Cochlear Macromechanical Modelling

by

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Submitted to the Department of Electrical Engineering and
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Abstract

Contemporary research into the manifestations and origins of nonlinear, active cochlear processes often takes place in a context in which linear, passive cochlear mechanics are poorly understood and poorly communicated. The distinctions among models of one-, two-, and three-dimensional fluid motion in the cochlear scala—models popularized by (among others) Zwislocki, Ranke, and Steele, respectively—are confounded by fuzzy use of terms such as "long-wave model" or "short-wave model." Models are frequently evaluated by comparing their place responses with experimentally observed frequency responses; their global impedance parameters are sometimes chosen solely to secure fit to some local measurement. And Steele's WKB (phase-integral) approach is treated, more often than not, as just another technique for solving cochlear dynamical equations, rather than as a conceptual framework yielding significant insight into cochlear phenomena.

In this thesis, I present cochlear dynamical equations for one-, two-, and three-dimensional fluid motion in a box-cochlea model, and I discuss the conditions under which such fluid motion is appropriately described as long wave, short wave, or as something in between. I describe the phase-integral approximate solution to these equations and discuss the utility of this framework for explaining cochlear phenomena. I develop generalized representations for both cochlear-partition impedance and cochlear-gain response that highlight the distinctions and similarities between the place response at a single frequency and the frequency response at a single place. The generalized representations clarify which aspects of partition impedance determine global phenomena, such as cochlear maps, and which aspects determine local features, such as magnitude-response peakiness and phase-response steepness.

Thesis Supervisor: William M. Siebert
Title: Ford Professor of Electrical Engineering
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Chapter 1

Problems in Cochlear Macromechanics

*Cochlear macromechanics* describes cochlear motion in terms of dynamical properties of cochlear fluids and the cochlear partition: the mass density and viscosity of the fluids, the stiffness and resistance of the partition. Cochlear macromechanical variables are the pressure in and velocity of the fluids and the pressure across and velocity of the partition. The dynamical interaction of cochlear fluids and partition and the variation in the partition’s mechanical properties from cochlear base to apex determine the unique pattern of partition motion in both place and frequency.

*Cochlear macromechanical modelling* is the building of cochlear models from elements characterized at the macromechanical level. Such models range from physically constructed scale models to various mathematical and computational approaches. The robustness and utility of a scientific model depend on knowing which aspects of the model are essential and which are ignorable; on being able to evaluate the model’s performance; and in clearly communicating the model’s structure, behavior, and performance.

Scientific modelling is more than an exercise in curve fitting with an arbitrary number of free parameters; it is an attempt to describe physical processes through the interaction of a few key attributes. The more a model focuses on essentials, the greater its utility. One very important aspect of scientific modelling is deciding what
to omit. A particular attribute might be omitted because its effect is insignificant or because the complexity of including it exceeds the resulting additional performance. On the other hand, determination that an attribute is essential sometimes occurs only after rejection of the possible alternatives. An essential attribute either must be included in the model, regardless of the cost of doing so, or the model must be rejected. For example, the box-cochlea model, the framework for the discussion here of cochlear modelling, represents the cochlea—a spirally coiled structure of three chambers of varying cross-sectional geometry filled with slightly compressible, slightly viscous fluid—by two chambers of constant rectangular cross section filled with incompressible, inviscid fluid. Previous investigations of this model have demonstrated that the varying geometric properties of the structure and the compressibility and viscosity of the fluid are not essential in describing cochlear partition motion.

A model's performance is judged by its agreement with measured data in a selected set of attributes. Because one facet of scientific endeavor is the attempt to destroy theoretical constructs and see what remains, the more difficult for measurements and model to agree across that set, the better the set for evaluating the model. For cochlear mechanics, the peakiness of the (displacement or velocity) magnitude response, the steepness of the angle response, and the cochlear map of best places or of best frequencies are particularly good to evaluate a cochlear model's performance.

Regardless of ignoring nonessential attributes, capturing essential ones, and excellent performance, a model which is poorly communicated to others has little utility for simulation or explanation: Numerous models differing only in small details from one another, and even more numerous publications about them contribute little to greater understanding of pertinent issues. Cochlear modelling needs a clear presentation of relevant issues, a straightforward formulation of their interaction, and a summary of how existing models emphasize and implement particular aspects among them.

Contemporary research in cochlear mechanism focusing on nonlinear, active processes and their manifestations—distortion products, otoacoustic emissions, nonlinear amplitude response—often occurs in a context in which linear, passive cochlear mechanics is poorly understood. In this chapter, I illustrate some confusing issues from
linear cochlear macromechanics.

1.1 Problems with Local and Global Attributes

A pair of problems in cochlear macromechanics has to do with the way a model's output is compared to experimental data and the way a model's parameters are chosen to suit certain conditions. The bulk of recent cochlear measurements are frequency domain measurements, but the output of cochlear models of the form studied here and elsewhere are functions of cochlear place. Measurements such as those from Békésy's pioneering experiments suggest that responses as a function of place at a single frequency and responses as a function of logarithmic frequency at a single place are similar in form. Other measurements such as those of the (inverse) cochlear map of best frequencies have the form of a linear function of log frequency. Is it appropriate for one to compare the measured responses at a place as a function of sinusoidal frequency to the modeled response at a frequency as a function of cochlear place, just because of the similarities in the form of the place and frequency responses, and because of the nature of cochlear maps?

There is, furthermore, a distinction to be made between global parameter variation, global features, and local features. An examples of a global parameter is the stiffness of the cochlear partition. Partition stiffness has been measured by Békésy and others, and reportedly varies by two or so orders of magnitude along the partition length; that is a global parameter that really isn't available for choice by the researcher. The cochlear map of best places as a function of frequency and the cochlear map of best frequencies as a function of place are both globally measured features. Examples of local measures are the peakiness of cochlear magnitude response near either the best place or best frequency is a local measurement and the steepness of the phase response near the same place or frequency. It is sometimes the case that a model's local performance, such as magnitude peakiness or angle steepness, is made to fit better the experimental data by adjusting global impedance parameters at the expense of fit to global measurements such as cochlear map.
Figure 1-1: Magnitude and angle comparisons between Rhode’s data and Allen’s two-dimensional fluid-flow model. The Rhode data was taken by Allen from Rhode [26]. From Allen [2, pg. 113].

An illustration of these problems is found in Allen’s 1978 model. The upper panel of Figure 1-1 shows the magnitude and angle of Rhode’s measured transfer ratio of basilar-membrane velocity to malleus velocity plotted as functions of frequency on a logarithmic scale [26]; the lower panel shows the model-generated magnitude and angle of the transfer ratio of basilar-membrane velocity to stapes velocity plotted as a function of place for a frequency of 1000 Hz [2].

Allen attempts to justify comparing the place-response output of his model with Rhode’s measurement of frequency selectivity on the basis of the cochlear map.
An immediately useful result is the approximate proportionality between log frequency and characteristic place. Note in [Figure 1-2] that one octave in frequency always corresponds to the same distance. For example, the distance between the places at 100 and 200 Hz is the same as the distance between the places for 5 and 10 kHz. Using this relationship, we may transform the velocity as a function of \( x \) for one frequency into the velocity as a function of frequency for one place. Since Rhode's (1971) data have already been plotted on a log-frequency scale, his data need only be scaled linearly along the ordinate and the abscissa in order to be compared to our computed results [2, pg. 112].

A linear relationship between best place and log frequency does not imply, as Allen suggests, that the response as a function of place for a particular frequency is the same or even proportional to the response as a function of frequency for a particular place. Even were the mapping well motivated, Rhode’s measurements were made in the most basal 0.2 cm of the cochlea where best frequencies ranged from 6.5–7.5 kHz, not in the cochlear region where 1000 Hz was the best frequency. (The comparison would seem to be better motivated for a frequency with a best place in the region Rhode measured; i.e., 8 kHz or so.)

Nor does it follow that the best place as a function of frequency and the best frequency as a function of place should be inverse functions of each other. Assuming that the transfer from stapes motion to partition motion is a function of place and frequency, then the partition velocity, \( v(x, t) \) has the form \( v(x, t) = \text{Re} \{ G(x, \omega) e^{j\omega t} \} \) when the input is \( \cos(\omega t) = \text{Re} \{ e^{j\omega t} \} \); \( G(x, \omega) \) is the cochlear gain. The loci of best places, \( x_b(\omega) \), are then the places for which \( \partial|G|/\partial x = 0 \); those of best frequencies, \( \omega_b(x) \), the frequencies for which \( \partial|G|/\partial \omega = 0 \). Without some compelling motivation, there is no reason to assume either that for some frequency \( \omega_0 \), the best frequency at the best place for \( \omega_0 \) is also \( \omega_0 \), or that for some place \( x_0 \), the best place for the best frequency at \( x_0 \) is also \( x_0 \); why should \( \omega_b(x_b(\omega_0)) = \omega_0 \), and why should \( x_b(\omega_b(x_0)) = x_0 \)? On the one hand, they must, in some sense, come close to being inverses of each other or someone who investigated both place and frequency responses.
Figure 1-2: Magnitude response curves for various frequencies of Allen's two-dimensional model. The abscissa is place along the cochlear length; the ordinate is 20 times the logarithm of the ratio of the magnitude of basilar membrane velocity to stapes velocity. The parameter values for the impedance representation are discussed in the text.

(like Békésy) would have noted a substantial difference between the two; but one needs a more substantial reason that the similarities between place-response and frequency-response shapes and that Békésy didn't note that they're different to believe the two to be the same. (See Siebert [36, pg. 246] for further discussion of this topic.)

Allen notes the difficulty in matching the peakiness of the magnitude plot and the steepness of the angle plot simultaneously. This illustrates that one needs to be concerned with the model's performance with regard to more than one local measure.

Finally, regarding parameter variation in the representation for cochlear partition dynamics, Allen used an impedance representation for the cochlear partition. The ratio of the pressure across the basilar membrane to basilar-membrane velocity was of the form:

\[
\frac{K(x)}{j\omega} + R(x) + j\omega M_0
\]

where the stiffness, \(K(x)\), and resistance, \(R(x)\) were given by

\[
K(x) = K_0 e^{-2\beta x}
\]

\[
R(x) = R_0 e^{-\beta x}
\]
The spatial variation parameter was $\beta = 1.7$ resulting in a change in stiffness by a factor of 100,000 over the length of the cochlea. Békésy's measured compliance changed by two orders of magnitude; Allen's, by five! While there is continuing debate about the reliability of Békésy's measurements, the difference between two and five orders of magnitude is so great that it should not be accepted without question. Additionally, the parameters $R_0$ and $\beta$ in the above impedance representation were different between the computations that yielded the data in Figures 1-1 and 1-2. While the parameter choice for Figure 1-2 results in frequencies from 100 Hz to 10 kHz having best places from one end of the cochlea to another, there is no such figure for the parameters used in Figure 1-1.

1.2 Problems with Computational Methods

The phase-integral approach to cochlear macromechanics was introduced by Steele as a technique for solving cochlear dynamical equations. [38, 41, 40] The approach is simple: if cochlear dynamical properties were constant along the cochlear length, then the cochlea would be well described as a kind of waveguide, but one in which the compliant material on one side (the partition) differs from that of the rigid walls on the other three. The distribution of velocity along the partition length would be a wave of the form

$$ V(x) = V_0 e^{-j k x} $$

where the wavenumber $k$ is determined by an eikonal equation relating a cochlear function describing fluid dynamics within the scala, $\tilde{H}(k)$, and the partition impedance, $Z_p(\omega)$.

$$ \frac{Z_p(\omega)}{2 j \omega \rho} + \tilde{H}(k) = 0 $$

Both the wavenumber $k$ and the cochlear function $\tilde{H}(k)$ are complex quantities; as such, the wave described by Equation 1.4 is not a uniform, constant amplitude, travelling wave. It is, however, a wave with constant rate of amplitude and phase accumulations determined by the imaginary and real parts of the wavenumber respectively. More generally, because the cochlear function is multi-valued, the formal solution
would be a superposition of modes:

$$V(x) = \sum_n V_n e^{-jk_n x}$$  \hspace{1cm} (1.6)

Ordinary partition impedances, such as those of a spring, dashpot, mass system, result in wavenumbers in the fourth quadrant. For any positive resistance, the imaginary part of the wavenumber will be negative, and the wave will attenuate linearly with increasing place. However, if the wavenumber is such that the imaginary part is relatively large, then the attenuation rate can be so large that the system is beyond cutoff, and there will be no observable propagating wave.

The phase-integral approach extends the approach above to the situation where partition dynamics vary with longitudinal position. The linear-in-place, complex phase for the non-varying impedance and geometry, $jkx$, is replaced by a phase integral, \(\int_0^x jk(\xi) d\xi\), where \(k(x)\) is a local solution to the eikonal equation.

The importance of the phase-integral technique goes beyond its utility in computing the response of a cochlear model. By constructing a framework which says that cochlear motion should have the form of a non-uniform (in amplitude and phase) travelling wave, the phase-integral solution constructs a model of reality that is consistent with observations of reality. Unlike the case of a model that's really nothing more than a simulation—e.g., a finite-difference implementation of fluid and partition dynamics—the phase-integral approach recognizes that the non-uniform travelling-wave nature of cochlear-partition displacement or velocity as an essential characterization of cochlear response.

Besides having the non-uniform propagating wave built into its structure, the phase-integral solution yields two other insights: there are no backward propagating waves, and higher-order modes are hard to observe. Since the forwardly propagating wave cuts off before reaching the helicotrema for all but the highest frequencies, there is no backward propagating mode. There is nothing in a simulation approach to say that one shouldn't expect both forward and backward propagating modes at any frequency. That higher-order modes should be hard to observe follows from considering the wavenumbers for the various modes. All higher-order modes have a
substantially larger (in magnitude) imaginary part than real part; their attenuation rate is far greater than their propagation rate. As such, even when a higher order mode is excited, it decays away rapidly, and any volume displacement associated with it is negligible.

1.3 Problems with Terminology

A less formal but, in some sense, more pervasive problem in cochlear modelling has to do with terminology. The models we examine for cochlear mechanism have, in general, some representation for the fluids in the cochlear spiral, and some representation for the cochlear-partition dynamics. The terminology distinguishing between the degrees of freedom afforded the motion of cochlear fluids—one-, two-, or three-dimensional—and the terminology describing regimes of motion and approximation within the two- and three-dimensional fluid-flow models—long-wave and short-wave—are sometimes confused.

For the one-dimensional problem, cochlear fluids simply flow backwards and forwards in the scala; transverse and vertical fluid motion—implicitly associated with partition motion—are assumed to be negligibly small. One model, frequently used for computational or physical implementations, of the one-dimensional cochlea is the cochlear transmission line. When cochlear fluids are assumed to move in two dimensions or in three dimensions approximate regimes of cochlear operation come into play. In the two dimensional case, there is a long-wave regime where vertical fluid motions are negligible, and the fluid dynamics approach those of the one-dimensional case; similarly for the three-dimensional case. For both two- and three-dimensions there is a contrasting regime, the short-wave regime, where cochlear fluid motion is limited to a region close to the moving partition. In that regime, fluid motions are short-wavelength, and fluid and partition motions are closely coupled to each other.

That the two- and three-dimensional models have long- and short-wave regimes of operation are distinct concepts from the ideas of long- and short-wave models of cochlear operation. A long-wave model would consist of taking the operational dy-
namics in either the two- or three-dimensional cochlear model and implementing those
dynamics to operate over all input frequencies; such a model might be implemented
in the form of a transmission line. That doesn’t mean that any transmission line im-
plementation is necessarily a “long-wave” model. The distinctions among one-, two-, and three-dimensional cochlear-fluid motion differ from the distinctions between long-
and short-wave approximate regimes of operation.

1.4 Summary

Continuing problems in cochlear macromechanics include issues of place and frequency
response, and of local and global parameter variation and response measurement.
For a particular choice of global parameters—a choice guided by measurements of
cochlear properties, particularly stiffness variation—a model should have the right
range of cochlear map variation, a global measurement. For the same choice of global
parameters, local measurements—peakiness of magnitude response and steepness of
angle response—should have appropriate values. Perhaps the single largest challenge
in cochlear modelling is meeting the global and local constraints simultaneously.

Another problem has to do with the utility of the phase-integral solution of Steele
and others as a framework for understanding cochlear dynamics. The phase-integral
method is more than just a mathematical method for solving cochlear dynamic equa-
tions; it is an approach to cochlear modelling that allows one to focus on essentials
and ignore useless details; it assumes from the beginning a form of cochlear motion
which is similar to observations.

Finally, a continuing problem in cochlear modelling has to do with terminology.
Because of the various models of differing numbers of dimensions of cochlear fluid
motion, and because of different approximate regimes within those models, the ter-
minologies about “one-dimensional,” “two-dimensional,” “three-dimensional,” “long-
wave,” “short-wave,” and “transmission-line” frequently are sometimes confused.

I hope to clear up some of the above issues in this document.
Chapter 2

Cochlear-Partition Motion

Our contemporary ideas of what goes on inside the ear when a sound occurs in the world are based on Georg von Békésy’s pioneering observations of cochlear-partition motion. Although Békésy’s observations have been revised and refined by better methodology and technology, what he saw, together with our knowledge of cochlear anatomy, remains the foundation for our understanding of cochlear motion.

The “classical” view of cochlear-partition motion is captured in four simple points. When a sinusoid is presented to the ear:

- Cochlear-partition motion is wavelike. For a fixed presentation frequency, the phase angle of the partition displacement is a monotonically decreasing function of distance from the base toward the apex.

- Cochlear-partition motion is place selective. For a fixed presentation frequency, the magnitude of the partition displacement or velocity has a well defined best place. The location of that place is a monotonically decreasing function of frequency; the best place for higher frequency sounds is nearer the base than for lower frequency ones.

- Cochlear-partition motion is frequency selective. At a fixed place along the cochlear length, the magnitude of the partition displacement has a well defined best frequency. The location of that frequency is a monotonically decreasing
function of place; the best frequency at basal locations is higher than that at more apical ones.

• At a fixed place, the phase angle of the partition displacement or velocity is a monotonically decreasing function of frequency.

The second and third points above are often characterized by saying that the cochlea exhibits tonotopic behavior—i.e., a unique mapping between places and frequencies. More recent measurements add a degree of subtlety to the above:

• Cochlear-partition motion in living tissue is more frequency selective than earlier measurements in cadavers indicated. It is as frequency selective as the firing-rate response of single auditory neurons.

• The tuning sharpness and phase-angle slope near the cochlear peak both depend on the level of the presented signal.

• Level-dependent effects depend on the biological viability of the preparation being observed.

There are numerous—to numerous to mention them all—researchers who have examined not only cochlear partition motion but also the cochlear microphonic, electrical recordings from hair cells (in various species), and neural data from the auditory nerve. Instead of summarizing the work of them all, I present details regarding the above points. First, I review Békésy’s observations; then, data from Khanna et al. establishing the peakiness of cochlear-partition response; and then, data from Rhode et al. demonstrating level dependence of peakiness and angle, non-linear behavior near the best frequency, and the necessity of biological viability for level dependent effects. Békésy’s is the work of a pioneer; his data were, for the most part, the first look his contemporaries had at cochlear motion. He received the 1960 Nobel Prize in Physiology for his ground breaking effort of enormous scope. The Khanna et al. reports describe greater frequency selectivity that included the effects of trauma and preparation death on the measurements. The Rhode et al. data provided examples
of cochlear-partition nonlinearity. The examples presented here are by no means exhaustive, nor are they presented as definitive of the historical record. Instead, they establish the context in which contemporary cochlear modelling efforts take place.

Cochlear modelling has global and local constraints to satisfy: Place responses are global, as are parameter variation along the cochlear partition; frequency responses are local. In this study, I explore how global parameter choices—the orders of magnitude of change in the partition stiffness over the cochlear length—affect local measurements. To that end, the work presented in the sequel is concerned less with fitting any of the curves seen in this chapter, and more with exploring how those curves come to be and what parameters one varies to change the features seen in any of those curves.

2.1 Békésy’s Observations

Békésy observed the motion of the cochlear partition in the cadavers of several species, including humans. With the cochlea still inside the excised temporal bone, he removed the middle ear, the round window membrane, and the stapes footplate but not the oval window membrane. He covered the round window with a rubber membrane and to the round-window rim attached a brass tube by which the sound stimulation was applied. This procedure reversed the roles of the oval and round windows from those occurring naturally [3, pg. 441]. Békésy ground or chipped away the bony structure of the cochlea until the partition was visible. The motion of the cochlear partition is such that for all but the lowest frequencies, there is an apical point beyond which there is negligible response to all but the most intense presentations; the most apical end responds only to the lowest frequency tones. By making his initial measurements at that end for those tones, then by grinding away that section of the bony cochlea and making his next observations more basalward, Békésy observed wide ranges cochlear-partition motion over a variety of places and frequencies. The preparation was under water during the observations. Békésy dropped silver crystals onto the optically transparent Reissner's membrane to make it visible; he reported
Figure 2-1: Cochlear-partition displacement at two times during one cycle of a 200 Hz sinusoid. The abscissa is the place along the cochlear partition; the ordinate, the displacement in arbitrary units. The time between the two curves corresponds to one quarter of a cycle—1.25 ms. From Békésy [3, pg. 462].

that, with the exception of very high frequency presentations, Reissner's membrane moved with the same magnitude and phase dependence as the basilar membrane and other partition structures [3, pg. 461].

Figure 2-1 illustrates the travelling-wave aspect of cochlear-partition displacement. Each curve shows the displacement at a differing instant during one cycle of a 200 Hz sinusoid. Other observable features in Figure 2-1 include the shortening of the wavelength at more apical locations and the peak in the displacement amplitude [3, pg. 462].

Figure 2-2 and Figure 2-3 show the place-responses magnitude for several frequencies; Figure 2-3 also shows the place-response phase angle. The magnitude curves in both figures have similar forms: There is a gradual increase in the displacement as the point of interest increases from the base to the apex; the displacement magnitude reaches a peak before the helicotrema at all but the lowest frequencies; on the apical side of the peak, the amplitude of motion decreases fairly rapidly to no motion at all. The distance between the location of the peak and the stapes decreases as frequency increases [3, pg. 447, pg. 462].

The lower panel in Figure 2-3 shows the displacement angle as a function of place for several frequencies. The angle's being a monotonically decreasing function of place is another statement that cochlear-partition motion has the form of a travelling wave propagating from base to apex. The displacement wavelength decreases from
Figure 2-2: Place-response magnitude for various frequencies of sinusoidal drive. The ordinate is the distance from the stapes; the ordinate, the partition displacement magnitude, normalized such that all curves have the same extent at the maximum-displacement point. From Békésy [3, pg. 448].

Figure 2-3: Place-response magnitude and angle for low frequency tones. The abscissa is the distance from the stapes. In the upper panel, the ordinate is the cochlear-displacement magnitude; in the lower panel, the ordinate is the cochlear-displacement phase angle. The labels on each curve are the sinusoidal frequency in Hz. It is likely that each magnitude curve was normalized such that all have the same maximum amplitude (as in Figure 2-2). From Békésy [3, pg. 462].
Figure 2-4: Best place as a function of sinusoidal frequency. The abscissa is the frequency in Hz on a logarithmic axis; the ordinate, the best place in millimeters from the stapes. From Békésy [3, pg. 440].

the stapes to the apex [3, pg. 462].

Figure 2-4 shows the location of the the best place as a function of frequency (logarithmic scale). Over a large frequency range the best-place location is an approximately linear function of the logarithm of the sinusoid frequency [3, pg. 440].

Figure 2-5 and Figure 2-6 show the displacement response to constant volume displacement at the stapes as a function of frequency. At any point along the partition, the response is frequency selective, having a maximum displacement at some particular frequency. Békésy did not publish a plot, similar to Figure 2-4, of the best frequencies as a function of place. A frequent assumption is that the best frequencies and best places are approximately inverse functions of each other, though that need not be the case [3, pg. 454, pg. 461] [36, pg. 246]. See also Chapter 1, pg. 14 of this work.

The phase curve in Figure 2-6 is inconsistent with the locally frequency selective properties being determined by a low-order lumped parameter system such as a simple resonator. Were the behavior that of a simple, second-order resonant system, the angle curve would go through a change of $\pi$ radians as the frequency increased through the resonant frequency; the curve exhibits neither of these properties. Furthermore,
Figure 2-5: Displacement magnitude as a function of frequency at several places along the cochlear partition. The abscissa is the sinusoidal stimulus frequency on a logarithmic scale; the ordinate, the relative magnitude—the fraction of maximum partition displacement magnitude—of cochlear-partition displacement. From Békésy [3, pg. 454].

Figure 2-6: Displacement magnitude and angle at a point 30 mm from the cochlear base. The abscissa is the sinusoidal stimulus frequency on a logarithmic scale; the ordinates are (right) the magnitude of partition displacement relative to the maximum displacement and (left) the phase angle of the displacement. From Békésy [3, pg. 461].
the observed tail of the angle response—linear vs. log frequency—above the best frequency is inconsistent with the observed motion’s being described by a low-order system [3, pg. 461].

That cochlear motion is described by low-order lumped-parameter resonance is a simplistic interpretation of Helmholtz’s idea, made at the time of the American Civil War, that certain aspects of hearing behavior would be explained if the basilar membrane were modelled as a set of resonating elements. The resonance would arise from the interaction of the tension in the membrane and the membrane mass. However, Helmholtz also suggested a coupling role and wave-like motion for the cochlear fluid.

Under these circumstances the parts of the membrane in unison with higher tones must be looked for near the round window, and those with the deeper, near the vertex of the cochlea, as Hensen also concluded from his measurements. That such short strings should be capable of corresponding with such deep tones, must be explained by their being loaded in the basilar membrane with all kinds of solid formations; the fluid of both galleries in the cochlea must also be considered as weighting the membrane, because it cannot move without a kind of wave motion in that fluid [10, 146].

Békésy noted a linear relation between the input, the volume displacement at the stapes, and the measured responses, the cochlear-partition volume displacement per unit length [3, pg. 447]. Excepting very large stapes deflections, the motion is linear over a large range of stapes deflections (Figure 2-7) [3, pp. 463–464].

2.2 The Observations of Khanna et al.

Khanna et al. measured the motion of the cochlear partition at several locations using laser homodyne interferometry [15, 13, 14, 16, 17]. They performed their experiments on anaesthetized cats. They restrained the cat’s head, exposed the external auditory meatus, and inserted a metal tube into the ear canal for delivery of the sound stimulus. They exposed the bulla and removed the round window membrane. They placed
Figure 2-7: Relationship of stapes amplitude and basilar membrane amplitude. The abscissa is the stapes maximum amplitude; the ordinate, the basilar-membrane maximum amplitude measured some 30 mm from the base. The tone frequency is 200 Hz. From Békésy [3, pg. 464].

Interferometry mirrors on the basilar membrane through the round window. The mirrors were flat gold crystals between 70 µm and 120 µm in width [16, pp. 38–41].

In laser homodyne interferometry, one attaches mirrors to the object whose velocity is being measured. One illuminates both the object and a reference mirror vibrating at a known frequency with coherent light. A photodetector measures the sum, the interference, of both signals, from which one determines the object velocity amplitude from the photodetector output. As applied by Khanna et al., homodyne interferometry is a constant output technique used to measure input amplitudes: one varies the signal frequency and measure the sound pressure at the eardrum necessary to yield a certain probe-mirror displacement [13, pp. 12–13].

Figure 2-8 shows the amount of pressure applied at the eardrum to yield a 1 Å displacement at each of two differing basilar-membrane locations. These measurements resemble threshold tuning curves of primary auditory afferent fibers as determined by Kiang et al. [12]: There is a broad, low-frequency tail over much of the frequency range; near the best frequency for each place, the response peaks sharply: there is a steep cutoff on the high-frequency side of the best frequency [16, pg. 48].

---

1 The first report of mechanical tuning being as sharp as neural tuning was made by Johnstone and Boyle [11].
Figure 2-8: Sound pressure at the eardrum required to produce a 1 Å basilar-membrane displacement. The abscissa is the stimulus frequency (logarithmic scale); the ordinate, the sound pressure in decibels required for the constant displacement response. Even though the probe mirror for the curve labelled ‘A’ was located more apical than the one for the curve labelled ‘B’, the best frequency for the curve ‘A’ is higher than that of curve ‘B’ due to deterioration of the physiological condition of the preparation. See the discussion in Section 2.4. From Khanna and Leonard [16, pg. 48].

2.3 The Measurements of Rhode et al.

Using the Mössbauer technique, Rhode et al. recorded basilar-membrane motion in response to tones and clicks. The recovered membrane velocities are nonlinearly related to input signal level. The nonlinearity is labile: It decreases in viability with increasing damage to the organism; it disappears shortly after the animal’s death. [29, 26, 28, 27, 30, 31, 33].

Rhode et al. made their measurements in the living ears of anaesthesized squirrel monkey and chinchilla. In both species, they exposed the ear canal and inserted a tube for delivery of the acoustic stimulus. They exposed the cochlear partition from the scala-tympani side, and applied the Mössbauer source—a piece of Co-57 foil approximately 60 μm square—to the basilar membrane [26, pp. 1219–1220] [31, pp. 1365–1366].

In the Mössbauer technique, one places a radioactive source on the moving object whose velocity is being measured. As the the object’s velocity changes, the emitted
radiation undergoes a Doppler shift. The emitted gamma rays are collected by a non-linear transducer; the counts are superposed to form a histogram. For a certain range of amplitudes, one can invert the transducer nonlinearity and recover the velocity waveform (or an approximation to it) [31, pg. 1365].

As evidence for nonlinear basilar membrane motion, Rhode and Robles [29] cite the following:

- For sinusoidal stimuli, a less than proportional increase in velocity amplitude near the best frequency as the stimulus amplitude is increased.
- For sinusoidal stimuli, dependence of the response phase on the stimulus amplitude.
- For click stimuli, a less than proportional decrease in the click-response tail amplitude as the stimulus amplitude is decreased.
- Linear behavior after the animal’s death.

Each curve in Figure 2-9 shows the measured velocity amplitude on a logarithmic scale vs. sound pressure level for a different frequency. Linear responses would parallel the dotted line. The place in question responded best to 8.5 kHz. Curves for frequencies at and near the best frequency show a compressive nonlinearity, while those for frequencies distant from the best frequency are almost linear [31, pg. 1366].

Each curve in Figure 2-10 shows the measured velocity phase vs. sound pressure level for a different frequency. The place in question responded best to 6.8 kHz. For increasing signal level, sinusoidal frequencies below the best frequency show decreasing phase lag with increasing signal level; those above, increasing phase lag [29, pg. 594].

Each panel in Figure 2-11 shows the recovered basilar membrane velocity vs. time in response to differing amplitude clicks. As the click level decreases, the response tail decays more quickly in time, and the response envelope becomes more symmetrical [30, pg. 931].
Figure 2-9: Intensity functions for basilar membrane velocity in chinchilla. The abscissa is the presented sound's sound-pressure level in dB; the ordinate, the recovered velocity in millimeters per second (logarithmic scale). Numerical labels at the ends of each curve show the frequency of sinusoidal presentation for that curve. From Robles, Ruggero, and Rich [31, pg. 1366].

Figure 2-10: Basilar membrane phase vs. sound pressure level in squirrel monkey. The abscissa is sound pressure level in dB; the ordinate, the phase in radians measured relative to the phase at 75 dB. Numerical labels at the ends of the curves show the frequency of sinusoidal presentation for that curve. From Rhode and Robles [29, pg. 594].
Figure 2-11: Basilar membrane velocity vs. time in response a click. The abscissa is time; the ordinate, the recovered basilar membrane velocity in mm/s. Numerical labels at the end of each plot show the attenuation of the click presentation. From Robles, Rhode, and Geisler [30, pg. 931].
Figure 2-12: Intensity functions for basilar membrane velocity in chinchilla at three different times. The abscissa is the SPL of the presented sound in dB; the ordinate, the recovered velocity in mm/s (logarithmic scale). Numerical labels at the ends of each curve show the frequency of sinusoidal presentation for that curve. Closed circles indicate the initial measurements; open circles were measured 3 hours later; closed squares, 8 hours later. From Robles, Ruggero, and Rich [31, pg. 1367].

The curves in Figure 2-12 are intensity functions at differing frequencies as in Figure 2-9, except now there are measurements at three different times. As the time into the experiment increases, the intensity functions become more linear at all frequencies. Presumably, the compressive nonlinearity near the best frequency depends upon the physiological viability of the animal’s ear [31, 1367]. Rhode and Robles report that the basilar-membrane motion becomes linear within one hour post mortem [29, pg. 590].

2.4 Discussion

The more recent results presented here show that in vivo cochlear-partition motion is more selective (in frequency) than Békésy’s measurements of cadavers indicated. Furthermore, those more recent results indicate that near the best frequency partition motion is compressively nonlinear: doubling the input velocity amplitude results in a less than doubling of the measured partition velocity amplitude. I wish to discuss these data from three perspectives:
• The newer results vary in detail from Békésy’s results. How are we to interpret Békésy’s results in light of the new measurements?

• The newer techniques involve placing a foreign object on the basilar membrane. How does the presence of such an object influence the cochlea’s mechanical behavior?

• The results suggest that the cochlea is an active, nonlinear device. How does one use passive, linear models to investigate active, nonlinear systems?

Objections to Békésy’s techniques and reports range from the sound levels used (too loud), the preparations used (too dead), and the manner of presentation of results (too impressionistic). In light of the newer results, one can state that Békésy’s results were obtained for cochlear-partition displacements well greater than those near the hearing threshold near the best frequency for each place. This is not to be unexpected, given that Békésy’s observations were made by light microscopy. The tone of earlier complaints that such signal levels were so high that mechanical nonlinearities might come into play is muted by noting contemporary observations showing compressive nonlinear effects near the best frequency at quite small signal levels.

The contemporary results demonstrate the role of biological viability in both sharper tuning and nonlinearity near the best frequency. The better results from the more recent experiments—those of Khanna et al., those of Sellick et al. [34], those of Wilson and Johnstone [45]—show much sharper tuning than in any of Békésy’s results. The Rhode et al. measurements—made using the Mössbauer technique and showing the return to linear behavior with decreasing biological viability—have been confirmed by Ruggero and Rich using a technique called laser velocimetry (originally applied by Nuttall et al.) [32] [23].

Questions concerning the preparation’s viability are not limited to Békésy’s use of cadavers. In Figure 2-8, the interferometry mirror for curve B was located more basal of the one for curve A, the best frequency for curve A is higher than that for curve B, a result inconsistent with the usual cochlear map; however, the curve labelled ‘A’ was measured before the curve labelled ‘B’. Khanna and Leonard explain the inconsistency
by damage to the preparation during the course of the experiment. Their measure of cochlear viability, the amplitude of the cochlear-microphonic response, decreased with increasing time into the experiment. Furthermore, they report that histological examination showed damage to the outer hair cells in the region where the mirrors were located. They report: "...changes in the mechanical response seen at the basilar membrane are due to changes in the mechanical response of the hair cells [17, pg. 56]."

The techniques of Khanna et al. and of Rhode et al. both place probe objects onto the cochlear partition. Does the very act of placing an object onto the cochlear partition change the mechanical situation there sufficiently such that any such results are suspect? Lewis raises such questions, noting changes in the cochlear input impedance—or, at least, changes in the auditory impedance seen at the eardrum—with the placement of objects on the cochlear partition [19, pp. 156–157]. A brief answer is that the return both of a more broad cochlear tuning and of linear behavior with the death of the animal suggests that active biological processes play a greater role in determining greater selectivity and nonlinearity than the presence of a piece of radioactive foil, a reflecting mirror, or a glass bead does.

Even though contemporary measures indicate nonlinear behavior in the most sensitive regions of cochlear response and the origin of both the nonlinearity and much of the selectivity through active biological processes, the work presented here concerns linear, passive models of cochlear systems. Such models, however, remain of great utility in understanding the origin of cochlear phenomenon. If a linear passive model can be made to have sharply frequency selective behavior for one choice of cochlear-partition dynamical parameters and more broadly selective behavior for another parameter set, it may be useful to consider the changes from one set to another as what happens for differing sound-pressure levels or status of biological viability.

One category missing from the recent results are place-response measurements. Although the similarities between place responses and frequency responses plotted against the logarithm of frequency are strong and striking, there remain differences. Frequency dependent measures of cochlear behavior can be misleading: They are local measurements in a distributed system. With that in mind, the observed nonlin-
ear behavior of cochlear responses plotted against frequency are likely to depend on cochlear dynamics not only at the local point being observed, but also in a nearby region and possibly along the entire cochlear length. To that extend, the lack of contemporary in vivo observations of place responses is a shame.
Chapter 3

Cochlear Dynamical Equations

Cochlear macromechanical dynamics are determined by two constraints: one on the fluid and the other on the partition. The fluid constraint is global: it describes the global fluid-pressure (or -velocity) distribution in terms of the global partition-velocity (or -pressure) distribution. When longitudinal-stiffness effects are ignorable, the partition constraint is local: it relates the local cochlear-partition velocity to the local pressure drop across the partition. Cochlear dynamical equations relate each value determined by the global constraint with that determined locally. Before taking a closer look at the box-cochlea model and the assumptions underlying it, I present an overview of cochlear dynamical equations and their solution.

The weighted-average (weighted, averaged across the cochlear-partition breadth) pressure in the cochlear fluid at each longitudinal point \( x \), \( P(x) \), is related to the cochlear-partition volume-velocity density, \( V(x) \), through a Green’s function, \( H(x, \xi) \):

\[
P(x) = -j\omega \rho \int_{-\infty}^{\infty} H(x, \xi)V(x - \xi) d\xi
\]  

(3.1)

The Green’s function describes the pressure at \( x \) due to a unit impulse of volume velocity located at \( \xi \). Equation 3.1 arises because the global velocity distribution is a superposition of weighted impulses. Recognizing that the spatial-domain convolution is equivalent to a multiplication in the spatial-frequency domain, an alternative expression for the pressure is:

\[
P(x) = -\frac{j\omega \rho}{2\pi} \int_{-\infty}^{\infty} \tilde{H}(x, k)\tilde{V}(k)e^{jkx} dk
\]  

(3.2)
I call the spatial-frequency-domain representation of the Green's function, \( \tilde{H}(x, k) \), the cochlear fluid function or, simply, the cochlear function. Frequently, the cochlear function is more easily determined than the Green's function. Although the details of \( \tilde{H}(x, k) \) depend on whether one is considering one-, two-, or three-dimensional fluid motion, rectangular or other scala cross section, representing local pressure through the global velocity distribution—either by the Green's function or by the cochlear function—follows from the assumed linearity of cochlear fluid dynamics.

When the motion of nearby points along the cochlear-partition length is coupled much more so by the cochlear fluid than by any longitudinal stiffness intrinsic to the partition itself, partition dynamics can be modelled as those of an acoustic admittance. While the fluid function describes the pressure at the partition in terms of the volume-velocity density everywhere along the partition length, the cochlear-partition admittance relates the local partition-velocity density to the pressure drop across the partition (twice the pressure in the fluid at the partition surface).

\[
V(x) = 2Y_p(x, \omega)P(x)
\]  
(3.3)

Many partition models represent the impedance, \( Z_p(x, \omega) = 1/Y_p(x, \omega) \), as having an exponentially decreasing stiffness, an exponentially changing resistance—either decreasing (see Allen [2]), constant (see Neely [21]), or increasing (see Siebert [37])—and a constant mass.

Cochlear-integral equations describe linear, passive, cochlear dynamics. The velocity equation equates the pressures determined both globally and locally.

\[
\frac{V(x)}{2j\omega\rho Y_p(x, \omega)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{H}(x, k)\tilde{V}(k)e^{jkx}dk = 0
\]  
(3.4)

A cochlear-integral equation in pressure similarly equate pressure determined globally by the velocity distribution and Green's function, \( G(x, \xi) \), and locally by partition dynamics.

\[
V(x) = -\frac{1}{j\omega\rho} \int_{-\infty}^{\infty} G(x, \xi)P(x - \xi)d\xi,
\]  
(3.5)

The frequency-domain equivalent to the above is

\[
V(x) = -\frac{1}{j2\pi\omega\rho} \int_{-\infty}^{\infty} \tilde{G}(x, k)\tilde{P}(k)e^{jkx}dk,
\]  
(3.6)
The cochlear-integral equation in pressure is
\[ j2\omega \rho Y_p(x, \omega) P(x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(x, k) \tilde{P}(k) e^{ikx} dk = 0 \] (3.7)

For spatially invariant cochlear geometry, \( \tilde{H}(x, k) = 1/\tilde{G}(x, k) \).

An approximate solution to integral equations of the above form is
\[ V(x) = jk(0) v_s A_s \left[ \frac{\tilde{H}'(k(x, \omega))}{\tilde{H}'(k(0, \omega))} \right]^{-1/2} e^{-j \int_0^x k(\xi, \omega) d\xi} \] (3.8)

where the wavenumber \( k(x, \omega) \) is the root of the eikonal equation
\[ \frac{1}{2j\omega \rho Y_p(x, \omega)} + \tilde{H}(k) = 0 \] (3.9)

and where \( v_s \) and \( A_s \) are the stapes velocity and area, respectively. For partition impedances associated with the partition dynamics of a stiffness that decreases monotonically with longitudinal position and a resistance, the resulting velocity distribution has several salient features: slow increase in the log magnitude and phase basalward of the best place due to the slow gain term \( \tilde{H}(k(x, \omega))^{-1/2} \), and a rapid accumulation of attenuation and phase more apical of that point due to the phase integral term \(-j \int_0^x k(\xi, \omega) d\xi \). Figure 3-1 shows the place-response log magnitude and phase for Neely’s impedance parameters and a frequency of 1600 Hz [21]. Results for other frequencies and other impedance choices are found in Section 3.5. The relationship of local measures of magnitude peakiness and angle steepness, and of global measures such as the cochlear map of best places or best frequencies, to the form and parameters of the cochlear-partition impedance is a primary focus of this work.

In the following section, I discuss the assumptions about the box-cochlea model, the modelling framework for the discussions in this work. In the immediately following section, I derive cochlear-fluid functions for one-, two-, and three-dimensional cochlear-fluid motion, and define cochlear-partition impedances; the latter task is straightforward for one- and two-dimensional fluid motion—it’s simply proportional to a point impedance—but involves some finesse in the three-dimensional case. In the next sections, I discuss an approximate solution to cochlear integral equations and application of boundary conditions to determining arbitrary constants. Finally,
Figure 3-1: Cochlear gain for Neely's 1978 impedance at 1600 Hz. The gain is the ratio of cochlear partition velocity to stapes velocity. The abscissa in both panels is cochlear place in cm. The ordinate in the upper panel is the log magnitude in dB; in the lower panel, it is the angle in $\pi$ radians.
this chapter concludes with a series of example results demonstrating this model’s agreement with previously published results by other researchers.

### 3.1 The Box-Cochlea Model

Actual cochlear dynamics involve the interaction of slightly viscous, slightly compressible fluids in a spiral of constantly changing cross section with a biologically viable cochlear partition. The partition’s dynamics involve not only its longitudinally varying stiffness, but also resistive and (possible) inertial attributes, and the effect of mechanically active outer-hair cells coupling the motions of the basilar and tectorial membranes. Regardless of what seems to be a need to include this myriad of details, the cochlear model which returns the most explanatory power at the expense of least complexity is the box-cochlea model illustrated in Figure 3-2. The box cochlea consists of two chambers of rectangular cross section, each filled with an incompressible, inviscid fluid of mass density $\rho$, separated by a partition with longitudinally varying mechanical parameters.

Figure 3-2 shows reference directions for motion in the box-cochlea model. The longitudinal variable is $x$; the vertical variable, $y$; the transverse variable, $z$. Pressures and velocities are denoted by complex amplitudes of sinusoidal steady-state quantities of radian frequency $\omega$. Time functions for the fluid pressure and particle velocity are:

\[
p(x, y, z, t) = \text{Re} \left\{ P(x, y, z)e^{i\omega t} \right\} \tag{3.10}
\]

\[
\vec{v}(x, y, z, t) = \text{Re} \left\{ \vec{V}(x, y, z)e^{i\omega t} \right\} \tag{3.11}
\]

where $\vec{V}(x, y, z) = V_x(x, y, z)\hat{x} + V_y(x, y, z)\hat{y} + V_z(x, y, z)\hat{z}$, and where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in the $x$, $y$, and $z$ directions respectively. The frequency dependence of $P(x, y, z)$ and $\vec{V}(x, y, z)$ are implicit; i.e., $P(x, y, z) = P(x, y, z, \omega)$ and $\vec{V}(x, y, z) = \vec{V}(x, y, z, \omega)$, etc.

I denote the pressure across the cochlear partition—the fluid pressure on the scala-vestibuli side less that on the scala-tympani side—by $P_d(x, z) = P(x, 0^+, z) - P(x, 0^-, z)$. Because cochlear-fluid pressure is assumed to be an odd function of the
vertical variable, $P_d(x, z) = 2P(x, 0^+, z)$. For a compliant partition, a positive pressure across the partition results in partition displacement toward scala tympani; for that reason, positive partition displacement is defined as being toward scala tympani. By continuity of the normal component of velocity at the partition surface, $V_p(x, z) = -V_g(x, 0, z)$. The partition volume-velocity density at each longitudinal point—the (complex amplitude of) volume displaced per unit time per unit length—is $V(x) = \int_0^b V_p(x, z)dz$. The normalized (unit area) transverse velocity distribution is denoted by $\phi(x, z) = V_p(x, z)/V(x)$.

This analysis of the box-cochlea model makes the following assumptions:

- Rectilinear geometry.
- Uniform, symmetric, rectangular scala with dimensions independent of longitudinal position.
- Rigid walls.
- Incompressible and inviscid fluid.
• Cochlear-partition geometry independent of longitudinal position.

• Transverse partition-velocity distribution independent of longitudinal position.

• Dynamically linear fluid and partition.

• Locally determined partition dynamics: negligible longitudinal stiffness.

With the exception of local partition dynamics, the above assumptions are not essential: The result of including a feature into the model is either negligible when compared to those of other features of the model, or the feature’s inclusion into the model increases the model’s complexity without substantially altering the model’s performance range. As an example of the first case, consider modelling cochlear-partition geometry. Cochlear anatomy is such that the moving part of the cochlear partition is narrower near the base than near the apex; but if the changes in partition dynamics along the cochlear length contribute more to the cochlear response than the change in shape, the changing shape is negligible. As an example of the second case, aspects of the model that change slowly—such as the cross-sectional area of the scala—can be introduced into the model at a cost of increased complexity; neither the additional explanatory power nor the improvement in response achieved by doing so is sufficient to motivate the change. The negligibility of such features should be quantified; however, omitting them for the nonce returns increased understanding given a modest investment in mathematical detail.

In the box-cochlea model, the uncoiled cochlear is squared up and represented as a box of breadth $b$, depth $d$, and length $l$. The surfaces at $z = 0$, $z = b$, and $y = b$ are rigid. The surface at $y = 0$ is rigid with the exception of the region from $z = z_1$ to $z = z_2$, the region of the moving cochlear partition. (In a more general formulation, $z_1$ and $z_2$ are functions of longitudinal position. For the bulk of this work, both $z_1$ and $z_2$ are treated as constants, and the partition is assumed to move only over the region $|z - b/2| < \epsilon b/2$, where $0 < \epsilon < 1$ is the fraction of the breadth that actually moves.) Steele and Zais studied a three-dimensional cochlear model with coiled geometry. They report "no significant difference present between the straight
[and] coiled models. The same is true in ... the phase plot and ... the plot of the wavelength of the traveling wave on the basilar membrane [42, pg. 1851]."

In the long-wave regime of cochlear partition motion (see Section 4), fluid flow in the scala is longitudinally directed and uniformly distributed across the cross-sectional area; volume velocity is proportional to point velocity with the proportionality being the cross-sectional area; velocity is independent of cross-sectional shape. In the short-wave regime, fluid motion is limited to a region near the moving cochlear partition. In that case, the scala walls seem substantially distant, and, again, cross-sectional shape is irrelevant. Finally, the comparison by Steele and Taber of a three-dimensional mathematical model with three-dimensional physical models led them to conclude: "...taper [and] cross-section shape ... do not have much effect [40, pg. 1017]."

Even though the basilar membrane is the stiffest element in the cochlear-partition complex [3, pg. 468], it is substantially more compliant than the bony cochlear walls or cochlear shelf. The walls and shelf are treated as being rigid [20].

The cochlear fluid is treated as incompressible and inviscid. Lighthill’s analysis of cochlear-energy flow [20] discusses compressible cochlear fluids. When the volume velocities at the oval and round windows are combined to yield even (push-push) and odd (push-pull) components, there is a travelling compression wave, albeit with a very fast velocity, for the even part of the pressure (in the vertical variable with respect to the partition surface) due to the push-push component of the source. Interference between forward and backward travelling components yields a standing wave in the scala, with pressure nulls at certain points for very high frequencies. In the incompressible case, both the even wave’s propagation velocity and wavelength become infinite. Because the compression wave is even with respect to the partition surface, there is no associated cochlear-partition displacement with the compression wave; there is, however, associated volume flow in the scala.

Treating cochlear fluids as inviscid does not mean that the model is incapable of accounting for frictional losses in the fluids; such losses are sometimes moved into the partition model. As noted by Neely, the loss term in the partition impedance can be made to represent both internal partition friction and viscous losses in the
boundary layer of fluid near the moving partition [22]. Ignoring nonlinear effects in
the cochlear-fluid description, amounts to assuming that the nonlinear terms in the
Navier-Stokes equations are small compared to the linear terms.

The partition velocity distribution across the partition breadth is assumed to have
a half-cosinusoidal shape [4, 20, 40]. The conclusions of Steele and Taber regarding
scala cross-sectional shape may be extended to the detailed shape of the transverse
velocity distribution.

The work of Rhode et al. demonstrated that partition motion is nonlinear in the
most sensitive regions. In that context, linear modeling efforts should be valid on
the basal side of place responses, on the low-frequency side of frequency responses.
However, the real utility of linear, passive cochlear macromechanical modelling lies in
modelling the dead cochlea. Displacements are assumed to be sufficiently small such
that nonlinear effects are negligible for both fluid and partition dynamics.

3.2 Cochlear-Fluid Functions

Linear, cochlear-fluid dynamics are described by cochlear-fluid functions. In this
section, I derive those functions for each of one-, two-, and three-dimensional cochlear-
fluid motion within the box-cochlea geometry.

When the size of longitudinally directed fluid motion is very much greater than
either vertically or transversely directed ones, cochlear-fluid dynamics are those of
a slug of fluid moving one-dimensionally, back-and-forth, in the scala cross section.
Zwislocki and others pioneered one-dimensional models in the late 1940s and early
1950s [49]; Zwislocki continues to explore their properties, particularly regarding sheer
motion between the basilar and tectorial membranes [50]. Other researchers have
investigated such models, including Peterson and Bogert [24], Zweig et al. [46], Hall
[8, 9], and Zwicker [47].

Two-dimensional cochlear-fluid motion occurs when transverse fluid motion is
ingorable compared to longitudinal and vertical fluid motions. Ranke introduced
two-dimensional models about the same time as Zwislocki was developing the trans-
mission-line model [25]. Others using two-dimensional models include Siebert [37] Allen [2], Neely [21], and Lesser and Berkley [18].

In the most general case, no rectilinear component of cochlear-fluid motion is ignored, the cochlear partition occupies only a fraction of the scala width, and fluid dynamics are three dimensional. Both the transverse-velocity distribution and vertical cochlear walls become constraining factors. Three-dimensional models were pioneered during the 1970s by Steele and Taber [39, 40]; others exploring such models include Siebert [35], and deBoer et al. [5, 6, 7].

### 3.2.1 One-Dimensional Motion

In the one-dimensional model, the pressure in and point velocity of the fluid are uniformly distributed across the scala cross section at each longitudinal position: $P(x, y, z) = P(x)$ and $\vec{V}(x, y, z) = V_x(x)\hat{x}$, where $\hat{x}$ is a unit vector in the $x$-direction. The moving partition occupies the entire breadth; its velocity is uniform in the transverse variable: $V_p(x, z) = V_p(x)$. The partition volume-velocity density is $V(x) = V_p(x)b$.

Consider the slab of fluid in the scala between $x$ and $x + \Delta$. Newton’s force law and conservation of mass determine the fluid dynamics. Newton’s law says that the force of the pressure drop between $x$ and $x + \Delta$ acting on the slab is equal to the inertial force.

$$P(x)bd - P(x + \Delta)bd = j\omega \rho V_x(x)bd\Delta$$

(3.12)

For an incompressible fluid, the cochlear-partition displacement accounts for the difference in volume flow through the cross sections at $x$ and $x + \Delta$.

$$V_x(x)bd - V_x(x + \Delta)bd = V_p(x)b\Delta$$

(3.13)

In the limit as $\Delta$ vanishes, these become

$$\frac{dP(x)}{dx} = -j\omega \rho V_x(x)$$

(3.14)

$$\frac{dV_x(x)}{dx} = \frac{-V_p(x)}{bd} = -\frac{V(x)}{bd}$$

(3.15)
The (spatial-)frequency-domain representations of the above are:

\[ jk \tilde{P}(k) = -j\omega \rho \tilde{V}_z(k) \]  \hspace{1cm} (3.16)
\[ jk \tilde{V}_z(k) = -\frac{\tilde{V}(k)}{bd} \]  \hspace{1cm} (3.17)

Using the value of \( \tilde{V}(k) \) from Equation 3.17 in Equation 3.16, the frequency-domain representation of the fluid pressure in terms of the frequency-domain representation of the partition volume velocity density is:

\[ \tilde{P}(k) = -\frac{j\omega \rho}{k^2bd} \tilde{V}(k) \]  \hspace{1cm} (3.18)

The space-domain representation is:

\[ P(x) = -\frac{j\omega \rho}{2\pi} \int_{-\infty}^{\infty} \tilde{H}(k) \tilde{V}(k) e^{jkx} dk \]  \hspace{1cm} (3.19)

where \( \tilde{H}_1(k) = 1/k^2bd \) is the one-dimensional cochlear-fluid function.

Let \( Y_p(x, \omega) \), where \( V(x) = Y_p(x, \omega) P_d(x) \), denote the partition acoustic admittance per unit length. Because the fluid pressure in the scala is assumed to be odd

---

The space-domain representation of a function \( F(x) \), and the frequency-domain representation of \( F(k) \), are mutually defined by:

\[ \tilde{F}(k) = \int_{-\infty}^{\infty} F(x) e^{-jkx} dx \]
\[ F(x) = \frac{1}{j2\pi} \oint \tilde{F}(k) e^{jkx} dk \]

which differs from the usual Laplace transform only in the complex frequency variable \( s \) of the Laplace transform being replaced by \( jk \). As such, regions of convergence of \( \tilde{F}(k) \) are strips in the complex \( k \)-plane; the contour of integration in the synthesis formula lies within a region of convergence of \( \tilde{F}(k) \). The real part of \( k \) is the rate of phase accumulation; the imaginary part, the rate of envelope accumulation. The frequency-domain representation of the space derivative is

\[ \frac{dF(x)}{dx} = \frac{1}{d\pi} \oint \tilde{F}(k) e^{jkx} dk \]
\[ = \frac{1}{j2\pi} \oint jk \tilde{F}(k) e^{jkx} dk \]
\[ \frac{dF(x)}{dx} \leftrightarrow jk \tilde{F}(k) \]

For simplicity of notation, the synthesis formula is denoted in the text by

\[ F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(k) e^{jkx} dk \]
with respect to the partition (due to the push-pull nature of the drive and mirror-image symmetry), the pressure drop across the partition is twice the fluid pressure on the positive-$y$ side; $P_d(x) = 2P(x)$. The fluid pressure is then

\[ P(x) = V(x)/2Y_p(x, \omega) \]  

(3.20)

Equating Equations 3.19 and 3.20 yields a homogeneous integral equation in the partition volume velocity density:

\[ \frac{V(x)}{2j\omega \rho Y_p(x, \omega)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{H}_1(k) \tilde{V}(k) e^{ikx} dk = 0 \]  

(3.21)

The corresponding integral equation in pressure is

\[ 2j\omega \rho Y_p(x, \omega) P(x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{P}(k) \tilde{H}_1(k) e^{ikx} dk = 0 \]  

(3.22)

### 3.2.2 Two-Dimensional Fluid Motion

For two-dimensional (in the $x$-$y$ plane) fluid motion, the fluid pressure and velocity are independent of transverse position; there are no transversely directed forces or fluid flows. Pressures associated with incompressible two-dimensional fluid flow satisfy Laplace's equation.

\[ \nabla^2 P(x, y) = 0 \]  

(3.23)

The normal velocity vanishes at the rigid walls at $y = \pm d$ and is continuous with the partition velocity at $y = 0$:

\[ -\frac{1}{j\omega \rho} \frac{\partial P(x, y)}{\partial y} \bigg|_{y=\pm d} = 0 \]  

(3.24)

\[ -\frac{1}{j\omega \rho} \frac{\partial P(x, y)}{\partial y} \bigg|_{y=0^+} = V(x, 0) = -V_p(x) \]  

(3.25)

A pressure which satisfies the above conditions is

\[ P(x, y) = -\frac{j\omega \rho}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh k(y - d)}{kb \sinh kd} \tilde{V}(k) e^{ikx} dk \]  

(3.26)

where

\[ V(x) = V_p(x) b = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(k) e^{ikx} dk \]  

(3.27)

\[ \tilde{V}(k) = \int_{-\infty}^{\infty} V(x) e^{-ikx} dx \]  

(3.28)
Equation 3.26 specifies the pressure at each point in the fluid in terms of the spatial frequency domain representation of the partition volume velocity density. At \( y = 0^+ \), that pressure is:

\[
P(x) = P(x, 0^+) = -\frac{j\omega p}{2\pi} \int_{-\infty}^{\infty} \tilde{H}_2(k) \tilde{V}(k) e^{jkr} \, dk
\]

(3.29)

The two-dimensional cochlear-fluid function is

\[
\tilde{H}_2(k) = \frac{\coth kd}{kb}
\]

(3.30)

As shown in Chapter 4 (pg. 81), when wavelengths are very long—when wavenumbers are very small, \( |k| \ll \pi/d \)—fluid velocity is essentially longitudinally directed and uniformly distributed in the scala cross section; any vertical fluid components associated with partition displacement are negligible compared to the longitudinal component of fluid velocity. In that case, the two-dimensional cochlear function is approximately

\[
\tilde{H}_2(k) \approx \frac{1}{k^2 bd} + \frac{d}{3b}
\]

(3.31)

In the long-wave regime, the two-dimensional cochlear function has the same \( k^2 bd \) form as the one-dimensional function does. What is the role of the constant offset \( d/3b \)? It will be shown in Section 4.3 that the offset plays the same role in the long-wave regime of two-dimensional motion that additional cochlear-partition mass plays in the one-dimensional case.

Alternatively, when wavelengths are short—when the real-part of the wavenumber is large, \( \text{Re}\{k\} \gg (2/d) \log 2 \)—the two-dimensional cochlear function is approximately

\[
\tilde{H}_2(k) \approx \frac{1}{kb}
\]

(3.32)

In the short-wave regime, longitudinal and vertical components of fluid motion are comparable, and fluid motion is limited to a region close to the partition. In the short-wave regime, fluid motion is closely coupled to partition motion.
### 3.2.3 Three-Dimensional Fluid Motion

As with the two-dimensional case, the pressure at any point within a chamber filled with incompressible fluid satisfies Laplace’s equation, this time in three dimensions:

\[
\tilde{\nabla}^2 P(x, y, z) = 0
\]  

(3.33)

The normal component of fluid velocity vanishes at the rigid walls and is continuous with velocity in the partition plane at \(y = 0\):

\[
-\frac{1}{j\omega \rho} \left. \frac{\partial P(x, y, z)}{\partial z} \right|_{z=0,b} = 0
\]

(3.34)

\[
-\frac{1}{j\omega \rho} \left. \frac{\partial P(x, y, z)}{\partial y} \right|_{y=\pm d} = 0
\]

(3.35)

\[
-\frac{1}{j\omega \rho} \left. \frac{\partial P(x, y, z)}{\partial y} \right|_{y=0} = V_y(x, 0, z) = -V_p(x, z)
\]

(3.36)

With \(\tilde{V}_n(k)\) being a spatial-frequency-domain representation of the longitudinally varying component weight of the transverse-velocity distribution, the velocity in the \(x-z\)-plane is represented as:

\[
V_y(x, 0, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{V}_n(k) \cos \left( \frac{n\pi z}{b} \right) e^{jkx} dk
\]

(3.37)

where

\[
V_n(x) = \frac{1}{b} \int_0^b V_y(x, 0, z) \cos \left( \frac{n\pi z}{b} \right) dz
\]

(3.38)

\[
\tilde{V}_n(k) = \int_{-\infty}^{\infty} V_n(x) e^{-jkx} dx
\]

(3.39)

The pressure in the fluid for \(y > 0\) is

\[
P(x, y, z) = -\frac{j\omega \rho}{2\pi} \times \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\tilde{V}_n(k) \cosh \left[ \sqrt{k^2 + (n\pi/b)^2} (y - d) \right]}{\sqrt{k^2 + (n\pi/b)^2} \sinh \left[ \sqrt{k^2 + (n\pi/b)^2} d \right]} \cos \left( \frac{n\pi z}{b} \right) e^{jkx} dk
\]

(3.40)

Representation of partition dynamics in the three-dimensional case is more subtle than for the one- and two-dimensional ones where partition motion is uniform across its breadth; both the pressure and the partition admittance density must be
appropriately defined. The normalized distribution of partition velocity across the partition breadth at location $x$, $\phi(x,z)$, is given by:

$$\phi(x,z) = -\frac{V_y(x,0,z)}{V(x)}$$  \hspace{1cm} (3.41)

The weighted-average pressure, $P(x)$, is defined such that the product of the conjugate of the volume-velocity density and the weighted-average pressure yields the power density flowing into the cochlear partition. The complex amplitude of the power flowing into the partition per unit length at longitudinal location $x$ is found by integrating the product of local pressure and conjugate point velocity across the transverse variable. The weighted-average pressure difference is then defined as that quantity which when multiplied by the conjugate volume velocity yields the same number.

$$W(x) = \int_0^b (-V_y^*(x,0,z))(2P(x,0,z))dz$$ \hspace{1cm} (3.42)

$$= V^*(x) \cdot 2 \int_0^b \phi^*(x,z)P(x,0,z)dz$$ \hspace{1cm} (3.43)

The weighted-average pressure at the cochlear partition is identified as

$$P(x) = 2 \int_0^b \phi^*(x,z)P(x,0,z)dz$$ \hspace{1cm} (3.44)

The partition admittance density is defined for the three-dimensional case as the ratio of the volume-velocity density to weighted-average pressure; $Y_p(x,\omega) = V(x)/P(x)$.

Substituting the fluid-pressure value at the partition surface into the expression for weighted-average pressure difference yields:

$$P(x) = -\frac{2j\omega \rho}{2\pi} \int_{-\infty}^{\infty} \tilde{H}_3(x,k)\tilde{V}(k)e^{jkx}dk$$ \hspace{1cm} (3.45)

where

$$\tilde{V}(k) = \int_{-\infty}^{\infty} V(x)e^{-jkx}dx$$ \hspace{1cm} (3.46)

$$V(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(k)e^{jkx}dk$$ \hspace{1cm} (3.47)

and

$$\Phi_n(x) = \int_0^b \phi(x,z)\cos\left(\frac{n\pi z}{b}\right)dz$$ \hspace{1cm} (3.48)

$$\phi(x,z) = \frac{1}{b} \sum_{n=-\infty}^{\infty} \Phi_n(x)\cos\left(\frac{n\pi z}{b}\right)$$ \hspace{1cm} (3.49)
and
\[
\tilde{H}_3(x, k) = \sum_{n=-\infty}^{\infty} |\Phi_n(x)|^2 \frac{\coth \sqrt{k^2 + (n\pi/b)^2} d}{\sqrt{k^2 + (n\pi/b)^2} b}
\] (3.50)

\[H_3(x, k)\] is the three-dimensional cochlear-fluid function.

The three-dimensional cochlear function has long- and short-wave approximate regimes similar to the two-dimensional cochlear function. The long-wave regime is appropriate for very small magnitude wavenumbers; when \(|k| \ll \pi/d\),

\[\tilde{H}_3(k) \approx \frac{1}{k^2 bd} + \tilde{H}_{3d}\] (3.51)

where \(\tilde{H}_{3d} > 1/3\) is the three-dimensional offset.

The short-wave approximation for the three-dimensional cochlear function is valid when \(\text{Re} \{k\} \gg n\pi/b\) for some \(n\); in that regime, the three-dimensional cochlear function is approximately:

\[\tilde{H}_3(k) \approx \frac{1}{kb} \left(1 + 2 \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2\right)\] (3.52)

For the three-dimensional cochlear function, the scale factor of \(1/kb\) is greater than unity, and the wavenumber argument must have a larger real part in order to yield the same value as the two-dimensional cochlear function in its short-wave regime. Section 4.4 discusses more details about the differences between the two- and three-dimensional short wave regimes. As with the two-dimensional case, however, one can envision the long-wave regime corresponding to longitudinally directed fluid velocity which is uniformly distributed in the scala cross section. For the three-dimensional case, there is a transition region between long- and short-wave regimes in which fluid motion is fully three dimensional. In the short-wave regime of the three-dimensional cochlear function, as with the two-dimensional case, fluid motion is limited to the region near and is closely coupled to the moving cochlear partition.
3.3 Phase-Integral Solution to Cochlear-Integral Equations

A solution to the cochlear-integral equations satisfies both the local dynamics imposed by the cochlear partition and the global ones imposed by the cochlear fluids’ moving within the scalae geometry. Given no spatial variation to cochlear-partition dynamics, the motion should consist of a superposition of modes, each having some constant complex wavenumber. In this section, I explore an extension of that idea to include the case of a slowly varying impedance, the phase-integral approximate solution. Application of the phase-integral (or WKB or WKBJ method) to the cochlear problem was introduced by Steele [38]; the approach presented here is based on Siebert’s work [35].

A cochlear-integral equation is of the form

\[ F(x)Q(x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(x, k)\tilde{Q}(k)e^{jkx}dk = 0 \]  \hspace{1cm} (3.53)

where

\[ \tilde{Q}(k) = \int_{-\infty}^{\infty} Q(x)e^{-jkx}dx \] \hspace{1cm} (3.54)
\[ Q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{Q}(k)e^{jkx}dk \] \hspace{1cm} (3.55)

and where

\[ \tilde{G}(x, k) = \int_{-\infty}^{\infty} G(x, \xi)e^{-jk\xi}d\xi \] \hspace{1cm} (3.56)
\[ G(x, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(x, k)e^{jk\xi}dk \] \hspace{1cm} (3.57)

Given \( F(x) \) and \( \tilde{G}(x, k) \), I desire to find \( Q(x) \). For the velocity-integral equation as previously developed, \( F(x) = 1/2j\omega \rho Y_p(x, \omega) \), \( \tilde{G}(x, k) = \tilde{H}(k) \), and \( Q(x) = V(x) \).

When \( F \) and \( G \) are independent of \( x \), \( Q(x) \) in the left-most term can be written in Fourier synthesis form, and Equation 3.53 becomes

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ F + \tilde{G}(k) \right] \tilde{Q}(k)e^{jkx}dk = 0 \] \hspace{1cm} (3.58)
The solution to Equation 3.58 is

\[ Q(x) = \sum_n Q_n e^{j k_n x} \quad (3.59) \]

where the wavenumbers are values of \( k \) for which \( F + \tilde{G}(k) \) vanishes, and the \( Q_n \) are determined by boundary conditions. The number of roots depends on the details of \( \tilde{G}(k) \). The solution is a superposition of modes, each term with complex wavenumber \( k_n \); the complex phase of each mode is \( k_n x = \int_0^x k_n d\xi \).

For the constant impedance case with a compliant and some resistance, the wavenumbers for forwardly propagating waves will fall in the fourth quadrant of the complex wavenumber plane. The magnitude and angle of the resulting velocity having such a wavenumber will both be linearly decreasing functions of the place variable \( x \).

If \( F \) and \( G \) change slowly with \( x \), an approximate solution is found by trying exponential forms with the complex phase being the integral of the wavenumber, now itself a function of place. Consider trial solutions to Equation 3.53 of the form \( Q(x) = Q_0 e^{j \psi(x)} \). Writing Equation 3.53 in the form of a convolution,

\[ F(x)Q(x) + \int_{-\infty}^{\infty} G(x, \xi)Q(x - \xi) d\xi = 0 \quad (3.60) \]

and substituting the trial solution into Equation 3.60 yields

\[ Q_0 \left[ F(x)e^{j \psi(x)} + \int_{-\infty}^{\infty} G(x, \xi)e^{j \psi(x-\xi)} d\xi \right] = 0 \quad (3.61) \]

For a non-trivial solution, the bracketed term must vanish.

A Taylor-series expansion of \( \psi(x - \xi) \) about \( \xi = 0 \), truncated to the quadratic term, is

\[ \psi(x - \xi) \approx \psi(x) - \psi'(x)\xi + \frac{1}{2} \psi''(x)\xi^2 \quad (3.62) \]

where primes indicate differentiation with respect to the independent variable \( x \). The exponential of the complex phase is

\[ e^{j \psi(x-\xi)} \approx e^{j \psi(x)} \cdot e^{-j \psi'(x)\xi} \cdot e^{\frac{j}{2} \psi''(x)\xi^2} \approx e^{j \psi(x)} \cdot e^{-j \psi'(x)\xi} \cdot \left( 1 - \frac{1}{2j} \psi''(x)\xi^2 \right) \quad (3.63) \]

(3.64)
where \( e^\theta \approx 1 + \theta \) for small \( \theta \). The convolution integral in Equation 3.60 becomes

\[
Q_0 \int_{-\infty}^{\infty} G(x, \xi) e^{i\psi(x)} d\xi
\approx Q_0 e^{i\psi(x)} \int_{-\infty}^{\infty} G(x, \xi) \left( 1 - \frac{1}{2j} \psi''(x) \xi^2 \right) e^{-j\psi'(x)\xi} d\xi
= Q_0 e^{i\psi(x)} \left[ \tilde{G}(\psi'(x)) + \frac{1}{2j} \psi''(x) \tilde{G}_{\psi'}(x, \psi'(x)) \right]
\]  
(3.65)

where subscripts denote partial differentiation with respect to the subscript variable, and where the Fourier identity

\[
\int_{-\infty}^{\infty} \xi^2 G(x, \xi) e^{-jk\xi} d\xi = -\tilde{G}_{kk}(x, k)
\]  
(3.67)

has been exploited. The convolution equation becomes

\[
Q_0 e^{i\psi(x)} \left[ F(x) + \tilde{G}(x, \psi'(x)) + \frac{1}{2j} \psi''(x) \tilde{G}_{\psi'}(x, \psi'(x)) \right] = 0
\]  
(3.68)

The phase derivative, \( \psi'(x) \), must satisfy

\[
F(x) + \tilde{G}(x, \psi'(x)) + \frac{1}{2j} \psi''(x) \tilde{G}_{\psi'}(x, \psi'(x)) = 0
\]  
(3.69)

Let \( k(x) \) be the solution to Equation 3.69 when the rightmost term is ignored; i.e., \( k(x) \) is the solution to

\[
F(x) + \tilde{G}(x, k(x)) = 0
\]  
(3.70)

Equation 3.70 is the eikonal equation in the wavenumber \( k(x) \). With \( \psi'(x) \) being the solution to Equation 3.69, let the difference between the \( k \) and \( \psi' \) be

\[
\Delta(x) = \psi'(x) - k(x)
\]  
(3.71)

For \( \Delta(x) \) sufficiently small, \( \tilde{G}(x, \psi'(x)) \) is well approximated by a Taylor series expanded about \( k(x) \) and truncated to the linear term:

\[
\tilde{G}(x, \psi'(x)) \approx \tilde{G}(x, k(x)) + \tilde{G}_{k}(x, k(x)) \Delta(x)
\]  
(3.72)

Approximating \( \tilde{G}_{\psi'}(x, \psi'(x)) \) by \( \tilde{G}_{kk}(x, k(x)) \), etc., the phase derivative equation, Equation 3.69 becomes:

\[
\tilde{G}(x, k(x)) \Delta(x) + \frac{1}{2j} \psi''(x) \tilde{G}_{kk}(x, k(x)) \approx 0
\]  
(3.73)
Exploiting the relationships amongst differentials

$$\psi''(x) \tilde{G}_{kk}(x, k(x)) = \frac{d}{dx} \tilde{G}_k(x, k(x)) - \tilde{G}_{zk}(x, l(x))$$

(3.74)

Equation 3.73 becomes

$$\tilde{G}_k(x, k(x)) \Delta(x) + \frac{1}{2j} \left[ \frac{d}{dx} \tilde{G}_k(x, k(x)) - \tilde{G}_{zk}(x, k(x)) \right] = 0$$

(3.75)

which is solvable for the phase-derivative error, $\Delta(x)$.

$$\Delta(x) = \frac{1}{2j} \frac{\tilde{G}_{zk}(x, k(x))}{\tilde{G}_k(x, k(x))} - \frac{1}{2j} \frac{d}{dx} \log \tilde{G}_k(x, k(x))$$

(3.76)

The phase $\psi(x)$ is then

$$\psi(x) = \int_{-\infty}^{x} \psi'(\xi) d\xi$$

(3.78)

$$= \int_{-\infty}^{x} (k(\xi) + \Delta(\xi)) d\xi$$

(3.79)

$$= \int_{-\infty}^{x} k(\xi) d\xi + \frac{1}{2j} \int_{-\infty}^{x} \frac{\tilde{G}_{zk}(\xi, k(\xi))}{\tilde{G}_k(\xi, k(\xi))} d\xi - \frac{1}{2j} \log \tilde{G}_k(x, k(x))$$

(3.80)

and the solution $Q(x)$ becomes

$$Q(x) \approx Q_0 \times \exp \left[ j \int_{-\infty}^{x} k(\xi) d\xi + \frac{1}{2} \int_{-\infty}^{x} \frac{\tilde{G}_{zk}(\xi, k(\xi))}{\tilde{G}_k(\xi, k(\xi))} d\xi - \frac{1}{2} \log \tilde{G}_k(x, k(x)) \right]$$

(3.81)

$$= Q_0 \left[ \tilde{G}_k(x, k(x)) \right]^{\frac{1}{2}} e^{\frac{j}{2} \int_{-\infty}^{x} k(\xi) d\xi}$$

(3.82)

The total solution will be a superposition of terms like the above, with $Q_n$ being the coefficient for each wavenumber $k_n(x)$. The $Q_n$ are determined to suit the given boundary conditions.

When $G(x, k)$ is independent of $x$, $G(x, k) = G(k)$, the partial derivative of $G$ with respect to $x$ vanishes, and the solution becomes

$$Q(x) = \sum_{n} Q_n [G'(k_n(x))]^{-\frac{1}{2}} e^{j \int_{-\infty}^{x} k_n(\xi) d\xi}$$

(3.83)

where $G'(k) = dG(k)/dk$. 

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3.4 Boundary Conditions

The cochlear-partition volume-velocity density is a superposition of modes, each having a wavenumber which is a solution to the eikonal equation:

\[
\frac{1}{2j\omega \rho Y_p(x, \omega)} + \ddot{H}(k) = 0
\]  

(3.84)

Because each cochlear function is an even function of its argument, the roots of the eikonal equation come in pairs, the members of which have opposite signs. The one-dimensional cochlear function simply has one pair of roots; the two- and three-dimensional ones, an infinity of roots. When the impedance, \( Z_p(x, \omega) = 1/Y_p(x, \omega) \) has a positive real part, the wavenumbers corresponding to forward propagating waves lie in the complex plane's fourth quadrant; those corresponding to backward propagating waves, in the second quadrant. In order to conform to the usual usage where forwardly propagating waves have positive real wavenumber, I replace the phase integral by its negative. I index the wavenumbers according to the size of the wavenumber's imaginary part at \( x = 0 \) for the positively propagating modes. The wavenumber with the least negative imaginary part at the origin is the first wavenumber, and the mode index increases for increasingly negative imaginary part.

\[
0 > \text{Im} \{ k_1(0) \} > \text{Im} \{ k_2(0) \} > \ldots > \text{Im} \{ k_n(0) \} > \ldots
\]  

(3.85)

with \( k_{-n}(x) = -k_n(x) \).

With the wavenumbers so numbered, the superposition of modes is expressed (without loss of generality) as:

\[
V(x) = \sum_{n=1}^{\infty} \left\{ V_n \left[ \frac{\ddot{H}'(k_n(x))}{\ddot{H}'(k_n(0))} \right]^{-1/2} e^{-j \int_0^x k_n(\xi) d\xi} + V_{-n} \left[ \frac{\ddot{H}'(k_{-n}(x))}{\ddot{H}'(k_{-n}(0))} \right]^{-1/2} e^{-j \int_x^0 k_{-n}(\xi) d\xi} \right\}
\]  

(3.86)

In principle, the modal weights \( V_{\pm n} \) can be determined by matching the specified boundary conditions: the stapes and helicotrema pressure-velocity relationships. An alternative that yields substantial insight is to equate the volume velocity of the stapes with that of the entire cochlear partition, plus, if necessary, that of the helicotrema. Because the cochlear fluid is modelled as being incompressible, any displacement of

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the stapes must correspond to a displacement of the cochlear partition or through
the helicotrema. Mathematically,

\[ v_s A_s = \int_0^\ell V(x)dx + U_h \]

\[ \approx \int_0^\ell V(x)dx \]  

(3.87)  

(3.88)

where the approximation is valid for frequencies where the response is cutoff well
before the helicotrema; \( v_s \) is the stapes velocity, \( A_s \) is the stapes area, and \( U_h \) is the
volume-velocity flow through the helicotrema. For the usual case of the only source
being that at the stapes, cochlear partition mode is described by a single, forwardly
propagating mode, the coefficient of which is easily determined. Excepting all but the
lowest frequencies, for the linear, passive cochlea driven at the stapes, there are no
backward modes to consider and no higher-order modes with which to be concerned.

When the sole source of volume displacement is at the stapes, and the place
response cuts off before the helicotrema, then no backward propagating modes are
excited. Without loss of generality, the partition velocity is written:

\[ V(x) = \sum_{n=1}^\infty V_n \left[ \frac{\hat{H}'(k_n(x))}{\hat{H}'(k_n(0))} \right]^{-1/2} e^{-j\int_0^x k_n(\xi)d\xi} \]

(3.89)

Using the approximation\(^2\)

\[ \int_{-\infty}^x f(\xi)e^{j\phi(\xi)}d\xi \approx \frac{1}{\phi'(x)}f(\xi)e^{j\phi(\xi)} \]

(3.90)

\(^2\)Consider an integral of the form

\[ \int_{-\infty}^x f(\xi)e^{j\phi(\xi)}d\xi \]

Let

\[ I(x) = \frac{f(x)}{j\phi'(x)}e^{j\phi(x)} \]

The derivative of \( I(x) \) is

\[ I'(x) = \frac{f'(x)}{j\phi'(x)}e^{j\phi(x)} + j\frac{f(x)\phi''(x)}{\phi'(x)^2}e^{j\phi(x)} + f(x)e^{j\phi(x)} \]

\[ = f(x)e^{j\phi(x)} \left[ 1 + j\frac{1}{\phi'(x)} \left( \frac{d}{dx} \log \frac{\phi'(x)}{f'(x)} \right) \right] \]

\[ \approx f(x)e^{j\phi(x)} \]

where the final approximation is justified when the magnitude of the \( j \) term is small compared to
one.
the volume velocity of the entire cochlear partition is written as:

$$U(0, \ell) = \sum_{n=1}^{\infty} V_n \left\{ \frac{1}{jk_n(0)} - \frac{1}{jk_n(\ell)} \left[ \frac{\tilde{H}'(k_n(\ell))}{\tilde{H}'(k_n(0))} \right]^{-1/2} e^{-j \int_0^\ell k_n(\xi) d\xi} \right\}$$

$$\approx \sum_{n=1}^{\infty} \frac{V_n}{jk_n(0)}$$

$$= \frac{V_1}{jk_1(0)} + \frac{V_2}{jk_2(0)} + \ldots$$

$$= \frac{V_1}{jk_1(0)} \left( 1 + \frac{V_2k_1(0)}{V_1k_2(0)} + \ldots \right)$$

$$\approx \frac{V_1}{jk_1(0)}$$

(3.91) (3.92) (3.93) (3.94) (3.95)

The first approximation is the cutoff approximation: the imaginary part of the integral is large and negative such that the magnitude of the exponential is tiny. The second approximation reflects the relative sizes of the imaginary parts of the various wavenumbers at the origin: Because the real part of each mode is tiny when \( x = 0 \), the magnitude of the first mode is smaller than that of the other modes, and each ratio in the sum is small. The gain of the cochlear-partition volume-velocity per unit length relative to the volume velocity of the stapes is then

$$G(x) = \frac{V(x)}{V_s A_s} = jk(0) \left[ \frac{\tilde{H}'(k(x))}{\tilde{H}'(k(0))} \right]^{-1/2} e^{-j \int_0^x k(\xi) d\xi}$$

(3.96)

where \( k(x) = k_1(x) \).

It is not, in fact, important that the volume-velocity source be located at the stapes end. As long as the velocity source is located external to the cochlear partition itself—a condition which leaves the pressure of the partition at the location of the source constrained—the volume displacement of the partition must be the same as that of the source. Similar arguments as used above lead to the conclusion that the velocity distribution corresponding to the forwardly propagating primary mode will be evoked. This is Békésy's paradoxical motion, and it explains why bone conduction leads to perceptible sound, even when the middle-ear bones are fused.
3.5 Examples

I close this chapter with three comparisons between this and other cochlear models. Figure 3-3 shows cochlear-partition velocity magnitude and angle relative to unit stapes velocity as determined by Neely for his 1978 two-dimensional model. As opposed to the presentation here, that work used a finite-difference approach to numerically implement Laplace’s equation in the fluid. Figure 3-4 also shows the cochlear gain, but as determined using the phase-integral approximate solution to the cochlear-integral equations developed here. Model parameters are the same in both cases. The figures are strikingly similar: The peakiness of each pair of magnitude curves is approximately the same; both sets show the same downward slope of the value at the best place. The angles shapes are similar, with similar overall behavior (reaching plateaus beyond the best place). The two sets of figures differ in only minor ways: The cochlear gain at the origin for the lower drive frequencies is higher in my results than in Neely’s; this is probably due to the volume-velocity approximation used which treats the cochlea as infinite instead of finite (Neely’s model incorporated dynamics and boundary conditions at the helicotrema). The exact value of the angle plateaus differ for each frequency, but those plateaus occur when the magnitudes are already down between 40 and 60 dB from the peak value; i.e., when the amplitude is between 1 and 0.1 percent of its peak value. The phase-integral model’s output doesn’t have the change in slope on the apical side of the peak that the Neely model does, but again, those differences occur well beyond the peak. In all, the behavior of both log magnitude and angle curves up to and around the best place is essentially the same.

Figure 3-5 shows results from Allen’s 1977 two-dimensional model. Allen used a Green’s function approach with a numerical implementation of the space-domain integral equation (as opposed to the spatial-frequency-domain approach used here). Figure 3-6 shows results computed using the phase-integral approximate solution with Allen’s 1977 parameters and the same stimulus frequency. As with the Neely case, the agreement between the results as presented by Allen and the result computed
Figure 3-3: Cochlear gain, the ratio of partition velocity to stapes velocity, as determined by Neely for his 1978 model. In both panels, the abscissa is place in centimeters. In the upper panel, the ordinate is the log magnitude of the gain in dB; in the lower one, the ordinate is the angle in radians. The numbers by each curve indicate the sinusoidal steady-state frequency in kHz.
Figure 3-4: Cochlear gain, the ratio of partition velocity to stapes velocity, determined using the phase-integral approximate solution to cochlear integral equations for Neely's 1978 model. In both panels, the abscissa is place in centimeters. In the upper panel, the ordinate is the log magnitude of the gain in dB; in the lower one, the ordinate is the angle in π radians. The numbers by each curve indicate the sinusoidal steady-state frequency in Hz.
Figure 3-5: Cochlear gain log magnitude and phase as determined by Allen for his 1977 model. The abscissa is place in centimeters. One ordinate is the cochlear gain log magnitude specified in decibels; each division is 10 dB with the tick labelled 0 (radians) corresponding to -30 dB. The labelled ordinate is the angle of the partition velocity relative to the stapes velocity in radians. Drive frequency was 1000 Hz.

using the phase-integral solution are strikingly similar. Both magnitude plots show the same increase in slope right before the best place, and a very steep cutoff on the more apical side; both angle plots curve steeply downward to a plateau at around $-6\pi$ radians, although Allen's angle curve falls off again after a short plateau. As with the Neely case, the differences occur in regions where the response is down between a factor of 100 and 1000 from that at the peak.

Figure 3-7 shows results from Siebert's 1974 two-dimensional model. Siebert used a short-wave approximation of the two-dimensional cochlear function developed here. Figure 3-8 shows results computed using the phase-integral approximate solution with Siebert's 1974 parameters. There are no major differences between the results as determined by Siebert using his short-wave model, and the output of the phase-integral model.
Figure 3-6: Cochlear gain for the phase-integral approximate solution using Allen’s 1977 model parameters. See the caption to Figure 3-5 for a description of the axes.

3.6 Summary

For the case where cochlear fluids can be treated as linear, inviscid, and incompressible, and for the case where the cochlear partition can be modelled as an impedance—i.e., longitudinal stiffness effects are negligible—then cochlear dynamics are described by integral equations equating the global distribution of trans-partition pressure as determined from the global distribution of partition velocity through the cochlear function with the local pressure as determined from the local velocity by the impedance relationship. An approximate solution to such dynamical equations is found by the phase-integral method, in which the form of the velocity distribution is assumed to have the property of being a non-uniform travelling wave. The details of such a method include forming a target function from the impedance, solving the eikonal equation to find the local wavenumber, then determining the cochlear-function-derivative term and the phase-integral from the wavenumber.
Figure 3-7: Cochlear gain log-magnitude and angle as determined by Siebert for his 1974 model. The abscissa is place in centimeters. The upper panel shows the normalized magnitude of the partition velocity; the lower panel, phase angle of the partition velocity relative to stapes velocity. The numbers by each curve indicate sinusoidal drive frequency in Hz.

Figure 3-8: Cochlear gain log-magnitude and angle for the phase-integral approximate solution using Siebert’s 1974 model parameters. The abscissa is place in centimeters. The upper panel shows the normalized magnitude of the partition velocity; the lower panel, phase angle of the partition velocity relative to stapes velocity. The numbers by each curve indicate sinusoidal drive frequency in Hz.
Chapter 4

Cochlear-Fluid Functions

As shown in Chapter 3, linear cochlear macromechanics are described by integral equations of the form

$$\frac{1}{2j\omega \rho Y(x, \omega)} V(x) + \int_{-\infty}^{\infty} \tilde{H}(x, k) \tilde{V}(k) e^{ikx} dk = 0$$  \hspace{1cm} (4.1)

where $V(x)$ is the volume velocity per unit length of the cochlear partition, $Y(x, \omega)$ is the partition’s acoustic admittance per unit length, and $\tilde{H}(x, k)$ is the cochlear function, the spatial Fourier transform of the Green’s function, at cochlear place $x$.

The gain of the phase-integral approximate solution to such equations, the ratio of volume-velocity density to stapes volume velocity, is

$$\frac{V(x)}{V_s A_s} \approx jk(0, \omega) \left[ \frac{\tilde{H}'(k(x, \omega))}{\tilde{H}'(k(0, \omega))} \right]^{-1/2} \exp \left( -j \int_0^x k(\xi, \omega) d\xi \right)$$  \hspace{1cm} (4.2)

where $V_s$ is the stapes velocity, $A_s$ is the stapes area, and the local complex wavenumber, $k(x, \omega)$, is the solution to the eikonal equation,

$$\frac{1}{2j\omega \rho Y(x, \omega)} + \tilde{H}(x, k) = 0$$  \hspace{1cm} (4.3)

The complex logarithm of the above gain is

$$\log \frac{V(x)}{V_s A_s} = \log(jk(0, \omega)) - \frac{1}{2} \log \left[ \frac{\tilde{H}'(k(x, \omega))}{\tilde{H}'(k(0, \omega))} \right] - j \int_0^x k(\xi, \omega) d\xi$$  \hspace{1cm} (4.4)

and the log-magnitude and angle of the gain are then

$$\log \left| \frac{V(x)}{V_s A_s} \right| = \log |k(0, \omega)| - \frac{1}{2} \log \left| \frac{\tilde{H}'(k(x, \omega))}{\tilde{H}'(k(0, \omega))} \right| - \int_0^x \text{Im} \{k(\xi, \omega)\} d\xi$$  \hspace{1cm} (4.5)
\[ \frac{V(x)}{V_s A_s} = \frac{\pi}{2} + k(0, \omega) - \frac{1}{2} \frac{\tilde{H}'(k(x, \omega))}{\tilde{H}'(k(0, \omega))} - \int_0^\infty \text{Re} \{k(\xi, \omega)\} \, d\xi \quad (4.6) \]

The velocity response depends on three factors: The wavenumber at the origin, \( k(0) \), determines the mapping from stapes velocity to partition velocity at the origin, \( V(0)/V_s A_s = jk(0) \); the derivative term, \( [\tilde{H}'(k(x, \omega))/\tilde{H}'(k(0, \omega))]^{-1/2} \), describes the slow increase in the gain and slow increase in the phase (as functions of place) when \( \left| \int_0^\infty \text{Im} \{k(\xi, \omega)\} \, d\xi \right| \ll 1 \); the phase-integral term, \( -j \int_0^\infty k(\xi, \omega) \, d\xi \), describes the rapid cutoff of the magnitude response and corresponding steepness of the phase angle above the best place.\(^1\) Note that the real part of the wavenumber is the rate of phase angle accumulation in the (complex) phase integral, while the imaginary part—negative for fourth-quadrant wavenumbers—determines the amount of attenuation.

The method of determining cochlear place response from a given frequency of sinusoidal drive is then as follows: Determine the value of the partition impedance, \( Z_p(x, \omega) \), at that frequency for each point; from that, determine the value of the target cochlear function, \( \tilde{H}_t(x, \omega) = -Z_p(x, \omega)/2 j \omega \rho ; \) solve the eikonal equation to determine the local wavenumber, \( \tilde{H}(k(x, \omega)) = \tilde{H}_t(x, \omega) \) for each point; then determine both \( \tilde{H}'(k(x, \omega)) \) and \( \int_0^\infty k(\xi) \, d\xi \) for each location. For a particular impedance and target function, the wavenumbers—and, hence, the cochlear-function derivatives and the phase integral—depend on details of the cochlear function. This chapter addresses those details.

Before proceeding with a discussion of the cochlear function as a mapping from one complex plane (that of the wavenumbers) to another (the cochlear function) for each of one-, two-, and three-dimensional fluid motion, I present an example using the impedance from Neely's 1978 thesis.

The method of this chapter is to consider each of the cochlear functions in terms of normalized functions of normalized arguments. Doing so facilitates understanding each function's behavior relative to that of the others; furthermore, it makes clear the influence of geometrical parameters. Each normalized cochlear function depends not

\(^1\)The above statements assume an "ordinary" impedance, one with a monotonically decreasing stiffness as a function of place, and one with a positive resistance.
only on the normalized wavenumber (normalized to unit cochlear depth), but also on
the aspect ratio of scala breadth to scala depth. The three-dimensional cochlear func-
tion also depends on the shape of the transverse partition velocity. Unless explicitly
noted, the aspect ratio is assumed to be unity; the transverse distribution of partition
velocity, a half-cosine occupying 1/4 the breadth and centered on the midline.

The wavenumbers of interest to this investigation, those corresponding to waves
propagating from base to apex, are in the fourth quadrant of the complex wavenumber
plane. Rectangular and polar forms of the wavenumber are used in this chapter to
explore the mapping behavior of the cochlear functions.

4.1 An Example

As an example of the behavior of the cochlear function and its role in solving the
eikonal equation, consider the particulars of the impedance used by Neely in his 1978
thesis for a frequency of 1600 Hz [21]. The Neely impedance is

\[ Z_p(x, \omega) = \frac{K_0 e^{-\alpha x}}{j\omega} + R_0 + j\omega M_0 \]  (4.7)

This model for partition dynamics is a second-order, lumped-parameter dynamic
system: a mass, a spring, and a dashpot. The partition stiffness, \( K_0 e^{-\alpha x} \), is an
exponentially decreasing function of place as measured; the partition resistance, \( R_0 \),
and mass, \( M_0 \), are assumed to be constant; there is no longitudinal stiffness. The real
and imaginary parts of the Neely impedance are shown in the upper panel of Figure 4-
1. Parameter values for the Neely impedance are shown in Table 5-1. Problems with
lumped-parameter impedance representations are discussed in Chapter 5.

Equation 4.3 can be rewritten as

\[ \tilde{H}(k) = -\frac{Z_p(x, \omega)}{2j\omega \rho} \]

\[ = \tilde{H}_t(x, \omega) \]  (4.8)

where \( \tilde{H}_t(x, \omega) \) will be called the target cochlear function (or just the target function).

For Neely’s impedance, the target function is:

\[ H_t(x, \omega) = \frac{K_0 e^{-\alpha x}}{2\omega^2 \rho} - \frac{M_0}{2\rho} + j \frac{R_0}{2\omega \rho} \]  (4.10)
Figure 4-1: Stages in solving the eikonal equation and determining cochlear gain. Each panel displays the real (left column) or imaginary (right column) part of the label quantity for Neely’s 1978 model parameters and a frequency of 1600 Hz. Abscissae are all cochlear place in cm. The real part of the logarithmic gain is the log-magnitude in dB; the imaginary part, the angle in π radians.
The positive component of the target-function real part is an exponentially decreasing function of position with inverse-square frequency dependence; the imaginary part is a constant in place with an inverse frequency dependence. The real and imaginary parts of the target function as a function of place for a frequency of 1600 Hz are shown in the second panel from the top of Figure 4-1. At \( x = 0 \), the value of the target function is \( H_t(0, 2\pi 1600) = 48.7 + j0.1 \); at \( x = 3.5 \), \( H_t(3.5, 2\pi 1600) = -0.70 + j0.1 \). The real-part of the target function is zero at \( x = 2.1 \).

For Neely's thesis model, fluid dynamics are two dimensional:

\[
\tilde{H}(k) = \tilde{H}_2(k) = \frac{\coth kd}{kb} \tag{4.11}
\]

Recall that the wavenumber is a complex value; the cochlear function, a complex function of a complex variable. Although the details of that mapping from various regions of the complex wavenumber plane will be discussed directly in Section 4.3, I present the results immediately. Figure 4-2 shows the mapping by \( \tilde{H}_2(k) \) into the complex cochlear-function plane from several different regions of the complex wavenumber plane. Plots on the left are of the wavenumber plane; plots on the right, the cochlear-function plane.

Panels A and B of Figure 4-2 show the long-wave regime of \( \tilde{H}_2 \). For small magnitude wavenumbers, \( |kd| \ll \pi \), \( \coth kd/kb \approx 1/k^2bd + d/3b \); quarter circles in the fourth quadrant of the \( k \) plane map into half circles in the first two quadrants of the \( \tilde{H} \) plane, with smaller radius wavenumber corresponding to larger radius cochlear function; this is the so-called long-wave regime for two-dimensional motion.

Panels C and D of Figure 4-2 show the transition between long- and short-wave behavior. For smaller magnitude wavenumbers, the mapping is the long-wave mapping as above. For wavenumbers with larger real parts, \( \text{Re} \{kd\} \gg 1 \), the mapping is the short-wave approximation: \( \coth kd/kb \approx 1/kb \); eighth circles in the \( k \) plane map into eighth circles in the \( \tilde{H} \) plane, with, again, smaller radius wavenumber corresponding to larger radius cochlear function. Fluid motion in the short-wave regime is for all practical purposes limited to the region immediate to the partition; both longitudinal and vertical fluid displacements are on the same order of magnitude, and are closely
Figure 4-2: Mapping from the complex wavenumber plane to the complex cochlear function plane by the two-dimensional cochlear function $\tilde{H}_2(k) = \coth kd/kb$, for Neely's thesis parameters ($b = d = 0.1$ cm). Abscissae are real part of wavenumber (left panels) or cochlear function (right panels); ordinates, imaginary parts. Panels A and B: long-wave regime; panels C and D: transition region; panels E and F: short-wave regime; panels G and H: pole region.
coupled to partition displacements.

Panels E and F of Figure 4-2 show the short-wave regime. The function $1/kb$ maps straight lines in the $k$ plane into circles in the $\tilde{H}$ plane; larger real wavenumber corresponds to a circle of larger radius in the cochlear-function plane; the imaginary part of the wavenumber, which controls accumulation of attenuation, becomes more negative as the real-part of the target function decreases exponentially. As the value of the target approaches $\text{Re}\{\tilde{H}_t(x, \omega)\} = 0$, the wavenumber approaches the poles and zeros of the two-dimensional kernel located along the imaginary $k$ axis.

Panels G and H of Figure 4-2 show the mapping near one of the imaginary-axis poles of $\tilde{H}_2(k)$ at $k = -30\pi$. In the vicinity of an imaginary axis pole, $\tilde{H} \propto 1/jk$, the two-dimensional cochlear function inverts and rotates its argument. Details of each of these mappings—e.g., derivations of the approximate forms—are discussed in Section 4.3, as are similar details for the mappings $\tilde{H}_1(k)$ and $\tilde{H}_3(k)$ in Sections 4.2 and 4.4, respectively.

The horizontal line in the right-hand panels of Figure 4-2 is the target-function contour in the $\tilde{H}$ plane for the Neely impedance at 1600 Hz; the curving line in the left-hand panels is the resulting corresponding wavenumber contour.

Figures 4-3 and 4-4 show the target and wavenumber contours and the several mappings from the panels in Figure 4-2 superimposed into the cochlear-function and wavenumber planes. The real and imaginary parts of the wavenumber are plotted as functions of place in the middle panels of Figure 4-1.

As stated in Equations 4.5 and 4.6, the cochlear gain depends upon the wavenumber at the origin, $jk(0, \omega)$, the derivative term, $[\tilde{H}'(k(x, \omega))/\tilde{H}'(k(0, \omega))]^{-1/2}$, and the phase-integral term, $\exp[-j \int_0^x k(\xi, \omega)d\xi]$. For the Neely impedance at 1600 Hz, the wavenumber at the origin is $k(0, 2\pi 1600) = 1.44 - j 0.001$. The panels next to the bottom of Figure 4-1 show the log magnitude and angle of the derivative and phase-integral terms. The log-magnitude of the derivative term is the upper curve; it increases slowly for increasing place, then decreases after the real part of the target function becomes negative. The lower curve shows the log magnitude of the phase-integral gain. Once the imaginary part of the phase integral becomes sufficiently
Figure 4-3: Cochlear-function target contour and cochlear-function mappings from Panels B, D, F, and H from Figure 4-2. The abscissa is the real part of the cochlear function; the ordinate, the imaginary part.
Figure 4-4: Wavenumber contour corresponding to the target-function contour in Figure 4-3, and wavenumbers mapped to the cochlear function plane from Panels A, C, E, and G. The abscissa is the real part of the wavenumber; the ordinate, the imaginary part (in cm⁻¹).
more negative than minus one, exponentiation of the phase integral causes the very steep decrease in log magnitude. When the log magnitude falls off exponentially, the ordinary magnitude falls off as the exponential of an exponential: what is effectively total cutoff occurs in a very short distance. The angle panel on the right shows the corresponding angle curves for the derivative and phase-integral terms. The upper curve is the angle of the derivative term: it goes through a small phase change about the point where the real part of the target function becomes negative. The lower curve is the angle of the phase integral term: it increases slowly at first, then more rapidly, then stops accumulating phase at all once the wavenumber reaches a point near the imaginary axis.\footnote{Such plateaus should more likely be considered as an artifact of the model, rather than as a predictor or correlate of actual cochlear behavior. Once the log-magnitude of the response is down more than, say, 60 or so dB, both responses should be treated with some skepticism.}

The resulting log magnitude and angle for the model are shown in the bottom panels of Figure 4-1. The basal part of the log-magnitude plot reflects the slow increase due to the cochlear-function derivative; the steep cutoff is due to the phase derivative term. The derivative term plays a lesser role in the angle curve: The small but abrupt phase shift seen in the angle curve of the derivative term is obscured by the phase integral contribution.

The method of determining the gain is similar for any other frequency and for any other ordinary impedance: Each target function contour leads to a wavenumber contour; the wavenumber contour results in values for the cochlear-function derivative and the phase integral, and, thus, the cochlear place response. The next sections discuss details of each of the one-, two-, and three-dimensional cochlear functions.

### 4.2 One-Dimensional Cochlear Fluid Motion

We begin consideration of cochlear functions with the simplest example: one-dimensional fluid flow. In one-dimensional fluid, partition velocity is uniformly distributed in the transverse direction, and the vertical component of fluid motion (due to partition motion) is so small as to be ignorable when compared to longitudinal fluid
motion.

The cochlear-fluid function for one-dimensional fluid flow is

\[
\tilde{H}_1(x, k) = \frac{1}{k^2 bd} \quad \text{(4.12)} \\
= \frac{d}{b (kd)^2} \quad \text{(4.13)}
\]

The normalized cochlear function for one-dimensional fluid flow is

\[
\tilde{F}_1(\mu, r) = r \frac{1}{\mu^2} \quad \text{(4.14)}
\]

where \( r = d/b \) is the cochlear aspect ratio, and where \( \mu = kd \) is the normalized wavenumber. In terms of the normalized one-dimensional cochlear function, the ordinary one-dimensional cochlear function is:

\[
\tilde{H}_1(x, k) = \tilde{F}_1(kd, d/b) \quad \text{(4.15)}
\]

The aspect ratio \( d/b \) plays a simple scaling role for the one-dimensional normalized cochlear function.

Representing the wavenumber \( \mu \) in the fourth quadrant of the complex plane by the polar form \( \mu = Re^{-j\theta} \), and with \( r = 1 \), \( \tilde{F}_1(\mu) \) becomes

\[
\tilde{F}_1(Re^{-j\theta}) = \frac{1}{R^2} e^{j2\theta} \quad \text{(4.16)}
\]

Small radii in the \( \mu \)-plane map to large radii in the \( \tilde{F} \)-plane, and angles measured clockwise in the \( \mu \)-plane map to angles measured counterclockwise in the \( \tilde{F} \)-plane. Figure 4-5 shows the application of the one-dimensional cochlear function to those polar loci of constant radius and constant angle.

For the one-dimensional cochlear function, the real and imaginary parts of the target function are equal when \( \mu = Re^{-j\pi/8} \); the real part of the target function vanishes when \( \mu = e^{-j\pi/4} \). The latter point explains why researchers investigating one-dimensional cochlear models must add substantial partition mass to their impedances in order to achieve sufficiently steep cutoffs. If one uses a massless partition model with one-dimensional fluid motion, the wavenumber angle will never exceed \( \pi/4 \), and the imaginary part of the wavenumber will never become more negative than the real
Figure 4-5: One-dimensional cochlear-function mapping from normalized wavenumbers of the form $\mu = Re^{-j\theta}$. The left panel shows fourth quadrant of the complex wavenumber plane: The abscissa is the wavenumber real part; the ordinate, the imaginary part. Loci of constant radius and angle are indicated on the plot; radii are spaced by equal multiplicative increments; angles, by equal additive increments. The panel on the right is the mapping of $H_1(\mu) = 1/\mu^2$ applied to the indicated contours. The abscissa is cochlear-function real part; the ordinate, imaginary part. Labels indicate the values of $R$ and $\theta$ for particular contours.
part is positive. In that case, one can only achieve an increased rate of attenuation at the cost of additional phase accumulation.

The one-dimensional cochlear function maps the entire fourth quadrant of the complex wavenumber plane to both the first and second quadrants of the complex cochlear-function plane, and it does so uniquely over those regions. Figure 4-6 shows the one-dimensional cochlear function with the Neely target function overlaid; it also shows the wavenumbers that were used in the original mapping, and those corresponding to the cochlear function. Figure 4-7 shows the stages of solving the eikonal equation and generating the cochlear gain for one-dimensional fluid flow. In the one-dimensional case, the wavenumber stays close to the origin even though the target function assumes the same values at the same places. Because of the relatively small wavenumber values—the contour stays close to the origin—the rates of both attenuation and phase accumulation are small compared to those for the two-dimensional case. The target contour for \( \Re \{ \tilde{H}(x, 2\pi 1600) \} > 0 \) corresponds to the wavenumber contour for \( \theta < \pi/4 \). Because the rates of attenuation and angle accumulation are slower for the one-dimensional case than for the two-dimensional one, the one-dimensional best place is located more apical than that for the two-dimensional case.

### 4.3 Two-Dimensional Cochlear Fluid Motion

In two-dimensional fluid motion, as in the one-dimensional case, the partition velocity is assumed to be distributed uniformly in the transverse directions; however, fluid motion is not restricted to be dominated by the longitudinally directed component. The two-dimensional cochlear fluid function,

\[
\tilde{H}_2(k) = \frac{\coth kd}{kb} = \frac{d \coth kd}{b kd} \quad (4.17)
\]

is represented in terms of the normalized two-dimensional cochlear function

\[
\tilde{F}_2(\mu, r) = r \frac{\coth \mu}{\mu} \quad (4.19)
\]
Figure 4-6: One-dimensional mapping, Neely's target function, and the resulting wavenumbers. The panel on the left is the wavenumber plane: the polar grid are loci of constant radius and angle. The panel on the right is the cochlear-function plane. The abscissae are the real parts; the ordinates, the imaginary parts. The wavenumber contour in the left panel corresponds to the target contour in the right panel.

by

\[ \tilde{H}_2(k) = \tilde{F}_2(kd, d/b) \] (4.20)

As with the one-dimensional function, the aspect ratio simply scales the value of the normalized two-dimensional function.

From the exponential representation for \( \tilde{F}_2(\mu) \),

\[ \tilde{F}_2(\mu) = \frac{1}{\mu} \frac{e^\mu + e^{-\mu}}{e^\mu - e^{-\mu}} \] (4.21)

one discerns that the normalized two-dimensional cochlear function has alternating poles and zeros along the imaginary axis. The zeros are located at \( \mu = \pm j(\omega n - 1)\pi/2 \); the poles at \( \mu = \pm jn\pi \) where \( n \) is a positive integer. There is a double pole at \( \mu = 0 \).

The poles and zeros of \( \tilde{F}_2(\mu) \) are apparent in Figure 4-8, which shows the value of \( \tilde{F}_2(\mu) \) for imaginary argument, \( \mu = j\mu_1, -10 < \mu_1 < 0 \). \( \tilde{F}_2 \) is strictly real along both \( \text{Im} \{\mu\} = 0 \) and \( \text{Re} \{\mu\} = 0 \). The roles of the poles and zero in distinguishing the effects of the two-dimensional cochlear function from those of the one-dimensional one are to make multiple regions in the \( \mu \) plane map into the same region of the \( \tilde{F} \)
Figure 4-7: Stages in solving the eikonal equation and determining cochlear gain for one-dimensional fluid flow. Each panel displays the real (left column) or imaginary (right column) part of the label quantity for Neely's 1978 model parameters and a frequency of 1600 Hz. Abscissae are all cochlear place in cm. The real part of the logarithmic gain is the log-magnitude in dB; the imaginary part, the angle in π radians.
Figure 4-8: Plot of \( \tilde{F}_2(\mu) \) vs. \( \mu \) for \( \mu \) purely imaginary and ranging from \(-j10\) to 0. The abscissa is the imaginary wavenumber; the ordinate, the value of the two-dimensional cochlear function.

plane, and to move the loci corresponding to \( \text{Re}\{\tilde{F}\} = 0 \) close to the imaginary axis (Figure 4-11).

By considering an infinite-series representation for \( \tilde{F}_2(\mu) \),

\[
\tilde{F}_2(\mu) = \frac{1}{\mu^2} + 2 \sum_{n=1}^{\infty} \frac{1}{\mu^2 + (n\pi)^2} \tag{4.22}
\]

one can determine approximations for \( \tilde{F}_2(\mu) \) for appropriate argument values. When the argument is small in magnitude, \(|\mu| \ll \pi\), \( \tilde{F}_2(\mu) \approx 1/\mu^2 + 1/3 \), the long-wave approximation. The upper panels in Figure 4-9 show the mapping from polar wavenumbers \( \mu = Re^{-j\theta} \) in the long-wave regime. For very small magnitude wavenumbers, \(|1/\mu^2| \gg 1/3\), the long-wave approximation reduces to the one-dimensional cochlear function. The one-dimensional model can be made to simulate the two-dimensional model in the long-wave regime by adding an appropriate amount of partition mass

\[3\sum_{n=1}^{\infty} 1/(n\pi)^2 = 1/6\]
term to the partition impedance. The mass one would add to the one-dimensional partition impedance corresponds to the inertia associated with vertical fluid motion which was ignored in the one-dimensional case.

By expressing the wavenumber in terms of its real and imaginary parts, $\mu = \mu_R + j\mu_I$, the two-dimensional function is

$$
\tilde{F}_2(\mu_R + j\mu_I) = \frac{\mu_R \sinh 2\mu_R + \mu_I \sin 2\mu_I + j(\mu_I \sinh 2\mu_R - \mu_R \sin 2\mu_I)}{(\mu_R^2 + \mu_I^2)(\cosh 2\mu_R - \cos 2\mu_I)} (4.23)
$$

$$
= \frac{\sinh 2\mu_R[(\mu_R + \mu_I \frac{\sin 2\mu_L}{\sinh 2\mu_R}) + j(\mu_I - \mu_R \frac{\sin 2\mu_L}{\sinh 2\mu_R})]}{\cosh 2\mu_R(\mu_R^2 + \mu_I^2)(1 - \frac{\cos 2\mu_L}{\cosh 2\mu_R})} (4.24)
$$

$$
\approx \frac{\mu_R - j\mu_I}{\mu_R^2 + \mu_I^2} (4.25)
$$

$$
= \frac{1}{\mu} (4.26)
$$

One can determine an approximate regime for $\tilde{F}_2(\mu)$ when the real part of the argument is large; i.e., when the wavenumber is sufficiently distant from the poles and zeros along the imaginary axis. When that real part is large—$\text{Re} \{\mu\} \gg 2 \log 2$—the hyperbolic terms in the expression above grow large and dominate the trigonometric terms, and $\tilde{F}_2(\mu) \approx 1/\mu$, the short wave approximation. The upper panel of Figure 4-9 also shows the mapping for larger radius wavenumbers, the short-wave mapping. When $\tilde{F}_2 \approx 1/\mu$, lines of constant radius and angle map into lines of constant radius and angle, with the radius of the inverse being the inverse of the argument radius, and with the angle of the inverse being the negative of the argument angle:

$$
1/Re^{-j\theta} = (1/R)e^{j\theta}.
$$

The lower panels of Figure 4-9 show the two-dimensional cochlear function applied to rectangular wavenumbers of the form $\mu = \mu_R + j\mu_I$, $\mu_R > 2$, $\mu_I < 0$. The mapping $1/\mu$ maps straight lines in the $\mu$ plane into circles in the $\tilde{F}$ plane. In the short-wave regime, lines of constant real part $\mu_R$ map into circles of radius $1/\mu_R$ centered at $1/2\mu_R$; similarly, lines of constant imaginary part $\mu_I$ map into circles of radius $1/|\mu_I|$ centered at $-1/2\mu_I$ ($\mu_I < 0$). Both long-wave and short-wave regimes of the two-dimensional cochlear function are apparent in Figure 4-10, which shows $\tilde{F}_2(\mu)$ vs. real valued wavenumbers plotted on log-log axes. In the long-wave regime, the slope of $\tilde{F}_2(\mu)$ vs. $\mu$ is -2; in the short-wave regime, -1.
Figure 4-9: Two-dimensional cochlear function mappings. The upper panels show the mapping from polar wavenumber ($\mu = Re^{-j\theta}$) contours to the cochlear function plane in both the long- and short-wave regimes; the lower panels, the mapping from rectangular wavenumber ($\mu = \mu_x - j\mu_y$) to the cochlear function plane in the short-wave regime. The left panels are the fourth quadrant of the complex wavenumber plane; the right panels, the complex cochlear-function plane. Radii and angles, real and imaginary parts as indicated.
Figure 4-10: Plot of $\tilde{F}(\mu)$ vs. $\mu$ (log-log axes). The abscissa is the real wavenumber argument; the ordinate, the two- or three-dimensional cochlear function, as labelled. The three-dimensional function was computed using a moving fraction $\epsilon = 0.25$. 
The two-dimensional cochlear function is multiple valued: there are differing regions in the $\mu$ plane that map into the same region of the $\tilde{F}$ plane. Illustrations of the mutli-valued behavior of the two-dimensional cochlear function is shown in Figure 4-11. Figure 4-11 shows the loci of points in the complex-wavenumber plane for which the real part of the cochlear function vanishes; $\text{Re}\left\{\tilde{F}_2(\mu)\right\} = 0$. For one-dimensional cochlear motion there was only one locus for which this condition occurred, $\text{Re}\left\{\mu\right\} = -\text{Im}\left\{\mu\right\}$; for two-dimensional motion, there is a locus that begins close to $\text{Re}\left\{\mu\right\} = -\text{Im}\left\{\mu\right\}$, but then turns and “falls into” the first zero at $\mu = -\pi/2$.

The next locus begins at the first non-zero pole location and loops down to the next zero. Each subsequent locus follows this pattern: it connects a pole and a zero. The interiors to these regions are those spaces in the normalized-wavenumber plane which map into the second quadrant of the normalized cochlear-function plane; i.e., $\text{Re}\left\{\tilde{F}_2(\mu)\right\} < 0$. Figure 4-11 also shows the loci which map into $\text{Re}\left\{\tilde{F}_2(\mu)\right\} = \text{Im}\left\{\tilde{R}_2(\mu)\right\}$. When the real part of the target function becomes negative (when the cochlear mass term becomes larger in magnitude than the cochlear stiffness term), the wavenumber will lie in one of those regions. Thus, two-dimensional cochlear models don’t necessarily require additional cochlear mass to achieve sufficient attenuation accumulation. Even in the absence of any cochlear mass, the wavenumber will tend to a point near the imaginary axis as the stiffness tends to zero with increasing place. Recall that for the same condition, the one-dimensional wavenumber would only reach the line $\text{Im}\left\{\mu\right\} = -\text{Re}\left\{\mu\right\}$.

Figure 4-12 shows the mapping of the two-dimensional cochlear function near the imaginary axis. The upper panels of Figure 4-12 show the mapping of polar wavenumbers near a pole of $\tilde{F}_2(\mu)$. The pole is located at $-j10$, and the wavenumbers are of the form $\mu = -j10 + \text{Re}^\theta$. Near the pole, the mapping becomes proportional to $1/j\mu$. The lower panels of Figure 4-12 show the mapping of polar wavenumber near a zero of $\tilde{F}_2(\mu)$. The zero is located at $-j19\pi/2$. Near the zero, the mapping is

\[4\text{In fact, the slight difference between the locus } \text{Re}\left\{\mu\right\} = -\text{Im}\left\{\mu\right\} \text{ that yields } \text{Re}\left\{\tilde{F}_1(\mu)\right\} = 0 \text{ and the one for which } \text{Re}\left\{\tilde{F}_2(\mu)\right\} = 0 \text{ near the } \mu\text{-plane origin is precisely due to the difference between the behavior of } 1/\mu^2 \text{ and } 1/\mu^2 + 1/3\]
Figure 4-11: Solid lines: Complex wavenumber loci which map into the imaginary axis of the cochlear-function plane; Dashed lines: Loci which map into the line in the cochlear-function plane where the real and imaginary parts of the cochlear function are equal. The abscissa is the wavenumber real part; the ordinate, the imaginary part.
proportional to $j\mu$.

### 4.4 Three-Dimensional Cochlear Fluid Motion

While one- and two-dimensional cochlear fluid motion differ because of the difference between having a double pole at the $k$-plane's origin and an infinity of poles and zeros along the $k$-plane's imaginary axis, two- and three-dimensional motion differ because the poles and zeros are located differently. Like the two-dimensional function, $\tilde{F}_3(\mu)$ has an infinity of poles and zeros along the $\mu$-plane imaginary axis; unlike the two-dimensional case, the locations are not uniformly spaced. The resulting mapping from wavenumber plane to cochlear-function plane is less regular along the imaginary axis. Two geometrical qualities, the aspect ratio and the fraction of moving partition width, determine details of the three-dimensional function and in a manner more complicated than the simple scaling action of the aspect ratio on both one- and two-dimensional cochlear functions.

The three-dimensional cochlear-fluid function is:

$$\tilde{H}_3(x) = \sum_{n=-\infty}^{\infty} |\Phi_n|^2 \coth \sqrt{\frac{k^2 + (\frac{n\pi}{b})^2}{k^2 + (\frac{n\pi}{b})^2}}$$

$$= \frac{d}{b} \sum_{n=-\infty}^{\infty} |\Phi_n|^2 \coth \sqrt{\frac{(kd)^2 + (n\pi\epsilon)^2}{(kd)^2 + (n\pi\epsilon)^2}}$$

(4.27)

(4.28)

Unless otherwise noted, the shape of cochlear partition displacement is assumed to be that of a half cosine.

$$\phi(z) = \begin{cases} \frac{\pi}{2\epsilon b} \cos \left[ \frac{\pi}{\epsilon b} (z - b/2) \right], & |z - b/2| < \epsilon(x)b/2 \\ 0, & \text{otherwise} \end{cases}$$

(4.29)

For such a shape, the partition shape coefficients, $\Phi_n(x)$ are:

$$\Phi_n(\epsilon) = \begin{cases} \frac{\cos^2(n\pi\epsilon/2)}{(1-n\epsilon)^2}, & n \text{ even, } n \neq 1/\epsilon \\ \pi^2/16, & n \text{ even, } n = 1/\epsilon \\ 0, & n \text{ odd} \end{cases}$$

(4.30)

Any shape which is symmetric about the partition centerline will have coefficients that vanish for odd indices.
Figure 4-12: Complex cochlear-function values near poles and zeros of the two-dimensional cochlear function. The upper panels show the mapping near the pole at \( \mu = -j10\pi \). The left panel shows wavenumber contours are of the form \( \mu = -j10 + Re^{j\theta} \); the right panel, the result of applying the two-dimensional cochlear function to those contours. The lower panels show the mapping near the zero at \( \mu = -j19\pi /2 \). The left panel shows the wavenumber contours centered on \(-j19\pi /2\); the right panel, the resulting cochlear function contours. Radii and angles are as labelled.
In terms of the normalized three-dimensional function, the ordinary three-dimensional function is

\[ \tilde{H}_3(k) = \tilde{F}_3(kd, d/b, \epsilon) \]  \hspace{1cm} (4.31)

where the normalized three-dimensional function is

\[ \tilde{F}_3(\mu, r, \epsilon) = |\Phi_0(\epsilon)|^2 \tilde{F}_2(\mu, r) + 2 \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2 \tilde{F}_2(\sqrt{\mu^2 + (n\pi r)^2}, r) \] \hspace{1cm} (4.32)

For three-dimensional motion, the aspect ratio, \(d/b\), appears not only as a scale factor (as the second argument to each of the \(\tilde{F}_2\)), but also in determining the wavenumber argument to \(\tilde{F}_2\) for each term in the summation. The moving-fraction width, \(\epsilon\), influences the weightings of the terms in the summation.

As with the two-dimensional case, one can expand the above into the series representation for \(\tilde{F}_2(\mu)\), yields:

\[ \tilde{F}_3(\mu, r, \epsilon) = r|\Phi_0(\epsilon)|^2 \left[ \frac{1}{\mu^2} + 2 \sum_{m=1}^{\infty} \frac{1}{\mu^2 + (m\pi)^2} \right] + 2r \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2 \left[ \frac{1}{\mu^2 + (n\pi r)^2} + 2 \sum_{m=1}^{\infty} \frac{1}{\mu^2 + (n\pi r)^2 + (m\pi)^2} \right] \] \hspace{1cm} (4.33)

in order to examine approximate regimes and determine pole and zero locations. As with the two-dimensional case, \(\tilde{F}_3(\mu)\) has a double pole at the \(\mu\)-plane origin and poles along the imaginary axis at \(\mu = \pm jm\pi\), where \(m\) is a positive integer; however, \(\tilde{F}_3(\mu)\) also has additional poles at \(\mu = \pm j\pi \sqrt{(nr)^2 + m^2}\), where both \(m\) and \(n\) are positive integers, and where \(|\Phi_n(\epsilon)|^2 \neq 0\); there is a zero somewhere between consecutive poles.

Poles and zeros of \(\tilde{F}_3(\mu)\) near \(\mu = 0\) are shown in Figure 4-13; the regular spacing of poles and zeros seen for the two-dimensional case has been replaced by a less regular spacing of poles and zeros for the three dimensional case. If \(|\Phi_n(\epsilon)|^2\) vanishes for odd terms, the poles will be limited to \(\mu = \pm j\pi \sqrt{(nr)^2 + m^2}\), where \(m\) is a positive integer, and where \(n\) is an even, positive integer.

As with the two-dimensional case, the three-dimensional cochlear function has long- and short-wave approximate regimes. When the magnitude of the normalized wavenumber is much less than \(\pi\), the three-dimensional normalized cochlear function is approximately

\[ \tilde{F}_3(\mu, r, \epsilon) \approx r|\Phi_0(\epsilon)|^2 \left[ \frac{1}{\mu^2} + \frac{1}{3} \right] + 2r \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2 \tilde{F}_2(n\pi r) \] \hspace{1cm} (4.34)
Figure 4-13: Plot of $\tilde{F}_2(\mu)$ vs. $\mu$ for $\mu$ purely imaginary and ranging from $-10$ to $0$. The abscissa is the imaginary wavenumber; the ordinate, the (real) value of the three-dimensional cochlear function.
Because $|\Phi_n| = 1$ ($\phi(z)$ is the normalized transverse velocity distribution, the zeroth component of the Fourier series will be unity), this simplifies to

$$\tilde{F}_3(\mu, r, \epsilon) \approx r \left[ \frac{1}{\mu^2} + \frac{1}{3} \right] + 2r \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2 \tilde{F}_2(n \pi r)$$

(4.35)

$$= r \left( \frac{1}{\mu^2} + \tilde{H}_{3d} \right)$$

(4.36)

where $\tilde{H}_{3d}$ is the offset. That offset is equal to or larger than that of the two-dimensional case. To model the three-dimensional long-wave regime with a one-dimensional model requires adding more cochlear-partition mass than for the two-dimensional case. As one would expect with an added degree of freedom of fluid motion, there is more inertia associated with transverse and vertical fluid motions than there was for just vertical fluid motion in the two-dimensional case.

The short-wave regime is found as follows: Because the shape coefficients decrease as a function of index, when the real part of the normalized wavenumber is sufficiently greater than $n \pi r$ for some $n$, the three-dimensional cochlear function is well approximated by

$$\tilde{F}_3(\mu, r, \epsilon) \approx \tilde{F}_2(\mu, r) \left( |\Phi_0(\epsilon)|^2 + 2 \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2 \right)$$

(4.37)

$$\approx \frac{1}{\mu} \left( |\Phi_0(\epsilon)|^2 + 2 \sum_{n=1}^{\infty} |\Phi_n(\epsilon)|^2 \right)$$

(4.38)

Long-, short-, and transitions regimes are apparent for the three-dimensional fluid function in Figure 4-10 (alongside the same mapping for two-dimensional cochlear fluid motion). Figure 4-14 shows the mapping of $\tilde{F}_3$ applied to polar-form wavenumbers. It shows the long-wave, transition-wave, and short-wave regimes. In the transition regime, the three-dimensional cochlear function is well approximated by neither the form $1/\mu^2$ nor by $1/\mu$. Because of that transition, the short-wave regime is associated with a much larger wavenumber real-part than for the two-dimensional function.

The three-dimensional fluid function, as was the case with the two-dimensional one, maps differing regions of the wavenumber plane into the same regions of the cochlear-function plane. As such, the discussion about poles and zeros of the two-dimensional function is applicable to the three-dimensional case. Because the poles
Figure 4-14: Three-dimensional cochlear-function mapping. The left panel shows wavenumbers of the form $\mu = R e^{-i\theta}$; the right panel, the result of applying $\tilde{F}_3(\mu)$ to those contours. The aspect ratio $r$ was one; the moving fraction $\epsilon$ was 0.25. The abscissa and ordinate in the left panel are real and imaginary parts of the wavenumber, respectively; in the right panel, they are the real and imaginary parts of the three-dimensional cochlear function. Values of $R$ and $\theta$ are as labelled.

and zeros aren’t uniformly distributed along the imaginary axis, there may be wavenumber-contour details near the imaginary-wavenumber axis that differ between the two- and three-dimensional cases; the overall picture of wavenumber behavior is the same. The $\mu$-plane contours corresponding to $\text{Re}\{\tilde{H}(\mu)\} = 0$ will connect adjacent poles and zeros. When the imaginary part of the wavenumber vanishes—i.e., when the real part of the target function vanishes—the wavenumber contour will have reached one of those loci. Thus, for target functions with decreasing real parts, one expects the wavenumber contour to reach out from the origin, sweep around quasi-circularly through the short-wave regime, then end up near the imaginary axis as the target-function real part either approaches zero (massless case) or passes through zero and becomes negative.

It is useful to consider physical aspects of cochlear fluid flow in order to understand the transition from long-wave to short-wave behavior. In the long-wave regime, the motion is longitudinal, with the entire scala being occupied by the moving slug of fluid, and with partition motion and it’s associated transverse and vertical fluid motions being negligible. As the wavelength decreases, either transverse or vertical motions
or both should become comparable to longitudinal motions, depending on the size of the moving fraction, the shape of the partition displacement, and the relative size of the breadth and depth. The transition between long- and short-wave behavior is, then, the transition from longitudinally directed fluid motion, uniformly distributed across the scala cross section, to locally concentrated three-dimensional fluid motion. As the wavelengths become shorter and shorter, the displacement of the partition seems uniformly distributed with respect to any transverse oscillations, and the fluid motion is effectively two-dimensional, although still concentrated about the partition.

The effect of geometric parameter variation on the three-dimensional cochlear function is examined in Figures 4-15, 4-16, and 4-17. Figure 4-15 shows the effect of varying the aspect ratio and moving fraction on $\tilde{F}_3$ for real arguments. Figure 4-16 shows the Fourier series coefficients for the half-cosine velocity distribution with varying widths. As is expected, the "width" of the Fourier series is inversely proportional to the width of the moving fraction. Figure 4-17 shows the interplay between the moving-fraction width and the aspect ratio. The ordering of the long-wave, transition, and short-wave regions depends upon the geometric properties. One sees that when the moving fraction of partition width is large ($= 1$), there is little or nothing to distinguish between two- and three-dimensional behavior. When the moving fraction is small, however, the transition from long- to short-wave behavior becomes more complicated depending on the aspect ratio. For the case of $\epsilon = 1/8$ and the breadth being very much greater than the height ($r = 4$), cochlear fluid motion goes from one-dimensional, to two-dimensional, to three-dimensional, then back to two-dimensional as the wavenumber decreases. Physically, because the width is so large compared to both the moving fraction and to the depth, the fluid motion is effectively two-dimensional at intermediate wavelengths. As the wavelength becomes smaller, becomes comparable to the width of the moving fraction, three dimensional motion becomes the rule. Finally, as the wavelength becomes even smaller, substantially smaller than the moving fraction, the motion become effectively uniform in the transverse dimension.

Figure 4-18 shows the three-dimensional cochlear function with the Neely tar-
Figure 4-15: Three-dimensional cochlear function for real argument with variation of moving fraction width and aspect ratio. Both panels show $\tilde{F}_3(\mu)$ vs. $\mu$ on log-log axes. In both panels, the abscissa is real wavenumber; the ordinate is $\tilde{F}_3(\mu)$. The left panel shows what happens when the moving fraction $\epsilon$ is varied while the aspect ratio is held constant at $r = 1$. The right panel shows what happens when the aspect ratio is varied while the moving fraction is held constant. Parameter values for $\epsilon$ and $r$ are as labelled.
Figure 4-16: Fourier-series coefficients for the half-cosine transverse velocity distribution for varying widths. In each panel, the abscissa is the coefficient index; the ordinate, the coefficient value. The label on each panel indicates the moving fraction of partition width.
Figure 4-17: Three-dimensional cochlear function mappings for various combinations of moving fraction and aspect ratio. In each panel, the abscissa is the real part of the cochlear function; the ordinate, the imaginary part. Each column has a fixed moving fraction; each row, a fixed aspect ratio. The fraction and aspect ratio are indicated by the attached numbers. Mapping is from the polar-wavenumber wedge shown on the left panel of Figure 4-14.
Figure 4-18: Three-dimensional mapping, Neely's target function, and the resulting wavenumbers. The panel on the left is the wavenumber plane: the polar grid are loci of constant radius and angle. The panel on the right is the cochlear-function plane. The abscissae are the real parts; the ordinates, the imaginary parts. The wavenumber contour in the left panel corresponds to the target contour in the right panel.

get function overlaid; it also shows the wavenumbers that were used in the original mapping, and those corresponding to the cochlear function. Figure 4-19 shows the stages in solving the eikonal equation and constructing the cochlear gain for threedimensional fluid motion with Neely's impedance parameters at 1600 Hz. The middle panels of Figure 4-19 show the real and imaginary parts of the wavenumber as a function of place. In the three-dimensional case, the wavenumber gets very large in both real and imaginary parts before returning to the vicinity of the imaginary axis. The next-to-bottom panels show the derivative and phase-integral contributions to the log-magnitude and angle of the gain. And the bottom panels show the log magnitude and phase angle of the velocity gain for three-dimensional fluid flow (and cochlear partition motion). For the three-dimensional case, the wavenumber components are substantially larger than those of the one- or two-dimensional cases, and both the attenuation and angle accumulation occur comparatively sooner. The best place is located more basal than for the one- or two-dimensional cases.
Figure 4-19: Stages in solving the eikonal equation and determining cochlear gain for three-dimensional fluid motion. Each panel displays the real (left column) or imaginary (right column) part of the label quantity for Neely’s 1978 model parameters and a frequency of 1600 Hz. Abscissae are all cochlear place in cm. The real part of the logarithmic gain is the log-magnitude in dB; the imaginary part, the angle in $\pi$ radians.
4.5 Summary

Cochlear functions map the complex wavenumber plane into the complex cochlear function plane. In order to determine the cochlear gain corresponding to a particular in pedance, one forms the target function in the cochlear function plane. Then, one inverts—numerically, graphically—the cochlear function to solve the eikonal equation. The gain function follows from values of the cochlear-function derivative and the phase integral of the wavenumber.

One-, two-, and three-dimensional fluid motions were considered. In the one-dimensional case, partition velocity is uniformly distributed in the transverse direction, so there are no transverse components of fluid motion; vertical components of fluid motion associated with partition up-and-down motion are assumed to be negligible compared to longitudinal motion of fluid within the scala. The two-dimensional case differs from the one-dimensional case in that vertical fluid motions are not assumed to be negligible; there is still no transverse component of partition or fluid motion. The three dimensional fluid-motion case assumes that the partition velocity and fluid both have a transverse component.

Cochlear functions were developed for the three cases of fluid motion. The long-wave approximate regimes of the two- and three-dimensional cochlear functions are similar to one-dimensional fluid motion with the exception that additional inertial terms due to vertical components were included. The short-wave approximate regimes correspond to the case where fluid motion is limited to the region near the moving cochlear partition, and the partition and fluid motions are closely coupled to each other.

The cochlear gain for Neely’s 1978 impedance parameters at 1600 Hz was examined for each of the one-, two- and three-dimensional cochlear functions. Differences amongst the three were explained in terms of the respective wavenumber contours.
Chapter 5

Partition Impedance
Representation and Generalized Cochlear Response

Cochlear-partition dynamics are often modelled as those of a low-order, lumped-parameter system; i.e., a rational-function impedance. However, the difficulty of cochlear models using that impedance to fit both global and local cochlear measurements raises questions as to whether the low-order, lumped-parameter model is the best way to go about mathematically capturing partition dynamics.

In the context of the integral-equation description of cochlear dynamics—of the eikonal equation, of the phase-integral approximation—each set of rational-function parameters corresponds to a set of trajectories in the cochlear-function plane. When the dynamics are those of a mass-spring-dashpot system, the impedance has the form:

\[ Z(x, \omega) = \frac{K(x)}{j\omega} + R(x) + j\omega M(x) \]  \hspace{1cm} (5.1)

and the target function, the form

\[ H_t(x, w) = \frac{K(x)}{2\omega^2 \rho} - \frac{M(x)}{2\rho} + j \frac{R(x)}{2\omega \rho} \]  \hspace{1cm} (5.2)

An alternative to the above is a viscoelastic description of partition dynamics. In its most general formulation, such a description opens up the mathematical represen-
tation for cochlear-partition dynamics. A simple case includes frequency dependent losses, for which the viscoelastic-impedance model becomes:

\[
Z(x, \omega) = -jZ_S(x, \omega) + Z_R(x, \omega) + j\omega M(x) = \frac{K(x)}{j\omega} + R(x, \omega) + j\omega M(x)
\] (5.3)

The central issue in cochlear macromechanical modelling is describing cochlear-partition dynamics. For a given geometry and the assumption of small fluid displacements (i.e., linearity), cochlear fluid dynamics are straightforwardly determined by specifying a fluid pressure which satisfies Laplace's equation in the fluid bulk and the boundary conditions at the walls in terms of the velocity of the partition. While arguments about the cochlear fluid problem involve details—viscosity, coiling, cochlear geometry—arguments about the cochlear-partition problem are arguments about essentials.

Each derivation of cochlear-integral equations assumed that partition dynamics were local; i.e., the coupling action of the cochlear fluid is the dominant mechanism by which motion at a distant point on the partition influences motion at the point of interest; the effect of any longitudinal forces must be negligible for the partition dynamics to be modelled by an acoustic impedance. Assuming partition dynamics to be local implies that transverse bending forces in the partition are substantially greater than longitudinal ones; i.e., there is no appreciable longitudinal coupling of partition motion through the partition itself. Even though Bekésy's report of circular basilar-membrane deformation under a point load [3] and Volčič's report showing that for a more viable preparation the deformation is more oblong than circular and oriented in the transverse direction [44] are in conflict, both speak to the issue of how to consider the effects of longitudinal stiffness. What matters is not whether static deformations show equal or unequal displacements in transverse and longitudinal directions; what matters is the relative forces in the longitudinal and transverse directions when the partition is moving. Longitudinal forces depend on bending along the longitudinal axis, and the amount of that bending depends on the wavelength.
To state convincingly that longitudinal-stiffness effects are negligible, one would have to determine how large longitudinal forces are compared to transverse ones. A means of estimating such would be to take a model in which longitudinal stiffness is ignored, solve it for a velocity distribution as a function of longitudinal position, then determine how large the longitudinal stiffness would have to be before longitudinal and transverse forces were comparable.

5.1 Rational-Function Impedance

The low-order, lumped-parameter representation of cochlear-partition dynamics is motivated by historical and physical reasons, as well as by reason of convenience. Békésy discusses how Helmholtz’s resonator theory required transverse tension in the basilar membrane ([3, pg. 471]). The local resonance would be like that of a piano string, with resonance between transverse tension and mass. Békésy shifted the discussion from basilar membrane tension to basilar membrane stiffness: “The essential physical characteristic for the displacement of fluid in the cochlea is that, under a given pressure change in one half of the cochlea, a certain volume of liquid per millimeter of length of the basilar membrane is forced into the other half, and this volume depends principally on the rigidity of the intervening partition [3, pg. 473].” I discuss Békésy’s partition stiffness measurements directly. With appropriate material assumptions, either a beam or orthotropic plate model for cochlear partition dynamics can be manipulated to yield a lumped parameter model [2, pg. 117]. Finally, the spring-dashpot-mass model is exceedingly familiar. The dynamics of such a system are taught in almost every introductory Physics class as an example of a resonant system.¹

The spring-dashpot-mass model of Figure 5-1 arises from assuming that the moving part of the cochlear partition occupies the entire scala breadth, b, and that trans-

¹“Resonance” is a term more appropriately describing such a system’s behavior when the quality factor \( Q(x) = \sqrt{K(x)M(x)/R(x)} \) is bigger than, say, five or ten. It is easy to call the system as “resonant” and call the frequency at which the admittance’s imaginary part vanishes the “resonant frequency” regardless of the system’s \( Q \)-factor, although doing so is often misleading.
Figure 5-1: A spring-dashpot-mass model for cochlear partition dynamics. $K(x)$, $R(x)$, and $M(x)$ are the stiffness, resistance, and mass per unit partition length.

verse velocity distribution is uniform (as in either the one- or two-dimensional model for cochlear fluid flow); each short longitudinal section is modelled as a moving mass whose motion is resisted by both a spring and a dashpot. The partition's specific acoustic impedance—pressure per unit (point) velocity—is

$$Z(x, \omega) = \frac{K(x)}{j\omega} + R(x) + j\omega M(x)$$

(5.5)

where $K(x)$, $R(x)$, and $M(x)$ are the stiffness, resistance, and mass per unit partition area. Let $Z_p(x, \omega)$ denote the inverse of the partition acoustic admittance density—$Z_p(x, \omega) = 1/Y_p(x, \omega)$—and call $Z_p(x, \omega)$ the partition impedance.\(^2\) For the above case, the partition impedance is the partition's specific acoustic impedance divided by the scala breadth $Z_p(x, \omega) = Z(x, \omega)/b$. For the more general case where the specific acoustic impedance varies across the partition breadth, the partition impedance is $Z_p(x, \omega)$ will be defined as $Z_p(x, \omega) = \int_0^b Z(x, z, \omega)|\phi(x, z)|^2 dz$, where $\phi(x, z)$ is the partition's normalized transverse velocity distribution—$V_p(x, z) = \phi(x, z)V(x)$—and $Z(x, z, \omega)$ is the specific acoustic impedance at each longitudinal and transverse point.

Békésy measured the stiffness of the partition both statically and dynamically. Figure 5-2 shows Békésy’s measured partition compliance as a function of distance from the stapes. The measured compliance increases by two orders of magnitude

\(^2\)Note that $Z_p(x, \omega)$ is neither an acoustic impedance nor a specific acoustic impedance. Being the inverse of an acoustic impedance density, $Z_p(x, \omega)$ has the units of specific acoustic impedance per unit area per unit length, i.e., dynes·s/cm\(^4\) in cgs units.
Figure 5-2: Cochlear partition compliance. The abscissa is the distance along the partition from the stapes. The left hand ordinate is the volume displacement per millimeter along the partition; the right hand ordinate is the maximum displacement (across the partition) for a head of 1 cm of water. From Békésy [3, pg. 476].

along the length of the partition in an exponential manner; that is, the static volume stiffness of the cochlear partition is approximated by a formula of the form

\[ K(x) = K_0 e^{-\alpha x} \]  

(5.6)

For a factor of 100 change (see Figure 5-2) and \( x \) in cm, \( \alpha \) must be about 1.3cm\(^{-1}\). Békésy stated quite unequivocally that this amount of change is species independent [3, pg. 483]. Békésy also stated that dynamic stiffness was somewhat smaller than statically measured stiffness.

Because there are no direct measurements regarding the lossiness of the cochlear partition, numeric values for the resistive term in lumped-parameter partition models are usually based either on assumptions regarding frictional loss in plate models or on some appropriate fit to the partition displacement or velocity data being modelled. Usually, the loss is assumed to be a simple resistance varying with longitudinal position, although both increasing and decreasing functions of place have been used.

When should cochlear partition mass be ignored? There are two conditions: that the partition elements' mass density is the same as that of the surrounding cochlear fluids, and that the enclosing boundary is sufficiently compliant. When both condi-
Table 5-1: Stiffness, resistance, and mass \((K(x), R(x), \text{ and } M(x))\) per unit area in rational-function models of cochlear-partition dynamics as used by various researchers. Values here are for the specific acoustic impedance relating pressure and point velocity, not partition impedance as defined in the text. Modified from Neely [21, pg. 117]

<table>
<thead>
<tr>
<th></th>
<th>(K(x))</th>
<th>(R(x))</th>
<th>(M(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allen [2]</td>
<td>(2 \times 10^9 e^{-3.4x})</td>
<td>(600e^{-1.7x})</td>
<td>0.10</td>
</tr>
<tr>
<td>Neely [21]</td>
<td>(10^9 e^{-2x})</td>
<td>200</td>
<td>0.15</td>
</tr>
<tr>
<td>Peterson and Bogert [24]</td>
<td>(1.72 \times 10^9 e^{-2x})</td>
<td>0</td>
<td>0.143</td>
</tr>
<tr>
<td>Siebert [37]</td>
<td>(2 \times 10^8 e^{-1.5x})</td>
<td>(5e^{2.28x})</td>
<td>0.01</td>
</tr>
</tbody>
</table>

In that case, the cochlear fluid dynamics capture the inertial aspect of cochlear partition dynamics, and no mass term should be included in the cochlear-partition impedance. If all elements in the partition have the same mass density as the cochlear fluid, a difference between the inertia of the cochlear masses and the cochlear fluids would occur only if the partition is rigid compared to the fluids. Apart from the immovable, bony shelf, partition structures are soft, cellular structures; their fluid can be considered to be contiguous with the fluid in the scalae.\(^3\)

Table 5-1 shows the parameter values for the rational-function impedances of several researchers. Table 5-2 shows the number of orders of magnitude of change in stiffness from the cochlear base to the apex in the models of Table 5-1 and several other models. The stiffness of every model varies more than Békésy’s observations.

Some researchers have begun to explore more complicated lumped-parameters models. In such, the above representation is maintained for the basilar membrane, but additional terms corresponding to motion of other partition structures are added; e.g., relative motion between the basilar and tectorial membranes, hair-cell stereocilia

\(^3\) Perhaps the best image conveying the above condition is that of a fluid-filled object in a water-filled aquarium. If the object is filled with water, but the volume defining boundary of the object is rigid—say rigid plastic of the same volume density as water—the object moves in the fluid according to the boundary conditions at its surface: no normal velocity, etc. If the object’s boundary is sufficiently compliant—a water-filled balloon in an aquarium—then where does the fluid dynamics end and the dynamics of the object begin. The boundary in that case is transparent, and the object moves with the same dynamics as the surrounding fluid.
<p>| Range of |
| Stiffness Variation |</p>
<table>
<thead>
<tr>
<th>(Orders of Magnitude)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allaire [1]</td>
</tr>
<tr>
<td>Allen [2]</td>
</tr>
<tr>
<td>Neely [21]</td>
</tr>
<tr>
<td>Peterson and Bogert [24]</td>
</tr>
<tr>
<td>Siebert [37]</td>
</tr>
<tr>
<td>Steele and Zais [42]</td>
</tr>
<tr>
<td>Zwislocki (1950) [49]</td>
</tr>
<tr>
<td>Zwislocki (1983) [48]</td>
</tr>
</tbody>
</table>

Table 5-2: The number of orders of magnitude change in stiffness from base to apex in various models.

<table>
<thead>
<tr>
<th>Structure</th>
<th>$K(x)$</th>
<th>$R(x)$</th>
<th>$M(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basilar Membrane</td>
<td>$4.5 \times 10^9 e^{-2.1x}$</td>
<td>26.7</td>
<td>0.375</td>
</tr>
<tr>
<td>Tectorial Membrane</td>
<td>$1.43 \times 10^9 e^{-2.1x}$</td>
<td>200</td>
<td>0.11</td>
</tr>
<tr>
<td>Organ of Corti</td>
<td>$1.42 \times 10^9 e^{-2.1x}$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 5-3: Stiffness, resistance, and mass ($K(x)$, $R(x)$, and $M(x)$, respectively) per unit area per unit width in the multiple lumped-parameter cochlear partition model of Zwislocki [48, pp. 109–110].

stiffness and resistance, etc. Such extensions complicate the form of $Z(x, \omega)$, and such models often have the unattractive characteristic of parameter tracking among the elements: not just one, but several parameters must vary in precisely the same manner with space. Table 5-3 shows the rational-function parameters for Zwislocki’s multiple-degree-of-freedom cochlear partition model.

## 5.2 Viscoelastic Plate Admittance

Another approach to describing the cochlear partition admittance is to consider the dynamics of a viscoelastic plate. Some forms for the complex Young’s modulus for a viscoelastic material are derived from polymer like chains of stiffness and loss, resulting in larger-order lumped-parameter formulae. An alternative is a loss-tangent
form in which the stiffness is a complex quantity of the form

\[ K(x) = \left[ \frac{K_R(x)}{j\omega} (1 + j\delta(x, \omega)) \right]^n \] (5.7)

One way to get at viscoelastic dynamics is to consider the static situation for an elastic plate, then generalize the rigidity parameter to be a complex quantity for the dynamic, viscoelastic case. With \( w \) as the \( y \)-directed displacement of a thin orthotropic plate in the \( x-z \) plane, and with \( p(x, z) \) as the pressure across the plate, the differential equation for the displacement is

\[ p(x, z) = D_z \frac{\partial^4 w(x, z)}{\partial x^4} + 2H \frac{\partial^4 w(x, z)}{\partial x^2 \partial z^2} + D_z \frac{\partial^4 w(x, z)}{\partial z^4} \] (5.8)

where \( H = D_{zz} + 2G_{zz} \); \( D_z, D_z, \) and \( D_{zz} \) are the flexural rigidities, and \( G_{zz} \) is the torsional rigidity of the plate [43]. When the pressure due to transverse curvature and rigidity is substantially greater than that of longitudinal curvature and rigidity, and and substantitally greater than that from the change of curvature in one direction relative to the other and its associated rigidity, the pressure is

\[ p(x, z) = D_z \frac{\partial^4 w(x, z)}{\partial z^4} \] (5.9)

as one would determine for a bending beam.

Let \( V(x) \) be the volume velocity flowing into the partition, the plate, per unit length; \( V(x) = - \int_0^b V_y(x, 0, z) dz \). Let \( \phi(x, z) \), be the normalized transverse velocity profile at each longitudinal point; \( \phi(x, z) = V_y(x, 0, z) / \int_0^b V_y(x, 0, z) dz \). Representing \( V_y(x, 0, z) \) in a cosine series,

\[ V_y(x, 0, z) = \sum_n V_n(x) \cos \frac{n\pi z}{b} \] (5.10)

where

\[ V_n(x) = \frac{1}{b} \int_0^b V_y(x, 0, z) \cos \frac{n\pi z}{b} dz \] (5.11)

\[ = -\frac{V(x)}{b} \int_0^b \phi(x, z) \cos \frac{n\pi z}{b} dz \] (5.12)

\[ = -V(x) \Phi_n(x) / b \] (5.13)
where

\[
\Phi_n(x) = \int_0^b \phi(x, z) \cos \frac{n\pi z}{b} \, dz
\]  
(5.14)

\[
\phi(x, z) = \frac{1}{b} \sum_n \Phi_n(x) \cos \frac{n\pi z}{b}
\]  
(5.15)

For sinusoidal motion, the complex amplitude of displacement is

\[
W_y(x, 0, z) = V_y(x, 0, z)/j\omega
\]  
(5.16)

\[
= -\frac{V(x)}{j\omega b} \sum_n \left( \frac{n\pi}{b} \right)^4 \Phi_n(x) \cos \frac{n\pi z}{b}
\]  
(5.17)

When transverse rigidity depends only on the longitudinal location, the plate equation becomes (for dynamic not static displacement):

\[
P(x, 0, z) = D_z(x, z) \frac{\partial^4 W_y(x, 0, z)}{\partial z^4}
\]  
(5.18)

\[
= -\frac{D_z(x)V(x)}{j\omega b} \sum_n \left( \frac{n\pi}{b} \right)^4 \Phi_n(x) \cos \frac{n\pi z}{b}
\]  
(5.19)

The weighted average-pressure difference of the cochlear integral equations is:

\[
P(x) = 2\int_0^b P(x, 0, z)\phi^*(x, z) \, dz
\]  
(5.20)

\[
= -\frac{2D_z(x)V(x)}{j\omega b} \int_0^b \sum_n \left( \frac{n\pi}{b} \right)^4 \Phi_n \cos \frac{n\pi z}{b} \cdot \sum_m \Phi_m \cos \frac{m\pi z}{b} \, dz
\]  
(5.21)

\[
= -\frac{2D_z(x)V(x)}{j\omega b} \sum_n \left( \frac{n\pi}{b} \right)^4 |\Phi_n(x)|^2
\]  
(5.22)

\[
= -\frac{2D_z(x)V(x)b}{j\omega} \int_0^b \left| \frac{\partial^2 \phi(x, z)}{\partial z^2} \right|^2 \, dz
\]  
(5.23)

When the velocity distribution is a half cosine over that part of the partition that moves;

\[
\phi(x, z) = \begin{cases} 
\frac{\pi}{2\epsilon(x)b} \cos \left( \frac{\pi}{\epsilon(x)b} (z - \frac{b}{2}) \right), & |z - \frac{b}{2}| < \frac{\epsilon(x)b}{2} \\
0, & \text{otherwise}
\end{cases}
\]  
(5.24)

the cochlear partition compliance per unit length is

\[
C_p(x) = \frac{V(x)}{P(x)}
\]  
(5.25)

\[
= \frac{j\omega b^4 \delta(x)}{16\pi^6 D_z(x)}
\]  
(5.26)
The stiffness component of the partition impedance \( Z_p(x, \omega) \) is

\[
K(x) = \frac{16\pi^6 D_z(x)}{j\omega b^4 e^3(x)}
\] (5.27)

In the viscoelastic case, one treats the rigidity as a complex quantity, depending both on place and frequency.

\[
D_z(x, \omega) = D_R(x, \omega)(1 + j\delta D(x, \omega))
\] (5.28)

and the “stiffness” and “resistance” components of the impedance become:

\[
Z_p(x) - M_0 = \frac{16\pi^6}{j\omega b^4 e^3(x)} D_R(x, \omega)(1 + j\delta D(x, \omega))
\] (5.29)

The above formulation suggests that partition stiffness depends upon the fifth-power of the moving fraction width. That would be true assuming that the transverse rigidity is independent of longitudinal location. However, rigidity of a uniform material depends on the depth; rigidity of a complex structure (such as the basilar membrane) depends on a number of other factors. The above should be taken, at best, as further motivation for exponential stiffness decrease for increasing position, and for generalized place and frequency dependence of the partition impedance real and imaginary parts.

### 5.3 The Generalized Target Function

The eikonal equation between the behavior of the cochlear function, \( \tilde{H}(k) \) and the target function derived from the cochlear impedance, \( \tilde{H}_t(x, \omega) = -Z(x, \omega)/2j\omega \rho \) is, for the case of the low-order, lumped parameter partition impedance representation with exponentially decreasing stiffness and resistance and with constant cochlear mass:

\[
\tilde{H}(k) = \frac{Z(x, \omega)}{2j\omega \rho}
\] (5.30)

\[
= \frac{K_0e^{-\alpha x}}{2\omega^2 \rho} - \frac{M_0}{2\rho} + \frac{R_0e^{-\beta x}}{2\omega \rho}
\] (5.31)

Taking the partition-mass term to the same side of the equation as the cochlear mass term—both correspond to inertial forces—and relabelling the functions from \( \tilde{H} \), the
cochlear function and its target, to \( \tilde{G} \), the modified cochlear function:

\[
\tilde{G}(k) = \tilde{H}(k) + \frac{M_0}{2\rho} \quad (5.32)
\]

\[
= \frac{K_0}{2\rho} \frac{1}{\omega^2 e^{\alpha x}} + j \frac{R_0}{2\rho} \frac{1}{\omega e^{\beta x}} \quad (5.33)
\]

\[
= \tilde{G}_t(x, \omega) \quad (5.34)
\]

Because of the assumed exponential dependence of both the decay of stiffness and resistance, I seek a power-function relationship between the real and imaginary parts of the modified cochlear function target:

\[
\text{Re} \left\{ \tilde{G}_t(x, \omega) \right\} = \frac{K_0}{2\rho} \frac{1}{\omega^2 e^{\alpha x}} \quad (5.35)
\]

\[
= \tilde{G}_0 \frac{1}{e^\gamma} = \text{Re} \left\{ \tilde{G}_t(\chi) \right\} \quad (5.36)
\]

\[
\text{Im} \left\{ \tilde{G}_t(x, \omega) \right\} = \frac{R_0}{2\rho} \frac{1}{\omega e^{\beta x}} \quad (5.37)
\]

\[
= \delta(\omega) \left[ \text{Re} \left\{ \tilde{G}_t(\chi) \right\} \right]^{\gamma} \quad (5.38)
\]

where

\[
\gamma = \frac{\beta}{\alpha} \quad (5.39)
\]

\[
\delta(\omega) = \frac{R_0/2\rho}{(K_0/2\rho)^{\gamma}} \omega^{2\gamma-1} \quad (5.40)
\]

\[
\tilde{G}_0 = \frac{K_0}{2\rho} \quad (5.41)
\]

For the generalized target-function representation, I define the generalized cochlear response as:

\[
G_1(\chi) = \left[ \frac{\tilde{H}(k_1(\chi))}{\tilde{H}(k_1(0))} \right]^{-1/2} e^{-j \int_0^\chi k_1(\xi) d\xi} \quad (5.42)
\]

where the generalized place is \( \chi = \alpha x + 2 \log \omega \). Both the cochlear place and frequency responses can be expressed in terms of the generalized cochlear response. The modified eikonal equation is simply:

\[
\tilde{G}(k) = \tilde{G}_t(\chi) \quad (5.43)
\]

First, each rational-function model of cochlear-partition impedance has a characteristic trajectory shape in the \( \tilde{G}_t \) plane determined solely by the ratio of exponential
decay rates. For example, the Neely partition model has \( \alpha = 2 \) and \( \beta = 0 \); the target function for each frequency is a straight line, parallel to the real axis. For the Allen impedance, \( \alpha = 3.4 \) and \( \beta = 1.7 \); thus, \( \gamma = 1/2 \), and the shape is that of the square-root function.

Second, each contour is a scaled version of each other according to the value \( \delta(\omega) \), the value of \( \text{Im} \{ \tilde{G}_t \} \) when \( \text{Re} \{ \tilde{G}_t \} = 1 \). For the Neely impedance, \( \delta(\omega) = 1000/\omega \); the family of curves is a set of straight lines parallel to the real \( \tilde{G}_t \)-plane axis, with those corresponding to higher frequencies being closer to the real axis than those corresponding to lower frequency ones. Because the power dependence of \( \delta(\omega) \) is \( 2/\gamma - 1 \), when \( \gamma = 1/2 \), \( \delta \) is independent of frequency; there is only one \( \tilde{G}_t \)-plane contour, only one resulting \( k \)-plane wavenumber contour, and only one underlying gain response.

Third, for positive values of \( K_0 \) and \( R_0 \), the \( \tilde{G}_t \) contours are always in the first quadrant of the \( \tilde{G}_t \) plane. The cochlear function is shifted along the real axis according to the value of the cochlear-partition mass. The role of cochlear mass in determining cochlear behavior is more easily examined.

Fourth, simple extensions of the above yields more exhaustive resentations for cochlear-partition dynamics. Both the real and the imaginary parts of the target modified cochlear function can be extended to arbitrary power functions of frequency, not just the integer powers -2 and -1 respectively; that is, the modified-target-function formulation is valid for viscoelastic cochlear-partition representations. For example, the imaginary part of the cochlear partition impedance can be represented by

\[
\text{Im} \{ Z(x, \omega) \} = \omega M(x) - \frac{K(x)}{\omega(\omega/\omega_K)^p} \tag{5.44}
\]

where \( K(x) \) continues to have the units of stiffness, and where \( \omega_K \) is a constant, the frequency at which the imaginary part reduces to the ordinary low-order, lumped-parameter stiffness. The power \( p \) is not necessarily an integer. Similarly, the real part of the partition impedance can be represented as

\[
\text{Re} \{ Z(x, \omega) \} = \frac{R(x)}{(\omega/\omega_R)^q} \tag{5.45}
\]
where \( R(x) \) has units of resistance, and where \( \omega_R \) is the frequency at which the real part reduces to the lumped-parameter resistance. With these definitions, and with the usual exponentially decreasing stiffness and resistance, the modified cochlear function target becomes:

\[
\tilde{G}_t(x, \omega) = \frac{K_0 e^{-\alpha x}}{2\omega^2 (\omega/\omega_K)^p \rho} + j \frac{R_0 e^{-\beta x}}{2\omega (\omega/\omega_R)^q \rho}
\]  

(5.46)

By manipulations similar to those above, \( \text{Im}\{\tilde{G}_t(x, \omega)\} = \delta(\omega) [\text{Re}\{\tilde{G}_t(x, \omega)\}]^\gamma \)

where

\[
\gamma = \frac{\beta}{\alpha}
\]  

(5.47)

\[
\delta(\omega) = \frac{R_0/2\rho}{(K_0/2\rho)^\gamma} \frac{w_R^q \omega^{(2+p)\gamma-(1+q)}}{w_K^q}
\]  

(5.48)

\[
\tilde{G}_0 = \frac{K_0}{2\rho}
\]  

(5.49)

As with the low-order, lumped-parameter case, the shape of the contour in the \( \tilde{G}_t \) plane depends only on \( \gamma = \beta/\alpha \). However, the scale factor \( \delta(\omega) \) is more complicated. Frequency independence of the scale factor—there being only one underlying contour—occurs if \( \gamma = (q+1)/(p+2) \). For example, for the Siebert viscoelastic impedance \[35\],

\[
Z(x, w) = \frac{K_0 e^{-\alpha x}}{j\omega} (1 + j\delta)
\]  

(5.50)

and \( p = 0 \) and \( q = 1 \).

Fifth, the relationship between place and frequency can be clarified. The substitution \( e^\chi = \omega^2 e^{\alpha x} \) replaces both the place and frequency variables with an arbitrary exponential dependence.

\[
\chi = 2\log \omega + \alpha x
\]  

(5.51)

One investigates the wavenumber contours corresponding to the target-function contours with \( \chi \) as the parameter. Exponential stiffness variation is captured by the exponential dependence on the generalized place variable \( \chi \); exploration of a variety of \( \gamma \) and \( \delta \) values covers the space of possibilities. The mapping of \( x \) and \( \log \omega \) into \( \chi \) underlies the ordinary concept of the cochlear map. As can be seen, such mapping has nothing necessarily to do with resonance between partition stiffness and partition
Table 5-4: Modified-target parameters for the rational-function impedances of various researchers (as shown in Table 5-1).

mass; it is the stiffness map that determines this first cochlear map. The role of cochlear mass in modifying such mapping is examined directly.

We will examine the impedances discussed previously – Neely’s, Allen’s, and Siebert’s rational-function impedances – to illustrate the utility and insight that comes from the modified-target representation. Modified-target parameters derived from the impedance parameters of Table 5-1 are shown in Table 5-4. (The parameters in Table 5-1 are specific acoustic parameters relating pressure to point velocity; the parameters in Table 5-4 are derived from the partition impedance, \( Z_p(x, \omega) = Z(x, \omega)/b \), hence the scale factor of 10 difference.)

### 5.4 Place and Frequency Responses

In terms of the generalized place variable, \( \chi \), the generalized cochlear gain is:

\[
G_1(\chi) = \left[ \frac{\tilde{H}(k_1(\chi))}{\tilde{H}(k_1(0))} \right]^{-1/2} e^{-j \int_0^\chi k_1(\xi) d\xi}
\]  
(5.52)

where \( k_1(\chi) \) is the root of the modified eikonal equation (Equation 5.43). The cochlear gain at an ordinary cochlear place and a particular frequency is expressed in terms of the generalized gain as:

\[
G(x, \omega) = jk(0, \omega) \left[ \frac{\tilde{H}(k(x, \omega))}{\tilde{H}(k(0, \omega))} \right]^{-1/2} e^{-j \int_0^x k(\xi, \omega) d\xi}
\]  
(5.53)

\[
= jk_1(2 \log \omega) \left[ \frac{\tilde{H}(k_1(\alpha x + 2 \log \omega))}{\tilde{H}(k_1(2 \log \omega))} \right]^{-1/2} e^{-j \int_{2 \log \omega}^{\alpha x + 2 \log \omega} k_1(\xi) d\xi}
\]  
(5.54)

\[
= jk_1(2 \log \omega) \left[ \frac{\tilde{H}(k_1(\alpha x + 2 \log \omega))/\tilde{H}(k_1(0))}{\tilde{H}(k_1(2 \log \omega))/\tilde{H}(k_1(0))} \right]^{-1/2} e^{-j \int_0^{\alpha x + 2 \log \omega} k_1(\xi) d\xi}
\]  
(5.55)
\[ = j k_1(2 \log \omega) \frac{G_1(\alpha x + 2 \log \omega)}{G_1(2 \log \omega)} \] (5.56)

The place response at a particular frequency follows directly from the above formula. At any particular frequency, the factors \( j k_1(2 \log \omega) \) and \( G_1(2 \log \omega) \) are both constants, scale factors modifying the response \( G_1(2 \log \omega + \alpha x) \). That response is just the generalized response shifted and scaled according to the mapping:

\[ x = \frac{\chi}{\alpha} - \frac{2}{\alpha} \log \omega \] (5.57)

Of course, one has to include the variation of \( \delta \) with frequency when appropriate. That is, if the \( \gamma \) is such that \( \delta \) is not independent of \( \omega \), the particular curve being mapped from \( \chi \) to \( \delta \) varies. (The generalized cochlear response is a function of \( \chi, \gamma, \delta \), and \( H_0: G(\chi) = G(\chi, \gamma, \delta, H_0).\))

### 5.5 Place Responses and the Cochlear Map of Best Place

What determines the cochlear map? It is often suggested that the cochlear map is "controlled" by the resonance between partition stiffness and partition mass. However, generalized responses exist even when the mass term \( H_0 \) is equal to zero. For example, Figure 5-3 shows the generalized response for the two-dimensional model for the case of a massless partition (and various \( \gamma \) and \( \delta \)). Although the responses vary greatly in selectivity, the mapping from generalized variable \( \chi \) to cochlear place \( x \) yields place selective responses for different frequencies even in the absence of cochlear mass. In that sense, partition mass—more generally resonance between partition stiffness and mass—doesn’t determine the cochlear map. In this section, I clarify just what does control the cochlear map of best place. The short answer is apparent from the curves in Figure 5-3: Part of the map is determined by the mapping between place and generalized place, and that depends on the rate constant \( \alpha \); another part of the map depends upon how much the best generalized place changes as a function of the loss parameter: \( \delta \); finally, the role of any cochlear mass isn’t to be ignored:
Figure 5-3: Log magnitude of the generalized response when the mass is zero. The abscissa in each panel is generalized place; the ordinate, the log magnitude of the generalized response in decibels relative to the leftmost point (arbitrarily labeled 0). The value of the rate parameter $\gamma$ is shown in each panel. The parameter $\delta$ varies exponentially within each panel from 0.001 (largest best place) to 10.0 (smallest best place, excepting the $\gamma = 1$ case, where the smallest best place belongs to the $\delta = 1$ curve and the next smallest best place to the $\delta = 10$ curve).

sufficient cochlear mass can eliminate any variation in best place with loss parameter and create a situation where best generalized place is independent of $\delta$. In that case, it is again the rate constant of the stiffness that determines the cochlear map.

If the best generalized place is at $\chi_b$, then the best cochlear place will be at

$$x_b = \frac{\chi_b}{\alpha} - \frac{2}{\alpha} \log \omega$$  \hspace{1cm} (5.58)

In general, $\chi_b = \chi_b(\delta)$; that is, the best place depends upon the loss parameter $\delta$. However, we have already seen that for the parameters of Allen's 1978 model, $\delta$ is independent of frequency; there is only one generalized cochlear response curve from which all the place and frequency responses are generated. Figure 5-4 shows the
stages in generating the ordinary place responses for different frequencies from the single generalized response for Allen’s 1977 impedance parameters. Panel A of Figure 5-4 shows the log magnitude of the generalized response for the modified target parameters for Allen’s model (Table 5-4). Panel B shows the mapping from place and frequency into the generalized place variable for several frequencies with the vertical line indicating the best place \( \chi_b \). Panels C and D of Figure 5-4 show the log-magnitude and angle of the ordinary cochlear-gain place response computed directly from the Allen impedance parameters for each frequency. The slight differences between one angle curve and another are reflections of sampling artifacts, not of each curve reflecting the one underlying shape. Because there is only one \( \delta \) and only one underlying gain curve, the best generalized place is independent of \( \delta \). In that case, Equation 5.58 says that the best place is a linear function of log frequency, and the slope is \(-2/\alpha\); there are no other parameters to adjust, no free variable to be introduced. When the best generalized place is independent of \( \delta \), the cochlear map of best place is given simply by Equation 5.58. For the Allen case in particular, the best generalized place is at \( \chi_b = 23.3 \); with \( \alpha = 3.4 \), the map of best places is

\[
x_b(f) = 6.85 - 0.59 \log(2\pi f)
\]  

(5.59)

Panel E of Figure 5-4 shows the best place as a function of frequency for the Allen parameters.

If there were only one underlying response for any choice of partition-impedance parameters, the game would be over: The only way to influence the cochlear map would be by choice of \( \alpha \), the rate constant of exponential stiffness decrease with place. But the manner in which partition stiffness decreases with increasing cochlear place is determined by nature, not by the one doing the modelling. The rate constant \( \alpha \) is not a free parameter whose value is adjusted by the researcher in order to fit some cochlear map.

What, then, is the role of the dependence of best generalized place on the loss parameter \( \delta \)? Continuing with the examination of previously studied models, Figure 5-5 shows the generation of the Neely place responses from the underlying generalized
Figure 5-4: Determination of ordinary place responses from the generalized place response for Allen's 1977 model parameters. Panel A: Generalized cochlear gain. The abscissa is generalized place; the ordinate, log magnitude in dB relative to the response at $\chi = 15.6$. Panel B: Generalized place mapping. The abscissa is generalized place; the ordinate, sinusoidal stimulus frequency. Horizontal lines: range of $\chi$ from $x = 0$ cm (left) to $x = 3.5$ cm (right) at the labeled frequency; vertical line: best generalized place ($\chi = 23.3$). Panels C and D: Ordinary place responses (partition volume velocity density relative to unit stapes volume velocity) for frequencies in Panel B. The abscissa in both panels is cochlear place in cm. The Panel C ordinate is the log-magnitude in dB; that in Panel D is angle in $\pi$ radians. Panel E: Best cochlear place as a function of sinusoidal stimulus frequency. The abscissa is the frequency in Hz; the ordinate, best place in cm.
responses. Panel A of Figure 5-5 shows the log magnitude of the generalized responses for the Neely parameters, the mappings from place and frequency to generalized place, and the locus of best place. The best generalized place is a decreasing function of \( \delta \), as is apparent in the magnitude figure, and is shown with the mappings from place and frequency to generalized place in Panel B.\(^4\) Panel C shows a closer view of the best generalized place as a function of \( \delta \), and a linear least-square fit to those data as a function of \( \log \delta \).

With \( \chi_b \) a function of \( \delta \), the cochlear map of best ordinary place becomes:

\[
x_b(\omega) = \frac{\chi_b(\delta(\omega))}{\alpha} - \frac{2}{\alpha} \log \omega \tag{5.60}
\]

With \( \chi_b(\delta) \) having the approximate form

\[
\chi_b(\delta) \approx \chi_0 - a \log \delta \tag{5.61}
\]

and with \( \log \delta(\omega) \) being proportional to \( \log \omega \)

\[
\log \delta = \log \delta_0 + (2\gamma - 1) \log \omega \tag{5.62}
\]

\[
\log \delta_0 = \log \left( \frac{R_0/2}{(K_0/2)^\gamma} \right) \tag{5.63}
\]

then the best place becomes

\[
x_b(\omega) = x_0 - \frac{a(2\gamma - 1) + 2}{\alpha} \log \omega \tag{5.64}
\]

where \( x_0 = (\chi_0 - a \log \delta_0)/\alpha \). Panels D and E of Figure 5-5 show the ordinary place response log-magnitude and angle for the sinusoidal frequencies shown in Panel B. Panel F shows the best place as a function of frequency by both formula and by measurement from the model output for Neely’s parameter set.

In Panel E of Figure 5-4, the slope of the best place function is \(-2/\alpha = 0.59\) as given by Equation 5.58. In Panel F of Figure 5-5, however, the slope of the best place function is \(-0.84\). A model with delta independent of frequency, or one in which \( \chi_b \) is independent of \( \delta \) would have to have a rate constant of \( \alpha = 2.36 \) to have

\(^4\)Recall that target contours closer to the real axis lead to wavenumber contours closer to the real axis, and those lead to a less rapid rate of attenuation accumulation. Therefore, larger \( \delta \) leads to more rapid attenuation, leads to smaller best place.
Figure 5-5: Determination of ordinary place responses from the generalized place responses for Neely's 1978 model parameters. Panel A: Generalized cochlear gain. The abscissa is generalized place; the ordinate, log magnitude in dB relative to the response at $\chi = 15.6$. Panel B: Generalized place mapping. The abscissa is generalized place; the ordinate, loss parameter $\delta(\omega)$. Horizontal lines: range of $\chi$ from $x = 0$ cm (left) to $x = 3.5$ cm (right) at the labeled frequency; curved line: best generalized places, $\chi_b(\delta)$, best place as a function of loss parameter. Panel C: Blow up of the best place shown in Panel B. The abscissa is the best generalized place; the ordinate, the loss parameter. Points: measured best place for given $\delta$; solid line: least-square fit to the data points. Panels D and E: Ordinary place responses (volume velocity density relative to unit stapes volume velocity) for the frequencies in Panel B. The abscissa in both panels is cochlear place in cm. The Panel D ordinate is the log-magnitude in dB; that in Panel E is angle in $\pi$ radians. Panel F: Best cochlear place as a function of sinusoidal stimulus frequency. The abscissa is the frequency in Hz; the ordinate, best place in cm. Solid line: predicted best places; points: measured best places (see text).
the same cochlear map of best place. So, the Neely model, with a rate constant of \( \alpha = 2 \) (three orders of magnitude stiffness variation), establishes a cochlear map of best place over the entire cochlear length within a frequency range of less than three orders of magnitude.

The Siebert 1974 model takes the above feature even further. Instead of having a resistance that is either decreasing (as with Allen) or constant (Neely), the Siebert resistance increases with increasing cochlear place. For the Siebert parameters, \( \gamma = -1.5 \); \( \tilde{G} \)-plane trajectories have the shape of one over the cube of the square root. The process of generating the ordinary cochlear place responses from the generalized place responses are shown in Figure 5-6. Panel A of Figure 5-6 shows the log-magnitude of the generalized response relative to the response at \( \chi = 12.9 \) for various values of the loss parameters. As well as showing the mapping from ordinary place and frequency to generalized place, Panel B shows the value of the best generalized place as a function of the loss parameter; it ranges much farther than for the Neely case. The best generalized place as a function of \( \delta \) and a linear, least-square fit to those data are shown in Panel C of Figure 5-6.

The Siebert place responses are shown in Panels D and E of Figure 5-6, and the best places as a function of frequency in Panel F. The straight line prediction of the best places was formed from the fit to the data in Panel C of Figure 5-6 by the same operations as used in estimating the Neely best places. The Siebert model achieves three decades of frequency variation map along the entire cochlear length with a rate constant of only 1.5, only 2.3 orders of magnitude variation.

We have already seen that when the loss parameter is independent of frequency, there is only one underlying gain curve, and the map from best generalized place to best cochlear place depends only on the stiffness of the partition; such was the case for the Allen parameter set. We have also seen that when the best generalized place depends on the loss parameter, then one can eek out a greater variation in best place for the same range of frequency using a smaller stiffness rate constant; both the Neely parameter set and Siebert parameter set displayed this property. Is the difference between the Neely \( \chi_b(\delta) \) and the Siebert \( \chi_b(\delta) \) due more to different
Figure 5-6: Determination of ordinary place responses from the generalized place responses for Siebert’s 1974 model parameters. Panel A: Generalized cochlear gain. The abscissa is generalized place; the ordinate, log magnitude in dB relative to the response at $\chi = 12.9$. Panel B: Generalized place mapping. The abscissa is generalized place; the ordinate, loss parameter $\delta(\omega)$. Horizontal lines: range of $\chi$ from $x = 0$ cm (left) to $x = 3.5$ cm (right) at the labeled frequency; curved line: best generalized places, $\chi_0(\delta)$, best place as a function of loss parameter. Panel C: Blow up of the best place shown in Panel B. The abscissa is the best generalized place; the ordinate, the loss parameter. Points: measured best place for given $\delta$; solid line: least-square fit to the data points. Panels D and E: Ordinary place responses (volume velocity density relative to unit stapes volume velocity) for the frequencies in Panel B. The abscissa in both panels is cochlear place in cm. The Panel D ordinate is the log-magnitude in dB; that in Panel E is angle in $\pi$ radians. Panel F: Best cochlear place as a function of sinusoidal stimulus frequency. The abscissa is the frequency in Hz; the ordinate, best place in cm. Solid line: predicted best places; points: measured best places (see text).
choices of the power parameter $\gamma$ or more to different choices of the mass parameter $H_0$?

Figure 5-7 shows the effect of the cochlear-mass term on the log-magnitude of the generalized response for two rate parameters $\gamma$ and two-dimensional fluid motion. The left column in Figure 5-7 shows the case where $\gamma = -1.5$ (Siebert parameters); the right column, $\gamma = 0.0$ (Neely parameters). Each row shows a differing value of the mass parameter $H_0$. When the mass parameter is small, both cases show the usual best case as a decreasing function of increasing loss parameter $\delta$. As the mass parameter increases, the location of the best places moves basalward for smaller loss parameters compared to the smaller mass case. For the largest mass shown, $H_0 = 1.0$, the three smallest values of loss parameter $\delta$ all have effectively the same best place for both rate parameter cases. The larger the cochlear mass is, the less effect the range of $\delta$ has on the location of the best place, and the ordinary cochlear map of best places ends up being determined, as in the Allen case, by the stiffness rate constant alone. For constant mass, the seeming effect is that resonance between stiffness and mass is determining the cochlear map, when in fact the cochlear mass and the resonance of the cochlear partition simply have the same form.

5.6 Frequency Response

It is worthwhile to note how the frequency selectivity at a place emerges from the model. The model's place selectivity arises from the competing actions of the slowly increasing cochlear-function-derivative term and of the phase-integral term, which decreases exponentially. The best place — where the magnitude slope is zero — corresponds to the slope of the magnitude of each of those terms being equal and opposite. For frequency selectivity at a place, similar factors—a slowly changing term and a term with a rapid cutoff—determine the response; however, in the case of the frequency response, the slow term comes from the wavenumber itself, the fast term cutoff comes from the ratio of the generalized place response at two different points,

As shown above, the ordinary response as a function of place and frequency ex-
Figure 5-7: Log magnitude of the generalized response for two differing rate parameters and three differing mass parameters for two dimensional fluid motion. In each panel, the abscissa is the generalized place; the ordinate, the log-magnitude of the generalized response. Each column was computed for a differing rate parameter, $\gamma$, as labeled; each row was computed for a differing mass parameter, $H_0$. The curves within each panel were determined for values of $\delta$ ranging exponentially from 0.001 (largest best place) to 10.0 (smallest best place).
pressed in terms of the generalized place response $G_1(\chi)$ is

$$G(x, \omega) = jk_1(\chi) \frac{G_1(2 \log \omega + \alpha x)}{G_1(2 \log \omega)}$$  \hspace{1cm} (5.65)

Using the complex logarithm to get at the log magnitude and angle responses yields:

$$\log G(x, \omega) = \log |k_1(\chi)| + j\pi/2 + \log G_1(2 \log \omega + \alpha x) - \log G_1(2 \log \omega)$$  \hspace{1cm} (5.66)

Because the real part of the complex logarithm is the log magnitude, one sees that
the log magnitude or angle response as a function of frequency at a particular place
is formed by the difference of the log magnitude or angle of the generalized place
response between $\chi = \alpha x + 2 \log \omega$ and $\chi = 2 \log \omega$. If the place is close enough to
the base such that $\alpha x$ is small, then the difference is constant over much of the slowly
changing part of the generalize place response, over that part due to $\tilde{H}(k(\chi))^{-1/2}$;
for those frequencies for which that difference is constant, the slow increase in the
log magnitude as a function of log frequency is due solely to the wavenumber term
$\log |k_1(\omega \log \omega)|$. When $\chi = \alpha x + 2 \log \omega$ and $\chi = 2 \log \omega$ are both on the fast cutoff
side of the log magnitude of the generalized place response, then the difference will
be a rapidly decreasing function of $\log \omega$. In fact, if the log magnitude in the cutoff
region is modelled as an exponentially decreasing function of generalized place, then
the log magnitude of the frequency response is an exponentially decreasing function
of log frequency; i.e., the magnitude falls off exponentially in frequency, not as some
power in frequency.

Figure 5-8 shows the construction of the frequency response from the underlying
generalized response for the case of the Allen parameters. The top panel shows the
log-magnitude and angle of the generalized cochlear response. The middle panel shows
the result of taking the difference $\log G_1(\chi + \chi_1) - \log G_1(\chi)$ for several different values
of $\chi_1$ corresponding to $\chi_1 = \alpha x$ for $x$ being 0.5, 1.0, 1.5 cm, etc. and for $\alpha = 3.4$.
The middle panel also shows the log-magnitude of the wavenumber term $jk_1(\chi)$. The
shape of the frequency response, like that of the place response, is a superposition
of one term which increases steadily over the entire frequency range, and one term
which is flat over most of the range and then sharply cuts off past a certain point. In
the place response, those terms were the cochlear-function derivative term, \(\tilde{H}'(k)^{-1/2}\), and the phase-integral term, \(\exp \int_0^\pi k(\xi)d\xi\). For the frequency response, the slow term is the wavenumber term \(jk_1(x)\); the cutoff term, the difference of the log gains. The bottom panel in Figure 5-8 shows the frequency responses for the Allen impedance parameters at places 0.5 cm apart. Similar computations would ensue for the Neely and Siebert parameters, but because each frequency corresponds to a different value of the loss parameter \(\delta(\omega)\), the difference corresponding to a particular value of \(\alpha x\) would be taken along a different curve for each frequency value.

An interesting feature of the frequency response is that, to the degree that actual cochleas act like cochlear models, one should be able to measure the stiffness at the origin and the rate of stiffness decrease from the frequency response. Defining

\[
\Gamma(x) = \log G_1(x)
\] (5.67)

\[\log G(x, \omega) = \log |k_1(2 \log \omega)| + j\frac{\pi}{2} + \log G_1(2 \log \omega + \alpha x) - \log G_1(2 \log \omega) \] (5.68)

Expanding \(\log G_1(\alpha x + 2 \log \omega)\) in a power series about \(2 \log \omega\) yields:

\[
\log G_1(2 \log \omega + \alpha x) \approx \log G_1(2 \log \omega) + \alpha x \Gamma'(2 \log \omega) \] (5.69)

\[
\log G(x, \omega) \approx \log |k_1(2 \log \omega)| + j\frac{\pi}{2} + \alpha x \Gamma'(2 \log \omega) \] (5.70)

for small \(\alpha x\). This suggests that in basal regions, when one examine the log magnitude of the frequency responses, the difference between measurements made at two different places should be proportional to the rate constant of exponential stiffness decrease. That is, the real part of the expression above for \(\Gamma'(x)\) is dominated by the constant slope of the cochlear function derivative.

To make this more specific, because the slow term in both place and frequency responses occurs in the long-wave regime, let's use the long-wave approximation for the cochlear function to better understand the above phenomenon. For sufficiently large target function (low frequencies, basal places), the cochlear function is approximately:

\[
\tilde{H}(k) = \frac{1}{k^2 A}
\] (5.71)
Figure 5-8: Construction of the frequency response from the generalized response. For each panel, the figure on the left is the log-magnitude, the one on the right, the angle, with both appropriately referenced. The upper panels show the generalized response computed for the Allen parameter set. The abscissa in both is the generalized place. The middle panel shows difference $\log G_1(\chi + \chi_1) - \log G_1(\chi)$ as a function of cochlear place $\chi$ for various values of $\chi_1$. The abscissa in both figures is generalized place. The additional curve in both figures is the log-magnitude (left) and angle (right) of the wavenumber $k_1(\chi)$. The bottom panel shows frequency responses computed directly for the Allen parameters. Each magnitude curve is a superposition of the wavenumber term and difference term shown above.
where \( A \) is the scala area (Section 4.3). In that regime, the cochlear-function derivative is

\[
\tilde{H}'(k) = -\frac{2}{k^3 A}
\]  

(5.72)

The logarithm of the generalized cochlear gain and its derivative are:

\[
\Gamma(\chi) = \log G_1(\chi) = -\frac{1}{2} \log \frac{\tilde{H}'(k_1(\chi))}{\tilde{H}'(k_1(0))} - j \int_0^\chi k_1(\xi) d\xi
\]

(5.73)

\[
\Gamma'(\chi) = \frac{d}{d\chi} \log G_1(\chi) = -\frac{1}{2} \frac{d}{d\chi} \log \tilde{H}'(k_1(\chi)) - j k_1(\chi)
\]

(5.74)

Solving the eikonal equation with

\[
\tilde{G}(\chi) = G_R e^{-\chi} + j \delta(G_R e^{-\chi})^\gamma
\]

(5.75)

the wavenumber is

\[
k_1(\chi) \approx (1/bdG_R)e^{\chi/2}(1 - j(\delta/2)(G_Re^{-\chi})^{\gamma-1})
\]

(5.76)

and \(-1/2 \frac{d}{d\chi} \log \tilde{H}'(k(\chi)) \approx 3/4\); that is, in the long wave (low-frequency, small place) regime, the difference between the (natural) log magnitude at two different places for the same frequency should be \((3/4)\alpha(x_2 - x_1)\), where \(x_2\) and \(x_1\) are the places. Then, rather straightforwardly, the rate constant of exponential stiffness decrease is

\[
\alpha = \frac{4}{3(x_2 - x_1)}(\log |G(x_2, \omega)| - \log |G(x_1, \omega)|)
\]

(5.77)

Similarly, the angle in the long-wave regime is due primarily the real part of the wavenumber:

\[
\angle G(x, \omega) \approx \frac{\pi}{2} - \frac{1}{\sqrt{G_0 A}} \alpha x \omega
\]

(5.78)

and from this one can estimate the stiffness \( K_0 \) at \( x = 0 \) by

\[
K_0 = 2A \left( \frac{\alpha x \omega}{\pi/2 - \angle G(x, \omega)} \right)^2
\]

(5.79)

### 5.7 Summary

Two important results follow from the generalized target function representation for the partition impedance. First, the nature of the cochlear map of best places is
clarified. The cochlear map is determined by the interactions of the best generalized place as a function of the loss parameter $\delta$, the dependence of $\delta$ on frequency, and on cochlear mass. Cochlear partition mass affects the map of best places in its making the best generalized place less strongly dependent on the loss parameter, not through resonance in the cochlear partition.

One should come away with a new view as to how to go about determining the appropriate impedance parameters. A strategy would be to compute the generalized responses for a sufficiently dense sampling of the parameters $H_0$, $\gamma$, and $\delta$. For each $H_0$-$\gamma$ pair, one would compute the responses for a number of $\delta$s, then attempt to determine $\chi_k(\delta)$. For all the $H_0$-$\gamma$ pairs (and for all dimensionalities of fluid flow), one expects to find a subset of parameters for which the direction of best place change, and the rate of that change as functions of $\delta$ are sufficient to achieve the cochlear map. The generalized responses of that limited set of parameters meeting that criterion would then be examined, as would prototype place and frequency responses. The first stage of this culling process involves meeting the global constraints of stiffness variation, given by nature, and the map of best places; the second stage involves meeting the local criterion: peakiness of magnitude response, steepness of angle response. It may well be that no combination of $H_0$, $\gamma$, and $\delta$ (and $G_0$ as well) is sufficient to represent observed behaviors. In that case, one can state unequivocally that a rational function partition-impedance model in a box-cochlea model for fluid flow is inadequate to describe observed cochlear-partition behavior.

The second larger result concerns the frequency response and an understanding of how it relates to the place response. The bottom line is that the longed for symmetry between place and log frequency simply doesn’t exist: frequency response curves are, for the most part, not the same thing as scaled and shifted place response curves. Although both share the same broad characteristics—magnitude: slow increase to peak followed by rapid cutoff; angle: slowly becoming more negative until near the best response, then falling off rapidly—the underlying reasons for the sameness is subtle. In the place response, the slow and fast terms are the cochlear-function derivative and the phase integral respectively; in the frequency response, they are
the wavenumber and the difference of the generalized gain.
Chapter 6

Suggestions for Further Research

The cochlear model used here readily lends itself to further investigation.

6.1 Alternative Geometries

The model used in this thesis focused almost exclusively on the uniform cochlear geometry situation, although the mathematics of the phase-integral approximation are more than happy to accommodate cochlear functions that vary with longitudinal position. It would be of great utility for someone to quantify the degree to which spatial variation along the cochlear length of geometric parameters is ignorable. Specifically, it would be nice to better understand the degree to which the broadening width of the cochlear partition influences cochlear response.

6.2 The Inverse Problem

Can one recover the partition impedance values given cochlear gain response? There are reasons to hope that one can do so, at least from place responses. The method used here was to determine the target function from the impedance, solve the eikonal equation to determine the wavenumber, then integrate the wavenumber and determine the cochlear-function derivative; finally, the last two were combined to generate the gain function. In trying to invert the process, it is clear that if one can determine the
wavenumbers, the mapping backward to cochlear function and then to impedance is straightforward. Can one determine the wavenumber from the gain function?

The cochlear-gain response as a function of place (at a particular frequency \( \omega \)) is

\[
G(x) = jk(0) \left[ \frac{\tilde{H}'(k(x))}{\tilde{H}'(k(0))} \right] \left[ \frac{1}{\tilde{H}'(k(0))} \right]^{-1/2} e^{-j \int_{0}^{x} k(\xi) d\xi} \tag{6.1}
\]

Denoting the logarithm of the gain as \( \Gamma(x) = \log G(x) \),

\[
\Gamma(x) = \log(jk(0, \omega)) - \frac{1}{2} \log \left[ \frac{\tilde{H}'(k(x))}{\tilde{H}'(k(0))} \right] - j \int_{0}^{x} k(\xi) d\xi \tag{6.2}
\]

The derivative of the log gain with respect to place is:

\[
\Gamma'(x) = -\frac{1}{2} \frac{\tilde{H}''(k(x))}{\tilde{H}'(k(x))} k'(x) - jk(x) \tag{6.3}
\]

Approximating the derivatives by forward differences yields:

\[
\frac{\Gamma(x + \Delta) - \Gamma(x)}{\Delta} = -\frac{1}{2} \frac{\tilde{H}''(k(x))}{\tilde{H}'(k(x))} \left( \frac{k(x + \Delta) - k(x)}{\Delta} \right) - jk(x) \tag{6.4}
\]

Given that one knows \( \Gamma(x) = \log G(x) \) at all places, the above is solved for \( k(x + \Delta) \), and coupled with \( G(0) = jk(0) \) to yield a means of solving for \( k(x) \).

\[
k(0) = -jG(0) \tag{6.5}
\]

\[
k(x + \Delta) = k(x) - 2 \frac{\Gamma(x + \Delta) - \Gamma(x) + jk(x) \Delta}{\tilde{H}''(k(x)) / \tilde{H}'(k(x))} \tag{6.6}
\]

An example of the output of the inverse algorithm is shown in Figure 6-1 for the case of the Neely parameters (Table 5-1). The algorithm does quite well up to the point where the imaginary part of the impedance vanishes. The real part estimate is quite good over the range where the imaginary part of the impedance is large in magnitude.

Because both the cochlear-function derivative and the phase integral derive from the same set of wavenumbers, it doesn’t follow that any old pair of log-magnitude and angle plots constitutes a cochlear place response. For example, in the long-wave regime, the linear, slow growth in log magnitude is due to the cochlear-function deriva-
tive, while the exponential increase in \( \xi \varepsilon \) (negative) angle is due to the wavenumber term; however, once the best place is neared, the wavenumber term dominates both
Figure 6-1: Cochlear-partition impedance recovered from the output of the phase-integral model computed using Neely’s 1977 parameters at 1600 Hz. The abscissa in both panels is cochlear place in centimeters. The upper panel shows the real part of the partition impedance, $R(x)/b$, in dynes·s/cm$^4$; the lower panel shows the imaginary part of the partition impedance, $K(x)/j\omega b - w^2M_0/b$ in dynes·s/cm$^4$. 

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log-magnitude and angle responses. Furthermore, the real and imaginary parts of the wavenumber are not independent of each other.

The upper panels in Figure 6-2 show the wavenumbers for the Neely 1978 parameters at 1600 Hz and the recovered wavenumbers using the inversion algorithm described above. The recovered wavenumbers (points) agree well with the values determined while computing the phase-integral approximate gain (solid line) up until the point where the real part of the wavenumber starts decreasing. The lower panels in Figure 6-2 show the result of trying to construct a log-gain function from the output of using the Neely model parameters at two differing frequencies. The bogus log-gain function was constructed by taking the log magnitude response for Neely’s 1978 model parameters and an input frequency of 1600 Hz and the angle response for Neely’s model and an input frequency of 1590 Hz. One would hope to recover an impedance which would yield the desired log magnitude and that angle. Such a technique would be a powerful tool in understanding what partition impedance is necessary to meet the local criteria regarding magnitude peakiness and angle steepness. As can be seen, however, the recovered wavenumbers are not realistically associated with a partition impedance having a decreasing stiffness and a positive resistance.

I would suggest that someone could examine the details of the above mappings to try to better understand just what can be gleaned from such an inversion algorithm. Because both the cochlear-function derivative and the phase integral are determined by the wavenumber, there is, one, a redundancy of wavenumber information in the resulting log gain; and, two, the relationships between real and imaginary parts of the log gain is probably somehow constrained. The situation must be similar to a Hilbert transform situation. A clearer understanding of the constraint imposed in the mapping from wavenumber through both cochlear-function derivative and phase integral should illustrate the manner in which log-magnitude peakiness and phase-angle steepness are related.
Figure 6-2: Wavenumbers recovered by the inversion algorithm. The left panels show the real part of the wavenumber; the right panels, the imaginary part. The abscissa in all panels is the cochlear place in cm; the ordinate is the real (left panels) or imaginary (right panels) part of the wavenumber in cm\(^{-1}\). The solid line is is the wavenumber found in solving the eikonal equation to compute the cochlear gain for Neely’s 1978 impedance parameters and a frequency of 1600 Hz; the points are the recovered wavenumber values computed by the algorithm described in the text. The upper panels show successful recovery of the wavenumbers from the log-gain for Neely’s impedance parameters at 1600 Hz. The lower panel is an example of failure of the recovery algorithm. The solid line is, again, the wavenumber contour for Neely’s 1978 parameters and an input frequency of 1600 Hz; the points are the output of the recovery algorithm when the log-magnitude of the input came for the 1600 Hz cochlear gain and the angle came for the 1590 Hz gain.
6.3 Sources and Terminations, Emissions and Reflections

This thesis has examined only the partition response for the case of an ideal volume-velocity source at the stapes. A more detailed model, which would be more appropriate for investigations of otoacoustic emissions and of cochlear input impedance would include source dynamics as well as a termination impedance at the helicotrema end.

In that case, instead of having solely a volume velocity constraint for the partition to satisfy, one would have both volume-velocity and pressure constraints. For example if the stapes source were characterized as an ideal volume-velocity source with a shunt admittance, then

- The pressure drop across the partition at \( x = 0 \) should be the same as the pressure drop across the shunt admittance.

- The volume flow through the ideal source should equal the sum of the volume flows through the shunt admittance and "through" the cochlea.

One could impose a termination at the helicotrema with similar constraints. However, as was shown previously, for all but the lowest frequencies, helicotrema dynamics will not come into play, because for most frequencies the cochlear wave cuts off well before the helicotrema is reached. For those low frequencies, as would be the case with an ordinary transmission line or waveguide, the partition velocity will consist of both forwardly and backwardly propagating modes. The termination at the helicotrema will be characterized by a reflection coefficient relating the impedance seen looking back into the cochlea and the impedance of the helicotrema.

One can continue such efforts to incorporate sources along the partition itself. Were there a source of pressure or volume velocity located at \( x = x_0 \), the longitudinal velocity distribution would have a forwardly propagating mode for \( x > x_0 \) and a backwardly propagating mode for \( x < x_0 \). If \( x_0 \) is basal to the cutoff place for the frequency of interest, the partition source will successfully couple energy into the partition, and waves will propagate both apically and basally of \( x_0 \). The apical
wave will propagate to the best place for the frequency of interest, then cutoff; the basal wave will propagate back to the stapes, where the termination due to stapes impedance will determine a reflection coefficient. The amount of energy that would leave the stapes due to a source along the partition depends upon the impedance match between the cochlear system and the stapes. For frequencies at which \( x = x_0 \) is in cutoff, little or no partition volume displacement would occur either to the left or right of \( x_0 \). However, volume displacement can't just disappear into the cochlear fluid, so any volume flow into the scala would have to be accounted for by volume flow through the helicotrema or volume flow out the basal end by means of the stapes.
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