A Study of the Mechanisms of Axisymmetric Vortex Breakdown

by

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A theoretical and numerical investigation of the mechanisms of axisymmetric vortex breakdown is conducted. Using a steady, inviscid theoretical model, tilting of the axial component of the vorticity vector into the negative azimuthal direction due to streamsurface divergence or adverse pressure gradients, is found to be the dominant mechanism leading to vortex breakdown. This result is then confirmed by numerically solving the incompressible, axisymmetric Navier-Stokes equations for vortex breakdown in a converging-diverging pipe. Theoretical and numerical analysis of viscous contributions to the production of negative azimuthal vorticity shows that, except at low Reynolds numbers, viscous contributions are negligible in the deceleration of the core flow upstream of breakdown. A susceptibility parameter is suggested which quantifies the susceptibility of a vortex to breakdown. This parameter, related to the total pressure loss in the vortex core, successfully correlates numerically simulated vortex breakdown results over a wide range of vortex inlet conditions and also agrees with experimental trends regarding breakdown susceptibility. Streamsurface divergence is shown to drive a supercritical vortex (supporting downstream-running waves only) towards criticality; thus, the production of negative azimuthal vorticity, which indicates amplification of streamsurface divergence, also accelerates the approach of criticality. Simulations of transient breakdown evolution reveal the trapping and amplification of waves which approach the critical location. Thus, breakdown is associated with wave trapping occurring when a vortex attains critical conditions. Finally, quantitative comparisons of numerical and experimental vortex breakdown show good agreement indicating the axisymmetric assumption is applicable in some physically-realizable breakdown flows.

Thesis Supervisor: Earll M. Murman,
Professor of Aeronautics and Astronautics
To my grandparents,

Florence and Louis Darmofal,

Irene and Anthony Jablonski.
Acknowledgments

It's 1 A.M. on a Monday evening, Ella is singing to me through my earphones, and the screen of my computer is filled with a large text window. I have been waiting for this moment for as long as I can remember and now that it's here, I find myself lacking the words to tell the many important people in my life how much their help, support, and love have meant to me. I apologize for my lack of eloquence. Also, I apologize for any omissions – so many people deserve mention that I'm sure to forget someone.

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On December 19, 1993, I will become the happiest man on the planet. I will be marrying the woman I have been in love with since the day we met. Although Claudia and I have been separated by 708 miles, our relationship has grown stronger with each day, with each "miss you" over the phone. If I have burned the midnight oil stronger and longer than others, she is the reason. I love you, TC, and I'll be home soon.

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Nomenclature

Roman symbols

$A$ $B + R$, total vorticity production
$B$ $S + T$, total inviscid term
$c$ phase velocity
$c_g$ group velocity
$H$ total pressure
$H'_0$ total pressure derivative with respect to streamfunction at axis
$p$ static pressure
$r$ radial coordinate
$R$ viscous vorticity production term
$Re$ Reynolds number
$Re_{sz}$ experimental Reynolds number
$S$ vortex stretching term
$s_\Psi$ streamline arc length
$T$ vortex tilting term
$t$ time
$t_b$ breakdown timescale
$t_c$ convective timescale
$u$ radial velocity
$v$ azimuthal velocity
$w$ axial velocity
$W_\infty$ freestream or edge axial velocity
$W_0$ axial velocity at axis
$z$ axial coordinate
$z_b$ breakdown location with respect to test section
$z_B$ axial location with respect to breakdown location
Greek symbols
\( \delta \)  
core size (approximately maximum swirl location)
\( \Delta w \)  
difference of freestream and axis axial velocity
\( \eta \)  
azimuthal vorticity
\( \gamma \)  
standing wave wavenumber
\( \gamma_0 \)  
smallest standing wave wavenumber
\( \Gamma \)  
circulation
\( \Omega \)  
Inlet swirl ratio
\( \Omega_{sz} \)  
Experimental inlet swirl ratio
\( \Omega_c \)  
Critical inlet swirl ratio
\( \Psi \)  
streamfunction
\( \theta \)  
azimuthal angle
\( \xi \)  
radial vorticity
\( \zeta \)  
axial vorticity

Subscripts and superscripts
\( ()_\infty \)  
free stream or edge
\( ()_0 \)  
inlet or axis
\( ()_{sz} \)  
experimental
\( ()_\nu \)  
calculated using viscous solver
\( ()_i \)  
calculated using inviscid theory
\( ()_i \)  
irrotational
\( ()_{j,k} \)  
computational node numbering
\( ()_q \)  
calculated using quasi-cylindrical equations
\( ()_r \)  
rotational
\( ()' \)  
differentiation with respect to the streamfunction
\( () \)  
perturbations
Chapter 1

Introduction

Why the bump goes bump.

C. Soberger

1.1 Vortex Breakdown Phenomenon

Vortex breakdown has been a widely researched fluid phenomenon since its discovery by Peckham and Atkinson[54] in 1957. In their investigation of the flow over "Gothic" wings, they noted "The condensation trail appeared to 'bell-out' before disappearing - as though the core was becoming more diffuse." Perhaps the most widely published photograph of vortex breakdown over delta wings, reproduced in Figure 1.1, was taken by Lambourne and Bryer[36]. Of particular interest in this photograph is the simultaneous occurrence of both the so-called axisymmetric and spiral modes of vortex breakdown.

Vortex breakdown is an important fluid phenomenon because its occurrence generally limits the lift generated by low aspect ratio wings at high angles of attack (such as the wings of fighter aircraft). For example, Figure 1.2 from Jarrah[30, 29] shows the variation in normal force versus angle of attack for static and pitching delta wings. In the static case, the maximum normal force coefficient occurs at approximately 32 degrees which coincides with the passing of vortex breakdown over the trailing edge of the delta wing. As angle of attack increases, the location of breakdown progresses
Figure 1.1: Axisymmetric and spiral modes of breakdown [36]

further forward on the wing and the lift typically decreases as well. This loss of lift, however, does not always coincide with the occurrence of vortex breakdown over the wing. Kegelman and Roos found that the loss of lift due to vortex breakdown is also a function of the leading-edge sweep angle[32]. Summarizing their results, Kegelman and Roos found that for leading-edge sweep above 70 degrees, the occurrence of breakdown over the wing coincides with the angle at which maximum lift is attained; however, below 70 degrees sweep, breakdown occurs over the wing well before the maximum lift angle. An additional, negative effect of vortex breakdown is the large-scale, unsteady flow which is produced downstream of breakdown; for fighter aircraft, this unsteady flow can result in large, unsteady forces acting on control surfaces and has been known to cause early fatigue of tails on F-18 aircraft[40].

During maneuvers, fighter aircraft are subjected to transient pitch variations which often encompasses angle of attack regions in which vortex breakdown occurs in the static flow. Figure 1.2 also shows the normal force coefficient for a delta wing undergoing a sinusoidal variation in angle of attack. A significant hysteresis effect exists for this unsteady maneuver. As the wing pitches up, the normal force coefficient continues increasing beyond the static maximum and attains an almost 30 percent increase at its maximum. During the pitchdown portion of the cycle, the normal force coefficient actually lags the breakdown static values. Jarrah[30] identified the reason for the
hysteresis effects during the pitchup as a lag in the location of vortex breakdown with respect to the static case; Jarrah noted that the breakdown lag with respect to the static case was approximately 7 to 10 convective timescales. In the pitchdown portion of the cycle, Jarrah attributed the normal force lag to a delay in a reformation of the leading-edge vortex. In a similar set of experiments, Reynolds and Abtahi[59] found lags in breakdown position ranging from 1 to 30 convective timescales during pitchdown motions.

Soon after the discovery of vortex breakdown in delta wing flows, researchers began to isolate the problem by considering the swirling flow through pipes[27]. The two most thorough studies of vortex breakdown in pipes were conducted by Sarpkaya[61, 62, 60] and later by Leibovich and his students[21, 20, 22]. Sarpkaya[61] classified three types of breakdown: bubble, spiral, and double helix. The bubble mode was the predominant mode at high swirl ratios while the spiral mode occurred for lower swirl ratios. The double helix mode was a rarely observed form which occurred at high swirls and low Reynolds numbers. The forms of breakdown were also observed to randomly alternate
amongst each other. Sarpkaya[60] also studied the effect of adverse pressure gradient by changing the pipe divergence of his experimental apparatus. In these experiments, Sarpkaya found that vortex breakdown occurred for lower swirl ratios as the adverse pressure gradient increased. Leibovich, using essentially the identical experimental set-up, also studied the vortex breakdown phenomenon. Faler and Leibovich[21] were the first to obtain measurements of the internal structure of an "axisymmetric" breakdown bubble. Although the bubble maintained reasonable axisymmetry in the upstream half, the rear half of the bubble was both unsteady and non-axisymmetric. Therefore, although the bubble mode of breakdown appears largely axisymmetric, non-axisymmetric effects still play some role in determining the exact behavior of the reversed flow region. Furthermore, Leibovich and Faler constructed ‘time-averaged’ streamlines in the breakdown bubble which showed the presence of a two-celled bubble structure with four stagnation points on the axis (see Chapter 6 for more detail on their results).

Recently, with the introduction of sophisticated, non-intrusive measurement techniques, several investigators have provided a much more detailed picture of the entire breakdown flowfield. Bornstein and Escudier[9] measured the internal structure of an axisymmetric breakdown and found several differences with the Faler and Leibovich measurements. In particular, they found only a single-celled bubble with an open tail and only two stagnation points on the axis. Detailed measurements have also been reported by [66, 13, 12]. The bubble structure observed in these experiments, as well as in the Faler and Leibovich and Bornstien and Escudier experiments, varied and the general conclusion which may be reached is that the exact form of breakdown appears to be very application specific.

The flow in an enclosed cylinder with a rotating endwall also produces breakdown bubbles. These flows have been studied experimentally by Vogel[68] and Escudier[17]. One particular advantage of these flows is that they tend to be highly axisymmetric. Also, the flows are a function of only two parameters: the Reynolds number based on rotation rate, and the cylinder aspect ratio. Furthermore, the boundary conditions are known exactly unlike experiments in tubes where the inflow and outflow conditions have
some uncertainty.

1.2 Theories of Vortex Breakdown

Many different theories have been suggested to explain vortex breakdown since its discovery. In the following, we briefly review these theories highlighting the important contributions they provide in understanding vortex breakdown. This review is by no means exhaustive and also reflects the views and opinions of the author. For further information (as well as different interpretations), several excellent reviews exist and should be consulted [26, 41, 42, 16].

1.2.1 Hydrodynamic Instability

The hydrodynamic instability theory of vortex breakdown hypothesizes that breakdown is the result of the instability of the upstream flow to infinitesimal disturbances. Leibovich presents a thorough review of general stability results for inviscid, swirling flows [42]. These results are mainly theoretical in nature. Recently, two independent efforts by Khorrami [33] and Mayer and Powell [50] mapped out the neutral curve of the q-vortex (see Appendix D) family of profiles for both viscous and inviscid disturbances.

In general, experimental measurements upstream of breakdown show the approach flows to be stable to all infinitesimal disturbances. Also, the form of breakdown, with its rapid change from a columnar state to a bubble or spiral and oscillations occurring only downstream, is not indicative of typical manifestations of hydrodynamic instabilities. Therefore, instability is generally not accepted as a reasonable explanation for vortex breakdown. Although instability mechanisms seem unimportant with regards to the onset of axisymmetric vortex breakdown, nonetheless, instability does seem to manifest itself downstream of breakdown. For example, experimental evidence exists
which suggests that the oscillations behind breakdown may be a result of hydrodynamic instability[22].

1.2.2 Boundary Layer Analogy

Upstream of breakdown, the gradients of the flow variables in the axial direction are much smaller than those in the radial direction. In this section of the flow, it is possible to describe the motion using a parabolized set of equations analogous to the boundary layer equations. These parabolized equations for swirling flows are generally referred to as the quasi-cylindrical equations. Hall[25] suggested that vortex breakdown is linked to the failure of the quasi-cylindrical equations in much the same way that separation is associated with the failure of the boundary layer equations. By numerically integrating the quasi-cylindrical equations in space, Hall found that the position of breakdown could be roughly predicted by the failure of the integration method. Also, Hall[26] was able to show that, due to the interaction of the swirl velocity with the radial pressure gradient, axial pressure gradients in the core are amplified with respect to axial pressure gradients in the freestream.

Even with this success, several serious drawbacks arise with this explanation of vortex breakdown. Firstly, the quasi-cylindrical equations have no mechanism to allow upstream influences. Also, the flow in the breakdown region itself cannot be calculated using this approach. Hall[26] points out that the boundary layer analogy also fails to describe the abruptness of vortex breakdown.

1.2.3 Wave Theories

Squire (as described in [26, 41, 42, 16]; original reference, [63]) first suggested that vortex breakdown may be explained by considering the waveguide nature of vortex
flows. A vortex may be divided into a supercritical state, one which allows only downstream propagation of waves, or a subcritical state, one which allows both upstream and downstream propagation of waves[5] analogous to supersonic and subsonic states in compressible flow. Squire suggested that breakdown occurs when the flow becomes critical as a result of disturbances present downstream being able to propagate upstream. These upstream propagating disturbances are hypothesized to lead to breakdown. Experimental evidence generally indicates that, with respect to axisymmetric waves, the flow upstream of vortex breakdown is always supercritical and the flow downstream of vortex breakdown is subcritical[42, 64]. Unfortunately, Squire provides no insight into where the critical state might occur. Also, the process by which an upstream disturbance might lead to breakdown is left unexplained. Since Squire’s initial suggestion, several other researchers have investigated the existence of waves on vortex flows and their possible connection with vortex breakdown[44, 58, 43, 45]. A more thorough review of wave theories of vortex breakdown is given in Chapter 5. Additionally, the reader may wish to consult Leibovich’s review[42].

1.2.4 Conjugate States

A theory mainly developed by Benjamin[5, 4] attempts to relate vortex breakdown to a jump from an initially supercritical state to a conjugate subcritical state. This conjugate state theory seeks an analogy between other well-known jump phenomenon such as shocks and hydraulic jumps. Benjamin found that a cylindrical (no axial variations) supercritical flow always possesses a cylindrical subcritical conjugate state. However, the jump to the subcritical conjugate state does not conserve axial momentum; therefore, Benjamin also hypothesized that breakdown must result in the formation of waves to account for a momentum loss. A difficulty with Benjamin’s hypothesis, which has been pointed out by previous authors[26, 42], is that the upstream flow need only be supercritical. The theory does not suggest when the jump to a conjugate state will occur. Furthermore, as noted with Squire’s theory, Benjamin does not suggest how the conditions for a jump to conjugate flow are developed. Several other researchers have
developed variants on Benjamin's conjugate state theory most notably are Landahl and Widnall[38] and Escudier and Keller[19].

1.2.5 Vorticity Dynamics

Brown and Lopez first recognized the importance of the swirl or azimuthal component of vorticity in the stagnation of the vortex core upstream of breakdown[11]. Using a steady inviscid flow model, they showed that a necessary but not sufficient condition for the stagnation of a vortex is

\[
\frac{v_0 \, \zeta_0}{w_0 \, \eta_0} > 1,
\]

where \(v_0, w_0\) are the azimuthal and axial velocity components and \(\eta_0, \zeta_0\) are the azimuthal and axial vorticity components. The subscript implies the velocity and vorticity values are those for some upstream location. If this condition is met, it is possible for a vortex to produce negative azimuthal vorticity, and, therefore, for breakdown to occur. Negative azimuthal vorticity production plays a key role in the description of vortex breakdown which is given in this thesis. Further discussion related to this topic is contained in Chapter 2.

1.3 The Applicability of the Axisymmetric Assumption

Numerous investigators have stressed the importance of non-axisymmetric effects in vortex breakdown with the most thorough arguments perhaps given by Leibovich[42]. In order for any theory of vortex breakdown to account for both the spiral and bubble form of breakdown, non-axisymmetric effects must be included. However, the flow upstream of breakdown tends to be axisymmetric to a large extent; furthermore, bubble breakdowns occur in real applications which, although not completely axisymmetric, are dominated by axisymmetric features[36, 61, 9, 17, 21]. The axisymmetric assumption
is strongest for flows with high levels of swirl for which the bubble breakdown mode is more commonly observed than the spiral breakdown mode[61, 20]. The validity of the axisymmetric assumption is also assessed a posteriori in Chapter 6 where numerical solutions to the axisymmetric equations are found to have good agreement with experimental flow measurements. Even at lower values of swirl, the mechanisms which lead to axisymmetric breakdown may be important in modifying the vortex core such that non-axisymmetric instabilities are present. Although this argument does not justify the assumption of axisymmetry, we believe that the fundamental mechanisms which lead to axisymmetric vortex breakdown are far from understood and a logical step to a complete explanation of vortex breakdown is to first have a rational explanation of axisymmetric vortex breakdown. This thesis is limited in scope to the purely axisymmetric form of breakdown although a framework leading to the possible explanation of spiral breakdown will be suggested in Chapter 7.

1.4 Previous Numerical Studies of Axisymmetric Breakdown

Many numerical studies of vortex breakdown have been performed using the axisymmetric, incompressible Navier-Stokes equations. The first successful simulation of axisymmetric vortex breakdown was by Lavan et al[39] in 1969. However, their main interests were not related to vortex breakdown. Kopecky and Torrance[34] present the first demonstrative calculations of vortex breakdown in a pipe assuming axisymmetry. Later, using more refined meshes than Kopecky and Torrance, Grabowski and Berger[23] calculated breakdown for a vortex embedded in an outer irrotational flow. All of these results unfortunately suffer from the breakdown bubble residing next to the inlet of the computational domain and raise questions concerning the accuracy of these simulations. In particular, Kopecky and Torrance’s inlet conditions were always found to be subcritical and some of Grabowski and Berger’s inlet profiles were also subcritical[41]. Menne[51] simulated breakdown in slightly diverging tubes similar to the pipe flow ex-
periments of Sarpkaya and Leibovich. These results were only qualitatively compared to the available experimental data; however, an overall qualitative agreement between the experimental and numerical results was found. Lopez\cite{47} calculated the flow in an enclosed cylinder with a rotating endwall and found excellent quantitative agreement with results from Escudier’s experimental work. Beran’s\cite{6,7} simulations of the steady equations are of particular note because they reveal a non-uniqueness in the governing equations. He found multiple solutions which closely resemble the semi-analytic work of Leibovich and Kribus\cite{45} who found multiple solutions of the steady, inviscid form of the governing equation. Also of particular interest is Beran and Culick’s\cite{7} use of a converging-diverging pipe geometry which served to isolate vortex breakdown from the inlet station. Additional simulations of vortex breakdown using the axisymmetric equations are \cite{35,24,53,52,65}.

These previous numerical studies generally focus only on the ability to calculate vortex breakdown. However, in this thesis, computational results are used to analyze previous theoretical results. In particular, our efforts will be focused on determining the essential mechanisms which lead to axisymmetric vortex breakdown.

1.5 A Conceptual Framework for Vortex Breakdown

In \cite{45}, Leibovich and Kribus suggest that a minimal set of effects which must be accounted for in any explanation of vortex breakdown are:

1. axial variation caused by pressure gradients,

2. the growth of large amplitude, axisymmetric waves,

3. energy transfer from axisymmetric waves to three dimensional modes.
This thesis aims at providing an explanation of the first two effects and thereby developing a rational theory for the axisymmetric form of vortex breakdown.

To illustrate the main hypothesis of this work, we consider first an analogy with supersonic decelerating flow. This analogy with vortex breakdown was first proposed by Bilanin[8] as an application of the work of Landahl[37]. Suppose we consider a steady supersonic decelerating flow that approaches very near to sonic conditions. If the wave properties of this flow are analyzed, two wave speeds will be found corresponding to an advancing wave with speed \(u + a\) and a receding wave with speed \(u - a\) (where \(u\) is the local axial velocity and \(a\) is the local speed of sound). As the flow approaches sonic conditions, the receding waves decelerate to near zero velocity; furthermore, according to linear theory, the amplitude of the waves will vary as \((u - a)^{-\frac{1}{2}}[37]\). This growth in wave amplitude is an example of the space-time focussing of waves. The infinite amplitude of these linear waves is avoided when non-linearities are introduced, however, the general trends are expected to be consistent with linear theory – namely, the deceleration, or so-called trapping, of waves at the sonic location along with a subsequent growth in wave amplitude. The two main mechanisms leading to this phenomenon are

1. the axial variation of wave speeds such that sonic conditions are approached,

2. the growth of large amplitude waves due to wave trapping.

These two items are analogous to the first two items which Leibovich and Kribus suggest are necessary in any description of vortex breakdown.

Next, we consider separately the two mechanisms described above in the context of vortex breakdown. The first element centers on describing the drive of the vortex core flow from supercritical towards subcritical conditions (i.e. the axial variations of the wave speeds). In a compressible flow, the wave speed of the base flow changes axially due to area variations, or, equivalently, pressure gradients. In vortical flows, flow divergence or pressure gradients play an analogous role. In particular, we will analyze the production of negative azimuthal vorticity which Brown and Lopez found to be
a necessary part of the core deceleration and show that negative azimuthal vorticity is created in response to flow divergence. Furthermore, the production of negative azimuthal vorticity amplifies the core deceleration relative to the outer flow and can be shown to drive the flow towards critical conditions. An important contrast with the work of Bilanin[8] is that although Bilanin first suggested wave trapping as an explanation for vortex breakdown, no attempt was made at identifying the mechanisms which lead to flow criticality.

In the second element of this theory, breakdown is associated with the trapping and amplification of a wave at the critical location. Similar to decelerating supersonic flow, as the propagation speed of the analogous receding wave decreases, these waves become trapped near the critical location, amplify, and eventually form a breakdown bubble. Bilanin used results from Landahl[37] and Whitham's kinematic wave theory[69] to show that wave trapping was at least possible; however, due to the inherent complexities of such an analysis, a definitive illustration of wave trapping leading to vortex breakdown was not obtained. However, using results from the numerical simulation of axisymmetric flows in pipes, we will be able to provide strong evidence that vortex breakdown is indeed an outcome of wave trapping phenomenon.

1.6 Approach

As mentioned above, the approach used in this study is to combine theoretical and numerical results with the eventual aim of analyzing various breakdown theories and, hopefully, arriving at an understanding of the phenomenon of vortex breakdown. The theoretical considerations developed in this thesis mainly rest on the previous work of other researchers. Furthermore, the numerical algorithm developed for the purposes of this investigation is only a slight variant of common computational techniques. The uniqueness in this thesis is the linking of theory with computation. Where as previous computational studies of vortex breakdown have only focussed on the ability to calculate
the phenomenon, in this thesis, emphasis is placed on examining breakdown theories using computational results. One of the powers of computational simulations is the ability to analyze flows which would not be possible to obtain experimentally. For example, vortex breakdown at lower swirl ratios typically occurs in the spiral mode, and the axisymmetric mode gradually becomes dominant with increasing swirl. If we tried to analyze vortex flows which are on the verge of axisymmetric breakdown, our attempts would inevitably be complicated by the appearance of the spiral mode of breakdown. Using an axisymmetric flow solver, we can calculate vortex flows which are at axisymmetric critical conditions without interference from the spiral mode. In this manner, computational techniques can be used to sort out the fundamental processes involved in axisymmetric vortex breakdown and this approach is employed throughout the remainder of this dissertation.

1.7 Thesis Overview

Before continuing with the main presentation, a brief overview of the thesis organization will be provided. Chapters 2-4 focus on the first element of the breakdown process, the production of azimuthal vorticity. Initially, we consider the production of negative azimuthal vorticity as the process by which external pressure gradients are amplified in the vortex core, and the link between azimuthal vorticity production and the approach of flow criticality will be delayed until Chapter 5. Chapter 2 describes the vorticity dynamics of a vortex flow as it approaches breakdown and illustrates how all physically-realizable vortices must produce negative azimuthal vorticity in the presence of an adverse pressure gradient. Specifically, we will find that the production of negative azimuthal vorticity is driven by the tilting of the axial component of vorticity due to streamsurface divergence. Chapter 3 investigates the modifying effects of viscosity on the inviscid vorticity dynamics mechanisms discussed in Chapter 2. Two viscous effects are identified which contribute to the production of azimuthal vorticity; however, scaling arguments and numerical results show that these production mechanisms are unimpor-
tant except at low Reynolds numbers. In Chapter 4, the susceptibility of a vortex to breakdown is discussed. First, the Brown and Lopez[11] velocity-vorticity helix angle condition for the eventual production of negative azimuthal vorticity is shown to be equivalent to requiring a total pressure deficit in the core with respect to the outer flow. Then, a parameter related to the core total pressure loss is proposed as a breakdown susceptibility measure. This parameter correlates a wide range of breakdown results and agrees with known experimental trends regarding vortex breakdown. The relation to previous breakdown indicators is discussed as well as possible applications of the susceptibility parameter.

Chapter 5 considers the second element of the breakdown process, the trapping and amplification of waves at the critical location. First, the production of negative azimuthal vorticity is shown to accelerate the approach of flow criticality. Then, a transient simulation of the breakdown process starting from a near critical vortex is described which illustrates the wave trapping nature of vortex breakdown. This chapter also discusses the impact of downstream boundaries on the breakdown process and finds that for flows near criticality, downstream effects can become important. In Chapter 6, detailed comparisons with experimental pipe flow results are made; the axisymmetric equations are found to be in good agreement with the experimental data. Then, using experimentally-observed conditions, the role of negative azimuthal vorticity production and wave trapping in vortex breakdown are again illustrated. Finally, we conclude this thesis with a summary of important contributions and recommendations for future work in Chapter 7.

A philosophy of this thesis is that the governing equations and the numerical algorithms provide the backbones of the work; however, the details of both are rather well known. Therefore, all relevant equations and a description of the numerical algorithms have been placed in appendices for reference. Appendix A presents the various forms of the governing equations used in the course of this dissertation. Appendix B describes the finite volume-based numerical algorithm used to solve the incompressible, Navier-Stokes equations. Results from a validation study of the numerical algorithm appear in
Appendix C; in this appendix, the flow in an enclosed cylinder with a rotating endwall is calculated and successfully compared with experimental and previous computational results. Appendix D contains a brief description of the q-vortex model which is used as an inlet condition for most of the numerical pipe flow calculations. Appendix E documents several non-unique solutions which were found during the course of this work; these solutions are in agreement with the previous calculations of Beran and Culick[7], and, as such, do not represent a new or significant contribution. Finally, Appendix F presents a preliminary investigation of transient effects in vortex breakdown; in particular, the timescales at which breakdown evolves and decays are studied in an attempt to develop some insight into possible lag mechanisms in unsteady breakdown flows such as pitching delta wings.
Chapter 2

The Role of Vorticity Dynamics in Axisymmetric Vortex Breakdown

Take care of the vorticity and the rest of the flow will take care of itself.

Sir W.R. Hawthorne

Vorticity dynamics plays an important role in the breakdown of a vortex. The interaction of the swirling component of the flow with the axial component is responsible for not only the significant axial velocities observed in the core of a leading-edge vortex but also the rapid stagnation of the core resulting in breakdown. In particular, the dynamics of the vorticity field, and the resultant implications on the velocity field, can be useful in describing the mechanisms responsible for the deceleration of the vortex core and vortex breakdown. In Section 2.1, azimuthal vorticity production is studied under the assumptions of steady, inviscid flow and the tilting of the axial component of vorticity due to flow divergence is found to be the dominant mechanism in the initial deceleration of the vortex core. Numerical results using a viscous flow solver are presented in Section 2.2 which confirm the previous steady, inviscid analytic. Finally, a summary of the results from this chapter is given in Section 2.3.
2.1 Vorticity Dynamics of Vortical Flows

A natural coordinate system to analyze axisymmetric vortical flows is a cylindrical coordinate system. The coordinate directions, \((r, \theta, z)\), the velocity components, \((u, v, w)\), and the vorticity components, \((\xi, \eta, \zeta)\) are shown in Figure 2.1. Upstream of breakdown, the radial velocity and the axial gradients are quite small. In this case, typical velocity and vorticity distributions are shown in Figure 2.2. These distributions are from the \(q\)-vortex model which is given by:

\[
\begin{align*}
    u_0(r) &= 0, \\
    \Gamma_\omega(r) &= \Omega \left[1 - \exp(-r^2)\right], \tag{D.1} \\
    w_0(r) &= 1 + \Delta w \exp(-r^2),
\end{align*}
\]

where \(\Omega\) is called the swirl ratio and \(\Delta w\) measures the freestream to core axial velocity difference. The circulation, \(\Gamma\), is defined as, \(\Gamma = v_r\). Many vortices have qualitatively
the same features upstream of breakdown[21]. In Figure 2.2, we note the axial velocity has a jet profile with $\Delta w = 1$, and, the corresponding azimuthal vorticity, $\eta$ (which is approximately $-\frac{\partial w}{\partial r}$ upstream of breakdown), must be positive for a jet flow. The swirl velocity, $v$, shows a core which rotates as a solid body (i.e. $v \propto r$) and an outer flow which behaves as a potential flow (i.e. $v \propto \frac{1}{r}$). The axial vorticity, $\zeta = \frac{1}{r} \frac{\partial (w r)}{\partial r}$, is concentrated in the core and, as $r$ increases, the vorticity goes to zero. The governing equations and other definitions relevant to the following discussion are given in detail in Appendix A and repeated where appropriate in the text.

The importance of the azimuthal vorticity was first recognized by Brown and Lopez[11]. They proposed that the production of negative azimuthal vorticity is necessary for the flow to decelerate on the axis. Their reasoning was that any velocity field may be broken
into an irrotational and rotational part,

\[ \ddot{u}(\vec{x}) = \ddot{u}_i(\vec{x}) + \ddot{u}_r(\vec{x}), \]  

(2.1)

where the irrotational portion contains no vorticity, \( \nabla \times \ddot{u}_i = 0 \), and the rotational velocity field is given by the Biot-Savart law

\[ \ddot{u}_r(\vec{x}) = -\frac{1}{4\pi} \int \frac{\vec{s} \times \vec{\omega}(\vec{x}')}{s^3} \, dV, \]  

(2.2)

where \( \vec{s} = \vec{x} - \vec{x}' \). Under these definitions, the vorticity is

\[ \vec{\omega} = \nabla \times \ddot{u} = \nabla \times \ddot{u}_r. \]

In vortex breakdown, the axial velocity near the axis stagnates and reverses. Brown and Lopez suggested that the stagnation of the core flow must be due mainly to the rotational component of velocity, \( u_r \). Assuming an axisymmetric flow, the rotational portion of the axial velocity at a point \((r, z)\) is given by

\[ w_r(r, z) = -\frac{1}{2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{r'(r - r')}{{(r - r')^2 + (z - z')^2}^{3/2}} \, dr' dz'. \]  

(2.3)

As described in Appendix A, \( w \) is the axial velocity (\( w_r \) being the rotational component of \( w \)), and \( \eta \) is the azimuthal or ‘ring’ vorticity. On the axis, Equation (2.3) simplifies to

\[ w_r(0, z) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{r'^2 \eta(r', z')}{[r'^2 + (z - z')^2]^{3/2}} \, dr' dz'. \]  

(2.4)

The contributions to the rotational axial velocity at \( r = 0 \) are entirely positive except for the sign of the azimuthal vorticity. Therefore, in order for \( w_r(0, z) \) to become negative, negative azimuthal vorticity must develop. Consequently, Brown and Lopez conclude that the production of negative azimuthal vorticity is a necessary part of the deceleration and ultimate breakdown of a vortical flow. The work herein supports this conclusion and seeks to understand how negative azimuthal vorticity is produced via the mechanisms of vortex stretching and tilting.
The role of vorticity dynamics in the breakdown process can be understood from the vorticity equation. By taking the curl of the momentum equation, Equation (A.1), the vorticity equation may be derived:

\[
\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{\omega}.
\]  

(2.5)

The first term on the right hand side of this equation represents the combined effects of vortex stretching and tilting. Viscous effects will be assumed negligible for the present discussion; justification for this assumption will be provided in Chapter 3. Dropping the viscous terms and writing out Equation (2.5) fully, we find:

\[
\frac{D\xi}{Dt} = \xi \frac{\partial u}{\partial r} + \zeta \frac{\partial u}{\partial z},
\]

(2.6)

\[
\frac{D\eta}{Dt} = \frac{\partial v}{\partial r} + \zeta \frac{\partial v}{\partial z} - \frac{v\xi}{r} + \frac{u\eta}{r},
\]

(2.7)

\[
\frac{D\zeta}{Dt} = \xi \frac{\partial w}{\partial r} + \zeta \frac{\partial w}{\partial z}.
\]

(2.8)

Below each equation, the letters 'S' and 'T' signify which terms are vortex stretching and vortex tilting, respectively. The following discussion of vortex stretching and tilting focuses on the \( \eta \) transport equation, Equation (2.7); however, similar arguments can confirm the proper identification (i.e. stretching or tilting) of the production terms in Equations (2.6) and (2.8).

An important point is that the relation between the azimuthal vorticity and the axial velocity, as expressed in Equations (2.2)-(2.4), is simply kinematic. However, the production of azimuthal vorticity, as expressed in Equation (2.7), is dynamic. Specifically, Equation (2.5) represents the conservation of angular momentum of a fluid element. The tilting and stretching term, \( \vec{\omega} \cdot \nabla \vec{u} \), is a result of a change in the shape of a fluid element, and, \( \nu \nabla^2 \vec{\omega} \) is the net viscous torque which is applied to a fluid element.
2.1.1 Vortex Stretching

The last term on the right hand side of Equation (2.7) was identified as a vortex stretching term. Consider a radial station, \( r \), with radial velocity, \( u \), and azimuthal vorticity, \( \eta \), as shown in Figure 2.3. A vortex ring may be imagined to exist with its length, \( l \), equal to the circumference, \( 2\pi r \). If the radial velocity is positive (negative), then the length of the vortex increases (decreases) due to the circumference change. Since the fractional change of the vorticity vector, \( \vec{\omega} \), is equal to the fractional change of the material element vector, \( \vec{l} \), which is initially aligned with the vorticity vector (see Batchelor[3] among others), we may write:

\[
\begin{align*}
\frac{d\vec{\omega}}{|\vec{\omega}|} &= \frac{d\vec{l}}{|\vec{l}|} \\
\frac{d\eta}{\eta}_s &= \frac{dl}{l} \\
\frac{1}{\eta} \frac{D\eta}{Dt}_s &= \frac{1}{l} \frac{Dl}{Dt} \\
\frac{1}{\eta} \frac{D\eta}{Dt}_s &= \frac{1}{2\pi r} \left( 2\pi \frac{Dr}{Dt} \right) \\
\frac{1}{\eta} \frac{D\eta}{Dt}_s &= \frac{u}{r}
\end{align*}
\]
\[ \Rightarrow \frac{D\eta}{Dt}_s = \frac{u\eta}{r} \]

where the subscript \( s \) indicates growth due to stretching. This mechanism plays an important role in the breakdown of a vortex. As pointed out by Brown and Lopez, the presence of negative azimuthal vorticity indicates a deceleration of the flow near the axis. From continuity, a decrease in the axial velocity must create a positive radial velocity. Thus, the possibility of a nonlinear feedback exists as a result of azimuthal vorticity stretching. Specifically, the following feedback could occur:

1. Consider a vortex with \( u > 0 \) and \( \eta < 0 \)

2. \( \frac{D\eta}{Dt}_s < 0 \Rightarrow |\eta| \) increases

3. Flow in the core (near the axis) decelerates

4. From continuity, fluid must be ejected from core \( \Rightarrow u \) increases

5. Growth rate of \( |\eta| \) increases.

For this set of events to occur, negative azimuthal vorticity must be present. Since vortices seen over delta wings are typically jet flows before breakdown, the azimuthal vorticity will generally be positive initially. Thus, in order for this stretching mechanism to become important, the azimuthal vorticity must first become negative through a different effect.

2.1.2 Vortex Tilting

Before breakdown, the vorticity vector consists mainly of axial and azimuthal components with the radial component typically near zero. The reduction of the azimuthal vorticity is accomplished through the tilting of the initially axial component of the vorticity vector into the azimuthal direction. Equation (2.7) contains three vortex tilting
terms. The second tilting term, $\zeta \pd{u}{z}$, is the tilting of the axial vorticity, $\zeta$, by an axial gradient of the swirl velocity. Figure 2.4 shows the tilting of the axial vorticity into the azimuthal direction. As in the discussion of the vortex stretching term in Section 2.1.1, the material line element initially coincides with the vorticity vector and therefore lies in the $z$-direction. Figure 2.4 (a) shows the rotation of the material line element as a result of an axial swirl gradient, $\pd{u}{z}$. In a flow with an axial gradient of the swirl velocity, the head of the material line element will travel in the azimuthal direction at a different speed than its tail. The result will be a tilting of the material line element into the azimuthal direction. Since the fractional change of the vorticity vector is proportional to the fractional change of the material line element vector, the axial swirl velocity gradient also results in the production of an azimuthal component of vorticity (see Figure 2.4 (b)). Using small angle approximations, this tilting mechanism can be derived as follows:

\[
\frac{d\vec{\omega}}{|\vec{\omega}|} = \frac{d\vec{l}}{|\vec{l}|}
\]

\[
\frac{d\vec{\omega}}{\zeta} = \frac{d\vec{l}}{dz}
\]

\[
\frac{d\eta}{\zeta} = \tan \phi
\]

\[
d\eta \approx \zeta \pd{u}{z} dt
\]

\[
\Rightarrow \left. \frac{D\eta}{Dt} \right|_{t_t} = \zeta \pd{u}{z}
\]
Figure 2.5: Inviscid creation of axial circulation gradient

The $t_z$ subscript indicates production due to the tilting of the axial vorticity component.

Similarly, the first and third tilting terms correspond to the rotation of radial vorticity, $\xi$, into the azimuthal direction. The combined effects of the tilting of the radial and axial vorticity components may be found by substituting Equations (A.13) and (A.15) into the tilting terms. After some cancellation, the result is:

$$\frac{D\Gamma}{Dt} = \frac{\partial}{\partial z} \left( \frac{\Gamma^2}{r^2} \right),$$

where $\Gamma$ is the circulation, $v r$. Thus, an axial change in circulation is analogous to the tilting of the axial and radial vorticity components into the azimuthal direction.

In a steady, inviscid flow, the circulation is only a function of the streamfunction, $\Gamma = \Gamma(\Psi)$, and the tilting contribution to the azimuthal vorticity production may be expressed as:

$$\frac{D\Gamma}{Dt} = -\frac{u}{r^2} \frac{d(\Gamma^2)}{d\Psi}.$$ 

Therefore, the tilting of axial and radial vorticity, or equivalently, the creation of an axial circulation gradient (Equation 2.9) in an inviscid flow occurs as a result of a streamtube divergence or non-zero radial velocity. Figure 2.5 depicts the creation of this circulation gradient. Thus, a non-zero radial velocity can create azimuthal vorticity simply as a result of the conservation of circulation. Furthermore, for physically-realistic situations, the distribution of circulation in a vortex before breakdown is such that $\frac{d(\Gamma^2)}{d\Psi} \geq 0$. In the case where $\frac{d(\Gamma^2)}{d\Psi} < 0$, the flow violates the Rayleigh criterion and is inviscidly unstable[42]. No evidence has been found of vortices which violate the Rayleigh criterion upstream.
of breakdown[42]. Hence, for the purposes of this investigation, all vortices upstream of breakdown will be assumed stable with \( \frac{d(\Gamma^2)}{d\theta} \geq 0 \). As a result, an outward radial velocity will result in a decrease of the azimuthal vorticity. This outward radial velocity could be created through simple potential flow effects. In a tube, a divergent section would create a positive radial velocity, or, the adverse pressure gradient at the trailing edge of a delta wing would have a similar effect. If the outward radial velocity persists, the tilting of vorticity will continually feed into the development of negative azimuthal vorticity.

### 2.1.3 Vorticity dynamics in flows with vortex breakdown

The process of vortex breakdown may then be interpreted in terms of vorticity dynamics as relates to the production of negative azimuthal vorticity. The mechanism is assumed to be purely inviscid and relies upon a radial outflow or, equivalently, an adverse pressure gradient, to drive the breakdown process. The divergent flow gives rise to the tilting of the axial vorticity vector into the negative azimuthal vorticity direction as a result of the conservation of circulation. The production of negative azimuthal vorticity indicates an amplified deceleration of the core flow relative to the outer flow. If the region of radial outflow or adverse pressure gradient continues long enough for the azimuthal vorticity to become negative, the stretching of the azimuthal vorticity becomes important denoting a further intensification of the deceleration of the core flow. Finally, the core flow stagnates and vortex breakdown occurs. As will be seen in Section 5.3, the final, abrupt stagnation process can be associated with the occurrence of flow criticality.

The interaction between vortex tilting and stretching during breakdown may be quantified under the assumptions of a steady, inviscid flow. As derived in Appendix A, the evolution of azimuthal vorticity in a steady, inviscid flow is governed by Equation (A.41),

\[
\frac{D\eta}{Dt} = -u \left[ \frac{1}{r^2} \frac{d(\Gamma^2)}{d\Psi} - \frac{1}{r} \eta \right].
\]  

(A.41)
This may then be rewritten as

\[ \frac{D\eta}{Dt} = -uB, \quad (2.10) \]

where the quantity \( B \) is defined as

\[ B = \frac{2}{r^2} \Gamma \Gamma' - \frac{1}{r} \eta. \quad (2.11) \]

The prime denotes a derivative with respect to the streamfunction. Following the developments of the previous sections, the tilting contribution to \( B \) is

\[ T = \frac{2}{r^2} \Gamma \Gamma', \quad (2.12) \]

and the stretching contribution is

\[ S = -\frac{1}{r} \eta. \]

The azimuthal vorticity may be written in terms of the circulation and the total pressure as shown in Equation (A.43),

\[ \eta = \frac{1}{r} \Gamma \Gamma' - r H'. \quad (A.43) \]

Thus, the stretching term may be written as

\[ S = -\frac{1}{r^2} \Gamma \Gamma' + H'. \quad (2.13) \]

By combining these two equations, \( B \) becomes

\[ B = \frac{1}{r^2} \Gamma \Gamma' + H'. \quad (2.14) \]

In vortex flows, the cores invariably have total pressure losses; hence, for practical applications, \( H' > 0 \). Furthermore, as discussed previously, the circulation distribution is such that \( \Gamma \Gamma' > 0 \). Thus, \( B > 0 \) and negative vorticity will always be produced in the presence of a positive radial velocity.

The steady, inviscid vorticity production terms, \( B, S, \) and \( T \), may be collapsed to a single parameter family of curves by a proper choice of non-dimensionalization.
Consider some location \( r = r_0 \) where the circulation and total pressure are given. The following non-dimensionalized variables may be defined:

\[
\tilde{T}, \tilde{S}, \tilde{B} = \frac{T, S, B}{H'}.
\]

Also, for convenience, we define

\[
\gamma^2 = \frac{\Gamma^\prime}{r_0^2 H'},
\]

which is constant along a streamsurface. As mentioned above, it is assumed that the total pressure and circulation are monotonically increasing functions of the streamfunction; therefore, \( \gamma^2 \) is assumed to be positive. In the non-physical situation where either \( \Gamma^\prime \) or \( H' \) are negative, then \( \gamma^2 \) is negative and \( \gamma \) becomes imaginary. Making these substitutions, we find

\[
\tilde{T} = 2/\tilde{r}^2, \quad (2.15)
\]

\[
\tilde{S} = -1/\tilde{r}^2 + 1, \quad (2.16)
\]

\[
\tilde{B} = 1/\tilde{r}^2 + 1, \quad (2.17)
\]

where \( \tilde{r} = \frac{1}{\gamma^2} \). When \( \tilde{r}^2 = 1 \), it follows that \( \tilde{S} = 0 \). Since \( S = -\eta/r \), the condition that \( \tilde{S} = 0 \) is equivalent to \( \eta = 0 \). Therefore, the azimuthal vorticity first reaches zero at \( \tilde{r} = 1 \) which is equivalent to \( r/r_0 \equiv r_c/r_0 = \gamma \). Thus, \( \gamma \) can be interpreted as the ratio of the critical streamsurface radial location, \( r_c \), at which the azimuthal vorticity first becomes zero to the initial streamsurface radial location, \( r_0 \). Since vortices upstream of breakdown are jet flows (\( \eta > 0 \)), this implies that \( \tilde{r} < 1 \) and, therefore, \( \gamma > 1 \) initially.

Figure 2.6 shows graphically the relation between \( \tilde{B}, \tilde{S}, \) and \( \tilde{T} \). From Equation (2.17) and Figure 2.6, the total azimuthal vorticity production term, \( \tilde{B} \), is always positive since \( \gamma^2 \) is positive. Thus, as noted before, an outward radial velocity will always produce a decrease in the azimuthal vorticity. Using this figure, it is now possible to give an interesting description of the axisymmetric breakdown process. Since \( \tilde{r} < 1 \) upstream of breakdown, at the first station where a non-zero radial velocity occurs, the stretching term is negative, and the tilting term is positive and at least twice the magnitude of
the stretching term. Thus, vortex tilting begins the breakdown process while vortex stretching initially plays a stabilizing as well as a secondary role. As the radial position of the streamurface increases, the azimuthal vorticity becomes negative at \( \hat{r} = 1 \), however, the tilting contribution is still the dominant mechanism in the production of azimuthal vorticity. Finally, at \( \hat{r} = \sqrt{3} \), the stretching contribution overtakes the tilting contribution. Equivalently, the ratio of the streamsurface radial location at which vortex stretching dominates to the initial streamsurface radial location is \( \sqrt{3} \gamma \), or \( \sqrt{3} \) times the critical radial location at which the azimuthal vorticity is first zero.

The radial and axial vorticity components in a steady, inviscid flow are:

\[
\begin{align*}
\xi &= -\frac{1}{r} \frac{\partial \Gamma}{\partial z} = u\Gamma', \\
\zeta &= \frac{1}{r} \frac{\partial \Gamma}{\partial r} = w\Gamma'.
\end{align*}
\]  

(2.18)
Similarly, the transport equations for the radial and axial vorticity are:

\[
\frac{D\xi}{Dt} = \frac{Du}{Dt}\Gamma', \\
\frac{D\zeta}{Dt} = \frac{ Dw}{Dt}\Gamma'. 
\]  
(2.19)

From Equation (2.18), the \((r, z)\) plane velocity and vorticity vector must remain aligned in a steady, inviscid, axisymmetric flow (see Batchelor[3], page 544),

\[
\frac{u}{w} = \frac{\xi}{\zeta}.
\]

Thus, as the radial surface diverges, the vorticity vector will remain tangent to the streamsurface and create a radial component of vorticity. This has implications upon the vorticity lines in the breakdown region. Since the vorticity lines must remain tangent to the streamsurface, no vortex lines can enter the breakdown bubble region. Therefore, any vorticity in the bubble region must come from downstream. If one includes viscous effects, vorticity could enter the bubble by diffusive processes. In an axisymmetric, viscous flow, the vorticity vector is always tangent to a surface of constant circulation; the important distinction between inviscid and viscous flows is that the circulation is constant along a streamsurface for inviscid flows. In the real flow, which is typically three-dimensional and locally unsteady, these conclusions do not hold; however, one might expect that the bubble form of breakdown in a real flow would approximate this type of behavior as the Reynolds number increased with the vortex lines passing around the breakdown region in general. If the flowfield is unsteady from some global effect, such as a delta wing undergoing pitch variations, these conclusions concerning vorticity lines are possibly invalid unless the pitch rate is small.

Finally, from Equation (2.19), the axial vorticity continuously decreases as breakdown is approached since the flow is stagnating (i.e. \(\frac{Dw}{Dt} < 0\)). This decrease results not only from the aforementioned tilting into both the azimuthal and radial directions but also from vortex line compression due to the axial flow deceleration. At breakdown, the axial vorticity approaches zero in the steady, inviscid limit. Therefore, the core flow just before breakdown is expected to contain near zero axial vorticity, positive radial
vorticity, and negative azimuthal vorticity.

2.1.4 Vortex tilting and axial pressure gradients

From the previous discussion, it is obvious that vortex tilting plays a large part in the initial deceleration of the vortex core leading to breakdown. Vortex tilting may also be linked to the axial pressure gradient. If one considers a vortical flow where the axial length scale, $L$, is much larger than the core length scale, $\delta$, a set of parabolized equations analogous to the boundary layer equations may be derived. The equations are commonly referred to as the quasi-cylindrical equations[26] and are given in full in Section A.2.1. The radial momentum equation for these parabolized equations is

$$
\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{\Gamma^2}{r^3} \tag{A.26}
$$

In boundary layer flows, the normal pressure gradient is zero; however, in a vortical flow, a radial pressure gradient exists matching the centripetal acceleration. Following Hall’s analysis of the quasi-cylindrical equations, Equation (A.26) is integrated inward from infinity to $r$ and differentiated in the axial direction to give

$$
\frac{\partial p}{\partial z} = \frac{dp_\infty}{dz} - \rho \int^\infty_r \frac{\partial}{\partial z} \left( \frac{\Gamma^2}{r^3} \right) dr \tag{2.20}
$$

The axial pressure gradient is thus the sum of a freestream and a circulation gradient term. The integral in Equation (2.20), may be related to vortex tilting using Equation (2.9):

$$
\frac{\partial p}{\partial z} = \frac{dp_\infty}{dz} - \rho \left. \int^\infty_r \frac{D\eta}{Dt} \right|_t dr.
$$

Therefore, the production of azimuthal vorticity is directly linked to the axial pressure gradient. As the azimuthal vorticity is decreased along a streamsurface due to the tilting of the axial vorticity component, the axial pressure gradient will increase. The total axial pressure gradient is the sum of this vortex tilting contribution and a freestream contribution. This link of the vorticity dynamics description of breakdown with the
‘boundary layer analogy’ or quasi-cylindrical theory of vortex breakdown shows that the two approaches both contain the same mechanism of vortex tilting which leads to the amplification of adverse pressure gradients and eventual breakdown of the flow. This amplification of the axial pressure gradient was first noted by Hall[26].

2.1.5 Relationship to Work of Brown and Lopez

A connection between the azimuthal production term, \( B \), and the results of Brown and Lopez[11] may also be made. Brown and Lopez, in considering the same steady, inviscid breakdown mechanism, derived an equation for the change in azimuthal vorticity with respect to a change in radius. By rewriting Equation (2.10), the same relation may be found:

\[
\begin{align*}
\frac{D\eta}{Dt} &= -uB, \\
\frac{Dt}{Dt} \frac{D\eta}{D\Gamma} &= -B, \\
\frac{D\eta}{Dr} &= -B, \\
\Rightarrow \frac{D\eta}{Dr} &= -\frac{1}{r^2}\Gamma' - H'.
\end{align*}
\] (2.21)

Finally, using the following identities derived by Brown and Lopez,

\[
H' = \frac{\eta_0}{r_0} \left( \frac{\alpha_0}{\beta_0} - 1 \right), 
\] (2.22)

\[
\Gamma' = \frac{r_0 \eta_0 \alpha_0}{\beta_0^n},
\] (2.23)

where \( \alpha_0 = \nu_0/\omega_0 \), \( \beta_0 = \eta_0/\zeta_0 \), and the subscript indicates an upstream, initial state, Equation (2.21) becomes

\[
\frac{D\eta}{Dr} = -\frac{\eta_0}{r_0} \left[ \left( \frac{\nu_0^2}{r_0^2} + 1 \right) \frac{\alpha_0}{\beta_0^n} - 1 \right].
\] (2.24)
This result is exactly the same as Brown and Lopez (which, of course, it must be). Finally, by plugging Equations (2.22) and (2.23) into the expression for \( \gamma \), we find

\[
\gamma^2 = \frac{\alpha_0/\beta_0}{\alpha_0/\beta_0 - 1}.
\]  

(2.25)

Several important distinctions exist between this work and that of Brown and Lopez. First, Brown and Lopez did not consider in detail the production of negative azimuthal vorticity. Therefore, the conclusion that the tilting of the axial vorticity drives the initial deceleration of the vortex core was not realized (except in the case of no azimuthal vorticity being present initially). Brown and Lopez, by relating their necessary condition to the upstream helix angle, did not recognize the connection with the circulation and total pressure gradient. In particular, as will be discussed in Chapter 4, the weakness of the upstream helix angle criterion was not noticed. In this investigation, a connection with Hall's boundary layer analogy is made which reveals that of the tilting of the axial vorticity vector can also be interpreted as the amplification of axial pressure gradients. Finally, Brown and Lopez did not consider flows with pressure gradients but rather investigated enclosed cylinders and trailing line vortex flows for which viscous effects play a substantial role in the onset of vortex breakdown. In contrast, this work focuses on the dominant role of adverse pressure gradients in typical applications. As will be shown in Chapter 3, viscous effects are of secondary importance in comparison to adverse pressure gradients except at low Reynolds numbers.

### 2.2 Numerical confirmation of vorticity dynamics

The interaction of vortex stretching and tilting in the process of vortex breakdown also has been confirmed numerically. Using the finite volume-based, axisymmetric, viscous, incompressible flow solver described in Appendix B, the flow through a converging-diverging pipe was calculated. This variable-area pipe was first calculated by Beran and Culick[7] using the steady form of the equations. The advantage of the convergent
Figure 2.7: Pipe geometry specification

The convergent section of the pipe is that it serves to isolate the inlet from the breakdown. Furthermore, the divergent section drives the flow towards breakdown providing the necessary adverse pressure gradient to initiate vortex tilting. If the pipe were of constant area, the breakdown would be caused by a complex interaction of viscous effects and vortex tilting, a mechanism which is not of interest for practical flows involving pressure gradients.

Figure 2.7 shows the specific pipe geometry. The pipe has a convergent section from $0 < z < z_t$, followed by a divergent section from $z_t < z < z_o$, and ends with a constant-area section from $z_o < z < z_{max}$. The inlet, throat, and outlet radii are $R_i$, $R_t$, and $R_o$, respectively. The converging-diverging sections are a half-period of a sinusoidal function. Specifically, the outer wall location, $R(z)$, is:

$$R(z) = \begin{cases} 
R_i + \frac{1}{3}(R_t - R_i)(1 - \cos[\pi(z/z_t)]) & \text{if } 0 < z < z_t \\
R_t + \frac{1}{3}(R_o - R_t)(1 - \cos[\pi(z - z_t)/(z_o - z_t)]) & z_t < z < z_o \\
R_o & z_o < z < z_{max}
\end{cases} \quad (2.26)$$

Constants for the three pipe geometries used throughout this work are given in Table 2.1. The distances have all been normalized by the core size, $\delta$. For the q-vortex inlet model as given by Equation (D.1), the maximum azimuthal velocity occurs at 1.121 times the core size. Further discussion of the q-vortex may be found in Appendix D. The only difference between the pipes are the outlet radius, $R_o$, with each pipe having a successively larger divergence. The grid spacing used in this study varied with the
<table>
<thead>
<tr>
<th></th>
<th>$z_t$</th>
<th>$z_o$</th>
<th>$z_{max}$</th>
<th>$R_t$</th>
<th>$R_t$</th>
<th>$R_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pipe 1</td>
<td>5</td>
<td>10</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.0</td>
</tr>
<tr>
<td>pipe 2</td>
<td>5</td>
<td>10</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.2</td>
</tr>
<tr>
<td>pipe 3</td>
<td>5</td>
<td>10</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.4</td>
</tr>
</tbody>
</table>

Table 2.1: Pipe geometry constants

Reynolds number, the swirl ratio, and the particular pipe. The Reynolds number, $Re$, and the swirl ratio, $\Omega$, defined in Section A.5, are

$$\Omega = \frac{\Gamma_{\infty}}{\delta W_{\infty}} \quad Re = \frac{W_{\infty} \delta}{\nu}, \quad (A.46)$$

where $\Gamma_{\infty}$ is the freestream circulation, $\delta$ is the core size, and $W_{\infty}$ is the freestream axial velocity.

The pipe length, $z_{max}$, was found by a trial and error procedure of reducing $z_{max}$ from a larger distance until the smallest value was attained which did not alter the steady-state results. For Pipe 1 with $Re = 1000$ and $\Omega = 1.49$, the computed axial velocities at $r = 0$ for $z_{max} = 30$ and 40 are shown in Figure 2.8. The solutions are nearly identical for $z < 25$. For $z > 25$, the solutions differ. However, the major interest of this study is in the first breakdown and the upstream flow; thus, the $z_{max} = 30$ solution is acceptable. A $z_{max} = 20$ solution was also computed, but, the results never converged to a steady state. Therefore, $z_{max} = 30$ was chosen for all simulations as a compromise between computational speed and accuracy of the downstream boundary condition model. As will be discussed in Section 5.4, lower Reynolds number flows are less sensitive to downstream effects; hence, $z_{max} = 30$ should be adequate for any simulation with $Re \leq 1000$.

If the flows did not contain a breakdown bubble, it was possible to use a grid with larger grid spacing and thus save on computational time. All of the grids employed in this work to simulate the converging-diverging pipe flow geometry used constant axial spacing, and, in the radial direction, the spacing was constant at a particular axial location (but varied globally with the local pipe radius). For the particular cases to be
Figure 2.8: Comparison of axial velocity for two pipe lengths: Pipe 1, $Re = 1000$, $\Omega = 1.49$.

<table>
<thead>
<tr>
<th>pipe</th>
<th>$Re$</th>
<th>$\Omega$</th>
<th>$j_{max}$</th>
<th>$\Delta r_0$</th>
<th>$k_{max}$</th>
<th>$\Delta z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pipe 1</td>
<td>100</td>
<td>1.49</td>
<td>31</td>
<td>0.067</td>
<td>151</td>
<td>0.200</td>
</tr>
<tr>
<td>pipe 1</td>
<td>400</td>
<td>1.49</td>
<td>31</td>
<td>0.067</td>
<td>301</td>
<td>0.100</td>
</tr>
<tr>
<td>pipe 1</td>
<td>600</td>
<td>1.49</td>
<td>31</td>
<td>0.067</td>
<td>401</td>
<td>0.075</td>
</tr>
</tbody>
</table>

Table 2.2: Grid spacing for vorticity dynamics results

discussed in this and Section 3.1, the grid spacings are given in Table 2.2. Figure 2.9 shows the grid for the pipe 1, $Re = 100$, $\Omega = 1.49$ calculation.

A brief description of the boundary conditions and other computational issues follows; however, the specifics of the numerical algorithm are given in Appendix B. For the case considered in this section, the inlet axial velocity is constant ($\Delta w = 0$). However, as a result of the convergent section, a jet flow will be present at the throat. The wall is modeled as an inviscid, slip boundary with the additional assumption that the flow is irrotational. The outlet boundary uses the quasi-cylindrical assumption (see Section A.2.1). At the axis, the viscous boundary conditions require zero rate of strain. No
artificial viscosity is added to stabilize the flow field for the results of this chapter; the actual physical dissipation terms at this Reynolds number have been sufficiently resolved and provide enough damping to stabilize the scheme.

In presenting the computational results, comparisons will be made between the calculation and the inviscid, analytic description of breakdown given in Section 2.1.3. In particular, the $B$, $S$, and $T$ contributions to the growth of negative azimuthal vorticity will be compared. For the inviscid, analytic terms, $B$, $S$, and $T$, the local streamfunction radius and the inlet values of $H'$ and $\Gamma'$ are needed. Then, the analytic terms are be scaled by the local radial velocity component to allow direct comparison to the production terms from the flow solver. The resulting definitions are:

$$T_i = uT,$$

$$S_i = uS,$$

$$B_i = uB,$$

where $T$, $S$, and $B$ are as defined in Equations (2.12)-(2.14) and the local radial velocity, $u$, is taken from the flow solver. Finally, the azimuthal vorticity may also be predicted from the streamfunction radial location and the upstream conditions using Equation (A.43):

$$\eta_i = \frac{1}{r} \frac{1}{\Gamma'} \Gamma' rH'.$$
The analogous contributions from the viscous flow calculated by the algorithm are:

\[ T_\nu = \Omega^2 \frac{1}{r^3} \frac{\partial (\Gamma^2)}{\partial z}, \]
\[ S_\nu = \frac{u\eta}{r}, \]
\[ B_\nu = T_\nu + S_\nu. \]

These terms are taken directly from the flow solver and use the same type of finite volume scheme to evaluate the necessary spatial derivatives. Also of interest are the viscous dissipation contributions to the production of azimuthal vorticity and the total production term incorporating the inviscid tilting and stretching mechanisms and the viscous dissipation effects. These terms are defined as

\[ R_\nu = \frac{1}{Re} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r\eta) \right) + \frac{\partial^2 \eta}{\partial z^2} \right], \]
\[ A_\nu = B_\nu + R_\nu. \]

Although \( T_i, S_i, B_i, \) and \( \eta_i \) have been derived from the analytic, inviscid theory, the streamfunction radial location and radial velocity are taken from the viscous flow solver; therefore, the results implicitly contain some viscous effects and should not be interpreted as being from a purely inviscid model. Several difficulties arose in the simulation of purely inviscid, axisymmetric flows which precluded the ability to compare viscous results directly with inviscid results. First, the inviscid flows were found to be extremely sensitive to the downstream boundary conditions which resulted in solutions which either never became steady, or, in some cases, diverged as a region of reversed flow formed at the outlet. This coupling of Reynolds number and downstream boundary effects is discussed further in Section 5.4. In addition, the effect of the artificial viscosity terms necessary to stabilize the numerical scheme became evident for inviscid flows. In particular, artificial smoothing was found generally to increase the likelihood of breakdown (i.e. increasing the smoothing coefficients resulted in breakdown at decreasing inlet swirl ratios for a given pipe geometry). This effect places some question on the proper interpretation of the numeric results. Given the difficulties in simulating and interpreting purely inviscid flows, only flows which could be resolved without artificial
viscosity (i.e. fairly low Reynolds numbers) were attempted.

**Case 1 : Pipe 1, \(Re = 600\), \(\Gamma = 1.49\) solution**

For the Pipe 1 geometry, at \(Re = 600\), the critical level of swirl at which breakdown first occurs is \(\Omega = 1.49\). This critical case is the first case which will be used to examine the role of vortex tilting and stretching in vortex breakdown; the additional cases appearing in Table 2.2 are presented in Section 3.1. The Reynolds number for Case 1 is much lower than that of typical delta wing vortices. For example, for a flat plate, sharp leading edge delta wing with 76 degree sweep and a root chord Reynolds number of \(3.8 \times 10^6\), experimental data gives a Reynolds number based on core size of approximately \(1.5 \times 10^4\) at 20.4 degrees angle of attack[67]. Note, the core is taken as the location of maximum swirl velocity. Therefore, the following comparisons of viscous results with inviscid theory would probably improve at typical flight conditions for which the Reynolds number based on core size might even be higher. Further discussion of the impact of viscous effects on vortex breakdown will be presented in Chapter 3.

Before going into a detailed description of the production of negative azimuthal vorticity through tilting of the axial vorticity, an overall view of the process is shown in Figure 2.10. In this figure, vorticity lines (i.e. lines which are everywhere tangent to
the vorticity) are plotted which are initiated from near the pipe throat at equidistant points about a circle of radius 0.15δ. A surface of constant circulation, \( \Gamma = 0.015 \), is also shown and the grey-scale shading of the surface indicates the magnitude of the azimuthal vorticity (black, \( \eta = 0.5 \); white, \( \eta = -2.0 \)). The rapid vorticity line divergence (as well as the constant circulation surface divergence) occurs at vortex breakdown. Upstream of breakdown, the vorticity lines are essentially axial in direction; however, as breakdown approaches, the tilting of the axially-aligned vorticity vector into the negative azimuthal direction begins. In the breakdown region, the vorticity vector is largely in the negative azimuthal direction. The shading also indicates the production of negative azimuthal vorticity as breakdown approaches. Finally, recall that the circulation may also be expressed using Stokes's theorem

\[
\Gamma = \nu r = \frac{1}{2\pi} \int \hat{\omega} \cdot \hat{n} dA,
\]

where \( \hat{n} \) is the vector normal to the chosen surface of integration with a circular boundary at \( r \). The simplest technique is to choose a planar surface (i.e. constant \( z \)) and then the circulation clearly becomes the flux of axial vorticity through a circular surface of radius \( r \). Therefore, a change in the circulation in the axial direction indicates a net flux of axial vorticity through a cylindrical surface of constant \( r \). As discussed above, an axial circulation gradient, and therefore a change in the flux of axial vorticity, can only occur from a radial stream surface divergence in an inviscid flow. This divergence and subsequent flux of the axial vorticity is also clearly seen in Figure 2.10.

\( \Psi, \eta, B_\nu, S_\nu, \text{ and } T_\nu \) results from the calculation are shown in Figure 2.11. The positive contours are solid black lines and the negative contours are hatched gray lines. The contour ranges are shown in parentheses above the individual figures and the contours are at eight equally-spaced values between these ranges. Also, note that the radial direction has been stretched relative to the axial direction in these figures.

From the streamline contours, the breakdown bubble is visible at approximately \( z = 9 \). The thick streamline, which initiates from the inlet at \( r = 0.39 \), is highlighted for reference because it is used later in the examination of flow property variations along

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a streamline. As proposed by Brown and Lopez, negative azimuthal vorticity is present and first appears at approximately $z = 8$ just upstream of the breakdown bubble. Also, a small region of positive azimuthal vorticity is present in the convergent section of the pipe at $z \approx 4$. This positive azimuthal vorticity is associated with the tilting of axial vorticity produced by the inward radial velocity and is the exact opposite of the effect which amplifies the core deceleration leading to breakdown.

The next three plots of Figure 2.11 show the inviscid production mechanisms, $B_\nu$, $S_\nu$, and $T_\nu$ in contours ranging from $(-0.8, 0.3)$. In comparing the stretching and tilting contributions upstream of breakdown, it becomes apparent that the tilting contribution is dominant nearly throughout the entire core deceleration except just before the breakdown bubble where the stretching mechanism becomes comparable in size. Also, comparing $B_\nu$ and $T_\nu$, the tilting mechanism is seen to correlate quite well with the total inviscid vorticity dynamics term. This behavior and relation of $B$, $S$, and $T$ was predicted in the inviscid theory in Section 2.1.3; therefore, at least qualitatively, the interaction of the vortex tilting and stretching mechanisms in a viscous flow is well described by a purely inviscid theory.

Further evidence of the relatively minimal impact of viscous effects on the initial breakdown process can be seen in Figure 2.12. The last three plots of the figure show contours of the total inviscid production term, $B_\nu$, the total production term, $A_\nu$, and the purely viscous production term, $R_\nu$. Comparison of $B_\nu$ and $A_\nu$ indicates only minimal differences with those differences appearing solely at the bubble itself where, as a result of the low speed and high gradients, the viscous effects become more important. This result is again brought out in the $R_\nu$ plot which shows only minimal viscous production effects which reach their largest magnitude at the rear of the breakdown bubble. Therefore, viscous effects are likely to be important in determining the flow in the bubble region; however, the onset of vortex breakdown appears to be an essentially inviscid phenomenon.

Figure 2.13 shows the inviscid theoretical predictions for the azimuthal vorticity
Figure 2.11: Vorticity dynamics: Pipe 1, $Re = 600$, $\Omega = 1.49$. Negative contours are dashed.
Figure 2.12: Viscous effects: Pipe 1, $Re = 600$, $\Omega = 1.49$. Negative contours are dashed.
Figure 2.13: Inviscid predictions: Pipe 1, $Re = 600, \Omega = 1.49$. Negative contours are dashed.
and the vorticity production terms, $\eta_i$, $B_i$, $S_i$, and $T_i$. Note that inside the bubble, the inviscid predictions are invalid because no upstream conditions on the circulation and total pressure distributions is available. The jagged appearance of the contours in the bubble region is a result of this effect. The azimuthal vorticity predictions are accurate only upstream of breakdown. In the region downstream of breakdown, the calculated vorticity, $\eta$, and the inviscidly-predicted vorticity, $\eta_i$, vary quite substantially. The reason for the discrepancy is that the functional form of the circulation and total pressure versus the streamfunction are different at the inlet and the outlet. Figure 2.14 compares the inlet and outlet distribution of $\Gamma^\prime$ versus $\Psi$ and Figure 2.15 compares the inlet and outlet $H^\prime$ distributions. Clearly, both the circulation and the total pressure have different inlet and outlet profiles.

The change in the total pressure and circulation distributions can be correlated with viscous effects near the breakdown bubble using the streamline plots in Figure 2.16. The streamline used in this plot initiates at $r = 0.39$ from the inlet and is highlighted in the streamline contour plots in Figures 2.11-2.13. The variation of a flow variable is plotted
Figure 2.15: Inlet and outlet distributions of $H'$

Figure 2.16: Upstream and downstream predicted $\eta_i$ and $\eta$ versus streamline length, $Re = 600$, $\Omega = 1.49$. Upstream $\eta_i$, dashed; downstream $\eta_i$, dotted; $\eta$, solid.
versus the distance along a streamline, $s_\Psi$, in the $(r, z)$ plane which can be defined as

$$s_\Psi = \int_0^t \sqrt{u^2 + w^2} \, dt,$$

where $t$ represents the time over which which a fluid element convects from its initial location. The upstream and downstream predicted $\eta_i$ are compared with the azimuthal vorticity from the solver. Until breakdown, the upstream and the numerical azimuthal vorticity values are in good agreement. However, at $s_\Psi \approx 10$, the upstream $\eta_i$ and $\eta$ values begin to disagree presumably due to the effects of viscous dissipation near the breakdown bubble. Using a downstream condition on the total pressure and circulation, the discrepancy behind breakdown can be corrected; however, the vorticity levels do not agree upstream of breakdown in this case.

Returning to the discussion of Figure 2.13, the production terms, $B_i$, $S_i$, and $T_i$, agree quite well with $B_\nu$, $S_\nu$, and $T_\nu$, in contrast to the comparisons of $\eta$ and $\eta_i$ discussed above. The reason for the agreement behind breakdown is that the production terms and the radial velocity are all approaching zero. In general, the inviscid terms are of larger magnitude than the numerically-calculated production terms which contain viscous effects. The production terms are compared more quantitatively in the streamline plots of Figures 2.17-2.19 (which use the same streamline as in Figure 2.16).

Finally, in Figure 2.20, the streamline variations of $B_\nu$, $S_\nu$, and $T_\nu$ are overlayed. The correlation of the tilting to the total production term in the initial stages of breakdown is excellent. As breakdown is approached, the stretching begins to become important and eventually is the dominant production mechanism just prior to breakdown.

Contours of all three vorticity components are plotted in Figure 2.21. Eight equally spaced contours are shown for each vorticity component with the ranges indicated above the plot. Note that the contour ranges are different for all three components. Far upstream of breakdown, the radial vorticity is negligible while the axial vorticity is approximately $2.0$ in the core. As breakdown is approached, the axial vorticity rapidly decreases while the radial vorticity increases. In general, the magnitude of the radial vorticity is much smaller than the magnitude of the axial vorticity except near the
Figure 2.17: $B_\nu$ and $B_i$ versus streamline length, $Re = 600$, $\Omega = 1.49$. $B_\nu$, solid; $B_i$, dotted.

Figure 2.18: $S_\nu$ and $S_i$ versus streamline length, $Re = 600$, $\Omega = 1.49$. $S_\nu$, solid; $S_i$, dotted.
Figure 2.19: $T_{\nu}$ and $T_i$ versus streamline length, $Re = 600$, $\Omega = 1.49$. $T_{\nu}$, solid; $T_i$, dotted.

Figure 2.20: $B_{\nu}$, $S_{\nu}$, and $T_{\nu}$ versus streamline length, $Re = 600$, $\Omega = 1.49$. Solid, $B_{\nu}$; dashed, $S_{\nu}$; dotted, $T_{\nu}$. 

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Figure 2.21: Vorticity components: Pipe 1, $Re = 600$, $\Omega = 1.49$. Negative contours are dashed.
breakdown bubble. In the bubble itself, the radial and axial vorticity components are both near zero; the radial vorticity is slightly negative while the axial vorticity is slightly positive. This behavior of the radial and axial vorticity is in good qualitative agreement with the expected behavior suggested from the inviscid theory.

Figure 2.22 shows streamsurfaces and circulation surfaces which are initiated from the same set of equidistant points at the inlet. The extent to which the two surfaces coincide is another measure of the inviscid nature of the flow. Also, the circulation surfaces are tangent to the vorticity vector everywhere and therefore give information direction of the vorticity vector in the meridional plane. From this plot, the streamsurfaces and circulation surfaces are in overall agreement except near the bubble which confirms the conclusion regarding the general inviscid nature of vortex breakdown.

An interesting feature of the azimuthal and axial vorticity components is their rapid growth at the end of the breakdown region. From approximately $z = 10$ to $z = 15$, the azimuthal vorticity reaches its highest positive values in the flow ($\eta \approx 1.3$). The axial vorticity grows rapidly starting at approximately $z = 10$. At $z \approx 15$, the axial vorticity
reaches a maximum of about $\zeta \approx 4.4$ (which is outside of the contour range in Figure 2.21) and then begins to drop again. This behavior can be seen in Figure 2.23 which is a plot of the axial vorticity along the vortex axis. The growth of the axial and azimuthal vorticity is a consequence of the conservation of circulation and the resulting tilting of the axial vorticity into the positive azimuthal direction as the streamsurfaces converge at the rear of the bubble. The positive tilting contribution, $T_\nu$, at $z \approx 13$ is clearly evident in Figure 2.11. Finally, note from Figure 2.23 that the axial vorticity just prior to breakdown drops rapidly to nearly zero which agrees with the behavior predicted by the inviscid theory.

Finally, the three vorticity component plots versus streamline length are overlayed in Figure 2.24. In front of breakdown ($s_\psi < 9$), the axial vorticity variations correspond one-to-one with the azimuthal vorticity variations. This effect is a direct result of the tilting of the axial vorticity into the azimuthal vorticity. In the entire process leading up to breakdown, the radial vorticity plays only a minor role merely assuring the tangency of the vortex vector with the streamsurface.
2.3 Conclusions

In this chapter, an inviscid, steady model describing the initial development of vortex breakdown using the fundamentals of vorticity dynamics was outlined. The central theme of this model is that an adverse pressure gradient, or equivalently, a streamsurface divergence is necessary to drive the vortex towards breakdown. The model shows that the initial stages of vortex breakdown are dominated by the tilting of the axial vorticity into the azimuthal direction. Furthermore, vortex stretching is shown to play a secondary role in the early stages of the breakdown process but becomes important at the final stages just upstream of breakdown. The effects of vortex tilting were shown to be related to the amplification of axial pressure gradients in the core of a vortex; this relation connects the vorticity dynamics description of breakdown with the boundary-layer analogy theory outlined by Hall[26].

A numerical simulation of swirling pipe flow was used to assess the validity of the vorticity dynamics description of breakdown. Comparison of the inviscid theory with viscous numerical results shows excellent agreement and supports the assumption that the onset of vortex breakdown is primarily an inviscid phenomenon. The simulated flow was driven to breakdown through an adverse pressure gradient imposed by a divergent
section of the pipe. The effects of vortex tilting were found to be dominant as hypothesized in the inviscid theory. In terms of the vorticity field, the initially axial-aligned vorticity vector was tilted into the azimuthal direction while the radial vorticity was small until just upstream of breakdown.

In concluding this discussion on the role of vorticity dynamics in vortex breakdown, we summarize the breakdown process:

1. An adverse pressure gradient exists which drives breakdown.

2. The adverse pressure gradient causes streamsurfaces to diverge and thereby tilts the axial vorticity into the azimuthal direction creating negative azimuthal vorticity.

3. This decrease in azimuthal vorticity further decreases the axial velocity by amplifying the axial pressure gradient, and, through continuity, increases the radial outflow.

4. Eventually, azimuthal vortex stretching becomes important and intensifies the deceleration.

5. The vortex core stagnates and breakdown occurs.
Chapter 3

The Effects of Viscosity on Vortex Breakdown

It is usual, however, in the first instance to neglect the tangential stresses altogether.

Sir Horace Lamb

In Chapter 2, a purely inviscid breakdown mechanism was proposed. The initiation of vortex breakdown relied upon the development of an axial circulation gradient which results in a tilting of the axial vorticity component into the azimuthal direction. In a truly inviscid, steady flow, this axial circulation gradient must be created by a radial divergence of a streamsurface. If viscous effects are considered, an axial circulation gradient may be created by diffusive processes. In this section, we examine in more detail the modifying effects of viscous dissipation upon the inviscid, vorticity dynamics interpretation of vortex breakdown. Therefore, this chapter is concerned with the viscous mechanisms which produce negative azimuthal vorticity; specifically, we hope to answer the question of whether or not viscous effects are significant enough to modify the basic inviscid breakdown mechanism discussed in Chapter 2. An order of magnitude analysis is used to show that, except at very low Reynolds numbers, viscous effects contribute negligibly to the production of azimuthal vorticity. Numerical experiments are conducted which also indicate the secondary role of viscosity in the breakdown process. Finally, we note that this chapter studies only one aspect by which viscous effects may contribute to the development of vortex breakdown (i.e. the production of azimuthal vorticity); in Section 5.4, a second possible aspect of viscous effects is discussed.
3.1 Viscous dissipation and vorticity dynamics

The fundamental behavior of viscous stresses is to oppose the tangential deformation of a fluid; this resistance to shearing strains is accomplished through the molecular diffusion of momentum between neighboring fluid elements. Until slightly upstream of breakdown, the quasi-cylindrical equations are an accurate model of the development of a vortex[26, 7]. Using the quasi-cylindrical equations as the starting point of this analysis, the modifying effects of viscosity upon the inviscid breakdown mechanism discussed in Chapter 2 are analyzed. The derivation of quasi-cylindrical equations is discussed in Section A.2.1. These equations result from simplifying the axisymmetric equations when changes in the radial direction are much larger than changes in the axial direction. The steady quasi-cylindrical equations are spatially parabolic allowing the axial direction to be treated as a marching direction. Therefore, in the following discussion, the radial operators appearing in the governing equations are thought of as acting on the local, radial distributions of the flow quantities to produce the flow at the next downstream location.

The steady, quasi-cylindrical form of the azimuthal vorticity equation is:

$$\frac{D\eta}{Dt} = \frac{1}{r^3} \frac{\partial (\Gamma^2)}{\partial z} + \frac{u\eta}{r} + \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r\eta) \right) \right],$$  \hspace{1cm} (A.31)

where the azimuthal vorticity is $\eta = -\frac{\partial w}{\partial r}$. The circulation equation is

$$u \frac{\partial \Gamma}{\partial r} + w \frac{\partial \Gamma}{\partial z} = \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) \right].$$

Rearranging this equation to give $\frac{\partial \Gamma}{\partial z}$ produces

$$\frac{\partial \Gamma}{\partial z} = -\frac{u \partial \Gamma}{w \partial r} + \frac{\nu}{w} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) \right],$$  \hspace{1cm} (3.1)

and substituting into the azimuthal vorticity equation gives

$$\frac{D\eta}{Dt} = -uB + \frac{2\nu\Gamma}{r^2w} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) \right] + \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r\eta) \right) \right].$$  \hspace{1cm} (3.2)
The term, $B$, is analogous to the inviscid vorticity production term given in Equation (2.11). The quasi-cylindrical form for $B$ used in Equation (3.2) is

$$B = \frac{1}{rw} \left[ \frac{1}{r^2} \frac{d(r^2)}{dr} + \frac{d}{dr} \left( \frac{1}{2} w^2 \right) \right].$$

From Equation (3.2), viscous effects are seen to enter in two distinct ways. In the second term, viscous dissipation of the radial circulation gradient produces azimuthal vorticity. As can be seen from Equation (3.1), the dissipation of the radial circulation gradient creates an effective axial circulation gradient which appears as a viscous version of the vortex tilting mechanism. The circulation gradient dissipation term can be zero only when the circulation is constant or the flow is in solid body rotation. In the constant circulation case, a fluid element undergoes equal and opposite viscous stresses such that the net viscous torque due to the circulation gradient is zero; this situation typically arises in the flow outside the core of a vortex where the circulation is generally constant. In the case of solid body rotation, no viscous stresses are produced on a fluid element; the flow at the axis in a vortex upstream of breakdown is typically in solid body rotation. Therefore, the circulation gradient viscous dissipation mainly affects the flow at the edge of the vortex core and, in general, this term will decrease the circulation in the axial direction resulting in a decrease in the azimuthal vorticity. As a result, this viscous effect tends to promote breakdown by producing negative azimuthal vorticity.

The other manner in which viscous effects alter the production of azimuthal vorticity is by acting directly on the azimuthal vorticity as given in the third term on the right-hand side of Equation (3.2). Since the azimuthal vorticity is mainly associated with the axial velocity distribution, this viscous effect is largely associated with the shearing strain resulting from jet or wake profiles in the vortex core. The corresponding shear stresses can act on a fluid element to produce a net viscous torque. In a jet-like core, this viscous effect will tend to reduce the magnitude of the velocity overshoot, therefore producing negative azimuthal vorticity. In a wake-like core, this viscous effect will tend to reduce the magnitude of the velocity deficit, therefore producing positive azimuthal vorticity. Thus, well upstream of breakdown, viscous dissipation acting on the azimuthal vorticity will decelerate the core flow; however, after the development of a wake flow,
this effect will tend to resist further core deceleration. Whether or not the overall effect is to promote breakdown is difficult to determine \textit{a priori}. Thus, when a vortex develops only in the presence of viscous effects, the occurrence of breakdown will be decided by a complicated relationship between the dissipation of circulation and azimuthal vorticity.

To simplify the following discussions, the individual terms which contribute to the RHS of Equation (3.2) are defined as:

\begin{align*}
B_{q_i} &= -uB, \\
B_{q_v} &= \frac{2\nu \Gamma}{r^2 w} \left[ \frac{\partial}{\partial r} \left( r \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) \right], \\
R_q &= \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r \eta)}{\partial r} \right) \right].
\end{align*}

An additional, useful definition is:

\begin{equation}
B_q = B_{q_i} + B_{q_v}.
\end{equation}

The quasi-cylindrical production terms have been taken from the simulations which use the axisymmetric Navier-Stokes equations; the direct use of the quasi-cylindrical equations is precluded because they fail upstream of breakdown[26]. Therefore, although these terms have been derived from the quasi-cylindrical equations, they implicitly contain effects from the full Navier-Stokes equations.

3.1.1 Case 2 : Pipe 1, $Re = 400, \Omega = 1.49$ solution

Case 2 is identical to Case 1 of Section 2.2 except for a decrease in the Reynolds number from 600 to 400. The purpose of this case is to better understand the modifying effects of viscous dissipation by direct comparison with the previous, slightly higher Reynolds number flow and by investigation of the quasi-cylindrical viscous effects discussed above. As in Case 1, the swirl level, $\Omega = 1.49$, is the critical level of swirl necessary to stagnate
the flow for this geometry and Reynolds number combination. Figures 3.1-3.3 are analogous to Figures 2.11-2.13 from Case 1. The contour levels and increments are identical in the corresponding plots to allow direct comparison of the two flowfields. The bubble location is nearly identical for both cases. The structure of the lower Reynolds number bubble is slightly more symmetrical with the rear of the bubble not ‘pinched in’ as much as in the $Re = 600$ flow. Comparison of the azimuthal vorticity levels shows a decrease of approximately 0.5 in magnitude (i.e. one contour level) from the high to low Reynolds number flows. This general decrease of vorticity levels also is apparent in the azimuthal vorticity production terms, $B_\nu$, $S_\nu$, and $T_\nu$. The viscous effects as measured by $R_\nu$ are only negligibly different at the chosen contour levels. As for the $Re = 600$ case, the inviscid predictions, ($\eta_i$, $B_i$, $S_i$, and $T_i$), match quite well with the viscous calculations. The same general trend is observed that the inviscid vorticity and vorticity production terms are larger in magnitude than the associated viscous terms. The radial and axial components of vorticity, shown in Figure 3.4, are nearly identical in both Reynolds cases. Therefore, we conclude that the differences between the two Reynolds number flows are minimal and that the flow appears to be headed towards an inviscid limit upstream of breakdown.

Next, the quasi-cylindrical approximations for the vorticity production terms are compared to the complete, non-parabolized values. Figure 3.5 are line contours of $B_q$, $B_{q_1}$, $B_{q_2}$, and $R_q$. Comparison of $B_q$ and $B_\nu$ show negligible differences upstream of breakdown. Likewise, the quasi-cylindrical approximation for the inviscid term, $B_{q_1}$, compares favorably with $B_i$. In general, all four values of $B$ (i.e. $B_\nu$, $B_i$, $B_q$, and $B_{q_1}$) are in very good agreement before the onset of breakdown.

Finally, the viscous effects identified from the quasi-cylindrical analysis are described. Note, the contour plots of $B_{q_2}$ and $R_q$ are over an order of magnitude smaller range than the values of $P_q$ and $B_{q_1}$. The values of $B_{q_2}$ are generally negative throughout the pipe and therefore produce a decrease in the amount of azimuthal vorticity. In the convergent section, an axial jet develops due to the radial inflow which tilts the axial vorticity into the positive azimuthal direction. However, this jet formation is resisted
Figure 3.1: Vorticity dynamics: Pipe 1, $Re = 400$, $\Omega = 1.49$. Negative contours are dashed.
Figure 3.2: Viscous effects: Pipe 1, $Re = 400$, $\Omega = 1.49$. Negative contours are dashed.
Figure 3.3: Inviscid predictions: Pipe 1, $Re = 400$, $\Omega = 1.49$. Negative contours are dashed.
Figure 3.4: Vorticity components: Pipe 1, $Re = 400$, $\Omega = 1.49$. Negative contours are dashed.
Figure 3.5: Quasi-cylindrical contributions: Pipe 1, $Re = 400$, $\Omega = 1.49$. Negative contours are dashed.
by viscous dissipation as characterized by \( R_q \). In this region of the flow, the \( R_q \) contours are negative; therefore, the azimuthal vorticity and the associated core axial velocity are reduced from this viscous effect. In the divergent section of the pipe, the azimuthal vorticity dissipation is positive due to the formation of a wake flow as breakdown is approached. Therefore, the combined effects of \( B_{q\nu} \) and \( R_q \) in this geometry will be to reduce the overall magnitude of the azimuthal vorticity production and, consequently, the magnitude of the azimuthal vorticity.

### 3.1.2 Case 3: Pipe 1, \( Re = 100, \Omega = 1.49 \) solution

In this case, the viscous effects are much more important. Specifically, at \( Re = 100 \), breakdown no longer occurs for \( \Omega = 1.49 \). The streamlines, azimuthal vorticity, and vorticity production terms are shown in Figure 3.6, analogous to Figures 2.11 and 3.1. At \( z \approx 10 \), the streamsurfaces have an evident divergence but the reversed flow, breakdown region does not appear. The levels of azimuthal vorticity are extremely low compared to the \( Re = 400 \) and \( Re = 600 \) solutions in which breakdown is exhibited. Similarly, the vorticity production terms, \( B_{\nu}, S_{\nu}, \) and \( T_{\nu} \) are much smaller than for Case 1 and Case 2. As shown in Figure 3.7, the radial vorticity levels have been reduced and the axial vorticity does not vary as largely as in the higher Reynolds solutions.

The quasi-cylindrical approximations are shown in Figure 3.8. First, we note that the quasi-cylindrical approximation, \( B_{q\nu} \), is an accurate approximation for \( B_{\nu} \) of Figure 3.6. The most important feature of this plot is the character of the viscous effects, \( B_{q\nu} \) and \( R_q \). The magnitude of the circulation dissipation effects in the convergent section are about 4 times the \( Re = 400 \) values which agrees with the relative Reynolds number scalings. Furthermore, the viscous dissipation acting on the azimuthal vorticity distribution, \( R_q \), is also greater and covers a larger area than the vorticity dissipation in the \( Re = 400 \) case. This larger viscous dissipation of the azimuthal vorticity distribution is the main factor for the delay of vortex breakdown at low Reynolds number. As previously discussed, the resistance of the vortex to shear, such as created by the
Figure 3.6: Vorticity dynamics: Pipe 1, $Re = 100$, $\Omega = 1.49$. Negative contours are dashed.
Figure 3.7: Vorticity components: Pipe 1, Re = 100, Ω = 1.49. Negative contours are dashed.
Figure 3.8: Quasi-cylindrical contributions: Pipe 1, $Re = 100$, $\Omega = 1.49$. Negative contours are dashed.
developing wake flow, increases with decreasing Reynolds number. Therefore, as vortex tilting created by the flow divergence drives the core flow to stagnation, increased viscous dissipation will retard the onset of vortex breakdown. Thus, lower Reynolds number flow will generally require larger swirl ratios for breakdown to occur.

3.2 Time Scale Analysis

In this section, an order of magnitude analysis is performed to assess the relative effects of viscosity in relation to the azimuthal vorticity growth due to inviscid vortex stretching and tilting. In this analysis, the estimates for the various scales will be based upon the core length and vorticity scales. The following approximations will be used in obtaining the appropriate core estimates:

\[
\begin{align*}
    u & \sim U \\
    \Gamma & \sim \Gamma_\infty \\
    w & \sim W_\infty \\
    r & \sim \delta \\
    \frac{\partial}{\partial r} & \sim \frac{1}{\delta} \\
    \eta & \sim \frac{W_\infty \Delta w}{\delta},
\end{align*}
\]

where \( \Delta w \) is a measure of the velocity difference between the core and the outer flow. The radial velocity scale, \( U \), must be approximated by some measure of the streamtube divergence imposed by the geometry or freestream pressure gradient.

The resulting approximation for \( B \) is

\[
B \sim \frac{W_\infty}{\delta^2} \left( \Omega^2 - \Delta w \right).
\]

Since we are interested in cases for which breakdown is possible, \( B \) must be greater than zero. Thus,

\[
\Omega^2 - \Delta w > 0.
\]
Then, substitution into the terms of Equation (3.2) gives the following estimates

\[
\begin{align*}
\mu B & \sim \left( \frac{W_\infty}{\delta} \right)^2 \left( \frac{U}{W_\infty} \right) \left( \Omega^2 - \Delta w \right) = I, \\
\frac{2
\nu \Gamma}{r^2 w} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \right] & \sim \left( \frac{W_\infty}{\delta} \right)^2 \left( \frac{1}{Re} \right) \left( \Omega^2 \right) = V_\Gamma, \\
\nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \eta \right) \right] & \sim \left( \frac{W_\infty}{\delta} \right)^2 \left( \frac{1}{Re} \right) (\Delta w) = V_\eta.
\end{align*}
\]

The three ratios which define the behavior of the flow are \( I/V_\Gamma, I/V_\eta, \) and \( V_\Gamma/V_\eta. \) These ratios are:

\[
\begin{align*}
I/V_\Gamma &= Re \left( \frac{U}{W_\infty} \right) \left( 1 - \frac{\Delta w}{\Omega^2} \right), \\
I/V_\eta &= Re \left( \frac{U}{W_\infty} \right) \left( \frac{\Omega^2}{\Delta w} - 1 \right), \\
V_\Gamma/V_\eta &= \frac{\Omega^2}{\Delta w}.
\end{align*}
\]

Recalling Equation (3.6), then \( V_\Gamma/V_\eta > 1; \) thus, the dominating viscous effect will be due to the dissipation of circulation. We stress here that these relations are only approximate. For example, in the \( Re = 100 \) solution discussed in Section 3.1.2, it was found that the viscous circulation contribution, \( B_{q\nu} \), was larger than the azimuthal vorticity dissipation, \( R_q \), for the convergent inlet section of the pipe (see Figure 3.8). This finding is in agreement with the expected behavior of \( V_\Gamma/V_\eta \). However, further downstream in the pipe divergence, the magnitude of \( R_q \) became larger than the magnitude of \( B_{q\nu} \) at \( r = 0.7 \) (the approximate location of maximum azimuthal vorticity). Near the axis, the circulation dissipation contribution still tended to remain larger than the azimuthal vorticity dissipation. Therefore, the conclusion that \( |B_{q\nu}| > |R_q| \) because \( V_\Gamma/V_\eta > 1 \) is not strictly accurate but rather indicates a general behavior of viscous, vortex flows which are susceptible to breakdown.

Another important aspect of the estimates in Equations (3.8)-(3.10) is their local nature. For example, consider the Case 3 results for which the inlet parameters were \( \Omega = 1.49 \) and \( \Delta w = 0. \) In this case, the ratio of \( V_\Gamma/V_\eta \) is infinite and the viscous effects due to the circulation gradient are expected to be much larger than the viscous effects.
dissipation due to the azimuthal vorticity distribution. This trend can be seen at the inlet of the pipe as shown in the contours of \( B_{qv} \) and \( R_q \) in Figure 3.8. At the inlet, the value of \( B_{qv} \) is approximately \(-0.06\) while the value of \( R_q \) is nearly zero. However, further downstream, the axial velocity difference as well as the swirl ratio will change and the relative magnitude of the viscous contributions should be approximated by appropriate measures of the local velocity field.

The condition for negligible viscous effects may be written

\[
Re \left( \frac{U}{W_\infty} \right) |\Omega^2 - \Delta w| / \left( \max \left[ \Omega^2, |\Delta w| \right] \right) >> 1. \tag{3.11}
\]

Assuming the circulation dissipation effects are larger, this reduces to \( I/V_T >> 1 \), or,

\[
Re \left( \frac{U}{W_\infty} \right) \left( 1 - \frac{\Delta w}{\Omega^2} \right) >> 1. \tag{3.12}
\]

Since the Reynolds number (based on core) is typically quite large in delta wings and other applications, inviscid effects will almost always dominate viscous effects. As mentioned in Section 2.2, for a flat plate, sharp leading-edge delta wing with 76 degree sweep and a root chord Reynolds number of \( 3.8 \times 10^6 \), the core Reynolds number is approximately \( 1.5 \times 10^4 \) at 20.4 degrees angle of attack[67]. The swirl ratio and velocity difference for this wing at 70% root chord are approximately, \( \Omega \approx 2 \) and \( \Delta w \approx 2 \). Therefore, the condition for negligible viscous effects for this particular case reduces to \( U/W_\infty >> 1.3 \times 10^{-4} \) which corresponds to a flow divergence of 0.007 degrees. This condition is obviously easily satisfied and vortex breakdown should be inviscidly dominated. Experimental evidence directly relating to vortex breakdown also overwhelmingly supports that breakdown location over a sharp leading-edge delta wing has little dependence on Reynolds number[36, 30].

Finally, it is possible to develop a breakdown timescale based upon the inviscid vorticity growth mechanisms when viscous effects are minimal. We define the breakdown timescale, \( t_b \), as the approximate time at which breakdown occurs as a fluid element
convects downstream. Specifically,

\[
\frac{D\eta}{Dt} = -uB, \\
(\eta_f - \eta_0)/t_b = -UB,
\]

where \( \eta_0 \) is the initial measure of the vorticity and \( \eta_f \) is the final vorticity value at which breakdown occurs. As above, \( \eta_0 \) is approximated by \( W_\infty \Delta w/\delta \). Since the axial velocity at the axis just prior to breakdown is by definition zero, then the final vorticity value may be approximated by \(-W_\infty/\delta \). Substituting in for the approximate value of \( B \) defined above, the breakdown timescale is defined as

\[
t_b = \left( \frac{\delta}{W_\infty} \right) \left( \frac{W_\infty}{U} \right) \left( \frac{1 + \Delta w}{\Omega^2 - \Delta w} \right). 
\]

Then, the ratio of \( t_b \) with the convective timescale, \( t_c \), is

\[
\frac{t_b}{t_c} = \left( \frac{\delta}{L} \right) \left( \frac{W_\infty}{U} \right) \left( \frac{1 + \Delta w}{\Omega^2 - \Delta w} \right), \tag{3.13}
\]

where

\[
t_c = \frac{L}{W_\infty},
\]

and \( L \) is the axial length scale related to the length over which the vortex is subjected to the flow divergence or adverse pressure gradient. When the ratio \( t_b/t_c \) is small, then breakdown is likely. From this approximate analysis, we conclude the following effects lower \( t_b/t_c \) and promote the occurrence of breakdown:

1. Increasing the ratio of the flow divergence length to the core size, \( L/\delta \),
2. Increasing the flow divergence, \( U/W_\infty \),
3. Increasing the swirl ratio, \( \Omega \),
4. Decreasing the axial velocity difference, \( \Delta w \).

The first and second effects increase the driving, adverse pressure gradient upon the vor-


tex while the third and fourth effects increase the vortex's susceptibility to the adverse pressure gradients.

3.3 Numerical Results

In order to confirm the previous order of magnitude analysis, a numerical experiment was conducted to assess the effects of viscosity on breakdown. The three pipe geometries described in Table 2.1 were used. The inlet profiles were q-vortices with constant axial velocity profiles ($\Delta w = 0$). The critical inlet swirl ratio, $\Omega_c$, at which breakdown first occurs was determined over a range of Reynolds numbers. Breakdown was identified by a stagnation point on the pipe axis. The critical inlet swirl ratio was determined to two decimal place accuracy. For example, if the $\Omega_c = 1.49$ for a particular pipe, then an inlet swirl ratio of 1.48 is the highest $\Omega$ tested which did not result in vortex breakdown. As before, no artificial viscosity was used to obtain these results and the grids for each pipe were similar to those described in Table 2.2.
For pipe 1, the critical swirl ratio becomes independent of Reynolds number when $Re > 300$ (see Figure 3.9). Detailed inspection of the flow reveals that local variations are still occurring at $Re > 300$; however, the global behavior has apparently reached the inviscid limit. For example, Figures 3.10 and 3.11 show the axial velocity at the centerline for $Re = 400, 500, 600$ and $Re = 700, 800, \text{ and } 1000$, respectively. The axial velocity upstream of breakdown appears to be asymptoting as the Reynolds number is increased; however, the post-breakdown behavior varies substantially with Reynolds number. As the Reynolds number increases, standing waves seem to be appearing in the solution. This behavior is similar to Benjamin's conjecture concerning the existence of waves downstream of breakdown[5]. The wave nature of vortex breakdown is discussed further in Chapter 5. Figure 3.12 provides an enlarged view of the centerline axial velocity upstream of breakdown for all of the pipe 1 cases. The axial velocity appears to be reaching an inviscid limit; the differences between each successively higher $Re$ case are continuously reduced. Therefore, although the flowfield behind breakdown is changing with increased Reynolds number, the initial breakdown process is increasingly unaffected.
Figure 3.11: Centerline axial velocity for pipe 1: $Re = 700, 800, 1000$

Figure 3.12: Enlargement of centerline axial velocity for pipe 1: $Re = 400-1000$
For the other two pipe geometries, the asymptote to an inviscid critical swirl ratio has still not been reached by $Re = 1000$. However, the trend appears to be the same and the critical swirl ratio should eventually asymptote to an inviscid limit. The general trend with increasing pipe divergence is a delay in attaining an inviscid limit for the critical swirl ratio.

Finally, the numerical results also show that the critical swirl ratio decreases with increasing pipe divergence. This trend was predicted in the timescale analysis from the previous section. In Equation (3.13), note that as $U/W_\infty$ increases, the swirl ratio at which $t_b/t_c < 1$ decreases. Therefore, both the analysis and the calculations show that a vortex is more likely to breakdown in a flow with larger divergence; conversely, a vortex with a lower swirl ratio requires a larger divergence to breakdown.

### 3.4 Conclusions

Using the quasi-cylindrical approximation, two viscous effects were identified which will alter the inviscid growth of azimuthal vorticity. The dissipation due to radial circulation gradients leads to a viscous equivalent of vortex tilting and reduces the azimuthal vorticity. The dissipation of the azimuthal vorticity distribution generally reduces the shear present between the freestream and core velocity and, as the core approaches stagnation, this viscous effect slows the inviscid production of negative azimuthal vorticity. The dissipation of azimuthal vorticity was found to result in the delay of vortex breakdown at low Reynolds numbers. However, using an approximate time scale analysis, it was found that both viscous effects will be negligible in practical applications. Also, an approximate breakdown timescale was developed which must be small compared to the convective timescale in order for breakdown to occur. Finally, a numerical investigation of the critical level of swirl needed to create breakdown in three pipes was conducted for a range of Reynolds numbers. The critical swirl ratio was found to asymptote at Reynolds numbers based on core radii of approximately 500 confirming that the rela-
tive effects of viscosity are negligible in practice. The axial velocity profiles upstream of breakdown were found to asymptote with increasing Reynolds numbers; however, downstream of breakdown, the axial velocity profiles develop wavelike behavior for increasing Reynolds number.
Chapter 4

The Susceptibility to Breakdown

The focus of this chapter is the development of a susceptibility measure for vortex breakdown. First, the Brown and Lopez criterion is shown to be a weak condition for vortex breakdown which is equivalent to requiring a total pressure loss in the vortex core. Then, using insight gained into the growth mechanisms of azimuthal vorticity from Chapter 2, a breakdown susceptibility measure is proposed in Section 4.2. This measure is found to correlate well with previously available trends from both experimental and numerical results. In Section 4.3, a numerical study of the susceptibility measure is conducted which further confirms the previous analysis. Finally, in Section 4.4, the implications of the susceptibility parameter on the control of vortex breakdown in delta wing flows is discussed.

4.1 The Brown and Lopez criterion

Brown and Lopez developed a necessary condition for breakdown by considering if a vortex could possibly develop negative azimuthal vorticity[11]. The original form of the Brown and Lopez criterion was given by the following constraint on the velocity-vorticity helix angle:

$$\frac{\omega_0}{w_0} > \frac{\eta_0}{\zeta_0}$$
where the subscript ‘0’ refers to an upstream condition. Powell[56] pointed out the weakness of the Brown and Lopez criterion by considering a direct application of Crocco’s equation for a steady, inviscid flow which gives

$$\nabla H = \vec{u} \times \vec{\omega}. \quad (A.42)$$

In the radial direction, this reduces to

$$\frac{\partial H}{\partial r} = v\zeta - w\eta.$$

Finally, substitution of this relation into the Brown and Lopez criterion gives

$$\frac{\partial H_0}{\partial r} = v_0\zeta_0 - w_0\eta_0 > 0,$$

assuming $w_0\zeta_0 > 0$. Thus, the total pressure must have a deficit somewhere in the core. Since this condition will be met by any vortex in the presence of viscous dissipation, the velocity-vorticity helix angle criterion, although necessary, is an extremely weak condition for the occurrence of vortex breakdown. The total pressure gradient requirement may be seen directly from Equation (A.43):

$$\eta = \frac{1}{r} \Gamma' - rH'.$$  \quad (A.43)

If one assumes that the circulation distribution is such that $\Gamma' > 0$, which is typical of vortices, then it follows that $H' > 0$ for the azimuthal vorticity to be negative at some $r$. In the case where $w_0 > 0$, this condition is identical to $\frac{\partial H_0}{\partial r} > 0$.

## 4.2 Breakdown susceptibility measures

A measure of the susceptibility of a vortex to breakdown may be developed using the steady, inviscid form of the azimuthal vorticity and the vorticity production term, $B$. $B$ maps the radial velocity to a growth in the azimuthal vorticity. The greater $B$, the
more rapid the decrease will be in azimuthal vorticity if a positive radial velocity exists. Although $B$ measures the rate at which the vorticity changes in an adverse pressure gradient, the eventual breakdown of the flow also depends on how far the initial state is from breakdown. Recall that the azimuthal vorticity is given by

$$
\eta = \frac{1}{r} r \Gamma' - r H',
$$

(A.43)

and $B$ is given by

$$
B = \frac{1}{r^2} \Gamma' + H'.
$$

(2.14)

Now, we consider the following thought experiment. Suppose at some upstream location, the distributions of $\Gamma'$ and $H'$ are given. We will treat these as independent parameters although, in reality, the circulation and total pressure will depend on each other. It is evident from the expression for $B$ that increasing $\Gamma'$ and/or $H'$ will result in a larger value of $B$ and therefore a more rapid decrease in the azimuthal vorticity. However, from the equation for $\eta$, increasing $\Gamma'$ also increases the azimuthal vorticity and therefore the axial jet in the core. As a result, changes in the circulation distribution $\Gamma'(\Psi)$ while holding $H'(\Psi)$ fixed tend to have opposing effects and probably will have only minor impact upon the susceptibility to breakdown. Increasing only $H'$ has the effect of decreasing the jet (i.e. decreasing $\eta$) while at the same time raising the rate at which negative azimuthal vorticity is produced. In the previous section, the Brown and Lopez necessary criterion for breakdown was found to be equivalent to requiring $H' > 0$; but the above discussion suggests that the magnitude of $H'$ may also be viewed as a measure of the likelihood of breakdown.

A convenient expression for $H'$ may be derived by assuming a vortex which is initially governed by the quasi-cylindrical equations described in Section A.2.1. The following non-dimensional form of $H'$ may be derived from Equation (A.34):

$$
H' = \frac{1}{r w} \frac{dH}{dr} = \frac{1}{r w} \left[ \frac{\Omega^2}{r^2} \frac{d}{dr} \left( \frac{1}{2} \Gamma^2 \right) + \frac{d}{dr} \left( \frac{1}{2} w^2 \right) \right],
$$

(4.1)
where the non-dimensionalization of $H'$ is

$$\tilde{H'} = \frac{H'}{W_\infty/\delta^2},$$

and the other non-dimensionalizations follow those described in Equation (A.45). Note that the bar notation for the dimensionless quantities has been dropped. This expression for $H'$ is applicable to flows upstream of breakdown where axial variations are small. Several important conclusions concerning the likelihood of breakdown may now be drawn from Equation (4.1).

1. Increasing the swirl ratio of a vortex increases its susceptibility to breakdown.

2. A wake flow is more susceptible to breakdown than a jet flow because $\frac{d}{dr} (w^2) > 0$ for vortices with an axial velocity deficit.

3. As the strength of a jet flow increases (i.e. the axial velocity overshoot increases), the vortex becomes less likely to breakdown because $\frac{d}{dr} (w^2)$ is constantly decreasing.

These conclusions agree quite well with known experimental and numerical results. For example, a swirl ratio greater than unity is a standard semi-empirical indication that breakdown is possible[26]. Also, several investigators have shown that increasing the swirl ratio leads to breakdown[18, 61]. Numerical simulations have also found the same dependence of breakdown on the swirl level[55, 15]. The stabilization of a vortex with a jetlike axial velocity profile was observed in previous computations[15, 31]. Of course, the second and third items can be simply interpreted as a result of Bernoulli's equation. In Section 4.3, additional numerical results provide further support that $H'$ is a suitable measure for the susceptibility to breakdown.

The value of $H'$ in the core of a vortex may be characterized by the value of $H'$ at
\( r = 0 \). Using the velocity profiles for a \( q \)-vortex as given in Appendix D, we find that

\[
H'_0 = 2 \left( \frac{\Omega^2}{1 + \Delta w} - \Delta w \right),
\]

(4.2)

where \( H'_0 \) is the value of \( H' \) at \( r = 0 \) for a \( q \)-vortex. Figure 4.1 is a contour map of \( H'_0 \) for the \((\Omega, \Delta w)\) space. The \( H'_0 = 0 \) contour demarks the boundary between vortices for which breakdown can or cannot occur. Previously, Brown and Lopez[11] have shown that breakdown did not occur for trailing line vortices with \( v/w - \eta/\zeta < 0 \) which is equivalent to \( H' < 0 \).
<table>
<thead>
<tr>
<th></th>
<th>$z_t$</th>
<th>$z_o$</th>
<th>$z_{max}$</th>
<th>$R_i$</th>
<th>$R_t$</th>
<th>$R_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pipe 4</td>
<td>5</td>
<td>20</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.2</td>
</tr>
<tr>
<td>pipe 5</td>
<td>5</td>
<td>20</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.4</td>
</tr>
</tbody>
</table>

Table 4.1: Pipe geometry constants for susceptibility simulations

4.3 **Numerical studies of breakdown susceptibility**

A numerical study of the susceptibility to breakdown was undertaken using the axisymmetric, swirling flow algorithm described in Appendix B. The pipe geometry for which the calculations were performed was previously given in Figure 2.7. The specifics of the pipe geometries are given in Table 4.1. Note that the length of the divergence section was increased from 5 to 15 core radii as compared to the pipe geometries in Chapters 2 and 3. The more gradual area variation was employed so that the position of breakdown would gradually vary with changes in the inlet profile instead of being constrained by the pipe throat almost immediately at the critical level of swirl. All of the simulations were performed on a grid with 30 cells in the radial direction and 150 cells in the axial direction. Finally, the Reynolds number for all of the solutions in this section was $Re = 300$. The effects of varying Reynolds numbers were considered in Chapter 3.

4.3.1 **Impact of $H'$ and $\Gamma \Gamma'$ variations on breakdown**

In this section, the relative importance of $H'$ and $\Gamma \Gamma'$ variations on vortex breakdown will be considered in a numerical investigation. To this end, we will use the total pressure gradient at the axis, $H'_0$, as a measure of the total pressure gradient throughout the core. For the circulation term, we define the following variable,

$$G' = \frac{1}{r^2} \Gamma \Gamma'.$$
<table>
<thead>
<tr>
<th></th>
<th>$H'_0$</th>
<th>$G'_0$</th>
<th>$\Omega$</th>
<th>$\Delta w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>2.6</td>
<td>2.6</td>
<td>1.140</td>
<td>0.00</td>
</tr>
<tr>
<td>Test 2</td>
<td>2.6</td>
<td>3.6</td>
<td>1.643</td>
<td>0.50</td>
</tr>
<tr>
<td>Test 3</td>
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<td>1.6</td>
<td>0.632</td>
<td>-0.50</td>
</tr>
<tr>
<td>Test 4</td>
<td>3.1</td>
<td>2.6</td>
<td>0.987</td>
<td>-0.25</td>
</tr>
<tr>
<td>Test 5</td>
<td>2.1</td>
<td>2.6</td>
<td>1.275</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 4.2: Inlet conditions for susceptibility test cases

The value at $r = 0$ will be denoted as $G'_0$ and will be taken as a measure of the circulation gradient throughout the core. For a $q$-vortex inlet condition, $H'_0$ is given by Equation (4.2) and $G'_0$ is given by

$$G'_0 = \frac{2\Omega^2}{1 + \Delta w}.$$  \hspace{1cm} (4.3)

Pipe 5 will be used for the simulations. The inlet conditions for the five test cases are given in Table 4.2. For Tests 1-3, the total pressure gradient is kept fixed while the circulation gradient is varied. For Test 1, 4, and 5, the circulation gradient is fixed while the total pressure gradient varies. Table 4.2 gives the necessary values of the swirl ratio and core velocity difference for the appropriate total pressure and circulation gradients.

Figure 4.2 is a blow-up of the $\Psi = 0$ streamsurface, which defines the outer edge of the bubble, for Test 1-3. Note that the radial distances are stretched in the figure. The location of breakdown is nearly identical for all three test cases. The variation of the circulation gradient has little effect on the initial location of breakdown in these simulations. Although the bubble location is relatively unaffected by changes in $G'_0$, the shape of the bubble does vary with the circulation distribution.

The bubble surfaces for Test 1, 4, and 5 are shown in Figure 4.3. In contrast to the previous cases, the location of the breakdown varies substantially in each of these cases. The breakdown is further upstream and the bubble is larger for the larger values of $H'_0$. For Test 4, the most upstream breakdown, the reversed flow region begins at $z \approx 12.4$. The Test 1 breakdown occurs approximately one core size downstream at $z \approx 13.7$ and
Figure 4.2: Bubble streamsurface for Test 1, 2, and 3

Figure 4.3: Bubble streamsurface for Test 1, 4, and 5
the Test 5 breakdown is approximately two cores behind that at $z \approx 15.8$.

Several other simulations were run to compare the impact of $H'_0$ and $G'_0$ on breakdown in different pipes and different inlet conditions. In all, the behavior was nearly identical. The total pressure gradient, measured by $H'_0$, controlled the location of breakdown while variations in $G'_0$ had negligible impact when the total pressure remained fixed. Therefore, as proposed above, the total pressure derivative with respect to the streamfunction, $H'$, is a useful measure of the susceptibility of a vortex to breakdown, and Equation (4.2) gives a useful characterization of $H'$ in a vortex core.

4.3.2 Parametric study of breakdown susceptibility

In the next set of simulations, a wide range of vortex inlet conditions were used in Pipes 4 and 5 and the location of the breakdown position was determined. The breakdown position, $z_b$, was defined as the point to within 0.1 core radii at which zero velocity occurred on the axis. Figure 4.4 shows the variation of breakdown position in pipe 4 versus the swirl ratio for three different core velocity differences, $\Delta \omega$. The swirl ratio was varied by increments of 0.025. The initial point on each curve (at the highest values of $z_b$) corresponds to the critical inlet swirl ratio to within $\pm 0.025$ for the given axial velocity profiles. At the high swirl ratio end, the tests were stopped for all axial profiles at $\Omega = 1.4$. Increasing the swirl ratio clearly moves the breakdown position forward in the pipe. Similarly, for the same swirl ratio, the wake axial velocity profile, $\Delta \omega = -0.1$, always suffered breakdown ahead of either of the other two profiles. Figure 4.5 more clearly shows the variation of breakdown position with the velocity profile. In this case, increasing the jet velocity improved the resistance to breakdown and resulted in a breakdown location downstream of the more wake-like profiles. Two swirl ratios are plotted in this figure with the higher swirl ratio always breaking down upstream of the lower swirl ratio.

Finally, these curves may be conveniently collapsed using the proposed susceptibility
Figure 4.4: $z_b$ vs. $\Omega$ for three core velocity profiles: pipe 4, $Re = 300$

Figure 4.5: $z_b$ vs. $\Delta w$ for different swirl ratios: pipe 4, $Re = 300$
measure, $H_0'$, as defined in Equation 4.2. Figure 4.6 is the variation of the breakdown location with the susceptibility measure, $H_0'$. The correlation of the breakdown location with $H_0'$ is excellent. For this geometry, the occurrence and location of breakdown has essentially been reduced to a single curve. Note that Figure 4.6 contains results not shown in Figures 4.4 and 4.5; these additional runs were performed to better define the curve and to provide a higher reliability for the use of $H_0'$ as an indicator of breakdown susceptibility over a larger range of swirl ratios and core velocities. In all, the solutions comprising the Figure 4.6 data range in swirl ratio between $(1.0, 1.625)$ and in axial velocity between $(-0.4, 0.4)$. For axial velocity differences larger than 0.4, the breakdown region tended to be near the downstream boundary and solutions with $\Delta w > 0.4$ were either impossible to converge or contained large axial gradients at the outlet. Therefore, the value of $\Delta w$ was limited to 0.4 for the results of Figure 4.6.

Similar simulations were run for the Pipe 5 geometry also. The qualitative results are the same as obtained in the Pipe 4 simulations. The variation of breakdown location with the swirl ratio for $\Delta w = -0.2$, 0.0, and 0.2 is shown in Figure 4.7. Again, the breakdown is seen to move upstream with increasing swirl ratio in all cases. The wakelike
axial profile, $\Delta w = -0.2$, proved the most susceptible to breakdown with breakdown moving further downstream for increasing core velocities. This trend is highlighted in Figure 4.8 also. Finally, Figure 4.9 shows the correlation of the breakdown location with the total pressure gradient, $H'_0$. The correlation is again excellent. Similar to Figure 4.6, additional cases were run to fill in the curve which do not appear in Figures 4.7 or 4.8. In this case, the solutions comprising the Figure 4.7 data range in swirl ratio between $(0.775, 2.150)$ and in axial velocity between $(-0.675, 1.0)$.

Finally, the correlation of the onset of breakdown with $H'_0$ was determined for Pipes 4 and 5. In these simulations, the axial velocity difference was varied from $\Delta w = -0.5$ to 1.0 in steps of 0.1. At a given value of $\Delta w$, the swirl ratio was progressively raised until breakdown first appeared at the critical inlet swirl ratio, $\Omega_c$. Figure 4.10 is a plot of the critical swirl ratio, $\Omega_c$, versus the axial velocity difference, $\Delta w$ with contours of constant $H'_0$ overlayed. The critical swirl variations are closely approximated by the constant $H'_0$ curves. For Pipe 4 and Pipe 5, the critical level of total pressure gradient is approximately 2.82 and 1.88, respectively. The large divergent section of Pipe 5 requires lower total pressure losses to breakdown the vortex. The results indicate that
Figure 4.8: $z_b$ vs. $\Delta w$ for different swirl ratios: pipe 5, $Re = 300$

Figure 4.9: $z_b$ vs $H_0'$: pipe 5, $Re = 300$
Figure 4.10: $\Omega_c$ versus $\Delta w$ overlayed on $H'_0$ contours. □ Pipe 4; ○ Pipe 5.

A critical inlet total pressure gradient exists for a given geometry. If the inlet total pressure gradient is greater than this critical value, the vortex will break down as long as the downstream boundary does not in some way inhibit the process. The effect of downstream conditions on vortex breakdown is discussed in Chapter 5.

### 4.4 Breakdown control in delta wing flows

The susceptibility measure, $H'_0$, gives some insight into possibilities for delaying breakdown. For example, two techniques for delaying breakdown could be to reduce the swirl ratio or to increase the axial jet. For delta wing flows, the swirl ratio could possibly be reduced by eliminating leading edge separation at some point on the wing and thereby stopping the entrainment of axial vorticity into the vortex. This idea has some difficulties, however, since reducing the swirl ratio would typically result in a lower axial jet as well as reduced vortex lift (although if breakdown is delayed the vortex lift may still increase overall). In the other technique, the axial jet may be strengthened by blowing
along the vortex axis; the obvious limitation of this technique is the need to locate the vortex axis. However, the vortex lift should definitely increase since breakdown is delayed while the swirl ratio is unaffected. This technique has been successfully applied by previous investigators\cite{14, 49}.

In addition, the susceptibility measure may be useful in a computational design optimization of delta wings. For example, suppose that the (rather simplistic) design goal is to maximize lift over a range of angles of attack and avoid the onset of breakdown if at all possible. For the moment, suppose that the delta wing is a flat plate and that the leading-edge sweep angle, \( \Lambda \), is the only design variable. For an efficient, first-cut design process, it may be possible to employ a conical flow solver and use the following cost functional

\[
C(\Lambda) = (C_l)^2 - \alpha_H(H_0')^2,
\]

where \( C_l \) is the sectional lift coefficient and \( \alpha_H \) determines the weighting of the total pressure gradient in comparison to the lift. The design process would seek to find the maximum of the cost functional, \( C(\Lambda) \). For large values of \( \alpha_H \), the total pressure gradient would be more important relative to the lift coefficient and the resultant design would sacrifice lift for a decreased susceptibility to breakdown. For small values of \( \alpha_H \), the lift coefficient would be maximized. More elegant design strategies may include the possibility of blowing high pressure air along the vortex axis. In this case, the design could take into account the necessary blowing energy, \( E \), and the direction of the blowing, \( \gamma \). A possible cost functional would then be

\[
C(\Lambda, \gamma, E) = (C_l)^2 - \alpha_H(H_0')^2 - \alpha_E(E)^2,
\]

where \( \alpha_E \) weights the energy needed to perform the blowing. After an initial design is completed using the conical techniques, a three-dimensional simulation could be performed to check and to further refine the design.
4.5 Conclusions

This chapter focused on the analysis and verification of a breakdown susceptibility measure for vortical flows. The weakness of the Brown and Lopez velocity-vorticity helix angle criterion was identified and the helix angle criterion was shown to be equivalent to requiring a total pressure loss, $H' > 0$. Further analysis suggested that the total pressure gradient, $H'$, may be a measure of the susceptibility of a vortex to breakdown. This measure was found to be in qualitative agreement with previously observed trends from experimental and numerical investigations. In Section 4.3, a numerical investigation of the applicability of $H'_0$ as a breakdown susceptibility measure for $q$-vortex velocity profiles was detailed. The total pressure gradient was found to be the significant controlling factor in the occurrence and location of breakdown. Further parametric studies revealed that the location of breakdown within two converging-diverging pipes could be well correlated for each pipe by $H'_0$ over a wide range of swirl ratios and axial velocity differences. A study of the onset of breakdown revealed that a critical level of total pressure gradient exists for a given geometry above which breakdown will occur. Finally, in Section 4.4, suggestions for the control of breakdown over a delta wing were made based on the susceptibility parameter, $H'$. In particular, a design approach was outlined which may be capable of reducing the susceptibility to vortex breakdown while optimizing the lift produced by a delta wing.
Chapter 5

Trapped Wave Theory and Vorticity Dynamics

...there is a rather loose analogy between a burst and a normal shock wave in a compressible flow, both being associated with an instability in decelerating flow...

Lambourne and Bryer

The study of wave motions has long been explored as a possible explanation for the occurrence of vortex breakdown. In this chapter, we explore the notion of vortex breakdown being related to a trapped wave. The main elements of this trapped wave theory as applies to vortex breakdown were first described in the thesis of Bilanin[8] and in a more general context by Landahl[37]. The theory suggests that vortex breakdown results from the trapping of waves at the critical location where the group velocity of long wave modes first becomes negative and allows upstream propagation. Therefore, a vital link in the trapped wave theory, missing in Bilanin's and most other wave theories of breakdown, is a description of the mechanism by which the flow becomes critical. In this chapter, we show that the vorticity dynamics description of vortex breakdown from Chapter 2, which reveals the importance of streamsurface divergence in the production of negative azimuthal vorticity, provides a link which describes the initial approach to flow criticality.

This chapter begins in Section 5.1 with a short review of wave theories and general wave properties of vortical flow: including the precise definition of flow criticality. In Section 5.2, we relate the production of negative azimuthal vorticity to the approach of
flow criticality. Section 5.3 outlines Bilanin's theoretical work on trapped wave theory and demonstrates the trapped wave nature of vortex breakdown with results from the time accurate numerical simulation of a transcritical flow. Finally, in Section 5.4, the effects of downstream boundaries are examined in the light of this vorticity dynamics and trapped wave theory of vortex breakdown.

5.1 Review of Wave Theories of Vortical Flows

As described in several vortex breakdown reviews[26, 41, 42, 16], Squire[63] first suggested that vortex breakdown may be explained by considering the waveguide nature of vortex flows. Since Squire's initial suggestion, many different theoretical and computational studies on the importance of wave motions in vortex breakdown have been conducted. In this section, we will briefly review some of the fundamental work on the wave motions of vortical flows and, in particular, stress the importance of flow criticality on all of these theories. The conjugate state description of vortex breakdown, which seeks to explain breakdown as a jump between a supercritical and subcritical state, analogous to a shock wave in compressible flow, is reviewed in Appendix E.

5.1.1 Definition of Criticality

A vortex acts as a waveguide as a result of the restoring action of centrifugal forces on a streamsurface perturbation. In general, a vortex may admit waves which travel in either the upstream or downstream direction. The definition of flow criticality set forth by Benjamin[5] classifies a vortex as supercritical when only downstream propagating waves are allowed and as subcritical when both downstream and upstream propagating waves are allowed. This definition is analogous to compressible flow where the two classes of flow are termed supersonic and subsonic, respectively; however, compressible flow is hyperbolic while vortical flows are dispersive. A dispersive waveguide is generally
defined as a system where the phase and group velocity vary nonlinearly with the wavenumber[70]. This distinction will be further illuminated in the following discussion of vortex flows.

To analyze wave motions, the vortex flow is divided into a steady, axisymmetric, mean flow and an unsteady, axisymmetric perturbation. The analysis of the perturbations is performed assuming inviscid flow. Also, the steady, mean flow is assumed to be columnar so that the mean velocity distributions are \([0, V(r), W(r)]\) in an \((r, \theta, z)\) coordinate system. The unsteady perturbation for the streamfunction is given by the following modal form,

\[
\psi = f(r) \exp\{i[k(z - ct)]\},
\]

where \(k\) is the axial wavenumber and \(c\) is the phase speed (assumed real). The phase of the waves is, \(\phi = k(z - ct)\). Therefore, the phase speed, \(c\), is the speed at which a constant value of phase, \(\phi\), propagates. The other flow properties assume an analogous modal form. Since this flow is dispersive, although individual wave modes travel at the phase velocity, \(c\), the energy of a wave packet travels at the group velocity, \(c_g = \frac{d}{dk} (ck)\). Generally, the phase and the group velocities for a given wavenumber are distinct.

Research on wave motions in vortex flows has focussed on three different interpretations of flow criticality. In order of complexity, these interpretations are:

1. A supercritical flow implies no standing waves \((c = 0)\) are admitted by the vortex while subcritical flows allow standing waves. A flow is subcritical when \(k\) is real for \(c = 0\).

2. A supercritical flow implies that the phase of the waves, \(\phi\), propagates only downstream while subcritical flows allow upstream travelling waves. This criterion is based on phase velocity, \(c\). A flow is subcritical when \(c < 0\) for real \(k\).

3. A supercritical flow implies energy travels only downstream while subcritical allows upstream travelling energy. This criterion is based on group velocity. A flow is subcritical when \(c_g < 0\) for real \(k\).
Benjamin[5] has shown that if standing waves exist with wavenumber, \( k_s > 0 \), then travelling waves exist with wavenumber, \( k_t \), in the range \( 0 \leq k_t < k_s \) such that their phase velocity is directed upstream. In addition, Leibovich[43] found several other properties of waves on vortex cores. Leibovich showed that the dispersion relation, which we will write \( c = c(k) \), has an upper branch, \( c_n^{(u)}(k) \), and a lower branch, \( c_n^{(l)}(k) \), for a given wavenumber \( k \). The subscript \( n \) refers to the particular mode which can be identified by \( n - 1 \) internal zeros. The mode number, \( n \), does not change the results and will be omitted in the following discussion. Leibovich showed that the two branches of the dispersion relations had the following bounds

1. for the lower branch, \( c^{(l)} < W_{\text{min}} \),

2. for the upper branch, \( c^{(u)} > W_{\text{max}} \),

where

\[
W_{\text{min}} \leq W(r) \leq W_{\text{max}}.
\]

Furthermore, Leibovich was also able to determine that

1. for the lower branch, \( k \frac{dc}{dk} > 0 \),

2. for the upper branch, \( k \frac{dc}{dk} < 0 \).

This implies that the minimum (maximum) value of \( c(k) \) occurs on the lower (upper) branch for zero wavenumber. Therefore, in terms of determining the criticality of the flow, long waves of the lower branch will be the first waves to have upstream phase propagation. Recall that the group velocity may also be written

\[
c_g = c + k \frac{dc}{dk}.
\]

Therefore, the group velocity may be bound by

1. for the lower branch, \( c^{(l)} \leq c_g^{(l)} < W_{\text{max}} \),
2. for the upper branch, \( W_{\text{min}} \prec c_g^{(u)} \prec c^{(u)} \).

Hence, upstream propagation of energy is only possible if the upstream propagation of the phase is possible. Also, since the group velocity approaches the phase velocity in the long wave limit, the slowest (or most negative) group velocity occurs at \( k = 0 \) and is equal to the phase velocity at \( k = 0 \). Therefore, criticality of the phase and group velocities occurs simultaneously and can be determined from the wavespeeds for long waves. Figure 5.1 represents Leibovich's findings pictorially. Finally, combining Benjamin's and Leibovich's results leads to the conclusion that all three criticality definitions are identical. Since the three criticality interpretations are coincident, we will focus on the standing wave classification which is the simplest of the three definitions to check. In this case, the streamfunction perturbation is defined as

\[
\psi = f(r) \exp(\gamma z),
\]

where \( \gamma \) is the axial wavenumber. As given by Hall[26], the governing equation for
stationary perturbations in an inviscid flow is:

\[
\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} + \left[ \gamma^2 - \frac{1}{W} \frac{d^2 W}{dr^2} + \frac{1}{rW} \frac{dW}{dr} + \frac{1}{r^3 W^2} \frac{d(\Gamma^2)}{dr} \right] f = 0, \tag{5.1}
\]

where \( \Gamma \) is the mean flow circulation. For flow in a pipe, the perturbation streamfunction is zero at \( r = 0 \) and at the pipe wall, \( r = R(z) \). An infinite, ordered set of eigenvalues exists for this Sturm-Liouville system such that \( \gamma_0^2 < \gamma_1^2 < \gamma_2^2 < \ldots \)\[5]. Thus, the flow is supercritical when \( \gamma_0^2 > 0 \) since then all eigenvalues of Equation (5.1) will be positive and only exponential solutions are possible. When \( \gamma_0^2 < 0 \), the perturbations are sinusoidally varying waves. In the supercritical case, \( \gamma_0 \) gives the slowest decay rate of disturbances in the upstream direction and is a measure for how much a vortex will be influenced by downstream effects. For subcritical flows, \( |\gamma_0| \) is the highest wavenumber possible for stationary waves on the mean flow. In the following presentation, the eigenvalues of several simulated pipe flows will be calculated to determine the flow criticality. The eigenvalues are calculated by discretizing Equation (5.1) with a second order accurate finite difference scheme and using an EISPACK eigenvalue solver on the resulting matrix.

An alternative technique exists to determine flow criticality which is mainly mathematical but will serve as the basis for proving that the production of azimuthal vorticity indicates an approach towards flow criticality. In this technique, we set the eigenvalue, \( \gamma \), to be zero. From Equation (5.1), this gives,

\[
\frac{d^2 f_c}{dr^2} - \frac{1}{r} \frac{df_c}{dr} + \left[ -\frac{1}{W} \frac{d^2 W}{dr^2} + \frac{1}{rW} \frac{dW}{dr} + \frac{1}{r^3 W^2} \frac{d(\Gamma^2)}{dr} \right] f_c = 0, \tag{5.2}
\]

where \( f_c(r) \) is the eigenmode when \( f_c = 0 \) and \( \frac{df_c}{dr} = \text{const at } r = 0 \) (or at \( r = R(z) \)). If \( f_c(r) \) contains an internal zero, then the mean flow is subcritical\[26\]. This follows from the Sturm-Liouville property that the nth eigenmode associated with the nth eigenvalue, \( \gamma_n \), contains \( n - 1 \) internal zeros. Therefore, if \( f_c \) contains an internal zero then \( \gamma_0^2 < 0 \) since the \( n = 0 \) eigenmode does not have an internal zero. In \[5\], Benjamin shows that Equation (5.2) may also be written

\[
\frac{d^2 f_c}{dy^2} - C(\Psi, y) f_c = 0, \tag{5.3}
\]

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\[ C(\Psi, y) = H''(\Psi) - \frac{1}{2y} I''(\Psi), \]  

(5.4)

where \( y = \frac{1}{2} r^2 \) and \( I = \frac{1}{2} \Gamma^2 \). Hall[26] proved that Equation (5.1), and therefore, Equation (5.3), holds for axial varying flows as long as the changes in \( z \) are much smaller than the changes in \( r \). Thus, the mean flow can be a slowly varying function of \( r \). Further implications of Equations (5.3) and (5.4) are discussed in Section 5.2.

5.1.2 Wave Theories of Breakdown

Squire's[63] theory of vortex breakdown centers on the flow becoming critical. When the flow is critical, Squire suggests that downstream disturbances can propagate upstream until the critical location is reached. At this critical location, the downstream disturbances are assumed to accumulate and grow in amplitude until vortex breakdown occurs. Squire's proposal has been criticized by Benjamin[5] because the waves which Squire considered in his analysis (i.e. infinitely long, standing waves) have downstream-directed group velocity. Therefore, Benjamin reasoned that Squire's upstream wave propagation mechanism was unlikely. Leibovich[42] has argued that Squire probably had in mind an evolving vortex and, in this case, breakdown would presumably occur where the vortex attained critical conditions. For this scenario, Leibovich contends that Benjamin's objections are less compelling. Hall[26] also criticized Squire's proposal because no explanation was offered as to how critical conditions were approached. Regardless of the accuracy of Squire's initial proposal, flow criticality is an integral part of all wave theories of vortex breakdown. In addition, analysis of experiments has shown that, with respect to axisymmetric waves, the flow upstream of vortex breakdown is always supercritical and the flow downstream of vortex breakdown is generally subcritical[42, 64].

Squire's theory of vortex breakdown deals solely with linear waves; several investigators have studied the effects of finite amplitude on waves in vortex flows[4, 57, 44, 58, 45]. Benjamin[4] first showed the existence of stationary solitary and cnoidal waves. In a fur-
ther investigation of finite amplitude effects, Leibovich[44] included the time-dependence and found that weakly non-linear waves in rotating fluids are governed by a Korteweg-de Vries (KdV) equation. The existence of solitary waves were confirmed experimentally by Pritchard[57]. For large amplitudes, the solitons can create stagnation points on the vortex axis with the general wave shape symmetric fore and aft. However, experimental and numerical evidence shows the axisymmetric bubble structure to be non-symmetric fore and aft with the bubble typically open downstream (see Section 6.3). One major difficulty with these finite amplitude theories is that the final position of vortex breakdown cannot be fixed since they deal with inviscid, constant area pipes. In addition, no explanation is offered as to the mechanisms by which these solitary waves are initially created. Nevertheless, the solitary wave solutions and their transient behavior have been useful in describing many of the phenomena observed in vortex breakdown experiments[21].

Building on the previous finite amplitude wave theories, Randall and Leibovich[58] included the effects of a slightly diverging pipe and viscosity. At the inlet of the divergent section, Randall and Leibovich assume the flow is critical; therefore, in the absence of viscosity or some other dissipative action, the solitary wave is found to propagate upstream to the inlet. By including viscous dissipation, Randall and Leibovich find that a solitary wave may become trapped at a location where viscous dissipation effects counterbalance pipe divergence effects and equate breakdown with the resulting stationary wave. However, from the analysis presented in Chapter 3, viscous effects were found to be negligible except at very low Reynolds numbers. Thus, a (self-admitted) weakness of the Randall and Leibovich trapped wave theory of vortex breakdown is its dependence on viscous dissipation. In [42], Leibovich offers the suggestion that a more likely form of dissipation is due to the growth of non-axisymmetric disturbance modes on the critical flow. Although this form of apparent dissipation seems likely to be more important than viscous dissipation, the nearly axisymmetric form of breakdown shows no signs of three-dimensional instabilities upstream of breakdown; therefore, although Leibovich's proposal may be an accurate model of the spiral mode of breakdown, a different mechanism seems necessary to explain the final location of the axisymmetric

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form of breakdown. A further discussion on the validity of Randall and Leibovich's trapped wave model is given in Section 6.4.

5.2 The Approach of Flow Criticality

Numerous experimental results have shown that a vortex upstream of breakdown is supercritical[26, 22, 64]. Therefore, a mechanism is needed which drives the flow towards criticality. We reconsider the second technique of determining flow criticality discussed in Section 5.1.1. In order for the flow to be critical, the eigenmode, \( f_c(r) \), for Equation (5.3) must have an internal zero. As demonstrated by Benjamin[5], the likelihood of an internal zero increases as \( C(\Psi, y) \) becomes increasingly negative making the solution vary more strongly. Therefore, we may interpret a decrease of \( C \) as an approach towards flow criticality.

We now proceed to show that, in general, streamsurface divergence results in a decrease of \( C \). At some upstream location, \( z = z_0 \), the distribution of \( H \) and \( I \) with respect to the streamfunction are assumed given. Next, we consider a downstream location, \( z_1 > z_0 \); by assumption, the flow is decelerating and, therefore, the streamsurfaces are diverging. The streamsurface divergence implies

\[
\Psi(z_1, y) \equiv \Psi_1 < \Psi_0 \equiv \Psi(z_0, y),
\]

for a given radial location, \( r = \sqrt{2y} \). If \(-H''\) and \(I''\) increase as \( \Psi \) decreases, then \( C(\Psi_1, y) < C(\Psi_0, y) \). Although it is not possible to show for all vortices that \(-H''\) and \(I''\) increase as \( \Psi \) decreases, some insight into the general behavior of these terms is possible. For this purpose, we will investigate the q-vortex model as given by Equations (D.1). Figures 5.2 and 5.3 show the distribution of \( H'' \) and \( I'' \) as a function of \( \Psi \) for \( \Omega = 1.50 \) and several magnitudes of the axial jet velocity. These distributions are qualitatively representative of most q-vortices although the specific details will vary with swirl ratio and jet size. Since q-vortices are reasonable models for a variety of vortex
flows[21], it is likely that distributions of \(H''\) and \(I''\) are also qualitatively representative of many naturally occurring vortices. From Figure 5.2, \(H''\) is a monotonically decreasing function of \(\Psi\). Therefore, a decrease in the local streamfunction results in an increase of \(-H''\). The value of \(I''\) increases for decreasing \(\Psi\), however, a minimum of \(I''\) is attained at a critical \(\Psi_{\text{min}}\) which depends on the specific velocity distributions. An important point to note is that \(|I''|\) is significantly larger for \(\Psi \to 0\) than elsewhere. Thus, we conclude that decreasing the streamfunction at a given radial location will increase \(I''\) for values inside the vortex core (approximately measured by \(\Psi_{\text{min}}\)), while mildly decreasing values of \(I''\) outside the vortex core. Therefore, since \(-H''\) and \(I''\) generally increase with decreasing streamfunction inside the vortex core, we conclude that streamsurface divergence drives a flow towards criticality.

In Chapter 2, the production of azimuthal vorticity and, as a result, the stagnation of the flow was found to be driven mainly by external streamsurface divergence or adverse pressure gradients. In regards to the approach of flow criticality, the important result of Chapter 2 is that the production of negative azimuthal vorticity amplifies the deceleration of the core flow and increases the streamsurface divergence relative to the
external flow. Therefore, the previous results of Chapters 2, 3, and 4, which pertain to various aspects of the production of negative azimuthal vorticity, are equally applicable to describing the approach of flow criticality.

The importance of vorticity production can be illustrated by considering the extreme case of an irrotational flow. As an irrotational swirling flow is subjected to adverse pressure gradients, no pressure amplification will occur in the vortex core due to vortex tilting. Therefore, the core streamsurface divergence will not be amplified. Furthermore, it is simple to show that an irrotational flow is always supercritical. Thus, breakdown associated with the flow becoming critical can never occur.

As an illustration of the connection between the production of negative azimuthal vorticity and the approach of flow criticality, we reconsider the swirling flow through the three pipe geometries described in Tables 2.1 and 2.2. The inlet swirl ratio for the simulations is $\Omega = 0.90$ which is below the swirl ratio for which breakdown first occurs in all three pipes. The inlet axial velocity is constant ($\Delta w = 0$). For these simulations, the Reynolds number was set to $Re = 1000$. This high Reynolds number flow
Figure 5.4: Azimuthal vorticity production and flow criticality for Pipes 1-3, $\Omega = 0.90$. 11 $\eta$ contours from (-0.8,0.1)
requires numerical smoothing to stabilize the scheme. The fourth difference smoothing coefficient in these simulations was \( \nu_4 = 0.001 \). Figure 5.4 shows \( \eta \) contours and the eigenvalue distribution, \( \gamma_0^2(z) \), for all three of the calculated pipe flows. The eigenvalues are calculated locally at each axial grid station – a process which is strictly valid only for gradual flow variations in \( z \). In the converging inlet section, the azimuthal vorticity increases for all three pipes. Similarly, the eigenvalues are increasing with \( \eta \) in this inlet section; therefore, the flow is become increasingly supercritical. In the divergent section of the pipe, the azimuthal vorticity decreases and, likewise, the eigenvalues begin to decrease. In the Pipe 1 flow, the azimuthal vorticity is positive throughout almost the entire computational domain. For Pipe 2, the azimuthal vorticity becomes clearly negative, and, Pipe 3 has the most negative values of \( \eta \). The azimuthal vorticity trends amongst the different pipes are also reflected in the criticality of the flows. In the largest pipe, Pipe 3, the flow nearly reaches criticality at \( z = 13 \) where \( \gamma_0^2 \approx 0 \). From Section 3.3 and Figure 3.9, we recall that vortex breakdown occurs in Pipe 3 for \( \Omega \geq 0.91 \) at \( Re = 1000 \). Therefore, as we will discuss in the next section, breakdown appears to be associated with flow criticality.

### 5.3 A Trapped Wave Theory of Vortex Breakdown

The theoretical elements of the trapped wave theory of vortex breakdown were presented by Bilanin[8] and, in an earlier, closely related work, by Landahl[37]. Similar to the presentations of Bilanin and Landahl, the trapped wave theory is explained in terms of the more familiar example of compressible flow. In the supersonic flow through a Laval nozzle, an oscillatory disturbance will produce two acoustic waves. One wave will travel at \( U + a \) where \( U \) is the flow velocity and \( a \) is the speed of sound. The other wave will travel at \( U - a \). Results from linear theory show that the wave amplitude of the slow wave will vary as \( (U - a)^{-\frac{1}{2}} \); therefore, if the flow is approaching sonic conditions, the wave amplitude will also increase. Upon reaching a sonic location, the wave becomes stationary and the amplitude tends toward infinity according to linear
theory. This phenomenon is an example of the space-time focusing of waves as a result of consecutive wave fronts catching up with each other at the sonic singularity. With the inclusion of non-linearities, this singularity takes the form of a shock which then propagates upstream.

Bilinik proposed that vortex breakdown is a result of the same type of trapped wave phenomenon as sonic flow. However, the analysis of vortex breakdown is complicated by the dispersive nature of vortical flows. In the following discussion, we outline Bilanin’s wave trapping model which is based upon the kinematic wave theory of Whitham[69] and Hayes[28]. Vortices admit infinitesimal wave solutions of the form

\[ \psi = a(z, t)f(r) \exp[i(kz - \omega t)], \]

where \( \psi \) is the streamfunction perturbation, \( a(z, t) \) is the amplitude, and \( f(r) \) is the eigenmode for the given wavenumber and frequency, \( k \) and \( \omega \), respectively. The wavenumber and frequency are related by a dispersion relation which gives \( \omega = \omega(k, z) \). The axial dependence is included in the dispersion relation to indicate the axially-varying nature of the mean flow. The group velocity is given by

\[ c_g = \frac{\partial \omega}{\partial k}. \]

For any dispersive, slowly varying medium, the wavenumber propagates according to

\[ \frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial z} = -\omega_z. \] (5.5)

The right-hand side uses the subscript notation to represent that the derivative is in the augmented space \((k, z)\); the partial notation on the left-hand side refers to derivatives in the propagation space \((z, t)\). Therefore, \( \omega_z \) is found simply by differentiating the mean flow dispersion relation, \( \omega(k, z) \), in the axial direction and evaluating this for the local wavenumber, \( k(z, t) \). Equation (5.5) is simply a statement of the conservation of waves. For a varying waveguide, the wavenumber must change in response to frequency changes. In a uniform waveguide, the wavenumber convects unchanged at the group
velocity.

The amplitude of the wave can be found from Whitham's[69] principle of the conservation of wave action. This conservation principle can be related to the conservation of energy. The wave action density, $A$, is proportional to the square of the wave amplitude. Specifically,

$$A = g a^2,$$

where $g = \frac{\partial G}{\partial \omega}$ and $G$ is the dispersion relation written in the form

$$G = G(\omega, k, z) = 0.$$

Then, for a linear system, the conservation of wave action may be stated as

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(c_g A) = 0. \quad (5.6)$$

Alternatively, Equation (5.6) may be written

$$\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial z} = -A(c_g)z. \quad (5.7)$$

Thus, for a flow in which the group velocity decreases, the wave action density necessarily increases as a result of the conservation of wave action. Generally, this increase of wave action density corresponds to an increase in the wave amplitude.

From Equations (5.5) and (5.7), the propagation speed for the wavenumber and amplitude of a given wave packet is given by the group velocity. An interesting point in the study of dispersive waves is that underlying their dispersive nature is a hyperbolic equation governing the propagation of wave packet wavelength and amplitude. Therefore, one might expect the same type of difficulties encountered in a hyperbolic flow at a singularity where the propagation velocity goes to zero. We have already noted that as the group velocity decreases, the wave action density must increase. Using the results of Landahl[37], Bilanin proposed that in a slowly varying vortex, space-time focusing of waves occurs when the group velocity goes to zero. Unfortunately, any more
Table 5.1: Pipe geometry constants with convergent outlet section

<table>
<thead>
<tr>
<th></th>
<th>$z_l$</th>
<th>$z_o$</th>
<th>$z_c$</th>
<th>$z_{\text{max}}$</th>
<th>$R_i$</th>
<th>$R_t$</th>
<th>$R_o$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pipe 6</td>
<td>5</td>
<td>10</td>
<td>25</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.0</td>
<td>1.8</td>
</tr>
<tr>
<td>Pipe 7</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>2.0</td>
<td>1.8</td>
<td>2.0</td>
<td>1.6</td>
</tr>
</tbody>
</table>

![Figure 5.5: Comparison of Pipe 6 and 7 geometries](image)

Preciseness concerning wave trapping theory is quite difficult without resorting to specific, contrived mean flows to provide the necessary dispersion relations\cite{8} or non-linear analysis\cite{58} both of which are beyond the scope of this work. In keeping with the spirit of this thesis, we will pursue the use of numerical simulations to investigate the possible connection of wave trapping to vortex breakdown.

The pipe geometry for which the simulations are calculated is nearly identical with Pipe 1 except the outlet has a convergent section employed to re-accelerate the flow. This convergent section will be used in conjunction with a study of downstream boundary effects on vortex breakdown in Section 5.4. The specific geometry is given by:

$$R(z) = \begin{cases} 
R_i + \frac{1}{2}(R_t - R_i)\{1 - \cos[\pi(z/z_l)]\} & \text{if } 0 < z < z_t \\
R_t + \frac{1}{2}(R_o - R_t)\{1 - \cos[\pi(z - z_t)/(z_o - z_t)]\} & z_t < z < z_o \\
R_o & z_o < z < z_c \\
R_o + \frac{1}{2}(R_c - R_o)\{1 - \cos[\pi(z - z_c)/(z_{\text{max}} - z_c)]\} & z_c < z < z_{\text{max}} 
\end{cases}$$

(5.8)

Constants for the two pipe geometries used in this chapter are given in Table 5.1. Figure 5.5 contains a comparison of the Pipe 6 and 7 geometries. In both pipes, we employ a grid of 31 radial nodes by 151 axial nodes. For the purposes of this section on trapped
wave theory, we will use Pipe 6 exclusively.

Reynolds numbers of 500, 1000, and 2000 have been used in the following simulations although the main concentration will be on the $Re = 1000$ solutions. For the aforementioned grid size, these Reynolds numbers require the use of artificial viscosity to stabilize the scheme. In the following results, $\nu_4 = 0.004$ was found to stabilize the scheme for all cases; note that this is higher than the results of Section 5.2. In those cases, the maximum axial velocity was lower than that observed in the following simulations due to the lower swirl ratio. The larger axial velocity raises the local cell Reynolds number and the scheme will therefore require higher artificial viscosity levels. Although it was possible to simulate some of the flows in this section at lower $\nu_4$ values, for consistency, $\nu = 0.004$ was used throughout. The inlet axial velocity profile was constant ($\Delta w = 0$).

In the first set of simulations, a swirl ratio sweep was run from $\Omega = 1.44$ to the $\Omega$ at which breakdown occurred for $Re = 500, 1000, \text{and} 2000$. Figures 5.6, 5.8, and 5.10 are plots of the minimum axial velocity at $r = 0$,

$$w_{\text{min}} = \min_{z}\{w(0, z)\},$$

versus the inlet swirl ratio. Breakdown appeared at $\Omega = 1.483, 1.51, \text{and} 1.528$ for increasing Reynolds number. The dependence of the breakdown swirl ratio on Reynolds number is counter to the results of Section 3.3 in which the onset of breakdown was found to be independent of Reynolds number for $Re > 500$; in Section 5.4, we will show that this phenomenon is related to the effect of viscosity on flow criticality and becomes important when downstream boundaries are located close to the critical location. In the course of these simulations, non-unique solutions were found which closely parallel the results of Beran and Culick[7]. These non-unique solutions are documented in Appendix E. The eigenvalue distributions for the standing wave equation, Equation (5.1), were calculated and appear in Figures 5.7, 5.9, and 5.11. In addition, the swirl ratio for which the flow first attains criticality is marked by a dashed line in the $w_{\text{min}}$ plots. From these results, flow criticality is seen to be nearly coincident with breakdown in all cases. Furthermore, the agreement between criticality and breakdown increases
with Reynolds number. These results suggest that the occurrence of flow criticality is a necessary condition for vortex breakdown as proposed in the trapped wave theory described above.

Figure 5.6 shows the eigenvalue distribution, $\gamma_0^2(z)$, for several swirl ratios at $Re = 500$. From this plot, $\gamma_0^2$ drops well below criticality. The discrepancy (although slight) between flow criticality and breakdown may be a result of several effects. First, physical and numerical dissipation may allow the flow to be subcritical without breakdown occurring by balancing the wave amplitude growth due to the inviscid singularity with amplitude decay due to dissipation[58]. The dissipative effects can be partially measured by observing the variation with Reynolds number of the minimum $\gamma_0^2$ attained before breakdown occurs from Figures 5.7, 5.9, and 5.11. With increasing Reynolds number, the minimum $\gamma_0^2$ is approaching zero. Therefore, dissipative effects tend to delay vortex breakdown beyond flow criticality. In Chapter 3, viscous effects were also found to delay vortex breakdown by decreasing the production of negative azimuthal vorticity. Another ramification of viscous effects will be discussed in Section 5.4. In addition to the dissipative effects, a second possible source of disagreement between breakdown and
Figure 5.7: Local eigenvalues for standing waves at $\Omega = 1.46, 1.47, 1.48, 1.482$, and $1.4825$. $Re = 500$.

Figure 5.8: Minimum axial velocity at $r = 0$ versus swirl ratio, $Re = 1000$. 
Figure 5.9: Local eigenvalues for standing waves at $\Omega = 1.49, 1.50, 1.502, 1.504, 1.506, 1.508, \text{ and } 1.509$. $Re = 1000$.

Figure 5.10: Minimum axial velocity at $r = 0$ versus swirl ratio, $Re = 2000$. 
Figure 5.11: Local eigenvalues for standing waves at $\Omega = 1.51, 1.52, 1.522, 1.524, 1.526,$ and $1.527$. $Re = 2000$.

flow criticality is in the eigenvalue calculation. The eigenvalues are found assuming the flow to have slow axial variations which introduces an error on the order of $(kL)^{-1}$ where $k$ is the wavenumber and $L$ measures the length scale of mean flow inhomogeneity. This assumption will be most inaccurate at low wavenumbers which, unfortunately, are the first wavenumbers to become subcritical[43]. Therefore, the local eigenvalues, although indicative of the wave propagation characteristics of the flow, may be slightly incorrect.

To illustrate the trapped wave nature of vortex breakdown, a transient flow simulation is necessary. We will use the $Re = 1000$, $\Omega = 1.504$ steady flow as the initial condition. As shown in Figure 5.9, at these conditions, the vortex is on the verge of criticality (i.e. $\gamma^2_0 \approx 0$). The inlet swirl ratio is then raised at $t = 0$ to $\Omega = 1.52$ initiating a disturbance which propagates throughout the domain. Since this swirl ratio is above that for which breakdown occurs ($\Omega = 1.51$), the final solution should contain breakdown and the transient will contain the breakdown evolution. A flow quantity which has been found useful in depicting the wave nature of the breakdown process is
the perturbation azimuthal vorticity, \( \tilde{\eta} \), defined by

\[
\tilde{\eta}(r, z, t) = \eta(r, z, t) - \frac{\partial}{\partial z} \eta(r, z),
\]

where \( \eta(r, z, t) \) is the total azimuthal vorticity and \( \eta(r, z) \) is the steady, base flow azimuthal vorticity. In this particular case, \( \tilde{\eta} \) is the azimuthal vorticity from the \( \Omega = 1.504 \) steady solution. We will focus on the azimuthal vorticity perturbation near the axis; since the azimuthal vorticity must be zero at \( r = 0 \), we instead will use the azimuthal vorticity perturbation at the \( j = 2 \) grid line, \( \tilde{\eta}_{2,k} \), which corresponds to a radial location of \( r \approx 0.066 \). The transient behavior of \( \tilde{\eta}_{2,k} \) from \( t = 0 \) to \( 578 \delta/W_\infty \) is shown in Figure 5.12. The initial perturbation is so small in comparison to the final perturbation that, on the scale of Figure 5.12, the initial stages of the breakdown evolution are undetectable. However, by \( t \approx 250 \), a small wavelike perturbation can be seen travelling forward and amplifying from \( z \approx 11 \). This wave is responsible for the first breakdown bubble. For \( 250 < t < 350 \), the wave continually moves upstream and amplifies, and a large positive azimuthal vorticity perturbation initiates just downstream of the first negative perturbation. At \( t \approx 350 \), a second spike of negative azimuthal vorticity appears which will be connected to the formation of second breakdown bubble. For \( t > 350 \), the flow slowly settles into a steady state and most of the transients have died away.

The features of the transient solution may be better observed by considering smaller time intervals. Figure 5.13 shows the initial stages of the wave evolution for \( 0 < t < 41 \). A small wave perturbation is convecting downstream and amplifies in the region from \( 5 < z < 10 \). This amplification is expected because the group velocity of the base flow is approaching zero (this follows from the eigenvalues in Figure 5.9 approaching zero); therefore, according to Equation (5.7), the wave action density must increase as the group velocity decreases. Since the wave action density, \( A \), is roughly proportional to the square of the wave amplitude, \( a \), we also expect an increase in wave amplitude. At later times, say \( t > 25 \), most of the perturbations have traversed the entire computational domain; however, a fairly long wavelength perturbation appears to be trapped at \( z \approx 12 \) which is slightly upstream of the critical location of the base flow.
Figure 5.12: Time evolution of $\tilde{\eta}_{2,\nu}$ from $t = 0$ to $578 \delta / W_\infty$ in steps of $4.1 \delta / W_\infty$
Figure 5.13: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 0$ to $41 \, \delta / W_\infty$ in steps of $0.41 \, \delta / W_\infty$

Figure 5.14: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 41$ to $124 \, \delta / W_\infty$ in steps of $0.82 \, \delta / W_\infty$
Figure 5.15: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 124$ to $206 \, \delta/W_\infty$ in steps of $0.82 \, \delta/W_\infty$.

Figure 5.16: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 206$ to $289 \, \delta/W_\infty$ in steps of $0.82 \, \delta/W_\infty$. 
Figure 5.17: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 289 \delta/W_\infty$ to $330 \delta/W_\infty$ in steps of $0.82 \delta/W_\infty$

Figure 5.18: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 330 \delta/W_\infty$ to $372 \delta/W_\infty$ in steps of $0.82 \delta/W_\infty$
Figure 5.14 shows the next stages of transient evolution for \(41 < t < 124\). The dominant feature in this portion of the evolution is the trapped wave. This wave is slowly amplifying and the wavelength is also slowly decreasing. By the end of this interval, all other transients have effectively disappeared. Figure 5.15, for \(124 < t < 206\), shows the trapped wave continually increasing in amplitude and decreasing in wavelength. Figure 5.16, for \(206 < t < 289\), again displays the continued growth of the trapped wave (note the scaling of the vorticity perturbation). For \(t \approx 275\), a large spike of positive azimuthal vorticity appears in the transient evolution (which we will interpret shortly). In Figure 5.17, this spike continues to grow and large amplitude oscillations are appearing behind the initial trapped wave. In addition, we note that the positive azimuthal vorticity spike, although generally growing, is also oscillating. Referring back to the long time history in Figure 5.12, this phase of the evolution is associated with the upstream propagation of the trapped wave. On the scale of Figure 5.17, this upstream propagation is difficult to detect. Finally, Figure 5.18, for \(330 < t < 372\), shows the formation of the second negative spike in azimuthal vorticity. For \(t > 372\), the transients have nearly died away and the breakdown bubble slowly settles into it steady location.

Many of the features of the vorticity perturbation time evolution may be interpreted by visualization of the streamsurfaces. Figure 5.19 shows the initial stages of the bubble formation. At \(t = 41\), only a slight perturbation is visible at \(z \approx 12.4\) By \(t = 165\), the streamfunction perturbation has increased further, and, at \(t = 223\), the \(\Psi = 0\) streamsurface has lifted of the axis to form a small region of reversed flow. At \(t = 254\), the bubble has become quite large and is moving upstream. The bubble formation from \(t = 0\) to \(t = 254\) is quite symmetric and strongly resembles the solitary wave solutions found by Leibovich[44]; it seems that the initial development of breakdown may indeed be describable by weakly nonlinear theory.

At \(t = 286\), an inner bubble is formed and the breakdown bubble loses its symmetry. This formation of an inner bubble corresponds with the large spike in positive azimuthal vorticity previously noted in Figure 5.16. The bubble is also continuing its upstream motion. Figure 5.20 shows the final stages of the bubble growth. For \(t = 290, 304\) and
Figure 5.19: Transient bubble growth – initial stages
Figure 5.20: Transient growth of bubble – final stages
309, the inner bubble is ejected from the rear of the breakdown region, another inner bubble forms, and this inner bubble is again ejected. This inner bubble formation and ejection process is responsible for the oscillation in the positive azimuthal vorticity spike discussed for Figure 5.16. After the inner bubble process completes, the bubble takes on an open-ended shape. At \( t = 355 \), the second bubble forms at \( z \approx 17.6 \). Finally, the steady result, which occurs gradually for \( t > 355 \), shows the first bubble has increased quite significantly in size and the second bubble has moved slightly upstream. Evidently, the unsteady bubble maybe smaller than the final steady bubble structure.

The transient solutions which contain the inner bubble are very similar to the experimental results of Faler and Leibovich[21] (see Figure 6.14 (b) for the experimental streamsurfaces). Faler and Leibovich noted that the bubble frequently migrated upstream and downstream (presumably in response to upstream disturbances). To compensate for this, their measurements were only taken during a time when the bubble remained stationary for 'several seconds'. In the computational simulations, the inner bubble formation and ejection process lasted approximately \( 60 \delta / W_\infty \). From Faler and Leibovich's least-squares fit to a q-vortex profile, \( \delta = 0.57 \text{cm} \) and \( W_\infty = 5.53 \text{cm/s} \) giving \( \delta / W_\infty \approx 0.1 \text{s} \). Therefore, if the transient inner bubble process scales similarly in their experiments and this computation, one would expect oscillations to last on the order of several seconds which corresponds to the time scale over which their measurements were taken. Thus, one possible interpretation of Faler and Leibovich's experimental results is that the experimentally-obtained streamsurfaces are essentially phase-averaged streamsurfaces obtained while the bubble structure was in a particular location. The use of the term, phase-average, is not meant to imply a periodic behavior but rather to distinguish from a time-averaged result. If Faler and Leibovich's results are truly time-averaged streamsurfaces, the bubble must be stationary long enough for any transient processes (related to bubble migration) to die off. Additional discussions concerning Faler and Leibovich's experimental bubble structure, as well as the variety of bubble structures observed in other experiments, may be found in Section 6.3.
5.4 Downstream Effects on Vortex Breakdown

Downstream effects on vortex breakdown will be the final subject discussed on the impact of wave propagation in vortical flows. In a series of experiments in swirling pipe flows, Escudier and Keller [18] noted that the breakdown region was affected by the introduction of downstream blockages if the vortex remained subcritical behind breakdown. When the flow returned to supercritical behind the breakdown, the introduction of blockages had only a local influence. This section will focus on similar effects related to downstream conditions and wave propagation.

To assess downstream effects on vortical flows, the flow in Pipe 7 is calculated at the same conditions of the Section 5.3 Pipe 6 calculations. Recall that Pipe 7 begins its contraction at $z = 20$; therefore, if the Pipe 6 flow is near critical at $z = 20$, we will expect upstream influences due to the change in outlet geometry for Pipe 7. The upstream extent of these effects is given by the lowest eigenvalue, $\gamma_0$, of the Pipe 6 flow at $z = 20$. As described in Section 5.1.1, steady perturbations in a supercritical flows decay as $\exp(-|\gamma_0|z)$. For subcritical flows, steady downstream perturbations do not decay.

The variation of the minimum axial velocity, $u_{\text{min}}$, with swirl ratio for both pipes and $Re = 500, 1000, \text{and } 2000$ are shown in Figure 5.21. The first trend to notice is that downstream effects are negligible for the $Re = 500$ flows. By comparing the eigenvalue distributions of the $Re = 500$ and $Re = 2000$ solutions for Pipe 6 in Figures 5.7 and 5.11, respectively, we observe that the region of criticality is more localized at the lower Reynolds number. At $Re = 500$, the flow quickly returns to a supercritical state for $z > 15$; however, at $Re = 2000$, the vortex does not return to a significantly supercritical state until near the pipe contraction at $z = 23$. In the context of downstream effects, viscous dissipation is important because the dissipation tends to return a subcritical flow to supercriticality. Thus, increased viscous dissipation tends to make the flow less sensitive to outlet conditions by effectively isolating the region of flow criticality. In flows with negligible downstream effects, such as Pipes 1-3 where the outlet was a constant
area section, the onset of breakdown should be independent of Reynolds number. This trend was indeed observed in the results of Section 3.3, Figure 3.9. In non-axisymmetric, high Reynolds number flows, three-dimensional unsteadiness, as is commonly observed behind breakdown, may act to return the flow to supercritical and provide a more effective mechanism by which the flow is isolated from downstream effects.

A second trend noticeable in the higher Re solutions is the $w_{\text{min}}$ offset observable throughout the entire $\Gamma$ range considered. Figure 5.22 compares the axial velocity at $r = 0$ for both pipes at $Re = 500$ and two swirl ratios of 1.44 and 1.482. Except in the immediate vicinity of the outlet, the flows are nearly identical. This situation is different for the $Re = 2000$ flows. Figure 5.23 compares the axial velocity at $r = 0$ for both pipes at $Re = 2000$ and two swirl ratios of 1.44 and 1.52. For $\Omega = 1.44$, which remains far from critical throughout the pipes, the axial velocity is almost identical. The apparent offset in $w_{\text{min}}$ at higher Reynolds numbers and lower swirls is largely due to the longer constant area section of Pipe 6 compared to Pipe 7. Thus, at low swirl, while the flow remains supercritical, minimal differences occur between the Pipe 6 and
Figure 5.22: Comparison of axial velocity at $r = 0$ for Pipe 6 and 7. $Re = 500$. 

(a) $\Omega = 1.44$

(b) $\Omega = 1.482$
Pipe 7 flows.

Unlike the $Re = 2000$ low swirl solution, the $\Omega = 1.52$ flow has significant differences in the axial velocity between pipes for $z > 10$ (see Figure 5.23 (b)). The increased outlet contraction clearly tends to accelerate the upstream flow. Since Pipe 7 contains a larger flow contraction, one would expect the streamfunction to be increased relative to the Pipe 6 streamfunction. This downstream effect will propagate further upstream as the flow approaches critical near the outlet section and result in the acceleration of the flow relative to the Pipe 6 flow.

The extent of the upstream influence from outlet geometry changes can be seen in Figure 5.24 for the near breakdown flows ($Re = 500, \Omega = 1.482$ and $Re = 2000, \Omega = 1.52$). In this figure, the difference in the streamfunction between the two pipe flows is defined as

$$\tilde{\Psi} = \Psi_7 - \Psi_6,$$

where $\Psi_6$ and $\Psi_7$ are the streamfunctions of Pipe 6 and 7, respectively. In the top plot of Figure 5.24 (a) and (b), the computational results for $\tilde{\Psi}$ are displayed from $10 < z < 20$.

The upstream decay of the streamfunction perturbation may be predicted from the smallest eigenvalue, $\gamma_0^2$, of the critical equation. For the Pipe 6 flows, $\gamma_0^2$ is plotted in Figures 5.7 and 5.11. For $Re = 500$ and $Re = 2000$, $\gamma_0(z = 20)$ is 0.82 and 0.45, respectively. Therefore, using the decay rates given by $\gamma_0$, the streamfunction perturbations may be predicted as:

$$\tilde{\Psi}_{\text{linear}}(r, z) = \tilde{\Psi}(r, 20) \exp(-\gamma_0|z - 20|),$$

where $\tilde{\Psi}(r, 20)$ is the difference of the streamfunctions for Pipe 6 and 7 at $z = 20$ as calculated using the numerical algorithm. The results from linearized theory are contained in the bottom plots of Figure 5.24 (a) and (b). Comparing the two computational results, the upstream influence of the outlet contraction is clearly larger for the higher Reynolds number flow which is nearer to criticality at $z = 20$. At $Re = 500$, the up-
Figure 5.23: Comparison of axial velocity at $r = 0$ for Pipe 6 and 7. $Re = 2000$. 

(a) $\Omega = 1.44$

(b) $\Omega = 1.52$
stream effects are confined to $z > 18$ except for a small perturbation around $z = 11$. However, at $Re = 2000$, the flow feels the downstream contraction nearly the entire length of the constant area section. Comparison of the computational solutions with linearized theory shows close agreement for both Reynolds numbers when $z > 18$. For $z < 18$, the $Re = 2000$ results show continued influence of the contraction beyond that predicted by linearized theory.

Several important implications of these results on downstream effects are summarized below.

1. From Section 5.3, as breakdown is approached, the flow is also approaching criticality. Therefore, downstream effects become increasingly important at conditions near breakdown. One immediate ramification is that experimental measurements of flows near criticality using intrusive techniques are likely to be extremely difficult since the introduction of a probe could result in the onset of breakdown.

2. For axisymmetric flows, viscous dissipation was found to return subcritical flows to a supercritical state. Thus, lower Reynolds number flows are less affected by downstream conditions than higher Reynolds number flows.

3. An interesting experiment related to non-unique flows (see Appendix E) and downstream effects could be conducted. For example, one could imagine a swirling pipe flow with geometry similar to the converging-diverging pipe used throughout this thesis. Initially, let the pipe have a straight outlet section and slowly raise the swirl until a steady, axisymmetric breakdown first appears. Then, insert a blockage device in the outlet and slowly move the blockage upstream letting the flow settle to a steady state after each movement. Eventually, breakdown will disappear due to the increased influence of the downstream blockage. Then, slowly retract the blockage occasionally allowing the flow to attain a steady state during the retraction. In light of the results of this section and Appendix E, non-unique flows may be observed.

4. Hall[26] suggested that the quasi-cylindrical equations could be used to calculate
Figure 5.24: Effects of downstream boundary on streamfunction perturbation. Comparison of computational results and linearized theory.
breakdown by associating breakdown with the failure of the equations analogous to the association of separation with the failure of the boundary layer equations. However, since the quasi-cylindrical equations have been parabolized, no mechanism exists to allow the upstream propagation of information. Therefore, for flows where downstream effects may be of importance, the calculation of breakdown location using the quasi-cylindrical equations is inappropriate. In flows with relatively small changes at downstream locations (where downstream must be appropriately defined in accordance with the application), breakdown location might be attainable from the failure of the quasi-cylindrical equations.

5. Although much of our attention has focused on physical downstream effects, numerical boundary conditions could also have a major impact on the accurate calculation of near critical flows. One remedy is to re-accelerate the flow at the outlet to supercritical values; although this re-acceleration will have upstream influences, the effects can be quantified where as numerical boundary condition effects may not be easily determined.

5.5 Conclusions

In this chapter, the importance of wave motions in vortex breakdown flows was investigated. In particular, a theory of vortex breakdown was developed combining elements of the vorticity dynamics interpretation of breakdown with Bilanin’s wave trapping theory[8]. In Section 5.2, the production of negative azimuthal vorticity, which amplifies streamsurface divergence in the vortex core, was linked with the reduction of the flow from a supercritical state to a critical state. Upon the occurrence of flow criticality, vortex breakdown was shown to be associated with the trapping of a wave at the critical location. Transient simulations of a wave packet perturbation on a transcritical vortex revealed that, as the dispersive wave packet travelled downstream, a wave would become stalled near the critical location. This trapped wave then slowly amplified and eventually formed a region of reversed flow. The initial, transient stages of breakdown
were extremely similar to the solitary wave solutions of Leibovich[44] indicating that the onset of breakdown may be describable by weakly nonlinear theories. Finally, in Section 5.4, downstream effects were found to be important in vortical flows near criticality.
Chapter 6

Comparisons of Experimental and Numerical Vortex Breakdown in Pipes

This chapter focuses on the computation of vortex breakdown for the swirling pipe flows first studied in experiments by Sarpkaya[61, 62, 60] and later by Leibovich and co-workers[20, 21, 22]. The two main purposes of this chapter and these calculations is to assess the validity of the axisymmetric assumption via quantitative comparison with experimental results and to further highlight the vorticity dynamics and wave trapping description of vortex breakdown. The only previous attempt at computationally modelling these pipe flows was performed by Menne[51]; however, in that work, the results were much more limited than the computations presented in this chapter. Furthermore, only one of Menne’s two cases corresponded to an experimentally-measured inlet condition. In this particular case, only the breakdown location could be compared and no detailed quantitative comparisons of the experimental and numerical flowfields were possible.

In Section 6.1, the experimental set-up is briefly described, the necessary computational parameters are given, and the impact of grid resolution is studied. Next, in Section 6.2, we investigate the correlation of breakdown location between the experimental and computational results. The computational flows are shown to be extremely sensitive to small variations in the inlet condition and evidence is provided that suggests the experimental flows are similarly sensitive. Section 6.3 contains detailed quantitative comparisons of the experimental and numerical axial and swirl velocity distributions.
In Section 6.4, an example of wave trapping occurring in this experimental pipe geometry is described. Finally, Section 6.5 describes an unsteady breakdown flow in which periodic vortex ring shedding occurs downstream of the breakdown location.

6.1 Preliminaries

The geometry of the pipe from the experimental studies by Sarpkaya and Leibovich and Faler is shown in Figure 6.1. The pipe begins with a constant area inlet section of radius, $r_t$, from $0 < z < z_i$. In the test section from $z_i < z < z_t$, the pipe linearly diverges to a radius $r_c$. The test section is followed by another constant area section ending at $z = z_c$. Finally, the pipe converges from $z_c < z < z_o$ and at the outlet the pipe radius is $r_o$. Upstream of the inlet section, a swirl guide vane system is used to impart swirl to the flow. This swirl generator is described in detail in the references[61, 21]. The pipe geometries in the simulations have the following axial lengths:

$$z_i/r_t = 1.76, \quad (z_t - z_i)/r_t = 13.33, \quad (z_o - z_c)/r_t = 10.50.$$ 

The radial sizes of the pipes are:

$$r_c/r_t = 1.33, \quad r_o/r_t = 0.787.$$
Table 6.1: Grid parameters and time step for numerical pipe flow simulations

<table>
<thead>
<tr>
<th>$Re_{ex}$</th>
<th>$\Omega_{ex}$</th>
<th>$(z_e - z_t)/r_t$</th>
<th>$z_{cl}/r_t$</th>
<th>$j_{max}$</th>
<th>$k_{cl}$</th>
<th>$k_{max}$</th>
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<td>2.62</td>
<td>6.30</td>
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<td>216</td>
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<tr>
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<td>2.62</td>
<td>11.55</td>
<td>41</td>
<td>151</td>
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<tr>
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<td>11.55</td>
<td>41</td>
<td>151</td>
<td>194</td>
<td>0.04</td>
</tr>
</tbody>
</table>

The inlet length corresponds to the location which Faler and Leibovich[21] (this reference will be referred to as FL1 in the text) identified as their inlet location. For completeness, the tube radius, $r_t$, was 1.905cm in the experiments. The diffuser test section has a divergence angle of 1.43° and is identical to the geometry used in the previous experiments. The length of the constant area section from $z_t < z < z_e$ varied with the particular case and the exact lengths are given in Table 6.1. The lengths were chosen to be as short as possible without changing the occurrence of breakdown. The shortening of the constant area section was performed to allow quicker calculations. For comparison, the length of the constant area section in the experiments was 42.57$r_t$ which is four times longer than the length used in most of the simulations. A simulation of the exact pipe used in the experiment would have nearly doubled the pipe's total length and, likewise, the number of nodes in the grid. In the cases where breakdown occurs near the end of the diffuser (at $z = z_t$), the length of the constant area section generally needed to be longer. If a shorter constant area section were used in these simulations, no breakdown would form due to the upstream influence of the convergent section. This effect is in accordance with the downstream effects discussed in Section 5.4. Note, the experimental Reynolds number and swirl ratio, $Re_{ex}$ and $\Omega_{ex}$, respectively, are based on the pipe diameter, the average axial speed, and circulation outside of the vortex core.

The geometry of the convergent section for the computations does not correspond to the exact experimental convergent section since the details of this portion of the apparatus were not described. The importance of the convergent section is to mainly reaccelerate the flow near the exit. In simulations attempted without the convergent section, the unsteady, iterative solution process often proved to be unstable. The diffi-
ulty corresponded to the development of reversed flow regions near the pipe outlet for which the parabolized numerical boundary condition is inappropriate. By attaching a convergent section, similar to that appearing in the experiment, the numerical boundary condition difficulty at the outlet was avoided by insureing an accelerating exit flow. The exact geometry of the convergent section is a half-period of a sinusoidal function. Except in the vicinity of the outlet, the specific shape of the convergent section had no effect on the rest of the flowfield and its proportions were chosen solely on the need for a smooth nozzle to accelerate the exit flow.

Grid spacing parameters for each of the solutions are listed in Table 6.1. The spacing in the axial direction is constant from 0 < z < z_{cl}. Beyond this clustered region, the spacing increases geometrically for each new node until the resulting cell aspect ratio reaches a user-set maximum. The number of radial points is controlled by the value of j_{max} and the node spacing is constant for a given axial station. Specifically, the grid spacing is

\[ \Delta z_k = \begin{cases} 
  \frac{z_{cl}}{(k_{cl} - 1)} & \text{if } k \leq k_{cl} \\
  \min[s_z \Delta z_{k-1}, A \Delta r_k] & k > k_{cl}.
\end{cases} \]

where k_{cl} is the number of nodes in the clustered region, s_z is the geometric spacing parameter, and A is the maximum cell aspect ratio. For the grids used here, s_z = 1.2 and A = 20. The specific choices for j_{max}, k_{cl}, and z_{cl}, as well as the resultant value of k_{max}, are given in Table 6.1.

All transient calculations were run using a four-stage Runge-Kutta time integration technique (see Section B.2.3 for further details). The time steps used in each calculation are given in Table 6.1. These time steps were found to be the largest time steps which did not exceed the stability limit.

Fourth difference artificial viscosity was also necessary to eliminate high frequency modes. The smoothing procedure is described in detail in Section B.2.2. The fourth difference smoothing coefficient, \( \nu_4 \), for all calculations in this chapter was 0.002 and was found by trial-and-error to be the smallest smoothing coefficient which eliminated
<table>
<thead>
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<th>$\Omega$</th>
<th>$Re$</th>
<th>$\Omega$</th>
<th>$\Delta w$</th>
<th>$\delta/r_t$</th>
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<td>590</td>
<td>2.25</td>
<td>1.33</td>
<td>0.208</td>
</tr>
</tbody>
</table>

Table 6.2: Inlet conditions for numerical pipe flow simulations

sawtooth modes in the solution. No second difference smoothing was needed.

Four different cases are calculated in the remainder of this chapter. From the data provided by Faler and Leibovich[20] (henceforth referred to as FL2), an approximate fit of the experimental data with a q-vortex is used as the inlet condition. The parameters for the q-vortex were calculated using the following procedure:

1. The axial velocity at $r = 0$ and outside of the core, $W_0$ and $W_\infty$, respectively, were determined. From this, $\Delta w = (W_0 - W_\infty)/W_\infty$.

2. The location of the maximum swirl, $r_{sw}/r_t$, and the maximum swirl velocity, $V_{max}$, were determined.

3. From the location of maximum swirl in the q-vortex, the core radius, $\delta$, was found from $\delta/r_t = 0.892r_{sw}/r_t$.

4. The swirl ratio of a q-vortex was calculated from $\Omega = 1.567V_{max}/W_\infty$.

5. The average axial velocity, $W_{ave}$, of a q-vortex in a pipe was calculated from $W_{ave}/W_\infty \approx 1 + \Delta w/(r_t/\delta)^2$.

6. The computational Reynolds number was found from $Re = (\delta/2r_t)(W_\infty/W_{ave})Re_{ax}$.

The results of this process for the three cases described in Section 6.2 appear in Table 6.2. For the $Re_{ax} = 2560$ experiment, a least squares fit to the inlet data is performed to allow a more accurate quantitative assessment of the experimental and computational results. This curve-fitting procedure is detailed in Section 6.3. Finally, the wall boundary layer must also be accounted for in the inlet condition. The boundary layers were added by
including a damping function which smoothly brings the velocities to zero at the pipe wall while not affecting the core flow. Specifically, the q-vortex is modified by

\[
\begin{align*}
    u_0(r) &= 0, \\
    v_0(r) &= v_q(r)D(r), \\
    w_0(r) &= w_q(r)D(r),
\end{align*}
\]

where \(v_q(r)\) and \(w_q(r)\) are the usual q-vortex profiles given by Equation (B.38) and the damping function, \(D(r)\), is

\[
D(r) = 1 - \exp[-\alpha(1 - r/r_t)].
\]

This form of \(D(r)\) gives the velocity components a linear variation with distance from the wall as \(r/r_t \to 1\). The constant \(\alpha\) sets the thickness of the boundary layer at the inlet. For the present simulations, \(\alpha = 14\); however, variations of \(\alpha\) from 8 to 16 made negligible differences in the calculated breakdown location.

Grid resolution studies were conducted using the \(Re_{ex} = 3220, \Omega_{ex} = 1.54\) test case. First, solutions for three different radial spacings were calculated such that \(j_{max} = 31, 41, \) and \(61\) which is equivalent to \(8.0, 10.8, \) and \(16.2\) cells per core radii. The axial velocity at \(r = 0\) for the three solutions is shown in Figure 6.2. All three solutions share the same general features, however, a great deal of spread occurs downstream of the first stagnation point at \(z \approx 15\). The numeric simulations do not actually attain a true stagnation point on the axis. For example, the \(j_{max} = 41\) solution local \(w(0, z)\) minimum at \(z \approx 15\) is 0.026. Many of the solutions obtained in these simulations of the Sarpkaya and Leibovich and Faler pipe flow experiments do not strictly contain at stagnation point. However, this upstream \(w(0, z)\) minimum is associated with a recirculation region which occurs off axis and, therefore, it will be referred to as a stagnation point in future discussions. This type of near stagnation point has also been observed experimentally[66].

For the initial breakdown, increasing the resolutions moves the breakdown consis-
Figure 6.2: Comparison of solutions for different radial grid resolutions
tently upstream. This general trend appears to be slowing with increasing grid resolution and the $j_{\text{max}} = 41$ and 61 solutions are in good agreement until just upstream of the second breakdown for $z > 20$. The streamsurfaces and the azimuthal vorticity distributions are compared in Figures 6.3 and 6.4 for the two fine grid solutions. Both the streamsurfaces and the vorticity distribution are in excellent agreement for $z < 22$. In the $j_{\text{max}} = 41$ solution, a second recirculation region is present which is found to actually consist of two bubbles in the finest grid solution ($j_{\text{max}} = 61$). However, for the first bubble and the upstream flow, the two solutions appear to be essentially the same. Since the assumption of axisymmetric flow is likely to be weakest in the region downstream of the initial breakdown, the additional grid spacing is probably unnecessary because any quantitative comparisons in this region should include non-axisymmetric effects. Therefore, $j_{\text{max}} = 41$ is used for most of the solutions. In the $Re_\infty = 2560$ solution, the higher grid resolution was used since quantitative comparisons were desired which would be as accurate as possible within the constraints of reasonable computation time.
Figure 6.4: Effect of radial grid resolution on η. 10 contours from (-1,1)

Three solutions also were calculated varying the number of nodes in the axial direction such that \( k_{cl} = 101, 151, \) and 201 which is approximately equivalent to 2.3, 3.5, and 4.7 cell per core radii (note \( z_{cl} = 11.55 \)). The axial velocity at \( r = 0 \) for the three solutions is shown in Figure 6.5. All three solutions are in good agreement, however, the \( k_{cl} = 101 \) solution lacks the resolution to correctly simulate the details of the flow in the breakdown region. Since the \( k_{cl} = 151 \) and 201 solutions are essentially the same, the axial grid resolution for the rest of the simulations is chosen to be \( k_{cl} = 151 \).

6.2 Calculation of Breakdown Location and its Sensitivity to Inlet Conditions

The three test cases described in Table 6.2 were calculated and the breakdown location of the computation and experiment are compared in Table 6.3. The breakdown location, \( z_b \),
Figure 6.5: Comparison of solutions for different axial grid resolutions
<table>
<thead>
<tr>
<th>$Re_{ex}$</th>
<th>$\Omega_{ex}$</th>
<th>Experimental</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>min $z_b/\tau_t$</td>
<td>mean $z_b/\tau_t$</td>
</tr>
<tr>
<td>3220</td>
<td>1.54</td>
<td>1.4</td>
<td>1.7</td>
</tr>
<tr>
<td>4540</td>
<td>1.28</td>
<td>3.7</td>
<td>4.1</td>
</tr>
<tr>
<td>6000</td>
<td>1.28</td>
<td>1.5</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Table 6.3: Comparison of experimental and numerical breakdown locations

is identified as the furthest upstream location of the reversed flow region and is measured with respect to the beginning of the divergent section of the pipe at $z_i$ (for $z_b = 0$, breakdown occurs at the beginning of the pipe divergence). The experimental and numerical results are in fair agreement with the mean experimental breakdown location always upstream of the numerical breakdown location. The location of breakdown in the computations were all steady; however, the flow downstream of the breakdown in the $Re_{ex} = 6000$ solution was found to be periodic. This unsteady, periodic breakdown flow is described in detail in Section 6.5. The general flow characteristics of the $Re_{ex} = 3220$ and $4540$ solutions are quite similar and are well characterized by the $Re_{ex} = 3220$ results of Section 6.1 (specifically, Figures 6.2, 6.3, and 6.4).

Several explanations of the breakdown location errors exist. The consistent downstream prediction of breakdown location may be due to insufficient grid resolution in the radial direction. These errors were found to result in moving the breakdown upstream with increasing resolution; however, this dependence of breakdown location on grid resolution was fairly weak (as seen in Figures 6.2 and 6.5). The breakdown location predictions may also be inaccurate due to error in the experimental data at the inlet. For example, in FL1, the maximum estimated errors in velocity measurements upstream of breakdown were expected to be ±0.4 cm/s. In the solutions considered in this chapter, the largest velocities are the axial velocities in the core which in the $Re_{ex} = 6000$ experiment were measured as approximately 30 cm/s. The swirl velocity in the core will be much smaller than the axial velocity since it must approach zero as the vortex axis is approached. In general, both the axial and swirl velocities were in the 10's of cm/s; therefore, from these considerations, one might expect errors in the velocity measurements to be about 0.75% to 4%. Furthermore, in areas of large radial
<table>
<thead>
<tr>
<th>$Re$</th>
<th>$\Omega$</th>
<th>$\Delta w$</th>
<th>$\delta/r_t$</th>
<th>$z_b/r_t$</th>
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<tr>
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<td>2.07</td>
<td>1.08</td>
<td>0.270</td>
<td>2.2</td>
</tr>
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</table>

$Re$ variations

<table>
<thead>
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$\Omega$ variations

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<th>$\Omega$</th>
<th>$\Delta w$</th>
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<th>$z_b/r_t$</th>
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<tbody>
<tr>
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<td>1.08</td>
<td>0.270</td>
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<td>0.270</td>
<td>1.1</td>
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$\Delta w$ variations

<table>
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<th>$\Delta w$</th>
<th>$\delta/r_t$</th>
<th>$z_b/r_t$</th>
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</thead>
<tbody>
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<td>1.06</td>
<td>0.270</td>
<td>1.7</td>
</tr>
<tr>
<td>403</td>
<td>2.07</td>
<td>1.10</td>
<td>0.270</td>
<td>2.8</td>
</tr>
</tbody>
</table>

$\delta/r_t$ variations

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$\Omega$</th>
<th>$\Delta w$</th>
<th>$\delta/r_t$</th>
<th>$z_b/r_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>403</td>
<td>2.07</td>
<td>1.08</td>
<td>0.265</td>
<td>2.0</td>
</tr>
<tr>
<td>403</td>
<td>2.07</td>
<td>1.08</td>
<td>0.275</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Table 6.4: Sensitivity of breakdown location to inlet variations: $Re_{ex} = 3220$, $\Omega_{ex} = 1.54$

gradients, such as occurs with the swirl velocity as $r \to 0$, FL1 sometimes had difficulty in tracking a signal which resulted in poor reproducibility and some data scatter. In particular, the swirl velocity measurements were not available for $r/r_t < 0.1$ for most of the experimental results. Thus, any misalignment of the vortex core with the pipe centerline or other non-uniformities would generally not be detectable in the results of FL1 and FL2. In addition to measurement errors, inaccuracies must be present in the approximation of the inlet vortex as a q-vortex. As a result, it seems likely to suspect that numerical accuracy, experimental errors, and modelling assumptions are likely to contribute to the errors in computed breakdown location.

To further assess the impact of experimental errors and vortex modelling on breakdown location, a parametric study of the $Re_{ex} = 3220$ test case was completed. In this study, $Re$, $\Omega$, $\Delta w$, and $\delta/r_t$ were independently varied by $\pm2\%$ and the new flows were calculated. The 2 percent error is in agreement with the expected measurement errors discussed above. The results of this parametric study are summarized in Table 6.4. The variation of the Reynolds numbers had no effect on the breakdown prediction. This agrees with the analysis presented in Chapter 3 which showed that viscous effects are typically negligible in the breakdown process.
Variations of the swirl ratio had a large effect on the breakdown location and produced breakdown at $z_b/r_t = 1.1$ and $3.3$ for the high and low swirl cases, respectively. These breakdown positions include the entire range of breakdown locations found experimentally for this flow. The variations of the velocity difference, $\Delta w$, also produced large deviations in breakdown location; however, these deviations were smaller than those produced by the swirl ratio variations. The relative effects of the swirl ratio and axial velocity difference are also predicted from the breakdown susceptibility measure, $H_0'$, developed in Chapter 4. Recall that

$$H_0' = 2 \left[ \frac{\Omega^2}{1 + \Delta w} - \Delta w \right].$$  \hspace{1cm} (4.2)

For the original test case, $H_0' = 1.96$. The low and high swirl cases have total pressure differentials of 1.80 and 2.12, respectively, or variations of $\pm 0.16$ from the original case. Similarly, the low and high $\Delta w$ cases have total pressure differentials of 2.04 and 1.88, respectively, or variations of $\pm 0.08$ from the original case. Thus, the susceptibility measure changes by twice as much for the swirl variations as the velocity difference variations. Comparing the corresponding breakdown locations, the $\Omega$ changes resulted in a variation of $\pm 1.1$ pipe radii whereas the $\Delta w$ changes resulted in an average variation of $\pm 0.55$. Thus, the swirl ratio changes had twice the effect on breakdown location as $\Delta w$ which suggests that the breakdown locations are behaving linearly with the susceptibility measure, $H_0'$.

Finally, the breakdown location also varies with the relative size of the core to the pipe radius, $\delta/r_t$. From Equation (3.13), the ratio of the breakdown timescale, $t_b$, to the convective timescale, $t_c$, is

$$\frac{t_b}{t_c} = \left( \frac{\delta}{L} \right) \left( \frac{W_\infty}{U} \right) \left( \frac{1 + \Delta w}{\Omega^2 - \Delta w} \right).$$ \hspace{1cm} (3.13)

Then, substituting for $L/\delta$ gives

$$\frac{t_b}{t_c} = \left( \frac{\delta}{r_t} \right) \left( \frac{r_t}{L} \right) \left( \frac{W_\infty}{U} \right) \left( \frac{1 + \Delta w}{\Omega^2 - \Delta w} \right).$$
Thus, decreasing $\delta/r_t$ lowers $t_b/t_c$ which increases the likelihood of breakdown. One possible interpretation of this effect is that the core is subjected to a given flow divergence, $W_\infty/U$, over a longer relative distance, $L/\delta$. As a result, the vortex experiences an overall larger decelerating force. This effect is indeed seen computationally (referring to Table 6.4). The smallest value of $\delta/r_t$, 0.265, results in the most upstream breakdown position, $z_b/r_t = 2.0$. Likewise, the largest value of $\delta/r_t$, 0.275, results in the most downstream breakdown position, $z_b/r_t = 2.3$.

This parametric study of breakdown sensitivity to inlet variations suggests that even small errors in inlet conditions, whether a consequence of experimental or modelling error, can result in large variations in breakdown prediction. A ramification of this result is that for pipe geometries similar to those used by Sarpkaya and Leibovich, the experimental results are likely to show significant variations due to any small variations in the swirl generation from experiment to experiment. Thus, although qualitative results will probably be the same, the quantitative details are likely to have significant variations. In fact, these variations can be seen in the experiments of Sarpkaya and Leibovich. Figure 6.6 contains a plot of the breakdown location versus experimental Reynolds number for $\Omega_{\infty} = 2.3$. These results show a variation in breakdown location of approximately twice the pipe radii between the two experiments.

The sensitivity of breakdown to inlet variations could also be connected with the large transient variations of breakdown position frequently observed in experiment. Any small inlet variations due to unsteadiness in the swirl generator could result in significant breakdown oscillations. Therefore, for the purposes of performing accurate experimental measurements, a pipe geometry should be designed such that the breakdown is as insensitive as possible to upstream flow perturbations. If the breakdown is relatively insensitive to inlet variations, quantitative comparisons with theory and computations will be easier since any accurate theory or computation must suffer the same sensitivity to inlet changes as the experimental flow. An insensitive pipe geometry would probably require a rapid divergent section to provide a strong adverse pressure gradient. For this type of geometry, it may also be necessary to suction off the wall boundary layer to
Figure 6.6: Comparison of Faler/Leibovich[20] and Sarpkaya[61] breakdown locations for \( \Omega_{ez} = 2.3 \).

prevent separation from clouding the interpretation of the results.

\section*{6.3 Detailed Comparison of Experimental/Numerical Results}

In this section, the experimental and numerical results for the \( Re_{ez} = 2560, \Omega_{ez} = 1.77 \) test case are quantitatively compared. The experimental velocity measurements of FL1 were taken at a fixed spatial location during time periods when the bubble remained stationary for several seconds. As discussed in Section 5.3, the resulting velocity data are not strictly time-averaged unless transients due to bubble motions have died off during the time at which the data was taken. The results of Section 5.3 indicate the possibility that the average flowfield measured in FL1 may contain transient effects due to the bubble motion. Further details on the experimental procedures and results are thoroughly described in FL1[21].
Table 6.5: Coefficients of least-square fit to experimental inlet conditions: $Re_{ez} = 2560$, $\Omega_{ez} = 1.77$.

As mentioned in Section 6.1, the inlet conditions for the computational results are a least-squares curve-fit of the experimental axial and swirl velocities. The resulting curve-fits have the form

$$v_0(\bar{r}) = a_1 \bar{r} + a_2 \bar{r}^2 + a_3 \bar{r}^3 + a_4 \bar{r}^4 + a_5 \bar{r}^5,$$

$$w_0(\bar{r}) = b_0 + b_1 \bar{r} + b_2 \bar{r}^2 + b_3 \bar{r}^3 + b_4 \bar{r}^4,$$

where $\bar{r} = r/r_t$ and the coefficients of the least-square fit are given in Table 6.5. These curve-fit swirl and axial velocity distributions have been non-dimensionalized by the maximum swirl velocity, $V_{max}$, and the average axial velocity, $W_{ave}$. In the experiment, the value of $V_{max} = 8.912$ cm/s and $W_{ave} = 5.676$ cm/s. The reference length scale is chosen as $\delta = r_t/3$ which is approximately the location of the maximum swirl velocity. Thus, the Reynolds number is $Re = W_{ave}\delta/\nu = Re_{ez}/6 = 427$. The swirl ratio, $\Omega$, is now simply defined as $\Omega = V_{max}/W_{ave} = 1.57$; however, using this value of swirl ratio resulted in a breakdown which occurred at the pipe inlet. Therefore, the inlet swirl ratio was adjusted in steps of 0.01 until the computed breakdown location roughly corresponded to the experimental breakdown location which was $z_b/r_t = -0.2$. The resulting swirl ratio which was used for the inlet condition was $\Omega = 1.47$ which corresponds to about 6% lower swirl ratio than the experimentally measured flow. The experimental and computational inlet velocity distributions are compared in Figure 6.7. The least-squares fit axial velocity distribution matches the experimental values quite reasonably. The computational inlet swirl velocity, although matching the functional dependence with the experimental results, is 6% lower due to the decrease of swirl ratio necessary to keep the breakdown from propagating entirely to the inlet. The discrepancy between the experiment and calculation is likely due to the cumulative effect of
Figure 6.7: Comparison of experimental/numerical inlet conditions. □ experiment; – calculation.

several small experimental and modelling errors (as discussed in Section 6.2), however, the only feasible approach was to adjust one parameter to account for the overall error. In this simulation, the swirl ratio, $\Omega$, was appropriately adjusted.

The general details of the recirculation region are shown in Figure 6.8. For these inlet conditions, the flow converges to a steady breakdown at $z_b/r_t = -1.0$ in contrast to the experimental value of $z_b/r_t = -0.2$ (note that an inlet swirl ratio of 1.48 resulted in the breakdown propagating downstream the computational inlet). A second breakdown region is also evident behind the initial breakdown region. In FL1, the experimental results are given in relation to the breakdown location and the same reference location is used in the following discussion. For convenience, we define

$$z_B = [(z - z_i) - z_b]/\delta.$$  

Thus, for locations upstream (downstream) of breakdown, $z_B$ is negative (positive).
The axial velocity at the pipe centerline upstream of breakdown is displayed in Figure 6.9. The experimental and numerical centerline axial velocity values are in excellent agreement. Several radial distributions of the axial velocity upstream and downstream of breakdown are shown in Figures 6.10 and 6.11, respectively. The experimental and numerical axial velocity distributions are in excellent qualitative agreement and are generally within 10% of each other upstream of breakdown. The flow outside the core is particularly well predicted. The deceleration of the core flow is noticeable beginning at \( z_B = -1.03 \). For the flow downstream of breakdown, the qualitative agreement is still good. In particular, the axial velocity profile at \( z_B = 0.16 \) is in surprisingly good agreement. The largest discrepancies exist at \( z_B = 2.36 \); the relatively larger error downstream of breakdown is not surprising since the experimental bubble was found to be unsteady and three-dimensional in this region.

The upstream and downstream swirl velocity distributions are shown in Figures 6.12 and 6.13, respectively. Qualitatively, the results are again in good agreement; however, quantitatively, the 6% decrease in swirl level is visible especially upstream of breakdown. Downstream of breakdown the quantitative results further degrade but the general trends appear to be captured by the axisymmetric flow solver. Thus, in some instances, vortex breakdown can be accurately modeled as an axisymmetric phenomenon.
Figure 6.9: Comparison of experimental/numerical axial velocity at $r = 0$ upstream of bubble. □ experiment; – calculation.

A closer view of the bubble structure for the computation and experiment is shown in Figure 6.14 (a) and (b). The experimental bubble structure was constructed from the average axial velocity profiles. As discussed above, these results are not strictly long-time averages and therefore the interpretation of the experimental structure is difficult. Unlike the experimental results, the computations only find two internal stagnation points in contrast to the four stagnation found in FL1. Also, the scales of the bubble images are the same allowing easy visual comparison of the relative bubble sizes. Clearly, the experimental bubble is considerably shorter than the numerical bubble; however, the bubble heights are nearly the same. Recall from Section 5.3, during the transient evolution of breakdown, both the inner recirculation region and the smaller bubble size were observed. This led to the speculation that perhaps the measurements of Faler and Leibovich were actually for transient bubble structures.

The computational breakdown structure appears to be more like the breakdown structure found by Bornstrein and Escudier[9] which is shown in Figure 6.14 (c). They found only two stagnation points in the bubble and a strong jet at the tail of the bubble
Figure 6.10: Comparison of experimental/numerical axial velocity distributions upstream of breakdown. □ experiment; - calculation.
Figure 6.11: Comparison of experimental/numerical axial velocity distributions downstream of breakdown. □ experiment; - calculation.
Figure 6.12: Comparison of experimental/numerical swirl velocity distributions upstream of breakdown. □ experiment; — calculation.
Figure 6.13: Comparison of experimental/numerical swirl velocity distributions downstream of breakdown. □ experiment; – calculation.
very similar to the calculations presented throughout this thesis. The computational bubble structure obtained in this thesis is also in agreement with the typical shape seen in other axisymmetric computations\cite{51, 35, 7}. Uchida\cite{66} also found a bubble structure similar to Bornstein and Escudier in the sense that only two stagnation points were observed. However, similar to Leibovich and Faler's result, Uchida detected large scale unsteadiness and three-dimensionality at the rear of the breakdown bubble. A different bubble structure altogether was attained experimentally by Brücker and Althaus\cite{13}; they observed large reversed flow throughout the entire bubble including the flow just downstream of the initial stagnation point. Similar to the Sarpkaya, Faler and Leibovich, and Uchida results, Brücker and Althaus also found non-axisymmetric effects in the bubble structure. From the above results and discussion, we conclude:

1. A unique 'axisymmetric' bubble structure does not exist and the details of the structure depend strongly on the particular flowfield.

2. Although the bubble breakdown is axisymmetric in appearance, non-axisymmetric effects are almost always observed in practice. However, the assumption of axisymmetry still allows accurate calculation of the bubble form of breakdown except towards the rear of the bubble where three-dimensional effects are often largest.

Finally, since the computational results are in qualitative and, in many cases, quantitative agreement with the experimental results, the azimuthal vorticity growth terms are investigated to determine if the mechanics of vortex breakdown are consistent with the description in Chapter 2. Figure 6.15 shows that a strong negative azimuthal vorticity develops upstream of breakdown. The initial development of the negative azimuthal vorticity is dominated by vortex tilting from the inlet until about $z = 4$ where the stretching becomes of equal importance. Figure 6.16 shows the viscous effects upon the azimuthal vorticity growth. Until just prior to breakdown, the viscous effects are negligible and the dominant vorticity production mechanism is due to inviscid tilting and stretching. Thus, as described in Chapter 2, vortex breakdown is essentially an inviscid, pressure gradient-driven phenomenon in which the tilting of the axial vorticity
Figure 6.14: Comparison of numerical and experimental bubble structures
Figure 6.15: Vorticity dynamics: $Re_{ex} = 2560, \Omega_{ex} = 1.777$
is responsible for the growth of negative azimuthal vorticity and the eventual stagnation of the core flow.

6.4 An Example of Wave Trapping

Wave trapping may also be illustrated for this experimental pipe geometry. For this purpose, we will investigate the $Re_{c_2} = 3220$ solution. First, a solution with near critical conditions is necessary. To obtain this critical solution, the inlet swirl ratio is lowered from the experimentally-observed inlet value until criticality conditions are just achieved. We could have also varied the jet size to alter the criticality. The inlet swirl necessary to bring the flow to critical conditions is $\Omega = 1.93$ (to within 0.01). The azimuthal vorticity, eigenvalues, and axial velocity at $r = 0$ of the critical flow are shown in Figure 6.17. The azimuthal vorticity contours show the characteristic decrease in azimuthal vorticity as a result of the pipe divergence. At $z \approx 60$, $\eta$ first becomes negative. As discussed in Chapter 2, the production of negative azimuthal vorticity is caused by streamsurface divergence in the outer flow and amplifies streamsurface divergence in the core region. Thus, as argued in Section 5.2, the production of negative azimuthal vorticities accelerates the approach to flow criticality. The eigenvalue distribution in Figure 6.17 (b) shows that criticality has been nearly achieved. We did not pursue solutions between swirl ratios of 1.93 and 1.94; however, from our previous experience in Section 5.3, we expect that criticality would be completely achieved for some intermediate swirl ratio. The deceleration of the core flow is visible in Figure 6.17 (c). From this plot, we note that although criticality has essentially been achieved, the core is still quite far from stagnation.

Next, we consider the transient development of vortex breakdown. Similar to the wave trapping simulation in Section 5.3, the inlet swirl is raised to the experimental value, $\Omega = 2.067$, at $t = 0$ and the evolution of the azimuthal vorticity perturbation is observed. The entire, long time evolution of the perturbation is shown in Figure 6.18
Figure 6.16: Viscous effects: $Re_{\epsilon z} = 2560$, $\Omega_{\epsilon z} = 1.777$
Figure 6.17: $Re_{ex} = 3220$ critical flow
Figure 6.18: Time evolution of $\dot{q}_{2,\kappa}$ from $t = 0$ to $1000 \delta/W_\infty$ in steps of $6 \delta/W_\infty$.

and the details of the breakdown evolution are given in Figures 6.19–6.21. The results show the distinctive wave trapping phenomenon occurring similar to the simulation of Section 5.3. Most of the initial perturbations are convected downstream with only slight modifications to the mean flow; however, one wave becomes trapped and slowly begins to grow in amplitude. As the wave amplitude grows, the wave begins to slowly move upstream. This propagation upstream was also visible in Section 5.3, however, since the experimental geometry has a much more gradual divergent section, the upstream propagation of the large amplitude wave continues for a longer time. In the Pipe 6, Section 5.3 results, the wave’s upstream propagation is stopped by the strong convergent section. In either pipe, as the bubble propagates upstream, the base flow is becoming increasingly supercritical. The eventual steady location of the breakdown bubble is.
Figure 6.19: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 0$ to $200 \, h/W_\infty$ in steps of $2 \, h/W_\infty$.

determined by a balance between nonlinear effects and the increasing supercriticality of
the base flow. This scenario is very similar to the description of breakdown professed
by Randall and Leibovich[58]; in their work, viscous dissipation was necessary in order
for the breakdown to reach a steady position. The hypothesis which led them to this
result was assuming that the flow upstream of breakdown was near critical; as can
be seen from these simulations, this assumption is only accurate in the initial stages
of breakdown. In this near critical case, our simulations reveal that the bubble does
propagate forward. However, as the breakdown moves upstream, the base flow becomes
increasingly supercritical; therefore, the analysis of Randall and Leibovich no longer
holds. Regardless of this inaccuracy, the trapped wave theory of Randall and Leibovich
does provide a model of the flow during the initial stages of breakdown; also, Leibovich
and Randall[46] have already derived an equation valid for arbitrary base flow wave
speeds and it seems possible that this approach may give an accurate representation of
the later stages of vortex breakdown.
Figure 6.20: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 200$ to $400 \, \delta / W_\infty$ in steps of $2 \, \delta / W_\infty$.

Figure 6.21: Time evolution of $\tilde{\eta}_{2,k}$ from $t = 400$ to $574 \, \delta / W_\infty$ in steps of $2 \, \delta / W_\infty$. 

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6.5 Unsteady, Periodic Flow in Breakdown Region

The simulation of the $Re_{ex} = 0000$, $\Omega_{ex} = 1.28$ test case never converged to a steady-state but rather became periodic. Figure 6.22 is a residual history plot which shows the development of a periodic residual. The period of the oscillations is 300 iterations which is equivalent to 15 $\delta/W_\infty$. The unsteady flow is a result of the periodic shedding of a vortex ring in the breakdown region. Contours of streamfunction and azimuthal vorticity highlighting the vortex ring shedding are shown in Figures 6.23 and 6.24, respectively. At $t = 0$, the nearly circular region of azimuthal vorticity is about to break away from the sheet leading to the stagnation point. By $t = 2.5$, the ring vortex has begun convecting downstream. By $t = 5$, the level of azimuthal vorticity is decreasing, and, by $t = 12.5$, a new ring is being formed and the process starts over again. In this unsteady vortex shedding, the breakdown location is unaffected and its location and the upst. eam flow are steady. This periodic vortex ring shedding bares a strong resemblance to the flow visualization by Escudier[16] in Figure 6.25 which was for vortex breakdown in a slit-tube arrangement. Therefore, this periodic vortex ring shedding process may be realizable in actual physical situations.
Figure 6.23: Streamfunction contours of unsteady vortex ring shedding downstream of breakdown for $Re_x = 6000$, $\Omega_x = 1.282$. 11 contours from (-0.1,0.1)
Figure 6.24: Azimuthal vorticity contours of unsteady vortex ring shedding downstream of breakdown for $Re_z = 6000$, $\Omega_z = 1.282$. 10 contours from (-2,2)
Finally, the time-averaged streamsurfaces and azimuthal vorticity contours are shown in Figures 6.26 and 6.27. The dominant stationary features from the unsteady contours remain in these time-averaged contours. As expected, the shedding structures are not observed in the time-averaged flow.
6.6 Conclusions

This chapter focused on the numerical simulation of the swirling pipe flows studied by Sarpkaya [61, 62, 60] and Leibovich [20, 21, 22]. The breakdown location in the simulations was found to be extremely sensitive to small variations in inlet conditions. This sensitivity to inlet conditions is conjectured to be the main reason for discrepancies between the results of Sarpkaya and Leibovich. In spite of this sensitivity, the numerically predicted breakdown locations were in good agreement with the experimental observations; however, due to this sensitivity, care must be taken when comparing and interpreting breakdown locations from different experiments or simulations.

Next, experimental and numerical breakdown flowfields were compared in detail. The experimental and numerical results were in good qualitative agreement throughout the flowfield and, in most instances, the velocity predictions were accurate to within 10%. The overall good agreement supports the claim that physically-realizable breakdown flows exist which are axisymmetric to good approximation. The bubble structure was steady and contained only two stagnation points with an open tail structure similar to the structure observed by Bornstein and Escudier [9] but differing from the Faler and Leibovich results [21]. As in the previous vorticity dynamics description of vortex breakdown in Chapter 2, the tilting of the axial vorticity into the azimuthal direction
initiated the breakdown process. Viscous effects were found to be negligible except around the bubble surface.

The transient simulation of breakdown evolution for the $Re_{ex} = 3220$ flow is described in Section 6.4. The wave trapping phenomenon is again observed as flow criticality is achieved. Randall and Leibovich's trapped wave model is shown to describe the initial stages of breakdown evolution. However, as the breakdown propagates forward, the increased supercriticality of the base flow is not accounted for in their theory. Therefore, the Randall and Leibovich trapped wave model must rely on viscous dissipation to slow wave propagation and amplitude growth.

Finally, the $Re_{ex} = 6000$ simulation never converged to a steady-state. Instead, the flow became periodic due to the shedding of a vortex ring from the slip layer emanating from the stagnation point at breakdown. The breakdown position remained steady throughout this shedding and only the flow downstream of the stagnation point was affected by the unsteadiness.
Chapter 7

Conclusions

7.1 Summary and Conclusions

The major goal of this thesis was to develop a thorough understanding of the essential mechanisms of axisymmetric vortex breakdown by conducting a combined theoretical and computational investigation. The study was divided into two essential parts:

1. the mechanisms which drive a vortex towards criticality, such that upstream propagation of waves becomes possible,

2. the trapping and amplification of waves which eventually leads to breakdown.

Chapters 2-4 deal with the first aspect of vortex breakdown, the mechanisms which drive the flow towards criticality. These mechanisms are first studied independently of their connection with flow criticality. In Chapter 2, the production of negative azimuthal vorticity is studied. This work extends the original investigation of Brown and Lopez concerning the necessity of negative azimuthal vorticity in vortex breakdown to consider, in detail, the mechanisms which produce negative azimuthal vorticity. This investigation revealed several important effects. First, the production of negative azimuthal vorticity always occurs for any physically-realizable vortex subjected to a streamtube divergence (or, equivalently, an external adverse pressure gradient). This process amplifies the deceleration of the core flow with respect to the external flow. In particular, tilting of the
axial vorticity vector was found to be the mechanism which initiates the core deceleration and vortex stretching plays only a minor role until just before breakdown. The tilting of the axial vorticity vector is shown to be equivalent to Hall's[26] amplification of axial pressure gradients due to swirl velocity gradients. The core just upstream of breakdown is shown to be nearly devoid of any axial vorticity and the vorticity vector is almost completely aligned in the negative azimuthal direction.

Chapter 3 investigates the effects of viscous dissipation upon the production of negative azimuthal vorticity. Two viscous effects are identified in Chapter 3 which contribute to the production of azimuthal vorticity: the effect of viscous dissipation on the circulation gradient, and the effect of viscous dissipation on the distribution of azimuthal vorticity. However, these viscous mechanisms of vorticity production are found to have negligible influence on the inviscidly-driven core stagnation process except at very low Reynolds numbers. In particular, for the pipe simulations which were calculated, viscous contributions did not alter the flows appreciably for core Reynolds numbers over 500.

Chapter 4 focuses on the development of a breakdown susceptibility measure. The Brown and Lopez velocity-vorticity helix angle criterion for breakdown is shown to require only a total pressure loss in the vortex core (i.e. $H' > 0$). However, based on azimuthal vorticity production arguments, the magnitude of $H'$ is suggested as a measure of the susceptibility of a vortex to breakdown. The general susceptibility trends described by $H'$ are in excellent agreement with accepted experimental trends concerning vortex breakdown. Parametric studies for one class of pipe flows revealed that the occurrence and location of vortex breakdown are well correlated by the value of $H'$ at $r = 0$.

In Chapter 5, the wave trapping process, which is hypothesized to lead to vortex breakdown, is examined. First, a connection is made between the production of negative azimuthal vorticity and the approach of flow criticality. Typical vortices, which are always supercritical upstream of breakdown, are shown to approach criticality as a
result of streamsurface divergence. Therefore, the rate at which criticality is approached is determined by the radial divergence (i.e. the radial velocity); this radial divergence is driven by external stream divergence (or adverse pressure gradients) and amplified in the core by the production of negative azimuthal vorticity. At flow criticality, a wave trapping process, first suggested by Bilanin[8], is found to describe the onset of breakdown. Elements of kinematic wave theory show that as the group velocity of the mean flow decreases, the amplitude of wave packets approaching the critical location must increase. Several numerical results show that the onset of vortex breakdown with increasing inlet swirl coincides with the onset of flow criticality. This wave trapping process is clearly illustrated by the numerical simulation of the evolution of vortex breakdown on a transcritical flow. The effects of downstream conditions on the onset of vortex breakdown were also studied. Theoretically, steady downstream disturbances have an exponentially decaying effect in supercritical flows and their decay rate decreases as flow criticality approaches. Numerical results show good agreement with these predictions from linear theory concerning the effect of downstream disturbances.

Chapter 6 serves the dual purpose of establishing the applicability (or lack thereof) of the axisymmetric assumption in pipe flows with breakdown and providing an additional example of the vorticity dynamics and wave trapping explanation of vortex breakdown. Comparisons of experimental[21] and numerical results for axisymmetric breakdown in a slightly diverging pipe showed both qualitative and quantitative agreement. Final breakdown location was extremely sensitive to small variations in inlet conditions and similar sensitivities were apparent in the two experimental results of Faler and Leibovich[21] and Sarpkaya[61]. Transient simulations of breakdown evolution on a near critical flow showed the proposed wave trapping phenomenon. The slow divergence in this pipe allowed the breakdown bubble to progress a large distance upstream before attaining its steady position.

Appendix A documents the various forms of the governing equations which appear in this dissertation. The numerical algorithm developed for this study is described in Appendix B and validated in Appendix C. The q-vortex model, used for the inlet
condition in most of the swirling pipe flow simulations, is briefly detailed in Appendix D. A short account of several non-unique solutions of the time-dependent equations is presented in Appendix E. Finally, Appendix F presents a preliminary study of transient effects of vortex breakdown. In particular, a parametric study is conducted of the timescales at which vortex breakdown develops and decays in response to inlet swirl variations. The implications of these timescales is interpreted as one of many possible lag mechanisms in unsteady breakdown flows.

7.2 Discussion

This combined model of vorticity dynamics and wave trapping has several important ramifications concerning previous theories of axisymmetric breakdown which are drawn out below.

1. Several important distinctions exist between this work and that of Brown and Lopez. These distinctions were discussed in Section 2.1.5 but are re-iterated here for completeness. First, Brown and Lopez did not consider in detail the production of negative azimuthal vorticity. Therefore, the conclusion that the tilting of the axial vorticity drives the initial deceleration of the vortex core was not realized (except in the case of no azimuthal vorticity being present initially). Brown and Lopez, by relating their necessary condition to the upstream helix angle, did not recognize the connection with the circulation and total pressure gradient. In particular, the equivalence of the helix angle criterion with a total pressure loss in the core and the subsequent weakness of the criterion were not noticed. In this investigation, a connection with Hall's boundary layer analogy is made which reveals that of the tilting of the axial vorticity vector can also be interpreted as the amplification of axial pressure gradients. Finally, Brown and Lopez did not consider flows with pressure gradients but rather investigated enclosed cylinders and trailing line vortex flows for which viscous effects play a substantial role in
the onset of vortex breakdown. In contrast, this work focuses on the dominant role of adverse pressure gradients and the secondary importance of viscous effects in typical applications.

2. An important result is that the occurrence of vortex breakdown coincides with the occurrence of flow criticality as suggested by Squire[63] over thirty years ago. The only apparent error with respect to Squire's interpretation is his suggestion that at criticality, upstream propagating disturbances travel forward to the critical location, amplify, and lead to breakdown. The wave trapping process illustrated in the evolutionary simulations resulted from downstream propagating waves becoming stalled at the critical location. These trapped waves then amplify and eventually move upstream such that the steady state position of breakdown is upstream (sometimes significantly) of the initial critical location. However, Squire's explanation, although lacking in rigor, probably does have some applicability to certain instances of vortex breakdown. The pipe flow simulations revealed that at inlet swirl ratios just lower than the critical inlet swirl ratio, the flow remains nearly critical unless re-accelerated (i.e. made more supercritical) by a pipe convergent section or by viscous effects. Therefore, if we imagine placing a small obstacle on the axis in the downstream portion of the flow, a local increase in the streamsurface divergence would occur. The upstream influence of this perturbation will die off exponentially at a rate given by the smallest eigenvalue of the spatial eigenvalue problem. If the upstream influence is large enough, the additional streamsurface divergence could drive the flow to criticality at which point vortex breakdown would occur. This type of behavior was observed by Lambourne and Bryer[36] when bodies of various thicknesses were inserted into a leading-edge vortex. Upstream influence of downstream conditions may also be important when considering the effects of trailing edge flaps on vortex breakdown over delta wings at high angles of attack. Again, Lambourne and Bryer mention that upward deflection of trailing edge flaps results in forward motion of breakdown and vice-versa. At near critical angles of attack, breakdown might be delayed or created by appropriate flap deflections. Regardless of the particular application, the importance of this discussion is that flows approaching critical
conditions are extremely sensitive to any perturbations in the flow, upstream or downstream.

3. Randall and Leibovich[58] developed a trapped wave theory of vortex breakdown which contains many of the elements of the trapped wave theory proposed herein. However, the admitted, fundamental weakness of their theory is its reliance upon viscous dissipation to fix the final breakdown position. According to the results of this dissertation, the probable error in their trapped wave model stems from their assumption that the base flow is near criticality. Although this is an accurate model for the initial stages of breakdown, at later stages, nonlinear effects (as shown by Randall and Leibovich) allow the wave to propagate upstream. As the wave propagates, the base flow typically becomes increasingly supercritical. Therefore, the wave propagation would be halted at the location where the growth of the bubble is balanced by the increasing supercriticality of the flow. In the companion paper to their trapped wave theory[46], Leibovich and Randall derive an equation valid for base flows which are far from critical; however, the implications of this result were never pursued.

4. A major stumbling block in almost every attempt at explaining vortex breakdown is the concentration on only one element of the problem. For example, most wave theories[63, 5, 44, 58] of vortex breakdown focus completely on the wave properties of the flow and only assume that criticality occurs at some point in the flow. However, the mechanisms which drive the flow to criticality are as important as the consequences of criticality occurring. In order for a thorough understanding of vortex breakdown to be developed, both effects must be considered. By analogy with the initial example of decelerating supersonic flow, the mechanisms which decelerate the flow (i.e. flow convergence) are equally as important as the mechanisms which lead to shock formation (i.e. space-time focussing of waves).
7.3 Contributions

This thesis makes several fundamental contributions concerning the mechanisms of axisymmetric breakdown. These contributions are enumerated below.

1. The work of Brown and Lopez concerning the necessity of negative azimuthal vorticity in breakdown flows was extended by considering the effects which produce negative azimuthal vorticity. Using vorticity dynamics considerations, the fundamental mechanism in the production of negative azimuthal vorticity is shown to be the tilting of the axial component of vorticity due to streamsurface divergence imposed by an external pressure gradient. The stretching of azimuthal vorticity becomes important only in the region just upstream of breakdown.

2. The description of vortex breakdown developed from vorticity dynamics considerations relies solely on inviscid mechanisms. A subsequent study of viscous effects found two dissipative effects which can lead to the production of azimuthal vorticity. However, using order of magnitude analysis and parametric numerical studies, viscous effects were found to have negligible influence on the breakdown process except at low Reynolds numbers. Thus, a fundamental contribution of this thesis is the demonstration that the mechanisms leading to vortex breakdown are mainly inviscid in nature.

3. Another contribution of this thesis is the recognition of a link between streamsurface divergence and the approach of flow criticality. For typical distributions of circulation and total pressure, streamsurface divergence drives the flow closer to criticality (assuming an initially supercritical vortex). Therefore, effects which increase streamsurface divergence, such as the production of negative azimuthal vorticity, also drive the flow closer to criticality. Hence, since negative azimuthal vorticity production amplifies the streamsurface divergence in the core relative to the outer flow, it also accelerates the approach of flow criticality.

4. A fourth fundamental contribution is the illustration of wave trapping leading to vortex breakdown as suggested by Bilanin[8]. Vortex breakdown occurs when the
flow reaches critical conditions as a result of the amplification of trapped waves in a process which is analogous to the amplification of trapped waves in a decelerating supersonic flow. Similar to decelerating supersonic flow near sonic conditions, a vortex near critical conditions is extremely sensitive to any flow perturbations. This wave trapping model has several important ramifications concerning many previous explanations of vortex breakdown (see the discussion in Section 7.2).

5. To the author's knowledge, the first quantitative comparisons were obtained between experimental and numerical pipe flow results for axisymmetric breakdown. Only one previous investigator[51] has attempted such calculations and his results were only for a single experimental condition and only the experimental and numerical breakdown locations were compared. In this dissertation, the experimental and numerical axial and swirl velocity distributions were found to be in good agreement upstream of breakdown and in the upstream half of the breakdown bubble. The computational bubble was approximately 50% larger than the experimentally-observed bubble. The overall good agreement between experimental and numerical results indicates that the bubble type of breakdown may be accurately calculated assuming axisymmetry except towards the rear of the bubble where three-dimensionality manifests itself.

7.4 Recommendations for Future Work

This thesis marks only the beginnings of the path to an understanding of vortex breakdown. A few ideas for future work are summarized below.

1. Perhaps the most obvious gap left in a complete understanding of vortex breakdown is the role of non-axisymmetric disturbances. Using techniques similar to the trapped wave simulations in Chapter 5, it may be possible to determine if the spiral mode of breakdown is also associated with wave trapping of non-axisymmetric modes or if spiral breakdown is more closely related to a hydrodynamic instability.
2. Although wave trapping was illustrated for the case of small inlet perturbations convected downstream on an axially-varying flow, as discussed in Section 7.2, breakdown may also occur on near critical flows as a result of downstream disturbances. Numerical simulations for this breakdown evolution should be studied. This would require some small modifications to allow introduction of perturbations at the pipe outlet.

3. A preliminary study of transient effects (see Appendix i') indicated the possibility of several lag mechanisms which could partially account for the lag observed in vortex breakdown in pitching delta wings. Since transient breakdowns appear to be well described by finite amplitude wave theories, a simple model may be possible which could account for hysteresis effects in unsteady pitching delta wing flows.

4. In Section 4.4, a computational technique for the optimization of delta wings with respect to breakdown susceptibility was described which should also be pursued. This optimization study would require the use of a conical flow solver as well as a three-dimensional solver to assess the validity of the suggested method. The effects of axial blowing could also be incorporated into the scheme with some modification of a typical conical flow solver.
Bibliography


Appendix A

Governing Equations

This appendix will derive the relevant equations for swirling, axisymmetric flows which are the basis for many of the computations and theoretical discussions contained in this work. Several secondary relations which will prove helpful in the body of the thesis are also derived. Finally, note that all equations derived in this section pertain to incompressible flows.

A.1 3-D Navier-Stokes Equations

The governing equations (in vector notation) for an incompressible, viscous fluid are:

\[
\begin{align*}
\frac{D\vec{u}}{Dt} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u}, \\
\nabla \cdot \vec{u} &= 0.
\end{align*}
\]  

(A.1) (A.2)

In the above equations, \( \vec{u}, p, \rho \) are the vector velocity field, the pressure, and the density, respectively. The kinematic viscosity, \( \nu \), is considered a constant for the purposes of this investigation. The total derivative notation is used above where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla.
\]
Thus, Equation (A.1) is a vector equation expressing the change in momentum of a fluid element due to pressure and viscous forces. Equation (A.2) is the continuity equation and expresses the conservation of mass for an incompressible fluid element.

Using the vector identity,

\[ \vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla q^2 - \vec{u} \times (\nabla \times \vec{u}), \]

where \( q^2 = \vec{u} \cdot \vec{u} \), Equation (A.1) may be also written:

\[ \nabla H = \vec{u} \times \vec{\omega} + \nu \nabla^2 \vec{u} - \frac{\partial \vec{u}}{\partial t}. \]  \hspace{1cm} (A.3)

In the previous equation, the total pressure, or head, is defined as, \( H = p/\rho + \frac{1}{2} q^2 \), and the vorticity is the curl of the velocity, \( \vec{\omega} = \nabla \times \vec{u} \).

For the particular application of swirling flows, it is helpful to define a cylindrical coordinate system where the position, velocity, and vorticity vector, \( \vec{x}, \vec{u}, \) and \( \vec{\omega} \), respectively, are defined as follows:

\[ \vec{x} = (r, \theta, z), \]
\[ \vec{u} = (u, v, w), \]
\[ \vec{\omega} = (\xi, \eta, \zeta). \]

The components of the vorticity vector in cylindrical coordinates are given by:

\[ \xi = \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \hspace{1cm} (A.4) \]
\[ \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \hspace{1cm} (A.5) \]
\[ \zeta = \frac{1}{r} \frac{\partial (rv)}{\partial r} - \frac{1}{r} \frac{\partial u}{\partial \theta}. \hspace{1cm} (A.6) \]
A.2 Axisymmetric Navier-Stokes Equations

Although vortical flows and, in particular, vortical flows with breakdown, are truly three-dimensional phenomena, many theoretical and numerical studies of these flows assume axisymmetry in order to facilitate both analysis and computation. Under the axisymmetric assumption, all \( \varphi \)azimuthal variations are zero, \( \frac{\partial}{\partial \varphi} = 0 \). Written out by components, the momentum equation, Equation (A.1), simplifies to:

\[
\frac{Du}{Dt} = \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} \right], \quad (A.7)
\]

\[
\frac{Dv}{Dt} + \frac{vw}{r} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2} \right], \quad (A.8)
\]

\[
\frac{ Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2} \right]. \quad (A.9)
\]

The total derivative, with the assumption of axisymmetry, takes the form:

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}. \quad (A.10)
\]

The conservation of mass, as expressed by Equation (A.2), simplifies to:

\[
\frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0. \quad (A.11)
\]

The streamfunction can be defined by:

\[
u = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad w = \frac{1}{r} \frac{\partial \Psi}{\partial r}. \quad (A.12)
\]

With the introduction of the streamfunction as given in Equation (A.12), the conservation of mass is automatically satisfied.

The assumption of axisymmetry also simplifies the vorticity components to:

\[
\xi = -\frac{\partial v}{\partial z}, \quad \text{(A.13)}
\]

\[
\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}. \quad \text{(A.14)}
\]
\[ \zeta = \frac{1}{r} \frac{\partial (rv)}{\partial r}. \]  \hspace{1cm} (A.15)

Note that by substituting Equation (A.12) in Equation (A.14), a Poisson equation relating the streamfunction and azimuthal vorticity is found:

\[ \eta = -\frac{1}{r} \frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right). \]  \hspace{1cm} (A.16)

The azimuthal vorticity transport equation may be derived by taking \( \frac{\partial}{\partial z} \) (Equation A.7) – \( \frac{\partial}{\partial r} \) (Equation A.9). This gives:

\[ \frac{D\eta}{Dt} = \frac{\partial}{\partial z} \left( \frac{v^2}{r} \right) + \frac{u\eta}{r} + \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (\eta)}{\partial r} \right) \right] + \frac{\partial^2 \eta}{\partial z^2}. \]  \hspace{1cm} (A.17)

Finally, the circulation may be used to replace the azimuthal velocity. Specifically, the circulation is defined as:

\[ \Gamma = vr. \]

With this definition, the vorticity components are:

\[ \xi = -\frac{1}{r} \frac{\partial \Gamma}{\partial z}, \]  \hspace{1cm} (A.18)

\[ \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \]  \hspace{1cm} (A.19)

\[ \zeta = \frac{1}{r} \frac{\partial \Gamma}{\partial r}, \]  \hspace{1cm} (A.20)

the azimuthal momentum equation becomes

\[ \frac{D\Gamma}{Dt} = \nu \left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) + \frac{\partial^2 \Gamma}{\partial z^2} \right], \]  \hspace{1cm} (A.21)

and, the azimuthal vorticity transport equation is:

\[ \frac{D\eta}{Dt} = \frac{1}{r^3} \frac{\partial (\Gamma^2)}{\partial z} + \frac{u\eta}{r} + \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (\eta)}{\partial r} \right) \right] + \frac{\partial^2 \eta}{\partial z^2}. \]  \hspace{1cm} (A.22)

For the numerical algorithm, a conservation form of the governing equations in the \((r, z)\) plane would allow a straightforward application of typical finite volume discretiza-
tion techniques to this axisymmetric, swirling flow problem. The Poisson equation for the vorticity, Equation (A.16), is easily written in conservation form as

$$\eta + \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial \Psi}{\partial z} \right) + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) = 0. \quad (A.23)$$

A conservation form of the azimuthal vorticity transport equation, Equation (A.22), may be found by substitution of the continuity equation. This gives the following conservation form:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial z} \left[ w \eta - \frac{\Gamma^2}{r^3} - \nu \frac{\partial \eta}{\partial z} \right] + \frac{\partial}{\partial r} \left[ w \eta - \nu \left\{ \frac{1}{r} \frac{\partial \eta}{\partial r} (r \eta) \right\} \right] = 0. \quad (A.24)$$

Finally, the circulation equation may be placed in a conservation form by first multiplying Equation (A.21) by \( r \), and then substituting in the continuity equation. The resulting conservation form for the circulation equation is

$$\frac{\partial}{\partial t} (r \Gamma) + \frac{\partial}{\partial z} \left[ r w \Gamma - \nu \left\{ \frac{\partial \Gamma}{\partial z} \right\} \right] + \frac{\partial}{\partial r} \left[ r w \Gamma - \nu \left\{ \frac{\partial \Gamma}{\partial r} - 2 \Gamma \right\} \right] = 0. \quad (A.25)$$

These three conservation form equations are then used in the development of a finite volume scheme for swirling, axisymmetric flows which is described in detail in Appendix B.

### A.2.1 Quasi-cylindrical Equations

Hall[26] introduced the quasi-cylindrical equations for use in the analysis of vortex flows. These equations are the companion equations to the boundary layer equations with the simplifying assumption being the neglect of axial gradients in comparison to radial gradients. The resulting set of equations is spatially parabolic. The unsteady, quasi-cylindrical equations are

$$\frac{\nu^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r}, \quad (A.26)$$
\[ \frac{Dv}{Dt} + \frac{uv}{r} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) - \frac{v}{r^2} \right], \quad (A.27) \]
\[ \frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) \right]. \quad (A.28) \]

The total derivative remains unchanged and is given by Equation (A.10). Similarly, the continuity equation is also unchanged and given by Equation (A.11). The major simplification in these equations from the axisymmetric, Navier-Stokes equations is in the radial momentum equation. The radial distribution of the pressure is controlled by the swirl velocity. Unlike the boundary layer equations, the radial (normal) pressure gradient through a vortex is not zero. Flows governed by this form of the radial momentum equation are said to be in radial equilibrium. Thus, any quasi-cylindrical flow is in radial equilibrium. A more restrictive subset of quasi-cylindrical (and thereby radial equilibrium) flows is a columnar vortex for which all properties are independent of the axial direction and the radial velocity is zero.

The circulation evolution equation under the quasi-cylindrical assumption is

\[ \frac{D\Gamma}{Dt} = \nu \left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) \right]. \quad (A.29) \]

The appropriate definition for the azimuthal vorticity is

\[ \eta = -\frac{\partial w}{\partial r}, \quad (A.30) \]

with the radial and axial vorticity components remaining unchanged and given by Equations (A.18) and (A.20), respectively. The quasi-cylindrical form of the azimuthal vorticity transport equation is

\[ \frac{D\eta}{Dt} = \frac{1}{r^3} \frac{\partial (r^2 \eta)}{\partial z} + \frac{u\eta}{r} + \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r\eta)}{\partial r} \right) \right]. \quad (A.31) \]

With the introduction of the streamfunction, the resulting form of the Poisson equation for the vorticity is

\[ \eta = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right). \quad (A.32) \]
Finally, the total pressure in quasi-cylindrical form is also simplified by the neglect of the radial velocity:

\[ H = \frac{p}{\rho} + \frac{1}{2}(v^2 + w^2). \] (A.33)

An often-used formula throughout the analysis will be the radial derivative of the total pressure in a quasi-cylindrical flow which is obtained by differentiation of Equation (A.33) and substitution of Equation (A.26) resulting in

\[ \frac{\partial H}{\partial r} = \frac{\Gamma \partial \Gamma}{r^2} + \frac{w \partial w}{\partial r}. \] (A.34)

### A.3 Axisymmetric Euler Equations

The next major simplification to the governing equations is the neglect of viscous effects by setting \( \nu = 0 \). The momentum equations, Equation (A.7)-(A.9), are:

\[ \frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \] (A.35)

\[ \frac{Dv}{Dt} + \frac{uv}{r} = 0, \] (A.36)

\[ \frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z}. \] (A.37)

Also, the circulation equation, Equation (A.21), is:

\[ \frac{D\Gamma}{Dt} = 0, \] (A.38)

and, the azimuthal vorticity transport equation is:

\[ \frac{D\eta}{Dt} = \frac{1}{r^2} \frac{\partial (\Gamma^2)}{\partial z} + \frac{w\eta}{r}. \] (A.39)
A.4 Steady, Axisymmetric Euler Equations

If a flow field is assumed to be steady, the time derivatives in the previous equations may be neglected. This simplifies the total derivative to:

$$\frac{D}{Dt} = u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}.$$ 

Furthermore, from Equation (A.38), the circulation is now dependent only on the streamfunction:

$$\Gamma = \Gamma (\Psi). \quad (A.40)$$

This allows the azimuthal vorticity transport equation, Equation (A.39), to be rewritten as:

$$\frac{D \eta}{Dt} = -u \left[ \frac{1}{r^2} \frac{d \Gamma^2}{d \Psi} - \frac{1}{r} \eta \right]. \quad (A.41)$$

Also, Equation (A.3), under the inviscid and time independent assumptions, becomes:

$$\nabla H = \vec{u} \times \vec{\omega}. \quad (A.42)$$

Taking the dot product of this equation and the velocity vector, the streamwise derivative of the total pressure is found to be identically zero. Therefore, both the head and the circulation are dependent only on the streamfunction,

$$H = H (\Psi).$$

The azimuthal vorticity may also be linked with the total pressure and circulation distributions by using the radial component of Equation (A.42) and the definition of the streamfunction, Equation (A.12).

$$\frac{\partial H}{\partial r} = v \zeta - w \eta,$$

$$r w \frac{d H}{d \Psi} = \frac{\Gamma}{r^2} \frac{\partial \Gamma}{\partial r} - w \eta,$$

$$r \frac{d H}{d \Psi} = \frac{\Gamma}{r} \frac{d \Gamma}{d \Psi} - w \eta,$$

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⇒ \eta = \frac{1}{2r} \frac{d(r^2)}{d\Psi} - r \frac{dH}{d\Psi} \tag{A.43}

Using the results from Equation (A.43), the azimuthal vorticity transport equation, Equation (A.41), may be written:

\[
\frac{D\eta}{Dt} = -u \left[ \frac{1}{2r^2} \frac{d(r^2)}{d\Psi} + \frac{dH}{d\Psi} \right]. \tag{A.44}
\]

### A.5 Non-dimensionalization of Axisymmetric Equations

The governing equations can be non-dimensionalized in many different ways. The choice used in this work serves to highlight the importance of the swirl ratio in the process of vortex breakdown. Specifically, the following non-dimensionalization will be used:

\[
\tilde{r} = \frac{r}{\delta}, \quad \tilde{z} = \frac{z}{\delta}, \quad \tilde{t} = \frac{t}{\delta/W_\infty}, \quad \tilde{u} = \frac{u}{W_\infty}, \quad \tilde{v} = \frac{v}{\Gamma_\infty/\delta}, \quad \tilde{w} = \frac{w}{W_\infty}, \quad \tilde{\xi} = \frac{\xi}{\Gamma_\infty/\delta^2}, \quad \tilde{\eta} = \frac{\eta}{W_\infty/\delta}, \quad \tilde{\zeta} = \frac{\zeta}{\Gamma_\infty/\delta^2}, \quad \tilde{\Gamma} = \frac{\Gamma}{\Gamma_\infty}, \quad \tilde{p} = \frac{p}{\rho W_\infty^2}, \quad \tilde{\Psi} = \frac{\Psi}{\rho W_\infty^2/\delta}. \tag{A.45}
\]

\(\delta, W_\infty,\) and \(\Gamma_\infty\) are the core size, the freestream velocity, and the freestream circulation, respectively. From this non-dimensionalization, two parameters enter directly in the governing equations: the swirl ratio, \(\Omega,\) and the Reynolds number, \(Re.\) They are defined as:

\[
\Omega = \frac{\Gamma_\infty}{\delta W_\infty}, \quad Re = \frac{W_\infty \delta}{\nu}. \tag{A.46}
\]

Dropping the overbar on the non-dimensionalized variables, Equation (A.21) becomes

\[
\frac{D\Gamma}{Dt} = \frac{1}{Re} \left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) + \frac{\partial^2 \Gamma}{\partial z^2} \right], \tag{A.47}
\]

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and, Equation (A.22) is:

\[
\frac{D\eta}{Dt} = \Omega^2 \frac{1}{r^3} \frac{\partial G^2}{\partial z} + \frac{w\eta}{r} + \frac{1}{Re} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r\eta) \right) + \frac{\partial^2 \eta}{\partial z^2} \right].
\]  

(A.48)

The non-dimensionalized form of the governing equations is used mainly for the numerical algorithm discussed in Appendix B. The use of non-dimensionalized variables will be clearly specified in the text when appropriate.
Appendix B

A Finite Volume Scheme for Axisymmetric, Navier-Stokes Equations

The axisymmetric, Navier-Stokes equations have been used by many investigators as a model for vortex breakdown in many previous numerical studies[34, 23, 35, 24, 6, 47, 7]. These methods spanned nearly the entire realm of possible numerical schemes from a streamfunction-vorticity approach to a primitive variable method using artificial compressibility. Also, some schemes used upwind differencing while others used central differencing. Furthermore, the exact choice of boundary conditions varied from researcher to researcher. The scheme presented here uses a streamfunction-vorticity formulation. The discretization is novel in that it uses a node-based finite volume scheme; however, in general, the code will be nearly identical to a code with a finite difference scheme using central differences.

Previous simulations of breakdown in a tube of varying radius were performed by Beran and Culick[7]. In their work, the equations were recast in a transformed coordinate system such that the outer boundary was again constant. Also, they solved only the steady form of the equations where as the time-dependent form are solved herein. The time-dependence is necessary to allow a study of the transient evolution of vortex breakdown flows. One final advantage of the finite volume scheme is that many of the methods popularized in time-marching finite volume schemes for the Euler or Navier-Stokes equations are directly applicable to this set of conservation equations and much of that experience has been transferred over to this axisymmetric, Navier-Stokes...
algorithm.

B.1 Governing Equations

The conservative form of governing equations is derived in Appendix A and are given by Equations (A.23), (A.24), and (A.25). These may be written more succinctly as:

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial z} + \frac{\partial G}{\partial r} = 0, \quad (B.1)
\]
\[
\frac{\partial F_\psi}{\partial z} + \frac{\partial G_\psi}{\partial r} = 0. \quad (B.2)
\]

The state and flux vectors for Equation (B.1) are defined by

\[
U = \begin{bmatrix} \eta \\ r \Gamma \end{bmatrix}, \quad F = \begin{bmatrix} w \eta - \Omega^2 \frac{r^2}{r^2} - \frac{1}{Re} \frac{\partial \eta}{\partial z} \\ wr \Gamma - \frac{1}{Re} \left( r \frac{\partial \Gamma}{\partial z} \right) \end{bmatrix}, \quad G = \begin{bmatrix} u \eta - \frac{1}{Re} \left( \frac{\partial \eta}{\partial r} \right) \left( r \eta \right) \\ ur \Gamma - \frac{1}{Re} \left( r \frac{\partial \Gamma}{\partial r} - 2 \Gamma \right) \end{bmatrix},
\]

and the flux vectors for Equation (B.2) are

\[
F_\psi = -u = \frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad G_\psi = w = \frac{1}{r} \frac{\partial \Psi}{\partial r}. \quad (B.3)
\]

The non-dimensionalization of the above equations is given in Section A.5.

For the purposes of the numerical scheme, the flux terms which contain first derivatives will be separated from the zeroth order terms such that the governing equations are:

\[
\frac{\partial U}{\partial t} + \frac{\partial F_0}{\partial z} + \frac{\partial F_1}{\partial z} + \frac{\partial G_0}{\partial r} + \frac{\partial G_1}{\partial r} = \eta. \quad (B.4)
\]

The split fluxes are given by:

\[
F_0 = \begin{bmatrix} w \eta - \Omega^2 \frac{r^2}{r^2} \\ wr \Gamma \end{bmatrix}, \quad F_1 = \begin{bmatrix} -\frac{1}{Re} \frac{\partial \eta}{\partial z} \\ -\frac{1}{Re} \left( r \frac{\partial \Gamma}{\partial z} \right) \end{bmatrix}. \quad (B.5)
\]
\[ G_0 = \begin{bmatrix} u\eta \\ ur\Gamma + \frac{1}{Re} (2\Gamma) \end{bmatrix}, \quad G_1 = \begin{bmatrix} -\frac{1}{Re} \left\{ \frac{\partial}{\partial t} (r\eta) \right\} \\ -\frac{1}{Re} \left( r \frac{\partial r}{\partial r} \right) \end{bmatrix}. \] (B.6)

B.2 Algorithm for Vorticity and Circulation Equations

The algorithm for the vorticity and circulation equations as given by Equation (B.4) is a straightforward adaptation of the common node-based schemes developed for the Euler or Navier-Stokes equations. One of the characteristics of this algorithm is the splitting of the time and spatial discretization into two distinct steps. This technique is known as the method of lines.

B.2.1 Spatial Discretization

Consider a control volume, \( C_{j,k} \), as shown in Figure B.1, surrounding the node \((j, k)\) such that the points \(a, c, e, g\) are located at face midpoints and the points \(b, d, f, h\) are located at cell centroids.

First, Equation (B.4) is integrated over the node \((j, k)\) control volume, \( C_{j,k} \). Applying the divergence theorem results in

\[ \frac{\partial}{\partial t} \int \int_{C_{j,k}} U \, dA + \oint_{S_{j,k}} F_0 \, dr - G_0 \, dz + \oint_{S_{j,k}} F_1 \, dr - G_1 \, dz = 0, \] (B.7)

where the \( S_{j,k} \) is the surface line contour taken in a counter-clockwise direction around \( C_{j,k} \). The residual vector may also be defined as

\[ R_{j,k} = \oint_{S_{j,k}} F_0 \, dr - G_0 \, dz + \oint_{S_{j,k}} F_1 \, dr - G_1 \, dz, \] (B.8)
where at the steady-state, $R_{j,k} = 0$. The transient behavior is

$$\frac{\partial}{\partial t} \int \int_{C_{j,k}} U \, dA = -R_{j,k}. \quad (B.9)$$

The line integrals in the residual vector are then divided into contributions from the four cells, $A, B, C, D$ such that

$$\Delta R_{0A} = \int_{j_{k-a-b-c-jk}} F_0 \, dr - G_0 \, dz, \quad (B.10)$$

$$\Delta R_{1A} = \int_{a-b-c} F_1 \, dr - G_1 \, dz. \quad (B.11)$$

Definitions analogous to Equations (B.10) and (B.11) are used for the flux contributions from cells $B, C, D$. The residual contribution from cell $A$ to node $(j, k)$ is

$$R_A = \Delta R_{0A} + \Delta R_{1A}. \quad (B.12)$$
The total residual from all surrounding cells to node \((j,k)\) is

\[
R_{j,k} = R_A + R_B + R_C + R_D
\]

The zeroth order flux vectors, \(F_0, G_0\), are assumed to vary linearly in space and match at cell faces. Thus, the line integral of the zeroth order fluxes cancel at the internal faces; consequently, the line integral in Equation (B.10) has been taken around the entire cell, \(jk - a - b - c - jk\). The first order flux vectors, \(F_1, G_1\), are assumed to be constant over a cell, therefore, the line integral in Equation (B.11) must be taken only over the outer surface \(a - b - c\). If this is not done, the integration of a constant around the entire closed path would be identically zero and thus contribute nothing to the total flux integral of node \((j,k)\).

\[
\Delta R_{0_A} \text{ is approximated as a quarter of the line integral around } S_A. \text{ Using a trapezoidal integration, the resulting numerical approximation for } \Delta H_{0_A} \text{ is}
\]

\[
\Delta R_{0_A} = \frac{1}{8} \left[ (F_{0,j,k} - F_{0,j+1,k})((s_{j,k} - s_{j+1,k+1})(r_{j+1,k+1} - r_{j+1,k}) - (G_{0,j,k} - G_{0,j+1,k})(z_{j+1,k+1} - z_{j+1,k}) \right. \\
+ \left. (F_{0,j+1,k} - F_{0,j+1,k+1})(r_{j+1,k} - r_{j,k}) - (G_{0,j+1,k} - G_{0,j+1,k+1})(z_{j+1,k+1} - z_{j,k}) \right].
\]

For the first-order flux term, \(\Delta R_{1_A}\), using a similar integration gives

\[
\Delta R_{1_A} = \frac{1}{2} [F_{1_A}(r_{j+1,k} - r_{j,k+1}) - G_{1_A}(z_{j+1,k} - z_{j,k+1})].
\]

The first-order derivatives which make up the viscous fluxes are calculated using a procedure similar to the calculation of \(\Delta R_{0_A}\). The resulting formulae for the derivatives of any scalar, \(s\), in cell \(A\) are

\[
\frac{\partial s}{\partial z} \bigg|_A = \frac{1}{2A_A} \left[ (s_{j,k} - s_{j+1,k+1})(r_{j+1,k+1} - r_{j+1,k}) - \\
(s_{j+1,k} - s_{j,k+1})(r_{j+1,k+1} - r_{j,k}) \right], \tag{B.13}
\]

\[
\frac{\partial s}{\partial r} \bigg|_A = -\frac{1}{2A_A} \left[ (s_{j,k} - s_{j+1,k+1})(z_{j+1,k+1} - z_{j+1,k}) - \\
(s_{j+1,k} - s_{j,k+1})(z_{j+1,k+1} - z_{j,k}) \right]. \tag{B.14}
\]
B.2.2 Numerical Smoothing

For low Reynolds number flows, the spatial discretization as described above is perfectly adequate. As the viscous terms become negligible at higher Reynolds number, the finite volume scheme admits sawtooth modes. These modes at best result in spatially-oscillatory solutions and in the worst case, no solution is possible because in marching to a steady-state, these sawtooth modes may become unstable somewhere in the computational domain. For these reasons, it is necessary to add artificial damping terms to the governing equations such that the sawtooth modes are damped while the rest of the flowfield is undisturbed.

A common practice in finite volume schemes is to add second and fourth difference smoothing. The second difference smoothing is used to dampen dispersive errors generated by regions of large gradients. Unless specifically stated, the second difference smoothing is not used for the results of this work. The fourth difference smoothing provides a background smoothing which eliminates sawtooth modes while still maintaining the accuracy of the scheme. The smoothing terms are incorporated by adding an additional term to the right-hand side of Equation (B.9). The resulting equation is

$$\frac{\partial}{\partial t} \int_{C_{j,s}} U \, dA = -R_{j,k} - D(U_{j,k}). \tag{B.15}$$

The artificial dissipation term, $D(U_{j,k})$, is separated into a second difference term and a fourth difference term

$$D(U_{j,k}) = -D_2(U_{j,k}) + D_4(U_{j,k}). \tag{B.16}$$

The basis of the smoothing terms is a second difference operator. The second difference operator in the $j$-direction at node $(j, k)$ acting on scalar $s_{j,k}$ is

$$\mathcal{D}_j^2 s_{j,k} = (\mathcal{D}_{A_j} + \mathcal{D}_{B_j} + \mathcal{D}_{C_j} + \mathcal{D}_{D_j}) s_{j,k}. \tag{B.17}$$
The contribution from the cells are defined as

\[ \mathcal{D}_{A_j} s_{j,k} = s_{j+1,k} - s_{j,k} = \mathcal{D}_{B_j} s_{j,k}, \]

\[ \mathcal{D}_{C_j} s_{j,k} = s_{j-1,k} - s_{j,k} = \mathcal{D}_{D_j} s_{j,k}. \]

This is simply a version of the classic central second difference operator which has been written in terms of individual contributions from cells. Similarly for the \( k \)-direction, the second difference operator is

\[ \mathcal{D}_k^2 s_{j,k} = (\mathcal{D}_{A_k} + \mathcal{D}_{B_k} + \mathcal{D}_{C_k} + \mathcal{D}_{D_k}) s_{j,k}. \]  

(B.18)

And, the contribution from the cells are defined as

\[ \mathcal{D}_{A_k} s_{j,k} = s_{j,k+1} - s_{j,k} = \mathcal{D}_{D_k} s_{j,k}, \]

\[ \mathcal{D}_{B_k} s_{j,k} = s_{j,k-1} - s_{j,k} = \mathcal{D}_{C_k} s_{j,k}. \]

The second difference smoothing, \( \mathcal{D}_2(U_{j,k}) \), uses a weighted form of the classic second difference operator defined in Equations (B.17) and (B.18). First, the second difference smoothing is broken into contributions from the surrounding cells such that

\[ \mathcal{D}_2(U_{j,k}) = (\mathcal{D}_{2A} + \mathcal{D}_{2B} + \mathcal{D}_{2C} + \mathcal{D}_{2D}) U_{j,k}, \]  

(B.19)

where the second difference smoothing contribution from cell \( A \) is

\[ \mathcal{D}_{2A}(U_{j,k}) = \kappa_{2A} \left( \frac{A}{\Delta t} \right)_{A} \left[ \mathcal{D}_{A_j} + \mathcal{D}_{A_k} \right] U_{j,k}. \]  

(B.20)

The \( \left( \frac{A}{\Delta t} \right)_{A} \) factor in Equation (B.20) is the ratio of the area to the time step in cell \( A \) and is needed to properly normalize the smoothing contribution. \( \kappa_{2A} \) is a cell-based switch on the azimuthal vorticity designed to turn on only in areas of high gradients. The switch implemented here employs a weighted second difference of the azimuthal
vorticity given by
\[ S_{j,k} = (\mathcal{D}_j^2 + \mathcal{D}_k^2) \eta_{j,k}, \]
where \( \eta_{j,k} \) is the azimuthal vorticity. The resulting cell-based switch, \( \kappa_{2_A} \), is
\[ \kappa_{2_A} = \nu_2 \sqrt{A_A} \left( |S_{j,k}| + |S_{j,k+1}| + |S_{j+1,k+1}| + |S_{j+1,k}| \right). \]
\( \nu_2 \) is a user-set coefficient scaling the second difference smoothing. The square root of the cell area, \( A_A \), is needed to normalize the switch. Since the velocity outside of the vortex core is approximately unity with the chosen non-dimensionalization, an order of magnitude estimate for the local vorticity is simply \( 1/\sqrt{A_A} \).

The fourth difference smoothing is a nested second difference operator and is constructed similarly to the second difference smoothing. The total fourth difference smoothing is first broken into contributions from the surrounding cells
\[ D_4(U_{j,k}) = (D_{4_A} + D_{4_B} + D_{4_C} + D_{4_D})U_{j,k}. \]
The fourth difference smoothing contribution from cell \( A \) is:
\[ D_{4_A}(U_{j,k}) = \kappa_{4_A} A \left( \frac{A}{\Delta t} \right) A \left[ \mathcal{D}_{A_1}(\mathcal{D}_j^2) + \mathcal{D}_{A_k}(\mathcal{D}_k^2) \right] U_{j,k}, \]
where the switch, \( \kappa_{4_A} \), is defined as
\[ \kappa_{4_A} = \max(0, \nu_4 - \kappa_{2_A}). \]
\( \nu_4 \) is a user-set coefficient which scales the fourth difference smoothing contribution. The form of \( \kappa_{4_A} \) was chosen to turn off in areas of large gradients where the fourth difference smoothing can become unstable.

Finally, any additional data which is necessary to complete the dissipation operators at a boundary is found from simple linear extrapolation in the computational space. In
other words, at the \( k_{\text{max}} + 1 \) location, the state vector is found from:

\[
U_{j,k_{\text{max}}+1} = 2U_{j,k_{\text{max}}} - U_{j,k_{\text{max}}-1}.
\]

Analogous extrapolations are performed at other computational boundaries.

B.2.3 Temporal Integration

The temporal integration employs a modified Runge-Kutta multi-stage integrator. The results of this thesis all use a four-stage integrator; however, the code has been written generically to handle any number of stages. Given a value of the state vector at iteration \( n \), the value of the state vector at iteration \( n + 1 \) is

\[
\begin{align*}
U_{j,k}^{(0)} &= U_{j,k}^{n}, \\
U_{j,k}^{(1)} &= U_{j,k}^{(0)} - \alpha_1 \left( \frac{\Delta t}{A} \right)_{j,k} \left[ R(U_{j,k}^{(0)}, \Psi_{j,k}^{(0)}) + D(U_{j,k}^{(0)}, \Psi_{j,k}^{(0)}) \right], \\
U_{j,k}^{(2)} &= U_{j,k}^{(0)} - \alpha_2 \left( \frac{\Delta t}{A} \right)_{j,k} \left[ R(U_{j,k}^{(1)}, \Psi_{j,k}^{(1)}) + D(U_{j,k}^{(1)}, \Psi_{j,k}^{(1)}) \right], \\
U_{j,k}^{(3)} &= U_{j,k}^{(0)} - \alpha_3 \left( \frac{\Delta t}{A} \right)_{j,k} \left[ R(U_{j,k}^{(2)}, \Psi_{j,k}^{(2)}) + D(U_{j,k}^{(2)}, \Psi_{j,k}^{(2)}) \right], \\
U_{j,k}^{(4)} &= U_{j,k}^{(0)} - \alpha_4 \left( \frac{\Delta t}{A} \right)_{j,k} \left[ R(U_{j,k}^{(3)}, \Psi_{j,k}^{(3)}) + D(U_{j,k}^{(3)}, \Psi_{j,k}^{(3)}) \right], \\
U_{j,k}^{n+1} &= U_{j,k}^{(4)}.
\end{align*}
\]

The streamfunction is to be obtained from the solution of the Poisson equation as described in Section B.3. The streamfunction, \( \psi_{j,k}^{(i)} \), is the streamfunction based on the solution of the Poisson equation using the state vector, \( U_{j,k}^{(i)} \). This scheme is fourth-order accurate in time when the multi-stage coefficients, \( \alpha_i \), are chosen to be \( \frac{1}{4} \), \( \frac{1}{3} \), \( \frac{1}{2} \), and 1, respectively. All calculations are performed using these coefficients for the four-stage scheme.

The residuals, \( R \), and the smoothing, \( D \), are functions of both the state vector and the streamfunction. In order to remain strictly fourth-order accurate in time, the
streamfunction must be solved for at every stage of the integration. In obtaining steady-state solutions, the solution does not need to proceed in a time accurate fashion. Thus, it is computationally more efficient to evaluate the smoothing only in the first two stages and the streamfunction only in the first stage. This scheme is written

\[
\begin{align*}
U^{(0)}_{j,k} &= U^n_{j,k}, \\
U^{(1)}_{j,k} &= U^{(0)}_{j,k} - \alpha_1 \left( \frac{\Delta t}{A} \right) \left[ R(U^{(0)}_{j,k}; \Psi^{(0)}_{j,k}) + D(U^{(0)}_{j,k}; \Psi^{(0)}_{j,k}) \right], \\
U^{(2)}_{j,k} &= U^{(0)}_{j,k} - \alpha_2 \left( \frac{\Delta t}{A} \right) \left[ R(U^{(1)}_{j,k}; \Psi^{(0)}_{j,k}) + D(U^{(1)}_{j,k}; \Psi^{(0)}_{j,k}) \right], \\
U^{(3)}_{j,k} &= U^{(0)}_{j,k} - \alpha_3 \left( \frac{\Delta t}{A} \right) \left[ R(U^{(2)}_{j,k}; \Psi^{(0)}_{j,k}) + D(U^{(1)}_{j,k}; \Psi^{(0)}_{j,k}) \right], \\
U^{(4)}_{j,k} &= U^{(0)}_{j,k} - \alpha_4 \left( \frac{\Delta t}{A} \right) \left[ R(U^{(3)}_{j,k}; \Psi^{(0)}_{j,k}) + D(U^{(1)}_{j,k}; \Psi^{(0)}_{j,k}) \right], \\
U^{n+1}_{j,k} &= U^{(4)}_{j,k}.
\end{align*}
\]

The largest time step is set by the stability limit of the discrete equations. Using the two-dimensional, convection-diffusion equation, an estimate for the stability limit of this scheme may be determined. The model convection-diffusion equation in two-dimensions is

\[
\frac{\partial u}{\partial t} + c_x \frac{\partial u}{\partial x} + c_y \frac{\partial u}{\partial y} = \frac{1}{Re} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right].
\]

The grid will be assumed to have constant spacing in the \(x\) and \(y\) directions given by \(\Delta x\) and \(\Delta y\). The finite volume discretization procedure described above is equivalent to central differences; therefore, the spatial differences will be approximated using 2nd order accurate central differences. Substitution of \(u(x, y, t) = \hat{u}(t) exp[i(\alpha x + \beta y)]\) into the spatial differences results in the following semi-discrete form of the convection-diffusion equation (after some manipulations)

\[
\frac{\partial \hat{u}}{\partial t} = \frac{z}{\Delta t} \hat{u},
\]
where
\[ z = -4 \left[ \nu_x \sin^2 \frac{\theta_x}{2} + \nu_y \sin^2 \frac{\theta_y}{2} \right] - i [\lambda_x \sin \theta_x + \lambda_y \sin \theta_y], \]

and
\[ \theta_x = \alpha \Delta x, \quad \nu_x = \frac{\Delta t}{Re \Delta x^2}, \quad \lambda_x = \frac{c_x \Delta t}{\Delta x}, \]
\[ \theta_y = \beta \Delta y, \quad \nu_y = \frac{\Delta t}{Re \Delta y^2}, \quad \lambda_y = \frac{c_y \Delta t}{\Delta y}. \]

The values of \( \theta_x, \theta_y \) are bounded by \( \pm \pi \) because the wavelength of the smallest wave representable on a mesh is \( 2 \Delta x \) or \( 2 \Delta y \). The four-stage Runge-Kutta time integration scheme applied to the semi-discrete model equation results in the following amplification,

\[ \hat{u}^{n+1}/\hat{u}^n = g(z) = 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \frac{1}{24} z^4. \]

When \( |g(z)| > 1 \), the scheme is unstable. The stability boundary is determined by the locus of points which satisfy \( |g(z)| = 1 \). Figure B.2 contains contours of the magnitude of the amplification factor in the complex \( z \) plane. For the algorithm to be stable, \( z \) must lie completely within the stability boundary, \( |g(z)| = 1 \), shown in Figure B.2. The exact stability limit for the scheme is a complex function of \( \nu_x, \nu_y, \lambda_x, \) and \( \lambda_y \) and is impractical to implement efficiently in a code. A more efficient technique which works quite well in practice is to limit the maximum extent of \( z \) in the real and imaginary directions to be within the stability limits. In the real direction, the maximum value of \( z \) occurs for waves where \( \theta_x = \theta_y = \pi \). In this case,

\[ \max |Re\{z\}| = 4(\nu_x + \nu_y). \]

To be within the stability boundary, this requires that

\[ 4(\nu_x + \nu_y) \leq 2.784. \quad (B.21) \]

Similarly, for the imaginary extent of \( z \), the maximum value occurs for \( \theta_x = \theta_y = \pi/2 \) and is given by

\[ \max |Im\{z\}| = \lambda_x + \lambda_y. \]

For the spatial Fourier operator to be within the stability boundary of the four-stage
Figure B.2: Amplification contours of four-stage Runge-Kutta scheme
scheme along the imaginary axis requires that

\[ \lambda_x + \lambda_y \leq 2\sqrt{2} \approx 2.828. \tag{B.22} \]

The two stability limits given by Equations (B.21) and (B.22) are necessary (but not sufficient) conditions for stability of the algorithm. In practice, both of these constraints are implemented by defining a maximum time step as

\[
(\Delta t_{\text{max}})^{-1} = \max \left[ \frac{|u|}{\Delta r} + \frac{|w|}{\Delta z}, \frac{4}{Re} \left( \frac{1}{\Delta r^2} + \frac{1}{\Delta z^2} \right) \right].
\]

In the above definition, the Cartesian velocities and grid spacings have been replaced by the appropriate cylindrical coordinate velocities and grid spacings. The time step used in the calculations is defined as

\[ \Delta t = CFL \cdot \Delta t_{\text{max}}, \]

where \( CFL \) is a modified definition of the classic Courant-Friedrichs-Lewy number. Many numerical experiments have shown that using a CFL of \( 2\sqrt{2} \) provides a quite accurate bound on the stability of the scheme for swirl ratios below approximately 1.40. However, for \( \Omega > 1.40 \), it was found necessary to reduce the CFL number. This more restrictive time step limitation could be due to centrifugal waves travelling faster than the local convective velocity, and, therefore violating the CFL condition. Vortex flows are waveguides which allow waves travelling at speeds differing from the convective speed. Leibovich has provided a review of the wave nature of vortex flows\[42\] and Chapter 5 also concentrates on wave mechanisms in vortical flows. These waves have been observed in the time-marching procedure to the steady state and their amplitude and speed increase with increasing swirl ratio. Therefore, it is conjecture that the stability limit at higher swirl ratio is connected to the large amplitude, fast-moving waves common in swirling flows.

Global time steps are used when a time accurate solution is required. The time step at every node is then defined as the minimum time step for stability over the entire
computational domain. Alternatively, the global time step can be set to a specific value which may violate the CFL condition locally. Most time accurate solutions are run with a constant time step which is chosen irrespective of insuring the CFL conditions are met over all cells. Although this time-marching procedure has local regions where the growth factor is larger than one, on the whole, the scheme is stable and as long as the region of unstable time stepping is small, the overall scheme tends to be stable. For steady-state solutions, the time accuracy is sacrificed for computational efficiency by using local time stepping. Using local time stepping, the time step is set such that the numerical algorithm is locally stable. This technique reduces the stiffness associated with varying convection speeds and grid sizes. As an example of the effectiveness of local time stepping versus global time stepping, Figure B.3 compares the results for a swirling pipe flow calculation at $Re = 100$ and $\Omega = 1.65$. The pipe geometry is that of Pipe 1 as described in Section 2.2. The local time step solution converges approximately three times faster than the global time step solution. Inspection of the solutions revealed no differences. Unless otherwise stated, all steady state solutions will be run using local time stepping.
B.3 Algorithm for Streamfunction Equation

The solution of the Poisson equation for the streamfunction, Equation (B.2), requires different treatment since the equation is a spatial elliptic problem with no time dependence. The spatial discretization is again performed using the same finite volume discretization technique described in Section B.2.1. Then, the solution of the resulting matrix equation is solved using and Alternating Direction Implicit (ADI) scheme with Successive Line Over-Relaxation (SLOR).

B.3.1 Spatial Discretization

Analogous to Equation (B.9), the Poisson equation may be integrated over the control volume and written

$$\int \int_{C_{j,k}} \eta \, dA = -R_\Psi. \quad (B.23)$$

The residual is the line integral of the fluxes, $F_\Psi, G_\Psi$, around $S_{j,k}$:

$$R_\Psi = \oint_{S_{j,k}} F_\Psi \, dr - G_\Psi \, dz. \quad (B.24)$$

These first-order fluxes are then approximated as before. The contribution to the nodal residual $R_{\Psi_{j,k}}$ from Cell A is

$$R_{\Psi_{A}} = \frac{1}{2} [F_{\Psi_{A}} (r_{j+1,k} - r_{j,k+1}) - G_{\Psi_{A}} (z_{j+1,k} - z_{j,k+1})].$$

The first-order fluxes are approximated using Equations (B.13) and (B.14). The source term is handled similarly to the unsteady term in Equation (B.9) and the resulting discretized equation is

$$\eta_{j,k} A_{j,k} = -R_{\Psi_{j,k}}.$$
This equation is a sparse, banded matrix equation for the streamfunction which may also be written

\[ R(\Psi_{j,k}, \Psi_{j\pm1,k}, \Psi_{j,k\pm1}, \Psi_{j\pm1,k\pm1}) = \eta_{j,k} A_{j,k}. \]  

(B.25)

### B.3.2 Inversion of Poisson Equation

The inversion of the matrix equation, Equation (B.25), can be done with many different methods. The technique chosen here, an ADI scheme with SLOR, ranges in the middle of both algorithmic complexity and computational efficiency. A complete description of this technique can be found in most introductory books on the numerical solution of partial differential equations\cite{1}. The ADI-SLOR procedure starts by sweeping through the \( j \)-direction. The resulting equations may be written in a tridiagonal form:

\[ L^J(\Psi^J_{j,k}, \Psi^J_{j,k\pm1}) = R^K(\Psi_{j\pm1,k}, \Psi_{j\pm1,k\pm1}) + \eta_{j,k} A_{j,k}. \]

This tridiagonal equation for \( \Psi^J_{j,k} \) may be solved very efficiently using Thomas’s algorithm. The initial values of the streamfunction on the right-hand side are given by a guess of the solution such that \( \Psi_{j,k} = \Psi^0_{j,k} \) where \( \Psi^0_{j,k} \) is the initial guess. The new values for the streamfunction are then over-relaxed using

\[ \tilde{\Psi}_{j,k} = \Psi_{j,k} + \omega(\Psi^J_{j,k} - \Psi_{j,k}), \]  

(B.26)

where \( \omega \) is the overrelaxation factor. When \( \omega \) is one, the non-accelerated version of the scheme results. Values of \( \omega \) less (greater) than one result in underrelaxation (overrelaxation) of the solution. Section B.3.3 details the choice of overrelaxation factors.

After an entire sweep through the \( j \)-direction is completed, a \( k \)-direction sweep follows. This equation is again tridiagonal having the form:

\[ L^K(\Psi^K_{j,k}, \Psi^K_{j\pm1,k}) = R^J(\tilde{\Psi}_{j,k\pm1}, \tilde{\Psi}_{j\pm1,k\pm1}) + \eta_{j,k} A_{j,k}. \]
The resulting values of $\Psi_{j,k}^K$ are then overrelaxed as before

$$\Psi_{j,k} = \tilde{\Psi}_{j,k} + \omega(\Psi_{j,k}^K - \tilde{\Psi}_{j,k}).$$

(B.27)

The original matrix equation, Equation (B.25), is considered to be inverted when the change from the initial to final value of $\Psi_{j,k}$ is small over a combined $j$ and $k$ sweep. A measure of this is the RMS change of the streamfunction defined as

$$\delta \Psi_{rms} = \sqrt{\frac{\sum_{j,k}(\Psi_{j,k} - \Psi_{j,k}^0)^2}{N}},$$

(B.28)

where $N$ equals the total number of points. If $\delta \Psi_{rms}$ is not below some threshold, then, $\Psi_{j,k}^0 \rightarrow \tilde{\Psi}_{j,k}$, and another ADI-SLOR iteration is performed.

### B.3.3 Overrelaxation Factors

The choice of overrelaxation factors plays an important part in the efficiency of the ADI-SLOI scheme. If $\omega$ is chosen greater than the optimum value, the convergence to a final answer will be hampered by over-estimating the changes in $\Psi_{j,k}$. Likewise, if $\omega$ is chosen less than the optimum value, the convergence to a final answer will be slowed by a less than optimum acceleration of the corrections to $\Psi_{j,k}$. The choice of overrelaxation factor has been found to be crucial to the overall efficiency of the algorithm since the Poisson equation must be solved at every time step and the resulting savings for even a single Poisson inversion can be substantial.

An automated procedure for choosing the overrelaxation factor has been implemented in this code. The first Poisson iteration proceeds with some initial value of $\omega$. The value of the RMS change is stored in $\delta \Psi_{rms}^0$. Then, the following procedure is used:

1. Perform a Poisson iteration with $\omega$
2. Check $\delta \Psi_{rms}$ for convergence
3. If \( (\delta \Psi_{rms}^0 - \delta \Psi_{rms})/\delta \Psi_{rms}^0 < r_\delta \) then \( \omega \to \max(1, r_\omega \omega) \).

4. Repeat until converged

The value of \( r_\delta \) is between \([0, 1]\). If the change in the streamfunction is increasing or not decreasing quickly enough, the SOR parameter is changed. \( r_\omega \) is less than one and reduces the initial value of \( \omega \). This simple procedure tends to reduce the acceleration parameter as convergence is reached. If this were not the case, the accelerated changes would constantly overcompensate and the solution could bounce between various values of \( \Psi_{j,k} \) without ever converging. This phenomenon was actually noticed in practice and was the main reason for the choice of the acceleration scheme. Comparing the constant \( \omega \) and varying \( \omega \) ADI-SLOR schemes, the varying \( \omega \) scheme typically converges at least 20% faster than the constant \( \omega \) scheme. The values of \( r_\delta, r_\omega, \omega \) which were found to be effective over a wide range of solutions were 0.05, 0.8, and 1.6, respectively.

### B.4 Steady Pipe Flow Boundary Conditions

One of the flowfields to be modelled is the swirling flow through a pipe of varying radius. The geometry definition for an axisymmetric, varying radius pipe is shown in Figure 3.4. The radius of the pipe is \( R(z) \). The inlet is located at \( z = 0 \) and the outlet is at \( z = z_{\text{max}} \).

#### B.4.1 Physical Boundary Conditions

**Axis boundary conditions**

The axis boundary conditions are \( u(0, z) = \Gamma(0, z) = \eta(0, z) = 0 \). \( u \) must be zero at the axis for a flow without sources of mass. \( \Gamma \) is zero due to the effect of viscosity which will
not permit infinite strain. If $\Gamma$ where to approach a finite value as the radius approach zero, then the shear would be infinite. For the same reason, the azimuthal vorticity, $\eta$ must also be zero on the axis. Setting $\Gamma$ and $\eta$ to zero on the axis is simply a Dirichlet boundary condition. Since $u(0, z) = 0$ for all $z$, this implies that $\Psi(0, z)$ is a constant from Equation (A.12). Therefore, the boundary condition on $u(0, z)$ is equivalent to a Dirichlet boundary condition on the streamfunction. For simplicity, $\Psi(0, z) = 0$ in these calculations. Re-iterating, the three boundary conditions at the axis are:

$$\begin{align*}
\Psi(0, z) &= 0, \\
\Gamma(0, z) &= 0, \\
\eta(0, z) &= 0. 
\end{align*} \quad (B.29)$$

Inviscid wall boundary conditions

Except for the comparisons with experiments in Chapter 6 and Appendix C, the wall flow is assumed to be inviscid and irrotational. The effects of boundary layers on the pipe wall have been neglected since they do not play a major role in the dynamics of vortex breakdown unless the flow separates. Although boundary layer separation and its interaction with vortex breakdown is important, it will not be modelled in this work since the main interest is the fundamental mechanics of breakdown.
Regardless of the assumption of inviscid flow, the wall streamfunction is a constant since the wall is the limiting streamline of the flow. Thus, the wall streamfunction is set equal to the wall streamfunction at the inlet: \( \Psi(R(z), z) = \Psi(R(0), 0) \). The circulation of an inviscid flow is constant along a particle path. Therefore, since the circulation is also constant along the wall and is determined by the inlet circulation: \( \Gamma(R(z), z) = \Gamma(R(0), 0) \). Finally, the assumption of an inviscid, irrotational flow implies that there is no vorticity. Hence, the boundary condition on \( \eta \) is \( \eta(R(z), z) = 0 \). Therefore, all of the boundary conditions at the wall are also Dirichlet boundary conditions. Re-iterating, the three boundary conditions at the wall are:

\[
\begin{align*}
\Psi(R(z), z) &= \Psi(R(0), 0), \\
\Gamma(R(z), z) &= \Gamma(R(0), 0), \\
\eta(R(z), z) &= 0.
\end{align*}
\]  

(B.30)

**Viscous wall boundary conditions**

In the calculations of Chapter 6 and Appendix C, a viscous boundary condition will be used at the wall. In this case, the presence of viscosity requires a no-slip boundary condition. Therefore, all components of the velocity are zero at the wall. This amounts to the following set of boundary conditions:

\[
\begin{align*}
\frac{\partial \Psi}{\partial r}(R(z), z) &= 0, \\
\frac{\partial \Psi}{\partial z}(R(z), z) &= 0, \\
\Gamma(R(z), z) &= 0.
\end{align*}
\]  

(B.31)

**Inlet boundary conditions**

At the inlet, \( \Gamma, \eta, \) and \( u \) will be set. The boundary conditions on the circulation and the azimuthal vorticity are straightforward Dirichlet boundary conditions. The radial
velocity boundary condition must be reformulated as a Neumann boundary condition on the streamfunction. From the definition of the streamfunction, recall that

$$u = -\frac{1}{r} \frac{\partial \Psi}{\partial z}.$$  

Therefore, given an inlet radial velocity distribution, a derivative boundary condition on the streamfunction at the inlet is applied. Re-iterating, the boundary conditions at the inlet are:

$$\frac{1}{r} \frac{\partial \Psi}{\partial z}(r, 0) = u_0(r),$$

$$\Gamma(r, 0) = \Gamma_0(r),$$

$$\eta(r, 0) = \eta_0(r),$$

where $u_0$, $\Gamma_0$, and $\eta_0$ are the inlet profiles of the radial velocity, circulation, and azimuthal vorticity. The specific choice of inlet profiles is described in Section B.4.3.

**Outlet boundary conditions**

The outlet boundary conditions assume the flow is quasi-cylindrical. Therefore, flows with large axial gradients at the outlet cannot be calculated. Also, flows with an upstream propagating wave, *i.e.* subcritical flows, cannot be properly modelled using this boundary condition. As a result, the flow at the outlet must be supercritical and must not contain large axial gradients. With the assumption of quasi-cylindrical flow, the governing equations at the outlet are Equations (A.29), (A.31), and (A.32).

**B.4.2 Numerical Boundary Conditions**

The axis and wall boundaries are located at $j = 1$ and $j = j_{\text{max}}$, respectively. Likewise, the inlet and outlet boundaries are located at $k = 1$ and $k = k_{\text{max}}$, respectively.
Axis boundary conditions

The numerical implementation of the Dirichlet, axis boundary conditions stated in Equation (B.29) is simple. These boundary conditions are applied at $j = 1$. The numerical, axis boundary conditions are:

\[
\begin{align*}
\Psi_{1,k} &= 0, \\
\Gamma_{1,k} &= 0, \\
\eta_{1,k} &= 0.
\end{align*}
\]

(B.33)

Inviscid wall boundary conditions

The numerical equivalent of the wall boundary conditions in Equation (B.30) must be applied at $j = j_{\text{max}}$. These numerical boundary conditions are:

\[
\begin{align*}
\Psi_{j_{\text{max}},k} &= \Psi_{j_{\text{max}},1}, \\
\Gamma_{j_{\text{max}},k} &= \Gamma_{j_{\text{max}},1}, \\
\eta_{j_{\text{max}},k} &= 0.
\end{align*}
\]

(B.34)

Viscous wall boundary conditions

The numerical boundary conditions for a no-slip boundary as given by Equation (B.31) require a little additional effort before implementation is possible. The circulation boundary condition is a straightforward Dirichlet condition,

\[
\Gamma_{j_{\text{max}},k} = 0.
\]

For the streamfunction, we note that the gradient of the streamfunction is identically zero in all directions at the wall. Therefore, the streamfunction is constant along the
wall which is obvious because the wall must be a streamsurface. Therefore, the stream-function viscous wall boundary condition is identical to the inviscid wall boundary condition,

\[ \Psi_{j_{\text{max}}, k} = \Psi_{j_{\text{max}}, 1}. \]

Finally, the vorticity boundary condition may be derived by a slight reformulation of the interior scheme for the Poisson equation given by Equations (B.23) and (B.24). These equations give

\[ - \int \int_{C_{j_{\text{max}}, k}} \eta \, dA = \int \int_{S_{j_{\text{max}}, k}} F_\Psi \, dr - G_\Psi \, dz = R_\Psi. \]

The integrals around the control volume at \( j_{\text{max}} \) contains only cells C and D. Recall from Equation (B.3), the Poisson fluxes are simply the radial and axial velocity components,

\[ F_\Psi = -u = \frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad G_\Psi = w = \frac{1}{r} \frac{\partial \Psi}{\partial r}. \]  

(B.3)

The residual can be approximated using a trapezoidal line integration around the alternative cell formed by points a-h-f-e-a. The values of \( F_\Psi \) and \( G_\Psi \) are zero at points a and e due to the no-slip boundary condition. At points f and h, the values of \( F_\Psi \) and \( G_\Psi \) are defined as the values of \( F_\Psi \) and \( G_\Psi \) in cells C and D, respectively. Then, trapezoidal integration gives

\[ -(\eta A)_{j_{\text{max}}, k} = \frac{1}{2} \left[ F_\Psi_C (r_h - r_e) - G_\Psi_C (z_h - z_e) \right. \]
\[ + \left. F_\Psi_D (r_a - r_f) - G_\Psi_D (z_a - z_f) \right] \]

(B.35)

which is the expression used for the viscous wall boundary condition on the azimuthal vorticity.

**Inlet boundary conditions**

The inlet boundary conditions given in Equation (B.32) must be applied at \( k = 1 \). For the radial velocity boundary condition, \( \frac{\partial \Psi}{\partial z} = -ru_0 \) must be satisfied at \( k = 1 \). This
equation may be integrated over the control volume around a node at the inlet

\[ \int \int_{C_{j,1}} \left( ru + \frac{\partial \Psi}{\partial z} \right) dr \, dz = 0 \]

This may be rewritten as

\[ \int \int_{C_{j,1}} ru \, dr \, dz + \oint_{S_{j,1}} \Psi \, dr = 0 \]

The control volume about a node at the inlet only contains cells \( A \) and \( D \) and this must be taken into account in a finite volume discretization. Then, approximating the integrals as before results in the following boundary condition on the streamfunction at the inlet:

\[
\begin{align*}
\Psi_{j,1}(r_{j-1,1} - r_{j+1,1}) & + \Psi_{j,-1,1}(r_{j-1,2} - r_{j,1}) + \\
\Psi_{j-1,2}(r_{j,2} - r_{j-1,1}) & + \Psi_{j,2}(r_{j+1,2} - r_{j-1,2}) + \\
\Psi_{j+1,2}(r_{j+1,1} - r_{j,2}) & + \Psi_{j+1,1}(r_{j,1} - r_{j+1,2}) + \\
8r_{j,1}u_{j,1}A_{j,1} & = 0, 
\end{align*}
\]  

(B.36)

where the inlet radial velocity is

\[ u_{j,1} = u_0(r_{j,1}). \]

The boundary conditions on the circulation and the azimuthal vorticity are simple Dirichlet boundary conditions:

\[
\begin{align*}
\Gamma_{j,1} & = \Gamma_0(r_{j,1}), \\
\eta_{j,1} & = \eta_0(r_{j,1}). 
\end{align*}
\]  

(B.37)
Outlet boundary conditions

The outlet boundary is governed by the quasi-cylindrical equations. The numerical approximation for these equations is simply a one-sided formulation of the scheme used for the interior calculations. In effect, the one-sided difference scheme only includes contributions from cells $B$ and $C$ and neglects any second differences in the axial direction.

### B.4.3 Inlet Vortex Model

The Batchelor trailing line vortex, also known as the q-vortex, will be used throughout this work for the vortex inlet conditions[2]. The q-vortex profiles have been found to match a wide-range of velocity profiles measured in experiments. The q-vortex velocity distributions are

\[
\begin{align*}
  u_0(r) &= 0, \\
  v_0(r) &= \frac{1}{r} \left[ 1 - \exp(-r^2) \right], \\
  w_0(r) &= 1 + \Delta w \exp(-r^2),
\end{align*}
\]

(B.38)

where $\Delta w$ is the difference between the axial velocity at the axis and in the farfield. The general level of the swirl is controlled by the swirl ratio, $\Omega$ and as a result of the chosen non-dimensionalization, it does not appear in the inlet distributions. If one neglects axial variations at the inlet, the azimuthal vorticity simplifies $\eta_0 = -\frac{du_0}{dr}$. Under this assumption, the azimuthal velocity for the q-vortex is

\[
\eta_0(r) = 2\Delta w r \exp(-r^2).
\]

(B.39)

Finally, the circulation for the q-vortex is

\[
\Gamma_0(r) = 1 - \exp(-r^2).
\]

(B.40)
Some of the general properties of the q-vortex model are described in Appendix D.

### B.5 Enclosed Cylinder with Rotating Endwall Boundary Conditions

The flow in an enclosed cylinder with a rotating endwall was investigated experimentally by Escudier[17]. The importance of this flow is that its boundary conditions are extremely simple and can be easily modelled in numerical simulations. The non-dimensionalization is chosen such that the Reynolds number is

\[
Re = \frac{\omega R^2}{\nu},
\]

where \(\omega\) is the angular velocity of the endwall. For this flow, the Reynolds number and the cylinder aspect ratio completely define the problem; therefore, for the numerical simulations, the swirl ratio, \(\Omega\), is set identically to one and does not effect the governing equations. At the axis, the boundary conditions are the same as for the steady, pipe flow calculations. The two stationary, solid wall boundaries at \(z = 0\) and at \(r = 1\) are modelled using a no-slip boundary condition as discussed for the steady pipe flow in Section B.4. The only variation of boundary condition not previously discussed is the rotating endwall at \(z = Z/R\). In this case, the flow is in solid body rotation with no radial and axial velocity. Therefore, the physical boundary conditions are

\[
\begin{align*}
\frac{\partial \Psi}{\partial r}(r, Z/R) & = 0, \\
\frac{\partial \Psi}{\partial z}(r, Z/R) & = 0, \\
\Gamma(r, Z/R) & = r^2.
\end{align*}
\]  

(B.41)

The resulting numerical boundary conditions on the streamfunction and the azimuthal vorticity are the previously discussed viscous wall boundary conditions applied to the
\[ z = \frac{Z}{R} \] wall. The circulation boundary condition is a simple Dirichlet condition,

\[ \Gamma_{j, k_{max}} = r_{j, k_{max}}^2. \]

### B.6 Unsteady Pipe Flow Boundary Conditions

Pipe flows for which the inlet circulation or inlet azimuthal vorticity distributions are time dependent require special consideration for the inviscid wall boundary conditions. For viscous wall boundary conditions, inlet unsteadiness does not alter the no-slip assumption and the analytic and numeric viscous wall boundary conditions described in Section B.4 are unchanged.

#### B.6.1 Physical Boundary Conditions

The physical boundary condition on the streamfunction is unchanged and the requirement of flow tangency (or, equivalently, the conservation of mass) gives

\[ \Psi(R(z), z, t) = \Psi(R(0), 0, t), \]

where the possible time dependence of the inlet streamfunction has been included. For the circulation, Equation (A.38) on the wall may be simplified to:

\[ \frac{\partial \Gamma}{\partial t} + q \frac{\partial \Gamma}{\partial s} = 0, \]

where \( q = \sqrt{u^2 + w^2} \) and the partial derivative \( \frac{\partial}{\partial s} \) is taken along the wall surface. Similarly, the azimuthal vorticity transport equation, Equation (A.39), becomes:

\[ \frac{\partial \eta}{\partial t} + q \frac{\partial \eta}{\partial s} = \frac{1}{r^3} \frac{q}{w} \frac{\partial (\Gamma^2)}{\partial s} + \frac{w \eta}{r}. \]
B.6.2 Numerical Boundary Conditions

The numerical implementation of the streamfunction boundary condition for unsteady flows is

$$\Psi_{j_{max},k}^{n+1} = \Psi_{j_{max},1}^{n+1}$$

where the superscript $n + 1$ represents the $n + 1$ iteration. For the circulation and azimuthal vorticity, a spatial residual operator is formed using first-order accurate upwind differences. These residuals are then used to integrate forward in time using the fourth-order Runge-Kutta scheme discussed in Section B.2.3.
Appendix C

Validation of Flow Solver

In this appendix, numerical results for the flow in an enclosed cylinder with a rotating endwall are compared with experiments and previous calculations. This flow was first studied experimentally in the context of vortex breakdown by Vogel[68] in 1968. Then, in 1985, more extensive experimental results including flow visualization were reported by Escudier[17]. Numerically, the flow has been successfully simulated by several investigators[48, 47, 65]. This flow is very amenable to numerical simulations because the boundary conditions are easily modeled unlike the idealizations that are necessary in pipe flows or trailing line vortex flows. Therefore, based on the successes of previous computations, this vortex breakdown flow was used as a test case to validate the numerical algorithm described in Appendix B. In particular, the numerical results from this algorithm will be compared to the experimental results of Escudier and the numerical results of Lopez[47].

The enclosed cylinder with rotating endwall is shown in Figure C.1. The dimensions of the cylinder have been scaled by the radius of the cylinder. Thus, the cylindrical boundary is at r = 1 and the pipe extends from z = 0 to H/R. The rotating wall is at z = H/R. The angular velocity of the wall, ω, has been used to non-dimensionalize the governing equations; therefore, the non-dimensional rotation rate is simply one. In the governing equations, the single non-dimensional parameter is the Reynolds number which is defined as

\[ Re = \omega R^2 / \nu. \]
Therefore, the entire flowfield is described uniquely by only two parameters, $H/R$ and $Re$. In the computations which follow, results for an aspect ratio of $H/R = 2.5$ are presented. Other aspect ratio cylinders were calculated during the course of this study, however, the $H/R - 2.5$ results are representative of the general trends. Four different Reynolds number solutions are considered: $Re = 1942, 1994, 2126$, and $2494$.

The computational grid was chosen to increase the grid resolution at the walls and at the centerline where it will be necessary to resolve boundary layers, an Ekman layer, and the vortex core. As suggested by Tsičverblit[65], grid spacing functions which meet these resolution goals are given by

\[
\begin{align*}
    r &= x - a \sin(2\pi x), \\
    z &= \frac{H}{R} [y - b \sin(2\pi y)],
\end{align*}
\]

where $x$ and $y$ range between 0 and 1, and increasing $a$ and $b$ leads to increased clustering near the boundaries of the domain. For the computations presented in this appendix, the grid spacing parameters were $a = 0.13$ and $b = 0.15$ with 61 nodes in both the radial and axial directions. In passing, note that clustering points near the two endwalls necessarily lessens the axial resolution in the center of the cylinder. Thus, the axial grid resolution in the breakdown region is affected by the clustering at the endwalls. However, the number of axial and radial nodes and the chosen clustering parameters were found to work well over a range of Reynolds numbers. Only minor efforts were made to optimize either the clustering or number of nodes since the results documented below indicate that the
essential physics of axisymmetric vortex breakdown can be successfully simulated with the chosen parameters.

Fourth difference artificial viscosity was used to stabilize the scheme with a coefficient, $\nu_4 = 0.001$, which was found to be the minimum damping coefficient for which the algorithm would converge. The scheme was marched to a steady-state using the local time-stepping procedure described in Section B.2.3 for a CFL of 2.0.

Figure C.2 compares experimental dye flow visualization with the numerically-generated streamfunctions for the four different Reynolds number solutions which were computed. The agreement is good for all the cases shown but is particularly good for the higher Reynolds number solutions. The main trend with higher Reynolds number is an increase in the bubble size. The increased agreement with increasing Reynolds number is therefore believed to be a better resolution of the bubble region (i.e. more cells contained in the bubble). Specifically, for the lower Reynolds number solutions, the length of the bubble near the stationary endwall is only a few cells long. If the grid resolution were increased, the bubble sizes might approach the experimental sizes. Evidence of the relatively coarse grids in the breakdown region can be seen from the jagged nature of the streamfunction contours; this jaggedness is especially evident in lowest Reynolds number bubble. The bubble locations are also in good agreement with the experiments and again are probably limited from higher accuracy only by the coarseness of the grids.

Another check on the quality of the numerical computations is the calculation of the vorticity integral as suggested by Lopez[47]. For the case of solid wall boundaries, the following identity can be derived,

$$I(\eta) = 2\pi \int_0^1 \int_{z=0}^{H/R} r^2 \eta dz dr = 0.$$  

This integral was evaluated using a trapezoidal integration for the five flows described above and the results are tabulated in Table C.1. The values of $I(\eta)$ are approximately $4.7 \times 10^{-5}$ for all Reynolds numbers. These results compare favorably with the vorticity integral calculation performed by Lopez[47]; for his computations, Lopez found the
Figure C 2: Comparison of Escudier experimental flow visualization [17] and numerical streamlines
<table>
<thead>
<tr>
<th>Re</th>
<th>$I(\eta) \times 10^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1942</td>
<td>4.641</td>
</tr>
<tr>
<td>1994</td>
<td>4.649</td>
</tr>
<tr>
<td>2126</td>
<td>4.716</td>
</tr>
<tr>
<td>2494</td>
<td>4.838</td>
</tr>
</tbody>
</table>

Table C.1: Evaluation of vorticity integral, $I(\eta)$

vorticity integral was approximately $2 \times 10^{-4}$. This test shows that the calculations are self-consistent and that the vorticity integral is correctly represented.

As a final check of the computations, the calculated azimuthal vorticity field for the $Re = 2494$ case will be compared to Lopez's calculated azimuthal vorticity field. Figure C.3 shows the comparison of the two fields. As can be seen from this figure, the details of the two vorticity fields are in excellent agreement. The quantitative details of other calculations which are not illustrated here also compare extremely well. Therefore, as a result of good agreement with both experiments and previous computations, as well as internal modelling consistencies, we conclude that the computational algorithm described in Appendix B is an accurate model for axisymmetric, incompressible, viscous flows.
Figure C.3: Comparison of $\eta$ contours with results from Lopez[47] for $Re = 2494$. Note dashed contours are positive. $\min(\eta) = -17.2$; $\max(\eta) = 4.4$. Contour spacing $c(i) = \min(\eta)(i/20)^3$ or $c(i) = \max(\eta)(i/20)^3$. 
Appendix D

Properties of the q-vortex

The q-vortex model[2] will be used throughout this work as the inlet model for the vortex. The specific velocity and circulation distributions were described in Section B.4.3 but are repeated here in a slightly different form:

\[ u_0(r) = 0, \]
\[ \Gamma_0(r) = \Omega \left[ 1 - \exp(-r^2) \right], \quad (D.1) \]
\[ w_0(r) = 1 + \Delta w \exp(-r^2). \]

The circulation in Equation (D.1) is now non-dimensionalized by the freestream axial velocity, hence, the swirl ratio, \( \Omega = \Gamma_\infty / (W_\infty \delta) \), appears as a parameter in the distribution unlike Equation (B.40).

In this appendix, the q-vortex model will be used to quantify the properties of vortices seen in experiments. The q-vortex has been found to be a good approximation for a wide variety of vortex flows[21]. In particular, this appendix will detail the radial distributions of \( H' \), \( B \), and \( \gamma \) for a range of circulation and axial velocity profiles. Three swirl ratios, \( \Omega = 0.91, 1.13, \) and \( 1.49 \), will be considered. These three swirl ratios were found to be the critical swirl levels in the pipe flow computations performed in Section 3.3.

Assuming the q-vortex is only a function of the radius as described in Equation (D.1), the pressure (and total pressure) may be found from the appropriate radial equilibrium.
relations as described in Section A.2.1. The resultant form of $H'$ is

$$H' = \frac{dH}{d\Psi} = \frac{1}{rw} \frac{dH}{dr} = \frac{1}{rw} \left[ \Omega^2 \frac{d}{r^2} \left( \frac{1}{2} \Gamma^2 \right) + \frac{d}{dr} \left( \frac{1}{2} \omega^2 \right) \right],$$

and, $B$ is

$$B = \frac{1}{rw} \left[ \frac{d}{dr} \left( \frac{1}{2} \Gamma^2 \right) + \frac{dH}{dr} \right] = \frac{1}{rw} \left[ \Omega^2 \frac{d(\Gamma^2)}{dr} + \frac{d}{dr} \left( \frac{1}{2} \omega^2 \right) \right].$$

An interesting observation is that under the assumption of radial equilibrium, the value of $H'$ for a given swirl ratio, $\Omega$, is identical to the value of $B$ at $\Omega/\sqrt{2}$. Under the same assumptions, $\gamma^2$ is

$$\gamma^2 = \frac{\Gamma \Gamma'}{r^2 H'}$$

$$= \frac{\Gamma \frac{dT}{dr}}{r^2 \frac{dH}{dr}}$$

$$= \frac{\Gamma \frac{dT}{dr}}{\Gamma \frac{dT}{dr} + r^2 \omega \frac{dw}{dr}}$$

$$\Rightarrow \gamma^2 = \frac{1}{1 + (r^2 \omega \frac{dw}{dr})/(\Gamma \frac{dT}{dr})}.$$  \hspace{1cm} (D.2)

Figures D.1, D.2, and D.3 show the distribution of $H'$ for the three swirl ratios and $\Delta w = 0.2, -0.1, 0.0, 0.1$ and 0.2. Likewise, Figures D.4, D.5, and D.6 show the analogous $B$ distributions and Figures D.7, D.8, and D.9 show the analogous $\gamma^2$ distributions. $H'$ reaches a maximum at the axis of the vortex and approaches zero for larger $r$. This behavior is expected since the total pressure loss is concentrate in the rotational core region. Also, the values of $H'$ decrease (increase) as the axial jet becomes stronger (weaker). Also, the larger swirl ratio vortices have a larger value of $H'$.

The distributions of $B$ are similar to the $H'$ distributions. The values of $B$ are largest at the axis and asymptote to zero for larger radii. This suggests that the core of the vortex is the most sensitive to adverse pressure gradients which drive the production
of azimuthal vorticity. This must be the case since the production of azimuthal vorticity relies on the tilting of the axial vorticity which is concentrated around at the axis.

The values of $\gamma^2$ are lowest at the axis. Increasing the swirl ratio lowers $\gamma^2$ (except for $\Delta w = 0$). We remark here that for $\Delta w = 0$, $\gamma^2 \equiv 1$ regardless of the swirl level. Thus, although the higher swirl vortices are much more susceptible to breakdown, $\gamma^2$ cannot distinguish between swirl ratios when the axial velocity profile is constant. Finally, the values of $\gamma^2$ increase (decrease) as the axial jet becomes stronger (weaker).
Figure D.1: $H'(r)$ for $\Omega = 1.49$

Figure D.2: $H'(r)$ for $\Omega = 1.13$
Figure D.3: $H'(r)$ for $\Omega = 0.91$

Figure D.4: $B(r)$ for $\Omega = 1.49$
Figure D.5: $B(r)$ for $\Omega = 1.13$

Figure D.6: $B(r)$ for $\Omega = 0.91$
Figure D.7: $\gamma^2(r)$ for $\Omega = 1.49$

Figure D.8: $\gamma^2(r)$ for $\Omega = 1.13$
Figure D.9: $\gamma^2(r)$ for $\Omega = 0.91$
Appendix E

Non-unique Solutions for the Time-Dependent Equations

In this appendix, we briefly document several non-unique solutions which were calculated during the course of this investigation. First, a short review of conjugate state theories, which are connected to non-uniqueness, is given in Section E.1. Then, in Section E.2, the non-unique results we have obtained are presented and their relationship to the previous non-unique solutions of Beran and Culick[7] is discussed.

E.1 A Review of Conjugate States and Non-uniqueness

In [5], Benjamin suggested that vortex breakdown is the transition between a cylindrical supercritical flow to a subcritical flow with waves. The two conjugate flows are related by requiring the same dependence of circulation and total pressure upon the streamfunction, $\Gamma'(\Psi)$ and $H'(\Psi)$, respectively. A major drawback of Benjamin's conjugate state theory is that it offers no criterion for the occurrence of vortex breakdown except that the flow be supercritical. Thus, as discussed by Hall[26], Benjamin's theory provides no further information than Squire's simpler explanation. Also, Leibovich[42] points out that flows behind vortex breakdown are almost always turbulent and unsteady and the assumption that the conjugate states share the same circulation and total pressure distribution is unjustifiable in practice. An important and often neglected result of Ben-
jamin's conjugate state theory is that two conjugate, cylindrical states cannot conserve momentum while maintaining $\tilde{\Gamma}'(\Psi)$ and $H'(\Psi)$. Two important conclusions may be drawn from this result:

1. Without changes in $\tilde{\Gamma}'$ and $H'$ through dissipation, a subcritical conjugate flow must contain waves. Therefore, a non-dissipative transition from a supercritical flow to a subcritical flow cannot be smooth and must contain waves.

2. If a cylindrical, subcritical conjugate flow exists behind breakdown, the flow must contain some form of dissipation to alter $\tilde{\Gamma}'$ and $H'$ such that momentum is still conserved.

It is interesting to note that for the pipe flow calculated in Section 2.2, the flow downstream of breakdown returned to a cylindrical state. Therefore, as discussed by Benjamin, the distributions of circulation and total pressure must change in order to conserve momentum. Figures 2.14 and 2.15 show that the distributions of $\tilde{\Gamma}'$ and $H'$ are indeed different upstream and downstream of breakdown.

A slightly different approach was taken by Landahl and Widnall[38] in their finite transition model for vortex breakdown. By assuming a particular radial dependence for the swirl and axial velocities, they derived a set of jump conditions which conserved mass, axial momentum, and angular momentum. The critical condition at which breakdown occurs is associated with a minimum in the energy flux; at this location, they postulate that any mechanism which further decreases the energy flux would require a drastic change in the flow state.

The existence of conjugate states as found by Benjamin also implies the possibility of non-unique solutions to the steady, inviscid form of the governing equations. These equations can be reduced to a single, elliptic differential equation known as the Bragg and Hawthorne Equation (BHE)[10]:

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = r^2 H'(\Psi) - \tilde{\Gamma}'(\Psi). \quad (E.1)$$
Hafez et al[24] have found multiple solutions of the BHE containing periodic wavetrains. Using a more analytic-based technique, Leibovich and Kribus[45] mapped four solution branches of the BHE: columnar to columnar solutions corresponding to Benjamin's conjugate states; solitary waves on a supercritical specifying flow; periodic wavetrains on a subcritical specifying flow; and solitary waves on the principal, subcritical conjugate flow of the supercritical, specifying flow.

Recently, Beran and Culick[7] found multiple solutions for the steady form of the axisymmetric Navier-Stokes equations. They calculated the flow through the same family of converging-diverging pipe geometries described by Equation (2.26) using a finite difference technique coupled with a continuation method. They found multiple solution branches for the same swirl ratio resembling solitary waves, periodic wavetrains, and cylindrical flows depending on the particular inlet swirl ratio. In the following sections, their results are compared with the non-unique solutions obtained for the time dependent method.

E.2 Non-unique solutions to the time-dependent equations

Pipe 6 and Pipe 7 flows are calculated for $Re = 500, 1000, \text{ and } 2003$. The results of the non-unique solutions are summarized in Figures E.1-E.3. The columnar branch of solutions is identical to the results presented in Sections 5.3 and 5.4. By starting the calculation at a low swirl value, $\Omega = 1.44$ in the cases to be discussed, the solutions were found to be columnar without breakdown. Then, the swirl ratio is slightly raised and, using the previous slightly lower swirl solution for an initial condition, the higher swirl flow is calculated. This maps out the columnar solution branch until, at some relatively larger swirl, this solution branch jumps to the branch containing breakdown. Then, the swirl ratio can be lowered and the solutions are found to remain on the breakdown branch until obtaining a second limit point at which the solutions jump
Figure E.1: Pipe 6 and 7 solutions for $Re = 500$. □ Pipe 6; ○ Pipe 7. Solid line, columnar; dashed line, breakdown.

back to the columnar solution branches. Also, after the first limit point for which the flow jumps from the columnar branch to the breakdown branch, the swirl may be raised. These solutions stay on the breakdown branch. At $Re = 2000$, the breakdown solution branches never settle to a steady state and the unsteady fluctuations never become periodic. We note that the difference in solutions due to geometry is in agreement with the trends discussed in Section 5.4.

Comparisons with Beran and Culick[7] shows overall good agreement. The pipe geometry used in their simulations does not have a convergent outlet section to re-accelerate the flow and some differences are expected. However, the results for the two branches are in qualitative as well as quantitative agreement. Beran and Culick found a solution branch composed of a periodic wavetrain which connects the columnar and breakdown branches at their respective limit points. These wavetrain solutions were never found in either the steady state solutions or the transient solutions computed in this work. Two possible explanations are: the solutions may not be possible for a geometry with a convergent downstream boundary, or, the solutions are not a stable
Figure E.2: Pipe 6 and 7 solutions for $Re = 1000$. □ Pipe 6; ○ Pipe 7. Solid line, columnar; dashed line, breakdown.

Figure E.3: Pipe 6 and 7 solutions for $Re = 2000$. □ Pipe 6; ○ Pipe 7. Solid line, columnar; dashed line, breakdown.
branch of the time-dependent equations.
Appendix F

A Preliminary Investigation of Lag Effects on Vortex Breakdown

In Section 1.1, we noted that force coefficients on pitching delta wings have been observed to undergo hysteresis effects such that the lift coefficient attained values greater than the maximum steady lift coefficient during the pitch-up motions. For pitch-down motions, the lift coefficient was lower than the steady lift coefficient for the same angle of attack. Jarrah[30, 29] related these effects to a lag in the breakdown position with respect to its steady location. In this appendix, we perform a preliminary study of two possible lag effects. First, several possible influences on the breakdown position for unsteady flows are:

1. The formation of vortex breakdown may occur at a timescale much different than the convective timescale.

2. The reformation of a columnar vortex from a state with vortex breakdown may occur at a timescale much different than the convective timescale.

3. Breakdown may propagate at a slow velocity in the sense that the transient position of breakdown will lag the steady position for even small reduced frequency maneuvers.

4. The unsteady external pressure which a vortex feels may be significantly different than the steady external pressure field and the breakdown will respond to its variations.
5. The vorticity shed from the leading edge will change and its entrainment into the vortex core will occur at some timescale which may be much different than the convective timescale.

We will focus on the first two possible lag mechanisms in this appendix.

First, the problem of breakdown formation timescales will be investigated. We already have some indication from the simulation in Section 5.3 that breakdown develops at a slow timescale relative to the convective timescale. In this simulation of breakdown evolution, the first occurrence of reversed flow was at \( t = 223\delta/W_\infty \). If we assume the convective lengthscale, \( L \), to be the length of the converging-diverging section, \( 10\delta \), then this is approximately 22 convective timescales between the initial perturbation and the occurrence of reversed flow. We note that after the first reversed flow appeared, the breakdown relatively rapidly increased to approximately its full size within about 4 additional convective timescales. In an effort to determine if this type of timescale for breakdown formation is typical of the evolutionary process, a parametric study was conducted. The flows through Pipe 6 were simulated with the initial condition being the steady flow for a supercritical (no breakdown) vortex with swirl ratio, \( 1.45 \leq \Omega_i \leq 1.508 \), and a jump in the inlet swirl ratio to \( 1.51 \leq \Omega_f \leq 1.56 \). The Reynolds number was 1000 for these flows. The time at which reversed flow first occurs is defined as \( \tau_n \) and is given in units of \( \delta/W_\infty \). The results of these simulations are summarized in Table F.1. The dominant effect is the nearness to criticality. The flows which jump to the lowest swirl ratio at which breakdown occurs, \( \Omega_f = 1.51 \), have timescales of approximately 1600 \( \delta/W_\infty \). This effect could result in significant delays if a delta wing pitches to an angle of attack just above the critical angle of attack. As \( \Omega_f \) is raised, the timescale quickly decreases; however, even the highest final swirl ratios, \( \Omega = 1.56 \), have \( \tau_n \approx 40 \). In terms of the convective timescales, these results show that a possible lag of 4 to 160 convective timescales is possible in the formation of vortex breakdown on an initially columnar vortex.
Table F.1: Variations of $\tau_n$ with $\Omega_i$ and $\Omega_f$ for $Re = 1000$, Pipe 6 flow

<table>
<thead>
<tr>
<th>$\Omega_f$</th>
<th>1.45</th>
<th>1.46</th>
<th>1.47</th>
<th>1.48</th>
<th>1.49</th>
<th>1.50</th>
<th>1.504</th>
<th>1.508</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.51</td>
<td>1547</td>
<td>1630</td>
<td>1679</td>
<td>1707</td>
<td>1717</td>
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<td>1.52</td>
<td>189</td>
<td>208</td>
<td>225</td>
<td>237</td>
<td>243</td>
<td>235</td>
<td>223</td>
<td>189</td>
</tr>
<tr>
<td>1.53</td>
<td>94</td>
<td>104</td>
<td>114</td>
<td>121</td>
<td>125</td>
<td>122</td>
<td>116</td>
<td>99</td>
</tr>
<tr>
<td>1.54</td>
<td>57</td>
<td>65</td>
<td>71</td>
<td>77</td>
<td>80</td>
<td>78</td>
<td>75</td>
<td>65</td>
</tr>
<tr>
<td>1.55</td>
<td>41</td>
<td>45</td>
<td>49</td>
<td>53</td>
<td>56</td>
<td>55</td>
<td>53</td>
<td>48</td>
</tr>
<tr>
<td>1.56</td>
<td>34</td>
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<td>38</td>
<td>41</td>
<td>42</td>
<td>42</td>
<td>41</td>
<td>38</td>
</tr>
</tbody>
</table>

Table F.2: Variations of $\tau_p$ with $\Omega_i$ and $\Omega_f$ for $Re = 1000$, Pipe 6 flow

<table>
<thead>
<tr>
<th>$\Omega_f$</th>
<th>1.46</th>
<th>1.48</th>
<th>1.49</th>
<th>1.50</th>
<th>1.51</th>
<th>1.52</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.46</td>
<td>862</td>
<td>934</td>
<td>962</td>
<td>967</td>
<td>958</td>
<td>943</td>
</tr>
<tr>
<td>1.45</td>
<td>339</td>
<td>373</td>
<td>386</td>
<td>386</td>
<td>379</td>
<td>369</td>
</tr>
<tr>
<td>1.44</td>
<td>232</td>
<td>252</td>
<td>259</td>
<td>259</td>
<td>254</td>
<td>248</td>
</tr>
<tr>
<td>1.43</td>
<td>188</td>
<td>202</td>
<td>207</td>
<td>207</td>
<td>203</td>
<td>199</td>
</tr>
<tr>
<td>1.42</td>
<td>165</td>
<td>178</td>
<td>181</td>
<td>182</td>
<td>179</td>
<td>175</td>
</tr>
<tr>
<td>1.41</td>
<td>152</td>
<td>163</td>
<td>167</td>
<td>167</td>
<td>165</td>
<td>162</td>
</tr>
</tbody>
</table>

A similar parametric study was conducted for the reformation of a vortex from a breakdown state. The timescale of interest, $\tau_p$, was defined to be the timescale at which no reversed flow remained in the computational domain. The results of this study are summarized in Table F.2. We note that as a result of the non-uniqueness described in Appendix E, the critical swirl ratio at which vortex breakdown appears is different than the critical swirl ratio at which vortex breakdown disappears. For this pipe, $\Omega_f = 1.46$ is the highest swirl ratio at which vortex breakdown is no longer possible. Again, the dominant effect is seen to be the nearness of $\Omega_f$ to the critical swirl ratio. For $\Omega = 1.46$, $\tau_p \approx 900$ while for $\Omega = 1.41$, $\tau_p \approx 160$. In relation to convective scales, this gives a lag in vortex reformation of between 16 and 90 convective timescales.

Recall from Section 1.1 that Jarrah[30, 29] found lags for the crossing of vortex breakdown over the trailing edge of between 7 and 10 convective timescales during pitch-up maneuvers. Also, Reynolds and Abtahi[59] found similar lags in breakdown.
position of between 1 to 30 convective timescales. Although the two effects studied in this appendix are certainly not the only causes of breakdown lags in pitching delta wings, the timescales are in the correct range as those observed experimentally and it seems likely that hysteresis effects could be partially explained by these mechanisms.
Appendix G

Code Listing

```c
*******************************************************************************
C     Include file for : Axisymmetric Incompressible (AI) NS solver
*******************************************************************************

C**   PARAMETERS
C     parameter(JDIM = 101, KDIM = 401, MAXDIM = KDIM)
C     parameter(MAXSTAGE = 5)
C
C    JDIM,KDIM                Max grid size in (y,z) space
C    MAXDIM                   max(JDIM, KDIM)
C    MAXSTAGE                 Max number of RK stages

*******************************************************************************
C**   SCALAR REALS
C    real res_poi,resmax_poi,resfrac
C    real res_max
C    real sor,sormax,sorfrac
C    real pi
C    real re,sw,Rei
C    real alpha
C    real cfl
C    real eps
C    real nu2,nu4
C    real elag
C    real blloc
C
C    common /realcom/ res_poi,resmax_poi,sor,pi,re,sw,alpha,resfrac,
C                     sormax,sorfrac,cfl,eps,nu2,nu4,res_max,Rei,
C                     elag,blloc,cpff
C
C    resmax_poi               Residual limit for Poisson solver
C    res_poi                   Current residual for Poisson solver
C    res_max                   Maximum residual of vorticity
C    sor                      Current over-relaxation for Poisson
C    sormax                   Maximum over-relaxation for Poisson
C    sorfrac                  Fraction to reduce sor
C    resfrac                  Minimum residual reduction
C    pi                       Self-explanatory
C    Re,Sw                    Reynolds, swirl number
C    Rei                      1/Re (zero if inviscid flow)
C    alpha                    Axial wake/jet perimeter
C    cfl                      cfl (if < 0 local else global)
C    eps                      Residual smoothing factor
C    nu2,nu4                   User set artificial damping
C    elag                     Lag parameter for vort bc
C    blloc                    Size of boundary layer
C    cpff                     Coefficient for farfield bc

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SCALAR INTGERS

integer jmax,kmax
integer iter.poi,niter.poi,iter,niter
integer ivort.type
integer ih.update
integer nstage
integer jresm,kresm
integer iwall

common /intcom/ jmax,kmax,niter.poi,iter,ivort.type,niter,
$ iter.poi,ih.update,nstage,jresm,kresm,iwall

jmax,kmax Grid size of current case
iter Current iteration number
niter Iteration limit
iter.poi Current Poisson iteration number
niter.poi Max iters for Poisson solver
ivort.type Inlet vortex type
ih.update # of iters for iplotter
nstage # of RK stages
jresm,kresm location of max residual, res_max
iwall iwall=1, viscous wall else inviscid

CHARACTER STRINGS

character*10 titl(MAXLIN)
character*80 datafile, histfile, gridfile

common /charcom/ datafile,histfile,gridfile,titl

titl titles for line plots
datafile/hist/gridfile filenames

ARRAYS

real s(JDIM,KDIM), e(JDIM,KDIM), g(JDIM,KDIM)
real u(JDIM,KDIM), v(JDIM,KDIM), w(JDIM,KDIM)
real r(JDIM,KDIM), z(JDIM,KDIM), ra(JDIM,KDIM)
real e0s(JDIM,KDIM), g0s(JDIM,KDIM)
real uinlet(JDIM), winlet(JDIM), ginlet(JDIM), einlet(JDIM)
real acell(JDIM,KDIM), anode(JDIM,KDIM)
real tsstep(JDIM,KDIM)
real pv(JDIM,KDIM)
real xit(MAXITER)
real cont(MAXCON)
integer i1lin(MAXLIN), isym(MAXLIN), nper(MAXLIN)
real plin(MAXLIN*MAXDIM,2)
real aana(MAXLIN), bbb(MAXDIM), ccc(MAXDIM), ddd(MAXDIM)
real ff0(JDIM,KDIM,2), ff1(JDIM,KDIM,2)
real ggg(JDIM,KDIM,2), ggg1(JDIM,KDIM,2)
real rrr(JDIM,KDIM,2)
real res_convect(2)
real hist(MAXITER,2)
real d2(JDIM+4,KDIM+4,2), diss(JDIM+4,KDIM+4)
real t4(JDIM+4,KDIM+4,2), d4(JDIM+4,KDIM+4,2)
real asmoo(JDIM+4,KDIM+4)
real rkcoef(MAXSTAGE)
integer uweval(MAXSTAGE), dseval(MAXSTAGE)
real wbc(KDIM), wbcos(KDIM), wr(KDIM)
real s_ave(JDIM,KDIM), e_ave(JDIM,KDIM), g_ave(JDIM,KDIM)
real u_ave(JDIM,KDIM), w_ave(JDIM,KDIM)

common /arraycom/ s,e,g,r,z,ra,u,v,w,acell,anode,e0s,g0s,
$ pv,xit,cont,i1lin,isym,nper,
$ plin,aaa,bbb,ccc,ddd,
$ ff0,ff1,gg,gg1,rrr,res_convect,
Streamfunction
Azimuthal vorticity
Radial, swirl & axial velocity
Circulation
Residuals
Radial and axial coordinates
Average cell radial coordinate
Cell and nodal areas
Plotting arrays
More plotting arrays
Arrays for tri-di matrix solver
Flux arrays
Residual array
Total convection residuals
Convection residual histories
Artificial diss. contributions
Temporary work array for dissipation
Area array for smoothing
Axial velocity diss. switch
RK multistage coefficients
Evaluate uw if =1 on ith stage
Evaluate art. dis. if =1 on ith stage
Timestep at node
Axial velocity at ffbc
Residual for ffbc
Current global timestep and time

(A)xisymmetric (I)ncompressible NS Solver (AI)

This is a bare-bones version of the code which does not include plotting routines and some random options.

program ai
include 'ai.inc'
character*1 answer

Get the set-up info
call setup

Initialize a few odds & ends
call init

Initialize geometry
call geom_init

Set init. conditions
call icond
call v_velocity
call inlet
call bc
iter = 0

Call main menu to see what to do
call menu(iopt)

Quit if iopt = 99
if (iopt.eq.99) then
  stop
endif

Begin the explicit iteration loop
100  continue
iter = iter + 1

Calculate new circulation & vorticity

if (mod(iter,20).eq.1) then
    write(*,901)
    write(*,900)
    write(*,901)
endif

write(*,910) iter,log10(res.poi),iter.poi,
        log10(res.convect(1)),log10(res.convect(2)),
        log10(res.max),jresm,kresm

)

format("-----------------------------------------------",
       "---------------------")

format(I5,T9,F9.5,T21,I5,T31,F9.5,T44,F9.5,T56,F9.5,T68,
       "",",",",",")

if (mod(iter,abs(niter)).ne.0) goto 100

It's either converged or hit the iteration limit

call menu(iopt)

Check to see if quitting...

if (iopt.ne.99) then
    goto 100
endif

end

*******************************************************************************

subroutine menu(i)

This is the main menu routine

*******************************************************************************

include 'ai.inc'
character*1 i answer

1000 continue

write(*,*),'****** MAIN MENU ******
write(*,*),(1) 'Change niter'
write(*,*),(2) 'Change cfl'
write(*,*),(3) 'Change resmax.poi'
write(*,*),(4) 'Change niter.poi'
write(*,*),(5) 'Change nu2,nu4'
write(*,*),(6) 'Change swirl ratio'
write(*,*),(7) 'Change Reynolds number'
write(*,*),(99) 'quit'
write(*,*),(0) 'Continue'
read(*,*), i

if (i.eq.0) then
    return
else if (i.eq.1) then
    write(*,*),'old niter = ',niter
    write(*,*),'Enter niter'
    read(*,*), niter
    goto 1000
else if (i.eq.2) then
    write(*,*),'old CFL = ',cfl
    write(*,*),'Enter new cfl'
    read(*,*), cfl

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else if (i.eq.3) then
    write(*,*),'old resmax_poi = ', resmax_poi
    write(*,*),'Enter resmax_poi'
    read(*,*), resmax_poi
    goto 1000
else if (i.eq.4) then
    write(*,*),'old niter.poi = ', niter.poi
    write(*,*),'Enter niter.poi'
    read(*,*), niter.poi
    goto 1000
else if (i.eq.5) then
    write(*,*),'old nu2,nu4 = ', nu2,nu4
    write(*,*),'Enter nu2,nu4'
    read(*,*), nu2,nu4
    goto 1000
else if (i.eq.6) then
    write(*,*),'old swirl ratio = ', sw
    write(*,*),'Enter swirl ratio'
    read(*,*), sw
    goto 1000
else if (i.eq.7) then
    write(*,*),'old Re = ', re
    write(*,*),'Enter Re'
    read(*,*), Re
    goto 1000
endif

If it gets here, then i = 99 and the user wishes to quit
return
end

INITIALIZATION Routines FOR AI

subroutine init

Initializes some odds & ends

include 'ai.inc'
p = acos(-1.)
if (Re.lt.0) then
    Re = 0.
else
    Re = 1./Re
endif

do j = 1,jmax
    do k = 1,kmax
        s_ave(j,k) = 0.
        e_ave(j,k) = 0.
        g_ave(j,k) = 0.
        u_ave(j,k) = 0.
        w_ave(j,k) = 0.
    endo
endo
time_sav = 0.
return
end

subroutine inlet

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Stores inlet profiles

```
include 'ai.inc'
do j = 1,jmax
    uinlet(j) = u(j,1)
    winlet(j) = w(j,1)
    ginlet(j) = g(j,1)
    einlet(j) = e(j,1)
endo
einlet(1) = 0.
ginlet(1) = 0.
uinlet(1) = 0.
return
end
```

subroutine bc
Stores fbc profile

```
include 'ai.inc'
do k = 1,kmax
    wbc(k) = w(jmax,k)
endo
return
end
```

subroutine icond
Initial flow conditions

```
include 'ai.inc'
*** Set initial condition on u,v,w

do j = 1,jmax
do k = 1,kmax
    zt = z(j,k)
    rt = r(jmax,1)*r(j,k)/r(jmax,k)
    u(j,k) = uinit(zt,rt)
    g(j,k) = ginit(zt,rt)
    w(j,k) = winit(zt,rt)
endo
endo
```

*** Find streamfunction by integrating mass flux

```
do k = 1,kmax
    s(1,k) = 0.      ! streamfunction is zero on axis
endo
```

```
do j = 2,jmax
    r1 = r(j ,k)
    r0 = r(j-1,k)
    dr = r1 - r0
    z1 = z(j ,k)
    z0 = z(j-1,k)
    dz = z1 - z0
    w1 = w(j ,k)
    w0 = w(j-1,k)
    u1 = u(j ,k)
    u0 = u(j-1,k)
    s(j,k) = s(j-1,k) + 0.5*
               ((r1*w1 + r0*w0)*dr
                - (r1*u1 + r0*u0)*dz )
```
Find vorticity by finite volume discretization

Zero vorticity

do k = 1,kmax
  do j = 1,jmax
    \( e(j,k) = 0 \).
  enddo
enddo

Distribute cell contributions to nodes

do k = 1,kmax-1
  do j = 1,jmax-1
    \( r1 = r(j,k) \)
    \( r2 = r(j,k+1) \)
    \( r3 = r(j+1,k+1) \)
    \( r4 = r(j+1,k) \)
    \( z1 = z(j,k) \)
    \( z2 = z(j,k+1) \)
    \( z3 = z(j+1,k+1) \)
    \( z4 = z(j+1,k) \)
    \( dr21 = r2 - r1 \)
    \( dr32 = r3 - r2 \)
    \( dr43 = r4 - r3 \)
    \( dz21 = z2 - z1 \)
    \( dz32 = z3 - z2 \)
    \( dz43 = z4 - z3 \)
    \( dz14 = z1 - z4 \)
    \( u_1 = u(j,k) \)
    \( u_2 = u(j,k+1) \)
    \( u_3 = u(j+1,k+1) \)
    \( u_4 = u(j+1,k) \)
    \( w_1 = w(j,k) \)
    \( w_2 = w(j,k+1) \)
    \( w_3 = w(j+1,k+1) \)
    \( w_4 = w(j+1,k) \)
    \( sc = (u_2 + u_1)*dr21 + (w_2 + w_1)*dz21 \)
    \( + (u_3 + u_2)*dr32 + (w_3 + w_2)*dz32 \)
    \( + (u_4 + u_3)*dr43 + (w_4 + w_3)*dz43 \)
    \( + (u_1 + u_4)*dz14 + (w_1 + w_4)*dz14 \)
    \( e(j,k) = e(j,k) + ec \)
    \( e(j,k+1) = e(j,k+1) + ec \)
    \( e(j+1,k+1) = e(j+1,k+1) + ec \)
    \( e(j+1,k) = e(j+1,k) + ec \)
  enddo
enddo

Normalize by nodal areas

do j = 1,jmax
  do k = 1,kmax
    \( u(j,k) = 0.125*e(j,k)/\text{area}(j,k) \) if 0.125 is for averaging & area
  enddo
enddo

Zero smoothing init contributions

do j = 1,jmax
  do k = 1,kmax
    \( \text{dissw}(j,k) = 0 \).
  enddo

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d2(j,k,1) = 0.
d2(j,k,2) = 0.
c
d4(j,k,1) = 0.
d4(j,k,2) = 0.
c
enddo
enddo
return
end

function uinit(zt,rt)
   Radial velocity initial condition

include 'ai.inc'

*** Q-vortex
if (ivort_type.eq.1) then
   uinit = 0.
*** Q-vortex for stability
else if (ivort_type.eq.2) then
   uinit = 0.
*** Q-vortex with wall boundary layer
else if (ivort_type.eq.3) then
   uinit = 0.
*** No vortex
else if (ivort_type.eq.4) then
   uinit = 0.
*** FL inlet vortex
else if (ivort_type.eq.5) then
   uinit = 0.
endif
return
end

function ginit(zt,rt)
   Circulation initial condition

include 'ai.inc'

*** Q-vortex
if (ivort_type.eq.1) then
   if (rt.eq.0) then
      ginit = 0.
   else
      ginit = 1 - exp(-rt**2)
   endif
*** Q-vortex for stability
else if (ivort_type.eq.2) then
   if (rt.eq.0) then
      ginit = 0.
   else
      ginit = 1 - exp(-rt**2)
   endif
*** Q-vortex with wall boundary layer

else if (ivort_type.eq.3) then
  c
  if (rt.eq.0) then
    ginit = 0.
  else
      rl = 1 - rt/r(jmax,1)
      ginit = (1 - exp(-(rt)**2)) * (1 - exp(-blloc*rl))
  endif
  c
  c*** No vortex
  c
else if (ivort_type.eq.4) then
  c
  ginit = 0.
  c
  c*** FL inlet condition
  c
else if (ivort_type.eq.5) then
  c
  if (alpha.gt.0) then
    rl = rt/r(jmax,1)
    csl1 = 0.
    csl2 = 68.70393
    csl3 = -178.6959
    csl4 = 182.5178
    csl5 = -72.52688
    vtmp = csl1 + csl2*rl + csl3*rl**2 + csl4*rl**3 + csl5*rl**4
    ginit = 3*rl*vtmp/8.912
  else
    rl = 3*rt/r(jmax,1)
    csl1 = 0.
    csl2 = 3.049149265
    csl3 = -3.300946
    csl4 = 1.669739
    csl5 = -0.4340289
    csl6 = 0.036479236
    vtmp = csl1 + csl2*rl + csl3*rl**2
    c $ + csl4*rl**3 + csl5*rl**4 + csl6*rl**5
    ginit = rl*vtmp
  endif
  c
end if
  c
return
end
  c

*****************************************************************************
  c
function winit(zt,rt)
  c
Axial velocity initial condition
  c
*****************************************************************************
  c
include 'ri.inc'
  c
  c*** Q-vortex
  c
  if (ivort_type.eq.1) then
    winit = 1 + alpha*exp(-(rt)**2)
  c
  c*** Q-vortex for stability
  c
  else if (ivort_type.eq.2) then
    winit = alpha + exp(-(rt)**2)
  c
  c*** Q-vortex with boundary layer
  c
  else if (ivort_type.eq.3) then
    rl = 1 - rt/r(jmax,1)
    winit = (1 + alpha*exp(-(rt)**2)) * (1 - exp(-blloc*rl))
  c
  c*** No vortex
  c
  else if (ivort_type.eq.4) then
    winit = 0.
  c
  c*** FL inlet condition
  c
  else if (ivort_type.eq.5) then

rl = rt/r(jmax,1) 
c1s1 = 13.141586 
c1s2 = 0.103614 
c1s3 = -151.8547 
c1s4 = 266.8924 
c1s5 = -157.5729 
wttmp = c1s1 + c1s2*rl + c1s3*rl**2 + c1s4*rl**3 + c1s5*rl**4
winit = wtmp/5.675972

cendif
creturn
cend

c******************************************************************************
c Geometry-related subroutines for AI
c******************************************************************************
csubroutine geom_init
cli
initalizes geometric variables
cli
inc include 'ai.inc'
ciii Initialize base grid parameters
c call geom_calc(jmax,kmax,JDIM,KDIM,r,z,anode,ace1,ra)
do j = 1,jmax-1
do k = 1,kmax-1
asmoo(j+1,k+1) = ace1(j,k)
endo
do j = 1,jmax-1
asmoo(j+1,1) = 2*asmoo(j+1,2) - asmoo(j+1,3)
asmoo(j+1,kmax+1) = 2*asmoo(j+1,kmax) - asmoo(j+1,kmax-1)
endo
do k = 1,kmax+1
asmoo(1,k) = 2*asmoo(2,k) - asmoo(3,k)
asmoo(jmax+1,k) = 2*asmoo(jmax,k) - asmoo(jmax-1,k)
endo
creturn
c******************************************************************************
c subroutine geom_calc(jmax,kmax,JDIM,KDIM,r,z,anode,ace1,ra)
c Calculate areas and average radius
cli
real r(JDIM,KDIM), z(JDIM,KDIM), ra(JDIM,KDIM)
real anode(JDIM,KDIM), ace1(JDIM,KDIM)
ciii Calculate cell areas & average radius
do j = 1,jmax-1
do k = 1,kmax-1
r1 = r(j,k) 
r2 = r(j,k+1)
r3 = r(j+1,k+1)
r4 = r(j+1,k)
ra(j,k) = 0.25*(r1+r2+r3+r4)
z1 = z(j,k) 
z2 = z(j,k+1)
z3 = z(j+1,k+1)
z4 = z(j+1,k)
dz31 = z3 - z1
dr31 = r3 - r1
dz42 = z4 - z2
dr42 = r4 - r2

acell(j,k) = 0.5*(dz31*dr42 - dr31*dz42)

**Calculate nodal areas**

```
do j = 1,jmax
   do k = 1,kmax
      anode(j,k) = 0.
   endo
endo

do j = 1,jmax-1
   do k = 1,kmax-1
      tmp = 0.25*acell(j,k)
      anode(j  ,k  ) = anode(j  ,k  ) + tmp
      anode(j  ,k+1) = anode(j  ,k+1) + tmp
      anode(j+1,k+1) = anode(j+1,k+1) + tmp
      anode(j+1,k  ) = anode(j+1,k  ) + tmp
   endo
endo
```

**IO ROUTINES FOR AI**

```
subroutine setup
   include 'ai.inc'
   character*80 tmpfile
   write(*,*) 'Enter AI set-up file'
   read(*,') tmpfile
   open(unit=3,form='formatted',status='old',file=tmpfile)
   read(3,'(a)') gridfile
   read(3,'(a)') datafile
   read(3,'(a)') histfile
   read(3,*) resmax.poi
   read(3,*) niter.poi
   read(3,*) sormax,sorfrac,
   end(3,*) resfrac
   read(3,*) ivort.type
   read(3,*) re,sw
   read(3,*) alpha
   read(3,*) cfl
   read(3,*) niter
   read(3,*) ih_update
   read(3,*) eps
   read(3,*) nstage
   read(3,*) (rkcoef(i),i=1,nstage)
   read(3,*) (uweval(i),i=1,nstage)
   read(3,*) (dsmal(i),i=1,nstage)
   read(3,*) nu2,nu4
   read(3,*) iwall
   read(3,*) elag
   read(3,*) blocal
   read(3,*) cpff
   close(3)
endo
```

```
subroutine rgrid
```
Read in the grid

include 'ai.inc'

open(1,file=gridfile,status="old")
read(1,*) jmax,kmax
if (JDIM.lt.jmax.or.KDIM.lt.kmax) then
  write(*,990) jmax,kmax,JDIM,KDIM
990  format('Dimensions too large: jmax = ',i4, ', kmax = ',i4,
$ 'JDIM = ',i4, ', KDIM = ',i4)
  stop
endif
read(1,*) ((z(j,k),r(j,k),j=1,jmax),k=1,kmax)
close(1)
c
return
end

Module for convection of circulation and vorticity

subroutine convect

Solve circulation and vorticity convection

include 'ai.inc'

Store the current circulation and vorticity

do j = 1,jmax
  do k = 1,kmax
    g0s(j,k) = g(j,k)
    e0s(j,k) = e(j,k)
  enddo
endo

c**** Calculate timestep
c
call timestep
c
c**** Update ffbc on axial velocity if trailing vortex
c
if (iwall.lt.0) call ffbc
c
c**** Update using a Runge-Kutta scheme
c
do n = 1,nstage
  if (uval(n).eq.1) then
    call poisson
    call uw_velocity
  endif
  call convect_calc
  if (dval(n).eq.1) call dis
  call convect_update(rkcoef(n))
  call v_velocity
cendo

c
return
end

subroutine timestep
c
Calculate timestep

c
include 'ai.inc'
if (cfl.gt.100) then
  do j = 1,jmax
    do k = 1,kmax

tstep(j,k) = cfl - 100.
enddo

ndel_sav = cfl - 100.
time_sav = time_sav + ndel_sav

else

do j = 2,jmax
  do k = 2,kmax
    dx = z(j,k) - z(j,k-1)
dy = r(j,k) - r(j-1,k)
tvisc = 0.25*Re*(dx*dy)**2/(dx**2+dy**2)
tconv = 1.0/4,(abs(w(j,k)/dx)+abs(u(j,k)/dy))
    tstep(j,k) = abs(cfl)*min(tvisc,tconv)
  enddo
enddo

*** If cfl > 0, use a global timestep ***
if (cfl.gt.0) then
  tmin = tstep(2,2)
do j = 2,jmax
    do k = 2,kmax
      if (tmin.gt.tstep(j,k)) tmin = tstep(j,k)
    enddo
enddo
do j = 2,jmax
  do k = 2,kmax
    tstep(j,k) = tmin
  enddo
ndel_sav = tmin
time_sav = time_sav + ndel_sav
else
  ndel_sav = tstep(2,2)
time_sav = time_sav + ndel_sav
endif

return
end

******************************************************************************

subroutine convect_update(ct)

Integrate in time

******************************************************************************

include 'ai.inc'
res_convect(1) = 0.
res_convect(2) = 0.
res_max = 0.
jresm = 1
kresm = 1
if (ivall.eq.2) then
  klimit = kmax-1
else
  klimit = kmax
endif

do j = 2,jmax-1
  do k = 2,klimit
 Add dissipation terms
r(r(j,k,1) = rrr(j,k,1) - d2(j,k,1) + d4(j,k,1)
r(j,k,2) = rrr(j,k,2) - d2(j,k,2) + d4(j,k,2)
tmp = tstep(j,k)/anode(j,k)
g(j,k) = g0a(j,k) - ct*tmp*rrr(j,k,1)/r(j,k)
e(j,k) = e0a(j,k) - ct*tmp*rrr(j,k,2)
res_convec(1) = res_convec(1) + rrr(j,k,1)**2
res_convec(2) = res_convec(2) + rrr(j,k,2)**2

if (res_max.lt.abs(rrr(j,k,2))) then
    res_max = abs(rrr(j,k,2))
    jresm = j
    kresm = k
endif

enddo

enddo

do ii = 1,2
    res_convert(ii) = sqrt(res_convert(ii))/float(jmax*kmax)
endo

**** Update boundaries

call bcond
return
end

*****************************************************************************************************************************************

c
subroutine convect_calc

Calculates the convect equation residuals

*****************************************************************************************************************************************

include 'ai.inc'

**** Calculate zeroth order fluxes

do k = 1,kmax
    ff0(1,k,1) = 0.
    ff0(1,k,2) = 0.
    gg0(1,k,1) = 0.
    gg0(1,k,2) = 0.
    rrr(1,k,1) = 0.
    rrr(1,k,2) = 0.
endo

do j = 2,jmax
    do k = 1,kmax
        rt = r(j,k)
        wt = w(j,k)
        ut = u(j,k)
        gt = g(j,k)
        et = e(j,k)

        ff0(j,k,1) = rt*wt*gt
        gg0(j,k,1) = (rt*ut+2.*Rei)*gt
        ff0(j,k,2) = wt*et - ((Sw*gt)**2)/(rt**3)
        gg0(j,k,2) = ut*et
        rrr(j,k,1) = 0.
        rrr(j,k,2) = 0.
    enddo
endo

**** Calculate first order fluxes in cells

do j = 1,jmax-1
    do k = 1,kmax-1
        dr31 = r(j+1,k+1) - r(j,k)
        dz31 = z(j+1,k+1) - z(j,k)

        dr42 = r(j+1,k) - r(j,k+1)
        dz42 = z(j+1,k) - z(j,k+1)

        df1 = (g(j+1,k+1)-g(j,k))*dr42  
        + (g(j,k+1)-g(j+1,k))*dr31
        + (g(j,k+1)-g(j+1,k))*dz31
        + (g(j+1,k+1)-g(j,k))*dz42
    enddo
endo
\[ df2 = (e(j+1,k+1)-e(j,k))*dr42 + (e(j,k+1)-e(j+1,k))*dr31 \]
\[ dg2 = (r(j+1,k+1)*e(j+1,k+1)-r(j,k)*e(j,k))*dz42 + (r(j,k+1)*e(j,k+1)-r(j+1,k)*e(j+1,k))*dz31 \]
\[ \text{tmp} = -0.5*\text{Rei/acell}(j,k) \]
\[ ff1(j,k,1) = \text{tmp}*df1*ra(j,k) \]
\[ gg1(j,k,1) = \text{tmp}*dg1*ra(j,k) \]
\[ ff1(j,k,2) = \text{tmp}*df2 \]
\[ gg1(j,k,2) = \text{tmp}*dg2*ra(j,k) \]

```c
enddo
```

```c
*** Calculate interior cell residuals and distribute to nodes
```
```c
do j = 1, jmax-1
do k = 1, kmax-1
  dr1 = r(j,k+1) - r(j+1,k)          
  dr2 = r(j+1,k+1) - r(j,k)          
  dr3 = -dr1                         
  dr4 = -dr2                         
  dz1 = z(j,k+1) - z(j+1,k)          
  dz2 = z(j+1,k+1) - z(j,k)          
  dz3 = -dz1                         
  dz4 = -dz2                         
  drA = -dr1                         
  dzA = -dz1                         
  drB = -dr2                         
  dzB = -dz2                         
  drC = -dr3                         
  dzC = -dz3                         
  drD = -dr4                         
  dzD = -dz4                         
```
```c
do ii = 1, 2
```
```c
  h0 = 0.125* ( ff0(j,k,ii)*dr1 - gg0(j,k,ii)*dz1 + 
                ff0(j,k+1,ii)*dr2 - gg0(j,k+1,ii)*dz2 + 
                ff0(j+1,k,ii)*dr3 - gg0(j+1,k+1,ii)*dz3 + 
                ff0(j+1,k,ii)*dr4 - gg0(j+1,k,ii)*dz4 )
```
```c
```c
  fit = ff1(j,k,ii)
  git = gg1(j,k,ii)
```c
```c
  h1A = fit*drA + git*dzA
  h1B = fit*drB + git*dzB
  h1C = fit*drC + git*dzC
  h1D = fit*drD + git*dzD
```
```c
*** Distribute to nodes
```
```c
  rrr(j,k,ii) = rrr(j,k,ii) + h0 + 0.5*h1A
  rrr(j,k+1,ii) = rrr(j,k+1,ii) + h0 + 0.5*h1B
  rrr(j+1,k,ii) = rrr(j+1,k,ii) + h0 + 0.5*h1C
  rrr(j+1,k,ii) = rrr(j+1,k,ii) + h0 + 0.5*h1D
```
```c
```c
enddo
```c
```
```c
*** Outlet bc is parabolized
```
```c
k = kmax
```
```c
```
```c
co j = 2, jmax-1
```
```c
dr1 = r(j+1,k) - r(j-1,k)          
```
```c
dr2 = r(j+1,k-1) - r(j,k)          
```
```c
dr3 = r(j,k-1) - r(j+1,k)          
```
```c
dr4 = r(j-1,k-1) - r(j+1,k)        
```
```c
dr5 = r(j-1,k) - r(j,k-1)          
```
```c
dr6 = r(j,k) - r(j-1,k-1)          
```
dz1 = z(j+1,k) - z(j-1,k) 
dz2 = z(j+1,k-1) - z(j,k) 
dz3 = z(j,k-1) - z(j+1,k) 
dz4 = z(j-1,k-1) - z(j+1,k-1) 
dz5 = z(j+1,k) - z(j,k-1) 
dz6 = z(j,k) - z(j-1,k-1) 
dz = z(j,k) - z(j,k-1) 
do ii = 1,2 
  h0 = 0.125*(ff0(j ,k ,ii)*dr1 - gg0(j ,k ,ii)*dz1 
$ + ff0(j+1,k ,ii)*dr2 - gg0(j+1,k ,ii)*dz2$ 
$ + ff0(j+1,k-1,ii)*dr3 - gg0(j+1,k-1,ii)*dz3$ 
$ + ff0(j ,k-1,ii)*dr4 - gg0(j ,k-1,ii)*dz4$ 
$ + ff0(j-1,k-1,ii)*dr5 - gg0(j-1,k-1,ii)*dz5$ 
$ + ff0(j-1,k ,ii)*dr6 - gg0(j-1,k ,ii)*dz6$ ) 
g1B = gg1(j ,k-1,ii) 
g1C = gg1(j-1,k-1,ii) 
h1 = 0.5*dz*(g1C - g1B) 
rrr(j,k,ii) = h0 + h1 
enddo 
return 
end 

******************************************************************************
subroutine bcond 
  Set boundary conditions on g,e only!!!
******************************************************************************
include 'ai.inc'

*** Axis 
do k = 1,kmax 
g(1,k) = 0. 
e(1,k) = 0. 
enddo 
if (iwall.eq.0) then 
*** Inviscid wall bc's 
do j = 1,jmax 
g(j,1) = ginlet(j) 
e(j,1) = einlet(j) 
enddo 
do k = 2,kmax 
g(jmax,k) = ginlet(jmax) 
e(jmax,k) = 0. 
enddo 
endif 
*** Viscous pipe bc's 
if (iwall.eq.1) then 
  ! viscous wall bc's 
do j = 1,jmax 
g(j,1) = ginlet(j) 
e(j,1) = einlet(j) 
enddo 
do k = 1,kmax 
g(jmax,k) = 0. 
enddo 
j = jmax-1 
do k = 1,kmax-1 
  dr31 = r(j+1,k+1) - r(j ,k ) 
dz31 = z(j+1,k+1) - z(j ,k )
c
\[ dr42 = r(j+1,k) - r(j,k+1) \]
\[ dz42 = z(j+1,k) - z(j,k+1) \]
\[ df = (s(j+1,k+1) - s(j,k)) \times dr42 \]
\[ + (s(j+1,k) - s(j,k)) \times dz42 \] $\$
\[ dg = (s(j+1,k+1) - s(j,k)) \times dz42 \]
\[ + (s(j+1,k+1) - s(j,k)) \times dz31 \] $\$
\[ tmp = 0.5 / \text{acell}(j,k) / \text{ra}(j,k) \]
\[ ffl(j,k,1) = \text{tmp} \times df \]
\[ ggl(j,k,1) = - \text{tmp} \times dg \]
endo
do k = 2, kmax - 1
c
fc = ffl(j,k-1,1)
fld = ffl(j,k,1)
gc = ggl(j,k-1,1)
gd = ggl(j,k,1)
rra = 0.5 \times (r(j+1,k)+r(j+1,k+1))
rrb = 0.5 \times (r(j+1,k)+r(j+1,k-1))
rrc = r(j,k-1)
rrd = r(j,k,1)
zza = 0.5 \times (z(j+1,k)+z(j+1,k+1))
zzb = 0.5 \times (z(j+1,k)+z(j+1,k-1))
zzc = 0.26 \times (z(j,k)+z(j,k-1)+z(j+1,k)+z(j+1,k-1))
zzd = 0.26 \times (z(j,k)+z(j+1,k)+z(j+1,k)+z(j+1,k))
$ \$
etmp = fc \times (rrd-rrb) - gc \times (zzd-zzb)
+ fd \times (rrb-rrc) - gd \times (zza-zzc)
etmp = -0.5 \times etmp / \text{anode}(j+1,k)
\[ e(j+1,k) = e0s(j+1,k) + \text{elag} \times (etmp-e0s(j+1,k)) \]
endo
c
fc = ffl(jmax-1,kmax-1,1)
gc = ggl(jmax-1,kmax-1,1)
rra = r(jmax,kmax)
rrb = 0.5 \times (r(jmax,kmax)+r(jmax,kmax-1))
rrc = r(jmax-1,kmax-1)
rrd = 0.5 \times (r(jmax,kmax)+r(jmax-1,kmax))
zza = z(jmax,kmax)
zzb = 0.5 \times (z(jmax,kmax)+z(jmax,kmax-1))
zzc = 0.26 \times (z(jmax-1,kmax-1)+z(jmax-1,kmax)
+ z(jmax ,kmax-1)+z(jmax ,kmax-1))
$ \$
zzd = 0.26 \times (z(jmax,kmax)+z(jmax,kmax))
etmp = fc \times (rrd-rrb+rra-rrc) - gc \times (zzd-zzb+zza-zzc)
etmp = -0.5 \times etmp / \text{anode}(jmax,kmax)
\[ e(jmax,kmax) = e0s(jmax,kmax) \]
+ \text{elag} \times (etmp-e0s(jmax,kmax))
c
endif
if (iwall.eq.2) then ! rotating pipe bc's
do k = 1, kmax
g(jmax,k) = 0.
e(jmax,k) = 0.
endo
do k = 2, kmax - 1
\[ dr = r(jmax,k) - r(jmax-1,k) \]
if (k.ne.2) then
\[ etmp1 = -2 \times s(jmax-1,k-1)/(r(jmax,k-1) \times dr \times 2) \]
else
  etmp1 = 0.
endif
etmp2 = -2*s(jmax-1,k)/(r(jmax,k)*dr**2)
etmp3 = -2*s(jmax-1,k+1)/(r(jmax,k+1)*dr**2)
etmp = 0.25*(etmp1+2*etmp2+etmp3)
e(jmax,k) = e0s(jmax,k) + e1ag*(etmp-e0s(jmax,k))
enddo
cdo j = 1,jmax
g(j,1) = 0.
e(j,1) = 0.

\[ g(j,k) = (\text{some expression}) \]
e(j,k) = 0.
do j = 2,jmax-1
dz = z(j,2) - z(j,1)
  if (j.ne.2) then
    etmp1 = -2*s(j-1,2)/(r(j-1,1)*dz**2)
  else
    etmp1 = 0.
  endif
etmp2 = -2*s(j,2)/(r(j,1)*dz**2)
etmp3 = -2*s(j+1,2)/(r(j+1,1)*dz**2)
etmp = 0.25*(etmp1+2*etmp2+etmp3)
e(j,1) = e0s(j,1) + e1ag*(etmp-e0s(j,1))
  if (j.ne.2) then
    etmp1 = -2*s(j-1,kmax-1)/(r(j-1,kmax)*dz**2)
  else
    etmp1 = 0.
  endif
etmp2 = -2*s(j,kmax-1)/(r(j,kmax)*dz**2)
etmp3 = -2*s(j+1,kmax-1)/(r(j+1,kmax)*dz**2)
etmp = 0.25*(etmp1+2*etmp2+etmp3)
e(j,kmax) = e0s(j,kmax) + e1ag*(etmp-e0s(j,kmax))
enddo
cendif
! trailing line vortex
do k = 2,kmax
e(jmax,k) = 0.
g(jmax,k) = g(jmax-1,k)
enddo
do j = 1,jmax
g(j,1) = ginlet(j)
e(j,1) = einlet(j)
enddo
cendif
creturnend
c********************************************
c         Smoothing subroutines
c********************************************
c subroutine diss
cc Calculate dissipation terms
c********************************************
cc include 'ai.inc'
c real kappa4,kappa2
cc*** Zero out smoothing and switch
do j = 1,jmax+4
do k = 1,kmax+4
d2(j,k,1) = 0.
d2(j,k,2) = 0.
t4(j,k,1) = 0.
t4(j,k,2) = 0.
dissw(j,k) = 0.
endo

*** Initialize d4

do j = 1,jmax
  do k = 1,kmax
    d4(j+2,k+2,1) = r(j,k)*g(j,k)
    d4(j+2,k+2,2) = e(j,k)
  enddo
endo

endo do j = 3,jmax+2
  do m = 1,2
    d4(j,2,m) = 2*d4(j,3,m) - d4(j,4,m)
    d4(j,1,m) = 3*d4(j,3,m) - 2*d4(j,4,m)
    d4(j,kmax+3,m) = 2*d4(j,kmax+2,m) - d4(j,kmax+1,m)
    d4(j,kmax+4,m) = 3*d4(j,kmax+2,m) - 2*d4(j,kmax+1,m)
  enddo
endo

endo do k = 1,kmax+4
  do m = 1,2
    d4(2,k,m) = 2*d4(3,k,m) - d4(4,k,m)
    d4(1,k,m) = 3*d4(3,k,m) - 2*d4(4,k,m)
    d4(jmax3,k,m) = 2*d4(jmax2,k,m) - d4(jmax1,k,m)
    d4(jmax4,k,m) = 3*d4(jmax2,k,m) - 2*d4(jmax1,k,m)
  enddo
endo

*** Calculate a low accuracy second difference for use with
the fourth difference and the switch

do j = 1,jmax+3
  do k = 1,kmax+3
    w1 = d4(j,k+1,2)
    u2 = d4(j,k,2)
    w3 = d4(j+1,k,2)
    u4 = d4(j+1,k+1,2)
    dissw(j,k+1) = dissw(j,k+1) + w4-u1 + w2-w1
    dissw(j,k ) = dissw(j,k ) + u3-w2 + w1-w2
    dissw(j+1,k ) = dissw(j+1,k ) + w2-w3 + w4-w3
    dissw(j+1,k+1) = dissw(j+1,k+1) + w1-w4 + w3-w4
  endo
endo

endo do m = 1,2
  u1 = d4(j,k+1,m)
  u2 = d4(j,k,m)
  u3 = d4(j+1,k,m)
  u4 = d4(j+1,k+1,m)
  d2(j,k+1,m) = d2(j,k+1,m) + u4-u1
  d2(j,k ,m) = d2(j,k ,m) + u3-u2
  d2(j+1,k,m) = d2(j+1,k,m) + u2-u3
  d2(j+1,k+1,m) = d2(j+1,k+1,m) + u1-u4
  t4(j,k+1,m) = t4(j,k+1,m) + u2-u1
  t4(j,k ,m) = t4(j,k ,m) + u1-u2
  t4(j+1,k,m) = t4(j+1,k,m) + u4-u3
  t4(j+1,k+1,m) = t4(j+1,k+1,m) + u3-u4
endo
endo

endo do j = 1,jmax+4
  do k = 1,kmax+4
    do m = 1,2
      d4(j,k,m) = 0.
  enddo
endo
From the low accuracy second difference, generate the 4th difference

do j = 2, jmax + 2
  do k = 2, kmax + 2
    jtt = max(j - 2, 2)
    ktt = max(k - 2, 2)
    kappa4 = asmo4(j - 1, k - 1) * nu4/tstep(j*t, ktt)
    do m = 1, 2
      uj1 = d2(j, k + 1, m)
      uj2 = d2(j, k, m)
      uj3 = d2(j + 1, k, m)
      uj4 = d2(j + 1, k + 1, m)
    enddo
    uk1 = t4(j, k + 1, m)
    uk2 = t4(j, k, m)
    uk3 = t4(j + 1, k, m)
    uk4 = t4(j + 1, k + 1, m)
  enddo
enddo

c*** Copy d4 values to proper location

do j = 1, jmax
  do k = 1, kmax
    do m = 1, 2
      d4(j, k, m) = d4(j + 2, k + 2, m)
    enddo
  enddo
enddo

c*** Finally, generate the second difference smoothing

do j = 1, jmax + 4
  do k = 1, kmax + 4
    do m = 1, 2
      d2(j, k, m) = 0.
    enddo
  enddo
enddo

c*** Copy values to proper location

do j = 1, jmax
  do k = 1, kmax
    t4(j + 1, k + 1, 1) = r(j, k) * g(j, k)
    t4(j + 1, k + 1, 2) = e(j, k)
  enddo
enddo

do j = 2, jmax + 1
  do k = 1, kmax + 1
    t4(j, 1, m) = t4(j, 2, m)
    t4(j, k + 2, m) = t4(j, kmax + 1, m)
  enddo
enddo

do k = 2, kmax + 1
  do m = 1, 2
    t4(j, k, m) = t4(j, k + 2, m)
  enddo
enddo

t4(1,k,m) = t4(2,k,m)
t4(jmax+2,k,m) = t4(jmax+1,k,m)

do j = 1,jmax+1
  do k = 1,kmax+1
    jj = j + 1
    kk = k + 1
    jtt = max(j-1,2)
    ktt = max(k-1,2)
    do m = 1,2
      swave = swfun(jj,kk,m)
      kappa2 = asmoo(j,k)*swave/tstep(jtt,ktt)
      u1 = t4(j ,k+1,m)
      u2 = t4(j ,k ,m)
      u3 = t4(j+1,k ,m)
      u4 = t4(j+1,k+1,m)
      d2(j ,k+1,m) = d2(j ,k+1,m) + kappa2*(u4-u1 + u2-u1)
      d2(j ,k ,m) = d2(j ,k ,m) + kappa2*(u3-u2 + u1-u2)
      d2(j+1,k ,m) = d2(j+1,k ,m) + kappa2*(u2-u3 + u4-u3)
      d2(j+1,k+1,m) = d2(j+1,k+1,m) + kappa2*(u1-u4 + u3-u4)
  enddo
endo
endo

*** Copy d2 into proper spot

do j = 1,jmax
  do k = 1,kmax
    do m = 1,2
      d2(j,k,m) = d2(j+1,k+1,m)
    enddo
  enddo
endo
endo
endo

do j = 1,jmax
  do m = 1,2
    d2(j,1 ,m) = 0.5*d2(j ,1,m)
    d2(j,kmax,m) = 0.5*d2(j,kmax,m)
    d4(j,1 ,m) = 0.5*d4(j ,1,m)
    d4(j,kmax,m) = 0.5*d4(j,kmax,m)
  enddo
endo
endo

do k = 1,kmax
  do m = 1,2
    d2(1 ,k,m) = 0.5*d2(1 ,k,m)
    d2(jmax,k,m) = 0.5*d2(jmax,k,m)
    d4(1 ,k,m) = 0.5*d4(1 ,k,m)
    d4(jmax,k,m) = 0.5*d4(jmax,k,m)
  enddo
endo
endo

real function swfun(j,k,m)

Returns the cell-based switch

include 'ai.inc'
s1 = abs(dissw(j ,k+1))
s2 = abs(dissw(j ,k ))
s3 = abs(dissw(j+1,k ))
s4 = abs(dissw(j+1,k+1))
swave = s1+s2+s3+s4

if (m.eq.1) then
  swfun = 0.
else
  swfun = nu2*sqrt(asmo(j-1,k-1))*swave
endif
return
end

******************************************************************************
Module for solution of Poisson's Equation for Streamfunction
******************************************************************************

subroutine poisson
Solve Poisson equation using SOR
******************************************************************************

include 'ai.inc'

*** Initialize iteration counter and sor parameter
iter_poi = 0
sor = sormax

*** Top of iteration loop
100 continue
iter_poi = iter_poi + 1
call poisson_kswEEP
call poisson_jswEEP

*** Check for convergence
if (iter_poi.ne.1) then
dr = (res_old - res_poi)/res_old
  if (dr.lt.resfrac) sor = max(sorfrac*sor,1.)
endif
res_old = res_poi
if (res_poi.gt.resmax_poi .and. iter_poi.lt.niter_poi) goto 100
return
end

******************************************************************************
subroutine poisson_kswEEP
Perform an iteration for Poisson's eqn in cylindrical
coordinates using a k-sweep (implicit in j-direction)
******************************************************************************

include 'ai.inc'

*** Set residual to zero
res_poi = 0.

*** Sweep in the k-direction (implicit j's)

*** k=1 boundary condition
if (iwall.eq.2) then
do j = 1,jmax
  s(j,1) = 0.
enddo
else
  Axis s = 0
  bbb(1) = 0.
ddd(1) = 1.
aaa(1) = 0.
ccc(1) = 0.
do j = 2,jmax-1
dr1 = r(j-1,1) * r(j+1,1)
\[ \text{dr}2 = r(j-1,2) - r(j,1) \]
\[ \text{dr}3 = r(j,2) - r(j-1,1) \]
\[ \text{dr}4 = r(j+1,2) - r(j-1,2) \]
\[ \text{dr}5 = r(j+1,1) - r(j,2) \]
\[ \text{dr}6 = r(j,1) - r(j+1,2) \]

\[
rhs = 8 \cdot \text{inlet}(j) \cdot r(j,1) \cdot \text{anode}(j,1) + s(j-1,2) \cdot \text{dr}3 + s(j,2) \cdot \text{dr}4 + s(j+1,2) \cdot \text{dr}5
\]

\[
\text{bbb}(j) = \text{dr}2
\]
\[
\text{ddd}(j) = \text{dr}1
\]
\[
\text{aaa}(j) = \text{dr}6
\]
\[
\text{ccc}(j) = -\text{rhs}
\]

\text{endo}

\text{if} (iwall.ge.0) \text{ then}

\[
\text{*** j = jmax is a streamsurface}
\]
\[
\text{bbb}(jmax) = 0.
\]
\[
\text{ddd}(jmax) = 1.
\]
\[
\text{aaa}(jmax) = 0.
\]
\[
\text{ccc}(jmax) = s(jmax,1)
\]

\text{else}

\[
\text{*** j = jmax, set wbc}
\]
\[
\text{bbb}(jmax) = -1.
\]
\[
\text{ddd}(jmax) = 1.
\]
\[
\text{aaa}(jmax) = 0.
\]
\[
\text{ccc}(jmax) = r(jmax,1) \cdot (r(jmax,1) - r(jmax-1,1)) \cdot wbc(1)
\]

\text{endif}

\text{call tridi(ccc,aaa,bbb,ccc,ddd,jmax)}

\text{do j = 1,jmax}
\[
\text{s}(j,1) = \text{ccc}(j)
\]
\text{endo}

\text{end}

\text{*** Sweep through interior k lines}

\text{do k = 2,kmax-1}

\[
\text{*** j=1 boundary condition is s = 0}
\]
\[
\text{bbb}(1) = 0.
\]
\[
\text{ddd}(1) = 1.
\]
\[
\text{aaa}(1) = 0.
\]
\[
\text{ccc}(1) = 0.
\]

\text{do j = 2,jmax-1}

\[
\text{dr}C = r(j-1,k) - r(j,k-1)
\]
\[
\text{dr}D = r(j,k+1) - r(j-1,k)
\]
\[
\text{dr}A = r(j+1,k) - r(j,k+1)
\]
\[
\text{dr}B = r(j,k-1) - r(j+1,k)
\]

\[
\text{dz}C = z(j-1,k) - z(j,k-1)
\]
\[
\text{dz}D = z(j,k+1) - z(j-1,k)
\]
\[
\text{dz}A = z(j+1,k) - z(j,k+1)
\]
\[
\text{dz}B = z(j,k-1) - z(j+1,k)
\]

\text{*** A contributions}

\[
\text{dr}42 = r(j+1,k) - r(j,k+1)
\]
\[
\text{dr}31 = r(j+1,k+1) - r(j,k)
\]
\[
\text{fa} = \text{dr}42 \cdot s(j+1,k+1) + \text{dr}31 \cdot s(j,k+1)
\]

\[
\text{dz}42 = z(j+1,k) - z(j,k+1)
\]
\[
\text{dz}31 = z(j+1,k+1) - z(j,k)
\]
\[ g_A = dz42* (s(j+1,k+1) ) + dz31* (s(j ,k+1) ) \]
\[ cA = 0.25/(aCell(j ,k )*ra(j ,k )) \]
\[ btmp = 0. \]
\[ dtmp = -cA*(dr42*drA + dz42*dzA) \]
\[ atmp = -cA*(dr31*drA + dz31*dzA) \]
\[ ctmp = cA* (fa*drA + gA*dzA) \]

**B contributions**
\[ dr42 = r(j+1,k-1) - r(j ,k ) \]
\[ dr31 = r(j+1,k ) - r(j ,k-1) \]
\[ fB = dr42* ( -s(j ,k-1) ) + dr31* ( -s(j+1,k-1) ) \]
\[ dz42 = z(j+1,k-1) - z(j ,k ) \]
\[ dz31 = z(j+1,k ) - z(j ,k-1) \]
\[ gB = dz42* ( -s(j ,k-1) ) + dz31* ( -s(j+1,k-1) ) \]
\[ cB = 0.25/(aCell(j ,k-1)*ra(j ,k-1)) \]
\[ dtmp = dtmp + cB*(dr31*drB + dz31*dzB) \]
\[ atmp = atmp + cB*(dr42*drB + dz42*dzB) \]
\[ ctmp = ctmp + cB* ( fB*drB + gB*dzB ) \]

**C contributions**
\[ dr42 = r(j ,k-1) - r(j-1,k ) \]
\[ dr31 = r(j ,k ) - r(j-1,k-1) \]
\[ fC = dr42* ( -s(j-1,k-1) ) + dr31* ( -s(j ,k-1) ) \]
\[ dz42 = z(j ,k-1) - z(j-1,k ) \]
\[ dz31 = z(j ,k ) - z(j-1,k-1) \]
\[ gC = dz42* ( -s(j-1,k-1) ) + dz31* ( -s(j ,k-1) ) \]
\[ cC = 0.25/(aCell(j-1,k-1)*ra(j-1,k-1)) \]
\[ btmp = btmp + cC*(dr31*drC + dz31*dzC) \]
\[ dtmp = dtmp + cC*(dr42*drC + dz42*dzC) \]
\[ ctmp = ctmp + cC* ( fC*drC + gC*dzC ) \]

**D contributions**
\[ dr42 = r(j ,k ) - r(j-1,k+1) \]
\[ dr31 = r(j ,k+1) - r(j-1,k ) \]
\[ fD = dr42* (s(j ,k+1) ) + dr31* (s(j-1,k+1) ) \]
\[ dz42 = z(j ,k ) - z(j-1,k+1) \]
\[ dz31 = z(j ,k+1) - z(j-1,k ) \]
\[ gD = dz42* (s(j ,k+1) ) + dz31* (s(j-1,k+1) ) \]
\[ cD = 0.25/(aCell(j-1,k )*ra(j-1,k )) \]
\[ btmp = btmp - cD*(dr42*drD + dz42*dzD) \]
\[ dtmp = dtmp - cD*(dr31*drD + dz31*dzD) \]
\[ ctmp = ctmp + cD* ( fD*drD + gD*dzD ) \]

**Source term**
\[ ctmp = ctmp + e(j,k )*anode(j,k) \]

**Assign coefficients**
\[ bbb(j) = btmp \]
\[ ddd(j) = dtmp \]
\[ aaa(j) = atmp \]
ccc(j) = -ctmp
enddo
if (iwall.ge.0) then
*** j = jmax is a streamsurface

bbb(jmax) = 0.
ddd(jmax) = 1.
aaa(jmax) = 0.
ccc(jmax) = s(jmax,1)
else
*** j = jmax, set wbc

bbb(jmax) = -1.
ddd(jmax) = 1.
aaa(jmax) = 0.
ccc(jmax) = r(jmax,k)*(r(jmax,k)-r(jmax-1,k))*wbc(k)
endif
*** Solve constant k-line

call tridi(ccc,aaa,bbb,ccc,ddd,jmax)
do j = 2,jmax-1

ds = ccc(j) - s(j,k)
res.poi = res.poi + ds**2
s(j,-) = s(j,k) + sor*ds
endo
ds1 = ccc(1)
s(jmax,k) = ccc(jmax)
endo
*** k=kmax boundary condition

k = kmax
if (iwall.eq.2) then
do j = 1,jmax
s(j,kmax) = 0.
endo
else
bbb(t) = 0.
ddd(1) = 1.
aaa(1) = 0.
ccc(1) = 0.
do j = 2,jmax-1

drB = r(j,k-1)-r(j,k)
dzB = z(j,k-1)-z(j,k)
drc = -drB
dzC = -dzB
*** B contributions

dz42 = z(j+1,k-1) - z(j,k)
dz31 = z(j+1,k) - z(j,k-1)
gB = dz42*(-s(j,k-1))
    + dz31*(-s(j+1,k-1))
$cB = 0.25/(acell(j,k-1)*x(j,k-1))
btmp = 0.
dtmp = cB*dzB*dz31
atmp = cB*dzB*dz42
ctmp = cB*dzB*gB
*** C contributions

dz42 = z(j,k-1) - z(j-1,k)
dz31 = z(j,k) - z(j-1,k-1)
gC = dz42*(-s(j-1,k-1))
   + dz31*(-s(j,k-1))

$ cC = 0.25/(acell(j-1,k-1)*ra(j-1,k-1))

c btmp = btmp + cC*dzc*dz31
dtmp = dtmp + cC*dzc*dz42
c tmp = ctmp + cC*dzc*gC

C*** Source term
C
ctmp = ctmp + e(j,k)*anode(j,k)
C
bbb(j) = btmp
ddd(j) = dtmp
aaa(j) = atmp
ccc(j) = -ctmp
C
enddo
C
if (iwall.ge.0) then
C*** j = jmax is a streamsurface
C
bbb(jmax) = 0.
ddd(jmax) = 1.
aaa(jmax) = 0.
ccc(jmax) = s(jmax,1)
C
else
C*** j = jmax, set wbc
C
bbb(jmax) = -1.
ddd(jmax) = 1.
aaa(jmax) = 0.
ccc(jmax) = r(jmax,k)*(r(jmax,k)-r(jmax-1,k))*wbc(k)
C
endif
C
call tridi(ccc,aaa,bbb,ccc,ddd,jmax)
C
do j = 1,jmax
   s(j,k) = ccc(j)
enddo
C
res.poi = sqrt(res.poi)/(float(jmax)*float(kmax))
C
return
end
C********************************************************************
C subroutine poisson_jsweep
C
C Performs an iteration for Poisson's eqn in cylindrical
C coordinates using a j-sweep (implicit in k-direction)
C********************************************************************
C
C include 'ai.inc'
C
C*** Set residual to zero
C
res.poi = 0.
C
C*** Sweep in j-direction (implicit k's)
C
C*** j=1 boundary condition (can be done explicitly)
C
do k = 1,kmax
   s(1,k) = 0.  ! streamfunction zero on axis
enddo
C
do j = 2,jmax-1
C
C*** k=1 boundary condition
C
if (iwall.eq.2) then
   bbb(1) = 0.
\[
\begin{align*}
\text{ddd}(1) &= 1. \\
\text{saa}(1) &= 0. \\
\text{ccc}(1) &= 0. \\
\text{else} \\
dr1 &= r(j-1,1) - r(j+1,1) \\
dr2 &= r(j-1,2) - r(j,1) \\
dr3 &= r(j,2) - r(j-1,1) \\
dr4 &= r(j+1,2) - r(j-1,2) \\
dr5 &= r(j+1,1) - r(j,2) \\
dr6 &= r(j,1) - r(j+1,2) \\
\text{rhs} &= 8 \times \text{u.inlet}(j) \times r(j,1) \times \text{anode}(j,1) \\
&+ s(j-1,1) \times dr2 + s(j-1,2) \times dr3 \\
&+ s(j+1,1) \times dr5 + s(j+1,1) \times dr6 \\
\text{endif} \\
\text{bbb}(1) &= 0. \\
\text{ddd}(1) &= dr1 \\
\text{saa}(1) &= dr4 \\
\text{ccc}(1) &= -\text{rhs} \\
\text{end} \\
\text{*** Interior cells} \\
do k = 2, \text{kmax}-1 \\
drC &= r(j-1,k) - r(j,k-1) \\
drD &= r(j,k+1) - r(j-1,k) \\
drA &= r(j+1,k) - r(j,k+1) \\
drB &= r(j,k-1) - r(j+1,k) \\
dzC &= z(j-1,k) - z(j,k-1) \\
dzD &= z(j,k+1) - z(j-1,k) \\
dzA &= z(j+1,k) - z(j,k+1) \\
dzB &= z(j,k-1) - z(j+1,k) \\
\text{*** A contributions} \\
dr42 &= r(j+1,k) - r(j,k+1) \\
dr31 &= r(j+1,k+1) - r(j,k) \\
fA &= dr42 * (s(j+1,k+1) + dr31 \times (s(j,k))) \\
\text{*** B contributions} \\
dr42 &= r(j+1, k-1) - r(j,k) \\
dr31 &= r(j+1, k) - r(j, k-1) \\
fB &= dr42 * (s(j+1, k) + dr31 \times (s(j, k-1))) \\
dz42 &= z(j+1, k) - z(j, k+1) \\
dz31 &= z(j+1, k+1) - z(j, k) \\
gA &= dz42 * (s(j+1, k+1) + dz31 \times (s(j, k))) \\
cA &= 0.25 / (\text{acell}(j,k) \times \text{ra}(j,k)) \\
\text{btmp} &= 0. \\
dtmp &= -cA * (dr42 \times drA + dz42 \times dzA) \\
atmp &= -cA * (dr31 \times drA + dz31 \times dzA) \\
ctmp &= cA \times (fA \times drA + gA \times dzA) \\
\text{btmp} &= \text{btmp} - cB \times (dr42 \times drB + dz42 \times dzB) \\
dtmp &= \text{dtmp} + cB \times (dr31 \times drB + dz31 \times dzB) \\
ctmp &= \text{ctmp} + cB \times (fB \times drB + gB \times dzB) 
\end{align*}
\]
C*** C contributions

dr42 = r(j , k-1) - r(j-1, k)
dr31 = r(j , k) - r(j-1, k-1)
fC = dr42*(
  -s(j-1, k-1)
  + dr31*(s(j-1, k))
)
dz42 = z(j , k-1) - z(j-1, k)
dz31 = z(j , k) - z(j-1, k-1)
gC = dz42*(
  -s(j-1, k-1)
  + dz31*(s(j-1, k))
)
cC = 0.25/(ace(j-1, k-1)*ra(j-1, k-1))
btmp = btmp - cC*(dr31*drC + dz31*dzC)
dtmp = dtmp + cC*(dr42*drC + dz42*dzC)
ctmp = ctmp + cC*(fC*drC + gC*dzC)

C*** D contributions

dr42 = r(j , k+1) - r(j-1, k+1)
dr31 = r(j , k+1) - r(j-1, k)
fD = dr42*(
  -s(j-1, k)
  + dr31*(s(j-1, k+1))
)
dz42 = z(j , k) - z(j-1, k+1)
dz31 = z(j , k+1) - z(j-1, k)
gD = dz42*(
  -s(j-1, k)
  + dz31*(s(j-1, k+1))
)
cD = 0.25/(ace(j-1, k)*ra(j-1, k))
dtmp = dtmp - cD*(dr31*drD + dz31*dzD)
ctmp = ctmp + cD*(dr42*drD + dz42*dzD)
ctmp = ctmp + cD*(fD*drD + gD*dzD)

C*** Source term

ctmp = ctmp + e(j,k)*anode(j,k)

C*** Assign coefficients

bbb(k) = btmp
ddd(k) = dtmp
aaa(k) = atmp
ccc(k) = -ctmp

c enddo

C*** k=kmax boundary condition

if (iwall.eq.2) then
  aaa(kmax) = 0.
ddd(kmax) = 1.
bbb(kmax) = 0.
ccc(kmax) = 0.
else
  k = kmax
drB = r(j , k-1)-r(j , k)
dzB = z(j , k-1)-z(j , k)
drC = -drB
dzC = -dzB

C*** B contributions

dz42 = z(j+1, k-1) - z(j , k)
dz31 = z(j+1, k) - z(j , k-1)
gB = dz42*(s(j+1, k))
  + dz31*(
    -s(j+1, k-1))
cB = 0.25/(ace(j , k-1)*ra(j , k-1))
btmp = -cB*dzB*dz42
dtmp = cB*dzB*dz31
atmp = 0.
ctmp = cB*dzB*GB

c*** C contributions

dz42 = z(j,k-1) - z(j-1,k )
dz31 = z(j ,k ) - z(j-1,k-1)
gC = dz42*(-s(j-1,k-1))
     + dz31*(s(j-1,k ))
c
cc = 0.25/(acell(j-1,k-1)*ra(j-1,k-1))
c
btmp = btmp - cC*dzC*dz31
dtmp = dtmp + cC*dzC*dz42
ctmp = ctmp + cC*dzC*gC

c*** Source term
c
ctmp = ctmp + e(j,k)*anode(j,k)
c
bbb(k) = btmp
ddd(k) = dtmp
aaa(k) = atmp
ccc(k) = -ctmp
c
endif
c*** Solve for constant j line
c
call tridi(ccc,aaa,bbb,ccc,ddd,kmax)
c
do k = 2,kmax-1
ds = ccc(k) - s(j,k)
res_poi = res_poi + ds*ds**2
s(j,k) = s(j,k) + sor*ds
dendo
s(j,1) = ccc(1)
s(j,kmax) = ccc(kmax)
c
endo

c*** j = jmax boundary condition
c
if (ivall.ge.0) then
do k = 2,kmax
s(jmax,k) = s(jmax,1) ! Outer boundary is a streamsurface
dendo
else
   do k = 1,kmax
      s(jmax,k) = s(jmax-1,k)
     + r(jmax,k)*(r(jmax,k)-r(jmax-1,k))**wbc(k)
   enddo
endif
c
res_poi = sqrt(res_poi)/(float(jmax)*float(kmax))
c
return
eend

*******************************************************************************
c subroutine uw_velocity
c*******************************************************************************
c Update velocities given the streamfunction

*******************************************************************************
c include 'ai.inc'
c
*** Zero all velocities
do j = 1,jmax
do k = 1,kmax
   u(j,k) = 0.
   w(j,k) = 0.
dendo
endo
c
*** Calculate cell terms and distribute to nodes
do j = 1,jmax-1
do k = 1,kmax-1
   dr42 = r(j+1,k ) - r(j ,k+1)
   dr31 = r(j+1,k+1) - r(j ,k )
   fA = dr42*(s(j+1,k+1)-s(j ,k ))
   + dr31*(s(j ,k+1)-s(j+1,k ))

   dz42 = z(j+1,k ) - z(j ,k+1)
   dz31 = z(j+1,k+1) - z(j ,k )
   gA = dz42*(s(j+1,k+1)-s(j ,k ))
   + dz31*(s(j ,k+1)-s(j+1,k ))

   fA = fA/ra(j,k)
   gA = gA/ra(j,k)

   u(j ,k ) = u(j ,k ) + fA
   u(j+1,k ) = u(j+1,k ) + fA
   u(j+1,k+1) = u(j+1,k+1) + fA
   u(j ,k+1) = u(j ,k+1) + fA

   w(j ,k ) = w(j ,k ) + gA
   w(j+1,k ) = w(j+1,k ) + gA
   w(j+1,k+1) = w(j+1,k+1) + gA
   w(j ,k+1) = w(j ,k+1) + gA
endo
endo

*** Normalize by area modal contributions and radius

do j = 1,jmax
   do k = 1,kmax
      u(j,k) = -0.125*u(j,k)/anode(j,k)
      w(j,k) = -0.125*w(j,k)/anode(j,k)
   endo
endo

*** u velocity on axis is zero

do k = 1,kmax
   u(1,k) = 0.
endo

*** Inviscid pipe

if (iwall.eq.0) then
   do j = 1,jmax
      u(j,1) = uinlet(j)
   endo
else if (iwall.eq.1) then
   do j = 1,jmax
      u(j,1) = uinlet(j)
   endo
   do k = 1,kmax
      u(jmax,k) = 0.
      w(jmax,k) = 0.
   endo
else if (iwall.eq.2) then
   do k = 1,kmax
      u(jmax,k) = 0.
      w(jmax,k) = 0.
   endo
   do j = 1,jmax
      u(j,1) = 0.
      w(j,1) = 0.
      u(j,kmax) = 0.
      w(j,kmax) = 0.
   endo
else if (iwall.lt.0) then
  do j = 1,jmax
    u(j,1) = uinlet(j)
  enddo
endif
return
end

******************************************************************************

Module for solution of pressure stuff
******************************************************************************

subroutine ffbc
 Update ffbc on pressure condition
******************************************************************************

include 'ai.inc'

*** Save old wvel
 do k = 1,kmax
   wbcs(k) = wbc(k)
 enddo

do n = 1,nstage

*** Calculate time for sub-iteration
 tmp = time_sav + (rkcoef(n)-1)*tdel_sav

*** Interior ffbc
 do k = 2,kmax-1

*** Central difference pressure
 z0 = z(jmax,k-1)
z1 = z(jmax,k )
z2 = z(jmax,k+1)
dpdp = (pffbc(z2,tmp)-pffbc(z0,tmp))/(z2-z0)

*** Upwind velocity
 w0 = w(jmax,k-1)
w1 = w(jmax,k )
dwdz = 0.5*(w1*w1-w0*w0)/(z1-z0)

 wrr(k) = dpdp + dwdz
enddo

*** Inlet, no changes
 wrr(1) = 0.

*** Outlet

*** Upwind pressure
 k = kmax
 z0 = z(jmax,k-1)
z1 = z(jmax,k )
dpdp = (pffbc(z1,tmp)-pffbc(z0,tmp))/(z1-z0)

*** Upwind velocity
 w0 = w(jmax,k-1)
w1 = w(jmax,k )
dwdz = 0.5*(w1*w1-w0*w0)/(z1-z0)

 wrr(k) = dpdp + dwdz

*** Integrate in time
 call bc_update(rkcoef(n))
enddo
return
end

subroutine bc_update(ct)
Integrate in time the ffbc

include 'ai.inc'
do k = 1,kmax
Add dissipation terms
tmp = tdel_sav
wbc(k) = wbc0s(k) - ct*tmp*wrr(k)
enddo
return
end

real function pffbc(z_tmpt, t_tmpt)
Pressure value at ffbc

include 'ai.inc'
w_tmpt = 1 - cpff*max(0.,z_tmpt-1.)
pffbc = 1 - 0.5*w_tmpt**2
return
end

Module for solution of the swirl velocity

subroutine v_velocity
Find v given circulation

include 'ai.inc'
do j = 2,jmax
    do k = 1,kmax
        v(j,k) = g(j,k)/r(j,k)
    enddo
enddo
do k = 1,kmax
    v(1,k) = 0.
enddo
return
end

subroutine tridi(u,a,b,c,d,imax)
solves a tri-diagonal matrix of length imax of the form:

\[
\begin{bmatrix}
  d & a & 0 & \cdots & 0 \\
  b & d & a & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & d & a & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & d & 0 & \cdots & d \\
\end{bmatrix}
\begin{bmatrix}
  u \\
  u \\
  \vdots \\
  u \\
  u \\
  u \\
\end{bmatrix}
= \begin{bmatrix}
  c \\
  c \\
  \vdots \\
  c \\
  c \\
  c \\
\end{bmatrix}
\]

where imax is the length of the column vectors.

real u(1),a(1),b(1),c(1),d(1)
do 10 i = 2,imax
\[ d(i) = d(i) - \frac{b(i)}{d(i-1)} \cdot a(i-1) \]
\[ c(i) = c(i) - \frac{b(i)}{d(i-1)} \cdot c(i-1) \]

10 \hspace{1em} \text{continue}
\hspace{1em} \text{u(imax) = c(imax)/d(imax)}
\hspace{1em} \text{do 20 i = imax-1,1,-1}
\hspace{1em} \hspace{1em} \text{u(i) = (c(i) - a(i) \cdot u(i+1))/d(i)}
\hspace{1em} \text{continue}
\hspace{1em} \text{return}
\hspace{1em} \text{end}

end