APPLICATIONS OF HYPERCUBE DIAGRAMS TO THE DESIGN OF SWITCHING CIRCUITS

by

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ABSTRACT

A method is presented for displaying two-valued, logical functions in a manner analogous to the graphical plots used for continuous functions, and applications of this technique to various aspects of switching circuit theory are described.

Functions of n variables are mapped on figures which represent n-dimensional hypercubes, with each point of the plot designating a condition of the variables for which the function has a value of unity.

Several forms of the hypercube diagrams are derived, and one particular type has been found to be the most satisfactory for general use.

A technique has been devised for reducing switching functions to a minimal sum-of-products or product-of-sums form, and this method appears to work very well in comparison with other procedures. The charts can also be applied to the factoring problem, and can serve as a helpful tool when a Shannon disjunctive network synthesis is being carried out.

Several simple tests for partial and total symmetry have been found, one of which does not require the use of diagrams. A test for certain other kinds of group invariance is also proposed, although further investigation will be required to ascertain its validity and to extend its usefulness.

Finally, the basis of a possible procedure for designing non-series-parallel networks is suggested.

Thesis Supervisor: Samuel H. Caldwell
Title: Professor of Electrical Engineering

ACKNOWLEDGMENT

The author is deeply indebted to Professor S. H. Caldwell, who supervised this thesis after having conceived the idea upon which it is based.

In teaching his Switching Circuits course, Professor Caldwell has always avoided giving the impression that the last word has been spoken on any topic. While describing what has already been accomplished in the field, he also points out the areas in which further investigation is necessary, thus presenting his subject in a dynamic and stimulating manner.

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CHAPTER I

INTRODUCTION

Combinational switching circuits are generally described by verbal statements, circuit diagrams, tables of combinations, the equations of Boolean algebra, or several of these. The subject of this thesis is a graphical type representation in which switching functions appear as patterns superposed on hypercube diagrams.

Similar configurations are described in The Design of Switching Circuits[1] on pp. 172-75 in connection with relay operating sequences, and on p.282 with regard to error checking. Professor Caldwell suggested that hypercube diagrams might be useful in the study of combinational circuits.

The topics investigated here include the construction of the charts in several forms, their general properties, a method for using them to simplify algebraic expressions, application to Shannon's disjunctive tree method, an embryo technique for designing bridge type circuits, a procedure for testing functions for partial and total symmetry, and a method for detecting group invariance.

Only applications to two-terminal, combinational circuits are discussed, although the general material concerning the hypercube diagrams can probably be made useful in other types of switching circuit problems.

Most of the work done here is of a sufficiently general nature to be applicable to systems employing

all kinds of switching elements. The parts in which non-series-parallel type relay circuits are considered constitute the exceptions, although it is possible that analagous configurations involving electronic and other type components may be devised.

When specific circuits are discussed, the concept of transmission will be employed. That is, open circuits have zero transmission and closed circuits have unity transmission. Parallel elements are represented by addition and series elements by multiplication.

The reader will be assumed to be familiar with the general concepts of switching circuit such as may be found in Shannon's 1938[2] and 1949[3] papers or in the <u>Design of Switching Circuits</u>[1].

CHAPTER II

CONSTRUCTION OF HYPERCUBE DIAGRAMS

A set of n two-valued variables may assume 2^n different conditions since each may take on the values 0 or 1, independent of the values of the others. One convenient and commonly used method for listing these conditions is to arbitrarily assign some order to the n variables and then list the combinations as binary numbers running from 0 to 2^n-1.

Another method, geometric in nature, is the one to be discussed in this paper. Again the variables are presented in some fixed order and each of the possible combinations is represented by a term consisting of a total of n ones and zeros, as in a binary number. Each of these terms is represented on a diagram by a point or node. Straight line segments are drawn connecting pairs of nodes whose corresponding terms differ in the value of only <u>one</u> variable

It is evident that each node will be connected in this manner to n other nodes with, each line representing a change in one of the n variables. The nodes are arranged in a pattern which is as symmetric as possible.

The configurations obtained for one, two and three variables are shown in figures 1, 2 and 3 at the top of the following page.

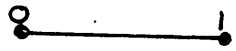


FIG. 1

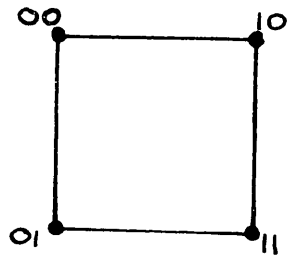


FIG. 2

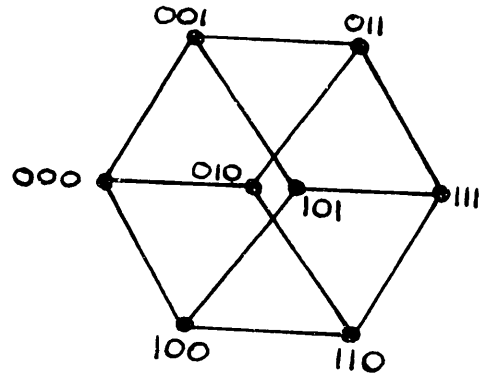


FIG. 3

It is interesting to note that the diagrams for n=1, n=2 and n=3 represent a line, a square and a cube respectively. Thus there appears to be a one to one correspondence between the number of variables and the number of dimensions involved in the diagrams. This idea will receive further attention in a later chapter.

At this point it might be well to mention that Fig. 3 is not as symmetric as it might be in that the lines are not all exactly equal and not all of the edges of the cube which should be parallel have been so drawn. The reasons for this departure from perfect symmetry become apparent from a study of Fig. 4 which is perfectly symmetric.

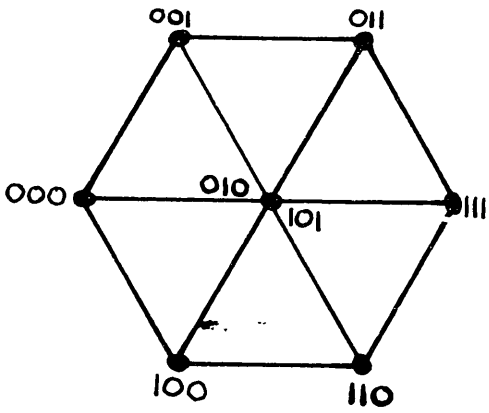


FIG. 4

Here the cube is drawn so that all the lines on the drawing are equal in length and the proper ones are parallel, with the result that points 010 and 101 are superposed, an obviously undesirable feature. Fig. 1 might also have been drawn as a pair of superposed points joined by an imaginary line perpendicular to the paper. This problem arises again in figures where $n>4$.

The methods used in constructing diagrams for $n>3$ are not essential to the further development of the subject and have, therefore, been relegated to the appendix.

The term hypercube (abbreviated HC) will be used here generically to denote the class of configurations including the preceding examples. Members of this class are described by their dimensions or the number of variables involved. Thus a line is a one-dimensional HC, a cube is a three-dimensional HC etc. Abbreviations such as 1D HC and 3D HC will be employed.

CHAPTER III

REPRESENTATION OF SWITCHING FUNCTIONS ON HYPERCUBE DIAGRAMS

Any switching function of n two-valued variables can be expanded into either of two canonic forms. That is, either as the product of a group of terms, each comprising the sum of all the n variables or their primes, or as the sum of terms, each consisting of the product of all the variables or their primes. (The prime of a variable is its inverse.)

An example of the product-of-sums expansion is:

$$f = x_3x_2x_1 + x_3'x_2x_1' + x_3x_2'x_1' + x_3x_2'x_1$$

One way of "plotting" f on a 3D HC is to replace unprimed terms with ones and primed terms with zeros, obtaining:

111 + 010 + 100 + 101

The nodes 111, 010, 100, and 101 are then marked.

The first figure in each term represents the value of x_3, the second x_2 and the third x_1. It is essential in any problem to adhere to one convention regarding which figure represents each variable.

The above example is illustrated in Fig. 5. (Marked nodes are indicated by circled x's.)

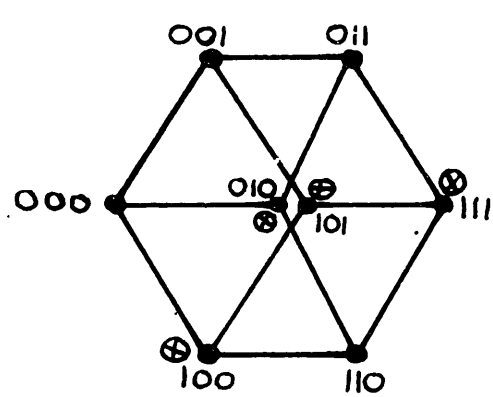


FIG. 5

Each marked node defines a condition of the three variables for which the plotted function is unity. At all other nodes the function is zero. This evidently means that the dual, or complement of f is represented by the unmarked nodes in Fig. 5. In every case either a function or its complement (or both) can be plotted by marking no more than half of the nodes of the appropriate hypercube.

The product-of-sums canonic form can also be used to plot functions on HC diagrams. The function previously mentioned can be written as:

$$f=(x_3 + x_2 + x_1)(x_3 + x_2 + x_1')(x_3 + x_2' + x_1')(x_3' + x_2' + x_1),$$

and then plotted by marking nodes 111, 110, 100 and 001. However, this type of plot does not seem to be as convenient to work with as the form derived from the sum-of-products and will not be used here. It should be observed that the nodes indicated above can be obtained from Fig. 5 by selecting the <u>unmarked</u> nodes and inverting every figure in each term. (Two points on the HC diagram with all of the corresponding variables opposite in value will be referred to hereafter as <u>dual</u> points. For example, 110 and 001 are a pair of dual points.)

It might be well to mention at this point the actual technique employed in mapping functions on the HC diagrams. Since marking the diagrams with pen or pencil would necessitate using a great many such figures, a more thrifty

and perhaps even more convenient method was devised whereby small pieces of cardboard with opposite faces colored in a contrasting manner serve as markers which can be readily moved about. No trouble has been encountered due to accidental displacement of markers caused by mild breezes from various sources. Those now in use measure about a quarter of an inch across but further experimentation will be necessary to determine the optimum shape.

CHAPTER IV

GENERAL PROPERTIES OF HYPERCUBE DIAGRAMS

In discussing hypercubes in a general manner it will be useful to refer to the four dimensional HC shown in Fig. 6 on p.11. Although specific references will be made to the $n=4$ case, all of the characteristics which will be pointed out here apply also to cases where $n \geqq 5$ and, perhaps in a degenerate manner, to the one, two, and three dimensional HC's also.

Reading from left to right, the four variables will be labelled x_4, x_3, x_2, and x_1 respectively.

Some properties which are revealed by a study of the HC diagram are:

1) Four lines emanate from each node, each line representing a change in one variable.

2) All lines are divided into four equal groups and members of each group are parallel to one another. One member of each group intersects each node. A closer examination reveals that each of these groups represents changes in a single variable. For example, the line connecting 0011 to 0111 is parallel to the line joining 1001 with 1101 and each of these lines indicates a change in x_3, the second variable.

3) Nodes symmetric to one another with respect to the geometric center of the HC diagram are dual points (definition on p. 7). Thus 1001 and 0110 are duals as are

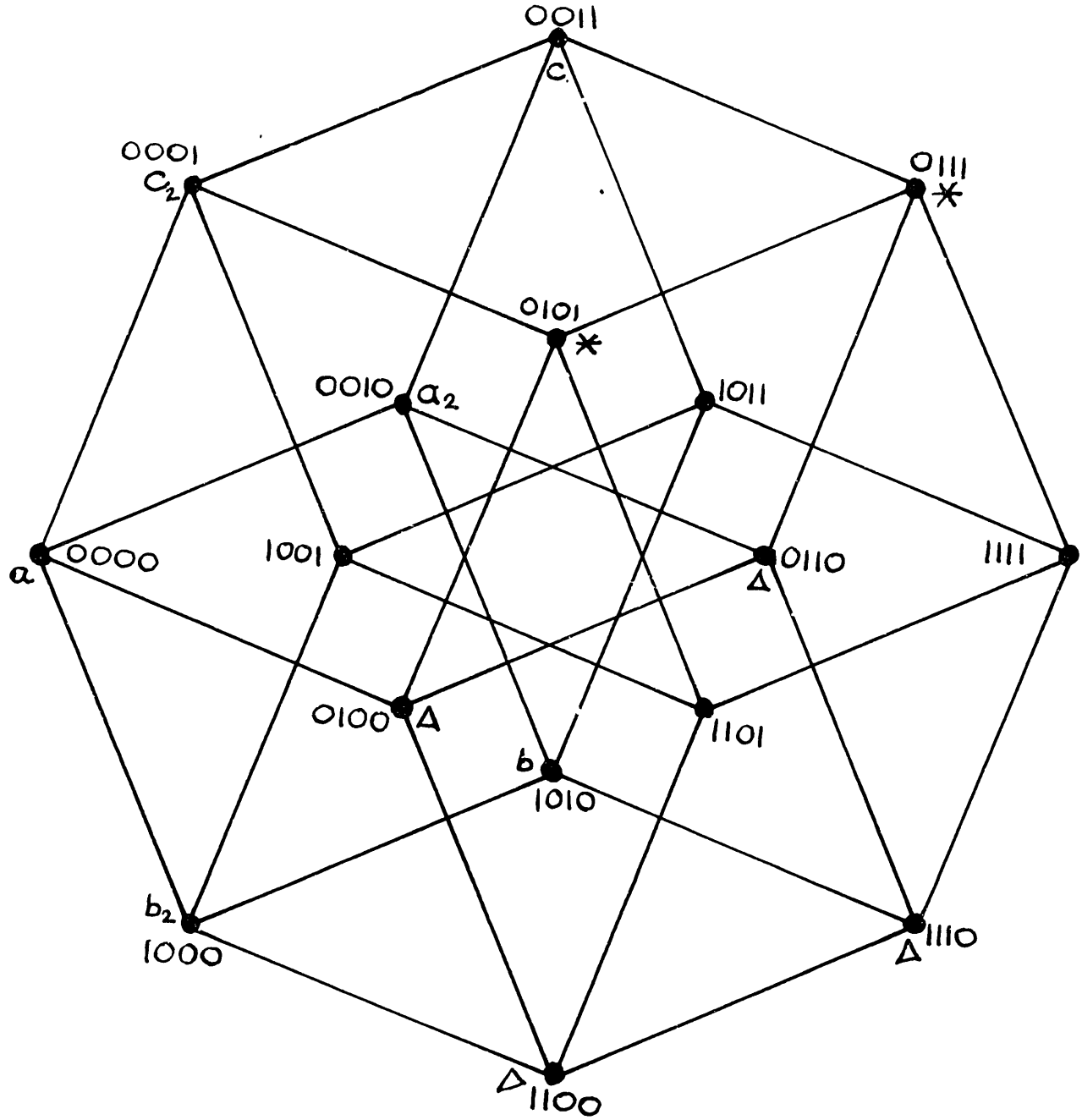
0111 and 1000.

Property 2 is a particularly useful feature since it enables us to transform the plot of a function of x_4, x_3, x_2, and x_1 into a plot of the identical function in the variables x_4, x_3, x_2' and x_1, for example, merely by shifting all the markers representing the original function along the parallel lines indicating a change in x_2. By so doing, all of the x_2's in the function are inverted. Thus, in Fig. 6, if the markers a, b, and c represent $f(x_4,x_3,x_2,x_1)$, then the markers a_2, b_2, and c_2 represent $f(x_4,x_3,x_2',x_1)$.

Sub-Hypercubes

Suppose that all but one variable are assigned constant values. Then the only possible nodes which can represent this state are the pair at which the n-1 variables have the appropriate values while the other variable is unrestricted. Such a pair will always be connected by a line representing a change in the nth variable. In Fig. 6 on p.11, for instance, the condition $x_4=0$, $x_3=1$ and $x_1=1$ is defined by the line bounded by 0101 and 0111 (the two starred nodes). The underlying principle is that the range of variation of a single two-valued variable consists of just two points . This is an example of a one dimensional hypercube contained in a four dimensional one.

FIG. 6


0011
c
0001
c_2
0111
*
0101
*
0010
a_2
1011
a
0000
1001
0110
Δ
1111
0100
Δ
1101
b
1010
b_2
1000
1110
Δ
Δ
1100

If two of the four variables are held constant, the remaining two will be able to take on four sets of values, thus including four nodes. This group of four points will be arranged in a square as indicated in the following example.

Hold x_3 at 1 and x_1 at 0. The nodes with these specifications are 0100, 1100, 0110 and 1110, all marked with triangles in Fig. 6. Here a 2D HC is contained within a 4D HC.

Following the same pattern, if only one variable is held constant and three are permitted to vary, the result will be a 3D HC. In Fig. 7, on p.14 all nodes with $x_3=1$ are connected together with double lines forming a 3D HC, which is, of course, an ordinary cube.

It seems then that if, in an n dimensional HC, k variables are held constant, the figure described will be an n-k dimensional HC. Figures generated in such a manner will be denoted as sub-hypercubes (sub-HC's). The limiting case of a single node is a zero dimensional sub-HC.

Counting Sub-Hypercubes

As a preliminary problem let us first find the number of cubes in a 4D HC. One cube will be generated each time a variable is assigned a constant value. There are four variables and two possible values for each, which leads us to the conclusion that the number of such cubes is $4x2=8$. This can be confirmed by careful counting.

A more general problem is to find the number of cubes in an nD HC (where $n > 3$). In order to generate a cube, n-3 of the variables must be assigned constant values. Out of n variables, the number of combinations of n-3 variables is $\frac{n!}{3!(n-3)!}$. Each of the n-3 variables may be assigned values of 0 or 1, so that with each combination, 2^{n-3} cubes can result. Therefore, the number of different cubes contained in an nD HC is $\frac{n!2^{n-3}}{3!(n-3)!}$. For $n=3$, this reduces to one and for $n=4$ the result is eight, as was previously shown.

The most general problem is that of calculating the number of h dimensional sub-HC's contained in an nD HC ($n > h$). An hD sub-HC results when n-h variables are held constant, and for every selection of n-h variables, 2^{n-h} sub-HC's result. Thus there are $\frac{n!2^{n-h}}{h!(n-h)!}$ sub-HC's of order h in every nD sub-HC.

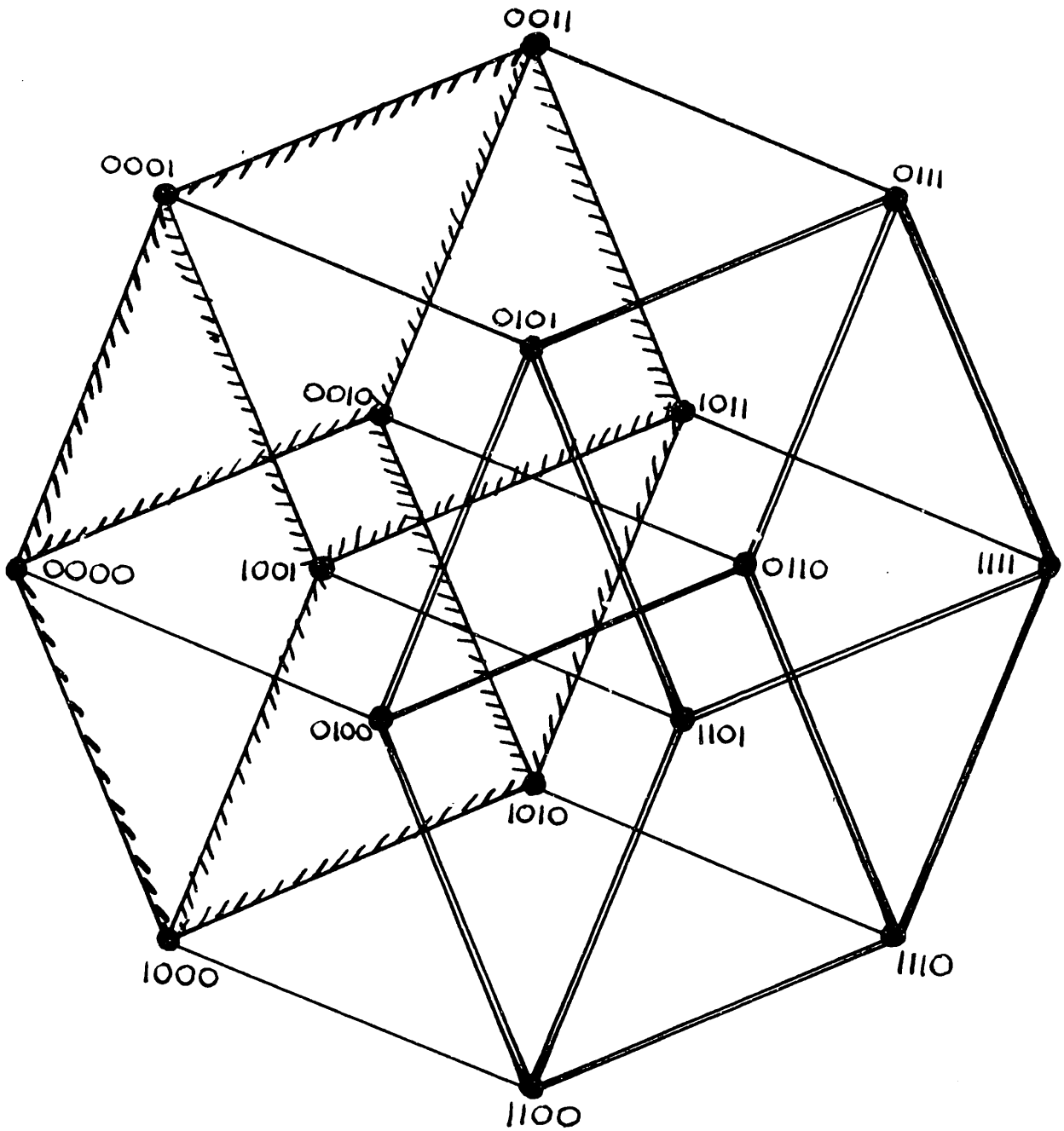
Where $h=n-1$, the above equation reduces to 2n. That is, the number of n-1 dimensional sub-HC's contained in an nD HC is equal to 2n.

Variables and Dimensions

The cases of one, two and three variables have corresponded to figures of one, two and three dimensions respectively. The generalization of this analogy to include the n dimensional figure will now be discussed.

Each variable permits movement in some direction on

FIG. 7


0011
0001
0111
0101
0010
1011
0000
1001
0110
1111
0100
1101
1010
1000
1110
1100

CHAPTER V

THE PROJECTION TYPE HYPERCUBE DIAGRAM

In the preceding sections and in the appendix, hypercube diagrams of, what might be termed, the "pictorial" type have been discussed. The figures shown for $n < 5$ are relatively simple and unambiguous, clearly presenting in a meaningful pattern all the possible combinations of n variables and permitting us to describe switching functions as parts of such configurations. Furthermore, when $n < 5$, the sub-HC's stand out clearly, a factor which is particularly important in the practical application of these diagrams, as will be seen later.

However, when n equals five or six, a study of figures A4 and A6 on pp.81,85 reveals that the complexity of the diagrams increases greatly, that double points, double lines and the like confuse matters seriously and that, although sub-HC's up to the 3D variety can still be plainly seen, the more complex sub-HC's cannot be picked out visually with any degree of confidence. These problems would, no doubt, be sharply increased in severity for cases where $n > 6$.

The difficulties due to ambiguous nodes and lines can be circumvented by distorting the diagrams slightly as in Fig. A5 (on p.82) so as to separate multiple points and lines and also to alter the positions of nodes which appear to lie on lines which do not actually intersect them.