Distinguishing open symplectic mapping tori via their wrapped Fukaya categories

by

Yusuf Barış Kartal

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Signature redacted

Author ....
Department of Mathematics
March 21, 2019

Signature redacted

Certified by ....
Paul Seidel
Professor
Thesis Supervisor

Signature redacted

Accepted by ....
Davesh Maulik
Chairman, Department Committee on Graduate Theses

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Abstract

The main goal of this thesis is to use homological methods as a step towards the classification of symplectic mapping tori. More precisely, we exploit the dynamics of wrapped Fukaya categories to distinguish an open version of symplectic mapping torus associated to a symplectomorphism from the mapping torus of the identity. As an application, we obtain pairs of diffeomorphic Weinstein domains with the same contact boundary and symplectic cohomology, but that are different as Liouville domains.

This work consists of two parts: in the first part, we define an algebraic model for the wrapped Fukaya category of the open symplectic mapping tori. This construction produces a category, called **the mapping torus category**, for a given dg-category over $\mathbb{C}$ with an autoequivalence. We then use the continuous dynamics of deformations of these categories to distinguish them under certain hypotheses. More precisely, we construct families of bimodules- analogous to flow lines- and use their different periodicity. The construction of the flow uses the geometry of the Tate curve and formal models for the graph of multiplication on $\mathbb{G}^{an}_{m,\mathbb{C}((q))}$.

The second part focuses on the comparison of mapping torus categories and the wrapped Fukaya categories of the open symplectic mapping tori. For this goal, we introduce the notion of "twisted tensor product" and prove a twisted Künneth theorem for the open symplectic mapping tori by using a count of quilted strips. In this part, we also give a large class of Weinstein domains whose wrapped Fukaya category satisfies the conditions for the theorem on mapping torus categories to hold.

Thesis Supervisor: Paul Seidel
Title: Professor

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Contents

List of Figures

Chapter 1

Dynamical invariants of mapping torus categories

1.1 Introduction

1.1.1 Motivation from symplectic geometry

Let M^{2n} be a Weinstein manifold and let ϕ be a symplectomorphism. For simplicity, assume ϕ acts as the identity on the boundary and it is exact with respect to boundary. Associated to this data one can construct the open symplectic mapping torus as

$$T_\phi^{2n+2} := (M \times \mathbb{R} \times S^1/(x,t,s) \sim (\phi(x), t+1, s)) \setminus \{[(x,t,s)] : t = 0, s = 1\} \quad (1.1)$$

This is a symplectic fibration over punctured torus $T_0 = T^2 \setminus \{*\}$ with monodromy as shown in Figure 2-1. It can be shown to carry a Liouville structure and its contact boundary at infinity is isomorphic to that of $T_0 \times M$, in other words the boundary of the mapping torus of identity.

One would like to distinguish the fillings T_ϕ and $T_0 \times M$, when ϕ is not Hamiltonian isotopic to identity. An attempt can be made as follows: Assume the fillings are the

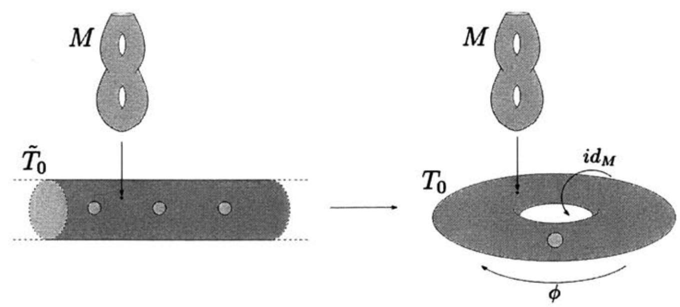

Figure 1-1: T_ϕ and its $\mathbb{Z}$-fold cover $\tilde{T}_0 \times M$

same. Consider the partial compactification

$$\overline{T}_\phi := M \times \mathbb{R} \times S^1/(x,t,s) \sim (\phi(x), t+1, s) \tag{1.2}$$

Assume we are able to identify $\overline{T}_\phi$ with $\overline{T}_{1_M} = T^2 \times M$. Every circle action on T^2 lifts to a circle action on $T^2 \times M$; however, this is not the case with $\overline{T}_\phi$. Indeed, the flow of the obvious lift of ∂_t at time $t = 1$ gives us the symplectomorphism

$$[(x,t,s)] \mapsto [(\phi^{-1}(x), t, s)] \tag{1.3}$$

which is different from the identity. In other words, it seems there are "more circle actions" on $T^2 \times M$ and its flux group is bigger. The first limitation of this approach is our inability to identify the partial compactifications. Second limitation is even if one successfully runs the above program and rigorously computes the flux groups, they would only be able to conclude fiberwise ϕ, the inverse of the symplectomorphism (1.3), is Hamiltonian isotopic to identity. We do not know how to conclude the same for ϕ acting on M.

1.1.2 Categorical construction and the statement of the main theorem

Instead, we follow the analogous idea, but we take a more algebraic route. Start with an A_∞-category $\mathcal{A}$ and an auto-equivalence, which we still denote by ϕ. We construct a category M_ϕ, called the mapping torus category, associated to ϕ. The definition of M_ϕ is inspired by mirror symmetry and it is constructed to be a model for the wrapped Fukaya category $\mathcal{W}(T_\phi)$, when $\mathcal{A} \simeq \mathcal{W}(M)$ and the auto-equivalence corresponds to an auto-equivalence of $\mathcal{W}(M)$ induced by the symplectomorphism.

More precisely, let $\mathcal{A}$ be an A_∞-category over $\mathbb{C}$ and ϕ be an auto-equivalence, i.e. an A_∞-functor $\phi : \mathcal{A} \to \mathcal{A}$ such that $H^*(\phi) : H^*(\mathcal{A}) \to H^*(\mathcal{A})$ is an equivalence. For simplicity assume $\mathcal{A}$ is a dg category and ϕ is a dg functor acting bijectively on objects and hom-complexes of $\mathcal{A}$. Based on this we can construct an A_∞ category M_ϕ over $\mathbb{C}$, and we call it **the mapping torus category of** ϕ.

Briefly, the construction goes as follows. Consider the universal cover of the Tate curve $\tilde{\mathfrak{T}}_0$ whose definition will be recalled in Section 1.2.1 (also see Figure 1-2). It is a nodal infinite chain of projective lines parametrized by $i \in \mathbb{Z}$, and it admits a translation automorphism $\mathfrak{tr}$ which moves one projective line to the next. Consider the bounded derived category of coherent sheaves supported on finitely many projective lines, denoted by $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$. We will construct a dg category $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ whose triangulated envelope is a dg enhancement of $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$. Moreover, it admits a strict dg autoequivalence, still denoted by $\mathfrak{tr}$, which lifts $\mathfrak{tr}_*$. Then, $\mathfrak{tr} \otimes \phi$ endows $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$ with a $\mathbb{Z}$-action, and we define the mapping torus category as

$$M_\phi := (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z} \tag{1.4}$$

The smash product with $\mathbb{Z}$, whose definition will be recalled in Section 1.4, corresponds geometrically to taking the quotient by the $\mathbb{Z}$-action.

The following example justifies the terminology "mapping torus category" from an algebro-geometric perspective:

Example 1.1.1. Let $\mathcal{A}$ be a dg model for $D^b(Coh(X))$, where X is a variety over $\mathbb{C}$ and $\phi = (\phi_X)_*$ for an automorphism $\phi_X \curvearrowright X$. Consider the algebraic space

$$\tilde{\mathfrak{T}}_0 \times X/(t,x) \sim (\mathfrak{tr}(t), \phi_X(x)) \tag{1.5}$$

We expect $tw^\pi(M_\phi)$- idempotent completed twisted (triangulated) envelope- to be a

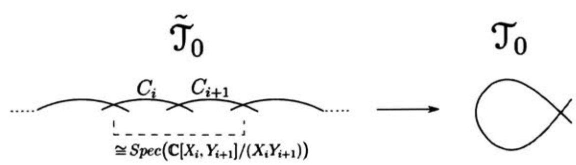

Figure 1-2: The nodal elliptic curve $\mathcal{T}_0$, and its $\mathbb{Z}$-fold covering $\tilde{\mathcal{T}}_0$

dg enhancement of bounded derived category of coherent sheaves on this algebraic space. We showed this in the case $X = Spec(\mathbb{C})$, when the construction gives the nodal elliptic curve $\mathcal{T}_0$ (see Figure 1-2). See Lemma 1.9.9 for this result. Note that this is an algebro-geometric version of the mapping torus and it provides another motivation for the categorical construction.

Remark 1.1.2. The informal mirror symmetry motivation for the construction of M_ϕ is as follows: one knows by [LP16] and [LP12] that the nodal elliptic curve $\mathcal{T}_0$ is mirror dual to T_0. T_ϕ is obtained as a quotient of $\tilde{T}_0 \times M$, where $\tilde{T}_0$ is an infinitely punctured cylinder that is covering T_0 (see also Figure 2-1). Heuristically, one can think of $\tilde{T}_0$ as a mirror to $\tilde{\mathcal{T}}_0$. Assume X is mirror to Weinstein manifold M^{2n}, and an automorphism of X, denoted by ϕ_X, corresponds to ϕ. A natural proposed mirror for T_ϕ is the algebraic space (1.5). M_ϕ is a straightforward categorification of the construction in Example 1.1.1.

Example 1.1.3. If $\phi = 1_{\mathcal{A}}$, M_ϕ is Morita equivalent to $\mathcal{C}oh(\mathcal{T}_0) \otimes \mathcal{A}$, where $\mathcal{C}oh(\mathcal{T}_0)$ is a dg model for $D^b(Coh(\mathcal{T}_0))$. Thus, the category of perfect modules over M_ϕ is equivalent to the category of perfect modules over $\mathcal{C}oh(\mathcal{T}_0) \otimes \mathcal{A}$.

We will assume the following conditions hold throughout the paper:

C.1 $\mathcal{A}$ is (homologically) smooth, see [KS09] for a definition

C.2 $\mathcal{A}$ is proper in each degree and bounded below, i.e. $H^*(hom_{\mathcal{A}}(x, y)) = 0$ is finite dimensional in each degree and vanishes for $* \ll 0$ for any $x, y \in Ob(\mathcal{A})$

C.3 $HH^i(\mathcal{A})$, the i^{th} Hochschild cohomology group of $\mathcal{A}$, is 0 for $i < 0$ and is isomorphic to $\mathbb{C}$ for $i = 0$

Based on this M_ϕ will be shown to satisfy **C.1-C.3** as well.

Now we can state our main theorem:

Theorem 1.1.4. *Let $\mathcal{A}$ be as above and assume further that $HH^1(\mathcal{A}) = HH^2(\mathcal{A}) = 0$. Assume M_ϕ is Morita equivalent to $M_{1_\mathcal{A}}$. Then, $\phi \simeq 1_\mathcal{A}$.*

1.1.3 Sketch of the proof

The proof goes as follows. Assume M_ϕ is Morita equivalent to $M_{1_\mathcal{A}}$. The notion of Morita equivalence will be recalled later in Definition 1.6.28, but we remark that this is equivalent to equivalence of derived categories for A_∞-categories over $\mathbb{C}$. To any categorical mapping torus one can associate a natural formal deformation (with curvature) over the topological local ring $R = \mathbb{C}[[q]]$. We denote this deformation by M_ϕ^R (resp. $M_{1_\mathcal{A}}^R$). Its explicit construction is as follows. There exists a natural smoothing of $\tilde{\mathfrak{T}}_0$, denoted by $\tilde{\mathfrak{T}}_R$ (see Figure 1-4). To this we associate a curved dg category, denoted by $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$, and then apply the same construction as (1.4) replacing $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ by $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. The deformations M_ϕ^R and $M_{1_\mathcal{A}}^R$ have no a priori relation to the Morita equivalence; however, $HH^2(M_\phi) \cong HH^2(M_{1_\mathcal{A}}) \cong \mathbb{C}$, under the assumptions of the theorem and the construction. Hence, there is only one formal deformation that is non-trivial at first order (up to reparametrization). Thus, we may assume without loss of generality that the Morita equivalence deforms to a Morita equivalence between M_ϕ^R and $M_{1_\mathcal{A}}^R$.

M_ϕ^R (resp. M_ϕ) carries a canonical $\mathbb{G}_m(R)$ (resp. $\mathbb{G}_m$)-action for which the infinitesimal action is sensible, and it gives a class $\gamma_\phi^R \in HH^1(M_\phi^R)$ (resp. $\gamma_\phi \in HH^1(M_\phi)$). The action can be considered as a family of M_ϕ^R-bimodules which is parametrized by the formal spectrum of $\mathbb{C}[t,t^{-1}][[q]]$, and which "follows" the class $1 \otimes \gamma_\phi^R$ along $t\partial_t$ direction. This family can be considered as a "short flow line" for $1 \otimes \gamma_\phi^R$, and we extend it to a "longer flow line", i.e. to a family over the formal spectrum of $\mathbb{C}[u,t][[q]]/(ut-q)$. This is the formal completion of $\{ut = 0\} \subset \mathbb{A}^2_\mathbb{C}$ and contains the formal spectrum $Spf(\mathbb{C}[t,t^{-1}][[q]])$ as a formal open subscheme, where the inclusion is induced by $t \mapsto t, u \mapsto qt^{-1}$. See Figure 1-3. To construct the extended

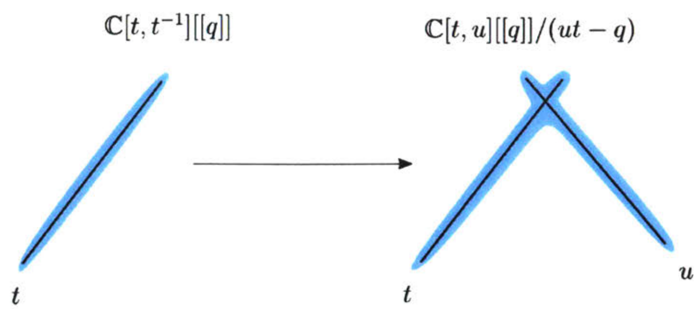

Figure 1-3: Inclusion of $\mathbb{C}[t,t^{-1}][[q]]$ into $\mathbb{C}[u,t][[q]]/(ut-q)$

family we consider a formal subscheme $\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(\mathbb{C}[u,t][[q]]/(ut-q))$ with the following properties:

1. it is flat over $Spf(\mathbb{C}[u,t][[q]]/(ut-q))$
2. it restricts to the graph of $\widehat{\mathbb{G}_{m,R}}$-action (see Remark 1.2.3) over the formal spectrum $Spf(\mathbb{C}[t,t^{-1}][[q]])$
3. it restricts to graph of composition of the inverse action with backwards translation $\mathfrak{tr}^{-1}$ over $Spf(\mathbb{C}[u,u^{-1}][[q]])$

In particular, we obtain the diagonal over the R-point $t=1$ and the graph of backwards translation over $u=1$. To turn this into a family of bimodules over M_ϕ^R, we first define a bimodule

$$\text{“}(\mathcal{F},\mathcal{F}') \mapsto hom_{\tilde{\mathfrak{T}}_R\times\tilde{\mathfrak{T}}_R}(q^*\mathcal{F}, p^*\mathcal{F}'\otimes\mathcal{G}_R)\text{”} \tag{1.6}$$

over $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ and show it naturally descends to $M_\phi^R = (\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}\otimes\mathcal{A})\#\mathbb{Z}$. After some other technical replacements, we obtain a family $\mathcal{G}_R^{sf}$ of bimodules over M_ϕ^R parametrized by $\mathbb{C}[u,t][[q]]/(ut-q)$ satisfying properties **G.1-G.3** below for $\gamma=\gamma_\phi^R$ and which restricts to "fiberwise ϕ" at $u=1$, i.e. to the bimodule corresponding to descent of auto-equivalence $1\otimes\phi$ on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ to M_ϕ^R. Hence, if we can show families constructed in this way correspond to each other under the Morita equivalence between M_ϕ^R and $M_{1_\mathcal{A}}^R$, this would imply the triviality of "fiberwise ϕ" and therefore triviality of ϕ, finishing the proof of the theorem.

For this, we would first need to show the classes γ_ϕ^R and $\gamma_{1_A}^R$ correspond to each other under the isomorphism $HH^1(M_\phi^R) \cong HH^1(M_{1_A}^R)$ induced by the Morita equivalence. To achieve this, we prove in Section 1.8 that these classes fall into natural rank 2 lattices inside $HH^1(M_\phi^R) \cong \mathbb{C}^2$ resp. $HH^1(M_{1_A}^R) \cong \mathbb{C}^2$ that are matched by the Morita equivalence, and show in Section 1.9 that the symmetries of $M_{1_A}^R$ induce $SL_2(\mathbb{Z})$ symmetry on the lattice. Hence, we can use these categorical symmetries to fix the initial Morita equivalence so that the classes γ_ϕ^R and $\gamma_{1_A}^R$ match.

Given this result, one would only need to prove a general theorem that the axioms

G.1 The restriction $\mathfrak{M}|_{q=0}$ is coherent. This is equivalent to its representability by an object of $tw^\pi(\mathcal{B}_0 \otimes \mathcal{B}_0^{op} \otimes \text{“}\mathcal{C}oh(A)\text{”})$. See Definition 1.6.8.

G.2 The restriction $\mathfrak{M}|_{t=1}$ is isomorphic to the diagonal bimodule over $\mathcal{B}$.

G.3 The family follows the class $1 \otimes \gamma \in HH^1(\mathcal{B}^e)$.

uniquely characterize the family once the class γ is chosen. This is achieved in Theorem 1.10.1, namely we show that two families satisfying **G.1**-**G.3** are quasi-isomorphic up to q-torsion. The proof of Theorem 1.10.1 relies on two things: the ideas in [Sei14], which we recall in Section 1.6.2, and the algebra/geometry of modules over $\mathbb{C}[u,t][[q]]/(ut-q)$ which carry connections along the derivation $t\partial_t - u\partial_u$. More explicitly, given two such family $\mathcal{G}_1$ and $\mathcal{G}_2$, we show the hom-complexes in the category of families involving them are chain complexes of $\mathbb{C}[u,t][[q]]/(ut-q)$-modules carrying such connections in each degree that commute with the differentials. Hence, degree 0 homomorphisms in the cohomological category give rank 1 modules with connection, and we show in Section 1.7 that such modules are free up to q-torsion. Following this line of ideas we prove the isomorphism $\mathcal{G}_1|_{t=1} \simeq \mathcal{G}_2|_{t=1}$ extends over $\mathbb{C}[u,t][[q]]/(ut-q)$ to an isomorphism up to q-torsion. This completes the proof.

Now, let us phrase the moral idea for the algebro-geometric minded reader. Consider the algebro-geometric torus given in Example 1.1.1. It has a natural deformation

$$\mathfrak{Y} = \tilde{\mathfrak{T}}_R \times X/(t,x) \sim (\mathfrak{tr}(t), \phi_X(x)) \tag{1.7}$$

which is a fibration over the formal smoothing $\mathfrak{T}_R = \tilde{\mathfrak{T}}_R/t \sim \text{tr}(t)$. Its generic fiber $\mathfrak{Y}_{\mathbb{C}((q))}$ (in the sense of Raynaud, see [Tem15, Section 5]) gives

$$\text{“}\mathbb{G}^{an}_{m,\mathbb{C}((q))} \times X/(t,x) \sim (qt, \phi_X(x))\text{”} \tag{1.8}$$

a rigid analytic version of $\mathbb{C}^* \times X/(t,x) \sim (q_0 t, \phi_X(x))$, where $|q_0| < 1$. There is an action of $\mathbb{G}^{an}_{m,\mathbb{C}((q))}$ on this rigid analytic space; however, it descends to an action of the elliptic curve $\mathbb{G}^{an}_{m,\mathbb{C}((q))}/q$ if and only if $\phi_X = 1_X$. In other words, the trivial mapping torus will be distinguished from the others in that the restriction of the graph of the action to $z = q \in \mathbb{G}^{an}_{m,\mathbb{C}((q))}$ is the diagonal of $\mathfrak{Y}_{\mathbb{C}((q))}$ while in general it is the graph of fiberwise ϕ_X. This action can be thought as analogous to the flow of a vector field. The uniqueness of the family is an analogue of the uniqueness of the flow of a vector field. This is more explicit if we consider the philosophy of Raynaud and realize rigid analytic spaces as formal schemes over $R = \mathbb{C}[[q]]$ up to admissable blow-ups in the special fiber $q = 0$. In particular, the family $\mathcal{G}^{sf}_R$ obtained from the graph

$$\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(\mathbb{C}[u,t][[q]]/(ut-q)) \tag{1.9}$$

morally corresponds to such a degeneration of the graph of action, restricted to a smaller annulus in $\mathbb{G}^{an}_{m,\mathbb{C}((q))}$ afterwards.

Remark 1.1.5. The proof can also be thought as an algebraic version of the argument in Section 1.1.1. The deformation M^R_ϕ is analogous to partial compactification $\overline{T}_\phi$ (see also [Sei02]). The Hochschild cohomology class γ^R_ϕ is an algebraic analogue of the (lift of) vector field ∂_t, and the family $\mathcal{G}^{sf}_R$ is the analogue of its flow. Hence, the restriction of this family to $u = 1$ is analogous to time 1-flow of ∂_t (time (-1)-flow to be precise), giving us “fiberwise ϕ” in both cases. The problem of concluding the triviality of ϕ from the triviality of fiberwise ϕ has an easy solution in categorical version.

1.1.4 Outline

In Sections 1.2 and 1.3 we review the construction of $\tilde{\mathfrak{T}}_0$, $\tilde{\mathfrak{T}}_R$, and present the dg model $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ and its deformation $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. In Section 1.4 we review the smash products and define M_ϕ and M_ϕ^R. Section 1.5 is dedicated to computation of Hochschild cohomology and its results will be referred in other computations later. In Section 1.6, we construct the family $\mathcal{G}_R^{sf}$ and prove it satisfies desired properties. This section also contains a brief review of families. In Section 1.7, we prove some results (such as freeness up to q-torsion) for finitely generated modules over $A_R = \mathbb{C}[u,t][[q]]/(ut-q)$ with connections along $t\partial_t - u\partial_u$. It can be omitted if one accepts the statements there. Sections 1.8 and 1.9 provide us the statements we need to fix the image of $\gamma_\phi \in HH^1(M_\phi) \cong \mathbb{C}^2$ under the Morita equivalence. In Section 1.8, we show that the classes that are obtained as the infinitesimal action of a $\mathbb{G}_m$ (resp. $\mathbb{G}_m(R)$)-action on M_ϕ (resp. M_ϕ^R) form a copy of $\mathbb{Z}^2$ inside $HH^1(M_\phi)$ (resp. $HH^1(M_\phi^R) \cong R^2$) generated by basis elements. This "cocharacter lattice" is obviously preserved under Morita equivalences, and Section 1.9 provides us symmetries of the categories acting transitively on primitive elements of the lattice. In Section 1.10, we finally conclude the proof of uniqueness(up to q-torsion) of families satisfying **G.1-G.3** and the proof of Theorem 1.1.4. In the final section, Section 1.11, we relate the growth rates of $rk(HH^*(M_\phi^R, \Phi_f^k))$, where Φ_f^k is the bimodule kernel of fiberwise ϕ^k, to growth rates for ϕ.

1.1.5 Generalizations and future work

We believe the following generalization of Theorem 1.1.4 holds:

Conjecture. *Assume $\mathcal{A}$ is as in Theorem 1.1.4. Let ϕ and ϕ' be two auto-equivalences satisfying the stated conditions and assume M_ϕ and $M_{\phi'}$ are Morita equivalent. Then ϕ and ϕ' have the same order.*

Indeed, we believe the order of ϕ would be the index of the subgroup of the lattice given by the elements for which the restriction of the corresponding family to "$u = 1$" is the diagonal. The reason we put $u = 1$ in quotation marks is the possibility that one needs to use a base different from $Spf(\mathbb{C}[u,t][[q]]/(ut-q))$ for the other lattice elements.

Let us finish by mentioning work in progress and applications. As mentioned, the categorical mapping torus is constructed as a model for $\mathcal{W}(T_\phi)$ and this will be shown in [Kar]. Assume this for the moment. As mentioned, if the symplectomorphism ϕ restricts to identity on the boundary, then $\partial T_\phi = \partial T_{1_M}$ as contact manifolds. If $\pi_1(M) = 1$, one can attach subcritical handles to T_ϕ to make the first cohomology vanish. This process does not change the wrapped Fukaya category due to results of [Iri13], [Cie02], [BEE12]; hence, the filings are still different as a corollary of Theorem 1.1.4. However, no argument involving symplectic flux can be used to distinguish the fillings since they have vanishing first cohomology.

Notational remarks

R will always denote $\mathbb{C}[[q]]$ with the q-adic topology. Similarly, $A_R = \mathbb{C}[u,t][[q]]/(ut-q)$ with the q-adic topology and $A = \mathbb{C}[u,t]/(ut) = A_R/(q)$. $Spf(B)$ denotes the formal spectrum of a complete topological ring B equipped with I-adic topology for an ideal $I \subset B$. This is a ringed space whose underlying topological space is $Spec(B/I)$(which is homeomorphic to $Spec(B/I^m)$ for any $m > 0$) and whose ring of global functions is the topological ring $\varprojlim B/I^n$. For more details see [Bos14]. Note, in our paper most formal affine schemes are completions of varieties along a closed subvariety.

Constructions/concepts over $R = \mathbb{C}[[q]]$ are implicitly assumed to be q-adically completed and continuous. This applies to categories over R, Hochschild cochains $CC^*(\mathcal{B})$ of such categories, and to tensor products of topological complete modules over R. For instance if M and N are such modules, $M \otimes N$ refers to $M \hat{\otimes}_R N$, which is the q-adic completion of $M \otimes N$. If M is over R and N is over $\mathbb{C}$, $M \otimes N$ refers to q-adic completion of $M \otimes_{\mathbb{C}} N$. We also mostly drop the subscripts of tensor products from the notation. Similarly, the base of products of schemes or formal schemes are written only when it is unclear (for instance $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$ refers to fiber product over $Spf(R)$).

We have elaborated on the definition of $\tilde{\mathcal{T}}_0$ in Section 1.2 (see also Figure 1-4). Indeed one can take

$$\mathcal{T}_0 := \tilde{\mathcal{T}}_0/t \sim \mathrm{tr}(t) \tag{1.10}$$

as the definition. For an explicit equation defining $\mathcal{T}_0$, see [LP12].

Given an ordinary algebra B, $\mathcal{C}_{dg}(B)$ denotes the dg category of chain complexes over B.

Given dg categories $\mathcal{B}$ and $\mathcal{B}'$, we can their tensor product category as a category with objects $Ob(\mathcal{B}) \times Ob(\mathcal{B}')$. Let $b \times b'$ denote the corresponding object of $\mathcal{B} \otimes \mathcal{B}'$ for given $b \in Ob(\mathcal{B}), b' \in Ob(\mathcal{B}')$. Morphisms satisfy

$$(\mathcal{B} \otimes \mathcal{B}')(b_1 \times b_1', b_2 \times b_2') = \mathcal{B}(b_1, b_2) \otimes \mathcal{B}'(b_1', b_2') \tag{1.11}$$

See [Kel06] for more details.

For a given A_∞-category $\mathcal{B}$, $tw(\mathcal{B})$ stands for the category of twisted complexes over $\mathcal{B}$ and $tw^\pi(\mathcal{B})$ stands for the split-closure (a.k.a. idempotent completion) of $tw(\mathcal{B})$. For a definition see [Sei08c, Chapter I.3,I.4]. $D^\pi(\mathcal{B})$ stands for the triangulated category $H^0(tw^\pi(\mathcal{B}))$. A dg/A_∞ enhancement of a triangulated category D is a dg/A_∞ category $\mathcal{B}$ such that D is equivalent to $H^0(\mathcal{B})$ as a triangulated category.

By generation, we mean split generation unless specified otherwise. See [Sei08c, Chapter I.4]. We used the notations $CC^*(\mathcal{B})$ and $CC^*(\mathcal{B},\mathcal{B})$ interchangeably. They both stand for the Hochschild complex of an A_∞-category $\mathcal{B}$. See [Sei15], [Sei13]. The notation $Bimod(\mathcal{B},\mathcal{B}')$ is used to mean the dg category of A_∞-bimodules over $\mathcal{B}$-$\mathcal{B}'$. There is a functor

$$\begin{gathered} Bimod(\mathcal{B},\mathcal{B}) \to \mathcal{C}_{dg}(\mathbb{C}) \\ \mathfrak{M} \mapsto CC^*(\mathcal{B},\mathfrak{M}) \end{gathered} \tag{1.12}$$

which is naturally quasi-isomorphic to Yoneda functor of the diagonal bimodule. In the case of an A_∞-algebra over $\mathbb{C}$, $CC^*(\mathcal{B},\mathfrak{M})$ has underlying graded vector space $\bigoplus_{i\geq 0} hom_{\mathbb{C}}(\mathcal{B}^{\otimes i},\mathfrak{M})[-i] = hom_{\mathbb{C}}(T\mathcal{B}[1],\mathfrak{M})$, where $T\mathcal{B}[1] = \bigoplus_{i\geq 0}\mathcal{B}^{\otimes i}[i]$(which is also defined in Section 1.6). We note that this direct sum means each degree of each summand is summed separately. Also, as remarked before the constructions take place in the category of completed R-modules in the case $\mathcal{B}$ is a curved category over R. For instance, $hom(\mathcal{B}^{\otimes i}, \mathfrak{M})$ only involves convergent sums of continuous homomorphisms and direct sums are assumed to be q-adically completed. For the differential of $CC^*(\mathcal{B}, \mathfrak{M})$, which involves $\mu_{\mathcal{B}}$ and $\mu_{\mathfrak{M}}$, see [Sei13, Remark 9.2].

For more homological algebra preliminaries see [Kel06], [Sei15],[Sei08c], [Sei13].

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1.2 The universal cover of the Tate curve

1.2.1 Reminder on the construction of $\tilde{\mathfrak{T}}_R$

We first review the construction of $\tilde{\mathfrak{T}}_R$ following [LP12]. We slightly change the notation. Recall R is $\mathbb{C}[[q]]$ endowed with q-adic topology.

Given $i \in \mathbb{Z}$, let $\bar{U}_{i+1/2}$ denote $Spec\big(\mathbb{C}[q][X_i, Y_{i+1}]/(X_iY_{i+1} - q)\big)$. It is a scheme over $Spec(\mathbb{C}[q])$, and it is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ as a scheme over $\mathbb{C}$. Moreover,

$$\bar{U}_{i+1/2}[X_i^{-1}] \cong Spec\big(\mathbb{C}[q][X_i, X_i^{-1}]\big) \tag{1.13}$$

is isomorphic to

$$\bar{U}_{i-1/2}[Y_i^{-1}] \cong Spec\big(\mathbb{C}[q][Y_i, Y_i^{-1}]\big) \tag{1.14}$$

as a scheme over $\mathbb{C}[q]$. Denote this scheme by $\bar{V}_i$. The isomorphism is given by the coordinate change $X_i \leftrightarrow Y_i^{-1}$. In other words, the coordinates X_i and Y_i satisfy

$X_iY_i = 1$ on $\bar{V}_i$.

By using the identifications $\bar{U}_{i+1/2}[X_i^{-1}] \cong \bar{U}_{i-1/2}[Y_i^{-1}]$, we can glue $\bar{U}_{i+1/2}$, $i \in \mathbb{Z}$. Hence, we obtain a scheme over $Spec(\mathbb{C}[q])$, which we denote by $\tilde{\mathfrak{T}}_{\mathbb{C}[q]}$. It is not Noetherian and it is covered by charts $\bar{U}_{i+1/2}$, $i \in \mathbb{Z}$.

Note, there is a $\mathbb{G}_{m,\mathbb{C}[q]}$-action over $\mathbb{C}[q]$ on this scheme. Locally, the action is given by

$$Y_{i+1} \mapsto tY_{i+1} \text{ and } X_i \mapsto t^{-1}X_i \tag{1.15}$$

where t is the coordinate of $\mathbb{G}_{m,\mathbb{C}[q]}$.

We will mainly be interested in

$$\tilde{\mathfrak{T}}_0 := \tilde{\mathfrak{T}}_{\mathbb{C}[q]}|_{q=0} = \tilde{\mathfrak{T}}_{\mathbb{C}[q]} \times_{Spec(\mathbb{C}[q])} Spec(\mathbb{C}[q]/(q)) \tag{1.16}$$

and its formal completion inside $\tilde{\mathfrak{T}}_{\mathbb{C}[q]}$. We denote this formal completion by

$$\tilde{\mathfrak{T}}_R := \tilde{\mathfrak{T}}_{\mathbb{C}[q]} \times_{Spec(\mathbb{C}[q])} Spf(R) \tag{1.17}$$

where the fiber product is taken with respect to the obvious morphism

$$Spf(R) \to Spec(\mathbb{C}[q]) \tag{1.18}$$

(Recall, $Spf(R)$ denotes the formal spectrum of the topological ring $R = \mathbb{C}[[q]]$.)

Let $U_{i+1/2} := \bar{U}_{i+1/2}|_{q=0}$ and $\tilde{U}_{i+1/2} := \bar{U}_{i+1/2} \times_{Spec(\mathbb{C}[q])} Spf(\mathbb{C}[[q]])$. In the coordinates above,

$$U_{i+1/2} = Spec(\mathbb{C}[X_i, Y_{i+1}]/(X_iY_{i+1})) \tag{1.19}$$

and

$$\tilde{U}_{i+1/2} = Spf(\mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q)) \tag{1.20}$$

respectively. In the latter, the formal spectrum is taken with respect to q-adic topology. Let

$$\tilde{V}_i := \tilde{U}_{i-1/2} \cap \tilde{U}_{i+1/2} = \bar{V}_i \times_{Spec(\mathbb{C}[q])} Spf(\mathbb{C}[[q]]) \tag{1.21}$$

Figure 1-4: The inclusion of $\tilde{\mathfrak{T}}_0$ into $\tilde{\mathfrak{T}}_R$

Notation. Let $j_{U_{i+1/2}}$, resp. j_{V_i} denote the inclusion of the open set $U_{i+1/2}$ resp. V_i. Similarly, let $j_{\tilde{U}_{i+1/2}}$ and $j_{\tilde{V}_i}$ denote the open inclusions into $\tilde{\mathfrak{T}}_R$.

Remark 1.2.1. It is easy to see that $\tilde{\mathfrak{T}}_0$ is an infinite chain of projective lines. Let C_i denote the projective line given as the union of $\{(X_{i-1}, Y_i) \in U_{i-1/2} : X_{i-1} = 0\}$ and $\{(X_i, Y_{i+1}) \in U_{i+1/2} : Y_{i+1} = 0\}$. Its affine charts have coordinates X_i and Y_i satisfying $X_iY_i = 1$ on the overlap $V_i := \bar{V}_i|_{q=0} \subset C_i$. See Figure 1-4.

Definition 1.2.2. Define the **translation automorphism** on $\tilde{\mathfrak{T}}_R$(resp. $\tilde{\mathfrak{T}}_0$) to be the automorphism given by the local transformations $\tilde{U}_{i-1/2} \to \tilde{U}_{i+1/2}$ (resp. $U_{i-1/2} \to U_{i+1/2}$) given by

$$X_i \mapsto X_{i-1}, Y_{i+1} \mapsto Y_i \tag{1.22}$$

on the coordinate rings. Denote both of them by $\mathbf{tr}$.

Remark 1.2.3. Restricting the $\mathbb{G}_{m,\mathbb{C}[q]}$-action in (1.15) along $Spf(R) \to Spec(\mathbb{C}[q])$, we obtain an action of $\widehat{\mathbb{G}_m} := Spf(\mathbb{C}[t, t^{-1}][[q]])$ on $\tilde{\mathfrak{T}}_R$ in the category of formal schemes over R. Similarly, restricting the $\mathbb{G}_{m,\mathbb{C}[q]}$-action along $0 : Spec(\mathbb{C}) \to Spec(\mathbb{C}[q])$, we obtain an action of $\mathbb{G}_m := \mathbb{G}_{m,\mathbb{C}}$ on $\tilde{\mathfrak{T}}_0$ in the category of schemes over $\mathbb{C}$.

1.2.2 Multiplication graph of $\tilde{\mathfrak{T}}_R$

Raynaud's insight provided a picture of (some) rigid analytic spaces over $\mathbb{C}((q))$ as generic fibers of formal schemes over $\mathbb{C}[[q]]$. In this view, the analytification of $\mathbb{G}_{m,\mathbb{C}((q))}$ can be obtained as the generic fiber of $\tilde{\mathfrak{T}}_R$. But, the analytification $\mathbb{G}^{an}_{m,\mathbb{C}((q))}$ is a group and this suggests finding a morphism of formal schemes

$$\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \to \tilde{\mathfrak{T}}_R \tag{1.23}$$

giving the group multiplication

$$\mathbb{G}_{m,\mathbb{C}((q))}^{an} \times \mathbb{G}_{m,\mathbb{C}((q))}^{an} \to \mathbb{G}_{m,\mathbb{C}((q))}^{an} \tag{1.24}$$

in the generic fiber. This could be possible after admissible blow-ups on the special fiber of $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$, but instead, we will write an explicit formal subscheme of $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$ over $Spf(R)$, which presumably gives the graph of multiplication when the generic fiber functor is applied. We emphasize that we will not show this and there will be no formal references to Raynaud's view or to rigid analytic spaces, as it is not needed for our purposes. Interested reader may see [Bos14] or [Tem15]) for more details.

Definition 1.2.4. Let $\mathcal{G}_{l,R}$ be the formal subscheme of $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$ locally given by the following equations

$$\begin{aligned} Y_i(1)Y_j(2) &= Y_{i+j}(3), & Y_j(2)X_{i+j}(3) &= X_i(1) \\ Y_i(1)X_{i+j}(3) &= X_j(2), & Y_j(2) &= X_i(1)Y_{i+j}(3) \\ Y_i(1) &= X_j(2)Y_{i+j}(3), & X_{i+j}(3) &= X_i(1)X_j(2) \end{aligned} \tag{1.25}$$

and by the equations

$$Y_i(1)Y_j(2)X_{i+j}(3) = 1 \quad X_i(1)X_j(2)Y_{i+j}(3) = 1 \tag{1.26}$$

Here, $X_i(1), Y_i(1)$ are the local coordinates of the first component, $X_i(2), Y_i(2)$ are of the second and $X_i(3), Y_i(3)$ are of the third. For fixed i and j, each of these equations make sense only on one chart of type $\tilde{U}_{k+1/2} \times \tilde{U}_{l+1/2} \times \tilde{U}_{m+1/2}$. Hence, $\mathcal{G}_{l,R}$ is the formal subscheme given on the chart $\tilde{U}_{k+1/2} \times \tilde{U}_{l+1/2} \times \tilde{U}_{m+1/2}$ by all the equations (1.25) and (1.26) for all i, j that make sense on this chart. If none of these makes sense (i.e. for all equations as above there is at least one local coordinate involved in the equation and that is not defined on the chart), we take the subscheme to be empty on that chart.

Example 1.2.5. For instance $Y_i(1)Y_j(2) = Y_{i+j}(3)$ makes sense on $\tilde{U}_{i-1/2} \times \tilde{U}_{j-1/2} \times \tilde{U}_{i+j-1/2}$ and $Y_j(2)X_{i+j}(3) = X_i(1)$ makes sense on $\tilde{U}_{i+1/2} \times \tilde{U}_{j-1/2} \times \tilde{U}_{i+j+1/2}$. The other equations that make sense on $\tilde{U}_{i-1/2} \times \tilde{U}_{j-1/2} \times \tilde{U}_{i+j-1/2}$ are $Y_j(2)X_{i+j-1}(3) = X_{i-1}(1)$ and $Y_i(1)X_{i+j-1}(3) = X_{j-1}(2)$.

Remark 1.2.6. There is an S_3-symmetry of the coordinates preserving equations, which would become more obvious after the coordinate change

$$X_i(3) \leftrightarrow Y_{-i}(3), X_{-i}(3) \leftrightarrow Y_i(3) \tag{1.27}$$

After the coordinate change, the symmetry is given by permuting the components of $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$.

We still need to check:

Lemma 1.2.7. *Equations (1.25) and (1.26) give a well-defined formal subscheme of* $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$.

Proof. We need to check the formal subschemes match in the intersections of charts $\tilde{U}_{k'+1/2} \times \tilde{U}_{l'+1/2} \times \tilde{U}_{m'+1/2}$ and $\tilde{U}_{k''+1/2} \times \tilde{U}_{l''+1/2} \times \tilde{U}_{m''+1/2}$. Assuming the intersection is non-empty and charts are different, we see that $k' \neq k''$, $l' \neq l''$ or $m' \neq m''$. Without loss of generality assume $l' \neq l''$, $l' = -1$ and $l'' = 0$. Hence, their intersection lives inside

$$\tilde{\mathfrak{T}}_R \times \tilde{V}_0 \times \tilde{\mathfrak{T}}_R = \tilde{\mathfrak{T}}_R \times (\tilde{U}_{-1/2} \cap \tilde{U}_{1/2}) \times \tilde{\mathfrak{T}}_R \tag{1.28}$$

Notice that the intersection of the subscheme defined on a specific chart $\tilde{U}_{k+1/2} \times \tilde{U}_{l+1/2} \times \tilde{U}_{m+1/2}$ with $\tilde{\mathfrak{T}}_R \times \tilde{V}_0 \times \tilde{\mathfrak{T}}_R$ is the same as the graph of the action

$$\tilde{\mathfrak{T}}_R \times Spf(\mathbb{C}[t, t^{-1}][[q]]) \to \tilde{\mathfrak{T}}_R \tag{1.29}$$

intersected with that chart. The action is still locally given by

$$Y_{i+1} \mapsto tY_{i+1} \text{ and } X_i \mapsto t^{-1}X_i \tag{1.30}$$

and we identify $\tilde{V}_0$ with $Spf(\mathbb{C}[t, t^{-1}][[q]])$ by putting $t = Y_0$.

Hence, the restriction of the graphs defined on $\tilde{U}_{k'+1/2} \times \tilde{U}_{l'+1/2} \times \tilde{U}_{m'+1/2}$ or $\tilde{U}_{k''+1/2} \times \tilde{U}_{l''+1/2} \times \tilde{U}_{m''+1/2}$ can be obtained by restricting the graph of the action above to $(\tilde{U}_{k'+1/2} \times \tilde{U}_{l'+1/2} \times \tilde{U}_{m'+1/2}) \cap (\tilde{U}_{k''+1/2} \times \tilde{U}_{l''+1/2} \times \tilde{U}_{m''+1/2})$. This implies they are the same. □

We will confine ourselves to $\mathcal{G}_{l,R} \cap \tilde{\mathfrak{T}}_R \times \tilde{U}_{-1/2} \times \tilde{\mathfrak{T}}_R$. Put $u = X_{-1}, t = Y_0$ and put $X_i = X_i(1), X'_i = X_i(3), Y_{i+1} = Y_{i+1}(1), Y'_{i+1} = Y_{i+1}(3)$. Moreover, we interchange the second and third coordinates to obtain a formal subscheme $\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(\mathbb{C}[u,t][[q]]/(ut-q))$, where the formal spectrum is taken with respect to q-adic topology. The topological algebra $\mathbb{C}[u,t][[q]]/(ut-q)$ will appear recurrently, so let us name it:

Notation. $A_R := \mathbb{C}[u,t][[q]]/(ut-q)$ with its q-adic topology and $A := \mathbb{C}[u,t]/(ut)$.

$\mathcal{G}_R$ is given by the equations

$$tY_{i+1} = Y'_{i+1}, tX'_i = X_i, Y_{i+1}X'_i = u \text{ on } \tilde{U}_{i+1/2} \times \tilde{U}_{i+1/2} \times Spf(A_R) \tag{1.31}$$

$$Y_{i+1} = uY'_i, X'_{i-1} = uX_i, Y'_iX_i = t \text{ on } \tilde{U}_{i+1/2} \times \tilde{U}_{i-1/2} \times Spf(A_R) \tag{1.32}$$

Remark 1.2.8. Equations (1.31) and (1.32) are merely translations of the equations (1.25) into new variables, and the equations (1.26) are not needed for the definition. $\mathcal{G}_R$ is covered by its open subschemes defined in (1.31) and (1.32).

Lemma 1.2.9. *$\mathcal{G}_R$ is flat over $A_R = \mathbb{C}[u,t][[q]]/(ut-q)$.*

Proof. We show this only for the formal subscheme of $\tilde{U}_{i+1/2} \times \tilde{U}_{i+1/2} \times Spf(A_R)$ defined by (1.31). The part defined by (1.32) is similar.

Notice the equations $tY_{i+1} = Y'_{i+1}, tX'_i = X_i, Y_{i+1}X'_i = u$ define a subscheme of

$$\begin{gathered}\mathbb{C}[X_i, Y_{i+1}][q]/(X_iY_{i+1} - q) \times_{\mathbb{C}[q]} \mathbb{C}[X'_i, Y'_{i+1}][q]/(X'_iY'_{i+1} - q) \\ \times_{\mathbb{C}[q]}\mathbb{C}[u,t][q]/(ut-q)\end{gathered} \tag{1.33}$$

whose formal completion along $q = 0$ gives (part of) $\mathcal{G}_R$. Indeed, it is isomorphic to the subscheme of $Spec(\mathbb{C}[X_i, Y_{i+1}, X'_i, Y'_{i+1}, u, t])$ given by the same equations(equations (1.31) imply $X_iY_{i+1} = X'_iY'_{i+1} = ut$). As $tY_{i+1} = Y'_{i+1}$ and $tX'_i = X_i$, we can see it as the subscheme of $Spec(\mathbb{C}[Y_{i+1}, X'_i, u]) \times Spec(\mathbb{C}[t])$ given by the equation $Y_{i+1}X_i' = u$. $Spec(\mathbb{C}[Y_{i+1}, X_i', u]/(Y_{i+1}X_i' - u))$ is flat over $\mathbb{C}[u]$; hence, $Spec(\mathbb{C}[Y_{i+1}, X_i', u, t]/(Y_{i+1}X_i' - u))$ is flat over $\mathbb{C}[u, t] \cong \mathbb{C}[u, t][q]/(ut - q)$ and so is its formal completion along $q = 0$. □

Remark 1.2.10. In the same way, we can show $\mathcal{G}_{l,R}$ is flat with respect to all three projections to $\tilde{\mathfrak{T}}_R$.

Notation. Let $\mathcal{G} := \mathcal{G}_R|_{q=0} \subset \tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0 \times Spec(A)$. It follows from Lemma 1.2.9 that $\mathcal{G}$ is flat over A.

1.3 A dg model for the universal cover of the Tate curve

1.3.1 The dg model $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$

In this section we construct a dg category $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ such that

$$D^{\pi}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \simeq D^b(Coh_p(\tilde{\mathfrak{T}}_0)) \tag{1.34}$$

where $Coh_p(\tilde{\mathfrak{T}}_0)$ is the abelian category of properly supported coherent sheaves $\tilde{\mathfrak{T}}_0$. We will take $Ob(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) := \{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$, where $\mathcal{O}_{\mathbb{C}_i}$ denotes the structure sheaf of the closed subvariety C_i and $\mathcal{O}_{C_i}(-1)$ denotes the structure sheaf twisted by a smooth point on C_i(it does not matter which). First we show

Lemma 1.3.1. *$\{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$ generates $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$ as a triangulated category.*

Proof. It is enough to show that every $\mathcal{F} \in Coh_p(\tilde{\mathfrak{T}}_0)$ is in the full subcategory generated by $\{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$. Let $i_n : C_n \to \tilde{\mathfrak{T}}_0$ denote the inclusion for a given $n \in \mathbb{Z}$. Consider $\mathcal{F} \to i_{n*}i_n^*\mathcal{F}$, where i_n^* refers to ordinary (not derived) pull-back. Note, i_{n*} does not need to be derived as i_n is affine. The sheaf $i_{n*}i_n^*\mathcal{F}$ is in the image $i_{n*}(D^b(Coh(C_n)))$, which is generated by $\mathcal{O}_{C_n}, \mathcal{O}_{C_n}(-1)$ as $C_n \cong \mathbb{P}^1$, $\mathcal{O}_{C_n} = i_{n*}\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{C_n}(-1) = i_{n*}\mathcal{O}_{\mathbb{P}^1}(-1)$. Hence, to finish the proof, we only need to show the kernel and the cokernel of the map $\mathcal{F} \to \bigoplus_{n\in\mathbb{Z}} i_{n*}i_n^*\mathcal{F}$ are in this category. But, both the kernel and the cokernel are finite direct sums of coherent sheaves supported on the nodes. Any such coherent sheaf can be filtered so that the subquotients are isomorphic to the structure sheaves of the nodes. Hence, they can be seen as iterated extensions of the structure sheaves of the nodal points, and the structure sheaf of the node is in $i_{n*}(D^b(Coh(C_n)))$(as the cokernel of a map $\mathcal{O}_{C_n}(-1) \to \mathcal{O}_{C_n}$). Hence, they are all in the triangulated subcategory generated by $\{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$. $\square$

To find an enhancement of $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$, we will closely follow [LS16]. First some generalities:

Let X be a separated scheme over $\mathbb{C}$, which is locally of finite type. Let $\{U_\alpha\}$ be an open cover, where the index set is ordered. Assume, every quasi-compact subset intersects only finitely many U_α. Let $\mathcal{F}$ be a sheaf on X; and for a given open subset $j : V \hookrightarrow X$ define

$$^V\mathcal{F} := j_!j^*(\mathcal{F}) \tag{1.35}$$

Also define

$$\mathcal{C}_!(\mathcal{F}) := \left\{ \cdots \to \prod_{\alpha_1<\alpha_2} \left(^{U_{\alpha_1}\cap U_{\alpha_2}}\mathcal{F} \right) \to \prod_{\alpha} \left(^{U_\alpha}\mathcal{F} \right) \right\} \xrightarrow{\simeq} \mathcal{F} \tag{1.36}$$

For the differential of this complex and exactness see [LS16] and references there-in. In our situation we will choose a cover so that triple intersections will be empty. The differential is given by maps

$$\left(^{U_{\alpha_1}\cap U_{\alpha_2}}\mathcal{F} \right) \to \left(^{U_{\alpha_1}}\mathcal{F} \right) \times \left(^{U_{\alpha_2}}\mathcal{F} \right) \tag{1.37}$$

on the factors, which are the differences of the natural maps $\left(^{U_{\alpha_1}\cap U_{\alpha_2}}\mathcal{F} \right) \to \left(^{U_{\alpha_i}}\mathcal{F} \right)$, $i = 1, 2$.

Now, assume the U_α are affine and their triple intersections are empty. We will modify the resolutions as follows: for each finite subset $I \subset \{\alpha\}$, fix a free resolution of $j_{U_I}^*\mathcal{F}$, where j_{U_I} is the inclusion of $U_I = \bigcap_{\alpha\in I} U_\alpha$. This extends to a double resolution over $\mathcal{C}_!(\mathcal{F})$, where $\mathcal{C}_!(\mathcal{F})$ is assumed to lie in the horizontal direction. Take its total complex to obtain a resolution of $\mathcal{F}$ by sums of sheaves of the form $j_!(E)$, where $j : V \to X$ is an open embedding and E is a vector bundle on V. We denote this bounded above complex of $\mathcal{O}_X$-modules by $R(\mathcal{F})$, suppressing the data of resolutions and maps between them in the notation.

From now on let $X = \tilde{\mathfrak{T}}_0$ and the covering be $\{U_{i+1/2}\}_{i\in\mathbb{Z}}$. Consider $\mathcal{O}_{C_i}(a)$, where $i, a \in \mathbb{Z}$. The complex $\mathcal{C}_!(\mathcal{O}_{C_i}(a))$, as a graded sheaf, is a shifted sum of $j_{V_i,!}\mathcal{O}_{V_i}$, $j_{U_{i-1/2},!}\mathcal{O}_{C_i\cap U_{i-1/2}}$ and $j_{U_{i+1/2},!}\mathcal{O}_{C_i\cap U_{i+1/2}}$. Note that to write it this way, we need to choose trivializations of $\mathcal{O}_{C_i}(a)|_{V_i}$, $\mathcal{O}_{C_i}(a)|_{U_{i-1/2}}$ and $\mathcal{O}_{C_i}(a)|_{U_{i+1/2}}$. Choose them together so that $\mathfrak{tr}$ moves the trivializations for $\mathcal{O}_{C_i}(a)$ to these for $\mathcal{O}_{C_i+1}(a)$. Under the natural isomorphism

$$U_{i+1/2} \cong Spec(\mathbb{C}[X_i, Y_{i+1}]/(X_iY_{i+1})) \tag{1.38}$$

$\mathcal{O}_{C_i\cap U_{i+1/2}}$ corresponds to the module $\mathbb{C}[X_i, Y_{i+1}]/(X_iY_{i+1}, Y_{i+1})$. Similarly, $\mathcal{O}_{C_i\cap U_{i-1/2}}$ corresponds to $\mathbb{C}[X_{i-1}, Y_i]/(X_{i-1}Y_i, X_{i-1})$. Let the free resolution of $\mathcal{O}_{V_i}$ be the trivial one. Also, let the other resolutions be

$$\ldots \xrightarrow{Y_i} \mathcal{O}(U_{i-1/2}) \xrightarrow{X_{i-1}} \mathcal{O}(U_{i-1/2}) \to \mathcal{O}(U_{i-1/2})/(X_{i-1}) \tag{1.39}$$

$$\ldots \xrightarrow{X_i} \mathcal{O}(U_{i+1/2}) \xrightarrow{Y_{i+1}} \mathcal{O}(U_{i+1/2}) \to \mathcal{O}(U_{i+1/2})/(Y_{i+1}) \tag{1.40}$$

The only non-zero horizontal arrow in the double resolution is

$$j_{V_i,!}\mathcal{O}_{V_i} \to j_{U_{i-1/2},!}\mathcal{O}_{U_{i-1/2}} \times j_{U_{i+1/2},!}\mathcal{O}_{U_{i+1/2}} \tag{1.41}$$

lifting

$$j_{V_i,!}\mathcal{O}_{C_i\cap V_i} \to j_{U_{i-1/2},!}\mathcal{O}_{C_i\cap U_{i-1/2}} \times j_{U_{i+1/2},!}\mathcal{O}_{C_i\cap U_{i+1/2}} \tag{1.42}$$

It is determined by an element in $\mathbb{C}[X_i, X_i^{-1}] \times \mathbb{C}[X_i, X_i^{-1}]$. Choose the horizontal arrows simultaneously for all i so that they are compatible with $\mathfrak{tr}$, in the sense above (i.e. the chosen arrows for C_i will move to C_{i+1} under $\mathfrak{tr}$).

In summary, applying the above procedure of finding double resolutions and totalizations, we find complexes of sheaves $R(\mathfrak{F})$ supported in non-positive degree and quasi-isomorphisms

$$R(\mathfrak{F}) \xrightarrow{\simeq} \mathfrak{F} \tag{1.43}$$

Definition 1.3.2. Let $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ be the full dg subcategory of complexes of $\mathcal{O}_{\tilde{\mathfrak{T}}_0}$-modules that is spanned by objects $R(\mathcal{O}_{C_i}(-1))$ and $R(\mathcal{O}_{C_i})$. We will denote these objects by $\mathcal{O}_{C_i}(-1)$ and $\mathcal{O}_{C_i}$ as well.

Proposition 1.3.3. *$tw^\pi(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})$ is a dg-enhancement of $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$.*

Proof. First, start by noting that $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$ is equivalent to $D^b_{coh,p}(\mathcal{O}_{\tilde{\mathfrak{T}}_0})$, the full subcategory of $D^b(\mathcal{O}_{\tilde{\mathfrak{T}}_0})$ spanned by objects whose hypercohomology sheaves are in $Coh_p(\tilde{\mathfrak{T}}_0)$. This can be shown using [Huy06, Cor 3.4,Prop 3.5] and the fact that $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$ is a union of subcategories equivalent to derived categories of properly supported coherent sheaves on open Noetherian subschemes of $\tilde{\mathfrak{T}}_0$. Hence, we will actually work with the latter category.

We need to show the natural map

$$Hom_{K(\mathcal{O}_{\tilde{\mathfrak{T}}_0})}(R(\mathfrak{F}), R(\mathfrak{F}')) \to Hom_{D(\mathcal{O}_{\tilde{\mathfrak{T}}_0})}(R(\mathfrak{F}), R(\mathfrak{F}')) \tag{1.44}$$

is an isomorphism. Here, $\mathfrak{F}$ and $\mathfrak{F}'$ are among $\{\mathcal{O}_{C_i}, \mathcal{O}_{C_i}(-1) : i \in \mathbb{Z}\}$ and $K(\mathcal{O}_{\tilde{\mathfrak{T}}_0})$ denotes the homotopy category of complexes of $\mathcal{O}_{\tilde{\mathfrak{T}}_0}$-modules.

First note

$$Hom_{K(\mathcal{O}_{\tilde{\mathfrak{T}}_0})}(R(\mathfrak{F}), \mathfrak{F}') \simeq Hom_{D(\mathcal{O}_{\tilde{\mathfrak{T}}_0})}(R(\mathfrak{F}), \mathfrak{F}') \tag{1.45}$$

To see this choose a resolution $\mathfrak{F}' \to I^\cdot$ by quasi-coherent sheaves that are injective as $\mathcal{O}_{\tilde{\mathfrak{T}}_0}$-modules. Then we know (see [Sta17, Tag 070G])

$$Hom_{D(\mathcal{O}_{\tilde{\mathfrak{T}}_0})}(R(\mathfrak{F}), \mathfrak{F}') \simeq Hom_{K(\mathcal{O}_{\tilde{\mathfrak{T}}_0})}(R(\mathfrak{F}), I^\cdot) \tag{1.46}$$

To show (1.45), we only need the hom complex

$$hom^\cdot(R(\mathfrak{F}), \mathfrak{F}' \to I^\cdot) \tag{1.47}$$

to be acyclic. But this is the totalization of a double complex supported on bidegrees that is in a fixed translate of the first quadrant. Moreover, the rows of this double complex are shifted direct sums of complexes of type $hom^{\cdot}(j_!(E), \mathcal{F}' \to I^{\cdot}) \simeq hom^{\cdot}(E, j^*(\mathcal{F}' \to I^{\cdot}))$, where j is the open embedding of either $U_{i+1/2}$ or V_i for some i, and E is a vector bundle on it. Hence, the rows are acyclic and (1.45) follows from the spectral sequence for the double complex.

Hence, we only need to show

$$Hom_{K(\mathcal{O}_{\tilde{\mathcal{F}}_0})}(R(\mathcal{F}), \mathcal{F}') \simeq Hom_{K(\mathcal{O}_{\tilde{\mathcal{F}}_0})}(R(\mathcal{F}), R(\mathcal{F}')) \tag{1.48}$$

or equivalently $hom^{\cdot}(R(\mathcal{F}), R(\mathcal{F}') \to \mathcal{F}')$ is acyclic. By Lemma 1.3.4 below the acyclicity of $hom^{\cdot}(j_!(E), R(\mathcal{F}') \to \mathcal{F}')$ is enough, where j and E are as in the above paragraph. Let U denote the domain of j.

Without loss of generality, assume E is the trivial line bundle on U. By the adjunction $j_! \vdash j^*$

$$hom^{\cdot}(j_!(E), R(\mathcal{F}') \to \mathcal{F}') \simeq hom^{\cdot}(E, j^*(R(\mathcal{F}') \to \mathcal{F}')) \simeq \Gamma(j^*(R(\mathcal{F}') \to \mathcal{F}')) \tag{1.49}$$

When $U = V_i$, $j^*(R(\mathcal{F}') \to \mathcal{F}')$ is an acyclic complex of coherent sheaves on U and Γ, the global sections functor, preserves its acyclicity. When $U = U_{i+1/2}$, $\Gamma(j^*R(\mathcal{F}'))$ can be obtained as the totalization of a double complex resolving the complex $\Gamma(j^*\mathcal{C}_!(\mathcal{F}'))$, whose explicit form is

$$\begin{aligned} &\{\Gamma(j_{V_i,!}j^*_{V_i}\mathcal{F}') \times \Gamma(j_{V_{i+1},!}j^*_{V_{i+1}}\mathcal{F}') \to \\ \Gamma(j_{V_i,!}j^*_{V_i}\mathcal{F}') \times \Gamma(j_{V_{i+1},!}j^*_{V_{i+1}}\mathcal{F}') &\times \Gamma(j_{U_{i+1/2},!}j^*_{U_{i+1/2}}\mathcal{F}')\} = \\ &\{\Gamma(j_{U_{i+1/2},!}j^*_{U_{i+1/2}}\mathcal{F}')\} = \{\Gamma(j^*_{U_{i+1/2}}\mathcal{F}')\} \end{aligned}$$

The equation holds as $\Gamma(j_{V_i,!}j^*_{V_i}\mathcal{F}') = 0$ for all i(which is true since $j^*_{V_i}\mathcal{F}'$ is locally free and $U_{i+1/2}$ is connected). This is still a resolution of $\Gamma(j^*_{U_{i+1/2}}\mathcal{F}')$. Hence, being the totalization of a double complex resolving $\Gamma(j^*\mathcal{C}_!(\mathcal{F}'))$, $\Gamma(j^*R(\mathcal{F}'))$ is another resolution of $\mathcal{F}'$ and

$$\Gamma(j^*(R(\mathcal{F}') \to \mathcal{F}')) \tag{1.50}$$

is an acyclic complex. This finishes the proof. ☐

Lemma 1.3.4. *Let $C^\cdot$, $D^\cdot$ be bounded above complexes of objects of an abelian category. Assume for each i, $hom^\cdot(C^i, D^\cdot)$ is acyclic. Then the total hom complex $hom^\cdot(C^\cdot, D^\cdot)$ is also acyclic.*

Remark 1.3.5. $\mathfrak{tr}_*$ gives an explicit dg quasi-equivalence of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ that acts bijectively on the objects and hom-sets. We denote this dg functor by $\mathfrak{tr}$ as well.

Remark 1.3.6. The complex $hom^\cdot_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}}(\mathcal{O}_{C_i}(a), \mathcal{O}_{C_{i'}}(b)) = 0$, for $|i - i'| \geq 2$ and $a, b \in \{0, 1\}$. Indeed, if $j_!(E) \neq 0$ and $j'_!(E') \neq 0$ appear in $R(\mathcal{O}_{C_i}(a))$ and $R(\mathcal{O}_{C_{i'}}(b))$ respectively, there is no way the domain of j or j' can contain the domain of the other; hence $Hom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(j_!(E), j'_!(E')) = 0$.

1.3.2 $\mathbb{G}_m$-action on $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$

Let $\mathcal{F} \in \{\mathcal{O}_{C_i}, \mathcal{O}_{C_i}(-1) : i \in \mathbb{Z}\}$. Put a $\mathbb{G}_m$-equivariant structure on $\mathcal{F}$. This makes every graded piece of $\mathcal{C}_!(\mathcal{F})$ naturally a $\mathbb{G}_m$-equivariant sheaf, and the differential is $\mathbb{G}_m$-equivariant. Moreover, the double complex resolving it can be made $\mathbb{G}_m$-equivariant as well in each bidegree, so that both differentials are $\mathbb{G}_m$-equivariant. Hence, $R(\mathcal{F}) \to \mathcal{F}$ is an equivariant resolution. Fix choices for each i so that $\mathfrak{tr}_*$ moves $\mathcal{O}_{C_i}$ to $\mathcal{O}_{C_{i+1}}$ as an equivariant sheaf and similarly for $\mathcal{O}_{C_i}(-1)$ as well as the resolutions. Hence, we obtain an action of $\mathbb{G}_m$ on hom-sets of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$, so that the differential and the multiplication are equivariant. In other words, there exists a $\mathbb{G}_m$-action at the chain level on this category.

Note, however the hom-sets $hom^\cdot_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(R(\mathcal{F}), R(\mathcal{F}'))$ are not rational as representations of $\mathbb{G}_m$. Instead, they are products of countably many rational representations at each degree. Inspired by this define:

Definition 1.3.7. Let $\mathcal{O}(\tilde{\mathfrak{T}}_0)^{eval}_{dg}$ be the dg-subcategory of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ with the same set of objects and with the morphisms given by the subspace of those in $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ that decompose into a finite sum of eigenvalues of $\mathbb{G}_m$-action.

Proposition 1.3.8. *The inclusion $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^{eval} \to \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ is a quasi-equivalence.*

This follows from a simple lemma whose proof we skip:

Lemma 1.3.9. *Let $(C^\cdot, d)$ be a chain complex satisfying*

- *There is a $\mathbb{G}_m$-action on each C^i and d is equivariant*
- *The induced action on $H^\cdot(C^\cdot)$ is rational, i.e. $H^\cdot(C^\cdot)$ admits a direct sum decomposition into eigenvalues of $\mathbb{G}_m$-action*
- *For each i, C^i has a product decomposition*

$$C^i = \prod_{k\in\mathbb{Z}} C^i\{k\} \tag{1.51}$$

into rational representations, such that $d : C^i \to C^{i+1}$ is a product of equivariant maps

$$d_k : C^i\{k\} \to C^{i+1}\{k\} \times C^{i+1}\{k+1\} \tag{1.52}$$

Let $C^\cdot_{eval}$ be the subcomplex of $C^\cdot$ spanned by eigenvalues of $\mathbb{G}_m$-action. Then the inclusion $C^\cdot_{eval} \to C^\cdot$ is a quasi-isomorphism.

1.3.3 A deformation of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$

We have constructed a deformation of $\tilde{\mathfrak{T}}_0$ in Section 1.2.1. In this subsection, we will use it to obtain a deformation of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ to a curved A_∞-category, which we denote by $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. We will manage this by deforming the double complex whose totalization gives $R(\mathcal{O}_{C_i}(-1))$ and $R(\mathcal{O}_{C_i})$ to a bigraded sheaf of $\mathcal{O}_{\tilde{\mathfrak{T}}_R}$-modules with two endomorphisms of degree $(1,0)$ and $(0,1)$.(In other words it deforms to an object that looks like a double complex except the differentials does not square to 0).

First a local model: Consider the resolutions

$$\{\ldots \xrightarrow{Y_i} \mathcal{O}(U_{i-1/2}) \xrightarrow{X_{i-1}} \mathcal{O}(U_{i-1/2})\} \to \mathcal{O}(U_{i-1/2})/(X_{i-1}) \tag{1.53}$$

$$\{\ldots \xrightarrow{X_i} \mathcal{O}(U_{i+1/2}) \xrightarrow{Y_{i+1}} \mathcal{O}(U_{i+1/2})\} \to \mathcal{O}(U_{i+1/2})/(Y_{i+1}) \tag{1.54}$$

and deform them to "complexes" of $\mathcal{O}_{\tilde{\mathfrak{T}}_R}$-modules given as

$$\{\ldots \xrightarrow{X_{i-1}} \mathcal{O}(\tilde{U}_{i-1/2}) \xrightarrow{Y_i} \mathcal{O}(\tilde{U}_{i-1/2}) \xrightarrow{X_{i-1}} \mathcal{O}(\tilde{U}_{i-1/2})\} \tag{1.55}$$

$$\{\ldots \xrightarrow{Y_{i+1}} \mathcal{O}(\tilde{U}_{i+1/2}) \xrightarrow{X_i} \mathcal{O}(\tilde{U}_{i+1/2}) \xrightarrow{Y_{i+1}} \mathcal{O}(\tilde{U}_{i+1/2})\} \tag{1.56}$$

The "differentials" do not square to 0 as $X_{i-1}Y_i = q$ and $X_iY_{i+1} = q$ in the corresponding rings.

This gives the data to deform the vertical differentials of the double complexes resolving $\mathcal{C}_!(\mathcal{O}_{C_i}(-1))$ and $\mathcal{C}_!(\mathcal{O}_{C_i})$. Deform the horizontal differential to

$$j_{\tilde{V}_i,!}\mathcal{O}_{\tilde{V}_i} \to j_{\tilde{U}_{i-1/2},!}\mathcal{O}_{\tilde{U}_{i-1/2}} \times j_{\tilde{U}_{i+1/2},!}\mathcal{O}_{\tilde{U}_{i+1/2}} \tag{1.57}$$

trivially. Let $R(\mathcal{O}_{C_i}(-1))_R$ and $R(\mathcal{O}_{C_i})_R$ denote the totalizations of these bigraded sheaves with degree $(0,1)$ and $(1,0)$ endomorphisms. They are graded sheaves with degree 1 endomorphisms, which squares to a degree 2 endomorphism that is a multiple of $q \in R = \mathbb{C}[[q]]$.

Definition 1.3.10. Let $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ be the curved dg category given by

- $Ob(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) = \{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$
- $hom_{\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}}(\mathcal{F}, \mathcal{F}') := hom^{\cdot}_{\mathcal{O}_{\tilde{\mathfrak{T}}_R}}(R(\mathcal{F})_R, R(\mathcal{F}')_R)$ for $\mathcal{F}, \mathcal{F}' \in Ob(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})$. The hom-"complex" is defined in the standard way similar to actual complexes, only note its differential does not square to 0
- The composition is composition of homomorphisms of "complexes"
- The curvature term is the degree 2 endomorphism obtained by squaring the differential of $R(\mathcal{O}_{C_i}(-1))_R$ and $R(\mathcal{O}_{C_i})_R$

It is easy to see that this is a curved dg category over $R = \mathbb{C}[[q]]$. For instance, the square of the differential of $hom_{\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}}(\mathcal{F}, \mathcal{F}')$ is simply the difference of composition with the differentials of $R(\mathcal{F})_R$ and $R(\mathcal{F}')_R$. It is also obvious that the specialization to $q=0$ gives $\mathcal{O}(\tilde{\mathcal{T}}_0)_{dg}$.

We now want to elaborate on the compatibility of this formal deformation with the geometric deformation above. We show "local" compatibility.

In general, if $\mathcal{B}$ is an algebra and $\mathcal{B}_R$ is a deformation of $\mathcal{B}$ over R, then we obtain a curved deformation of the category $\mathcal{B}^{mod}$, A_∞-modules over $\mathcal{B}$. It is the category of curved modules, which is given by the same data as a semi-free A_∞-module over $\mathcal{B}_R$ but the A_∞-module equation is satisfied only up to $O(q)$. Hence, we obtain a deformation of the category of finitely generated modules as a subcategory of the deformation of $\mathcal{B}^{mod}$.

Assume $\mathcal{B}_R$ is commutative and apply this to $U = Spec(\mathcal{B})$ and to $Spf(\mathcal{B}_R)$. This way we obtain a recipe to produce formal deformations of (generating A_∞-models of) $D^b(Coh(U))$ such that $\mathcal{O}_U$ deforms to $\mathcal{O}_{U_R}$ Thus, the inclusion functor from the full subcategory spanned by $\mathcal{O}_U$ deforms to an A_∞-functor from the algebra $\mathcal{O}_{U_R}$. Call such a deformation a good deformation.

Now our compatibility result is:

Proposition 1.3.11. *For each $i \in \mathbb{Z}$, there exists*

- *A dg enhancement $\mathcal{C}oh(U_{i+1/2})$ of $D^b(Coh(U_{i+1/2}))$*
- *A good deformation $\mathcal{C}oh(U_{i+1/2})_R$ of $\mathcal{C}oh(U_{i+1/2})$*
- *A dg enhancement $\mathcal{C}oh(V_i)$ of $D^b(Coh(V_i))$*
- *A_∞-functors $j^*_{U_{i+1/2}} : \mathcal{O}(\tilde{\mathcal{T}}_R)_{cdg} \to \mathcal{C}oh(U_{i+1/2})_R$*
- *A_∞-functors $j^*_{U_{i+1/2},V_i} : \mathcal{C}oh(U_{i+1/2})_R \to \mathcal{C}oh(V_i)_R$, $j^*_{U_{i-1/2},V_i} : \mathcal{C}oh(U_{i-1/2})_R \to \mathcal{C}oh(V_i)_R$ where $\mathcal{C}oh(V_i)_R$ is the trivial deformation of $\mathcal{C}oh(V_i)$*

such that at $q=0$, $j^*_{U_{i+1/2}}$ specializes to a lift of the natural functor

$$j^*_{U_{i+1/2}} : D^b(Coh_p(\tilde{\mathcal{T}}_0)) \to D^b(Coh(U_{i+1/2})) \tag{1.58}$$

and similarly $j^*_{U_{i+1/2},V_i}$ and $j^*_{U_{i-1/2},V_i}$. Moreover, everything can be chosen in a $\mathfrak{tr}$*-equivariant way.*

This proposition can be proven using constructions similar to these in Section 1.3.1 and it will be useful in order to write localization maps for Hochschild cohomology. These maps will be written as deformations of maps induced by restriction functors in Section 1.5.

Remark 1.3.12. The deformations of $j^*_{U_{i\pm 1/2}}$ and $j^*_{U_{i\pm 1/2},V_i}$ in Prop 1.3.11 can be chosen so that

$$j^*_{U_{i+1/2},V_i} \circ j^*_{U_{i+1/2}} \simeq j^*_{U_{i-1/2},V_i} \circ j^*_{U_{i-1/2}} \tag{1.59}$$

Remark 1.3.13. This deformation is compatible with $\mathfrak{tr}$ and there is an obvious strict auto-equivalence acting on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. This auto-equivalence deforms the translation auto-equivalance of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$. We denote it by $\mathfrak{tr}$ as well.

Remark 1.3.14. The hom-sets of $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ are graded complete vector spaces over R and there is an action of $\mathbb{G}_m(R) = R^*$ on hom-sets deforming the action in Section 1.3.2. Moreover, the completed base change of $\mathcal{O}(\tilde{\mathfrak{T}}_0)^{eval}_{dg}$ to $R/\mathbb{C}$ is a non-full curved dg subcategory, inheriting the curved dg category structure from $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. We denote it by $\mathcal{O}(\tilde{\mathfrak{T}}_R)^{eval}_{cdg}$. Its inclusion into $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ clearly deforms the inclusion $\mathcal{O}(\tilde{\mathfrak{T}}_0)^{eval}_{dg} \to \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$, which is a quasi-equivalence by Prop 1.3.8. It is clear that for all $\mathcal{F}$, $\mathcal{F}'$ and j, $hom^j_{\mathcal{O}(\tilde{\mathfrak{T}}_R)^{eval}_{cdg}}(\mathcal{F}, \mathcal{F}')$ is a completed rational representation of $\mathbb{G}_m(R) = R^*$, i.e. it is the q-adic completion of a representation of R^* with an eigenvalue decomposition.

1.4 The construction of the mapping torus

1.4.1 Smash products and the construction

In this section, we define the mapping torus category and its canonical deformation associated to a pair $(\mathcal{A}, \phi)$. Let us first remind the reader of smash products:

Definition 1.4.1. Let $\mathcal{B}$ be a dg category and G be a discrete group. Assume G acts on $\mathcal{B}$ by auto-equivalences that are bijective on $Ob(\mathcal{B})$ and on hom-sets. Moreover, assume composition of the auto-equivalences associated to $g_1, g_2 \in G$ is equal to the auto-equivalence associated to g_1g_2. Define $\mathcal{B}\#G$ to be the dg-category such that

- $Ob(\mathcal{B}\#G) := Ob(\mathcal{B})$
- $hom_{Ob(\mathcal{B}\#G)}(b_1, b_2) := \bigoplus_{g\in G} hom_{\mathcal{B}}(g(b_1), b_2)$ as a chain complex. We will denote $f \in hom_{\mathcal{B}}(g(b_1), b_2)$ by $f \otimes g$ when it is considered as an element of $hom_{Ob(\mathcal{B}\#G)}(b_1, b_2)$.
- $(f' \otimes g') \circ (f \otimes g) := (f' \circ g'(f)) \otimes (g'g)$

Remark 1.4.2. When $\mathcal{B}$ is taken to be an ordinary algebra, Definition 1.4.1 gives the well-known semi-direct product construction. Indeed, it is possible to recover Definition 1.4.1 by applying this construction to the total algebra of $\mathcal{B}$.

Remark 1.4.3. Under similar assumptions, Definition 1.4.1 generalizes verbatim to curved dg algebras.

Let $(\mathcal{A}, \phi)$ be as in Section 1.1, i.e. $\mathcal{A}$ is a dg category satisfying **C.1-C.3** and ϕ is a strict auto-equivalence. Note the conditions **C.1-C.3** are not yet necessary. The auto-equivalence $\mathfrak{tr} \otimes \phi$ generates a $\mathbb{Z}$-action on $(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})$ satisfying the assumptions of Definition 1.4.1.

Definition 1.4.4. Define M_ϕ to be the dg category $(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}$. Similarly, define M_ϕ^R to be the curved dg algebra $(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A})\#\mathbb{Z}$, where the tensor product is over $\mathbb{C}$(and q-completed) and the $\mathbb{Z}$-action is generated by $\mathfrak{tr} \otimes \phi$ acting on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A}$.

Remark 1.4.5. The tensor product of a curved dg category with an uncurved dg category is defined in a way analogous to tensor product of dg categories. Note the curvature $\mu^0_{\mathfrak{F}\times a}$ of an element $\mathfrak{F} \times a \in Ob(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A}) = Ob(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \times Ob(\mathcal{A})$ is $\mu^0_{\mathfrak{F}} \otimes 1_a$, where $\mu^0_{\mathfrak{F}}$ is the curvature of $\mathfrak{F}$.

Remark 1.4.6. The $\mathbb{G}_m(\mathbb{C})$ (resp. $\mathbb{G}_m(R)$) action in Section 1.3.2(resp. Remark 1.3.14) induces an action on $(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})$ (resp. $(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A})$); which is compatible with $\mathfrak{tr}\otimes\phi$ as the action on $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ (resp. $(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})$ is chosen to be compatible with $\mathfrak{tr}$. Hence, it descends to an action on M_ϕ (resp. M_ϕ^R). Similarly, this action is not rational (resp. completed rational); however, we can pass to non-full quasi-equivalent (resp. quasi-equivalent at $q = 0$) subcategories on which action is rational (resp. completed rational).

1.4.2 Bimodules over $\mathcal{B}\#G$ and over M_ϕ

Let us make some general remarks about the dg bimodules over $\mathcal{B}\#G$, where $(\mathcal{B}, G)$ is as in Definition 1.4.1. Let $G_\Delta = \{(g,g) : g \in G\} \subset G \times G$ and consider its action on $\mathcal{B}^e = \mathcal{B} \otimes \mathcal{B}^{op}$. One can then consider modules over $(\mathcal{B}^e)\#G_\Delta$. Concretely, any such module is given by

- A $\mathcal{B}$-$\mathcal{B}$ bimodule $\mathfrak{M}$
- For each $g \in G$, $b_1, b_2 \in Ob(\mathcal{B})$ chain isomorphisms

$$c_{b_1,b_2}(g) : \mathfrak{M}(b_1, b_2) \to \mathfrak{M}(g(b_1), g(b_2)) \tag{1.60}$$

such that $c_{g'(b_1),g'(b_2)}(g) \circ c_{b_1,b_2}(g') = c_{b_1,b_2}(g \circ g')$, $c_{b,b}(1_b) = 1_{g(b)}$ and satisfying

$$g(f.m.f') = g(f).g(m).g(f') \tag{1.61}$$

for any $f' \in hom_{\mathcal{B}}(b_1, b_2)$,$m \in \mathfrak{M}(b_2, b_3)$,$f \in hom_{\mathcal{B}}(b_3, b_4)$, where $g(m)$ denotes $c_{b_2,b_3}(g)(m)$.

Now construct the $\mathcal{B}\#G$-$\mathcal{B}\#G$ bimodule $\mathfrak{M}\#G$ as follows

- $\mathfrak{M}\#G(b_1, b_2) = \bigoplus_{g\in G} \mathfrak{M}(g(b_1), b_2)$ as a complex. Let $m \otimes g$ denote $m \in \mathfrak{M}(g(b_1), b_2)$ when it is considered as an element of $\mathfrak{M}\#G(b_1, b_2)$
- Given $g_1, g_2 \in G$, $m \in \mathfrak{M}(g_1(b_1), b_2)$, $f \in hom_{\mathcal{B}}(g_2(b_2), b_3)$

$$(f \otimes g_2)(m \otimes g_1) = fg_2(m) \otimes g_2g_1 \tag{1.62}$$

- Given $g_1, g_2 \in G$, $f \in hom_{\mathcal{B}}(g_1(b_1), b_2)$, $m \in \mathfrak{M}(g_2(b_2), b_3)$

$$(m \otimes g_2)(f \otimes g_1) = mg_2(f) \otimes g_2 g_1 \tag{1.63}$$

The simplest example is the diagonal bimodule of $\mathcal{B}$. In that case, the process clearly gives the diagonal bimodule of $\mathcal{B}\#G$.

This construction can be seen as a base change under the map

$$(B^e)\#G_\Delta \to (\mathcal{B}\#G)^e \cong (\mathcal{B}^e)\#(G \times G) \tag{1.64}$$

sending $(b \otimes b') \otimes g \mapsto (b \otimes g) \otimes (b' \otimes g^{-1}) \in (\mathcal{B}\#G)^e$, which corresponds to $(b \otimes b') \otimes (g, g) \in (\mathcal{B}^e)\#(G \times G)$. To see this, one may simply prove this construction gives a left adjoint to the restriction map of modules under this map.

Also, note the functoriality of this construction in the dg category of dg bimodules. In particular, it sends exact triangles into exact triangles and quasi-isomorphisms into quasi-isomorphisms.

To use this to produce bimodules over M_ϕ, we first need to produce bimodules over $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ satisfying the above invariance condition.

Definition 1.4.7. Given a complex E of $\mathcal{O}_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}$-modules we can define the corresponding $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$-bimodule as

$$\mathfrak{M}_E : (\mathfrak{F}, \mathfrak{F}') \mapsto \text{“}RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}}(q^*(\mathfrak{F}), p^*(\mathfrak{F}') \overset{L}{\otimes}_{\mathcal{O}_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}} E)\text{”} \tag{1.65}$$

where q, p are projections to first and second factor respectively.

Remark 1.4.8. To remove the quotation marks in the definition(i.e. to make it more precise), replace $\mathfrak{F}$ by $R(\mathfrak{F})$ and $p^*(\mathfrak{F}') \overset{L}{\otimes}_{\mathcal{O}_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}} E$ by a (K-)injective resolution $I_{\mathfrak{F}'}$ of $p^*(R(\mathfrak{F}')) \otimes_{\mathcal{O}_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}} E$ that is functorial in $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$(and thus $RHom$ by Hom). As we noted, we will often omit the subscripts of tensor product from the notation. To see the existence of such a resolution see $R(\mathfrak{F}) \mapsto p^*(R(\mathfrak{F}')) \otimes_{\mathcal{O}_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}} E$ as a dg functor from $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ to chains on the sheaves on $\tilde{\mathfrak{T}}_0$. The latter has functorial K-injective resolutions since sheaves of $\mathcal{O}_{\tilde{\mathfrak{T}}_0}$-modules has functorial injective resolutions. See the construction in [Spa88].

To endow it with a $\mathbb{Z}_\Delta$-action(i.e. with maps c_{b_1,b_2} as above) fix an isomorphism

$$E \simeq (\mathfrak{tr} \times \mathfrak{tr})_*(E) \tag{1.66}$$

and assume the injective resolution $I^{\cdot}_{\mathcal{F}'}$ of $p^*(R(\mathcal{F}'))\overset{L}{\otimes}E$ is carried to the injective resolution $I^{\cdot}_{\mathfrak{tr}_*\mathcal{F}'}$ of $(\mathfrak{tr}\times\mathfrak{tr})_*(p^*(R(\mathcal{F}'))\otimes E) \simeq p^*(R(\mathfrak{tr}_*\mathcal{F}'))\otimes(\mathfrak{tr}\times\mathfrak{tr})_*E \simeq p^*(R(\mathfrak{tr}_*\mathcal{F}'))\otimes E$ under $(\mathfrak{tr} \times \mathfrak{tr})_*$. Then $(\mathfrak{tr} \times \mathfrak{tr})_*$ gives us chain isomorphisms

$$hom^{\cdot}(q^*R(\mathcal{F}), I^{\cdot}_{\mathcal{F}'}) \simeq hom^{\cdot}(q^*R(\mathfrak{tr}_*\mathcal{F}), I^{\cdot}_{\mathfrak{tr}_*\mathcal{F}'}) \tag{1.67}$$

which is the desired $\mathbb{Z}_\Delta$-action. In the following, the isomorphisms $E \simeq (\mathfrak{tr} \times \mathfrak{tr})_*(E)$ will be obvious.

Definition 1.4.9. We can produce another $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$-bimodule out of the complex of $\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0$ modules E. Namely define $\mathfrak{M}'_E$ by

$$\mathfrak{M}'_E : (\mathcal{F}, \mathcal{F}') \mapsto \text{“}RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}}(q^*(\mathcal{F}) \overset{L}{\otimes}_{\mathcal{O}_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}} E, p^*(\mathcal{F}'))\text{”} \tag{1.68}$$

Remark 1.4.8 applies in this case too. A quasi-isomorphism as in (1.66) would be sufficient to endow $\mathfrak{M}'_E$ with a $\mathbb{Z}_\Delta$ equivariant structure We will not use this fact and we skip the technical details.

Assume in addition we have a bimodule $\mathfrak{N}$ over $\mathcal{A}$ such that

$$\mathfrak{N} \simeq (\phi \otimes \phi)_*(\mathfrak{N}) \tag{1.69}$$

strictly (via a dg-bimodule map that acts as chain isomorphisms for each pairs of objects). Hence, we have a $\mathbb{Z}_\Delta$-equivariant structure on $\mathfrak{N}$, i.e. an equivariant structure with respect to $\phi \otimes \phi$. We can endow the $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$-bimodule $\mathfrak{M} \otimes \mathfrak{N}$ with a $\mathbb{Z}_\Delta$-equivariant structure (with respect to $\mathfrak{tr} \otimes \phi$); hence obtain a bimodule over

$M_\phi = (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}$ using the recipe above. In particular, the diagonal bimodule of $\mathcal{A}$ is an example of such an $\mathfrak{N}$.

As an application of these ideas let us prove:

Proposition 1.4.10. *M_ϕ is a smooth category whenever $\mathcal{A}$ is.*

Proof. Consider the normalization $\pi : \mathbb{P}^1 \times \mathbb{Z} \to \tilde{\mathfrak{T}}_0$. Throughout this proof let $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal of $\tilde{\mathfrak{T}}_0$, and let $\tilde{\mathcal{O}}_\Delta$ denote $(\pi \times \pi)_*(\mathcal{O}_{\Delta_{\mathbb{P}^1\times\mathbb{Z}}})$, where $\Delta_{\mathbb{P}^1\times\mathbb{Z}}$ is the diagonal of $\mathbb{P}^1 \times \mathbb{Z}$. We have a short exact sequence of sheaves on $\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0$

$$0 \to \mathcal{O}_\Delta \to \tilde{\mathcal{O}}_\Delta \to \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{x_{j+1/2}} \boxtimes \mathcal{O}_{x_{j+1/2}} \to 0 \tag{1.70}$$

Here, $x_{j+1/2}$ is the node in the chart $U_{i+1/2}$, and the map $\mathcal{O}_\Delta \to \tilde{\mathcal{O}}_\Delta$ comes as the push-forward of $\mathcal{O}_{\tilde{\mathfrak{T}}_0} \to \pi_*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{Z}})$ under the diagonal map. Using Beilinson's resolution of diagonal of $\mathbb{P}^1$(at each component separately) and exactness of affine push-forward $(\pi \times \pi)_*$ we obtain a resolution

$$0 \to \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{C_i}(-1) \boxtimes \mathcal{O}_{C_i}(-1) \to \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{C_i} \boxtimes \mathcal{O}_{C_i} \to \tilde{\mathcal{O}}_\Delta \to 0 \tag{1.71}$$

This implies the sheaf $\mathcal{O}_\Delta$ is quasi-isomorphic to twisted complex

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{C_i}(-1) \boxtimes \mathcal{O}_{C_i}(-1) \to \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{C_i} \boxtimes \mathcal{O}_{C_i} \to \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{x_{j+1/2}} \boxtimes \mathcal{O}_{x_{j+1/2}} \tag{1.72}$$

We could apply $E \mapsto \mathfrak{M}_E$ to (1.72); however, $\mathfrak{M}_{E'\boxtimes E''}$ is not quasi-isomorphic to a Yoneda bimodule. Inspired by [Lun10], we will instead apply $E \mapsto \mathfrak{M}'_E$ to $\mathcal{O}_\Delta^\vee$(i.e. to derived dual of $\mathcal{O}_\Delta$) and to dual of the resolution (1.72). First notice,

$$\begin{aligned}\mathfrak{M}'_{\mathcal{O}_\Delta^\vee}(\mathcal{F},\mathcal{F}') = {}&\text{“}RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}}(q^*(\mathcal{F}) \overset{L}{\otimes}_{\mathcal{O}_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}} \mathcal{O}_\Delta^\vee, p^*(\mathcal{F}'))\text{”} \simeq \\ &\text{“}RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}}(q^*(\mathcal{F}), R\mathcal{H}om_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}(\mathcal{O}_\Delta^\vee, p^*(\mathcal{F}')))\text{”} \simeq \\ &\text{“}RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}}(q^*(\mathcal{F}), \mathcal{O}_\Delta \otimes p^*(\mathcal{F}')))\text{”} \simeq \text{“}RHom_{\tilde{\mathfrak{T}}_0}(\mathcal{F},\mathcal{F}')\text{”}\end{aligned} \tag{1.73}$$

Here, the quotation marks are used to omit the resolutions that are necessary for (dg) functoriality of the corresponding expression from the notation (see Remark 1.4.8). As a result of (1.73), $\mathfrak{M}_{\mathcal{O}_\Delta^\vee}$ is quasi-isomorphic to diagonal bimodule. The only non-trivial step is the quasi-isomorphism between second and third rows and this follows from Lemma 1.4.11(let X be $\tilde{\mathfrak{T}}_0$ and f be the diagonal embedding). Taking the derived duals, we find $\mathcal{O}_\Delta^\vee$ is quasi-isomorphic to twisted complex

$$\bigoplus_{j\in\mathbb{Z}} \mathcal{O}^\vee_{x_{j+1/2}} \boxtimes \mathcal{O}^\vee_{x_{j+1/2}} \to \bigoplus_{i\in\mathbb{Z}} \mathcal{O}^\vee_{C_i} \boxtimes \mathcal{O}^\vee_{C_i} \to \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{C_i}(-1)^\vee \boxtimes \mathcal{O}_{C_i}(-1)^\vee \tag{1.74}$$

Notice the derived duals of coherent sheaves are quasi-isomorphic to bounded complexes of coherent sheaves, thanks to the Gorenstein property.

Applying $E \mapsto \mathfrak{M}'_E$, we find $\mathfrak{M}'_{\mathcal{O}_\Delta^\vee}$ is quasi-isomorphic to

$$\bigoplus_{i\in\mathbb{Z}} \mathfrak{M}'_{\mathcal{O}_{C_i}(-1)^\vee \boxtimes \mathcal{O}_{C_i}(-1)^\vee} \to \bigoplus_{i\in\mathbb{Z}} \mathfrak{M}'_{\mathcal{O}^\vee_{C_i} \boxtimes \mathcal{O}^\vee_{C_i}} \to \bigoplus_{j\in\mathbb{Z}} \mathfrak{M}'_{\mathcal{O}^\vee_{x_{j+1/2}} \boxtimes \mathcal{O}^\vee_{x_{j+1/2}}} \tag{1.75}$$

Note we are secretly using the fact that

$$\mathfrak{M}'_{\bigoplus_{m\in\mathbb{Z}} E_m} \simeq \bigoplus_{m\in\mathbb{Z}} \mathfrak{M}'_{E_m} \tag{1.76}$$

for $\{E_m\}$ satisfying: given $\mathcal{F}, \mathcal{F}' \in Coh_p(\tilde{\mathfrak{T}}_0)$ there exists only finitely many E_m whose support intersects $supp(q^*(\mathcal{F})) \cup supp(p^*(\mathcal{F}'))$.

Note also that the sheaves involved in expressions (1.70) and (1.71) can be made $(\mathfrak{tr} \times \mathfrak{tr})_*$-equivariant in an obvious way so that the maps can be chosen to be compatible with these $\mathbb{Z}_\Delta$-equivariant structures. This does apply to their duals as well; hence, the bimodule $\mathfrak{M}'_{\mathcal{O}_\Delta^\vee} \otimes \Delta_{\mathcal{A}} \simeq \Delta_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}}$ is quasi-isomorphic to a twisted complex of bimodules

$$\left\{ \bigoplus_{i\in\mathbb{Z}} \mathfrak{M}'_{\mathcal{O}_{C_i}(-1)^\vee \boxtimes \mathcal{O}_{C_i}(-1)^\vee} \to \bigoplus_{i\in\mathbb{Z}} \mathfrak{M}'_{\mathcal{O}^\vee_{C_i} \boxtimes \mathcal{O}^\vee_{C_i}} \to \bigoplus_{j\in\mathbb{Z}} \mathfrak{M}'_{\mathcal{O}^\vee_{x_{j+1/2}} \boxtimes \mathcal{O}^\vee_{x_{j+1/2}}} \right\} \otimes \Delta_{\mathcal{A}} \tag{1.77}$$

compatibly with the $\mathbb{Z}_\Delta$-action.

Assume $E_i = E_i' \boxtimes E_i''$, where $E_i', E_i'' \in Coh_p(\tilde{\mathfrak{T}}_0)$ satisfying $E'_{i+1} = \mathfrak{tr}_* E_i'$ and

$E''_{i+1} = \mathfrak{tr}_* E''_i$, as in (1.74). Then $\mathfrak{M}'_{E'_i \boxtimes E''_i}$ is a right $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg} \otimes \mathcal{O}(\tilde{\mathfrak{I}}_0)^{op}_{dg}$-module (i.e. a functor from $\mathcal{O}(\tilde{\mathfrak{I}}_0)^{op}_{dg} \otimes \mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$ to chains over $\mathbb{C}$) represented by

$$E'^{\vee}_i \times E''_i \in Ob(\mathcal{O}(\tilde{\mathfrak{I}}_0)^{op}_{dg} \otimes \mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}) \tag{1.78}$$

where $E'^{\vee}_i$ is-again- the derived dual of E'_i. This is essentially stating

$$RHom_{\tilde{\mathfrak{I}}_0 \times \tilde{\mathfrak{I}}_0}(q^*\mathfrak{F} \overset{L}{\otimes} (E'_i \boxtimes E''_i), p^*\mathfrak{F}') \simeq RHom_{\tilde{\mathfrak{I}}_0}(\mathfrak{F}, E'^{\vee}_i) \otimes_{\mathbb{C}} RHom_{\tilde{\mathfrak{I}}_0}(E''_i, \mathfrak{F}') \tag{1.79}$$

Hence, $\bigoplus_{i \in \mathbb{Z}} \mathfrak{M}'_{E'_i \boxtimes E''_i} \otimes \Delta_{\mathcal{A}}$, with its obvious $(\mathfrak{tr}_* \otimes \phi) \otimes (\mathfrak{tr}_* \otimes \phi)$-equivariant structure, descends to

$$\left(\bigoplus_{i \in \mathbb{Z}} \mathfrak{M}'_{E'_i \boxtimes E''_i} \otimes \Delta_{\mathcal{A}} \right) \# \mathbb{Z} \tag{1.80}$$

which is quasi-isomorphic to a twisted complex we informally denote by

$$\text{“} h_{E'^{\vee}_0 \times E''_0} \otimes \Delta_{\mathcal{A}} \text{”} \tag{1.81}$$

where $h_{E'^{\vee}_0 \times E''_0}$ is the contravariant Yoneda functor associated to $E'^{\vee}_0 \times E''_0$. To see “$h_{E'^{\vee}_0 \times E''_0} \otimes \Delta_{\mathcal{A}}$” is quasi-isomorphic to a twisted complex over M^e_ϕ one may find a twisted complex $X = (X, \delta, \pi)$ over $\mathcal{A}^e$ that is quasi-isomorphic to $\Delta_{\mathcal{A}}$ and apply descent to an infinite equivariant sum and obtain

$$\left(\bigoplus_{i \in \mathbb{Z}} \mathfrak{M}'_{E'_i \boxtimes E''_i} \otimes (\phi \otimes \phi)^i(X) \right) \# \mathbb{Z} \tag{1.82}$$

which can be represented by a twisted complex of objects “$E'^{\vee}_0 \times a' \times E''_0 \times a''$”.

Hence, Δ_{M_ϕ}, which can be obtained by descent from $\Delta_{\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg} \otimes \mathcal{A}}$ can be represented by a twisted complex as the latter is $\mathbb{Z}_\Delta$-equivariantly quasi-isomorphic to (1.77). □

Lemma 1.4.11. *Let X,Y,Z be (locally Noetherian) Gorenstein varieties over $\mathbb{C}$, $f : X \hookrightarrow Y$ be a closed embedding and $p : Y \to Z$ be a flat map. Assume $p \circ f$ is also flat. Then, for any coherent sheaf $\mathfrak{F}$ on Z, there exists a natural isomorphism in the* *derived category*

$$\mathcal{O}_X \otimes_{\mathcal{O}_Y} p^*(\mathcal{F}) \xrightarrow{\simeq} R\mathcal{H}om_Y(\mathcal{O}_X^\vee, p^*\mathcal{F}) \tag{1.83}$$

Here $\mathcal{O}_X = Rf_*\mathcal{O}_X = f_*\mathcal{O}_X$ *and* $\mathcal{O}_X^\vee = R\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$.

Proof. We will drop the subscript $\mathcal{O}_Y$ of the tensor product and $\otimes$ refers to derived tensor product as usual. However, notice in this case flatness of X over Z implies $\mathcal{O}_X \otimes p^*(\mathcal{F}) \simeq \mathcal{O}_X \overset{L}{\otimes} p^*(\mathcal{F})$. In particular $\mathcal{O}_X \overset{L}{\otimes} p^*(\mathcal{F})$ is a bounded complex of coherent sheaves.

We also remark that $\mathcal{O}_X^{\vee\vee} \simeq \mathcal{O}_X$ over Y, thanks to the Gorenstein property (see [Har66, Section V,Theorem 9.1]). In other words, we have an isomorphism in the derived category

$$\mathcal{O}_X \to R\mathcal{H}om_Y(\mathcal{O}_X^\vee, \mathcal{O}_Y) \tag{1.84}$$

which induces $\mathcal{O}_X \otimes p^*\mathcal{F} \to R\mathcal{H}om_Y(\mathcal{O}_X^\vee, \mathcal{O}_Y) \otimes p^*\mathcal{F}$. Our asserted quasi-isomorphism is the composition of the natural maps

$$\mathcal{O}_X \otimes p^*\mathcal{F} \to R\mathcal{H}om_Y(\mathcal{O}_X^\vee, \mathcal{O}_Y) \otimes p^*\mathcal{F} \to R\mathcal{H}om_Y(\mathcal{O}_X^\vee, \mathcal{O}_Y \otimes p^*\mathcal{F}) \tag{1.85}$$

Whether (1.85) gives a quasi-isomorphism is a local question; thus, we can assume X, Y, Z to be Noetherian(and even affine). First, let us compute $\mathcal{O}_X^\vee$ using Duality theorem [Har66, Section VII,Theorem 3.3]. Let D_Y be a dualizing complex on Y and D_X be a dualizing complex on X. Assume D_Y and D_X are related by f in an appropriate sense, i.e. $f^A D_Y \simeq D_X$ in the notation of [Har66]. One can define corresponding dualizing functors as $\mathbb{D}_Y(\mathscr{E}) = R\mathcal{H}om_Y(\mathscr{E}, D_Y)$ and $\mathbb{D}_X(\mathscr{E}) = R\mathcal{H}om_X(\mathscr{E}, D_X)$. Then [Har66, Section VII,Theorem 3.3] states that $Rf_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ Rf_*$. If we apply this to $\mathcal{F} = \mathcal{O}_X \in Coh(X)$, we obtain

$$R\mathcal{H}om_Y(Rf_*\mathcal{O}_X, D_Y) \simeq Rf_* R\mathcal{H}om_X(\mathcal{O}_X, D_X) \simeq Rf_* D_X \tag{1.86}$$

As X and Y are Gorenstein, D_X and D_Y are quasi-isomorphic to shifted line bundles.

Hence,

$$\mathcal{O}_X^\vee = R\mathcal{H}om_Y(Rf_*\mathcal{O}_X, \mathcal{O}_Y) \simeq D_Y^{-1} \otimes Rf_*D_X \tag{1.87}$$

Moreover, a functor $f^! : D^+_{coh}(Y) \to D^+_{coh}(X)$ satisfying

$$Rf_*R\mathcal{H}om_X(\mathscr{E}, f^!\mathscr{E}') \xrightarrow{\simeq} R\mathcal{H}om_Y(Rf_*\mathscr{E}, \mathscr{E}') \tag{1.88}$$

for every $\mathscr{E} \in D^-_{qcoh}(X), \mathscr{E}' \in D^+_{coh}(X)$ is constructed in the proof of [Har66, Section VII, Corollary 3.4] and it also satisfies $f^! \simeq \mathbb{D}_X \circ Lf^* \circ \mathbb{D}_Y$. This implies

$$\begin{gathered} R\mathcal{H}om_Y(\mathcal{O}_X^\vee, p^*\mathcal{F}) \simeq R\mathcal{H}om_Y(D_Y^{-1} \otimes Rf_*D_X, p^*\mathcal{F}) \simeq \\ R\mathcal{H}om_Y(Rf_*D_X, D_Y \otimes p^*\mathcal{F}) \simeq Rf_*R\mathcal{H}om_X(D_X, f^!(D_Y \otimes p^*\mathcal{F})) \end{gathered} \tag{1.89}$$

Note, we take $\mathscr{E} = D_X$ and $\mathscr{E}' = D_Y \otimes p^*\mathcal{F}$ for the last isomorphism.

Now we assert,

$$f^!(D_Y \otimes p^*\mathcal{F}) \simeq (pf)^*\mathcal{F} \otimes D_X \tag{1.90}$$

Indeed,

$$\begin{aligned} f^!(D_Y \otimes p^*\mathcal{F}) \simeq \mathbb{D}_X Lf^* R\mathcal{H}om_Y(D_Y \otimes p^*\mathcal{F}, D_Y) \simeq \\ \mathbb{D}_X Lf^* R\mathcal{H}om_Y(p^*\mathcal{F}, \mathcal{O}_Y) \simeq \mathbb{D}_X R\mathcal{H}om_X((pf)^*\mathcal{F}, \mathcal{O}_X) \simeq \\ (pf)^*\mathcal{F} \otimes D_X \end{aligned}$$

The last identity holds due to Gorenstein property. The identity

$$Lf^* R\mathcal{H}om_Y(p^*\mathcal{F}, \mathcal{O}_Y) \simeq R\mathcal{H}om_X((pf)^*\mathcal{F}, \mathcal{O}_X) \tag{1.91}$$

can be proven using flatness of p and pf. Namely let $E \xrightarrow{\simeq} \mathcal{F}$ be a locally free resolution. $R\mathcal{H}om_Y(p^*\mathcal{F}, \mathcal{O}_Y) \simeq p^*E^\vee \simeq Lp^*E^\vee$ is bounded below. Still

$$Lf^*Lp^*E^\vee \simeq L(pf)^*E^\vee \simeq (pf)^*E^\vee \simeq (pf)^*\mathcal{F}^\vee \tag{1.92}$$

Combining (1.89) and (1.90), we see that

$$R\mathcal{H}om_Y(\mathcal{O}_X^\vee, p^*\mathcal{F}) \simeq Rf_*((pf)^*\mathcal{F}) \simeq Rf_*(\mathcal{O}_X) \otimes p^*\mathcal{F} = \mathcal{O}_X \otimes p^*\mathcal{F} \tag{1.93}$$

This finishes the proof. □

1.5 Hochschild cohomology of the mapping torus categories

1.5.1 Hochschild cohomology of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ and $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$

In this section we will compute the Hochschild cohomology of the mapping torus categories. For this we first need the Hochschild cohomology of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ and $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. Let $\mathcal{C}oh(\tilde{\mathfrak{T}}_0)$ be a dg enhancement for the bounded derived category of coherent sheaves on $\tilde{\mathfrak{T}}_0$. This clearly restricts to a dg enhancement of $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$. We will denote it by $\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)$. Similarly, let $\mathcal{C}oh(U_{i+1/2})$ and $\mathcal{C}oh(U_{i+1/2} \cap U_{j+1/2})$ be dg enhancements of corresponding derived categories. Then there are pull-back maps

$$\mathcal{C}oh(\tilde{\mathfrak{T}}_0) \to \mathcal{C}oh(U_{i+1/2}) \to \mathcal{C}oh(U_{i+1/2} \cap U_{j+1/2}) \tag{1.94}$$

which are A_∞-functors but without loss of generality one can choose the enhancements so that they become dg-functors. Hence, $\mathcal{C}oh(U_{i+1/2})$ and $\mathcal{C}oh(U_{i+1/2} \cap U_{j+1/2})$ can be considered as bimodules over $\mathcal{C}oh(\tilde{\mathfrak{T}}_0)$. Moreover, the diagonal bimodule $\mathcal{C}oh(\tilde{\mathfrak{T}}_0)$ is quasi-isomorphic to the homotopy limit

$$holim\Big(\prod_i \mathcal{C}oh(U_{i+1/2}) \to \prod_{i<j} \mathcal{C}oh(U_{i+1/2} \cap U_{j+1/2})\Big) \tag{1.95}$$

of bimodules(by this notation we mean the homotopy limit of the big diagram involving $\mathcal{C}oh(U_{i+1/2})$ and $\mathcal{C}oh(U_{i+1/2} \cap U_{j+1/2})$; however, (1.95) can also be realized as the cocone of these products). That $\mathcal{C}oh(\tilde{\mathfrak{T}}_0)$ is quasi-isomorphic to (1.95) holds since the triple intersections are empty. Also as the double intersections for $|j-i| \geq 2$ are empty we have

$$\mathcal{C}oh(\tilde{\mathfrak{T}}_0) \simeq holim\Big(\prod_i \mathcal{C}oh(U_{i+1/2}) \to \prod_j \mathcal{C}oh(U_{j-1/2} \cap U_{j+1/2})\Big) \tag{1.96}$$

as bimodules over $\mathcal{C}oh(\tilde{\mathfrak{T}}_0)$ and its full subcategory $\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)$.

Apply the functor

$$\begin{aligned} Bimod(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)) &\longrightarrow \mathcal{C}_{dg}(\mathbb{C}) \\ \mathcal{B} &\longmapsto CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{B}) \end{aligned}$$

where $\mathcal{C}_{dg}(\mathbb{C})$ is the category of chains over $\mathbb{C}$. This functor can be seen as a Yoneda functor and hence it preserves the limits. This implies

$$CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh(\tilde{\mathfrak{T}}_0)) \simeq$$
$$holim\Big(\prod_i CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh(U_{i+1/2})) \to \prod_j CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh(U_{j-1/2} \cap U_{j+1/2}))\Big) \tag{1.97}$$

We can easily identify the chain complexes

$$CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)) \cong CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh(\tilde{\mathfrak{T}}_0)) = CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)) \tag{1.98}$$

Moreover we have

Lemma 1.5.1. *Let $U \subset \tilde{\mathfrak{T}}_0$ be a quasi-compact open subvariety. Given a dg model $\mathcal{C}oh(U)$ and restriction(pull-back) functor $\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0) \to \mathcal{C}oh(U)$, the induced chain map*

$$CC^*(\mathcal{C}oh(U), \mathcal{C}oh(U)) \to CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh(U)) \tag{1.99}$$

is a quasi-isomorphism.

Proof. This follows from Lemma 1.5.2 and Lemma 1.5.3. ☐

Lemma 1.5.2. *Let $U \subset \tilde{\mathfrak{T}}_0$ be an open quasi-compact subvariety. Then there exists a line bundle $\mathcal{L}$ and a section $s \in \Gamma(\mathcal{L})$ such that $U = \{s \neq 0\}$ and for any such $(\mathcal{L}, s)$*

the localization of $\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)$ *at the natural transformation*

$$s : 1_{\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)} \to (\cdot) \otimes \mathcal{L} \tag{1.100}$$

is quasi-equivalent to $\mathcal{C}oh(U)$.

Proof. See [Sei08a] for the definition of localization and the proof of a similar statement. Note, the existence of such a pair $(\mathcal{L}, s)$ holds for general U only because we are on a curve. But, we only need it for $U = U_{i+1/2}$ or V_j in which case there are obvious pairs $(\mathcal{L}, s)$. □

Lemma 1.5.3. *Let* $\mathcal{B}$ *be a dg category,* Φ *be an auto-equivalence and* $T : 1 \to \Phi$ *be a natural transformation. Consider the localization functor* $\mathcal{B} \to T^{-1}\mathcal{B}$. *Then,*

$$CC^*(T^{-1}\mathcal{B}, T^{-1}\mathcal{B}) \simeq CC^*(\mathcal{B}, T^{-1}\mathcal{B}) \tag{1.101}$$

Proof. We will not include the proof here. For motivation, one can consider the case $\mathcal{B}$ is an ordinary commutative algebra and $T = f \in \mathcal{B}$. In this case, it is obvious that $RHom_{\mathcal{B}_f^e}(\mathcal{B}_f, \mathcal{B}_f) \cong RHom_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}_f)$. □

We can summarize this discussion as

$$\begin{gathered} CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0)) \simeq \\ holim\Big(\prod CC^*(\mathcal{C}oh(U_{i+1/2})) \to \prod CC^*(\mathcal{C}oh(U_{j-1/2} \cap U_{j+1/2}))\Big) \end{gathered} \tag{1.102}$$

For the moment let $\mathcal{C}oh(U_{i+1/2})_R$, $\mathcal{C}oh(V_i)_R$ denote some curved deformations compatible with the deformation of $\tilde{\mathfrak{T}}_0$ to $\tilde{\mathfrak{T}}_R$(and its restriction to corresponding open subsets). Note the compatibility here is in a loose sense, see the notion of good deformation in Section 1.3.3 for instance. Most importantly, we need restriction functors (1.94) to deform so that the map in (1.94) deforms as well. The chain complexes

$$CC^*(\mathcal{C}oh(U_{i+1/2})_R) \text{ and } CC^*(\mathcal{C}oh(V_i)_R) \tag{1.103}$$

deform the complexes $CC^*(\mathcal{C}oh(U_{i+1/2}))$ and $CC^*(\mathcal{C}oh(V_i))$ respectively. Similarly the complex $CC^*(\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)_R)$ deforms $CC^*(\mathcal{C}oh_p(\tilde{\mathcal{T}}_0))$, where $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)_R$ is a curved deformation of $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)$ extending the one for $\mathcal{O}(\tilde{\mathcal{T}}_0)_{dg}$. We can write the map

$$\begin{gathered} CC^*(\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)_R) \to \\ holim\Big(\prod CC^*(\mathcal{C}oh(U_{i+1/2})_R) \to \prod CC^*(\mathcal{C}oh(U_{j-1/2} \cap U_{j+1/2})_R)\Big) \end{gathered} \tag{1.104}$$

following similar steps as before, for instance by deforming the maps in (1.96); and thus, (1.97). Note that the analogue of Lemma 1.5.1 can be shown by a semi-continuity/q-adic filtration argument. Namely

Lemma 1.5.4. *If one has a chain map*

$$f : C^*_R \to C'^*_R \tag{1.105}$$

of complexes of complete topological torsion free vector spaces over R which deforms a quasi-isomorphism $C^* \xrightarrow{\simeq} C'^*$, then f itself is a quasi-isomorphism.

Moreover, using the semi-continuity and deformability of the maps such as 1.97 and 1.99, we prove

$$\begin{gathered} CC^*(\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)_R) \simeq \\ holim\Big(\prod CC^*(\mathcal{C}oh(U_{i+1/2})_R) \to \prod CC^*(\mathcal{C}oh(U_{j-1/2} \cap U_{j+1/2})_R)\Big) \end{gathered} \tag{1.106}$$

Now, let us turn to the questions about the Hochschild cohomology of $\mathcal{C}oh(U_{i+1/2})$, $\mathcal{C}oh(V_i)$ as well as their deformations.

First note we can as well compute the Hochschild cohomology of perfect complexes $\mathcal{P}erf(U_{i+1/2}) \subset \mathcal{C}oh(U_{i+1/2})$ and $\mathcal{P}erf(V_i) \subset \mathcal{C}oh(V_i)$ as well as their deformations. It is possible to show that the restriction maps induce isomorphisms

$$CC^*(\mathcal{C}oh(U)) \xrightarrow{\simeq} CC^*(\mathcal{P}erf(U)) \tag{1.107}$$

and this implies by semi-continuity

$$CC^*(\mathcal{C}oh(U)_R) \xrightarrow{\simeq} CC^*(\mathcal{P}erf(U)_R) \tag{1.108}$$

where U is $U_{i+1/2}$ or V_i and $\mathcal{P}erf(U)_R$ is the corresponding deformation. See [AG15, Appendix F] for an Ind-completed version of (1.107). Alternatively one can identify Hochschild cohomologies of $\mathcal{C}oh(U)$, resp. $\mathcal{P}erf(U)$ with derived self-endomorphisms of the diagonal of U in the category $D^bCoh(U \times U)$, resp. $D(QCoh(U \times U))$, which are known to match.

As U is affine, $CC^*(\mathcal{P}erf(U)) \simeq CC^*(\mathcal{O}(U))$ and similar for the deformations. Notice that we use the fact that we can deform the functors

$$\mathcal{O}(U) \to \mathcal{P}erf(U) \to \mathcal{C}oh(U) \tag{1.109}$$

which was imposed for "good deformations". Let $\mathcal{O}(\tilde{U})$ and $\mathcal{O}(U)_R$ both denote the corresponding deformation of the algebra $\mathcal{O}(U)$. More explicitly

$$\mathcal{O}(\tilde{U}_{i+1/2}) = \mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q) \tag{1.110}$$

$$\mathcal{O}(\tilde{V}_i) = \mathbb{C}[X_i, X_i^{-1}][[q]] = \mathbb{C}[Y_i^{-1}, Y_i][[q]] \tag{1.111}$$

In summary

$$CC^*(\mathcal{C}oh(U_{i+1/2})_R) \simeq CC^*(\mathcal{O}(\tilde{U}_{i+1/2})) \tag{1.112}$$

$$CC^*(\mathcal{C}oh(V_i)_R) \simeq CC^*(\mathcal{O}(\tilde{V}_i)) \tag{1.113}$$

where the Hochschild cohomologies are computed over R. Now, using [Fn07, Appendix, Theorem 2] one can show:

Lemma 1.5.5.

$$CC^*(\mathcal{O}(U_{i+1/2})) \simeq \mathbb{C}[X_i, Y_{i+1}, X_i^*, Y_{i+1}^*, \beta_{i+1/2}]/(X_iY_{i+1}) \tag{1.114}$$

where the latter dga is the quotient of the free (super-commutative) graded algebra gen-

erated by the variables $X_i, Y_{i+1}, X_i^*, Y_{i+1}^*, \beta_{i+1/2}$ *with degrees* $|X_i| = |Y_{i+1}| = 0, |X_i^*| = |Y_{i+1}^*| = 1, |\beta_{i+1/2}| = 2$ *as a graded algebra. Its differential* d *satisfies*

$$d(X_i) = d(Y_{i+1/2}) = d(\beta_{i+1/2}) = 0 \tag{1.115}$$

$$d(X_i^*) = Y_{i+1}\beta_{i+1/2}, d(Y_{i+1}^*) = X_i\beta_{i+1/2} \tag{1.116}$$

Using an R-relative version of the same theorem, we can prove:

Lemma 1.5.6.

$$CC^*(\mathcal{O}(\tilde{U}_{i+1/2})) \simeq \mathbb{C}[X_i, Y_{i+1}, X_i^*, Y_{i+1}^*, \beta_{i+1/2}][[q]]/(X_iY_{i+1} - q) \tag{1.117}$$

where the degrees of the variables are the same and the differential still satisfies (1.115) and (1.116). We note that in (1.117) the q*-adic completion of the free graded algebra* $\mathbb{C}[X_i, Y_{i+1}, X_i^*, Y_{i+1}^*, \beta_{i+1/2}]$ *is taken separately at each degree.*

It is now easy to calculate the cohomology of the above dga's:

Lemma 1.5.7. *The cohomology of* $CC^*(\mathcal{O}(U_{i+1/2}))$ *can be computed as*

$$HH^*(\mathcal{O}(U_{i+1/2})) = \begin{cases} \mathcal{O}(U_{i+1/2}) = \mathbb{C}[X_i, Y_{i+1}]/(X_iY_{i+1}) & * = 0 \\ \mathcal{O}(U_{i+1/2})\langle X_iX_i^*\rangle \oplus \mathcal{O}(U_{i+1/2})\langle Y_{i+1}Y_{i+1}^*\rangle & * = 1 \\ \mathbb{C}\langle\beta^k\rangle \cong \mathcal{O}(U_{i+1/2})/(X_i, Y_{i+1}) & * = 2k, k \geq 1 \\ \frac{\mathcal{O}(U_{i+1/2})\langle X_iX_i^*\beta^k, Y_{i+1}Y_{i+1}^*\beta^k\rangle}{((X_iX_i^* - Y_{i+1}Y_{i+1}^*)\beta^k)} & * = 2k+1, k \geq 1 \end{cases} \tag{1.118}$$

which can be written concisely as the graded commutative algebra

$$\frac{\mathbb{C}[X_i, Y_{i+1}, X_iX_i^*, Y_{i+1}Y_{i+1}^*, \beta_{i+1/2}]}{(X_iY_{i+1}, X_i\beta_{i+1/2}, Y_{i+1}\beta_{i+1/2}, (X_iX_i^* - Y_{i+1}Y_{i+1}^*)\beta_{i+1/2})} \tag{1.119}$$

Note, the cohomology groups (1.118) are not free over $\mathcal{O}(U_{i+1/2})$ unless $* = 0$. For instance, in the second line of (1.118) $Y_{i+1}(X_iX_i^*) = 0$ still holds and $\mathcal{O}(U_{i+1/2})\langle X_iX_i^*\rangle \oplus \mathcal{O}(U_{i+1/2})\langle Y_{i+1}Y_{i+1}^*\rangle \cong \mathbb{C}[X_i] \oplus \mathbb{C}[Y_{i+1}]$.

Lemma 1.5.8. *The cohomology of $CC^*(\mathcal{O}(\tilde{U}_{i+1/2}))$ can be computed as*

$$HH^*(\mathcal{O}(\tilde{U}_{i+1/2})) = \begin{cases} \mathcal{O}(\tilde{U}_{i+1/2}) = \mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q) & * = 0 \\ \mathcal{O}(\tilde{U}_{i+1/2})\langle X_iX_i^* - Y_{i+1}Y_{i+1}^* \rangle & * = 1 \\ \mathbb{C}\langle \beta^k \rangle \cong \mathcal{O}(\tilde{U}_{i+1/2})/(X_i, Y_{i+1}) & * = 2k, k \geq 1 \\ 0 & * = 2k+1, k \geq 1 \end{cases} \tag{1.120}$$

which can be written concisely as the graded commutative algebra

$$\frac{\mathbb{C}[X_i, Y_{i+1}, X_iX_i^* - Y_{i+1}Y_{i+1}^*, \beta_{i+1/2}][[q]]}{(X_iY_{i+1} - q, X_i\beta_{i+1/2}, Y_{i+1}\beta_{i+1/2}, (X_iX_i^* - Y_{i+1}Y_{i+1}^*)\beta_{i+1/2})} \tag{1.121}$$

where the q-completion is taken in each degree separately.

The Hochschild cohomology of $\mathcal{O}(V_i)$ and $\mathcal{O}(\tilde{V}_i)$ can be computed using the same theorem or Hochschild-Kostant-Rosenberg isomorphism. We have

$$CC^*(\mathcal{O}(V_i)) \simeq \mathbb{C}[X_i, X_i^{-1}, X_i^*] \text{ and } CC^*(\mathcal{O}(\tilde{V}_i)) \simeq \mathbb{C}[X_i, X_i^{-1}, X_i^*][[q]] \tag{1.122}$$

Here, $|X_i| = 0, |X_i^*| = 1$ and the differential vanishes. In the latter, the q-adic completion is taken separately at each degree.

To compute the Hochschild cohomology of $\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ we also need the localization maps

$$HH^*(\mathcal{O}(U_{i+1/2})) \to HH^*(\mathcal{O}(V_i)), HH^*(\mathcal{O}(U_{i+1/2})) \to HH^*(\mathcal{O}(V_{i+1})) \tag{1.123}$$

$$HH^*(\mathcal{O}(\tilde{U}_{i+1/2})) \to HH^*(\mathcal{O}(\tilde{V}_i)), HH^*(\mathcal{O}(\tilde{U}_{i+1/2})) \to HH^*(\mathcal{O}(\tilde{V}_{i+1})) \tag{1.124}$$

They all vanish when $* \geq 2$ for the right hand side vanish. For the others identify

$$\mathcal{O}(V_i) \cong \mathcal{O}(U_{i+1/2})_{X_i}, \mathcal{O}(V_{i+1}) \cong \mathcal{O}(U_{i+1/2})_{Y_{i+1}} \tag{1.125}$$

$$\mathcal{O}(\tilde{V}_i) \cong (\mathcal{O}(\tilde{U}_{i+1/2})_{X_i})[[q]], \mathcal{O}(\tilde{V}_{i+1}) \cong (\mathcal{O}(\tilde{U}_{i+1/2})_{Y_{i+1}})[[q]] \tag{1.126}$$

The identification gives the localization maps (1.123) and (1.124) for $* = 0$.

For degree $* = 1$ we have

$$HH^1(\mathcal{O}(U_{i+1/2})) \to HH^1(\mathcal{O}(V_i)) \qquad HH^1(\mathcal{O}(U_{i+1/2})) \to HH^1(\mathcal{O}(V_{i+1}))$$
$$X_iX_i^* \mapsto X_iX_i^*, Y_{i+1}Y_{i+1}^* \mapsto 0 \qquad X_iX_i^* \mapsto 0, Y_{i+1}Y_{i+1}^* \mapsto Y_{i+1}Y_{i+1}^*$$

and

$$HH^1(\mathcal{O}(\tilde{U}_{i+1/2})) \to HH^1(\mathcal{O}(\tilde{V}_i)) \qquad HH^1(\mathcal{O}(\tilde{U}_{i+1/2})) \to HH^1(\mathcal{O}(\tilde{V}_{i+1}))$$
$$X_iX_i^* - Y_{i+1}Y_{i+1}^* \mapsto X_iX_i^* \qquad X_iX_i^* - Y_{i+1}Y_{i+1}^* \mapsto -Y_{i+1}Y_{i+1}^*$$

To see this, for instance for $HH^1(\mathcal{O}(\tilde{U}_{i+1/2})) \to HH^1(\mathcal{O}(\tilde{V}_i))$, see $X_iX_i^* - Y_{i+1}Y_{i+1}^*$ as the derivation $X_i\partial_{X_i} - Y_{i+1}\partial_{Y_{i+1}}$ acting on $\mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q)$. As mentioned above, $\mathcal{O}(\tilde{V}_i) = \mathbb{C}[X_i^{\pm}][[q]] \cong \mathbb{C}[X_i^{\pm}, Y_{i+1}][[q]]/(X_iY_{i+1} - q)$ and the derivation acts as $X_i^m \mapsto mX_i^m$, which is exactly the action of $X_i\partial_{X_i}$ on $\mathbb{C}[X_i^{\pm}][[q]]$. The others follow from similar considerations.

To compute the limits, we need one extra information. Namely, we identify $\mathcal{O}(V_i)$ with $\mathbb{C}[X_i^{\pm}]$ and $\mathbb{C}[Y_i^{\pm}]$ and the coordinates satisfy $X_iY_i = 1$. Basic calculus would tell us that the derivation corresponding to $X_iX_i^*$ acts the same as $-Y_iY_i^*$.

Now, we are ready to compute the Hochschild cohomology of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ and $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ in low degree. First, recall we can see the homotopy limit as the right derived functor of the limit functor.

Remark 1.5.9. For conceptual ease, we will think of above data and localization maps as defining sheaves on $\tilde{\mathfrak{T}}_0$ and $\tilde{\mathfrak{T}}_R$. We emphasize there is no need to pass to sheaves and one can merely work with diagram categories. However, this is the basis of many ideas we have used. Then, the desired (homotopy) limits can be thought as (right derived) global sections of these sheaves. For instance, for $\tilde{\mathfrak{T}}_0$ consider the sheaf that assigns

$$U \mapsto CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0), \mathcal{C}oh(U)) \tag{1.127}$$

for $U = U_{i+1/2}$ or $U_{j-1/2} \cap U_{j+1/2}$. The restriction maps are induced by the pull-back maps for the inclusions $U_{i-1/2}\cap U_{i+1/2}\to U_{i+1/2}$. By (1.97) and (1.98), the global sections of this sheaf compute the Hochschild cohomology of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$. See also (1.102). One can replace $CC^*(\mathcal{C}oh_p(\tilde{\mathfrak{T}}_0),\mathcal{C}oh(U))$ by explicit supercommutative dga as in (1.114), but this will not be necessary since cohomology level information is sufficient to compute the cohomology of the global sections as we will see.

Lemma 1.5.10. *Cohomology of these sheaves are isomorphic to*

$$\mathcal{O}_{\tilde{\mathfrak{T}}_0} \text{ resp. } \mathcal{O}_{\tilde{\mathfrak{T}}_R} \text{ for } * = 0 \tag{1.128}$$

$$\bigoplus_{i\in\mathbb{Z}}\mathcal{O}_{C_i} \text{ resp. } \mathcal{O}_{\tilde{\mathfrak{T}}_R} \text{ for } * = 1 \tag{1.129}$$

$$\bigoplus_{i\in\mathbb{Z}}\mathcal{O}_{x_{i+1/2}} \text{ resp. } \bigoplus_{i\in\mathbb{Z}}\mathcal{O}_{x_{i+1/2}} \text{ for } * = 2k, k\geq 1 \tag{1.130}$$

$$\bigoplus_{i\in\mathbb{Z}}\mathcal{O}_{x_{i+1/2}} \text{ resp. } 0 \text{ for } * = 2k+1, k\geq 1 \tag{1.131}$$

To relate the global sections(a.k.a. the limits of relevant diagrams) of these sheaves to desired homotopy limit, we can use the Grothendieck spectral sequence.

More precisely, let $\mathcal{CC}^*$, resp. $\mathcal{CC}^*_R$ denote the homotopy sheaves on $\tilde{\mathfrak{T}}_0$, resp. $\tilde{\mathfrak{T}}_R$ mentioned in Remark 1.5.9. We combine (1.102), (1.106), the invariance of Hochschild cohomology under passing to twisted complexes and Remark 1.5.9, and we apply Grothendieck spectral sequence to obtain two spectral sequences

$$E_2^{pq}=H^p(\mathcal{HH}^q)\Rightarrow HH^{p+q}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \tag{1.132}$$

$$E_2^{pq}=H^p(\mathcal{HH}^q_R)\Rightarrow HH^{p+q}(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \tag{1.133}$$

Here, $\mathcal{HH}^q$, resp. $\mathcal{HH}^q_R$, denotes the q^{th} hypercohomology of $\mathcal{CC}^*$, resp. $\mathcal{CC}^*_R$, which are listed in Lemma 1.5.10. The spectral sequence degenerates in E_2 page (since $H^p=0$ unless $p=0,1$) and we can easily compute

Proposition 1.5.11.

$$HH^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) = \begin{cases} \mathbb{C} & * = 0 \\ \prod_{i\in\mathbb{Z}} \mathbb{C}\langle Y_i Y_i^* \rangle & * = 1 \\ \prod_{i\in\mathbb{Z}} \mathbb{C}\langle \beta_{i+1/2} \rangle & * = 2 \end{cases} \tag{1.134}$$

$$HH^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) = \begin{cases} R & * = 0 \\ R & * = 1 \\ \prod_{i\in\mathbb{Z}} \mathbb{C}\langle \beta_{i+1/2} \rangle & * = 2 \end{cases} \tag{1.135}$$

Moreover, $HH^1(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})$ is generated by a class locally given by the derivation $Y_{i+1}Y_{i+1}^* - X_i X_i^* = Y_{i+1}\partial Y_{i+1} - X_i \partial X_i$ and $qHH^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) = 0$ for $* \geq 2$.

Definition 1.5.12. Let $\gamma_{\mathcal{O}} \in HH^1(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})$(resp. $\gamma_{\mathcal{O}}^R \in HH^1(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})$) denote the class locally given by $Y_{i+1}Y_{i+1}^* - X_i X_i^*$. Note $\gamma_{\mathcal{O}}$ corresponds to $(Y_i Y_i^*)_i$, where each $Y_i Y_i^*$ is considered as a vector field on $C_i \subset \tilde{\mathfrak{T}}_0$.

As we will see $\gamma_{\mathcal{O}}$ and $\gamma_{\mathcal{O}}^R$ can be obtained as the "infinitesimal action" corresponding to $\mathbb{G}_m$-action mentioned in Section 1.3.2. See Prop 1.6.50, for instance.

1.5.2 Hochschild cohomology of M_ϕ

Let us return to the main problem of computing $HH^*(M_\phi)$. The simple idea is as follows: Given two dg/A_∞ categories(possibly with curvature) $\mathcal{B}_1$ and $\mathcal{B}_2$, we have a map

$$CC^*(\mathcal{B}_1, \mathcal{B}_1) \otimes CC^*(\mathcal{B}_2, \mathcal{B}_2) \to CC^*(\mathcal{B}_1 \otimes \mathcal{B}_2, \mathcal{B}_1 \otimes \mathcal{B}_2) \tag{1.136}$$

Moreover, this is a quasi-isomorphism under certain compactness conditions on $\mathcal{B}_i$, for instance if both are smooth. In addition, given dg category $\mathcal{B}$ with a strict action of the discrete group G, we can compute $HH^*(\mathcal{B}\#G, \mathcal{B}\#G)$ as the derived invariants of the complex $CC^*(\mathcal{B}, \mathcal{B}\#G)$.

Let us first start with a few remarks on $HH^*(\mathcal{B}\#G, \mathcal{B}\#G)$. Let $\mathcal{B}$ be a dg category with a strict action of discrete group G. Let $\mathfrak{M}$ be a bimodule over $\mathcal{B}\#G$. Then we have

$$RHom_{(\mathcal{B}\#G)^e}(\mathcal{B}\#G, \tilde{\mathfrak{M}}) \cong RHom_{(\mathcal{B})^e\#G_\Delta}(\mathcal{B}, \tilde{\mathfrak{M}}) \tag{1.137}$$

This is true since $\mathcal{B}\#G$, as a bimodule over $\mathcal{B}\#G$ can be obtained as a base change under

$$\mathcal{B}^e\#G_\Delta \to (\mathcal{B}\#G)^e \tag{1.138}$$

i.e. it is isomorphic to the induced representation $Ind_{G_\Delta}^{G\times G}(\mathcal{B})$. Hence,

$$RHom_{(\mathcal{B}\#G)^e}(\mathcal{B}\#G, \tilde{\mathfrak{M}}) \cong RHom_G(\mathbb{C}, RHom_{(\mathcal{B})^e}(\mathcal{B}, \tilde{\mathfrak{M}})) \tag{1.139}$$

Here, $RHom_G(\mathbb{C}, \cdot)$ is the derived invariants functor on $D(Rep(G))$. Let $G = \mathbb{Z}$ and C^* be a representation of G, where the generator $1 \in \mathbb{Z}$ acts by $\eta \curvearrowright C^*$. Then, we can construct a chain model for the derived invariants as

$$cocone(C^* \xrightarrow{\eta - 1_{C^*}} C^*) = cone(C^* \xrightarrow{\eta - 1_{C^*}} C^*)[-1] \tag{1.140}$$

Assume $G = \mathbb{Z}$ and the generator $1 \in \mathbb{Z}$ acts on $\mathcal{B}$ by the strict auto-equivalence ψ. Let ψ_* denote the auto-equivalence induced on $CC^*(\mathcal{B}, \tilde{\mathfrak{M}})$. Note the action on $\tilde{\mathfrak{M}}$ is by $t \otimes t^{-1} \in (\mathcal{B}\#\mathbb{Z})^e$ where $t \in \mathcal{B}\#\mathbb{Z}$ denotes the generator of $\mathbb{Z}$ in $\mathcal{B}\#\mathbb{Z}$. We have

$$\begin{array}{ccc} CC^*(\mathcal{B}\#\mathbb{Z}, \tilde{\mathfrak{M}}) & & \\ \downarrow i^* & & \\ CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) & \xrightarrow{\psi_* - 1} & CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) \end{array} \tag{1.141}$$

where i^* is induced by $i : \mathcal{B} \to \mathcal{B}\#\mathbb{Z}$ and where the composition is 0 in cohomology. Presumably, one can write an explicit h

$$\begin{array}{ccc} CC^*(\mathcal{B}\#\mathbb{Z}, \tilde{\mathfrak{M}}) & & \\ \downarrow i^* & \searrow^{h} & \\ CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) & \xrightarrow{\psi_* - 1} & CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) \end{array} \tag{1.142}$$

satisfying $d(h) = (\psi_* - 1) \circ i^*$. However, instead of appealing to this we remark that

(1.141) can be completed to a natural strictly commutative square

$$\begin{array}{ccc} CC^*(\mathcal{B}\#\mathbb{Z}, \tilde{\mathfrak{M}}) & \xleftarrow{\simeq} & C(\mathcal{B}, \tilde{\mathfrak{M}}) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow{\scriptstyle 0} \\ CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) & \xrightarrow{\psi_*-1} & CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) \end{array} \tag{1.143}$$

where $C(\mathcal{B}, \tilde{\mathfrak{M}})$ can be naturally obtained from various hom-sets and Hochschild complexes via natural replacement procedures in derived categories. In other words, (1.143) amounts to writing h as in (1.142) in the derived category. As a result, we have natural map (in the derived category over the base ring, which is R or $\mathbb{C}$)

$$CC^*(\mathcal{B}\#\mathbb{Z}, \tilde{\mathfrak{M}}) \to cocone(CC^*(\mathcal{B}, \tilde{\mathfrak{M}}) \xrightarrow{\psi_*-1} CC^*(\mathcal{B}, \tilde{\mathfrak{M}})) \tag{1.144}$$

(1.144) is a quasi-isomorphism by the previous remarks(such as (1.137) or that (1.140) computes the derived invariants). Moreover, (1.143) and (1.144) generalize to the curved case as well and (1.144) is still a quasi-isomorphism by Lemma 1.5.4. We prefer to notationally pretend that the quasi-isomorphism (1.144) is a chain map.

Using the remarks above we can prove

Proposition 1.5.13. *Let $\mathcal{A}$ be a dg category that satisfies Conditions **C.1-C.3**. Then*

$$\begin{aligned} CC^*(M_\phi, M_\phi) \simeq cocone\big(&CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \otimes CC^*(\mathcal{A}, \mathcal{A}) \\ &\xrightarrow{tr_*\otimes\phi_*-1} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \otimes CC^*(\mathcal{A}, \mathcal{A})\big) \end{aligned} \tag{1.145}$$

i.e. $CC^*(M_\phi, M_\phi)$ is given by the derived invariants of the $\mathbb{Z}$-action on

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \otimes CC^*(\mathcal{A}, \mathcal{A}) \tag{1.146}$$

Proof. We noted (1.144) is a quasi-isomorphism. As a special case, we obtain the quasi-isomorphism

$$\begin{aligned} CC^*(M_\phi, M_\phi) \simeq cocone\big(&CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}) \\ &\xrightarrow{tr_*\otimes\phi_*-1} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z})\big) \end{aligned} \tag{1.147}$$

We can write

$$(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})_{(\mathrm{tr}\otimes\phi)^n} \tag{1.148}$$

as a bimodule over $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$. Here, $(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})_{(\mathrm{tr}\otimes\phi)^n}$ denotes the diagonal bimodule of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$ twisted by $(\mathrm{tr} \otimes \phi)^n$ on the right (i.e. $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$ acts on the right by the composition of $(\mathrm{tr} \otimes \phi)^n$ and the right action on the diagonal bimodule). If $n \neq 0$

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})_{(\mathrm{tr}\otimes\phi)^n}) \simeq 0 \tag{1.149}$$

which follows from

$$RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})_{\mathrm{tr}^n}) = 0 \tag{1.150}$$

unless $n = 0$. The reason (1.149) follows from (1.150) is due to a calculation very similar to the calculation below.

We will not prove (1.150) here but simply mention that its proof is based on showing

$$RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})_{\mathrm{tr}^n}) \simeq RHom_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}(\mathcal{O}^\vee_{graph(\mathrm{tr}^n)}, \mathcal{O}^\vee_{\Delta_{\tilde{\mathfrak{T}}_0}}) \tag{1.151}$$

which is 0 as the graph of tr^n and the diagonal are disjointly supported. For the equivalence one does not need to fully develop Fourier-Mukai theory for compactly supported coherent sheaves on $\tilde{\mathfrak{T}}_0$. Instead, we can write resolutions of $\mathcal{O}_{\Delta_{\tilde{\mathfrak{T}}_0}}$ and $\mathcal{O}_{graph(\mathrm{tr}^n)}$ by infinite direct sums of exterior products of compactly supported sheaves (such as $\mathcal{O}_{C_i} \boxtimes \mathcal{O}_{C_i}$) such that direct sums satisfy some finiteness property (as in (1.72) and (1.74)). We can make the comparison in (1.151) (i.e. compare the homomorphisms of coherent sheaves and induced bimodules) for these exterior tensor products first, and then use this to deduce (1.151).

In summary

$$\begin{aligned} CC^*(M_\phi, M_\phi) \simeq cocone\big(&CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}) \\ &\xrightarrow{\mathrm{tr}_*\otimes\phi_*-1} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A})\big) \end{aligned} \tag{1.152}$$

Now, consider the natural map

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})\otimes CC^*(\mathcal{A},\mathcal{A}) \to CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}) \tag{1.153}$$

We would like to show this gives a quasi-isomorphism. Notice

$$\begin{aligned} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}) \simeq \\ RHom_{(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A})^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}) \simeq \\ RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, RHom_{\mathcal{A}^e}(\mathcal{A}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}) \simeq \\ RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, CC^*(\mathcal{A},\mathcal{A})\otimes\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \end{aligned}$$

The last quasi-isomorphism is due to smoothness of $\mathcal{A}$. The Künneth map

$$\begin{aligned} RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})\otimes CC^*(\mathcal{A},\mathcal{A}) \to \\ RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, CC^*(\mathcal{A},\mathcal{A})\otimes\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \end{aligned} \tag{1.154}$$

is obvious. Clearly, this map strictly commutes with $\mathbb{Z}$-actions; hence, it induces a map between derived $\mathbb{Z}$-invariants of left and right hand sides. We want to show this map is a quasi-isomorphism. The conditions **C.1**,**C.2** imply that $CC^*(\mathcal{A},\mathcal{A})$ has finite dimensional cohomology in each degree. Moreover,

$$RHom_{\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^e}(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \tag{1.155}$$

has bounded below cohomology. This is sufficient to show that the map above induces a quasi-isomorphism. This finishes the proof. □

Corollary 1.5.14. *For $\mathcal{A}$ satisfying the conditions **C.1-C.3**, we have isomorphisms*

$$HH^0(M_\phi) \cong \mathbb{C}, HH^1(M_\phi) \cong HH^1(\mathfrak{T}_0) \oplus HH^1(\mathcal{A})^\phi \cong \mathbb{C}^2 \oplus HH^1(\mathcal{A})^\phi \qquad (1.156)$$

as vector spaces.

Proof. This follows from Prop 1.5.11 and Prop 1.5.13. □

Recall $\mathfrak{T}_0$ denotes the nodal elliptic curve over $\mathbb{C}$.

Corollary 1.5.15. *If $HH^1(\mathcal{A}) = HH^2(\mathcal{A}) = 0$, then $HH^1(M_\phi) \cong \mathbb{C}^2$ and $HH^2(M_\phi) \cong \mathbb{C}$.*

Remark 1.5.16. The analogue of Prop 1.5.13 holds for M_ϕ^R as well. In other words,

$$\begin{aligned} CC^*(M_\phi^R, M_\phi^R) \simeq cocone\big(&CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A},\mathcal{A}) \\ &\xrightarrow{tr_*\otimes\phi_*-1} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A},\mathcal{A})\big) \end{aligned} \qquad (1.157)$$

where $\otimes$ denotes the q-adic completion of the tensor product over $\mathbb{C}$. The proof works similarly. One can alternatively use the semi-continuity (Lemma 1.5.4) since the Künneth map and the map in (1.144) admit natural deformations over R.

Definition 1.5.17. Let $\gamma_\phi \in HH^1(M_\phi)$(resp. $\gamma_\phi^R \in HH^1(M_\phi^R)$) denote the class obtained by "descent" of $\gamma_{\mathcal{O}} \otimes 1$(resp. $\gamma_{\mathcal{O}}^R \otimes 1$).

Similar to $\gamma_{\mathcal{O}}$ and $\gamma_{\mathcal{O}}^R$, these classes come as the infinitesimal action of $\mathbb{G}_m$. This will be shown in Cor 1.6.51 for $\gamma_{\mathcal{O}}^R$.

1.6 A family of endo-functors of M_ϕ

1.6.1 Introduction

In this section, we will use $\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(A_R)$, resp. $\mathcal{G} := \mathcal{G}_R|_{q=0} \subset \tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0 \times Spec(A)$ to define explicit modules over $M_\phi^R \otimes M_\phi^{R,op} \otimes A_R$, resp. $M_\phi \otimes M_\phi^{op} \otimes A$, i.e."families of bimodules parametrized by A_R, resp. A". We can see them as bimodules over M_ϕ^R, resp. M_ϕ, taking values in A_R(resp. A)-modules.

First, define a bimodule over $\mathcal{O}(\tilde{\mathfrak{I}}_R)_{cdg}$ with values in A_R-modules by the formula

$$(\mathcal{F}, \mathcal{F}') \mapsto hom^{\cdot}_{\mathcal{O}_{\tilde{\mathfrak{I}}_R \times \tilde{\mathfrak{I}}_R}}(q^*(R(\mathcal{F})_R), p^*(R(\mathcal{F}')_R) \otimes_{\mathcal{O}_{\tilde{\mathfrak{I}}_R \times \tilde{\mathfrak{I}}_R}} \mathcal{O}_{\mathcal{G}_R}) \tag{1.158}$$

Here, as before q and p are projections onto first and second factor respectively. Recall, the $R(\mathcal{F})_R$ and $R(\mathcal{F}')_R$ are "pseudo-complexes" of sheaves, i.e. graded sheaves whose d^2 is divisible by $q \in R$. See Definition 1.6.3 and Subsection 1.3.3. Tensor product is taken in each factor and hom-complex is as in ordinary complexes. Homomorphisms are over $\mathcal{O}_{\tilde{\mathfrak{I}}_R \times \tilde{\mathfrak{I}}_R}$; hence, we obtain an A_R-module, which is flat by Lemma 1.2.9. Denote the A_R-semi-flat bimodule defined by (1.158) by $\mathcal{G}_R^{pre}$.(A pseudo-complex is A_R-semi-flat if it is flat over A_R in each degree. Similarly, A_R-semi-flatness of a bimodule $\mathfrak{M}$ means each $\mathfrak{M}(L, L')$ is an A_R-semi-flat pseudo-complex, and the bimodule maps are A_R linear.) The only subtlety with semi-flatness of 1.158 is that it involves infinite products of flat A_R-modules. However, this does not cause a problem for the flatness of these infinite products can be shown explicitly, or alternatively one can use [Cha60, Theorem 2.1].

Similarly define a bimodule over $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$ with values in A-modules by

$$(\mathcal{F}, \mathcal{F}') \mapsto hom^{\cdot}_{\mathcal{O}_{\tilde{\mathfrak{I}}_0 \times \tilde{\mathfrak{I}}_0}}(q^* R(\mathcal{F}), p^* R(\mathcal{F}') \otimes_{\mathcal{O}_{\tilde{\mathfrak{I}}_0 \times \tilde{\mathfrak{I}}_0}} \mathcal{O}_{\mathcal{G}}) \tag{1.159}$$

This bimodule is the restriction of the bimodule $\mathcal{G}_R^{pre}$ defined by (1.158) to $q = 0$. It is again A-semi-flat. Denote it by $\mathcal{G}^{pre}$.

Both $\mathcal{G}_R$ and $\mathcal{G}$ are invariant under the action of $\mathfrak{tr} \times \mathfrak{tr} \times 1$. This implies $\mathcal{G}_R^{pre}$ and $\mathcal{G}^{pre}$ satisfy the invariance condition (i.e. carry a $\mathbb{Z}_\Delta$-equivariant structure) in Section 1.4.2. So does the A_R (resp. A)-valued $\mathcal{O}(\tilde{\mathfrak{I}}_R)_{cdg} \otimes \mathcal{A}$(resp. $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg} \otimes \mathcal{A}$)-bimodule $\mathcal{G}_R^{pre} \otimes_{\mathbb{C}} \Delta_{\mathcal{A}}$ (resp. $\mathcal{G}^{pre} \otimes_{\mathbb{C}} \Delta_{\mathcal{A}}$). Recall that $\mathbb{Z}_\Delta$ is the diagonal action corresponding to action generated by $\mathfrak{tr} \times \phi$ on $\mathcal{O}(\tilde{\mathfrak{I}}_R)_{cdg} \otimes \mathcal{A}$ (resp. $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg} \otimes \mathcal{A}$).

Definition 1.6.1. Let $\mathcal{G}_R$ denote the A_R-valued M_ϕ^R-bimodule obtained by descent of $\mathcal{G}_R^{pre} \otimes_{\mathbb{C}} \Delta_{\mathcal{A}}$ as in Section 1.4.2. Similarly, let $\mathcal{G}$ denote the A-valued M_ϕ-bimodule obtained by descent of $\mathcal{G}^{pre} \otimes_{\mathbb{C}} \Delta_{\mathcal{A}}$.

Remark 1.6.2. Clearly, $\mathcal{G} = \mathcal{G}_R|_{q=0}$. Also, $\mathcal{G}_R$ and $\mathcal{G}$ are semi-flat (over A_R, resp. A) as well, i.e. $\mathcal{G}_R(L, L')$ is flat over A_R in each degree (and same for $\mathcal{G}(L, L')$).

1.6.2 Review of generalities on families of objects and their infinitesimal change

In this subsection, we will recall how to make notions such as families of (bi)modules and their infinitesimal change precise. We will mostly follow the first section of [Sei14]. We will write it for curved algebras over R; however, it works for curved categories over other pro-finite local rings as well(hence for uncurved categories). Contrary to most of the rest of the paper we will work with A_∞-algebras/categories and modules, instead of dg algebras/categories. These can be considered as a special case of A_∞-algebras. The only major difference is in the homomorphisms between them; for instance, homomorphisms of A_∞-modules are automatically derived. We used the notation $RHom$ to remove any ambiguity before, but below the hom-complexes are complexes of A_∞-morphisms.

First a preliminary definition:

Definition 1.6.3. A pseudo-complex over the local ring $R = \mathbb{C}[[q]]$ is a graded(and complete in each degree) R-module C^* and a degree 1 endomorphism, "the differential", d such that d^2 is a multiple of $q \in R$. Pseudo-complexes form a curved dg category, where the homomorphisms of a given degree are given by graded module homomorphisms and the curvature element is d^2. We denote this category by $\mathcal{C}_{cdg}(R)$.

Definition 1.6.4. Similarly, we can form a curved category of pseudo-complexes over A_R, which we denote by $\mathcal{C}_{cdg}(A_R)$. Let $\mathcal{C}^{sf}_{cdg}(A_R)$ denote the full (curved)subcategory of $\mathcal{C}_{cdg}(A_R)$ spanned by pseudo-complexes that are q-adically complete and topologically free (i.e. q-adic completion of a free A_R-module) in each degree and whose restrictions to $q = 0$ give K-projective complexes of A-modules.

Definition 1.6.5. Let $\mathcal{B}$ be a curved A_∞-algebra over $R = \mathbb{C}[[q]]$. A family of (right) modules parametrized by $Spf(A_R)$ is an A_∞-homomorphism $\mathcal{B}^{op} \to \mathcal{C}^{sf}_{cdg}(A_R)$. In other words it is a module $\mathfrak{M}$ over $\mathcal{B}$ such that each $\mathfrak{M}(b)^i$ is a topologically free A_R-module, $\mathfrak{M}(b)|_{q=0}$ is K-projective over A and the structure maps are A_R-linear and continuous. Families of left modules and bimodules are defined similarly.

Definition 1.6.6. If $\mathcal{B}_0$ is an uncurved category over $\mathbb{C}$, then a family over it is defined similarly as a functor from $\mathcal{B}_0^{op}$ to K-projective complexes of A-modules.

Remark 1.6.7. $\mathcal{G}_R$ fails to be "semi-K-projective" (i.e. each $\mathfrak{M}(b)|_{q=0}$ is K-projective) but we will pass to a semi-free replacement of it satisfying K-projectivity condition. The phrase "A_R-valued" bimodule/module refers to such a bimodule/module with complete A_R-linear structure as above, where freeness/K-projectivity conditions are dropped. In other words, a given module $\mathfrak{M}$ is A_R-valued if $\mathfrak{M}(b)$ is an A_R-module in each degree for every object b, and the structure maps of the module are linear over A_R.

Now, let us make the condition **G.1** precise:

Definition 1.6.8. Given an A_∞-category $\mathcal{B}_0$ over $\mathbb{C}$, define a coherent twi-family of $\mathcal{B}_0$-modules parametrized by A to be a twisted complex of objects $b \otimes M \in ob(\mathcal{B}_0 \otimes \mathcal{C}_{dg}(A))$, where $b \in ob(\mathcal{B}_0)$ and M is a K-projective complex of flat A-modules whose cohomology is bounded and coherent (finitely generated) over A. In other words, this is the category spanned by such objects $b \otimes M$, $b' \otimes M'$ with hom-sets $\mathcal{B}_0(b, b') \otimes hom^{\cdot}_A(M, M')$. There exists a Yoneda functor from the category of twi-families to $(\mathcal{B}_0)^{mod}_A$, the category of families of $\mathcal{B}_0$-modules parametrized by $Spec(A)$ with A-linear morphisms, see Definition 1.6.17. This functor can be shown to be cohomologically fully faithful. A family that is quasi-isomorphic to an object in the image of the idempotent completion of coherent twi-families is called a coherent family.

Remark 1.6.9. Coherent twi-families are analogous to families of twisted complexes defined in [Sei14], except we allow (K-projective replacements) of coherent sheaves M that are not just vector bundles over the base curve. See [Sei14, Section 1f]. Yoneda lemma- for this Yoneda functor- can be shown in a way similar to well known Yoneda lemma for A_∞-(bi)modules. When we write $b \otimes M$ for a finitely generated module

M over A, we will mean a K-projective replacement of M. We will elaborate more on this later (to replace in a way compatible with $\mathfrak{tr}$ and action). We denote the replacements by $\mathcal{G}^{pre,sf}, \mathcal{G}_R^{pre,sf}, \mathcal{G}^{sf}, \mathcal{G}_R^{sf}$, etc.

Lemma 1.6.10. *K-projective replacements of families exist, and they are unique up to quasi-isomorphism of families.*

Proof. The existence follows from the existence of functorial K-projective replacement functors on $\mathcal{C}_{dg}(A)$ that extend to $\mathcal{C}_{cdg}(A_R) \to \mathcal{C}_{cdg}^{sf}(A_R) \subset \mathcal{C}_{cdg}(A_R)$. See the construction in [Spa88]. Their uniqueness follow from a length filtration argument similar to [Sei14, Lemma 1.10]. More precisely, one only needs to show that the homomorphisms from $\mathfrak{M}^{sf}$ to $\mathfrak{M}'$ is acyclic when $\mathfrak{M}^{sf}$ is K-projective at $q = 0$, and $\mathfrak{M}'|_{q=0}$ is acyclic. As $hom(\mathfrak{M}, \mathfrak{M}')$ deforms $hom(\mathfrak{M}|_{q=0}, \mathfrak{M}'|_{q=0})$, and acyclicity of the latter implies that of the former, we can focus on $hom(\mathfrak{M}|_{q=0}, \mathfrak{M}'|_{q=0})$. The length filtration argument, and K-projectivity of $\mathfrak{M}|_{q=0}$ implies the result. □

Remark 1.6.11. Presumably, one may modify the definition of a family as a functor from $\mathcal{B}_0$ to Ind-coherent sheaves over A and realize coherent families as compact objects of category of such. However we do not need this.

Now, we state a lemma:

Lemma 1.6.12. *If $\mathfrak{M}$ and $\mathfrak{M}'$ are coherent families and $\mathcal{B}_0$ satisfies conditions **C.1-C.2** (see Subsection 1.1.2), then $(\mathcal{B}_0)_A^{mod}(\mathfrak{M}, \mathfrak{M}')$ is cohomologically bounded below and finitely generated over A.*

Proof. This follows from the analogous statement for coherent twi-families and Yoneda lemma. □

Corollary 1.6.13. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ are families over a curved category $\mathcal{B}$ over R such that their restrictions to $q = 0$ are coherent, and $\mathcal{B}|_{q=0}$ satisfies **C.1-C.2**. Then $\mathcal{B}_{A_R}^{mod}(\mathfrak{M}, \mathfrak{M}')$ is cohomologically bounded below and cohomologically finitely generated over A_R.*

Proof. This follows from Lemma 1.6.12 and 1.10.3. □

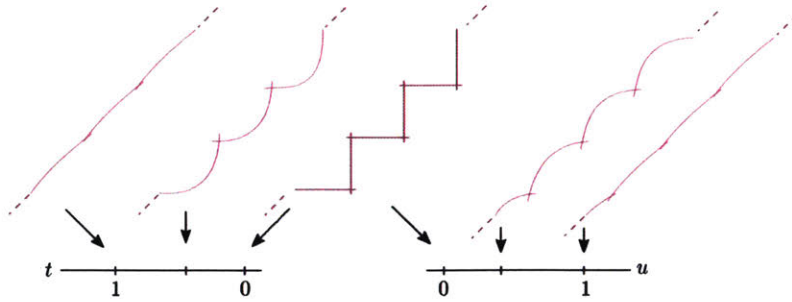

Figure 1-5: The graph $\mathcal{G}$ shown separately over t and u axes of $Spec(\mathbb{C}[t,u]/(tu))$

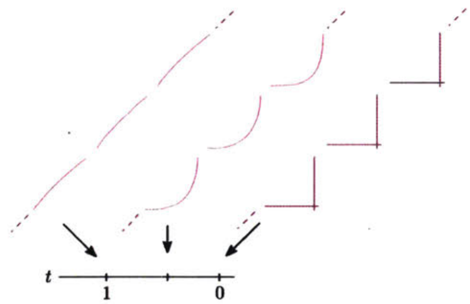

Figure 1-6: The relative partial normalization $\tilde{\mathcal{G}}_t$ which can also be seen as a degeneration of $\mathbb{G}_m$ action on $\mathbb{P}^1 \times \mathbb{Z}$

Remark 1.6.14. Definition 1.6.5 and 1.6.6 are obvious generalization of definition of families of modules over smooth curves in [Sei14]. Moreover, one can define pushforward of families along $Spec(\mathbb{C}[t]) \to Spec(A)$ etc. and the pushforward of (a direct summand of) a family of modules coming from a family of twisted complexes is obviously coherent. For instance, when $\mathcal{B}_0 = \mathbb{C}$, a vector bundle over $Spec(\mathbb{C}[t])$ gives such a family that pushes forward to a coherent family.

Proposition 1.6.15. *The family $\mathcal{G}^{sf}_R|_{q=0} \simeq \mathcal{G}^{sf}$ is coherent.*

Proof. The proof is similar to proof of Prop 1.4.10; thus, we skip some details.

First, apply Lemma 1.4.11 when $X = \mathcal{G}$, $Y = \tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0 \times Spec(A)$, $Z = \tilde{\mathfrak{T}}_0$ and p is the second projection. It implies the family $\mathcal{G}^{pre,sf}$ is ($\mathbb{Z}_\Delta$-equivariantly) quasi-isomorphic to

$$(\mathcal{F}, \mathcal{F}') \mapsto \text{“}RHom_{\tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0}(q^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{G}}^{\vee}, p^*\mathcal{F}')\text{”} \tag{1.160}$$

As before (e.g. (1.65), (1.68) and (1.73)), we put quotation marks since we use a suitable enhancement of (1.160) to a dg functor (see also Remark 1.4.8). Also note that $RHom_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0}(\cdot,\cdot)$ in (1.160) is an abbreviation for

$$RHom_{\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0\times Spec(A)/Spec(A)}(\cdot,\cdot) \tag{1.161}$$

We can use the notation of Section 1.4 and denote this family by $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee}$. Tensoring $\mathcal{O}_{\mathcal{G}}$ with the (pull-back of) short exact sequence $0\to A\to A/u\oplus A/t\to A/(u,t)\to 0$, we obtain a quasi-isomorphism

$$\mathcal{O}_{\mathcal{G}}^\vee \simeq cocone(\mathcal{O}_{\mathcal{G}}^\vee|_{t=0}\oplus \mathcal{O}_{\mathcal{G}}^\vee|_{u=0}\to \mathcal{O}_{\mathcal{G}}^\vee|_{t=u=0}) \tag{1.162}$$

where $\mathcal{O}_{\mathcal{G}}^\vee|_{t=0}$ refers to push-forward of derived restriction of $\mathcal{O}_{\mathcal{G}}^\vee$ along $Spec(\mathbb{C}[u])\to Spec(A)$(similar for others). This quasi-isomorphism is compatible with natural $\mathbb{Z}_\Delta$-actions and it implies

$$\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee} \simeq cone(\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{t=u=0}}\to \mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{t=0}}\oplus \mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{u=0}}) \tag{1.163}$$

$\mathbb{Z}_\Delta$-equivariantly. Hence, it is sufficient to prove $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{t=0}}$, $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{u=0}}$ and $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{t=u=0}}$ descend to coherent families over M_ϕ, after tensoring with $\mathcal{A}$. That $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{t=u=0}}$ is coherent follows from the others. Also, the proof for $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{t=0}}$ is almost the same as the proof for $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{u=0}}$; hence, we prove coherence only for the latter. $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{u=0}}$- as a family over $Spec(\mathbb{C}[u,t]/(ut))$- can be seen as the push-forward of the family $\mathfrak{M}'_{\mathcal{O}^\vee_{\mathcal{G}|_{u=0}}}$ considered as a family of bimodules over $Spec(\mathbb{C}[t])$.

Consider the subscheme $\mathcal{G}|_{u=0}\subset\tilde{\mathfrak{T}}_0\times\tilde{\mathfrak{T}}_0\times Spec(\mathbb{C}[t])$, where we identify $\mathbb{C}[t]$ with $A/(u)$. We proceed similar to Prop 1.4.10. The subscheme $\mathcal{G}_t:=\mathcal{G}|_{u=0}$ is given by

$$tY_{i+1}=Y'_{i+1}, tX'_i=X_i, Y_{i+1}X'_i=0 \text{ on } U_{i+1/2}\times U_{i+1/2}\times Spec(\mathbb{C}[t]) \tag{1.164}$$

$$Y_{i+1}=X'_{i-1}=0, Y'_iX_i=t \text{ on } U_{i+1/2}\times U_{i-1/2}\times Spec(\mathbb{C}[t]) \tag{1.165}$$

It is flat over $\mathbb{C}[t]$ by Lemma 1.2.9 and can be seen as a flat degeneration of the graph of $Spec(\mathbb{C}[t,t^{-1}])$-action on $\tilde{\mathfrak{I}}_0$. Consider the normalization $\pi : \mathbb{P}^1 \times \mathbb{Z} \to \tilde{\mathfrak{I}}_0$, where

$$\pi|_{\mathbb{P}^1 \times \{n\}} : \mathbb{P}^1 \times \{n\} \to C_n \tag{1.166}$$

is an isomorphism. We can see X_i, Y_i as coordinates of $\mathbb{P}^1 \times \{i\}$ satisfying $X_iY_i = 1$. Let $\tilde{\mathcal{G}}_t \subset \mathbb{P}^1 \times \mathbb{Z} \times \mathbb{P}^1 \times \mathbb{Z} \times Spec(\mathbb{C}[t])$ denote a natural flat degeneration of the graph of $\mathbb{G}_m = Spec(\mathbb{C}[t,t^{-1}])$-action on $\mathbb{P}^1 \times \mathbb{Z}$ given by

$$Y_i \mapsto tY_i, X_i \mapsto t^{-1}X_i \tag{1.167}$$

More precisely, $\tilde{\mathcal{G}}_t$ is given by

$$Y_i' = tY_i \text{ on } Spec(\mathbb{C}[Y_i, Y_i', t]) \cong \{X_i \neq 0\} \times \{X_i' \neq 0\} \times Spec(\mathbb{C}[t]) \tag{1.168}$$

$$X_i = tX_i' \text{ on } Spec(\mathbb{C}[X_i, X_i', t]) \cong \{Y_i \neq 0\} \times \{Y_i' \neq 0\} \times Spec(\mathbb{C}[t]) \tag{1.169}$$

$$X_iY_i' = t \text{ on } Spec(\mathbb{C}[X_i, Y_i', t]) \cong \{Y_i \neq 0\} \times \{X_i' \neq 0\} \times Spec(\mathbb{C}[t]) \tag{1.170}$$

The domains on the right side are considered as subsets of

$$\mathbb{P}^1 \times \{i\} \times \mathbb{P}^1 \times \{i\} \times Spec(\mathbb{C}[t]) \tag{1.171}$$

See Figure 1-6 for a picture of $\tilde{\mathcal{G}}_t$, and Figure 1-5 for a picture of $\mathcal{G}$, where $\mathcal{G}_t = \mathcal{G}|_{u=0}$ and $\mathcal{G}|_{t=0}$ are drawn separately.

It is easy to check that $\pi \times \pi \times 1$ restricts to a morphism $\tilde{\pi} : \tilde{\mathcal{G}}_t \to \mathcal{G}_t$. It is an isomorphism over the part of $\mathcal{G}_t$ in $U_{i+1/2} \times U_{i-1/2} \times Spec(\mathbb{C}[t])$. The part of $\mathcal{G}_t$ in $U_{i+1/2} \times U_{i+1/2} \times Spec(\mathbb{C}[t])$ has coordinate ring

$$\begin{gathered}\mathbb{C}[X_i, Y_{i+1}, X_i', Y_{i+1}', t]/(X_iY_{i+1}, X_i'Y_{i+1}', Y_{i+1}X_i', Y_{i+1}' - tY_{i+1}, X_i - tX_i') \\ \cong \mathbb{C}[Y_{i+1}, X_i', t]/(Y_{i+1}X_i')\end{gathered} \tag{1.172}$$

and the part of $\tilde{\mathcal{G}}_t$ over it has coordinate ring

$$\mathbb{C}[X_i, X_i', t]/(X_i - tX_i') \times \mathbb{C}[Y_{i+1}, Y_{i+1}', t]/(Y_{i+1}' - tY_{i+1}) \cong \mathbb{C}[X_i', t] \times \mathbb{C}[Y_{i+1}, t] \tag{1.173}$$

The map induced on coordinate rings from the former to the latter is given by

$$X_i' \mapsto (X_i', 0), Y_{i+1} \mapsto (0, Y_{i+1}), t \mapsto (t, t) \tag{1.174}$$

Hence, it is isomorphic to normalization map of an affine nodal curve relative to $\mathbb{C}[t]$. This description implies the map $\mathcal{O}_{\mathcal{G}_t} \to \tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{G}}_t}$ corresponding to $\tilde{\mathcal{G}}_t \to \mathcal{G}_t$ is injective with cokernel $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{(x_{i+1/2}, x_{i+1/2})} \boxtimes \mathbb{C}[t]$, where $x_{i+1/2}$ still denotes the nodal point in $U_{i+1/2}$. In other words, we have a short exact sequence

$$0 \to \mathcal{O}_{\mathcal{G}_t} \to \tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{G}}_t} \to \bigoplus_{i\in\mathbb{Z}} (\mathcal{O}_{x_{i+1/2}} \boxtimes \mathcal{O}_{x_{i+1/2}}) \boxtimes \mathbb{C}[t] \to 0 \tag{1.175}$$

This is the analogue of (1.70) in Prop 1.4.10. Moreover, using smoothness of $\mathbb{P}^1 \times \{i\}$, we can resolve $\mathcal{O}_{\tilde{\mathcal{G}}_t}|_{\mathbb{P}^1\times\{i\}\times\mathbb{P}^1\times\{i\}\times Spec(\mathbb{C}[t])}$ by sheaves of type $E \boxtimes E'$, where E and E' are coherent. More precisely, it is quasi-isomorphic to a complex of sheaves of type $E \boxtimes E'$(we do not need to consider its direct summands as $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by exterior products, but it would not affect us.) Concretely, one can use

$$\mathcal{O}(-1) \boxtimes \mathcal{O}(-1) \boxtimes \mathbb{C}[t] \xrightarrow{X\boxtimes Y'\boxtimes 1 - Y\boxtimes X'\boxtimes t} \mathcal{O} \boxtimes \mathcal{O} \boxtimes \mathbb{C}[t] \tag{1.176}$$

(1.175) and resolution maps from (1.176) can be made invariant under $\mathfrak{tr} \times \mathfrak{tr} \times 1$ and hence $\tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{G}}_t}$ is quasi-isomorphic to a finite complex of sheaves of type $\bigoplus_{i\in\mathbb{Z}} (\mathfrak{tr}^i \tilde{E}' \boxtimes \mathfrak{tr}^i \tilde{E}') \boxtimes \mathbb{C}[t]$, where $\tilde{E}'$ and $\tilde{E}''$ are push-forwards of compactly supported coherent sheaves on the normalization $\mathbb{P}^1 \times \mathbb{Z}$ (hence, isomorphic to twisted complexes over $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$). This complex is the analogue of (1.71) in Prop 1.4.10. The same holds for $\mathcal{O}_{\mathcal{G}_t}$ by (1.175) and for $\mathcal{O}_{\mathcal{G}_t}^\vee$ by taking duals.

Let $\mathbf{E} = \bigoplus_{i\in\mathbb{Z}} (\mathfrak{tr}^i \tilde{E}'^\vee \boxtimes \mathfrak{tr}^i \tilde{E}'^\vee) \boxtimes \mathbb{C}[t]$. A $\mathbb{C}[t]$-relative version of the idea in the proof of Prop 1.4.10 shows that $\mathfrak{M}'_{\mathbf{E}} \otimes \mathcal{A}$, as a family over $Spec(\mathbb{C}[t])$ descends to a family of bimodules over M_ϕ that is representable by a (family of) twisted complexes(See [Sei14]). Hence, its push-forward along $Spec(\mathbb{C}[t]) \to Spec(A)$ is coherent by Remark 1.6.14 and the same holds for $\mathfrak{M}'_{\mathcal{O}_{\mathcal{G}}^\vee|_{u=0}} \simeq \mathfrak{M}'_{\mathcal{O}_{\mathcal{G}_t}^\vee}$, which finishes the proof. $\square$

Remark 1.6.16. One can presumably show that $\mathcal{G}$ is representable by a family of twisted complexes and $\mathcal{G}_R$ is representable by an (unobstructed) twisted complex over $M_\phi^R \otimes M_\phi^{R,op} \otimes A_R$, as it is a deformation of $\mathcal{G}$. However, we do not need this.

Before making infinitesimal change precise, we need a few more definitions:

Definition 1.6.17. Let $\mathcal{B}_{A_R}^{pre}$ be the category whose objects are families of modules over $\mathcal{B}$ parametrized by $Spf(A_R)$ and morphisms $\mathfrak{M} \to \mathfrak{M}'$ are pre-module homomorphisms $\mathfrak{M} \to \mathfrak{M}'$ over $\mathcal{B}$ in the sense of [Sei08c]. This is a dg category. Let $\mathcal{B}_{A_R}^{mod}$ (or simply $\mathcal{B}_{A_R}$ abusing the notation) denote the subcategory where the pre-module homomorphisms are the A_R-linear ones. One can define such categories for left modules and bimodules similarly.

Remark 1.6.18. The superscripts "pre" in Definition 1.6.17 and in $\mathcal{G}_R^{pre}$ are unrelated.

Remark 1.6.19. More explicitly, a morphism of $\mathcal{B}_{A_R}^{pre}$ can be defined as a sequence of R-linear maps

$$\begin{aligned} f^1 &: \mathfrak{M} \to \mathfrak{M} \\ f^2 &: \mathfrak{M} \otimes \mathcal{B} \to \mathfrak{M}[-1] \\ &\dots \end{aligned}$$

One obtains a morphism of $\mathcal{B}_{A_R}^{mod}$ if A_R-linearity is imposed.

Remark 1.6.20. Notice the hom-sets of $\mathcal{B}_{A_R}^{pre}$ have the structure of an $A_R \otimes A_R^{op}$-module, which comes from the algebra maps

$$A_R \to \mathcal{B}_{A_R}^{pre}(\mathfrak{M}, \mathfrak{M}) \tag{1.177}$$

for each $\mathfrak{M}$. The algebra map sends $a \in A_R$ to $f = (f^1 = a, 0, 0, \dots)$, i.e. to the multiplication by a.

Now, we will make "the infinitesimal change of the family" precise, still following

[Sei14]. For simplicity let us confine ourselves to A_∞-algebras, keeping in mind that one can pass to them from A_∞-categories via constructions similar to total algebra.

Let $\mathcal{B}$ be a curved A_∞-algebra over $R = \mathbb{C}[[q]]$. Let $B(\mathcal{B})$ denote the graded q-adic completion of $T(\mathcal{B}[1]) = \bigoplus_{n\geq 0} \mathcal{B}[1]^{\otimes n}$. Recall $B(\mathcal{B})$ is a (co-unital) coalgebra (in the category of q-adically complete, graded R-modules) and one can see the A_∞-structure as a coderivation μ of degree 1 satisfying $\mu \circ \mu = \frac{1}{2}[\mu, \mu] = 0$ and such that $\mu \circ \epsilon$ is a multiple of $q \in R$, where $\epsilon : R \to B(\mathcal{B})$ is the natural coaugmentation given by inclusion of $\mathcal{B}^{\otimes 0} = R$.

A right A_∞-module structure on graded complete R-module $\mathfrak{M}$ is given by a degree one endomorphism of the comodule $\mathfrak{M} \otimes B(\mathcal{B})$ satisfying co-Leibniz rule with μ. In other words, it is a dg comodule over the dg coalgebra $(B(\mathcal{B}), \mu)$. See [KS09] for details. Note again the tensor product denotes the completed tensor product over R. In this language, the morphisms of $\mathcal{B}^{pre}_{A_R}$ are comodule homomorphisms of $\mathfrak{M} \otimes B(\mathcal{B})$ and the differential on the hom-set is given by the commutator with the endomorphism corresponding to the A_∞-structure.

Remark 1.6.21. If the module $\mathfrak{M}$ is A_R-valued- e.g. if it is a family of modules- then the comodule $\mathfrak{M} \otimes B(\mathcal{B})$ has an A_R-linear structure commuting with the comodule structure maps.

To define the infinitesimal change let us introduce an auxiliary notion:

Definition 1.6.22. Let $\mathfrak{M}$ be a family of modules over $\mathcal{B}$(or more generally an A_R-valued $\mathcal{B}$-module). A pre-connection on $\mathfrak{M}$ along the derivation $D_{A_R} = t\partial_t - u\partial_u$ (see Section 1.7) is an element $\not{D} \in (\mathcal{B}^{pre}_{A_R})^0(\mathfrak{M}, \mathfrak{M})$ such that $[\not{D}, a] = D_{A_R}(a).1_{\mathfrak{M}}$ for every $a \in A_R$ considered as an element of $(\mathcal{B}^{pre}_{A_R})^0(\mathfrak{M}, \mathfrak{M})$.

In other words, a pre-connection is a comodule endomorphism of $\mathfrak{M} \otimes B(\mathcal{B})$ that satisfies Leibniz rule (with respect to natural A_R-linear structure on $\mathfrak{M} \otimes B(\mathcal{B})$). Families of modules (over A_∞-categories over $\mathbb{C}$) parametrized by $Spec(A)$ and pre-connections on them along D_A can be defined analogously.

Remark 1.6.23. The endomorphisms of the comodule $\mathfrak{M} \otimes B(\mathcal{B})$ can be shown to be in one to one correspondence with the pre-module endomorphisms in the sense of

[Sei08c]. Hence, $\not{D}$ can be defined as a sequence of maps

$$\begin{aligned} \not{D}^1 &: \mathfrak{M} \to \mathfrak{M} \\ \not{D}^2 &: \mathfrak{M} \otimes \mathcal{B} \to \mathfrak{M}[-1] \\ &\dots \end{aligned}$$

such that $\not{D}^1$ is a connection of the graded module $\mathfrak{M}$(up to sign) and $\not{D}^i$, $i \geq 2$ are A_R-linear and there is no further constraint. This is how they are defined in [Sei14] and this approach shows that A_R-semi-freeness implies existence of pre-connections.

Definition 1.6.24. Given a pre-connection $\not{D}$ on $\mathfrak{M}$ its deformation class is $def(\not{D}) := d(\not{D}) \in (\mathcal{B}^{pre}_{A_R})^1(\mathfrak{M}, \mathfrak{M})$, i.e. the differential of $\not{D}$ in the dg category $\mathcal{B}^{pre}_{A_R}$. It is A_R-linear due to commutation relation in Definition 1.6.22. Its class in $H^1(\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M}))$ will be denoted by $Def(\not{D})$.

Remark 1.6.25. Two pre-connections on $\mathfrak{M}$ differ by an element of $(\mathcal{B}_{A_R})^0(\mathfrak{M}, \mathfrak{M})$; hence, the classes of their differentials are the same in the category $\mathcal{B}_{A_R}$. We denote it by $Def(\mathfrak{M})$ as well.

Let us show the naturality of this class:

Lemma 1.6.26. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two families of modules with pre-connections $\not{D}$ and $\not{D}'$ resp. Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a closed morphism in $\mathcal{B}^0_{A_R}$. Then the images of $Def(\mathfrak{M})$ and $Def(\mathfrak{M}')$ coincide under the natural maps*

$$H^1(\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M})) \xrightarrow{f \circ (\cdot)} H^1(\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M}')) \xleftarrow{(\cdot) \circ f} H^1(\mathcal{B}_{A_R}(\mathfrak{M}', \mathfrak{M}')) \tag{1.178}$$

Proof. Consider the pre-module homomorphism

$$\not{D}' \circ f - f \circ \not{D} : \mathfrak{M} \to \mathfrak{M}' \tag{1.179}$$

It is A_R-linear, as f is A_R-linear; hence, it falls into category $\mathcal{B}_{A_R}$. Its differential is equal to

$$d(\not{D}') \circ f - f \circ d(\not{D}) = def(\not{D}') \circ f - f \circ def(\not{D}) \tag{1.180}$$

Hence, the $\mathcal{B}_{A_R}$-classes of $def(\not{D}')\circ f$ and $f\circ def(\not{D})$ are the same. □

Corollary 1.6.27. *Let $f:\mathfrak{M}\to\mathfrak{M}'$ be a quasi-isomorphism of families with pre-connections. Then $Def(\mathfrak{M})$ corresponds to $Def(\mathfrak{M}')$ under the natural isomorphism*

$$H^1(\mathcal{B}_{A_R}(\mathfrak{M},\mathfrak{M}))\cong H^1(\mathcal{B}_{A_R}(\mathfrak{M}',\mathfrak{M}')) \tag{1.181}$$

We also want to show naturality of deformation classes under Morita equivalences. Let $\mathcal{B}'$ be another curved A_∞-algebra and let X be a $\mathcal{B}$-$\mathcal{B}'$-bimodule. For a definition of A_∞-bimodules see [Sei13]. One can also see a bimodule as a graded complete module over R such that the bicomodule $B(\mathcal{B})\otimes X\otimes B(\mathcal{B}')$(again tensor product is over R and completed) has a differential compatible with the coderivations of $B(\mathcal{B})$ and $B(\mathcal{B}')$. Such a bimodule X induces a dg functor

$$\mathcal{B}^{mod}\to\mathcal{B}'^{mod} \tag{1.182}$$

between the categories of right modules as well as dg functors

$$\mathcal{B}^{pre}_{A_R}\to\mathcal{B}'^{pre}_{A_R} \tag{1.183}$$

$$\mathcal{B}^{mod}_{A_R}\to\mathcal{B}'^{mod}_{A_R} \tag{1.184}$$

between the categories of families. It is given by $(\cdot)\otimes_{\mathcal{B}}X$. See [Abo10] for a definition. Note also, (1.183)(resp. (1.184) is $A_R\otimes A_R$(resp. A_R)-linear.

Definition 1.6.28. X is called a Morita equivalence if there exists a $\mathcal{B}'$-$\mathcal{B}$ bimodule Y such that

$$Y\otimes_{\mathcal{B}}X\simeq\mathcal{B}' \text{ in the dg category of bimodules over } \mathcal{B}' \tag{1.185}$$

$$X\otimes_{\mathcal{B}'}Y\simeq\mathcal{B} \text{ in the dg category of bimodules over } \mathcal{B} \tag{1.186}$$

In this case, $\mathcal{B}$ and $\mathcal{B}'$ are called Morita equivalent.

If X is a Morita equivalence, then the induced functors (1.182),(1.183) and (1.184) are quasi-equivalences. For a definition of $X \otimes_{\mathcal{B}'} Y$ and $Y \otimes_{\mathcal{B}} X$ see [Abo10]. One can also define them as cotensor products of corresponding dg bicomodules, clarifying the module structure. As the transformation (1.183) is $A_R \otimes A_R$-linear and strictly unital, it sends an endomorphism $\not{D}$ satisfying the commutation rule $[\not{D}, a] = D(a).1$ to such an endomorphism of the image. In other words, it produces a pre-connection of the image and clearly $def(\not{D})$ is sent to the deformation class of the image. Hence, we have proved

Corollary 1.6.29. *Let X be a $\mathcal{B}$-$\mathcal{B}'$-bimodule admitting an "inverse" Y as above and thus inducing an equivalence $\Phi_X : \mathcal{B}^{mod}_{A_R} \to \mathcal{B}'^{mod}_{A_R}$. Then, for a given family (with connection) $\mathfrak{M} \in Ob(\mathcal{B}^{mod}_{A_R})$, the deformation class $Def(\mathfrak{M})$ is sent to $Def(\Phi(\mathfrak{M}))$ under*

$$\Phi_X : H^1(\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M})) \to H^1(\mathcal{B}'_{A_R}(\Phi(\mathfrak{M}), \Phi(\mathfrak{M}))) \tag{1.187}$$

Now, let us make the meaning of infinitesimal change precise following [Sei14]. First, recall there exists a natural map

$$CC^*(\mathcal{B}, \mathcal{B}) \to CC^*(\mathcal{B}^{mod}, \mathcal{B}^{mod}) \tag{1.188}$$

inducing a chain map

$$CC^*(\mathcal{B}, \mathcal{B}) \to \mathcal{B}^{mod}(\mathfrak{N}, \mathfrak{N}) \tag{1.189}$$

for every $\mathcal{B}$-module $\mathfrak{N}$. Seen as a map $T\mathcal{B}[1] \otimes \mathfrak{N} \to \mathfrak{N}$, the latter is given by explicit formulas

$$-\sum \mu^i_{\mathfrak{N}}(1_{\mathfrak{N}} \otimes 1_{\mathcal{B}}^{\otimes r} \otimes g^j \otimes 1_{\mathcal{B}}^{\otimes s}) \tag{1.190}$$

where g^j denotes the components of a cochain $g \in CC^*(\mathcal{B}, \mathcal{B})$ (note again the Kozsul signs or see (1.19) in [Sei14], up to possible differences in signs). Using the same formula, we have

$$CC^*(\mathcal{B} \otimes A_R, \mathcal{B} \otimes A_R) \to \mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M}) \tag{1.191}$$

for every family of $\mathcal{B}$-modules. Moreover, any cochain $\gamma \in CC^*(\mathcal{B}, \mathcal{B})$ induces a cochain in $CC^*(\mathcal{B}\otimes A_R, \mathcal{B}\otimes A_R)$; hence, we have a chain map

$$\Gamma_{\mathfrak{M}} : CC^*(\mathcal{B},\mathcal{B}) \to \mathcal{B}_{A_R}(\mathfrak{M},\mathfrak{M}) \tag{1.192}$$

given by the formula (1.190). Indeed, we can simply treat $\mathfrak{M}$ as a $\mathcal{B}$-module to compute the class. Then, the induced A_∞-module endomorphism on $\mathfrak{M}$ is A_R-linear.

Remark 1.6.30. There are analogues of (1.188) and (1.189) for left modules and bimodules, which also generalizes to families as in (1.192). For instance, given a $\mathcal{B}$-$\mathcal{B}$-bimodule (or more generally $\mathcal{B}''$-$\mathcal{B}$-bimodule) $\mathfrak{M}$, we have the map

$$CC^*(\mathcal{B},\mathcal{B}) \to hom^*_{Bimod(\mathcal{B},\mathcal{B})}(\mathfrak{M},\mathfrak{M}) \tag{1.193}$$

that maps $g \in CC^*(\mathcal{B},\mathcal{B})$ to $-\sum \mu_{\mathfrak{M}}^{i'|1|i}(1_{\mathcal{B}}^{\otimes i'}|1_{\mathfrak{M}}|1_{\mathcal{B}}^{\otimes r}\otimes g^j \otimes 1_{\mathcal{B}}^{\otimes s})$ with similar sign conventions. The image of g will be denoted by $\Gamma_{\mathfrak{M}}(1\otimes g)$.

Definition 1.6.31. Let $\gamma \in CC^1(\mathcal{B},\mathcal{B})$ be a closed cochain and $\mathfrak{M}$ be a family of right $\mathcal{B}$ modules (resp. $\mathcal{B}$-$\mathcal{B}$-bimodules) admitting a pre-connection. We will say $\mathfrak{M}$ follows $[\gamma]$ (resp. $1\otimes\gamma$) if $Def(\mathfrak{M}) = [\Gamma_{\mathfrak{M}}(\gamma)] \in H^1(\mathcal{B}_{A_R}(\mathfrak{M},\mathfrak{M}))$ (resp. $Def(\mathfrak{M}) = [\Gamma_{\mathfrak{M}}(1\otimes\gamma)]$).

Remark 1.6.32. $CC^*(\mathcal{B},\mathcal{B})$ is quasi-isomorphic to endomorphisms of the diagonal bimodule. $\mathfrak{N}\otimes_{\mathcal{B}}\mathcal{B}\simeq\mathfrak{N}$, for any right module $\mathfrak{N}$; thus, we have a natural map

$$CC^*(\mathcal{B},\mathcal{B}) \simeq hom^*_{\mathcal{B}^e}(\mathcal{B},\mathcal{B}) \to hom^*_{\mathcal{B}}(\mathfrak{N}\otimes_{\mathcal{B}}\mathcal{B}, \mathfrak{N}\otimes_{\mathcal{B}}\mathcal{B}) \simeq hom^*_{\mathcal{B}}(\mathfrak{N},\mathfrak{N}) \tag{1.194}$$

It is possible to show this map is $\pm\Gamma_{\mathfrak{N}}$ in cohomology (the notation $\Gamma_{\mathfrak{N}}$ is used for $\mathcal{B}$-modules in general not only families over A_R). It follows in the setting of Corollary 1.6.29 that if a family $\mathfrak{M}$ follows γ then $\Phi(\mathfrak{M})$ follows the class corresponding to γ under the quasi-isomorphism $CC^*(\mathcal{B},\mathcal{B}) \simeq CC^*(\mathcal{B}',\mathcal{B}')$. The same holds for families of left modules and bimodules.

Now, we want to prove the cohomology groups of hom-complexes between families following the same class admit connections along D_{A_R} (see Definition 1.7.2 for the notion of connections along A_R-modules). First some preliminaries:

Definition 1.6.33. Let $E = (E^{\cdot}, d)$ be a chain complex of A_R-modules(resp. A-modules). A pre-connection on $(E^{\cdot}, d)$ is a choice of connections $D_{E^i} : E^i \to E^i$ along D_{A_R}, (resp. D_A) for each i. We will denote a pre-connection by $\not{D}_E$, or simply by $\not{D}$.

We will mostly drop "resp. D_A" keeping in mind that the definitions and proofs would go through analogously.

Definition 1.6.34. Define the Atiyah class $at(\not{D}) : E^{\cdot} \to E^{\cdot}[1]$ of a pre-connection to be the differential of $\not{D} : E^{\cdot} \to E^{\cdot}$ considered as an R-linear map. More precisely,

$$at(\not{D}) := d \circ \not{D} - \not{D} \circ d \tag{1.195}$$

It is a chain map over A_R.

Remark 1.6.35. The cohomology class $[at(\not{D})] \in hom^{\cdot}_{A_R}(E^{\cdot}, E^{\cdot})$ is independent of the choice of pre-connection. Denote it by $At(E^{\cdot})$.

We also include the following, which is proven in [Sei14].

Lemma 1.6.36. *$At(E^{\cdot}) = 0$ if and only if one can find a pre-connection that is a chain map over R.*

Proof. The only if part is clear. Let us prove the if part. If $At(E^{\cdot}) = 0$, that means there exists a pre-connection such that $at(\not{D}) = d(c)$ for some $c \in hom^0_{A_R}(E^{\cdot}, E^{\cdot})$ such that $at(\not{D}) = d(c)$. Thus, $\not{D} - c$ is a degree 0 chain map, which is still a pre-connection since c is linear over A_R. □

We will call such a connection a homotopy connection. Let us note a general lemma that will be of use later:

Lemma 1.6.37. *Let C^* be a chain complex of complete A_R-modules whose cohomology groups are finitely generated over A_R in each degree. Assume $t - 1 \in A_R$ acts* *injectively on C^* (e.g. when it is flat over A_R). Also, assume C^* carries a homotopy connection, i.e. a pre-connection that is also a chain map. Then*

$$H^i(C^*/(t-1)C^*) \cong H^i(C^*)/(t-1)H^i(C^*) \tag{1.196}$$

Proof. First, note $t-1$ acts injectively on any finitely generated A_R-module N that carries a connection along D_{A_R}. To see this consider

$$N_0 = \{x \in N : (t-1)^n x = 0 \text{ for some } n \geq 0\} \subset N \tag{1.197}$$

It is invariant under the connection D_N on N. As it is still finitely generated, there exists $n_0 \geq 0$ such that $(t-1)^{n_0} N_0 = 0$. Given $x \in N_0$

$$0 = D_N((t-1)^{n_0}x) = n_0(t-1)^{n_0-1}tx + (t-1)^{n_0}D_N(x) = n_0(t-1)^{n_0-1}x \tag{1.198}$$

as $(t-1)^{n_0-1}tx = (t-1)^{n_0}x + (t-1)^{n_0-1}x = (t-1)^{n_0-1}x$ and $D_N(x) \in N_0$. Thus, $(t-1)^{n_0-1}N_0$. By induction

$$(t-1)^{n_0-1}N_0 = (t-1)^{n_0-2}N_0 = \cdots = N_0 = 0 \tag{1.199}$$

Now consider the long exact sequence associated to short exact sequence

$$0 \to C^* \xrightarrow{t-1} C^* \to C^*/(t-1)C^* \to 0 \tag{1.200}$$

Putting $N = H^i(C^*)$, we see $H^i(C^*) \xrightarrow{t-1} H^i(C^*)$ is injective; hence, the induced map

$$H^i(C^*)/(t-1)H^i(C^*) \to H^i(C^*/(t-1)C^*) \tag{1.201}$$

is an isomorphism. □

Now let us prove a crucial result, again following [Sei14]:

Proposition 1.6.38. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two families of $\mathcal{B}$-modules with pre-connections.*

Assume there exists a class $[\gamma] \in HH^1(\mathcal{B}, \mathcal{B})$ *such that both* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *follow* $[\gamma]$. *Then,* $\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M}')$ *has Atiyah class* 0*, and it admits a homotopy connection. Moreover, the homotopy connection can be chosen to be compatible with the composition possibly up to homotopy.*

Proof. Recall $\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M}')$ can be thought as the comodule homomorphisms and the map $\not{D}_{\mathfrak{M}'} \circ (\cdot) - (\cdot) \circ \not{D}_{\mathfrak{M}}$ gives a pre-connection on this complex. Its differential is given by

$$d(\not{D}_{\mathfrak{M}'}) \circ (\cdot) - (\cdot) \circ d(\not{D}_{\mathfrak{M}}) = def(\not{D}_{\mathfrak{M}'}) \circ (\cdot) - (\cdot) \circ def(\not{D}_{\mathfrak{M}}) \tag{1.202}$$

which is cohomologous to

$$\Gamma_{\mathfrak{M}'}(\gamma) \circ (\cdot) - (\cdot) \circ \Gamma_{\mathfrak{M}}(\gamma) \tag{1.203}$$

However, as γ is a closed class it induces a natural transformation and this cocycle is null-homotopic. Indeed, $\Gamma_{\mathfrak{M}}(\gamma)$ is the degree 0 part of restriction of a Hochschild cocycle $\Gamma(\gamma)$ to $\mathfrak{M}$ where

$$\Gamma : CC^*(\mathcal{B}, \mathcal{B}) \to CC^*(\mathcal{B}^{mod}, \mathcal{B}^{mod}) \tag{1.204}$$

is a chain map and $\Gamma(\gamma)$ has only degree 0 and 1 parts possibly non-vanishing. It is what Seidel denotes in [Sei14] by γ^{mod}, up to sign. As γ is closed, $\Gamma(\gamma)$ too is closed. The vanishing of the differential implies

$$\Gamma_{\mathfrak{M}'}(\gamma) \circ (\cdot) - (\cdot) \circ \Gamma_{\mathfrak{M}}(\gamma) \tag{1.205}$$

is equal to differential of degree 1 part $\pm\Gamma(\gamma)^1$. Hence, the Atiyah class vanishes and by Lemma 1.6.36, the complex admits a homotopy connection.

For the compatibility with the composition, first correct the pre-connections $\not{D}_{\mathfrak{M}}$ and $\not{D}_{\mathfrak{M}'}$ by A_R-linear cochains bounding $def(\not{D}_{\mathfrak{M}}) - \Gamma_{\mathfrak{M}}(\gamma)$ and $def(\not{D}_{\mathfrak{M}'}) - \Gamma_{\mathfrak{M}'}(\gamma)$ so that (1.202) and (1.203) would actually be equal. Second, note the pre-connection

$\not{D}_{\mathfrak{M}'} \circ (\cdot) - (\cdot) \circ \not{D}_{\mathfrak{M}}$ is automatically compatible with the composition. To make it into a homotopy connection as in Lemma 1.6.36, we have to correct it by a cochain bounding the Atiyah class and $\Gamma(\gamma)^1$ is a natural such choice. The closedness of $\Gamma(\gamma)$ implies

$$\begin{aligned} &\pm \Gamma(\gamma)^1(\cdot) \circ (\cdot) \pm (\cdot) \circ \Gamma(\gamma)^1(\cdot) \pm \Gamma(\gamma)^1(\cdot \circ \cdot) = \\ &\pm \mu^2(\Gamma(\gamma)^1, \cdot) \pm \mu^2(\cdot, \Gamma(\gamma)^1) \pm \Gamma(\gamma)^1(\mu^2(\cdot, \cdot)) = 0 \end{aligned} \tag{1.206}$$

Hence, the pre-connection $\not{D}_{\mathfrak{M}'} \circ (\cdot) - (\cdot) \circ \not{D}_{\mathfrak{M}}$ corrected by $\Gamma(\gamma)^1$ is still compatible with composition μ^2 and is a homotopy connection. □

Corollary 1.6.39. *The cohomology groups $H^i(\mathcal{B}_{A_R}(\mathfrak{M}, \mathfrak{M}'))$, considered as A_R-modules, admit connections along D_{A_R}.*

As we mentioned, the notion of following a class $[\gamma]$ measures infinitesimal change on a family. Now, we will give a recipe to compute the class which a family follows by using $\mathbb{G}_m$-actions.

Remark 1.6.40. The heuristic is as follows: let M be a manifold and G be a Lie group acting smoothly on M. Then, to any $X \in Lie(G)$, one associates a vector field $X^\#$ on M obtained as $X^\#_m = \frac{d(exp(tX).m)}{dt}|_{t=0}$. Smooth equivariant maps relate infinitesimal action on both sides.

We start with some generalities. First, let us define a class of categories and equivariant modules on which one can make sense of the infinitesimal action. As we will follow a formal approach, it will not make a big difference to work over $\mathbb{C}$ or over R.

Definition 1.6.41. Let $\mathcal{B}_0$ be an A_∞-category over $\mathbb{C}$ with a strict action of $\mathbb{G}_m(\mathbb{C})$(i.e. it acts on hom-sets and differentials and compositions are equivariant). Call this action pro-rational if one can choose a product decomposition for each $hom^i_{\mathcal{B}_0}(b, b')$ into countably many rational representations of $\mathbb{G}_m$ such that the decompositions together satisfy the following: if we restrict the differential or one factor of the composition into a finite subproduct it factors through a finite subproduct of the target. Similarly, call a strict action of $\mathbb{G}_m(R)$ on a curved A_∞-category $\mathcal{B}$ over R pro-rational if each

$hom^i_{\mathcal{B}}(b,b')$ admits a product decomposition into R-free completed rational representations (i.e. q-adic completion of rational representations of $\mathbb{G}_m(R)$). Same condition for differential and compositions is imposed (and no extra condition on curvature is needed).

Example 1.6.42. The guiding example is the following: consider the abelian category $Rat(\mathbb{G}_m)$ of rational representations of $\mathbb{G}_m = \mathbb{G}_{m,\mathbb{C}}$. Let $\mathcal{B}_0$ be the dg category of unbounded complexes over $Rat(\mathbb{G}_m)$. Then given such complexes $C^\cdot, D^\cdot$

$$hom^i_{\mathcal{B}_0}(C^\cdot, D^\cdot) = \prod_{n\in\mathbb{Z}} hom_{Rat(\mathbb{G}_m)}(C^n, D^{n+i}) \tag{1.207}$$

Clearly, this decomposition satisfies desired property. One can give analogous example for pseudo-complexes.

The following can be thought as a special case of this example:

Lemma 1.6.43. *$\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$ and $\mathcal{O}(\tilde{\mathfrak{I}}_R)_{cdg}$ are pro-rational.*

Proof. $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$ is a subcategory of chain complexes of sheaves on $\tilde{\mathfrak{I}}_0$. Each degree of these complexes is equipped with a $\mathbb{G}_m$-equivariant structure, and hom's of these complexes are given by products of hom's of these $\mathbb{G}_m$-equivariant sheaves. A closer examination shows each factor in these products is a rational representation, and the product decomposition is compatible with differentials and compositions. The proof is the same for $\mathcal{O}(\tilde{\mathfrak{I}}_R)_{cdg}$. □

Definition 1.6.44. A strictly equivariant module $\mathfrak{M}$ over a (uncurved/curved) category with strict pro-rational action of $\mathbb{G}_m(\mathbb{C})/\mathbb{G}_m(R)$ is called pro-rational if each $\mathfrak{M}^i(b)$ admits a product decomposition such that the module maps satisfy a similar local finiteness as above. Similarly, a family of modules $\mathfrak{M} \in \mathcal{B}_{A_R} = \mathcal{B}^{mod}_{A_R}$ is called strictly pro-rational if it admits a strict pro-rational $\mathbb{G}_m(R)$-equivariant structure as a $\mathcal{B} \otimes A_R$-module. Here $z \in \mathbb{G}_m(R)$ acts on A_R by $t \mapsto zt, u \mapsto z^{-1}u$. In other words it is pro-rational as a $\mathcal{B}$-module and each $\mathfrak{M}^i(b)$ is equivariant as an A_R-module.

Similar definitions make sense for bimodules and tri-modules and so on.

By Section 1.6.3, we have the following (which is similar to Lemma 1.6.43).

Lemma 1.6.45. *A_R, resp. A-valued bimodules $\mathcal{G}_R^{pre}$, resp. $\mathcal{G}^{pre}$ are pro-rational.*

Remark 1.6.46. Putting a pro-rational action on a category is essentially a special case of enriching the category in the complexes/pseudo-complexes in pro-completion of the category of rational(or completed rational) representations. For a rational representation W, we can formally differentiate the $\mathbb{G}_m$-action and obtain an operator $(z\partial_z)^{\#}$ associated to $z\partial_z \in Lie(\mathbb{G}_m)$. If $\mathbb{G}_m$-action on $v \in W$ is by $z \mapsto z^m$, the $(z\partial_z)^{\#}$ action is by m. The pro-completion process remembers the cofiltration of representations and hence we can formally define $(z\partial_z)^{\#}$-operator on the vector spaces underlying pro-objects. As the morphisms between pro-objects are compatible, they intertwine with $(z\partial_z)^{\#}$. Similar statements hold for ind-completion; hence, we can formally define this operator on direct sums of pro-objects of rational representations.

Remark 1.6.47. M_ϕ and M_ϕ^R are not pro-rational in the sense above, neither are $\mathcal{G}_R$ and $\mathcal{G}$ for their definition includes direct sums. However, the complexes involved are direct sums of pro-rational representations; hence, we can define infinitesimal action of $z\partial_z$ at each component separately, and we will use this. It is straightforward to define the notion of "ind-pro rational" as direct sums of pro-rational representations. More generally, such direct sums would be included (and infinitesimal action would be built-in), if we used sub-representations of products of rational representations that are invariant under $(z\partial_z)^{\#}$ (which is already defined on the product). Note, the morphisms of the latter category are assumed to be not only $\mathbb{G}_m$-equivariant, but also compatible with $(z\partial_z)^{\#}$. It is closed under products, direct sums and tensor products.

Lemma 1.6.48. *Given a strictly pro-rational curved A_∞ category $\mathcal{B}$ over R,*

$$(z\partial_z)^{\#} : hom^i_{\mathcal{B}}(b, b') \to hom^i_{\mathcal{B}}(b, b') \tag{1.208}$$

defines a Hochschild cocycle of degree 1. *Same holds for uncurved categories over* $\mathbb{C}$.

Proof. One only needs to check the closedness of the class, which can be proven using explicit computation. Alternatively, one can turn the $\mathbb{G}_m(R)$-action on $\mathcal{B}$ into an action on the bar construction $B(\mathcal{B})$. The A_∞ structure μ can be considered as a coderivation, which is clearly equivariant; hence the action is by dg coalgebra automorphisms. The bar construction is not a product of rational representations, but one can still differentiate the action to obtain a meaningful coderivation $(z\partial_z)^\#$. Let z act by $\rho_z \curvearrowright B(\mathcal{B})$. Differentiate the relation $\rho_z \circ \mu = \mu \circ \rho_z$ to obtain $[\mu, (z\partial_z)^\#] = 0$. □

Lemma 1.6.49. *Let $\mathfrak{M}$ be an A_R-semi-free strictly pro-rational family of $\mathcal{B}$-modules. Then, $\mathfrak{M}$ admits a natural pre-connection and its deformation class is the image of $[(z\partial_z)^\#] \in HH^1(\mathcal{B},\mathcal{B})$ under the natural map $HH^1(\mathcal{B},\mathcal{B}) \to H^1(\mathcal{B}_{A_R}(\mathfrak{M},\mathfrak{M}))$. In other words, $\mathfrak{M}$ follows $(z\partial_z)^\#$.*

Proof. $z \in \mathbb{G}_m(R)$ acts as an operator $\eta_z \in \mathfrak{M}(b,b') \to \mathfrak{M}(b,b')$ and it is possible to differentiate it by the remarks above. Moreover, $\eta_z(ax) = (z.a)\eta_z(x)$ for any $a \in A_R, x \in \mathfrak{M}(b,b')$ and differentiating this relation, we obtain

$$(z\partial_z)^\#_{\mathfrak{M}}(ax) = a(z\partial_z)^\#_{\mathfrak{M}}(x) + (z\partial_z)^\#_{A_R}(a)x \tag{1.209}$$

where $(z\partial_z)^\#_{\mathfrak{M}}(x) =$ "$(\frac{d\eta_z(x)}{dz}|_{z=1})$" and $(z\partial_z)^\#_{A_R}(a) =$ "$(\frac{d(z.a)}{dz}|_{z=1})$" denote the corresponding infinitesimal actions. As it will be remarked in Section 1.7,

$$(z\partial_z)^\#_{A_R}(a) = D_{A_R}(a) \tag{1.210}$$

hence, $(z\partial_z)^\#_{\mathfrak{M}}(x)$ is a pre-connection with no higher maps. We denote it by $\not{D}_{\mathfrak{M}}$.

Now similarly differentiate $\mathbb{G}_m(R)$-action on $\mathfrak{M} \otimes B(\mathcal{B})$. We obtain a coderivation acting on $\mathfrak{M}\otimes B(\mathcal{B})$. Let us project it to $\mathfrak{M}$ and extend as a comodule homomorphism. This way we obtain the pre-connection $\not{D}_{\mathfrak{M}}$ seen as a comodule endomorphism of $\mathfrak{M} \otimes B(\mathcal{B})$.

Now, its differential: first note the differential on dg comodule $\mathfrak{M} \otimes B(\mathcal{B})$ can be written as the sum $\delta_{\mathfrak{M}} + \delta_{\mathcal{B}} \curvearrowright \mathfrak{M} \otimes B(\mathcal{B})$. Here, $\delta_{\mathfrak{M}}$ is the extension of the structure maps of the module as a comodule endomorphism. More explicitly, it is given by

$$\delta_{\mathfrak{M}} = \sum \mu^i_{\mathfrak{M}} \otimes 1_{\mathcal{B}} \otimes 1_{\mathcal{B}} \otimes \cdots \otimes 1_{\mathcal{B}} \tag{1.211}$$

$\delta_{\mathcal{B}}$ denotes the remaining terms, i.e.

$$\delta_{\mathcal{B}} = \sum 1_{\mathfrak{M}} \otimes 1_{\mathcal{B}} \otimes \cdots \otimes \mu^j_{\mathcal{B}} \otimes \cdots \otimes 1_{\mathcal{B}} \tag{1.212}$$

The extension of an operator $\not{D}_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ to $\mathfrak{M} \otimes B(\mathcal{B})$ is via the formula $\not{D}_{\mathfrak{M}} \otimes 1_{\mathcal{B}} \otimes 1_{\mathcal{B}} \otimes \ldots 1_{\mathcal{B}}$ on the each summand $\mathfrak{M} \otimes \mathcal{B}^{\otimes i}$. Let $\overline{\not{D}_{\mathfrak{M}}}$ denote this operator (only for the rest of this proof). The differential of it is given by the commutator

$$(\delta_{\mathfrak{M}} + \delta_{\mathcal{B}}) \circ \overline{\not{D}_{\mathfrak{M}}} - \overline{\not{D}_{\mathfrak{M}}} \circ (\delta_{\mathfrak{M}} + \delta_{\mathcal{B}}) = [\delta_{\mathfrak{M}}, \overline{\not{D}_{\mathfrak{M}}}] + [\delta_{\mathcal{B}}, \overline{\not{D}_{\mathfrak{M}}}] \tag{1.213}$$

By the formulas above, $[\delta_{\mathcal{B}}, \overline{\not{D}_{\mathfrak{M}}}] = 0$. Hence,

$$def(\not{D}_{\mathfrak{M}}) = \delta_{\mathfrak{M}} \circ \overline{\not{D}_{\mathfrak{M}}} - \overline{\not{D}_{\mathfrak{M}}} \circ \delta_{\mathfrak{M}} \tag{1.214}$$

which is a comodule homomorphism whose composition with the projection $\mathfrak{M} \otimes B(\mathcal{B}) \to \mathfrak{M}$ is given by

$$\mu^i_{\mathfrak{M}}(\not{D}_{\mathfrak{M}} \otimes 1_{\mathcal{B}} \cdots \otimes 1_{\mathcal{B}}) - \not{D}_{\mathfrak{M}} \circ \mu^i_{\mathfrak{M}} \tag{1.215}$$

on the summand $\mathfrak{M} \otimes \mathcal{B}[1]^{\otimes i-1}$. For a fixed i, equivariance of $\mu^i_{\mathfrak{M}}$ and the fact that $\not{D}_{\mathfrak{M}}$ is obtained as the infinitesimal action of $z\partial_z \in Lie(\mathbb{G}_m)$ implies

$$\mu^i_{\mathfrak{M}} \circ (\not{D}_{\mathfrak{M}} \otimes 1_{\mathcal{B}}^{\otimes i-1}) + \sum_j \mu^i_{\mathfrak{M}} \circ (1_{\mathfrak{M}} \otimes 1_{\mathcal{B}}^{\otimes j} \otimes (z\partial_z)^{\#} \otimes 1_{\mathcal{B}}^{\otimes i-j-2}) = \not{D}_{\mathfrak{M}} \circ \mu^i_{\mathfrak{M}} \tag{1.216}$$

Hence, the deformation class(projected to $\mathfrak{M}$) is given by

$$-\sum_j \mu^i_{\mathfrak{M}} \circ (1_{\mathfrak{M}} \otimes 1_{\mathcal{B}}^{\otimes j} \otimes (z\partial_z)^{\#} \otimes 1_{\mathcal{B}}^{\otimes i-j-2}) \tag{1.217}$$

The Lemma follows from the definition of $\Gamma_{\mathfrak{M}}(\gamma)$ given by formula (1.190). □

Most of the results/definitions above follow similarly for left modules and bimodules. Indeed, we will use the results for a semi-free, semi K-projective replacement of $\mathcal{G}_R$, which is a family of bimodules.

1.6.3 The deformation class of $\mathcal{G}_R$

To apply the results above, first we should clarify the $\mathbb{G}_m(R)$-action on $\mathcal{G}_R$. First, the graph $\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(A_R)$ (resp. $\mathcal{G} \subset \tilde{\mathfrak{T}}_0 \times \tilde{\mathfrak{T}}_0 \times Spec(A)$) is invariant under the action of $\mathbb{G}_m(R)$ (resp. $\mathbb{G}_m(\mathbb{C})$) which is trivial on the first factor, as in Remark 1.2.3 on the second factor, and by $z : t \mapsto zt, u \mapsto z^{-1}u$ on the third. This is clear from the defining equations (1.31) and (1.32).

Hence, there is an action of $\mathbb{G}_m(R)$(resp. $\mathbb{G}_m(\mathbb{C})$) on $\mathcal{G}_R^{pre}(\mathcal{F},\mathcal{F}')$(resp. $\mathcal{G}^{pre}(\mathcal{F},\mathcal{F}')$) compatible with the action of the same group on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}^{op} \otimes A_R$(resp. $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^{op} \otimes A$) that is trivial on the first factor, as in Remark 1.3.14(resp. Section 1.3.2) in the second factor, and by $z : t \mapsto zt, u \mapsto z^{-1}u$ on the third. A similar action exists for A_R(resp. A)-valued M_ϕ^R (resp. M_ϕ)-bimodule $\mathcal{G}_R$(resp. $\mathcal{G}$). The strict pro-rationality is obvious in the former case. In the latter, we do not have pro-rationality. However, by Remark 1.6.47, it is still sensible to formally differentiate the action and the results above are valid since we have direct sums of pro-rational representations.

However, as mentioned the bimodules above are not semi-free over A_R(resp. A) and they do not satisfy K-projectivity condition. This is easy to resolve by passing to equivariant semi-free replacements that are K-projective over A at $q = 0$. More generally, consider a curved A_∞-category $\mathcal{B}$ with strict and pro-rational action of $\mathbb{G}_m(R)$. Let $\mathfrak{M}$ be a bimodule over $\mathcal{B}$ with a pro-rational action. We may weaken pro-rationality to "differentiability" of the action (see Remarks 1.6.46 and 1.6.47). Then the bar construction gives us a bimodule. To its objects, it assigns a complex which in each degree is given by a product of expressions such as

$$\mathcal{B}^{i_k}(b_k,\cdot)\otimes\mathcal{B}^{i_{k-1}}(b_{k-1},b_k)\otimes\cdots\otimes\mathfrak{M}^{i_l}(b_{l-1},b_l)\otimes\cdots\otimes\mathcal{B}^{i_1}(b_1,b_2)\otimes\mathcal{B}^{i_0}(\cdot,b_1) \quad (1.218)$$

Its differential and structure maps can also be given by explicit expressions involving the structure maps of $\mathfrak{M}$ and A_∞-products of $\mathcal{B}$(alternating sums of expressions such as $1_\mathcal{B}\otimes 1_\mathcal{B}\otimes\cdots\otimes\mu_{\mathfrak{M}}^{p|1|q}\otimes\cdots\otimes 1_\mathcal{B}$) and it also defines a strictly equivariant $\mathcal{B}$-bimodule, where the infinitesimal action can be formally defined again (i.e. by differentiating each component of the tensor product and applying the Leibniz rule, see also Remark 1.6.47). We gave it for bimodules just as an illustration, and this can be done for left modules, right modules, trimodules, and so on. In particular, we can use this construction to replace $\mathcal{G}_R^{pre}$ and $\mathcal{G}_R$ with A_R-semi-free, semi-K-projective families of bimodules over $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ and M_ϕ^R in a canonical way. It is compatible with $\mathfrak{tr}$ in the former case. Let us denote these replacements by $\mathcal{G}_R^{pre,sf}$ and $\mathcal{G}_R^{sf}$.

To find Hochschild cohomology classes that are followed by $\mathcal{G}_R^{pre,sf}$ and $\mathcal{G}_R^{sf}$, we can apply Lemma 1.6.49. The following proposition relates the infinitesimal action cocycle $(z\partial_z)^\#$ to previously defined Hochschild classes.

Proposition 1.6.50. *The infinitesimal action cocycle $(z\partial_z)^\#$ defined in Lemma 1.6.48 of $\mathbb{G}_m(R)$-action on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ has the class $\gamma_\mathcal{O}^R$.*

Proof. This follows from local to global techniques of Section 1.5. For instance, consider the isomorphism (1.106). It is based on maps

$$\begin{array}{ccc} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg},\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) & \longrightarrow & CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg},\mathcal{C}oh(U)_R) \\ & & \uparrow \simeq \\ & & CC^*(\mathcal{C}oh(U)_R,\mathcal{C}oh(U)_R) \end{array} \quad (1.219)$$

where $U=U_{i+1/2}$ or V_i and $\mathcal{C}oh(U)_R$ is a curved deformation of $\mathcal{C}oh(U)$. We can replace $\mathcal{C}oh(U)$ by the image of the restriction functor from $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ and use a strictly $\mathbb{G}_m(R)$-equivariant model such that this functor would be strictly equivariant too. Moreover, it is easy to see that the images of cocycles $(z\partial_z)^\#$ in lower right and upper left complexes correspond in cohomology. Similar statements hold for the restriction maps

$$CC^*(\mathcal{C}oh(U_{i+1/2})) \to CC^*(\mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q)) \tag{1.220}$$

and so on. Hence, it is enough to prove the infinitesimal action cocycle $(z\partial_z)^\#$ is the same as local building blocks of $\gamma_{\mathcal{O}}^R$. In other words, we wish to show $Y_{i+1}Y_{i+1}^* - X_iX_i^*$ corresponds to cocycle $(z\partial_z)^\#$ for the action

$$z : Y_{i+1} \mapsto zY_{i+1}, X_i \mapsto z^{-1}X_i \tag{1.221}$$

under the Hochschild-Kostant-Rosenberg isomorphism of [Fn07, Appendix, Theorem 2]. Examining this isomorphism, we can see $Y_{i+1}Y_{i+1}^* - X_iX_i^*$ corresponds to derivation $Y_{i+1}\partial_{Y_{i+1}} - X_i\partial_{X_i}$; i.e., to $(z\partial_z)^\#$ for the given action. □

Corollary 1.6.51. *Consider the $\mathbb{G}_m(R)$-action on M_ϕ^R. The infinitesimal action cocycle $(z\partial_z)^\#$ has the same class as γ_ϕ^R.*

Proof. Follows from a similar examination of the isomorphism in Prop 1.5.13. □

Corollary 1.6.52. *The family $\mathcal{G}_R^{pre,sf}$ of $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$-bimodules follows $1\otimes\gamma_R^{\mathcal{O}} \in HH^1(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}^e)$. Similarly, the family of M_ϕ^R-bimodules $\mathcal{G}_R^{sf}$ follows $1\otimes\gamma_R^\phi \in HH^1((M_\phi^R)^e)$.*

Proof. This follows from the remarks at Section 1.6.3 about the $\mathbb{G}_m(R)$-action on $\mathcal{G}_R^{pre}, \mathcal{G}_R$ as well as on families $\mathcal{G}_R^{pre,sf}, \mathcal{G}_R^{sf}$ by applying Lemma 1.6.49. □

Remark 1.6.53. Results similar to Corollary 1.6.52 hold for $\mathcal{G}_R^{pre,sf}|_{q=0}$, which is a semi-free replacement of $\mathcal{G}^{pre}$ and for $\mathcal{G}_R^{sf}|_{q=0}$, which is a semi-free replacement of $\mathcal{G}$.

1.7 Modules over A_R

Recall $A_R = \mathbb{C}[u,t][[q]]/(ut-q)$ and $A = A_R/(q) = \mathbb{C}[u,t]/(ut)$.

Definition 1.7.1. Let D_A, resp. D_{A_R} denote the derivation $t\partial_t - u\partial_u$ on A, resp. A_R.

Note that D_{A_R} is R-linear on A_R and it can be seen as the infinitesimal action of the $\mathbb{G}_m$-action given by $z : t \mapsto zt, u \mapsto z^{-1}u$.

Definition 1.7.2. Let M be a topological module over A_R. A connection on M along D_{A_R} is a linear map $D_M : M \to M$ satisfying $D_M(fm) = D_{A_R}(f)m + fD_M(m)$. We will often refer to it simply as a connection on M.

A connection can be seen as an infinitesimal version of an equivariant structure with respect to the action above.

Remark 1.7.3. A_R is a Noetherian ring. Moreover, finitely generated modules over A_R are automatically complete with respect to q-adic topology.

Let us first prove:

Proposition 1.7.4. *Let M be a finitely generated module over A_R, which can be endowed with a connection D_M. Assume $M/(t-1)M$ is a q-torsion module over $A_R/(t-1) \cong R$. Then, M is q-torsion.*

Proof. We can assume $M/(t-1)M = 0$, by replacing M by q^iM for $i \gg 0$. Consider M/uM. It is a finitely generated module over $\mathbb{C}[t]$ and carries a connection along the derivation $t\partial_t$ on $\mathbb{C}[t]$. As it vanishes at $t = 1$, it has to be torsion over $\mathbb{C}[t]$ and as it carries a connection its annihilator is invariant under the action $z \in \mathbb{G}_m : t \mapsto zt$. Hence, $ann_{\mathbb{C}[t]}(M/uM) = (t^{n-1})$ for some n, implying $t^nM \subset utM = qM \subset tM$. This shows t-adic and q-adic topologies on M coincide and M is t-adically complete as well. Hence, we can see M as a module over $\mathbb{C}[u][[t]]$ that is finitely generated over $A_R = \mathbb{C}[u,t][[ut]] \subset \mathbb{C}[u][[t]]$.

Given $s \in M$, consider $\mathbb{C}[u][[t]].s \subset M$. It is an A_R submodule and A_R is Noetherian; hence, $\mathbb{C}[u][[t]].s \cong \mathbb{C}[u][[t]]/ann(s)$ is finitely generated over A_R as well, where $ann(s) := ann_{\mathbb{C}[u][[t]]}(s)$. Now dividing by u again, this implies

$$\mathbb{C}[u][[t]]/(ann(s) + u\mathbb{C}[u][[t]]) \tag{1.222}$$

is a module over $\mathbb{C}[[t]]$ that is finitely generated over $\mathbb{C}[t] = A_R/(u)$. By the classification of finitely generated modules over PIDs, we see that this module is indeed

t-torsion, i.e. there exists N' such that

$$t^{N'} \in ann(s) + u\mathbb{C}[u][[t]] \tag{1.223}$$

In other words $ann(s)$ contains an element that is of the form $t^{N'} + O(u)$.

Now, pick a set of generators $s_1, \dots, s_r$ and N_i such that there exists an element in $ann(s_i)$ that is of the form $t^{N_i} + O(u)$. The product of these annihilating elements is of the form $t^N + O(u) \in \mathbb{C}[u][[t]]$ and annihilates M.

Let $\hat{M}$ denote the completion of M with respect to the ideal $(u) \subset \mathbb{C}[u][[t]]$. It is a finitely generated module over $\mathbb{C}[[u,t]]$ and it also carries a connection along $t\partial_t - u\partial_u$. Hence, its annihilator J over $\mathbb{C}[[u,t]]$ satisfy the conditions of Lemma 1.7.5 below. Moreover, by the above paragraph, an element of the form $t^N + O(u)$ is in J. Hence, J cannot be contained in (0) or (u) and the prime ideals belonging to J contain t. This implies there exists N_1 such that $t^{N_1} \in J$. In other words, t^{N_1} annihilates $\hat{M}$.

We want to use this to show $t^{N_1}M = 0$, implying M is q-torsion.

Our approach is using [AM69, Theorem 10.17], namely the kernel of the (u-adic) completion map $M \to \hat{M}$ is the set of elements of M annihilated by some element of $1 + (u)$. By above, $t^{N_1}M$ is in the kernel and we see that M is annihilated by an element of the form $t^{N_1}(1+O(u))$. Consider $t^{N_1}M/t^{N_1+1}M$. It is annihilated by t and an element of the form $1+O(u)$. Hence, we can get rid of multiples of t in $1+O(u)$ and see that there exists a polynomial $f(u)$ such that $1+uf(u)$ annihilates $t^{N_1}M/t^{N_1+1}M$. $t^{N_1}M/t^{N_1+1}M$ is a finitely generated module over $\mathbb{C}[u]$ with a connection along $-u\partial_u$. Hence, by the classification of modules over PIDs, it has to be finite direct sum of copies of $\mathbb{C}[u]$ and of $\mathbb{C}[u]/(u^l)$, for various $l \geq 1$. Thus, that it is annihilated by $1 + uf(u)$ implies $t^{N_1}M/t^{N_1+1}M = 0$. In other words,

$$t^{N_1}M = t^{N_1+1}M = t^{N_1+2}M = \dots \tag{1.224}$$

But recall M is complete in t-adic topology i.e. the completion map

$$M \to \lim_{\leftarrow} M/t^n M \tag{1.225}$$

is an isomorphism. Thus, $M \simeq M/t^{N_1}M$ and $t^{N_1}M = 0$. This implies $q^{N_1}M = 0$, finishing the proof. □

Lemma 1.7.5. *Consider the action of the group $\mathbb{C}^*$ on $\mathbb{C}[[u,t]]$, where $z \in \mathbb{C}$ acts by $t \mapsto zt, u \mapsto z^{-1}u$. Consider an ideal J that is invariant under the action, or equivalently $(t\partial_t - u\partial_u)(J) \subset J$. If J is a prime ideal, then it is one of $(0), (u), (t), (u,t)$. If J is an arbitrary invariant ideal, then the prime ideals belonging to J(its prime components) are among $(0), (u), (t), (u,t)$(thus $\sqrt{J}$ is the intersection of some of $(0), (u), (t), (u,t)$).*

Proof. The second statement is an easy corollary of the first: namely assume $J = \bigcap \mathfrak{q}_i$ is a a minimal primary decomposition, which exist by [AM69, Theorem 7.13]. Then prime radicals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ are invariants of J, by [AM69, Theorem 4.5]. J is fixed under the action of $\mathbb{C}^*$ on $\mathbb{C}[[u,t]]$. Hence, so are its prime components.

Now assume J is prime. As the Krull dimension of $\mathbb{C}[[u,t]]$ is 2, the height of J can be 0,1 or 2. If it is 0, $J = (0)$. If it is 2, $J = (u,t)$ as $\mathbb{C}[[u,t]]$ is a local ring. Hence, assume J has height 1. As $\mathbb{C}[[u,t]]$ is a UFD, J is principal(see [Har77, Prop 1.12A]). Take a prime $f \in J$ such that $J = (f)$. The invariance of J under $\mathbb{C}^*$-action implies that for all $z \in \mathbb{C}^*$, $z.f$ is a generator as well; hence, it differs from f by a unit of $\mathbb{C}[[u,t]]$.

Partially order the monomials as

$$t^a u^b \leq t^c u^d \text{ iff } a \leq c \text{ and } b \leq d \tag{1.226}$$

As the units of $\mathbb{C}[[u,t]]$ are of the form $\alpha + O(u,t)$, where $\alpha \in \mathbb{C}^*$, the set of non-zero monomials of f that are minimal with respect to this order does not change when we multiply it with a unit. Moreover, the coefficients of the minimal monomials are multiplied by the same constant $\alpha \in \mathbb{C}^*$. On the other hand, the $\mathbb{C}^*$-action acts on the monomial $t^i u^j$ by z^{i-j}. Thus, the difference $i-j$ has to be the same for all minimal monomials. But if $i - j = i' - j'$, then either $t^i u^j \leq t^{i'} u^{j'}$ or $t^{i'} u^{j'} \leq t^i u^j$. Hence, there can be only one minimal non-zero monomial of f. Call it $t^i u^j$. As all the other monomials are divisible by it, $t^i u^j$ differs from f by a unit; hence, $J = (t^i u^j)$. As J

is prime, it is either (t) or (u). This finishes the proof of the lemma. □

We wish to use Prop 1.7.4 to prove some properties of modules of higher rank. For that we need another lemma:

Lemma 1.7.6. *Let M be a finitely generated module over A_R, which can be endowed with a connection D_M. Then $M^\vee = Hom_{A_R}(M, A_R)$ is free over A_R.*

Proof. $M^\vee$ is finitely generated and admits a connection as well. Assume $M^\vee \neq 0$. Consider the local ring $(A_R)_{(u,t)} \subset \mathbb{C}[[u,t]]$. Note, we do not take its q-completion. This is a Noetherian local ring whose (u,t)-adic completion is $\mathbb{C}[[u,t]]$. Hence, by [AM69, Cor 11.19], they have the same Krull dimension, which is 2. Thus,

$$depth((A_R)_{(u,t)}) \leq dim((A_R)_{(u,t)}) = 2 \tag{1.227}$$

by [Eis95, Prop 18.2]. As u, t is a regular sequence (i.e. u is not a zero divisor on A_R and t is not a zero divisor on $A_R/(u)$), $depth((A_R)_{(u,t)}) = 2$.

Moreover, $(A_R)_{(u,t)}$ is a regular local ring, hence it has finite global dimension (see [Eis95, Cor 19.6]). Thus, we can apply the Auslander-Buchsbaum formula ([Eis95, Thm 19.9]) to every finitely generated module and obtain

$$depth(M') + pd(M') = depth((A_R)_{(u,t)}) = 2 \tag{1.228}$$

where $pd, depth$ are over $(A_R)_{(u,t)}$. Let $M' := (M^\vee)_{(u,t)}$. Clearly, u, t is a regular sequence for $M' \cong Hom_{A_R}(M, (A_R)_{(u,t)})$; hence, if $M' \neq 0$, it has depth 2 and projective dimension 0. Thus, it is projective. As $(A_R)_{(u,t)}$ is local, this implies $M' = (M^\vee)_{(u,t)}$ is free.

In particular, this implies that the $A = \mathbb{C}[u,t]/(ut)$-module $(M^\vee)_0 = M^\vee/qM^\vee$ is free around $(0,0)$, i.e. $(M^\vee/qM^\vee)_{(u,t)}$ is free over $A_{(u,t)}$. Using the connection on $M^\vee$, one can show the freeness of $((M^\vee)_0)_t$, resp. $((M^\vee)_0)_u$ over $\mathbb{C}[t,t^{-1}]$, resp. $\mathbb{C}[u,u^{-1}]$. This is sufficient to conclude $(M^\vee)_0$ is free.

This implies the freeness of $M^\vee$ as well: choose a basis $A^n \xrightarrow{\cong} (M^\vee)_0$ and lift it to a linear map $A_R^n \to M^\vee$. A simple semi-continuity argument would show this is an isomorphism as well, finishing the proof. ◻

Remark 1.7.7. The proof implies the freeness of any finitely generated module with connection for which u, t is a regular sequence. However, we do not need this.

We can use Prop 1.7.4 and Lemma 1.7.6 to prove:

Proposition 1.7.8. *Let M be a finitely generated module over A_R that can be endowed with a connection. Then, M is free up to q-torsion.*

Proof. Consider the natural map $M \to M^{\vee\vee}$. This map is compatible with the connections; thus, both its kernel and cokernel are q-torsion by Prop 1.7.4. $M^{\vee\vee}$ is free by Lemma 1.7.6, implying M is free up to q-torsion. ◻

Remark 1.7.9. One can use Prop 1.7.8 to produce $M^{\vee\vee} \hookrightarrow M$ with q-torsion cokernel.

1.8 A rank 2 lattice inside $HH^1(M_\phi^R)$

In this section, we will find a subgroup of $HH^1(M_\phi^R)$ that contains γ_ϕ^R, that is isomorphic to $\mathbb{Z}^2$ and that is preserved under Morita equivalences. The basic idea is as follows:

Given an A_∞-category $\mathcal{B}$, one can define the derived Picard group as the functor from commutative rings to groups sending

$$DPic : T \mapsto \{T\text{-semifree, invertible } \mathcal{B}^e \otimes T\text{-modules}\}/\text{quasi-isomorphism} \quad (1.229)$$

Here we call a $\mathcal{B}^e \otimes T$-module $\mathfrak{M}$ invertible if there exists another $\mathcal{B}^e \otimes T$-module $\mathfrak{N}$ such that

$$\mathfrak{M} \otimes_{\mathcal{B}} \mathfrak{N} \simeq \mathfrak{N} \otimes_{\mathcal{B}} \mathfrak{M} \simeq \mathcal{B} \otimes T \quad (1.230)$$

In other words it is a "family" of invertible $\mathcal{B}$-bimodules parametrized by T. The group structure is given by $(\mathfrak{M}, \mathfrak{M}') \mapsto \mathfrak{M} \otimes_{\mathcal{B}} \mathfrak{M}'$ See [Kel04] for a infinitesimal and derived version of it. In [Kel04], the author also argues to show that the Lie algebra of this group is isomorphic to $HH^1(\mathcal{B})$, with the Gerstenhaber bracket as the Lie bracket. This group functor is obviously Morita invariant. Hence, it is natural to look at its coroots, i.e. maps $\mathbb{G}_m \to DPic$ and the induced image of $(z\partial_z) \in Lie(\mathbb{G}_m)$. This subset will be a lattice in our case. However, we will not formally refer to this group functor again. Instead, we will simply use of group like families of bimodules, whose definition is close to definition above.

In the case of mapping tori, notice another $\mathbb{G}_m(\mathbb{C})$, resp. $\mathbb{G}_m(R)$-action on M_ϕ, resp. M_ϕ^R, this time rational, resp. completed rational(which we will informally refer as another $\mathbb{G}_m/\widehat{\mathbb{G}_m}$-action). By definition $\mathcal{B}\#\mathbb{Z}$ is automatically equipped with an extra $\mathbb{Z}$-grading. Recall the morphisms from b_1 to b_2 are

$$\bigoplus_{g\in\mathbb{Z}} hom_{\mathcal{B}}(g(b_1), b_2) \tag{1.231}$$

and we declare $hom_{\mathcal{B}}(g(b_1), b_2)$ to be the degree g part in this extra grading. In particular, M_ϕ and M_ϕ^R carry this extra grading and we let $z \in \mathbb{G}_m$ act by z^g on degree g part.

Remark 1.8.1. When $\mathcal{A} = \mathbb{C}$ and $\phi = 1_{\mathbb{C}}$, this new action corresponds to twist by line bundles in $Pic^0(\mathfrak{T}_R/R)$.

It is easy to see that the new and old $\mathbb{G}_m/\widehat{\mathbb{G}_m}$-actions commute strictly. Hence, we have an action of $\mathbb{G}_m \times \mathbb{G}_m/\widehat{\mathbb{G}_m} \times_{SpfR} \widehat{\mathbb{G}_m}$ on M_ϕ/M_ϕ^R. We want to organize them in group-like families of bimodules. First, let us give meaning to this more general notion of families, mimicking [Sei14] and [Kel04]. We will work over R, as everything is similar over $\mathbb{C}$.

Definition 1.8.2. Let $\mathcal{B}$ be a curved A_∞-category over R. Let T be a topologically finitely generated complete commutative R-algebra (examples are given by formal completions of affine subschemes of affine varieties). A family of $\mathcal{B}$-modules/bimodules parametrized by T is an A_∞-module/bimodule $\mathfrak{M}$ over $\mathcal{B}$ such that each $\mathfrak{M}(b)$ resp. $\mathfrak{M}(b,b')$ is a free T-module (in each degree), $\mathfrak{M}(b)/q\mathfrak{M}(b)$ resp. $\mathfrak{M}(b,b')/q\mathfrak{M}(b,b')$ is K-projective over T/qT and the module maps are T-linear. The morphisms between such families can be defined in a analogous way to Definition 1.6.17 and Remark 1.6.19. We denote the category of families(with T-linear morphisms) by $\mathcal{B}_T^{mod}$ or simply by $\mathcal{B}_T$.

Contrary to previous section, we will assume T is the ring of functions on a formal affine scheme that is smooth over R. We can define pre-connections in a similar way:

Definition 1.8.3. A pre-connection ∇ on the family $\mathfrak{M}$ of right $\mathcal{B}$-modules parametrized by T is a $B(\mathcal{B})$-comodule map

$$\mathfrak{M} \otimes B(\mathcal{B}) \rightarrow \Omega^1_{T/R} \otimes \mathfrak{M} \otimes B(\mathcal{B}) \tag{1.232}$$

that satisfies the Leibniz rule with respect to T.

This can again be seen as a collection of maps

$$\begin{aligned} \nabla^1 : \mathfrak{M}(b_0) &\rightarrow \Omega^1_{T/R} \otimes \mathfrak{M}(b_0) \\ \nabla^2 : \mathfrak{M}(b_1) \otimes \mathcal{B}(b_0, b_1) &\rightarrow \Omega^1_{T/R} \otimes \mathfrak{M}(b_0) \\ \nabla^3 : \mathfrak{M}(b_2) \otimes \mathcal{B}(b_1, b_2) \otimes \mathcal{B}(b_0, b_1) &\rightarrow \Omega^1_{T/R} \otimes \mathfrak{M}(b_0) \\ &\dots \end{aligned}$$

such that ∇^1 satisfies the Leibniz rule and ∇^i are T-linear for $i > 1$. See [Sei14]. The deformation class

$$def(\nabla) \in (\mathcal{B}_T^{mod})^1(\mathfrak{M}, \Omega^1_{T/R} \otimes \mathfrak{M}) \tag{1.233}$$

has a similar definition, i.e. as the differential of ∇. The class of $def(\nabla)$ is independent of the choice of pre-connection and will again be denoted by $Def(\mathfrak{M})$. Everything works for bimodules in a similar way.

Now we define group-like families. The moral of the definition is simple: For a (formal) affine group G the transformations "$G \rightarrow DPic(\mathcal{B})$" are the families parametrized by G and group-like families are those corresponding to group homomorphisms "$G \rightarrow DPic(\mathcal{B})$". More explicitly, let $T = \mathcal{O}(G_0)[[q]] = \mathcal{O}(G)$, where G_0 is an affine algebraic group over $\mathbb{C}$ and $G = G_0 \times_{\mathbb{C}} Spf(R)$. T is a Hopf algebra in the category of complete R-modules. Let

$$m, f_1, f_2 : G \times G \to G \tag{1.234}$$

denote multiplication, first and second projection respectively.

Definition 1.8.4. A group-like family of invertible bimodules parametrized by T is a family $\mathfrak{M}$ such that $f_1^*\mathfrak{M} \otimes_{\mathcal{B}} f_2^*\mathfrak{M} \simeq m^*\mathfrak{M}$.

This condition can be phrased in terms of the Hopf algebra structure of T. In particular, it is easy to see that a (strict) G-action on $\mathcal{B}$ gives a group-like family.

Example 1.8.5. We mentioned an action of $\widehat{\mathbb{G}_m \times \mathbb{G}_m} := \widehat{\mathbb{G}_m} \times_{Spf(R)} \widehat{\mathbb{G}_m}$ on M_ϕ^R. This was a pointwise action; however, it is easy to see it as a group-like family parametrized by $\mathbb{C}[z_1^\pm, z_2^\pm][[q]]$. Denote it by ρ_{uni}. Similarly, there is group-like family of bimodules over M_ϕ parametrized by $\mathbb{G}_m \times \mathbb{G}_m$.

Lemma 1.8.6. *The restriction of the family parametrized by $\widehat{\mathbb{G}_m \times \mathbb{G}_m}$ to two different R-points are different. In other words, "$\widehat{\mathbb{G}_m \times \mathbb{G}_m}(R) \to DPic_{M_\phi^R}(R)$" is injective.*

Proof. Assume the family is trivial over an element of $\widehat{\mathbb{G}_m \times \mathbb{G}_m}(R) = R^* \times R^*$. This implies there exists $z_1, z_2 \in R^*$ such that the bimodule corresponding to z_1 for the action in Remark 1.4.6 is quasi-isomorphic to bimodule corresponding to z_2 for the action coming from extra grading.

First, let us show $z_1 = 1$. Pick a smooth R-point $p \in \tilde{\mathfrak{T}}_R$ such that the restriction to $q = 0$ is on $C_0 \subset \tilde{\mathfrak{T}}_0$. We can represent $\mathcal{O}_p$ as a deformation of a mapping cone $Cone(\mathcal{O}_{C_0}(-1) \to \mathcal{O}_{C_0})$; hence, by a twisted complex of this form, which we also denote by $\mathcal{O}_p$ (it is an unobstructed object over $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$, the differential of this twisted complex may include other terms that are $O(q)$). Hence, we obtain a subcategory $\mathcal{O}_p \otimes \mathcal{A} = \{\mathcal{O}_p \otimes a : a \in ob(\mathcal{A})\} \subset tw^\pi(M_\phi^R)$. The z_1-action moves this subcategory to $\mathcal{O}_{z_1.p} \otimes \mathcal{A}$. Moreover, the morphism $\mathcal{O}_{C_0}(-1) \to \mathcal{O}_{C_0}$ is of degree 0, in the extra grading; hence, z_2 fixes $\mathcal{O}_p \otimes \mathcal{A}$. It is not hard to prove $\mathcal{O}_p \otimes \mathcal{A}$ is orthogonal to $\mathcal{O}_{z_1.p} \otimes \mathcal{A}$ unless $z_1.p = p$. Thus, $z_1 = 1$.

Hence, by assumption the bimodule ${}_{z_2}(M_\phi^R)_1$ corresponding to z_2-action is quasi-isomorphic to diagonal bimodule. The bimodule ${}_{z_2}(M_\phi^R)_1$ has the same underlying pseudo-complexes but the M_ϕ^R action is twisted by z_2 on the right (i.e. $f.x.f' = z_2(f)xf'$). This bimodule can be obtained from $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A}$ as in Subsection 1.4.2 and the z_2 twist amounts to changing the $\mathbb{Z}_\Delta$-equivariant structure, where the new $\mathbb{Z}_\Delta$-equivariant structure is given by $g.m = z_2^g g(m)$. The complex $CC^*(M_\phi^R, {}_{z_2}(M_\phi^R)_1)$ can be computed to be the derived invariants of

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A}, \mathcal{A}) \tag{1.235}$$

as in Prop 1.5.13, but again the $\mathbb{Z}$-action on $CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})$ is different from the one in Prop 1.5.13 by z_2(i.e. it acts by $z_2(tr_*)$ in the first component).

This complex has no negative degree cohomology and its cohomology in degree 0 is isomorphic to R, fixed by the previous action. Hence, The derived invariants of it in degree 0 is 0 with respect to new action, unless $z_2 = 1$. Therefore, $HH^0(M_\phi^R, {}_{z_2}(M_\phi^R)_1) = 0$ unless $z_2 = 1$; thus $z_2 = 1$. □

Remark 1.8.7. For a group-like family $\mathfrak{M}$, the deformation class

$$Def(\mathfrak{M}) = [def(\nabla)] \in H^1(\mathcal{B}_T^{e,mod})(\mathfrak{M}, \Omega^1_{T/R} \otimes \mathfrak{M}) \simeq HH^*(\mathcal{B}, \Omega^1_{T/R} \otimes \mathcal{B}) \tag{1.236}$$

induces a linear map

$$\mathfrak{g} \to HH^1(\mathcal{B}, \mathcal{B}) \tag{1.237}$$

where $\mathfrak{g} = Lie(G/R) = Lie(G_0)[[q]]$.

Lemma 1.8.8. *The map $R^2 = Lie(\widehat{\mathbb{G}_m \times \mathbb{G}_m}) \to HH^1(M_\phi^R, M_\phi^R)$ induced as in Remark 1.8.7 is an isomorphism.*

Proof. We know both sides are isomorphic to R^2. It is enough to show the restriction $\mathbb{C}^2 \to HH^1(M_\phi, M_\phi)$ is an isomorphism. This map is given by the deformation class of a group-like family parametrized by $\mathbb{G}_m \times \mathbb{G}_m$. The restriction of the family to $\mathbb{G}_m \times \{1\}$ corresponds to the action in Remark 1.4.6. This restricted family carries a $\mathbb{G}_m$-equivariant structure; where $\mathbb{G}_m$ action on $M_\phi \otimes M_\phi^{op} \otimes \mathcal{O}(\mathbb{G}_m)$ is trivial on the first factor, by $z : t \mapsto zt$ on $\mathcal{O}(\mathbb{G}_m)$ and as in Remark 1.4.6 on the second. By a version of Lemma 1.6.49, we can show the restricted family follows $1 \otimes \gamma_\phi \in HH^1(M_\phi^e, M_\phi^e)$. Its restriction to diagonal bimodule gives $\pm\gamma_\phi \in HH^1(M_\phi, M_\phi)$; hence, the image of $(z\partial_z, 0) \in Lie(\mathbb{G}_m \times \mathbb{G}_m)$ is $\pm\gamma_\phi$. The sign depends on the identification of $RHom^*_{M_\phi^e}(M_\phi^e, M_\phi^e)$ with $HH^*(M_\phi, M_\phi)$.

Similarly, consider the restriction to $\{1\} \otimes \mathbb{G}_m$. The infinitesimal action $\gamma_2 := (0, z\partial_z)^\#$ is a 1-cocycle on M_ϕ^R that acts by $n \in \mathbb{Z}$ on degree n morphisms with respect to extra grading($\gamma_2^1(f) = nf$ for $|f| = n$, $\gamma_2^i = 0$ for $i \neq 1$) and as before the family restricted to $\{1\} \times \mathbb{G}_m$ follows $\pm\gamma_2$ (it follows $1 \otimes \gamma_2$ to be precise, and the restriction to diagonal gives $\pm\gamma_2$).

We want to show γ_ϕ and γ_2 are independent. By the proof of Prop 1.5.13, we have a long exact sequence

$$\begin{aligned} \dots \to HH^0(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}) &\xrightarrow{tr_* \otimes \phi_* - 1 = 0} HH^0(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}) \to \\ HH^1(M_\phi) &\to HH^1(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}) \to \dots \end{aligned} \tag{1.238}$$

$\gamma_\phi \in HH^1(M_\phi)$ maps to $\gamma_{\mathcal{O}} \otimes 1$ and the image is non-zero. On the other hand, the map on the lower line is the equivalent to

$$HH^1((\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}) \to HH^1(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}, (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}) \tag{1.239}$$

and the restriction of γ_2 to $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$ is zero since the functor $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A} \to (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}$ maps to degree 0 part (with respect to extra grading). Thus, γ_ϕ and γ_2 are independent and this finishes the proof. □

Lemma 1.8.9. *Let $\mathcal{B}$ and $G = G_0 \times_{\mathbb{C}} Spf(R)$ be as above and let ρ and ρ' be two group-like families of bimodules parametrized by G. Further assume $\mathcal{B}|_{q=0}$ is smooth and proper in each degree and $HH^0(\mathcal{B}) = R$. Assume ρ and ρ' can be represented as objects of $tw^\pi(\mathcal{B}^e \otimes \mathcal{O}(G))$, where the tensor product is over R and completed as usual(more precisely, ρ and ρ' are direct summands of families of twisted complexes in the sense of [Sei14]). Then there is a natural formal subgroup scheme $S \subset G$ with*

R-points $\{x \in G(R) : \rho_x \simeq \rho'_x\}$. *Moreover, S is closed.*

Proof. By choosing a minimal model for $\mathcal{B}|_{q=0}$ and considering the corresponding deformation, we may assume $\mathcal{B}(b, b')$ is a bounded below complex of finite rank free R-modules. Hence, choosing representatives for ρ and ρ' by twisted complexes, we can assume

$$hom^{\cdot}_{(\mathcal{B}^e)^{mod}_{\mathcal{O}(G)}}(\rho, \rho') =: hom^{\cdot}(\rho, \rho') \tag{1.240}$$

(and other hom-complexes among ρ and ρ') is a bounded below complex of finitely generated $\mathcal{O}(G)$-modules. Here, $(\mathcal{B}^e)^{mod}_{\mathcal{O}(G)}$ is the category of families of $\mathcal{B}$-bimodules parametrized by $\mathcal{O}(G)$.

Given a bounded below complex C^* of finitely generated, free $\mathcal{O}(G)$-modules, we can define "a closed locus of points such that $rkH^0(C^*_x) \geq m$" as follows: consider

$$\dots \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \to \dots \tag{1.241}$$

Let $C^0 \cong \mathcal{O}(G)^r$. We wish to define the locus of points x where

$$rk(d_x^{-1}) + rk(d_x^0) \leq r - m \tag{1.242}$$

where rk is the matrix rank. This is the same as

$$\bigcup_{a+b=r-m+2} \{rk(d_x^{-1}) < a \text{ and } rk(d_x^0) < b\} \tag{1.243}$$

and hence can be defined by the ideal

$$\bigcap_{a+b=r-m+2} (I_a(d^{-1}) + I_b(d^0)) \tag{1.244}$$

where $I_k(d^i)$ denotes the ideal generated by $k \times k$ minors of d^i, where d^i is considered as an $\mathcal{O}(G)$-valued matrix (or alternatively one can realize it as

$$\bigcap_{\alpha+\beta=r-m+1} \{rk(d_x^{-1}) < \alpha \text{ or } rk(d_x^0) < \beta\} \tag{1.245}$$

as in [GL87]). Borrowing the terminology of [GL87], denote the (formal) subscheme defined by the ideal (1.244) by $S_m(C^*)$. Let

$$\begin{aligned} S_1 = S_1(hom^\cdot(\rho,\rho)) \cap S_1(hom^\cdot(\rho',\rho)) \cap S_1(hom^\cdot(\rho,\rho')) \cap S_1(hom^\cdot(\rho',\rho'))\setminus \\ \big(S_2(hom^\cdot(\rho,\rho)) \cup S_2(hom^\cdot(\rho',\rho)) \cup S_2(hom^\cdot(\rho,\rho')) \cup S_2(hom^\cdot(\rho',\rho'))\big) \end{aligned} \tag{1.246}$$

which deforms the locus of points $x \in G_0 = G|_{q=0}$ where $H^0(hom^\cdot(\rho_x, \rho'_x))$ etc. are of rank 1. Note as we are working with formal schemes, this is not immediately a defining condition. Note also we do not assert R-flatness of S_1. S_1 is an (formal) open subscheme of the closed intersection of $S_1(hom^\cdot(\rho,\rho'))$ and so on.

Now, let us define S as a formal open subscheme of S_1. Roughly, we want to define a subscheme whose K-points satisfy the property that the composition

$$H^0(hom^\cdot(\rho'_x,\rho_x)) \otimes_{\mathcal{O}(G)} H^0(hom^\cdot(\rho_x,\rho'_x)) \to H^0(hom^\cdot(\rho_x,\rho_x)) \tag{1.247}$$

is surjective, where $R/q \subset K$. Let $x \in S_1(K)$ be a K-point, where $R/q = \mathbb{C} \to K$ is a field extension. Let $v \in hom^0(\rho_x, \rho'_x)$ be a closed element generating $H^0(hom^\cdot(\rho_x,\rho'_x)) = K$ We can extend v to $\tilde{v}$ defined on a neighborhood of $x \in S_1$ such that $d^0(\tilde{v}) = 0$. The existence of such a $\tilde{v}$ follows from Lemma 1.8.10. Pick $\tilde{v}'$, a closed section of $hom^0(\rho',\rho)$ satisfying the same property. The composition $\tilde{v}' \circ \tilde{v}$ generates the cohomology $H^0(hom^\cdot(\rho_x,\rho_x))$ and by the proof of Lemma 1.8.10 it is invertible in a neighborhood of x in S_1. Same holds for the composition $\tilde{v} \circ \tilde{v}'$ as well. Hence, there exists a neighborhood Ω of x in S_1 such that $\rho|_\Omega \simeq \rho'|_\Omega$. Define S to be the union of open subsets $\Omega \subset S_1$ such that $\rho|_\Omega \simeq \rho'|_\Omega$.

We want to characterize S functorially. A formal scheme $\mathfrak{X}$ over $R = \mathbb{C}[[q]]$ be seen as an ind-scheme, presented as the colimit of $\mathfrak{X} \times_R \mathbb{C}[[q]]/(q^{n+1})$. See [Sta17, Tag 0AIT], [hs] or [hh]. More explicitly, we can realize it as a functor over $Alg_{\mathbb{C}}$, algebras over $\mathbb{C}$ or even better over $Alg_{R,f}$ algebras over R such that q maps to a nilpotent element (such as $R/(q^m)$). Given $T \in Alg_{R,f}$, the R-linear maps $Spec(T) \to S \subset G$ are maps $Spec(T) \to G$ such that ρ_T and ρ'_T are locally isomorphic over T. Indeed, if we have such a map $f : Spec(T) \to S$ and a point $x \in Spec(T)$ we can choose a neighborhood $f(x) \in \Omega \subset S$ such that $\rho_\Omega \simeq \rho'_\Omega$ and thus $\rho_{f^{-1}(\Omega)} \simeq \rho'_{f^{-1}(\Omega)}$; hence they are locally isomorphic. On the other hand, if we assume ρ_T and ρ'_T are locally isomorphic, we can easily check the rank conditions so that f factors through $S_1 \subset G$. Local surjectivity is also easy to check; thus, it is actually a map into S.

From this functorial description, it is easy to see that S is a subgroup functor of G over R(hence it has a flat unit section over R). Clearly, it is locally closed. But locally closed subgroup schemes(such as $S|_{q=0}$) are actually closed. Hence, we are done. ☐

Lemma 1.8.10. *Let U, V, W be finite dimensional vector spaces over $\mathbb{C}$ and*

$$U \xrightarrow{A(q,s)} V \xrightarrow{B(q,s)} W \tag{1.248}$$

be a family of matrices parametrized by a formal scheme $\mathfrak{X}'$ over R(we do not assume R-flatness). Assume $B \circ A = 0$ identically and restrict to locus

$$\text{“}\mathfrak{X} := \bigcup_{a+b=r+1} \{rkA < a, rkB < b\} \setminus \bigcup_{a+b=r} \{rkA < a, rkB < b\}\text{”} \tag{1.249}$$

where $r = dimV$. Then given a point $x = (q = 0, s = s_0)$ of $X = \mathfrak{X}|_{q=0}$ there exists a neighborhood Ω of x inside $\mathfrak{X}$(X and $\mathfrak{X}$ has the same underlying topological space) and a section $v(q, s)$ defined over Ω(i.e. a family of vectors in V parametrized by Ω) which restricts to a generator of $ker(B(0, s))/Im(A(0, s))$ for all $(0, s) \in \Omega$.

Proof. The locus $\mathfrak{X}$ is essentially the locus of points at which the cohomology $kerB/ImA$ is exactly of rank 1. Let $x = (0, s_0)$ satisfy $rkA(x) < a, rkB(x) < b$ for $a + b = r + 1$. When we perturb x, $nullity(B)$ may only decrease and $rk(A)$ may only increase, but if this happens the rank of cohomology decreases too. Thus, rkA and rkB are constant in a neighborhood of x inside $\mathfrak{X}$. In other words, it has a neighborhood $\Omega \subset \mathfrak{X}$ on which $a \times a$ minors of $A(q, s)$ and $b \times b$ minors of $B(q, s)$ all vanish and there exists an $b - 1 \times b - 1$ minor of B which is invertible on Ω. By row and column operations we can assume this minor is the upper-left principal minor and the upper-left $b - 1 \times b - 1$ square submatrix of B is the identity matrix. By more row and column operations we can assume the rest of the entries of B are 0. Hence, $kerB$ has the simple description as the column vectors with vanishing first $a-1$-entries. This implies there exists $v(q,s),(q,s)\in\Omega$ such that $B(q,s)v(q,s)=0$ and $v(0,s_0)\notin Im(A(0,s))$. $Im(A(0,s_0))$ is generated by columns of $A(0,s_0)$ and the condition $v(0,s_0)\notin Im(A(0,s_0))$ can be phrased as the columns of $[A(0,s_0),v(0,s_0)]$ generate the subspace of vectors with vanishing first $a-1$ entries. Hence, it is an open condition and by further shrinking Ω we can ensure $v(0,s)$ generates the cohomology at $(0,s)\in\Omega$. □

Remark 1.8.11. Note the statement of the Lemma 1.8.9 does not immediately imply flatness of S over R; however, we believe this to be true.

Remark 1.8.12. There is a possibility that it is unnecessary to assume the representability by objects $tw^{\pi}(\mathcal{B}^e\otimes\mathcal{O}(G))$ as it may be a corollary of smoothness of $\mathcal{B}|_{q=0}$.

Proposition 1.8.13. *Let ρ be a group-like family of invertible bimodules over M_ϕ^R parametrized by $\widehat{\mathbb{G}_m}$. There exists a homomorphism $\eta:\widehat{\mathbb{G}_m}\to\widehat{\mathbb{G}_m\times\mathbb{G}_m}$ of formal group schemes over R such that ρ is the pull-back of the family ρ_{uni}(see Example 1.8.5) under η.*

Proof. Let $G=\widehat{\mathbb{G}_m\times\mathbb{G}_m}$ and recall ρ_{uni} denote the family in Example 1.8.5. Pulling back ρ resp. ρ_{uni} under projections $G\times\widehat{\mathbb{G}_m}\to\widehat{\mathbb{G}_m}$ resp. $G\times\widehat{\mathbb{G}_m}\to G$ we obtain two group-like families on $G\times\widehat{\mathbb{G}_m}$, which we denote by ρ' resp. ρ''. Apply Lemma 1.8.9 to ρ' and ρ'' to obtain a formal subgroup scheme S of $G\times\widehat{\mathbb{G}_m}$, "the locus of points such that $\rho'_y\simeq\rho''_y$". Hence,

$$S(R)=\{(x,x')\in G(R)\times\widehat{\mathbb{G}_m}(R):\rho_{uni,x}\simeq\rho_{x'}\}\tag{1.250}$$

By Lemma 1.8.6, the map $S(R)\to\widehat{\mathbb{G}_m}(R)$ is injective. It is easy to prove a version of Lemma 1.8.6 for the special fiber $q=0$ by using the same idea; thus, $S|_{q=0}(\mathbb{C})=S(\mathbb{C})\to\mathbb{G}_m(\mathbb{C})$ is injective as well.

By the functorial description of S, the Lie algebra (i.e. $R[\epsilon]/(\epsilon^2)$-points that specialize to identity at $\epsilon = 0$) of S has a description as $Lie(G) \times_{HH^1(M_\phi^R)} Lie(\widehat{\mathbb{G}_m})$ and by Lemma 1.8.8 the map $Lie(S) \to Lie(\widehat{\mathbb{G}_m})$ is an isomorphism. Similarly, $Lie(S|_{q=0}) \to Lie(\widehat{\mathbb{G}_m}|_{q=0})$ is an isomorphism.

Combined with the injectivity statement above, this shows $S|_{q=0} \to \mathbb{G}_m$ is an isomorphism. Being a formal closed subscheme of a formal affine scheme, $S = Spf(B)$ for some quotient B of $\mathcal{O}(G \times \widehat{\mathbb{G}_m})$ and the map $S \to \widehat{\mathbb{G}_m}$ corresponds to an algebra map $\mathbb{C}[z^\pm][[q]] \to B$ inducing an isomorphism $\mathbb{C}[z^\pm] \to B/qB$. Thus, the map $\mathbb{C}[z^\pm][[q]] \to B$ is surjective and S can be seen as a formal affine subscheme of $\widehat{\mathbb{G}_m}$. One can prove surjectivity of $\mathbb{C}[z^\pm][[q]] \to B$ by lifting an element $b \in B$ step by step. This uses the fact that $\bigcap_n q^n B = 0$, which follows from q-adic completeness of B.

Thus, let $B = \mathbb{C}[z^\pm][[q]]/I$. The identity morphism $Spf(R) \to \widehat{\mathbb{G}_m}$ factors through S; hence, $I \subset (z-1)$. Moreover, the $R[\epsilon]/\epsilon^2$-points of S that specialize to identity at $\epsilon = 0$ are in correspondence with such points of $\widehat{\mathbb{G}_m}$. Thus, $I \subset ((z-1)^2)$. S is a subgroup of $\widehat{\mathbb{G}_m}$ over $Spf(R)$, thus the comultiplication

$$\Delta : \mathbb{C}[z^\pm][[q]] \to \mathbb{C}[z^\pm][[q]] \otimes \mathbb{C}[z^\pm][[q]] \cong \mathbb{C}[z_1^\pm, z_2^\pm][[q]], z \mapsto z \otimes z \cong z_1 z_2 \quad (1.251)$$

should induce a map $B \to B \otimes B$. Let $f(z) = g(z)(z-1)^2 \in I$.

$$\begin{aligned}\Delta(f(z)) = \Delta(g(z))(z_1z_2 - 1)^2 \in I \otimes \mathbb{C}[z^\pm][[q]] + \mathbb{C}[z^\pm][[q]] \otimes I \subset \\ ((z_1-1)^2, (z_2-1)^2) \subset \mathbb{C}[z_1^\pm, z_2^\pm][[q]]\end{aligned} \quad (1.252)$$

As $(z_1z_2 - 1)^2 = (z_1z_2 - z_2 + z_2 - 1)^2 \equiv 2(z_1 - 1)(z_2 - 1)(mod((z_1-1)^2, (z_2-1)^2))$,

$$\Delta(g(z))(z_1 - 1)(z_2 - 1) \in ((z_1-1)^2, (z_2-1)^2) \quad (1.253)$$

Thus, $\Delta(g(z)) \in (z_1 - 1, z_2 - 1)$. Thus, $g(z) \in (z-1)$ and $f(z) \in ((z-1)^3)$; hence, $I \subset ((z-1)^3)$. Inductively, $I \subset (z-1)^n$ for all n. This shows $I = 0$ and the embedding $S \to \widehat{\mathbb{G}_m}$ is an isomorphism.

Hence, we have a diagram $G \leftarrow S \xrightarrow{\cong} \widehat{\mathbb{G}_m}$ of groups and inverting the isomorphism a group homomorphism $\eta : \widehat{\mathbb{G}_m} \to G$. The pull-back of ρ_{uni} along this map is the quasi-isomorphic to pull-back of ρ'' along $\eta \times 1 : \widehat{\mathbb{G}_m} \to G \times \widehat{\mathbb{G}_m}$, which factors through S; hence, it is (locally) quasi-isomorphic to pull-back of ρ' and thus to the family ρ on $\widehat{\mathbb{G}_m}$(there is no non-trivial line bundle on $\widehat{\mathbb{G}_m}$ so local isomorphism is the same as isomorphism). This completes the proof. □

Definition 1.8.14. For a curved A_∞-category $\mathcal{B}$ over $R = \mathbb{C}[[q]]$, define $L(\mathcal{B}) \subset HH^1(\mathcal{B}, \mathcal{B})$ to be the set of $< z\partial_z, Def(\rho) >$ for all group-like families ρ parametrized by $\widehat{\mathbb{G}_m}$. In other words, it is the set of images of $(z\partial_z)^\#$ under the map $Lie(\widehat{\mathbb{G}_m}) \to HH^1(\mathcal{B}, \mathcal{B})$ induced by the deformation class of a group-like family ρ.

Corollary 1.8.15. *$L(M^R_\phi) \subset HH^1(M^R_\phi, M^R_\phi) \cong R^2$ is a subgroup isomorphic to $\mathbb{Z}^2$ spanned by a basis of the free module R^2.*

Proof. By Prop 1.8.13, the deformation class of a group-like family parametrized by $\widehat{\mathbb{G}_m}$ can be computed as the pull-back of $Def(\rho_{uni})$ under a group homomorphism $\widehat{\mathbb{G}_m} \to G \cong \widehat{\mathbb{G}_m} \times \widehat{\mathbb{G}_m}$. It is easy to classify such maps as the rank 2 coroot lattice inside R^2(the proof is exactly the same as $\mathbb{G}_{m,\mathbb{C}}$) and this observation together with Lemma 1.8.8 concludes the proof. □

1.9 Two relative spherical twists of the trivial mapping torus

To prove the theorem, we need to modify the Morita equivalence $M_\phi \simeq M_{1_\mathcal{A}}$ such that the induced isomorphism $HH^1(M_\phi, M_\phi) \cong HH^1(M_{1_\mathcal{A}}, M_{1_\mathcal{A}})$ carries γ_ϕ to $\gamma_{1_\mathcal{A}}$. It is easy to show $M_{1_\mathcal{A}} \simeq \mathcal{A} \otimes (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \# \mathbb{Z})$ and $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \# \mathbb{Z}$ is Morita equivalent to wrapped Fukaya category of a punctured torus by [LP16]; thus, it must have a $SL_2(\mathbb{Z})$-symmetry, which we would expect to act transitively on the primitive lattice points of $HH^1(M_{1_\mathcal{A}}, M_{1_\mathcal{A}})$. We will not use action coming through this Morita equivalence and instead will write spherical twists that act in the desired way. The first twist is more general, but the second one only exists for the trivial mapping torus.

1.9.1 The twist of M_ϕ^R along a smooth point

We first find a self-Morita equivalence of the category M_ϕ^R that sends γ_ϕ^R to $\gamma_\phi^R \pm \gamma_2^R$ and that fixes γ_2^R. Here $\gamma_2^R \in HH^1(M_\phi^R, M_\phi^R)$ denotes the R-relative version of the Hochschild cocycle γ_2 defined in Lemma 1.8.8 of Section 1.8. Namely, γ_2^R is the infinitesimal action of second $\widehat{\mathbb{G}_m}(R)$-action(which has weight equal to extra grading) and it forms a basis of $HH^1(M_\phi^R, M_\phi^R) \cong R^2$ together with γ_ϕ^R.

In this subsection, we will not officially refer to twists by spherical functors as defined for instance in [AL17]. However, for those interested we remark that what we construct is equivalent to using twist by the functor

$$\begin{aligned} &\mathcal{A}[[q]] \to M_\phi^R \\ &\text{“} a \mapsto a \otimes \mathcal{O}_x \text{”} \end{aligned} \tag{1.254}$$

where $\mathcal{O}_x$ is the structure sheaf of a smooth R-section of $\mathfrak{T}_R = \tilde{\mathfrak{T}}_R/\mathbb{Z}$. After showing existence of right and left Morita adjoints, one can write the spherical twist and conclude that it is an equivalence at $q = 0$ using [AL17]. Then it is easy to show that invertible bimodules over M_ϕ deform only to invertible bimodules over M_ϕ^R. The spherical twist/cotwist by the structure sheaf of a smooth point p on a curve C is simply $(\cdot) \otimes \mathcal{O}_C(p)$ resp $(\cdot) \otimes \mathcal{O}_C(-p)$, so we will use this directly.

Let $p = p_0$ be a smooth R-point of $\tilde{\mathfrak{T}}_R$ supported on C_0. Let $p_i = \mathfrak{tr}^i(p)$ and consider the line bundle $\mathcal{L}_p = \mathcal{O}\big(\sum_{i\in\mathbb{Z}} p_i\big)$. Define a bimodule over $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ by the rule

$$\tilde{\Lambda}_p : (\mathcal{F}, \mathcal{F}') \mapsto hom^{\cdot}_{\mathcal{O}_{\tilde{\mathfrak{T}}_R}}(R(\mathcal{F})_R, R(\mathcal{F}')_R \otimes \mathcal{L}_p) \tag{1.255}$$

Recall $R(\mathcal{F})_R$,$R(\mathcal{F}')_R \otimes \mathcal{L}_p$ are pseudo-complexes of $\mathcal{O}_{\tilde{\mathfrak{T}}_R}$-modules and we are taking their hom pseudo-complexes as usual. This defines an (unobstructed for tautological reasons) $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$-bimodule. It is the bimodule corresponding to (non-existent) functor $(\cdot) \otimes \mathcal{L}_p$ and we will pretend as if it is this functor. One can make its restriction to $q = 0$ into an actual functor by extending $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ with similar resolutions $R(\mathcal{O}_{C_i}(a))$ of $\mathcal{O}_{C_i}(a)$, for all $a \in \mathbb{Z}$. Call this bigger category $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^{super}$. The line bundle $\mathcal{L}_p$ is invariant under $\mathfrak{tr}_*$ and thus the $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A}$ bimodule $\tilde{\Lambda}_p \otimes \mathcal{A}$ has an obvious

$\mathbb{Z}_\Delta$-equivariant structure. We can thus descent it to a bimodule

$$\Lambda_p = (\tilde{\Lambda}_p \otimes \mathcal{A})\#\mathbb{Z} \tag{1.256}$$

over M_ϕ^R. Its restriction $\Lambda_p|_{q=0}$ still does not induce an A_∞-functor; however, we can construct $M_\phi^{super} := (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}^{super} \otimes \mathcal{A})\#\mathbb{Z} \supset M_\phi$(which is equivalent to a full subcategory of $tw^\pi(M_\phi)$) on which $\Lambda_p|_{q=0}$ acts as an A_∞-functor ("$(\cdot) \otimes \mathcal{O}(p)$"). It is easy to see this functor is a quasi-equivalence with a quasi inverse defined by the similar formula "$(\cdot) \otimes \mathcal{O}(-p)$"(more precisely by using $\mathcal{L}_p^{-1}$ in place of $\mathcal{L}_p$ in the definition of $\tilde{\Lambda}_p$). This implies $\Lambda_p|_{q=0}$; thus, Λ_p is invertible.

Fix $p \in \tilde{\mathfrak{T}}_R(R)$ as above. Let ρ temporarily denote the group like family $\rho_{uni}|_{\mathbb{G}_m \times \{1\}}$, which corresponds to $\widehat{\mathbb{G}_m}(R)$-action on M_ϕ^R defined earlier in Remark 1.4.6. Using Λ_p we can define a new group like family

$$\text{“}\Lambda_p \circ \rho \circ \Lambda_p^{-1}\text{”} :\simeq z \mapsto \Lambda_p \otimes_{M_\phi^R} \rho_z \otimes_{M_\phi^R} \Lambda_p^{-1} \tag{1.257}$$

The reason composition is in quotation marks is again that we have "quasi-functors" instead of actual functors. However, we will abuse the notation and simply use composition symbol. By Prop 1.8.13 this family can be seen as the restriction of ρ_{uni} along a cocharacter of $\widehat{\mathbb{G}_m \times \mathbb{G}_m}$. We wish to compute this cocharacter.

Consider instead the group-like family

$$\Lambda_p \circ \rho \circ \Lambda_p^{-1} \circ \rho^{-1} : z \mapsto \Lambda_p \otimes_{M_\phi^R} \rho_z \otimes_{M_\phi^R} \Lambda_p^{-1} \otimes_{M_\phi^R} \rho_z^{-1} \tag{1.258}$$

It can be seen as the composition of Λ_p and $\rho \circ \Lambda_p^{-1} \circ \rho^{-1}$. Given $z \in \widehat{\mathbb{G}_m}$, we can compute $\rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1} \simeq \Lambda_{z.p}^{-1}$. Hence,

$$\Lambda_p \circ \rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1} \simeq \Lambda_p \circ \Lambda_{z.p}^{-1} \tag{1.259}$$

Lemma 1.9.1. *$\Lambda_p \circ \Lambda_{z.p}^{-1}$ is the bimodule obtained by replacing $\mathcal{L}_p$ by $\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$ at the* *beginning. In other words, first consider*

$$(\mathcal{F}, \mathcal{F}') \mapsto hom^{\cdot}_{\mathcal{O}_{\tilde{\mathfrak{I}}_R}}(R(\mathcal{F})_R, R(\mathcal{F}')_R \otimes \mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}) \tag{1.260}$$

then take its exterior product with $\mathcal{A}$ and descent to M_ϕ^R.

Proof. (Sketch) Instead of showing this leads to $\Lambda_p \circ \Lambda_{z.p}^{-1}$ for individual z, one can specialize to $q = 0$ and compare bimodules over M_ϕ. Extending M_ϕ as above, both $\Lambda_p \circ \Lambda_{z.p}^{-1}$ and the bimodule corresponding to $\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$ can be realized as actual A_∞-functors and both are enhancements of "$(\cdot) \otimes \mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$". Hence, the specializations to $q = 0$ are the same. As families of bimodules over deformation M_ϕ^R, they are group-like which correspond to coroots of $\widehat{\mathbb{G}_m \times \mathbb{G}_m}$ speacialing to same coroots of $\mathbb{G}_m \times \mathbb{G}_m$. Hence, by discreteness of the coroots we conclude they are the same. ☐

Now let us turn our attention to the line bundle

$$\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1} = \mathcal{O}_{\tilde{\mathfrak{I}}_R}\left(\sum_{i\in\mathbb{Z}}(p_i - z.p_i)\right) \tag{1.261}$$

The equivariant structure of the induced bimodule comes from the obvious isomorphism

$$\mathbf{tr}_*\mathcal{O}_{\tilde{\mathfrak{I}}_R}\left(\sum_{i\in\mathbb{Z}}(p_i - z.p_i)\right) \cong \mathcal{O}_{\tilde{\mathfrak{I}}_R}\left(\sum_{i\in\mathbb{Z}}(p_i - z.p_i)\right) \tag{1.262}$$

Lemma 1.9.2. *$\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$ admits a trivialization $G \in \Gamma(\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1})$ such that under the isomorphism $\mathcal{O}_{\tilde{\mathfrak{I}}_R} \xrightarrow{\cong} \mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$ the equivariant structure $\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1} \to \mathbf{tr}_*(\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1})$ identifies with $\mathcal{O}_{\tilde{\mathfrak{I}}_R} \xrightarrow{z^{-1}} \mathbf{tr}_*\mathcal{O}_{\tilde{\mathfrak{I}}_R} = \mathcal{O}_{\tilde{\mathfrak{I}}_R}$.*

Proof. A section of $\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$ is a rational function on $\tilde{\mathfrak{I}}_R$ with simple poles at p_i and zeroes at $z.p_i$, for all $i \in \mathbb{Z}$. Denote the Y_0-coordinate of the smooth point $p = p_0 \in \tilde{U}_{1/2}$ by $y_0 \in R^*$. We can find a section using convergent infinite products. Namely, consider the chart $\tilde{U}_{i+1/2} = Spf(\mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q))$. Define the rational function $\tilde{G}_{i+1/2}$ on $\tilde{U}_{i+1/2}$ by the formula

$$\tilde{G}_{i+1/2} := \prod_{j=0}^{\infty} \frac{1 - q^j z y_0 X_i}{1 - q^j y_0 X_i} \prod_{j=0}^{\infty} \frac{1 - q^j z^{-1} y_0^{-1} Y_{i+1}}{1 - q^j y_0^{-1} Y_{i+1}} \tag{1.263}$$

Its convergence is obvious by q-adic completeness. On $\tilde{U}_{i+1/2}$ it is

$$\frac{1 - zy_0X_i}{1 - y_0X_i}\frac{1 - z^{-1}y_0^{-1}Y_{i+1}}{1 - y_0^{-1}Y_{i+1}} \tag{1.264}$$

up to invertible functions. Using the relations $X_iY_i = 1$, $X_{i-1} = qX_i$ and $Y_{i+1} = qY_i$ we can compare

$$\frac{\tilde{G}_{i+1/2}}{\tilde{G}_{i-1/2}} = \frac{(1 - zy_0X_i)/(1 - y_0X_i)}{(1 - z^{-1}y_0^{-1}Y_i)/(1 - y_0^{-1}Y_i)} = z \tag{1.265}$$

on $\tilde{U}_{i-1/2} \cap \tilde{U}_{i+1/2}$. Now define $G = G(z) \in \Gamma(\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1})$ locally by the formula

$$G_{i+1/2} := z^{-i}\tilde{G}_{i+1/2} \tag{1.266}$$

This gives a trivialization of $\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$ and $G \circ \mathrm{tr} = z^{-1}G$ as a rational function. Hence, the equivariant structure turns into $\mathcal{O}_{\tilde{\mathfrak{T}}_R} \xrightarrow{z^{-1}} \mathrm{tr}_*\mathcal{O}_{\tilde{\mathfrak{T}}_R} = \mathcal{O}_{\tilde{\mathfrak{T}}_R}$ under the identification $\mathcal{L}_p \otimes \mathcal{L}_{z^{-1}.p}^{-1} \cong \mathcal{O}_{\tilde{\mathfrak{T}}_R}$. □

Hence, the bimodule

$$(\mathcal{F}, \mathcal{F}') \mapsto hom_{\mathcal{O}_{\tilde{\mathfrak{T}}_R}}(R(\mathcal{F})_R, R(\mathcal{F}')_R \otimes \mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}) \tag{1.267}$$

identifies with the diagonal bimodule of $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. Moreover, its $\mathbb{Z}_\Delta$-equivariant structure is

$$\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}(\mathcal{F}, \mathcal{F}') \xrightarrow{z.\mathrm{tr}} \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}(\mathrm{tr}\mathcal{F}, \mathrm{tr}\mathcal{F}') \tag{1.268}$$

(while $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{tr}} \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}(\mathrm{tr}\mathcal{F}, \mathrm{tr}\mathcal{F}')$ is the $\mathbb{Z}_\Delta$-equivariant structure descending to diagonal). If we descent $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A}$ with respect to $\mathbb{Z}_\Delta$ equivariant structure $z.\mathrm{tr} \otimes 1_{\mathcal{A}}$, we obtain the bimodule

$$_z(M_\phi^R)_1 \cong {}_1(M_\phi^R)_{z^{-1}} \tag{1.269}$$

It is the bimodule with same underlying pseudo-complexes and right action as M_ϕ^R but with left action twisted by action of z by extra grading(i.e. $(f \otimes h)(m \otimes g) = z^h fh(m) \otimes hg$). Hence, depending on the convention this is the bimodule $\rho_{uni,(1,z)}$ or

$\rho_{uni,(1,z^{-1})}$. Everything above can be done relative to $z \in \widehat{\mathbb{G}_m}$ and we conclude

Corollary 1.9.3. *$\Lambda_p \circ \rho \circ \Lambda_p^{-1} \circ \rho^{-1}$ is quasi-isomorphic to $\rho_{uni}|_{\{1\}\times\widehat{\mathbb{G}_m}}$ or its composition with the antipode $z \mapsto z^{-1} : \{1\} \times \widehat{\mathbb{G}_m} \to \{1\} \times \widehat{\mathbb{G}_m}$.*

Denote $\rho_{uni}|_{\{1\}\times\widehat{\mathbb{G}_m}}$ temporarily by ρ_2

Corollary 1.9.4. $\Lambda_p \circ \rho \circ \Lambda_p^{-1} \simeq \rho \circ \rho_2^{\pm}$.

Taking their deformation classes we find

Corollary 1.9.5. *Under the automorphism of $HH^1(M_\phi^R)$ induced by Λ_p, γ_ϕ^R corresponds to $\gamma_\phi^R \pm \gamma_2^R$.*

Now we want to show Λ_p fixes γ_2^R and γ_2. For this we will again examine $\rho_2 \circ \Lambda_p \circ \rho_2^{-1}$. A systematic approach would be first proving Λ_p is the same as the twist by

$$\begin{gathered} \mathcal{A}[[q]] \to M_\phi^R \\ \text{“}a \mapsto a \otimes \mathcal{O}_p\text{”} \end{gathered} \tag{1.270}$$

as mentioned above and then showing its conjugate by $\rho_{2,z}$ is the same as the twist by the composition of the spherical functor with conjugation by $\rho_{2,z}$. For instance, in the case $\mathcal{A} = \mathbb{C}$, this is given as the twist by "$\rho_{2,z}(\mathcal{O}_p) = \mathcal{O}_p$"; hence, it is the same. However, we take a simpler approach.

Lemma 1.9.6. $\rho_{2,z} \circ \Lambda_p \circ \rho_{2,z}^{-1} \simeq \Lambda_p$

Proof. By Corollary 1.9.3

$$\Lambda_p \circ \rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1} \simeq \rho_{2,z}^{\pm} \tag{1.271}$$

Hence, it is sufficient to show

$$\Lambda_p^{-1} \circ \Lambda_p \circ \rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1} \circ \Lambda_p \simeq \Lambda_p \circ \rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1} \tag{1.272}$$

Clearly, the former is quasi-isomorphic to $\rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1} \circ \Lambda_p$, which is simply $\Lambda_{z.p}^{-1} \circ \Lambda_p$. By the proof of Lemma 1.9.1, $\Lambda_{z.p}^{-1} \circ \Lambda_p$ is given by the descent of the bimodule corresponding to the same line bundle namely $\mathcal{L}_p \otimes \mathcal{L}_{z.p}^{-1}$, with the same $\mathbb{Z}_\Delta$-equivariant structure. Hence

$$\Lambda_{z.p}^{-1} \circ \Lambda_p \simeq \Lambda_p \circ \Lambda_{z.p}^{-1} \tag{1.273}$$

We have shown before that $\Lambda_p \circ \Lambda_{z.p}^{-1} \simeq \Lambda_p \circ \rho_z \circ \Lambda_p^{-1} \circ \rho_z^{-1}$. This completes the proof. □

Corollary 1.9.7. *The induced action of* Λ_p *on* $HH^1(M_\phi^R)$ *fixes* γ_2^R.

Proof. Compare the deformation classes of $\Lambda_p \circ \rho_2 \circ \Lambda_p^{-1} \simeq \rho_2$. □

Remark 1.9.8. Notice we can twist the family $\mathcal{G}_R^{sf}$ by

$$\mathcal{G}'_R := \text{“}\Lambda_p \circ \mathcal{G}_R^{sf} \circ \Lambda_p^{-1}\text{”} \tag{1.274}$$

to obtain a family that follows $1 \times (\gamma_\phi^R \pm \gamma_2^R)$. It is easy to see that the family satisfies Properties **G.1-G.3** with $\gamma = \gamma_\phi^R \pm \gamma_2^R$. One can attempt to use "convolutions of the families of bimodules relative to $Spf(A_R)$" to produce families following other classes in $L(M_\phi^R) \subset HH^1(M_\phi^R, M_\phi^R)$. However, we do not know how to show property **G.1** for the new family.

1.9.2 The twist of $M_{1_\mathbb{C}}$ along the "structure sheaf"

The second twist is more restrictive. We can still work the the curved algebra $M_{1_\mathbb{C}}^R$; however, we will not do this. We find a self Morita equivalence of $M_{1_\mathcal{A}}$ that fixes $\gamma_{1_\mathcal{A}}$ and that carries γ_2 to $\gamma_2 \pm \gamma_{1_\mathcal{A}}$. It is sufficient to do this for $\mathcal{A} = \mathbb{C}$. In the following γ_1 will denote $\gamma_{1_\mathbb{C}}$.

We can work as in the previous subsection. However, we find it conceptually relieving to relate $M_{1_\mathbb{C}}$ to algebraic geometry. Hence, we wish to start by sketching a proof of a weaker version of the claim in Example 1.1.1. Namely:

Lemma 1.9.9. $tw^\pi(M_{1_\mathbb{C}})$ *is a dg enhancement of* $D^b(Coh(\mathcal{T}_0))$, *where* $\mathcal{T}_0$ *is the nodal elliptic curve.*

Proof. $\mathcal{T}_0$ can be realized as "$\tilde{\mathcal{T}}_0/(x \sim \mathrm{tr}(x))$" and we have a projection map $\pi : \tilde{\mathcal{T}}_0 \to \mathcal{T}_0$ (denoted by π only throughout this proof). Choose dg-models $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)$ and $\mathcal{C}oh(\mathcal{T}_0)$ for $D^b(Coh_p(\tilde{\mathcal{T}}_0))$ and $D^b(Coh(\mathcal{T}_0))$ such that

- There exists a dg functor $\pi_* : \mathcal{C}oh_p(\tilde{\mathcal{T}}_0) \to \mathcal{C}oh(\mathcal{T}_0)$ enhancing the push-forward by π
- $\mathfrak{tr}$ induces a strict action $\mathfrak{tr}_*$ on $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)$
- $\pi_* \circ \mathfrak{tr}_* = \pi_*$ (strictly)

We can further assume $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)$ has the objects $\{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$ and denote $\mathfrak{tr}_*$ by $\mathfrak{tr}$ following the previous convention. We can also assume there exists a zigzag of strictly $\mathbb{Z}$-equivariant dg quasi-equivalences relating $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)$ and $\mathcal{O}(\tilde{\mathcal{T}}_0)_{dg}$. Hence, $\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)\#\mathbb{Z} \simeq \mathcal{O}(\tilde{\mathcal{T}}_0)_{dg}\#\mathbb{Z}$. The relation $\pi_* \circ \mathfrak{tr} = \pi_*$ implies π_* descends to

$$\begin{gathered} \mathcal{C}oh_p(\tilde{\mathcal{T}}_0)\#\mathbb{Z} \to \mathcal{C}oh(\mathcal{T}_0) \\ \mathcal{F} \longmapsto \pi_*(\mathcal{F}) \end{gathered} \tag{1.275}$$

Let $f \in \mathcal{C}oh_p(\tilde{\mathcal{T}}_0)(\mathfrak{tr}^g\mathcal{F}, \mathcal{F}')$ and consider f as an element of $(\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)\#\mathbb{Z})(\mathcal{F}, \mathcal{F}')$(recall we denoted it by $f \otimes g$). It is sent to

$$\pi_*(f) \in \mathcal{C}oh(\mathcal{T}_0)(\pi_*(\mathfrak{tr}^g\mathcal{F}), \pi_*(\mathcal{F}')) = \mathcal{C}oh(\mathcal{T}_0)(\pi_*(\mathcal{F}), \pi_*(\mathcal{F}')) \tag{1.276}$$

under the new functor. Denote the new functor by π_* as well.

The induced functor between homotopy categories of twisted envelopes is essentially surjective. This follows from the fact that the push-forward of $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}$ under the normalization map $\mathbb{P}^1 \to \mathcal{T}_0$ generates $D^b(Coh(\mathcal{T}_0))$. See [LP16].

To conclude the proof, we need to check (cohomological) fully faithfulness of the functor

$$\begin{gathered} \mathcal{C}oh_p(\tilde{\mathcal{T}}_0)\#\mathbb{Z} \to \mathcal{C}oh(\mathcal{T}_0) \\ \mathcal{F} \longmapsto \pi_*(\mathcal{F}) \end{gathered} \tag{1.277}$$

We do this only for $\mathcal{F} = \mathcal{F}' = \mathcal{O}_{C_0}$ as the others are similar. First notice

$$\begin{gathered} (\mathcal{C}oh_p(\tilde{\mathcal{T}}_0)\#\mathbb{Z})(\mathcal{O}_{C_0}, \mathcal{O}_{C_0}) = \\ \mathcal{C}oh_p(\tilde{\mathcal{T}}_0)(\mathcal{O}_{C_{-1}}, \mathcal{O}_{C_0}) \oplus \mathcal{C}oh_p(\tilde{\mathcal{T}}_0)(\mathcal{O}_{C_0}, \mathcal{O}_{C_0}) \oplus \mathcal{C}oh_p(\tilde{\mathcal{T}}_0)(\mathcal{O}_{C_1}, \mathcal{O}_{C_0}) \end{gathered} \tag{1.278}$$

and its cohomology is

$$RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_{-1}}, \mathcal{O}_{C_0}) \oplus RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_0}, \mathcal{O}_{C_0}) \oplus RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_1}, \mathcal{O}_{C_0}) \qquad (1.279)$$

A simple local computation(i.e. calculating local hom's and their global sections) reveals $RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_{-1}}, \mathcal{O}_{C_0})$ is one dimensional in every positive odd degree and 0 in other degrees. Same holds for $RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_1}, \mathcal{O}_{C_0})$. On the other hand, $RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_0}, \mathcal{O}_{C_0})$ can be calculated to be one dimensional in degree 0 and two dimensional in positive even degrees.

We can compute local hom's of $\pi_*\mathcal{F}$ and $\pi_*\mathcal{F}'$ on the étale chart

$$\pi : Spec(\mathbb{C}[X_0, Y_1]/(X_0Y_1)) \to \mathfrak{T}_0 \qquad (1.280)$$

and see that $RHom_{\mathfrak{T}_0}(\pi_*\mathcal{O}_{C_0}, \pi_*\mathcal{O}_{C_0})$ is 1 dimensional in degree 0 and 2 dimensional in higher degrees. Hence, degrees match up and it is hidden in the local computation that the map

$$\begin{aligned} RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_{-1}}, \mathcal{O}_{C_0}) \oplus RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_0}, \mathcal{O}_{C_0}) \oplus RHom_{\mathcal{O}_{\tilde{\mathfrak{T}}_0}}(\mathcal{O}_{C_1}, \mathcal{O}_{C_0}) \\ \to RHom_{\mathfrak{T}_0}(\pi_*\mathcal{O}_{C_0}, \pi_*\mathcal{O}_{C_0}) \end{aligned} \qquad (1.281)$$

is an isomorphism. □

Notice, $\mathcal{O}_{\mathfrak{T}_0}$ is a 1-spherical object of $D^b(Coh(\mathfrak{T}_0))$ in the sense of [ST01, Definition 2.9]. In other words,

- $RHom_{\mathfrak{T}_0}(\mathcal{O}_{\mathfrak{T}_0}, \mathcal{F})$ and $RHom_{\mathfrak{T}_0}(\mathcal{F}, \mathcal{O}_{\mathfrak{T}_0})$ are finite dimensional for all $\mathcal{F} \in D^b(Coh(\mathfrak{T}_0))$
- $RHom_{\mathfrak{T}_0}(\mathcal{O}_{\mathfrak{T}_0}, \mathcal{O}_{\mathfrak{T}_0}) = \mathbb{C} \oplus \mathbb{C}[-1]$
- $RHom^j_{\mathfrak{T}_0}(\mathcal{F}, \mathcal{O}_{\mathfrak{T}_0}) \times RHom^{1-j}_{\mathfrak{T}_0}(\mathcal{O}_{\mathfrak{T}_0}, \mathcal{F}) \to RHom_{\mathfrak{T}_0}(\mathcal{O}_{\mathfrak{T}_0}, \mathcal{O}_{\mathfrak{T}_0}) \cong \mathbb{C}$, composition map, is a non-degenerate pairing

The first and second conditions are immediate and the third one follows from Serre duality. Note, we are ignoring the condition they call (K1)) as it can be arranged by choosing an appropriate representative of $\mathcal{O}_{\mathfrak{T}_0}$ in the enhancement.

Hence, by [ST01, Proposition 2.10], there exists a (quasi-)equivalence $T_{\mathcal{O}_{\mathfrak{T}_0}}$ of (an enhancement of) $D^b(Coh(\mathfrak{T}_0))$ fitting into an exact triangle

$$RHom_{\mathfrak{T}_0}(\mathcal{O}_{\mathfrak{T}_0},\cdot)\otimes\mathcal{O}_{\mathfrak{T}_0}\to(\cdot)\to T_{\mathcal{O}_{\mathfrak{T}_0}}(\cdot)\to RHom_{\mathfrak{T}_0}(\mathcal{O}_{\mathfrak{T}_0},\cdot)\otimes\mathcal{O}_{\mathfrak{T}_0}[1] \tag{1.282}$$

Thus by Lemma 1.9.9, there exists an object of $tw^{\pi}(M_{1_{\mathbb{C}}})$ corresponding to $\mathcal{O}_{\mathfrak{T}_0}$ and a self Morita equivalence of M_{ϕ}, which we denote by $\Lambda_{\mathcal{O}}$. This is the second twist we are looking for. Next we examine its effect on γ_1 and γ_2.

Remark 1.9.10. The actions ρ_1 and ρ_2 induce $\mathbb{G}_m$-actions on $D^b(Coh(\mathfrak{T}_0))$. ρ_1 is already induced by the geometric action in Remark 1.2.3, hence its induced action on $D^b(Coh(\mathfrak{T}_0))$ comes from the action $\mathbb{G}_m \curvearrowright \mathfrak{T}_0$ making $\tilde{\mathfrak{T}}_0\to\mathfrak{T}_0$ equivariant. In other words, it is the action of $Aut^0(\mathfrak{T}_0)\cong\mathbb{G}_m$. To describe induced ρ_2-action first note $\Lambda_p\curvearrowright D^b(Coh(\mathfrak{T}_0))$ is simply tensoring with $\mathcal{O}_{\mathfrak{T}_0}(p)$, where we use p to denote the image of $p_0\in\tilde{\mathfrak{T}}_0\to\mathfrak{T}_0$ as well. Hence, $\Lambda_p\circ\Lambda_{z.p}^{-1}$ acts by $\mathcal{O}_{\mathfrak{T}_0}(p-z.p)$. By Section 1.9.1, this action is the induced action of $\rho_2^{\pm}$. In other words, ρ_2-action induces the action of $Pic^0(\mathfrak{T}_0)$ on $D^b(Coh(\mathfrak{T}_0))$. In summary, ρ_{uni} induces the action of $Aut^0(\mathfrak{T}_0)\times Pic^0(\mathfrak{T}_0)\cong\mathbb{G}_m\times\mathbb{G}_m$ on $D^b(Coh(\mathfrak{T}_0))$.

Consider $\rho_{1,z}\circ\Lambda_{\mathcal{O}}\circ\rho_{1,z}^{-1}$. This is the twist by the 1-spherical object $\rho_{1,z}(\mathcal{O}_{\mathfrak{T}_0})$. By Remark 1.9.10, $\rho_{1,z}(\mathcal{O}_{\mathfrak{T}_0})\simeq\mathcal{O}_{\mathfrak{T}_0}$; hence, $\rho_{1,z}\circ\Lambda_{\mathcal{O}}\circ\rho_{1,z}^{-1}\simeq\Lambda_{\mathcal{O}}$. In other words, ρ_1 commutes with $\Lambda_{\mathcal{O}}$ and by taking deformation classes, we conclude the map induced by $\Lambda_{\mathcal{O}}$ sends γ_1 to itself.

On the other hand, consider the commutator $\rho_{2,z}\circ\Lambda_{\mathcal{O}}\circ\rho_{2,z}^{-1}\circ\Lambda_{\mathcal{O}}^{-1}$. As z varies, this gives a group like family, determined by a cocharacter of ρ_{uni}, thanks to Prop 1.8.13. We want to determine this cocharacter. A quick calculation shows that for any two smooth points $q,q'\in\mathfrak{T}_0$, $\Lambda_{\mathcal{O}}(\mathcal{O}_{\mathfrak{T}_0}(q-q'))=\mathcal{O}_{\mathfrak{T}_0}(q-q')$. Hence,

$$\begin{aligned}\rho_{2,z}\circ\Lambda_{\mathcal{O}}\circ\rho_{2,z}^{-1}\circ\Lambda_{\mathcal{O}}^{-1}(\mathcal{O}_{\mathfrak{T}_0})=\rho_{2,z}\circ\Lambda_{\mathcal{O}}\circ\rho_{2,z}^{-1}(\mathcal{O}_{\mathfrak{T}_0})=\\ \rho_{2,z}\circ\Lambda_{\mathcal{O}}(\mathcal{O}_{\mathfrak{T}_0}(p-z^{\mp}.p))=\rho_{2,z}(\mathcal{O}_{\mathfrak{T}_0}(p-z^{\mp}.p))=\mathcal{O}_{\mathfrak{T}_0}\end{aligned} \tag{1.283}$$

This implies the second component of the cocharacter of $Aut^0(\mathfrak{T}_0)\times Pic^0(\mathfrak{T}_0)$ vanishes(since it fixes $\mathcal{O}_{\mathfrak{T}_0}$). On the other hand, for the smooth point $p\in\mathfrak{T}_0$, $\Lambda_{\mathcal{O}}(\mathcal{O}_p)=$

$\mathcal{O}_{\mathfrak{T}_0}(-p)[1]$. If we apply $\rho_{2,z} \circ \Lambda_{\mathcal{O}} \circ \rho_{2,z}^{-1} \circ \Lambda_{\mathcal{O}}^{-1}$ to $\Lambda_{\mathcal{O}}(\mathcal{O}_p)$ we obtain $\mathcal{O}_{\mathfrak{T}_0}(-z^{\pm}.p)[1]$. This shows, $Aut^0(\mathfrak{T}_0)$ component is the cocharacter of weight ± 1. This implies

$$\rho_{2,z} \circ \Lambda_{\mathcal{O}} \circ \rho_{2,z}^{-1} \circ \Lambda_{\mathcal{O}}^{-1} = \rho_1^{\pm} \tag{1.284}$$

and thus $\Lambda_{\mathcal{O}} \circ \rho_2 \circ \Lambda_{\mathcal{O}}^{-1} = \rho_2 \circ \rho_1^{\mp}$. By taking the deformation classes we conclude:

Corollary 1.9.11. *The map induced by $\Lambda_{\mathcal{O}}$ on $HH^1(M_{1_{\mathcal{C}}})$ sends γ_1 to γ_1 and γ_2 to $\gamma_2 \mp \gamma_1$.*

To conclude the section, we have found two self-Morita equivalences Λ_p and $\Lambda_{\mathcal{O}}$ of $M_{1_{\mathcal{A}}}$ that acts on $HH^1(M_{1_{\mathcal{A}}})$ by the matrices $\left[\begin{smallmatrix} 1 & 0 \\ \pm 1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & \mp 1 \\ 0 & 1 \end{smallmatrix}\right]$ respectively in $\{\gamma_1, \gamma_2\}$ basis. The action of any self-Morita equivalence has to preserve the lattice $L(M_\phi) := L(M_\phi^R)|_{q=0} \subset HH^1(M_\phi)$. It is easy to show these matrices generate the group $SL(L(M_\phi)) \cong SL_2(\mathbb{Z})$. Indeed, it is a classical fact that $SL_2(\mathbb{Z})$ is generated by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. See [Ser73] for instance. The latter matrix can easily be obtained as

$$\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \tag{1.285}$$

Corollary 1.9.12. *The group of self-Morita equivalences of $M_{1_{\mathcal{A}}}$ act transitively on primitive vectors of the lattice $L(M_{1_{\mathcal{A}}}) \cong \mathbb{Z}^2$.*

1.10 Uniqueness of family of bimodules and the proof of the main theorem

In this section, we will use the previous sections to conclude the proof of Theorem 1.1.4. In other words we will prove:

Theorem 1.1.4. *Let $\mathcal{A}$ and ϕ be as in Section 1.1, i.e. satisfying **C.1**-**C.2** and so on. Assume further that $HH^1(\mathcal{A}) = HH^2(\mathcal{A}) = 0$. If M_ϕ is Morita equivalent to $M_{1_{\mathcal{A}}}$, then $\phi \simeq 1_{\mathcal{A}}$.*

To prove this theorem, we will give a characterization of the family of bimodules $\mathfrak{G}_R^{sf}$. Let us first work in a more general setting. Let $\mathcal{B}_0$ be an A_∞- category and $\mathcal{B}$

be a curved deformation over $R = \mathbb{C}[[q]]$. Let $\mathfrak{M}$ be a family of bimodules over $\mathcal{B}$ parametrized by $Spf(A_R)$ and let $\gamma \in HH^1(\mathcal{B})$. Consider the properties:

G.1 The restriction $\mathfrak{M}|_{q=0}$ is a coherent family. This is equivalent to its representability by an object of $tw^\pi(\mathcal{B}_0 \otimes \mathcal{B}_0^{op} \otimes \text{"}\mathcal{C}oh(A)\text{"})$. See Definition 1.6.8.

G.2 The restriction $\mathfrak{M}|_{t=1}$ is isomorphic to the diagonal bimodule over $\mathcal{B}$.

G.3 The family follows the class $1 \otimes \gamma \in HH^1(\mathcal{B}^e)$.

The semi-freeness of $\mathfrak{M}$ is implied by the family assumption. The property **G.1** is a technical one. However, notice the similarity of properties **G.2** and **G.3** to an initial value problem. We show

Theorem 1.10.1. *Assume $\mathcal{B}_0$ is smooth, proper in each degree and $HH^0(\mathcal{B}_0) = \mathbb{C}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two families of bimodules satisfying **G.1-G.3**. Then $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic up to q-torsion. In other words, there are maps*

$$f_1 : \mathfrak{M} \to \mathfrak{M}', f_2 : \mathfrak{M}' \to \mathfrak{M} \tag{1.286}$$

in the category $(\mathcal{B}^e)^{mod}_{A_R}$ of families of bimodules such that $f_2 \circ f_1 \simeq q^n 1_{\mathfrak{M}}$ and $f_1 \circ f_2 \simeq q^n 1_{\mathfrak{M}'}$.

Proof. Let $Hom(\mathfrak{M}, \mathfrak{M}')$ denote $H^0((\mathcal{B}^e)^{mod}_{A_R})(\mathfrak{M}, \mathfrak{M}')$ throughout the proof. First, notice that it is finitely generated over A_R. To see this consider the complex

$$(\mathcal{B}^e_0)^{mod}_A(\mathfrak{M}|_{q=0}, \mathfrak{M}'|_{q=0}) \tag{1.287}$$

of A-modules. Here, $(\mathcal{B}^e_0)^{mod}_A$ is the category of families of $\mathcal{B}_0$-bimodules parametrized by $Spec(A)$, which can be defined analogously. As stated in Lemma 1.6.12, the condition **G.1** implies this complex has cohomology that is finitely generated over A in each degree. Thus, by Lemma 1.6.13 or 1.10.3 the same holds for the complex

$$(\mathcal{B}^e)^{mod}_{A_R}(\mathfrak{M}, \mathfrak{M}') \tag{1.288}$$

and $Hom(\mathfrak{M}, \mathfrak{M}')$ is finitely generated.

Second, by Prop 1.6.38, the complex $(\mathcal{B}^e)^{mod}_{A_R}(\mathfrak{M}, \mathfrak{M}')$ admits a homotopy connection along $\not{D}_{A_R}$; thus, so is its cohomology. In particular, the A_R-modules $Hom(\mathfrak{M}, \mathfrak{M}')$, $Hom(\mathfrak{M}', \mathfrak{M})$ and so on carry connections along A_R.

Applying Lemma 1.6.37 to this complex we see that

$$Hom(\mathfrak{M}, \mathfrak{M}')/(t-1)Hom(\mathfrak{M}, \mathfrak{M}') \cong H^0((\mathcal{B}^e)^{mod}(\mathfrak{M}|_{t=1}, \mathfrak{M}'|_{t=1})) \tag{1.289}$$

Here we are also using the fact that the restriction of $(\mathcal{B}^e)^{mod}_{A_R}(\mathfrak{M}, \mathfrak{M}')$ to $t = 1$ gives the hom-complex $(\mathcal{B}^e)^{mod}(\mathfrak{M}, \mathfrak{M}')$ and this follows from semi-freeness of families over A_R. However, by condition **G.2**, $H^0((\mathcal{B}^e)^{mod}(\mathfrak{M}|_{t=1}, \mathfrak{M}'|_{t=1}))$ is simply the self-endomorphisms of the diagonal; which is computed by Hochschild cohomology. Hence, the assumption $HH^0(\mathcal{B}_0) = \mathbb{C}$ implies $H^0((\mathcal{B}^e)^{mod}(\mathfrak{M}|_{t=1}, \mathfrak{M}'|_{t=1})) \cong R$.

In summary $Hom(\mathfrak{M}, \mathfrak{M}')$ is a finitely generated A_R-module with a connection whose restriction to $t = 1$ is isomorphic to R. Hence, by Prop 1.7.8, it is free of rank 1, up to q-torsion. In other words, there is a map $Hom(\mathfrak{M}, \mathfrak{M}') \to A_R$ with q-torsion kernel and cokernel; hence, there exists an $f \in Hom(\mathfrak{M}, \mathfrak{M}')$ and $k \in \mathbb{N}$ satisfying the following: for every $x \in Hom(\mathfrak{M}, \mathfrak{M}')$, there exists a unique $a \in A_R$ such that $q^k x = af + y$ for some q-torsion element y (by increasing n, we can ensure y vanishes, assume this holds). The same is true for $Hom(\mathfrak{M}', \mathfrak{M})$, $Hom(\mathfrak{M}, \mathfrak{M})$ and $Hom(\mathfrak{M}', \mathfrak{M}')$. Choose such an elements $f \in Hom(\mathfrak{M}, \mathfrak{M}')$, $g \in Hom(\mathfrak{M}', \mathfrak{M})$ with the same $k \in \mathbb{N}$.

Moreover, the composition map

$$Hom(\mathfrak{M}', \mathfrak{M}) \otimes_{A_R} Hom(\mathfrak{M}, \mathfrak{M}') \to Hom(\mathfrak{M}, \mathfrak{M}) \tag{1.290}$$

(again the tensor product is q-adically completed) has kernel and cokernel that are q-torsion. To see this consider the cokernel C. By the compatibility of the connection with composition in Prop 1.6.38, the image and the cokernel carry connections along D_{A_R}. Moreover, the restriction of composition map to $t = 1$ gives the composition of families restricted to $t = 1$; hence, it is an isomorphism and $C/(t-1)C = 0$. Using

Prop 1.7.4 we see that C is q-torsion.

Hence, there exists an m such that $q^m 1_{\mathfrak{M}}$ is in the image of (1.290). By increasing m, we can ensure an element of the form $a(g \otimes f)$ maps to $q^m 1_{\mathfrak{M}}$. Similarly, we can ensure there exists an element of the form $a'(f \otimes g)$ that maps to $q^m 1_{\mathfrak{M}'}$ under composition. Hence, $q^m ag = ag \circ f \circ a'g = q^m a'g$. Letting $f_1 = f$, $f_2 = q^m ag$ proves the statement of the theorem. □

Remark 1.10.2. A version of Theorem 1.10.1 for families over smooth complex curves is proven in [Sei14, Prop 1.21]. We follow a similar idea.

Lemma 1.10.3. *Let C^* be a complex of q-adically complete A_R-modules that are free of q-torsion. Assume the cohomology of the complex $C^*|_{q=0} = C^*/qC^*$ of A-modules is finitely generated over A in each degree. Then, $H^*(C^*)$ is finitely generated over A_R in each degree.*

Proof. (Sketch) Pick $y_1, \dots, y_n \in C^i/qC^i$ that are closed and whose classes generate $H^i(C^*/qC^*)$ as an A-module. Consider the module $A < y_1, \dots, y_n > \subset C^i/qC^i$ and consider its submodule of elements x such that there exists an $\tilde{x} \in C^i$ that deform x and satisfying $d(\tilde{x}) = 0$. This submodule is finitely generated over A as well and we can find closed elements $\tilde{x}_1, \dots, \tilde{x}_m \in C^i$ whose restrictions to $q = 0$ generate this submodule of deforming elements. Now, it is easy to see the cohomology classes of $\tilde{x}_1, \dots, \tilde{x}_m$ generate $H^i(C^*)$ over A_R. □

Proposition 1.10.4. *The family $\mathcal{G}_R^{sf}$ satisfies the conditions **G.1-G.3** for $\gamma = \gamma_\phi^R$.*

Proof. We have already shown **G.1** in Prop 1.6.15 and **G.3** in Corollary 1.6.52. See also Remark 1.6.53. To see **G.2**, notice $\mathcal{G}_R|_{t=1} \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$ is the diagonal by defining equations (1.31) and (1.32). Hence, it induces the diagonal bimodule of $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$, which descends to diagonal bimodule of M_ϕ^R. □

Remark 1.10.5. Similarly, by (1.31) and (1.32), $\mathcal{G}_R|_{u=1} \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$ is the graph of $\mathfrak{tr}^{-1}$. Hence, the bimodule $\mathcal{G}_R^{pre}$ is quasi-isomorphic to

$$(\mathcal{F}, \mathcal{F}') \mapsto \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}(\mathcal{F}, \mathfrak{tr}(\mathcal{F}')) \tag{1.291}$$

Since we take the smash product with action generated by $\mathfrak{tr} \otimes \phi$, the bimodule induced on $M_\phi^R = (\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A})\#\mathbb{Z}$ is given by

$$(\mathcal{F} \otimes a, \mathcal{F}' \otimes a') \mapsto M_\phi^R(\mathcal{F} \otimes a, (1 \otimes \phi^{-1})(\mathcal{F}' \otimes a')) = M_\phi^R(\mathcal{F} \otimes a, \phi_f^{-1}(\mathcal{F}' \otimes a')) \quad (1.292)$$

where ϕ_f is the "fiberwise ϕ" functor, which will be defined in Section 1.11.

Before going back to main theorem, let us state some lemmas in abstract deformation theory:

Lemma 1.10.6. *Let $\mathcal{B}$ and $\mathcal{B}'$ be Morita equivalent A_∞ categories over $\mathbb{C}$. Let $\mathcal{B}_R$ be a (possibly curved) deformation of $\mathcal{B}$ over $R = \mathbb{C}[[q]]$. Then there exists a (possibly curved) deformation $\mathcal{B}'_R$ of $\mathcal{B}'$ over R such that the initial Morita equivalence extends to a Morita equivalence of $\mathcal{B}_R$ and $\mathcal{B}'_R$.*

Next result is a versality statement, which is a version of [Sei15, Lemma 3.5] and indeed follows from [Sei15, Lemma 3.9].

Lemma 1.10.7. *Let $\mathcal{B}$ be an A_∞-category such that $HH^2(\mathcal{B}) = \mathbb{C}$. Then any two (curved) deformations $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$ over $R = \mathbb{C}[[q]]$ that are non-trivial in the first order are related by a base change by an automorphism f_q of R that specialize to identity at $q = 0$. In other words, $\mathcal{B}_1 = f_q^*\mathcal{B}_2$.*

Corollary 1.10.8. *Assume $\mathcal{B}$ and $\mathcal{B}'$ are Morita equivalent. Let $\mathcal{B}_R$ and $\mathcal{B}'_R$ be respective curved deformations over $R = \mathbb{C}[[q]]$ that are non-trivial in the first order. Assume $HH^2(\mathcal{B}) \cong HH^2(\mathcal{B}') \cong \mathbb{C}$. Then there exists an automorphism f_q of R specializing to identity at $q = 0$ such that initial Morita equivalence extends to a Morita equivalence of $\mathcal{B}_R$ and $f_q^*\mathcal{B}'_R$.*

Proof. This follows from Lemma 1.10.6 and 1.10.7. Cf. [Sei15, Cor 3.6]. □

Let us go back to proof of main theorem.

Theorem 1.1.4. *Let $\mathcal{A}$ be as in Section 1.1 and assume further that $HH^1(\mathcal{A}) = HH^2(\mathcal{A}) = 0$. Assume M_ϕ is Morita equivalent to $M_{1_\mathcal{A}}$. Then, $\phi \simeq 1_\mathcal{A}$.*

Proof. The Morita equivalence gives an isomorphism $HH^1(M_\phi) \cong HH^1(M_{1_A})$. Moreover, it gives a correspondence of group-like families parametrized by $\mathbb{G}_m$ and the correspondence is compatible with taking deformation classes. This implies that the isomorphism carries $L(M_\phi) := L(M_\phi^R)|_{q=0} \subset HH^1(M_\phi)$ onto $L(M_{1_A}) \subset HH^1(M_{1_A})$. The primitive class $\gamma_\phi \in L(M_\phi) \cong \mathbb{Z}^2$ is carried to another primitive class in $L(M_{1_A})$. By Corollary 1.9.12, there exists a self-Morita equivalence of M_{1_A} that carries every primitive class to every other primitive class. In particular, we can find one that carries image of γ_ϕ to γ_{1_A} and composing the initial Morita equivalence with the latter, we can assume the isomorphism of Hochschild cohomologies induced by the equivalence $M_\phi \simeq M_{1_A}$ maps γ_ϕ to γ_{1_A}.

By Corollary 1.10.8, the Morita equivalence extends to a Morita equivalence of M_ϕ^R and $f_q^* M_{1_A}^R$ for some automorphism f_q of R specializing to identity at $q = 0$. For simplicity assume $f_q = 1_R$.

Under this equivalence γ_ϕ^R corresponds to a deformation of γ_{1_A}(i.e. to an element $\gamma_{1_A}^R + O(q)$). This element also has to be in the discrete lattice $\mathbb{Z}^2 \cong L(M_{1_A}^R) \subset HH^1(M_\phi^R)$; hence, it is $\gamma_{1_A}^R$.

Consider two families of bimodules over M_ϕ^R and $M_{1_A}^R$, which we denoted by $\mathcal{G}_R^{sf}$. To avoid confusion, let us now denote them by $\mathcal{G}_\phi$ and $\mathcal{G}_1$ respectively. They both satisfy the properties **G.1-G.3** on their domains(**G.3** is satisfied for the class γ_ϕ^R and $\gamma_{1_A}^R$ respectively). The Morita equivalence gives rise to a correspondence of bimodules and families of bimodules. See (1.184) in Section 1.6.2. By Corollary 1.6.29 and Remark 1.6.32, the family over M_ϕ^R corresponding to $\mathcal{G}_1$ satisfies **G.3** for the class corresponding to $\gamma_{1_A}^R$, i.e. for γ_ϕ^R by the paragraph above. In other words, it follows $1 \otimes \gamma_\phi^R$. Denote this family over M_ϕ^R by $\mathcal{G}_1'$. That it satisfies **G.1** essentially follows from the fact that the Morita equivalence between M_ϕ and M_{1_A} induces a quasi-equivalence between $tw^\pi(M_\phi)$ and $tw^\pi(M_{1_A})$. That it satisfies **G.2** is clear.

Hence, by Theorem 1.10.1 the families $\mathcal{G}_\phi$ and $\mathcal{G}_1'$ are the same up to q-torsion. In particular, consider their restriction to R-point $u = 1$ of $Spf(A_R)$. By Remark 1.10.5, $\mathcal{G}_1$ restricts to diagonal; hence, $\mathcal{G}_1'$ restricts to diagonal of M_ϕ^R. By the same remark, $\mathcal{G}_\phi$ restricts to kernel Φ_f^{-1} of the ϕ_f^{-1} that will be defined more carefully in Section 1.11. Hence, Φ_f^{-1} is quasi-isomorphic to diagonal bimodule of M_ϕ^R up to q-torsion(thus, so is Φ_f by invertibility).

Let $p = p_0 \in \tilde{\mathfrak{I}}_R$ be a smooth R-point supported on C_0. As remarked in the proof of Lemma 1.8.6, there exist an unobstructed object of $tw^\pi(\mathcal{O}(\tilde{\mathfrak{I}}_R)_{cdg})$ given as a deformation of a cone of "$\mathcal{O}_{C_0}(-1) \to \mathcal{O}_{C_0}$". Note that it is easier to define as an unobstructed module rather than a twisted complex. Hence, we have an unobstructed object "$\mathcal{O}_p \otimes a$" $\in tw^\pi(M_\phi^R)$ for each $a \in ob(\mathcal{A})$ and a full (uncurved) subcategory $\{\mathcal{O}_p\} \otimes \mathcal{A} \subset tw^\pi(M_\phi^R)$. ϕ_f acts on this subcategory and the restriction of the bimodule Φ_f to it is given by

$$(\mathcal{O}_p \otimes a, \mathcal{O}_p \otimes a') \mapsto M_\phi^R(\mathcal{O}_p \otimes a, \phi_f(\mathcal{O}_p \otimes a')) \tag{1.293}$$

In other words, it is the bimodule corresponding to action of ϕ_f on $\{\mathcal{O}_p\} \otimes \mathcal{A}$. By above, it is quasi-isomorphic to diagonal bimodule up to q-torsion.

As these are uncurved categories, we can invert q. The category $\{\mathcal{O}_p\} \otimes \mathcal{A}$ becomes $\{\mathcal{O}_p\}_K \otimes \mathcal{A} \simeq K[t] \otimes \mathcal{A}$, where $K = \mathbb{C}((q))$ and t is a variable of degree 1. Note slight sloppiness of notation about q-adic completions of $\{\mathcal{O}_p\} \otimes \mathcal{A}$. On this category the diagonal $K[t] \otimes \mathcal{A}$ and $K[t] \otimes \Phi$ acts the same way. Lemma 1.10.9 concludes the proof. □

Lemma 1.10.9. *Let Φ and Ψ be self Morita equivalences(we can assume Φ and Ψ are just bimodules over $\mathcal{A}$). Assume $K[t] \otimes \Phi$ and $K[t] \otimes \Psi$ are quasi-isomorphic as $K[t] \otimes \mathcal{A}$-bimodules(where $deg(t) = 1$). Then Φ and Ψ are quasi-isomorphic.*

Proof. Consider the algebra maps $K \xrightarrow{i} K[t] \xrightarrow{p} K$. We have a functor

$$\begin{gathered} Bimod(K[t], K[t]) \to Bimod(K, K) \\ M \mapsto K \overset{L}{\otimes}_{K[t]} M \end{gathered} \tag{1.294}$$

where the bimodule structure on the right is induced by the inclusion. Geometrically this map would be "(p^*, i_*)". It sends the diagonal bimodule of $K[t]$ to diagonal bimodule of K. We can define a similar functor

$$\begin{aligned} Bimod(K[t]\otimes\mathcal{A},K[t]\otimes\mathcal{A}) &\to Bimod(\mathcal{A},\mathcal{A}) \\ \mathfrak{M} &\longmapsto K\overset{L}{\otimes}_{K[t]}\mathfrak{M} \end{aligned} \tag{1.295}$$

sending $K[t]\otimes\Phi$ and $K[t]\otimes\Psi$ to Φ and Ψ respectively. This finishes the proof. $\square$

1.11 Growth rates and another dynamical invariant

In the previous section, we have exploited the uniqueness of the family $\mathcal{G}_R^{sf}$ to distinguish trivial mapping tori from the others. However, $\mathcal{G}_R^{sf}$ encodes more and we can use it to produce more invariants of the tori. As mentioned in Remark 1.10.5, we can extract "fiberwise ϕ" by restricting this family to R-point $u=1$. Let us define it more carefully.

Let ψ be an auto-equivalence of $\mathcal{A}$ that commutes with ϕ. For simplicity assume ψ is a strict dg autoequivalence (i.e. acts bijectively on objects and hom-sets and its higher components vanish) and it commutes with ϕ strictly.

Definition 1.11.1. Under these assumptions, ψ induces auto-equivalences of M_ϕ and M_ϕ^R(again bijective on objects and hom-sets) given by descent of $1\otimes\psi$ acting on $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\otimes\mathcal{A}$, resp. $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}\otimes\mathcal{A}$ to their smash product with $\mathbb{Z}$, namely M_ϕ, resp. M_ϕ^R. Denote this autoequivalence by ψ_f and corresponding M_ϕ resp. M_ϕ^R-bimodule by Ψ_f.

Intuitively, this autoequivalence corresponds to application ψ on each fiber of "the fibration $M_\phi\to\mathfrak{T}_0$"; hence the name fiberwise ψ. This Section is about description of growth of $HH^*(M_\phi^R,\Phi_f^k)$.

Remark 1.11.2. Let Ψ be the $\mathcal{A}$-bimodule corresponding to ψ, i.e.

$$\Psi:(a,a')\mapsto\mathcal{A}(a,\psi(a')) \tag{1.296}$$

Due to the strict commutation assumption, Ψ is naturally $\mathbb{Z}_\Delta$-equivariant with the action is generated by

$$\mathcal{A}(a, \psi(a')) \to \mathcal{A}(\phi(a), \phi(\psi(a'))) = \mathcal{A}(\phi(a), \psi(\phi(a'))) \tag{1.297}$$

We can obtain Ψ_f by descent of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \Psi$, resp. $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \Psi$. In particular, we can assume $\Phi_f = (1 \otimes \Phi)\#\mathbb{Z}$ in the sense of Section 1.4.

Lemma 1.11.3. *Assume $\mathcal{A}$ is a smooth dg category. Then*

$$\begin{aligned} CC^*(M_\phi, \Psi_f) \simeq cocone\big(&CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \otimes CC^*(\mathcal{A}, \Psi) \\ &\xrightarrow{tr_* \otimes \phi_* - 1} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \otimes CC^*(\mathcal{A}, \Psi)\big) \end{aligned} \tag{1.298}$$

i.e. the derived invariants of

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}, \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) \otimes CC^*(\mathcal{A}, \Psi) \tag{1.299}$$

Proof. The proof of Prop 1.5.13 works in this case. Namely, we would need to replace $CC^*(\mathcal{A}, \mathcal{A})$ by $CC^*(\mathcal{A}, \Psi)$ and so on. □

Corollary 1.11.4. *Assume $\mathcal{A}$ is a smooth dg category. Then*

$$\begin{aligned} CC^*(M_\phi^R, \Psi_f) \simeq cocone\big(&CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A}, \Psi) \\ &\xrightarrow{tr_* \otimes \phi_* - 1} CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A}, \Psi)\big) \end{aligned} \tag{1.300}$$

i.e. the derived invariants of

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A}, \Psi) \tag{1.301}$$

Proof. All the maps leading to quasi-isomorphism in Lemma 1.11.3 can be written over R, as remarked in Section 1.5. Hence, the result follows from Lemma 1.5.4 and 1.11.3. Note we use q-adically completed tensor products again. □

By Proposition 1.5.11 and Künneth formula, the cohomology of

$$CC^*(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}) \otimes CC^*(\mathcal{A}, \Psi) \tag{1.302}$$

is isomorphic to

$$(R \oplus R[-1] \oplus HH^{\geq 2}(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})) \otimes HH^*(\mathcal{A}, \Psi) \tag{1.303}$$

up to q-torsion. $HH^{\geq 2}(\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}, \mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg})$ is also q-torsion by Proposition 1.5.11. Hence, it is $HH^*(\mathcal{A}, \Psi) \oplus HH^*(\mathcal{A}, \Psi)[-1]$. By Corollary 1.11.4, its derived invariants compute $HH^*(M_\phi^R, \Psi_f)$. In other words

$$\begin{aligned} HH^*(M_\phi^R, \Psi_f) \cong R \otimes HH^*(\mathcal{A}, \Psi)^\phi \oplus \\ R \otimes (HH^*(\mathcal{A}, \Psi)^\phi \oplus HH^*(\mathcal{A}, \Psi)/(\phi - 1))[-1] \oplus \\ R \otimes HH^*(\mathcal{A}, \Psi)/(\phi - 1)[-2] \end{aligned}$$

Letting $\psi = \phi^k$, this relates growth of $HH^*(M_\phi^R)$ to the growth of invariant part of $HH^*(\mathcal{A}, \Phi^k)$. We also recover

$$\begin{aligned} HH^*(\mathcal{A}, \Psi)^\phi \oplus \\ (HH^*(\mathcal{A}, \Psi)^\phi \oplus HH^*(\mathcal{A}, \Psi)/(\phi - 1))[-1] \oplus \\ HH^*(\mathcal{A}, \Psi)/(\phi - 1)[-2] \end{aligned}$$

out of $HH^*(M_\phi^R, \Psi_f)$ up to q-torsion(the input is up to q-torsion). By Section 1.10 this data is an invariant of the pair (M_ϕ, γ_ϕ), when $\psi = \phi^k$, where $k \in \mathbb{Z}$. Hence, we obtain:

Proposition 1.11.5.

$$\begin{aligned} HH^*(\mathcal{A}, \Phi^k)^{\phi} \oplus \\ (HH^*(\mathcal{A}, \Phi^k)^{\phi} \oplus HH^*(\mathcal{A}, \Phi^k)/(\phi - 1))[-1] \oplus \\ HH^*(\mathcal{A}, \Phi^k)/(\phi - 1)[-2] \end{aligned}$$

is an invariant of the pair (M_ϕ, γ_ϕ). *In other words, if* M_ϕ *is Morita equivalent to* $M_{\phi'}$ *such that* γ_ϕ *corresponds to* $\gamma_{\phi'}$ *under the induced isomorphism between Hochschild cohomologies, then the graded vector spaces given above are isomorphic. Note* $\boldsymbol{\Phi^k}$ *denotes the self-convolution of the bimodule* Φ *k-times.*

Chapter 2

Distinguishing open symplectic mapping tori via their wrapped Fukaya categories

2.1 Introduction

This paper is a sequel to [Kar18]. In [Kar18], we have constructed a category M_ϕ that is supposed to model the wrapped Fukaya category of an open symplectic mapping torus and we have exploited the dynamics of these categories to distinguish them. In this paper, we prove the equivalence of M_ϕ with the wrapped Fukaya category and give example applications of the main theorem, such as construction of non-deformation equivalent Liouville domains with the same topology and symplectic cohomology.

More precisely, let M be a Weinstein domain with vanishing first and second Betti numbers and let ϕ be a compactly supported (i.e. $\phi|_{\partial M} = 1_M$), exact symplectomorphism acting on M. Consider the completion $\widehat{M}$. One can define **the open symplectic mapping torus of** ϕ as

$$\widehat{T}_\phi := (\mathbb{R} \times S^1 \setminus \mathbb{Z} \times \{1\}) \times \widehat{M}/(s,\theta,x) \sim (s+1,\theta,\phi(x)) \tag{2.1}$$

There is an obvious projection map $\pi : \widehat{T}_\phi \to \widehat{T}_0$, where $\widehat{T}_0$ is the punctured 2-torus. π

is a symplectic fibration with a flat symplectic connection and with fibers isomorphic to $\widehat{M}$. The symplectic form is

$$\{\omega_{\widehat{M}}\} + \pi^*\omega_{\widehat{T}_0} \tag{2.2}$$

Here, $\omega_{\widehat{M}}$, resp. $\omega_{\widehat{T}_0}$ is the symplectic form on $\widehat{M}$, resp. $\widehat{T}_0$, and $\{\omega_{\widehat{M}}\}$ denotes fiberwise $\omega_{\widehat{M}}$.

$\widehat{T}_0$ can be seen as the completion of a torus with one boundary component. This domain will be denoted by T_0, and its $\mathbb{Z}$-fold covering space corresponding to covering

$$(\mathbb{R} \times S^1 \setminus \mathbb{Z} \times \{1\}) \to \widehat{T}_0 \tag{2.3}$$

will be denoted by $\tilde{T}_0$. One can build a Weinstein domain

$$T_\phi := \tilde{T}_0 \times M/(s, \theta, x) \sim (s+1, \theta, \phi(x)) \tag{2.4}$$

whose completion gives $\widehat{T}_\phi$. See Figure 2-1.

We will later prove (Proposition 2.2.4) that T_ϕ carries a natural Liouville structure that is deformation equivalent to a Weinstein structure. Moreover, $\partial T_\phi = \partial(T_0 \times M)$ as contact manifolds. Our main result is about distinguishing the fillings T_ϕ and $T_0 \times M$. More precisely:

Theorem 2.1.1. *Suppose M satisfies Assumption 2.1.2 below, and ϕ induces a non-trivial action on $\mathcal{W}(M)$. Then, T_ϕ and $T_0 \times M$ have inequivalent wrapped Fukaya categories. In particular, they are not graded symplectomorphic.*

By assumptions on $H^1(M)$ and $H^2(M)$, $K_M = \bigwedge_{\mathbb{C}}^n T^*M$ has a canonical trivialization (that is unique up to homotopy); hence, $\mathcal{W}(M)$ and $SH^*(M)$ can be $\mathbb{Z}$-graded. Moreover, K_{T_0} can be trivialized using the double cover $\mathbb{R}^2 \setminus \mathbb{Z}^2 \to T_0$, and this induces a natural trivialization on K_{T_ϕ} and a $\mathbb{Z}$-grading on $\mathcal{W}(T_\phi)$. Theorem 2.1.1 distinguishes the wrapped Fukaya categories with this particular grading. The conclusion of this is that there is no exact symplectomorphism between the two domains that preserves the trivializations up to homotopy (i.e. they are not graded symplectomorphic).

$\mathcal{W}(M)$ and $SH^*(M)$ can be defined with coefficients in $\mathbb{Z}$, but we assume they are defined over $\mathbb{C}$. The assumption we need for Theorem 2.1.1 is:

Assumption 2.1.2. *$\mathcal{W}(M)$ is cohomologically proper and bounded below in each degree (see Assumption 2.1.7), $SH^*(M)$ vanishes for $* < 0, * = 1, * = 2$, and $SH^0(M) = \mathbb{C}$.*

There are many examples of symplectic manifolds satisfying Assumption 2.1.2. For instance:

Example 2.1.3. Let X be a smooth hypersurface in $\mathbb{CP}^7$ of degree greater than or equal to 9 and $D \subset X$ be a transverse hyperplane section. Let $M = X \setminus D$ and let ϕ be the square of a Dehn twist along a spherical Lagrangian (one can find such Lagrangians easily by considering degenerations of M into varieties with quadratic singularities). Then by Theorem 2.1.1, T_ϕ and $T_0 \times M$ are not graded symplectomorphic. On the other hand, ϕ is smoothly isotopic to identity by an unpublished work of Giroux (see [May09, §5.3] and [Sie16]). Thus, T_ϕ and $T_0 \times M$ are diffeomorphic.

Example 2.1.4. Similarly, let X be a smooth hypersurface in $\mathbb{CP}^5$ of degree greater than or equal to 7, and M be complement of a transverse hyperplane section. Let ϕ be the eighth power of a Dehn twist. One can show using [Kry07] and [KK05] that ϕ is smoothly isotopic to identity (see remarks at the end of [KK05, Section 3.1]). Hence, again we obtain a Weinstein domain T_ϕ that is different from $T_0 \times M$ as a graded Liouville domain, but they are the same as smooth manifolds.

That these manifolds satisfy Assumption 2.1.2 is proven in Section 2.5 (see Corollary 2.5.6). The reason ϕ acts non-trivially in either case is that when $\mathcal{W}(M)$ is $\mathbb{Z}$-graded, τ_L , the Dehn twist along a spherical Lagrangian L, acts on L (considered as an object of $\mathcal{W}(M)$) as shift by $1 - n$. One can consider the cases where an A_n-configuration of Lagrangian spheres is embedded into M to produce more sophisticated examples (i.e. examples where action of ϕ is different from a shift).

As these two examples demonstrate, T_ϕ is a construction that turns exotic symplectomorphisms (i.e. symplectomorphisms that are isotopic to 1 in $Diff(M, \partial M)$

but act non-trivially on $\mathcal{W}(M)$) into exotic Liouville structures. In particular, we can use Theorem 2.1.1 to obtain pairs of diffeomorphic, but not (graded) symplectomorphic Liouville domains for every even $n \geq 4$. Indeed, as we explain now, it is possible to produce non-symplectomorphic examples as well.

Assume $\pi_1(M) = 1$ and $n > 1$. One can attach subcritical handles along the same isotropic spheres on the boundary of T_ϕ and $T_0 \times M$ to obtain Weinstein manifolds M_1 and M_2 satisfying $\pi_1(M_1) = \pi_1(M_2) = 1$. Moreover, attaching subcritical handles does not change the derived equivalence class of the wrapped Fukaya category. A proof of this statement can be found in [GPS18, Cor 1.22], where one uses the Weinstein property for generation as in [GPS18, Cor 1.21] (see also [Cie02], [Iri13] and [BEE12]). Combining Theorem 2.1.1 with this fact, we obtain:

Corollary 2.1.5. *M_1 and M_2 give different exact fillings of $\partial M_1 = \partial M_2$.*

Notice, after handle attachment, the trivialization of the canonical bundle is unique up to homotopy. Hence, Corollary 2.1.5 produces non-symplectomorphic fillings (which is stronger than not being graded symplectomorphic).

The proof of Theorem 2.1.1 is in two steps. The first is to define an algebraic model M_ϕ for $\mathcal{W}(T_\phi)$, and prove an analogue of Theorem 2.1.1. This is achieved in [Kar18]. More precisely, we have proven:

Theorem 2.1.6. *[Kar18, Theorem 1.3] Suppose Assumption 2.1.7 is satisfied. Assume ϕ is not equivalent to the identity functor $1_\mathcal{A}$. Then, M_ϕ and $M_{1_\mathcal{A}}$ are not Morita equivalent. In particular, they are not derived equivalent.*

In the statement of Theorem 2.1.6, $\mathcal{A}$ is an A_∞-category over $\mathbb{C}$, ϕ denotes an auto-equivalence of $\mathcal{A}$, and M_ϕ is constructed based on this data (one can assume $\mathcal{A}$ is dg and ϕ is strict for the construction of M_ϕ and for the proof of the theorem). Theorem 2.1.6 is an obvious algebraic analogue of Theorem 2.1.1.

Assumption 2.1.7. *$\mathcal{A}$ is (homologically) smooth (see [KS09] for a definition), proper in each degree and bounded below, i.e. $H^*(hom_\mathcal{A}(x,y)) = 0$ is finite dimensional in each degree and vanishes for $* \ll 0$ for any $x, y \in Ob(\mathcal{A})$. Moreover, $HH^i(\mathcal{A})$, the*

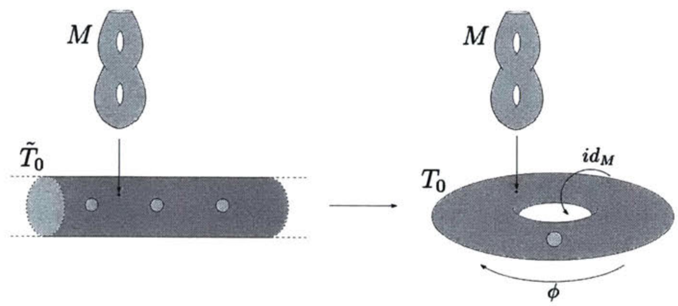

Figure 2-1: T_ϕ and its $\mathbb{Z}$-fold cover $\tilde{T}_0 \times M$

i^{th} Hochschild cohomology group of $\mathcal{A}$, is 0 for $i < 0, i = 1, i = 2$ and is isomorphic to $\mathbb{C}$ for $i = 0$.

The second step in the proof of Theorem 2.1.1 is the comparison of M_ϕ with $\mathcal{W}(T_\phi)$, and this is the goal of this paper. In other words, we prove:

Theorem 2.1.8. *M_ϕ is Morita equivalent to $\mathcal{W}(T_\phi)$, if $\mathcal{A} = \mathcal{W}(M)$ and the auto-equivalence ϕ is induced by the given symplectomorphism (which was also denoted by ϕ).*

Theorem 2.1.1 is clearly implied by Theorem 2.1.6 and Theorem 2.1.8. More precisely, since M is Weinstein it is non-degenerate in the sense of [Abo10]. Therefore, it is smooth by [Gan12, Theorem 1.2] and its Hochschild cohomology is isomorphic to $SH^*(M)$ by [Gan12, Theorem 1.1]. Hence, if $\mathcal{A} = \mathcal{W}(M)$ and the auto-equivalence ϕ is induced by the given symplectomorphism, then Assumption 2.1.7 follows from Assumption 2.1.2. Hence, $\mathcal{W}(T_\phi) \simeq M_\phi$ is different from $\mathcal{W}(T_0 \times M) = \mathcal{W}(T_{1_M}) \simeq M_{1_\mathcal{A}}$ by Theorem 2.1.6.

Remark 2.1.9. The proof of Theorem 2.1.6 uses dynamical properties of (deformations) of categories M_ϕ. From one perspective, it may be seen as the comparison of a categorical version of Flux groups of M_ϕ and $M_{1_\mathcal{A}}$. However, the dynamics is not visible at a geometric level alone; hence, one has to exploit dynamics of Fukaya categories. Moreover, Corollary 2.1.5 gives examples of simply connected diffeomorphic fillings that are distinguished by categorical dynamics. As they have vanishing first cohomology, one cannot expect to use any form of flux. Therefore, the dynamics is only visible at the level of Fukaya categories.

Remark 2.1.10. As we will show later (Lemma 2.5.8), it is possible to give examples that cannot be distinguished by their symplectic cohomology groups either. Indeed, this is true for Example 2.1.3 and Example 2.1.4 if we assume the degree of the hypersurface X is at least 14, resp. 10.

Remark 2.1.11. Given a pair of strictly commuting auto-equivalences ϕ and ϕ' on $\mathcal{A}$, one can generalize the mapping torus category M_ϕ to double monodromy mapping torus category $M_{\phi,\phi'}$ (as a twisted tensor product with respect to group action/extra grading by $\mathbb{Z}\times\mathbb{Z}$, see Section 2.3 for $\mathbb{Z}$-action case). In this case, $\mathcal{W}(T_\phi)$ with different choice of gradings will correspond to $M_{\phi[m],[n]}$, where $[m]$ and $[n]$ denote the shift functors. Moreover, one can attempt to generalize Theorem 2.1.6 to classify $M_{\phi[m],[n]}$ in terms of the order of ϕ modulo translation (this requires a simple modification of the technique of [Kar18], as well as some minor technical checks). Hence, this would imply that T_ϕ and $T_0 \times M$ are not symplectomorphic if ϕ does not act as a shift. Moreover, this would produce infinitely many different Liouville domains (that are diffeomorphic in the examples above). We do not plan to write this generalization separately.

2.1.1 Summary of the proof of Theorem 2.1.8

To prove Theorem 2.1.8, we need to give a simpler description of M_ϕ. We claim it is a "twisted tensor product" of $\mathcal{O}(\mathcal{T}_0)_{dg}$ and $\mathcal{A}$. Here, $\mathcal{O}(\mathcal{T}_0)_{dg}$ is a dg model for the derived category of coherent sheaves on the nodal elliptic curve $\mathcal{T}_0$ over $\mathbb{C}$. This claim is proven in Section 2.3. After introducing the notion of twisted tensor products, the claim follows from the definition of M_ϕ. We remind the definition of M_ϕ from [Kar18] for the convenience of the reader.

By the results of [LP16], $\mathcal{O}(\mathcal{T}_0)_{dg}$ is also quasi-equivalent to $\mathcal{W}(T_0)$. However, the notion of twisted tensor product requires extra gradings on $\mathcal{O}(\mathcal{T}_0)_{dg}$ and $\mathcal{W}(T_0)$. We define the extra gradings, and we show in Section 2.3.3 that $\mathcal{O}(\mathcal{T}_0)_{dg}$ and $\mathcal{W}(T_0)$ are quasi-equivalent as categories with extra grading as well (by reproving the equivalence of these categories via the gluing formula of [GPS18]).

Hence, Theorem 2.1.8 reduces to the following:

Theorem 2.1.12. *$\mathcal{W}(T_\phi)$ is quasi-equivalent to twisted tensor product of $\mathcal{W}(T_0)$ and $\mathcal{W}(M)$.*

Note, the notion of twisted tensor product and twisted bimodule is similar to the notion that can be found in [BO08] and [GNW15] and perhaps its appearance should not be as surprising for Künneth type theorems for symplectic fibrations.

This is a twisted version of Künneth theorem for wrapped Fukaya categories. Untwisted versions of this theorem are proven in [Gan12], [Gao17], and recently in [GPS18]. Indeed, for simplicity we also start by proving a Künneth theorem in the untwisted case and for the compact Fukaya category. Namely:

Theorem 2.1.13. *$\mathcal{F}(T_0 \times M)_p$ is quasi-equivalent to tensor product of $\mathcal{F}(T_0)$ and $\mathcal{F}(M)$.*

Here, $\mathcal{F}(T_0 \times M)_p$ is full subcategory of $\mathcal{F}(T_0 \times M)$ spanned by product type Lagrangian branes. To define the "Künneth" functor, we use the same count of pseudo-holomorphic quilts as in [Gan12] and [GPS18] (see Figures 2-10 and 2-15) and obtain a functor

$$\mathcal{F}(T_0 \times M)_p \to Bimod(\mathcal{F}(T_0), \mathcal{F}(M)) \tag{2.5}$$

In Section 2.4.2.1, we detail the definition of (2.5). Then we define maps of $\mathcal{F}(T_0)$-$\mathcal{F}(M)$ bimodules, similar to continuation maps to show the image of objects of $\mathcal{F}(T_0 \times M)_p$ are quasi-isomorphic to Yoneda bimodules, and we use a TQFT argument and Yoneda lemma to prove fully faithfulness. In Section 2.4.2.2, we present necessary modifications to apply the same idea in the twisted case. The algebra of the twisted case is taken care of in Section 2.3, and Section 2.4.2.2 consists of modifications on the labeling for quilts and modifications needed on the perturbation data for issues such as compatibility with A_∞-structure on $\mathcal{W}(T_\phi)$ as well as analytic issues arising from the use of infinite type covering spaces. We would like to emphasize Section 2.4.2.1 and Section 2.4.2.2 are not the main goal and they are written with the purpose of conveying the idea on how to prove fully faithfulness before the compactness issues with wrapped Fukaya categories arise. Indeed, Theorem 2.1.13 is rather uninteresting on its own due to lack of a generation statement for $\mathcal{F}(T_0 \times M)$ by product type objects.

In Section 2.4.3, we explain the modifications required for wrapped Fukaya categories. We start by introducing a version of $\mathcal{W}^2$ of [Gan12], $\mathcal{W}^s$ of [Gao17] and $\mathcal{W}^{prod}$ of [GPS18]. These are versions of wrapped Fukaya category defined using split-type data on the products. T_ϕ is not a product; however, its conical end can be identified with that of the product and we define $\mathcal{W}^2(T_\phi)$ using data that is of product type on the conical end. This takes care of compactness issues related to conical end. After one takes care of compactness one could define a functor

$$\mathcal{W}(T_\phi) \to Bimod_{tw}(\mathcal{W}(T_0), \mathcal{W}(M)) \tag{2.6}$$

using a similar count of quilts, where $Bimod_{tw}$ is the category of twisted bimodules defined in Section 2.3. More precisely, we describe a set of "twisted product type" Lagrangians generating $\mathcal{W}(T_\phi)$ and define the functor on these Lagrangians. It will be remarked that their image under (2.6) are equivalent to twisted Yoneda bimodules. These bimodules are defined in Section 2.3 and their span is equivalent to the twisted tensor product (in fact, their span can be taken to be the definition of twisted tensor product).

The fully faithfulness argument we use for compact Fukaya categories works almost verbatim, and we obtain a quasi-equivalence from $\mathcal{W}(T_\phi)$ to twisted tensor product, proving Theorem 2.1.12.

Finally, we would like to mention another possible proof of Theorem 2.1.12. Recently, a gluing formula for wrapped Fukaya categories appeared in [GPS18]. One can also cut T_ϕ into Liouville sectors that are products. Then one can use gluing formula, ordinary Künneth theorem for sectors (again proven in [GPS18]) and the framework of twisted tensor products given in Section 2.3 to give another proof of Theorem 2.1.12. We sketch this in Appendix A.

2.1.2 Outline

In Section 2.2, we start by investigating Liouville and Weinstein structures on T_ϕ. In other words, we show T_ϕ carries a natural Liouville structure that is deformation equivalent to a Weinstein structure. We give a description of cocores of this Weinstein manifold, giving us generators by [CDGG17], or by [GPS18, Theorem 1.9].

In Section 2.3, we set up the algebra of twisted tensor products and bimodules. We then demonstrate how one can realize M_ϕ as a twisted tensor product and comment on the extra gradings on $\mathcal{W}(T_0)$. In particular, we prove equivalence of $\mathcal{W}(T_0)$ and $\mathcal{O}(\mathfrak{T}_0)_{dg}$ as extra graded categories.

Section 2.4 is devoted to proof of Theorem 2.1.12. We start by giving detailed proofs in simpler cases (such as untwisted and twisted Künneth theorem for compact Lagrangians) to better illustrate the idea.

In Section 2.5, we give a large class examples of symplectic manifolds satisfying Assumption 2.1.2, which let us apply Theorem 2.1.1, and construct exotic Liouville manifolds as in Corollary 2.1.5.

Acknowledgments

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2.2 Structures on the mapping torus

2.2.1 Liouville structure

Let $\widehat{M}$ and $\widehat{T}_0$ denote the completions of M and T_0. Let λ_M and Z_M denote the Liouville form and vector field on the completion $\widehat{M}$ as well. We assume ϕ is exact, i.e. there exists a smooth function K with compact support in the interior of Liouville domain M such that $\phi_*(\lambda_M) = \lambda_M + dK$.

Proposition 2.2.1. *$\widehat{T}_\phi$ has a natural Liouville structure.*

Proof. We first try the following ansatz for the Liouville form:

$$\lambda_{T_\phi} = \{\lambda_s\} + \pi^* \lambda_{T_0} \tag{2.7}$$

where λ_{T_0} is a choice of Liouville form on T_0 and $\{\lambda_s\}$ refers to a family of Liouville forms on M parametrized by the first coordinate of $\widehat{T}_0$ (and extend them to completions). We will construct λ_s as $\lambda_M + dK_s$, where $\{K_s\}$ is a smooth family of compactly supported functions on M as above (extended by 0 to $\widehat{M}$). If we take the parameter s in $\mathbb{R}$, we have to show the ansatz 2.7 induces a 1-form on $\widehat{T}_\phi$, i.e.

$$\phi_* \lambda_s = \lambda_{s+1} \tag{2.8}$$

We will choose $\{\lambda_s\}$ to be constant near every $s \in \mathbb{Z}$ (indeed over $(s - \epsilon, s + \epsilon)$ for a fixed ϵ such that the hole of the domain T_0 have s-component in this interval).

For (2.8) to hold, we need

$$d(\phi_*(K_s) + K) = dK_{s+1} \tag{2.9}$$

on $\widehat{M}$ which would be implied by

$$\phi_*(K_s) + K = K_{s+1} \tag{2.10}$$

(2.10) gives us enough data to define family $\{K_s\}$. Namely, fix a small $\epsilon > 0$ (one may assume it is large enough to cover the s-component of the hole of T_0). Let $\rho : (-\epsilon, 1+\epsilon) \to [0,1]$ be a function such that $\rho(s) = 0$ for $x \in (-\epsilon, \epsilon)$ and $\rho(x) = 1$ for $x \in (1-\epsilon, 1+\epsilon)$. Define $K_s = \rho(s)K$, for $s \in (-\epsilon, 1+\epsilon)$. The equality $\phi_*(K_s) + K = K_{s+1}$ holds for $s \in (-\epsilon, \epsilon)$ and we can extend K_s to all $s \in \mathbb{R}$ using (2.10).

Unfortunately, λ_{T_ϕ} is not a primitive for the original symplectic form. More explicitly

$$d(\lambda_{T_\phi}) = \{d_M(\lambda_s)\} + d(\lambda_{T_0}) + \rho'(s)dsdK = \omega_{T_\phi} + \rho'(s)dsdK \qquad (2.11)$$

Here, d_M is the exterior derivative along the fiber direction and $d(\lambda_{T_0})$ is used to mean $\pi^* d(\lambda_{T_0})$. We are implicitly using the coordinates $s \in (0,1)$ and the fact that $\rho'(s)$ vanishes near $s = 0, 1$. We can correct the form λ_{T_ϕ} as

$$\lambda_{T_\phi} + \rho'(s)Kds \qquad (2.12)$$

and its derivative is clearly ω_{T_ϕ}. Moreover, (2.12) looks like λ_s near the conical ends of fibers and $\lambda_M + \lambda_{T_0}$ near the puncture; hence, it is a Liouville form. □

2.2.2 Weinstein structures on T_ϕ

Definition 2.2.2. A triple (M, λ_M, f_M) is called Weinstein if (M, λ_M) is a Liouville manifold with Liouville vector field Z_M and f_M is a proper (generalized) Morse function on M such that

$$Z_M(f_M) \geq \epsilon(|Z_M|^2 + |df_M|^2) \qquad (2.13)$$

for some $\epsilon > 0$ (and for some Riemannian metric). If a pair (Z_M, f_M) satisfies (2.13), Z_M is called gradient-like for f_M and f_M is called Lyapunov for Z_M (see [CE12] for more details).

Assume (M, ω_M) is Weinstein, with Weinstein structure (M, λ_M, Z_M, f_M). We aim to make T_ϕ Weinstein, possibly with respect to a deformation equivalent Liouville structure.

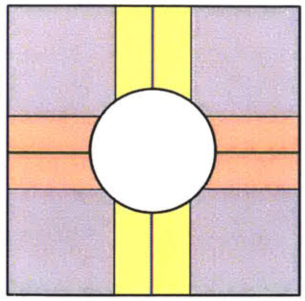

Figure 2-2: Handlebody decomposition of T_0

Fix a Weinstein structure on T_0 such that the handlebody decomposition is as in Figure 2-2. The yellow and orange strips (i.e. the vertical and horizontal strips respectively) are the 1-handles, and the blue and yellow curves (i.e. the vertical and horizontal curves) are the cocores. We denote this Weinstein structure by $(\lambda_{T_0}, Z_{T_0}, f_{T_0})$. The ansatz for the Weinstein structure on T_ϕ is the following: Let $\{\lambda_z : z \in \tilde{T}_0\}$ be a family of Liouville structures on M that descent to mapping torus (in other words $\lambda_{(s+1,t)} = (\phi^{-1})^*\lambda_{(s,t)}$). Assume

$$\lambda_z = \lambda_M + dK_z \tag{2.14}$$

for a family of functions K_z with support uniformly contained in a compact subset of $M \setminus \partial M$. Let $\{f_z : z \in T_0\}$ be a family of functions on the fibers of $\tilde{T}_0 \to T_0$ (i.e. a family of functions on M parametrized by M that descent to T_ϕ). Assume near the critical points of f_{T_0}, λ_z and f_z does not depend on $z \in \tilde{T}_0$ and form a Weinstein structure on M. Then,

$$\lambda_{T_0} + C^{-1}\lambda_z \tag{2.15}$$

$$f_{T_0} + C^{-1}f_z \tag{2.16}$$

is the ansatz for the Weinstein structure. First, notice (2.15) is a Liouville form for large enough C. To see this consider its differential:

$$\omega_{T_0} + C^{-1}\omega_M + C^{-1}\nabla\lambda_z \tag{2.17}$$

where ∇ is the natural flat symplectic connection of local system of symplectic manifolds $T_\phi \to T_0$ (in other words, locally it is differentiation in the base direction). Take $(n+1)^{th}$ exterior power to obtain

$$C^{-n}\omega_{T_0} \wedge \omega_M^n + O(C^{-n-1}) \tag{2.18}$$

To ensure it is Liouville, we could assume that λ_z is constant near ∂T_0 (the middle circle in Figure 2-2). However, this is not the best option for other purposes. Instead, we arrange it to be constant over a neighborhood of the part of ∂T_0 bounding gray and orange areas. We enlarge this area slightly to include part of yellow strip as well. That it is pointing outward over the rest will follow from the computation below.

Consider the mapping torus as a fibration over this Weinstein domain. Let T denote the yellow (i.e. vertical) middle strip in Figure 2-2. The monodromy ϕ is forgotten if we take out the pre-image of T. In other words, the complement is a product $(T_0 \setminus T) \times M$. Hence, the mapping torus can be constructed topologically by gluing $\overline{T_0 \setminus T} \times M$ and $T \times M$. We identify the left boundary of T (times M) by id_M, but we need to twist the right boundary by ϕ.

Now, we demand the family (λ_z, f_z) to be constant and equal to (λ_M, f_M) over a small neighborhood of the orange and gray area in Figure 2-2 (i.e. in a neighborhood of $T_0 \setminus T$). Here, we use a trivialization of the local system of symplectic manifolds over this area. To construct a 1-form and a function over the 1-handle T, we need to construct a family $\{(\lambda_z, f_z) : z \in T\}$ that is constant near right and left boundary of T and that interpolates between (λ_M, f_M) and $(\phi^{-1})^*(\lambda_M, f_M)$.

The 1-handle T can be identified with $[-1,1] \times [-1,1]$ in the qp-plane such that

$$f_T = (p^2 - \epsilon q^2)/2, \omega_T = dpdq \tag{2.19}$$

$$\lambda_T = \frac{pdq + \epsilon qdp}{1-\epsilon}, Z_T = \frac{p\partial_p - \epsilon q \partial_q}{1-\epsilon} \tag{2.20}$$

Note, we can simply assume $\epsilon \in (0,1)$ is $1/2$ as we will not let it vary.

As $(\phi^{-1})^*\lambda_M = \lambda_M + dK$, for some compactly supported function K, we can

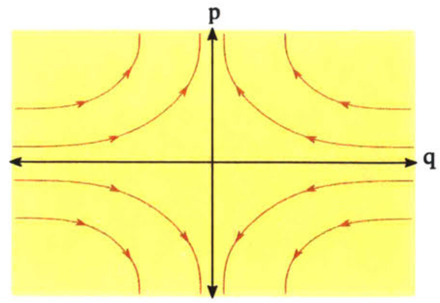

Figure 2-3: One handle T in qp-coordinates

interpolate between λ_M and $(\phi^{-1})^*\lambda_M$ by a family λ_q (it only depends on the q-coordinate) such that $\lambda_q = \lambda_M$ on $q \in [-1, \delta]$ for some small δ and $\lambda_q = (\phi^{-1})^*\lambda_M$ on $q \in [1-\delta, 1]$. Similarly, we choose a family f_q of functions on M such that $f_q = f_M$ for $q \in [-1, \delta]$ and $f_q = (\phi^{-1})^* f_M$ for $q \in [1-\delta, 1]$. Define

$$\lambda_C = \lambda_T + C^{-1}\lambda_q \tag{2.21}$$

$$f_C = f_T + C^{-1} f_q \tag{2.22}$$

Then we have

$$\omega_C := d\lambda_C = dpdq + C^{-1}\omega_M + C^{-1}dq \wedge \nabla_q(\lambda_q) \tag{2.23}$$

which is symplectic for large C as we remarked before. We need to compute symplectic dual Z_C of λ_C. Write

$$Z_C = Z_T + Z_q + Z_{corr} \tag{2.24}$$

where Z_q is the Liouville vector field corresponding to λ_q (Z_{corr} is a correction term not a Liouville vector field). Then

$$\begin{aligned} i_{Z_{corr}}\omega_C = i_{Z_C}\omega_C - i_{Z_T}\omega_C - i_{Z_q}\omega_C = \\ \lambda_C - \lambda_T + \frac{\epsilon}{1-\epsilon}C^{-1}q\nabla_q(\lambda_q) - C^{-1}\lambda_q + C^{-1}i_{Z_q}\nabla_q(\lambda_q)dq = \\ \frac{\epsilon}{1-\epsilon}C^{-1}q\nabla_q(\lambda_q) + C^{-1}i_{Z_q}\nabla_q(\lambda_q)dq \end{aligned} \tag{2.25}$$

Clearly,

$$C^{-1}i_{Z_q}\nabla_q(\lambda_q)dq = C^{-1}i_{Z_q}\nabla_q(\lambda_q)i_{\partial_p}\omega_C \tag{2.26}$$

Define

$$Z_{corr,1} = Z_{corr} - C^{-1} i_{Z_q} \nabla_q(\lambda_q)\partial_p \tag{2.27}$$

Then

$$i_{Z_{corr,1}}\omega_C = \frac{\epsilon}{1-\epsilon} C^{-1} q \nabla_q(\lambda_q) \tag{2.28}$$

Write $Z_{corr,1} = g\partial_p + v_f$, where v_f is in fiber direction (there is no $g_1\partial_q$ as this would produce $-g_1 dp$ term, which cannot be eliminated). (2.28) implies

$$g i_{\partial_p}\omega_C + i_{v_f}\omega_C = \frac{\epsilon}{1-\epsilon} C^{-1} q \nabla_q(\lambda_q) \tag{2.29}$$

In other words

$$g dq + C^{-1} i_{v_f}\omega_M - C^{-1} i_{v_f}\nabla_q(\lambda_q) dq = \frac{\epsilon}{1-\epsilon} C^{-1} q \nabla_q(\lambda_q) \tag{2.30}$$

Using the natural splitting of tangent spaces into horizontal and vertical directions, we conclude

$$g = C^{-1} i_{v_f}\nabla_q(\lambda_q) \text{ and } i_{v_f}\omega_M = \frac{\epsilon}{1-\epsilon} q \nabla_q(\lambda_q) \tag{2.31}$$

The symplectic dual of $\nabla_q \lambda_q$ is clearly $\nabla_q Z_q$. Hence,

$$v_f = \frac{\epsilon}{1-\epsilon} q \nabla_q Z_q \text{ and } g = C^{-1} q \frac{\epsilon}{1-\epsilon} i_{\nabla_q Z_q}(\nabla_q \lambda_q) \tag{2.32}$$

(the latter term actually vanishes). To sum up

$$Z_{corr} = v_f + O(C^{-1})\partial_p \tag{2.33}$$

and thus

$$Z_C = Z_T + Z_q + \frac{\epsilon}{1-\epsilon} q \nabla_q Z_q + O(C^{-1})\partial_p \tag{2.34}$$

As we mentioned, we do not let ϵ vary, but for sufficiently large C, Z_T dominates $O(C^{-1})\partial_p$ near the upper and lower boundary of T. Hence, it is pointing outward there. On the other hand, Z_q are all the same near ∂M, so it is pointing outward on

$T \times \partial M$ as well. In short, the form is Liouville over the 1-handle T.

Now, let us examine f_C. First, the only critical point of f_T is at $(0,0)$. For large enough C, df_T dominates $C^{-1}df_q + C^{-1}dq \wedge \partial_q f_q$ away from $(0,0)$. Near $(0,0)$, f_q is constant and equal to the Morse function f_M. Hence, the only critical points of f_C live over $q = p = 0$ and they are all non-degenerate.

Moreover,

$$Z_C(f_C) = Z_T(f_T) + O(C^{-1}) \tag{2.35}$$

since $v_f(f_T) = 0$. Hence, away from the critical point $(0,0)$, $Z_T(f_T)$ dominates the other terms and Lyapunov property (2.13) is satisfied.

Near $(0,0)$, λ_q, f_q are constant in q; hence, by (2.25) $Z_{corr} = 0$, $Z_q = Z_M, f_q = f_M$. This implies

$$Z_C(f_C) = Z_T(f_T) + C^{-1}Z_M(f_M) \tag{2.36}$$

From this, Lyapunov property is clear.

By gluing the "Weinstein structures" on $\overline{T_0 \setminus T} \times M$ and $T \times M$, we obtain:

Proposition 2.2.3. *There exist a Weinstein structure on T_ϕ that is of the form*

$$\lambda_{T_\phi} = \lambda_{T_0} + C^{-1}\lambda_z \tag{2.37}$$

$$f_{T_\phi} = f_{T_0} + C^{-1}f_z \tag{2.38}$$

where λ_z is a family of Liouville forms, f_z is a family of functions on M, both are locally constant (in z) outside one handle T and around the critical point of T. This is Weinstein for all sufficiently large C.

Recall how we made the original symplectic structure on T_ϕ Liouville. We found a primitive of the form

$$\lambda'_{T_\phi} = \lambda_{T_0} + \lambda_s + \rho'(s)Kds \tag{2.39}$$

where $\lambda_s = \lambda_M + \rho(s)d_M(K)$.

Turning C parameter on would effect these only by

$$\lambda'_{T_\phi,C} = \lambda_{T_0} + C^{-1}\lambda_s + C^{-1}\rho'(s)Kds \tag{2.40}$$

Now, for large enough C, the Liouville structure (2.37) and (2.40) are linearly interpolated by Liouville forms. Hence, they are deformation equivalent. (2.40) is clearly deformation equivalent to (2.39). In summary:

Proposition 2.2.4. *T_ϕ with its standard symplectic structure is Liouville and the corresponding Liouville form is deformation equivalent to Liouville form of a Weinstein structure.*

2.2.3 Generators for $\mathcal{W}(T_\phi)$

Now, we will write an explicit set of generators for $\mathcal{W}(T_\phi)$. As shown in [CDGG17], the cocores of a Weinstein manifold generate its wrapped Fukaya category. The cocores of the Weinstein structure in Proposition 2.2.3 can be described as follows: The cocores of T_0 with the chosen structure are given by green and purple curves in Figure 2-2 (i.e. the dividing horizontal and vertical curves), which we denote by L_{gr}, and L_{pur} respectively. Fix lifts of these curves to $\mathbb{Z}$-fold cover $\tilde{T}_0 \to T_0$, and denote them by $\tilde{L}_{gr}, \tilde{L}_{pur}$.

Definition 2.2.5. Let $L' \subset M$ and $L \subset T_0$ be cylindrical Lagrangians with a fixed lift $\tilde{L} \subset \tilde{T}_0$ of the latter. Let $L \times_\phi L'$ denote the image of $\tilde{L} \times L'$ under the projection map $\tilde{T}_0 \times M \to T_\phi$.

It is easy to see the cocores of critical handles of (2.38) are among the Lagrangians $L_{gr} \times_\phi L'$, $L_{pur} \times_\phi L'$. More precisely, if L' is a cocore disc for M, moving it along green and purple curves in Figure 2-2 gives us the cocores of T_ϕ.

It is easy to see that by careful choices $L_{gr} \times_\phi L'$ and $L_{pur} \times_\phi L'$ can be forced to stay as exact Lagrangians throughout the Liouville deformations involved. Hence, we have proven

Corollary 2.2.6. *$\mathcal{W}(T_\phi)$ is generated by objects of the form $L_{gr} \times_\phi L'$ and $L_{pur} \times_\phi L'$, where L' is a cocore for M.*

2.3 Mapping torus categories and twisted tensor products

2.3.1 Twisted tensor product, twisted bifunctors and bimodules

Let A and A' be ordinary algebras. Assume A carries an extra $\mathbb{Z}$-grading and A' carries an automorphism. Following [BO08], we can define $A \otimes_{tw} A'$ as the algebra with underlying vector space $A \times A'$ and with multiplication

$$(a_1 \otimes a_1').(a_2 \otimes a_2') = a_1 a_2 \otimes \phi^{-|a_2|}(a_1') a_2' \tag{2.41}$$

where $|a_2|$ is the degree of a_2 in the extra grading. Hence, one can describe a right module over $A \otimes_{tw} A'$ as vector space M with a right A-module structure

$$(m, a) \mapsto \mu^{1|1;0}(m|a;) \tag{2.42}$$

and a right A'-module structure,

$$(m, a') \mapsto \mu^{1|0;1}(m|;a') \tag{2.43}$$

satisfying

$$\mu^{1|0;1}(\mu^{1|1;0}(m|a;)|;a') - \mu^{1|1;0}(\mu^{1|0;1}(m|;\phi^{|a|}(a'))|a;) = 0 \tag{2.44}$$

for any $m \in M, a \in A, a' \in A'$. This is the same as saying $(m.a).a' = (m.\phi^{|a|}(a')).a$.

The definition of such bimodules extends to A_∞-categories immediately. Namely, let $\mathcal{B}$ and $\mathcal{B}'$ be two A_∞-categories. Assume $\mathcal{B}$ carries an extra $\mathbb{Z}$-grading (so that $\mu_{\mathcal{B}}$ preserves the degree) and $\mathcal{B}'$ is endowed with a strict automorphism ϕ without higher maps.

Definition 2.3.1. A (right-right) twisted A_∞-bimodule $\mathfrak{M}$ over $\mathcal{B}$-$\mathcal{B}'$ is given by an assignment

$$(L, L') \in ob(\mathcal{B}) \times ob(\mathcal{B}') \mapsto \mathfrak{M}(L, L') \tag{2.45}$$

and maps

$$\begin{aligned}\mathfrak{M}(L_0, \phi^g L_0') \otimes \mathcal{B}(L_1, L_0)^{g_1} \otimes \mathcal{B}(L_2, L_1)^{g_2} \otimes \cdots \otimes \mathcal{B}(L_m, L_{m-1})^{g_m} \otimes \\ \mathcal{B}'(L_1', L_0') \otimes \cdots \otimes \mathcal{B}'(L_n', L_{n-1}') \to \mathfrak{M}(L_m, L_n')[1-m-n]\end{aligned} \tag{2.46}$$

where $\mathcal{B}(L_1, L_0)^{g_1}$ denotes the degree g_1-part of $\mathcal{B}(L_1, L_0)$ in the extra grading, and $g = \sum g_i$. We will denote these maps by $\mu_{\mathfrak{M}} = \mu_{\mathfrak{M}}^{1|m;n}$, omitting L_i, L_j' and degrees from the notation. These maps are required to satisfy

$$\begin{aligned}\sum \pm \mu_{\mathfrak{M}}(\mu_{\mathfrak{M}}(m|x_1, \dots x_i; \phi^{right}(x_1', \dots x_j'))|x_{i+1}, \dots x_m; x_{j+1}', \dots, x_n') + \\ \sum \pm \mu_{\mathfrak{M}}(m|x_1 \dots \mu_{\mathcal{B}}(\dots) \dots, x_m; x_1' \dots x_n') + \\ \sum \pm \mu_{\mathfrak{M}}(m|x_1 \dots x_m; x_1', \dots \mu_{\mathcal{B}'}(\dots) \dots x_n') = 0\end{aligned}$$

where ϕ^{right} denotes $\phi^{|x_{i+1}|+|x_{i+2}|+\dots+|x_m|}$, i.e. ϕ applied as many times as the total degree of x_s's on the right, and $\phi^{right}(x_1', \dots x_j')$ means ϕ^{right} is applied to each $x_1', \dots, x_j'$ separately, rather than a higher component of ϕ (we use this notation in order to shorten the expression).

A (pre-bimodule) homomorphism f from a twisted bimodule $\mathfrak{M}$ to another $\mathfrak{M}'$ is defined to be a collection of maps

$$\begin{aligned}f^{1|m;n} : \mathfrak{M}(L_0, \phi^g L_0') \otimes \mathcal{B}(L_1, L_0)^{g_1} \otimes \mathcal{B}(L_2, L_1)^{g_2} \otimes \cdots \otimes \mathcal{B}(L_m, L_{m-1})^{g_m} \otimes \\ \mathcal{B}'(L_1', L_0') \otimes \cdots \otimes \mathcal{B}'(L_n', L_{n-1}') \to \mathfrak{M}'(L_m, L_n')[-m-n]\end{aligned} \tag{2.47}$$

The differential of the pre-bimodule homomorphism f is

$$\begin{aligned}\sum \pm \mu_{\mathfrak{M}}(f(m|x_1, \dots x_i; \phi^{right}(x_1', \dots x_j'))|x_{i+1}, \dots x_m; x_{j+1}', \dots, x_n') + \\ \sum \pm f(\mu_{\mathfrak{M}}(m|x_1, \dots x_i; \phi^{right}(x_1', \dots x_j'))|x_{i+1}, \dots x_m; x_{j+1}', \dots, x_n') + \\ \sum \pm f(m|x_1 \dots \mu_{\mathcal{B}}(\dots) \dots, x_m; x_1' \dots x_n') + \\ \sum \pm f(m|x_1 \dots x_m; x_1', \dots \mu_{\mathcal{B}'}(\dots) \dots x_n')\end{aligned}$$

(when this is 0, we say f is a homomorphism of twisted bimodules). Twisted bimodules form a dg category denoted by $Bimod_{tw}(\mathcal{B}, \mathcal{B}')$. The composition is similar to composition of pre-bimodule homomorphisms (see [Sei08c]); however, with a similar twisting rule (see Note 2.3.2).

Note 2.3.2. The general rule in defining "twisted A_∞-object" is that when swapping morphisms x of $\mathcal{B}$ and x' of $\mathcal{B}'$, one acts on x' by $\phi^{|x|}$ (i.e. $x \otimes x' \mapsto \phi^{|x|}(x') \otimes x$) (for instance, this is the case with composition of pre-bimodule maps etc.).

Note 2.3.3. We have not specified signs here, but any set of sign conventions for ordinary A_∞-bimodules can be used here (in particular signs that one can obtain by unfolding Koszul signs in bar constructions).

Remark 2.3.4. One can weaken the assumption that ϕ is a strict auto-equivalence without higher maps. Namely, one can assume ϕ is bijective on objects and ϕ^1 (the first component of ϕ) is bijective on hom-sets. In other words, the action of ϕ is an isomorphism of the coalgebra $T\mathcal{B}'[1]$. In this case, the rule in swapping morphisms x and x' becomes

$$\begin{gathered}\text{“}x \otimes (x'_1 \otimes \dots x'_n) \longmapsto \\ \textstyle\sum_{i_1+\dots+i_k=n} \pm((\phi^{|x|})^{i_1}(x'_1, \dots x'_{i_1}) \otimes \dots \otimes (\phi^{|x|})^{i_k}(x'_{n-i_k+1} \dots x'_n)) \otimes x\text{”}\end{gathered} \tag{2.48}$$

In other words, as before $\phi^{|x|}$ is applied to $x'_1 \otimes \dots x'_n$ while moving x to its right; however, this time $\phi^{|x|}$ is considered to be an automorphism of the dg coalgebra $T\mathcal{B}'[1]$.

Note in this situation, definition of twisted Yoneda bimodules (see Example 2.3.5) becomes more subtle. Namely, one has to choose a quasi-equivalence $s : \mathcal{B}' \to \mathcal{B}'^{str}$ such that there is a strict auto-equivalence ϕ^{str} on $\mathcal{B}'^{str}$ without higher maps and such that $\phi^{str} \circ s$ and $s \circ \phi$ are strictly equivalent (not only homotopic). In this case, a length filtration argument shows that

$$Bimod_{tw}(\mathcal{B}, \mathcal{B}'^{str}) \to Bimod_{tw}(\mathcal{B}, \mathcal{B}') \tag{2.49}$$

is cohomologically fully faithful. One could define Yoneda functors in $Bimod_{tw}(\mathcal{B}, \mathcal{B}')$

to be the images of Yoneda bimodules in $Bimod_{tw}(\mathcal{B}, (\mathcal{B}')^{str})$. Hence, we prefer to avoid the situation ϕ has higher maps. This more relaxed notion could be useful as it allows us to work with minimal models.

Example 2.3.5. (Twisted Yoneda bimodule) Let $(L, L') \in ob(\mathcal{B}) \times ob(\mathcal{B}')$. Define the twisted Yoneda bimodule as

$$\bigoplus_{r\in\mathbb{Z}} \mathcal{B}(\cdot, L)^r \otimes \mathcal{B}'(\cdot, \phi^{-r}(L')) \tag{2.50}$$

To define structure maps, one uses A_∞-structures of $\mathcal{B}$ and $\mathcal{B}'$, but twists $\mathcal{B}'$ component by the degree of elements of $\mathcal{B}$ on its right. More explicitly, the bimodule structure is given by:

$$(y \otimes y'|x_1, \dots, x_m; x_1', \dots x_n') \longmapsto \begin{cases} \pm\mu^1_{\mathcal{B}}(y) \otimes y' \pm y \otimes \mu^1_{\mathcal{B}'}(y') & m = n = 0 \\ \pm\mu_{\mathcal{B}}(y, x_1, \dots, x_m) \otimes \phi^{-left}(y'), & m = 0, n \neq 0 \\ \pm y \otimes \mu_{\mathcal{B}'}(y', x_1', \dots, x_n'), & m = 0, n \neq 0 \\ 0, & m \neq 0, n \neq 0 \end{cases} \tag{2.51}$$

where $\phi^{-left} = \phi^{-|x_1|-\dots-|x_m|}$. We will denote twisted Yoneda bimodules by $h_L \otimes_{tw} h_{L'}$, or simply by $h_L \otimes h_{L'}$ when the twisting is trivial. Notice that

$$h_L \otimes_{tw} h_{L'} \simeq h_{L\langle 1\rangle} \otimes_{tw} h_{\phi^{-1}(L')} \tag{2.52}$$

where $L\langle 1\rangle$ is the "shift" of L defined by $\mathcal{B}(\cdot, L\langle 1\rangle)^r = \mathcal{B}(\cdot, L)^{r+1}$ (one may enlarge $\mathcal{B}$ by adding these objects, and make it closed under such shifts).

Definition 2.3.6. Assume $\mathcal{B}$ and $\mathcal{B}'$ are dg categories. Define the twisted tensor product $\mathcal{B} \otimes_{tw} \mathcal{B}'$ to be the dg category satisfying

1. $ob(\mathcal{B} \otimes_{tw} \mathcal{B}') = ob(\mathcal{B}) \times ob(\mathcal{B}')$
2. $hom(L_1 \times L_1', L_2 \times L_2') = \bigoplus_{r\in\mathbb{Z}} \mathcal{B}(L_1, L_2)^r \otimes \mathcal{B}'(L_1', \phi^{-r}(L_2'))$ as chain complexes

with the composition defined by (2.41) (but with Koszul signs).

Remark 2.3.7. If A and A' denote the total algebras of $\mathcal{B}$ and $\mathcal{B}'$ respectively, one can equivalently define $\mathcal{B} \otimes_{tw} \mathcal{B}'$ to be the dg category with total algebra $A \otimes_{tw} A'$ (and the set of idempotents $e_L \otimes e_{L'}$ where $L \in ob(\mathcal{B}), L' \in ob(\mathcal{B}')$).

Remark 2.3.8. Given a model for the tensor product of A_∞ algebras such as the model in [Lod07], one can presumably define its twisted version and a Yoneda embedding. However, we will bypass this by considering the full subcategory of $Bimod_{tw}(\mathcal{B}, \mathcal{B}')$ spanned by twisted Yoneda bimodules. This is equivalent to giving an explicit model by Yoneda Lemma (see Lemma 2.3.12).

From now on, assume the categories $\mathcal{B}$ and $\mathcal{B}'$ are cohomologically unital with units denoted by e_L and $e_{L'}$. Further assume ϕ acts freely on objects of $\mathcal{B}'$, e_L is homogeneous of degree 0 (in the extra grading) and ϕ sends $e_{L'}$ to $e_{\phi(L')}$ for all L'. Once we have cohomological unitality, these can be arranged easily.

We would like to investigate the structure of the category $Bimod_{tw}(\mathcal{B}, \mathcal{B}')$. Most of the following proofs are standard (up to remembering the rule of twisting $x \otimes x' \to \phi^{|x|}(x') \otimes x$ and $x' \otimes x \to x \otimes \phi^{-|x|}(x')$). Nevertheless, we will include them for the convenience of the reader.

First, let us prove something for graded twisted bimodules over graded algebras:

Lemma 2.3.9. *Let A and A' be graded algebras equipped with an extra grading and an automorphism ϕ respectively. Let M be a twisted graded (right-right) A-A'-bimodule. Then there exists a bar type resolution of M as a twisted bimodule consisting of shifted direct sums of $M \otimes A^{\otimes m} \otimes A'^{\otimes n}$.*

Proof. First, consider the bar resolution of M with respect to A. It is given by

$$\{M \otimes A^{\otimes m}\} \to M \tag{2.53}$$

with the map $M \otimes A \to M$ being $m \otimes a \mapsto ma$. One can endow this resolution with an A'-action making it $A \otimes_{tw} A'$-linear. For instance, define $(m \otimes a).a' := m\phi^{|a|}(a') \otimes a$. Now, apply the standard bar construction to each term in the resolution to obtain a double resolution of type $M \otimes A^{\otimes m} \otimes A'^{\otimes n}$. Likewise, one can equip these terms with an A-action making the rows linear over $A \otimes_{tw} A'$ as well. By taking the total complex, we obtain what we desire. $\square$

We would also like to prove independence of $Bimod_{tw}(\mathcal{B}, \mathcal{B}')$ from the quasi-equivalence type. Namely:

Lemma 2.3.10. *Let $f : \mathcal{B} \to \underline{\mathcal{B}}$ be a quasi-equivalence of extra graded A_∞ categories. Let $f' : \mathcal{B}' \to \underline{\mathcal{B}'}$ be a quasi-equivalence of A_∞-categories that are equipped with strict auto-equivalences ϕ and $\underline{\phi}$ without higher maps. Assume f is compatible with the extra grading and f' strictly commutes with given auto-equivalences. Then, there is an induced dg quasi-equivalence*

$$F : Bimod_{tw}(\underline{\mathcal{B}}, \underline{\mathcal{B}'}) \to Bimod_{tw}(\mathcal{B}, \mathcal{B}') \tag{2.54}$$

Proof. For simplicity assume all A_∞-categories have only one object (hence, they are A_∞-algebras). The induced map is standard and twisting does not effect the definition. Namely, if $\mathfrak{M} \in Bimod_{tw}(\underline{\mathcal{B}}, \underline{\mathcal{B}'})$, then define $F(\mathfrak{M})$ to be the bimodule with the same underlying chain complex, and with structure maps

$$\begin{gathered}(m|x_1, \dots, x_m; x_1', \dots, x_n') \mapsto \\ \sum \pm \mu_{\mathfrak{M}}(m|f^{i_1}(x_1, \dots, x_{i_1}), f^{i_2}(x_{i_1+1}, \dots) \dots; f'^{j_1}(\dots), \dots)\end{gathered} \tag{2.55}$$

Likewise, for a pre-bimodule homomorphism g, $F(g)$ is defined by

$$\begin{gathered}(m|x_1, \dots, x_m; x_1', \dots, x_n') \mapsto \\ \sum \pm g(m|f^{i_1}(x_1, \dots, x_{i_1}), f^{i_2}(x_{i_1+1}, \dots) \dots; f'^{j_1}(\dots), \dots)\end{gathered} \tag{2.56}$$

It is easy to see F is a dg functor. To see F is cohomologically fully faithful, filter the hom complexes by the total length. Then, the induced map between associated graded complexes is clearly a quasi-isomorphism.

To see it is essentially surjective, one can construct a quasi inverse G as

$$\text{“}\mathfrak{M} \otimes_{\mathcal{B}} \underline{\mathcal{B}} \otimes_{\mathcal{B}'} \underline{\mathcal{B}'}\text{”} \tag{2.57}$$

In other words, as a complex, $G(\mathfrak{M})$ is obtained as shifted sums of

$$\mathfrak{M}\otimes\mathcal{B}^{\otimes m}\otimes\underline{\mathcal{B}}\otimes\mathcal{B}'^{\otimes n}\otimes\underline{\mathcal{B}'} \tag{2.58}$$

similar to untwisted case. The only difference is, when defining the structure maps (and action of the functor G on morphisms), one has to take twisting into account. For instance, $\mu_{\mathfrak{M}}(m|;b')\otimes b$ term in differential of $m\otimes b\otimes b'$ is replaced by $\mu_{\mathfrak{M}}(m|;\phi^{|b|}(b'))\otimes b$. Then $FG(\mathfrak{M})$ is quasi isomorphic to bimodule given by

$$\text{“}\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}'\text{”} \tag{2.59}$$

as a twisted $\mathcal{B}$-$\mathcal{B}'$-bimodule, where the quasi-isomorphism is induced by f and f' seen as maps of $\mathcal{B}$-$\mathcal{B}$, resp. $\mathcal{B}'$-$\mathcal{B}'$ bimodules $\mathcal{B}\to\underline{\mathcal{B}}$, resp. $\mathcal{B}'\to\underline{\mathcal{B}'}$. There exists a natural map

$$\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}'\to\mathfrak{M} \tag{2.60}$$

of twisted bimodules and one can filter the cone of (2.60) by the total length. The E_1-page of the corresponding spectral sequence is (a union of the summands of) the standard bar resolutions of Lemma 2.3.9, for $A=H^*(\mathcal{B})$, $A'=H^*(\mathcal{B}')$ and $M=H^*(\mathfrak{M})$; hence, it is acyclic (to make sure the E_1-page agrees with the standard bar resolution in Lemma 2.3.9, one can construct (2.57) and (2.59) by first taking $\otimes_{\mathcal{B}}\underline{\mathcal{B}}$ and then $\otimes_{\mathcal{B}'}\underline{\mathcal{B}'}$). This implies, the natural map from (2.59) to $\mathfrak{M}$ is a quasi-isomorphism and we are done. ☐

The bimodules we will encounter in Section 2.4 will fall into span of twisted Yoneda bimodules. However, the following is a natural corollary of the proof of Lemma 2.3.10 and we include it here:

Corollary 2.3.11. *The category $Bimod_{tw}(\mathcal{B},\mathcal{B}')$ is generated by twisted Yoneda bimodules (in the sense that every object is quasi-isomorphic to a (homotopy) colimit of finite complexes of twisted Yoneda bimodules).*

Proof. The resolution (2.59) can be seen as an infinite resolution by twisted Yoneda bimodules. More precisely, let $(\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}')^{\leq n}$ denote the submodule of $\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}'$ spanned by chains of length less than $n+1$ (this is a submodule since the structure maps of the bimodule are not increasing the length). This is clearly a finite complex of infinite sums of (shifted) twisted Yoneda bimodules. Moreover, $\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}'$ is a (homotopy) colimit of $(\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}')^{\leq n}$, since homotopy colimits of injective inclusions can be taken as the ordinary colimit (one can describe the homotopy colimit as a cone of two direct sums, with an induced map into (2.59), then the induced map is a chain equivalence, since the statement that homotopy colimit is the same as the limit is true at chain level). Therefore, $\mathfrak{M}\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}'}\mathcal{B}'$ is a colimit of twisted Yoneda bimodules. □

As expected, we also have the following:

Lemma 2.3.12. *[Yoneda Lemma] The chain complexes $hom(h_L\otimes_{tw}h_{L'},\mathfrak{M})$ and $\mathfrak{M}(L,L')$ are quasi-isomorphic with a quasi-isomorphism given by*

$$\gamma_{L,L'}: hom(h_L\otimes_{tw}h_{L'},\mathfrak{M})\to\mathfrak{M}(L,L') \tag{2.61}$$

$$f\longmapsto f^{1|0;0}(e_L\otimes e_{L'}) \tag{2.62}$$

Proof. The proof of this is similar to [Sei08c, Lemma 2.12]. Namely, one writes a quasi-inverse

$$\lambda:\mathfrak{M}(L,L')\to hom(h_L\otimes_{tw}h_{L'},\mathfrak{M}) \tag{2.63}$$

similar to [Sei08c, (1.25)]. For instance, assume $\mathcal{B}$ and $\mathcal{B}'$ are dg categories and $\mathfrak{M}$ is a twisted dg bimodule (i.e. has vanishing higher structure maps). Let $d\in\mathfrak{M}(L,L')$. $\lambda(d)$ given by

$$\lambda(d)^{1|0;0}(b\otimes b')=\mu_{\mathfrak{M}}(\mu_{\mathfrak{M}}(d|b;)|;b') \tag{2.64}$$

$$\lambda(d)^{1|i;j}=0 \text{ if } i\neq 0 \text{ or } j\neq 0 \tag{2.65}$$

defines a right quasi-inverse to $\gamma_{L,L'}$. To see λ is a quasi-isomorphism, one can apply the same length filtration spectral sequence argument in [Sei08c, Lemma 2.12] (more precisely, one has to show exactness of another bar resolution for twisted bimodules: for this one can simply take the dual of the resolution in Lemma 2.3.9 or follow its proof to construct the other bar resolution). One can generalize the map λ to the general A_∞ case (note one has to take twisting into account, the rule is as always $b\otimes x' \to \phi^{-|b|}(x')\otimes b$ etc.) and apply the same proof; however, we take the following route:

Alternative to using more general λ, one can choose quasi-equivalences from $\mathcal{B}$ and $\mathcal{B}'$ to dg categories $\underline{\mathcal{B}}$ and $\underline{\mathcal{B}'}$ carrying an extra grading and a strict auto-equivalence respectively such that the quasi-equivalences are strictly compatible with extra grading, resp. strictly commute with given auto-equivalences (one can also assume the chosen cohomological units map to strict units, but this is not necessary). Then, the induced map

$$Bimod_{tw}(\underline{\mathcal{B}},\underline{\mathcal{B}'}) \to Bimod_{tw}(\mathcal{B},\mathcal{B}') \tag{2.66}$$

is an equivalence by Lemma 2.3.10. Hence, Yoneda lemma holds in the essential image of dg bimodules. As twisted Yoneda bimodules over $\underline{\mathcal{B}}$-$\underline{\mathcal{B}'}$ are dg and their image under (2.66) are quasi-isomorphic to twisted Yoneda bimodules, the essential image of dg bimodules is all $Bimod_{tw}(\mathcal{B},\mathcal{B}')$ by Corollary 2.3.11. This finishes the proof. □

2.3.2 Mapping torus category as a twisted tensor product

Let $\tilde{\mathfrak{T}}_0$ denote the universal cover of the nodal elliptic curve $\mathfrak{T}_0$, which is an infinite chain of projective lines. See Figure 2-4. $\tilde{\mathfrak{T}}_0$ carries a translation automorphism denoted by $\mathfrak{tr}$. It moves every projective line to the next (to the right in the figure) and generates the group of Deck transformations of $\tilde{\mathfrak{T}}_0 \to \mathfrak{T}_0$, where $\mathfrak{T}_0$ is the nodal elliptic curve over $\mathbb{C}$, which can as well be defined by $\mathfrak{T}_0 := \tilde{\mathfrak{T}}_0/(y\sim \mathfrak{tr}(y))$. In [Kar18], we have constructed a dg category $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ such that

$$H^0(tw^\pi(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}))\simeq D^b(Coh_p(\tilde{\mathfrak{T}}_0)) \tag{2.67}$$

where $Coh_p(\tilde{\mathfrak{I}}_0)$ is the abelian category of coherent sheaves with proper support on $\tilde{\mathfrak{I}}_0$. The objects of $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$ correspond to sheaves $\mathcal{O}_{C_i}(-1)$ and $\mathcal{O}_{C_i}$. We use $\mathcal{O}_{C_i}(-1)$ and $\mathcal{O}_{C_i}$ to denote the corresponding objects of $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$ as well. Push-forward along $\mathfrak{tr}$ induces a strict dg auto-equivalence of $\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}$, which we still denote by $\mathfrak{tr}$.

Recall the following construction from [Kar18]: let $\mathcal{A}$ be a dg category, and let ϕ be a strict dg auto-equivalence of $\mathcal{A}$. Define **the mapping torus category** M_ϕ as the dg category with objects

$$ob(M_\phi) := ob(\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}) \times ob(\mathcal{A}) \tag{2.68}$$

and with morphisms

$$M_\phi(\mathcal{F} \times a, \mathcal{F}' \times a') = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}(\mathcal{F}, \mathfrak{tr}^{-n}(\mathcal{F}')) \otimes \mathcal{A}(a, \phi^{-n}(a')) \tag{2.69}$$

for $\mathcal{F}, \mathcal{F}' \in ob(\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg})$ and $a, a' \in ob(\mathcal{A})$. (2.68) and (2.69) can be written concisely as

$$M_\phi := (\mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg} \otimes \mathcal{A}) \# \mathbb{Z} \tag{2.70}$$

To define the mapping torus category for a more general A_∞-category $\mathcal{A}$ with a strict quasi-equivalence ϕ (possibly with higher components), one has to find a dg category $\mathcal{A}^{str}$, a strict dg auto-equivalence ϕ^{str} on $\mathcal{A}^{str}$ and a quasi-equivalence $\mathcal{A} \to \mathcal{A}^{str}$ that commutes strictly with ϕ and ϕ^{str}.

Remark 2.3.13. In [Kar18], hom-sets were defined as

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{O}(\tilde{\mathfrak{I}}_0)_{dg}(\mathfrak{tr}^n(\mathcal{F}), \mathcal{F}') \otimes \mathcal{A}(\phi^n(a), a') \tag{2.71}$$

instead of (2.69). It is easy to identify (2.69) and (2.71) as chain complexes and under this identification, one can describe the product structure on M_ϕ by (2.74). Hence, the definitions are equivalent, but (2.69) is better suited for description of M_ϕ as a twisted tensor product.

Let $\mathcal{O}(\mathfrak{T}_0)_{dg}$ denote the category with objects $\mathcal{O}_{C_0}(-1), \mathcal{O}_{C_0}$ and morphisms

$$\mathcal{O}(\mathfrak{T}_0)_{dg}(\mathcal{F},\mathcal{F}') = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}(\mathcal{F}, \mathfrak{tr}^{-n}(\mathcal{F}')) \tag{2.72}$$

$\mathcal{O}(\mathfrak{T}_0)_{dg}$ is quasi-equivalent to $M_{1_{\mathbb{C}}}$.

Endow $\mathcal{O}(\mathfrak{T}_0)_{dg}$ with an extra $\mathbb{Z}$-grading by setting $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}(\mathcal{F}, \mathfrak{tr}^{-n}(\mathcal{F}'))$ to be the degree n morphisms of $\mathcal{O}(\mathfrak{T}_0)_{dg}(\mathcal{F},\mathcal{F}')$.

Proposition 2.3.14. *M_ϕ can be seen as the twisted tensor product of $\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{A}$.*

Proof. This becomes a tautology once one recalls the product structure on M_ϕ and $\mathcal{O}(\mathfrak{T}_0)_{dg}$. For instance, given

$$\begin{aligned} \alpha_1\otimes f_1 &\in \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}(\mathcal{F}, \mathfrak{tr}^{-m}(\mathcal{F}'))\otimes\mathcal{A}(a,\phi^{-m}(a'))\subset M_\phi(\mathcal{F}\times a, \mathcal{F}'\times a') \\ \alpha_2\otimes f_2 &\in \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}(\mathcal{F}', \mathfrak{tr}^{-n}(\mathcal{F}''))\otimes\mathcal{A}(a',\phi^{-n}(a''))\subset M_\phi(\mathcal{F}'\times a', \mathcal{F}''\times a'') \end{aligned} \tag{2.73}$$

the product in M_ϕ is defined as

$$(\alpha_2\otimes f_2)(\alpha_1\otimes f_1) = \pm\mathfrak{tr}^{-m}(\alpha_2)\alpha_1\otimes\phi^{-m}(f_2)f_1 \tag{2.74}$$

and the product of α_1 and α_2 in $\mathcal{O}(\mathfrak{T}_0)_{dg}$ is defined as $\mathfrak{tr}^{-m}(\alpha_2)\alpha_1$ ($\pm$ is simply the Koszul sign coming from switching f_2 and α_1). We can easily identify hom-complexes of M_ϕ and $\mathcal{O}(\mathfrak{T}_0)_{dg}\otimes_{tw}\mathcal{A}$, and this description shows product structures coincide (see Definition 2.3.6). □

A natural question one can ask is the dependence of quasi-equivalence type of M_ϕ on the dg model $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$. One knows any other dg model for $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$ is quasi-equivalent to $\mathcal{O}(\tilde{\mathfrak{T}}_0)$ by the main result of [LO10]. Moreover, one can improve (zigzags) of quasi-equivalence(s) to make it strictly $\mathfrak{tr}$-equivariant. More precisely:

Lemma 2.3.15. *Consider pairs $(\mathcal{B},\psi)$, where $\mathcal{B}$ is an A_∞-category, ψ is an auto-equivalence acting bijectively on objects and hom-sets, and acting freely on objects. Let $(\mathcal{O}',\mathfrak{tr}')$ be another model for $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$ with the same set of objects as $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$*

and with a strict lift $\mathfrak{tr}'$ *of* $\mathfrak{tr}_*$*. Then, there exists a zigzag of* A_∞*-quasi-equivalences between* $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ *and* $(\mathcal{O}', \mathfrak{tr}')$ *through pairs* $(\mathcal{B}, \psi)$*.*

A very simple proof of lemma can be given by using a push-out description of $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$. We will explain this in this section. See Note 2.3.19. Lemma 2.3.15 would also follow for instance from a statement that any two dg lifts of the Fourier-Mukai transform $\mathfrak{tr}$ are naturally quasi-isomorphic: indeed, one can prove an analog of [Gai13, Theorem 4.6.2] to develop a Fourier-Mukai theory for properly supported coherent sheaves on $\tilde{\mathfrak{T}}_0$, namely a large class of quasi-functors of $tw^\pi(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})$ (such as these which shift the cohomological support of the sheaf by a bounded amount) can be represented fully-faithfully by coherent sheaves. Hence, any two dg lifts of $\mathfrak{tr}$ would be quasi-equivalent, and together with uniqueness of dg enhancements, this would imply desired statement (note $\tilde{\mathfrak{T}}_0$ is a union of Noetherian schemes and analogous local results can also be used to prove these assertions).

Lemma 2.3.15 implies the extra grading on $\mathcal{O}(\mathfrak{T}_0)_{dg} = \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\#\mathbb{Z}$ is independent of the chosen dg model for which $\mathfrak{tr}$ lifts as a strict dg auto-equivalence. Hence:

Lemma 2.3.16. *M_ϕ does not depend on the chosen dg model $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ or on the chosen strictification $(\mathcal{A}^{str}, \phi^{str})$*

Proof. The twisted tensor product of dg categories is equivalent to span of twisted Yoneda bimodules. Changing the model for $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ or the strictification for $(\mathcal{A}, \phi)$ does not change this span by Lemma 2.3.10. □

Now, we want to express $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ as a homotopy push-out, as this will be used in Section 2.3.3. This will also give a proof of Lemma 2.3.15.

Consider the normalization map

$$\pi_N : \mathbb{P}^1 \times \mathbb{Z} \to \tilde{\mathfrak{T}}_0 \tag{2.75}$$

In the notation of [Kar18], $\mathbb{P}^1 \times \{i\}$ is the component that maps to $C_i \subset \tilde{\mathfrak{T}}_0$, $0 \in \mathbb{P}^1$ maps to the nodal point $x_{i-1/2} \in \tilde{\mathfrak{T}}_0$ and $\infty \in \mathbb{P}^1$ maps to the nodal point $x_{i+1/2} \in \tilde{\mathfrak{T}}_0$. We also assume $\mathfrak{tr}$ lifts to normalization as $(y, i) \mapsto (y, i+1)$ (and is still denoted by

$\mathfrak{tr}$). Choose a dg enhancement for $D^b(Coh(\mathbb{P}^1))$ and take the subcategory spanned by $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_0$ and $\mathcal{O}_\infty$. Denote it by $\mathcal{O}(\mathbb{P}^1)_{dg}$. Without loss of generality enlarge the category $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ by adding objects corresponding to nodes $\mathcal{O}_{x_{i+1/2}}$ in a $\mathfrak{tr}$-equivariant way. This does not change the twisted envelope obviously and it causes $\mathcal{O}(\mathfrak{T}_0)_{dg} = \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\#\mathbb{Z}$ to enlarge in its twisted envelope as well (together with the natural extra grading). One can choose the enhancement $\mathcal{O}(\mathbb{P}^1)_{dg}$ so that

1. There is a dg functor Ξ_0 from $\mathcal{O}(\mathbb{P}^1)_{dg}$ to $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ lifting the push-forward of $\mathbb{P}^1 \to C_0 \subset \tilde{\mathfrak{T}}_0$
2. There are dg functors $i_0, i_\infty : \mathbb{C} \to \mathcal{O}(\mathbb{P}^1)_{dg}$ lifting the push-forward of $\{0\} \to \mathbb{P}^1$ and $\{\infty\} \to \mathbb{P}^1$
3. Compositions $\Xi_0 \circ i_0, \Xi_0 \circ i_\infty : \mathbb{C} \to \mathcal{O}(\mathbb{P}^1)_{dg} \to \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ are strictly related by $\mathfrak{tr}$, i.e. $\mathfrak{tr} \circ \Xi_0 \circ i_0 = \Xi_0 \circ i_\infty$

Define $\Xi_i := \mathfrak{tr}^i \circ \Xi_0$. These are dg functors lifting the push-forward of $\mathbb{P}^1 \to C_i \subset \tilde{\mathfrak{T}}_0$. Taking $\mathbb{Z}$-many pairwise orthogonal copies of this dg category, we obtain a dg model for properly supported coherent sheaves on $\mathbb{P}^1 \times \mathbb{Z}$, denoted by $\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}$.

There is a dg lift of the push-forward of normalization map (2.75) which we also denote by π_N. Let Pt_∞ denote the dg category consisting of infinitely many copies of $\mathbb{C}$ indexed by $i+1/2, i \in \mathbb{Z}$. Denote its objects by $*_{i+1/2}$. Pt_∞ has an auto-equivalence mapping $*_{i-1/2}$ to $*_{i+1/2}$, which we still denote by $\mathfrak{tr}$. The collection of functors

$$i_0 : (\mathbb{C})_{i+1/2} =: \mathbb{C} \to \mathcal{O}(\mathbb{P}^1)_{dg} \cong \mathcal{O}(\mathbb{P}^1 \times \{i+1\})_{dg} \tag{2.76}$$

(i.e. i_0 used to map $*_{i+1/2}$ to $(i+1)^{th}$ $\mathbb{P}^1$) gives a functor

$$Pt_\infty \to \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \tag{2.77}$$

which is essentially push-forward of $0 \in \mathbb{P}^1$ at each component. Denote this functor by i_0 as well. Similarly, the collection of functors $i_\infty : (\mathbb{C})_{i+1/2} \to \mathcal{O}(\mathbb{P}^1 \times \{i\})_{dg}$ gives

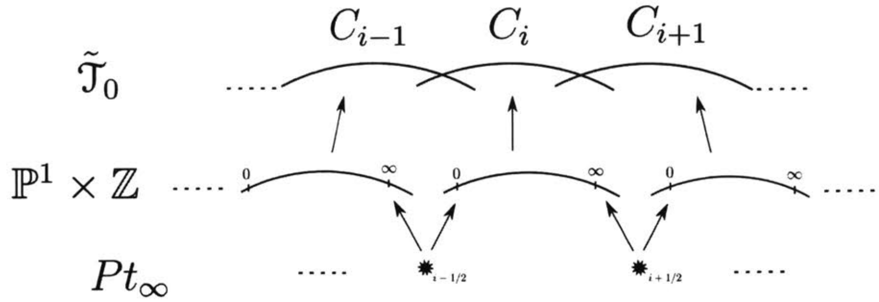

Figure 2-4: A schematic picture of diagram (2.79)

a functor

$$Pt_\infty \to \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \tag{2.78}$$

which is essentially push-forward of $\infty \in \mathbb{P}^1$ at each component (to a different $\mathbb{P}^1$ though). Denote it by i_∞. Clearly, $\pi_N \circ i_0 = \pi_N \circ i_\infty$. In other words, we have a strictly commutative diagram

$$\begin{array}{ccccc} & & \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} & & & & \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \\ & \nwarrow & & \nearrow & \\ & & Pt_\infty & & \end{array} \tag{2.79}$$

Thus, we have an induced map

$$hocolim\big(Pt_\infty \rightrightarrows \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}\big) \to \mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \tag{2.80}$$

One can define the homotopy coequalizer above as $\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \coprod_{(Pt_\infty \coprod Pt_\infty)} Pt_\infty$, i.e. by gluing $\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}$ and Pt_∞ along $i_0 \coprod i_\infty : (Pt_\infty \coprod Pt_\infty) \to \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}$ and $id \coprod id$ (note it is not the same as colimit of (2.79), see Figure 2-4 for a schematic picture). See [GPS18] for the definition of homotopy push-outs.

For convenience, let us spell out a description of this coequalizer via Grothendieck construction, following [GPS18] (more precisely, we give an equivalent, slightly modified version that works for coequalizer diagrams, this version is equivalent to one given in [Tho79]). Consider the category $\mathcal{G}r$ with objects

$$ob(\mathcal{G}r) = ob(\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}) \coprod ob(Pt_\infty) \tag{2.81}$$

We define the morphisms to be

$$hom(X, X') := \begin{cases} \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}(X, X'), & \text{if } X, X' \in \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \\ Pt_\infty(X, X'), & \text{if } X, X' \in Pt_\infty \\ \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}(i_0(X), X') \oplus & \\ \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}(i_\infty(X), X'), & \text{if } X \in Pt_\infty, X' \in \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \\ 0, & \text{if } X \in \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}, X' \in Pt_\infty \end{cases} \tag{2.82}$$

In other words, $\mathcal{G}r$ is a category that contains Pt_∞ and $\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}$ as full subcategories, and contains additional morphisms corresponding to maps $i_0(X) \to X'$ and $i_\infty(X) \to X'$. In particular, if we let X' to be $i_0(X)$, resp. $i_\infty(X)$, then $\mathcal{G}r(X, X')$ contains morphisms corresponding to identity. Denote the family of these morphisms by C. The homotopy coequalizer can be defined as

$$hocolim\big(Pt_\infty \rightrightarrows \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}\big) := C^{-1}\mathcal{G}r \tag{2.83}$$

For the definition of localization, see [GPS17] or proof of Lemma 2.3.26.

Now we prove:

Lemma 2.3.17. *(2.80) is a quasi-equivalence.*

Proof. One way to see is direct computation: namely, write the Grothendieck construction for the diagram, then check that the induced functor from the localization is a quasi-equivalence. This is a cohomology level check, in the sense that one can localize after taking the cohomology. The localization of the cohomological category of $\mathcal{G}r$ has an explicit description in terms of sequences of morphisms and their formal inverses, and it is not hard to check in this case that the induced functor into $H^0(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})$ is an equivalence.

Another option is to use the fact "*Coh* sends colimits to colimits". As it was explained to us by Vivek Shende, this follows from [GR17, Theorem A.1.2]. See also [Nad16, Corollary 2.5] for the explanation on how the statement follows from [GR17, Theorem A.1.2].

More precisely, [GR17, Theorem A.1.2] states that the contravariant functor $X \mapsto IndCoh(X)$, $f \mapsto f^!$ restricted to category of affine, Noetherian schemes with closed embeddings sends push-outs to pull-back squares (in a category of dg categories). Unfortunately, this does not immediately apply to our situation; however, we can use it easily.

First, assume the statement that $X \mapsto IndCoh(X)$, $f \mapsto f_*$ sends push-outs to push-outs for Noetherian, projective schemes with closed embeddings hold. Then the same holds with $IndCoh$ replaced by Coh. Let $Ev(n)$ denote the full subcategory of $\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}$ spanned by sheaves on even indexed curves $C_{-2n}, C_{-2n+2}, \dots, C_{2n}$. In other words,

$$Ev(n) = \bigsqcup_{i=-2n,-2n+2,\dots,2n} \mathcal{O}(\mathbb{P}^1)_{dg} \tag{2.84}$$

Let Ev denote the union of all $Ev(n)$. Similarly let $Od(n)$ denote the subcategory corresponding to curves with odd index $-2n+1, -2n+3, \dots, 2n-1$ and Od denote their union. Let Pt_n denote the subcategory of Pt_∞ spanned by points with index $-2n+1/2, -2n+3/2, \dots, 2n-1/2$ (i.e. the points of intersection of curves in $Ev(n)$ and $Od(n)$). Finally let $\tilde{\mathfrak{T}}_0(n)$ denote the reduced subvariety of $\tilde{\mathfrak{T}}_0$ given by the union of $C_{-2n}, C_{-2n+1}, \dots, C_{2n}$. Clearly, $\tilde{\mathfrak{T}}_0(n)$ is a push-out of curves involved in $Ev(n)$ and $Od(n)$ along Pt_n. As we assume "push-outs to push-outs" hold for Noetherian projective schemes with closed embeddings, we have

$$Od(n) \sqcup_{Pt_n} Ev(n) \simeq \mathcal{C}oh(\tilde{\mathfrak{T}}_0(n)) \tag{2.85}$$

where $\mathcal{C}oh(\tilde{\mathfrak{T}}_0(n))$ is a dg-model for $Coh(\tilde{\mathfrak{T}}_0(n))$. Moreover, the colimit of $\mathcal{C}oh(\tilde{\mathfrak{T}}_0(n))$ gives a dg model for properly supported coherent sheaves on $\tilde{\mathfrak{T}}_0$, which is derived equivalent to $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$. Hence,

$$\begin{aligned}\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \simeq hocolim_n \mathfrak{Coh}(\tilde{\mathfrak{T}}_0(n)) \simeq hocolim_n (Od(n) \sqcup_{Pt_n} Ev(n)) \simeq \\ (hocolim_n Od(n)) \sqcup_{(hocolim_n Pt_n)} (hocolim_n Ev(n)) = Od \sqcup_{Pt_\infty} Ev\end{aligned} \tag{2.86}$$

The second to the last equivalence is an abstract category theory statement. It is easy to see that $Od \sqcup_{Pt_\infty} Ev$ is equivalent to $hocolim\big(Pt_\infty \rightrightarrows \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}\big)$. Hence, the claim follows.

Now, we need to show why "push-outs to push-outs" hold for projective Noetherian schemes, at least in our specific case. We only need to show the functor $X \mapsto IndCoh(X)$, $f \mapsto f^!$ sends push-out diagrams of projective Noetherian schemes along closed embeddings to pull-back diagrams, as "push-outs to push-outs" follows in the same way as [Nad16, Corollary 2.5]. The basic idea is to combine the statement for affine push-outs with Zariski descent.

Let X be the union of curves $C_{-2n}, C_{-2n+2}, \ldots, C_{2n}$ and Y be the union of curves $C_{-2n+1}, C_{-2n+3}, \ldots, C_{2n-1}$. Let Z be the union of nodal points in the intersections of $C_{-2n}, C_{-2n+1}, \ldots, C_{2n}$. Then $\tilde{\mathfrak{T}}_0(n) = X \sqcup_Z Y$. Let $\{U_i\}$ be an affine open cover over $\tilde{\mathfrak{T}}_0$. Assume $\{U_i\}$ is closed under intersections and the index set is ordered so that $i \leq j$ if and only if $U_i \subset U_j$. Let $U_i^X = U_i \cap X$, $U_i^Y = U_i \cap Y$ and $U_i^Z = U_i \cap Z$. These give affine open covers of X, Y and Z. By Zariski descent for $IndCoh$ (see [Gai13])

$$IndCoh(\tilde{\mathfrak{T}}_0(n)) \simeq holim IndCoh(U_i) \tag{2.87}$$

Indeed, this can be stated by saying that the functor $V \mapsto IndCoh(V), f \mapsto f^!$ from the category of schemes with open embeddings sends colimits to limits (note $f^* = f^!$ for open embeddings). Similar descent statement hold for X, Y and Z. Moreover,

$U_i = U_i^X \sqcup_{U_i^Z} U_i^Y$ and they are affine so push-outs to pull-back hold for them. Thus,

$$\begin{aligned} IndCoh(\tilde{\mathfrak{T}}_0(n)) \simeq holimIndCoh(U_i) \simeq holimIndCoh(U_i^X \sqcup_{U_i^Z} U_i^Y) \simeq \\ holim(IndCoh(U_i^X) \times_{IndCoh(U_i^Z)} IndCoh(U_i^Y)) \simeq \\ holim(IndCoh(U_i^X)) \times_{holim(IndCoh(U_i^Z))} holim(IndCoh(U_i^Y)) \simeq \\ IndCoh(X) \times_{IndCoh(Z)} IndCoh(Y) \end{aligned}$$

This completes the proof. □

Hence, (2.83) generates another enhancement for $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$ and (2.80) is $\mathbb{Z}$-equivariant. Taking smash products with respect to $\mathbb{Z}$-action (see [Kar18, Section 4]), we obtain

Corollary 2.3.18. *There is a quasi-equivalence $(C^{-1}\mathcal{G}r)\#\mathbb{Z} \to \mathcal{O}(\mathfrak{T}_0)_{dg}$ that is compatible with extra gradings.*

Notice, $(C^{-1}\mathcal{G}r)\#\mathbb{Z}$ does not depend on the choice of enhancement made for $D^b(Coh(\mathbb{P}^1))$ by [LO10].

Note 2.3.19. Lemma 2.3.15 also follows from these considerations. Namely, given any other model $(\mathcal{O}', \mathfrak{tr}')$ for $\mathcal{O}(\tilde{\mathfrak{T}}_0)$ with a strict auto-equivalence $\mathfrak{tr}'$ lifting $\mathfrak{tr}_*$, we can choose a dg functor similar to Ξ_0 and define Ξ_i by composing with $\mathfrak{tr}'^i$. Assume the chosen model $\mathcal{O}(\mathbb{P}^1)_{dg}$ is minimal. We then obtain a diagram similar to (2.79), corresponding Grothendieck construction and a functor to $\mathcal{O}'$. This is strictly compatible with translation. Hence, there is a quasi-equivalence from the explicit localization of the Grothendieck construction to $\mathcal{O}'$ that is strictly compatible with translation. The localization of Grothendieck construction does not depend on $\mathcal{O}'$; hence, it gives us a zigzag as promised in Lemma 2.3.15.

By $\mathbb{Z}$-equivariance, one can also realize $(C^{-1}\mathcal{G}r)\#\mathbb{Z}$ as a localization of $\mathcal{G}r\#\mathbb{Z}$. This localization carries an extra grading by Lemma 2.3.26 and it is quasi-equivalent to $(C^{-1}\mathcal{G}r)\#\mathbb{Z}$ (hence to $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}\#\mathbb{Z} = \mathcal{O}(\mathfrak{T}_0)_{dg}$) since the localization map $\mathcal{G}r\#\mathbb{Z} \to (C^{-1}\mathcal{G}r)\#\mathbb{Z}$ is compatible with extra grading.

Notice $\mathcal{G}r\#\mathbb{Z}$ is equivalent to Grothendieck construction for

$$Pt_\infty\#\mathbb{Z} \rightrightarrows \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})\#\mathbb{Z} \tag{2.88}$$

which is defined similar to (2.82) by replacing hom-sets with the hom-sets of smash product. This construction is equivalent to a dg category with objects $ob(\mathcal{O}(\mathbb{P}^1)_{dg})\coprod\{*\}$ and with morphisms

$$hom(X,X') := \begin{cases} \mathcal{O}(\mathbb{P}^1)_{dg}(X,X'), & \text{if } X,X' \in \mathcal{O}(\mathbb{P}^1)_{dg} \\ \mathbb{C}, & \text{if } X = X' \in \{*\} \\ \mathcal{O}(\mathbb{P}^1)_{dg}(i_0(X),X')\oplus & \\ \mathcal{O}(\mathbb{P}^1)_{dg}(i_\infty(X),X'), & \text{if } X \in \{*\}, X' \in \mathcal{O}(\mathbb{P}^1)_{dg} \\ 0, & \text{if } X \in \mathcal{O}(\mathbb{P}^1)_{dg}, X' \in \{*\} \end{cases} \tag{2.89}$$

In other words, this is a dg category consisting of (some) coherent sheaves on $\mathbb{P}^1$, an extra object $*$ and morphisms $* \to \mathcal{F}$ corresponding to morphisms $\mathcal{O}_0 \to \mathcal{F}$ and $\mathcal{O}_\infty \to \mathcal{F}$. This category is the Grothendieck construction for the diagram $\mathbb{C} \rightrightarrows \mathcal{O}(\mathbb{P}^1)_{dg}$, which is defined similar to $\mathcal{G}r$. Let us denote this category by $\mathcal{G}r\#\mathbb{Z}$ as well.

$\mathcal{O}(\mathcal{T}_0)_{dg}$ is obtained by localizing $\mathcal{G}r\#\mathbb{Z}$ at two morphisms from $*$ corresponding to identity maps of $\mathcal{O}_0$ and $\mathcal{O}_\infty$. Denote these morphisms by c_0 and c_∞. This process geometrically corresponds to identifying 0 and ∞ on $\mathbb{P}^1$.

The corresponding extra grading is given by setting the summand

$$\mathcal{O}(\mathbb{P}^1)_{dg}(i_\infty(X),X') \subset hom(X,X') \tag{2.90}$$

to be the degree 1-part and the remaining expressions in (2.89) to be of degree 0. The extra grading descends to localization $\mathcal{O}(\mathcal{T}_0)_{dg}$ (see Lemma 2.3.26) and it clearly matches the extra grading coming from smash product with respect to $\mathbb{Z}$-action.

Remark 2.3.20. This extra grading on (2.89) comes from the identification with $\mathcal{G}r\#\mathbb{Z}$ via $* \mapsto x_{-1/2}$ and $\mathbb{P}^1 \mapsto \mathbb{P}^1 \times \{0\} \subset \mathbb{P}^1 \times \mathbb{Z}$. In a different identification, one can set elements of $\mathcal{O}(\mathbb{P}^1)_{dg}(i_0(X), X')$ to be of degree -1 and the rest to be of degree 0. The descriptions become equivalent after localization, and we will go with the former.

2.3.3 Extra grading on $\mathcal{W}(T_0)$

By [CDGG17] and [GPS18, Theorem 1.9], $\mathcal{W}(T_0)$ is generated by Lagrangians that lift under the covering map $\tilde{T}_0 \to T_0$. We will only consider these Lagrangians as objects of $\mathcal{W}(T_0)$, and we fix a lift for each object of $\mathcal{W}(T_0)$. For a Lagrangian $L \subset T_0$, denote the lift by $\tilde{L}$. Given $r \in \mathbb{Z}$, let $\tilde{L}\langle r\rangle$ denote another lift of L obtained by shifting $\tilde{L}$ by r in the positive direction.

The chain complexes $CW(L_1, L_0)$ are generated by Hamiltonian chords from L_1 to L_0 for a fixed Floer datum. This chord lifts to a path from $\tilde{L}_1$ to $\tilde{L}_0\langle -r\rangle$, for a unique r. We define the extra grading by letting this chord to be of degree r. This should not be confused with the original grading of $\mathcal{W}(T_0)$.

We now wish to compare the categories $\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{W}(T_0)$, while taking their extra gradings into account. Recall:

Lemma 2.3.21. *[Kar18, Lemma 9.9] $\mathcal{O}(\mathfrak{T}_0)_{dg}$ generates a dg model for $D^b(Coh(\mathfrak{T}_0))$.*

Theorem 2.3.22. *[LP16, Theorem B.(ii)] $\mathcal{W}(T_0)$ is a dg model for $D^b(Coh(\mathfrak{T}_0))$. Indeed, $tw(\mathcal{W}(T_0))$ is a dg enhancement of $D^b(Coh(\mathfrak{T}_0))$, and one can choose the equivalence so that L_{gr} and L_{pur} (green and purple curves in Figure 2-2) correspond to $\widetilde{\mathcal{O}}_{\mathfrak{T}_0}$ and $\mathcal{O}_x$ respectively. Here $\widetilde{\mathcal{O}}_{\mathfrak{T}_0}$ is the push-forward of structure sheaf under normalization map and $\mathcal{O}_x$ is the structure sheaf of the singular point.*

Choose suitable generators for $\mathcal{W}(T_0)$ (L_{gr}- the green curve in Figure 2-2 and the diagonal curve not shown in the picture). One can rephrase [LP16, Theorem B.(ii)] as:

Corollary 2.3.23. *$\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{W}(T_0)$ are quasi-equivalent A_∞-categories. Moreover, under this quasi-equivalence $\widetilde{\mathcal{O}}_{\mathfrak{T}_0}$ corresponds to L_{gr}, $\widetilde{\mathcal{O}}_{\mathfrak{T}_0}(-1)$ corresponds to the curve that wraps around the torus once (which would be the diagonal curve in*

Figure 2-2), and $\mathcal{O}_x$-the structure sheaf of the node, as an object of $tw(\mathcal{O}(\mathfrak{T}_0)_{dg})$-corresponds to L_{pur}.

However, we need the comparison of categories $\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{W}(T_0)$ as A_∞-categories with extra grading. In other words, we need to relate these categories by zigzags of quasi-equivalences which respect extra gradings.

Remark 2.3.24. By choosing homotopy transfer data to be of degree 0, one can construct minimal models with extra grading. Moreover, the constructed quasi-equivalences and homotopies all respect the extra grading as well. See [Mar06] for more about transferring A_∞-structures. It is easy to see that such a minimal model is unique up to a gauge equivalence that preserves the extra grading. Hence, one can equivalently define the notion of equivalence for extra graded A_∞-categories as the graded quasi-equivalence of their minimal models.

It is easy to see the extra gradings on $\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{W}(T_0)$ match at a cohomological level (up to a minor modification of the quasi-equivalence between them, for instance using symmetries of $\mathcal{W}(T_0)$). However, this does not directly imply that they are equivalent, for the same reason that two gauge equivalent minimal A_∞-structures on a extra graded vector space (i.e. a doubly graded vector space, note that A_∞-maps are of degree 0 in the second grading) are not necessarily equivalent via a gauge equivalence that respects the grading.

Nevertheless, one can prove:

Lemma 2.3.25. *$\mathcal{O}(\mathfrak{T}_0)_{dg}$ and $\mathcal{W}(T_0)$ are quasi-equivalent as A_∞-categories with an extra grading.*

We do not know how to prove this using the approach in [LP16]. However, one can prove Theorem 2.3.22 and Corollary 2.3.23 using other approaches. One is via stop removal (see [Syl16] and [GPS18]). Yankı Lekili has informed of this approach.

The other is via gluing formula of [GPS18]. In other words, we will give a description of $\mathcal{W}(T_0)$ as a homotopy coequalizer similar to description of $\mathcal{O}(\mathfrak{T}_0)_{dg}$ in Section 2.3.2. $\mathcal{W}(T_0)$ will be described as a localization of an intermediate category $\mathcal{G}r_s$ with extra grading. To obtain extra grading on $\mathcal{W}(T_0)$, we need:

Lemma 2.3.26. *Let $\mathcal{B}$ be a category with extra grading and C be a set of homogeneous morphisms. Then the category $C^{-1}\mathcal{B}$ can be endowed with an extra grading such that the localization map respects gradings of $\mathcal{B}$ and $C^{-1}\mathcal{B}$. Moreover, If $\mathcal{B}_1$ and $\mathcal{B}_2$ are quasi-equivalent as extra graded categories, and C_1 and C_2 correspond to each other in cohomology under the (zigzag of) quasi-equivalence(s), then $C_1^{-1}\mathcal{B}_1$ is quasi-equivalent to $C_2^{-1}\mathcal{B}_2$ as an extra graded category.*

Proof. First, let us remind the definition of localization following [GPS17]: consider the set of cones $\mathcal{B}_C$ of elements of C. Take the Lyubashenko-Ovsienko/Drinfeld quotient of $\mathcal{B}$ by $\mathcal{B}_C$ (see [LO06], [Syl16], [GPS17]). In general, this specific model allows one to endow the quotient $\mathcal{B}/\mathcal{B}'$ by an extra grading when $\mathcal{B}$ is an extra graded category and $\mathcal{B}' \subset \mathcal{B}$ is a full subcategory. In our case, one has to enlarge $\mathcal{B}$ within $tw(\mathcal{B})$ by adding cones of C_1, resp. C_2. However, the extra grading also extends to this larger subcategory as the morphisms of C are all homogeneous. Hence, the first claim follows.

For the second claim assume without loss of generality that there is a quasi-equivalence $\mathcal{B}_1 \to \mathcal{B}_2$ that respects the extra gradings and that carries C_1 to C_2 strictly (for instance assume $\mathcal{B}_2$ is minimal). Then, enlarge both categories by adding cones. The quasi-equivalence extends to a (graded) quasi-equivalence of enlarged categories as well (that sends cones of C_1 to cones of C_2). The natural functor from the localization with respect to cones of C_1 to localization with respect to cones of C_2 preserves the extra gradings. Hence, there is an induced quasi-equivalence $C_1^{-1}\mathcal{B}_1 \to C_2^{-1}\mathcal{B}_2$ that preserves the gradings. □

To apply gluing formula, decompose T_0 into Liouville sectors T (the 1-handle shown in yellow in Figure 2-2) and $\overline{T_0 \setminus T}$. In other words, cut T_0 into sectors along the 1-handle T. The finite boundary of these sectors correspond to side edges of T. See Figure 2-2 or Figure 2-6 for a clearer picture (we are being sloppy about the notation as T_0 previously referred to Liouville domain rather than its completion, similarly with T). As a Liouville sector, T is equivalent to $T^*[0,1]$; hence, has a wrapped Fukaya category equivalent to $\mathbb{C}$. It is generated by L_{pur}- the purple curve

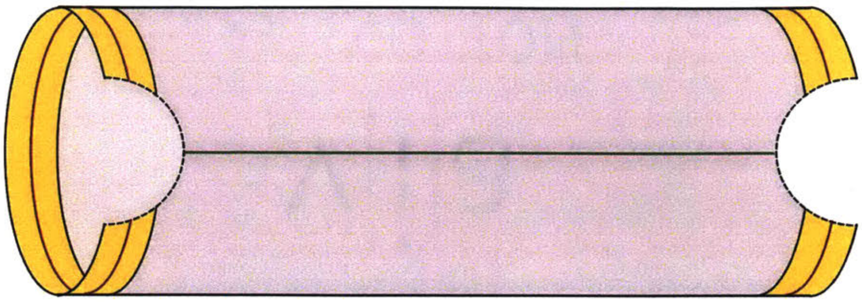

Figure 2-5: The sector $N = \overline{T_0 \setminus T}$ and two inclusions of T into N

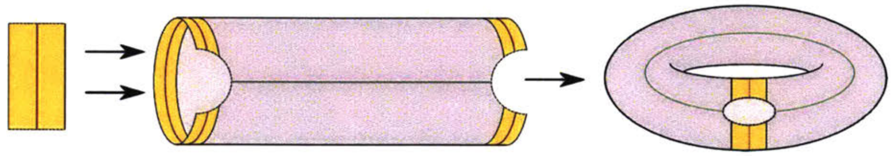

Figure 2-6: The inclusions of sector T into $N = \overline{T_0 \setminus T}$ and the inclusion of N into T_0

in Figure 2-2.

On the other hand, $\overline{T \setminus T_0}$ is equivalent to a cylinder with two stops at its boundary; hence, it is easy to see that its wrapped Fukaya category is equivalent to $D^b(Coh(\mathbb{P}^1))$ (we will also use the letter N to refer to $\overline{T \setminus T_0}$ as a Liouville sector). Indeed, as generators one can take the green curve in Figure 2-5 and another curve that wraps around once without intersecting the green curve (the diagonal of Figure 2-2). Clearly, the subcategory spanned by them is equivalent to Kronecker quiver, and one can write the curves partially winding the stops as cones of these two generators. The curves partially winding the stops are the purple, vertical curves in Figure 2-5- the small Lagrangian linking discs of the stops in terminology of [GPS18]. As these curves together with the green curve generate the partially wrapped Fukaya category, one has the desired equivalence with (a dg model for) $D^b(Coh(\mathbb{P}^1))$. Alternatively, this partially wrapped category is equivalent to Fukaya-Seidel category of Landau-Ginzburg model $(\mathbb{C}, z + z^{-1})$, which is well known to be a mirror to $\mathbb{P}^1$. Green and purple curves in Figure 2-5 **are sufficient as generators.**

We glue T and $N = \overline{T_0 \setminus T}$ along the sector given by a neighborhood of their shared edges (represented by two yellow sectors in Figure 2-5). This sector is isomorphic to $T^*[0,1] \coprod T^*[0,1]$. In summary, we have a pushout diagram by [GPS18, Theorem

1.20, Corollary 1.21]:

$$\begin{array}{ccc} \mathcal{W}(\overline{T_0 \setminus T}) = \mathcal{W}(N) & \longrightarrow & \mathcal{W}(T_0) \\ \uparrow & & \uparrow \\ \mathcal{W}(T^*[0,1]) \coprod \mathcal{W}(T^*[0,1]) & \longrightarrow & \mathcal{W}(T^*[0,1]) \simeq \mathcal{W}(T) \end{array} \tag{2.91}$$

Even though the inclusion of two yellow sectors into the 1-handle T is not an isomorphism, it induces an equivalence between their wrapped Fukaya categories. In other words, the lower horizontal arrow in (2.91) can be seen as the identity on each component. Hence, we can write this gluing diagram as

$$\mathbb{C} \simeq \mathcal{W}(T^*[0,1]) = \mathcal{W}(T) \rightrightarrows \mathcal{W}(N) \to \mathcal{W}(T_0) \tag{2.92}$$

where $\mathcal{W}(T_0)$ is equivalent to homotopy coequalizer of

$$\mathcal{W}(T) \rightrightarrows \mathcal{W}(N) \tag{2.93}$$

A pictural representation of (2.92) is given by Figure 2-6 (Figure 2-6 can also be seen as a coequalizer diagram; however, from the perspective of [GPS17], [GPS18] this picture is slightly informal, as the maps of sectors in Figure 2-6 are not global inclusions, but rather like "étale maps" for sectors).

Let j_0 and j_1 denote both the inclusions $T \rightrightarrows N = \overline{T_0 \setminus T}$ and induced functors $\mathcal{W}(T) \rightrightarrows \mathcal{W}(N)$ (assume j_0 correspond to left inclusion for instance). To make statement about homotopy coequalizer precise, consider the category $\mathcal{G}r_s$ with objects

$$ob(\mathcal{W}(T)) \coprod ob(\mathcal{W}(N)) \tag{2.94}$$

and with morphisms

$$hom(X,X') := \begin{cases} \mathcal{W}(N)(X,X'), & \text{if } X,X' \in \mathcal{W}(N) \\ \mathcal{W}(T)(X,X') = \mathbb{C}, & \text{if } X = X' \in \mathcal{W}(T) \\ \mathcal{W}(N)(j_0(X),X') \oplus \mathcal{W}(N)(j_1(X),X'), & \text{if } X \in \mathcal{W}(T), X' \in \mathcal{W}(N) \\ 0, & \text{if } X \in \mathcal{W}(N), X' \in \mathcal{W}(T) \end{cases} \tag{2.95}$$

As before, $\mathcal{G}r_s$ can be seen as the Grothendieck construction for (2.93). The rightmost arrow in (2.92) induces a functor

$$\mathcal{G}r_s \to \mathcal{W}(T_0) \tag{2.96}$$

(2.92) is a homotopy coequalizer diagram means (2.96) is a localization at two morphisms $L_{pur} \to j_0(L_{pur})$ and $L_{pur} \to j_1(L_{pur})$ corresponding to identities of $j_0(L_{pur})$ and $j_1(L_{pur})$ (we are abusing the notation and denote the purple curves in copies of T in Figure 2-6 by L_{pur} as well). Denote the set of these two morphisms by C_s.

Grade $\mathcal{G}r_s$ as before: let morphisms of type $\mathcal{W}(N)(j_0(X),X')$ be of degree 1 and the remaining components be of degree 0. This extra grading descends to localization $C_s^{-1}\mathcal{G}r_s$ by Lemma 2.3.26. To see the extra grading obtained by localizing $\mathcal{G}r_s$ matches the previously given one on $\mathcal{W}(T_0)$, one only has to show the map 2.96 respects the extra grading. For instance, let L, resp. L' denote the image of $j_1(X)$, resp. $X' \subset N$ in T_0 (fix lifts $\tilde{L}$ and $\tilde{L}'$ for L and L'). One can arrange the lifts so that the image of a chord from $j_1(X)$ to X' lifts to a chord from $\tilde{L}$ to $\tilde{L}'$ (hence, degree 0 in the previously given extra grading). This implies the image of any chord from $j_0(X)$ to X' lifts to a chord from $\tilde{L}$ to $\tilde{L}'\langle -1\rangle$ (hence, degree 1). The other cases are easier.

One can prove Theorem 2.3.22 by showing $\mathcal{G}r$ (defined by (2.81) and (2.82)) is equivalent to $\mathcal{G}r_s$ (defined by (2.94) and (2.95)) and the sets of morphisms C and C_s^{-1} correspond under the equivalence (in cohomology). Hence, the homotopy colimits $C^{-1}\mathcal{G}r$ and $C_s^{-1}\mathcal{G}r_s$ are equivalent. Moreover, the equivalence preserves the extra gradings on $\mathcal{G}r$ and $\mathcal{G}r_s$; hence, the induced equivalence also preserves the extra grading by Lemma 2.3.26. This proves Lemma 2.3.25.

Remark 2.3.27. The definition of wrapped Fukaya categories in [Abo10] and [GPS17] are not the same and one has to use continuation functors to show their equivalence (which extends continuation maps, see also Note 2.4.24). However, it is easy to see that continuation maps actually respect the extra grading as the extra degree depends only on the relative homotopy type of the chord (alternatively, if one wants to prove derived equivalence of these categories using invertible bimodules, one would endow the bimodule with a compatible extra grading, inducing a functor between "graded twisted complexes").

Remark 2.3.28. Stop removal approach we mentioned above also gives a description of $\mathcal{W}(T_0)$ as a quotient (of $D^b(Coh(\mathbb{P}^1))$, the situation is the same algebraically). Hence, one can presumably apply proof of Lemma 2.3.26 to this quotient to prove Lemma 2.3.25.

Remark 2.3.29. Let A be a bigraded ordinary algebra (we consider this as an graded algebra with an extra grading). It is well known that the space of minimal A_∞-structures (ignoring the extra grading) on A is controlled by a part of dgla $CC^*(A, A)$ (see [Sei15, §3a] for instance). Moreover, one can extract the space of A_∞-structures modulo gauge from minimal L_∞-models for this dgla. Hence, once the complex has small cohomology, one has a better control over this space (e.g. [Sei15, Lemma 3.2]).

Similarly, the space of A_∞-structures that is of degree 0 with respect to extra grading (modulo gauge equivalence that respects extra grading) is controlled by part of $CC^*(A, A)^{\mathbb{G}_m}$, where $\mathbb{G}_m$-action is induced by the rational $\mathbb{G}_m$-action on A corresponding to extra grading. Hence, one can possibly prove the existence of a gauge equivalence respecting the extra grading via a cohomological comparison between $CC^*(A, A)^{\mathbb{G}_m}$ and $CC^*(A, A)$, at least in our case where the Hochschild cohomology is small (see [LP12],[Kar18, §5]).

Remark 2.3.30. Another option is this: [LP16] proves equivalence of wrapped Fukaya category of the n-fold cover of T_0 and coherent sheaves on the n-fold cover of $\mathfrak{T}_0$. It is easy to see compatibility of this equivalence by the Deck transformations.

n-fold coverings endow both sides with $\mathbb{Z}/n$-gradings that is induced by the original extra $\mathbb{Z}$-grading. Hence, this comparison for all n, implies the original gradings are the same.

Remark 2.3.31. Homological mirror symmetry for T_0 and $\mathcal{T}_0$ with extra $\mathbb{Z}/n$-grading can be seen as homological mirror symmetry for their n-fold covers. Similarly, equivalence with extra $\mathbb{Z}$-grading can informally be thought as mirror symmetry between $\tilde{\mathcal{T}}_0$ and $\tilde{T}_0$.

2.4 Künneth and twisted Künneth theorems

In this section, we define an A_∞-functor

$$\mathcal{W}(T_\phi) \to Bimod_{tw}(\mathcal{W}(T_0), \mathcal{W}(M)) \tag{2.97}$$

using count of quilted strips, and show it is full and faithful. We start by defining the functor in the untwisted case, for the Fukaya category of compact Lagrangians. In other words, we start by writing an A_∞-functor

$$\mathcal{F}(T_0 \times M) \to Bimod(\mathcal{F}(T_0), \mathcal{F}(M)) \tag{2.98}$$

Starting with ordinary Fukaya category allows us to convey the basic TQFT argument without compactness issues related to wrapping. Then, we will indicate the necessary modifications for twisted version, still for the compact Fukaya category. However, there will be a compactness problem related to use of infinite cover $\tilde{T}_0$. We will find appropriate Floer data, for which Gromov compactness holds.

Then, we will explain how to do the same for wrapped Fukaya categories. We start by defining a category $\mathcal{W}^2(T_\phi)$ that is analogous to category $\mathcal{W}^2$ in [Gan12] and $\mathcal{W}^{prod}$ in [GPS18]. It is equivalent to $\mathcal{W}(T_\phi)$ by an argument similar to [GPS18].

As a special case of Corollary 2.2.6, $\mathcal{W}(T_0 \times M)$ is split generated by Lagrangians of type $L \times L'$. Hence, we will restrict attention to only these objects, we will prove their images are quasi-isomorphic to Yoneda bimodules, and that (2.98) is fully faithful on these objects.

2.4.1 Quilted strips

Moduli of n-quilted strips is defined in [Ma'15], and it controls A_∞ n-modules. Their main strata can be identified with n-parallel lines with markings in $\mathbb{C}$ with fixed distance from each other (up to conformal equivalence). For $n = 3$, this is used in [Gan12] to define functors from a version of wrapped Fukaya category on $M \times M^-$ to bimodules over $\mathcal{W}(M)$. Indeed, defining a functor (2.98) is equivalent to defining a left-right-right $\mathcal{W}(T_0 \times M)$-$\mathcal{W}(T_0)$-$\mathcal{W}(M)$-trimodule. We would like to exploit similar ideas to define (2.98). Let us start by describing moduli of quilted strips first:

Definition 2.4.1. Let $\mathbf{d} = (d_1, d_2, d_3) \in \mathbb{Z}^3_{\geq 0}$. A 3-quilted strip with $\mathbf{d}$-markings is

- a pair of strips r_1, r_2 biholomorphic to $\mathbb{R} \times [0, 1]$
- d_1-markings on the upper boundary of r_1, and d_2-markings on the upper boundary of r_2
- d_3-markings on the lower boundary of r_1 and r_2
- an identification of r_1 and r_2 preserving the incoming/outgoing ends of the strip and mapping lower markings of r_1 to lower markings of r_2

The isomorphisms of such quilted strips are given by isomorphisms of both strips commuting with the identification (i.e. by simultaneous isomorphisms of r_1 and r_2).

Definition 2.4.2. Let $\mathcal{Q}(\mathbf{d}) = \mathcal{Q}(d_1, d_2, d_3)$ denote the moduli space of 3-quilted strips up to isomorphism.

The identification of r_1 and r_2 is uniquely determined up to translation. When $d_3 > 0$, it is uniquely determined. When $d_3 = 0$, different identifications give different elements. It is not hard to identify $\mathcal{Q}(d_1, d_2, d_3)$ with the space of 3-quilted lines in [Ma'15]. We will indeed picture these objects as in Figure 2-7 which is similar to

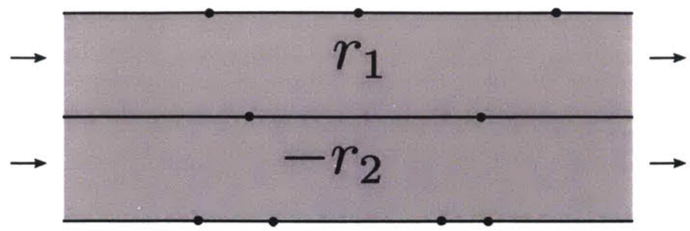

Figure 2-7: An element of $\mathcal{Q}(3,4,2)$

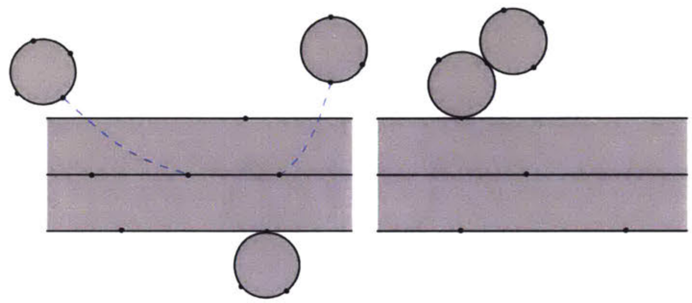

Figure 2-8: An element of $\partial\overline{\mathcal{Q}(5,5,7)}$

[Gan12] and [GPS18]. In this figure, $-r_2$ is the strip r_2 with conjugate holomorphic structure. This quilted surface can be folded to obtain a single strip, thanks to global identification of r_1 and r_2. The complement of the markings in the quilted strip $r = (r_1, r_2) \in \mathcal{Q}(\mathbf{d})$ will be denoted by $\mathcal{S}_r^q$. The complement of markings in the folding of the strip will be denoted by $\mathcal{S}_r^f$ (the superscript q stands for quilted and f stands for folded). The family of these surfaces form universal bundles over $\mathcal{Q}(\mathbf{d})$, denoted by $\mathcal{S}^q$ and $\mathcal{S}^f$ respectively. The complement of the markings in r_1 and r_2 will be denoted by $\mathcal{S}_r^{(1)}$ and $\mathcal{S}_r^{(2)}$ respectively (hence, $\mathcal{S}_r^f = \mathcal{S}_r^{(1)} \cap \mathcal{S}_r^{(2)}$).

$\mathcal{Q}(\mathbf{d})$ admits a natural compactification described in detail in [Ma'15]. We denote this compactification by $\overline{\mathcal{Q}(\mathbf{d})}$. We will not give all the details, and instead give an example of a boundary element in Figure 2-8. Note, however, to describe the boundary structure/gluing we have to restrict the strip-like ends. We demand the strip like ends on the discs to be rational, i.e. extend to a biholomorphic map from the whole strip to the disc. Moreover, for the incoming ends on discs, we impose the other end of the strip to converge to the outgoing marking on the same component. For the ends on the quilted component, we choose standard ends on the left and right. To choose ends for the markings on the upper boundary of r_1 and r_2 (i.e. lower boundary

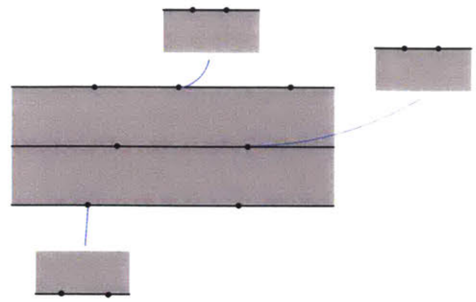

Figure 2-9: A more convenient description of boundary elements of $\mathcal{Q}(\mathbf{d})$ for the purposes of gluing

of $-r_2$), we consider the natural embedding of the strip into upper half-plane so that upper boundary maps to real axis, and that does not change the strip width. This embedding is determined up to translation, and we demand the incoming strip-like ends corresponding to upper markings of r_1 and r_2 to be as before, i.e. rational, and such that the extended map from

$$Z = \mathbb{R} \times [0, 1] \tag{2.99}$$

would have other end converging to ∞ (of the ambient half-plane). The ends for the middle markings are similar.

To explain gluing picturally, consider elements of boundary strata as in Figure 2-9. Then gluing a hyperplane to the strip is essentially taking a large half-disc in that hyperplane, taking out a small half-disc from the edge of the strip, and gluing the large half-disc after rescaling.

The reason we choose specific strip-like ends is because we need to keep global identification of r_1 and r_2 (i.e. folding). If we allowed arbitrary ends, the identification would require Riemann mapping theorem, and the bottom markings of r_1 and r_2 would no longer match.

2.4.2 Künneth functor for Fukaya category of compact Lagrangians

For definitions and conventions of Fukaya category $\mathcal{F}(M)$, we follow [Sei08c]. In this section, we define a Künneth functor

$$\mathfrak{M} : \mathcal{F}(T_0 \times M)_p \to Bimod(\mathcal{F}(T_0), \mathcal{F}(M)) \tag{2.100}$$

where $\mathcal{F}(T_0 \times M)_p$ is the full subcategory of $\mathcal{F}(T_0 \times M)$ spanned by product type Lagrangians (this is important in the proof of fully faithfulness, but not for defining the Künneth functor). Defining a Künneth functor (2.100) is equivalent to defining a left-right-right trimodule over $\mathcal{F}(T_0 \times M)_p$-$\mathcal{F}(T_0)$-$\mathcal{F}(M)$. Hence, we will denote $\mathfrak{M}(L'')(L, L')$ by $\mathfrak{M}(L'', L, L')$ or by $\mathfrak{M}_{L''}(L, L')$, where $L \subset T_0, L' \subset M, L'' \subset T_0 \times M$ (the reason we choose this input order is that we use tri-module $\mathfrak{M}$ to define functor (2.100)).

Given compact, exact Lagrangians (with a brane structure) $L'' \subset T_0 \times M$, $L \subset T_0$, and $L' \subset M$, choose a Floer datum (as defined in [Sei08c]) for the pair $(L \times L', L'')$ satisfying conditions to be specified. Then, $\mathfrak{M}(L'', L, L')$ is generated by Hamiltonian chords in $T_0 \times M$ from $L \times L'$ to L''. The maps

$$\begin{aligned} CF(L''_{p-1}, L''_p) \otimes CF(L''_{p-2}, L''_{p-1}) \otimes \cdots \otimes CF(L''_0, L''_1) \otimes \mathfrak{M}(L''_0, L_0, L'_0) \otimes \\ CF(L_1, L_0) \otimes \dots CF(L_m, L_{m-1}) \otimes CF(L'_1, L'_0) \cdots \otimes CF(L'_n, L'_{n-1}) \\ \to \mathfrak{M}(L''_p, L_m, L'_n)[1 - m - n - p] \end{aligned}$$

defining the bimodule structure, as well as the A_∞-functor (2.100) are obtained by counting pseudo-holomorphic quilts as in Figure 2-10. More precisely, for each stable quilted strip r, we choose a family of almost complex structures J_r parametrized by the folded strip $\mathcal{S}_r^f$, and a $C^\infty(T_0 \times M)$-valued 1-form K_r on $\mathcal{S}_r^f$. K_r vanishes in directions tangent to $\partial \mathcal{S}_r^f$. The choice is required to satisfy consistency condition similar to [Sei08c]. On the boundary $\overline{\mathcal{Q}(\mathbf{d})}$, the data should coincide up to infinite order with the data obtained via gluing. Notice we do not choose separate perturbation data

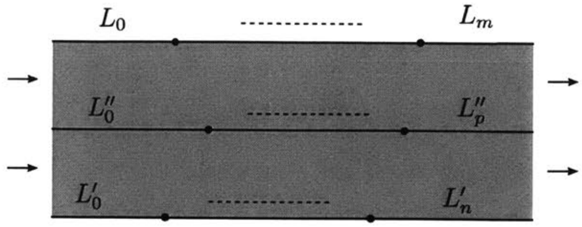

Figure 2-10: The quilts defining bimodule/functor structures on $\mathfrak{M}$

(family of almost complex structures and Hamiltonian valued 1-form) on $\mathcal{S}_r^{(1)}$ and $\mathcal{S}_r^{(2)}$ (for T_0 and M respectively). Indeed, we impose the condition that the restriction of perturbation data on $\mathcal{S}_r^f$ to $\mathbb{R}\times[2/3,1]\cap\mathcal{S}_r^f$ splits as product of restrictions of data on $((\mathbb{R}\times[2/3,1])\cap\mathcal{S}_r^{(1)})\times T_0$ and $((\mathbb{R}\times[2/3,1])\cap\mathcal{S}_r^{(2)})\times M$. Hence, on the $(\mathbb{R}\times[2/3,1])$, one can think of the components as separate holomorphic maps into T_0 and M. On the other hand, we assume it restricts to perturbation data that defines Fukaya category of $T_0\times M$ on $((\mathbb{R}\times[0,1/3])\cap\mathcal{S}_r^f)\times T_0\times M$. As we will see, this condition is more relevant to the twisted case. In the untwisted case for compact Fukaya category, one can presumably restrict to product type data over all $\mathcal{S}_r^f$.

As for the Floer data: for the pair (L_{i+1},L_i) we choose almost complex structure and Hamiltonian on T_0 appropriate for the definition of $HF(L_{i+1},L_i)$. Similarly, for the pair (L'_{i+1},L'_i), resp. (L''_i,L''_{i+1}) we make the same choices that are used to define $HF(L'_{i+1},L'_i)$, resp. $HF(L''_i,L''_{i+1})$. For the Floer data on the ends of the strip, the same condition on perturbation data is imposed. It is of product type on $\mathbb{R}\times[2/3,1]\cap Z_\pm$ and of type compatible with $T_0\times M$ on $\mathbb{R}\times[0,1/3]\cap Z_\pm$ (i.e. of type defining Fukaya category of $T_0\times M$).

Given a chord x from $L_0\times L'_0$ to L''_0, and generators $x''_i\in CF(L''_{i-1},L''_i)$, $x'_i\in CF(L'_i,L'_{i-1})$, and $x_i\in CF(L_i,L_{i-1})$ the coefficient of $y\in\mathfrak{M}(L''_p,L_m,L'_n)$ is given by the count of zero dimensional moduli of (r,u), where $r\in\mathcal{Q}(m,n,p)$ and $u:(u_1,u_2):\mathcal{S}_r^f\to T_0\times M$ satisfying

- $(du-X_{K_r})^{0,1}=0$ on the complement of markings of r
- u converges to x on the left end, to y on the right end, and to x''_i on the marking between labels L''_{i-1} and L''_i

- u_1, resp. u_2 converges to x_i, resp. x_i' on the top marking between L_{i-1} and L_i, resp. L_{i-1}' and L_i'

One can show the trimodule equations/A_∞-functor and bimodule equations are satisfied using standard compactness and gluing arguments. Moreover, the generators of $\mathfrak{M}(L'', L, L')$ are graded as usual, and the moduli spaces defining $\mathfrak{M}$ are oriented (see [Gan12],[GPS18]). Thus, $\mathfrak{M}$ can be defined over $\mathbb{Z}$ (for our purposes we define it over $\mathbb{C}$) and is $\mathbb{Z}$-graded.

2.4.2.1 Fully faithfulness

In this section, we give a proof of fully faithfulness of $\mathfrak{M}$ as a functor

$$\mathcal{F}(T_0 \times M)_p \to Bimod(\mathcal{F}(T_0), \mathcal{F}(M)) \tag{2.101}$$

One can prove this by explicit identifications of the generators as in [Gan12], [GPS18]. However, we prefer a different method.

We start by writing a quasi-isomorphism between the bimodule $\mathfrak{M}(L \times L', \cdot, \cdot)$ and the Yoneda bimodule corresponding to (L, L').

First, let us indicate a possible way which we do not follow, but which we derive the idea from. Observe that one can see $L \times L'$ as the geometric composition of $L \times \{*\}$ and $\{*\} \times L'$. Then, one can use a version of quilts with Y-ends as in [LL13]. More precisely, one can attempt to count quilts as in Figure 2-11 with correct labeling to define a quasi-isomorphism from $\mathfrak{M}(L \times L')$ to Yoneda bimodule corresponding to (L, L'). The disadvantage of this approach is that after folding this quilt (as in [LL13]), the perturbation data cannot be continued over the point corresponding to Y-end. Hence, we would need different arguments for compactness.

Instead, we take this as inspiration and define continuation maps directly. For this purpose, we can count quilts as shown in Figure 2-12. To clarify, the green asterisk is an unconstrained point, and it increases the dimension of moduli of such quilted surfaces by one (and it rigidifies the surface when there is no marking). Hence, the labeling does not change on the left or right of green asterisk, the holomorphic maps

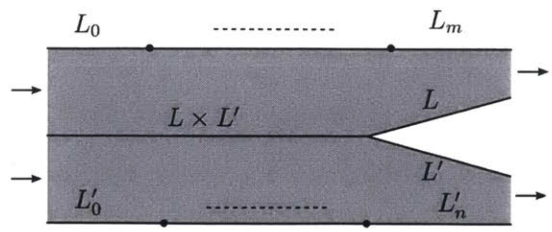

Figure 2-11: A quasi-isomorphism from $\mathfrak{M}(L\times L')$ to Yoneda bimodule corresponding to (L, L')

and perturbation data are defined at the asterisk, and so on. For this count to define a map from $\mathfrak{M}(L \times L') = \mathfrak{M}(L \times L', \cdot, \cdot)$, we must specify the right Floer data on the left, and right. On the left we use Floer data used for the incoming (left) end in Figure 2-10, that are used for generators of $\mathfrak{M}(L \times L', L_0, L'_0)$. On the right, we use product type Floer data. Notice, if you use product type data to define bimodules by counting same quilts as in Figure 2-10 but with no markings on the seam, we would obtain Yoneda bimodule (see [Gan12, Proposition 9.4]). Hence, by compactness and gluing theorems, this defines a bimodule map from $\mathfrak{M}(L \times L')$ to $h_L \otimes h_{L'}$ (More precisely, one has to ensure the data remains $\mathfrak{M}$, resp. Yoneda type on the left, resp. right when the strip approaches to boundary of its moduli. For instance, one can assume the data becomes $\mathfrak{M}$/Yoneda type if the horizontal distance to the asterisk is larger than 1). Denote this bimodule map by $\eta_{L,L'}$. To see it is a quasi-isomorphism, we can construct an explicit quasi-inverse by counting the same quilts as in Figure 2-12, only after interchanging the types of Floer data used on the left and right end. Denote this bimodule map from $h_L \otimes h_{L'}$ to $\mathfrak{M}(L \times L')$ by $\gamma_{L,L'}$. To see this is a quasi-inverse, one can describe the composition of $\eta_{L,L'}$ with $\gamma_{L,L'}$ by a count of quilted strips as in Figure 2-12, with the same type of Floer data at the ends (the type defining $\mathfrak{M}$ or product type, depending on the direction of the composition).

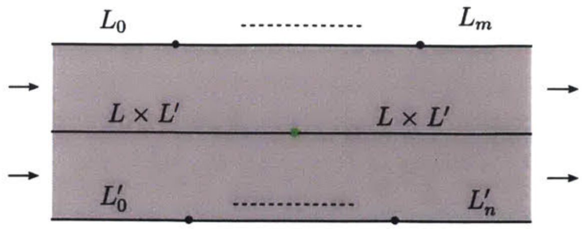

Figure 2-12: Another quasi-isomorphism from $\mathfrak{M}(L \times L')$ to Yoneda bimodule corresponding to (L, L'). Note the green asterisk is an unconstrained point

Now, we have a diagram:

$$\begin{array}{ccc} CF(L_1 \times L'_1, L_2 \times L'_2) & \xrightarrow[\mathfrak{M}]{} & hom(\mathfrak{M}(L_1 \times L'_1), \mathfrak{M}(L_2 \times L'_2)) \\ & & \downarrow \gamma^*_{L,L'} \\ & & hom(h_{L_1} \otimes h_{L'_1}, \mathfrak{M}(L_2 \times L'_2)) \\ & & \downarrow \mathcal{Y}=\mathcal{Y}_{L_1,L'_1,\mathfrak{M}(L_2 \times L'_2)} \\ & \searrow^{\Gamma} & \mathfrak{M}(L_2 \times L'_2, L_1, L'_1) \end{array} \tag{2.102}$$

$\gamma^*_{L,L'}$ is a quasi-isomorphism, since $\gamma_{L,L'}$ is. $\mathcal{Y}_{L_1,L'_1,\mathfrak{M}(L_2 \times L'_2)}$ is the Yoneda map, which is also a quasi-isomorphism. It is given by insertion of (cohomological) units of L_1 and L'_1, and we abbreviate it by $\mathcal{Y}$. Hence, to show $\mathfrak{M}$ is a fully faithful functor, we only need to show Γ is a quasi-isomorphism.

For this purpose, we describe the composition. It is pictured in Figure 2-13. Again, the green asterisks are unconstrained, and merely stabilize the components they are in. Gluing the half discs correspond to unit insertion, i.e. $\mathcal{Y}$. Middle component corresponds to $\gamma^*_{L,L'}$, and the leftmost component to the functor $\mathfrak{M}$. The input is in the seam of leftmost component. After gluing and folding, this can be described as in Figure 2-14. The green asterisk stabilize the otherwise unstable disc. It is easy to see this map is a quasi-isomorphism. Indeed, a left-quasi-inverse can be obtained by flipping the same surface horizontally (and interchanging positive and negative ends), and by gluing these we obtain a count of stable pseudo-holomorphic discs with one incoming and one outgoing end. Modify the perturbation data to a translation invariant one without changing its restriction to strip-like ends. This holomorphic map would need to be constant due to its rigidity. Same argument works for a right

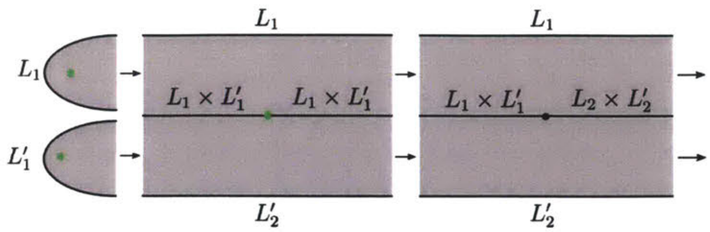

Figure 2-13: The composition Γ

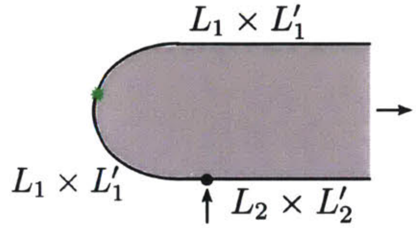

Figure 2-14: The composition Γ after folding and gluing

quasi-inverse. Hence, we have proven:

Proposition 2.4.3. *The functor $\mathfrak{M}$ defined by count of pseudo-holomorphic quilts as in Figure 2-10 is cohomologically fully faithful.*

Remark 2.4.4. Following a trick in [GPS18], one can make the choice of Floer data defining $\mathfrak{M}$ to be of product type. This way $\mathfrak{M}(L_1 \times L_1') \simeq h_{L_1} \otimes h_{L_1'}$ is immediate and we can avoid maps η and γ. In this case, one has to replace the condition that the data is of type defining $T_0 \times M$ over $(\mathbb{R} \times [0, 1/3] \cap \mathcal{S}_r^f) \times (\widehat{\tilde{T}_0} \times \widehat{M})$ by the same condition for the data on a smaller, bounded neighborhood of the markings in the middle seam. One can take the neighborhood to be points of distance less than $1/3$ to a marking on the seam. One further imposes that the data is of product type outside a slightly larger neighborhood of middle markings (such as points of distance less than $2/3$).

2.4.2.2 Modifications needed for the twisted case

In this section, we will show how to define a fully faithful twisted Künneth functor

$$\mathcal{F}(T_\phi)_{p,l} \to Bimod_{tw}(\mathcal{F}(T_0)_l, \mathcal{F}(M)) \tag{2.103}$$

$\mathcal{F}(T_\phi)_{p,l}$ denotes the full subcategory of $\mathcal{F}(T_\phi)$ spanned by product type Lagrangians $L \times_\phi L'$ (i.e. images of Lagrangians of type $\tilde{L} \times L' \subset \tilde{T}_0 \times M$ under projection to T_ϕ, where $\tilde{L}$ is a fixed lift of a compact Lagrangian in T_0). $\mathcal{F}(T_0)_l$ is the full subcategory of $\mathcal{F}(T_0)$ spanned by Lagrangian branes that lift to cover $\tilde{T}_0$. Fix a lift for each such Lagrangian (given $L \subset T_0$, let $\tilde{L} \subset \tilde{T}_0$ denote the lift and $\tilde{L}\langle r\rangle$ denote $\mathfrak{tr}^r(\tilde{L})$ as before). We endow $\mathcal{F}(T_0)_l$ with an extra grading as in Section 2.3.3. More precisely, the extra degree of a chord from L_1 to L_0 is the unique r such that chord lifts to a path from $\tilde{L}_1$ to $\tilde{L}_0\langle -r\rangle$.

$\mathcal{F}(T_\phi)_{p,l}$, resp. $\mathcal{F}(T_0)_l$ does not generate $\mathcal{F}(T_\phi)$, resp. $\mathcal{F}(T_0)$; however, our main goal is to prove the same result for wrapped Fukaya categories. As shown in Corollary 2.2.6, resp. [LP16], Lagrangians of type $L \times_\phi L'$, resp. Lagrangians that lift to $\tilde{T}_0$ generate the respective wrapped Fukaya category. We will define the analogue of (2.103) on the full subcategory of wrapped Fukaya category spanned by generators $L \times_\phi L'$, but this- together with fully faithfulness- implies the "twisted Künneth theorem" for the whole wrapped category. The reason we start by ordinary Fukaya category rather than wrapped Fukaya category is purely expository. Twisted case involves a different compactness argument, and we prefer to explain it for ordinary Fukaya category first to separate this from compactness issues related to wrapping.

Before technicalities, we would like to mention that the labeling for quilts defining $\mathfrak{M}$ are as in Figure 2-15. To clarify, the asymptotic conditions we put on the markings on the seam are given by chords from L''_i to L''_{i+1} in T_ϕ. These chords has infinitely many lifts to chords from $\pi^{-1}(L''_i)$ to $\pi^{-1}(L''_{i+1})$; however, once the labeling and other asymptotic conditions are fixed, the lift is uniquely determined (this also uses the fact that we only consider L'' that lifts under π). For instance, assume the labeling is as in Figure 2-15. If we know the chord from $\tilde{L}_0\langle -g\rangle \times L'_0$ to $\pi^{-1}(L''_0)$, this determines the component of $\pi^{-1}(L''_0)$ to which that part of the seam maps. Hence, this determines the lift of the chord from L''_0 to L''_1, determining the component of $\pi^{-1}(L''_1)$ that the seam maps to, and so on.

We want equivalence of Figure 2-15 to Figure 2-16; hence, we will choose the Floer data and perturbation data accordingly. One can compare this to the following: the

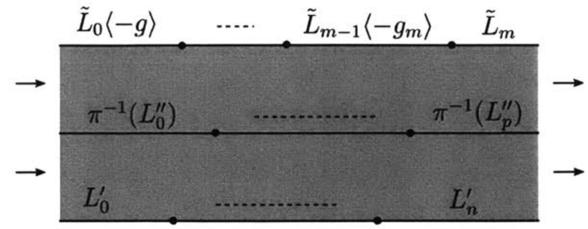

Figure 2-15: Labeling for quilted strips defining $\mathfrak{M}$ in twisted case

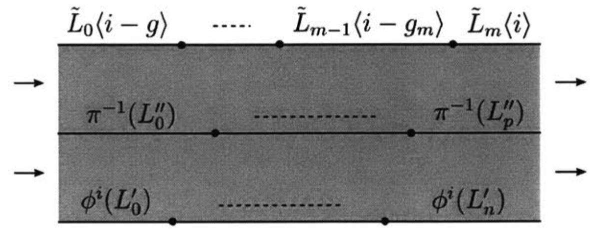

Figure 2-16: Equivalent labeling for quilted strips defining $\mathfrak{M}$

symplectomorphism ϕ does not a priori induce a strict auto-equivalence of the Fukaya category. To make it strict, one needs to add quasi-isomorphic objects $[L', n]$ for each object $L' \subset M$ and $n \in \mathbb{Z}$ (and let ϕ act by $[L', n] \mapsto [\phi(L'), n+1]$). This allows one to choose Floer data invariant under ϕ. We will abuse the notation and keep denoting the objects of $\mathcal{W}(M)$ by letters such as L' and their images under the induced strict autoequivalence by $\phi(L')$. The reason this is emphasized is that for Figure 2-15 and Figure 2-16 to be equivalent, we need similar invariance conditions under $\mathfrak{tr} \times \phi$. For this purpose, it still suffices to add quasi-isomorphic objects $[L', n]$ for each $L' \subset M$.

Given L'', L, L' and $i \in \mathbb{Z}$, choose Floer data for the pair $(\tilde{L}\langle i-g\rangle \times \phi^i(L'), \pi^{-1}(L''))$. We impose the condition that the data for $\tilde{L}\langle -g\rangle \times L'$ and $\tilde{L}\langle i-g\rangle \times \phi^i(L')$ are related by $(\mathfrak{tr} \times \phi)^i$. Hence, the chords from $\tilde{L}\langle i-g\rangle \times \phi^i(L')$ to $\pi^{-1}(L'')$ can be identified with the chords from $\tilde{L}\langle -g\rangle \times L'$ to $\pi^{-1}(L'')$. Let $\mathfrak{M}(L'', L, L')$ be generated by these chords.

Similarly, choose the perturbation data for the labeling as in Figure 2-16 for each i, in a way that it is related by $(\mathfrak{tr} \times \phi)^i$ to perturbation data for Figure 2-15. These two conditions are important not only for bimodule equations, but also to obtain maps (2.46).

We have two types of almost complex structures/Hamiltonians on $\tilde{T}_0 \times M$: one is product type data where we assume the datum on $\tilde{T}_0$ comes from T_0 under the projection $\tilde{T}_0 \to T_0$. The other is the data that come from the data on T_ϕ under the covering map $\tilde{T}_0 \times M \to T_\phi$. We call the former product type data and the latter T_ϕ-type data.

Now, we impose the following: after folding the strip, the perturbation data will be of product type on $\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f$, and it will be of T_ϕ-type on $\mathbb{R} \times [0, 1/3] \cap \mathcal{S}_r^f$. The reason we impose this is to ensure that in the compactification the stable discs on the seam will be lifts of pseudo-holomorphic curves in T_ϕ, and on the upper and lower boundaries they will correspond to pseudo-holomorphic curves in T_0 and M respectively.

Assume compactness for moduli spaces of stable pseudo-holomorphic quilted strips hold. Using equivalence of Figure 2-15 and Figure 2-16, one can easily check that the count of such quilted strips defines a functor (2.103). In other words, when there are no markings in the middle seam the twisted bimodule equations (2.46) are satisfied, and when we allow inputs in the middle, this count defines an A_∞-functor to the category of twisted bimodules with morphisms as in (2.47).

The proof of fully faithfulness is essentially the same as in Section 2.4.2.1. Namely, one has to show the equivalence of $\mathfrak{M}_{L\times_\phi L'}$ with the twisted Yoneda bimodule $h_L \otimes_{tw} h_{L'}$. A geometric description of twisted Yoneda bimodule is given by:

Lemma 2.4.5. *Consider labeled quilted strips as in Figure 2-15 with no markings in the middle seam (i.e. $p = 0$) and $L_0'' = L \times_\phi L'$. Endow the upper strips $\mathcal{S}_r^{(1)}$ with Floer data coming from T_0 and lower strips $\mathcal{S}_r^{(2)}$ with Floer data coming from M (i.e. endow the folded strip with product type data that is invariant under $\mathfrak{tr}$). Assume the data is chosen in such a way that it only depends on the stable components $\mathcal{S}_r^{(1)}$ and $\mathcal{S}_r^{(2)}$ when there are markings on both lower and upper boundaries (as r varies over a single moduli) and assume it is translation invariant on unstable components (strips). Assume the data for labellings as in Figure 2-15 and Figure 2-16 are related by $(\mathfrak{tr} \times \phi)^i$ as before. Then the count of such strips gives the twisted Yoneda bimodule $h_L \otimes_{tw} h_{L'}$.*

Proof. For the proof in untwisted case, see [Gan12, Prop 9.4]. For the twisted case, call this bimodule $\mathfrak{N}$. The generators of $\mathfrak{N}(L_0, L_0')$ are given by chords from $\tilde{L}_0 \times L_0' \to \pi^{-1}(L \times L')$. As we are using product type Floer data for $T_0 \times M$, the graded vector space generated by these chords can be identified with $CF(L_0, L)^r \otimes CF(L_0', \phi^{-r}(L'))$, where $CF(L_0, L')^r$ refers to degree r part in extra grading. Hence, as a graded vector space, we have an identification of $\mathfrak{N}(L_0, L_0')$ with $h_L \otimes_{tw} h_{L'}$ (see (2.50)). That $\mathfrak{N}$ and $h_L \otimes_{tw} h_{L'}$ have the same structure maps follows from the same proof as [Gan12, Prop 7.3, Prop 9.4] together with identification of strips in Figures 2-15 and 2-16. $\square$

Remark 2.4.6. The reason we make the assumptions on the stable/unstable components of $\mathcal{S}_r$ is to apply the trick in [Gan12, Prop 7.3]. Namely, for instance, if there are markings on both upper and lower boundaries, then a rigid quilted strip would restrict to pseudo-holomorphic maps $\mathcal{S}_r^{(1)} \to T_0$ and $\mathcal{S}_r^{(2)} \to M$. But the quilted strips $\mathcal{S}_r$ with these components are parametrized by $\mathbb{R}$ and as the data only depends on the components, the quilted strip itself cannot be rigid. One can presumably, dismiss this condition, as it is likely that we can write bimodule quasi-isomorphisms between bimodules defined using this type of data and more general data similar to $\gamma_{L,L'}$. The condition roughly means that the data on $\mathcal{S}_r$ comes from data on the moduli spaces of stable discs defining $\mathcal{F}(T_0)$ and $\mathcal{F}(M)$.

Hence, using the count of pseudo-holomorphic strips labeled as in Figure 2-15 with no markings and one unconstrained point on the middle seam (cf. Figure 2-12) one can define a quasi-isomorphism between twisted Yoneda functor and $\mathfrak{M}(L \times_\phi L')$ analogously to Section 2.4.2.1. More precisely, choose the data on compactified moduli of quilted strips as in Figure 2-12 inductively on the strata. On the lower strata, assume the data on quilted strip components (without asterisk) are of $\mathfrak{M}$ or Yoneda type (i.e. as defined in Lemma 2.4.5). Then extend to higher dimensional strata after gluing.

To obtain analogue of diagram (2.102), one also needs an analogue of $\mathcal{Y}$, which is provided by Lemma 2.3.12. The rest of the proof of fully faithfulness is the same.

Now, we turn back to the question of compactness. For this, we need to impose

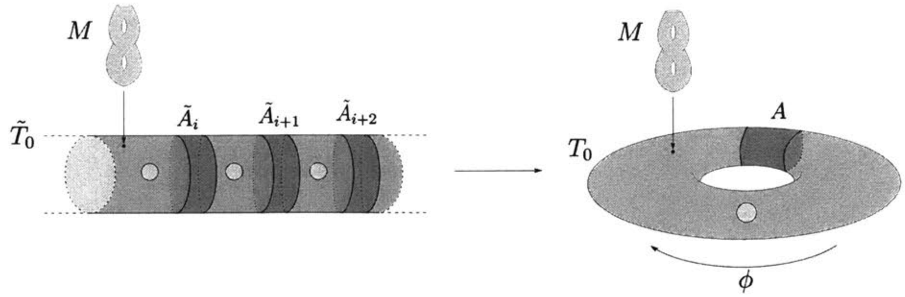

Figure 2-17: The annulus $A \subset T_0$ and its lifts to $\tilde{T}_0$

further conditions on perturbation data as we will explain now. $\tilde{T}_0 \times M$ is not of finite type; therefore, even in the case of Fukaya category of compact Lagrangians, we have to show pseudo-holomorphic curves of fixed (finite) boundary conditions, fixed asymptotic conditions and bounded energy do not escape to infinity. Presumably, one can adapt geometric boundedness arguments in [Sik94], [Gro15] and dissipative Floer data in [Gro15]. However, there is a simpler solution, which can actually be seen as a baby version of i-boundedness of [Gro15]. Choose an annulus A as in Figure 2-17. In particular it satisfies:

1. A lifts to $\tilde{T}_0$
2. Every path connected subset of $\tilde{T}_0$ that is not contained in a finite subdomain crosses infinitely many lifts of A on both boundary components.

Fix a trivialization of fibration $T_\phi \to T_0$ over A, i.e. identify it with $A \times M$ over A. Let $\tilde{A}_i$ denote the lifts of A. We will choose the Floer and perturbation data on $\tilde{T}_0 \times M$ to be of product type over $\tilde{A}_i \times M$ (we do not have to say with respect to which trivialization of $\tilde{A}_i \times M \to \tilde{A}_i$, as they all differ by $1 \times \phi^k$ and this does not effect the product type assumption).

The projection of a given a pseudo-holomorphic map into $\tilde{T}_0 \times M$ is pseudo-holomorphic over the annuli $\tilde{A}_i$. This is due to product type assumption. Hence, we need a lower bound on the energy of pseudo-holomorphic curves mapping into annuli. This is achieved in the following lemma:

Lemma 2.4.7. *Let $A = [s_-, s_+] \times S^1$ be an annulus with coordinates $s, t \in \mathbb{R}$ (where t-coordinate is 1-periodic). Endow A with the symplectic structure $\omega_A = ds \wedge dt$ and* *a compatible almost complex structure. Let Σ be a closed, connected Riemann surface with (non-empty) boundary and β be a closed one form on Σ that has integral periods (in other words $\int_C \beta \in \mathbb{Z}$ for every 1-cycle $C \subset \Sigma$). Let $h = as$, where a is a fixed element of $\mathbb{Q}_+$. Assume $v : (\Sigma, \partial\Sigma) \to (A, \partial A)$ satisfies the equation*

$$(dv - X_h \otimes \beta)^{0,1} = 0 \tag{2.104}$$

Then, the topological energy of v is non-negative and takes values in $(s_+ - s_-)(\mathbb{Z} + a\mathbb{Z}) \subset (s_+ - s_-)\mathbb{Q}$. In particular, it is at least $(s_+ - s_-)gcd(1, a) > 0$ if $Im(v)$ is not contained in $\partial A = \{s_-, s_+\} \times S^1$.

Proof. We compute the topological energy using Stokes theorem. In other words

$$E^{top}(v) = \int_\Sigma v^*\omega_A - d(v^*(h).\beta) = \int_{\partial\Sigma} v^*sdt - v^*(h).\beta \tag{2.105}$$

Let $\partial\Sigma_+$, resp. $\partial\Sigma_-$ denote $\partial\Sigma \cap v^{-1}(s_+)$, resp. $\partial\Sigma \cap v^{-1}(s_-)$. Define

$$n := \int_{\partial\Sigma_+} dt = \int_{\partial\Sigma_-} dt \in \mathbb{Z} \text{ and } m := \int_{\partial\Sigma_+} \beta = \int_{\partial\Sigma_-} \beta \in \mathbb{Z} \tag{2.106}$$

The equalities of integrals hold since dt and β are closed. They are integers since we choose β with integral periods. Then as s and h are constant on either of $\Sigma_\pm$

$$E^{top}(v) = (s_+ - s_-)(n - ma) \tag{2.107}$$

This proves the first claim.

Non-negativity holds since

$$E^{top}(v) = E^{geo}(v) = \int_\Sigma \frac{1}{2}\|dv - X_h \otimes \beta\|^2 \geq 0 \tag{2.108}$$

in this case ($E^{geo} - E^{top} = \int_\Sigma v^*h.d\beta$, but $d\beta = 0$).

Finally, if $E^{top}(v) = E^{geo}(v) = 0$, then $dv = X_h \otimes \beta$ over all Σ. Hence, image of v is contained in a single X_h-orbit. This has to be be on the boundary of A. □

Note 2.4.8. To apply Lemma 2.4.7, one can let the perturbation term to be $K = h\beta + \gamma$, where γ is a 1-form on Σ and h, β are as in Lemma 2.4.7 (since $X_K = X_h \otimes \beta$ in this case). Moreover, if we assume Σ is simply connected, then the condition that β has integral periods becomes automatic, and the condition on the perturbation term K becomes infinitesimal. In other words, one can let $K \in \Omega^1(\Sigma, C^\infty(A))$ to satisfy the following conditions:

- $\beta_0 := \partial_s(K)$ is a closed 1-form on Σ (i.e. $d_\Sigma(\partial_s(K)) = d_A(\partial_s(K)) = 0$)
- $K - s\beta_0$ is a 1-form on Σ (i.e. $d_A(K - s\beta_0) = 0$)

Recall s is a coordinate on A and not on Σ. We do not need to fix a in case Σ is simply connected, but if we fix a, one recovers β as β_0/a.

Definition 2.4.9. For a simply connected Riemann surface S, we call the perturbation data (K, j) (or simply the perturbation term K) **a-sloppy**, if K satisfies two conditions listed in Note 2.4.8. In the case of Floer data (on strip-like ends), we also assume β is standard (this means $\beta = d\epsilon_2$, if the strip-like end has coordinates $(\epsilon_1, \epsilon_2) \in \mathbb{R}_\pm \times [0, 1]$, this assumption is relevant when such data depends on a).

Fix small $a \in \mathbb{Q}_+$ so that all Hamiltonian chords in $\tilde{T}_0$ and T_ϕ over chords of $h = as$ are non-degenerate. To summarize all the conditions on perturbation/Floer data (K, J) (where $K \in \Omega^1(\mathcal{S}_r^f, C^\infty(\widehat{\tilde{T}_0} \times \widehat{M}))$ and J is a family of almost complex structures on $\widehat{\tilde{T}_0} \times \widehat{M}$ parametrized by $\mathcal{S}_r^f$):

1. over $(\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f) \times \widehat{\tilde{T}_0} \times \widehat{M}$, (K, J) is of product type
2. over $(\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f) \times \widehat{\tilde{T}_0} \times \widehat{M}$, the $\tilde{T}_0$-component of the datum is the pullback (under $\tilde{T}_0 \to T_0$) of restriction of a datum on $(\mathbb{R} \times [2/3, 1]) \cap \mathcal{S}_r^{(1)}$
3. over $(\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f) \times \widehat{\tilde{T}_0} \times \widehat{M}$, $\widehat{M}$-component is the restriction of a datum on $(\mathbb{R} \times [2/3, 1]) \cap \mathcal{S}_r^{(2)}$
4. over $(\mathbb{R} \times [0, 1/3] \cap \mathcal{S}_r^f) \times \widehat{\tilde{T}_0} \times \widehat{M}$, (K, J) is $\mathfrak{tr} \times \phi$-invariant (i.e. it is pulled back from $\widehat{T}_\phi$)

5. over $\mathcal{S}_r^f \times \tilde{A}_i \times \widehat{M}$, (K, J) is of product type

6. the $\widehat{\tilde{T}_0}$ component of K is a-sloppy over $\mathcal{S}_r^f \times \tilde{A}_i \times \widehat{M}$

7. $|X_K| < C$ and $|d_S(K)| < C$ over $\tilde{T}_0 \times M \subset \widehat{\tilde{T}_0} \times \widehat{M}$, where $d_S(K)$ is the exterior derivative of K in the $\mathcal{S}_r^f$ direction and C is a positive constant

8. (only for compact Fukaya category) K is supported in $\mathcal{S}_r^f \times (int(\tilde{T}_0) \times int(M))$

When making choices of data for families of strips, we assume there exists a uniform constant C as above; however, we do not fix C for all choices of data (in other words, there could be another choice of data over the same family with a larger constant). Note that not fixing the constant is important in the contractability of such choices.

Remark 2.4.10. When we are working with the compact Fukaya category, we can assume everything takes place in the interior of given Liouville domains; hence, we mostly used the compact parts $\tilde{T}_0$, T_0, M etc., rather than their completions. The last condition in particular, implies the pseudo-holomorphic maps to the Liouville completion (with boundary and asymptotic conditions in the interior) stay in the interior (this follows from Lemma 2.4.19, if one assumes that almost complex structure is cylindrical near the boundary).

For a clarification on the product type assumption, see Definition 2.4.17. We defer the question of existence and contractability of such data to Section 2.4.3.The norms $|X_K|$ and $|d_S(K)|$ at a point $x \in \mathcal{S}_r^f$ are taken with respect to metric induced by the almost complex structure at x. Given consistent choices of strip-like ends, one has thin-thick decomposition of the (quilted) Riemann surfaces in the moduli (see for instance [Sei08c], [Gro15]). For instance, if we glue two Riemann surfaces along one end of each, the thick part of the glued surface is the complement of other strip-like ends and the part corresponding to glued ends of initial surfaces. Moreover, we can make the choices of ends such that:

Assumption 2.4.11. *The thick part has area uniformly bounded by a constant D_1.*

The consistency of choice of data (J, K) means that near the boundary of moduli, (J, K) and the data obtained by gluing from the lower dimensional strata matches up to infinite order (see [AS10], [Abo10]). The curvature R vanishes over the strip-like ends (see (2.109) in Lemma 2.4.13); however, this does not imply vanishing over the thin part if we only assume asymptotic consistency. On the other hand, we can assume:

Assumption 2.4.12. *The difference of (perturbation terms of) (J, K) and data obtained by gluing is exponentially (C^1) small in terms of the gluing length. In particular, the curvature has uniformly bounded integral over the thin parts as well (call this uniform bound of integral D_2).*

Under these assumptions, we have:

Lemma 2.4.13. *The curvature is bounded and for a given pseudo-holomorphic strip u, $E^{top} > E^{geo} - D$ for a uniform constant D.*

Proof. Recall that the curvature R has the following form

$$R = (\partial_s K(\partial_t) - \partial_t K(\partial_s) + \{K(\partial_s), K(\partial_t)\})dsdt = d_S(K) + \{K, K\} \tag{2.109}$$

in local coordinates (see [Sei12, 5.14] for instance). Clearly

$$|\{K, K\}| = |\omega(X_K, X_K)| \leq |X_K|^2 \tag{2.110}$$

Hence, $|R| < C^2 + C$. Moreover, R vanishes over the strip-like ends (this follows easily from (2.109)), and the integral of R is bounded by D_2 over the thin parts by Assumption 2.4.12. Let $(\mathcal{S}_r^f)^{thick}$, resp. $(\mathcal{S}_r^f)^{thin}$ is the thick part, resp. thin part of $\mathcal{S}_r^f$ (we are being slightly informal in thin-thick decomposition of $\mathcal{S}_r$ as $\mathcal{S}_r$ is actually a quilted surface, one has to take care of parts near the upper and lower boundary of $\mathcal{S}_r$ before folding). We have

$$|E^{geo} - E^{top}| \leq \left| \int_{(\mathcal{S}_r^f)^{thick}} u^*R \right| + \left| \int_{(\mathcal{S}_r^f)^{thin}} u^*R \right| < D_1(C^2 + C) + D_2 \tag{2.111}$$

which proves the claim. ☐

Remark 2.4.14. One can weaken the assumption on the choices of strip-like ends and K. Namely, one can allow the bound C to vary over the strip so that the curvature R would be uniformly bounded by a positive function over the strip with finite integral.

Choose such perturbation data consistently over all moduli spaces $\mathcal{Q}(\mathbf{d})$. Then we have:

Corollary 2.4.15. *The moduli of pseudo-holomorphic strips with fixed labeling and asymptotic conditions on the strip-like ends is compact.*

Proof. It is enough to show that images of all strips are contained in a fixed bounded region in $\tilde{T}_0 \times M$. In other words, they do not go to ends of $\tilde{T}_0$ component. First, the topological energy depends only on the asymptotic conditions on the marked points, see [Sei12, (5.13)] for instance. Hence, by Lemma 2.4.13, the geometric energy is bounded.

Moreover, by Lemma 2.4.7, there exists an $\epsilon > 0$ such that each time the folded strip passes through the pre-image of $\tilde{A}_i$, the topological energy increases at least by ϵ. Indeed, the geometric energy in $\tilde{A}_i$ component is the same as topological energy in that component. The total geometric energy is the sum of geometric energies in both components. Hence, the total geometric energy increases at least by ϵ. Since the total geometric energy is bounded, the curve can cross only finitely many of $\tilde{A}_i$, finishing the proof. ☐

Remark 2.4.16. As mentioned, to apply the argument in Section 2.4.2.1, one defines maps $\gamma_{L,L'}$ and $\eta_{L,L'}$ between twisted Yoneda bimodule and $\mathfrak{M}_{L\times_\phi L'}$. For this purpose, one still has to choose the perturbation data on folded strip so that compactness holds. A simple modification of Conditions (1)-(8) above gives rise to such data: namely, for instance for counts defining γ assume the data satisfies Conditions (1)-(8) on the right end, and it is of product type and $\mathfrak{tr}$-invariant (i.e. satisfies (1)-(3)) on the left end (in addition to intersection of the folded strip with $\mathbb{R} \times [2/3, 1]$). We do not need to (and cannot) impose $\mathfrak{tr} \times \phi$-invariance over all $\mathbb{R} \times [0, 1/3]$. Conditions (5)-(8) remain unchanged. Modifications for counts defining $\eta_{L,L'}$ are similar. It is also similar for counts of stable discs with one auxiliary marking that define units. For instance, we assume a-sloppy conditions for counts defining units of $\mathcal{F}(T_0)$.

2.4.3 Modifications needed for wrapped Fukaya categories

In this section, we describe the necessary modifications to Section 2.4.2.1 to prove:

Theorem 2.1.12. *$\mathcal{W}(T_\phi)$ is quasi-equivalent to twisted tensor product of $\mathcal{W}(T_0)$ and $\mathcal{W}(M)$.*

For the definition of wrapped Fukaya categories we follow [Abo10]. In other words, we use Hamiltonians that are quadratic at infinity.

Throughout this section, let $\mathcal{W}(T_0)$ denote split generating subcategory (of the big wrapped Fukaya category) spanned by L_{gr} and L_{pur}. The fixed lifts endow this category with an extra grading as before. Let $\mathcal{W}(M)$ denote a split generating category spanned by cocores of the Weinstein structure used in Section 2.2.2 and their iterates by ϕ. Assume $\mathcal{W}(M)$ is made ϕ-equivariant in the way we explained. Let $\mathcal{W}(T_\phi)$ denote the category spanned by $L \times_\phi L'$ where $L \in \mathcal{W}(T_0)$ and $L' \in \mathcal{W}(M)$.

To prove Theorem 2.1.12, we need to define an analog of (2.103), i.e. an A_∞-functor

$$\mathcal{W}(T_\phi) \to Bimod_{tw}(\mathcal{W}(T_0), \mathcal{W}(M)) \tag{2.112}$$

This will use count of quilted strips as in Figure 2-15. The main issue to be addressed is compactness. Apart from preventing pseudo-holomorphic curves from escaping to left and right ends of $\widehat{\tilde{T}}_0 \times \widehat{M}$ (which we showed how to handle in Section 2.4.2.2), one has to prevent curves from escaping to conical end. To solve this problem, we follow [GPS18]. Namely, we define a category $\mathcal{W}^2(T_\phi)$, that is quasi-equivalent to $\mathcal{W}(T_\phi)$ and define an A_∞-functor

$$\mathcal{W}^2(T_\phi) \to Bimod_{tw}(\mathcal{W}(T_0), \mathcal{W}(M)) \tag{2.113}$$

using the count of quilted strips as in Figure 2-15 (or equivalently as in Figure 2-16).

The category $\mathcal{W}^2(T_\phi)$ is analogous to $\mathcal{W}^2$ of [Gan12], $\mathcal{W}^s$ of [Gao17] and $\mathcal{W}^{prod}$ of [GPS18]. The basic idea is the following: even though T_ϕ is not a product, its conical end can be identified with the conical end of the product. Hence, one can talk about the product type data on the conical end.

More precisely:

$$\begin{gathered}\widehat{T}_\phi \setminus T_\phi = \big((\widehat{\tilde{T}}_0 \times \widehat{M}) \setminus (\tilde{T}_0 \times M)\big)/(\mathfrak{tr} \times \phi) = \\ \big((\widehat{\tilde{T}}_0 \setminus \tilde{T}_0) \times \widehat{M}\big)/(\mathfrak{tr} \times \phi) \cup \big(\widehat{\tilde{T}}_0 \times (\widehat{M} \setminus M)\big)/(\mathfrak{tr} \times \phi)\end{gathered} \tag{2.114}$$

$(\widehat{\tilde{T}}_0 \setminus \tilde{T}_0)$ is isomorphic to infinitely many copies of $\widehat{T}_0 \setminus T_0$ and $\mathfrak{tr}$ moves one to the next. Hence

$$\big((\widehat{\tilde{T}}_0 \setminus \tilde{T}_0) \times \widehat{M}\big)/(\mathfrak{tr} \times \phi) \cong (\widehat{T}_0 \setminus T_0) \times \widehat{M} \tag{2.115}$$

Moreover, ϕ acts trivially on $(\widehat{M} \setminus M)$; hence

$$\big(\widehat{\tilde{T}}_0 \times (\widehat{M} \setminus M)\big)/(\mathfrak{tr} \times \phi) \cong \widehat{T}_0 \times (\widehat{M} \setminus M) \tag{2.116}$$

The intersection of these subsets is isomorphic to $(\widehat{T}_0 \setminus T_0) \times (\widehat{M} \setminus M)$ with the obvious embeddings. Hence, the conical end of T_ϕ can be written as the union of products (2.115) and (2.116).

As before, we refer the reader to [Sei08c] for the conventions about Floer data and perturbation data. As a quick reminder, given a Riemann surface with boundary Σ and symplectic manifold X, a perturbation data is a pair (J, K), where J is a family of compatible almost complex structures on X parametrized by Σ and K is a $C^\infty(X)$ valued 1-form on Σ that vanishes along the directions tangent to $\partial\Sigma$.

For a product of symplectic manifolds, we have a special type of data:

Definition 2.4.17. Let $X \times Y$ be a product of connected symplectic manifolds and (J, K) be a perturbation datum on $X \times Y$ with domain Σ. We call J is of product type over $\Sigma \times X \times Y$ if it can be written as a direct sum of families of almost complex structures on X and on Y parametrized by Σ. Similarly, we call K is of product type over $\Sigma \times X \times Y$ if it can be written as $K_1 + K_2$, where $K_1 \in \Omega^1(\Sigma, C^\infty(X))$

and $K_2 \in \Omega^1(\Sigma, C^\infty(Y))$. Clearly, the decomposition of J is unique. Moreover, the decomposition of K is unique up to addition of a 1-form on Σ. In other words, one can also decompose K as $K = (K_1 + \gamma) + (K_2 - \gamma)$ for a $\gamma \in \Omega^1(\Sigma)$, and that covers all possible decompositions. We still refer to K_1 and K_2 as the components of K.

(J, K) is called product type over $\Sigma \times X \times Y$ if both J and K are of product type over $\Sigma \times X \times Y$.

For the wrapped Floer homology we need:

Definition 2.4.18. Let (X, λ_X) be a Liouville domain and $\widehat{X}$ be its completion. Let Σ be a Riemann surface with boundary components labeled by Lagrangians such that λ_X vanishes on these Lagrangians. Let (J, K) be a perturbation datum on $\widehat{X}$ with domain Σ. We call (J, K) cylindrical if $J_s|_{\widehat{X} \setminus X}$ is invariant under Liouville flow for each $s \in \Sigma$ and $K|_{\Sigma \times (\widehat{X} \setminus X)}$ can be written as $r^2\gamma + \eta$ where $\gamma, \eta \in \Omega^1(\Sigma)$, $d\gamma \leq 0$ and r is the Liouville parameter (over the strip-like ends, assume $\gamma = dt$ as usual). The negativity (subclosed) assumption can be stated as

$$d_\Sigma(K - \eta) = (\partial_s(K - \eta)(\partial_t) - \partial_t(K - \eta)(\partial_s))dsdt \leq 0 \tag{2.117}$$

where $s + it$ are holomorphic coordinates on Σ. For the restrictions to strip-like ends of Σ (once they are chosen), we make the standard assumption on J and K. For instance, K can be written as r^2dt on the conical end over the strip-like ends (or as wr^2dt when the end is weighted for the purposes of rescaling, see [Abo10, Definition 4.1]).

The integrated maximum principle (see [AS10, Lemma 7.2], [Abo10, Lemma B.1], [Gan12, Appendix A.2]) applies to cylindrical perturbation data. More precisely, let Σ be a closed Riemann surface with corners such that $\partial\Sigma$ can be written as union of smooth curves $\partial_l\Sigma$ and $\partial_n\Sigma$. Choose cylindrical perturbation data on X with domain Σ (we assume $K|_{\partial_l\Sigma} = 0$ but we do not assume $K|_{\partial_n\Sigma}$ vanishes). Let $L \subset \widehat{X} \setminus int(X)$ be a cylindrical Lagrangian such that $\lambda_X|_L = 0$. Let $u : \Sigma \to \{r \geq r_0\}$ be a map satisfying

1. $(du - X_K)^{0,1} = 0$

2. $u(\partial_n \Sigma) \subset \{r = r_0\}$

3. $u(\partial_l \Sigma) \subset L$

Then, we have:

Lemma 2.4.19. *The image of u is contained in $\{r = r_0\}$.*

Combining Definitions 2.4.17 and 2.4.18, we obtain the following definition:

Definition 2.4.20. Let (J, K) be a Floer data on $\widehat{T_\phi}$ with domain Σ. We call (J, K) of $\mathcal{W}^2(T_\phi)$**-type** if it is of product type on $\Sigma \times (\widehat{T_0} \setminus T_0) \times \widehat{M}$, resp. $\Sigma \times \widehat{T_0} \times (\widehat{M} \setminus M)$ and their components are cylindrical (on the conical ends). Here, $(\widehat{T_0} \setminus T_0) \times \widehat{M}$, and $\widehat{T_0} \times (\widehat{M} \setminus M)$ are seen as subsets of $\widehat{T_\phi}$.

Lemma 2.4.19 applies to $\mathcal{W}^2(T_\phi)$-type data as well. More precisely, for curves mapping to conical end $\widehat{T_\phi} \backslash T_\phi$, one can apply Lemma 2.4.19 to components separately and conclude that the curve is contained in a fixed compact subset. See Lemma 2.4.27 for a proof in a more general case.

The existence and contractability of such data may not seem obvious at first. Hence, we prefer to include a proof here:

Lemma 2.4.21. *$\mathcal{W}^2(T_\phi)$-type data exists and the set of such data is contractible.*

Proof. For the existence, put data on $\Sigma \times (\widehat{T_0} \setminus T_0) \times \widehat{M}$ and $\Sigma \times \widehat{T_0} \times (\widehat{M} \setminus M)$ that agree on the intersection and that satisfy the assumptions (product type and cylindrical on conical ends). Then extend it to the rest of $\Sigma \times \widehat{T_\phi}$.

For the connectedness of such data, one needs to preserve the product type assumption. Hence, given two such data (J, K) and (J', K'), first construct a datum (J_i, K_i) that agree with (J, K) in the first components of $(\widehat{T_0} \backslash T_0) \times \widehat{M}$ and $\widehat{T_0} \times (\widehat{M} \backslash M)$ and that agree with (J', K') in the second. Then one can connect (J_i, K_i) to both data along the data that is product type on the conical end and satisfying the required conditions (A simple illustration of the idea is the following: to connect two metrics

$g_1 \times g_2$ and $g_1' \times g_2'$ on the product along product type metrics, one first connects $g_1 \times g_2$ to $g_1 \times g_2'$ then $g_1 \times g_2'$ to $g_1' \times g_2'$. Indeed, this together with retraction of the space of metrics onto almost complex structures let us connect product type almost complex structures as well).

The higher connectivity and contractability are similar. Indeed, one can do the same for a family of data parametrized by a topological space X. Namely, first extend the family from $X \cong X \times \{0\}$ to a family of data parametrized by $X \times [0,1]$, where the first components are the same over $X \times \{1\}$. Then, use the contractability of data on the second component to extend this family to cone of X (i.e. to $X \times [0,2]/X \times \{2\}$). $\square$

Now we are ready to define $\mathcal{W}^2(T_\phi)$:

Definition 2.4.22. Let $\mathcal{W}^2(T_\phi)$ (or $\mathcal{W}^2$ in short) be the category with objects $L \times_\phi L'$ (see Section 2.2.3 for the definition of $L \times_\phi L'$), where $L \in \mathcal{W}(T_0)$ and $L' \in \mathcal{W}(M)$. Fix a Floer data for the pairs of objects and consistent choices of perturbation data for finite sequences of objects, all assumed to be $\mathcal{W}^2(T_\phi)$-type. The hom complex $CW_{\mathcal{W}^2}(L_1 \times_\phi L_1', L_2 \times_\phi L_2')$ is generated by the Hamiltonian chords from $L_1 \times_\phi L_1'$ to $L_2 \times_\phi L_2'$. The A_∞-structure is defined by the count of holomorphic stable discs as usual.

Remark 2.4.23. We are ignoring the standard rescaling problem here. To be more precise, one needs to choose weights and time shifting maps for each stable disc. See [Abo10, Definition 4.1]. The rescaling uses the Liouville vector field defined in Section 2.2.1 (which matches with the product Liouville form for $\widehat{T}_0 \times \widehat{M}$ on the conical end of $\widehat{T}_\phi$ under the natural identification).

It is standard that this defines a $\mathbb{Z}$-graded A_∞-category over $\mathbb{C}$ for a generic choice of data. See [Gan12] and [Gao17] for more details.

Note 2.4.24. One still has to compare $\mathcal{W}(T_\phi)$ and $\mathcal{W}^2(T_\phi)$, which is hard in the definition of wrapped Fukaya category with quadratic Hamiltonians. On the other hand, [GPS18] uses a different definition and gives a proof of equivalence of these categories in the untwisted case (i.e. for $\phi = 1_M$, $T_\phi = T_0 \times M$). Their proof applies verbatim in the twisted case. More precisely, they define $\mathcal{W}$ and $\mathcal{W}^2$ ($\mathcal{W}^{prod}$ in their notation) using localization and the same definition can be carried in the case of $\mathcal{W}^2(T_\phi)$ as well. In other words, one can first define an ordered version of $\mathcal{W}^2(T_\phi)$ using $\mathcal{W}^2(T_\phi)$-type almost complex structures (with vanishing perturbation term) and then localize with respect to a set of continuation elements. Then the same proof for the equivalence of $\mathcal{W}^{prod}$ and $\mathcal{W}$ applies (one also has to cylindrize $L \times_\phi L'$ as in [GPS18], but this also takes place outside the interior of $\widehat{T_\phi}$ which cannot be distinguished from the same manifold when $\phi = 1$). What remains is the comparison of different definitions of wrapped Fukaya categories, and the comparison of different definitions of $\mathcal{W}^2$. One can write a functor from the ordered version to $\mathcal{W}$ defined using quadratic Hamiltonians by counting stable discs with one auxiliary inner circle tangent to output point (inner circle is not a seam, the disc maps to the same space, but the domain of the pseudo-holomorphic map varies over the multiplihedra). Then, one can show this factors through a quasi-equivalence from the localization of ordered Fukaya category. We skip this proof here. Alternatively, everything we do in this section can be carried out using localization definitions (see Remark 2.4.32).

We will address the compactness of the moduli of pseudo-holomorphic (folded, quilted) strips later in this section. If we assume compactness holds, definition of (2.113) and proof of fully faithfulness is the same as Section 2.4.2.1. In particular, one defines continuation morphisms $\gamma_{L,L'}$ and $\eta_{L,L'}$ using the count of quilted strips Figure 2-12 (except one has to modify the labeling as in Figure 2-15 and choose data so that this would be equivalent to Figure 2-16).

Moreover, the essential image of (2.113) is generated by twisted Yoneda bimodules. This completes the proof of Theorem 2.1.12.

What remains is to give a class of Floer data for which the moduli of pseudo-holomorphic quilted strips is compact. The conditions will be a combination of conditions in Section 2.4.2.1 and conditions ensuring compactness in the context of wrapped Floer homology.

As remarked before the conical end $\widehat{\tilde{T}_0} \times \widehat{M} \setminus \tilde{T}_0 \times M$ is a union of infinitely many copies of the conical end $\widehat{T_0} \setminus T_0 \cong \mathbb{R}_{>0} \times S^1$. Denote closure of these components by $\tilde{B}_i, i \in \mathbb{Z}$ (ordered in a way that $\mathfrak{tr}(\tilde{B}_i) = \tilde{B}_{i+1}$).

Definition 2.4.25. Given a Riemann surface Σ (such as S_r^f), we call perturbation data (J, K) of $\mathcal{W}^2$-type, if it is of product type over $\Sigma \times \tilde{B}_i \times \widehat{M}$ for each i and over $\Sigma \times \widehat{\tilde{T}_0} \times (\widehat{M} \setminus M)$ such that $\tilde{B}_i$, resp. $(\widehat{M} \setminus M)$ components are cylindrical.

Remark 2.4.26. It is clear that $\mathcal{W}(T_\phi)$ type data can be seen as $\mathcal{W}^2$-type data that is invariant under $\mathfrak{tr} \times \phi$.

As mentioned before, one way of proving compactness results required for wrapped Floer homology is to use integrated maximum principle (see [AS10, Lemma 7.2], [Abo10, Lemma B.1], [Gan12, Appendix A.2]) and the assumptions in Definition 2.4.25 ensure that no solution can escape to conical end as in the case of $\mathcal{W}^2(T_\phi)$-type data. More precisely:

Lemma 2.4.27. *Let $u : \Sigma \to \widehat{\tilde{T}_0} \times \widehat{M}$ be a solution to perturbed Cauchy-Riemann equation with $\mathcal{W}^2$-type perturbation data. Assume the strip-like ends converge to chords in a compact region and the boundary components of Σ map to Lagrangians of type $\tilde{L} \times L' \subset \widehat{\tilde{T}_0} \times \widehat{M}$ such that L, L' are cylindrical and the Liouville form vanishes over them outside a compact subset. Then, u is contained in a subset of type $\{r_1 \leq R\} \times \{r_2 \leq R\}$. Here, r_1 and r_2 are Liouville parameters and R is constant that depends only on the asymptotic and boundary conditions.*

Proof. As before, this follows by considering components of $u = (u_1, u_2)$ separately. Namely, consider the map $u_1 : \Sigma \to \widehat{\tilde{T}_0}$. It is not pseudo-holomorphic, but the part that maps to cylindrical end is pseudo-holomorphic due to $\mathcal{W}^2$-type assumption. Hence, it is contained in a subset $\{r_1 \leq R\}$ by Lemma 2.4.19. The other component is similar. □

Note that the subset $\{r_1 \leq R\} \times \{r_2 \leq R\} \subset \widehat{\tilde{T}_0} \times \widehat{M}$ is not compact. However, one can address the problem of escaping to left and right ends of $\widehat{\tilde{T}_0}$ as in Lemma 2.4.7 and Note 2.4.8.

We would like to write a list of conditions on Floer and perturbation data. There is a technical subtlety about the perturbation data; namely, one needs to apply rescaling trick to define the trimodule $\mathfrak{M}$ (similar to [Abo10, Definition 4.1] for instance). Hence, the following set of conditions is precise only for the Floer data (the strip without any marked points) defining the differential and we will postpone necessary modification for the interested reader to Section 2.4.3.1. In the following, $\mathcal{S}_r$ denotes the unstable strip; however, the reader who wishes to ignore rescaling issue may take $\mathcal{S}_r$ to be elements of general moduli spaces $\mathcal{Q}(\mathbf{d})$ and $\overline{\mathcal{Q}(\mathbf{d})}$ (hence there will still be phrases about making consistent choices of data even though the careful reader may wish to think this in the context of Section 2.4.3.1).

Fix a small $a \in \mathbb{Q}_+$ so that all Hamiltonian chords in $\hat{\hat{\tilde{T}}}_0$ and $\widehat{T_\phi}$ projecting to chords of $h = as$ are non-degenerate. The conditions on perturbation/Floer data (J, K) are:

1. over $(\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f) \times \hat{\hat{\tilde{T}}}_0 \times \widehat{M}$, (K, J) is of product type
2. over $(\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f) \times \hat{\hat{\tilde{T}}}_0 \times \widehat{M}$, the $\hat{\hat{\tilde{T}}}_0$-component of the datum is the pullback (under $\hat{\hat{\tilde{T}}}_0 \to \widehat{T}_0$) of restriction of a datum on $((\mathbb{R} \times [2/3, 1]) \cap \mathcal{S}_r^{(1)}) \times \widehat{T}_0$
3. over $(\mathbb{R} \times [2/3, 1] \cap \mathcal{S}_r^f) \times \hat{\hat{\tilde{T}}}_0 \times \widehat{M}$, $\widehat{M}$-component is the restriction of a datum on $((\mathbb{R} \times [2/3, 1]) \cap \mathcal{S}_r^{(2)}) \times \widehat{M}$
4. over $(\mathbb{R} \times [0, 1/3] \cap \mathcal{S}_r^f) \times \hat{\hat{\tilde{T}}}_0 \times \widehat{M}$, (K, J) is $\mathcal{W}^2(T_\phi)$-type
5. over $\mathcal{S}_r^f \times \tilde{A}_i \times \widehat{M}$, (K, J) is of product type
6. the $\hat{\hat{\tilde{T}}}_0$ component of K is a-sloppy over $\mathcal{S}_r^f \times \tilde{A}_i \times \widehat{M}$
7. $|X_K| < C$ and $|d_S(K)| < C$ over $\tilde{T}_0 \times M \subset \hat{\hat{\tilde{T}}}_0 \times \widehat{M}$, where $d_S(K)$ is the exterior derivative of K in the $\mathcal{S}_r^f$ direction and C is positive constant
8. (only for the wrapped Fukaya category) K is of $\mathcal{W}^2$-type

The first and second conditions are related to $\mathcal{W}^2(T_\phi)$-$\mathcal{W}(T_0)$-$\mathcal{W}(M)$-trimodule structure as in Section 2.4.2.2 (i.e. we need them to define a functor $\mathcal{W}^2(T_\phi) \to Bimod_{tw}(\mathcal{W}(T_0), \mathcal{W}(M))$).

As before, the norms $|X_K|$ and $|d_S(K)|$ at a point $x \in \mathcal{S}_r^f$ are taken with respect to metric induced by the almost complex structure at x.

We still assume Assumption 2.4.11 and Assumption 2.4.12. Also, when making choices of data for families of strips, we assume: there exists a uniform constant C as in (7); however, we do not fix C for all choices of data (in other words, there could be another choice of data parametrized by the same family with larger constant). Note that not fixing the constant is important in the contractability of such choices.

Lemma 2.4.28. *For sufficiently large C, the space of Floer/perturbation data satisfying conditions (1)-(8) is non-empty. Moreover, this space is contractible.*

Proof. We only consider the case where there are no markings in the upper and lower boundary as more general case is similar. Hence, let r denote this quilt.

Due to translation invariance of Floer data, one has to construct data parametrized by the interval $[0,1]$ that is $\mathfrak{tr} \times \phi$-invariant on $[0,1/3]$ and is product type and $\mathfrak{tr}$-invariant on $[2/3,1]$. Hence, construct $\mathcal{W}^2(T_\phi)$-type data on $[2/3,1]$. One has to make sure it satisfies condition (5) and (6). But this is simply another product type assumption, namely the data is product type on the subproduct $\tilde{A}_i \times \widehat{M}$ and $\tilde{A}_i$ component satisfies various assumptions. Condition (7) is a pointwise condition and can also be incorporated easily. One has to make extensions from data on

$$[0,1/3] \times (\bigcup \tilde{A}_i \times \widehat{M} \cup \bigcup \tilde{B}_i \times \widehat{M} \times \hat{\tilde{T}}_0 \times (\widehat{M} \setminus M)) \tag{2.118}$$

to $[0,1/3] \times \hat{\tilde{T}}_0 \times \widehat{M}$ while keeping K small on the rest. This is a reason we have to take large C: a-sloppy condition already puts a constraint on how small K may become (similarly, that the Floer data is r^2dt on the boundary is a constraint).

To put $\mathfrak{tr}$-invariant product data on $[2/3,1]$ is simpler: simply repeat what we did before to construct data on $[2/3,1] \times \hat{\tilde{T}}_0$. Then, take product with data on $\widehat{M}$ satisfying bound conditions similar to (7).

To extend the data on $[0,1/3] \cup [2/3,1]$, while keeping product type conditions on $\tilde{A}_i \times \widehat{M}$, $\tilde{B}_i \times \widehat{M}$ and $\hat{\tilde{T}}_0 \times (\widehat{M} \setminus M)$, one has to go through "zigzags" over the subproducts $\tilde{A}_i \times \widehat{M}$, $\tilde{B}_i \times \widehat{M}$ and $\hat{\tilde{T}}_0 \times (\widehat{M} \setminus M)$. This is explained in the proof of

Lemma 2.4.21 (this is similar to connectivity of product type metrics on $X \times Y$: to connect $g_X \times g_Y$ to $g'_X \times g'_Y$, connect both to $g_X \times g'_Y$). The condition (7) is convex in K; hence, (approximately) piecewise-linear interpolations of the perturbation term does not break the condition (maybe at the expense of increasing C slightly).

Hence, we can construct Floer data satisfying these conditions. The proof works for perturbation data on quilted strips with markings as well. Moreover, the existence proof works in families as well, implying the existence of consistent choices and contractability of such choices. □

It is easy to see that (asymptotically) consistent choices of Floer/perturbation data (with uniform C) exists. The contractability of such choices can also be shown in a similar way. Moreover, the choice can be made so that

Assumption 2.4.29. *The difference of (perturbation terms of) (J, K) and data obtained by gluing is exponentially (C^1) small in terms of the gluing parameter.*

As remarked before, this assumption lets us bound the integral of the curvature

$$R = (\partial_s K(\partial_t) - \partial_t K(\partial_s) + \{K(\partial_s), K(\partial_t)\})dsdt = d_S(K) + \{K, K\} \tag{2.119}$$

over the thin parts (recall the remarks about thin-thick decomposition) and the bound can be chosen uniformly over a single moduli space.

Make a consistent choice of Floer/perturbation data satisfying this assumption. Collecting everything we have:

Lemma 2.4.30. *The solutions to perturbed Cauchy-Riemann equation (for such perturbation data) with given boundary conditions and asymptotic conditions are contained in a fixed compact region of $\widehat{\tilde{T}_0} \times \widehat{M}$. Hence, the moduli of stable-quilted strips is compact.*

Proof. That the curves are contained over a finite subdomain of $\widehat{\tilde{T}_0}$ follows similar to Corollary 2.4.15. That they do not escape to infinity on the conical end follows from Lemma 2.4.27. □

It is standard to show the 0 and 1-dimensional moduli of quilted strips is cut-out transversally for a generic choice; hence, they form smooth compact manifolds with corners. Thus, by counts of zero dimensional moduli, one can define the trimodule $\mathfrak{M}$, the twisted bimodule maps and other relevant maps. By standard gluing results, TQFT argument given in Section 2.4.2, and generation result Corollary 2.2.6, one concludes the proof of Theorem 2.1.12. More precisely, the functor

$$\mathcal{W}(T_\phi) \simeq \mathcal{W}^2(T_\phi) \to Bimod_{tw}(\mathcal{W}(T_0), \mathcal{W}(M)) \tag{2.120}$$

defined by $\mathfrak{M}$ is full and faithful and the essential image of Lagrangians of type $L \times_\phi L'$ split generate the split closure of triangulated envelope of twisted Yoneda bimodules.

Remark 2.4.31. Similar to Remark 2.4.16, modifications are needed to define the maps $\gamma_{L,L'}$ and $\eta_{L,L'}$ used in Section 2.4.2.1 so that compactness holds for the relevant count. For instance, for counts defining $\gamma_{L,L'}$, we assume the data satisfies Conditions (1)-(8) on the right end and it satisfies (1)-(3) on the left end (in addition to intersection of the folded strip with $\mathbb{R} \times [2/3, 1]$). We do not need to (and cannot) impose $\mathfrak{tr} \times \phi$-invariance over all $\mathbb{R} \times [0, 1/3]$. Conditions (5)-(8) remain unchanged (except for the rescaling issue that will be addressed in the next subsection). Modifications for stable discs defining units are similar; for instance, we assume a-sloppy condition for counts defining units of $\mathcal{W}(T_0)$.

Remark 2.4.32. As mentioned, [GPS17] defines the wrapped Fukaya category in a different way: they first define an ordered A_∞-category and then localize with respect to a set of morphisms called continuation maps. This definition is attributed to the work of Abouzaid and Seidel. Everything, we did can be done in this language, this adds minor complication to the algebra, yet simplifies the analysis substantially. For instance, due to absence of perturbation term, none of the conditions related to K would be needed. Moreover, there is no rescaling issue in this definition, making Section 2.4.3.1 redundant.

2.4.3.1 Rescaling problem for quilted strips

So far, we have deliberately ignored the (almost standard) rescaling problem one can encounter in definition of wrapped Fukaya category. This section addresses this problem for the quilted strips defining $\mathfrak{M}$. It is slightly technical and the reader may skip if they prefer. We will assume familiarity with rescaling trick as in [Abo10].

Recall that to define the products on the wrapped Fukaya category, one has to rescale the output Lagrangian by the Liouville flow and identify the Floer chain complexes (see [Abo10, (3.4)]). Let Σ be the domain curve (i.e. an open Riemann surface with boundary and fixed strip-like ends). The amount of rescaling is determined by a function $\rho_\Sigma : \partial\Sigma \to [1, \infty)$ that is equal to a constant (called the weight) near the boundary punctures. Indeed, perturbation data in the context of wrapped Fukaya category involves a consistent choice of **time shifting maps** ρ_Σ as Σ varies (where the signed sum of weights is assumed to be 0).

As mentioned, for $\mathcal{W}^2(T_\phi)$, one does the rescaling by the Liouville vector field introduced in Section 2.2.1, which restricts to product type Liouville structure over the cylindrical end.

However, rescaling is not compatible with the conditions (5) and (6) as the Liouville flow distorts the annuli $\tilde{A}_i$ and the rescaled data no longer satisfies these. Hence, we would like to explain the necessary modifications on these conditions to make data invariant under rescaling.

Another issue about rescaling is the Liouville vector field we use to rescale. One may try the product type Liouville vector field on $\widehat{\tilde{T}_0} \times \widehat{M}$, but then condition (4) will not be preserved. Similarly, the Lagrangian labeling on the middle seam belong to T_ϕ, but under such a rescaling it will lose this property. To deal with this problem, we will not modify (4) but rather choose the Liouville vector field we rescale with carefully. We will let it vary over the domain curve.

First, the function ρ of Section 2.2.1 (not to be confused with time shift map ρ_Σ) can be chosen such that the Liouville vector field defined by (2.12) is product type over $A \times M \subset T_\phi$ (i.e. it can be written as a sum of vector fields on A and M, under the trivialization of $T_\phi \to T_0$ over the annulus $A \subset T_0$) with T_0 component given by the restriction of Z_{T_0}. Let $Z_{\tilde{T}_\phi}$ denote the lift of this vector field to $\tilde{T}_0 \times M$. By definition, the restriction of $Z_{\tilde{T}_\phi}$ to $\tilde{A}_i \times M$ has the same $\tilde{A}_i \subset \tilde{T}_0$ component as the product type Liouville vector field given as the sum of Liouville vector fields corresponding to pull-back of λ_{T_0} and corresponding to λ_M. Denote the latter (product type) vector field on $\tilde{T}_0 \times M$ by $Z_{\tilde{T}_0} + Z_M$. As the $\tilde{T}_0$ components of $Z_{\tilde{T}_\phi}$ and $Z_{\tilde{T}_0} + Z_M$ are the same (over $\tilde{A}_i$), their linear interpolation is through Liouville vector fields that is of product type over $\tilde{A}_i$. Moreover, the flow of any Liouville vector field of type $(1-e)Z_{\tilde{T}_\phi} + e(Z_{\tilde{T}_0} + Z_M)$ (where $e \in [0,1]$) acts on $\tilde{A}_i \times M$ in the same way and preserves the product decomposition. We will denote the corresponding Liouville vector fields on the completions by the same notation.

The conditions (5) and (6) will be weakened to hold on a varying family of annuli over the strip. More precisely, let Σ be an open Riemann surface with boundary and fixed strip-like ends and let $f_\Sigma : \Sigma \to (0,\infty)$ be a map that is constant over the strip-like ends (hence bounded and bounded away from 0). For instance, one can choose a time shifting map $\rho_\Sigma : \partial\Sigma \to [1,\infty)$ which extends to a map $\Sigma \to [1,\infty)$ that is constant over the strip-like ends. Let $Z_{\tilde{T}_0}$ denote the Liouville vector field on $\widehat{\tilde{T}_0}$ corresponding to pull-back of λ_{T_0} under $\tilde{T}_0 \to T_0$ and let ψ^ρ denote the time $\log(\rho)$ Liouville flow of $Z_{\tilde{T}_0}$. Let

$$\tilde{A}_i^\Sigma \subset \Sigma \times \tilde{T}_0 \subset \Sigma \times \widehat{\tilde{T}_0} \tag{2.121}$$

denote the pre-image of $\Sigma \times \tilde{A}_i$ under $\psi^f : \Sigma \times \widehat{\tilde{T}_0} \to \Sigma \times \widehat{\tilde{T}_0}$. We repress f from the notation. For $\Sigma = \mathcal{S}_r^{(1)}$, we choose f to be an extension of a time-shifting map (denoted by $\rho_r^{(1)} : \mathcal{S}_r^{(1)} \to [1,\infty)$). Also, denote $\tilde{A}_i^\Sigma$ by $\tilde{A}_i^r$.

The condition we would like to replace (5) with is:

(5′) over $\tilde{A}_i^r \times \widehat{M} \subset \mathcal{S}_r^f \times \widehat{\tilde{T}_0} \times \widehat{M}$, (K, J) is of product type

The meaning of this for J is clear. For K, this means it can be written as a sum of $K_1 \in \Omega^1(\tilde{A}_i^r)$ (such that K_1 vanishes in $\widehat{\tilde{T}_0}$-directions) and $K_2 \in \Omega^1(\mathcal{S}_r^{(2)}, C^\infty(\widehat{M}))$ (see Remark ??).

Note 2.4.33. Before modifying the condition (6), we would like to make a clarification similar to Remark ??. As remarked the Floer data for the definitions of wrapped Fukaya category contains another piece of data: a time-shift map $\rho_\Sigma : \partial\Sigma \to [1,\infty)$. A time-shift map for the quilted strip can be defined to be a pair of functions $\rho_r^{(1)} : \partial\mathcal{S}_r^{(1)} \to [1,\infty)$ and $\rho_r^{(2)} : \partial\mathcal{S}_r^{(2)} \to [1,\infty)$ (that are constant on the boundaries of strip-like ends) such that the restrictions of $\rho_r^{(1)}$ and $\rho_r^{(2)}$ are the same over the lower boundaries of $\mathcal{S}_r^{(1)}$ and $\mathcal{S}_r^{(2)}$ (i.e. over the middle seam), and over the right and left ends. In this case, the rescaling is by $\log(\rho_r^{(1)})Z_{\tilde{T}_0}$ on the upper boundary of quilted strip $\mathcal{S}_r$, by $\log(\rho_r^{(2)})Z_M$ on the lower boundary of $\mathcal{S}_r$ (i.e. upper boundary of $\mathcal{S}_r^{(2)}$), and by $\log(\rho_r^{(1)})Z_{\tilde{T}_\phi} = \log(\rho_r^{(2)})Z_{\tilde{T}_\phi}$ on the seam.

We choose extensions of time-shift maps to $\mathcal{S}_r^{(1)}$ and $\mathcal{S}_r^{(2)}$ such that they are the same on the left/right strip-like ends and on $(\mathbb{R}\times[0,1/3])\cap\mathcal{S}_r^{(1)} = (\mathbb{R}\times[0,1/3])\cap\mathcal{S}_r^{(2)}$. The asymptotic consistency condition for such pairs of maps as r varies over the moduli is clear. We assume the choices are made consistently. Note, we have to assume the equality of sum of input and output weights. This is automatic when perturbation term is of type $H\gamma$ such that $d\gamma = 0$, but not in our case. We also assume that $\rho_r^{(i)}$ are bounded (uniformly over the moduli), and we make the consistent choices so that over the thin parts the difference between $\rho^{(i)}$ and the extended time shift maps one can obtain via gluing are exponentially close in terms of gluing length (this also requires us to make choices of extensions of time shift maps to the stable discs that are used to define A_∞-structure on $\mathcal{W}(T_0)$, $\mathcal{W}(M)$ and $\mathcal{W}(T_\phi)$).

As mentioned, the rescaling is made by an interpolation of $Z_{\tilde{T}_0} + Z_M$ and $Z_{\tilde{T}_\phi}$ over the left and right strip-like ends. We would like to include this as part of the data. More precisely, consider the left or right strip-like end of $\mathcal{S}_r^f$, which we denote by $\mathcal{S}_{r,\pm}^f$. Let $w = \log(\rho_r^{(1)}) = \log(\rho_r^{(2)})$ denote the weight on this end. We choose a translation invariant family of vector fields Z_r^{rsc} (we omit left/right end from the notation) on $\widehat{\tilde{T}}_0 \times \widehat{M}$ parametrized by $\mathcal{S}_{r,\pm}^f$ (equivalently a family parametrized by $[0,1]$) such that

1. Z_r^{rsc} is of type $w.((1-e)Z_{\tilde{T}_\phi} + e(Z_{\tilde{T}_0} + Z_M))$ at all points of $\mathcal{S}_{r,\pm}^f$, where $e \in [0,1]$

2. Z_r^{rsc} restricts to $w.(Z_{\tilde{T}_0} + Z_M)$ on $(\mathbb{R}\times[2/3,1])\cap\mathcal{S}_{r,\pm}^f$

3. Z_r^{rsc} restricts to $wZ_{\tilde{T}_\phi}$ on $(\mathbb{R} \times [0, 1/3]) \cap \mathcal{S}^f_{r,\pm}$

4. over the subset $\tilde{A}_i \times \widehat{M}$, $\tilde{T}_0$ component of Z_r^{rsc} is given by $wZ_{\tilde{T}_0}$

Let

$$\psi^{rsc} : \mathcal{S}^f_{r,\pm} \times \widehat{\tilde{T}_0} \times \widehat{M} \to \mathcal{S}^f_{r,\pm} \times \widehat{\tilde{T}_0} \times \widehat{M} \tag{2.122}$$

denote the time-1 flow of this vector field. Observe,

$$(\psi^{rsc})^{-1}(\mathcal{S}^f_{r,\pm} \times \tilde{A}_i \times \widehat{M}) = ((\mathcal{S}^f_{r,\pm} \times \widehat{\tilde{T}_0}) \cap \tilde{A}_i^r) \times \widehat{M} \tag{2.123}$$

This is true because for any $s \in \mathcal{S}^f_{r,\pm}$, over (shifts of) $\{s\} \times \tilde{A}_i \times \widehat{M}$ the vector field $(Z_r^{rsc})_s$ splits into components and the first component matches $\log(\rho_r^{(1)}) Z_{\tilde{T}_0} = wZ_{\tilde{T}_0}$. This piece of data (i.e. Z_r^{rsc}) will be used later.

Now, we go back to replacing condition (6). We would like to replace this condition with the condition that $\tilde{A}_i^r$ component can be obtained from an a-sloppy perturbation data on $\mathcal{S}_r^{(1)} \times \tilde{A}_i \times \widehat{M}$ by pulling-back via $\psi^{\rho_r^{(1)}}$. More precisely:

Definition 2.4.34. A Floer/perturbation datum on $\tilde{A}_i^\Sigma \subset \Sigma \times \widehat{\tilde{T}_0}$ is called **distorted** a**-sloppy** if it can be obtained by pulling-back an a-sloppy datum on $\Sigma \times \tilde{A}_i$ via ψ^f (we take $f = \rho_r^{(1)}$, when $\Sigma = \mathcal{S}_r^{(1)}$).

A Floer/perturbation datum on $\tilde{A}_i^\Sigma$ clearly means a 1-form on $\tilde{A}_i^\Sigma \subset \Sigma \times \widehat{\tilde{T}_0}$ that vanishes in vertical (i.e. $T(\widehat{\tilde{T}_0})$) directions and an almost complex structure in vertical tangent bundle (in other words, it is restriction of a datum on $\widehat{\tilde{T}_0}$ with domain Σ).

Unfortunately, distorted a-sloppy condition is not sufficient. If $\mathcal{S}_r$ is the strip without any markings on the horizontal boundaries and on the seam, then the rescaling is uniform (in $\tilde{T}_0$-direction). Hence, one can identify pseudo-holomorphic quilts for distorted a-sloppy data with pseudo-holomorphic quilts for a-sloppy data by applying ψ^{rsc} (extended by translation). The argument in Corollary 2.4.15 gives the desired compactness result. For more general quilts, we need to add a geometric boundedness condition.

For the definitions we refer the reader to [Gro15]. Briefly, one calls an almost complex structure b-bounded if the corresponding metric has injectivity radius bounded below by $\frac{1}{b}$ and sectional curvature bounded above by b. Groman defines geometrically bounded perturbation data (in the presence of a perturbation term) as the data for which the Gromov construction is geometrically bounded (see [Gro15, Section 5.2]). Recall for a given Riemann surface Σ (endowed with an area form compatible with the holomorphic structure), symplectic manifold X and perturbation datum on X with domain Σ, the Gromov construction is a symplectic structure and an almost complex structure on $\Sigma \times X$ such that the solutions $\Sigma \to X$ of the perturbed equation correspond to pseudo-holomorphic sections of $\Sigma \times X \to \Sigma$. For a map $u : \Sigma \to X$, we denote its graph by $(1, u)$. Define:

Definition 2.4.35. A perturbation datum on $\tilde{A}_i^\Sigma$ is b-bounded if the corresponding Gromov graph construction is b-bounded. It is geometrically bounded if it is b bounded for some b.

Remark 2.4.36. To clarify, to define the compatible symplectic structure and the metric on the Gromov construction, [Gro15] uses an extra parameter related to bounds C of $\{K, K\}$ and $d_S(K)$ in (7). However, it is possible to make uniform choices of data so that there is such a bound (that works for the choice). Indeed, one can simply use the bound C.

One can presumably obtain geometrically bounded datum on $\tilde{A}_i^\Sigma$ from such a datum on $\Sigma \times \tilde{A}_i$ by rescaling as well. As the rescaling constant is bounded, geometric boundedness constant b would change only by a constant.

Most important feature of b-bounded data is the monotonicity. We refer to [AL94] and [Gro15] for details. In general, if X is a symplectic manifold with a geometrically bounded almost complex structure, then there are constants $r(b), \epsilon(b)$ such that for any pseudo-holomorphic map $u : \Sigma \to X$ from a compact Riemann surface Σ, $r < r(b)$, $x \in \Sigma$ such that $u^{-1}(B(u(x); r)) \cap \partial\Sigma = \emptyset$, the energy of u on $u^{-1}(B(u(x); r))$ is greater than or equal to $\epsilon(b) r^2$. Applying this to Gromov construction, we obtain similar lower bounds.

Let b be a large positive number. The replacement for (6) is:

(6′) The $\tilde{T}_0$-component of (K, J) is distorted a-sloppy and b-bounded over $\tilde{A}_i^r$ for infinitely many positive and negative $i \in \mathbb{Z}$

The constants a and b do not depend on the component i. We require a constant b that works for a given choice of data parametrized by the moduli space $\overline{\mathcal{Q}(\mathbf{d})}$; however, we allow larger b for other choices. This is similar to assumption we made for C. We relaxed the condition from all $i \in \mathbb{Z}$ to infinitely many $i \in \mathbb{Z}$ so that space of choices would be (weakly) contractible. We still need the following uniformity assumption for the families of data:

Assumption 2.4.37. *Consider a family of data parametrized by a topological space X. At each point $x \in X$, there exists a neighborhood $x \in U \subset X$ and a subset $\Lambda \subset \mathbb{Z}$ that is unbounded on both sides such that (6′) holds at each point of U and for each i in Λ.*

Remark 2.4.38. Assumption 2.4.37 is called uniform i-boundedness in [Gro15]. On the $\mathfrak{tr} \times \phi$-invariant and $\mathfrak{tr}$-invariant parts, (6′) necessarily holds for all i.

Assume we make (asymptotically) consistent choices of data satisfying conditions (1)-(8) with the exception of (5) and (6). Assume instead (5′) and (6′). We have:

Lemma 2.4.39. *The solutions to perturbed Cauchy-Riemann equation with fixed boundary conditions and asymptotic conditions (on the ends) are contained in a fixed compact region of $\widehat{\tilde{T}}_0 \times \widehat{M}$. Hence, the moduli of stable quilted strips is compact.*

Proof. We follow the proof of [Gro15, Theorem 6.3] to make the necessary modifications to Lemma 2.4.30. The steps of [Gro15, Theorem 6.3] will be recalled as well for convenience of the reader.

First, there is no escape to conical ends as Lemma 2.4.27 still applies. Hence, we only need to show that the projection to $\widehat{\tilde{T}}_0$ is contained in a finite subdomain. Equivalently, the projection crosses through only finitely many $\tilde{A}_i$.

As before the topological energy is fixed and by Lemma 2.4.13, the geometric energy is bounded. Let E denote a bound on the geometric energy. Consider the thin-thick decomposition obtained by gluing, where the thick part has area uniformly bounded over the moduli and the strips of the thin part are endowed with standard metric. We first show there is a fixed compact subset of $\widehat{\tilde{T}_0}$ that contains the projection of the image of the thick part. Let Σ_T denote the thick part. Assume Σ_T is connected for simplicity, and extend it where it is glued to finite or semi-infinite strips of the thin part by fixed length (say 1). Denote the extended Riemann surface with boundary by Σ'_T. The area of Σ'_T is also bounded over the moduli, say by a constant $Area'_T$. Hence, the energy of the graph (in the Gromov construction) of any solution has energy bounded above by $Area'_T + E$. The boundary of Σ'_T naturally decomposes into two 1-manifolds with boundary. One is the boundary with Lagrangian labeling (denote this part by $\partial_l \Sigma'_T$), and the other is the shared boundary with the strips of the thin part (denote this part by $\partial_f \Sigma'_T$).

There is a compact subset $K_1 \subset \tilde{T}_0$ that contains the Lagrangians labeling the boundary of $\partial_l \Sigma'_T$. Assume there is no compact set containing the image of the thick part (for all solutions); hence, the solutions extend either to the left or to the right end of $\widehat{\tilde{T}_0}$. Without loss of generality, assume it is the right end. Hence, there exists an n_0 such that for any i_0 there exists a solution (u, v) (where u is the $\tilde{T}_0$-component) and points $x_i \in \Sigma_T$ $(i = 0, \ldots, i_0)$ such that $(x_i, u(x_i)) \in \tilde{A}^r_{n_0+i}$. Assume n_0 is sufficiently large so that $(\Sigma'_T \times K_1) \cap \tilde{A}^r_{n_0+i} = \emptyset$ for all $i \geq 0$.

One can assume there is a $\delta > 0$ such that the distance of $(x_i, u(x_i))$ to boundaries of $\tilde{A}^r_{n_0+i}$ is at least δ (δ is independent of i_0 etc.). The distance of x_i with $\partial_f \Sigma'_T$ is larger than 1, assume this is larger than δ. Also, the distance of $(1 \times u)(x_i)$ with $\Sigma'_T \times K_1$ is larger than δ. Hence, $(1, u)^{-1}(B((1, u)(x_i), \delta)) \cap \partial \Sigma'_T = \emptyset$. This implies that the energy of $(1, u)$ on $(1, u)^{-1}(B((1, u)(x_i), \delta))$ is at least $\epsilon(b)\delta^2$ and the energy of $(1, u)$ on Σ'_T is at least $\epsilon(b)\delta^2 i_0$. As $\epsilon(b), \delta$ do not depend on i_0, and the energy of $(1, u)$ (or rather $(1, u, v)$) on Σ'_T is bounded above, we have an upper bound on such i_0. Hence, there is a compact subset of $\widehat{\tilde{T}_0}$ that contains the image of the thick part (the reason it does not escape to conical end is integrated maximum principle as explained in the first paragraph). Call this compact set K.

Now the thin part: strip-like ends have boundary contained in the compact set K. Its end attached to thick part and the Hamiltonian chord it converges on its infinite end are also contained in K. As the Hamiltonian is a-sloppy there (or distorted a-sloppy but the distorting Liouville flow is uniform on the strip), it is contained in a slightly larger compact subset K' that depends on K, the energy bound and the bounds on the distortion. The proof of this is similar to Corollary 2.4.15 and follows from Lemma 2.4.7.

The same argument does not immediately apply to other components of the thin part since we impose asymptotic consistency rather than strict consistency (see Remark 2.4.40 however). On the other hand, although the map does not satisfy perturbed Cauchy-Riemann equation with respect to glued data, the proof of Lemma 2.4.7 still provides the same lower bound on the topological energy with respect to glued data (up to a constant related to homogeneous rescaling). Call this lower bound ϵ'. Moreover, we still have Assumption 2.4.29; therefore, the difference of the chosen perturbation term on these components and the one one could obtain by gluing are exponentially (C^1-) small in terms of gluing length. Hence, the difference of the topological energies computed with respect to these perturbation terms converges to 0 as one increases the gluing length. In particular, for large gluing length (i.e. close to the strata), this difference becomes less than $\epsilon'/3$. Similarly, the exponential decay conditions near the strata of moduli of strips ensure that the integral of the curvature term (of the chosen data) over the component of the thin part converges to 0 near the strata (the curvature is 0 for the glued data, it is small for the data exponentially close in terms of the gluing length). Hence, close to the strata the difference between geometric and topological energies for the original data is less than $\epsilon'/3$. In summary, the geometric energy with respect to chosen data still increases by a fixed constant $\epsilon'/3$ each time it passes through one of $\tilde{A}_i^r \times \widehat{M}$ and once an energy bound is fixed, these components of the thin part are also contained in a fixed region. This completes the proof. □

Remark 2.4.40. One other way to bound the thin parts other than the strip-like ends is the following: we assume the consistency of the data only asymptotically. However, this is for the purposes of transversality and one does not need this when gluing time-shift maps $(\rho_r^{(1)}, \rho_r^{(2)})$. In other words, we can assume that this part of the data is the same as what one would obtain by gluing (near the strata of $\overline{\mathcal{Q}(\mathbf{d})}$). Hence, this part cannot escape to infinity for the same reason as the strip-like ends.

Existence and contractability of such consistent choices is similar to Lemma 2.4.21, Lemma 2.4.28. One only has to incorporate the argument in [Gro15, Theorem 4.6].

Remark 2.4.41. The argument in [Gro15, Theorem 4.6] is not necessary while constructing the data on $\mathfrak{tr}\times\phi$ and $\mathfrak{tr}$-invariant parts: while interpolating through $\mathfrak{tr}\times\phi$, resp. $\mathfrak{tr}$-invariant data, the bound b can be kept small on compact subsets; hence, on all $\tilde{A}_i^r$.

The rest is standard: one can identify the Floer chain groups with respect to data satisfying conditions (1)-(8) with labeling (L_0, L_0', L_0'') (where $L_0 \subset \widehat{T}_0$ with a fixed lift, $L_0'' \subset \widehat{M}$ and $L_0'' \subset \widehat{T_\phi}$ as before) with Floer chain groups with respect to rescaled data where conditions (5),(6) are generalized by (5′),(6′) with labeling $(\psi^{\rho_r^{(1)},*}(L_0), \psi^{\rho_r^{(2)},*}(L_0'), \psi^{rsc,*}(L_0''))$. Here, $\psi^{\rho_r^{(1)}}$, resp. $\psi^{\rho_r^{(2)}}$ denotes the time $\log(\rho_r^{(1)}) = w$, resp. time $\log(\rho_r^{(2)}) = w$ flow of $Z_{\tilde{T}_0}$, resp. Z_M. Note $\psi^{rsc,*}(L_0 \times L_0')$ is equal to product of rescalings of L_0 and L_0' (by Z_{T_0}, resp. Z_M by weight $\rho^{(1)} = \rho^{(2)} = w$). Moreover, $\psi^{rsc,*}(L_0'')$ is equal to rescaling by the Liouville vector field on T_ϕ by the same weight. Hence, one can also thought of the rescaling labeling on the folded strip: $(L_0 \times L_0', L_0'')$ turns into $(\psi^{rsc,*}(L_0 \times L_0'), \psi^{rsc,*}(L_0''))$.

An important point in the identification of Floer chain complexes is the following: since the rescaling vector field is time dependent on the ends of quilted strips, its time $(-\log(w))$ flow, $(\psi^{rsc})^{-1}$, does not map the chords of the original time dependent Hamiltonian (say H) to chords of $\psi^{rsc,*}H$. However, one can still realize the image of the chord under $(\psi^{rsc})^{-1}$ as the chords of an Hamiltonian satisfying $\mathcal{W}^2$ (and other) condition(s). Hence, the complex formed with this alternative Hamiltonian is naturally identified by ψ^{rsc} to complex $\mathfrak{M}(L'', L, L')$.

To see the images of these chords are given as chords of other time dependent Hamiltonians, let Z_t, $t \in [0,1]$ denote the time dependent family of Liouville vector fields of type $(1-e)Z_{\tilde{T}_\phi} + e(Z_{\tilde{T}_0} + Z_M)$ that we choose for the conical end (i.e. $w^{-1}Z^{rsc}$). Let ψ_t denote the time $\log(w)$-flow of Z_t. Then, $\psi_0^{-1} \circ \psi_t =: g_t$ is a symplectic isotopy.

Indeed, since Z_t can be written as $Z_0 + X_{m_t}$ (all Z_t can be obtained by extending $Z_{\tilde{T}_0} + Z_M + (1-e)X_{\rho(s)K}$, see Section 2.2.1 for the notation), g_t is an Hamiltonian isotopy.

Let $H = H_t$ be the time dependent family of Hamiltonians (so that the perturbation term is given by $K = H_t dt$) and ϕ_t denote its Hamiltonian flow. ψ^{rsc} is given by $\psi_t = \psi_0 g_t$ at t and thus it maps the trajectory $\{\phi_t\psi_0^{-1}(x) : t \in [0,1]\}$ of $\{H_t\}$ onto $\{\psi_0 g_t \phi_t \psi_0^{-1}(x)\}$. Since g_t is also Hamiltonian, $g_t\phi_t$ is an Hamiltonian isotopy. The conjugate of an Hamiltonian isotopy by ψ_0 is an Hamiltonian isotopy (since ψ_0 only rescales the symplectic form). In other words, $\psi_0 g_t \phi_t \psi_0^{-1}$ is Hamiltonian and $\psi_0 g_t \phi_t \psi_0^{-1}(x)$ is a trajectory of this Hamiltonian.

One has to check $\mathcal{W}^2$-condition still holds for the Hamiltonian corresponding to isotopy $g_t\phi_t = \psi_0^{-1}\psi_t\phi_t$ and this could be checked directly from the vector field generating the isotopy. Since, $\psi_0^{-1}\psi_t$ is identity on $\hat{\tilde{T}}_0 \times (\widehat{M} \backslash M)$, Hamiltonian is the same as $\{H_t\}$ on this part. Over $\tilde{B}_i \times \widehat{M}$, the vector fields Z_t have a product decomposition as $Z_{\tilde{T}_0} + (Z_M + \mu(s)X_K)$ (μ is a function that is constant near the puncture, note here K is as in Section 2.2.1 and this expression is not global). In particular, $\tilde{B}_i$ component is independent of t. Hence, the isotopies ψ_0, ψ_t and ϕ_t also have product decomposition, where the first two have the same $\tilde{B}_i$ component and ϕ_t has $\tilde{B}_i$-component equal to that of a cylindrical perturbation term as in Definition 2.4.18. Thus, the composition $\psi_0^{-1}\psi_t\phi_t$ has $\tilde{B}_i$-component generated by a cylindrical perturbation term (recall that quadratic condition was built into our definition of cylindrical perturbation term). Similarly, one can check the perturbation term for rescaled labeling still satisfies distorted a-sloppy condition. Indeed, this is implied by the assumption that $Z_{\tilde{T}_\phi}$ is of product type near $\tilde{A}_i \times M$ and it has the same $\tilde{T}_0$ component as $Z_{\tilde{T}_0} + Z_M$.

After the identification, define $\mathfrak{M}$ using the perturbation data satisfying modified conditions and labeling given by $\psi^{\rho_r^{(1)},*}(L_i)$ on the upper boundary of $\mathcal{S}_r^{(1)}$, by $\psi^{\rho_r^{(2)},*}(L_j')$ on the upper boundary of $\mathcal{S}_r^{(2)}$ (i.e. lower boundary of the quilted strip $\mathcal{S}_r$), and by $\psi_{Z_{T_\phi}}^{\rho_r^{(1)},*}(L_k'') = \psi_{Z_{T_\phi}}^{\rho_r^{(2)},*}(L_k'')$ on the seam (where $\psi_{Z_{T_\phi}}^{\rho}$ denotes time $\log(\rho)$-flow of the Liouville vector field on T_ϕ).

Rescaling is similar for counts defining bimodule maps from $\mathfrak{M}_{L\times_\phi L'}$ to twisted

Yoneda bimodules and will not be written separately.

2.5 Examples of symplectic manifolds satisfying Assumption 2.1.2 and applications

In this section, we give a large class of examples satisfying Assumption 2.1.2. We search for examples among Liouville manifolds with periodic Reeb flow since it is easier to compute the Conley-Zehnder indices. More specifically, we will confine ourselves to complements of smooth ample divisors. For Assumption 2.1.2, we need

1. Vanishing first and second Betti numbers

2. Reeb orbits with sufficiently large degree

Let us start by addressing (1):

Lemma 2.5.1. *Let X be a smooth and projective variety and $D \subset X$ be a smooth, connected hypersurface that is given as a transverse hyperplane section of a projective embedding. Further assume:*

1. $b_1(X) = b_1(D) = 0$

2. $b_2(X) = b_2(D) = 1$

Let $M = X \setminus ND \simeq X \setminus D$, where ND is a tubular neighborhood of D. Then, $b_1(M) = b_2(M) = 0$.

Proof. First note $H^*(X, D) \cong H^*(M, \partial M)$ by excision. Consider the long exact sequence

$$\begin{aligned} H^0(X,D) \to H^0(X) \xrightarrow{\cong} H^0(D) \to H^1(X,D) \to H^1(X) \xrightarrow{\cong} H^1(D) \\ \to H^2(X,D) \to H^2(X) \hookrightarrow H^2(D) \dots \end{aligned} \tag{2.124}$$

($H^2(X) \to H^2(D)$ is injective since $H^2(X)$ is one dimensional and this map carries a Kähler class on X to one on D). Thus, $H^1(M, \partial M) = H^1(X, D) = 0$, and

$H^2(M, \partial M) = H^2(X, D) = 0$. This implies $H^1(M) \to H^1(\partial M)$ is an isomorphism, and $H^2(M) \to H^2(\partial M)$ is injective by a similar long exact sequence. Hence, it is sufficient to prove $H^1(\partial M) = H^2(\partial M) = 0$.

Consider Serre spectral sequence for the fibration $S^1 \hookrightarrow \partial M \to D$ given by

$$E_2^{pq} = H^p(D, \{H^q(S^1)\}) \Rightarrow H^{p+q}(\partial M) \tag{2.125}$$

where $\{H^q(S^1)\}$ denotes the local system formed by the cohomology of each fiber. $\partial M \to D$ is the circle bundle of a complex line bundle; thus, it is an oriented bundle and $\{H^1(S^1)\}$ is the trivial local system (i.e. constant sheaf). Same holds for $\{H^0(S^1)\}$ easily. This implies $p = 1, q = 0$ and $p = 1, q = 1$ terms in the E_2-page vanish; thus, $H^1(\partial M)$ and $H^2(\partial M)$ can be obtained as the cohomology of

$$H^0(D, H^1(S^1)) \xrightarrow{d_2} H^2(D, H^0(S^1)) \tag{2.126}$$

where d_2 is the differential of E_2-page. Both groups are of rank 1, and to finish the proof we only need $d_2 \neq 0$.

By assumption, there exist a projective embedding $X \hookrightarrow \mathbb{P}^N$ and a hyperplane $\mathcal{H} \subset \mathbb{P}^N$ such that $X \pitchfork \mathcal{H}$ and $D = X \cap \mathcal{H}$. Then, $N_{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^N}(\mathcal{H})|_{\mathcal{H}}$ restricts to $N_D \cong \mathcal{O}_X(D)|_D$. Let $S_{\mathcal{H}}$ and $S_D \cong \partial M$ denote the corresponding circle bundles. There is a similar spectral sequence for $S_{\mathcal{H}}$ as well and we have a diagram

$$\begin{array}{ccc} H^0(D, H^1(S^1)) & \xrightarrow{d_2} & H^2(D, H^0(S^1)) \\ \uparrow & & \uparrow \\ H^0(\mathcal{H}, H^1(S^1)) & \xrightarrow{d_2} & H^2(\mathcal{H}, H^0(S^1)) \end{array} \tag{2.127}$$

by naturality. In previous considerations, we can replace X by $\mathbb{P}^N$ and D by $\mathcal{H}$, and conclude

$$H^0(\mathcal{H}, H^1(S^1)) \xrightarrow{d_2} H^2(\mathcal{H}, H^0(S^1))$$

computes $H^1(S_{\mathcal{H}})$ and $H^2(S_{\mathcal{H}})$. But we know $H^1(\mathbb{P}^N \setminus \mathcal{H}) \cong H^1(S_{\mathcal{H}})$ (we proved $H^1(M) \cong H^1(\partial M)$, apply this to the case $X = \mathbb{P}^N, D = \mathcal{H}$). Hence, $H^1(S_{\mathcal{H}}) = 0$. A simpler way to see this is: topologically $\mathcal{O}_{\mathbb{P}^N}(\mathcal{H})|_{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^N}(-\mathcal{H})|_{\mathcal{H}}$ which is just the circle bundle of the tautological bundle over $\mathcal{H}$. It is easy to see this is homeomorphic to

$$S^{2N-1} \to S^{2N-1}/S^1$$

hence the total space has vanishing H^1.

In summary, the lower horizontal arrow in (2.127) cannot vanish. The right vertical arrow cannot vanish since it carries at least one Kähler class to one on D. Left vertical arrow is an isomorphism by connectedness of D. All the groups in (2.127) are 1-dimensional; thus, the composition does not vanish and neither does the upper horizontal d_2. This completes the proof. □

To make sure $SH^1(M) = SH^2(M) = 0$, we need Reeb orbits to be of sufficiently large degree. More precisely, we will use the spectral sequence in [Sei08b] whose E_1-page is given by:

$$E_1^{pq} = \begin{cases} H^q(M) & p = 0 \\ H^{p(1-\mu)+q}(\partial M) & p < 0 \\ 0 & p > 0 \end{cases} \tag{2.128}$$

and which converges to symplectic cohomology of M (note that we made a degree shift on $SH^*(M)$ by n so that $SH^*(M) \cong HH^*(\mathcal{W}(M))$ by [Gan12, Theorem 1.1]). Here, $\mu \in 2\mathbb{Z}$ is a Conley-Zehnder type index defined in [Sei08b].

Assume we are in the setting of Lemma 2.5.1 and $K_X = \mathcal{O}(mD)$. Then we have:

Lemma 2.5.2. $\mu = -2m - 2$.

Proof. Recall how μ is defined: given a Liouville domain M with 1-periodic Reeb flow at its contact boundary, choose a trivialization of K_M. Let x be a Reeb orbit on ∂M. We obtain a trivialization of x^*TM as a symplectic bundle. Hence, the Reeb flow defines a path in $Sp(2n)$ and the class of this path in $\pi_1(Sp(2n)) \cong \mathbb{Z}$ is the number $\mu/2$.

To compute μ, identify a neighborhood of D with a neighborhood of zero section of $\mathcal{N}_D$ (the holomorphic normal bundle of D). Let $d \in D$ and let F denote a fiber of

$\mathcal{N}_D$. Then $T_X|_F \cong F \times (T_{D,d} \oplus \mathcal{N}_{D,d}) =: E$ as a symplectic bundle. In other words, it is the trivial bundle with fiber $T_{D,d} \oplus \mathcal{N}_{D,d}$. Under this trivialization, the circle action induced by Reeb vector field is

$$\begin{aligned} F \times (T_{D,d} \oplus \mathcal{N}_{D,d}) &\longrightarrow F \times (T_{D,d} \oplus \mathcal{N}_{D,d}) \\ (\alpha, v, v') &\longmapsto (z^{-1}\alpha, v, z^{-1}v') \end{aligned} \tag{2.129}$$

If we trivialize using a section Ω of K_X, the section can be chosen to have vanishing order m along D. Hence, the dual section has vanishing order $(-m)$-along D. Therefore, the map

$$\begin{aligned} f : SF \times (T_{D,d} \oplus \mathcal{N}_{D,d}) &\longrightarrow SF \times (T_{D,d} \oplus \mathcal{N}_{D,d}) = E|_{SF} \\ (\alpha, v, v') &\longmapsto (\alpha, v, \alpha^{-m}v') \end{aligned} \tag{2.130}$$

is the new trivialization (symplectic trivialization obtained by using Ω). Here, SF is the unit circle in F. The right hand side is considered to be the restricted bundle, and the left hand side is considered to be a trivial bundle, and the trivialization map is the framing (if E' is a vector bundle over F', then a trivialization is a map $F' \times V \to E'$ for a vector space V). The S^1-action is by z^{-1} on the right hand side. In other words,

$$z : (\alpha, v, \alpha^{-m}v') \mapsto (z^{-1}\alpha, v, z^{-1}\alpha^{-m}v') \tag{2.131}$$

or

$$f(\alpha, v, v') \mapsto f(z^{-1}\alpha, v, z^{-m-1}v') \tag{2.132}$$

More diagrammatically

$$\begin{array}{ccc} SF \times (T_{D,d} \oplus \mathcal{N}_{D,d}) & \longrightarrow & E|_{SF} \\ \downarrow & & \downarrow \\ SF \times (T_{D,d} \oplus \mathcal{N}_{D,d}) & \longrightarrow & E|_{SF} \end{array} \qquad \begin{array}{ccc} (\alpha, v, v') & \stackrel{f}{\longmapsto} & (\alpha, v, \alpha^{-m}v') \\ \downarrow & & \downarrow \\ (z^{-1}\alpha, v, z^{-m-1}v') & \stackrel{f^{-1}}{\longleftarrow} & (z^{-1}\alpha, v, z^{-1}\alpha^{-m}v') \end{array} \tag{2.133}$$

Hence, the path in $U(n) \subset Sp(2n)$ induced by the circle action is

$$S^1 \longrightarrow U(T_{D,d} \oplus \mathcal{N}_{D,d}) \tag{2.134}$$

$$z \longmapsto id_{T_{D,d}} \oplus z^{-m-1} id_{\mathcal{N}_{D,d}}$$

If we compose this map with det, we obtain a map of degree $-m-1$. Thus, $\mu/2 = -m-1$, and $\mu = -2m-2$. □

Combining the spectral sequence (2.128) and Lemmas 2.5.1 and 2.5.2, we obtain:

Corollary 2.5.3. *Assume $m > 0$. Then, $SH^*(M)$ vanishes for $* < 0$ or $* = 1, 2$ and it is 1 dimensional for $* = 0$.*

We also have:

Lemma 2.5.4. *Assume $m > 0$. Then, $\mathcal{W}(M)$ is proper in each degree, i.e. $HW(L, L')$ is finite dimensional in each degree and bounded below for any pair of objects of $\mathcal{W}(M)$.*

Proof. Consider the generating subcategory of $\mathcal{W}(M)$ spanned by cocores. One can arrange the cocores to be cylindrical; hence, their intersections with the contact boundary (with periodic Reeb flow) are Legendrian submanifolds. Let L_0 and L_1 be two such Lagrangians. The generators of $CW(L_0, L_1)$ are given by

1. finitely many chords in the interior
2. finitely many chords in the contact end of length less than 1. Note each such chord lives on a unique Reeb orbit because of periodicity
3. chords obtained by concatenating a chord of length less than 1 with the Reeb orbit it lives on k times (where $k \in \mathbb{Z}_{\geq 0}$)

Let x be an orbit living on a Reeb orbit, and assume y is obtained by concatenating the Reeb orbit k-times. A straightforward calculation shows $deg(y) = deg(x) - k\mu$, where μ is as before (i.e. as in [Sei08b]). By Lemma 2.5.2, $\mu = -2m - 2 < 0$. Thus, the degree of y grows as one increases k. In other words, there are only finitely many generators of $CW(L_0, L_1)$ of degree less than d (for every d). This completes the proof. □

Combining the results of this section, we have:

Proposition 2.5.5. *Assume the pair (X, D) satisfies the assumptions of Lemma 2.5.1 and $K_X \cong \mathcal{O}(mD)$ such that $m > 0$. Then M (the Liouville domain corresponding to $X \setminus D$) satisfies Assumption 2.1.2.*

Corollary 2.5.6. *Let X be a smooth hypersurface in $\mathbb{CP}^{n+1}$ (for $n \geq 4$) of degree at least $n + 3$ (i.e. of general type) and D be a transverse hyperplane section. Let $\widehat{M} = X \setminus D$. Then M satisfies Assumption 2.1.2.*

Proof. This follows from Lefschetz hyperplane theorem and Proposition 2.5.5. □

Remark 2.5.7. As commented in the Section 2.1, powers of Dehn twists act non-trivially on $\mathcal{W}(M^{2n})$, when $n > 1$; hence giving us applications of the Theorem. However, the least trivial examples are when ϕ is (pseudo-)isotopic to identity relative to ∂M. We are not aware of such examples when $n = dim(M)/2$ is odd, but powers of Dehn twists give such examples when n is even. Indeed, the order of a Dehn twist in mapping class group divides $4|\Phi_{2n+1}|$, where Φ_{2n+1} is the group of homotopy spheres of dimension $2n + 1$ (see [Kry07],[KK05]).

Now, we will show that T_ϕ and $T_0 \times M$ cannot be distinguished by their symplectic cohomology for a large class of examples provided by Proposition 2.5.5. More precisely:

Lemma 2.5.8. *Let (X, D) be as in Proposition 2.5.5 and ϕ be an even power of a Dehn twist along a spherical Lagrangian in M. Assume $n = dim_{\mathbb{C}}(X) = dim_{\mathbb{R}}(X)/2$ is even. Then, $SH^*(T_\phi) \cong SH^*(T_0 \times M)$ as vector spaces, if $m+1 > n$. In particular, this holds if X is an hypersurface in $\mathbb{CP}^{n+1}$ $(n \geq 4)$ of degree larger than $2n + 1$.*

Proof. First, note that one can recover $SH^*(T_\phi)$ as a vector space from $SH^*(M)$ and action of ϕ on $SH^*(M)$. This follows for instance by combining [Kar18, Prop 5.13], [Gan12, Theorem 1.1], and Theorem 2.1.12. Hence, it is sufficient to show that ϕ acts trivially on $SH^*(M)$ if $m+1>n$.

One obtains the spectral sequence (2.128) by using the length filtration on the Reeb orbits. Notice that ϕ acts trivially on $p=0$ terms (i.e. on $H^*(M)$) by Picard-Lefschetz formula, and it acts trivially on $p<0$ terms since it is compactly supported (and continuation maps defining ϕ action on $SC^*(M)$ are length decreasing).

Since M is a Weinstein domain of dimension $2n$, it has the homotopy type of an n-dimensional CW complex, and its cohomology is supported in degree $0,\dots,n$. Hence, $p=0^{th}$ column of (2.128) is supported in degrees $0,\dots,n$. Similarly, the cohomology of ∂M is supported in degrees $0,\dots,2n-1$; therefore, for $p<0$, (p,q) term can be non-zero only if

$$2n-1\geq(1-\mu)p+q\geq 0 \tag{2.135}$$

which is equivalent to

$$2n-1+\mu p\geq p+q\geq \mu p \tag{2.136}$$

In other words, p^{th} column is supported in degree $\mu p,\dots,2n-1+\mu p$. By assumption, $\mu(p-1)>(2n-1+\mu p)+1$; hence, terms of $(p-1)^{th}$ column and p^{th} column do not interact. Same holds with $(-1)^{th}$ column and 0^{th} column as well. Hence, the spectral sequence degenerates in E_1-page and the action of ϕ on each term is trivial. This implies that ϕ acts trivially on $SH^*(M)$ (in summary, one can filter the complex $SC^*(M)$ by length and the action of ϕ is trivial on the cohomology of associated graded. Moreover, orbits of different length differ at least by degree 2, implying the desired result). □

Remark 2.5.9. Presumably, when the degree of the hypersurface X is sufficiently large, $SH^*(T_\phi)$ and $SH^*(T_0\times M)$ agree as BV-algebras as well. Indeed, we strongly believe for any finitely many set of BV_∞-operations, one can increase the degree of the hypersurface to produce examples where $SH^*(T_\phi)$ and $SH^*(T_0\times M)$ are isomorphic with an isomorphism respecting these operations (for instance, one can produce examples where symplectic cohomologies are the same as A_n-algebras). We do not yet know how to prove $SH^*(T_\phi)$ and $SH^*(T_0 \times M)$ are the same (or different) as BV_∞-algebras; however, we believe it is not possible to prove a statement that would imply Theorem 2.1.1 or Corollary 2.1.5 just by computing closed string invariants.

Appendix A

Proof of Theorem 2.1.12 using the gluing formula for wrapped Fukaya categories

One can give an alternative proof of Theorem 2.1.12 using the gluing formula in [GPS18]. The proof is easy after the algebraic setup given in Section 2.3, and we sketch this proof in this appendix. A notational remark: in [GPS17] Liouville sectors are defined with their infinite ends; however, we omit the completions from the notation throughout this section similar to Section 2.3.3 (i.e. we write M instead of $\widehat{M}$, T_0 instead of $\widehat{T_0}$ etc.).

Recall the notation from Section 2.3.3: T denotes the 1-handle that is shown in yellow in Figure 2-2 and N denotes $\overline{T_0 \setminus T}$ (more precisely, one has to consider the completions). See also Figure 2-5. As a sector, T is isomorphic to $T^*[0,1]$ and $\mathcal{W}(T) \simeq \mathbb{C}$. Similarly, N is equivalent to a cylinder with one stop on each boundary components and $\mathcal{W}(N)$ is derived equivalent to $D^b(Coh(\mathbb{P}^1))$. To calculate $\mathcal{W}(\widehat{T_\phi})$, decompose T_ϕ into two sectors $T\times M$ and $N\times M$. In other words, $T_\phi = T\times M \cup N\times M$ and these subsectors intersect on $(T^*[0,1]\times M)\sqcup(T^*[0,1]\times M) = T^*[0,1]\times(M\sqcup M)$. Since M is a Weinstein domain, $M\sqcup M$ and horizontal completions of $T\times M$, $N\times M$ are Weinstein. Moreover, T_ϕ is a Liouville domain (as opposed to a more general Liouville sector). Therefore, the assumptions of [GPS18, Cor 1.21] are satisfied, and one has a homotopy push-out diagram similar to (2.91):

$$\begin{array}{ccc} \mathcal{W}(N \times M) & \longrightarrow & \mathcal{W}(T_\phi) \\ \uparrow & & \uparrow \\ \mathcal{W}(T^*[0,1] \times M) \coprod \mathcal{W}(T^*[0,1] \times M) & \longrightarrow & \mathcal{W}(T^*[0,1] \times M) \simeq \mathcal{W}(T \times M) \end{array} \tag{A.1}$$

Therefore, one has a homotopy coequalizer diagram

$$\mathcal{W}(T^*[0,1] \times M) \rightrightarrows \mathcal{W}(N \times M) \to \mathcal{W}(T_\phi) \tag{A.2}$$

The map $\mathcal{W}(N \times M) \to \mathcal{W}(T_\phi)$ is induced by the inclusion and the maps $\mathcal{W}(T^*[0,1] \times M) \rightrightarrows \mathcal{W}(N \times M)$ are induced by $j_0 \times 1_M$ and $j_1 \times \phi$ (recall j_0, j_1 were used to denote both inclusion maps from T into N shown in Figure 2-5 and the functors induced by these inclusions). By the Künneth theorem [GPS18, Theorem 1.5], $\mathcal{W}(T^*[0,1] \times M) \simeq \mathcal{W}(T^*[0,1]) \otimes \mathcal{W}(M)$ and $\mathcal{W}(N \times M) \simeq \mathcal{W}(N) \otimes \mathcal{W}(M)$ (note we again need Weinstein property for M for the Künneth map to be essentially surjective). Under these quasi-equivalences (A.2) can be identified with the diagram

$$\mathcal{W}(T^*[0,1]) \otimes \mathcal{W}(M) \rightrightarrows \mathcal{W}(N) \otimes \mathcal{W}(M) \to \mathcal{W}(T_\phi) \tag{A.3}$$

where the arrows are $j_0 \otimes 1_{\mathcal{W}(M)}, j_1 \otimes \phi$. Moreover, as in Section 2.3.3, $\mathcal{W}(T^*[0,1]) \simeq \mathbb{C}$ and $\mathcal{W}(N) \simeq \mathcal{O}(\mathbb{P}^1)_{dg}$ and j_0, j_1 turn into i_0, i_∞ under these identifications. In summary, we have a homotopy coequalizer diagram

$$\mathcal{W}(M) \rightrightarrows \mathcal{O}(\mathbb{P}^1)_{dg} \otimes \mathcal{W}(M) \to \mathcal{W}(T_\phi) \tag{A.4}$$

where the arrows $\mathcal{W}(M) \rightrightarrows \mathcal{O}(\mathbb{P}^1)_{dg} \otimes \mathcal{W}(M)$ are given by $i_0 \otimes 1_{\mathcal{W}(M)}$ and $i_\infty \otimes \phi$. As the situation is symmetric, one can swap i_0 and i_∞ or replace ϕ by ϕ^{-1} (different identifications may lead to this).

Let us now describe M_ϕ as a similar homotopy pushout. Let $\mathcal{A} \simeq \mathcal{W}(M)$. Recall the diagram (2.79) and the quasi-isomorphism (2.80) from the homotopy coequalizer to $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ are strictly $\mathfrak{tr}$ equivariant. Hence, there exists a quasi-equivalence

$$\big(hocolim(Pt_\infty \rightrightarrows \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg}) \otimes \mathcal{A}\big)\#\mathbb{Z} \xrightarrow{\simeq} \big(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}\big)\#\mathbb{Z} = M_\phi \tag{A.5}$$

Following [GPS18], we described the homotopy coequalizer as a localization of the Grothendieck construction (see (2.83)). Hence, M_ϕ is equivalent to $((C^{-1}\mathcal{G}r)\otimes\mathcal{A})\#\mathbb{Z}$. It is easy to see that localization commutes with tensoring with $\mathcal{A}$, i.e.

$$(C^{-1}\mathcal{G}r) \otimes \mathcal{A} \simeq (C \otimes 1)^{-1}(\mathcal{G}r \otimes \mathcal{A}) \tag{A.6}$$

where $C \otimes 1$ is the set of morphisms $\{(c \otimes 1_{L'}) : c \in C, L' \in ob(\mathcal{A})\}$ (in the absence of strict units, choose a ϕ-equivariant set of cohomological units). Moreover, as C is $\mathfrak{tr}$-invariant, localization commutes with smash product as well. Hence,

$$M_\phi \simeq (C \otimes 1)^{-1}\big((\mathcal{G}r \otimes \mathcal{A})\#\mathbb{Z}\big) \tag{A.7}$$

It is easy to see $\mathcal{G}r \otimes \mathcal{A}$ is the Grothendieck construction for the diagram

$$Pt_\infty \otimes \mathcal{A} \rightrightarrows \mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \otimes \mathcal{A} \tag{A.8}$$

and $(\mathcal{G}r \otimes \mathcal{A})\#\mathbb{Z}$ is the Grothendieck construction for

$$(Pt_\infty \otimes \mathcal{A})\#\mathbb{Z} \rightrightarrows (\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \otimes \mathcal{A})\#\mathbb{Z} \tag{A.9}$$

Hence, by (A.7), M_ϕ is the homotopy coequalizer of the diagram (A.9). The $\mathbb{Z}$ action is still by $\mathfrak{tr} \otimes \phi$; however, translation carries components of Pt_∞, resp. $\mathbb{P}^1 \times \mathbb{Z}$ to different components. Hence,

$$(Pt_\infty \otimes \mathcal{A})\#\mathbb{Z} \simeq \mathcal{A} \text{ and } (\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})_{dg} \otimes \mathcal{A})\#\mathbb{Z} \simeq \mathcal{O}(\mathbb{P}^1)_{dg} \otimes \mathcal{A} \tag{A.10}$$

If there were no $\mathcal{A}$ in (A.9), the arrows would become i_0 and i_∞ under the identification (A.10), as remarked in Section 2.3.3. On the other hand, as $\mathbb{Z}$-action is given by $\mathfrak{tr}\otimes\phi$, the arrows in (A.9) become different under the identification (A.10). More precisely, one of them becomes $i_0 \otimes 1_{\mathcal{A}}$ and the other one becomes $i_\infty \otimes \phi$. Hence, we have a coequalizer diagram

$$\mathcal{A} \rightrightarrows \mathcal{O}(\mathbb{P}^1)_{dg} \otimes \mathcal{A} \to M_\phi \tag{A.11}$$

where the arrows $\mathcal{A} \rightrightarrows \mathcal{O}(\mathbb{P}^1)_{dg} \otimes \mathcal{A}$ are given by $i_0 \otimes 1_{\mathcal{A}}$ and $i_\infty \otimes \phi$. Notice under different identifications of $Pt_\infty \# \mathbb{Z} \simeq \mathbb{C}$ and $\mathcal{O}(\mathbb{P}^1 \times \mathbb{Z})\#\mathbb{Z} \simeq \mathcal{O}(\mathbb{P}^1)$, these arrows could turn into $i_0 \otimes \phi^{-1}$ and $i_1 \otimes 1_{\mathcal{A}}$ either.

By (A.4) and (A.11), both $\mathcal{W}(T_\phi)$ and M_ϕ are the homotopy coequalizers of equivalent diagrams. Hence, they are equivalent, completing other proof of Theorem 2.1.12.

Note A.0.1. One can see the commuting of smash products and localization in two ways: the first is writing explicit zigzags using the definition in [GPS18]. More precisely, let $\mathcal{B}$ be a dg category with a strict $\mathbb{Z}$ action and let C be a $\mathbb{Z}$-invariant set of morphisms. Then, by adding cones to $\mathcal{B}$ (and extending the action), the problem turns into showing that $\#\mathbb{Z}$ and quotient by a $\mathbb{Z}$-invariant subcategory commutes (i.e. $(\mathcal{B}/\mathcal{B}_0)\#\mathbb{Z} \simeq (\mathcal{B}\#\mathbb{Z})/(\mathcal{B}_0\#\mathbb{Z})$), which can be achieved using the explicit model in [LO06], and [Syl16]. The hom-complexes for $(\mathcal{B}\#\mathbb{Z})/(\mathcal{B}_0\#\mathbb{Z})$ may look larger. To show it is equivalent to $(\mathcal{B}/\mathcal{B}_0)\#\mathbb{Z}$, one has to first extend the category $\mathcal{B}$ to a quasi-equivalent category by adding objects (g, b) (for all $g \in \mathbb{Z}, b \in ob(\mathcal{B}_0)$) equivalent to gb. Then the quotient of extended categories (with objects added to $\mathcal{B}_0$ as well) is quasi-equivalent to $\mathcal{B}/\mathcal{B}_0$. Smash product with this quasi-equivalent category gives $(\mathcal{B}\#\mathbb{Z})/(\mathcal{B}_0\#\mathbb{Z})$.

The second way is to see $\mathcal{B}\#\mathbb{Z}$ as another colimit. Namely, consider the diagram of categories given by one category, $\mathcal{B}$, and endofunctors $g \in \mathbb{Z}$. Then the corresponding Grothendieck construction (as in [Tho79]) is exactly $\mathcal{B}\#\mathbb{Z}$. In this situation, one does not need to localize with respect to corresponding set of morphisms, as they are already invertible, and one can easily show colimit property. Then, it is easy to see that $(C^{-1}\mathcal{B})\#\mathbb{Z}$ and $C^{-1}(\mathcal{B}\#\mathbb{Z})$ can be characterized by the same universal property.

Bibliography

[Abo10] Mohammed Abouzaid. A geometric criterion for generating the Fukaya category. *Publ. Math. Inst. Hautes Études Sci.*, (112):191–240, 2010.

[AG15] D. Arinkin and D. Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Math. (N.S.)*, 21(1):1–199, 2015.

[AL94] Michèle Audin and Jacques Lafontaine, editors. *Holomorphic curves in symplectic geometry*, volume 117 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.

[AL17] Rina Anno and Timothy Logvinenko. Spherical DG-functors. *J. Eur. Math. Soc. (JEMS)*, 19(9):2577–2656, 2017.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[AS10] Mohammed Abouzaid and Paul Seidel. An open string analogue of Viterbo functoriality. *Geom. Topol.*, 14(2):627–718, 2010.

[BEE12] Frédéric Bourgeois, Tobias Ekholm, and Yasha Eliashberg. Effect of Legendrian surgery. *Geom. Topol.*, 16(1):301–389, 2012. With an appendix by Sheel Ganatra and Maksim Maydanskiy.

[BO08] Petter Andreas Bergh and Steffen Oppermann. Cohomology of twisted tensor products. *J. Algebra*, 320(8):3327–3338, 2008.

[Bos14] Siegfried Bosch. *Lectures on formal and rigid geometry*, volume 2105 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[CDGG17] B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini, and R. Golovko. Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors. *ArXiv e-prints*, December 2017.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back*, volume 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012. Symplectic geometry of affine complex manifolds.

[Cha60] Stephen U. Chase. Direct products of modules. *Trans. Amer. Math. Soc.*, 97:457–473, 1960.

[Cie02] Kai Cieliebak. Handle attaching in symplectic homology and the chord conjecture. *J. Eur. Math. Soc. (JEMS)*, 4(2):115–142, 2002.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Fn07] Christian Frø nsdal. Quantization on curves. *Lett. Math. Phys.*, 79(2):109–129, 2007. With an appendix by Maxim Kontsevich.

[Gai13] Dennis Gaitsgory. ind-coherent sheaves. *Mosc. Math. J.*, 13(3):399–528, 553, 2013.

[Gan12] Sheel Ganatra. *Symplectic Cohomology and Duality for the Wrapped Fukaya Category*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Gao17] Y. Gao. Functors of wrapped Fukaya categories from Lagrangian correspondences. *ArXiv e-prints*, December 2017.

[GL87] Mark Green and Robert Lazarsfeld. Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. *Invent. Math.*, 90(2):389–407, 1987.

[GNW15] Lauren Grimley, Van C. Nguyen, and Sarah Witherspoon. Gerstenhaber brackets on hochschild cohomology of twisted tensor products, 2015.

[GPS17] S. Ganatra, J. Pardon, and V. Shende. Covariantly functorial wrapped Floer theory on Liouville sectors. *ArXiv e-prints*, June 2017.

[GPS18] S. Ganatra, J. Pardon, and V. Shende. Structural results in wrapped Floer theory. *ArXiv e-prints*, September 2018.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[Gro15] Y. Groman. Floer theory and reduced cohomology on open manifolds. *ArXiv e-prints*, October 2015.

[Har66] Robin Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[hh] Saal Hardali (https://mathoverflow.net/users/22810/saal hardali). Are all formal schemes *really* ind-schemes? MathOverflow. URL:https://mathoverflow.net/q/236351 (version: 2016-04-16).

[hs] Zoran Skoda (https://mathoverflow.net/users/35833/zoran skoda). Functorial point of view for formal schemes. MathOverflow. URL:https://mathoverflow.net/q/33079 (version: 2010-07-23).

[Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[Iri13] Kei Irie. Handle attaching in wrapped Floer homology and brake orbits in classical Hamiltonian systems. *Osaka J. Math.*, 50(2):363–396, 2013.

[Kar] Yusuf Barış Kartal. Distinguishing open symplectic mapping tori via their wrapped Fukaya categories.

[Kar18] Y. B. Kartal. Dynamical invariants of mapping torus categories. *ArXiv e-prints*, September 2018.

[Kel04] Bernhard Keller. Hochschild cohomology and derived Picard groups. *J. Pure Appl. Algebra*, 190(1-3):177–196, 2004.

[Kel06] Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.

[KK05] Louis H. Kauffman and Nikolai A. Krylov. Kernel of the variation operator and periodicity of open books. *Topology Appl.*, 148(1-3):183–200, 2005.

[Kry07] Nikolai A. Krylov. Relative mapping class group of the trivial and the tangent disk bundles over the sphere. *Pure Appl. Math. Q.*, 3(3, Special Issue: In honor of Leon Simon. Part 2):631–645, 2007.

[KS09] M. Kontsevich and Y. Soibelman. Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. In *Homological mirror symmetry*, volume 757 of *Lecture Notes in Phys.*, pages 153–219. Springer, Berlin, 2009.

[LL13] YankıLekili and Max Lipyanskiy. Geometric composition in quilted Floer theory. *Adv. Math.*, 236:1–23, 2013.

[LO06] Volodymyr Lyubashenko and Sergiy Ovsienko. A construction of quotient A_∞-categories. *Homology Homotopy Appl.*, 8(2):157–203, 2006.

[LO10] Valery A. Lunts and Dmitri O. Orlov. Uniqueness of enhancement for triangulated categories. *J. Amer. Math. Soc.*, 23(3):853–908, 2010.

[Lod07] Jean-Louis Loday. The diagonal of the stasheff polytope, 2007.

[LP12] Yanki Lekili and Timothy Perutz. Arithmetic mirror symmetry for the 2-torus, 2012.

[LP16] Yanki Lekili and Alexander Polishchuk. Arithmetic mirror symmetry for genus 1 curves with n marked points, 2016.

[LS16] Valery A. Lunts and Olaf M. Schnürer. New enhancements of derived categories of coherent sheaves and applications. *J. Algebra*, 446:203–274, 2016.

[Lun10] Valery A. Lunts. Categorical resolution of singularities. *J. Algebra*, 323(10):2977–3003, 2010.

[Ma'15] Sikimeti Ma'u. Quilted strips, graph associahedra, and A_∞ n-modules. *Algebr. Geom. Topol.*, 15(2):783–799, 2015.

[Mar06] Martin Markl. Transferring A_∞ (strongly homotopy associative) structures. *Rend. Circ. Mat. Palermo (2) Suppl.*, (79):139–151, 2006.

[May09] M. Maydanskiy. Exotic symplectic manifolds from Lefschetz fibrations. *ArXiv e-prints*, June 2009.

[Nad16] David Nadler. Wrapped microlocal sheaves on pairs of pants. *arXiv e-prints*, page arXiv:1604.00114, Mar 2016.

[Sei02] Paul Seidel. Fukaya categories and deformations. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 351–360. Higher Ed. Press, Beijing, 2002.

[Sei08a] Paul Seidel. A_∞-subalgebras and natural transformations. *Homology Homotopy Appl.*, 10(2):83–114, 2008.

[Sei08b] Paul Seidel. A biased view of symplectic cohomology. In *Current developments in mathematics, 2006*, pages 211–253. Int. Press, Somerville, MA, 2008.

[Sei08c] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[Sei12] Paul Seidel. Fukaya A_∞-structures associated to Lefschetz fibrations. I. *J. Symplectic Geom.*, 10(3):325–388, 2012.

[Sei13] Paul Seidel. Lectures on categorical dynamics and symplectic topology, 2013. URL: `http://math.mit.edu/~seidel/937/lecture-notes.pdf`. Last visited on 2017/08/30.

[Sei14] Paul Seidel. Abstract analogues of flux as symplectic invariants. *Mém. Soc. Math. Fr. (N.S.)*, (137):135, 2014.

[Sei15] Paul Seidel. Homological mirror symmetry for the quartic surface. *Mem. Amer. Math. Soc.*, 236(1116):vi+129, 2015.

[Ser73] J.-P. Serre. *A course in arithmetic.* Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[Sie16] K. Siegel. Squared Dehn twists and deformed symplectic invariants. *ArXiv e-prints*, September 2016.

[Sik94] Jean-Claude Sikorav. Some properties of holomorphic curves in almost complex manifolds. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 165–189. Birkhäuser, Basel, 1994.

[Spa88] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[ST01] Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[Sta17] The Stacks Project Authors. *stacks project.* ***http://stacks.math.columbia.edu****, 2017.*

[Syl16] Zachary Sylvan. On partially wrapped Fukaya categories. arXiv e-prints, *page arXiv:1604.02540, April 2016.*

[Tem15] Michael Temkin. Introduction to Berkovich analytic spaces. In Berkovich spaces and applications, *volume 2119 of* Lecture Notes in Math., *pages 3–66. Springer, Cham, 2015.*

[Tho79] R. W. Thomason. Homotopy colimits in the category of small categories. Math. Proc. Cambridge Philos. Soc., *85(1):91–109, 1979.*