very narrow, that is, with δ small compared with a, p, and λ, it seems quite reasonable to assume that
essentially all of the current flows only along the tape. In other words, in this case $K_{||}$ is the major component of current density, whereas $K_{\perp}$ is small. If the point of view is taken that the fields are produced by the currents which flow, with the tape narrow and current flowing primarily in the direction of the tape, the specific distribution of current across the tape will affect to only a small degree the fields even in the near neighborhood of the wire and to a much less degree the fields on adjacent and far away turns. Thus, if some reasonable assumptions are made concerning this current distribution, it is to be expected that only small errors will be made in the field expressions. The assumption of small $K_{\perp}$ — actually, it is taken to be zero — is not very radical since one would expect that for narrow tapes the perpendicular or transverse currents on the "outside" and "inside" of the tape very nearly cancel each other in magnitude and phase so that the total transverse current density, $K_{\perp}$, is exceedingly small. Further, if a solution is obtained for the helix problem on the assumption of zero $K_{\perp}$ which leads to a nonzero value of $E_{\perp}$ on the tape, then for a narrow tape only a small transverse current density would be required to cancel this finite $E_{\perp}$; and this would perturb the zero $K_{\perp}$ solution only slightly.

The above may be expressed in a somewhat more rigorous fashion in the following manner. It is clear from the discussion in Section III.2 that the current density components $K_{||}$ and $K_{\perp}$ may be represented as

$$K_{||} = e^{-jhz} \sum_m K_{||m} e^{-jm(\frac{2\pi z}{P} - \theta)}$$  \hspace{1cm} (1)$$

$$K_{\perp} = e^{-jhz} \sum_m K_{\perp m} e^{-jm(\frac{2\pi z}{P} - \theta)}$$  \hspace{1cm} (2)$$

where $K_{||m}$ and $K_{\perp m}$ are the Fourier coefficients of the current density expansions. Assume for the moment that exact solutions for the current
density distribution on the tape are available in terms of a Fourier expansion with period $\delta$. Then these would be

$$K_{\|} = e^{\jmath hz} \sum_n K_{\|n} e^{-\jmath n \frac{2\pi}{\delta} (z - \frac{p}{2\pi} \theta)} ,$$

(3)

$$K_{\perp} = e^{\jmath hz} \sum_n K_{\perp n} e^{-\jmath n \frac{2\pi}{\delta} (z - \frac{p}{2\pi} \theta)} ,$$

(4)

for $\frac{p}{2\pi} \theta + \delta > z > \frac{p}{2\pi} \theta$, and zero elsewhere. The superscript $\delta$ is used to distinguish the Fourier coefficients in these expansions from those in (1) and (2). Equating (1) to (3) and (2) to (4), multiplying both sides by $e^{\jmath q \frac{2\pi}{p} (z-\theta)}$, and integrating over the interval $p$, one obtains finally

$$K_{\|m} = \frac{\delta}{p} \sum_n K_{\|n} \tilde{M}_{nm} ,$$

(5)

$$K_{\perp m} = \frac{\delta}{p} \sum_n K_{\perp n} \tilde{M}_{nm} ,$$

(6)

where

$$\tilde{M}_{nm} = \frac{e^{j2\pi(m\frac{p}{p} - n\frac{p}{\delta})}}{e^{j2\pi(m\frac{p}{p} - n\frac{p}{\delta})} - 1} ,$$

(7)

all for $m \neq 0$; and

$$K_{\|0} = \frac{\delta}{p} K_{\|0} ,$$

(8)

$$K_{\perp 0} = \frac{\delta}{p} K_{\perp 0} .$$

(9)

(Note (III.3-12) and $\tilde{M}_{nm} = 1$ for $\frac{n}{\delta} = \frac{m}{p}$.) If $\delta$ is small compared with $p$, then for $m \neq 0$ but small

$$\tilde{M}_{nm} = \frac{e^{j\frac{2\pi}{p} \frac{\delta}{\delta - n}}}{e^{j\frac{2\pi}{p} \frac{\delta}{\delta - n}} - 1} \approx \frac{\frac{2\pi}{p} \frac{\delta}{\delta - n}}{-j2\pi n} = -\frac{\frac{2\pi}{p}}{n} \frac{\delta}{\delta - n} ,$$

(10)

for $n \neq 0$; and

$$\tilde{M}_{0m} \approx 1 ,$$

(11)
where \( \approx \) means approximate equality. It can be seen that for small \( m \) the dominant terms in the series for \( K_{\|m} \) and \( K_{\perp m} \) are \( K_{\|0}^{5} \) and \( K_{\perp 0}^{5} \) since the higher order terms are smaller by \( \frac{5}{p} \), to say nothing of the \( \frac{1}{n} \) factor. Since series (3) and (4) are assumed to be convergent, little error should be made in using the dominant terms \( K_{\|0}^{5} \) and \( K_{\perp 0}^{5} \) to calculate \( K_{\|m} \) and \( K_{\perp m} \) at least for small \( m \). For large \( m \), except for \( \frac{m}{p} \) near \( \frac{n}{5} \) --- here the numerators of the \( \tilde{\tau}_{\|m} \) factors may be small ---, the use of \( K_{\|0}^{5} \) and \( K_{\perp 0}^{5} \) should still lead to a fair approximation for \( K_{\|m} \) and \( K_{\perp m} \). Further, the field representations should be influenced to only a minor degree by the values of \( K_{\|0}^{5} \) and \( K_{\perp 0}^{5} \) for large \( m \). \( K_{\|0}^{5} \) and \( K_{\perp 0}^{5} \) are related to the average components of the parallel and transverse current density distribution on the tape, with \( K_{\perp 0}^{5} \) proportional to the total current in the tape direction and \( K_{\perp 0}^{5} \) exceedingly small for narrow tapes as noted previously. Now of course, \( K_{\|0}^{5} \) and \( K_{\perp 0}^{5} \) are not known exactly, but it appears clear from the above that if some reasonable assumptions are made concerning them, rather good representations for the fields should be obtained.

If an inexact current distribution on the tape is assumed, the tangential electric field can no longer be made zero everywhere on the tape, and this boundary condition can be only approximately satisfied. This may be done in several ways. One could, for example, require the average value, or better the mean square value, of the tangential electric field on the tape to be a minimum, with the propagation constant \( h \) which gives this minimum taken as the solution. However, another procedure is used here which leads to a somewhat simpler determinantal equation for calculative purposes than the above possibilities and which appears to be a quite adequate approximation. In this it is required that \( E_{\|} \) be zero along the center of the tape; in other words, one of the boundary conditions is matched exactly along a line. As noted before, for a narrow
tape the dominant current density is $K_{||}$, and, loosely speaking, it is $E_{||}$ which forces this current to flow along the tape. Thus, if the most important boundary condition is completely satisfied on a line, one may hope to obtain a reasonably good approximation to the exact case where this condition must be satisfied over a surface. Ignoring the boundary condition for $E_{\perp}$ on the tape is not very serious for a narrow tape since, as already pointed out, if the approximate solution leads to a finite $E_{\perp}$ there, only a very small $K_{\perp}$, which alters the solution only slightly, will neutralize it. The satisfaction of the $E_{||} = 0$ condition merely along a line may seem like only a fair approximation, but if this condition is met, it can be expected that $E_{||}$ is also zero, or very nearly so, on a surface in the neighborhood of this line which is almost a narrow tape. Since the exact cross sectional shape of the conductor should affect the characteristics of the solution only slightly, particularly if the largest transverse dimension of this shape is small compared with $a$, $p$, and $\lambda$, it may be concluded that this approximation also exerts at most only a small influence on the final result. It should be mentioned that this approximation method, which proceeds by assuming a quasi-static distribution of the magnetic (or electric) field under conditions where such an assumption is quite valid and then by matching the electric (or magnetic) field condition along a line, has often been used in boundary value problems.\(^{36}\)

In order to apply the approximation procedure described above, the expression for $E_{||}$ is needed for $r = a$ in terms of $K_{||^m}$ and $K_{\perp^m}$. Since

$$K_{\theta,z} = e^{-jhz} \sum_m K_{\theta,z^m} e^{-jm(\frac{2\pi}{p}z-\phi)}$$  \hspace{1cm} (III.2-13,14)

$$K_{||,\perp} = e^{-jhz} \sum_m K_{||,\perp^m} e^{-jm(\frac{2\pi}{p}z-\phi)}$$  \hspace{1cm} (1,2)
and since

\[ K_{\parallel} = K_z \sin \psi + K_\theta \cos \psi, \quad (II.4-13) \quad K_\perp = K_z \cos \psi - K_\theta \sin \psi, \quad (II.4-11) \]

there results immediately from orthogonality

\[ K_{\parallel m} = K_{zm} \sin \psi + K_{\theta m} \cos \psi, \quad (12) \]

\[ K_{\perp m} = K_{zm} \cos \psi - K_{\theta m} \sin \psi, \quad (13) \]

and

\[ K_{zm} = K_{\parallel m} \sin \psi + K_{\perp m} \cos \psi, \quad (14) \]

\[ K_{\theta m} = K_{\parallel m} \cos \psi - K_{\perp m} \sin \psi. \quad (15) \]

Also

\[ E_{\parallel} = E_z \sin \psi + E_\theta \cos \psi, \quad (16) \]

\[ E_{\perp} = E_z \cos \psi - E_\theta \sin \psi. \quad (17) \]

Using (14) and (15) in (III.2-16) and (III.2-17) to obtain \( E_{\theta}^e \) and \( E_z^e \) in terms of \( K_{\parallel m} \) and \( K_{\perp m} \), and then using (16) to obtain \( E_{\parallel}^e \), one obtains finally

\[
E_{\parallel}^e = \sum_{m} \left\{ \left[ \frac{n_m^2 - m_h a (1 + \frac{r}{n}) \cot \psi + \frac{m_h a^2}{\eta_m^2} \frac{\cot^2 \psi}{\eta_m^2} \right] I_m(\eta_m) K_m(\eta_m \frac{r}{a}) \right. \\
+ \frac{k^2 a^2}{\omega} \cot^2 \psi \left. I_m(\eta_m) K_m'(\eta_m \frac{r}{a}) \right\} e^{-jm(\frac{2\pi}{p} - \theta)} \]

\[
= \sum_{m} \left\{ \left[ \frac{n_m^2 \cot \psi + m_h a (1 - \frac{a}{r} \cot^2 \psi) - \frac{a m_h a^2}{\eta_m^2} \frac{\cot^2 \psi}{\eta_m^2} \right] I_m(\eta_m) K_m(\eta_m \frac{r}{a}) \right. \\
- \frac{k^2 a^2}{\omega} \cot \psi I_m'(\eta_m) K_m(\eta_m \frac{r}{a}) \left. \right\} e^{-jm(\frac{2\pi}{p} - \theta)}. \quad (18) \]

For narrow tapes a reasonable assumption is that the magnitude of the current density \( K_{\parallel} \) is constant across the tape. For this
III.5

\[ K_{\|} = e^{-jhz} \sum_{m} K_{\| m} e^{-j m(2\pi z-\delta)} A e^{-jhz} e^{j(h-\beta_{\|})(z-\frac{p}{2\pi} \theta)} \left( \frac{p}{2\pi} \theta + \delta \right) z > \frac{p}{2\pi} \theta , \]

\begin{align*}
&= 0, \text{ elsewhere,} \\
&= (19)
\end{align*}

with \( A \) an undetermined constant. The factor \( e^{j(h-\beta_{\|})(z-\frac{p}{2\pi} \theta)} \) is included for some generality to account for a possible linear phase shift of the current density across the tape, with \( \beta_{\|} \) real, positive, and independent of \( \theta \) and \( z \). The point \( z = 0, \theta = 0 \) is chosen somewhat arbitrarily as point (1) in Fig. III-1b. Note that the form of the assumed current distribution has been chosen so that the phase variation of the current density in \( z \) for constant \( \theta \) is dependent on \( \beta_{\|} \) alone. The current is taken to flow in the positive \( \theta \) and \( z \) directions along the tape. The constant \( A \) may be related to the amplitude of the total current \( |I| \) which flows in the direction of the tape by

\[ |I| = | \int_{\text{tape}} K_{\|} \, dl | \]  \hspace{1cm} (20)

where \( l \) is measured perpendicular to the tape edges. Using the right side of (19) in (20) and assuming the constant phase front of the current density is perpendicular to the tape edges -- for this \( \beta_{\|} = h \sin^2 \psi \) (see Section III.12) --, one obtains after some transformations and change of variable

\[ A = \frac{|I|}{\delta \cos \psi} \]  \hspace{1cm} (21)

The assumption of some other phase variation for the current density in place of the one used above makes very little difference if \( \delta \) is small. Using (21) in (19), multiplying both sides of (19) by \( e^{-j m(2\pi z-\delta)} \), and integrating on \( z \) from 0 to \( p \), one obtains
\[ 1 K_{m}^{\parallel} = \left| \frac{1}{P \cos \Psi} \right| e^{j(h - \beta_{m} + \frac{2\pi}{P})\frac{\delta}{2}} \sin \left[ (h - \beta_{m} + \frac{2\pi}{P})\frac{\delta}{2} \right] \] 
\[ = \left| \frac{1}{P \cos \Psi} \right| e^{j(h - \beta_{m} + \frac{2\pi}{P})\frac{\delta}{2}} 1 D_{m}^{D} . \] (22)

The subscript 1 is used to distinguish the current density Fourier coefficients of this approximation from those of another type which are derived shortly, and \( 1 D_{m} \) is defined in an obvious manner. Since \( K_{\perp} \) is taken to be zero,

\[ 1 K_{1m}^{\perp} = 0 . \] (23)

A more reasonable assumption than a constant for the variation of the current density is one for which the distribution approximates that on an isolated narrow thin tape. With this approximation \( K_{\parallel} \) is taken as becoming infinitely large in an inverse square root manner as the tape edges are approached, with \( K_{\perp} \) again taken to be zero. (See Section III.12.) For this assumption

\[ K_{\parallel} = e^{-jhz} \sum_{m} K_{m}^{\parallel} e^{-jm(\frac{2\pi}{P}z - \theta)} \approx A e^{-jhz} e^{j(h - \beta_{m})(z - \frac{\theta}{2\pi})} \frac{\delta}{\sqrt{(z - \frac{\theta}{2\pi})(\delta - z + \frac{\theta}{2\pi})}} . \]  
\[ \text{for} \quad \frac{\theta}{2\pi} + \delta > z > \frac{\theta}{2\pi} , \] (24)

0, elsewhere.

The comments following (19) apply here also, except that in this case \( A \) has a slightly different value. Using the right side of (24) in (20) and the same assumptions employed to derive (21), one obtains for this

\[ A = \frac{|1|}{\pi \delta \cos \Psi} . \] (25)

Using (25) in (24) and proceeding in the usual fashion, one obtains

\[ 2 K_{m}^{\parallel} = \left| \frac{1}{P \cos \Psi} \right| e^{j(h - \beta_{m} + \frac{2m\pi}{P})\frac{\delta}{2}} J_{0} [(h - \beta_{m} + \frac{2m\pi}{P})\frac{\delta}{2}] = \left| \frac{1}{P \cos \Psi} \right| e^{j(h - \beta_{m} + \frac{2\pi}{P})\frac{\delta}{2}} 2 D_{m}^{D} . \] (26)
where $J_0$ is the ordinary Bessel function defined in Section A.4, and $\delta^2 D_m$ is defined in an obvious manner. In obtaining (26) it is necessary, after making a change of variable, to use

$$
\int_0^1 \frac{\cos bx}{\sqrt{1 - x^2}} \, dx = \frac{\pi}{2} J_0(b).
$$

(27)

(See reference 3, page 48.) In (26) the subscript 2 is used to distinguish the results of this approximation from those of the previous constant density assumption. Note that $1K_{\delta m}$ and $2K_{\delta m}$ are similar in form and not too unlike in actual value since $\frac{\sin z}{z}$ and $J_0(z)$ are much the same, at least for small $z$. This confirms the remarks made previously, and more will be said of this later. Since $K_1$ is taken to be zero, for this approximation also

$$
2K_{1m} = 0.
$$

(28)

Insertion of (22) or (26) and (23) or (28) in (18) yields for $E_\parallel^e$ for either approximate current distribution

$$
E_\parallel^e \approx j e^{-jhz} \frac{\sin \psi \tan \psi}{\omega \epsilon_0} |I| e^{-j(\frac{\epsilon}{2\pi}) \frac{\delta}{2}} \sum_m \left[ \frac{2}{\eta_m^2 a^2} \cot^2 \psi \right] I_m(\eta_m) K_m(\eta_m \frac{r}{a}) + \frac{a}{r} \frac{m^2 \epsilon_0 \delta}{\eta_m^2} \frac{\cot^2 \psi}{\eta_m^2} I_m'(\eta_m) K_m'(\eta_m \frac{r}{a}) \right] 1, 2D_m e^{-j m \frac{\pi}{2}} e^{-jm(\frac{2\pi}{P} z - \theta)}
$$

(29)

If $E_\parallel^e$ is required to be zero along the center of the tape, the substitutions $r = a$ and $z = \frac{P}{2\pi} \theta + \frac{\delta}{2}$ in (29) yield

$$
E_\parallel^e(r = a, z = \frac{P}{2\pi} \theta + \frac{\delta}{2}) = 0 \propto j e^{-j(\frac{h_0}{2\pi}) \frac{\delta}{2}} \frac{\sin \psi \tan \psi}{\omega \epsilon_0} |I| \times \sum_m \left\{ \left( \eta_m - \frac{m^2 a \cot \psi}{\eta_m} \right) I_m(\eta_m) K_m(\eta_m) + k^2 a^2 \cot \psi I_m'(\eta_m) K_m'(\eta_m) \right\} 1, 2D_m.
$$

(30)
Since the right side of (30) consists of a complex factor whose magnitude is independent of \( h \), times a real series, it is clear that if (30) is to be satisfied, the series must be zero. Since

\[
(\eta_m - \frac{m a \cot \psi}{\eta_m})^2 = \eta_m^2 - 2m a m \cot \psi + \frac{m^2 a^2 \cot^2 \psi}{\eta_m^2} = h^2 a^2 - k^2 a^2 + k^2 a^2 \frac{m^2 \cot^2 \psi}{\eta_m^2},
\]

(31)

there results finally for the approximate determinantal equation for the narrow tape helix for either assumed current distribution

\[
0 \approx \sum_m \left\{ (h^2 a^2 - k^2 a^2 + k^2 a^2 \frac{m^2 \cot^2 \psi}{\eta_m^2}) I_m(\eta_m) K_m(\eta_m) + k^2 a^2 \cot^2 \psi I'_m(\eta_m) K'_m(\eta_m) \right\} I_m D_m.
\]

(32)

### III.6 General Solution of the Approximate Determinantal Equation

In order to obtain useful numerical results some procedure is required for determining those values of \( h \), real and in magnitude greater than \( k \), which satisfy the determinantal equation (III.5-32) for specific values of \( k a \), \( \cot \psi \), and \( \delta \). The procedure used here is a combination of analytical, numerical, and graphical methods which are described in the following and in Appendix C.

For the purpose of simplifying the determinantal equation somewhat, it is assumed that \( \beta_\parallel = h \). This is equivalent to assuming that the constant phase front of the current distribution on the tape occurs for constant \( z \). Although this would be expected to be a sufficiently satisfactory assumption for large \( \psi \), it might be thought to be quite poor for small \( \psi \). However, it will be shown later that the more realistic assumption of a current distribution constant phase front which is perpendicular to the tape edges makes very little difference even for small \( \psi \) if the tape is narrow. With \( \beta_\parallel = h \), \( D_m \) becomes
\[ I_m^D(\beta_m = h) = \frac{\sin \frac{mx}{m^2}}{mx} \],

where the parameter \( x \) is defined as

\[ x = \frac{n^2}{p} \]

and should not be confused with the coordinate axis. Only the constant magnitude current density assumption is considered for the moment. It is shown in Section III.12 that the use of \( I_m^D \) alters the final result only in a minor way.

Since the determinantal equation is in the form of an infinite series, a means of summing this series must be found. Further, examination shows that for large \( m \) the terms vary like \( \frac{\sin \frac{mx}{m^2}}{m^2x} \), which means that the series converges very slowly and thus is not very suitable for numerical work in its present form. To improve this a well-known procedure is used in which a series, whose terms for large \( m \) are essentially equal to those of the series whose sum is desired, is added to and then subtracted term by term from the latter. If the sum of the terms in the series which is added and subtracted is known, the desired series is transformed to the sum of a known function plus a remainder series which is more rapidly convergent than the original one. The increase in the rate of convergence depends on how rapidly the terms in the subtracted series approach those of the original.

Using the representations noted in (A.2-22) and (A.2-23), one has to a very good approximation for \(|m| > 1\)

\[ I_m^I(\eta_m)K_m^I(\eta_m) \approx \frac{1}{2} \frac{1}{(m^2 + \eta_m^2)^{1/2}} \],

\[ I_m^I(\eta_m)K_m^I(\eta_m) \approx -\frac{1}{2} \frac{(m^2 + \eta_m^2)^{1/2}}{\eta_m^2} \].

\[ (3) \]

\[ (4) \]
Using (1), (3), and (4) in (III.5-32), in the manner described above, separating out the \( m = 0 \) term since (3) and (4) are not sufficiently good approximations for this, and converting the sum over negative \( m \) to a sum over positive \( m \) results in

\[
0 \approx \xi^2 a^2 I_0(\xi a) K_0(\xi a) + k^2 a^2 \cot^2 \psi I'_0(\xi a) K'_0(\xi a)
\]

\[
+ \frac{1}{2} \sum_{m=1}^{\infty} \left\{ \left( h^2 a^2 - k^2 a^2 + k^2 a^2 \frac{m^2 \cot^2 \psi}{\eta^2_m} \right) \frac{1}{(m^2 + \eta^2_m)^{1/2}} - k^2 a^2 \cot^2 \psi \frac{1}{\eta^2_m} \right\} \sin \frac{mx}{m}
\]

\[
+ \frac{1}{2} \sum_{m=1}^{\infty} \left\{ \left( h^2 a^2 - k^2 a^2 + k^2 a^2 \frac{m^2 \cot^2 \psi}{\eta^2_m} \right) \frac{1}{(m^2 + \eta^2_m)^{1/2}} - k^2 a^2 \cot^2 \psi \frac{1}{\eta^2_m} \right\} \sin \frac{mx}{m}
\]

\[
+ \sum_{m=1}^{\infty} R(\eta^2_m) + \sum_{m=1}^{\infty} R(\eta^2_m)
\]

(5)

In (5),

\[
\eta = \xi a = \sqrt{h^2 a^2 - k^2 a^2}
\]

(6)

(see II.3-17),

\[
\eta^2_m = \left[ m^2 \cot^2 \psi + 2mh a \cot \psi + h^2 a^2 - k^2 a^2 \right]^{1/2},
\]

(7)

\[
\eta^2_m = \left[ m^2 \cot^2 \psi - 2mh a \cot \psi + h^2 a^2 - k^2 a^2 \right]^{1/2},
\]

(8)

with \( m \geq 1 \), and

\[
R(\eta^2_m) = \left\{ \left( h^2 a^2 - k^2 a^2 + k^2 a^2 \frac{m^2 \cot^2 \psi}{\eta^2_m} \right) \left[ I'_m(\eta^2_m) K_m(\eta^2_m) - \frac{1}{2} \frac{1}{(m^2 + \eta^2_m)^{1/2}} \right] \right. 
\]

\[
+ \left. k^2 a^2 \cot^2 \psi \left[ I'_m(\eta^2_m) K_m(\eta^2_m) + \frac{1}{2} \frac{1}{(m^2 + \eta^2_m)^{1/2}} \right] \right\} \sin \frac{mx}{m}
\]

(9)

Because of the symmetrical position of \( h \) and \( m \) in (5), it can be observed that if (5) has any solutions for \( h \) positive, it will also have identical solutions for \( h \) negative and of the same magnitude. This confirms a remark
made in Section III.4, at least to this approximation. Since only $h > k$
need be considered in view of the above, from (7)

$$\eta_m > \left[ m^2 \cot^2 \psi + 2mka \cot\psi \right]^{\frac{1}{2}}. \tag{10}$$

Because of the limitation $\frac{\cot \psi}{2} > ka$ and since $m > 1$ in (10), some consid-
eration shows that for any practical value of $\psi$, $\eta_m$ is sufficiently
large so that (3) and (4) are exceedingly good approximations for all $ka,$
$ha,$ and $m$. This in turn means that all the $R(\eta_m)$ terms are very small
compared with their corresponding terms in the first sum in (5), and,
therefore, the sum $\sum_{m=1}^{\infty} R(\eta_m)$ can be omitted with negligible error. How-
ever, since $\eta_m$ can become very small — it is zero at the edges of the
$m$th forbidden region — the sum $\sum_{m=1}^{\infty} R(\eta_m)$ must not be omitted. Fortu-
nately, it turns out that, in general, only one term of this series is
significant, and even this occurs only near the boundary of a forbidden
region. Dropping $\sum_{m=1}^{\infty} R(\eta_m)$ and simplifying the expressions in the
braces $\{\}$ in the first two sums of (5) by combining terms in $k^2a^2,$ one
obtains

$$0 \approx \xi^2 a^2 I_o(\xi a)K_o(\xi a) + k^2 a^2 \frac{\cot^2 \psi}{2} I'_o(\xi a)K'_o(\xi a)$$

$$+ \left( \frac{h^2 a^2 - k^2 a^2 \csc^2 \psi}{2} \right) \sum_{m=1}^{\infty} \left[ \frac{1}{(m^2 + \eta_m^2)^{\frac{1}{2}}} + \frac{1}{(m^2 + \eta_m^2)^{\frac{1}{2}}} \right] \frac{\sin mx}{mx} + \sum_{m=1}^{\infty} R(\eta_m). \tag{11}$$

Both $\frac{1}{(m^2 + \eta_m^2)^{\frac{1}{2}}}$ and $\frac{1}{(m^2 + \eta_m^2)^{\frac{1}{2}}}$ go like $\frac{1}{m \csc \psi}$ for large $m$ so that after
adding and subtracting $\frac{1}{m \csc \psi}$ from each term in the brackets $[\,]$ in
the second sum in (11), there results

$$0 \approx \xi^2 a^2 I_o(\xi a)K_o(\xi a) + k^2 a^2 \frac{\cot^2 \psi}{2} I'_o(\xi a)K'_o(\xi a)$$

$$+ \left( \frac{h^2 a^2 - k^2 a^2 \csc^2 \psi}{2} \right) \frac{1}{\csc \psi} \sum_{m=1}^{\infty} \frac{\sin mx}{mx}$$

$$+ \left( \frac{h^2 a^2 - k^2 a^2 \csc^2 \psi}{2} \right) \sum_{m=1}^{\infty} \left[ \frac{1}{(m^2 + \eta_m^2)^{\frac{1}{2}}} + \frac{1}{(m^2 + \eta_m^2)^{\frac{1}{2}}} \right] \frac{2}{\csc \psi} \frac{\sin mx}{mx} + \sum_{m=1}^{\infty} R(\eta_m). \tag{12}$$
From (1.6-5) the dominant term in \( \frac{1}{x} \sum_{m=1}^{\infty} \frac{\sin mx}{m^2} \) is \( \ln \frac{e}{x} \). For \( x \leq 0.1 \) this term alone represents the series to better than 0.01%, and for \( x \) as large as 0.5 the error is only 0.2%. Thus, for narrow tapes it is sufficient to use just the dominant term, and (12) becomes

\[
0 \approx \xi^2 a^2 I_0(\xi a)K_0(\xi a) + k^2 a^2 \cot^2 \psi I_0'(\xi a)K_0'(\xi a) + \frac{(h^2 a^2 - k^2 a^2 \csc^2 \psi)}{\csc \psi} \ln \frac{e}{x} \\
+ \frac{(h^2 a^2 - k^2 a^2 \csc^2 \psi)}{2} \sum_{m=1}^{\infty} \left[ \frac{1}{(m^2 + \eta_m^2)^2} + \frac{1}{m \csc \psi} \frac{\sin mx}{m^x} \right] + \sum_{m=1}^{\infty} R(n_m)
\]

(13)

Since the terms in the first series in (13) vary as \( \frac{\sin mx}{m^x} \) for large \( m \), (13) might be used as it stands for numerical computation. However, in view of the wide range of \( ha \) and \( ka \) which must be investigated and because of the rather inconvenient form of the terms, a suitable approximation for this series is highly desirable. This may be derived by approximating the \( \frac{\sin mx}{m^x} \) term by a simple algebraic factor, which agrees exceedingly well with \( \frac{\sin mx}{m^x} \) even for relatively large \( m \) with \( x \) small, and by changing the summation to an integration. For small \( x \) this process leads to a quite good approximation which is derived in Appendix C. The final expression is rather complicated and is therefore not repeated here. The entire calculative procedure for finding the values of \( h \) which satisfy (13) is completely described in Appendix C with some examples and is consequently not repeated here either.

It will be recalled that (13) is the modified expression for \( E_0 \) at the center of the tape. If the point of view is taken that this field is the result of the current which flows, a physical interpretation can be given for the various terms in (13). The first two terms are from the \( m = 0 \) term of the series and give the average field from the current in all the turns. Note that the first two terms alone are the determinantal equation for the zeroth mode of the sheath helix. The third term in (13) is logarithmic in the tape width and comes from the large order terms in
(III.5-32). It is essentially the field resulting from the current in the near neighborhood of the point at the center of the tape. Although not so obvious, the fourth term in (13) may be interpreted as the portion of the field at the center of the tape contributed by the adjacent turns, whereas the final term is interpreted as the rest of the field resulting from the turns very far away. This interpretation for the final term is confirmed by noting that for \( m = 1 \), at least, the field determined from \( R(\eta_1) \) near \( \eta_1 = 0 \) has the same character, namely logarithmic, as when the field on a turn resulting from the current in turns very far away is simply calculated by using the usual dipole far field radiation expressions. Except for this result, the simple calculation has little interest and is not shown here.

Since the first two terms in (13) give precisely the determinantal equation for the zeroth mode of the sheath helix, it is evident from the discussion in Section II.5(d) that the sum of these is essentially zero over a wide range of frequencies for \(|h|a = ka \csc \psi \) or \( \frac{ka}{|h|a} = \sin \psi \).

For this value of \(|h|a\) the other terms in (13) are also zero with the exception of the final remainder term, which is very small except near \( \eta_m = 0 \). Consequently, to this approximation, the single wound narrow tape helix has solutions which are very near those for the zeroth mode of the sheath helix except for the effect of the forbidden regions. This will be seen more clearly in the next section where a particular case is discussed. An interesting aspect of (13) is that it shows quite clearly why the infinitesimal thin wire diameter solution of reference 17 and the usual zeroth mode sheath helix solution of references 7, 9, 10, and 11, which is discussed in the previous chapter, agree so well in the range of frequencies where the asymptotic solution \(|h|a = ka \csc \psi \) is valid.
III.7 Numerical Results for $\psi = 10^0$ and $x = 0.1$; Comparison with Published Experimental Results

Calculations for the case of $\psi = 10^0$ and $x = 0.1$ have been carried through in detail, and the propagation constants which result are shown in Figs. III-4 and III-5. $\psi = 10^0$ was chosen as a representative value for which the solutions would exhibit their general properties. $x = 0.1$ was chosen since it was felt that this value is sufficiently small so that the approximations should be quite valid. Since the tape width appears to make only a small difference in the results, this choice of $x$ is not a significant restriction. As in the sheath helix case, several waves can exist at a particular frequency. In Figs. III-4 and III-5 the subscript on the $h$ values refers to the tape solutions, the numerical subscript 0, 1, 2 refers to the dominant character of the fields for $r \gg a$ associated with a particular solution, and the prime superscript is used to distinguish between solutions with the same numerical subscript.

Since a solution is now made up of an entire set of space harmonics, it becomes difficult to speak of modes in the same sense as was used in the sheath helix case in the previous chapter. However, if one considers the field structure for $r \gg a$, it is usually found that one of the space harmonics of a wave is much larger than all the others, and the number of this space harmonic might be considered as a mode number. This has been done, in so far as possible, by labeling the various $h_n$ solutions, as will be evident soon.

In Fig. III-4 the solid and dotted lines refer to waves which would be observed if one were located at $z \gg 0$ or $z \ll 0$, respectively, with a source at $z = 0$. As in the sheath helix case the reasons for this are understood best by considering the source-present or driven helix, and this is done in the following chapter. Note that Fig. III-4b is essentially symmetrical about the $\frac{ka}{\cot \psi}$ axis, so that for every wave in the
FIG. III-4a
\[ \frac{ka}{\cot \psi} \quad \text{vs.} \quad \frac{h_t}{\cot \psi} \]
for
\[ \psi = 10^\circ, \quad x = 0.1 \]
The Narrow Tape Helix

FIG. III-4b
(Similar to FIG. III-4b)
positive z direction there is a similar wave in the negative z direction. However, it is physically clear, as is also evident from the field expressions, that these waves "rotate" in opposite \( \theta \) directions. Fig. III-4b and the solutions shown there are to be interpreted in the following manner.

If an observation point is located at \( z\gg 0 \) so that the radiation field from a source at \( z = 0 \) is negligible, then waves with propagation constants \(-|h''_{t1}|\), \(|h_{t0}|\), and \(|h'_{t1}|\) could be detected at low frequencies. As the frequency increases, a point is reached where the \(-|h''_{t1}|\) and \(|h_{t0}|\) waves have propagation constants of the same magnitude, but their group velocities are zero. The usual group velocity concept and its relationship to the axial power flow are completely valid here, as will be proved in Section III.9.

\[
\frac{V_g}{V_0} = \frac{d(k\alpha)}{d(\cot \psi)}
\]

is, of course, given by \( \frac{d(k\alpha)}{d(\cot \psi)} \). No power is propagated by the \(-|h''_{t1}|\) and \(|h_{t0}|\) waves at this point, \( k\alpha = 0.77 \), and this evidently corresponds to a standing wave. However, the \(|h'_{t1}|\) wave is also present, and as the frequency is increased further, it is the only wave which exists until \( k\alpha = 1.61 \), after which the \(-|h''_{t2}|\) wave appears. The \(|h'_{t1}|\) and \(-|h''_{t2}|\) waves exist together until \( k\alpha = 1.64 \), at which point standing waves occur again. From \( 2.26 > k\alpha > 1.64 \) no "free mode" propagation occurs; for \( 2.49 > k\alpha > 2.26 \) only the \(|h'_{t2}|\) wave exists; while for \( k\alpha > 2.49 \) there are no "free mode" waves. If one is located at \( z\ll 0 \), the action is similar to the above except that the dotted lines in Fig. III-4b are traced.

By using (III.2-17), (III.5-11), (III.5-12), (III.5-22), and (III.6-1) with \( k_{m} = 0 \), the following expression for \( E_{z}^{e} \) to this approximation can be readily obtained:

\[
E_{z}^{e} \approx j \left| \frac{e^{-jhz}}{\pi \alpha a} \right| \sum_{m} \left[ \text{nh}a + \frac{2}{\alpha} \tan \psi \right] I_{m}(\eta_{m}) K_{m}(\eta_{m} \alpha) \frac{\sin m \pi x}{m \pi x} e^{jmx} e^{-jm(\frac{2}{p}z - \theta)}
\]

or

\[
E_{z}^{e} \approx \sum_{m} a_{m} e^{-jh_{m}z}
\]
FIG. III-5

\[ \frac{ka}{\theta^*} \text{ vs. } ka \]

for

\[ \psi = 10^\circ, x = 0.1 \text{ for the Narrow Tape Helix} \]

\[ \psi = 10^\circ, x' = 0.1 \text{ for the Narrow Gap Helix} \]

(See Section III.10)

and for

\[ \psi = 10^\circ \text{ for the Sheath Helix} \]

from Reference 10

Sheath Helix, n=0

Narrow Gap Helix

Narrow Tape Helix

|h_{t1}| |h_{t2}| |h_{t3}|
which defines \( A_m \). The ratio of the phase velocity, \( \frac{m}{V_o} \), of the various space harmonics of a wave to the velocity, \( V_o \), of a uniform plane wave in the medium is given by

\[
\frac{m}{V_o} = \frac{ka}{ha} = \frac{ka}{m \cot \psi + ha} = \frac{ka}{m + \frac{ha}{\cot \psi}}.
\] (3)

Note that \( \frac{m}{V_o} \) can be positive or negative. Fig. III-6 shows the values of \( \frac{m}{V_o} \) for \( \beta \geq \beta_0 = -2 \) for the various waves. Fig. III-7 shows the \( E_z \) space harmonic amplitude ratios in decibels at \( r = 2a \); that is, it is a plot of \( 20 \log_{10} \left( \frac{A_m}{A_n} \right) \). The \( \beta_1^2 \) wave is not shown in this latter plot since it is of minor interest, and since for it the \( m = +2 \) harmonic is by far the dominant one. All the space harmonics which are of sufficient amplitude to appear in Fig. III-7 are plotted there. It is evident that even for \( r = 2a \) only a few space harmonics are significant.

In Section III.9 it is shown, from a consideration of the power flow associated with each wave, that the amplitudes of the \( \beta_0^1 \) and \( \beta_1^1 \) waves can be expected to be much less than the amplitude of the \( \beta_0^1 \) wave over most of the range \( ka < 0.75 \). Although the actual amplitudes depend on the configuration of the source and can be determined only by solving the driven helix problem, within relatively broad limits the \( \beta_0^1 \) wave is the dominant one at low frequencies. Assume that one were to measure \( E_z \) at \( r = 2a \) with a small detecting dipole. If Figs. III-6 and III-7 are considered together, it is evident that for \( ka < 0.5 \), at least, a space harmonic wave with no angular variation and with a phase velocity ratio like \( \frac{V}{V_o} \) would be observed. For \( 0.77 > ka > 0.5 \) a space harmonic wave whose phase velocity ratio is like \( -\frac{V}{V_o} \) and whose angular variation is like \( e^{-j\theta} \) would be observed, although this would probably be a region of some confusion. For \( 1.5 > ka > 0.77 \) a space harmonic wave whose phase velocity ratio is like \( -\frac{V}{V_o} \) and whose angular variation is
Sheath Helix Solution for $n=0$, $nh > 0$ (See Figs. II-6a and III-5)

\[ \frac{-\psi}{v} \quad \text{vs. ka} \]
\[ \psi = 10^\circ, \quad x = 0.1 \quad z > 0 \]

The Narrow Tape Helix

Sheath Helix Solution for $n=1$, $nh > 0$ (See Fig. II-10a)

\[ \frac{-\psi}{v} \quad \text{vs. ka} \]
\[ \psi = 10^\circ, \quad x = 0.1 \quad z > 0 \]

The Narrow Tape Helix
like \( e^{-j\theta} \) would be observed, although near the upper limit of this region there would be some mixing of space harmonics of almost equal amplitude. Over most of the range \( 1.6l > ka > 1.5 \) the dominant space harmonic has an angular variation like \( e^{-j2\theta} \), although it can be expected that this region would again be one of some confusion because of the mixture of space harmonics of nearly equal amplitude. For \( 2.26 > ka > 1.6l \) no "free mode" wave propagation would be observed, while for most of the range \( 2.49 > ka > 2.26 \) a space harmonic wave whose phase velocity ratio is like \( \frac{2V}{V_0} \left| h'_t t_2 \right| \) and whose angular variation is like \( e^{-j2\theta} \) would be observed. However, the \( m = -2 \) and \( m = -3 \) space harmonics are not very different in amplitude over all of this region, and it would probably be one of some confusion.

In Fig. III-6 the phase velocity ratios of some sheath helix modes which were obtained from solutions of (II.4-3) for \( \psi = 10^\circ \) are shown. Over restricted frequency ranges most of the dominant space harmonic waves associated with the tape helix have phase velocities which correspond quite closely to the phase velocities of particular sheath helix modes. Thus, if \( E_{z}^e \) around a helix were observed in the above fashion, one might conclude that sheath helix modes exist but only over restricted frequency ranges. This was precisely the sort of action noted in reference 24. One of the reasons for undertaking the analysis reported here was to attempt to explain this phenomenon. Although an exact comparison is not possible because of a lack of data, it is thought that the above offers an explanation of why such action occurs. Although only the \( E_{z}^e \) field is discussed above, the other field components have space harmonic amplitudes at \( r = 2a \) which act in a similar, although not identical, fashion. For \( r >> a \) the ratios \( \left| \frac{A_m}{A_n} \right| \) are much larger than at \( r = 2a \), and the transitions between the dominant space harmonics are more abrupt. These transitions occur in \( E_{z}^e \) at \( r = 2a \) and in all the field components for \( r >> a \) near \( ka = 0.5, 1.5, \) and \( 2.45 \), as shown in Fig. III-7. For \( r << a \)
FIG. III-7a
Space Harmonic Amplitudes
of $E_z^e$ for the $|h_{to}|$ Wave
at $r = 2a$ vs. $ka$
$\Psi = 10^\circ, x = 0.1$

The Narrow Tape Helix $z > 0$

FIG. III-7b
Space Harmonic Amplitudes
of $E_z^e$ for the $|h_{t1}^l|$ and $|h_{t2}^l|$ Waves at $r = 2a$ vs. $ka$
$\Psi = 10^\circ, x = 0.1$

The Narrow Tape Helix $z > 0$

FIG. III-7c
Space Harmonic Amplitudes
of $E_z^e$ for the $-|h_{t1}^l|$ Wave
at $r = 2a$ vs. $ka$
$\Psi = 10^\circ, x = 0.1$

The Narrow Tape Helix $z > 0$
the higher order space harmonics also damp out rapidly although for \( r \ll a \) only the zeroth space harmonic may be significant. Indeed, for \( r = 0 \) only that harmonic is nonzero for all the waves.

The numbering of the space harmonics in Figs. III-6 and III-7 is for \( z > 0 \) using the solid line solutions of Fig. III-4b. If \( z < 0 \) is considered, the dotted line solutions of Fig. III-4b are used, and the numbering of the space harmonics is reversed since the field rotates in the opposite \( \theta \) direction. For \( ka < 0.5 \) the dominant space harmonics in the \( |h'_{t1}| \) and \(-|h''_{t1}| \) waves are those for which \( m = -1 \) and \( m = +1 \), respectively, and these have phase velocity ratios which are very close to unity. It would appear that these waves are essentially perturbed uniform plane waves which are circularly or elliptically polarized, and rotating and traveling in opposite directions; however, sufficiently accurate values of \( \eta_{t1} \) were not available for confirming that the field components approach the proper form. If this supposition is correct, it explains why at low frequencies the \( |h'_{t1}| \) and \( |h''_{t1}| \) waves would be excited to very small amplitudes only by any finite source.

From Figs. III-5 and III-6c it can be seen that for \( ka < 0.4 \) or
\[
\frac{ka}{\cot \psi} < 0.07
\]
the dominant \( m = 0 \) harmonic has a phase velocity ratio which is somewhat less than that given by the sheath helix zeroth mode solution. References 22 and 23 have reported measurements which agree quite closely with the zeroth mode sheath helix solution for
\[
0.05 > \frac{ka}{\cot \psi} > 0.003
\]
if the mean radius of the physical helix is used as the radius of the sheath helix. The theory presented here for the narrow tape helix thus appears to predict phase velocity ratios which are too low for \( \frac{ka}{\cot \psi} \) small. However, increasing the tape width, \( \delta \), changes the phase velocity ratio in a direction so as to make for better agreement. It seems quite likely that if a wider tape were assumed, with some deterioration in the validity of the approximations, closer agreement
FIG. III-8
\[ \frac{v_p}{v_0} \text{ vs } ka \]
\[ \psi = 10^\circ, x = 0.1 \]

The Narrow Tape Helix
\[ z > 0 \]

- Measured Points for a \( \psi = 13^\circ \) Round Wire Helix from reference 25
- Measured Points for a \( \psi = 13^\circ \) Round Wire Helix from reference 32
- Measured Points, ascribed to author of reference 25 in reference 32

Similar measured data in references 25 and 32 for \( 1.5 > ka > 0.8 \)

FIG. III-9
\[ \frac{v_p}{v_0} \text{ vs. } ka \]
\[ \psi = 10^\circ, x = 0.1 \]

The Narrow Tape Helix
\[ z > 0 \]
(See text concerning a, b, c)

- Measured Points for a \( \psi = 13^\circ \) Round Wire Helix from reference 25
- Measured Points for a \( \psi = 13^\circ \) Round Wire Helix from reference 32
- Measured Points, ascribed to author of reference 25 in reference 32

Similar measured data in references 25 and 32 for \( 1.5 > ka > 0.8 \)
between the tape solution and the experimentally measured phase velocity ratios would occur. This matter is considered further in Section III.10. Although reference 22 reported no wave other than the usual "slow" one for a wide range of frequencies (but for \( \frac{ka}{\cot \psi} < 0.04 \)), exciting conditions, and terminating conditions, reference 23 reported a "fast wave", one with a phase velocity ratio near unity, coexisting with the "slow" wave. It would appear that this "fast" wave was either the \( m = -1 \) or \( m = +1 \) space harmonic of the \( |h_{+1}^1| \) or \( -|h_{-1}^1| \) wave (see Fig. III-6b and 6d). It was pointed out in reference 23 that the "fast" wave disappeared (or became of sufficiently small amplitude so as not to be observed) when the exciting arrangement was altered.

Instead of considering the propagation constant and phase velocity of the waves referred to the \( z \) axis, it is possible to refer these to a line measured along or parallel to the helix itself. Since this line is given by \( \frac{2\pi}{P} z - \theta = \text{constant} \), it is evident that the propagation constant \( \beta \) along this line is given by

\[
\beta = h \sin \psi .
\]

A phase velocity ratio \( \frac{\nu_{\beta}}{\nu_o} \) can now be defined in the usual manner so that

\[
\frac{\nu_{\beta}}{\nu_o} = \frac{k}{\beta} = \frac{ka}{ha \sin \psi} .
\]

Fig. III-8 shows \( \frac{\nu_{\beta}}{\nu_o} \) for the different waves as a function of \( ka \). Also shown are some experimental points given in references 25 and 32. These were obtained by moving a search probe along the surface of a round wire helix. The experimental technique is described in detail in reference 32. Reference 33 shows experimental curves for other helices for \( 13.6^\circ < \psi < 6.5^\circ \). All of these exhibit essentially the same characteristics shown by the experimental points in Fig. III-8 for \( \psi = 13^\circ \). In particular, as the frequency is increased, the ratio \( \frac{\nu_{\beta}}{\nu_o} \) at first remains near unity over a
relatively wide range; then, a break occurs in which there is evidently the mixing of several waves; and, afterwards, the ratio \( \frac{v_\theta}{v_o} \) is observed to be less than unity and to increase towards unity.

Although the agreement between the values predicted from the theory and those measured experimentally is rather good, some possible limitations should be noted. The most obvious of these is the fact that the experimental data is for \( \psi = 13^\circ \), whereas the theoretical curves in Fig. III-8 are for \( \psi = 10^\circ \). Calculations have not been carried through in detail, but some careful estimates indicated that for \( \psi = 13^\circ \) the theoretical \( \frac{v_\theta}{v_o} \) curve from the \( h_{t1}^l \) wave agrees even more closely with the experimental points. The difference is small, being of the order of \( 1\% \) for \( 1.2 > ka > 0.8 \). The theoretical \( \frac{v_\theta}{v_o} \) curve from the \( h_{to}^l \) wave is altered even less for \( 0.7 > ka > 0.4 \). Although the theory applies to a narrow tape helix, whereas the experimental helix was a round wire one, this difference also seems to change the results an insignificant amount for \( ka > 0.5 \). A more serious objection to comparing the experimental and theoretical results is that the former were obtained from a helix of finite length, whereas the latter apply to a helix of infinite length. However, since the experimental data refer to the average phase velocity obtained from measurements over the third to sixth turn of a seven turn helix, and since reference 25 noted that in this region the end effects were rather small, it would appear that comparisons are still valid. Although radiation from a finite length helix alters the phase distribution from that which exists on an infinite length helix, one would expect that over the center portion of a helix of several turns radiation effects would be minor.

Reference 25 shows no \( h_{t1}^l \) wave for \( ka < 0.8 \), but it is of interest to note that reference 32 requires, in addition to an \( h_{to}^l \) wave, an outward traveling wave of quite small amplitude and with the phase
velocity ratios shown in Fig. III-8 to obtain agreement with experimentally measured phase velocities for \(0.8 > ka > 0.5\). No \(-h_{t1}''\) wave has been reported, and it is thought that this wave may normally be of sufficiently small amplitude for \(ka < 0.7\) so as to avoid detection. Further, for \(ka < 0.5\) the theoretical solutions for the \(h_{t1}'\) and \(h_{t2}''\) waves are quite dependent on the assumption of a helix of infinite length, and it is to be expected that these solutions would be altered for a finite length helix.

One other possible shortcoming of the theory should be noted. References 25 and 32 note some evidence for a wave of the \(h_{t2}''\) type, but traveling outward rather than inward and existing for \(1.6 > ka > 1.3\). The theory does not predict such a wave. No measurements have been reported for \(ka > 1.7\) so that no experimental confirmation of the existence of the \(h_{t2}''\) "free mode" wave is possible.

Although the theoretical results do not appear to agree completely with all the observed results, the rather close correspondence between theory and experiment is rather satisfying and suggests that the approximations used are reasonably valid. It should be mentioned that when these results -- essentially Fig. III-5 -- were derived, it was thought that they were quite new and original. However, at the time this report was being written, the writer's attention was called to reference 51 in which very similar results were obtained through the integral equation approach. This approach, which leads to results that are equivalent to those obtained by the characteristic function method used in this chapter, is discussed in the next chapter.

III.8 Other Roots; Generalization for Other Values of Ψ

It is interesting to inquire whether other "free modes" besides those discussed in the previous section exist, or, in other words, are there any other values of h which satisfy (III.6-11) or (III.6-13)? The following discussion shows the unlikelihood of any other roots for the
approximate determinantal equation, although it certainly cannot be thought of as a rigorous proof. Consider the right side of (III.6-11) or (III.6-13) plotted as a surface, say \( f(ha, ka) \), versus the coordinates \( ka \) and \( ha \) with \( x \) and \( \psi \) constant, and \( h \gg k \). The \( ka \) versus \( ha \) curve then becomes the locus of the intersection of this surface, \( f(ha, ka) \), with the plane \( f(ha, ka) = 0 \). The first two terms in (III.6-11) merely form the sheath helix zeroth mode solution which is known to have only one locus or line of intersection with the \( f(ha, ka) = 0 \) plane (see Sections II.5(d) and B.4, and Fig. II-9), and for all but small values of \( ka \) this occurs for \( ha \approx + ka \csc \psi \). The first series in (III.6-11) is an essentially positive quantity for \( x \) small, and it is clear that the entire term is zero only for \( ha = + ka \csc \psi \). In general, the major effect of this term is to make the locus approach the \( ha = + ka \csc \psi \) line more closely for small values of \( ka \). The last series in (III.6-11) is small everywhere except for \( ha = \mid m \mid \cot \psi \pm ka \), and, even here, it is significant only for \( \mid m \mid = 1 \). In this latter case \( R(\eta) \) becomes logarithmically large so that the surface \( f(ha, ka) \) always crosses zero near both boundaries of the \( \mid m \mid = 1 \) forbidden region, at least for small \( ka \). The higher order terms, \( R(\eta_m) \), may cause additional zero crossings of \( f(ha, ka) \) near \( ha = \mid m \mid \cot \psi \pm ka \), but this would appear to happen, if at all, only for quite small values of \( \mid m \mid \). The above remarks and those of Sections C.2, 3, and 4 are the reasons for believing that no other real roots, besides those shown, exist for the approximate determinantal equation for the narrow tape helix. The question of complex or pure imaginary roots has already been considered in Section III.1.

The preceding discussion concerning the existence of the roots suggests a simple method whereby the "free mode" solutions can be obtained without carrying out the lengthy calculative process described in Section C.1. This method consists merely of drawing the lines
\[ \frac{ka}{\cot \psi} = \sin \psi \, \frac{|h|}{\cot \psi} \] in Fig. III-2 in the allowed regions and then joining these lines smoothly with lines drawn along the boundaries of the \( |m| = 1 \) forbidden regions. Fig. III-4 illustrates the meaning of this procedure. The simple method will result in values of \( |h| \) for the \( |n_{to}| \) wave which are too large for \( ka \) small. These may be improved for \( ka \) not too small by averaging them with the values obtained in this region from the zeroth mode sheath solution alone. Fig. III-5 indicates why this procedure seems valid. Although the method described above misses the \( |h|^n_{t2} \) wave and undoubtedly predicts incorrect band limits for the "free mode" solutions between the higher order allowed regions for small \( \psi \), its simplicity is appealing and its accuracy is probably sufficient for many purposes for all practical values of \( \psi \). The effect of the tape width is not specifically considered in this simple construction although the averaging procedure mentioned above accounts for the influence of this parameter in an approximate manner. It is worthwhile noting that some experiments performed by Mr. L. Stark of these laboratories indicate that the simple procedure described above gives reasonably accurate predictions of the performance of the \( |h_{to}| \) wave. The writer wishes to thank Mr. Stark for this information.

III.9 Power Flow; Relative Axial Electric Field; Power Loss

It is useful first to prove that the group velocity and total average real axial power flow for "free mode" waves in the tape helix case have the same algebraic sign as they do in the sheath helix case. The proof follows the steps given by (II.6-13) through (II.6-21) since these formulae are quite general and apply to the tape helix case also. The boundary conditions for the tape helix at \( r = a \) require that \( E_t = 0 \) on the tape and that \( (H^1 - H^0)_t = 0 \) between the tape; and these conditions are invariant with frequency. Consequently, the line integral in
(II.6-21) is zero, and (II.6-22) and the usual definition of group velocity follow directly. The group velocities of the space harmonics of a given wave are identical since they are all associated with the same wave. This is evident from

\[
\frac{v_g}{v_o} = \frac{m v_g}{v_o} = \frac{d(ha + m \cot \psi)}{d(ka)} = \frac{d(ha)}{d(ka)} \quad .
\]

Fig. III-9 shows the group velocities of the various waves for the particular case of \( \psi = 10^\circ \) and \( x = 0.1 \) discussed in Section III.7. These were obtained from an enlarged plot of the type shown in Fig. III-4 using (1). Because of the approximations involved, the values of \( \frac{v_g}{v_o} \) at the points a, b, and c in Fig. III-9 must be considered doubtful; in particular, point a may be at \( \frac{v_g}{v_o} = 1 \).

The average real axial power flow associated with the "free modes" of the tape helix can be calculated in the usual fashion using the field expressions (III.2-15) through (III.2-20) and (II.6-3). The power flow in the different waves can be calculated separately. The reasons for this are given in Section II.6 for the sheath helix case and are still valid here. Using the field expressions in (II.6-3), one obtains after considerable manipulation, including a change of variable and an interchange of the summation and integration,

\[
P^i_z = n a^2 \sum \left\{ \frac{\mu_m^2}{\eta_m^2} \int_0^1 \left[ I_m(\eta m x) + \frac{m^2}{\eta_m^2} I_m^2(\eta m x) \right] d\eta m x \right. \\
+ \left. \frac{\mu_m^2}{\eta_m} \int_0^1 I_m(\eta m x) I_m^2(\eta m x) d\eta m x \right\} ,
\]

where

\[
M_m^i = \frac{\eta_m^2 h_m^a}{\omega \sigma a} \frac{k_m^2 |\kappa_{zm}|^2}{m^2 h_m^2 + \frac{m^2 k_m^2}{\eta_m} \frac{m^2 k_m^2}{k_m^2} + \frac{2 m h_m^2}{\omega \sigma a} \frac{k_m^2 \text{Re}(\kappa_{zm} \kappa_{zm})}{\kappa_{zm}^2} ,
\]

(3)
In (3), (4), and (6) below, the argument of the modified Bessel functions is \( \eta_m \), and the functional notation is omitted for convenience. For \( P_z^e \) identical expressions result, except that \( K_m \) and \( I_m \) are interchanged everywhere, and the integration on \( x \) is from 1 to \( \infty \). The integrations can be performed using the formulae of Section A.3.

The exact expressions for \( \kappa_{zm} \) and \( \kappa_{0m} \) are, of course, not available, and in order to proceed one assumes for the narrow tape that the only component of current density is \( K_\parallel \) with \( K_\perp = 0 \). From (III.5-14) and (III.5-15) there results

\[
|\kappa_{zm}|^2 \approx |\kappa_{\parallel m}|^2 \sin^2 \psi \quad ,
\]

\[
|\kappa_{0m}|^2 \approx |\kappa_{\parallel m}|^2 \cos^2 \psi \quad ,
\]

\[
\text{Re}(\kappa_{0m} \kappa_{zm}^*) \approx |\kappa_{\parallel m}|^2 \sin \psi \cos \psi \quad .
\]

Substituting (5) in (3) and (4) and then these in (2), performing a similar operation to obtain an expression for the external power flow, carrying out the integrations, adding \( P_z^i \) and \( P_z^e \) to obtain \( P_z \), and then combining terms and using the Wronskian relationship to simplify somewhat, one obtains finally

\[
P_z \approx \frac{na^2 \sin^2 \psi}{\omega \varepsilon a} \sum_m h_m a \left[ K_m \right. \left. + \frac{1}{\eta_m} \left( \frac{I_{m}^{i}}{K_{m}^{i}} + \frac{K_{m}^{i}}{I_{m}^{i}} \right) \right]
\]

\[+ \frac{k^2 a^2 \cot^2 \psi}{\eta_m} \left\{ - \frac{K_{m}^{i}}{I_{m}^{i}} + \frac{1}{2} \left( 1 + \frac{m^2}{\eta_m} \right) \left( \frac{I_{m}^{i}}{K_{m}^{i}} + \frac{K_{m}^{i}}{I_{m}^{i}} \right) \right\}
\]

\[+ \frac{m q_m (h_m^2 a^2 + k^2 a^2)}{\eta_m^4} \cot \psi \right] |\kappa_{\parallel m}|^2 \quad .
\]

where \( q_m \) is given by

\[
q_m = \eta_m^2 - m h_m a \cot \psi = k^2 a^2 + m h a \cot \psi \quad .
\]
Total Average Axial Power

Flow vs. ka

$\Psi = 10^0, x = 0.1$

The Narrow Tape Helix

$z > 0$

\[
\frac{P_s}{\sqrt{\frac{E}{\epsilon}}} |I|^2
\]

\[\begin{align*}
800 \\
600 \\
400 \\
200 \\
100 \\
80 \\
60 \\
40 \\
20 \\
10.0 \\
8.0 \\
6.0 \\
4.0 \\
2.0 \\
1.0 \\
0.8 \\
0.6 \\
0.4 \\
0.2
\end{align*}\]
Some examination shows that exclusive of the \(|K_{m,n}|^2\) factor the higher order terms in (6) decrease only like \(\frac{1}{|m|}\). Of course, with the \(|K_{m,n}|^2\) factor the series converges but too slowly to be useful for computation. In order to improve the convergence, (6) is transformed in the same manner as the approximate determinantal equation. The details are shown in Section C.5 along with other data useful for calculating the value of (6).

For the particular case of \(\psi = 10^0\) and \(x = 0.1\), with the constant current density constant \(z\) phase front approximation, the calculations described in Section C.5 gave the results shown in Fig. III-10. \(\sqrt{\mu \epsilon}\) is the intrinsic impedance of the medium and is 120\(\mu\) ohms for free space. If the helix is excited by a finite source, for example, a voltage generator in series with the tape, the equivalent circuit may be drawn as in Fig. III-11. \(R_{h_{t1}}(ka), R_{h_{t2}}(ka)\), etc. represent the impedances of the "free mode" waves which are essentially resistive and are functions of \(ka\); and \(Z_{\text{local}}(ka)\) represents the impedance resulting from the local radiation and induction fields near the source. The real average power orthogonality of the "free mode" waves is sufficient to show that the equivalent circuit can be drawn as in Fig. III-11, although it can also be deduced in a more rigorous fashion from the results of the next chapter concerning the source-present helix problem. Although not described in Chapter II, similar results apply to the sheath helix case, as can be realized from (II.7-17) and the discussion in Section II.7. The impedances

![Diagram](image-url)
or admittances of the "free mode" waves result from the contributions of the poles of the integrand, whereas $Z_{\text{local}}(ka)$ or $Y_{\text{local}}(ka)$ results from the contribution of the branch cut integration. Reference 2, page 423, gives an excellent description of similar results obtained for the somewhat simpler case of the common open two wire transmission line.

Although the actual values of $R_{|h_{t0}|}(ka)$, $R_{|h'^{1}_{t1}|}(ka)$, etc., may be defined in several ways, and although the currents which flow through them will depend to a considerable extent on the actual configuration of the source, it seems clear that the ratio $\frac{P_z}{|I|^2}$ is at least a measure of the relative ease with which the different waves are excited by any finite source.

It is therefore evident from Fig. III-10 that the $|h'^{1}_{t1}|$ and $|h'^{1}_{t1}|$ waves are excited to much smaller amplitudes than the $|h_{t0}|$ wave for the particular case of $\psi = 10^\circ$ and $x = 0.1$ for $ka < 0.7$. It would appear that these same conclusions are applicable to other values of $\psi$ and $x$. This explains the remarks made in Section III.7 concerning the amplitudes of the different waves. From Figs. III-10 and III-11 it can be seen that the points of zero group velocity correspond to zero values for $R_{|h_{t0}|}(ka)$, $R_{|h'^{1}_{t1}|}(ka)$, etc., and explains, perhaps, the meaning of these resonance points. However, it can be expected that for a physical helix, that is, for one in which the wire has a finite conductivity, the characteristics of the power flow curves or the values of $R_{|h_{t0}|}(ka)$, etc., will be modified at these points.

In traveling-wave tubes using helices, the magnitude of the axial electric field at the center of the helix is often of considerable interest.\textsuperscript{9} From the expressions for $E_z^i$ and $P_z$ for the sheath helix obtained from (II.4-17) and (II.6-6) and the solutions for the zeroth sheath helix mode given in reference 10, $|E_z^i(r=0)|^2/P_z$ for the sheath helix is readily calculated. This ratio can be determined more easily, perhaps, using curves available in reference 9. To within the
approximations used here, for the narrow tape helix it is evident from (III.7-1) that

$$|E_z^i(r=0)| \approx \frac{|I|}{\mu_0 c a} \xi a^2 \tan \psi K_0(\xi a).$$  (8)

By substitution of the values of $|h_{to}| a$ in (8) and by use of the values of $\frac{P_z}{\sqrt{|I|^2}}$ given in Fig. III-10, the ratio $|E_z^i(r=0)|^2/P_z$ for the $|h_{to}|$ wave for the narrow tape helix for which $\psi = 10^\circ$ and $x = 0.1$ can be obtained. Fig. III-12 shows the ratios of the magnitudes of the axial electric fields for $r = 0$ in the sheath and narrow tape helices, assuming the total average power flow, the mediums, and the pitch angle are the same for both cases. Note that the axial electric field at $r = 0$ given by the narrow tape helix approximation is somewhat smaller than that predicted by the sheath helix model over the range of $ka$ where the axial phase velocity is relatively constant. Calculations have been carried out only for the $|h_{to}|$ wave. Reference 22 noted, as interpreted by this writer, that the measured axial electric field at $r = 0$ in a helix at $ka = 0.2$ was from 1.6 to $3.4$ db less than predicted by the sheath model. The agreement with the value given in Fig. III-12 by the narrow tape approximation is quite good. However, it is felt that this agreement is somewhat,
although not entirely, coincidental since the helix on which the experiments were performed was a round wire helix whose pitch angle was $5^\circ$ and whose diameter was 13% of the mean helix diameter and 50% of its pitch.

In concluding this section, an estimate of the power loss resulting from the finite conductivity of the conductor is given. It is assumed that the tape is sufficiently thick so that currents flowing on the "inside" and "outside" of the tape do not interact. In other words, the frequency and conductivity or both are taken to be sufficiently high so that the skin depth is much smaller than the tape thickness and the tape width.\(^1\) It is assumed further that the current divides equally between the "inside" and "outside" of the tape. This is essentially equivalent to neglecting the interaction of adjacent turns or to assuming that the current distribution is similar to what it would be on a straight infinitely long tape. This assumption is certainly in error, but it should not be too serious if the tape width is small. It is also assumed that the transverse current is negligible and that only one wave has any significant amplitude at a particular frequency. Using the above assumptions and the constant current density approximation, one readily finds the power loss per unit axial length of the helix to be given by

$$P_L \approx \frac{|I|^2}{2\delta \sin \psi} \sqrt{\frac{\alpha_t}{2\sigma_t}},$$

where $\mu_t$ and $\sigma_t$ are the conductivity and permeability, respectively, of the tape material. The inverse square root density approximation leads to the following approximate expression for the loss:

$$P_L \approx \frac{1}{\pi} \ln \left| \frac{5 \cos \psi}{\Delta} \right| \frac{|I|^2}{5 \sin 2\psi} \sqrt{\frac{\alpha_t}{2\sigma_t}}.$$  \hspace{1cm} (10)

In this, $\Delta$ is the tape thickness, and it is assumed that the power loss only up to within a distance $\Delta$ of the edges need be considered. This is admittedly quite crude, but such an assumption is necessary to avoid
obtaining an infinite power loss and can be partly justified by considering an elliptical rather than a flat tape conductor. For \( \cos \psi \approx 1 \) and for \( 1000 \gg \frac{g}{\Delta} \gg 10 \), (9) and (10) differ only by a factor near 2, so that (9) is probably adequate for the present purposes. For a circular wire helix with the above assumptions one obtains

\[
P_L \approx \frac{I^2}{4 \pi b \sin \psi} \sqrt{\frac{\omega \mu_t}{2 \sigma_t}},
\]

(11)

where \( b \) is the radius of the wire.

If the loss is small, an axial attenuation constant can be defined in the usual way as \( \frac{1}{2} \)

\[
\text{Attenuation Constant} = \frac{1}{2} \frac{P_L}{P_z}.
\]

Calling \( C \) the value of \( \frac{P_z}{\sqrt{\frac{\mu}{\varepsilon} I^2}} \) obtained from Fig. III-10 or a similar plot for other pitch angles, using (9) in (12), and taking \( \mu = \mu_t \), one obtains after some manipulation

\[
\text{Attenuation Constant} \approx \frac{1}{C_X} \left( \frac{1}{(\mu a \sin \psi)^2} \right) \sqrt{\frac{ka}{\sqrt{2 \sigma_t \sqrt{\frac{\mu}{\varepsilon}}}}}.
\]

(13)

in nepers per unit length. \( x \) is the parameter defined by (III.6-2). As an example, for free space where \( \sqrt{\frac{\mu}{\varepsilon}} = 120\pi \) ohms, for \( x = 0.1 \), \( ka = 0.2 \), \( \psi = 10^\circ \), \( \sigma_t = 5.6 \times 10^7 \) mhos/meter for copper, and \( a = 0.035 \) meters, (13) gives an attenuation constant of 0.0182 nepers/meter. No check with any experimental data has been made. In the above the common assumption has been made that the loss is small so that the propagation characteristics are affected only slightly.\( ^1 \)

\( ^2 \) Reference 52 has pointed out that at extremely low frequencies the loss will considerably alter the characteristics from those found for the lossless case.
The Narrow Gap Approximation

III.10 Boundary Conditions; Derivation and Solution of Approximate Determinantal Equation

In the previous sections an approximate treatment of the tape helix when the tape is narrow has been given. It is possible to use similar methods to determine the propagation and other characteristics of the helix when the tape is quite wide, or when the gap width, \( \delta' \), shown in Fig. III-1 is assumed to be small compared with \( a, p, \) and \( \lambda \). In this case it is obviously no longer possible to assume that \( K_1 \) is zero. However, since \( \delta'<<\lambda \), the distribution of the electric field in the gap is essentially quasi-static, and some reasonable assumptions concerning this should lead to field expressions which are nearly correct.

Since \( E_{\parallel} \) must be zero at the tape edges, it seems clear from physical considerations that it must be quite small everywhere in the gap. Therefore, a fair approximation should result if \( E_{\parallel} \) is taken to be zero in the gap. However, since \( E_{\parallel} \) must also be zero on the tape if it is a perfect conductor, the assumption that \( E_{\parallel} = 0 \) in the gap means that \( E_{\parallel} = 0 \) everywhere for \( r = a \). Because the tape is assumed to be of infinitesimal thickness, \( E_1 \) in the gap must become infinitely large in an inverse square root manner as the tape edges are approached, and a distribution which acts in this manner alone should be a good approximation. However, it might be assumed instead that \( E_1 \) in the gap is constant in magnitude, and this leads to results which are quite similar to those obtained from the more realistic inverse square root distribution. If an approximate tangential electric field distribution in the gap is assumed, the requirement that the tangential magnetic field be continuous at all points through the gap can no longer be satisfied. The property which characterizes the narrow gap helix is that no current cross the gap from turn to turn; for this \( K_1 (\text{gap}) = 0 \), or \( H_{\parallel} \) must be continuous
through the gap. Although this boundary condition might be approximately satisfied in several ways, the simplest expression results if this condition is imposed only along a line in the center of the gap. The similarity between the assumptions concerning $E_\|$, $K_\perp$, and $E_\perp$ in the gap for the narrow gap case, and those concerning $K_\perp$, $E_\|$, and $K_\|$, respectively, on the tape for the narrow tape case should be noted. From this the arguments for the validity of the approximations used in the narrow gap case are quite analogous to those given in Section III.5 for the narrow tape case and need not be repeated here. Section III.12 contains some further comments about boundary conditions near thin edges.

It is clear that $E_\perp$ at $r = a$ can be expressed in the form

$$E_\perp(r=a) = e^{-jhz} \sum_m \varepsilon_{im} e^{-jm\left(\frac{2\pi}{p}z-c\right)}, \quad (1)$$

where the $\varepsilon_{im}$ are the Fourier coefficients of the perpendicular electric field expansion. If $E_\perp$ in the gap is assumed to be constant in magnitude, then

$$E_\perp(r=a) = e^{-jhz} \sum_m \varepsilon_{im} e^{-jm\left(\frac{2\pi}{p}z-\theta\right)} \approx A e^{-jhz} e^{j(h-\beta_n)\left[z-\frac{P\theta}{2\pi}-(p-\delta')\right]}$$

for $\frac{P\theta}{2\pi} + p > z > \frac{P\theta}{2\pi} - p + \delta'$,

$$0, \text{ elsewhere.} \quad (2)$$

$\beta_n$ has exactly the same meaning here as in (III.5-19) with which (2) should be compared, and the point $z = 0$, $\theta = 0$ is again chosen as point (1) in Fig. III-1b. If $E_\perp$ in the gap is assumed to vary in an inverse square root manner, then

$$E_\perp(r=a) = e^{-jhz} \sum_m \varepsilon_{im} e^{-jm\left(\frac{2\pi}{p}z-\theta\right)} \approx A e^{-jhz} e^{j(h-\beta_n)\left[z-\frac{P\theta}{2\pi}-(p-\delta')\right]}$$

$$\frac{1}{\left[\left(z-\frac{P\theta}{2\pi}-(p-\delta')\right)\left[\delta' - z + \frac{P\theta}{2\pi} + (p-\delta')\right]\right]^{1/2}}$$

for $\frac{P\theta}{2\pi} + p > z > \frac{P\theta}{2\pi} - p + \delta'$,

$$0, \text{ elsewhere.} \quad (3)$$
(3) should be compared with (III.5-24). A in (2) has a different value than A in (3). These can be related to the amplitude of the voltage across the gap in each case by

$$|V| = |\int_{\text{gap}} E_{\perp}(r=a) \, dl| .$$

(4) is similar to (III.5-20). Proceeding exactly as in Section III.5, assuming the constant phase front of the electric field is perpendicular to the gap edges, although this makes very little difference if $\delta'$ is small, one finds that

$$1_{L_m} = \frac{|V|}{p} \cos \psi \left( -j \frac{2\pi}{p} \delta' \right) e^{j(h-h_m') + \frac{2\pi}{p} \delta'} 1_{D_m}$$

(5)

for the constant magnitude gap field, and

$$2_{L_m} = \frac{|V|}{p} \cos \psi \left( -j \frac{2\pi}{p} \delta' \right) e^{j(h-h_m') + \frac{2\pi}{p} \delta'} 2_{D_m}$$

(6)

for the inverse square root gap field. (5) and (6) should be compared with (III.5-22) and (III.5-26). $1_{D_m}$ and $2_{D_m}$ are identical to $1_{D_m}$ and $2_{D_m}$ except that $\delta$ in the latter is replaced by $\delta'$. The subscripts 1 and 2 have the same significance as in Section III.5.

Since $E_\parallel$ is taken to be zero everywhere for $r = a$, after some reduction (III.5-18) gives, because of the orthogonality of the space harmonics,

$$\kappa_{nm} \approx -\nu_m \kappa_{lm} ,$$

(7a)

where

$$\nu_m = \frac{\left[ (\eta_m^2 + mh_m a \tan \psi)(\eta_m^2 - mh_m a \cot \psi) - k^2 a^2 \eta_m^2 \right] I_m(\eta_m) K_m(\eta_m)}{\left[ (\eta_m^2 - mh_m a \cot \psi)^2 + k^2 a^2 \eta_m^2 \right] \cot \psi}$$

(7b)

Using (III.2-16), (III.2-17), (III.5-14), (III.5-15), and (III.5-17), one can readily show that

$$E_{\perp}(r=a) = j \frac{e^{-jhz}}{\omega \epsilon_0 a} \sin \psi \cos \psi \sum_m \left( a_m \kappa_{nm} + b_m \kappa_{lm} \right) e^{-j \frac{2\pi}{p} z - \theta} ,$$

(8)
where
\[ a_m = \left[ (\eta_m^2 - m_h a \cot \psi)(\eta_m^2 + m_h a \tan \psi) \frac{I_m(\eta_m)K_m(\eta_m)}{\eta_m^2} - k^2a^2 \frac{I_m(\eta_m)K_m(\eta_m)}{\eta_m^2} \right], \]
\[ b_m = \left[ (\eta_m^2 + m_h a \tan \psi)^2 \frac{I_m(\eta_m)K_m(\eta_m)}{\eta_m^2} + k^2a^2 \frac{\tan^2 \psi}{\eta_m^2} \frac{I_m(\eta_m)K_m(\eta_m)}{\eta_m^2} \right] \cot \psi. \]

Substitution of (7a) in (8) gives
\[ E_{1}(r=a) \approx j \frac{e^{-jhz}}{\omega \epsilon a} \sin \psi \cos \psi \sum_m (b_m - a_m \nu_m) K_{1m} e^{-jm(\frac{2\pi}{P} z - \theta)} \] (11)

But \( E_{1}(r=a) \) is also expressed by (1) so that from the values of \( e_{1m} \) given by (5) or (6) the following approximation for \( K_{1m} \) results:
\[ K_{1m} = -j \frac{\omega \epsilon a |\psi|}{p \sin \psi \cos \psi} e^{j(h\delta_m \frac{\delta}{2})} e^{-jm \frac{\eta_m^2 \delta'}{p}} \sum_{m'} \frac{1}{b_m - a_m \nu_m} \] (12)

From (7b), (9), and (10) one finds after considerable manipulation that
\[ \frac{1}{b_m - a_m \nu_m} = \frac{\cos \psi \sin^3 \psi}{k^2a^2 \eta_m^2 I_m(\eta_m)K_m(\eta_m)} \left[ (h^2a^2 - k^2a^2 + k^2a^2 \frac{2\cot^2 \psi}{\eta_m^2} I_m(\eta_m)K_m(\eta_m) \right] \] (13)

Since
\[ K_1 = e^{-jhz} \sum_m K_{1m} e^{-jm(\frac{2\pi}{P} z - \theta)} \] (III.5-2)

if one uses (12) in (III.5-2) and requires that \( K_1 = 0 \) at the center of the gap, or where \( z = \frac{p}{2\pi} \theta + p - \frac{\delta'}{2} \), the following approximate determinantal equation for the narrow gap case is obtained:
\[ K_1(z=\frac{p}{2\pi} \theta + p - \frac{\delta'}{2}) = 0 \approx -j \frac{\omega \epsilon a |\psi|}{p \sin \psi \cos^2 \psi} e^{j(h\delta_m \frac{\delta}{2} - \frac{hp}{2\pi} \theta - hp)} \sum_m \frac{1}{b_m - a_m \nu_m} \] (14)

The right side of (14) consists of a complex factor whose magnitude is independent of \( h \) times a real series; and if (14) is to be satisfied, the series must be zero. Substituting (13) in (14), multiplying by \( \frac{1}{4} \)
for convenience, and dropping all other multiplying factors which do not affect the solution, one obtains

\[ 0 \approx \sum_m \frac{1}{-\eta_{m}^{2} I_{m}'(\eta_{m}) K_{m}'(\eta_{m}) I_{m}(\eta_{m}) K_{m}(\eta_{m})} \left[ h^{2} a^{2} - k^{2} a^{2} + k^{2} a^{2} \frac{m^{2} \cot^{2} \psi}{\eta_{m}^{2}} I_{m}(\eta_{m}) K_{m}(\eta_{m}) + k^{2} a^{2} \cot^{2} \psi I_{m}'(\eta_{m}) K_{m}'(\eta_{m}) \right]_{1,2} D_{m}^{'} \quad (15) \]

It should now be noted that except for \(-\eta_{m}^{2} I_{m}'(\eta_{m}) K_{m}'(\eta_{m}) I_{m}(\eta_{m}) K_{m}(\eta_{m})\), (15) is identical to the approximate determinantal equation for the narrow tape helix given in (III.5-32) if the gap width, \(s'\), in the former is equal to the tape width, \(s\), in the latter. But further, as can be seen from (A.2-9) and Table A-III, the factor \(-\eta_{m}^{2} I_{m}'(\eta_{m}) K_{m}'(\eta_{m}) I_{m}(\eta_{m}) K_{m}(\eta_{m})\) is exceedingly close to unity for \(\eta_{m} > 0\) and \(|m| > 1\), and it can be put equal to unity in (15) for \(|m| > 1\) with negligible error. Only the \(m = 0\) term is appreciably altered by this factor, and even this occurs only for \(\eta_{0}\) and, therefore, \(ka\) small. It is evident that the discussion in Sections III.6, C.1, C.2, C.3, and C.4 concerning the numerical solution of (III.5-32) is directly applicable to (15) also.

If the constant field constant \(z\) phase front distribution is assumed so that

\[ D_{m}^{'} = \frac{\sin mx^{'} / \eta_{m}^{'} x^{'}}, \quad (16) \]

where \(x^{'}\) is given by

\[ x^{'} = \frac{\pi s^{'}}{p}, \quad (17) \]

(see (III.6-1) and (III.6-2)), the numerical results available from the narrow tape helix calculations can be used with only minor modifications. The effects of using \(2D_{m}^{'}\) and some other assumption concerning the gap field phase front are considered in Section III.12 and are shown to be small. The \(\frac{ka}{|h_{to}| a}\) ratios which result for the narrow gap helix for \(\psi = 10^\circ\) and \(x = 0.1\) are shown in Fig. III-5. Note that only for the \(|h_{to}|\) wave for \(ka < 0.3\) are the results for the narrow gap and narrow
tape case distinguishable. It is evident from the previous discussion, (15), and (III.6-11) or (III.6-13) that as \( \delta \) becomes smaller, the \( \frac{ka}{h_{to}a} \) solution for the narrow gap case for small \( ka \) approaches the asymptotic value \( \sin \psi \) more closely. This also occurs for the narrow tape case as \( \delta \) is made smaller. It appears, therefore, that as the tape width is increased from a very small value to a very large one, the \( \frac{ka}{h_{to}a} \) ratio for small \( ka \) increases, approaches a maximum, and then decreases. The decrease occurs at a slower rate than the increase for a proportionate change in tape width because of the \(-ln_{0}^{2}I_{0}(\eta_{o})K_{0}(\eta_{o})I_{0}(\eta_{o})K_{0}(\eta_{o})\) factor which divides the \( m = 0 \) term in the wide tape determinantal equation.

It also seems quite possible that the maximum \( \frac{ka}{h_{to}a} \) ratio for small \( ka \) differs only slightly from the zeroth mode sheath helix solution and may occur in the region where \( \delta \approx \delta' \approx \frac{D}{d} \). This action of the \( |h_{to}| \) wave solution may explain why the experimentally measured values of \( \frac{ka}{h_{to}a} \) for small \( ka \) agree so well with the zeroth mode sheath solution since the helices on which experiments were performed were wound of round wire whose diameter was about 50% of the pitch. The radial extension of the wire in the helices used for the measurements may also have been an influencing factor. This factor is obviously not accounted for in the tape helix theory.

Since the \( \frac{ka}{h_{to}a} \) solutions for both the narrow tape and narrow gap helices are essentially identical except for the \( |h_{to}| \) wave for small \( ka \), the various phase and group velocities are also very nearly alike. Consequently, the curves of Figs. III-4, III-6, III-8, and III-9 are quite indicative of the results which are obtained for \( \psi = 10^\circ \) and \( x' = 0.1 \), and specific curves for the narrow gap case are not shown. Many of the remarks in Sections III.7 and III.8 concerning the narrow tape helix apply to the narrow gap helix as well. It can be expected that the relative amplitudes of the space harmonics for the two cases will be somewhat
altered although the general characteristics are certainly maintained.
This is evident from the fact that the dominant harmonic is determined
primarily by the values of \( \eta_m^* \), and these will be nearly the same in both
cases. Incidentally, it is clear from (III.2-17), (1), (5), and (6)
that
\[
E_z^e \approx \frac{|V|}{\rho} e^{j(h-\beta_0)z} \sum_m e^{-jm \frac{\pi}{p} 5'} \sum_{1,2}^{D_m} \frac{K_m(\frac{\eta_m}{a})}{K_m(\frac{\eta_m}{m})}.
\]
(18)

\( E_z^e \) is given by an identical expression except that \( I_m \) replaces \( K_m \). Calculations for numerical values of \( E_z^e \) for the narrow gap case have not
been performed.

III.11 Power Flow; Tape Current; Power Loss

It is possible to derive an approximate expression for the real aver-
age axial power flow for the narrow gap helix in a manner quite analogous
to that used in Section III.9 for the narrow tape helix. Here, however,
since both components of surface current density occur, the procedure is
somewhat more involved. (III.9-2), (III.9-3), (III.9-4), and their coun-
terparts for the power flow for \( r > a \) are still exactly correct. Further,
from (III.5-14) and (III.5-15) it is readily shown that
\[
|\kappa_{zm}|^2 = |\kappa_{zm}|^2 \sin^2 \psi + |\kappa_{zm}|^2 \cos^2 \psi + 2\sin \psi \cos \psi \Re(\kappa_{zm} \bar{\kappa}_{zm}), \quad (1)
\]
\[
|\kappa_{zm}|^2 = |\kappa_{zm}|^2 \cos^2 \psi + |\kappa_{zm}|^2 \sin^2 \psi - 2\sin \psi \cos \psi \Re(\kappa_{zm} \bar{\kappa}_{zm}), \quad (2)
\]
\[
\Re(\kappa_{zm} \bar{\kappa}_{zm}) = (|\kappa_{zm}|^2 - |\kappa_{zm}|^2) \sin \psi \cos \psi + (\cos^2 \psi - \sin^2 \psi) \Re(\kappa_{zm} \bar{\kappa}_{zm}). \quad (3)
\]

(1), (2), and (3) are exact and may be compared with (III.9-5) in which
terms containing \( \kappa_{zm} \) were dropped. In order to proceed further, the
approximate forms for \( \kappa_{zm} \) and \( \kappa_{zm} \) for the narrow gap case must be used.
From (III.10-1) and (III.10-11) one obtains
\[
\kappa_{zm} \approx -\frac{j}{\sin \psi \cos \psi} \frac{e_{zm}}{b_m - a_m \gamma_m}. \quad (4)
\]
For purposes of convenience the explicit approximate form of $\varepsilon_{m}$ is not inserted here. The approximate value of $K_{m}$ is given by (III.10-7).

Using these in (1), (2), and (3), one obtains the following:

$$|\kappa_{zm}|^{2} \approx \frac{\omega^{2} \varepsilon_{m}^{2} |\varepsilon_{m}|^{2}}{(b_{m} - a_{m})^{2} \cos^{2} \psi} (\nu_{m} \cot \psi)^{2}, \quad (5a)$$

$$|\kappa_{zm}|^{2} \approx \frac{\omega^{2} \varepsilon_{m}^{2} |\varepsilon_{m}|^{2}}{(b_{m} - a_{m})^{2} \sin^{2} \psi} (\nu_{m} \tan \psi)^{2}, \quad (5b)$$

$$\text{Re}(\kappa_{zm}) \approx \frac{\omega^{2} \varepsilon_{m}^{2} |\varepsilon_{m}|^{2}}{(b_{m} - a_{m})^{2} \sin \psi \cos \psi} (\nu_{m} \tan \psi) (\nu_{m} \cot \psi). \quad (5c)$$

If (5) is substituted in (III.9-3) and (III.9-4), the results obtained after some manipulation are

$$K_{m} \approx \frac{\omega \varepsilon_{m} a_{m}}{(b_{m} - a_{m})^{2} \cos^{2} \psi} (\nu_{m} \tan \psi)^{2} \frac{k_{m}^{2}}{\nu_{m}^{2}} K_{m}, \quad (6)$$

$$I_{m} \approx -\frac{2m(b_{m}^{2} + k_{m}^{2})}{(b_{m} - a_{m})^{2}} \omega \varepsilon_{m} |\varepsilon_{m}|^{2} (\nu_{m} + \tan \psi)^{2} \csc^{2} \psi \frac{k_{m}^{2}}{\nu_{m}^{2}} I_{m}, \quad (7)$$

$K_{m}$ and $I_{m}$ take the same form except that $I_{m}$ and $K_{m}$ are interchanged. The argument of the modified Bessel functions is $\eta_{m}$, and the functional notation is omitted for convenience as usual. In (6) and (7) $q_{m}'$ is given by

$$q_{m}' = \frac{\nu_{m} \cot \psi}{\nu_{m} + \tan \psi} - m_{m} a \cot \psi \quad . \quad (8a)$$

This quite complicated function fortunately reduces to a much simpler form, and after considerable manipulation it becomes

$$q_{m}' = -k_{m}^{2} \cot^{2} \psi \frac{\eta_{m}^{2} I_{m}^{1} K_{m}}{q_{m}' I_{m}^{1}} \quad , \quad (8b)$$

where $q_{m}$ is given as before by

$$q_{m} = \frac{\eta_{m}^{2} - m_{m} a \cot \psi}{ \nu_{m} \tan \psi} \quad . \quad (III.9-7)$$
The ratio \( \frac{(v_m + \tan \psi)^2}{(b_m - \alpha_m v_m)^2} \) also complicates (6) and (7) considerably, but, here again, some simplification occurs. Using (III.10-7b) and (III.10-13), one obtains

\[
\frac{(v_m + \tan \psi)^2}{(b_m - \alpha_m v_m)^2} = \frac{\sin^4 \psi}{k_m a_I^2 K_m^2} \cdot \frac{q_m^2}{\eta_m^4}.
\] (9)

Substituting (9) in (6) and (7) and then these in (III.9-2), performing a similar operation to obtain an expression for \( P_z^e \), carrying out the integrations using the formulae of Section A.3, adding \( P_z^i \) and \( P_z^e \) to obtain \( P_z \), and then combining terms using the Wronskian relationship to simplify somewhat, one obtains finally

\[
P_z \approx \pi a^2 \frac{\omega \varepsilon_a}{k_m a_I^4} \sin^2 \psi \tan^2 \psi \sum_m \frac{q_m^2}{\eta_m^4} \left[ \frac{I_m K_m}{I_m'^2 K_m^2} \left( \frac{q_m^2}{\eta_m^4} \left( \frac{1}{2} \left( \frac{I_m}{I_m} + \frac{K_m}{K_m} \right) + \frac{1}{24} \right) - \frac{1}{4} \left( \frac{K_m}{K_m} \right) \left( \frac{I_m'}{I_m} + \frac{K_m'}{K_m} \right) \right) + \frac{k_m^2 a^2 \cot^2 \psi}{\eta_m^4} \left( \frac{1}{2} \frac{1}{\eta_m} \frac{K_m'}{K_m} + \frac{1}{24} \left( 1 + \frac{m^2}{\eta_m^2} \right) \left( \frac{I_m'}{I_m} + \frac{K_m'}{K_m} \right) \right) \right] \right] \| \varepsilon_{1m} \|^2.
\] (10)

It should be noted that (10) has a form which is very similar to (III.9-6), the approximate power flow equation for the narrow tape case. No numerical calculations have been carried through using (10), but the procedure described in Section C.5 can obviously be used for such calculations. If one of the approximate forms for \( \varepsilon_{1m} \) (III.10-5) or (III.10-6), is substituted in (10), it can be expressed in the form

\[
P_z = \sqrt{\frac{\varepsilon}{\mu}} |V|^2 D. \]

In this, \( D \) is a nondimensional number depending on the value of the series. Since the equivalent circuit for the narrow tape helix shown in Fig. III-11 applies to the narrow gap case also, \( D \) as a function of \( ka \) will have its poles and zeros interchanged with those of
the function \( \frac{P_z}{\sqrt{\frac{\mu}{\varepsilon}} |I|^2} \) shown in Fig. III-10.

In the narrow tape case the current \(|I|\), defined by (III.5-20), is the amplitude controlling factor. In the narrow gap case the voltage \(|V|\), defined by (III.10-4), performs this role. It may prove useful to relate these in at least an approximate fashion so that the two cases can be compared. The following calculation uses (III.5-20) to obtain \(|I|\) assuming \(|V|\) is known, although a quite analogous calculation using (III.10-4) could be used to go in the other direction. Using (4) with (III.10-7) and (III.10-13) in (III.5-1), and inserting this in (III.5-20), one obtains

\[
|I| \approx \frac{\omega a \sin \psi \cos \psi}{k_2^2} \left| \sum_m P_m \varepsilon \int_0^{(p-d) \cos^2 \psi} e^{-j\left(h + \frac{P}{2 \omega a^2}\right) \sec \psi} \, dz \right|. \tag{11}
\]

The fact that the integration proceeds along a line perpendicular to the tape edges has been used in (11). Also, \( P_m \) is given by

\[
P_m = \frac{1}{\eta_m^2} \sum \left[ m^2 \left( \cot \psi + \tan \psi \right) + \xi^2 a^2 - k_2^2 a^2 \left( \frac{m^2}{\eta_m^2} + \frac{m' m'}{\eta_{m'}^2} \right) \right]. \tag{12}
\]

Using (III.10-5) or (III.10-6) in (11), performing the integration, using trigonometric identities involving the sum and differences of angles, and dropping unimportant phase factors give after some manipulation

\[
1, 2 |I| = \frac{\omega a |V|}{2 \eta \kappa a^2 \cot \psi} \left| \sum_m P_m 1, 2 D_m \frac{(\cos \psi + \cos \psi') \sin \psi' + (\sin \psi' - \sin \psi') \cos \psi'}{m + \tau} \right|
\]

\[
- j \sum_m P_m 1, 2 D_m \frac{(\sin \psi + \sin \psi') \sin \psi' + (\cos \psi - \cos \psi') \cos \psi'}{m + \tau}.
\tag{13}
\]

In (13) \( x' \) is given by (III.10-17), and \( y' \), and \( z' \), and \( \tau \) by

\[
y' = 2\pi \left( 1 - \frac{x'}{\eta} \right) \cos^2 \psi + x', \tag{14}
\]

\[
z' = \pi \left( 1 - \frac{x'}{\eta} \right) \tau \cos^2 \psi, \tag{15}
\]

\[
\tau = \tan^2 \psi \left( 1 + h a \cot \psi \right). \tag{16}
\]
The convergence of (13) can be improved by the usual procedure of adding and subtracting asymptotic forms. Only the constant field constant \( z \) phase front distribution is considered here. Since \( \frac{2\varphi}{m} \cos \psi \) approaches \( \frac{2\varphi}{m} \) as \( \lambda m \) becomes increasingly large, it can be shown, using (III.10-16), that (13) becomes

\[
\int |I| = \frac{\omega e a |V|}{2\pi a_0^2 \csc \psi} \left\{ \left[ \frac{2\varphi_o}{x} - \frac{2\varphi}{x} \cos \psi \right] S(y',x') + \sum_{s} \left[ \sum_{s} \right] \sin z' + \left[ \sum_{s} \right] \cos z' \right\}

- j \left\{ \left[ \sum_{s} \right] \sin z' + \left[ \frac{2\varphi}{x} \cos \psi \right] S(y',x') + \sum_{s} \left[ \sum_{s} \right] \right\} \tag{17}
\]

In (17)

\[
S(y',x') = S_2(y'x') - S_2(y'x) - S_2(2x') \tag{18}
\]

where \( S_2(x) \) is defined by (A.7-1); also,

\[
\sum_1 = \sum_{m=1}^{\infty} \left( \frac{P_m}{m+\tau} - \frac{P_m}{m-\tau} + \frac{\varphi a m \sin m y'}{m \cos \psi} \right) \cos \frac{my'}{m x'} \tag{19}
\]

\[
\sum_2 = \sum_{m=1}^{\infty} \left( \frac{P_m}{m+\tau} - \frac{P_m}{m-\tau} + \frac{\varphi a m \sin m x'}{m \cos \psi} \right) \cos \frac{mx'}{m x'} \tag{20}
\]

\[
\sum_3 = \sum_{m=1}^{\infty} \left( \frac{P_m}{m+\tau} + \frac{P_m}{m-\tau} \right) \sin \frac{my'}{m x'} \tag{21}
\]

\[
\sum_4 = \sum_{m=1}^{\infty} \left( \frac{P_m}{m+\tau} + \frac{P_m}{m-\tau} \right) \sin \frac{mx'}{m x'} \tag{22}
\]

In addition,

\[
P_0 = \frac{\xi^2 \alpha^2 I_0(\xi a)K_0(\xi a) - k^2 \alpha^2 I_0'(\xi a)K_0'(\xi a)}{\xi^2 \alpha^2 I_0(\xi a)K_0(\xi a) - k^2 \alpha^2 I_0'(\xi a)K_0'(\xi a)} \tag{23}
\]

\[
+ P_m = \frac{1}{\gamma_m P_m' K_m'(\gamma_m)} \left\{ \mbox{h\alpha}(\cot \psi + \tan \psi) + \xi^2 \alpha^2 - k^2 \alpha^2 \left[ \frac{I_m'(\gamma_m)K_m' \gamma_m}{I_m(\gamma_m)K_m(\gamma_m)} \right] \right\}, \tag{24}
\]

\[
- P_m = \frac{1}{\gamma_m P_m' K_m'(\gamma_m)} \left\{ \mbox{h\alpha}(\cot \psi + \tan \psi) + \xi^2 \alpha^2 - k^2 \alpha^2 \left[ \frac{I_m'(\gamma_m)K_m' \gamma_m}{I_m(\gamma_m)K_m(\gamma_m)} \right] \right\}, \tag{25}
\]
with \( \eta_m \) and \( \eta_m \) given as before by (III.6-7) and (III.6-8). Although no numerical work has been performed using (17), it is believed to be in a form such that numerical results can be readily obtained. The \( S_2 \) functions are quite simple and highly convergent. Although the sums (19) through (22) may be slightly tedious to calculate, the terms in the parentheses decrease at least like \( \frac{1}{m^2} \) so that not too many terms would be required for reasonable accuracy. It would appear that the excellent approximations for the modified Bessel function products discussed in Section A.2 are useful for calculating \( \eta_m^P \). Since the sums are in the nature of correction terms, even relatively large errors in their values may not be serious.

The use of assumptions very similar to those discussed at the end of Section III.9 makes possible the calculation of the loss resulting from the finite conductivity of the tape in the narrow gap case. The power loss per unit axial length of the helix is given by

\[
P_L = \frac{1}{2\rho} \sqrt{\frac{\omega L_t}{2\sigma_t}} \left[ \int_{\text{one turn}} \left( |H^1_{\theta}|^2 + |H^1_z|^2 + |H^2_{\theta}|^2 + |H^2_z|^2 \right) dA \right]. \tag{26}
\]

Using the approximations for \( \kappa_{\eta m} \) and \( \kappa_{\eta m} \) given in Section III.10, solving for \( \kappa_{\theta m} \) and \( \kappa_{z m} \), inserting these in the appropriate field expressions from (III.2-15) through (III.2-20), and then these in (26), one might obtain an explicit expression for \( P_L \). This is clearly a very tedious task, and it was not thought worthwhile carrying through the calculation for this report. If the inverse square root gap voltage distribution is assumed, the series expressing \( P_L \) may not converge. If such is the case, it may be necessary to use the artifice suggested in Section III.9 following (III.9-10) or to consider the results obtainable from the constant gap voltage distribution as adequate.

Although numerical results are not available for the quantities
III.12

considered in this section, with the exception of $P_L$, it should be relatively easy to obtain such results from the formulae which are given. These have been included here for purposes of comparison with the narrow tape case and for reference in case of future interest.

Further Consideration of Approximations

III.12 Effects of the Amplitude and Phase Approximations

In order to derive an approximate determinantal equation for the narrow tape case, it was necessary to make an assumption concerning the current density distribution on the tape. Similarly, for the narrow gap case the distribution of the electric field in the gap was assumed. Both constant amplitude and inverse square root approximations were considered although numerical results for only the former have been presented so far. Also, some simplifications concerning the phase of the current density on the tape or the electric field in the gap were made. In this section these matters are discussed further.

Before doing this some remarks about the inverse square root distribution are of interest. It would be expected that such a distribution would be a very good approximation for particular components very near the edges of the tape since in any small region the wave equation is closely approximated by Laplace's equation.$^{36, 38}$ This expectation is confirmed from general considerations of the behavior of an electromagnetic field in the neighborhood of a sharp edge.$^{53}$ Such considerations show that, in general,

$$E_{\parallel} = O(R^{\frac{3}{2}}), \quad (1a) \quad H_{\perp} = O(1), \quad (1b)$$

$$K_{\perp} = O(1), \quad (1c) \quad \text{for } R \to 0,$$

where $R$ is the distance from the edge, and $O$ is the usual order symbol. All other field components become singular like $R^{-\frac{1}{2}}$ as $R$ approaches zero, or
\[ E_\perp = 0(R^{-\frac{1}{2}}), \quad (2a) \]
\[ H_\perp = 0(R^{-\frac{1}{2}}), \quad (2b) \]
\[ K_\parallel = 0(R^{-\frac{1}{2}}), \quad (2c) \quad \text{for } R \to 0. \]

For an electromagnetic wave normally incident on a surface with an edge of infinitesimal thickness, it can be shown that an even greater restriction than (1c) applies; namely,
\[ K_\perp = 0(R^2), \quad (3) \]
and it seems likely that (3) applies to waves of arbitrary incidence as well.\textsuperscript{53} The above remarks and equations (III.5-3) through (III.5-10) show that, at least for narrow tapes, with the inverse square root distribution \( K_{\parallel m} \) and \( K_{\perp m} \) are approximated to an even better degree than indicated in Section III.5. It would appear that for relatively wide tapes also the error is not large. Of course, similar remarks hold for the narrow gap approximation.

From using the inverse square root and constant \( \pi \) phase front approximation so that
\[ \sum_{m=1}^{\infty} J_0(mx) = J_0(mx), \quad (4) \]
(compare with (III.6-1)), an equation exactly like (III.6-12) results except that \( J_0(mx) \) replaces \( \sum_{m=1}^{\infty} \frac{\sin mx}{mx} \) everywhere. It should be recalled that \( J_0(0) = 1 \) and \( J_0(-mx) = J_0(mx) \). The third term in this equation is \( \sum_{m=1}^{\infty} J_0(mx) \) instead of \( \frac{1}{x} \sum_{m=1}^{\infty} \frac{\sin mx}{m^2} \), aside from the multiplier independent of \( x \). From (A.7-10) the dominant term in this Schäfsmilch series is \( \ln \frac{2}{x} \), and for \( x = 0.1 \) this term alone represents the series to better than 0.01%. For the constant density assumption the equivalent term which represents \( \frac{1}{x} \sum_{m=1}^{\infty} \frac{\sin mx}{m^2} \) is \( \ln \frac{0}{x} \). Since \( x = \frac{n^2}{\rho} \), it is evident that the only change in the local field term in the approximate determinantal equation resulting from the different assumed distributions is a change in the effective width of the tape. For example, \( x = 0.10 \) for the constant
amplitude distribution is equivalent to $x = 0.0735$ for the inverse square root distribution.

Since the fourth term in (III.6-12) represents the contribution to $E_h$ from adjacent turns, it would be expected from physical considerations that the assumed current distribution would have only a minor influence on this term. Although a reduction in effective tape width in a similar ratio to that noted above appears desirable, as is evident from Fig. C-2, because of the rapid convergence of the term in the brackets $[\ ]$ in the fourth term in (III.6-12), the effect of this change is generally negligible. Since $J_0(mx)$ almost equals $\frac{\sin mx}{mx}$ for small $m$ and $x$, the $R(\eta_m)$ terms are not altered to any appreciable degree by either choice of current density distribution. Thus, the only difference in the solutions for the approximate determinantal equation caused by the different amplitude but constant $z$ phase front approximations is a slight change in the tape width to which such solutions apply in the ratio $\sqrt{2}$.

If (4) is used in place of (III.6-1) to calculate the power flow from (III.9-6), some consideration of (C.5-2) shows that one change is to replace $\left(\frac{\sin mx}{mx}\right)^2$ by $J_0^2(mx)$ in each series. For small $x$ this causes essentially no change in the values of the series. The only other change is to replace $\ln \frac{2.211}{x}$ by $\ln \frac{2}{x}$ since $\sum_{m=1}^{\infty} \frac{\sin^2 mx}{m^3}$ is replaced by $\sum_{m=1}^{\infty} \frac{J_0^2(mx)}{m}$; and from (A.7-13) $\ln \frac{2}{x}$ represents this Bessel function sum quite closely for small $x$. Consequently, as far as the power flow is concerned, the only difference caused by the different amplitude but constant $z$ phase front approximations is a change in the effective tape width in the ratio $\sqrt{2}$.

This ratio is slightly different from that required for similarity in the determinantal equation, although for small $x$ the numerical effect is not significant.

In order to simplify the sums encountered in the approximate determinantal and power flow equations, it has been assumed, so far, that
\( \beta_\parallel = h \), or that the constant phase front of the current density on the narrow tape is in a plane of constant \( z \). Although this should be quite satisfactory for a narrow tape, a better approximation, and one which could possibly be applied to relatively wide tapes, is one in which the constant phase front is perpendicular to the tape edges. From (III.5-19) or (III.5-24) and Fig. III-1b it can be shown that this requires that

\[
\frac{hp}{2\pi} \theta_1 - \beta_\parallel z_1 + \beta_\parallel \frac{D}{2\pi} \theta_1 = \frac{hp}{2\pi} \gamma_2 - \beta_\parallel z_2 + \beta_\parallel \frac{D}{2\pi} \theta_2 ,
\]

where

\[
z_2 - z_1 = 5 \cos^2 \psi ,
\]

\[
a(\theta_1 - \theta_2) = 5 \sin \psi \cos \psi .
\]

Using (6) and (7) in (5) results in

\[
\beta_\parallel = h \sin^2 \psi .
\]

If (8) is used instead of \( \beta_\parallel = h \), by the same procedure whereby (III.6-13) is derived from (III.5-32), it can be shown that with the constant amplitude assumption the approximate determinantal equation for the narrow tape helix becomes

\[
0 \approx \left[ \xi^2 a^2 I_o (\xi a) K_o (\xi a) + k^2 a^2 \cot^2 \psi I'_o (\xi a) K'_o (\xi a) \right] \frac{\sin x \Delta}{x \Delta} \\
+ \frac{(h^2 a^2 - k^2 a^2 \csc^2 \psi)}{\csc \psi} (\cos \Delta \ln x + \frac{h^2 a^2 - k^2 a^2 \csc^2 \psi}{2}) \sum_{m=1}^{\infty} \left[ -\frac{1}{(m^2 + \eta_m^2)^2} \frac{\sin x (m-\Delta)}{x (m+\Delta)} \frac{\sin \eta mx}{\eta mx} \\
+ \frac{1}{(m^2 + \eta_m^2)^2} \frac{\sin x (m+\Delta)}{x (m-\Delta)} \frac{1}{m \csc \psi} \cos x \Delta \right] \frac{\sin \eta mx}{\eta mx} + \sum_{m=1}^{\infty} R_1 (\eta_m) .
\]

\( R_1 (\eta_m) \) is like \( R (\eta_m) \) in (III.6-13) except that \( \frac{\sin \eta mx}{\eta mx} \) is replaced by \( \frac{\sin x (m-\Delta)}{x (m-\Delta)} \). This should make little difference in the value of this series even for \( x \) relatively large. \( \Delta \) here is given by

\[
\Delta = ha \sin \psi \cos \psi .
\]

Note that for \( \Delta = 0 \), (9) becomes equal to (III.6-13). (9) was investigated
in some detail for the particular case of $\psi = 10^0$ and $x = 0.1$, and it was found that the roots of (9) are insignificantly different from those of (III.6-13). How this comes about can be realized from the following argument. For small $ka$ the solutions occur for small $ha$ and, consequently, small $\Delta$. In this case $\frac{\sin x \Delta}{x \Delta}$, $\cos x \Delta$, and $\frac{\sin x (m \Delta)}{x (m \Delta)}$ differ by a negligible amount from 1, 1, and $\frac{\sin mx}{mx}$, respectively. For medium and large values of $ka$, although $ha$ becomes quite large where roots occur, $\Delta$ is still sufficiently small so that $\frac{\sin x \Delta}{x \Delta}$ and $\cos x \Delta$ are both very close to unity. Further, although $\frac{\sin x (m \Delta)}{x (m \Delta)}$ can differ considerably from $\frac{\sin mx}{mx}$, this occurs only for relatively large $m$ and in opposite directions for the $+$ and $-$ terms so that these tend to compensate. Finally for large $ka$ the solutions occur very near $|h| a = ka \csc \psi$, and quite large changes in $x$ and $\Delta$ influence this solution only slightly. It therefore appears likely that even for $\psi$ smaller than $10^0$ -- note the influence of $\sin \psi$ on $\Delta$ -- and $x$ larger than 0.1, (III.6-13) rather than (9) can be used to obtain results of sufficient accuracy.

For the inverse square root approximation with (8), it is not possible to reduce the approximate determinantal equation to a form like (9) since simple addition formulae do not exist for Bessel functions as they do for trigonometric functions. However, from (A.5-9) it is evident that

$$J_o \left[ x (m \Delta) \right] = J_o (x m) J_o (x \Delta) + 2 \sum_{n=1}^{\infty} (-1)^n J_n (x m) J_n (x \Delta) .$$

(11)

Although the details will not be repeated here, if (11) is used in place of $J_o (xm)$ in the current density distribution factor, it can be shown that the form of the resulting determinantal equation is similar to (9). Furthermore and most important, the effect of a change in $x \Delta$ occurs, as it does in (9), to order $x^2 \Delta^2$. Because of this it is probably sufficient to put $\Delta = 0$ in this case also. The effect of using (8) rather than

$P = h$ on the power flow has not been considered in detail. However, it
is clear that for small $x$, at least, the effect is negligibly small.

Although the influence of the amplitude and phase distribution on only the narrow tape solution is considered in the previous discussion, it should be obvious, in view of the results of Section III.10, that the conclusions apply to the narrow gap case as well. The fact that different amplitude and phase distributions alter the "free mode" solutions for the tape helix only slightly is a reminder that the powerful variational techniques for solving electromagnetic wave boundary value problems might prove useful here. Although such methods have been used in a variety of problems, none of these appear to be of the type where both TE and TM waves are required on an open transmission system. The possibility of applying variational procedures to obtain solutions for such systems would appear to be worthy of future investigation.

Related Problems

III.13 Multiwire Helices

The methods used in the previous sections can be applied to the problem of a helix wound with several wires or tapes. Although it is possible to derive an exact formal expression for the determinantal equation in this case as in Section III.3 for the single tape helix, this expression is of little practical use. Consequently, the assumptions discussed at length in Section III.5 are used. The constant amplitude constant $z$ phase front current density approximation for narrow tapes is used in the analysis given in this section, although it should be clear that any of the other approximations, including the narrow gap one, lead to practically identical results.

The developed view of a four wire helix as an example of a multiwire helix is shown in Fig. III-13. One of the wires has been darkened as an aid in following the windings, but this has no other significance.
\[
\cot \psi = \frac{2\pi a}{p}
\]

Multiwire Helix

FIG. III-13

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Chart of Forbidden Regions for Multiwire Helix for Zeroth Mode
Fig. III-13 is to be compared with Fig. III-1b. In a multiwire helix each wire forms a helix of pitch $p$, pitch angle $\psi$, and there are $N$ wires uniformly spaced in the pitch distance. $\delta$ and $\delta'$ are taken, as before, to be the tape width and gap width, respectively, in the axial or $z$ direction. It is to be expected from the results of the sheath helix theory and from the general theory of multiwire transmission systems that several different types of "free modes" occur on a multiwire helix. These are distinguished by the relative phase of the currents flowing in the different wires at any given cross-section of constant $z$, and one may anticipate that the propagation constants for the different modes are dissimilar functions of frequency. As before, the tapes are assumed to be perfect conductors, and the medium for $r > a$ is taken to be the same as that for $r < a$ and lossless.

(a) The Zeroth Mode

If the currents in all the tapes are in phase at a cross-section of constant $z$ and all have the same amplitude, the system can be rotated through an angle of any multiple of $\frac{2\pi}{N}$ with no change in the fields. Further, if the system is shifted axially a distance $\frac{p}{N}$, the new fields can differ from the previous ones by only a constant factor. Proceeding exactly as in Section III.2, one can readily show that the electric Hertzian potential must be of the form

$$
\Pi_{\infty}^z = e^{-jhz} \sum_{m} A_{mN}^i e^{\frac{2\pi}{N} (\eta_{mN} \frac{z}{a} - \frac{2\pi}{p} z - \theta)}
$$

(1)

with an identical form for the magnetic Hertzian potential. The summation index $m$ runs, as before, by integer values $-2, -1, 0, 1, 2, \ldots$.

Some consideration shows that the exact field expressions in this case are those given by (III.2-15) through (III.2-20) if there $m$ is replaced by $mN$. The significance of the above may, perhaps, be further realized from the expansion for the approximate current distribution. Since each
III.13(a)

of the windings is a uniform helix, (III.5-1) applies, and with the assumptions taken here

\[ e^{-jhz} \sum_m K_m \ e^{-jm(2\pi z - \theta)} \approx \frac{|I|}{\delta \cos \psi} e^{-jhz} \left\{ \begin{array}{ll}
\frac{2\pi z}{p} + 2x > \theta \geq \frac{2\pi z}{p} \\
\vdots \\
\frac{2\pi z}{p} + n \frac{2\pi}{N} + 2x > \theta > \frac{2\pi z}{p} + n \frac{2\pi}{N} \\
\vdots \\
\frac{2\pi z}{p} + (N-1) \frac{2\pi}{N} + 2x > \theta > \frac{2\pi z}{p} + (N-1) \frac{2\pi}{N} \\
0, \text{ elsewhere,}
\end{array} \right. \]

where \( x = \frac{n_0}{p} \), as before. Proceeding in the usual way, one obtains

\[ K_m = \frac{|I|}{p \cos \psi} e^{-jmx \sin \frac{mx}{m_N}} N, \ m = \ldots -2N, -N, 0, N, 2N, \ldots \] \hspace{1cm} (3a)

or

\[ K_m = \frac{|I|}{p \cos \psi} e^{-jmn_x \sin \frac{mN_x}{m_N}} N, \ m = \ldots -2, -1, 0, 1, 2, \ldots \] \hspace{1cm} (3b)

The subscript \( l \) has the same meaning as in (III.5-22) and (III.6-1) with which (3) should be compared. The difference in the sign of the exponential is the result of a different choice for the coordinate origin. \( |I| \) is the amplitude of the total current flowing in the direction of the tape on each one. From (3) it is again evident, as already deduced from considerations of symmetry, that the summations in the field and current density representations proceed in steps of \( N \), or in successive integer steps of \( m \) if \( m \) is replaced by \( mN \).

Since only the \( mN^{th} \) harmonics appear in the field representations with \( m = \ldots -2, -1, 0, 1, 2, \ldots \), the forbidden region restriction is correspondingly modified. From an analysis similar to that given in Section III.14 one obtains the plot shown in Fig. III-14 for the case being considered here. This should be compared with Fig. III-2, and the obvious relationships noted. The limitation given by (III.4-8) can be generalized, and one has

\[ ka < \frac{N}{2} \cot \psi, \]
or
\[ p < N \frac{\lambda}{2}. \] (4)  

If in Fig. III-3 the abscissa is labeled \( \frac{ka}{N \cot \psi} \) rather than \( \frac{ka}{\cot \psi} \), it can be used as it stands for \( N > 1 \). Instead of doing this, one might change the scale so that the point \( \frac{1}{2} \) in Fig. III-3 becomes \( \frac{N}{2} \) with the abscissa label remaining \( \frac{ka}{\cot \psi} \).

If (3) is substituted in (III.5-18) with \( K_m = 0 \), and it is required that \( E_n = 0 \) for \( r = a \) at the center of all the tapes, that is, at
\[ \theta = \frac{2\pi}{p} z + x, \ldots, \frac{2\pi}{p} z + x + \frac{2\pi}{N}, \ldots, \frac{2\pi}{p} z + x + \frac{(N-1)2\pi}{N}, \] the following single approximate determinantal equation results from the satisfaction of this boundary condition at each point:

\[ 0 \approx \sum_{m=-1,0,1} \left\{ \left[ h^2 a^2 - k^2 a^2 + k^2 a^2 \frac{m^2 N^2 \cot^2 \psi}{\eta_m^2} \right] I_m \left( \eta_m \right) K_m \left( \eta_m \right) \right. \\
+ k^2 a^2 \cot^2 \psi \left( \eta_m \right) I'_m \left( \eta_m \right) K'_m \left( \eta_m \right) \left\} \frac{\sin m\pi x}{m\pi x}. \] (5)  

(5) should be compared with (III.5-32). An unimportant multiplying factor has been dropped in deriving (5). If \( N \) is now allowed to become increasingly large with \( Nx \) approaching a finite limit \( (n > Nx > 0) \), it is evident from (A.2-22), (A.2-23), and (III.2-4) that all the terms in (5) for \( m \neq 0 \) become of order \( \frac{1}{N} \). In the limit of \( N = \infty \), the only finite term remaining in (5) is the term for which \( m = 0 \). This is exactly the determinantal equation, (II.5-23), for the zeroth mode of the sheath helix, and the relationship between this sheath mode and the multwire helix mode being considered here is thereby established. Note that in the limit of \( N = \infty \), the only forbidden region restriction which remains for any finite value of \( ka \) is merely that \( |h|a > ka \).

If \( N \) is large, then one may write
\[ |I| = \frac{|K_m|}{N} p \cos \psi, \] (6)  
where \( |K_m| \) is the magnitude of the surface current density in the direction
of the wires. Substitution of (6) in (3), (3) in (III.5-14) and (III.5-15) with \( \kappa_m = 0 \), and the resulting expressions for \( \kappa_{zm} \) and \( \kappa_{sm} \) in (III.2-15) through (III.2-20) yields, for \( r \neq a \) as \( N \) becomes infinitely large since the \( m \neq 0 \) terms in the field representations become exponentially small, exactly the sheath helix field expressions given by (II.4-15) through (II.4-20) with \( n = 0 \). For \( r = a \) the higher harmonic amplitudes become increasingly large although of increasingly rapid variation as \( N \) increases. It is clear that the proper solutions for the sheath system are obtained by determining the limiting expressions for \( r \neq a \) as \( N \) becomes increasingly large.

For finite \( N \) use of the procedure described in Section III.6 transforms (5) to the following form:

\[
0 \approx \xi^2 a^2 I_0(\xi a)K_0(\xi a) + k^2 a^2 \cot^2 \psi J_0(\xi a)K_0(\xi a) \\
+ \left( a^2-k^2 \csc^2 \psi \right) \sum_{m=1}^{\infty} \left[ \frac{1}{(m^2N^2+\mu_{mn}^2)^{1/2}} - \frac{1}{(m^2N^2+\mu_{mn}^2)^{1/2}} \right] \frac{\sin mNx}{mNx} + \sum_{m=1}^{\infty} R(\eta) \quad (7)
\]

Since \( N \gg 2 \) for a multiwire helix, the second series in (7) contains no terms with modified Bessel functions of first order. From this and the discussions in Sections C.3 and C.4 it is evident that for the multiwire helix in this mode no \( h \) root exists corresponding to the \( |h_{t1}^m| \) solution for \( N = 1 \). Also, the portion of the \( |h_{t1}^m| \) solution along the \( |m| = 1 \) (really \( |m|N = 1 \)) boundary disappears; in fact, the forbidden region of this order no longer occurs. Since \( \pi > N \pi > 0 \), it can be seen that the first series in (7) is essentially \( \frac{1}{N} \) times as large as the corresponding term in (III.6-11). In view of all this, the simple procedure suggested in Section III.8 with the omission of the step for putting in the solutions along the \( |m|N = 1 \) boundary will give quite accurate results. Further, even for \( N \) as small as 2 it is perhaps better to use the sheath solution in the low frequency region rather than the average value as
suggested in Section III.8. The power flow for the multiwire helix in this mode can be readily calculated using (3) and the procedures described in Sections III.9 and C.5.

(b) The Higher Modes

It is simplest and it appears sufficient to consider only a particular case in detail here. The mode to be discussed is the one for which the currents in the tapes at a cross section of constant $z$ differ in phase progressively by $\frac{2\pi}{N}$ radians but have the same amplitude. The meaning of this is made clear in Fig. III-15 for $N = 2, 3, 4$ where the relative phase of the currents at a particular instant of time is indicated by the vectors in the conventional manner for simple harmonic varying quantities. Since each winding is a uniform helix, with the assumptions taken here one has

$$e^{-jhz} \sum_m K_m e^{-jm(\frac{2\pi}{N} z - \theta)} \approx$$

$$\frac{|I|}{\cos \psi} e^{-jhz} \frac{2\pi}{p} z + 2x > \frac{2\pi}{p} z, \quad \ldots$$

$$\frac{|I|}{\cos \psi} e^{-jhz} e^{-jm \frac{2\pi}{N}} \frac{2\pi}{p} z + 2x + n \frac{2\pi}{p} z + n, \quad \ldots$$

$$\frac{|I|}{\cos \psi} e^{-jhz} e^{-j(N-1)\frac{2\pi}{N}} \frac{2\pi}{p} z + 2x + (N-1)\frac{2\pi}{N} > \frac{2\pi}{p} z + (N-1)\frac{2\pi}{N}, \quad (8)$$

0, elsewhere.

From (8)

$$1K_m = \frac{|I|}{p \cos \psi} e^{-jmx} \sin \frac{mx}{\sin \frac{mx}{e} \frac{-j(m+1)2\pi}{N} - l}, \quad (9a)$$

or

$$1K_m = \frac{|I|}{p \cos \psi} e^{-jmx} \sin \frac{mx}{\sin \frac{mx}{N}, \quad n = -l + nN \quad (9b)}$$

Note that the $m = -1$ harmonic always exists, although the absence or
Relative Phase of Currents

FIG. III-15a

N = 2

FIG. III-15b

N = 3

FIG. III-15c

N = 4

FIG. III-16

Chart of Forbidden Regions for Multiwire Helix for First Mode
presence of the higher harmonics is determined by \( N \). Since many of the space harmonics are absent, the forbidden region restriction is modified. It is readily determined that in this case the \( \frac{\h a}{\cot \psi} \), \( \frac{ka}{\cot \psi} \) plane appears as in Fig. III-16. Although the order of the harmonics present has been determined from an approximate distribution, the same results hold for the exact case if the currents in the tapes vary in phase in the assumed manner.

Substitution of (9) in (III.5-18) with \( K_{lm} = 0 \) and imposition of the boundary condition that \( E^e_r = 0 \) for \( r = a \) at the center of all the tapes lead to the following single approximate determinantal equation:

\[
0 \simeq \sum_n \left\{ (h^2 a^2 k^2 a^2 + k^2 a^2 m^2 \cot^2 \psi) I_m(\eta_m) K_m(\eta_m) + k^2 a^2 \cot^2 \psi I'_m(\eta_m) K'_m(\eta_m) \right\} \frac{\sin mx}{\eta_m},
\]

where \( m = -1 + nN \) and \( n = \ldots -2, -1, 0, 1, 2, \ldots \). If now \( N \) is allowed to become increasingly large with \( N \) approaching a finite limit, it is evident from (A.2-22), (A.2-23), and (III.2-4) that all the terms in (10) for \( n \neq 0 \) or \( m \neq -1 \) become of order \( \frac{1}{N} \), so that in the limit of \( N = \infty \) only the term for \( m = -1 \) remains. If the substitution

\[
\frac{\h a}{\cot \psi} = \frac{\h a}{\cot \psi} + 1 \quad (11a)
\]

or

\[
h = \frac{\h a}{\frac{2\pi}{p}} \quad (11b)
\]

is made in (10), some consideration of (III.5-31) shows that with

\[
\gamma_{-1}^2 = \h^2 a^2 - k^2 a^2 \text{ playing the role of } \xi^2 a^2, \text{ and } \h a \text{ playing the role of ha, the remaining term in (10) becomes precisely the determinantal equation for the sheath helix, (II.4-6), with } n = 1 \text{ there. The relationship of the transformation implied by (11) to a shift in the abscissa axis of Fig. III-16 should be noted. Using (6) and (9) and proceeding exactly as in Part (a) of this section, one can readily show that for } N = \infty \text{ the general field expressions become the sheath helix field expressions for } n = 1 \text{ with the correspondence between the various quantities noted above.}
The relationship between the sheath helix mode for \( n = 1 \) and the particular assumed multiwire helix mode considered here is consequently established. Note again that in the limit of \( N = \infty \) the only forbidden region restriction which remains for any finite value of \( ka \) is that \( |k|a > ka \).

The manner in which the solutions for all the higher modes on multiwire helices can be determined is now evident. It is assumed that the phases of equal amplitude currents in the wires or tapes differ by \( n(s \frac{2\pi}{N}) \) at any cross section of constant \( z \), and proceeding in a manner which should be very clear by now, one obtains an approximate determinantal equation. In this, \( s \) is essentially the sequence number, which can be positive or negative and is fixed for any particular mode; \( n \) refers to a particular tape and runs from \( 0 \) to \( N - 1 \); whereas \( N \) is the number of conductors. In the limit of \( N = \infty \), through the use of a transformation quite similar to that given by (11) for the case considered here, the sheath helix equations are obtained. For finite \( N \) the approximate determinantal equation can be solved by the methods already described in sufficient detail, and the other properties of the system can also be readily determined. Since the summation in any of the determinantal equations for the higher modes is no longer symmetrical in the summation index, the loci of roots are no longer symmetrically disposed in the \( ka \), \( ka \) plane. This would also be expected from the results given in Chapter II concerning the higher modes on the sheath helix.

III.14 The Tape Ring System

In the sheath helix system it is possible to allow \( \Psi \) to be zero and so obtain what is called in Chapter II the sheath ring. Of course, for a physical helix the smallest pitch angle is limited by the tape width or wire diameter. However, a relatively realistic system does correspond to the sheath ring, and this can be readily analyzed by the
methods already described. The tape ring system is considered to consist
of an infinite series of circular rings of radius \( a \) coaxial with the \( z \)
axis and uniformly spaced a distance \( p \) apart. Each ring has a width \( \delta \in \)
the axial direction, and the edges are separated by a distance \( \delta' \) so that
\( \delta + \delta' = p \). The tape of which each ring is made is taken to have infinitesimal
thickness in the radial direction and is assumed to be perfectly conduct-
ing. The assumptions concerning the medium in which the tape ring
system is immersed are identical to those already made for the tape helix.

By an argument quite similar to that of Section III.2 it is readily
determined that the electric Hertzian potential can be written as

\[
\Pi_\zeta e = e^{-jhz} e^{j\theta} \sum \frac{A_m^i}{K_n^{\infty}} (\frac{2\pi}{p})^{1/2} \frac{2\pi}{p} \zeta, 
\]

with a similar form for the magnetic Hertzian potential. (1) should be
compared with (III.2-7). \( \eta_m \) is given by (III.2-4). In this, \( \eta \)
loses its significance as a pitch angle and becomes only a convenient parameter.
In the tape ring system the \( \Theta, z \) constraint which is imposed on the field
representations for uniform helices is no longer present. In (1), \( n \) is
not dependent on \( m \), and it is clear from physical considerations that
independent solutions may exist for each \( n \). It is also clear that in (1)
\( n \) may be positive or negative, so that solutions with \( \Theta \) dependence like
\( \sin n\Theta \) or \( \cos n\Theta \) may be obtained by linear combination. The current dens-
ity can also be expressed in a form similar to (1); namely,

\[
K_\Theta, z = e^{-jhz} e^{j\Theta} \sum m K_{\Theta, zm} e^{-j\Theta} \frac{2\pi}{p} \zeta. 
\]

The considerable similarity between the tape ring problem and the tape
helix problem is quite obvious, and use may readily be made of this. It
is found that the field expressions given by (III.2-15) through (III.2-20)
are valid in the case here if one changes \( e^{j\Theta} \) there to \( e^{j\Theta} \) here, the
order of the modified Bessel functions \( m \) there to \( n \) here, and \( m \) there, if
it occurs by itself as an ordinary algebraic multiplier, to \( n \) here. Thus, for example, \( E^e_\theta \) for the tape ring system becomes

\[
E^e_\theta = j \frac{e^{-jzh}}{\omega a} e^{jna} \sum_m \left\{ -\frac{a}{r} n_h a I_n(\eta_m)K_n(\eta_m \frac{r}{a}) K_m \right. \\
+ \left. \frac{1}{\eta_m} \left[ a^2 n_h^2 a^2 I_n(\eta_m)K_n(\eta_m \frac{r}{a}) + a^2 n_h^2 a^2 I_n(\eta_m)K_n(\eta_m \frac{r}{a}) \right] K_{zm} \right\} e^{-jm \frac{2\pi}{p} z}.
\]

(3)

If the rings are narrow so that \( K_\theta \) and \( K_z \) can be considered to act like \( K_\parallel \) and \( K_\perp \), respectively, in the narrow tape helix case, one can readily obtain

\[
l, z K_{zm} = \frac{|I|}{p} e^{jnx} \quad l, z D_m(\beta_n = h) \quad \text{,} \quad \text{(ha)}
\]

\[
l, z K_{zm} = 0 \quad \text{.} \quad \text{(hb)}
\]

The notation has exactly the same significance as that used in Section III.5 and should need no further explanation. The \( \beta_\parallel = h \) assumption would appear to be the most reasonable one to use here. Substituting (4) in (3) and requiring that \( E^e_\theta = 0 \) at \( r = a \) along the center of the rings lead to the following approximate determinantal equation:

\[
0 \approx \sum_m \frac{1}{\eta_m^2} \left[ a^2 n_h^2 a^2 I_n(\eta_m)K_n(\eta_m) + a^2 n_h^2 a^2 I_n(\eta_m)K_n(\eta_m) \right] l, z D_m(\beta_n = h). \quad \text{(5)}
\]

If now \( p \) is allowed to become increasingly small, since \( x \) approaches a finite limit \( (n > x > 0) \), it is evident from (A.2-22), (A.2-23), and (III.2-4) that all the terms in (5) for \( m \neq 0 \) become of order \( p \). Thus, in the limit of \( p = 0 \), the only finite term remaining in (5) is the one for which \( m = 0 \), and this is the determinantal equation, (II.5-1), for the sheath ring. The sheath ring field expressions follow from the tape ring field expressions by a similar substituting and limiting process.

If the rings are wide so that the gaps are narrow, a procedure
quite like that of Section III.10 leads to the following not surprising
form for the approximate determinantal equation:

\[ 0 \approx \sum_m \frac{1}{-4n^2r^2_{m}(\eta_m)K_n(\eta_m)I_n(\eta_m)K_n(\eta_m)\eta_m^2} \left[ n^2a^2I_n(\eta_m)K_n(\eta_m) \right] + k^2a^2r^2_{m}(\eta_m)K_n(\eta_m) \right] \sqrt{D_m^2(\theta_1 = h)} \quad (6) \]

No calculations have been performed using (5) or (6), but it is clear
that these equations can be solved by the methods already described at
length. Since the order of the modified Bessel functions in (5) and (6)
remains fixed, it may be possible to simplify the procedure. Note that
a forbidden region restriction quite like the one in force for the zeroth
mode multiwire helix applies here also, and that in the limiting case of
\( p = 0 \) the only restriction which remains for any finite ka is that \( |h| \geq ka \).

**Summary**

The symmetrical properties of a uniform helix require the field solu-
tions to take a certain form. Through the use of these characteristics
and the usual conditions imposed on the field components, an exact formal
solution for the tape helix problem is obtained. Although the completion
of this solution does not appear feasible, the form of the field represen-
tations leads to the conclusion that "free mode" solutions cannot exist
in certain regions.

In order to obtain useful results the cases of a narrow tape helix
and a narrow gap helix are solved by making appropriate approximations
of the boundary conditions. The propagation constants of the "free modes"
for both of these cases are shown to be practically identical except for
small values of ka. Both inward and outward traveling waves exist for
particular values of ka in a fashion reminiscent of the sheath helix
solutions. The anomalous behavior of the propagation constant in the
neighborhood of \( ka = 1 \) is derived directly from the theory, and excellent agreement with experimental data is obtained. Expressions for the power flow and other useful quantities are derived, and numerical results are presented for a particular case.

The multiwire helix problem is solved subject to similar approximations to those used previously, and the problem of the tape ring system is treated similarly.

Throughout the analysis particular emphasis is placed on numerical results. Relatively simple means for obtaining such results are fully described and examples are given.
CHAPTER IV

THE INTEGRAL EQUATION SOLUTION

The fields surrounding a helical line on which a current of exponential form is assumed to flow can be determined from a Hertzian vector potential expressed in integral form. This has been done elsewhere but is reviewed briefly here. (Section IV.1) Several different methods of approximating the boundary conditions are considered and shown to yield results essentially equivalent to those obtained for the narrow tape case in the previous chapter. (Section IV.2)

The electric field along a thin wire can be expressed as an integral involving the current. Although this expression has been derived in many places, it is reviewed briefly also and then applied to the helix. (Section IV.3) If the helix is lossless and no sources are present, the results are practically identical to those obtained above.

The case of the driven helix is then considered, and an expression for the current at any distance from the source valid to within the thin wire approximation is obtained. (Section IV.4(a)) The significance of this solution is considered, and the "free mode" portion is discussed in detail. The characteristics of the approximate determinantal equation in the lossless case are investigated, and it is shown that roots are possible for only real values of $h$. The limiting case of $\Psi = 90^\circ$ is considered, and the effect of loss in the helix wire is discussed briefly. (Section IV.4(b))
IV.1 Derivation of the Fields from an Assumed Current Distribution

In this section the fields surrounding a helix carrying an assumed current distribution are derived from an integral expression for the vector potential, and the results are compared with those obtained in the previous chapter. No originality is claimed for the following derivation for, in fact, the method is exactly like that used in reference 16. Here, however, contrary to the procedure used in reference 16, the propagation constant is considered to be an unknown which must be determined by the boundary conditions. The work of references 17 and 51 is also pertinent.

It is assumed that a current of the form

$$I(s') = |I| e^{-j\beta s'}$$  \hspace{1cm} (1)

flows along a helical line defined by

$$x' = a \cos \theta'$$  \hspace{1cm} (2)

$$y' = a \sin \theta'$$  \hspace{1cm} (3)

$$z' = s' \sin \psi = \frac{P}{2\pi} \theta'$$  \hspace{1cm} (4)

$s'$ is the distance measured along the helical line, and $a$ and $\psi$ are as shown in Figs. II-2 and II-3. The prime refers to points on the helix, and $\theta'$ is the cylindrical angular coordinate which can be considered to vary from $-\infty$ to $+\infty$ as implied in (4). $\beta$ in (1) is taken to be real. As is well-known, an electric Hertzian potential $\overline{\Pi}$ can be written for the case here as

$$\overline{\Pi} = -j \frac{|I|}{\mu \omega \epsilon} \int_{-\infty}^{\infty} \frac{a_{\parallel} e^{-j\beta s'} e^{-jKR}}{R} ds' .$$  \hspace{1cm} (5)

All the field components can be derived from this potential alone, that is, with $\overline{\Pi}^* = 0$, through the use of (II.3-2) and (II.3-4). $a_{\parallel}$ is given by (II.1-3), or, equivalently, by
\[ \bar{\mathbf{a}}_n = -\bar{\mathbf{a}}_x \cos \psi \sin \theta' + \bar{\mathbf{a}}_y \cos \psi \cos \theta' + \bar{\mathbf{a}}_z \sin \psi . \quad (6) \]

\( \bar{\mathbf{a}}_x \) and \( \bar{\mathbf{a}}_y \) are unit vectors in the \( x \) and \( y \) directions. In (5) \( R \) is given by

\[ R = \left[ (x-x)^2 + (y-y)^2 + (z-z)^2 \right]^{\frac{1}{2}}, \quad (7a) \]

\[ R = \left[ r^2 + a^2 - 2ar \cos(\theta-\theta') + (z - \frac{D}{2\pi} \theta')^2 \right]^{\frac{1}{2}}, \quad (7b) \]

where \( x, y, z \) or \( r, \theta, z \) define the point of observation. In order to insure convergence of the integral in (5), the medium may be assumed to be slightly lossy, after which the loss may be considered to approach zero. Since

\[ \bar{\mathbf{a}}_x = \bar{a}_r \cos \theta - \bar{a}_\theta \sin \theta, \quad (8) \]

\[ \bar{\mathbf{a}}_y = \bar{a}_r \sin \theta + \bar{a}_\theta \cos \theta, \quad (9) \]

one obtains for (6)

\[ \bar{\mathbf{a}}_n = \bar{a}_r \cos \psi \sin(\theta-\theta') + \bar{a}_\theta \cos \psi \cos(\theta-\theta') + \bar{a}_z \sin \psi. \quad (10) \]

Using (7b) and (10) in (5) and making a change of variable, first to \( \theta' \) and then to \( \xi \), where

\[ \xi = \theta' - \frac{2\pi}{p} z, \quad (10) \]

one obtains for (5)

\[ \bar{\Pi} = -j \frac{|I|}{4\pi \omega c} \sec \psi e^{-j\beta z \sec \psi} \int_{-\infty}^{\infty} e^{-j\beta a \xi} \sec \psi \left[ \bar{a}_r \cos \psi \sin(\rho - \xi) \right. \]

\[ + \bar{a}_\theta \cos \psi \cos(\rho - \xi) \]

\[ + \bar{a}_z \sin \psi \left\{ \frac{e^{-j\bar{R}}}{R} \right\} d\xi. \quad (11) \]

In (11)

\[ \rho = \theta - \frac{2\pi}{p} z, \quad (12) \]
IV.1

and

$$R = \left[ r^2 + a^2 - 2ar \cos(\rho - \xi) + \xi^2 a^2 \tan^2 \psi \right]^{1/2}. \quad (13)$$

From (A.5-2) one obtains

$$\frac{e^{-jKR}}{R} = \int_0^\infty J_0 \left[ x \sqrt{r^2 + a^2 - 2ar \cos(\rho - \xi)} \right] \frac{e^{-a|\xi| \tan\psi \sqrt{x^2 - k^2}}}{\sqrt{x^2 - k^2}} \, x \, dx, \quad (14)$$

where the integration near the point $x = k$ must be treated as noted there. Also, from (A.5-7) there results

$$J_0 \left[ x \sqrt{r^2 + a^2 - 2ar \cos(\rho - \xi)} \right] = \sum_{m=-\infty}^{\infty} j_m(xr) j_m(xa) e^{jm(\rho - \xi)}. \quad (15)$$

Using (14) and (15) in (11), one obtains after interchanging the operations of summation and integration

$$\overline{\Pi} = -j \frac{[l]}{l_{\text{moe}}} a \sec \psi e^{-j\beta \sec \psi} \sum_m \int_0^\infty x \, dx \, j_m(xr) j_m(xa) \int_{-\infty}^{\infty} e^{-j\beta a \xi \sec \psi} x \left[ \bar{a}_r \cos \psi \sin(\rho - \xi) + \bar{a}_\theta \cos \psi \cos(\rho - \xi) + \bar{a}_z \sin \psi \right] e^{jm(\rho - \xi)} e^{-\tan \psi |\xi| \sqrt{x^2 - k^2}} \, d\xi. \quad (15)$$

At this stage it is useful to recall that $\beta$ can be related to $h$, the propagation constant along the $z$ axis, through

$$\beta = h \sin \psi. \quad (III.7-4)$$

Carrying out the integration on $\xi$ in (15), which is easily performed using (A.8-6), and then using (III.7-4), one finds that

$$\overline{\Pi} = -j \frac{[l]}{l_{\text{moe}}} a^2 \cot \psi e^{-jhz} \sum_m \int_0^\infty x \, dx \, j_m(xr) j_m(xa) \left\{ \frac{e^{j(m+1)\rho}}{a^2 x^2 + \eta_m^2 + 1} - \frac{e^{j(m-1)\rho}}{a^2 x^2 + \eta_{m-1}^2} + \bar{a}_\theta \left( \frac{e^{j(m+1)\rho}}{a^2 x^2 + \eta_{m+1}^2} + \frac{e^{j(m-1)\rho}}{a^2 x^2 + \eta_{m-1}^2} \right) + \bar{a}_z \tan \psi \left( \frac{2e^{j\eta_m \rho}}{a^2 x^2 + \eta_m^2} \right) \right\}. \quad (16)$$

In (16) $\eta_m$ is exactly the quantity defined in (III.2-4) and used throughout the previous chapter. From (A.5-1) one obtains
\[ \int_0^\infty \frac{X^j_m(X^m)r^m}{X^2 + \eta^2} \, dX = I_m(\eta X)K_m(\eta \frac{r}{a}) \quad , \quad r > a, \quad (17) \]

so that the final result for \( \vec{\Pi} \) becomes after some rearrangement and the use of (12)

\[
\vec{\Pi} = -j \frac{\Pi}{2\pi \omega} \cot \psi \, e^{-jhz} \sum_m \left\{ -ja_a \left[ I_{m-1}(\eta X)K_{m-1}(\eta \frac{r}{a}) - I_{m+1}(\eta X)K_{m+1}(\eta \frac{r}{a}) \right] + \bar{a}_0 \left[ I_{m-1}(\eta X)K_{m-1}(\eta \frac{r}{a}) + I_{m+1}(\eta X)K_{m+1}(\eta \frac{r}{a}) \right] \right\} e^{-j\frac{2\pi}{P}(z-\epsilon)}. \quad (18)
\]

Since (17) applies only for \( r > a \), a superscript \( e \) is affixed to \( \vec{\Pi} \). An identical formula applies for \( r < a \), except that the \( I_m \) and \( K_m \) functions are interchanged everywhere.

For purposes of comparison it is sufficient to find \( \vec{E}^e_{\parallel} \). By substitution of (18) in (II.3-3), which is simplified by the use of (II.3-5), \( E^e_z \) and \( E^e_\theta \) can be found. \( E^e_{\parallel} \) is then obtained by inserting these in (III.5-16). In the process of deriving \( E^e_z \) and \( E^e_\theta \), the following quite simple result occurs:

\[ \nabla \cdot \vec{\Pi}^e = - \frac{\Pi}{2\pi \omega} \cot \psi \, e^{-jhz} \sum_m I_m(\eta X)K_m(\eta \frac{r}{a}) e^{-j\frac{2\pi}{P}(z-\epsilon)}. \quad (19) \]

In order to simplify the expressions resulting from the various operations, it is necessary to make use of the recurrence forms given in Section A.1. There results finally after considerable manipulation

\[ E^e_{\parallel} = j \frac{\Pi \sin \psi \tan \psi}{\kappa \omega a} e^{-jhz} \sum_m \left\{ -a_m^2 a_m^2 (1 - a_m^2) \cot \psi + \frac{a_m^2 a_m^2}{\kappa^2} \right\} I_m(\eta X)K_m(\eta \frac{r}{a}) \]

\[ + \kappa^2 a_m^2 \cot^2 \psi \left[ I_m^2(\eta X)K_m^2(\eta \frac{r}{a}) \right] e^{-j\frac{2\pi}{P}(z-\epsilon)}. \quad (20) \]

It should now be recognized that (20) agrees exactly with (III.5-29) if in the latter one puts \( \delta = 0 \). But this implies from (III.5-1) and
(III.5-22) or (III.5-26) that

$$K_{\parallel} = \lim_{\delta \to 0} \frac{1}{p \cos \psi} \sum_m e^{-j m \frac{2\pi}{p} z - \theta},$$

with $K_{\parallel} = 0$, of course. (21) can be recognized as the nonconvergent Fourier series expansion for an impulse of integrable area $\frac{|I|}{\cos \psi}$ along the line $\frac{2\pi}{p} z = \theta$. Thus, the equivalence and relationship between the fields derived by the characteristic function approach of the previous chapter and those obtained from the integral expression (5) are established. It is obvious that the other field component expressions which can be obtained from (18) agree with those given in the previous chapter if (21) is used there.

It is worthy of note that (18) can be obtained from (11) by using (A.5-4) in place of (A.5-2), and then (A.5-8) in place of (A.5-7). In the final evaluation it is then not necessary to use (17), although it is necessary to recognize and interpret a nonconvergent integral as the representation of the unit impulse described by (II.7-2).

IV.2 Approximate Matching of the Boundary Conditions; Comparison with the Narrow Tape Case

Since the series representations for the field components obtained in the previous section are those resulting from an assumed current flowing along a line, it is obvious from physical considerations that they must become divergent as the line is approached. From the asymptotic forms for the products of the modified Bessel functions it is readily shown that the representations converge for $r > a$ and $r < a$ and are conditionally convergent for $r = a$ if $\frac{2\pi}{p} z - \theta \neq 0$. Reference 17 discusses the matter of convergence of these series in considerable detail. If the current does not flow on a helical line, but rather on the surface of a small diameter perfectly conducting wire, the fields should be only slightly different from those found for the line current. Thus,
if the condition that the tangential electric field be zero on the entire surface of the wire is imposed, a quite good approximation to the physical situation should result. However, this too is quite difficult to do, and further approximations must be made. One of these, which is implicit in the fact that current is assumed to flow only along the line, is that the current around the wire and the tangential electric field condition in that direction can be ignored. The reasons for this are precisely those given in Section III.5 for ignoring $K_{\perp}$ and $E_{\perp}$ in the narrow tape case.

Several methods of approximately matching the tangential electric field boundary condition on the wire in the wire direction are now considered. In the absence of any source this field component must be zero, and the application of this requirement results in a determinantal equation for the "free mode" propagation constants.

In the following it is assumed that the wire diameter is $2b$ and that the axis of the wire coincides with the helical line defined by (IV.1-2), (IV.1-3), and (IV.1-4). It is also assumed that $b$ is much smaller than $a$, $p$, and $\lambda$. One method of satisfying the electric field boundary condition in an approximate manner is to require the value of $E_{\parallel}$ for $r = a$ averaged over the wire to be zero. Although $E_{\parallel}$ is divergent for $r = a$ and $\theta = \frac{2\pi}{p}z$, the divergence is only logarithmic and, consequently, integrable. If it is assumed that the phase of the current on the wire is essentially constant in a plane of constant $z$, one obtains on putting $[E_{\parallel}^{E}(r=a)]_{\text{average}}$ equal to zero after dropping unimportant constants

$$0 \approx \sum_{m} \left\{ (\eta_{m} - \frac{m b \cot \psi}{\eta_{m}})^{2} I_{m}(\eta_{m})K_{m}(\eta_{m}) + k^{2} a^{2} \cot^{2} \psi I_{m}^{1}(\eta_{m})K_{m}^{1}(\eta_{m}) \right\} P_{m}, \quad (1)$$

where

$$P_{m} = \frac{1}{L_{mb} \sec \psi} \int_{0}^{\frac{2\pi}{p} (z+b \sec \psi)} e^{-\frac{jm(2\pi z - \phi)}{p}} \, d\phi = \frac{\sin(m \frac{2\pi b}{p} \sec \psi)}{m \frac{2\pi b}{p} \sec \psi}. \quad (2)$$
2b \sec \psi \text{ is merely the maximum wire dimension in the } z \text{ direction. Comparing (1) with (III.5–32) and (2) with (III.6–1) shows that if } 2b \sec \psi \text{ is considered equivalent to } 5, \text{ (1) is exactly like the determinantal equation obtained for the narrow tape case for the approximations implied in (III.5–32) and (III.6–1). Instead of averaging over the wire, one might require the average value of } E^e_{m} \text{ at } \theta = \frac{2\pi r}{P} (z_b \sec \psi) \text{ and } r = a \text{ to be zero. In this case an equation like (1) results except that in place of } \frac{1}{ \overline{J}_m } \text{ a factor } 2 \overline{J}_m \text{ occurs, where}

$$2 \overline{J}_m = \cos m \left( \frac{2\pi r}{P} \sec \psi \right). \quad (3)$$

If (3) is used in place of the \( \frac{\sin mx}{m} \) distribution factor, some consideration of the results of Section III.6 and (A.6–4) indicates that this approximation merely alters the effective diameter of the wire.

Finally, one might impose the condition \( E^e_{m}(r = a + b, \theta = \frac{2\pi z}{P}) \approx 0 \). In this case one has from (IV.1–20)

$$0 \approx \sum_m \left\{ \left[ \frac{2}{m} \frac{2}{m} \cos^2 \psi \right] I_m(\eta_m) K_m(\eta_m(1 + \Delta)) + \frac{1}{4m} \frac{1}{4m} \frac{m^2}{m^2} \frac{2}{m} \frac{2}{m} \cot^2 \psi \right\}$$

where now \( \Delta = \frac{b}{a} \). Although the presence of \( \Delta \) in the algebraic multiplier as well as in the argument of \( K_m \) and \( K'_m \) in (4) complicates matters somewhat, it is still possible to make a quite satisfactory approximation if \( \Delta \) is small. Some examination shows that if \( \Delta \ll 1 \), it has negligible effect on the value of the algebraic multiplier, and, consequently, it can be put equal to zero there. Using the approximations for \( I_m' \) and \( K_m' \) given in (A.2–11) and (A.2–15) (ignoring again for justifiable reasons the presence of \( \Delta \) in an algebraic multiplier), and adding and subtracting this asymptotic form for \( I_m' K_m' \), one obtains from (4)

$$0 \approx \xi^2 a^2 I_0(\xi a) K_0(\xi a(1 + \Delta)) + k^2 a^2 \cot^2 \psi I_0'(\xi a) K_0'(\xi a(1 + \Delta)) + \sum_{\eta_m} \frac{I_m(\eta_m) K_m(\eta_m(1 + \Delta))}{\eta_m} + \sum_{m=1}^\infty A(\eta_m). \quad (5)$$
In (5), \( \sum_{m=1}^{\infty} a(\eta_m) \) is a remainder series whose characteristics are very similar to those of \( \sum_{m=1}^{\infty} R(\eta_m) \) in (III.6-5). A remainder series similar to \( \sum_{m=1}^{\infty} R(\eta_m) \) in (III.6-5) has been dropped from (5) since, as noted following (III.6-10), if only \( h > k \) is considered, such a series has negligible effect. Using the approximations for \( I_m \) and \( K_m \) given by (A.2-12) and (A.2-13), it can be shown that

\[
I_m(\eta_m)k_m\left[|m|\cot \psi (1+\Delta)\right] \approx \frac{1}{2} e^{-|m|\Delta \csc \psi} .
\]

In (6) terms of order \( \Delta \) in an algebraic multiplier and terms of order \( \Delta^2 \) in the exponential are ignored. This is not necessary, but it simplifies the rest of the development and is sufficiently accurate for small \( \Delta \). Since \( \eta_m \) approaches \(|m|\cot \psi \) as \(|m| \) increases, proceeding in the usual fashion, one obtains for (5)

\[
0 \approx \xi^2 a^2 I_o(\xi a)K_o[\xi a(1+\Delta)] + k^2 a^2 \cot^2 \psi I'_o(\xi a)K'_o[\xi a(1+\Delta)]
+ \left( h^2 a^2 - k^2 a^2 \csc^2 \psi \right) \sum_{m=1}^{\infty} \frac{e^{-m\Delta \csc \psi}}{m} + \left( h^2 a^2 - h a^2 \csc^2 \psi \right) \sum_{m=1}^{\infty} \left\{ I_m(\eta_m)k_m[\eta_m(1+\Delta)] \right\}
+ I_m(\eta_m)k_m[\eta_m(1+\Delta)] - \frac{e^{-m\Delta \csc \psi}}{m \csc \psi} \right\} + \sum_{m=1}^{\infty} a(\eta_m) .
\]

Since

\[
\sum_{m=1}^{\infty} \frac{e^{-mz}}{m} = - \ln(1-e^{-z}) = \ln \frac{1}{z} + \frac{z}{2} - \frac{z^2}{12} + \ldots ,
\]

(see reference \( k \), equation 601), it is clear that (7) becomes essentially equivalent to (II.6-13). It is therefore evident that the solutions of the two equations are practically equal except for a very minor difference equivalent to a small change in the cross sectional dimensions of the conductor.

At the end of Section III.7 reference was made to a recently published work which presented results similar to those obtained in Chapter III for the propagation constants of the "free mode" waves on a narrow
tape helix.\textsuperscript{51} The method of solution described in reference 51 is very nearly like that indicated in Section IV.1 and in this section, and the approximate determinantal equation given there is almost identical to (4) here. The procedure for solving the equation presented there is a numerical and graphical one related to the one used here, although in the work of reference 51 it appears that the series are summed directly rather than after some transformations as done here. These latter transformations make the calculations much simpler, of course, and allow the generalizations discussed in Section III.6 to be made. Only one case is presented in reference 51, and this is very near the case considered here. Also, it is shown there that agreement with the experimental results given in reference 27 can be obtained. However, reference 51 does not point out that the $z_h^{(1)}$ wave is an inward rather than an outward traveling wave, and no mention is made of the fact that "free mode" solutions exist between the higher order forbidden regions. The matters of power flow, space harmonic phase velocity, group velocity, power loss, etc., are not considered there, and, as already noted, no attempt is made to simplify the calculative procedure. Of course, the very similar results of references 16 and 17 have already been noted, although the manner in which the boundary conditions are approximated in those references did not lead to the more complete results which are given in this report.

The Source-Present Problem

IV.3 Integral Equation for $\mathbf{E}$

In the analysis which is given in this section, it is necessary to start with the integral expression for the electric field on the surface of a conductor of small cross sectional area in terms of the total current in the conductor and the various parameters. Although this has been
derived in several places, it is useful for purposes of completeness to
review the derivation here.\textsuperscript{1,6,5,1,56} From the expression for the elec-
tric Hertzian potential in the form\textsuperscript{1,56}
\[ \mathbf{\Pi}(x) = -\frac{j}{4\mu_0 e} \int_{V'} \mathbf{J}(x') \frac{e^{-jkR}}{R} \, dv' \] (1)
it is readily shown from (II.3-3), with \( \mathbf{\Pi}^* = 0 \), that
\[ \mathbf{E}(x) = -\frac{j}{4\mu_0 e} \int_{V'} \left\{ \left[ \mathbf{J}(x') \cdot \nabla' \right] \nabla' + k^2 \mathbf{J}(x') \right\} \frac{e^{-jkR}}{R} \, dv' . \] (2)
Here, \( x \) and \( x' \) stand for the three coordinates at the points of observa-
tion and integration, respectively, \( \mathbf{J}(x') \) is the vector current density
per unit area flowing in the volume \( V' \), \( R \) is given by (IV.1-7a), and \( \nabla' \)
is
\[ \nabla' = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} . \] (3)
In deriving (2) use is made of the fact that
\[ \nabla' f(R) = -\nabla f(R) , \] (4)
where \( f(R) \) is a function of \( R \), and \( \nabla \) is
\[ \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} . \] (5)
If the current is assumed to flow in a wire of small transverse dimen-
sions so that the current only in the direction of the wire is signifi-
cant, it is evident that (2) is very closely approximated by
\[ \mathbf{E}(x) \approx -\frac{j}{4\mu_0 e} \int_{s'} \left\{ \left[ I(s') \overrightarrow{a}_s' \cdot \nabla' \right] \nabla' + k^2 I(s') \overrightarrow{a}_s' \right\} \frac{e^{-jkR}}{R} \, ds' . \] (6)
where \( \overrightarrow{a}_s' \) is a vector along the axis of the wire, \( s' \) is the distance
measured along the axis of the wire, and \( I(s) \) is the total current flow-
ing through a cross section of the wire at \( s' \). It is assumed that the
transverse dimensions of the wire are small compared with the radius of
curvature, which is considered to be continuous, and with the wavelength.
Since $\mathbf{a}' \cdot \nabla' = \frac{d}{ds'}$, where $\frac{d}{ds'}$ means the derivative in the direction of $s'$, that is, along $\mathbf{a}'$, and since $\nabla'$ outside the brackets [] in (6) operates on $\frac{e^{-jkR}}{R}$ only, (6) becomes

$$E(x) \approx -\frac{1}{j2\pi \omega c} \left\{ -\nabla \int_{s'} I(s') \frac{d}{ds'} \left( \frac{e^{-jkR}}{R} \right) ds' + k^2 \int_{s'} \mathbf{a}' I(s') \frac{e^{-jkR}}{R} ds' \right\}. \quad (7)$$

Integrating the first term in (7) by parts, one obtains

$$\int_{s'} I(s') \frac{d}{ds'} \left( \frac{e^{-jkR}}{R} \right) ds' = \left[ I(s') \frac{e^{-jkR}}{R} \right]_{s_1}^{s_2} - \int_{s_1}^{s_2} \frac{e^{-jkR}}{R} \frac{dI(s')}{ds'} ds', \quad (8)$$

where $s_1$ and $s_2$ are the extremities of the wire. If (8) is inserted in (7), an approximate expression for the electric field resulting from the current flowing in a wire of finite length is obtained. If the wire is assumed to be infinite in length so that $s_2$ and $s_1$ are at plus and minus infinity, respectively, and if $I(s')$ is bounded, then the first term in (8) vanishes. Note that this also occurs if $s_1 = s_2'$, that is, if the wire describes a closed circuit, or if $I(s_1) = I(s_2') = 0$. For a circuit of infinite length (7) becomes

$$E(x) \approx -\frac{1}{j2\pi \omega c} \left\{ \int_{-\infty}^{\infty} \frac{dI(s)}{ds} \nabla \left( \frac{e^{-jkR}}{R} \right) ds' + k^2 \int_{-\infty}^{\infty} \mathbf{a}' I(s') \frac{e^{-jkR}}{R} ds' \right\}. \quad (9)$$

If the point of observation is far away from the current carrying conductor, it is clear that (9) gives a very good approximation to the true electric field. It is perhaps not so clear that (9) is still a very good approximation at the surface of the wire within the assumptions made so far, although the following brief remarks may make this more evident. The fields at the surface of the conductor resulting from the current in the far away parts of the conductor are obviously affected only slightly by the shape of the wire as long as its transverse dimensions are small. On the other hand, the fields at the surface of the conductor resulting from the current in its adjacent parts
are very nearly those which would occur if the wire were straight, and if
the wire is assumed to be of circular cross section, the current can be
assumed to flow on the axis of the wire without altering the external
fields. This whole matter is considered in reference 54 where the reasons
for taking (9) as a quite good approximation for the electric field even
on the surface of the conductor are explained in great detail. In view
of the above, if the point of observation is on the surface of the con-
ductor at a point $s$, with $s$ the distance measured along a line on the
surface of the wire essentially parallel to its axis and with $\overrightarrow{a}$ a unit
vector along this line, then the electric field at $s$ and in the direction
of the wire is from (9)

$$E_x(s) = \overrightarrow{a} \cdot \vec{E}(x) \approx \frac{-j}{\mu_0 \varepsilon} \left\{ \int_{-\infty}^{\infty} \frac{\partial I(s)}{\partial s'} \left( \frac{e^{-jkR}}{R} \right) ds' + k^2 \int_{-\infty}^{\infty} \overrightarrow{a} \cdot \overrightarrow{a'} I(s') e^{-jkR} \frac{ds'}{R} \right\} \quad (10a)$$

Integrating the first term in (10a) by parts, one may also write it as

$$E_y(s) \approx \frac{-j}{\mu_0 \varepsilon} \left\{ -\int_{-\infty}^{\infty} I(s') \frac{\partial^2}{\partial s \partial s'} \left( \frac{e^{-jkR}}{R} \right) ds' + k^2 \int_{-\infty}^{\infty} \overrightarrow{a} \cdot \overrightarrow{a'} I(s') e^{-jkR} \frac{ds'}{R} \right\} \quad (10b)$$

The above development is, of course, quite intimately related to the ap-
proach extensively used in recent years in the analysis of linear anten-
nae. 54 A point which should be emphasized is that the theory considers
the current to flow along a line coincident with the axis of the wire,
but the fields are calculated and the boundary conditions satisfied on a
line lying on the surface of the conductor.

The analysis can now be applied to the helix. The point of observa-
tion is defined by

$$x = (a+b) \cos \theta , \quad (11)$$
$$y = (a+b) \sin \theta , \quad (12)$$
$$z = s \sin \psi_o = \frac{D}{2} \theta , \quad (13)$$

whereas the line along which the current is assumed to flow is defined
by (IV.1-2), (IV.1-3), and (IV.1-4). \( b \) is taken as the radius of the wire which is assumed to be circular in cross section, and \( b \) is considered to be much smaller than \( a, p, \) or \( \lambda \). As a consequence of this, \( \psi_o \) may be considered equal to \( \psi \). In (10), \( \bar{a}_s \) is \( a_s \) given by (IV.1-6), and with the assumptions just given, a similar expression results for \( \bar{a}_s \). It is found that to this approximation

\[
\bar{a}_s' \cdot \bar{a}_s \approx \sin^2 \psi + \cos^2 \psi \cos(\theta-\theta') .
\]  

(14)

With \( R \) measured from the axis of the wire to the point of observation given by (11), (12), and (13),

\[
R = \left[ a^2 (1 + \frac{b}{\bar{a}})^2 + a^2 - 2a^2 (1 + \frac{b}{\bar{a}}) \cos(\theta-\theta') + a^2 \tan^2 \psi (\theta-\theta')^2 \right]^\frac{1}{2} ,
\]  

(15)

where \( \psi \approx \psi_o \) is taken as a satisfactory approximation here also. In view of the symmetrical position of \( \theta \) and \( \theta' \) in (15) it is evident that \( \frac{\partial}{\partial s} = -\frac{\partial}{\partial s'} \). Using this in (10a), integrating the first term by parts, and substituting (14) result in

\[
E^e_s(s) \approx \frac{-1}{4\pi co} \int_{-\infty}^{\infty} \left\{ \frac{\partial^2 I(s')}{\partial s'^2} + k^2 \left[ \sin^2 \psi + \cos^2 \psi \cos(\theta-\theta') \right] I(s') \right\} e^{\frac{-jkR}{R}} ds' ,
\]  

(16)

where the \( e \) superscript is affixed for obvious reasons. If \( I(s) \) is assumed to be of the form given by (IV.1-1), (16) takes the form given in reference 6 and elsewhere. Finally, if the procedure described in Section IV.1 is used to evaluate (16), the result obtained is almost exactly like (IV.1-20), with \( r = (a+b) = a(1+\Delta) \) and \( \frac{2\pi}{p} z = \theta \) there. This is to be expected in view of the relationship between the development leading to (IV.1-20) and that given in this section. There is a slight difference in the algebraic multiplier of the first term — the term in the brackets \( [ \ ] \) —, but this is a coefficient of order \( \Delta = \frac{b}{a} \) which is negligible. If \( E^e_s(s) \) is now required to be zero on the assumption that the conductor has infinite conductivity, it is apparent from
the discussion in Section IV.2 that the "free mode" solutions are those considered in Chapter III. Since $E_\parallel$ is required to be zero only along a line on the surface, it seems clear that it will not be zero on a circular wire of radius $b$ whose axis is the helix line, but rather on some conductor whose shape and cross sectional dimensions are slightly different. However, this matter is a minor one and need not arouse much concern. It would appear, in view of the correspondence noted above, that the integral equation for the helix problem, first written down over fifty years ago, has been solved at least to the order of approximation which it implies.

IV.4 The Gap Source

(a) Application of the Source; the "Free Modes"

If the transverse dimensions of the helix wire are sufficiently small so that it can be considered to be a one dimensional conductor, then the current function can be written in the form of a Fourier integral as

$$I(s) = \int_{-\infty}^{\infty} I(\beta) e^{-j\beta s} \, d\beta.$$  \hspace{1cm} (1)

The component of electric field along the wire, $E_\parallel(s)$, is taken to be

$$E_\parallel(s) = \frac{E}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{\beta l}{2}}{\beta} e^{-j\beta s} \, d\beta.$$  \hspace{1cm} (2)

It is assumed, as before, that $E_\parallel(s)$ is the same around the periphery of the wire at any point $s$. In this case $E_\parallel(s)$ is zero everywhere except in a gap, or for $|s| < \frac{l}{2}$, where it has a constant magnitude. It is assumed further that the distribution described by (2) is maintained by an impressed electric field which is the negative of (2), and that this is the result of a voltage source placed in series with the helix wire at $s = 0$, or at $x = a, y = 0, z = 0$. In (1) and (2) $\beta$ is considered as a general complex variable, and the integration in (1) proceeds along the real axis with any necessary indentations which must be determined.
The remarks of Section II.7(b) are pertinent here. If the conductor is assumed to be perfect, the current in the presence of the source may be determined by substituting (1) and (2) in (IV.3-16), or

\[
\frac{E}{n} \int_{-\infty}^{\infty} \frac{\sin \frac{\beta l}{2}}{\beta} e^{-j\beta s} d\beta \approx \frac{-j}{n \omega} \int_{-\infty}^{\infty} \beta^2 I(\beta) e^{-j\beta s'} d\beta + k^2 \left[ \sin^2 \psi + \cos^2 \psi \cos (\beta - \epsilon) \right] \int_{-\infty}^{\infty} I(\beta) e^{-j\beta s'} ds' \frac{e^{-jkr}}{R} ds'. \tag{3}
\]

Interchanging the order of the integrations results in

\[
\int_{-\infty}^{\infty} \left\{ \frac{E}{n} \frac{\sin \frac{\beta l}{2}}{\beta} e^{-j\beta s} + \frac{I(\beta)}{\omega} \left[ \int_{-\infty}^{\infty} \left( -\beta^2 + k^2 \left[ \sin^2 \psi + \cos^2 \psi \cos (\beta - \epsilon) \right] \right) e^{-j\beta s'} \frac{e^{-jkr}}{R} ds' \right] \right\} d\beta \approx 0. \tag{4}
\]

(4) will be satisfied if the term in the braces \{ \} is zero. Now the integral with respect to \( s' \) has already been evaluated since it is essentially the expression for \( E_0(s) \) obtained from (IV.3-16). Thus, solving (4) for \( I(\beta) \), inserting this in (1), changing the variable to \( h \) which is related to \( \beta \) by (III.7-4), and using (IV.3-13), one finally obtains

\[
I(s) \approx -j \sqrt{2 \pi} e^{-jhz} \int_{-\infty}^{\infty} \frac{\sin \left[ \left( h \sin \psi \frac{l}{2} \right) \right]}{\left( h \sin \psi \frac{l}{2} \right)} dh \cdot
\]

\[
\sum_m \left\{ k^2 a^2 - k^2 a^2 + \frac{m^2}{\eta_m^2} k^2 a^2 \cot^2 \psi \right\} \frac{I_m(\eta_m)K_m[\eta_m(1+\Delta)]}{I_m(\eta_m)K_m[\eta_m(1+\Delta)]} \tag{5}
\]

In (5), \( \eta_m \) is again the quantity defined by (III.2-4), and \( \Delta = \frac{b}{a} \). \( h \) is now considered as a generalized complex variable, and the path of integration along the real axis must be properly indented around the singularities on that axis. \( V = EI \) and is the voltage of the source. (5) gives \( I(s) \) as a function of \( z \), where \( z \) is related to \( s \) through
(IV.3-13). A multiplier, \( \frac{1}{1+\Delta} \), of the term \( \frac{m^2}{\eta^2_m} k^2 a^2 \cot^2 \psi \) in the denominator of (5) has been placed equal to unity in accordance with the argument following (IV.2-4). (5) is reminiscent of the expressions encountered in Section II.7 except that here the denominator of the integrand is an infinite series. It is clear from the discussion there, and this matter is considered in further detail below, that the "free modes" correspond to the poles of the integrand of (5) or the zeros of the infinite series in the denominator. This is, of course, the usual determinantal equation which has already been discussed at length. In view of the remarks made in Section IV.2 it would appear that one could substitute the approximate determinantal equation of the narrow tape helix for the denominator of the integrand in (5) with only little error.

In order to decide how the indentations around the singularities on the real axis must be made, the disposition of these singularities when the medium is slightly lossy is considered. Since the development leading to (II.7-32), which describes how the "free mode" zeros move off the real axis in terms of the group velocity, is still valid here, the location is readily ascertained. It is evident from (5) and (III.2-4) that there are an infinite number of branch points in the integrand of (5), as opposed to just two which occur in the simpler expressions derived in Section II.7 for the sheath system. Using the notation of that section and assuming that the medium is slightly lossy, one finds that the branch points are located where \( \eta_m = 0 \), or where

\[
ha = \pm ka - m \cot \psi = \pm k_o a \sqrt{1 - \frac{j \frac{\nu_o \mu_o}{k_o}}{k_o}} - m \cot \psi ,
\]

with \( m \) taking on all integer values including zero. Fig. IV-1 shows how these branch points are located in the \( ha \) plane. It is more convenient to consider the \( ha \) rather than the \( h \) plane here. In addition to the branch point locations, the points at which the roots of the determinantal
The Small Wire Helix, $\psi = 10^\circ$
equation occur for $\psi = 10^\circ$ as calculated for the narrow tape helix are shown. The manner in which these vary as ka varies is indicated in Figs. IV-la through IV-le which are only approximately to scale. These plots should be compared with Fig. III-4. For a lossless medium the poles and branch points move into the real axis, and the contour of integration along the real axis is indented in an obvious fashion. Note that for $ka = \frac{\cot \psi}{2}$ the branch points lie in juxtaposition across the real axis.

The following discussion parallels a similar one in Section II.7(b). Because of the presence of the $e^{-jhz}$ factor in the integrand of (5), it vanishes on a circle of infinite radius in the third and fourth quadrants of the h or ha plane for $z > 0$. This assumes that the denominator of the integrand in (5) does not cause any convergence difficulties as $|h|$ becomes infinite. Such difficulty is encountered for $\psi = 90^\circ$ as noted in Part (b) of this section, although examination of the denominator of the integrand in (5) -- the approximate determinantal equation -- by the use of the more convergent forms obtained by transformation indicates that for $\psi < 90^\circ$ no trouble should be encountered. However, an explicit proof of this has not been constructed. Nevertheless, one can argue from physical considerations that the exact current representation would be convergent and that it would not be much different from (5) over most of the range of h. Thus, it seems quite proper to consider (5) as a good approximation for $I(s)$ and to assume some convergence factor for large $|h|$ if this is necessary. Continuing then, with the deformation of the contour noted above, the integral in (5) becomes a sum of integrals around the poles of the integrand plus integrals along the branch cuts which are taken to extend downward from $+ka - m \cot \psi$ with $+\infty > m > -\infty$. For $z$ very large the contributions from the branch cut integrations become small like $\frac{1}{z}$, whereas the contributions from the poles have a $z$ variation like $e^{-jhz}$ with $h$ real. The reasons for calling the various waves
inward and outward traveling waves should now be clear. Obviously, an
argument similar to the above holds for $z < 0$ if the contour of integra-
tion is swept into the upper half plane.

The current associated with the various "free mode" waves can be
obtained by evaluating the residues of the integrand at its poles. If
it is assumed that $k$ is sufficiently small so that for any of the $h$
roots \[
\frac{\sin[(h \sin \psi)^{1/2}]}{(h \sin \psi)^{1/2}}
\] can be considered equal to unity, and if the
infinite series determinantal equation in the denominator of the inte-
grand in (5) is called $f_1^1(ha)$, it is readily found that for $z >> 0$

\[
I_{f.m.}(s) \approx -2\pi k a \sqrt{\frac{\varepsilon}{\mu}} \sum_{h \text{ roots}} \frac{e^{-jhz}}{\frac{df_1^1(ha)}{d(ha)}}.
\]  

(7)

The $f.m.$ subscript refers to the fact that these are the "free mode"
waves. Although it is not possible to give a simple expression for
\[
\frac{df_1^1(ha)}{d(ha)}\]
, it should be noted that solving for the roots in the graphi-
cal manner described in Section C.4 leads immediately to a numerical
value for \[
\frac{df_1^1(ha)}{d(ha)}\]. In view of the comments of Section IV.2 one can
use Figs. C-4, C-5, \(f(ha) = f_1^1(ha)\) and similar ones which are
available, but not presented here, for determining the amplitude of
the "free mode" waves. Note from Fig. C-4 that \[
\frac{df(ha)}{d(ha)}\] near the bound-
aries of the $|m| = 1$ region, in particular where \(f(\mid h_{t2}^1\mid a)\) and \(f(\mid h_{t2}^2\mid a)\)
equal zero, is extremely large so that the amplitudes of these waves, at
least for small values of $ka$, are very small. This confirms the remarks
made in Sections III.7 and III.9 concerning these waves. If the ampli-
tudes of the currents associated with the different "free mode" waves are
calculated by the procedure indicated above, the power flow and the vari-
cous field components associated with these waves can be determined from
formulae which have already been given.
The input admittance of the infinite helix can be written as

\[ Y(s = \frac{1}{2} \text{ or } z = \frac{1}{2} \sin \psi) \approx -\frac{I(s = \frac{1}{2} \text{ or } z = \frac{1}{2} \sin \psi)}{V}, \quad (8) \]

the minus sign occurring since the applied voltage is opposite to the induced voltage. The input admittance contains the contributions from the poles which are essentially real terms plus the contributions from the branch cuts which contain both real and imaginary parts. The equivalent circuit for the helix shown in Fig. III-11 is an immediate consequence of (5) and (8), and these equations form a rigorous basis for the remarks made in Section III.9 in connection with the equivalent circuit. The real impedances \( R |h_{t0}|(ka), R |h_{t1}|(ka), \) etc., are the characteristic impedances of the "free mode" waves. As is well-known, such impedances can be defined in several ways — that is, on a voltage-current basis, on a power-current basis, or on a power-voltage basis.\(^2\) One has then

\[ R |h_{t0}|(ka) = \frac{|V|}{|I|}, \quad R |h_{t1}|(ka) = \frac{2P_z}{|I|^2}, \quad R |h_{b0}|(ka) = \frac{|V|^2}{2P_z}. \quad (9) \]

If \(|V|\) is taken as the amplitude of the source voltage, \(|I|\) as the amplitude of the "free mode" current wave, and \(P_z\) as the total average real axial power flow, it is possible to calculate the \( R |h_{t0}|(ka) \) values given by the first two equations in (9) for \( \psi = 10^\circ \) from available data. This includes Fig. III-10 and Figs. C-4, C-5, and similar curves for other \(ka\).

It was found that for \(ka \leq 1.2\), \( R |h_{t0}|(ka) \approx 2R |h_{t0}|(ka) \) for all the

waves with an error of less than 20\% even in the worst case. For \(ka > 1.2\) no definite regularity was noted. To complete the solution for the input admittance one must perform the integration along the branch cuts. This is usually a quite difficult operation, and no attempt was made to carry it out here. It is to be expected from analogy with similar problems that the imaginary part resulting from such a calculation will depend to a
considerable extent on the assumed shape of the wire, whereas the real part, as well as the "free mode" wave amplitudes resulting from the pole residues, will not be so dependent. Also, one would expect the imaginary and real parts of the resultant of the branch cut integrations to approach infinity and a finite limit, respectively, as the gap length approaches zero.

It has been assumed so far that the determinantal equation of the helix has only pure real h roots. Although it has not been possible to prove this for the exact case, as discussed in Section III.4, it can be proved for the approximate cases. Taking the denominator of (5) as an example, one has

$$\sum_m \left\{ \left[ \frac{a^2 - k^2 a^2}{\eta_m^2} + \frac{m^2}{\eta_m^4} k^2 a^2 \cot^2 \psi \right] I_m(\eta_m) K_m(\eta_m (1+\Delta)) \right\} = 0 , \quad (10)$$

where for reference

$$\eta_m = \left[ m^2 \cot^2 \psi + 2mha \cot \psi + h^2 a^2 - k^2 a^2 \right]^{\frac{1}{2}} . \quad (III.2-4)$$

In Section B.1, from the fact that the complex Maxwell field equations are the Fourier transforms of the time dependent field equations, it is deduced that the propagation constant for any "free mode" wave satisfies the relationship \( \tilde{h}(\omega) = -h(-\omega) \), where the tilde, as usual, means complex conjugate. Although the discussion there considers a field component which is represented by only one term, a quite similar procedure leads to an identical result if the field components are represented by infinite series. It should be emphasized now that only lossless systems are being considered so that \( k^2 = \omega^2 \mu \epsilon \) is pure real. Since the left side of (10) equals zero, this implies that its complex conjugate does also. Thus, taking the complex conjugate of (10), replacing \( \bar{h}(\omega) \) with \(-h(-\omega)\), and then replacing \(-h(-\omega)\) by \(h'(\omega)\) result in an equation identical to
(10) with \( H(\omega) \) in place of \( h(\omega) \). Consequently, for every \( h(\omega) \) root there is an identical \( H(\omega) \) root. But this requires that for such roots
\[
h(\omega) = H(\omega) = -h(-\omega) = \hat{h}(\omega),
\]
(11)
from which it can be immediately concluded that (10) has roots only for real values of \( h \), and not for complex or pure imaginary values. It should be clear that a similar proof applies to all the approximate determinantal equations given for the helix problem in this and the previous chapter. Incidentally, the use of the above procedure simplifies the proof given in Section B.1 leading to (B.1-13).

The true significance of the realness of the \( h \) roots as well as the solution given by (5) should be emphasized. The point is that to the order of the approximations which have been made, the solution of the infinite helix source-present problem assumes a form quite similar to those available for more rigorously solvable open boundary problems. Among these are the infinite straight wire, the dielectric rod, the dielectric slab, the two wire line, the sheath helix (given here), and others."2,39,48 In common with these the lossless infinite helix supports purely propagating waves, or "free modes", but none in which the propagation constant \( h \) is pure imaginary or complex. Radiation from an infinite driven helix is represented by a branch cut integration — really an infinite number of such integrations — or a continuum of exponential waves and not by discrete simple exponentially damped terms. Radiation from a finite helix occurs as a result of the end effects, as in finite linear antennas and dielectric rods, and the function of the "free modes" is to transport energy between the source and the ends.2,39 This is not to say that the radiation properties of a finite helix cannot be determined approximately by the assumption of the existence of a simple outward traveling current wave. Indeed, it has been found that the far-field radiation patterns of finite helices calculated on the basis of such an assumption are in good
agreement with experimentally measured patterns, at least over restricted frequency regions.\textsuperscript{25,27,30} Nevertheless, as in the theory of linear antennae, the need for making assumptions about the current distribution on a finite helix should be considered as only a temporary expedient until a more exact solution becomes available.

In connection with this, it should be clear from the foregoing development in this chapter that one can write down the integral equation for the current on a driven helix of finite length. That this can be done has also been noted in reference 52. The solution of this equation would provide a solution of the helical antenna problem which should prove of great interest and use.\textsuperscript{25} The occurrence of finite limits in the integral makes the problem considerably more difficult than the infinite helix problem. However, the partial solution of the latter problem presented here and the application of techniques which have proved useful in solving the integral equations occurring in linear antenna theory may together provide an avenue of approach worthy of future consideration.

(b) **Limiting Case of** $\Psi = 90^\circ$; **Effect of Loss in Wire**

If $\Psi = 90^\circ$ -- cot $\Psi = 0^\circ$ and $s = z$ --, the approximate expression for the current obtained from (5) becomes

$$I(z) \approx -jWw_0a^2 \int_{-\infty}^{\infty} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \sum_m \frac{I_m(\eta_0)K_m[\eta_0(1+\Delta)]}{\eta_0^2} \, dh,$$

where

$$\eta_0 = \xi a = \sqrt{k^2 a^2 - k^2 a^2}$$  \hspace{1cm} (III.6-6)

But from (A.5-8)

$$\sum_m I_m(\xi a)K_m[\xi a(1 + \frac{b}{a})] = K_0(\xi b),$$

so that from (12)
I(z) \approx -j\omega\epsilon b^2 \int_{-\infty}^{\infty} \frac{1}{\xi^b} \frac{\sin \frac{h l}{2}}{\xi^b K_0(\xi^b)} e^{-jhz} dh \quad (14)

(14) gives an approximation for the current distribution on an infinite straight perfectly conducting circular cylindrical wire of radius b excited by a gap source centered at z = 0. From (II.7-17) and (II.7-34) it is clear that the exact expression for the current in this circumstance is given by

I(z) = -j\omega\epsilon b^2 \int_{-\infty}^{\infty} \frac{K_1(\xi^b)}{\xi^b K_0(\xi^b)} \frac{\sin \frac{h l}{2}}{\xi^b} e^{-jhz} dh \quad (15)

where it should be recalled that b here plays the role of a there. (14) and (15) are identical except that \frac{1}{\xi^b} appears in the numerator of the integrand of the approximate expression, whereas \frac{1}{\xi^b} appears in this position in the exact expression. Since \frac{1}{\xi^b} is the dominant term in the expansion of \frac{K_1(\xi^b)}{\xi^b}, (14) should give a fair approximation to (15) if kb is small.

The integrand of (14) diverges for |h| large, and some further approximations and assumptions are required before any comparisons can be made. No such comparisons are made here, and the only purpose of the above is to show that the approximate current expression agrees in a certain fashion with the exact expression in the limit of \psi = 90^\circ. As a matter of fact, in reference 58, page 165, except for a change of notation and the use of \lambda = 0 there, (14) appears as the solution for the driven infinite conductor problem. The matter of divergent integrals in such circumstances is discussed in reference 58, page 147. Incidentally, (15) is derived in Section 10.16 of reference 2, and a solution valid for thin wires is given.

So far in the analysis in this chapter it has been assumed that the
wire is perfectly conducting. If the concept of surface or internal
impedance is introduced, the effect of finite conductivity may be deter-
mined.\footnote{1,2,58} It is necessary to assume that the current distribution in
the wire is the same as that in a straight infinitely long circular wire
of radius b. In this case the term $Z_1 I(s)$ is added to the left side of
(3) where $Z_1$ is given below. Proceeding as before, one obtains in place
of (5)

$$I(s) \approx -j \omega a e^2 \int_{-\infty}^{\infty} \frac{\sin \left( h \sin \psi \frac{\xi}{2} \right)}{(h \sin \psi)^{\frac{\xi}{2}}} e^{-jhz} dh.$$ 

The determinantal equation is modified in this case by the addition of
the term $jZ_1 \frac{2 \omega a e^2}{\sin \psi}$. With the usual assumption that the displacement
current in the wire is negligible compared with the conduction current\footnote{1,2,58}

$$Z_1 = \frac{1}{2\pi b} \left( j \frac{\omega \mu_t}{\sigma_t} \right)^{\frac{1}{2}} I_0 \left( \frac{(j \omega \mu_t \sigma_t)^{\frac{1}{2}} b}{I_1[\frac{(j \omega \mu_t \sigma_t)^{\frac{1}{2}} b]} \right),$$

where $\mu_t$ and $\sigma_t$ are the permeability and conductivity of the wire
material. If the skin effect is relatively complete, an excellent ap-
proximation for (17) is

$$Z_1 \approx \frac{1}{2\pi b} \left( j \frac{\omega \mu_t}{\sigma_t} \right)^{\frac{1}{2}}.$$ 

It is obvious that with a wire of finite conductivity the "free
mode" propagation constants are no longer pure real but have a small
imaginary part. It is probably simpler to use the usual approximation
methods discussed in Section III.9 in order to determine the attenuation
constant, rather than to attempt to solve the determinantal equation in-
cluding loss. However, if the losses are small and if the lossless
determinantal equation -- call this \( f_1(ha) \) as before -- has been solved numerically or in some fashion so that \( \frac{df_1(ha)}{d(ha)} \) is available, then by the usual first order perturbation assumption

\[
(ha)_{\text{loss}} \approx -\frac{J^{2.1}}{2} \frac{2\omega a^2}{\sin \Psi} \left. \frac{df_1(ha)}{d(ha)} \right|_{ha=h_t a} + (h_t a),
\]

(19)

where \( h_t a \) is a solution of the lossless determinantal equation.

It will be recalled that for a lossless single wire helix immersed in an infinite homogeneous lossless medium no "free mode" or exponentially propagated wave occurs for \( p > \frac{\lambda}{2} \). For a lossy wire with \( p = \infty \), that is, an infinite straight wire, exponential propagation occurs although with a complex propagation constant.\(^1\)\(^,\)\(^59\) However, if the wire is lossless in this case, \( p = \infty \), true exponential propagation does not occur, and the variation of the current with distance is exponential as a first approximation only.\(^2\)\(^,\)\(^58\) It is evident that single wire helices wound of lossless and lossy wires act in a somewhat different fashion for \( p > \frac{\lambda}{2} \), although it would be expected that for \( p < \frac{\lambda}{2} \) the difference is quite small.

It appears that the forbidden region restriction must be modified somewhat for lossy wires, and an investigation of the determinantal equation with loss -- the denominator of the integrand of (16) -- is required. No such investigation was attempted for this report, and it is suggested that this problem is worthy of further analysis.

**Summary**

From the integral expression for the Hertzian potential, the fields surrounding a helical line carrying a current of exponential form is determined. The results are equivalent to those obtained by the procedure used in Chapter III. Several different ways of approximating the
boundary conditions also lead to essentially identical results.

By using the integral expression for the electric field along a wire in terms of the current which flows, a formula for the current produced on an infinite helical wire by a series voltage source is obtained. Although the complete evaluation of this is not attempted, the "free mode" portion is examined. It is shown that for $\psi = 90^\circ$ the expression for the current reduces essentially to the known proper form for this case. Finally, the influence of loss in the wire is indicated.
APPENDIX A

MATHEMATICAL RELATIONS

Many of the formulae and expressions which are used throughout this report are given in the following sections for convenient reference. Those relationships which are considered well-known are stated with only a reference to their origin. Others, which are considered not so well-known, are discussed in more detail, and their derivation indicated and in some cases carried through.

Modified Bessel Functions

A.1 Differential Equation; Recurrence Formulae; Wronskian

The modified Bessel functions $I_m(z)$ and $K_m(z)$ of order $m$ and argument $z$ are the independent solutions of the differential equation

$$z \frac{d}{dz}(z \frac{dy}{dz}) - (z^2 + m^2)y = 0,$$

(1)

where $y$ is either $I_m(z)$ or $K_m(z)$. In this and the following sections $m$ is taken to be a positive integer, and $z$ is considered real and positive, although many of the relationships are still valid for $m$ noninteger and $z$ complex. Also in the following, the functional notation is often omitted wherever it is convenient and where no confusion results; in addition, a prime is taken to mean differentiation with respect to the independent variable $z$. The recurrence formulae for the $I_m$ and $K_m$ functions are

$$I_{m-1} - I_{m+1} = \frac{2m}{z} I_m,$$

$$K_{m-1} - K_{m+1} = -\frac{2m}{z},$$

(2)
\[ I_{m-1} + I_{m+1} = 2I'_{m} \quad , \quad K_{m-1} + K_{m+1} = -2K'_{m} \quad , \] 
\[ zI'_{m} + mI_{m} = zI_{m-1} \quad , \quad zK'_{m} + mK_{m} = -zK_{m-1} \quad , \] 
\[ zI'_{m} - mI_{m} = zI_{m+1} \quad , \quad zK'_{m} - mK_{m} = -zK_{m+1} \quad . \] 

From the above one obtains
\[ I'_0 = I_1 \quad , \quad K'_0 = -K_1 \quad , \] 
\[ I_{-m} = I_m \quad , \quad K_{-m} = K_m \quad . \] 

The Wronskian gives the relationship between the independent solutions, and for these
\[ W(I_m, K_m) = I'_m K_m - I_m K'_m = -\frac{1}{z} \quad . \] 

Many additional useful formulae which may be derived from the above are given in reference 3, Chapter III, and reference 4, page 174.

A.2 Series Expansions and Approximations

The \( I_m(z) \) function is defined by the series expansion
\[ I_m(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{m+2n}}{n!(n+m)!} \quad , \] 
where \( ! \) denotes the factorial. For small \( z \)
\[ I_m(z) \approx \frac{z^m}{2^m m!} \quad . \] 

For large \( z \) where \( z \gg m \) the following asymptotic series are useful:
\[ I_m(z) \approx \frac{e^\frac{z}{2}}{\sqrt{2\pi z}} \left[ 1 - \frac{(m^2-1)}{1! \cdot 8z} + \frac{(m^2-1)(m^2-3^2)}{2! \cdot (8z)^2} \cdot \ldots \right] \quad , \] 
\[ I'_m(z) \approx \frac{e^\frac{z}{2}}{\sqrt{2\pi z}} \left[ 1 - \frac{(m^2+1+3)}{1! \cdot 8z} + \frac{(m^2-1)(m^2+3^5)}{2! \cdot (8z)^2} - \frac{(m^2-1)(m^2-3^2)(m^2+3^7)}{3! \cdot (8z)^3} \cdot \ldots \right] \quad , \] 
where \( \approx \) means approximate equality.
The \( K_m(z) \) function is defined by the series expansion

\[
K_m(z) = (-1)^{m+1} I_m(z) \ln \frac{ze^\gamma}{2} + \frac{1}{2} \sum_{n=0}^{m-1} \frac{(-1)^n(m-n-1)!}{n!} (\frac{z}{2})^{2n-m}
+ \frac{(-1)^m}{2} \sum_{n=0}^{\infty} \frac{1}{n!(m+n)!} (\frac{z}{2})^{2n+m} (s_n + s_{n+m}),
\]

where \( \gamma \) is Euler's constant

\[
\gamma = 0.5772 \ldots
\]

\[
s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \quad n > 1 \text{ with } s_0 = 0,
\]

and \( \ln \) means logarithm to the base \( e \). (5) is valid for all positive integer \( m \) including \( m = 0 \), but for \( m = 0 \) the finite sum is omitted. Thus, for small \( z \)

\[
K_0(z) \approx -\ln \frac{ze^\gamma}{2},
\]

\[
K_m(z) \approx \frac{(m-1)!}{2} (\frac{z}{2})^m, \quad m \geq 1.
\]

For large \( z \) where \( z \gg m \) the following asymptotic series are useful:

\[
K_m(z) \approx (\frac{\pi}{2z})^\frac{1}{2} e^{-z} \left[ 1 + \frac{(\ln^2-1)}{1! (8z)} + \frac{(\ln^2-1)(\ln^2-3^2)}{2! (8z)^2} + \ldots \right],
\]

\[
K'_m(z) \approx - (\frac{\pi}{2z})^\frac{1}{2} e^{-z} \left[ 1 + \frac{(\ln^2+1x3)}{1! (8z)}
+ \frac{(\ln^2-1)(\ln^2+3x5)}{2! (8z)^2} + \frac{(\ln^2-1)(\ln^2-3^2)^2(\ln^2+5x7)}{3! (8z)^3} + \ldots \right].
\]

Graphs of these functions are shown on pages 2142 and 224 of reference 45 (\( \frac{2}{\pi} K_0 \) and \( \frac{2}{\pi} K_1 \), and \( I_m \), respectively, with \( m \) given values of 0 through 6 for values of the argument from 0 to 6). In addition, Appendix I of reference 9 shows how well, at least for \( m \) equal to 0 and 1, \( I_m \) and \( K_m \) can be approximated for small and large \( z \) by just a few terms of (1) and (5), and (3) and (10), respectively.
The asymptotic formulae (3), (4), (10), and (11) above are useful only for $z \gg m$. When the argument is comparable to the order with both large, the following formulae are more satisfactory:

\[ I_m(z) \approx \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{e^{(z^2 + m^2)^{\frac{1}{2}}}}{(z^2 + m^2)^{\frac{1}{2}}} \frac{1}{m^{\frac{1}{2}}} \frac{z^m}{[m + (z^2 + m^2)^{\frac{1}{2}}]^m} \left[ 1 + O\left(\frac{1}{m}\right)\right], \quad (12) \]

\[ K_m(z) \approx \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{e^{-(z^2 + m^2)^{\frac{1}{2}}}}{(z^2 + m^2)^{\frac{1}{2}}} \frac{1}{m^{\frac{1}{2}}} \frac{z^m}{[m + (z^2 + m^2)^{\frac{1}{2}}]^m} \left[ 1 + O\left(\frac{1}{m}\right)\right], \quad (13) \]

where $O\left(\frac{1}{m}\right)$ means that the next term in the expansion is of order $\frac{1}{m}$ compared with the first when $m$ is large. In reference 40 these formulae are derived using limiting forms of associated Legendre functions and integral relationships between these and the modified Bessel functions, and it is shown how the higher order terms may be obtained. It is claimed there that the first term alone in the above formulae gives three figure accuracy if $m$ or $z$ is larger than 10. These forms may be more simply derived by using integral definitions of the modified Bessel functions and the method of steepest descend for integration. This procedure is shown in detail on page 554 of reference 41, but note there a slightly different definition for $K_m(z)$ than the one used here.

Similar formulae for the derivatives equivalent to (12) and (13) seem not to have been given in the literature. Although they might be derived using the recurrence forms given in (A.1-3), or from the formulae of reference 40, it proves much simpler again to use the integral definitions of the modified Bessel functions and the method of steepest descend for integration. The details of this calculation will not be shown but the results are to first order

\[ I'_m(z) \approx \frac{(z^2 + m^2)^{\frac{1}{2}}}{z} I_m(z), \quad (11,) \]
\[ K'_m(z) \approx -\frac{1}{z} \left( \frac{z^2 + m^2}{z^2 + m^2} \right)^{\frac{1}{2}} K_m(z) \quad . \tag{15} \]

No estimate can be given of the accuracy of these approximations, but they appear to be somewhat less accurate than (12) and (13).

If one form of the integral definition of \( K_m(z) \) is used (see reference 3, page 181, Eq. (5)),

\[ K_m(z) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-zcoshu+mu} du \quad ; \tag{16} \]

and if the method of steepest descents is used to evaluate the integral, where the path of steepest descent is the real axis, the following form is obtained:

\[ K_m(z) \approx (\frac{\pi}{2})^{\frac{1}{2}} \frac{1}{(z^2 + m^2)^{\frac{1}{4}}} \left[ m + (z^2 + m^2)^{\frac{1}{2}} \right]^{\frac{1}{2}} z^m \left[ 1 - \frac{1}{(z^2 + m^2)^{\frac{1}{2}}} \left( \frac{1}{8} - \frac{5}{24} \frac{m^2}{z^2 + m^2} \right) + \cdots \right] . \tag{17} \]

Similarly, since

\[ K'_m(z) = -\frac{1}{2} \int_{-\infty}^{\infty} coshu e^{-zcoshu+mu} du \quad , \tag{18} \]

from the method of steepest descents the following form results:

\[ K'_m(z) \approx -\left( \frac{\pi}{2} \right)^{\frac{1}{2}} \frac{1}{(z^2 + m^2)^{\frac{1}{4}}} \left[ m + (z^2 + m^2)^{\frac{1}{2}} \right]^{\frac{1}{2}} z^m \left( \frac{1}{z^2 + m^2} \right)^{\frac{1}{2}} \times \]

\[ \left[ 1 + \frac{1}{(z^2 + m^2)^{\frac{1}{2}}} \left( \frac{3}{8} - \frac{7}{24} \frac{m^2}{z^2 + m^2} \right) + \cdots \frac{1}{m} \right] . \tag{19} \]

(17) and (19) give the next order approximations for \( K_m(z) \) and \( K'_m(z) \) where the first order approximations are given in (13) and (15). It will be noted from (17) and (19) that (15) is correct to the first order but not to the second. The details of the derivation of (17) and (19) are not shown here since reference 11 mentioned above gives an adequate explanation of the procedure. Similar results for \( I_m(z) \) and \( I'_m(z) \)
could be obtained in an identical manner, but an even simpler procedure is available. As will be pointed out below, the second order term in the expansion for \( I_m(z)K_m(z) \) is of order \( \frac{1}{m^2} \) compared with the first; consequently, one obtains

\[
I_m(z) \approx \frac{1}{l} \frac{e^{(z^2 + m^2)^{1/2}}}{(2\pi)^{1/2}} \frac{z^m}{m^{1/2}} \left[ \frac{1}{m + (z^2 + m^2)^{1/2}} \right] \times \left[ 1 + \frac{1}{l} \left( \frac{1}{2} - \frac{5}{24} \frac{m^2}{z^2 + m^2} \right) + O\left( \frac{1}{m^2} \right) \right]. \tag{20}
\]

It is readily found, using (A.1-8), (17), (19), and the fact that the second term in the expansion of \( I_m(z)K_m(z) \) is of order \( \frac{1}{m^2} \) compared with the first, that the second order term in \( I_m(z)K_m(z) \) is also of order \( \frac{1}{m^2} \) compared with the first; consequently, one obtains

\[
I_m'(z) \approx \frac{1}{l} \frac{e^{(z^2 + m^2)^{1/2}}}{(2\pi)^{1/2}} \frac{z^m}{m^{1/2}} \left[ \frac{1}{m + (z^2 + m^2)^{1/2}} \right] \times \left[ 1 - \frac{1}{l} \left( \frac{3}{8} - \frac{7}{24} \frac{m^2}{z^2 + m^2} \right) + O\left( \frac{1}{m^2} \right) \right]. \tag{21}
\]

(20) and (21) give the next order approximations for \( I_m(z) \) and \( I_m'(z) \) where the first order approximations are given by (12) and (14). It will be noted from (20) and (21) that (14) is correct to the first order but not to the second.

Throughout the analysis in this report the products \( I_m(z)K_m(z) \) and \( I_m'(z)K_m'(z) \) appear quite frequently. From (12), (13), (14), and (15) the following approximations are obtained:

\[
I_m(z)K_m(z) \approx \frac{1}{l} \frac{1}{(z^2 + m^2)^{1/2}}, \tag{22}
\]
\[ I'_m(z)K'_m(z) \approx -\frac{1}{2} \frac{(z^2+m^2)^{\frac{1}{2}}}{z^2}. \]  

(23)

The excellence of the approximation (22) has been noted previously (see references 42 and 43), but (23) seems not to have been reported before. An independent numerical check made for this report gave results which may be summarized as follows:

<table>
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<th>m</th>
<th>[ \frac{1}{I'_mK'_m} \left( 1 - \frac{1}{2(z^2+m^2)^{\frac{1}{2}}} \right) ] Approx. Max.</th>
<th>Approximate z for Maximum Error</th>
</tr>
</thead>
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<tr>
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<td>-0.0149</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>-0.0114</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>-0.0067</td>
<td>2.0</td>
</tr>
<tr>
<td>4</td>
<td>-0.0038</td>
<td>3.0</td>
</tr>
<tr>
<td>5</td>
<td>-0.0026</td>
<td>3.0</td>
</tr>
</tbody>
</table>

In preparing the above table, calculations of \( I'_mK'_m \) were made to five decimal places using tables in reference 3 and the appropriate forms and modifications of (A.1-2) through (A.1-5). Calculations were carried to a large enough argument to insure that the maximum error had been obtained. No calculations were made for \( m \) greater than 5, but in view of the character of the separate asymptotic forms and the functions, and the trend exhibited in Table A-I, it seems quite safe to say that the representation given by (22) is much better than one percent for all arguments for \( m \) greater than 5. For \( m \) equal to zero (22) is an unsatisfactory representation unless the argument is relatively large; only for arguments larger than about 4 is the error less than one percent. Further, as the argument approaches zero for \( m \) equal to zero, the error becomes increasingly large. On the other hand, for \( m \) not equal to zero, the
error becomes increasingly small as the argument approaches zero, the
representation (22) being exact for an argument of zero. It is of inter-
est to note that the next order approximation to \( I_m(z)K_m(z) \) is given by

\[
I_m(z)K_m(z) \approx \frac{1}{2(z^2 + m^2)^{3/2}} \left[ 1 + \frac{1}{6} \frac{1}{z^2 + m^2} - \frac{3}{4} \frac{m^2}{(z^2 + m^2)^2} + \frac{5m^4}{6(z^2 + m^2)^3} + O\left(\frac{1}{m^4}\right) \right].
\] (24)

This is obtained from reference 10 and is the basis of a statement made
above. Although this is a considerably better approximation than (22),
particularly for small \( m \), (22) is much simpler and is an adequate repre-
sentation for many purposes.

A numerical check of the representation given by (23) yielded re-
sults which may be summarized as follows:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \frac{I_m K_{m-1}}{I_m K_m} + \frac{1}{2} \frac{1}{z^2 + m^2} )</th>
<th>Approx. Max.</th>
<th>Approximate ( z ) for Maximum Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.036</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.012</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.0050</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0031</td>
<td>2.0</td>
<td></td>
</tr>
</tbody>
</table>

In preparing the above table, calculations of \( I_m K_{m-1} \) were made to five
decimal places using tables in reference 3 and the approximate forms and
modifications of (A.1-2) through (A.1-5). Calculations were carried to
a large enough argument to insure that the maximum error had been ob-
tained. No calculations were made for \( m \) greater than 4, but in view of
the character of the separate asymptotic forms and the functions, and
the trend exhibited in Table A-II, it seems quite safe to say that the
representation given by (23) is much better than one percent for all
arguments for \( m \) greater than \( h \). Curiously enough, it will be noted from the above tables that (23) is a better approximation than (22), if judged on a percentage basis, at least for \( m \) values of 1, 2, 3, and 4. For \( m \) equal to zero (23) is an unsatisfactory representation unless the argument is relatively large; only for arguments larger than about 6 is the error less than one percent. From (A.1-6)

\[
I'_0K'_0 = -I'_1K'_1, \tag{25}
\]

so that in this case (22) may be used for approximate calculations. A point of some interest concerning the behavior of the approximations (22) and (23) for zero argument is the following:

\[
\lim_{z \to 0} \left[ I'_m(z)K'_m(z) - \frac{1}{2} \frac{1}{(z^2+m^2)^{1/2}} \right] = 0, \quad m \gg 1, \tag{26}
\]

whereas

\[
\lim_{z \to 0} \left[ I'_m(z)K'_m(z) + \frac{1}{2} \frac{(z^2+m^2)^{1/2}}{z^2} \right] = -\frac{1}{4} \frac{1}{m(m^2-1)}, \quad m \gg 2, \tag{27}
\]

but

\[
\left[ I'_1(z)K'_1(z) + \frac{1}{2} \frac{(z^2+1)^{1/2}}{z^2} \right] \approx \frac{3}{16} + \frac{1}{4} \ln \frac{ze^\nu}{2} \tag{28}
\]

for \( z \) small.

An interesting and useful relationship which may be deduced from (22) and (23) is

\[
-4z^2I'_m(z)K'_m(z)I'_m(z)K'_m(z) \approx 1. \tag{29}
\]

Although this might be expected to be a good approximation only for large \( m \) and \( z \), it turns out to be quite good for \( m \gg 1 \) and \( z \gg 0 \). A numerical check of (29) yielded results which may be summarized as follows:
In preparing the above table, calculations of \( I_{m}^{K_{m}^{m}} \) were made to five decimal places using tables in reference 3 and the appropriate forms and modifications of (A.1-2) through (A.1-5). Calculations were carried to a large enough argument to insure that the maximum error had been obtained. No calculations were made for \( m \) greater than \( h \), but in view of the character of the separate asymptotic forms and the functions, and the trend exhibited in Table A-III, it seems quite safe to say that the representation given by (29) is much better than one percent for all arguments for \( m \) greater than \( h \). For \( m \) equal to zero (29) is an unsatisfactory approximation unless the argument is relatively large; only for arguments larger than about 5 is the error less than one percent.

References 42 and 44 give an extensive listing of tables of modified Bessel functions which are available or in preparation at the present time.

### Integral Expressions

For the \( I_{m}(z) \) function, starting with (A.1-1), multiplying through by \( \frac{dz}{dz_{m}} \) and integrating, one obtains

\[
\int_{0}^{z} z I_{m}^{2}(z) dz = \frac{1}{2} (z^{2-2m}) I_{m}^{2}(z) - \frac{z}{2} I_{m}^{2}(z) = \frac{z}{2} \left[ I_{m}^{2}(z) - I_{m+1}^{2}(z) \right] - mzI_{m}(z)I_{m+1}(z),
\]

where the second form is obtained from the first using (A.1-5).
Similarly, for the $K_m(z)$ function, starting with (A.1-1), multiplying through by $\frac{dK_m}{dz}$, and integrating, one obtains

$$\int_z^\infty zK_m^2(z)dz = -\frac{1}{2}(z^2 + m^2)K_m^2(z) + \frac{2}{2} K_m(z) = \frac{z^2}{2}[K_{m+1}^2(z) - K_m^2(z)] - mK_m(z)K_{m+1}(z).$$

(2)

The above integrals are valid for all positive integer $m$ including $m = 0$. These integrals are also given on page 70 of reference 46, although the method of derivation there is somewhat different from that indicated above.

The following integrals may be determined from inspection:

$$\int_0^z I_m(z)I'_m(z)dz = \frac{1}{2}\int_0^z \frac{d}{dz} I_m^2(z)dz = \frac{1}{2} I_m^2(z), \quad m \geq 1,$$

(3a)

whereas

$$\int_0^z I_0(z)I'_0(z)dz = \frac{1}{2}\int_0^z \frac{d}{dz} I_0^2(z)dz = \frac{1}{2}[I_0^2(z) - 1],$$

(3b)

and

$$\int_z^\infty K_m(z)K'_m(z)dz = \frac{1}{2}\int_z^\infty \frac{d}{dz} K_m^2(z)dz = -\frac{1}{2} K_m^2(z), \quad m \geq 0.$$  

(4)

From the above equations and the recurrence forms given in (A.1-2) through (A.1-6) the following integral may be obtained:

$$\int_0^z \left[ I_m^2(z) + \frac{m^2}{z^2} I_m^2(z) \right] zdz = \frac{m}{2} I_m^2(z) + \frac{3}{4} \int_0^z zI_{m+1}^2(z)dz + \frac{1}{4} \int_0^z zI_{m-1}^2(z)dz =$$

$$= mI_m^2(z) + \frac{2}{2} \left[ I_{m+1}^2(z) - I_m^2(z) \right] + (m+1)zI_m(z)I_{m+1}(z) =$$

$$= zI_m(z)I'_m(z) + \frac{z^2}{2} I_m^2(z) - \frac{1}{2}(z^2 + m^2)I_m^2(z).$$

(5)

Similarly, for the $K_m(z)$ function,
\[
\int_{z}^{\infty} \left[ K_{m}^{2}(z) + \frac{m^2}{z^2} K_{m+n}^{2}(z) \right] z dz = -\frac{m^2}{z^2} K_{m}(z) + \frac{3}{4} \int_{z}^{\infty} z K_{m+1}^{2}(z) dz + \frac{1}{4} \int_{z}^{\infty} z K_{m-1}^{2}(z) dz = \\
- m K_{m}^{2}(z) + \frac{3}{2} \left[ K_{m}^{2}(z) - K_{m+1}^{2}(z) \right] + (m+1) z K_{m}(z) K_{m+1}(z) = \\
- z K_{m}(z) K_{m}'(z) + \frac{3}{2} K_{m}^{2}(z) + \frac{1}{2} (z^2 m^2) K_{m}^{2}(z).
\]

(5) and (6) are valid for all positive integer \(m\) including \(m = 0\).

It will be noted that the integrand in (5) and (6) is the sum of two quantities. Not only does this form naturally arise in the evaluation of the power flow equation of the helix, but also it results in the cancellation of a finite series which occurs if each term is evaluated separately since the series comes in with a positive sign from one but with a negative sign from the other. However, this finite sum and its origin have some intrinsic interest and are briefly mentioned here.

From (A.1-5) and (A.1-3)
\[
\int_{0}^{z} \frac{I_{m}(z)}{z} dz = \frac{1}{m} \int_{0}^{z} I_{m}(z) I_{m}(z) - \frac{1}{m} \int_{0}^{z} I_{m}(z) I_{m+1}(z) dz = \\
- \frac{I_{m}^{2}(z)}{2m} - \frac{1}{m} \int_{0}^{z} I_{m}(z) \left[ 2I_{m}(z) - I_{m+1}(z) \right] dz.
\]

Evaluating the first part of the last integral as before, and noting that the last part is the same as before but reduced in order, one obtains finally after successive substitution and integration
\[
\int_{0}^{z} \frac{I_{m}^{2}(z)}{z} dz = \frac{I_{m}^{2}(z)}{2m} - \frac{1}{m} \left\{ \sum_{n=0}^{m} (-1)^{m-n} \left( \frac{2-5}{2} \right) n I_{n}^{2}(z) + \frac{1}{z} (-1)^{m+1} \right\},
\]

where
\[
5_{\text{on}} = \begin{cases} 
1 & n=0 \\
0 & n \neq 0 
\end{cases}
\]

Similarly, proceeding as above, one obtains
\[
\int_{z}^{\infty} \frac{K_{m}(z)}{z} dz = - \frac{K_{m}^{2}(z)}{2m} + \frac{m}{m} \left\{ \sum_{n=0}^{m} (-1)^{m-n} \left( \frac{2-5}{2} \right) n K_{n}^{2}(z) \right\}.
\]
(8) and (10) are valid for all positive integer \( m \) excluding \( m = 0 \). These formulae are of some interest since they enable the integrals to be evaluated, at least for integer \( m \), in a simple manner. Reference 46, page 71, gives solutions for these integrals which are valid for all \( m \), noninteger as well as integer, but they involve derivatives with respect to the order. However, it should be noted that reference 3, page 137, gives a formula related to (8) and (10) above for Bessel functions of the type discussed in Section A.14.

**Bessel Functions**

### A.14 Differential Equation; Recurrence Formulae; Wronskian; Relationship to Modified Bessel Functions

The Bessel functions \( J_m(z) \), \( N_m(z) \), \( H_m^{(1)}(z) \), and \( H_m^{(2)}(z) \) of order \( m \) and argument \( z \) are solutions of the differential equation

\[
z \frac{d}{dz} \left( z \frac{d}{dz} y \right) + (z^2 - m^2)y = 0,
\]

(1)

where \( y \) is either \( J_m(z) \), \( N_m(z) \), \( H_m^{(1)}(z) \), or \( H_m^{(2)}(z) \). Any two of these may be taken as the two independent solutions of (1). All of these functions satisfy the same recurrence formulae

\[
\begin{align*}
z_{m-1} + z_{m+1} &= \frac{2m}{z} z_m, \\
z_{m-1} - z_{m+1} &= 2z'_m, \\
z_m z'_m + m z_m &= z_{m-1}, \\
z_m z'_m - m z_m &= -z_{m+1}.
\end{align*}
\]

(2) \hspace{1cm} (3) \hspace{1cm} (4) \hspace{1cm} (5)

From the above one obtains

\[
\begin{align*}
z'_o &= -z_1, \\
z'_m &= (-1)^m z_m.
\end{align*}
\]

(6) \hspace{1cm} (7)

Here \( z_m(z) \) represents any of the Bessel functions of order \( m \) and argument \( z \), \( J_m(z) \), \( N_m(z) \), \( H_m^{(1)}(z) \), or \( H_m^{(2)}(z) \). \( H_m^{(1)}(z) \) and \( H_m^{(2)}(z) \) are linear.
combinations of \( N_m(z) \) and \( J_m(z) \) and are defined by

\[
H_m^{(1)}(z) = J_m(z) + j N_m(z) , \tag{8}
\]

\[
H_m^{(2)}(z) = J_m(z) - j N_m(z) , \tag{9}
\]

where \( j \) is as usual \( \sqrt{-1} \).

The Wronskian gives the relationship between the independent solutions, and for \( J_m(z) \) and \( N_m(z) \) this is

\[
\psi(J_m, N_m) \equiv J_m N'_m - N_m J'_m = \frac{2}{\pi z} . \tag{10}
\]

From (8), (9), and (10) the Wronskian of the other solutions can readily be obtained. The ordinary Bessel functions are related to the modified Bessel functions described in Sections A.1 and A.2 in the following manner:

\[
I_m(z) = e^{-j \frac{\pi m}{2}} J_m(jz) , \tag{11}
\]

\[
J_m(z) = e^{-j \frac{\pi m}{2}} I_m(jz) , \tag{12}
\]

\[
K_m(z) = j \frac{\pi}{2} e^{j \frac{\pi m}{2}} H_m^{(1)}(jz) = j \frac{\pi}{2} e^{-j \frac{\pi m}{2}} H_m^{(1)}(jz) , \tag{13}
\]

\[
H_m^{(2)}(z) = j \frac{\pi}{2} e^{j \frac{\pi m}{2}} K_m(jz) . \tag{14}
\]

It is also of interest to note that

\[
J_m(z e^{j\pi}) = e^{jm\pi} J_m(z) , \tag{15}
\]

\[
H_m^{(1)}(z e^{j\pi}) = - e^{-jm\pi} H_m^{(2)}(z) , \tag{16}
\]

\[
H_m^{(2)}(z e^{-j\pi}) = - e^{jm\pi} H_m^{(1)}(z) . \tag{17}
\]

In the formulae given so far in this appendix, \( m \) has been in general restricted to real positive integers, including zero, and \( z \) has been confined to real positive values, although many of the relationships are true for less restrictive values of \( m \) and \( z \). For example, (11) through (17) are valid for \( z \) complex if \( -\frac{\pi}{2} < \arg z < \frac{\pi}{2} \) and \( m \) integer. Actually, even many of these formulae are valid for less restrictive values of \( m \)
and \( z \), but these limits are adequate for the intended application.

Many additional useful formulae including the various series expansions and asymptotic forms are given in reference 3, Chapter III. (Note the use of \( Y_m(z) \) there instead of \( N_m(z) \) as above.)

A.5 Infinite Integrals; Addition Formulae

On page 429 of reference 3 the following infinite integral is given as a special case of a more general result:

\[
\int_0^\infty \frac{X}{X^2+t^2} J_\mu(bX) J_\mu(aX) dX = I_{\mu}(bt)K_{\mu}(bt) .
\]  

(1)

In this it is required that \( a > b > 0 \) and that the real part of \( t \) be greater than zero. This is given as equation (5) in the above reference with \( \mu = \nu \), (\( \mu \) need not be integer for (1) to be valid but only integer values of \( \mu \) are used here) and a slight change of notation has been made. The derivation of (1) is given in detail in the reference and will not be repeated here; it is accomplished through the use of contour integration.

An infinite integral of considerable use in radiation problems is the following:

\[
e^{-jkR/R} = \int_{C_1} J_0(\lambda r) e^{-|z|\sqrt{X^2-k^2}} \frac{X dX}{\sqrt{X^2-k^2}} ,
\]

(2)

where

\[
R = \sqrt{r^2+z^2} ,
\]

(3)

and the contour \( C_1 \) is the positive real axis indented above \( X = k \) as in Fig. A-1. \( k \) is considered as the limiting point of a complex number in the fourth quadrant whose imaginary part approaches zero. In using (2) the sign of \( \sqrt{X^2-k^2} \) is chosen so that its real part is zero or positive. With this (2) represents outgoing waves if \( e^{j\omega t} \) is used as the time varying factor.

Still another form of (2) is often of use. One can show that
The following addition formulae are often useful:

\[ J_0 \left( \sqrt{r^2+a^2-2ra \cos \theta} \right) = \sum_{m=-\infty}^{\infty} J_m (\chi r) J_m (\chi a) e^{im\theta} \]  

(7)
$$K_0(x \sqrt{r^2+a^2-2ra \cos \theta}) = \sum_{m=-\infty}^{\infty} K_m(xr)I_m(xa) e^{jm\theta} .$$

(8)

If one puts $x = 1$, $\theta = \pi$, $r = x$, and $a = y$ in (7), then

$$J_0(x+y) = J_0(x)J_0(y) + 2 \sum_{m=1}^{\infty} (-1)^m J_m(x)J_m(y) .$$

(9)

(7), (8), and (9) are special cases of more general formulae discussed in Chapter XI of reference 3.

Trigonometric and Bessel Function Sums

A.6 Trigonometric Sums

As in reference 36 trigonometric sums of order $n$ are defined as follows:

$$S_n(x) = \sum_{m=1}^{\infty} \frac{\sin mx}{m^n} ,$$

(1)

$$C_n(x) = \sum_{m=1}^{\infty} \frac{\cos mx}{m^n} .$$

(2)

$S_1(x)$ and $C_1(x)$ are well-known sums and are given by

$$S_1(x) = \sum_{m=1}^{\infty} \frac{\sin mx}{m} = \frac{x^n}{2^n} , \quad 2\pi > x > 0 ,$$

(3)

(see reference 1, equation 115.08), and

$$C_1(x) = \sum_{m=1}^{\infty} \frac{\cos mx}{m} = - \ln \left[ 2 \sin \frac{x}{2} \right] , \quad 2\pi > x > 0 ,$$

(4)

(see reference 1, equation 603.2).

It will be noted that $\frac{dS_1(x)}{dx} = C_1(x)$; thus, expanding $- \ln [2 \sin \frac{x}{2}]$ in a power series about $x = 0$ (see reference 1, equation 603.1) and integrating term by term, one obtains

$$S_2(x) = \sum_{m=1}^{\infty} \frac{\sin mx}{m^2} = \int_0^x C_1(x)dx = x \ln \frac{x}{2} + \frac{1}{72} x^3 + \frac{1}{114,400} x^5 + \frac{1}{1,270,080} x^7 + \ldots$$

(5)

for $2\pi > x > 0$.

The expansion of $S_2(x)$ about $\pi$ is also useful and can be shown to be

$$S_2(\pi+x) = - x \ln 2 + \frac{x^3}{24} + \frac{x^5}{960} + \frac{x^7}{20,160} + \ldots$$

(6)
It will now be noted that

\[
\frac{d}{dx} S_n(x) = C_{n-1}(x) , \quad (7) \quad \frac{d}{dx} C_n(x) = - S_{n-1}(x) , \quad (8)
\]

both for \(n \geq 2\). Therefore, from (3) and (4) all the \(C_n(x)\) and \(S_n(x)\) can be readily determined. The odd order \(C_n(x)\) (with the exception of \(C_1(x)\)) and the even order \(S_n(x)\) are given by infinite series, whereas the even order \(C_n(x)\) and the odd order \(S_n(x)\) are given by finite sums. The sums

\[
\sum_{n=1}^{\infty} \frac{1}{n} \] which occur in these functions are tabulated in reference 47 (page 280 in Vol. I and page 244 in Vol. II) for \(100 \geq n \geq 2\); for \(n\) even, these sums are related to the Bernoulli numbers and may be expressed in fractions of various even powers of \(\pi\). (See reference 4, equations 45 and 47.1.) Several of the \(C_n(x)\) and \(S_n(x)\) or related functions are given in reference 2, page 53.

A.7 **Bessel Function Sums**

Series of the form

\[
\sum_{m=1}^{\infty} a_m J_n(mx)
\]

are known as Schlömilch series and are discussed in considerable detail in Chapter XIX of reference 3. A method for evaluating such series is indicated there and is used for some particular cases below. (See also reference 36.)

Starting with (A.7-4), calling this \(g(z)\) here so that

\[
g(z) = \sum_{m=1}^{\infty} \frac{\cos mz}{m} = - \ln \left[ 2 \sin \frac{z}{2} \right], \quad (2)
\]

putting \(z = x \sin \theta\), and integrating with respect to \(\theta\) from 0 to \(\frac{\pi}{2}\), one obtains

\[
\int_0^{\frac{\pi}{2}} g(x \sin \theta) d\theta = \sum_{m=1}^{\infty} \frac{1}{m} \int_0^{\pi} \cos(mx \sin \theta) d\theta = \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{J_m(mx)}{m}, \quad (3)
\]
since
\[ J_0(z) = \frac{2}{\pi} \int_0^{\pi/2} \cos(\theta \sin \phi) d\theta. \tag{4} \]

(See reference 3, page 21.) From (2) and (3)
\[ \int_0^{\pi} g(x \sin \theta) d\theta = -\int_0^{\pi} \ln \left[ 2 \sin \left( \frac{x \sin \theta}{2} \right) \right] d\theta = \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{J_m(mx)}{m}. \tag{5} \]

Expanding the integrand \( \ln \left[ 2 \sin \left( \frac{x \sin \theta}{2} \right) \right] \),
\[ \ln \left[ 2 \sin \left( \frac{x \sin \theta}{2} \right) \right] = \ln 2 + \ln \sin \left( \frac{x \sin \theta}{2} \right) = \]
\[ \ln 2 + \ln \frac{x \sin \theta}{2} - \frac{\left( \frac{x \sin \theta}{2} \right)^2}{6} - \frac{\left( \frac{x \sin \theta}{2} \right)^4}{120} - \frac{\left( \frac{x \sin \theta}{2} \right)^6}{2835} + \ldots, \tag{6} \]

where the expansion \( \ln \sin u \) is given in reference 4, equation 603.1, and is obtained by integrating the expansion of \( \cot u \), one has finally
\[ \int_0^{\pi} g(x \sin \theta) d\theta = -\int_0^{\pi} \left[ \ln x + \ln \sin \theta - \frac{x^2}{2n} \sin^2 \theta - \frac{1}{2n} \sin^4 \theta - \frac{x^6}{15n^3} \sin^6 \theta + \ldots \right] d\theta. \tag{7} \]

Now,
\[ \int_0^{\pi} \ln \sin \theta d\theta = -\frac{\pi}{2} \ln 2, \tag{8} \]
and
\[ \int_0^{\pi} \sin^{2n} \theta d\theta = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \quad n \geq 1, \tag{9} \]

(see reference 4, equations 868.1 and 854.1, respectively), so, finally,
\[ \sum_{m=1}^{\infty} \frac{J_m(mx)}{m} = \ln \frac{2}{x} + \frac{x^2}{48} + \frac{x^4}{7680} + \frac{x^6}{580,625} + \ldots \quad x > 0. \tag{10} \]

Proceeding in a similar way, using \( g(z) = C_0(z) = \sum_{m=1}^{\infty} \frac{\cos \pi z}{m^2} = \frac{\pi^2}{6} - \frac{z^2}{4}(2\pi - z) \),
one may obtain
\[ \sum_{m=1}^{\infty} \frac{J_m(mx)}{m^2} = \frac{\pi^2}{6} - x + \frac{x^2}{6}. \tag{11} \]

Another useful sum may be obtained in the following manner: in (10) let
\( x = 2y \sin \varphi \); then integrating both sides with respect to \( \varphi \) from 0 to \( \pi \),
using (8), (9), and the relationship

\[ J_0^2(my) = \frac{1}{\pi} \int_0^\infty J_0(2my \sin \phi) d\phi \quad (12) \]

(see reference 3, page 32), one obtains

\[ \sum_{m=1}^\infty \frac{J_0^2(my)}{m} = \ln \frac{2}{y} + \frac{\gamma^2}{24} + \frac{\gamma^4}{1280} + \frac{\gamma^6}{151200} + \cdots, \quad \frac{\pi}{2} > y > 0. \quad (13) \]

Missellaneous

### A.8 Integrals

In obtaining an approximation to some infinite sums encountered in Chapter III of this report, the following real integrals are useful:

\[ \int \frac{dx}{(ax^2 + bx + c)^{1 \over 2}} = \frac{1}{a} \ln \left[ \frac{1}{2a} (ax^2 + bx + c)^{1 \over 2} + 2ax + b \right], \quad (1) \]

and

\[ \int \frac{x^2 dx}{(ax^2 + bx + c)^{1 \over 2}} = \left( \frac{x}{2a} - \frac{3b}{4a^2} \right) \left( ax^2 + bx + c \right)^{1 \over 2} + \frac{3b^2 - 4ac}{8a^2} \int \frac{dx}{(ax^2 + bx + c)^{1 \over 2}}, \quad (2) \]

where it is required that \( a > 0 \) and \( (ax^2 + bx + c) > 0 \) over the range of integration. These are relatively simple integrals which are obtained by completing the square of the quantity inside \( ( \cdot)^{1 \over 2} \) in the denominator of the integrands and then using, after a change of variable, the appropriate one of the following elementary forms:

\[ \int \frac{du}{(u^2 + t^2)^{1 \over 2}} = \ln \left[ u + (u^2 + t^2)^{1 \over 2} \right], \quad (3) \]

\[ \int \frac{udu}{(u^2 + t^2)^{1 \over 2}} = \frac{1}{2} (u^2 + t^2)^{1 \over 2} \quad (4) \]

and

\[ \int \frac{u^2 du}{(u^2 + t^2)^{1 \over 2}} = \frac{u}{2} (u^2 + t^2)^{1 \over 2} - \frac{t^2}{2} \ln \left[ u + (u^2 + t^2)^{1 \over 2} \right]. \quad (5) \]
(See reference 4, equations 200.01 and 202.01.) (5) can be obtained from (3) by some alterations in form and integration by parts. Reference 4, page 70, gives (1) and (2) and many other similar integrals.

Another useful integral is

\[ \int_{-\infty}^{\infty} e^{-j\xi t} e^{-z|\xi|} d\xi = \frac{2z}{t^2 + z^2}, \]

which is obtained by separating the range of integration and performing an elementary integration. It is considered real here, whereas \( z \) may be complex, but it is necessary that Re\( (z) > 0 \).
APPENDIX B

SHEATH HELIX DETERMINANTAL EQUATION

In the ensuing discussion the sheath helix determinantal equation derived in Section II.4 is considered in detail. Following an investigation of the properties of the solution, in particular the characteristics of the propagation constant $h$, the special and relatively simple cases of $\psi = 0^\circ$ and $\psi = 90^\circ$ are analyzed; after which the general case is examined.

Properties of the Solution

B.1 Characteristics of the Propagation Constant $h$

As shown in Section II.4, the determinantal equation for the sheath helix is

$$\frac{I_n'(\xi a)K_n'(\xi a)}{I_n(\xi a)K_n(\xi a)} = -\frac{(\xi^2a^2 + nha \cot \psi)^2}{k^2a^2\xi^2a^2 \cot^2 \psi}, \quad (\text{II.4-8}), \quad (1)$$

where

$$\xi = \sqrt{h^2 - k^2} \quad . \quad (2)$$

Since the system is lossless, $k$ is real. If $h$ is real and in magnitude larger than $k$, then $\xi$ is real, and the right side of (1) is real and negative. The left side is real and negative also, and a solution may therefore exist. If $h$ is real and in magnitude less than $k$, then $\xi$ is pure imaginary, $+j|\xi|$, and $\xi^2$ is real but negative. In this case the right side of (1) is real and positive, whereas the left side becomes
\[
\frac{J_n'(\xi |a|)H_n^{(2)}(\xi |a|)}{J_n(\xi |a|)H_n^{(2)}(\xi |a|)}
\]  \hspace{2cm} (3)

(See Section A.4.) The imaginary part of this is proportional to the Wronskian \( W(J_n, N_n) \), and since the right side of (1) is pure real, this Wronskian must be zero for a solution to exist. But this is impossible, so \( h \) real and less than \( k \) in magnitude is not possible. This approach was used in Appendix D of reference 39 for determining the characteristics of the roots in a dielectric rod problem. It was also used in reference 19 in the sheath helix case in a manner essentially identical to the above. It should be noted that the above proof applies directly for all \( n \) with \( 90^\circ > \psi > 0^\circ \), and for \( \psi = 0^\circ \) with \( n \neq 0 \). In this last case the determinantal equation takes the form shown in Section B.2.

The remaining possibilities for \( h \) may be divided into two cases, \( h \) complex with nonzero real and imaginary parts, and \( h \) pure imaginary. If \( h \) is pure imaginary so that \( h = \pm j|h| \), \( \xi^2 \) is pure real and negative, and again \( \xi = \pm j|\xi| \). For the special cases of \( n = 0, \psi \neq 0^\circ \) or \( 90^\circ \), and \( n \neq 0, \psi = 0^\circ \), the right side of (1) is still pure real. But the left side is again like (3), and if a solution is to exist, \( W(J_n, N_n) \) must be zero. This is not possible so that in these special cases the impossibility of \( h \) being pure imaginary is proved directly.

In order to derive some results concerning the characteristics of \( h \) for nonspecial values of \( n \) and \( \psi \), one proceeds in the following manner. Since the complex Maxwell field equations (II.3-1) and (II.3-2) are merely the Fourier transforms of the time-dependent Maxwell equations, it can be proved directly that

\[
\bar{E}(\omega) = \hat{E}(\omega)
\]  \hspace{2cm} (4)

where the tilde means complex conjugate, and the functional dependence on \( \omega \) is emphasized.\(^{39}\) This relationship holds for all the complex field
vectors and must be true for all \( r, \theta, \) and \( z \). For the sheath helix for \( n \neq 0 \) and \( \psi \neq 0^\circ \) or \( 90^\circ \) it can be shown that the required dependence of the fields on \( \theta \) is like \( e^{-jn\theta} \); it cannot be \( \sin \theta \) or \( \cos \theta \) alone since the boundary conditions cannot be satisfied for these. The complex electric field vector, for example, must therefore be written as

\[
\mathbf{E}_n(\omega)(r, \theta, z, \omega) = \mathbf{E}_n(\omega)(r, \omega)e^{-jn(\omega)\theta},
\]

where the \( \theta \) and \( z \) dependence are separated out, and where the functional dependence of \( h \) and \( n \) on \( \omega \) is emphasized. \( h \) is also a function of \( n \), and \( n \) must be an integer and real. Now,

\[
\mathbf{E}_n(-\omega)(r, \theta, z, -\omega) = \mathbf{E}_n(-\omega)(r, -\omega)e^{-jn(-\omega)\theta},
\]

and

\[
\mathbf{E}_n(\omega)(r, \theta, z, \omega) = \mathbf{E}_n(\omega)(r, \omega)e^{jn(\omega)\theta}.
\]

Since (4) must be true for all \( \theta \) and \( z \), it follows from (6) and (7) for \( \theta = z = 0 \), using (4), that

\[
\mathbf{E}_n(-\omega)(r, -\omega) = \mathbf{E}_n(\omega)(r, \omega).
\]

From (4), (6), (7), and (8) for \( \theta = 0, \ z \neq 0 \),

\[
\tilde{h}_n(\omega)(\omega) = -h_n(-\omega)(-\omega),
\]

and from the same equations for \( \theta \neq 0, \ z = 0 \),

\[
n(-\omega) = -n(\omega).
\]

An examination of the determinantal equation (1) shows that if \( \omega \) is replaced by \( -\omega \), the equation remains unchanged — note that \( I_n = I_{-n} \) and \( K_n = K_{-n} \) — and, consequently, the solutions must be the same for

\[
n(\omega)h_n(\omega)(\omega) = n(-\omega)h_n(-\omega)(-\omega).
\]

Using (10) and (11) results in

\[
h_n(\omega)(\omega) = -h_n(-\omega)(-\omega).
\]
But (9) and (12) together give

\[ \tilde{h}_n(\omega) = h_n(\omega) \]  \hspace{1cm} (13)

From (13) it is clear that at least for nonspecial values of \( n \) and \( \psi \), the \( h \) roots of the determinantal equation must be pure real; they cannot be complex or pure imaginary.

It can be seen that the determinantal equation remains unchanged for a given \( \omega \) if the sign of \( h \) and \( n \) are changed simultaneously. This coincides with the physically apparent requirement that if a wave like \( e^{-jhz}e^{-jn\theta} \) is possible, then a wave like \( e^{+jhz}e^{+jn\theta} \) should likewise be a solution. Note that the proof for the realness of \( h \) is not dependent on whether more than one solution exists for a given \( n \). If this is the case, the proof is unaltered when the roots are ordered in the proper manner.

If \( n = 0 \), \( \psi \neq 0^\circ \) or \( 90^\circ \), (9) still applies so that

\[ \tilde{h}_0(\omega) = -h_0(-\omega) \]  \hspace{1cm} (14)

Examination of the determinantal equation shows that it remains unchanged for

\[ h_o^2(\omega) = h_o^2(-\omega) \]  \hspace{1cm} (15)

Thus,

\[ h_o(-\omega) = \pm h_o(\omega) \]  \hspace{1cm} (16)

or, using (14),

\[ \tilde{h}_o(\omega) = \pm h_o(\omega) \]  \hspace{1cm} (17)

From this it is concluded that the \( h \) roots for \( n = 0 \) and \( \psi \neq 0^\circ \) or \( 90^\circ \) may be pure real or pure imaginary but not complex. However, it has already been shown that \( h \) cannot be pure imaginary so that here \( h \) must also be pure real.

If \( n \neq 0 \) but \( \psi = 0^\circ \), the determinantal equation becomes a function of \( n^2 \) alone, and linearly polarized fields are possible. Nevertheless, (9) still applies, and it is necessary that
\[ \overline{h}_{|n|}(\omega) = - h_{|n|}(-\omega) \]  
(18)

The determinantal equation remains unchanged for
\[ h_{|n|}^2(\omega) = h_{|n|}^2(-\omega) \]  
(19)
or
\[ h_{|n|}(-\omega) = \pm h_{|n|}(\omega) \]  
(20)

Using (20) and (18) results in
\[ \overline{h}_{|n|}(\omega) = \pm h_{|n|}(\omega) \]  
(21)

Thus, \( h \) for \( n \neq 0 \) but \( \psi = 0^\circ \) may be pure real or pure imaginary but not complex. However, it has already been proved that for \( \psi = 0^\circ \) and \( n \neq 0 \), \( h \) cannot be pure imaginary. In this case also, \( h \) must be pure real.

An examination of the above shows that it has been proved that for all \( n \), including \( n = 0 \), the \( h \) roots must be real and in magnitude larger than \( k \) for all \( \psi \) except \( \psi = 0^\circ \) and \( 90^\circ \). Further, for \( n \neq 0 \) and \( \psi = 0^\circ \) the same restrictions on \( h \) apply. The only cases not yet covered are those for \( \psi = 90^\circ \) for all \( n \) and the one for \( \psi = 0^\circ, n = 0 \). Although these cases might be considered as limiting ones of the general ones, some precautions must be noted. In particular, it should be pointed out that in these special cases, ordinary waveguide modes can exist since the sheath system boundary conditions can be met with \( E_\theta^1 = E_z^1 = 0 \).

Some consideration of the boundary conditions shows that the \( TE_{om} \) modes can exist for \( \psi = 0^\circ \), whereas the \( TM_{nm} \) modes can exist for \( \psi = 90^\circ \). In these cases the fields are nonzero only for \( r < a \), and \( h \) is real and in magnitude less than \( k \), or it is imaginary. This matter is discussed somewhat further in the following and in Chapter II.

The conclusion which is drawn from the above discussion is that in general for all \( \psi \) and \( n \) the determinantal equation has roots only for real values of \( h \) which are larger in magnitude than \( k \), and that no other roots need be looked for since, indeed, they cannot exist.
The writer would like to thank Professor R. B. Adler for pointing out how some of the proofs which he has given in reference 39 concerning the properties of propagation constants could be applied to the sheath helix problem.

Special Cases

B.2 The Sheath Ring, $\psi = 0^\circ$

For $\psi = 0^\circ$ the determinantal equation is

$$\frac{I_n(\zeta a)K_n'\zeta a) - \frac{n^2 h a^2}{2 a^2 \zeta^2 a^2}}{I_n \zeta a)K_n(\zeta a)} = \frac{n^2 h a^2}{2 a^2 \zeta^2 a^2} \ldots (II.5-1)$$  

As noted in Section II.5(b), this can be obtained as the limiting value of the general determinantal equation for the sheath helix for $\psi$ approaching and finally equaling zero, or it can be obtained by solving the Maxwell equations again, subject to the boundary conditions for this case which are given by (II.5-2) through (II.5-4). It can be seen that (1) is an even function of $h$ so that for every positive $h$ root a negative $h$ root of the same magnitude exists.

For $n = 0$ (1) becomes

$$\frac{I_0(\zeta a)K_1(\zeta a)}{I_0(\zeta a)K_0(\zeta a)} = 0$$ 

Considering the left side of (2) as a function of real $\zeta a$, one readily finds that it can be zero only for $\zeta a = 0$. Consequently, one possible solution might be $h = n k$, and the fields would be those for a TEM wave, that is, with $E_x = H_z = 0$. However, such a solution clearly cannot satisfy the boundary conditions since, if any current flows on the sheath ring at all, $H_z$ must exist. It is evident that (2) can be satisfied for $I_1(\zeta a) = 0$ or for imaginary values of $\zeta$. This corresponds to the TE_{om} modes in the interior of a perfectly conducting wall circular waveguide. For these, $E_z = 0$ identically, and there is no $a$ component of current.
density on the wall so that such modes can exist inside a sheath ring.

For \(|n| \geq 1\), using \(h^2 = \zeta^2 + k^2\) and \(\zeta a = z\) for convenience where \(z\) should not be confused with the coordinate axis, (1) can be rearranged to obtain

\[
n^2 + \frac{k^2a^2}{z^2} \left[ n^2 + \frac{z^2}{I_n(z)K_n'(z)} \frac{I_n'(z)K_n'(z)}{I_n(z)K_n(z)} \right] = 0 \equiv f_n(z),
\]

which defines \(f_n(z)\). A function \(g_n(z)\) is also defined as

\[
\frac{1}{z^2} \left[ n^2 + \frac{z^2}{I_n(z)K_n'(z)} \frac{I_n'(z)K_n'(z)}{I_n(z)K_n(z)} \right] = g_n(z),
\]

so that (3) becomes

\[
n^2 + k^2a^2g_n(z) = f_n(z) = 0.
\]

Note that \(f_n(z)\) and \(g_n(z)\) are even functions of \(n\). For a given \(n\) and \(ka\), the zeros of \(f_n(z)\) are the required values of \(z\) or \(\zeta a\) from which the values of \(h\) which satisfy the determinantal equation are obtained.

As shown in Section B.1, it is necessary that \(h\) be real and \(|h| > k\) so that it is only necessary to examine (3) or (5) for \(z\) real and \(z > 0\).

\(f_n(z)\) is a rather complicated transcendental equation, and, in general, its roots can be found only by graphical or successive approximation procedures. If the approximations (A.2-22) and (A.2-23) for \(I_nK_n\) and \(I_n'K_n'\) are used in (5), it is readily found that solutions exist only for \(|n| = ka\) independent of \(h\). Although this turns out to be a rather good approximation, particularly for \(|n| \geq 2\), more exact solutions are desired.

In order to find the zeros of \(f_n(z)\) the function \(g_n(z)\) must be investigated. It is found that \(g_1(z)\) and \(g_2(z)\) vary with \(z\) in the manner shown in Fig. B-1. \(g_n(z)\) for \(n \geq 3\) is not shown since it has a character which is very similar to \(g_2(z)\). Using the series expansions given in Section A.2, one can show that for \(z\) small
\[ g_1(z) = \ln \frac{ze^{\gamma}}{2} + \ldots, \quad (6) \]

\[ g_2(z) = -\frac{1}{3} - \frac{z^2}{2}(\ln \frac{ze^{\gamma}}{2} + \frac{11}{72}) + \ldots, \quad (7) \]

\[ g_n(z) = -\frac{n^2}{n^2-1} + \frac{5n^2-2}{2(n^2-1)^2(n^2-4)} z^2 + \ldots, \quad n \geq 3, \quad (8) \]

where \( \gamma \) is given by (A.2-6). (6), (7), and (8) are the dominant terms in \( g_n(z) \) for \( z \) near zero. For \( z \) large and much larger than \( n \) it can be shown from the asymptotic forms that

\[ g_n(z) \approx -1 + \frac{1}{2z^2} \quad (9) \]

for all \( n \geq 1 \). As can be seen in Fig. B-1a, \( g_1(z) \) increases monotonically as \( z \) increases for small \( z \), attains a maximum of \(-0.969\) for \( z \) near 3, and then decreases monotonically approaching the \(-1\) asymptote from above as \( z \) becomes increasingly large. The \( g_2(z) \) function varies in a somewhat similar manner except that it has a finite intercept for \( z = 0 \), and its maximum is closer to \(-1\) and occurs for a larger value of \( z \). Using (8) and (9) for \( z \) small and large, respectively, and calculating in the middle range, it is possible to show that \( g_n(z) \) is similar to \( g_2(z) \); and this has been done in detail for \( 7 \geq n \geq 3 \). For all of these, \( 7 \geq n \geq 3 \), starting at \(-\frac{n^2}{n^2-1}\) for \( z = 0 \), \( g_n(z) \) first increases monotonically for small \( z \); increases to a value which is somewhat less than \(-1\) and which approaches \(-1\) more closely as \( n \) increases; and then decreases monotonically approaching the \(-1\) asymptote from above as \( z \) becomes increasingly large. Unfortunately, it has not been possible to construct a general proof that \( g_n(z) \) as a function of \( z \) varies in this manner for all \( n \).

However, in view of the definitely known character of \( g_n(z) \) for all \( n \) near \( z = 0 \) and \( \infty \), the general properties of the modified Bessel functions and their derivatives, and the known character of \( g_n(z) \) for \( 7 \geq n \geq 1 \), it seems quite certain that \( g_n(z) \) varies in a similar manner for all \( n \).
With the character of \( g_n(z) \) known, \( f_n(z) \) can be obtained and its zeros located. A convenient way of deriving \( f_n(z) \) from \( g_n(z) \) is to alter only the ordinate values of the \( g_n(z) \) curve by the scale factor \( k^2a^2 \) and then to translate this resulting curve an amount \( n^2 \) in the ordinate direction. Fig. B-2 shows how this scheme is applied for \( |n| \geq 2 \). The curves there are not to scale, and the effect of \( \Delta_n \), which is defined below, is exaggerated; however, they do show the qualitative behavior of \( f_n(z) \). The \( |n| = 1 \) case is discussed separately below. Some consideration shows that for a particular \( n \) there are four regions of interest as \( ka \) varies. These are shown in Fig. B-2. It can be seen that for \( \sqrt{n^2-1} > ka > 0 \), \( f_n(z) \) has no zeros; for \( |n| > ka > \sqrt{n^2-1} \), there is one zero; for \( |n| + \Delta_n > ka > |n| \) where \( \Delta_n \) is exceedingly small --- \( \Delta_2 \approx 0.01 \) and becomes smaller as \( |n| \) increases --- there are two zeros; whereas for \( ka > |n| + \Delta_n \), there are no zeros again. \( \Delta_n \) is defined as the maximum value by which \( ka \) may be increased from a value of \( |n| \) so that \( f_n(z) \) still has two zeros in the sense that for \( ka = |n| + \Delta_n \) the two zeros coalesce and become one. Of course, \( \Delta_n \) is even in \( n \). Note that for \( ka \) just barely larger than \( |n| \) the additional zero which appears in \( f_n(z) \) occurs for a very large value of \( z \). As \( ka \) increases, this zero approaches the first zero which is a continuation of the one which appears for \( ka > \sqrt{n^2-1} \) until the two coalesce and disappear. The reason for showing some portions of the curves as solid lines and other portions as dotted is to aid in tracing the root values of \( z \) or \( \xi a \) and \( ha \) which result from the respective parts of the curves. A comparison of Figs. II-4 and II-5 with Fig. B-2 should make this clear. The case for \( |n| = 1 \) is quite similar to the above except that since \( g_1(z) \) goes like \( \ln z \) for small \( z \), there is always one zero in \( f_1(z) \) for \( 1 > ka > 0 \). Note that the bounds obtained for the \( |n|^{th} \) mode now apply for \( |n| = 1 \) also. The curves in Fig. B-2b, c, d apply for \( |n| = 1 \) if they are assumed to have \( \ln z \)
dependence for small $z$ rather than a finite value there. Again, for $1 + \Delta_1 > k a > 1$, where $\Delta_1 \approx 0.015$, there are two zeros in $f_1(z)$ while for $k a > 1 + \Delta_1$ there are none.

Using the above analysis, one can locate the zeros of $f_n(z)$. The corresponding values of $\zeta a$ and then $k a$ can be determined as a function of $k a$ for a given $n$. Actually, the situation is somewhat more complicated than indicated above. A discussion of this matter is given in Sections II.5 and II.7. Figs. II-4 and II-5 show the results obtained from the foregoing analysis.

**B.3 The Sheath Tube, $\psi = 90^\circ$**

In this case the boundary conditions (II.2-1) through (II.2-7) become

$$E_i^i = E_i^e = E_z^i = E_z^e = 0, \quad (II.5-9), (1)$$

$$E_i^i = E_i^e = E_\theta^i = E_\theta^e, \quad (II.5-10), \quad (2)$$

$$H_i^i = H_i^e = H_z^i = H_z^e, \quad (II.5-11), \quad (3)$$

for $r = a, 2 a > \theta > 0$, and $-\infty > z > \infty$. Rather than consider the determinant equation (B.1-1) and its limiting value for $\psi = 90^\circ$, it seems best to rederive the necessary conditions from the beginning, using (II.3-18) through (II.3-21) in (II.3-7) through (II.3-12) to obtain the field components, and substituting these in (1), (2), and (3) above to impose the boundary conditions.

For $n = 0$ the field solutions, if any, become a set of TE and TM waves separately, or, possibly, a TEM wave. If a TE wave is assumed so that $E_z = H_\theta = 0$, it is necessary that

$$-\xi^2 B_{0}^{i1}(\xi a) = -\xi^2 B_{0}^{e1}(\xi a)$$

(4)

from (3), and

$$j \omega \xi B_{0}^{i1}(\xi a) = -j \omega \xi B_{0}^{e1}(\xi a)$$

(5)
from (2). If it is assumed that $\zeta \neq 0$ — this would be a TEM wave — and that $B^1_0$ and $B^e_0$ are not zero, (4) and (5) together require the Wronskian $\mathcal{W}(I_0, K_0)$ to be zero, which is impossible. In fact, it may be argued that if $H_\theta = 0$, no $z$ component of current density $K_z$, which is the only component allowed by the boundary conditions, can exist. Consequently, there is no solution in this case. If a TM wave is assumed, $H_z = E_\theta = 0$, and it is necessary that

$$-\zeta^2 A^i_0 I_0(\zeta a) = -\zeta^2 A^e_0 K_0(\zeta a) = 0$$

from (1). (6) cannot be satisfied, assuming $A^i_0$ and $A^e_0$ are not zero, for $\zeta > 0$ because of the characteristics of the modified Bessel functions. The only possibility which remains for $n = 0$ is that $\zeta = 0$, $h = \pm k$, and a TEM wave exists. This is a wave which is identical to the axially symmetric mode on the outside of a perfectly conducting infinite cylindrical wire; there is no current density in the $\theta$ direction in this case so that the boundary conditions correspond to (1) through (3) above.

(See reference 1, page 528.) However, as noted in reference 2, page 276, such a wave can never be set up "on a single wire except with an infinite amount of power properly supplied over an entire plane perpendicular to the wire." Although this mode is a solution to the Maxwell equations with the prescribed boundary conditions, it is not a "free mode" in the sense that it can be excited by any finite source.

For $|n| > 1$ the boundary conditions give

$$-\zeta^2 I_n(\zeta a)A^i_n = -\zeta^2 K_n(\zeta a)A^e_n = 0$$

from (1),

$$-\zeta^2 I_n(\zeta a)B^i_n = -\zeta^2 K_n(\zeta a)B^e_n$$

from (3), and

$$-\frac{n h}{a} I_n(\zeta a)A^i_n + j \omega \mu I_n'(\zeta a)B^i_n = -\frac{n h}{a} K_n(\zeta a)A^e_n + j \omega \mu K_n'(\zeta a)B^e_n$$

(9)
from (2). For \( \xi \neq 0 \), and \( A_n^i \) and \( A_n^e \) zero or not, (8) and (9) together require the Wronskian \( \mathcal{W}(I_n, K_n) \) to be zero, which is impossible. If \( B_n^i \) and \( B_n^e \) are zero with \( \xi \neq 0 \), (7) requires that \( A_n^i \) and \( A_n^e \) be zero since the characteristics of the modified Bessel functions prevent (7) from being satisfied in any other way. It may therefore be concluded from the above that if any finite solutions are possible for \( \psi = 90^\circ \) and \( |n| > 1 \), they can only be TEM waves with \( \xi = 0 \) and \( n = \pm k \). A set of solutions which satisfies the boundary conditions are the higher order, \( |n| > 1 \), symmetrical component waves, infinite in number in this limiting case, which exist on an infinite wire circular "cage" transmission line. It is evident that such waves require that only finite power be supplied to the system since their far field variation is like \( \frac{1}{r|n|+1} \). These waves are of interest since they happen to be the limit, as \( \psi \) approaches \( 90^\circ \), of a set of modes which occur on the sheath helix.

In addition to the above, where only real values of \( \xi \) are considered, solutions exist for \( \psi = 90^\circ \) for imaginary \( \xi \) corresponding to the TM\( _{n,m} \) modes in the interior of a hollow perfectly conducting wall circular waveguide. These are discussed briefly in Section II.7(f).

The General Case

B.4 The Sheath Helix

For \( \psi \neq 0^\circ \) or \( 90^\circ \) the determinantal equation, repeated here for convenience, is

\[
\frac{I_n'(\xi a)K_n'(-\xi a)}{I_n(\xi a)K_n(-\xi a)} = -\frac{(\xi^2 a^2 + nha \cot \psi)^2}{k^2 a^2 \xi^2 a^2 \cot^2 \psi}.
\]  

(B.1-1), (II.4-8)

As noted in Section II.5 and later here, no generality is lost if a positively wound helix is assumed; and in the following \( \psi \) is taken so that \( 90^\circ > \psi > 0^\circ \), \( 0 < \cot \psi < \infty \).
For $n = 0$ (B.1-1) becomes, with $\xi a = z$,

$$\frac{I_1(z)K_1(z)}{I_0(z)K_0(z)} = \frac{z^2}{k^2 a^2 \cot^2 \psi}.$$  \hspace{1cm} (II.5-28)  \hspace{1cm} (1)

The left side of (1) is a monotonically increasing function of $z$ which starts from $z = 0$ with infinite positive slope and approaches $+1$ from smaller values as $z$ becomes increasingly large. The right side of (1) is a parabolic shaped curve which starts from $z = 0$ with zero slope and becomes increasingly large as $z$ increases. The roots of (1) can be obtained by plotting or calculating the right and left sides of (1) as functions of $z$ and determining the point where both sides are equal. A typical example is shown in Fig. B-3. It is clear from the characteristics of the functions that for any finite value of $ka \cot \psi$ there is always one, but only one, value of $z$ for which the right and left sides of (1) are equal. Since (1) is an even function of $h$, there are two $h$ roots such that $h = \pm |h_0|$. Rather complete results obtained by the above

$\begin{array}{|c|c|c|c|c|c|c|}
\hline
z & 0 & 1.0 & 2.0 & 3.0 & 4.0 & 5.0 \\
\hline
\text{Left side of (1)} & & & & & & \\
\hline
\text{Right side of (1)} & & & & & & \\
\text{FIG. B-3} & \text{Determination of Roots of (1)} & \text{For} \; ka \cot \psi = \sqrt{2} \\
\hline
\end{array}$

graphical process or some similar one have been presented in several places in the literature. In reference 9, Chapter III, Fig. 3.2, a curve of $\frac{ka \cot \psi}{h_0 \lambda}$ in terms of the notation used here is shown along with other useful data. The solution for the $n = 0$ case is relatively simple in that the results can be easily presented as a function of the
parameter $k \cot \psi$. This is not so for $n \neq 0$, as will be understood shortly. In reference 10, curves of $\frac{ka}{\mathcal{H}_o} \alpha$ versus $ka$ are shown, and these are reproduced for a few typical values in Fig. II-8. The results are also discussed further in Section II.5.

To obtain solutions for the determinantal equation for $n \neq 0$ it turns out to be most convenient to solve for $\cot \psi$. From (B.1-1) one obtains

$$\cot \psi = \frac{-n\hbar \pm \sqrt{-z k^2 a^2 \frac{I_n^I}{n n}}}{\left[ n^2 + \frac{k^2 a^2}{z} \right] \left( n^2 + \frac{z}{\mathcal{H}_o} \frac{I_n^I}{n n} \right)}.$$  \hspace{1cm} (2)

The argument of the modified Bessel functions is $z = \xi a$, and the functional notation is omitted from now on for convenience. In obtaining the denominator of (2) the relationship $\hbar^2 a^2 = \xi^2 a^2 + k^2 a^2$ is used.

Note that the denominator of (2) is the function $f_n(z)$ defined by (B.2-3).

The $\pm$ in the numerator of (2) results from the fact that (B.1-1) is a quadratic function of $\cot \psi$, and both signs must be considered in looking for solutions. It can be seen that (B.1-1) and (2) are not functions of $\hbar^2$ but rather of $\hbar$. This is somewhat different from the usual "free mode" problem where the determinantal equation is a function of $\hbar^2$ alone.

The difference is a direct result of the "skewness" of the boundary conditions and means that the proper sign of $\hbar$ must be chosen in seeking solutions. Thus, for waves with increasing phase retardation in the positive $z$ (coordinate) direction, where the $z$ dependence of the solutions is $e^{-\imath \hbar z}$, $ha = +\sqrt{\xi^2 a^2 + k^2 a^2}$ is chosen; on the other hand, for waves with increasing phase retardation in the negative $z$ direction, $ha = -\sqrt{\xi^2 a^2 + k^2 a^2}$ is taken. Since the factor $n\hbar$ appears in (B.1-1) and (2) and since $n$ takes on all positive and negative values, it is clear that if (2) is solved for $ha = +\sqrt{\xi^2 a^2 + k^2 a^2}$ for all $n$, then
the solutions for $h_a = -\sqrt{5 a^2 + k^2 a^2}$ are directly available with the $\pm n$ solutions for positive $h$ corresponding to the $\mp n$ solutions for negative $h$. It is also evident from the form of (B.1-1) and the symmetry of the system that it is immaterial whether a positively wound, $\cot \psi > 0$, or negatively wound, $\cot \psi < 0$, sheath helix is assumed since this merely interchanges the numbering of the solutions. However, as soon as a positively wound helix is chosen, which is done here, and the boundary conditions thereby fixed, the solution must be carried through on this basis. The reason for emphasizing this point is that it means that solutions need be looked for only where the right side of (2) is positive.

If $h_a = +\sqrt{5 a^2 + k^2 a^2}$ is chosen, (2) becomes

$$\cot \psi = \frac{-n \sqrt{z^2 + k^2 a^2} \pm \sqrt{z^2 + k^2 a^2} \frac{I_{n+1}^{k'}}{I_{n}^{k}}}{n^2 + \frac{k^2 a^2}{z^2} \left(n^2 + z^2 \frac{I_{n}^{k}}{I_{n}^{k'}}\right)}.$$  

(3)

$I_n$, $K_n$ and $I_n'$ are always positive, whereas $K_n'$ is always negative, so that the quantity under the radical in the numerator of the right side of (3) is always positive. As noted in Section B.1, only $z > 0$ and real need be considered. In solving (3) the procedure is as follows: $n$ and $ka$ are fixed, and the right side of (3) is plotted as a function of $z$; the intersection of this curve with a previously chosen fixed value of cot $\psi$ yields a value of $z$, and therefore $h_a$, which is a solution of (3) and also (B.1-1). By repeating this process for the complete range of $ka$, keeping $n$ and cot $\psi$ fixed, the values of $h_a$ as a function of $ka$ which are a solution of the determinantal equation are obtained. This procedure is now continued for other $n$ and cot $\psi$ until the entire range of these parameters is covered. Not only is this method a relatively convenient one for numerical work, but also for a qualitative examination of the characteristics of the solutions. The writer is indebted to Professor L. J. Chu for indicating the possibilities of this approach.
Examination of (3) shows that it can be written as

\[ \cot \psi = \frac{1}{\frac{1}{2} \left| n \right| \sqrt{z^2 + k^2 a^2} \pm \sqrt{-z^2 k^2 a^2 \frac{T'_{n} K}{T_{n} K_n}}} \left[ n \sqrt{z^2 + k^2 a^2} \pm \sqrt{-z^2 k^2 a^2 \frac{T'_{n} K}{T_{n} K_n}} \right] \]

With \( n > 0 \) or \( n = +|n| \) (4) becomes

\[ \cot \psi = -\frac{1}{\frac{1}{2} \left| n \right| \sqrt{z^2 + k^2 a^2} \pm \sqrt{-z^2 k^2 a^2 \frac{T'_{n} K}{T_{n} K_n}}} \left[ \left| n \right| \sqrt{z^2 + k^2 a^2} \pm \sqrt{-z^2 k^2 a^2 \frac{T'_{n} K}{T_{n} K_n}} \right] \]

whereas for \( n < 0 \) or \( n = -|n| \) (4) becomes

\[ \cot \psi = -\frac{1}{\frac{1}{2} \left| n \right| \sqrt{z^2 + k^2 a^2} \pm \sqrt{-z^2 k^2 a^2 \frac{T'_{n} K}{T_{n} K_n}}} \left[ -\left| n \right| \sqrt{z^2 + k^2 a^2} \pm \sqrt{-z^2 k^2 a^2 \frac{T'_{n} K}{T_{n} K_n}} \right] \]

with the \( \pm \) order in (5) and (6) corresponding to that in (4). From (5) it is clear that with the plus sign, the right side of (5) is surely negative, whereas with the minus sign, it can be positive. Thus for \( n > 0 \), solutions need be looked for only with the minus sign in the numerator of (3) or (4). From (6) the fact that its right side can be positive with both the plus and minus signs is clear — it is surely positive with the minus sign — so that for \( n < 0 \) both cases must be investigated. Note that the right side of (5) with the minus sign is the negative of the right side of (6) with the plus sign. It should be mentioned that the factored forms (5) and (6) were used for investigation in reference 19, whereas the form given by (3) is preferred here, for the most part, because of the rather complete knowledge of the denominator of the right side of (3), the function \( f_n(z) \). However, the factored forms are useful, particularly for an investigation of the roots for \( n < 0 \) with the minus sign in (4).

From the above, for \( n > 0 \) solutions occur only for
\[
\cot \psi = \frac{-|n| \sqrt{z^2 + k^2 a^2} - \sqrt{-z^2 k^2 a^2 \frac{\mathbb{T}_{n K}^2}{\mathbb{T}_{n n}^2}}}{\left[ n^2 + \frac{k^2 a^2}{z^2} (n^2 + z^2 \frac{\mathbb{T}_{n K}^2}{\mathbb{T}_{n n}^2}) \right]}
\]

and, therefore, only where the denominator of (7) is negative. Since the denominator of (7) is \( f_n(z) \),

\[
\cot \psi = \frac{-|n| \sqrt{z^2 + k^2 a^2} - \sqrt{-z^2 k^2 a^2 \frac{\mathbb{T}_{n K}^2}{\mathbb{T}_{n n}^2}}}{f_n(z)}
\]

It is clear that \(-\sqrt{z^2 + k^2 a^2}\) is a monotonically decreasing function of \( z \), starting at \(-ka\) for \( z = 0 \) and going like \(-z\) for large \( z \). Using the expansions valid for small and large \( z \) given in Section A.2 and filling in the middle range by calculations, one can show that \(-\sqrt{-z^2 \frac{\mathbb{T}_{n K}^2}{\mathbb{T}_{n n}^2}}\) is a monotonically decreasing function of \( z \), starting at \(-|n|\) for \( z = 0 \) and going like \(-z\) for large \( z \). The numerator of (8) therefore appears as in Fig. B-1; it is a quantity which starts at \(-2|n|ka\) for \( z = 0 \) and approaches an asymptotic value of \(-(ka+|n|)z\) from below as \( z \) becomes increasingly large. The characteristics of the numerator are the same for all \( n \). In the following it is assumed momentarily that \( n \geq 2 \); the \( n = 1 \) case is slightly different and is considered separately below. The function \( f_n(z) \) for \( n \geq 2 \) has already been shown in Fig. B.2. Some examination shows that in the region \( \sqrt{n^2-1} > ka > 0 \), the right side of (8) is negative so that no solution exists. For \( n \geq ka \sqrt{n^2-1} \) -- actually, it is even more limited as noted by (9b) below -- the right side of (8) appears as shown in Fig. B-5a. It is a monotonically increasing function of \( z \), becoming infinitely large at the zero of \( f_n(z) \), and it is negative for \( z \) larger than this value. Thus, for \( ka \) in this region one and only one value of \( z \) and \( n \) satisfy the determinantal equation for a given \( ka, \psi \), and \( n \geq 2 \). But more, it is evident from inspection of the numerator and denominator
See text concerning \( n = 1 \)
of (8) that if, defining \( n \psi_{\text{min}} \),

\[
\cot \psi < \cot n \psi_{\text{min}} = \frac{-2n \frac{ka}{n^2(1 - \frac{k^2}{n^2})}}{\frac{2ka}{n(n - 1)}} = \frac{2ka}{n^2 - 1}, \quad n \geq 2, \quad (II.5-31), (9a)
\]

or if, defining \( n' \alpha \),

\[
ka < n' \alpha = \frac{(n^2 - 1) + \sqrt{(n^2 - 1)^2 + (n^2 - 1)n^2 \cot^2 \psi}}{n \cot \psi}, \quad (II.5-30), \quad (9b)
\]

no solution can exist. Thus, for a given \( \psi \) and \( n \geq 2 \) there is a low frequency "cut-off" or "divergence" frequency; for wavelengths longer than those given by (9b) the mode does not exist. In the region \( n' \alpha > n' \alpha \) it can be seen from a consideration of Fig. B-5a that for a given \( \alpha \) and \( n \), for \( \psi \) increasing, \( \cot \psi \) decreasing, the roots become smaller. Therefore, for a given \( \alpha \) and \( n \) the value of \( x \alpha \) decreases as \( \psi \) increases.

The curves in Fig. B-5 as well as similar curves in Figs. B-4 and B-6 through B-9 are, of course, not to scale and are used here merely to illustrate how information about the roots is obtained. Also, portions of the curves are shown as solid lines or as lines dotted in various ways to aid in tracing the root values of \( z \) or \( \zeta \alpha \) and \( \alpha \) which result from the respective parts of the curves. Thus, Fig. B-5 and Figs. II-10 and II-11 are related, whereas Figs. B-6 through B-9 lead to Figs. II-13 and II-14. The parts of the graphs in Figs. B-5 and B-7 through B-9 for which solutions occur for \( \cot \psi < 0 \) are shown shaded to indicate that these are discarded since a positively wound helix is being considered here.

For the region \( n + \Delta > ka > n \) (\( \Delta \) is used in the same sense as in Section B.2) the numerator of (8) is unaltered and still appears as in Fig. B-4. The function \( f_n(z) \) is again as in Fig. B-2. In this range of \( \alpha \) there are two zeros in \( f_n(z) \) so that the right side of (8) appears as in Fig. B-5b. There are one, two, or three roots, depending on the value
of \cot \psi . The smaller root occurs for a \( z \) less than the first zero of \( f_n(z) \), the second occurs for a \( z \) somewhat larger than the second zero of \( f_n(z) \), and the third root occurs for a \( z \) larger in value than the second root. The second and third roots coalesce for a particular value of \( \cot \psi \). It should be realized from the fact that \( \Delta_n \) is exceedingly small, and decreases for increasing \( n \), that the branch of the function which yields the second and third roots occurs for exceedingly large values of \( \cot \psi \). As \( ka \) increases over a very slight range, the two branches approach each other so that for \( ka = n + \Delta_n \) the right side of (8) appears as in Fig. B-5c. If \( \cot \psi \) is kept at a large constant value, some consideration shows that as \( ka \) approaches \( n + \Delta_n \), the value of \( z \) for the first root increases slightly, that for the second root decreases rapidly, while that for the third root increases somewhat — all assuming \( \cot \psi \) is sufficiently large to have three roots. As \( ka \) increases even more, the right side of (8) becomes like the curve shown in Fig. B-5d; it becomes a monotonically increasing function of \( z \) so that in this range of \( ka \), or for \( ka \) slightly larger than \( n + \Delta_n \) and increasing, for all \( \psi \) excluding 0° and 90°, of course, only one root occurs. In passing from the form shown in Fig. B-5c to that in Fig. B-5d, the roots for \( \cot \psi \) sufficiently large near the first zero of \( f_n(z) \) approach each other, coalesce, and then disappear.

In the above discussion it was assumed that \( n \geq 2 \). However, the case for \( n = 1 \) is quite similar, as in the discussion for the \( \psi = 0^\circ \) case in Section B.2. Since the \( g_1(z) \) function goes like \( \ln z \) for \( z \) small, for \( ka > 0 \) the \( f_1(z) \) function goes like \( (1 + ka^2 \ln z) \) so that the \( \cot \psi \) curves in Fig. B-5 have a zero intercept for \( z = 0 \) for \( n = 1 \). Consequently, in the range \( 1 > ka > 0 \) for \( 90^\circ > \psi > 0^\circ \) there is always one and only one root of the determinantal equation for \( n = 1 \). Aside from this difference all the other remarks made above concerning the solutions for
n > 2 also apply for n = 1. In other words, if the limit (9b) is interpreted properly for n = 1, then it may be said that the general character of the roots, as a function of n and \( \cot \psi \) for all \( n > 1 \), is quite similar.

Using the results of the preceding analysis, one can determine the dependence of \( h \) on \( k a \), \( \cot \psi \), and \( n \) for \( n > 0 \) — really \( n h > 0 \) since it was assumed in the above that \( h > 0 \). This is shown in Figs. II-10 and II-11, and the solutions are discussed further in Section II.7. As noted there and in Section II.7, the situation is somewhat more complicated than indicated here.

The solutions for \( n < 0 \) are now considered. As pointed out previously, for \( n < 0 \) both the plus and minus signs in (3) or (4) must be investigated. Also, for the minus sign with \( n < 0 \) the right side of (4) is surely positive, and from (6) this is

\[
\cot \psi = \frac{z^2}{|n| \sqrt{2^2 + k^2 a^2} + \sqrt{-2^2 k^2 a^2 \frac{\Gamma_n^1 K_n^1}{n^2 n}}}
\]

(10)

It is more convenient in this case to use the factored form of (4) given by (10). It can be seen that the denominator of (10) is merely the negative of the numerator of (3) which has already been shown in Fig. B-1.

Thus, the denominator of (10) is a monotonically increasing function of \( z \) which starts at \( 2 |n| k a \) for \( z = 0 \) and approaches the asymptotic value of \( (k a + |n|) z \) from above. The numerator of (10) is \( z^2 \) which is, of course, a monotonically increasing parabolic shaped function of \( z \). Some consideration shows that the right side of (10) is a monotonically increasing function of \( z \) which starts at zero for \( z = 0 \) and approaches the asymptotic value of \( \frac{z}{(k a + |n|)} \) from below as \( z \) becomes increasingly large. This is shown in Fig. B-6. Thus, for any given \( k a \), \( n \), and \( \cot \psi \) there is always one, but only one, root for the determinantal equation from this branch.

It is clear that for \( k a \) and \( n \) fixed, the value of the \( z \) root decreases as
FIG. B-6
\[ \cot \psi \text{ vs. } z \]
\[ \frac{z}{(ka+|n|)} \]

\[ \begin{align*}
ka &> 0 \\
n &< -1 \\
\text{with minus sign in (3)} \\
\text{and } h &> 0
\end{align*} \]

FIG. B-7a
\[ \cot \psi \text{ vs. } z \]
\[ \frac{z}{1-ka} \]

\[ \begin{align*}
1 &\geq ka > 0 \\
n &< -1 \\
\text{with plus sign in (3) and } h &> 0
\end{align*} \]

FIG. B-7b
\[ \cot \psi \text{ vs. } z \]
\[ \frac{1+\Delta}{1-ka} \]

\[ \begin{align*}
1 &\geq ka > 1 \\
n &< -1 \\
\text{with plus sign in (3)} \\
\text{and } h &> 0
\end{align*} \]
cot \psi \) decreases. Further, for a given \( \cot \psi \) as \( k \) increases with \( \eta \) fixed, \( z \) increases, and if \( k \eta > \eta \), the root is given approximately by 
\[ z \approx k \eta \cot \psi. \]
For a given \( \cot \psi \) with \( \eta \) fixed as \( k \eta \) decreases, it may 
be shown that \( z \approx \eta \cos \psi \) becomes an increasingly better approximation 
for the root so that 
\[ h = \sqrt{z^2 + k^2 \eta^2} \approx \eta \cot \psi. \]
The preceding holds for all \( \eta < 0 \) so that no special discussion is required for any particular \( \eta \).

As noted in the discussion following (6), for \( \eta < 0 \) with the plus sign 
in (3) or (4), there results a form which is precisely the negative of 
the form for \( \eta > 0 \) with the minus sign — the only allowable case for \( \eta > 0 \). 
Therefore, \( \cot \psi \) is given for \( \eta < 0 \) with the plus sign by the negative of 
the right side of (7) or (8). If in Fig. B-5 the curves for \( \cot \psi < 0 \) are 
now taken to be the curves for \( \cot \psi > 0 \) and vice versa, at least for 
\( |\eta| > 2 \), a consideration of the character of the roots can be carried on 
as before. However, it is somewhat clearer to discuss the cases for the 
different \( \eta \) separately here, and rather than use the curves in Fig. B-5,
curves more suitable for a discussion of the roots for \( \eta < 0 \) are shown.

For \( \eta = -1 \) with the plus sign in (3) or (4), for \( 1 > k \eta > 0 \) the \( \cot \psi \) 
curve appears as in Fig. B-7a. For a given \( k \) and \( \cot \psi \), especially for 
large \( \cot \psi \), there is a root for a value of \( z \) slightly larger than the 
zero of \( f_1(z) \), and another root for \( z \approx (1-k) \eta \cot \psi \) as determined from 
the asymptotic character of the \( \cot \psi \) curve. As \( \cot \psi \) decreases for a 
given \( k \eta \), it is clear that these roots move together, coalesce, and 
disappear. This also occurs for a given \( \cot \psi \) as \( k \eta \) increases. For \( \cot \psi \) 
increasing with \( k \eta \) fixed, the root near the zero of \( f_1(z) \) decreases in 
value slightly, whereas the other increases nearly linearly. For a 
given \( \cot \psi \) for which two roots exist, as \( k \eta \) becomes exceedingly small, 
the value of \( h \eta \) from the root for small \( z \) approaches \( k \eta \) in such a manner 
that \( \frac{k \eta}{h \eta} \) approaches unity; the value of \( h \eta \) from the other root approaches 
\( \cot \psi \). For \( 1 + \Delta_1 > k \eta > 1 \) the \( \cot \psi \) curve appears as in Fig. B-7b.
All with plus sign in (3) and $h > 0$
Here, for sufficiently large cot$\psi$ two roots exist. As cot$\psi$ decreases with ka fixed, these roots move together, coalesce, and disappear. Further, for a fixed cot$\psi$, as ka increases, the same action occurs so that for $ka > 1 + \Delta_1$ no roots result from this branch.

For $n = -2$ with the plus sign in (3) or (4), the numerator of the function which cot$\psi$ equals is the negative of the function shown in Fig. B-4 with $|n| = 2$, and the denominator is $f_{\psi_2}(z)$. For $\sqrt{3} > ka > 0$ the cot$\psi$ curve appears as in Fig. B-8a.

Examination by means of the series expansions valid for small $z$ shows that for any finite ka the cot$\psi$ curve decreases from its value at $z = 0$ and then increases approaching the $\frac{z}{2-ka}$ asymptote from above. For a fixed cot$\psi$ which is not too small there is for very small ka only one root near $z \approx 2 \cot\psi$. As ka increases with cot$\psi$ fixed at its previous value, the magnitude of this $z$ root decreases. For $\frac{ka \psi_2}{1 - \frac{k^2a^2}{3}}$ another root occurs, and for $\psi > 0$ this occurs in the region where $ka < \sqrt{3}$. Since for $2 > ka > \sqrt{3}$ the cot$\psi$ curve appears as in Fig. B-6b, and for $2 + \Delta_2 > ka > 2$ as in Fig. B-6c, it is clear that as ka increases, the two roots approach each other, coalesce, and disappear so that no roots result from this branch for $ka > 2 + \Delta_2$. Although the coalescence and disappearance must occur for $ka > 2 + \Delta_2$, it can happen for much smaller values of ka. It is evident from Fig. B-6 that for larger values of cot$\psi$, ka must approach $2 + \Delta_2$ more closely before the roots disappear.

For $n \leq -3$ with the plus sign in (4) the numerator of the function which cot$\psi$ equals is the negative of the function shown in Fig. B-4, and the denominator is $f_{\psi_n}(z)$. Consequently, for $\sqrt{n^2 - 1} > ka > 0$ the cot$\psi$ curve appears as in Fig. B-9a. Here, the cot$\psi$ curve can be monotonically increasing from its value at $z = 0$, or it can decrease and then increase as for the $n = -2$ case. By using the series expansions valid
All with plus sign in (3) and $h > 0$
for small $z$, it is found that for

$$ka < \left[ \frac{n^2(n^2-4)}{n^4 + 6n^2 - 4} \right]^{\frac{1}{4}}$$  \hspace{1cm} (11)

the cot $\psi$ curve increases monotonically. Therefore, if (11) is satisfied, for

$$\cot \psi > \frac{2ka}{|n|(1 - \frac{k^2 a^2}{n^2 - 1})}$$  \hspace{1cm} (12)

there is one and only one root; but if cot $\psi$ is insufficiently large to satisfy (12), there are no roots. If $ka$ is larger than the limit given by (11), the cot $\psi$ curve at first decreases as $z$ increases, and there can be two roots even if (12) is not satisfied. If, however, the limit (12) is still satisfied, there is only one root. It is evident from some consideration of (4) and Fig. B-9a that the value of cot $\psi$ obtained by satisfying (11) and (12) with equal signs is very near the value below which only one root can occur. For a fixed cot $\psi$ as $ka$ increases, two roots occur if

$$ka > \frac{(n^2-1)}{|n|\cot \psi} \left[ \sqrt{1 + \frac{n^2 \cot^2 \psi}{n^2 - 1}} - 1 \right]$$  \hspace{1cm} (II.5-32), (13)

This assumes that $ka$ is sufficiently large so that the limit given by (11) is passed. As $ka$ increases, for $|n| > ka > \sqrt{n^2 - 1}$ and for $|n| + \Delta_n > ka > |n|$ the cot $\psi$ curves appear as in Figs. B-9b and c. The action here is similar to that of the $n = -2$ case, and the comments made there for these ranges, if 2 is replaced by $|n|$, apply equally well here. Thus, the $n \leq -3$ cases are rather similar to the $n = -2$ case except for the peculiarity described by (11).

From the results of the foregoing analysis the dependence of $ha$ on $ka$, cot $\psi$, and $n$ for $n < 0$ -- really $nh < 0$ since it was assumed that $h > 0$ -- can be obtained. This is shown in Figs. II-13 and II-14, and the solutions are discussed further in Section II-5. As noted there and
in Section II.7, the situation is somewhat more complicated than indicated above. In concluding this section it should be pointed out again that the function \( f_n(z) \) and related modified Bessel functions have been thoroughly investigated over the entire range of \( z \) only for \( 7 \geq n \geq 1 \), so that it may be said that the results here have been confirmed only for these values. However, in view of the known form of the functions for large and small \( z \) for all \( n \) and the general properties of modified Bessel functions and their derivatives, it seems quite certain that the results are valid for all \( |n| \geq 1 \).
APPENDIX C

TAPE HELIX DETERMINANTAL EQUATION

AND POWER FLOW

In this appendix a means for readily calculating the value of the series appearing in the approximate determinantal equation for the narrow tape helix, (III.6-13), is described. Also, the power flow equation, (III.9-6), is put in a form such that calculations may be more easily performed, and useful tables for carrying out these calculations are given.

Tape Helix Determinantal Equation

C.1 Method of Approximating the Series

In Section III.6 after several manipulations the approximate tape helix determinantal equation is put in a form where one of the terms is a series like \( \sum_{m=1}^{\infty} f(m) \), where \( f(m) \) is a relatively complicated algebraic function of the summation index \( m \). In order to avoid summing this series it is desirable to find a closed form which equals or at least approximates it sufficiently well. The method whereby this is done here is a quite well-known elementary one but warrants a few words of explanation.

The series \( \sum_{m=1}^{\infty} f(m) \) may, of course, be thought of as the summation of the ordinate values of the function \( f(m) \) regarded as a continuous function of \( m \) at the abscissa values \( m = 1, 2, 3, \ldots \), where it is assumed that \( f(m) \) is such that the series is convergent. From Fig. C-1 it is clear that in the region where \( f(m) \) is increasing, \( N + 1 > m > n \),
FIG. C-1
Approximation of Sum by Integral

FIG. C-2
Approximations for Distribution Factor
\[ \int_{n}^{N+1} f(m) \, dm > \sum_{m=n}^{N} f(m) > \int_{n}^{N+1} f(m-1) \, dm, \] \hspace{1cm} (1) \]

whereas in the region where \( f(m) \) is decreasing, \( N+1 > m > n' \),

\[ \int_{n'}^{N+1} f(m-1) > \sum_{m=n'}^{N} f(m) > \int_{n'}^{N+1} f(m) \, dm. \] \hspace{1cm} (2) \]

From Fig. C-1 and (1) and (2) it would appear that for all cases a rather fair approximation to the sum would be given by

\[ \sum_{m=n}^{N} f(m) \approx \int_{n}^{N+1} f(m - \frac{1}{2}) \, dm = \int_{n-\frac{1}{2}}^{N+\frac{1}{2}} f(m) \, dm. \] \hspace{1cm} (3) \]

If \( f(m) \) is a linear function of \( m \), this relationship is exact; and if \( f(m) \) is a not too rapidly varying function, quite good results may be obtained if some precautions are taken. Consider the sum \( \sum_{m=1}^{\infty} \frac{1}{m} = \frac{\pi^2}{6} = 1.645 \). As an approximation, using (3) gives \( \sum_{m=1}^{\infty} \frac{1}{2m^2} \approx \int_{1/2}^{\infty} \frac{dm}{m^2} = 2 \) which is quite poor. If, however, it is realized that the difficulty here is the very rapid variation of \( f(m) \) in the neighborhood of \( m = 1 \) so that it is better to write \( \sum_{m=1}^{\infty} f(m) = f(1) + \sum_{m=2}^{\infty} f(m) \), then a quite good approximation results. For the above example, \( \sum_{m=2}^{\infty} \frac{1}{2m^2} \approx 1 + \frac{1}{2} \sum_{3}^{\infty} \frac{dm}{m^2} = 1.667 \),

which is in error only by slightly more than one percent. As other examples, \( \sum_{m=1}^{\infty} \frac{1}{m^3} = 1.202 \), whereas \( \sum_{m=1}^{\infty} \frac{1}{m^3} \approx 1 + \int_{1/2}^{\infty} \frac{dm}{m^3} = 1.222 \); and

\[ \sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{90} = 1.082, \] whereas \( \sum_{m=1}^{\infty} \frac{1}{m^4} \approx 1 + \int_{1/2}^{\infty} \frac{dm}{m^4} = 1.099. \) From these it is evident that by this simple procedure approximations for sums may be obtained which are quite accurate.

**C.2 Approximation of the Series**

The series for which an approximation is desired is

\[ \sum_{m=1}^{\infty} \left[ \frac{1}{m^{2} + \frac{1}{4} m^2} + \frac{1}{m^{2} + \frac{3}{4} m^2} - \frac{2}{m \csc \psi} \right] \frac{\sin mx}{mx}. \] \hspace{1cm} (1) \]
where
\[ \eta_m = \left[ m^2 \cot^2 \psi + 2m \alpha \cot \psi + \frac{h^2 a^2}{c^4} - k^2 a^2 \right]^{1/2}, \]  
(III.6-7)

and
\[ \eta_m = \left[ m^2 \cot^2 \psi - 2m \alpha \cot \psi + \frac{h^2 a^2}{c^4} - k^2 a^2 \right]^{1/2}. \]  
(III.6-8)

In attempting to use the procedure discussed in the previous section, integrals of the form
\[ \int_0^\infty \frac{1}{m} \frac{\sin mx}{\sin \frac{mx}{m} \eta_m} \, dm \]  
must be evaluated. It does not appear possible to evaluate these in any simple form, and the following approximation method which is particularly suitable for small \( x \) is used. Since the quantity in the brackets \([ \ ]\) in (1) converges like \( \frac{1}{m^2} \) for large \( m \), only small error will be made if some reasonable approximation which allows the integration to be performed and which is satisfactory for small values of \( m \) is used for \( \frac{\sin mx}{\sin \frac{mx}{m}} \). In fact, for \( x = 0.1 \), for example, \( \frac{\sin mx}{\sin \frac{mx}{m}} = 0.84 \) for \( m = 10 \), so that this factor might be taken equal to unity as a sufficiently good approximation in this case. However, for generality and for use where somewhat larger values of \( x \) than 0.1 are assumed, \( \frac{\sin mx}{\sin \frac{mx}{m}} \), or rather an approximation to it, is retained. An expression which has analytical simplicity and also approximates \( \frac{\sin mx}{\sin \frac{mx}{m}} \) quite well, even for relatively large values of \( mx \), is
\[ \left[ 1 - \left( \frac{mx}{q^2} \right)^2 \right], \]  
where \( q \) is a number equal to or somewhat less than one. A plot which illustrates this is shown in Fig. C-2. It will be noted that for \( q = 1.0 \) the simple approximate form gives values which are in fair agreement with those of \( \frac{\sin mx}{\sin \frac{mx}{m}} \) for \( m > mx > 0 \). However, since the dominant terms in the series occur for small \( m \), a value of \( q = 0.8 \) appears to be a better choice. For this the simple form deviates from \( \frac{\sin mx}{\sin \frac{mx}{m}} \) by only 10\% for \( mx \) as large as 0.6\%, which for \( x = 0.1 \) would correspond to \( m = 19 \). Now, because of the relatively rapid convergence of the terms multiplying \( \frac{\sin mx}{\sin \frac{mx}{m}} \) in the series, good results would be obtained if \( x \) is small by summing up only to \( m = \frac{\pi}{x} \). Since the higher
terms alternate in sign with every sign reversal of \( \sin mx \) and are quite small in any case, the error would be quite small if this were done. In using the \( \left[1 - \frac{(mx)^2}{(q \pi)^2}\right] \) form this means that it is proper to integrate on \( m \) up to where this term becomes zero, and, in fact, it is improper to integrate beyond this point since for larger \( m \) the simple form begins to deviate considerably from \( \frac{\sin mx}{mx} \). A plot of \( J_0(0.735mx) \) is also shown in Fig. C-2. This is discussed in Section III.12.

In order to show how the above is used and to obtain some estimate of the sort of accuracy which may be expected, the procedure is applied to sums of the type \( \sum_{m=1}^{\infty} \frac{\sin mx}{m^x} \). In this case the approximation becomes

\[
\sum_{m=1}^{\infty} \frac{\sin mx}{m^x} \approx \sum_{m=1}^{\infty} \frac{\sin mx}{m^x} \approx \frac{\sin x}{x} + \sum_{m=2}^{\infty} \frac{1}{m-1} \left[1 - \frac{(mx)^2}{(q \pi)^2}\right] \approx \frac{\sin x}{x} + \int^\pi_{\pi/2} \frac{1}{m-1} \left[1 - \frac{(mx)^2}{(q \pi)^2}\right] dm.
\]

(2)

Carrying out the integration indicated in (2) and comparing the results with the exact values available from the \( S_n(x) \) functions of Section A.6, one finds that for \( 5 < n \leq 2 \) and \( x = 0.1, 0.3, 0.5 \), the approximation available from (2) gives less than 2% error in all cases. Even for \( n = 1 \) the error is less than 8%. It therefore seems quite reasonable to expect that when applied to (1), this approximation procedure will yield results which are sufficiently accurate for the calculations required here.

In substituting an integration for the summation to find an approximation for (1), one encounters no difficulty with the terms \( \frac{1}{(m^2+n^2)^2} \) and \( \frac{2}{m \csc \psi} \); these are monotonically decreasing functions of \( n \), and the integration proceeds without difficulty. This is unfortunately not the case for the \( \frac{1}{(m^2+n^2)^2} \) factor. As a continuous function of \( m \) for certain values of \( m_n \), \( k_m \), and \( \cot \psi \), \( (m^2+n^2) \) can become negative, so that in this range the integration becomes meaningless and must be excluded. In an allowed region,
\( n_m^2 > 0 \) for all integer \( m \), so that \( (m^2 + n_m^2)^2 \) in this case. Since \( (m^2 + n_m^2) \) is a second degree polynomial in \( m \), it can have at most one minimum. It is therefore clear that \( (m^2 + n_m^2) \) can become negative only between two adjacent integers. Now, even though \( (m^2 + n_m^2) \) is not negative, it may be quite small between two adjacent integers so that \( \frac{1}{(m^2 + n_m^2)^2} \) is quite large. If a continuous integration were performed in this case, a large and false contribution would thus be made to the representation of the sum. It is evident that this difficulty may be avoided by the procedure of removing as discrete terms the values of \( \frac{1}{(m^2 + n_m^2)^2} \) on either side of the region where \( (m^2 + n_m^2) \) is small or negative. If \( (m^2 + n_m^2) \) becomes negative, it must first become zero, and this will occur for \( m \) values given by

\[
12^m = \frac{h a \cos \psi \pm \sqrt{k^2 a^2 - h a^2 \sin^2 \psi}}{\csc \psi}.
\]  

(3)

For \( k a > h a \sin \psi \), \( (m^2 + n_m^2) \) will have real zeros. If \( k a < h a \sin \psi \), then \( (m^2 + n_m^2) \) will not become zero but will be a minimum at

\[
3^m = h a \cos \psi \sin \psi.
\]  

(4)

Since \( (m^2 + n_m^2) \) can be zero only between two adjacent integers and since \( 3^m \) is the arithmetic mean of \( 1^m \) and \( 2^m \), it is clear that the first integer below a zero or minimum of \( (m^2 + n_m^2) \) will be \( m_1 \), where \( m_1 < 3^m \). It should be recalled that only positive values of \( h \) where \( h > k \) need to be considered.

Using the results of the previous discussion, taking \( f(m) \) to be the expression in the brackets \( [\cdot] \) in (1) for the moment and

\[
q(m) = \left[ 1 - \left( \frac{mx}{qm} \right)^2 \right],
\]  

(5)

one obtains as the approximation for (1)
\[ \sum_{m=1}^{\infty} f(m) \frac{\sin mx}{mx} \approx \sum_{m=1}^{\frac{m-q}{x}} f(m) \frac{\sin mx}{mx} \approx f(1) \frac{\sin x}{x} + \sum_{m=2}^{\frac{m-q}{x}} f(m) g(m), \]

\[ \approx f(1) \frac{\sin x}{x} + \int_{\frac{3}{2}}^{\frac{q}{x}} \frac{1}{\left[ m^2 + n^2 \right]^{\frac{1}{2}}} g(m) dm + \int_{\frac{3}{2}}^{\frac{m-1}{2}} \frac{1}{\left[ m^2 + n^2 \right]^{\frac{1}{2}}} g(m) \, dm \]

\[ + \frac{1}{\sqrt{m_1^2 + n_1^2}} g(m_1) + \frac{1}{\left[ (m_1+1)^2 + n_1^2 \right]^{\frac{1}{2}}} g(m_1+1) + \int_{m_1+\frac{3}{2}}^{\frac{q}{x}} \frac{1}{\left[ m^2 + n^2 \right]^{\frac{1}{2}}} g(m) \, dm \]

\[ - \frac{2}{\csc \psi} \int_{\frac{3}{2}}^{\frac{q}{x}} \frac{1}{m} g(m) \, dm. \] (6)

The first three integrals in (6) may be evaluated using the formulae given in Section A.3, and the last is quite elementary. With \( \varphi_m, \pi_m, \) and \( g(m) \) defined as before, and

\[ \xi(m) = \left( \varphi_m^2 + m^2 \right)^{\frac{1}{2}}, \] (7)

\[ \xi(m) = \left( \pi_m^2 + m^2 \right)^{\frac{1}{2}}, \] (8)

\[ \Lambda = \frac{\left[ 3h^2 a^2 \cos^2 \psi - \xi^2 a^2 \right]}{2} \sin^2 \psi, \] (9)

\[ \Omega(m) = 2 \csc \psi \left[ \xi(m) + m \csc \psi \right], \] (10)

\[ \Omega(m) = 2 \csc \psi \left[ \xi(m) + m \csc \psi \right], \] (11)

\[ \Phi(m) = \frac{\sin^2 \psi}{2} \left[ m - 3h^2 a^2 \sin \psi \cos \psi \right], \] (12)

\[ \Phi(m) = \frac{\sin^2 \psi}{2} \left[ m + 3h^2 a^2 \sin \psi \cos \psi \right], \] (13)

the final result becomes

\[ \xi^2 a^2 = h^2 a^2 - k^2 a^2, \] (III.6–6)
\[
\sum_{m=1}^{\infty} f(m) \frac{\sin mx}{mx} \approx \left[ -\frac{1}{\xi'(1)} + \frac{1}{\xi'(0)} - 2 \sin \psi \right] \frac{\sin x}{x} \\
+ \sin \psi \left[ 1 - \Lambda(\frac{x}{q^n})^2 \right] \ln \left[ \frac{\Omega'(q^n)}{\Omega'(\frac{3}{2})} \right] - (\frac{x}{q^n})^2 \left[ \Phi'(\frac{q^n}{x}) \xi'(\frac{q^n}{x}) - \Phi'(\frac{3}{2}) \xi'(\frac{3}{2}) \right] \\
+ \left\{ \sin \psi \left[ 1 - \Lambda(\frac{x}{q^n})^2 \right] \ln \left[ \frac{\Omega'(m_1 - \frac{3}{2})}{\Omega'(\frac{3}{2})} \right] - (\frac{x}{q^n})^2 \left[ \Phi'(m_1 - \frac{1}{2}) \xi'(m_1 - \frac{1}{2}) - \Phi'(\frac{3}{2}) \xi'(\frac{3}{2}) \right] \right\}^{*1} \\
+ \left\{ \sin \psi \left[ 1 - \Lambda(\frac{x}{q^n})^2 \right] \ln \left[ \frac{\Omega'(m_1 + \frac{3}{2})}{\Omega'(m_1 + \frac{3}{2})} \right] - (\frac{x}{q^n})^2 \left[ \Phi'(m_1 + \frac{3}{2}) \xi'(m_1 + \frac{3}{2}) - \Phi'(\frac{3}{2}) \xi'(\frac{3}{2}) \right] \right\}^{*2} \\
+ \frac{1}{\xi'(m_1)} g(m_1) + \left[ \frac{1}{\xi'(m_1 + 1)} g(m_1 + 1) \right]^{*3} - 2 \sin \psi \left[ \ln \left( \frac{2}{3} \frac{q^n}{x} - \frac{1}{2} + \frac{1}{6} \frac{q^n}{x} \right)^2 \right].
\]

(111)

Since the \( m = 1 \) term is included separately, it is not always necessary to include all of the terms in (111). This is indicated by \( *_{1,2,3} \) where the meaning of these is as follows:

\( *_1 \) means omit this term if \( m_1 = 0, 1, \) or 2 but include if \( m_1 \geq 3 \),

\( *_2 \) means omit this term if \( m_1 = 0 \) or 1 but include if \( m_1 \geq 2 \),

\( *_3 \) means omit this term if \( m_1 = 0 \) but include if \( m_1 \geq 1 \).

\( m_1 \) is the first integer less than \( \frac{3}{2} m \) where the latter is given by (11). For \( x \) small the \( \frac{\sin x}{x} \) factor in the first term in (11) may be considered equal to unity. Further, if \( x \) is small, the terms multiplied by \( (\frac{x}{q^n})^2 \) are usually quite small compared with the others in (111) and may be omitted with little error. If this is the case, (111) becomes considerably simpler. It will be noted that, strictly speaking, the \( g(m) \) factor in the \( *_2 \) and \( *_3 \) terms should be \( \frac{\sin mx}{mx} \). However, for small \( x \), for the range of \( m_1 \) normally encountered, negligible error results if \( g(m) \) is used.

In view of the rather formidable nature of (111) it may be questioned
whether it is more useful than the series it approximates. Some experience indicates that (14) is relatively easy to use and is more convenient to evaluate than the series. Many of the terms and expressions in (14) are similar, which makes for relative simplicity in evaluation. Further, the use of (14) avoids the not inconsiderable question of where to stop evaluating terms in (1). It is difficult to estimate how closely (14) represents the series. However, in a typical case which was considered in detail, \( x = 0.1 \) and \( \psi = 10^0 \), for many values of \( ka \) and \( ha \), (14) agreed with its series representation to better than 10% at nearly all points, with the error being possibly somewhat less than this in view of the truncation of the series after a seemingly sufficient number of terms. For many points the error was less than 5%, and often it was as small as 1%. It cannot be claimed that these results would hold in general, but it appears quite probable that for \( x \) small and for all practical values of \( \psi \), the representation is in error by no more than 10% in the worst case. In view of the fact that this term is in the nature of a correction term for the determinantal equation, this accuracy is entirely adequate.

Another possible way of approximately evaluating (1) is to expand

\[
\frac{1}{(m^2 + \frac{\psi}{m})^2} \quad \text{and} \quad \frac{1}{(m^2 + \frac{\psi}{m})^2}
\]

in a series of inverse powers of \( m \) and to inter-

change the summation on \( m \) with the one resulting from this expansion.

There results an infinite series of the \( S_n(x) \) functions of Section A.6 of which only the first few may be necessary for a good approximation, and for which highly convergent forms are available. Unfortunately, this scheme works only if

\[
m^2 \csc^2 \psi > 2mha \cot \psi + \xi^2 a^2,
\]

and this limits the range over which \( ha \) may be investigated for a given \( ka \) and \( \psi \). By adding the \( m = 1 \) terms separately and applying this process to terms with \( m \geq 2 \), the range may be extended. This adds only a little
to the total necessary range, and if the process is extended any further, it becomes no more convenient than summing the original series directly. Without discussing the matter in any further detail, it should be mentioned that this method was tried in the range where the inequality (15) is satisfied for \( m = 1 \) for the particular case of \( x = 0.1 \) and \( \psi = 10^0 \), and solutions for the narrow tape helix determinantal equation were obtained which agreed to within 2% with those obtained by the more general procedure.

C.3 The Remainder Term

In the narrow tape helix determinantal equation, (III.6-13) a remainder term appears which is written as

\[
\sum_{m=1}^{\infty} R(\eta_m) = \sum_{m=1}^{\infty} \left\{ (h^2 a^2 - k^2 a^2 \cot^2 \psi) \left[ I_m(\eta_m)K_m(\eta_m) \frac{1}{2} \frac{1}{m^2 + \eta_m^2} \right] + k^2 a^2 \cot^2 \psi \left[ I_m(\eta_m)K_m(\eta_m) \frac{1}{2} \frac{1}{m^2 + \eta_m^2} \right] \right\} \sin mx.
\]

In general, because of the excellence of the representations (III.6-3,4) for the Bessel function products, this remainder is exceedingly small. However, for \( ka \) and \( \psi \) fixed, as \( ha \) is allowed to vary, \( \eta_m \) becomes small near and finally zero at the edges of the forbidden regions, or where

\[
ha = m \cot \psi \pm ka.
\]

It should be emphasized again that only positive values of \( h \) and \( m \) need to be considered. From (1), by using the series expansions given in Section A.2, expressions useful for small values of \( \eta_m \) may be obtained. For \( R(\eta_1) \), with \( \eta_1 \) small,

\[
R(\eta_1) \approx \left\{ \frac{k^2 a^2 \cot^2 \psi}{2} \left[ \ln \left(\frac{\eta_1 e^y}{2}\right) \left(1 + \frac{\eta_1^2}{4}\right) + \frac{3}{4} \frac{7}{2} \eta_1^2 \right] + \frac{1}{4} \frac{\eta_1^2}{\ln \left(\frac{\eta_1 e^y}{2}\right)} \right\} \sin x.
\]
(See (A.2-6) for \( \gamma \) and (III.6-6) for \( \xi^2a^2 \).) Comparison with exact values shows that (3) is a quite accurate representation for \( 0.4 > \eta_1 > 0 \) as given; for \( 0.7 > \eta_1 > 0.4 \) the term \( \xi^2a^2 \) should be dropped for best accuracy.

In general, larger values of \( \eta_1 \) than 0.7 need not be considered with this approximation. (3) tends to give too large a remainder, but the error is small. For \( R(\eta_2) \), with \( \eta_2 \) small,

\[
R(\eta_2) \approx -\frac{1}{12} k^2 a^2 \cot^2 \psi - \eta_2 \left( \frac{k^2 a^2 \cot^2 \psi \ln\left(\frac{2e^\psi}{2}\right)}{2} + \frac{1}{20} (\xi^2 a^2 - \frac{1}{2} k^2 a^2 \cot^2 \psi) \sin 2x \right) \frac{\sin 2x}{2x}.
\]

(4)

Comparison with exact values shows that (4) is a sufficiently good representation for \( \eta_2 < 0.6 \), which is large enough for most purposes. (4) also gives too large a remainder but the error is small, particularly for \( \eta_2 < 0.4 \). For \( R(\eta_m) \) for \( m > 3 \), with \( \eta_m \) small,

\[
R(\eta_m) \approx \left\{ -k^2 a^2 \cot^2 \psi - \frac{1}{2m(m^2-1)} + \eta_m \frac{(7m^2-1)}{2m^3(m^2-1)} \right\} \times \frac{\sin mx}{mx}.
\]

(5)

It will be noted that for \( \eta_1 \) approaching zero

\[
R(\eta_1 \to 0) \approx \frac{k^2 a^2 \cot^2 \psi}{2} \left( \ln\left(\frac{3}{2}\right) + \frac{3}{4} \right) \sin \frac{x}{x},
\]

(6)

whereas for \( \eta_m \) equal to zero for \( m > 2 \)

\[
R(\eta_m = 0) = - \frac{1}{2m(m^2-1)} k^2 a^2 \cot^2 \psi \frac{\sin mx}{mx}.
\]

(7)

Thus, near the \( |m| = 1 \) forbidden region the remainder term is logarithmically large, whereas near the other forbidden regions the remainder term is finite.

If \( \Delta \) is the amount by which a given ha differs from its value at the edge of a forbidden region, so that

\[
ha = m \cot \psi \pm ka \pm \Delta,
\]

(8)

it is readily shown that
\[ \eta_m^2 = 2 \Delta \, ka + \Delta^2 . \]  

Consequently, for a given \( ka \), \( \eta_m \) varies in a similar way on either side of a forbidden region. Fig. C-3 shows a plot of \( \eta_m \) vs. \( \Delta \) for various \( ka \). It will be noted that \( \Delta \) need not be very large before \( \eta_m \) is sufficiently large so that all the \( R(\eta_m) \) terms are quite small. Further, at the \( n^{th} \) boundary, where \( ha = n \cot \psi \pm ka \), \( \eta_m \) becomes

\[ \eta_m^2 = (m-n)^2 \cot^2 \psi + 2ka(m-n)\cot \psi . \]  

It can be seen that even for \( \psi \) relatively large, with \( |m-n| \geq 1 \) and \( ka < \frac{\cot \psi}{2} \), \( \eta_m \), in general, is sufficiently large so that \( R(\eta_m) \) is negligibly small. The result of all this is that only near the \( m^{th} \) forbidden region is it necessary to calculate \( \sum_{m=1}^{\infty} R(\eta_m) \), and in nearly
every case only the $m^{th}$ term need be calculated there. Also, the simple
forms given in (3), (4), and (5) are sufficient for this purpose. At
worst, if the boundaries of the forbidden regions occur close together --
this happens for $k_0$ quite near \( \cot \frac{\psi}{2} \) -- it may be necessary to calculate
$R(\eta_m)$ and $R(\eta_{m-1})$ or $R(\eta_{m+1})$. If this is so, experience indicates that
(3), (4), and (5) are still sufficiently accurate approximations.

C.4 Calculations for a Typical Case

It will be recalled that the approximate determinantal equation for
the narrow tape helix whose roots are desired is

\[
0 \approx \sum_m \left\{ (h^2 a^2 - k_0^2 a^2 + k_0^2 \frac{m^2 \cot^2 \psi}{\eta_m^2}) I_m(\eta_m) K_{m-1}(\eta_m) + k_0^2 a^2 \cot^2 \psi I'_m(\eta_m) K'_m(\eta_m) \right\}_1,2 D_m,
\]

(III.5-32)

By simplifying $1,2 D_m$ and by using asymptotic forms of the Bessel functions
and then asymptotic forms of these in turn, it is shown in Section III.6
that (III.5-32) may be written to within a very small error as

\[
0 \approx \xi^2 a^2 I_0(\xi a) K_0(\xi a) + k_0^2 a^2 \cot^2 \psi I'_0(\xi a) K'_0(\xi a) + \frac{(h^2 a^2 - k_0^2 a^2 \csc^2 \psi)}{\csc \psi} \ln \frac{e}{\xi}
\]

\[
+ \frac{(h^2 a^2 - k_0^2 a^2 \csc^2 \psi)}{2} \sum_{m=1}^{\infty} \left[ \frac{1}{(m^2 + \eta_m^2)^2} + \frac{1}{(m^2 + \eta_m^2)^2} - \frac{2}{m \csc \psi} \right] \frac{\sin mx}{mx}
\]

\[
+ \sum_{m=1}^{\infty} R(\eta_m)
\]

(III.6-13)

In Section C.2 a means of representing the second series in (III.6-13) by
a more convenient form is given, and in Section C.3 it is shown that gen-
erally only one term in the last series in (III.6-13) is important.

Further, this last occurs only occasionally, and when it does, quite sim-
ple expressions are available for determining the value of the term. By
these means the determinantal equation, (III.5-32), is expressed in a
way such that calculations may be performed in a fairly simple manner.
This is in contrast to the original equation which converges very slowly and contains modified Bessel functions in such a form as to make calculations essentially prohibitive.

The procedure for determining the roots is now as follows. Fixed values of $\psi$, $x$, and $ka$ are chosen and substituted in (III.6-13) so that it becomes a function of $ha$ alone, say $f(\text{ha})$. The zeros of $f(\text{ha})$ are then determined most conveniently and with adequate accuracy by calculating and plotting $f(\text{ha})$ as a function of $ha$ over a sufficiently wide range of this variable. The process is then repeated for other values of $ka$, with $\psi$ and $x$ still fixed, until the range $\frac{\cot \psi}{2} > ka > 0$ is covered. For other values of $x$, with $\psi$ constant, the calculations would, in principle, have to be repeated, although the effect of varying $x$ over a relatively wide range may be determined with sufficient accuracy without this additional work. However, for other values of $\psi$, it is necessary to repeat the entire process, although it appears that certain generalizations are possible (see Section III.8).

Calculations have been carried through in detail using the above procedure for the particular case of $\psi = 10^\circ$ and $x = 0.1$. Graphs of $f(\text{ha})$ vs. $ha$ for $ka = 0.5$ and 2.45 are shown in Figs. C-4 and C-5 to indicate how the process works. It will be noted that for $ka = 0.5$, in addition to a zero at $ha = 2.33$, $f(\text{ha})$ has zeros exceedingly close to the $|m| = 1$ forbidden region as a result of the influence of the $R(\eta_1)$ term. For $ka = 2.45$ there is only one zero in $f(\text{ha})$ at $ha = 14.3$. It is worthwhile noting that in this case the $R(\eta_m)$ terms make a significant difference. However, by using $R(\eta_2)$ alone near the $|m| = 2$ boundary, and $R(\eta_3)$ alone near the $|m| = 3$ boundary, the slope and value of $f(\text{ha})$ are sufficiently well determined at these points so that a smooth curve drawn through them agrees exceedingly well with values obtained in the middle range where $R(\eta_2)$ and $R(\eta_3)$ must both be included in $f(\text{ha})$. 
Even without this agreement, the value of $h_0$ at the zero crossing would be affected only slightly by relatively large changes in the value of $f(h_0)$. Of course, this applies throughout the range of $h_0$ and is one favorable aspect of this procedure.

In addition to $h_0 = 0.5$ and 2.45, detailed calculations were performed for $h_0 = 0.05, 0.10, 0.30, 0.75, 0.80, 1.0, 1.2, 1.45, 1.63, 1.68, 2.25, 2.30, 2.38, 2.5$, all for $\psi = 10^\circ$ and $x = 0.1$. In this manner all the zeros of $f(h_0)$ were determined for $\frac{\cot \psi}{2}$ (equal to 2.83 in this case) $h_0 > 0$. The final results are shown in Figs. III-4 and III-5, and the solutions are discussed in Section III.7.

**Power Flow for the Narrow Tape Helix**

### C.5 Transformation of the Series; Useful Data for Calculations

Equation (III.9-6) gives an expression for the power flow, assuming that $|\kappa_m| \neq 0$ and $|\kappa_m| = 0$. If the constant magnitude current density and constant a phase front approximations are used, one obtains from (III.5-22) and (III.6-1)

$$|\kappa_m|^2 \approx \frac{|I|^2}{p^2 \cos^2 \psi} \frac{\sin^2 mx}{m^2 x^2}.$$  \hspace{1cm} (1)

Using (1) in (III.9-6), separating out the zeroth term and converting the sum over negative $m$ to one over positive $m$, using the approximation (A.2-22) for $I_m K_m$, since this appears only as a multiplier for $|m| > 1$, adding and subtracting the dominant terms from the series, and then again from these dominant terms, one obtains finally
FIG. C-4

- In allowed regions $f(h)$ is increasingly positive for $ha > 11.84$.
- $ka = 0.5$
- $\psi = 10^\circ$
- $x = 0.1$

Logarithmic singularities in $f(h)$ along $|m| = 1$ boundary.

FIG. C-5

- In allowed regions $f(h)$ is increasingly negative for $ha < 3.89$.
- $ka = 2.5x$
- $\psi = 10^\circ$
- $x = 0.1$

- In allowed regions $f(h)$ is increasingly positive for $ha > 19.45$. 

There are two graphs showing the behavior of $f(h)$ vs. $ha$ for different values of $ka$, $\psi$, and $x$. The graphs indicate the regions where $f(h)$ is either increasing or decreasing with $ha$ and note the boundaries for these regions.
\[ P_z \approx \frac{1}{\lambda n' \sqrt{\varepsilon}} \left\{ K_o(\xi a)I_o(\xi a) \left[ h_a f_o^{I} + k^2 a^2 \cot^2 \psi f_o^{II} \right] \right. \\
+ \sum_{m=1}^{\infty} \frac{\sin^2 m\pi x}{m^2 \pi^2} \frac{1}{2(m^2 + \eta_m^2)^2} \left[ h_m a(q_m^2 + f_m^I k^2 a^2 \cot^2 \psi + f_m^I) + m \frac{q_m^m}{\eta_m^m} (h_m^2 a^2 + k^2 a^2) \cot \psi - h_m \right] \\
+ \sum_{m=1}^{\infty} \frac{\sin^2 m\pi x}{m^2 \pi^2} \frac{1}{2(m^2 + \eta_m^2)^2} \left[ h_m a(q_m^2 - f_m^I k^2 a^2 \cot^2 \psi - f_m^I) - m \frac{q_m^m}{\eta_m^m}(h_m^2 a^2 + k^2 a^2) \cot \psi + h_m \right] \\
+ \frac{h_a}{\csc \psi} \ln \frac{2.241}{x} + \frac{h_a}{\csc \psi} \frac{1}{2} \left( \sum_{m=1}^{\infty} \frac{\sin^2 m\pi x}{m^2 \pi^2} \left[ \frac{1}{(m^2 + \eta_m^2)^2} + \frac{1}{m^2 \pi^2} - \frac{2}{m \csc \psi} \right] \right). \tag{2} \]

In (2) the functions \( f_o^{I} \) and \( f_o^{II} \) are given by

\[ f_o^{I}(\xi a) = \xi a \left[ \frac{I_o'(\xi a)}{I_o(\xi a)} + \frac{K_o'(\xi a)}{K_o(\xi a)} \right], \tag{3} \]

\[ f_o^{II}(\xi a) = \frac{1}{\xi a} \left[ - \frac{K_o'(\xi a)}{K_o(\xi a)} + \frac{I_o'(\xi a)}{I_o(\xi a)} \right]. \tag{4} \]

where the argument of the modified Bessel functions is \( \xi a \). \( \pm f_m^{I} \) and \( \pm f_m^{II} \) are given by

\[ \pm f_m^{I}(\eta_m) = \frac{1}{\eta_m^2} \left[ \frac{I_m'(\eta_m)}{I_m(\eta_m)} + \frac{K_m'(\eta_m)}{K_m(\eta_m)} \right], \tag{5} \]

\[ \pm f_m^{II}(\eta_m) = \frac{1}{\eta_m^2} \left[ - \frac{K_m'(\eta_m)}{K_m(\eta_m)} + \frac{I_m'(\eta_m)}{I_m(\eta_m)} \right]. \tag{6} \]

for \( m \geq 1 \), where the argument of the modified Bessel functions is \( \eta_m \).

\( \xi a \) and \( \eta_m \) are given by (III.6-6, 7, 8). From (II.2-2) and (III.9-7)

\[ \pm h_m a = h a \pm m \cot \psi, \tag{7} \]

\[ \pm q_m = \xi a^2 \pm m h a \cot \psi, \tag{8} \]

with \( m \geq 1 \). In the process of transforming (2), the sum \( \frac{1}{2} \sum_{m=1}^{\infty} \frac{\sin^2 m\pi x}{m^2 \pi^2} \) appears. By using the methods described in Section A.6 it can be readily
shown that
\[
\frac{1}{x^2} \sum_{m=1}^{\infty} \frac{\sin^2 mx}{m^3} = \ln \frac{1}{2x} + \frac{3}{2} + \frac{x^2}{36} + \frac{x^4}{2700} + \ldots \text{ for } 2\pi > x > 0. \tag{9}
\]

The first two terms in (9) represent the total to better than 0.01% for \(x \leq 0.1\) and account for the \(\ln \frac{2.211}{x}\) term in (2). In the preceding only the constant current density approximation is considered. It is shown in Section III.12 that the inverse square root approximation for the current density alters the power flow equation only slightly.

(2) was used to calculate the power flow for the particular case of \(\psi = 10^0\) and \(x = 0.1\) using the values of \(h_{1a}\) obtained as the solutions for the approximate determinantal equation. The results are shown in Fig. III-10 where the dimensionless ratio \(\frac{P_z}{\sqrt{\frac{11}{\varepsilon} |I|^2}}\) is plotted as a function of \(ka\). Although (2) is a rather complicated function, the following experience obtained from the calculations for \(\psi = 10^0\) and \(x = 0.1\) indicates that some simplifications are possible. Over most of the range of \(ka\) for the \(h_{1a}\) wave only the first and fourth terms of (2) are significant. As the point of zero group velocity is approached, the third and fifth terms become important. For the \(h_{11}\) wave only the third term is significant over most of the range of \(ka\), and in this the \(m = 1\) term gives by far the major contribution. As the point of zero group velocity is approached, the first, fourth, and fifth terms begin to make significant contributions. For the \(h_{11}'\) wave, for the part of the solution very near the \(m = -1\) forbidden region (see Figs. III-2 and III-4) only the third term in (2) is important, and the \(m = 1\) term gives the major contribution. As the \(h_{11}'\) solution deviates from the \(m = -1\) forbidden region boundary, the contribution of the third term becomes small but not insignificant, whereas the first, fourth, and fifth terms of (2) contribute the major portion. This situation persists
up to the zero group velocity point. For the $\pm |h''_{t2}|$ wave the third term in (2) is most significant, and in this the $m = 2$ term is quite dominant.

For the $\pm |h'_{t2}|$ wave the third term is the largest part near the boundaries of the forbidden regions, and in this the $m^{th}$ term is the most important where $m = 2$ or 3. Between the boundaries the first, fourth, and fifth terms make the major contributions. It is of interest to note, as may be readily shown using the values of $f^I_m$ and $f^{II}_m$ valid for small $\rho_m$ given shortly, the $P_z$ becomes infinitely large as $\rho_m$ approaches zero if $|I|$ is assumed finite. It is therefore evident that if the helix is assumed to be excited by a finite source, $|I|$ must be zero where $\rho_m$ is zero.

Some consideration shows that the second and third terms in (2) make significant contributions only when $\rho_m$ is small, and then only the $m^{th}$ term is important. If calculations are made only for $h_t > 0$, as described above, the second term in (2) can be ignored completely if the proper interpretation is given to the negative values of $P_z$ obtained for the $\pm |h''_{t1}|$ and $\pm |h''_{t2}|$ waves. In calculating the values of $P_z$ from (2) for $\psi = 10^\circ$ and $x = 0.1$, all the series were summed up to $m = 7$ to insure that all the significant terms were considered. From the experience obtained from this calculation it appears that for other values of $\psi$ the series can be truncated before such large values of $m$, although this will depend to a considerable extent on the particular value of $\psi$. The performance of the various terms for $\psi = 10^\circ$ and $x = 0.1$ described above should act as a guide in reducing the labor required to obtain numerical results from (2) for other values of $\psi$. Of course, the solutions of the determinantal equation for other pitch angles will be required, but the comments of Section III.8 may prove useful for this.

Although it may be possible to discard several of the terms in (2), some values of the functions $f^I_m(z)$ and $f^{II}_m(z)$ will undoubtedly still be required. Table C-1 of these function for $7 > m > 0$ was prepared to aid
in the calculation of the special case described previously and is presented here for possible future use. It is believed that the entries in Table C-1 are correct to within three places in the last significant figure shown, although no check has been made and the errors may be larger. More significant figures than are generally required are shown, and experience indicates that fewer significant figures are often quite adequate for the required accuracy. Any of the well-known interpolation formulae may be used to determine the values of the functions for arguments other than those shown, except possibly near the turning point in $z^3 f^I_m(z)$ for $m \geq 1$. Here, it may be necessary to calculate the value of the function from its original definition in terms of the modified Bessel functions. Note that $z^2 f^I_o(z)$, $z^2 f^I_1(z)$, and $z^3 f^I_3(z)$ are shown instead of the functions themselves; this was done to simplify the entries in Table C-1.

By the use of the formulae of Section A.2, the following approximations for small $z$ which give three figure accuracy for the range shown can be derived:

\[ f^I_0(z) \approx 1 + \frac{z^2}{2} + \frac{1}{2\ln \frac{ze^\gamma}{2}}, \quad (10) \]

and

\[ z^2 f^I_0(z) \approx \frac{z^2}{4} (1 - \frac{z^2}{8}) + \frac{1}{16} \frac{z^2}{\ln \frac{ze^\gamma}{2}}, \quad (11) \]

for $z < 0.1$;

\[ z^4 f^I_1(z) \approx 1 + \frac{z^2}{2} (1 + \ln \frac{ze^\gamma}{2}), \quad (12) \]

and

\[ z^3 f^I_2(z) \approx \frac{1}{z} + \frac{3}{8} z - \frac{z}{2} \ln \frac{ze^\gamma}{2}, \quad (13) \]

for $z < 0.2$;

\[ z^2 f^I_m(z) \approx 1 - \frac{z^2}{2(m^2 - 1)}, \quad (14) \]

and

\[ z^3 f^I_m(z) \approx \frac{m^2}{z} + \frac{m^2}{2} \frac{z}{m^2 - 1}, \quad (15) \]

for $z < 0.3$ for $m = 2$, and for $z < 0.5$ for $7 \geq m \geq 3$. $\gamma$ is given by (A.2-6).

(14) and (15) become increasingly better approximations as $m$ increases and can be used with good accuracy for larger $z$ in this case. For large $z$ the following approximations apply:
\[ z^{\text{I}}_{m}(z) \approx \frac{1}{2} + \frac{\text{Im}^2 \cdot 1}{8z^2}, \quad (16) \]

\[ z^{\text{II}}_{m}(z) \approx \frac{z}{2} + \frac{\text{Im}^2 \cdot 2}{8z}, \quad (17) \]

for \( 7 \geq m \geq 1 \). These usually give three figure accuracy or better for the functions for all values of \( z \) larger than shown in Table C-I and are most accurate for the smaller values of \( m \). Since the functions for \( m = 0 \) are defined slightly differently, the approximations for large \( z \) for these become

\[ f^{\text{I}}_{0}(z) \approx \frac{1}{2} - \frac{1}{8z^2}, \quad (18) \]

\[ z^{2}f^{\text{II}}_{0}(z) \approx \frac{1}{2} - \frac{5}{8z^2}. \quad (19) \]

In the course of preparing Table C-I, the need for the ratios \( \frac{\text{I}^{m}(z)}{K^{m}(z)} \) and \( \frac{\text{Im}^{m}(z)}{K^{m}(z)} \) arose. These were calculated using tables which were available and the recurrence formulae given in Section A.1. The results are shown in Table C-II and are included here since they are not only of some interest in the tape helix calculations, but also of use in the calculations connected with the sheath helix case discussed in Chapter II. The entries are believed to be correct to within three places in the last significant figure shown. However, some spot checks revealed a few points where the errors were larger (these were corrected), and no guarantee can be made that all errors have been eliminated.
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ACKNOWLEDGEMENT

The writer wishes to express his gratitude to Professor L. J. Chu for his many helpful suggestions in the supervision of this work. He would also like to thank Professor R. B. Adler for the time he spent in numerous discussions, Miss E. Pauline Leighton and Mrs. Barbara Levine for carrying through some rather tedious calculations, and Miss Joyce Littlefield, Mrs. Pauline Cochary, and his wife, Elaine, for their considerable assistance in the final preparation of this report. The writer would finally like to thank the Research Laboratory of Electronics, Massachusetts Institute of Technology, for providing the facilities necessary for the conduct of this research.
BIBLIOGRAPHY AND REFERENCES


50. Morse and Feshbach, Methods of Theoretical Physics, Technology Press, Cambridge, Mass., 1948.


BIOGRAPHICAL NOTE

Samuel Sensiper was born on April 26, 1919, in Elmira, New York. He received the Bachelor of Science degree from the Massachusetts Institute of Technology in 1939 and the Electrical Engineering degree from Stanford University in 1941. At Stanford he was a Teaching and Research Assistant. He was with the Sperry Gyroscope Company from 1941 until 1947. Mr. Sensiper was awarded the United States Navy Certificate of Commendation for his work during World War II. He is also the holder of four issued patents, one held jointly. In 1947 he returned to the Massachusetts Institute of Technology where he was an Industrial Electronics Fellow during 1947 and 1948. He became a Research Assistant at the Research Laboratory of Electronics in 1949.

Mr. Sensiper is a Senior Member of the Institute of Radio Engineers and has been a member of the Antennas and Waveguides Committee of that organization since 1949. He is also an Associate Member of Sigma Xi, and a member of Eta Kappa Nu, the American Association for the Advancement of Science, and the Federation of American Scientists. He is married and has no children.