SCATTERING THEORY ON COMPACT MANIFOLDS WITH BOUNDARY

by

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Abstract

This thesis proves trace-type formulas in two different settings. In chapter one, we use *b*-geometric methods to give a proof of a classical result in one-dimensional scattering theory. Working on $\mathbf{R}$, consider the Laplacian $\Delta = D_x^2$ and let $V \in C_c^\infty(\mathbf{R})$, $V \geq 0$. Then, in a distributional sense,

$$\mathcal{F}\mathrm{Tr}[\cos(t\sqrt{\Delta+V}) - \cos(t\sqrt{\Delta})](\lambda) = -\frac{i}{2}\frac{d}{d\lambda}\log\det S(\lambda),$$

where $\mathcal{F}$ denotes the Fourier transform and S is the scattering matrix.

The main result of this thesis is a trace-type formula for $\cos(t\sqrt{\Delta})$ on smooth, *b*-Riemannian manifolds (following Melrose) with exact *b*-metrics. These are compact manifolds with boundary and *b*-metric (and *b*-tangent space) which gives them complete ends of infinite volume. By changing coordinates on such a manifold, one gets a manifold with an infinite "cylindrical" end. The Laplacian has continuous spectrum of high multiplicity as well as possible discrete spectrum. Using some methods from scattering theory, we compute the *b*-trace of $\cos\sqrt{\Delta}t$, a regularized integral of the restriction of the kernel to the diagonal:

$$\mathcal{F}[b\text{-}\mathrm{Tr}(\cos\sqrt{\Delta}t)](\lambda) = \pi\sum_{\lambda_j^2\in\mathrm{ppSpec}\,\Delta}\delta(\lambda-\lambda_j) + \frac{1}{2i}\frac{d}{d\lambda}\log\det\Psi(\lambda)$$
$$+\frac{\pi}{4}\sum_{\lambda_k^2\in\mathrm{spec}\,\Delta_{\partial X}}\delta(\lambda-\sigma_k) + c\frac{\pi}{4}\delta(\lambda).$$

Here $\Psi(\lambda)$ is the scattering matrix, obtained from the leading behavior at the boundary, of the generalized eigenfunctions of Δ associated with λ^2 and c is the number of connected components of the boundary of X.

Thesis Supervisor: Dr. Richard Melrose

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Chapter 0

Introduction

0.1 Background

Trace formulas may be used in spectral theory to get information about the spectrum of an operator. Below, we discuss several cases where the trace of an operator is given in terms of an object of interest in spectral or scattering theory.

Recall the definition of the trace of an operator. A continuous linear operator $A : H \to H$, where H is a Hilbert space, is trace class if for all orthonormal systems $\{e_j\}$, $\{f_j\}$ in H, $\sum_j |(Ae_j, f_j)| < \infty$. If A is trace class, then the trace of A is defined to be $\mathrm{Tr}(A) = \sum(Ae_j, e_j)$, where $\{e_j\}$ is a complete orthonormal system in H. If $H = L^2(M)$, then Lidskii's theorem tells us that

$$\mathrm{Tr}(A) = \int_M A(p,p)$$

where $A(p,p')$, $p, p' \in M$ is the Schwartz kernel of A.

Consider, for example, the case of a smooth compact Riemannian manifold M without boundary. The Laplacian, Δ, has only discrete spectrum of finite multiplicity.

Consider the operator $\cos(t\sqrt{\Delta})$ which satisfies

$$\begin{aligned}(D_t^2 - \Delta)\cos(t\sqrt{\Delta}) &= 0\\ \cos(t\sqrt{\Delta})_{|_{t=0}} &= Id\\ D_t\cos(t\sqrt{\Delta})_{|_{t=0}} &= 0.\end{aligned} \tag{0.1}$$

If $\rho \in C_c^\infty(\mathbf{R})$, then $\langle\cos(t\sqrt{\Delta}), \rho(t)\rangle$ is trace class and

$$\mathrm{Tr}\langle\cos(t\sqrt{\Delta}), \rho(t)\rangle = \frac{1}{2}\sum_{\substack{\sigma_j^2 \in \operatorname{spec}\Delta \\ \sigma_j \geq 0}} \hat{\rho}(\sigma_j) + \hat{\rho}(-\sigma_j);$$

that is, in the distributional sense, with $\mathcal{F}$ denoting the Fourier transform,

$$\mathcal{F}\mathrm{Tr}(\cos(t\sqrt{\Delta}))(\lambda) = \pi \sum_{\sigma_j^2 \in \operatorname{spec}\Delta} \delta(\lambda - \sigma_j) + \pi c\delta(\lambda), \tag{0.2}$$

where c is the number of connected components of M. This fact can be used to prove Weyl's law for the asymptotic distribution of eigenvalues:

$$N(\lambda) \sim c_n \mathrm{Vol}(M)\lambda^n + O(\lambda^{n-1}) \text{ as } \lambda \to \infty$$

where $N(\lambda)$ = number of $\sigma_j^2 < \lambda^2$ (see **[Hörmander 68]**; or **[D-G]** for an improved remainder term).

Another example of this type of trace formula can be found in one-dimensional scattering theory. Here, we consider the operators $\Delta = D_x^2$ and $\Delta + V(x)$, where $V(x) \in C_c^\infty(\mathbf{R})$ is a compactly supported potential, and, for simplicity, $V \geq 0$. The scattering matrix, $S(\lambda)$, relates solutions of the perturbed equation $(\Delta + V)f_\lambda = \lambda^2 f_\lambda$ to solutions of the model $\Delta g_\lambda = \lambda^2 g_\lambda$. A classical theorem says that

$$\mathcal{F}\mathrm{Tr}[\cos(t\sqrt{\Delta + V}) - \cos(t\sqrt{\Delta})](\lambda) = -\frac{i}{2}\frac{d}{d\lambda}\log\det S(\lambda). \tag{0.3}$$

Note that we must subtract $\cos(t\sqrt{\Delta})$ as a way of regularizing, since $\cos(t\sqrt{\Delta + V})$ is not trace class even in a distributional sense. Here $\log\det S(\lambda)$ is the analog of the counting function for the eigenvalues in the previous case.

A proof of this equality can be found in Chapter 1 of this thesis.

0.2 Problem

This thesis proves a trace formula in a new setting, that of b-Riemannian manifolds, as discussed by **[Melrose]**. Let X be a smooth compact manifold with boundary ∂X and an exact b-metric (see chapter 2 for definition). Then, if x is an appropriate boundary defining function ($x \geq 0$, $\{x = 0\} = \partial X$), the Laplacian on X is given, near the boundary, by $(xD_x)^2 + \Delta_{\partial X} + xQ$, where $\Delta_{\partial X}$ is the Laplacian on ∂X obtained from the restriction of the metric on X to ∂X, and Q is a differential operator of at most second order tangent to the boundary. This operator has continuous spectrum of high multiplicity, as well, perhaps, as discrete spectrum.

As in the previously mentioned cases, we can define and construct $\cos(t\sqrt{\Delta})$. Like the one-dimensional scattering case, $\cos(t\sqrt{\Delta})$ is not trace class even after being paired with a smooth, compactly supported function in t. We must use a regularization of the trace, the b-trace defined in **[Melrose]**: if A is an operator with continuous kernel $A(p,p')$,

$$b\text{-Tr}_\nu A = \lim_{\epsilon \downarrow 0}\Big[\int_{x>\epsilon} A(p,p) + \log\epsilon \int_{\partial X} A(p,p)_{|p\in\partial X}\Big], \tag{0.4}$$

where $\nu \in C^\infty(\partial X, {}_+N\partial X)$ is a trivialization of the normal bundle of X and x is a boundary defining function with $dx \cdot \nu = 1$ at the boundary of X. The problem here is to find the analogue of the right hand sides of (0.2) and (0.3). This thesis proves

Theorem 0.2.1 *Let $\Psi(\lambda)$ be the scattering matrix defined in chapter two. Then*

$$\mathcal{F}b\text{-}Tr(\cos(t\sqrt{\Delta}))(\lambda) = \frac{1}{2i}\frac{d}{d\lambda}\log\det\Psi(\lambda) - +\frac{\pi}{4}\sum_{\sigma_k^2\in\operatorname{spec}\Delta_{\partial X}}\delta(\lambda-\sigma_k)$$
$$+\frac{\pi}{4}c\delta(\lambda) + \pi\sum_{\lambda_j^2\in\operatorname{ppSpec}\Delta}\delta(\lambda-\lambda_j)$$

where c is the number of connected components of ∂X.

The scattering matrix relates the leading behavior of solutions of $\Delta f_\lambda = \lambda^2 f_\lambda$ to solutions of $((xD_x)^2 + \Delta_{\partial X})g_\lambda = \lambda^2 g_\lambda$. It is defined in Chapter 2.

0.3 Organization of Thesis

This thesis is organized into 5 chapters. As previously mentioned, the first chapter gives a proof of the trace formula for the one-dimensional scattering case. The remaining chapters are devoted to proving Theorem 0.2.1.

We include chapter 1 because it uses b-geometric methods and is similar in method to the proof for the manifold with boundary. However, the rest of the thesis is independent of this chapter.

We outline the proof, given in chapters 2-5, of Theorem 0.2.1.

The second chapter defines the wave group $U(t)$ on X.

$$U(t) : \dot{C}^\infty(X) \times \dot{C}^\infty(X) \ni (u_0, u_1) \mapsto (u(t), D_t u(t)) \in \dot{C}^\infty(X) \times \dot{C}^\infty(X)$$

with

$$\begin{aligned} (D_t^2 - \Delta)u(t) &= 0 \\ u(t)_{|_{t=0}} &= u_0 \\ D_t u(t)_{|_{t=0}} &= u_1 \end{aligned} \tag{0.5}$$

and gives some of its properties. For $\rho \in \mathcal{S}(\mathbf{R})$ we will calculate

$$b\text{-Tr}U(\rho) = b\text{-Tr}\langle U(t), \rho(t)\rangle = 2b\text{-Tr}\langle \cos(t\sqrt{\Delta}), \rho(t)\rangle,$$

which will give us what the b-trace of $\cos(t\sqrt{\Delta})$ is in a distributional sense. This chapter also defines the scattering matrix $\Psi(\lambda)$ and explores a few of its properties.

The third chapter proves the existence for $\vec{u}$ in a subset of the finite energy space of

$$\lim_{t\to\infty} U_0(-t)\chi(t\sqrt{x})U(t)\vec{u},$$

where $U_0(t)$ is the wave group on the model manifold $\hat{X} = [0, \infty) \times \partial X$ and χ is a cut-off function which is 1 in a neighborhood of the boundary of X and 0 outside a slightly larger neighborhood. We define this limit to be $M_+\vec{u}$ and extend it to a slightly larger class of initial data. The operator M_+ separates initial data corresponding to discrete spectrum of the Laplacian from that corresponding to continuous spectrum; in fact, we show that

$$U(t) = \tilde{M}_+ U_0(t) M_+ + U(t) P_d \tag{0.6}$$

where P_d is projection onto the span of the L_b^2 eigenfunctions of Δ and $\tilde{M}_+$ is an operator defined in a way analogous to the definition of M_+. This Moeller wave operator is analogous to the M_+ which appears in Chapter 1.

The fourth chapter uses the results of the third to begin to calculate b-$\mathrm{Tr}(\cos t\sqrt{\Delta})$. It calculates the contribution from the discrete spectrum and reduces the contribution from the continuous spectrum to integrals of $U_0(\rho)M_+$ and $\tilde{M}_+U_0(\eta)$ over the corners of $X \times \tilde{X}$.

The final chapter actually calculates the contribution of the continuous spectrum to the b-trace, using the results of chapters 3 and 4.

Chapter 1

One-Dimensional Scattering

This is an explanation of how to use *b*-geometric methods to prove a trace formula for one-dimensional scattering theory. It assumes some knowledge of scattering theory (see, for example, **[Zworski]** or **[D-T]** for one-dimensional scattering, and **[Thoe]** for a discussion of the wave groups and wave operators in $\mathbf{R}^3$) and some b-geometry (see **[Melrose]**).

Let $V \in C_c^\infty(\mathbf{R})$, $V \geq 0$. For $t \in \mathbf{R}$ let $U_V(t)$ be the wave group,

$$U_V(t) : C_c^\infty(\mathbf{R}) \times C_c^\infty(\mathbf{R}) \ni (u_0, u_1) \mapsto (u(t), D_t u(t)) \in C_c^\infty(\mathbf{R}) \times C_c^\infty(\mathbf{R}) \quad (1.1)$$

with:

$$(D_t^2 - D_x^2 - V(x))u(t) = 0$$

$$\begin{aligned} u(0) &= u_0 \\ D_t u(0) &= u_1. \end{aligned} \quad (1.2)$$

We denote the special case $V \equiv 0$ by $U_0(t)$. Let $S(\lambda)$ be the scattering matrix, which is defined by (1.12-1.16). We want to prove the following classical theorem:

Theorem 1.0.1 *For all $\rho \in C_c^\infty(\mathbf{R})$,*

$$Tr(\langle U_V(t) - U_0(t), \rho(t) \rangle) = (-i/2\pi)\langle \frac{d}{d\lambda} \log \det S(\lambda), \hat{\rho}(\lambda) \rangle.$$

Let $M_+ = \lim_{t\to\infty} U_0(-t)U_V(t)$ be the wave operator. Then $U_0(t)M_+ = M_+U_V(t)$, and thus

$$\mathrm{Tr}\langle U_V(t) - U_0(t), \rho(t)\rangle = \mathrm{Tr}[M_+^{-1}, \langle U_0(t), \rho(t)\rangle M_+]. \tag{1.3}$$

If the operators in the commutant were trace class, then the trace would be zero. This is not the case, however.

What the commutant formula suggests is that we do something which looks more complicated: work on a compact manifold with boundary and b-metric, where we can use the ideas that relate the b-trace of a commutant to the indicial families of the operators involved. We get such a manifold by compactifying the real line to the interval $J = [-1,1]$, (by sending $y \in \mathbf{R}$ to $(e^y+1)/(e^y-1)$). Under this mapping, $\langle U_0(t), \rho(t)\rangle$ and $\langle U_V(t), \rho(t)\rangle$ go to elements of $\Psi_b^{-\infty}([-1,1])$. While M_+^{-1} and M_+ do not go to elements of the small b-calculus, they almost become elements of the overblown b-calculus ${}^{ob}\Psi([-1,1])$. For the purposes of this note, the overblown calculus will be defined as follows:

$$A \in {}^{ob}\Psi([-1,1]) \iff A = A' + A''R^*, \tag{1.4}$$

where A', $A'' \in \Psi_b([-1,1])$, and $R : J \to J$, $R(x) = -x$.

Using a (known) formula relating the b-trace of the commutator of two operators in the small b-calculus (when they are sufficiently smooth) to their indicial families, we develop a similar formula for elements of the overblown b-calculus. Then, since the indicial family of M_+ can be calculated in terms of the scattering matrix, this will give b-$\mathrm{Tr}(\langle U_V - U_0, \rho.\rangle) = \mathrm{Tr}(\langle U_V - U_0, \rho\rangle)$, as desired.

What we are doing here, really, is using the fact that for "well behaved" operators A and B on X a compact manifold with boundary, b-$\mathrm{Tr}[A,B]$ depends only on the kernels of A and B near the corners of $X \times X$. The indicial family captures the leading behavior of the kernels at the corners.

Unfortunately, making this rigorous requires some digressions off the subject of scattering theory and into b-geometry.

1.1 Some Lemmas for the Overblown Calculus

In this section we prove some lemmas we will need for a b-geometric proof of the trace formula.

Lemma 1.1.1 *If* $A \in {}^{ob}\Psi^m([-1,1])$, $B \in {}^{ob}\Psi^{m'}([-1,1])$ *then* $AB \in {}^{ob}\Psi^{m+m'}([-1,1])$.

Proof: This follows easily from composition for elements of Ψ_b. Let

$$A = A' + A''R^*,\ B = B' + B''R^*, \tag{1.5}$$

where A', $A'' \in \Psi_b^m([-1,1])$, B', $B'' \in \Psi_b^{m'}([-1,1])$. Then

$$\begin{aligned} A \circ B &= (A' + A''R^*) \circ (B' + B''R^*) \\ &= (A'B' + A''R^*B''R^*) + (A'B'' + A''(R^*B'R^*))R^* \\ &= C' + C''R^*, \end{aligned}$$

where C' and C'' are in $\Psi_b^{m+m'}([-1,1])$ by the usual composition formula.

Q.E.D.

Defining the indicial family (in larger sense than usual):

Recall the usual definition of the indicial family of $A \in \Psi_b^m(X)$, for X a smooth compact manifold with boundary: If $\kappa(A)$ is the kernel of A lifted to X_b^2, and ν is a trivialization of the normal bundle of X, then

$$I_\nu(A,\lambda) = \int_0^\infty s^{-i\lambda}\kappa(A)_{|_{ff}}\frac{ds}{s}$$

where s is the projective coordinate x/x' at the front face (ff) of X_b^2, and $\nu \cdot dx = 1$ at the boundary, and similarly for x'.

We wish to define an indicial family for an element of the overblown calculus in a way which reduces to the usual indicial family for elements of the small b-calculus. The indicial family for an element A of ${}^{ob}\Psi([-1,1])$ is a two-by-two matrix (one entry for each corner of $[-1,1]\times[-1,1]$) of functions depending on λ and will be denoted $\mathbf{I}_\nu(A,\lambda)$. For our purposes, we will require that $R^*\nu=\nu$.

If $A'\in\Psi_b([-1,1])$, let $I_\nu^{-1,-1}$ and $I_\nu^{1,1}$ denote the usual indicial families at the diagonal corners $(-1,-1)$ and $(1,1)$ respectively. Then, for $A\in{}^{ob}\Psi([-1,1])$, with $A=A'+A''R^*$ as usual, let $I_\nu^{\pm1,\pm1}(A,\lambda)=I_\nu^{\pm1,\pm1}(A',\lambda)$. Let

$$I_\nu^{-1,1}(A,\lambda)=I_\nu^{-1,-1}(AR^*,\lambda)=I_\nu^{-1,-1}(A'',\lambda) \tag{1.6}$$

and

$$I_\nu^{1,-1}(A,\lambda)=I_\nu^{1,1}(AR^*,\lambda)=I_\nu^{1,1}(A'',\lambda). \tag{1.7}$$

Then, finally, define $\mathbf{I}_\nu(A,\lambda)$ to be:

$$\mathbf{I}_\nu(A,\lambda)=\begin{pmatrix} I_\nu^{-1,-1}(A,\lambda) & I_\nu^{-1,1}(A,\lambda)\\ I_\nu^{1,-1}(A,\lambda) & I_\nu^{1,1}(A,\lambda)\end{pmatrix}. \tag{1.8}$$

Notice that if $A\in\Psi_b([-1,1])$, then $I_\nu(A,\lambda)$ is a diagonal matrix since A vanishes at the off-diagonal corners. Also, notice that requiring $R^*\nu=\nu$ has the pleasant effect that, for example, $I_\nu^{\pm1,\pm1}(A,\lambda)=I_\nu^{\mp1,\mp1}(R^*AR^*,\lambda)$.

This definition for the indicial family has the advantage that

Lemma 1.1.2 *If A, $B\in{}^{ob}\Psi([-1,1])$, then*

$$\mathbf{I}_\nu(AB,\lambda)=\mathbf{I}_\nu(A,\lambda)\mathbf{I}_\nu(B,\lambda).$$

Proof: The proof relies on the composition formula for the overblown calculus and the fact that if P, $Q\in\Psi_b$ then $I_\nu(PQ,\lambda)=I_\nu(P,\lambda)I_\nu(Q,\lambda)$. This implies the proposition directly in the case that A and B are both in the small calculus.

Consider $I_\nu^{-1,-1}(AB,\lambda)$, the upper left hand entry.

$$\begin{aligned} I_\nu^{-1,-1}(AB,\lambda) &= I_\nu^{-1,-1}(A'B' + A''(R^*B''R^*),\lambda) \\ &= I_\nu^{-1,-1}(A',\lambda)I_\nu^{-1,-1}(B',\lambda) + I_\nu^{-1,-1}(A'',\lambda)I_\nu^{-1,-1}(R^*B''R^*,\lambda) \\ &= I_\nu^{-1,-1}(A',\lambda)I_\nu^{-1,-1}(B',\lambda) + I_\nu^{-1,1}(A'',\lambda)I_\nu^{1,-1}(B'',\lambda). \end{aligned}$$

The calculations for the other entries are similar.

Q.E.D.

We need to define the b-trace of $A \in {}^{ob}\Psi^m([-1,1])$. Recall, for X a compact manifold with boundary, ϕ continuous, $\nu \in C^\infty(\partial X, {}_+N\partial X)$ a trivialization of the normal bundle of X, and $x \in C^\infty(X)$ a boundary defining function with $dx \cdot \nu = 1$ at ∂X,

$${}^\nu\int_X \phi = \lim_{\epsilon \downarrow 0}\left[\int_{x>\epsilon}\phi + \log\epsilon \int_{\partial X}\phi_{|\partial X}\right]$$

(**[Melrose]**). Now, define, for $A \in {}^{ob}\Psi^m([-1,1])$ with $m < -1$

$$\begin{aligned} b\text{-Tr}_\nu(A) &= {}^\nu\textstyle\int_{[-1,1]} A_{|\Delta_b} \\ &= {}^\nu\textstyle\int_{[-1,1]}(A' + A''R^*)_{|\Delta_b} \\ &= b\text{-Tr}_\nu(A') + \text{Tr}(A''R^*). \end{aligned}$$

Now we can find a formula for the b-trace of the commutator of two elements of the overblown calculus. Fortunately, with this definition of indicial families it has a nice form.

Lemma 1.1.3 *If* $A \in {}^{ob}\Psi^{-\infty}([-1,1])$, $B \in {}^{ob}\Psi^m([-1,1])$, *then*

$$b\text{-}Tr_\nu([A,B]) = -\frac{1}{2\pi i}\,Tr\int(\frac{d}{d\lambda}\mathbf{I}_\nu(A,\lambda))\mathbf{I}_\nu(B,\lambda)d\lambda.$$

Proof: As one should expect by now, the proof uses a similar formula for the b-trace of the commutator of elements of the small b-calculus. Writing A and B in the usual fashion,

$$\begin{aligned} b\text{-Tr}_\nu([A,B]) &= b\text{-Tr}_\nu[A' + A''R^*, B' + B''R^*] \\ &= b\text{-Tr}_\nu([A',B'] + [A',B''R^*] + [A''R^*,B'] + [A''R^*,B''R^*]) \end{aligned}$$

Consider the terms one at a time. By the result from the small b-calculus, the first term is

$$b\text{-Tr}_\nu[A', B'] = -\frac{1}{2\pi i}\text{Tr}\int(\frac{d}{d\lambda}\mathbf{I}_\nu(A', \lambda))\mathbf{I}_\nu(B', \lambda)d\lambda.$$

The second and third terms both give zero. Consider, for example, the second term, $[A', B''R^*]$:

$$b\text{-Tr}_\nu[A', B''R^*] = \lim_{\epsilon\to 0}\Big(\int_{-1+\epsilon<x<1-\epsilon}\int_J A'(x,x')B''(x',-x) \\ - \int_{-1+\epsilon<x<1-\epsilon}\int_J A'(x,x')B''(x',-x) - (\log\epsilon)\gamma\Big)$$

where γ is the constant which makes the limit converge. However,

$$\lim_{\epsilon\to 0}\int_{x<-1+\epsilon}\int_J A'(x,x')B''(x',-x) = 0,$$

since $B''(x',-x)$is rapidly vanishing near $(-1,-1)$ and $A'(x,x')$ is rapidly vanishing at the boundaries away from the diagonal corners. The same is true at the other corners, so

$$\lim_{\epsilon\to 0}\Big[\int_{-1+\epsilon<x<1-\epsilon}\int_J A'(x,x')B''(x',-x) - \int_J\int_{-1+\epsilon<x'<1-\epsilon} A'(x,x')B''(x',-x)\Big] \\ = \int_J\int_J A'(x,x')B''(x',-x) - \int_J\int_J A'(x,x')B''(x',-x) \\ = 0.$$

The third term likewise vanishes.

The fourth term is

$$b\text{-Tr}_\nu[A''R^*, B''R^*] \\ = -\frac{1}{2\pi i}\int(\frac{d}{d\lambda}I_\nu^{1,-1}(A,\lambda))I_\nu^{-1,1}(B,\lambda) + (\frac{d}{d\lambda}I_\nu^{1,-1}(A,\lambda))I_\nu^{-1,1}(B,\lambda)d\lambda,$$

as shown below:

$$b\text{-Tr}_\nu[A''R^*, B''R^*] \\ = b\text{-Tr}_\nu(A''R^*B''R^* - B''R^*A''R^*) \\ = b\text{-Tr}_\nu(A''(R^*B''R^*) - (R^*B''R^*)A'' + R^*B''R^*A'' - B''R^*A''R^*) \\ = b\text{-Tr}_\nu[A'', R^*B''R^*] + b\text{-Tr}_\nu(R^*B''R^*A'' - B''R^*A''R^*). \tag{1.9}$$

For the first term in (1.9) we can use the result from the small calculus to get:

$$\begin{aligned} b\text{-Tr}_\nu[A'', R^* B'' R^*] &= -\frac{1}{2\pi i}\text{Tr}\int (\frac{d}{d\lambda}\mathbf{I}_\nu(A'',\lambda))\mathbf{I}_\nu(R^* B'' R^*,\lambda)d\lambda \\ &= -\frac{1}{2\pi i}\int (\frac{d}{d\lambda}I_\nu^{-1,-1}(A'',\lambda))I_\nu^{-1,-1}(R^* B'' R^*,\lambda) \\ &\quad + (\frac{d}{d\lambda}I_\nu^{1,1}(A'',\lambda))I_\nu^{1,1}(R^* B'' R^*,\lambda)d\lambda \\ &= -\frac{1}{2\pi i}\int (\frac{d}{d\lambda}I_\nu^{-1,1}(A,\lambda))I_\nu^{1,-1}(B,\lambda) \\ &\quad + (\frac{d}{d\lambda}I_\nu^{1,-1}(A,\lambda))I_\nu^{-1,1}(B,\lambda)d\lambda. \end{aligned}$$

Since we have chosen the trivialization of the normal bundle, ν such that $R^*\nu = \nu$,

$$b\text{-Tr}_\nu(R^* B'' R^* A'') = b\text{-Tr}_\nu(B'' R^* A'' R^*)$$

and the second term in (1.9) is zero.

Q.E.D.

1.2 Application to Scattering Problem

Finally, we return to the one-dimensional scattering problem outlined earlier. We will use the notation $U(t)$, $U_0(t)$ and M_+ for the wave groups and wave operator whether they are on $\mathbf{R}\times\mathbf{R}$ or $[-1,1]\times[-1,1]$.

Unfortunately, M_+ is not wholly contained in the overblown calculus. If we let

$$M_+ = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{1.10}$$

then m_{11}, m_{21}, and $m_{22} \in {}^{ob}\Psi([-1,1])$, but $m_{12} \in {}^{ob}\Psi([-1,1]) + \mathcal{A}_{phg}^{(0,0,0)}$, where $\mathcal{A}_{phg}^{(0,0,0)}$ is the space of operators whose kernels are polyhomogeneous conormal distributions with "trivial" expansions at the boundary. Since

$$M_+^{-1} = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix} \tag{1.11}$$

M_+^{-1} is very similar. However, since we are only interested in the diagonal terms in $[M_+^{-1}, \langle U_0, \rho\rangle M_+]$, and since m_{21} vanishes rapidly at the boundaries of X_b^2, the same argument used to prove the b-trace formula for elements of the small b-calculus works in this case. Therefore, the formula for the b-trace of the commutator of two elements of ${}^{ob}\Psi([-1,1])$ is applicable here.

To apply the b-trace formula, we need to know the indicial families of $\langle U_0, \rho\rangle$ and M_+. We can, without loss of generality, assume that ρ is even, since the diagonal entries in $U_V(t) - U_0(t)$ are even in t. This assumption makes $\mathbf{I}(\langle U_0, \rho\rangle, \lambda)$ have the nice form $\mathbf{I}(\langle U_0, \rho\rangle, \lambda) = \hat{\rho}(\lambda) Id$, where Id is the 4×4 identity matrix (the kernel of $U_0(t)$ is a 2×2 matrix).

We need to fix a little more notation. Suppose $\mathrm{supp}(V) \subset [a, b]$. Using the notation of **[Zworski]**, let $\phi_+(x, \lambda)$ satisfy

$$\begin{aligned} D_x^2\phi_+ + V\phi_+ &= \lambda^2\phi_+ \\ \phi_+(x,\lambda) &= e^{-ix\lambda} \text{ if } x > b \end{aligned} \tag{1.12}$$

and $\phi_-(x, \lambda)$ satisfy

$$\begin{aligned} D_x^2\phi_- + V\phi_- &= \lambda^2\phi_- \\ \phi_-(x,\lambda) &= e^{-ix\lambda} \text{ if } x < a \end{aligned} \tag{1.13}$$

Then there is a distribution X and a function Y such that

$$-i\lambda\phi_+(x,\lambda) = \hat{X}(-\lambda)\phi_-(x,\lambda) + \hat{Y}(\lambda)\phi_-(x,-\lambda) \tag{1.14}$$

and

$$i\lambda\phi_-(x,\lambda) = \hat{X}(\lambda)\phi_+(x,\lambda) + \hat{Y}(\lambda)\phi_+(x,-\lambda). \tag{1.15}$$

The scattering matrix is

$$S(\lambda) = \begin{pmatrix} \frac{i\lambda}{\hat{X}(\lambda)} & \frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \\ \frac{\hat{Y}(-\lambda)}{\hat{X}(\lambda)} & \frac{i\lambda}{\hat{X}(\lambda)} \end{pmatrix} \tag{1.16}$$

Lemma 1.2.1 *Using the notation above, we have*

$$\mathbf{I}(M_+,\lambda)=\frac{1}{2}\begin{pmatrix} \frac{i\lambda}{\hat{X}(\lambda)}+1 & \frac{-i}{\hat{X}(\lambda)}+\frac{1}{\lambda}+i\pi\delta(\lambda) & \frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} & \frac{-\hat{Y}(\lambda)}{\lambda\hat{X}(\lambda)}+i\pi\delta(\lambda) \\ \frac{-i\lambda^2}{\hat{X}(\lambda)}+\lambda & \frac{i\lambda}{\hat{X}(\lambda)}+1 & \frac{-\lambda\hat{Y}(\lambda)}{\hat{X}(\lambda)} & \frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \\ \frac{\hat{Y}(-\lambda)}{\hat{X}(\lambda)} & \frac{-\hat{Y}(-\lambda)}{\lambda\hat{X}(\lambda)}-i\pi\delta(\lambda) & \frac{i\lambda}{\hat{X}(\lambda)}+1 & \frac{-i}{\hat{X}(\lambda)}+\frac{1}{\lambda}-i\pi\delta(\lambda) \\ \frac{-\lambda\hat{Y}(-\lambda)}{\hat{X}(\lambda)} & \frac{\hat{Y}(-\lambda)}{\hat{X}(\lambda)} & \frac{-i\lambda^2}{\hat{X}(\lambda)}+\lambda & \frac{i\lambda}{\hat{X}(\lambda)}+1 \end{pmatrix}$$

Proof: Since the calculations are similar for all the entries, we will calculate only a few. Letting $\mathbf{I}(M_+,\lambda)=(b_{ij})$, we will calculate $b_{11}=I^{-1,-1}(m_{11},\lambda)$, $b_{41}=I^{1,-1}(m_{21},\lambda)$ and $b_{21}=I^{-1,-1}(m_{12},\lambda)$.

If we work on $\mathbf{R}\times\mathbf{R}$,

$$I^{-1,-1}(M_+,\lambda)=\int e^{-is\lambda}\lim_{x'\to-\infty}M_+(s+x',x')ds$$

and

$$I^{1,-1}(M_+,\lambda)=\int e^{-is\lambda}\lim_{x'\to-\infty}M_+(-s-x',x')ds.$$

To compute $I^{-1,-1}(m_{11},\lambda)$ and $I^{1,-1}(m_{21},\lambda)$, consider, for $x_0<a$

$$M_+\begin{pmatrix}\delta(x-x_0)\\ 0\end{pmatrix}=\frac{1}{2}M_+\begin{pmatrix}\delta(x-x_0)+\delta(x-x_0)\\ D_x\delta(x-x_0)-D_x\delta(x-x_0)\end{pmatrix} \tag{1.17}$$

Since, for $t>0$, the support of $U_V(t)\,(\delta(x-x_0),D_x\delta(x-x_0))$ does not meet the support of V,

$$M_+\begin{pmatrix}\delta(x-x_0)\\ D_x\delta(x-x_0)\end{pmatrix}=\begin{pmatrix}\delta(x-x_0)\\ D_x\delta(x-x_0)\end{pmatrix}.$$

Now consider $(\delta(x-x_0),-D_x\delta(x-x_0))$. For t sufficiently large,

$$U_V(t)\begin{pmatrix}\delta(x-x_0)\\ -D_x\delta(x-x_0)\end{pmatrix}=\begin{cases}\begin{pmatrix}f(x+x_0+t)\\ D_xf(x+x_0+t)\end{pmatrix} & \text{for } x<a\\ \begin{pmatrix}g(x-x_0-t)\\ -D_xg(x-x_0-t)\end{pmatrix} & \text{for } x>b\end{cases}$$

where $\hat{f}(\lambda) = \frac{\hat{Y}(\lambda)}{\hat{X}(-\lambda)}$ and $\hat{g}(\lambda) = \frac{i\lambda}{\hat{X}(\lambda)}$. Since, for $a < x < b$, $U_V(t)(u_0, u_1)(x) \to 0$ as $t \to \infty$, we get

$$U_0(-t)U_V(t)\begin{pmatrix} \delta(x-x_0) \\ -D_x\delta(x-x_0) \end{pmatrix} \to \begin{pmatrix} f(x+x_0)+g(x-x_0) \\ D_xf(x+x_0)-D_xg(x-x_0) \end{pmatrix}$$

as $t \to \infty$. Since we have $g(r) = 0$ for $r > (b-a)$ and $f(r) = 0$ for $r < 2a$, by letting $x_0 \to -\infty$, we get

$$\begin{aligned} I^{-1,-1}(m_{11},\lambda) &= \tfrac{1}{2}\textstyle\int e^{-is\lambda}(g(s)+\delta(s))ds \\ &= \tfrac{1}{2}(\hat{g}(\lambda)+1) \\ &= \tfrac{1}{2}(\tfrac{i\lambda}{\hat{X}(\lambda)}+1). \end{aligned}$$

Similarly, we get

$$\begin{aligned} I^{1,-1}(m_{21},\lambda) &= \frac{1}{2}\int e^{-is\lambda}(-D_sf(-s))ds \\ &= -\frac{1}{2}\int \lambda e^{-is\lambda}(f(-s))ds \\ &= -\frac{1}{2}\lambda\hat{f}(-\lambda) \\ &= -\frac{1}{2}\frac{\lambda\hat{Y}(-\lambda)}{\hat{X}(\lambda)} \end{aligned}$$

Computing $I^{-1,-1}(m_{12},\lambda)$ is somewhat more difficult. For $x_0 < a$, consider $M_+(0,\delta(x-x_0))$. For $0 \le t < a - x_0$,

$$U_V(t)\begin{pmatrix} 0 \\ \delta(x-x_0) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} iH(x-x_0+t)-iH(x-x_0-t) \\ \delta(x-x_0+t)+\delta(x-x_0-t) \end{pmatrix}$$

For large t,

$$U_V(t)\begin{pmatrix} 0 \\ \delta(x-x_0) \end{pmatrix} = \begin{cases} 0 \text{ if } x < x_0 - t \\ \frac{1}{2}\begin{pmatrix} iH(x-x_0+t) \\ \delta(x-x_0+t) \end{pmatrix} \text{ if } x_0 - t < x < 2a - x_0 - t \\ \frac{1}{2}\begin{pmatrix} F(x+x_0+t) \\ D_xF(x+x_0+t) \end{pmatrix} \text{ if } 2a - x_0 - t < x < a \\ \frac{1}{2}\begin{pmatrix} G(x-x_0-t) \\ D_xG(x-x_0-t) \end{pmatrix} \text{ if } x > b \end{cases}$$

where $\hat{G}(\lambda) = -\frac{i}{\hat{X}(\lambda)}$. As in the previous case, we get

$$\begin{aligned} I^{-1,-1}(m_{12}, \lambda) &= \tfrac{1}{2} \int e^{-is\lambda}(iH(s) + G(s))ds \\ &= \tfrac{1}{2}(\tfrac{1}{\lambda} + i\pi\delta(\lambda) - \tfrac{i}{\hat{X}(\lambda)}) \end{aligned}$$

Q.E.D.

Finally, we can prove the trace formula theorem.

Theorem: For all $\rho \in C_c^\infty(\mathbf{R})$,

$$\mathrm{Tr}(\langle U_V(t) - U_0(t), \rho(t)\rangle) = (-i/2\pi)\langle \frac{d}{d\lambda} \log\det S(\lambda), \hat{\rho}(\lambda)\rangle.$$

Proof: The proof has now been essentially reduced to linear algebra.

$$\begin{aligned} \mathrm{Tr}(\langle U_V(t) - U_0(t), \rho(t)\rangle) &= b\text{-}\mathrm{Tr}[M_+^{-1}, \langle U_0, \rho\rangle M_+] \\ &= \tfrac{-1}{2\pi i}\mathrm{Tr} \int \tfrac{d}{d\lambda}(\mathbf{I}(M_+^{-1}, \lambda))\mathbf{I}(\langle U_0, \rho\rangle, \lambda)\mathbf{I}(M_+, \lambda)d\lambda \\ &= \tfrac{-1}{2\pi i}\mathrm{Tr} \int \hat{\rho}(\lambda)\tfrac{d}{d\lambda}(\mathbf{I}(M_+, \lambda)^{-1})\mathbf{I}(M_+, \lambda)d\lambda \\ &= \tfrac{1}{2\pi i} \int \hat{\rho}(\lambda)\tfrac{d}{d\lambda} \log\det(\mathbf{I}(M_+, \lambda))d\lambda \end{aligned}$$

The final equality comes from the fact that

$$\mathrm{Tr}(A^{-1}(t)\frac{d}{dt}A(t)) = \frac{d}{dt}\log\det A(t)$$

for A an invertible matrix depending on $t \in \mathbf{R}$. Then, since $\det(\mathbf{I}(M_+, \lambda)) = \det S(\lambda)$, we are done.

Q.E.D.

Chapter 2

Wave Group and Scattering Matrix

This chapter gives some technical background, defining some terms and fixing notation for much of the thesis.

Let X be a smooth compact manifold with boundary ∂X and an exact b-metric g. A b-metric is a metric on the interior of X with the property that there exists a boundary defining function x ($x \in C^\infty(X)$, with $x \geq 0$ and $\partial X = \{x = 0\}$) such that near the boundary,

$$g(x,y) = (\frac{dx}{x})^2 + g'(x,y) \tag{2.1}$$

where $y \in \partial X$, $g' \in C^\infty(X; T^*X \otimes T^*X)$, and $g'_{|\partial X}$ is a metric on ∂X. Throughout this thesis, x or x', when it is a coordinate function on X, will refer to this boundary defining function, and, unless stated otherwise, $y, y' \in \partial X$.

A technical note: Both the b-trace and the definition of the scattering matrix depend on a choice of boundary defining function x. We may initially choose any boundary defining function x so that at the boundary g has the form (2.1) (although this condition essentially fixes x at the boundary up to a constant positive multiple at each component of the boundary). After this, however, we must be consistent and use this fixed boundary-defining function. We do not indicate this dependence on choice of boundary-defining function in our notation.

A b-differential operator of order m is differential operator of order m on X which is tangent to the boundary; ie, at the boundary, it is given by

$$\sum_{j+|\alpha|\leq m} a_{\alpha,j}(x,y)(xD_x)^j D_y^\alpha,$$

where the $a_{j,\alpha}$ are smooth coefficients. We denote the space of such operators $\mathrm{Diff}_b^m(X)$. Associated to the metric g is the Laplacian Δ, a second crder b-differential operator, which, near the boundary, is given by

$$\Delta = (xD_x)^2 + \Delta_{\partial X} + xQ$$

where $\Delta_{\partial X}$ is the Laplacian on the boundary ∂X associated with $g'_{|\partial X}$ and Q is a b-differential operator of order at most two.

Also associated with the b-metric is a b-density. The space of distributions which are square integrable with respect to this density is denoted by $L_b^2(X)$. Analogously, for $m \geq 0$ an integer,

$$H_b^m(X) = \{u \in L_b^2(X) : Pu \in L_b^2 \text{ for all } P \in \mathrm{Diff}_b^m(X)\}.$$

For $a \in \mathbf{R}$, we have the weighted space

$$x^a H_m^b(X) = \{u : u = x^a v, v \in H_b^m(X)\}.$$

Another space which we will often use is $\dot{C}^\infty(X)$, the space of smooth functions on X which vanish, with all derivatives, at the boundary.

2.1 The Wave Group

In this section we define the wave group on X and give some of its properties. We define the wave group, $U(t)$, for $t \in \mathbf{R}$ to be:

$$U(t) : \dot{C}^\infty(X) \times \dot{C}^\infty(X) \ni (u_0, u_1) \mapsto (u(t), D_t u(t))$$

with

$$
\begin{aligned}
(D_t^2 - \Delta)u(t) &= 0 \\
u(t)_{|_{t=0}} &= u_0 \\
D_t u(t)_{|_{t=0}} &= u_1
\end{aligned}
\tag{2.2}
$$

Note that the distance from a point in the interior of X to a point on the boundary in infinite. Since we are interested in the b-tangent bundle (tangent vectors with finite length with respect to our b-metric), the manifold is complete, and the existence and uniqueness of the wave group is standard. Below, we state some usual facts about the wave group, and show that the kernel at the corner of $X \times X$ is of a special form.

We can think of $U(t)$ as an operator with $(D_t^2 - \Delta)U(t) = 0$, and $U(t)_{|_{t=0}} = Id_2$, where Id_2 the 2×2 identity matrix; or, equivalently, as a distribution $U(t,p,p')$, with $(D_t^2 - \Delta)U(t,p,p') = 0$, $U(0,p,p') = Id_2\delta_{p'}(p)$. If p' is in a compact set $K \subset X$ with $K \cap \partial X = \emptyset$, then the existence of such a kernel for finite time is well-known, as are some of its properties:

- Finite propagation speed: $U(t,p,p') = 0$ if $d(p,p') > t$; note that in particular this means that for p' in the interior of X, and $p \in \partial X$, $U(t,p,p') = 0$ for all t such that $|t| < \infty$.
- $U(t,p,p')$ is unique in that $\int_{p'} U(t,p,p')\vec{u}(p')$ is the unique solution to the Cauchy problem with initial data $\vec{u}$ supported in K.
- The singular support of $U(t,p,p') \subset \{(t,p,p') :$ there is a geodesic of length $|t|$ joining p to $p'\}$; moreover, $WF(U(t,p,p')) \subset \{(t,\tau,p,\xi,p',\eta) : \tau^2 = |\xi|^2$ and $(p,\xi) = \Phi^t(p',\eta)$, where Φ^t is the flow of the Hamilton vector field associated to $|\xi|$ on $T^*X\}$

(see, for example, [D-G], [Duistermaat], or [Hörmander]).

We are particularly interested in $U(t,p,p')$ for p' (and p) near the boundary of X. Consider p' in a neighborhood of the boundary. In particular, let V be a neighborhood of the boundary with $V \cong [0,a)_x \times \partial X$, and then let's restrict ourselves to $p' = (x',y') \in [0,a/2) \times \partial X \cong \tilde{V}$. Then we can expect (and we will show this is true below) that for some $T > 0$, $U(t,p,p')$ has support in $[0,a) \times \partial X$ for $t < T$; this allows us to use these local coordinates. Let $p = (x,y)$ and $p' = (x',y') \in \tilde{V}$, where, as usual, x, x' are boundary defining functions and $y, y' \in \partial X$. As previously, we want, for small t, $U(t,x,y,x',y')$ to be a solution to

$$\begin{gathered}(D_t^2 - \Delta)U(t,x,y,x',y') = 0,\\ U(0,x,y,x',y') = Id_2 x\delta(x-x')\delta_{y'}(y).\end{gathered} \tag{2.3}$$

The extra x comes from the fact that the density on X has a factor of $1/x$ in it.

We introduce the space X_b^2, which is $X \times X$, blown up at the corner (p,p'), $p,p' \in \partial X$. Really, this means we introduce polar coordinates in the boundary defining functions near the boundary: $r = x + x'$, $\tau = \frac{x-x'}{x+x'}$, where $r \in [0,a)$ and $\tau \in [-1,1]$. Then we use the fact that $C^{-\infty}(X^2) = C^{-\infty}(X_b^2)$. We lift the problem to X_b^2, and then if we can find a solution on X_b^2, we map it back to X^2 to get a solution there.

Using projective coordinates $s = x/x', x'$ and $y, y' \in \partial X$, this lift turns the problem into

$$(D_t^2 - \Delta_{s,y})U(t,sx',y,x',y') = 0 \tag{2.4}$$

$$U(0,x,y,x',y') = Id_2\delta(s-1)\delta_{y'}(y).$$

If in local coordinates $g = g_{00}(\frac{dx}{x})^2 + 2\sum_{j>0} g_{0j}\frac{dx}{x}dy_j + \sum_{i,j>0} g_{ij}dy_i dy_j$, then

$$\Delta_{s,y} = \sqrt{g}^{-1}[sD_s g^{00}\sqrt{g}sD_s + \sum_{j>0} sD_s g^{0j}\sqrt{g}D_{y_j} + \sum_{j>0} D_{y_j}\sqrt{g}g^{j0}D_{y_j} + \sum_{i,j>0} D_{y_i}\sqrt{g}g^{ij}D_{y_j}],$$

where the coefficients depend on y and sx', g is the determinant of (g_{ij}) and (g^{ij}) is the inverse of (g_{ij}). That is, $\Delta_{s,y}$ is a Laplacian on $[0,\infty) \times \partial X$, depending on a parameter x'. Thus, we need only solve

$$\begin{gathered}(D_t^2 - \Delta_{s,y})\tilde{U}(t,s,y,y';x') = 0\\ \tilde{U}(0,x,y,y';x') = Id_2\delta(s-1)\delta_{y'}(y)\end{gathered} \tag{2.5}$$

in a neighborhood of $s = 1$, where it is non-degenerate, with x' as a parameter. Since for small t, by the usual properties of solutions to the wave equation, the support of $\tilde{U}(t)$ stays in a small neighborhood of $s = 1$, we may map this back to $X \times X$ to get a solution to (2.3) for small time. We also get that $\tilde{U}(t, s, y, y'; x')$ has the same properties as before:

- Finite propagation speed: $\tilde{U}(t, s, y, y'; x') = 0$ if $d_{x'}((s, y), (1, y')) > t$;

- $\tilde{U}(t, s, y, y'; x')$ is unique in the following sense: if we map it back to X^2 and apply it to $\vec{u}$ with support in $\tilde{V}$, for small time it gives the unique solution to the Cauchy problem for initial data $\vec{u}$;

- The singular support of $\tilde{U}(t, s, y, y'; x') \subset \{(t, s, y, y'; x')$ there is a geodesic (depending on x') of length $|t|$ joining (s, y) to $(1, y')\}$; moreover, $WF(\tilde{U}(t, s, y, y'; x')) \subset \{(t, \tau, s, \xi_0, y, \xi', y', \eta'; x') : \tau^2 = |\xi|^2$ and $(s, y, \xi_0, \xi') = \Phi^t(1, y, \eta_0, \eta')$, any η_0, where $\xi = (\xi_0, \xi')\}$.

The uniqueness of $U(t)$ for small time means we can patch together these two pieces to get a global solution for small time. Then existence and uniqueness of $U(t)$ for small t is enough to show the existence for all t, since uniqueness implies the group property: for $t, s \in \mathbf{R}$, $U(t)U(s) = U(s+t)$ since both sides solve the same differential equation with the same initial conditions.

This construction tells us that $\tilde{U}(t, s, y, y'; 0) = \tilde{U}_0(t, s, y, y'; x)$, where $\tilde{U}_0$ is the corresponding operator on the model manifold $[0, \infty) \times \partial X$ with product metric $(\frac{dx}{x})^2 + g'_{|\partial X}$.

This discussion tells us that we expect the b-trace of $U(t)$ to have singular support contained in $\{t\}$ such that there is a closed geodesic of length $|t|$ on X. In addition, it tells us the singularity at $t = 0$ which will be of interest in the future (although not really in this thesis).

The wave front set properties are enough to show that

$$U(t) : \dot{C}^\infty(X) \times \dot{C}^\infty(X) \to \dot{C}^\infty(X) \times \dot{C}^\infty(X).$$

Moreover, if we define,

$$\|(u_0, u_1)\|_E^2 = \|\mathrm{grad} u_0\|_{L_b^2}^2 + \|u_1\|_{L_b^2}^2,$$

then, for $(u_0, u_1) \in \dot{C}^\infty(X) \times \dot{C}^\infty(X)$

$$\|U(t)(u_0, u_1)\|_E = \|(u_0, u_1)\|_E$$

for all t. The space of functions with finite energy is

$$S_E(X) = \{(u_0, u_1) : \mathrm{grad} u_0 \in L_b^2, u_1 \in L_b^2\}.$$

The energy "norm" is only a seminorm on $S_E(X)$; note that $\|(c, 0)\|_E = 0$, for c a constant. We define a Hilbert space H_E which is given by

$$H_E(X) = S_E(X)/\{(c, 0) : c \in \mathbf{C}\}.$$

Note that the equivalence classes of $\dot{C}^\infty(X) \times \dot{C}^\infty(X)$ are dense in $H_E(X)$, and then we can extend $U(t) : H_E(X) \to H_E(X)$. Additionally, $U(t) : (c, 0) \mapsto (c, 0)$ for c a constant, and we may write $U(t) : S_E(X) \to S_E(X)$.

We check that the properties of the wave front set are enough to show that if $\rho \in C_c^\infty(\mathbf{R})$, then

$$U(\rho) = \langle U(t), \rho(t) \rangle \in \Psi_b^{-\infty}(X). \tag{2.6}$$

First, $U(\rho)$ is smooth because there is no point in the wave front set of U with $\tau = 0$, where τ is the cotangent variable corresponding to t. Second, we need to check that $U(\rho)$ is rapidly decreasing at $lb \sqcup rb$ of X_b^2. This follows from the second part of the discussion of constructing $U(t)$, since for all finite time the support of $\tilde{U}(t, s, y, y'; x')$ does not reach $s = 0$ or $s = \infty$, and anything starting from the interior cannot reach the boundary in finite time.

2.2 The Scattering Matrix

Let $\{\phi_j\}$ be a set of orthonormal eigenfunctions of $\Delta_{\partial X}$ on ∂X and let $\{\sigma_j^2\}$ be the corresponding eigenvalues in nondecreasing order. Then if $(\Delta - r)f = 0$ and $f \in x^{-\epsilon}H_b^\infty(X)$ for any $\epsilon > 0$, then at the boundary f is a linear combination of

$$\begin{aligned} x^{i\sqrt{r-\sigma_j^2}}\phi_j, &\quad r \geq \sigma_j^2, \\ x^{-i\sqrt{r-\sigma_k^2}}\phi_k, &\quad r > \sigma_k^2, \\ \log x\phi_m, &\quad \text{if } r = \sigma_m^2, \\ x^{\sqrt{\sigma_l^2-r}}\phi_l, &\quad r > \sigma_l^2, \end{aligned} \tag{2.7}$$

and a term of order $x \log x$ at worst. The space of generalized eigenfunctions of Δ associated to λ^2 is $f \in x^{-\epsilon}H_b^\infty(X)$ for any $\epsilon > 0$ with $f \notin L_b^2(X)$ and $(\Delta - \lambda^2)f = 0$. The dimension of the space, which we call $m_c(\lambda)$ for the continuous multiplicity of λ, is given by

$$m_c(\lambda) = \text{number of } \sigma_j^2 \leq \lambda^2,\ \sigma_j \geq 0.$$

For $\lambda > \sigma_k \geq 0$, we are particularly interested in the generalized eigenfunctions $\Phi_{k,\lambda}^+, \Phi_{k,\lambda}^- \in x^{-\epsilon}H_b^\infty(X)$ for any $\epsilon > 0$, where

$$(\Delta - \lambda^2)\Phi_{k,\lambda}^\pm = 0 \tag{2.8}$$

and, at the boundary, for ∂X connected,

$$\begin{aligned} 2\Phi_{k,\lambda}^+ &\sim x^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k + \sum_{0\leq\sigma_m\leq\lambda} x^{-i\sqrt{\lambda^2-\sigma_m^2}}S_{mk}(\lambda)\phi_m \\ 2\Phi_{k,\lambda}^- &\sim x^{-i\sqrt{\lambda^2-\sigma_k^2}}\phi_k + \sum_{0\leq\sigma_m\leq\lambda} x^{i\sqrt{\lambda^2-\sigma_m^2}}T_{mk}(\lambda)\phi_m. \end{aligned} \tag{2.9}$$

If ∂X is not connected, and σ_k^2 is an eigenvalue of ∂X_i, the ith boundary component, then at ∂X_i we have an expansion of $\Phi_{k,\lambda}^\pm$ as in(2.9), where the sums above are taken over $\sigma_m^2 \in \operatorname{spec} \Delta_{\partial X_i}$. If ∂X_j is a different boundary component, then, at ∂X_j

$$2\Phi^+_{k,\lambda} \sim \sum_{\substack{0 \le \sigma_n \le \lambda \\ \sigma_n^2 \in \operatorname{spec} \Delta_{\partial X_j}}} x^{-i\sqrt{\lambda^2 - \sigma_n^2}} S_{nk}(\lambda)\phi_n$$

$$2\Phi^-_{k,\lambda} \sim \sum_{\substack{0 \le \sigma_n \le \lambda \\ \sigma_n^2 \in \operatorname{spec} \Delta_{\partial X_j}}} x^{i\sqrt{\lambda^2 - \sigma_n^2}} T_{nk}(\lambda)\phi_n. \tag{2.10}$$

The existence of such generalized eigenfunctions is discussed in **[Melrose]**, Chapter 6. The S_{mk}'s and T_{mk}'s will, with a few other bits thrown in, give us the entries of the scattering matrix.

Consider the boundary pairing B defined in **[Melrose]** (Chapter 6): for f, g a sum of terms of the form (2.7) and possibly terms of lower order,

$$B(f,g) = \frac{1}{i}\int_X \left([\Delta - r](\chi f)\overline{(\chi g)} - (\chi f)\overline{[\Delta - r](\chi g)}\right),$$

where $\chi \in C^\infty(X)$, $\chi \equiv 1$ near ∂X and $\chi \equiv 0$ away from a product neighborhood of the boundary. This pairing depends only on the leading parts of f, g and the Laplacian at the boundary. Taking $r = \lambda^2$, we know that $B(f_1, f_2) = 0$ for all $f_i \in x^{-\epsilon} H_b^\infty$ for any $\epsilon > 0$ such that f_i is the leading part, at the boundary, of a generalized eigenfunction corresponding to λ^2.

Using this boundary pairing gives us

$$\begin{aligned} 0 &= B(\Phi^+_{k,\lambda}, \Phi^+_{l,\lambda}) \\ &= \frac{1}{2}[-\delta_{kl}\sqrt{\lambda^2 - \sigma_k^2} + \sum_{0 \le \sigma_m < \lambda} \sqrt{\lambda^2 - \sigma_m^2} S_{mk}(\lambda)\overline{S_{ml}}(\lambda)] \end{aligned} \tag{2.11}$$

and, using the $\Phi^-_{k,\lambda}$

$$\delta_{kl}\sqrt{\lambda^2 - \sigma_k^2} = \sum_{0 \le \sigma_m < \lambda} \sqrt{\lambda^2 - \sigma_m^2} T_{mk}(\lambda)\overline{T_{ml}}(\lambda) \tag{2.12}$$

Then, if we set

$$\Psi(\lambda) = (\Psi_{jk}(\lambda)),$$

with $j, k \leq m_c(\lambda)$, and

$$\Psi_{jk}(\lambda) = \begin{cases} \left(\frac{\sqrt{\lambda^2-\sigma_k^2}}{\sqrt{\lambda^2-\sigma_j^2}}\right)^{1/2} S_{jk}(\lambda) \text{ if } \lambda > \max(|\sigma_k|, |\sigma_j|) \\ \left(\frac{\sqrt{\lambda^2-\sigma_k^2}}{\sqrt{\lambda^2-\sigma_j^2}}\right)^{1/2} T_{jk}(-\lambda) \text{ if } \lambda < -\max(|\sigma_k|, |\sigma_j|) \end{cases} \tag{2.13}$$

then $\Psi(\lambda)$ is a unitary matrix whose dimension changes when λ crosses a point in the spectrum of $\Delta_{\partial X}$. This will be our scattering matrix.

Equations (2.11) and (2.12) tell us a bit more information. We know that $|S_{mk}(\lambda)|$ and $|T_{mk}(\lambda)|$ are bounded, and if $\sigma_k > \sigma_m \geq 0$, then $S_{mk}(\sigma_k) = \lim_{\lambda\downarrow\sigma_k} S_{mk}(\lambda) = 0$, and $T_{mk}(\sigma_k) = \lim_{\lambda\downarrow\sigma_k} T_{mk}(\lambda) = 0$. Also, because of the method of construction, and the algebraic relations above, near $\lambda^2 = \sigma_l^2$, $S_{mk}(\lambda)$ and $T_{mk}(\lambda)$ are smooth functions of $\sqrt{\lambda^2 - \sigma_l^2}$.

The boundary pairing also tells us that

$$\begin{aligned} 0 &= 2B(\Phi^+_{k,\lambda}, \Phi^-_{l,\lambda}) \\ &= -\overline{T_{kl}}(\lambda)\sqrt{\lambda^2 - \sigma_k^2} + S_{lk}(\lambda)\sqrt{\lambda^2 - \sigma_l^2}, \end{aligned}$$

which in turn shows that $\Psi(-\lambda) = \Psi^*(\lambda)$.

We are particularly interested in $\frac{1}{i}\log\det\Psi(\lambda)$, the argument of the determinant of $\Psi(\lambda)$, especially its derivative in λ, since that is what appears in the right-hand side of Theorem 0.2.1. We may take $\lim_{\lambda\downarrow 0}\frac{1}{i}\log\det\Psi(\lambda) \in [0, 2\pi)$ (actually, with this choice it is 0 or π), although this is not crucial. What we need to know is what happens to the argument of the determinant when we cross $\lambda = \sigma_k$.

Consider the matrix $\tilde{\Psi}_k(\sigma_k) = (\Psi_{ij}(\lambda))_{|\sigma_k}$, where $\sigma_i = \sigma_k = \sigma_j$. It is a unitary matrix; moreover,

$$\tilde{\Psi}_k(\sigma_k) = \tilde{\Psi}_k(-\sigma_k) = \tilde{\Psi}_k^{-1}(\sigma_k).$$

Therefore, $\tilde{\Psi}_k(\sigma_k)$ has only 1 and -1 for eigenvalues. Suppose for a moment that the dimension of the eigenspace with eigenvalue σ_k is 1 and $\sigma_k \neq 0$. Then $\tilde{\Psi}_k(\sigma_k) = \pm 1$. If $\tilde{\Psi}_k(\sigma_k) = 1$, then it is not unreasonable to expect the argument of $\det\Psi(\lambda)$ to be continuous at $\lambda = \sigma_k$. If, however, $\tilde{\Psi}_k(\sigma_k) = -1$, then the argument of $\det \Psi(\lambda)$ should change by π at $\lambda = \sigma_k$. This is meant to motivate the following: If $\sigma_k^2 \in \operatorname{spec} \Delta_{\partial X}$, $\sigma_k^2 \neq 0$, then at $\lambda = \sigma_k$ we take $\frac{1}{i} \log \det \Psi(\lambda)$ to decrease by π times the multiplicity of -1 as an eigenvalue of $\tilde{\Psi}_k(\sigma_k)$. At $\lambda = 0$, we take the argument of the determinant to decrease by 2π times the multiplicity of -1 as an eigenvalue of $\tilde{\Psi}_0(0)$.

The multiplicity of -1 as an eigenvalue of $\tilde{\Psi}_k(\sigma_k)$ can be interpreted as the dimension of the space of generalized eigenfunctions associated to $\lambda^2 = \sigma_k^2$ which behave like $\log x\phi$ at the boundary of X, where ϕ is a boundary eigenfunction ($\Delta_{\partial X}\phi = \sigma_k^2\phi$). If $\sigma_k > 0$, -1 being an eigenvalue of $\tilde{\Psi}_k(\sigma_k)$ means there is a linear combination $F(\lambda)$ of $\Phi_{j,\lambda}^+$ $\sigma_j = \sigma_k$, with $F(\lambda) \to 0$ as $\lambda \downarrow \sigma_k$. The leading behavior at the boundary of $\lim \lambda \downarrow \sigma_k F(\lambda)/\sqrt{\lambda^2 - \sigma_k^2}$ is given by $\log x\phi$, where ϕ is an eigenfunction for the boundary Laplacian and is an eigenfunction with eigenvalue -1 of $\tilde{\Psi}_k(\sigma_k)$.

2.3 General Notation

The following are some notational conventions which we will observe throughout.

X is our smooth compact manifold with boundary ∂X. We denote by $\tilde{X}$ the model manifold which is $[0,\infty)_x \times \partial X$, with the product metric $(\frac{dx}{x})^2 + g'_{|\partial X}$. The generalized eigenfunctions $\Phi_{k,\lambda}^{\pm}$ on X are as described in (2.8) and (2.9). For any $\epsilon > 0$ and $\lambda \neq \sigma_i$, the $\{\Phi_{k,\lambda}^+\}$ or the $\{\Phi_{k,\lambda}^-\}$ span the part of $\operatorname{null}(\Delta - \lambda^2, -\epsilon)$ perpendicular to the L_b^2 eigenfunctions. The corresponding generalized eigenfuctions on the model $\tilde{X}$ are

$$\Phi_{k,\lambda}^{0,+} = \frac{1}{2} x^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k \text{ and } \Phi_{k,\lambda}^{0,+} = \frac{1}{2} x^{-i\sqrt{\lambda^2-\sigma_k^2}}\phi_k. \tag{2.14}$$

For $\lambda \neq \sigma_i$, altogether these span the generalized eigenfunctions of $(xD_x)^2 + \Delta_{\partial X}$ on $[0,\infty) \times \partial X$.

We denote the wave group on the model manifold by $U_0(t)$, which is given by

$$U_0(t) = \begin{pmatrix} D_t F(t) & F(t) \\ D_t^2 F(t) & D_t F(t) \end{pmatrix} \tag{2.15}$$

where

$$F(t,x,y,x',y') = \frac{1}{\pi} \sum_{\sigma_k \geq 0} \int_{\sigma_k}^{\infty} (e^{i\lambda t} - e^{-i\lambda t})[\Phi_{k,\lambda}^{0,+}(x,y)\overline{\Phi_{k,\lambda}^{0,+}(x',y')} + \Phi_{k,\lambda}^{0,-}(x,y)\overline{\Phi_{k,\lambda}^{0,-}(x',y')}] \frac{d\lambda}{\sqrt{\lambda^2 - \sigma_k^2}}$$

if ∂X is connected. If the boundary isn't connected, we get a similar operator on each connected component of $\tilde{X}$, with the sum taken over the eigenvalues of boundary Laplacian of that component.

Chapter 3

The Moeller Wave Operator

We continue to use the notation $U(t)$ for the wave group on X, and use the notation $U_0(t)$ for the wave group on the product manifold $\tilde{X} = \partial X \times [0, \infty)$.

In this section we define an operator, M_+, which maps $H_E(X)$ to $H_E(\tilde{X})$, and has the following properties, among others:

1. $\text{Null}(M_+) = H_{pp}(X) =$ span of the L_b^2 eigenfunctions of Δ
2. M_+ is a partial isometry
3. $M_+U(t) = U_0(t)M_+$ for all $t \in R$.

The first two properties show that M_+ is an isometry from the orthocomplement of the discrete spectral subspace of Δ onto its range. The advantage of such an operator is that, as in the one-dimensional scattering case, $U_0(t)M_+ = M_+U(t)$, and then $\tilde{M}_+U_0(t)M_+ = P_cU(t)$, where $\tilde{M}_+$ is an analogous operator mapping $H_E(\tilde{X}) \rightarrow H_E(X)$, and P_c is projection onto the part corresponding to the continuous spectrum. Then we will use arguments similar to those from the formula for the b-trace of the commutator of elements of the small b-calculus to calculate $b\text{-Tr}\langle U(t)P_c, \rho(t)\rangle$.

If ∂X is connected, let $\chi \in C^\infty(X)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ near $x = 0$, and $\chi \equiv 0$

away from a neighborhood of the boundary. We find a subspace of $S_E(X)$, dense in the energy seminorm, so that

$$\lim_{t\to\infty} U_0(-t)\chi(x\sqrt{t})U(t)$$

exists on this subspace, and show it is continuous. Then we extend by continuity to the rest of H_E to define M_+. If ∂X is not connected, a similar construction would define operators $M_{+,i}$ corresponding to connected components ∂X_i of ∂X. In general we limit ourselves to the case of ∂X connected, although the changes involved to switching to a manifold with disconnected boundary are primarily notational.

3.1 Preliminaries

The proof requires a number of lemmas which rely heavily on the spectral measure of the Laplacian constructed in **[Melrose]**. There it is shown that the spectral measure of Δ, dE, is

$$dE = \sum_k 2\lambda\delta(\lambda^2 - (\lambda_k')^2)P_k d\lambda + \sum_{\sigma_j^2\in spec(\Delta_{\partial X})} (\lambda^2 - \sigma_j^2)_+^{-\frac{1}{2}} F_j d\lambda \tag{3.1}$$

for $\lambda > 0$, taking λ^2 as the spectral variable, which is convenient later. Here $(\lambda_k')^2$ are the eigenvalues of Δ, all of finite multiplicity, and P_k is projection onto the kth eigenspace. The F_j are smooth functions of $(\lambda^2 - \sigma_j^2)^{\frac{1}{2}}$, valued in $\Psi_{b,os}^{-2,\hat{\mathcal{E}}}$, where $\mathcal{E} = (E_{lb}, E_{rb})$ and $E_{lb} = E_{rb}$ is the smallest C^∞ index set containing $(\pm i\sqrt{\lambda^2 - \sigma_k^2}, 0)$, for $\sigma_k^2 < \lambda^2$ and $(\sqrt{\sigma_l^2 - \lambda^2}, 0)$, for $\sigma_l^2 > \lambda^2$. The range of the F_j is in the null space of $\Delta - \lambda^2$. Let

$$dE_c = \sum_{\sigma_j^2\in spec(\Delta_{\partial X})} (\lambda^2 - \sigma_j^2)_+^{-\frac{1}{2}} F_j d\lambda$$

denote the part of the spectral measure corresponding to the continuous spectrum.

Recall the notation fixed in the previous chapter: $\{\phi_k\}$ is an orthonormal set of eigenfunctions for $\Delta_{\partial X}$ corresponding to the eigenvalues $\{\sigma_k^2\}$. Then, for $\lambda > \sigma_k \geq 0$

$\Phi^+_{k,\lambda}$ are the "generalized" eigenfunctions with fixed leading behavior at the boundary; $\Phi^{0,\pm}_{k\lambda}$ are corresponding generalized eigenfunctions on the model manifold $\tilde{X}$.

We will need a more explicit formulation of F_j. From the formal self-adjointness of dE and the fact that the range of F_j is in the null space of ${}^b\Delta - \lambda^2$, we get, for p, $p' \in X$,

$$\sum_{\sigma_j^2 \in spec(\Delta_{\partial X})} (\lambda^2 - \sigma_j^2)_+^{-\frac{1}{2}} F_j(p, p', \lambda) = \sum_{k,l < m_c(\lambda)} d_{kl}(\lambda) \Phi^+_{k,\lambda}(p) \overline{\Phi^+_{l,\lambda}(p')} \tag{3.2}$$

where $m_c(\lambda) =$ the number of $\sigma_k^2 \leq \lambda^2$ with $\sigma_k \geq 0$ is the continuous multiplicity of λ,

$$d_{kl}(\lambda) = \sum_{\sigma_j^2 \in spec(\Delta_{\partial X})} (\lambda^2 - \sigma_j^2)_+^{-\frac{1}{2}} c_{jkl}(\lambda)$$

is a self-adjoint matrix, and

$$\sum_{k,l \leq m_c(\lambda)} c_{jkl}(\lambda) \Phi^+_{k,\lambda}(p) \overline{\Phi^+_{l,\lambda}(p')}$$

is C^∞ in $(\lambda^2 - \sigma_j^2)^{\frac{1}{2}}$, with values in $x^{-\epsilon} H_b^\infty(X) \otimes (x')^{-\epsilon} H_b^\infty(X)$.

Using the notation above,

$$U(t) = \begin{pmatrix} \int \cos(\lambda t) dE & \int \frac{i}{\lambda} \sin(\lambda t) dE \\ \int \lambda i \sin(\lambda t) dE & \int \cos(\lambda t) dE \end{pmatrix}. \tag{3.3}$$

We define a set of "good" initial data, $S_1(X) = \{\vec{u} \in H_E(X)$, $dE\vec{u}$ is compactly supported in $\lambda, \mathrm{supp}(dE_c\vec{u}) \cap \{\sigma_j\} = \emptyset$ for all j, and $dE_c u$ is C^∞ in λ with values in $x^{-\epsilon} H_b^\infty(X)$ for any $\epsilon > 0\}$.

Lemma 3.1.1 *For every $\epsilon > 0$ and every $\vec{u} \in S_E(X)$, there exists $\vec{v} \in S_1(X)$ such that $\|\vec{u} - \vec{v}\|_E^2 < \epsilon$.*

Proof: Since for every $\vec{u} \in S_E(X)$ there is a $\vec{u'} \in \dot{C}^\infty(X) \times \dot{C}^\infty(X)$ such that $\|\vec{u} - \vec{u'}\|_E < \epsilon$, without loss of generality we can assume that $\vec{u} = (u_0, u_1) \in \dot{C}^\infty(X) \times \dot{C}^\infty(X)$.

The proof involves finding v_0 and v_1 independently. Since the proof is essentially the same for both, we will construct only v_0.

Using λ^2 as the spectral variable, we have

$$\|\text{grad}u_0\|_{L_b^2}^2 = \int \lambda^2 dE(p,p',\lambda)\overline{u_0(p)}u_0(p')$$

Consider first the part of the spectral measure corresponding to the discrete spectrum. Since $\sum_k \lambda_k^2 \|P_k u_0\|_{L^2}^2 < \infty$, clearly there is a K such that

$$\sum_{k>K} \lambda_k^2 \|P_k u_0\|_{L^2}^2 < \frac{\epsilon^2}{N}.$$

Now consider the part of the spectral measure corresponding to the continuous spectrum. Since $\int_0^\infty \int_X \int_X \lambda^2 dE_c(p,p',\lambda)\overline{u_0(p)}u_0(p') < \infty$, there is an R such that

$$\int_{R-1}^\infty \int_X \int_X \lambda^2 dE_c(p,p',\lambda)\overline{u_0(p)}u_0(p') < \frac{\epsilon^2}{N}.$$

Let $\rho \in C_c^\infty(R)$, $\rho(\lambda) \equiv 1$ for $\lambda < (R-1)$, and $\rho(\lambda) \equiv 0$ for $\lambda > R$. We will use ρ to get compact support. Let

$$M = \max_{\substack{\sigma_j,\sigma_k<R \\ |\lambda-\sigma_k|<1}} \iint \lambda F_j(p,p',\lambda)\overline{u_0(p)}u_0(p').$$

This is possible since F_j is continuous in $(\lambda^2 - \sigma_j^2)^{\frac{1}{2}}$, and hence continuous in λ, and since $u_0 \in \dot{C}^\infty(X)$. Pick a $\delta > 0$ such that $\sqrt{\delta} < \epsilon^2/NRM(m_c(R))^2$. Let $\eta \in C^\infty(R)$ be such that $0 \le \eta \le 1$, $\eta(t) \equiv 0$ if $|t| < \frac{\delta}{2}$, and $\eta(t) \equiv 1$ if $|t| > \delta$. Then, for $0 \le l \le m_c(R)$, let

$$\tilde{v}_l(\lambda) = \rho(\lambda) \prod_{\sigma_i<R} \eta(\lambda^2 - \sigma_i^2) \int_X \overline{\Phi_{l,\lambda}^+(p)}u_0(p)$$

Let $\tilde{v}_l = 0$ if $l > m_c(R)$.

Finally, let

$$v_0(p) = \sum_{k \le K, (\lambda_k')^2 \in \text{spec}\,\Delta} P_k u_0 + \int \sum_{k,l \le m_c(\lambda)} d_{kl}(\lambda)\Phi_{k,\lambda}^+(p)\tilde{v}_l(\lambda)d\lambda.$$

Note that dEv_0 has the desired support properties, and that $dE_c v_0$ is C^∞ in λ with values in $x^{-\epsilon}H_b^\infty(X)$. Checking that v_0 is suitably close to u_0, we find

$$\begin{aligned}
&\|\mathrm{grad}(u_0 - v_0)\|^2_{L^2_b} \\
&= \sum(\lambda'_k)^2\|P_k(u_0 - v_0)\|_{L^2} + \int \lambda^2 \int_X \int_X dE_c(p,p',\lambda)\overline{u_0(p)}u_0(p') \\
&\leq 2\textstyle\sum_{k>K}(\lambda'_k)^2\|P_k u_0\|_{L^2} \\
&\quad + 2\int_R^\infty \lambda^2 \sum(\lambda^2 - \sigma_j^2)_+^{-\frac{1}{2}} \sum_{k,l\leq m_c(\lambda)} c_{jkl}(\lambda) \int \Phi^+_{k,\lambda}(p)\overline{u_0(p)} \int \overline{\Phi^+_{l,\lambda}(p')}u_0(p') \\
&\quad + 2\int_{|\lambda^2-\sigma_i^2|<\delta} \rho(\lambda)\lambda^2 \sum(\lambda^2 - \sigma_j^2)_+^{-\frac{1}{2}} \textstyle\sum_{k,l\leq m_c(\lambda)} c_{jkl}(\lambda) \int \Phi^+_{k,\lambda}(p)\overline{u_0(p)} \int \overline{\Phi^+_{l,\lambda}(p')}u_0(p') \\
&\leq 2(\tfrac{\epsilon^2}{N} + \tfrac{\epsilon^2}{N}) + \delta^{\frac{1}{2}} RM(m_c(R))^2 \leq 6\tfrac{\epsilon^2}{N}
\end{aligned}$$

so it is enough to choose $N > 24$.

Q.E.D.

3.2 Definition of M_+

In this section we define M_+ for ∂X connected. A slight generalization works for ∂X not connected. We have a series of lemmas leading to the proof that

$$\lim_{t\to\infty} U_0(-t)\chi(x\sqrt{t})U(t)\vec{u} \in S_E(X)$$

exists for $\vec{u} \in S_1(X)$, which approximates $S_E(X)$ in the energy norm. From this it is a fairly easy step to existence of the limit for all $\vec{u} \in H_E(X)$.

Let

$$\begin{aligned}
u_{LT}(t) = \; & \tfrac{1}{2}\int\{\textstyle\sum_{k,l}(e^{i\lambda t}\Phi^{0,+}_{k\lambda} + \sum_m e^{-i\lambda t}\Phi^{0,-}_{m,\lambda}S_{mk})d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle\}d\lambda \\
& + \tfrac{1}{2}\int \tfrac{1}{\lambda}\{\textstyle\sum_{k,l}(e^{i\lambda t}\Phi^{0,+}_{k,\lambda} - \sum_m e^{-i\lambda t}\Phi^{0,-}_{m,\lambda}S_{m,k})d_{k,l}(\lambda)\langle u_1, \Phi^+_{l,\lambda}\rangle\}d\lambda
\end{aligned} \tag{3.4}$$

Strictly speaking, $u_{LT}(t)$ is defined on $[0,\infty) \times \partial X$, but by multiplying it by $\chi(x\sqrt{t})$ we can also consider it to be defined on X, at least for large t.

Lemma 3.2.1 *For $\vec{u} \in S_1(X)$,*

$$\lim_{t\to\infty} \left\| \chi(x\sqrt{t})U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \chi(x\sqrt{t})\begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix} \right\|_E = 0$$

Proof: Recall

$$U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \int \cos(\lambda t) dE(u_0) + \int \frac{i}{\lambda}\sin(\lambda t) dE(u_1) \\ \int \lambda i \sin(\lambda t) dE(u_0) + \int \cos(\lambda t) dE(u_1) \end{pmatrix}$$

This is rather naturally divided into four terms and the proof is essentially the same for all four, so we will concentrate on proving:

$$\lim_{t\to\infty} \| \mathrm{d}\,\chi(x\sqrt{t}) \int \cos(\lambda t) dE(u_0)$$
$$- \mathrm{d}\,\chi(x\sqrt{t})\frac{1}{2}\int \{\sum_{k,l} [e^{i\lambda t}\Phi^{0,+}_{k\lambda} + \sum_m e^{-i\lambda t}\Phi^{0,-}_{m,\lambda} S_{mk}(\lambda)\phi_m] d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle\} d\lambda)\|_{L^2_b}$$
$$= 0$$

We have

$$\int \cos(\lambda t) dE(u_0) = \sum \cos(\lambda'_k t) P_k u_0 + \int \cos(\lambda t) dE_c u_0$$

Consider first the part corresponding to the discrete spectrum.

$$\begin{aligned} \lim_{t\to\infty} \| \mathrm{d}\,\chi(x\sqrt{t}) \textstyle\sum \cos(\lambda'_k t) P_k u_0 \|_{L^2_b} &\le \lim_{t\to\infty} \textstyle\sum ((\lambda'_k)^2 + c) \int_{x < \frac{a}{\sqrt{t}}} |P_k u_0|^2 \\ &= 0 \end{aligned}$$

since $P_k u_0 \in L^2_b(X)$ for all k. Note that it follows from this (and the corresponding estimates for the other parts), after we show that M_+ exists, that the L^2_b eigenfunctions of Δ are in the null space of M_+.

The part corresponding to the continuous spectrum is

$$\int \cos(\lambda t) dE_c u_0 = \frac{1}{2}\int (e^{i\lambda t} + e^{-i\lambda t}) \sum_{k,l} d_{k,l}(\lambda)\Phi^+_{k,\lambda}\langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda.$$

Since $\langle u_0, \Phi^+_{l,\lambda}\rangle$ is non-zero for only a finite number of l and for a compact set in λ (and thus we are summing over only finite k and l), it suffices to consider a general term corresponding to one k and one l. We have

$$\chi(x\sqrt{t})\int[e^{i\lambda t}+e^{-i\lambda t}]d_{k,l}(\lambda)\Phi^+_{k,\lambda}\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda$$
$$=\chi(x\sqrt{t})\int[e^{i\lambda t}+e^{-i\lambda t}]d_{k,l}(\lambda)\Bigg[\Phi^{0,+}_{k,\lambda}+\sum_{0\leq\sigma_m<\lambda}\Phi^{0,-}_{m,\lambda}S_{m,k}$$
$$+\sum_{\sigma_n>\lambda}x^{\sqrt{\sigma_n^2-\lambda^2}}S_{n,k}\phi_n+O(x\log x)\,\Big]\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda$$

Consider first

$$\left|\chi(x\sqrt{t})\int d_{k,l}(\lambda)e^{-i\lambda t}\Phi^{0,+}_{k,\lambda}\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda\right|$$
$$=\frac{1}{2}\left|\chi(x\sqrt{t})\int e^{-i\lambda t}x^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k D_\lambda\left(d_{k,l}(-t+\log x\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}})^{-1}\langle u_0,\Phi^+_{l,\lambda}\rangle\right)d\lambda\right|$$
$$\leq C\chi(x\sqrt{t})t^{-\frac{1}{2}+\epsilon}(\log x)^{-\frac{1}{2}-\epsilon}. \tag{3.5}$$

Since the same is true of $\chi(x\sqrt{t})\Delta\int d_{k,l}(\lambda)e^{-i\lambda t}\Phi^{0,+}_{k,\lambda}\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda$, with a different constant C, we get that

$$\lim_{t\to\infty}\|\text{grad}\,\chi(x\sqrt{t})\int d_{k,l}(\lambda)e^{-i\lambda t}\Phi^{0,+}_{k,\lambda}\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda\|_{L^2_b}=0.$$

A similar argument shows

$$\lim_{t\to\infty}\|\text{grad}\,\chi(x\sqrt{t})\int d_{k,l}(\lambda)e^{i\lambda t}\Phi^{0,-}_{m,\lambda}S_{m,k}\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda\|_{L^2_b}=0.$$

It remains to show that

$$\lim_{t\to\infty}\|\text{grad}\chi(x\sqrt{t})\int\cos(\lambda t)[\sum_{\sigma_n>\lambda}x^{\sqrt{\sigma_n^2-\lambda^2}}S_{n,k}\phi_n+O(x\log x)]\langle u_0,\Phi^+_{l,\lambda}\rangle d\lambda\|_{L^2_B}=0.$$

First notice that on the support of $\langle u_0,\Phi^+_{l,\lambda}\rangle$,

$$\sum_{\sigma_n>\lambda}x^{\sqrt{\sigma_n^2-\lambda^2}}S_{n,k}\phi_n=O(x^\epsilon)$$

for some $\epsilon > 0$. Then we have

$$\|\mathrm{grad}\chi(x\sqrt{t}) \int \cos(\lambda t)[\sum_{\sigma_n > \lambda} x^{\sqrt{\sigma_n^2-\lambda^2}} S_{n,k}\phi_n + O(x \log x)]\langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda\|^2_{L^2_b}$$
$$\leq C \int_{x<\frac{a}{\sqrt{t}}} x^{2\epsilon}$$
$$\rightarrow_{t\rightarrow\infty} 0$$

Q.E.D.

Now think of $u_{LT}(t)$ as being defined on $[0,\infty) \times \partial X$. We need to know that as t goes to infinity, $\chi(x\sqrt{t})u_{LT}(t)$ behaves like $u_{LT}(t)$.

Lemma 3.2.2 *For $\vec{u} \in S_1(X)$, and $u_{LT}(t)$ defined in (3.4),*

$$\lim_{t\rightarrow\infty} \left\| \begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix} - \chi(x\sqrt{t}) \begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix} \right\|_E = 0.$$

Proof: As in the previous lemma, we will show only that

$$\lim_{t\rightarrow\infty} \|\mathrm{grad}(1-\chi(x\sqrt{t})) \int \{(\sum_k e^{i\lambda t}\Phi^{0,+}_{k\lambda} + \sum_{k,m} e^{-i\lambda t}\Phi^{0,-}_{m,\lambda} S_{mk}(\lambda))$$
$$\times \sum_l d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle\} d\lambda)\|_{L^2_b} = 0 \quad (3.6)$$

since the other terms are similar. As before, it is enough to consider a term corresponding to one k and one l.

The main thing to notice is that for $\lambda \in \mathrm{supp}\langle u_0, \Phi^+_{l,\lambda}\rangle$ and for $x \in \mathrm{supp}(1 - \chi(x\sqrt{t}))$

$$t + \frac{\lambda \log x}{\sqrt{\lambda^2 - \sigma_k^2}} \neq 0$$

for large t. This allows us to use an integration by parts argument.

For $x \in \mathrm{supp}(1 - \chi(x\sqrt{t}))$ and for t large,

$$|\int e^{i\lambda t}\Phi^{0,+}_{k\lambda} d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda|$$
$$= |\int (t + \tfrac{\lambda \log x}{\sqrt{\lambda^2-\sigma_k^2}})^{-1} D_\lambda(e^{i\lambda t}\Phi^{0,+}_{k\lambda}) d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda|$$
$$= |\int e^{i\lambda t}\Phi^{0,+}_{k\lambda} D_\lambda(t + \tfrac{\lambda \log x}{\sqrt{\lambda^2-\sigma_k^2}})^{-1} d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda| \quad (3.7)$$
$$= |\int e^{i\lambda t}\Phi^{0,+}_{k\lambda} (D_\lambda(t + \tfrac{\lambda \log x}{\sqrt{\lambda^2-\sigma_k^2}})^{-1})^2 d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda|$$
$$\leq Ct^{-1-\epsilon}(1 + |\log x|)^{-1+\epsilon}$$

A similar argument shows that

$$(1-\chi(x\sqrt{t}))\left|\Delta\int e^{i\lambda t}\Phi^{0,+}_{k\lambda}d_{k,l}(\lambda)\langle u_0,\Phi^{+}_{l,\lambda}\rangle d\lambda\right| \leq Ct^{-1-\epsilon}(1+|\log x|)^{-1+\epsilon}.$$

Since

$$|\mathrm{grad}(1-\chi(x\sqrt{t})| \leq C\sqrt{t}$$

and

$$|\Delta(1-\chi(x\sqrt{t})| \leq Ct,$$

the previous inequalities are enough to show (3.6).

Q.E.D.

We will need an energy decay estimate for a variety of reasons, including proving that the null space of M_+ is only the span of the L^2_b eigenfunctions. Since it uses some results of the previous lemmas, we include it here.

For $a > 0$, but small enough to for $x = a$ to make sense, define the energy of $\vec{u}$ in $x > a$, $E_{x>a}(\vec{u})$, to be

$$(E_{x>a}(\vec{u}))^2 = \int_{x>a} |\mathrm{grad}u_0|^2 + \int_{x>a} |u_1|^2.$$

Let $S_{pp} \subset S_E$ be the span of the L^2_b eigenfunctions.

Lemma 3.2.3 *Let $\vec{u} \in S_E \cap S_{pp}^{\perp}$. Then, for any $a > 0$,*

$$\lim_{t\to\infty} E_{x>\frac{a}{\sqrt{t}}}(U(t)\vec{u}) = 0.$$

Proof: Notice that we can reduce this to the case $\vec{u} \in S_1(X) \cap S_{pp}^{\perp}$ by the following: Given $\vec{v} \in S_E \cap S_{pp}^{\perp}$, we want to find, for any $\epsilon > 0$, a T such that for $t > T$, $E_{x>\frac{a}{\sqrt{t}}}(U(t)\vec{v}) < \epsilon$. By Lemma 3.1.1, we can find $\vec{u} \in S_1(X) \cap S_{pp}^{\perp}$ such that

$$\|\vec{u}-\vec{v}\|_E < \frac{\epsilon}{2}.$$

Then

$$\begin{aligned} E_{x>\frac{a}{\sqrt{t}}}(U(t)\vec{v}) &= E_{x>\frac{a}{\sqrt{t}}}(U(t)(\vec{v}-\vec{u}) + U(t)\vec{u}) \\ &= E_{x>\frac{a}{\sqrt{t}}}(U(t)(\vec{v}-\vec{u})) + E_{x>\frac{a}{\sqrt{t}}}(U(t)\vec{u}) \\ &\leq \tfrac{\epsilon}{2} + E_{x>\frac{a}{\sqrt{t}}}(U(t)\vec{u}) \end{aligned}$$

since $U(t)$ is unitary in the energy norm.

Let $\vec{u} \in S_1(X)$. As usual, we make use of the spectral measure, and will limit ourselves to showing

$$\lim_{t\to\infty} \int_{x>\frac{a}{\sqrt{t}}} |\text{grad} \int \cos(\lambda t) dE u_0|^2 = 0.$$

Remembering that we only need concern ourselves with the continuous part of the spectral measure, we have

$$2\int \cos(\lambda t) dE u_0 = \int [e^{i\lambda t} + e^{-i\lambda t}] \sum_{k,l} d_{k,l}(\lambda) \Phi^+_{k,\lambda} \langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda$$

We work with a single k and l.

Near the boundary,

$$\Phi^+_{k,\lambda} = \Phi^{0,+}_{k,\lambda} + \sum_{0\leq\sigma_m<\lambda} \Phi^{0,-}_{m,\lambda} S_{m,k} + \sum_{\sigma_n>\lambda} x^{\sqrt{\sigma_n^2-\lambda^2}} \tilde{S}_{n,k}\phi_n + f(x,y)$$

where $f(x,y) \in x^{\frac{1}{2}} H_b^\infty(X)$.

We have already shown that, for b small enough so that X has the product structure for $x < b$,

$$\int_{b>x>\frac{a}{t}} \left| \text{grad} \int [e^{i\lambda t} + e^{-i\lambda t}] \sum_{k,l} d_{k,l}(\lambda) [\Phi^{0,+}_{k,\lambda} + \sum_{0\leq\sigma_m<\lambda} \Phi^{0,-}_{m,\lambda} S_{m,k}] \langle u_0, \Phi^+_{l,\lambda}\rangle d\lambda \right|^2 \to 0$$

For $x < b$, for $\lambda \in \text{supp}\langle u_0, \Phi^+_{j,\lambda}\rangle$, there is an $\epsilon > 0$ such that

$$\Phi^+_{k,\lambda} - (\Phi^{0,+}_{k,\lambda} + \sum_{0\leq\sigma_m<\lambda} \Phi^{0,-}_{m,\lambda} S_{m,k}) = x^\epsilon g_\lambda(x,y),$$

where $g_\lambda(x,y) \in H_b^\infty(X)$ is smooth in λ on the support of $\langle u_0, \Phi^+_{k,\lambda}\rangle$. Certainly,

$$\begin{aligned} &\int_{b>x>\frac{a}{\sqrt{t}}} \left| \text{grad} \int \cos(\lambda t) x^\epsilon g_\lambda(x,y) d_{k,l}(\lambda) \langle u_0, \Phi^+_{j,\lambda}\rangle d\lambda \right|^2 \\ &\leq \int_{x<b} \left| \text{grad} \int \cos(\lambda t) x^\epsilon g_\lambda(x,y) d_{k,l}(\lambda) \langle u_0, \Phi^+_{j,\lambda}\rangle d\lambda \right|^2. \end{aligned}$$

Since $g_\lambda(x,y)d_{k,l}(\lambda)\langle u_0, \Phi^+_{j,\lambda}\rangle$ is compactly supported and L^2 in λ,

$$\left|\text{grad} \int \cos(\lambda t) g(x,y) d_{k,l}(\lambda)\langle u_0, \Phi^+_{j,\lambda}\rangle d\lambda\right|$$

is smooth in t and x, y, for $x < b$ and L^2 in t. Then

$$\lim_{t\to\infty} \int_{x<b} \left|\text{grad} x^\epsilon \int \cos(\lambda t) g(x,y) d_{k,l}(\lambda)\langle u_0, \Phi^+_{j,\lambda}\rangle d\lambda\right|^2 = 0.$$

Now consider

$$\int_{x>b} \left|\text{grad} \int \cos(\lambda t) d_{k,l}(\lambda)\Phi^+_{k,\lambda}\langle u_0, \Phi^+_{j,\lambda}\rangle d\lambda\right|^2.$$

Since $\Phi^+_{k,\lambda}$ is smooth on $x > b$, we can use the same argument to say

$$\lim_{t\to\infty} \int_{x>b} \left|\text{grad} \int \cos(\lambda t) d_{k,l}(\lambda)\Phi^+_{k,\lambda}\langle u_0, \bar{\Phi}^+_{j,\lambda}\rangle d\lambda\right|^2 = 0.$$

Q.E.D.

The preceding lemmas show

Proposition 3.2.1 *For $\vec{u} \in S_1(X)$,*

$$M'_+\vec{u} = \lim_{t\to\infty} U_0(-t)\chi(x\sqrt{t})U(t)\vec{u} \in S_E(\tilde{X})$$

exists and is given by

$$M'_+\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \int\{\sum_{k,l}(\Phi^{0,+}_{k,\lambda} + \sum_m \Phi^{0,-}_{m,\lambda}S_{mk})d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle\}d\lambda \\ +\int \frac{1}{\lambda}\{\sum_{k,l}(\Phi^{0,+}_{k,\lambda} - \sum_m \Phi^{0,-}_{m,\lambda}S_{m,k})d_{k,l}(\lambda)\langle u_1, \Phi^+_{l,\lambda}\rangle\}d\lambda \\ \\ \int \lambda\{\sum_{k,l}(\Phi^{0,+}_{k,\lambda} - \sum_m \Phi^{0,-}_{m,\lambda}S_{mk})d_{k,l}(\lambda)\langle u_0, \Phi^+_{l,\lambda}\rangle\}d\lambda \\ +\int\{\sum_{k,l}(\Phi^{0,+}_{k,\lambda} + \sum_m \Phi^{0,-}_{m,\lambda}S_{m,k})d_{k,l}(\lambda)\langle u_1, \Phi^+_{l,\lambda}\rangle\}d\lambda \end{pmatrix}$$

Moreover, $\|M'_+\vec{u}\|_E = \|(I - P_{pp})\vec{u}\|_E$, where $I - P_{pp}$ is projection off of S_{pp}.

Proof: First, we rewrite $U_0(-t)\chi(x\sqrt{t})U(t)$:

$$U_0(-t)\chi(x\sqrt{t})U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = U_0(-t)\begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix}$$
$$- U_0(-t)(1-\chi(x\sqrt{t}))\begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix}$$
$$- U_0(-t)\chi(x\sqrt{t})\left(\begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix} - U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right).$$

The first term is an element of $S_E(X)$ independent of t:

$$U_0(-t)\begin{pmatrix} u_{LT}(t) \\ D_t u_{LT}(t) \end{pmatrix} = \begin{pmatrix} u_{LT}(0) \\ D_t u_{LT}(0) \end{pmatrix}.$$

By Lemma 3.2.1, the limit as t goes to infinity of the second term is 0 in $H_E(X)$. The third term is $0 \in H_E(X)$ in the limit by Lemma 3.2.2. This means that as an element of $S_E(X)$, the last two terms have as limit $(c, 0)$, where c is a constant. However, a slight strengthening of (3.5) and (3.7), along with the corresponding estimates for the other terms involved, gives us that these two terms are actually both 0 (say, in $H_b^1(\tilde{X}) \times L_b^2(\tilde{X})$) in the limit.

The statement on the energy follows directly from Lemma 3.2.3.

Q.E.D.

Then, to define M_+ on $[\vec{v}] \in H_E(X)$, take a sequence of $\vec{u_j} \in S_1(X)$ converging in the energy norm to $\vec{v}$. Then define

$$M_+[\vec{v}] = \lim_{j\to\infty}[M_+'\vec{u_j}].$$

3.3 Some properties of M_+

We call $[\vec{u}] \in H_E(\tilde{X})$ "left-moving" if

$$\lim_{t\to\infty} E_{x>a} U_0(t)\vec{u} = 0$$

for any $a > 0$. "Right-moving" elements of $H_E(\tilde{X})$ can be defined similarly, changing only $x > a$ to $x < a$. Using methods similar to Lemma 3.2.1 one can show that $H_E(\tilde{X})$ can be separated into left-moving and right-moving pieces.

Lemma 3.3.1 *The range of M_+ is the left-moving elements of $H_E(\tilde{X})$.*

Proof: Clearly, from the definition of M_+ we have that the range of M_+ is contained in the left-moving elements of $H_E(\tilde{X})$. To prove the converse, we construct $\tilde{M}_+$, something like a left inverse to M_+, which is non-zero on the left-moving elements of $H_E(\tilde{X})$. Of course, since M_+ is not one-to-one, we cannot expect a true inverse; only one off of the kernel of M_+.

Consider $\vec{v} = (v_0, v_1) \in S_1(\tilde{X})$, where $\vec{v}$ is left-moving. Let $\chi(x)$ be a smooth function which is 1 near $x = 0$ and 0 away from a neighborhood of that boundary. We have

$$\begin{aligned} u_0(t) = &\frac{1}{\pi}\sum_{\sigma_k\geq 0}\int_{\lambda>\sigma_k}\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}(e^{i\lambda t}+e^{-i\lambda t})(\Phi^{0,+}_{k,\lambda}\langle v_0,\Phi^{0,+}_{k,\lambda}\rangle+\Phi^{0,-}_{k,\lambda}\langle v_0,\Phi^{0,-}_{k,\lambda}\rangle)d\lambda \\ &+\frac{1}{\pi}\sum_{\sigma_k\geq 0}\int_{\lambda>\sigma_k}\frac{1}{\sqrt{\lambda^2-\sigma_k^2}}(e^{i\lambda t}-e^{-i\lambda t})(\Phi^{0,+}_{k,\lambda}\langle v_1,\Phi^{0,+}_{k,\lambda}\rangle+\Phi^{0,-}_{k,\lambda}\langle v_1,\Phi^{0,-}_{k,\lambda}\rangle)d\lambda. \end{aligned}$$

By calculations similar to those for the general manifold X, we get that the long time behavior of $\chi(x\sqrt{t})u_0(t)$ is given by

$$\begin{aligned} &\frac{1}{\pi}\sum_{\sigma_k\geq 0}\int_{\lambda>\sigma_k}\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}(e^{i\lambda t}\Phi^{0,+}_{k,\lambda}\langle v_0,\Phi^{0,+}_{k,\lambda}\rangle+e^{-i\lambda t}\Phi^{0,-}_{k,\lambda}\langle v_0,\Phi^{0,-}_{k,\lambda}\rangle)d\lambda \\ &+\frac{1}{\pi}\sum_{\sigma_k\geq 0}\int_{\lambda>\sigma_k}\frac{1}{\sqrt{\lambda^2-\sigma_k^2}}(e^{i\lambda t}\Phi^{0,+}_{k,\lambda}\langle v_1,\Phi^{0,+}_{k,\lambda}\rangle-e^{-i\lambda t}\Phi^{0,-}_{k,\lambda}\langle v_1,\Phi^{0,-}_{k,\lambda}\rangle)d\lambda. \end{aligned}$$

In fact, if $\vec{v}$ is left-moving, this is the same as the long-time behavior of $u_0(t)$.

For t sufficiently large, we can consider $\chi(x\sqrt{t})u_0(t)$ to be defined on X, and it approaches a solution to the wave equation on X. Indeed, simply by replacing $\Phi^{0,+}_{k,\lambda}$ by $\Phi^{+}_{k,\lambda}$ and $\Phi^{0,-}_{k,\lambda}$ by $\Phi^{-}_{k,\lambda}$ in the above equation, we get a solution to the wave equation on X which has the same long-term behavior as $\chi(x\sqrt{t})u_0(t)$. Thus we have

$$\lim_{t\to\infty} U(-t)\chi(x\sqrt{t})U_0(t)\vec{v} = \begin{pmatrix} \frac{1}{\pi}\sum\int_{\lambda>\sigma_k}\{\Phi^{+}_{k,\lambda}(\lambda\langle v_0,\Phi^{0,+}_{k,\lambda}\rangle + \langle v_1,\Phi^{0,+}_{k,\lambda}\rangle) \\ +\Phi^{0,-}_{k,\lambda}(\lambda\langle v_0,\Phi^{0,-}_{k,\lambda}\rangle - \langle v_1,\Phi^{0,-}_{k,\lambda}\rangle)\}\frac{d\lambda}{\sqrt{\lambda^2-\sigma_k^2}} \\ \\ \frac{1}{\pi}\sum\int_{\lambda>\sigma_k}\{\Phi^{+}_{k,\lambda}(\lambda\langle v_0,\Phi^{0,+}_{k,\lambda}\rangle + \langle v_1,\Phi^{0,+}_{k,\lambda}\rangle) \\ -\Phi^{0,-}_{k,\lambda}(\lambda\langle v_0,\phi^{0,-}_{k,\lambda}\rangle - \langle v_1,\Phi^{0,-}_{k,\lambda}\rangle)\}\frac{\lambda d\lambda}{\sqrt{\lambda^2-\sigma_k^2}} \end{pmatrix} \tag{3.8}$$

Define this to be $\tilde{M}'_+\vec{v} \in S_E(X)$. Define $\tilde{M}_+$ on $H_E(\tilde{X})$ by continuity as for M_+. Note that $\tilde{M}_+M_+ = I - P_{pp}$ and that $M_+\tilde{M}_+ = P_L$, where P_L is projection onto the left-moving elements of $H_E(X)$.

Q.E.D.

A note about the Schwartz kernel of $\tilde{M}_+$: The Schwarz kernel of $\tilde{M}_+$ is defined up to a constant. Since we really want $\tilde{M}_+$ to be $\tilde{M}'_+$ on $S_1(\tilde{X})$, this choice will fix the constant. We need to determine the kernels of M'_+ and $\tilde{M}'_+$ so that for $\vec{u} \in \dot{C}^\infty(X)\times\dot{C}^\infty(X)$,

$$\tilde{M}'_+M'_+\vec{u} = (1-P_d)\vec{u}$$
$$M'_+U(t)\vec{u} = U_0(t)M'_+\vec{u}$$

not just in H_E but in S_E. This would fix the constant. Set $M'_+ = (m_{ij})$, $i,j = 1,2$. Then let $\tilde{M}' = (m^*_{ij})$ (up to the constant question, this follows from the fact that $\tilde{M}_+$ is M^*_+, with the adjoint taken with respect to the energy norm). Try giving M'_+ the kernel which appears, paired with (v_0, v_1), in (3.8) (that is, just "erase" the v_i). Then we can check that these kernels have the desired properties. (Note that (3.8) above gives us the constants $d_{kl}(\lambda)$ which we did not know before.) We use these kernels for the kernel of M_+ and $\tilde{M}_+$.

Lemma 3.3.2 *As defined above, M_+ has the following properties:*

1. *$Null(M_+) = H_{pp}(X) =$ span of the L^2_b eigenfunctions of $\Delta \subset H_E(X)$*

2. *M_+ is a partial isometry*

3. *$M_+U(t) = U_0(t)M_+$ for all $t \in R$*

Proof: (1) The proof of Lemma 3.2.1 shows that the L^2_b eigenfunctions are in the kernel of M_+. The energy estimate of Lemma 3.2.3 shows that nothing else can be in the kernel.

(2) We show that $\|M_+\vec{v}\|_E = \|\vec{v}\|_E$ for $\vec{v}$ orthogonal to $H_{pp}(X)$. This follows essentially immediately from the definition of $M_+\vec{v}$. Take a sequence $\vec{u_j} \in S_1(X)$ going to $\vec{v}$ in the energy norm, and orthogonal to $H_{pp}(X)$. Then

$$\|M'_+\vec{u_j}\|_E = \|\vec{u_j}\|_E$$

and

$$\|M_+\vec{v}\|_E = \lim_{j\to\infty} \|M'_+\vec{u_j}\|_E = \|\vec{v}\|_E.$$

(3) To show that $M_+U(t) = U_0(t)M_+$, we use the definition of M'_+ to write, for $\vec{u} \in S_1(X)$,

$$\begin{aligned}
M'_+U(t)\vec{u} &= \lim_{s\to\infty} U_0(-s)\chi(x\sqrt{s})U(s)U(t)\vec{u} \\
&= \lim_{s\to\infty} \Big(U_0(t)U_0(-t-s)\chi(x\sqrt{s+t})U(s+t) \\
&\quad + U_0(-s)(\chi(x\sqrt{s+t}) - \chi(x\sqrt{s}))U(s+t)\vec{u}\Big) \\
&= \lim_{s\to\infty} \Big[U_0(t)U_0(-s)\chi(x\sqrt{s})U(s)\vec{u} \\
&\qquad + U_0(-s)(\chi(x\sqrt{s+t}) - \chi(x\sqrt{s}))U(s+t)\vec{u}\Big] \\
&= U_0(t)M'_+\vec{u}.
\end{aligned}$$

The corresponding fact for M_+ follows by a limiting argument.

Q.E.D.

Chapter 4

Reduction of the b-Trace to Integrals at the Corners

Thus far we have shown that

$$U(t) = \tilde{M}_+ U_0(t) M_+ + P_d U(t)$$

where P_d is projection onto the L_b^2 eigenvalues. Pairing with $\rho \in C_c^\infty(\mathbf{R})$, we get

$$U(\rho) = \langle U(t), \rho(t) \rangle = \tilde{M}_+ U_0(\rho) M_+ + P_d U(\rho). \tag{4.1}$$

We know that $U(\rho) \in \Psi_b^{-\infty}(X)$, so we may take the b-trace of $U(\rho)$. In the first section of this chapter we show that we may take the b-Trace of each of the terms on the right hand side of (4.1). The second section calculates the contribution of the discrete spectrum. The remaining sections work with $b\text{-Tr}(\tilde{M}_+ U_0(\rho) M_+)$, putting it into a form that, in the final chapter, we use to calculate the contribution of the continuous spectrum.

4.1 Continuity of Summands

In practice, since we are concerned only with the diagonal entries of $U(\rho)$, we may take ρ to be even, since the diagonal entries of $U(t)$ are even in t. This makes both $U(\rho)$ and $U_0(\rho)$ diagonal matrices.

Lemma 4.1.1 *For $\rho \in \mathcal{S}(\mathbf{R})$, ρ even, and P_1, $P_2 \in \mathit{Diff}_b^m(X)$, the Schwartz kernel of $P_1\tilde{M}_+U_0(\rho)M_+P_2$ is continuous.*

We will need another lemma to prove this one.

Lemma 4.1.2 *For $\vec{u} \in H_E(X)$ such that $\Delta\vec{u} \in H_E(X)$,*

$$M_+\Delta\vec{u} = \Delta_{\tilde{X}} M_+\vec{u}.$$

Proof: We work on a dense subset of $H_E(X)$. Let $\vec{u} \in S_1(X)$, and note that

$$U(t)\Delta\vec{u} = \Delta U(t)\vec{u}$$

since they satisfy the same differential equation and initial conditions. Therefore,

$$\begin{aligned} M_+\Delta\vec{u} &= \lim_{t\to\infty} U_0(-t)\chi(x\sqrt{t})U(t)\Delta\vec{u} \\ &= \lim_{t\to\infty} U_0(-t)\chi(x\sqrt{t})\Delta U(t)\vec{u} \\ &= \lim_{t\to\infty}[\Delta_{\tilde{X}}U_0(-t)\chi(x\sqrt{t})U(t)\vec{u} + U_0(-t)[\chi(x\sqrt{t})\Delta - \Delta_{\tilde{X}}\chi(x\sqrt{t})]U(t)\vec{u}] \\ &= \Delta_{\tilde{X}}M_+\vec{u} + \lim_{t\to\infty} U_0(-t)[\chi(x\sqrt{t})\Delta - \Delta_{\tilde{X}}\chi(x\sqrt{t})]U(t)\vec{u}. \end{aligned}$$

This reduces the proof to showing that

$$\lim_{t\to\infty} \|(\chi(x\sqrt{t})\Delta - \Delta_{\tilde{X}}\chi(x\sqrt{t}))U(t)\vec{u}\|_E = 0.$$

We have

$$(\chi(x\sqrt{t})\Delta - \Delta_{\tilde{X}}\chi(x\sqrt{t}))U(t)\vec{u} = (\chi(x\sqrt{t})\Delta - \chi(x\sqrt{t})\Delta_{\tilde{X}})U(t)\vec{u} - [\Delta_{\tilde{X}}, \chi(x\sqrt{t})]U(t)\vec{u}$$

Since $\Delta = \Delta_{\tilde{X}} + xQ$, where Q is a b-differential operator of order at most two, the first term tends to zero as t goes to infinity and the support of $\chi(x\sqrt{t})$ shrinks toward the boundary. The support of $[\Delta_{\tilde{X}}, \chi(x\sqrt{t})]$ is contained in the region $a\sqrt{t} \leq x \leq b/\sqrt{t}$, a region in which the energy of $U(t)\vec{u}$ is tending to zero. For this reason, the second term tends to zero as t goes to infinity.

Q.E.D.

Proof of Lemma 4.1.1 : To show that the diagonal entries of $P_1\tilde{M}_+U_0(\rho)M_+P_2$ are continuous, consider pairing it with

$$(0, x\delta_{p_0}(p)) \otimes (0, x'\delta_{p_0'}(p')),$$

and show that this varies continuously with p_0 and $p_0' \in X$. This will show that the lower right entry is continuous. Since the upper left entry is actually the same, this is sufficient.

Choose an N such that $-\frac{n+2}{2} - m + 2N > 1$, and use Lemma 4.1.2 to write

$$\tilde{M}_+U_0(\rho)M^+ = (1+\Delta)^{-N}\tilde{M}_+(1+\Delta_{\tilde{X}})^{2N}U_0(\rho)M^+(1+\Delta)^{-N}.$$

We have

$$(1+\Delta)^{-N}P'x'\delta_{p_0'}(p') \in H_b^{-\frac{n+2}{2}-m+2N}(X),$$

so that $(1+\Delta)^{-N}P'(0, x'\delta_{p_0'}(p'))$ is in the finite energy space, and thus is in the domain of M_+. Applying M_+, we get

$$M_+(1+\Delta)^{-N}(0, \delta_{p_0'}(p')) \in H_E(\tilde{X})$$

varying continuously with p_0'. Applying $(1+\Delta_{\tilde{X}})^{2N}U_0(\rho)$, we get another element of $H_E(\tilde{X})$, which again depends continuously on p_0'. Then

$$\tilde{M}_+(1+\Delta_{\tilde{X}})^N U_0(\rho)M_+(1+\Delta)^{-N}\delta_{p_0'}(p') \in H_E(X),$$

and

$$(1+\Delta)^{-N}\tilde{M}_+(1+\Delta_{\tilde{X}})^N U_0(\rho)M_+(1+\Delta)^{-N}\delta_{p_0'}(p') \in H_E(X)$$

depends continuously on p_0' and has for its second entry a continuous function. Thus it may be paired (continuously) with $x\delta_{p_0}$.

Q.E.D.

Restrict ourselves to even ρ. Since

$$P_dU(\rho) = U(\rho) - \tilde{M}_+U_0(\rho)M_+, \tag{4.2}$$

and the right hand side is continuous for $\rho \in C_c^\infty(\mathbf{R})$, we have that $P_dU(\rho)$ is continuous for $\rho \in C_c^\infty(\mathbf{R})$. However, note that for $\hat{\rho} \in C_c^\infty(\mathbf{R})$,

$$P_dU(\rho) = \frac{1}{2}\begin{pmatrix} \sum_{\lambda_k^2 \in \mathrm{ppSpec}\,\Delta} \hat{\rho}(\lambda_k) f_k \overline{f_k} & 0 \\ 0 & \sum_{\lambda_k^2 \in \mathrm{ppSpec}\,\Delta} \hat{\rho}(\lambda_k) f_k \overline{f_k} \end{pmatrix},$$

where the f_k are the corresponding L_b^2 eigenfunctions, is in $\Psi_b^{-\infty}(X)$. Since $\tilde{M}_+U_0(\rho)M_+$ is continuous for all $\rho \in \mathcal{S}(\mathbf{R})$, we have that both summands in the right-hand side of (4.2) are continuous for $\hat{\rho}$ compactly supported. In the following calculations, it will often be convenient to assume that the Fourier transform of ρ, rather than ρ itself, is compactly supported.

4.2 Contribution of the Discrete Spectrum

Now that we know that $P_dU(\rho)$ is smooth, we can easily calculate its b-trace. Note that for $\hat{\rho} \in C_c^\infty(\mathbf{R})$, $\sum \hat{\rho}(\lambda_k) f_k \overline{f_k}$ is actually trace class, so we have $b\text{-Tr}(P_dU(\rho)) = \mathrm{Tr}(P_dU(\rho))$. We have

$$\begin{aligned} b\text{-Tr}(P_dU(\rho)) &= b\text{-Tr}\frac{1}{2}\begin{pmatrix} \sum_{\lambda_k^2 \in \mathrm{ppSpec}\,\Delta} \hat{\rho}(\lambda_k) f_k \overline{f_k} & 0 \\ 0 & \sum_{\lambda_k^2 \in \mathrm{ppSpec}\,\Delta} \hat{\rho}(\lambda_k) f_k \overline{f_k} \end{pmatrix} \\ &= \sum_{\lambda_k^2 \in \mathrm{ppSpec}\,\Delta} \hat{\rho}(\lambda_k) \int_X |f_k|^2 \\ &= \sum_{\lambda_k^2 \in \mathrm{ppSpec}\,\Delta} \hat{\rho}(\lambda_k) \end{aligned}$$

since $\|f_k\|_{L_b^2} = 1$.

4.3 Restatement of the Problem for $P_cU(\rho)$

The main trick we will use now is, as mentioned earlier, to rewrite $b\text{-Tr}P_cU(\rho)$ as the b-Trace of a commutator, and then use methods similar to those in **[Melrose]** to calculate the answer in terms of leading behavior at the boundary.

To make our operators nicer and simplify calculations, we will make some further restrictions on ρ. As previously noted, we may assume that ρ is even. Secondly, we will work with ρ such that $\hat{\rho} \in C_c^\infty(\mathbf{R})$. Since such functions are dense in $\mathcal{S}(\mathbf{R})$, this will be sufficient for our use. Finally, since this is a linear problem in ρ, it suffices to prove the identity for both ρ such that $\hat{\rho}(\lambda) \equiv 0$ in a neighborhood of $\lambda = 0$ and for ρ such that $\text{supp}(\hat{\rho}) \subset [-\frac{\sigma_J}{2}, \frac{\sigma_J}{2}]$, where σ_J is the first non-zero eigenvalue of $\Delta_{\partial X}$. Summarizing, we are working with ρ such that

1. ρ is even
2. $\hat{\rho} \in C_c^\infty(\mathbf{R})$
3. $\text{supp}(\hat{\rho}) \subset [-\frac{\sigma_J}{2}, \frac{\sigma_J}{2}]$, where σ_J is the first non-zero eigenvalue of $\Delta_{\partial X}$ OR $\hat{\rho}(\lambda)$ is 0 in a neighborhood of the origin.

The last condition seems to make a difference only in the proofs of Lemmas 4.4.1 and 5.2.4.

Assuming that ρ is even we get that

$$U_0(\rho) = \frac{1}{2\pi} \sum_{\sigma_k \geq 0} \phi_k \overline{\phi_k} \int_{\sigma_k}^{\infty} \hat{\rho}(\lambda)[(\frac{x}{x'})^{i\sqrt{\lambda^2-\sigma_k^2}} + (\frac{x}{x'})^{-i\sqrt{\lambda^2-\sigma_k^2}}]\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}} d\lambda \cdot Id \tag{4.3}$$

where Id is the identity for 2×2 matrices. We can use one more trick to make our operators nicer. Given an even ρ such that $\hat{\rho} \in C_c^\infty(\mathbf{R})$, pick an $\eta \in \mathcal{S}(\mathbf{R})$ such that η is even and $\hat{\eta} \equiv 1$ on the support of $\hat{\rho}$. Then, by examination of equation 4.3, we can see that $U_0(\rho * \eta) = U_0(\rho)$. We also have that $U_0(\rho * \eta) = U_0(\rho)U_0(\eta)$, since

$$\begin{aligned} U_0(\rho)U_0(\eta) &= \int U_0(t)\rho(t)dt \int U_0(s)\eta(s)ds \\ &= \int U_0(t-s)\rho(t-s)dt \int U_0(s)\eta(s)ds \\ &= \iint U_0(t)\rho(t-s)\eta(s)dsdt \\ &= U_0(\rho * \eta) \end{aligned}$$

Then we can rewrite

$$\tilde{M}_+ U_0(\rho) M_+ = (\tilde{M}_+ U_0(\eta))(U_0(\rho) M_+).$$

Then, using the fact that $(U_0(\rho)M_+)(\tilde{M}_+U_0(\eta))$ is $\mathbf{R}^+$-invariant, and thus its b-trace is 0 (see **[Melrose]**), we write

$$b\text{-}\mathrm{Tr}(P_c U(\rho)) = b\text{-}\mathrm{Tr}(\tilde{M}_+ U_0(\rho) M_+) = b\text{-}\mathrm{Tr}[\tilde{M}_+ U_0(\eta), U_0(\rho) M_+]. \qquad (4.4)$$

This is very similar to the trick we used to calculate the normalized trace of the wave group in the one-dimensional case. However, here our operators are a little more complicated, so we can't use the results of **[Melrose]** directly in this instance; we will, however, follow the general idea of his proof, putting in additional calculations where necessary.

We put in the extra $U_0(\eta)$ so that we have the commutator of two fairly well-behaved operators; this makes a number of proofs practically identical for both operators. In practice we require that η satisfy conditions 1 and 2 above, and also that it have the property that either $\mathrm{supp}(\hat{\eta}) \subset [-\frac{3\sigma_J}{4}, \frac{3\sigma_J}{4}]$ (if $\mathrm{supp}\hat{\rho} \subset [-\frac{\sigma_J}{2}, \frac{\sigma_J}{2}]$) or $\hat{\eta} = 0$ in a neighborhood of the origin (if $\hat{\rho}$ has the same property).

4.4 Reduction to Corners

This section shows that the only contribution to $b\text{-}\mathrm{Tr}[\tilde{M}_+ U_0(\eta), U_0(\rho) M_+]$ comes from the kernels of $\tilde{M}_+ U_0(\eta)$ and $U_0(\rho) M_+$ near the corners of $X \times \tilde{X}$ and $\tilde{X} \times X$ respectively. This result is fairly easy for the b-Trace of the commutator of two elements of

$\Psi_b^{-\infty}$, since their kernels decay rapidly at $lb \sqcup rb$. We will use instead Lemma 4.4.1 below, which says something about the decay of the kernels of $\tilde{M}_+ U_0(\eta)$ and $U_0(\rho)M_+$ at the boundaries away from the corners.

Here, as often, we use the same letters to refer both to operators and to their Schwartz kernels. Recall that M_+ is a 2×2 matrix of operators; we will call the entries m_{ij}, $i = 1,2$, $j = 1,2$ with the usual conventions, and then $\tilde{M}_+ = (m^*_{ij})$. Since $U_0(\rho) = A \cdot Id$, where Id is the 2×2 identity matrix and $A \in \Psi_b^{-\infty}$, we will abuse notation and write $U_0(\rho)$ for both $A \cdot Id$ and A.

Lemma 4.4.1 *For any $\tilde{\chi} \in C^\infty(X \times \tilde{X})$ with $\chi \equiv 1$ near the corners of $X \times \tilde{X}$,*

$$\int_X \int_{\tilde{X}} |(1 - \tilde{\chi})(m^*_{ij} U_0(\eta))(p, p')(U_0(\rho) m_{ji})(p', p)| < \infty$$

Proof: We prove this for ∂X connected, although the general case is only notationally more difficult.

Since the kernels of all the operators involved are continuous, we only need to worry about their behavior near the boundaries. Because of the cut-off function, we need only concern ourselves with the behavior of the kernels on a compact set away from the corners. We begin with the case $i = j$, the simpler of the two cases. First we show that in a compact set away from the corners, $U_0(\rho)m_{11}$ decays as $1/\log x$ near the boundary $x = 0$. A similar proof gives decay of $1/\log x'$ at the $x' = 0$ and $x' = \infty$ boundaries away from the corners. The same proof works for $m^*_{jj} U_0(\eta)$. Multiplying the two together with a factor of $(1 - \tilde{\chi})$, we will get something which decays as $c|\log x|^{-2}$ or $c|\log x'|^{-2}$ at the respective boundaries, and is thus integrable.

Using the formula for the kernel of m_{11} obtained in Chapter 3, we have

$$U_0(\rho)m_{11} = \frac{1}{\pi} \sum_{\sigma_k \in spec \Delta_{\partial X}} \int_{\sigma_k}^{\infty} \hat{\rho}(\lambda)(\Phi^{0,+}_{k,\lambda} \overline{\Phi^{+}_{k,\lambda}} + \Phi^{0,-}_{k,\lambda} \overline{\Phi^{-}_{k,\lambda}}) \frac{\lambda}{\sqrt{\lambda^2 - \sigma_k^2}} d\lambda.$$

For simplicity, we will prove the estimate only for

$$\sum_{\sigma_k \in spec \Delta_{\partial X}} \int_{\sigma_k}^{\infty} \hat{\rho}(\lambda) \Phi^{0,+}_{k,\lambda} \overline{\Phi^{+}_{k,\lambda}} \frac{\lambda}{\sqrt{\lambda^2 - \sigma_k^2}} d\lambda,$$

since the other part is analogous. Since $\hat{\rho}$ is compactly supported, it suffices to consider a term corresponding to one k. Expanding $\Phi^{+}_{k,\lambda}$ at the boundary $x = 0$, we get

$$\int_{\sigma_k}^{\infty} \Phi^{0,+}_{k,\lambda}\overline{\Phi^{+}_{k,\lambda}}\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda$$

$$= \int_{\sigma_k}^{\infty} \Phi^{0,+}_{k,\lambda}\left[\overline{\Phi^{0,+}_{k,\lambda}} + \sum_{\sigma_m \le \lambda} \overline{S_{mk}(\lambda)\Phi^{0,-}_{m,\lambda}} + \sum_{\sigma_m > \lambda} \overline{\tilde{S}_{mk}(\lambda)x^{\sqrt{\sigma_m^2-\lambda^2}}\phi_m} + O(x\log x)\right]\frac{\hat{\rho}(\lambda)\lambda d\lambda}{\sqrt{\lambda^2-\sigma_k^2}}. \tag{4.5}$$

Dealing with the terms one at a time, we have, for the first term,

$$\int_{\sigma_k}^{\infty} \hat{\rho}(\lambda)\Phi^{0,+}_{k,\lambda}\overline{\Phi^{0,+}_{k,\lambda}}\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda = \int_{\sigma_k}^{\infty} \hat{\rho}(\lambda)\Phi^{0,+}_{k,\lambda}\frac{1}{\log x}\overline{(D_\lambda x^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k)}d\lambda$$

$$= \frac{i}{\log x}\hat{\rho}(\sigma_k)\phi_k\overline{\phi_k} + \frac{1}{\log x}\int_{\sigma_k}^{\infty} D_\lambda(\hat{\rho}(\lambda)\Phi^{0,+}_{k,\lambda})\overline{\Phi^{+}_{k,\lambda}}d\lambda.$$

Since the last integral is bounded, the left hand side is bounded by a constant times $\frac{1}{\log x}$ near $x = 0$ and away from the corners.

The second term of 4.5 is slightly more difficult. Consider a term corresponding to just one m (there are finitely many relevant ones):

$$\int_{\max \sigma_k,\sigma_m}^{\infty} \hat{\rho}(\lambda)\Phi^{0,+}_{k,\lambda}\overline{S_{mk}(\lambda)\Phi^{0,-}_{m,\lambda}}\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda$$

$$= \int_{\max \sigma_k,\sigma_m}^{\infty} \hat{\rho}(\lambda)\Phi^{0,+}_{k,\lambda}\overline{S_{mk}(\lambda)\phi_m}\frac{1}{\log x}D_\lambda(x^{i\sqrt{\lambda^2-\sigma_k^2}})\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda$$

$$= -\frac{i}{\log x}\left(\Phi^{0,+}_{k,\lambda}\hat{\rho}(\lambda)\overline{S_{mk}(\lambda)\Phi^{0,-}_{m,\lambda}}\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)_{|\max \sigma_m,\sigma_k}$$

$$- \frac{1}{\log x}\int_{\max \sigma_k,\sigma_m}^{\infty} D_\lambda\left(\hat{\rho}(\lambda)\overline{S_{mk}(\lambda)}\Phi^{0,+}_{k,\lambda}\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)\overline{\Phi^{0,-}_{m,\lambda}}d\lambda.$$

Using the fact that $\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}S_{m,k}(\lambda)$ is continuous, we get that the second summand behaves like $\frac{1}{\log x}$ in the region in question.

The third term is much the same. Considering a term corresponding to one m, we get

$$\int_{\sigma_k}^{\sigma_m} \hat{\rho}(\lambda)\Phi_{k,\lambda}^{0,+}\overline{\tilde{S}_{mk}(\lambda)\phi_m} x^{\sqrt{\sigma_m^2-\lambda^2}} \frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}} d\lambda$$

$$= \int_{\sigma_k}^{\sqrt{\sigma_m^2-\delta^2}} \hat{\rho}(\lambda)\Phi_{k,\lambda}^{0,+}\overline{\tilde{S}_{mk}(\lambda)\phi_m} x^{\sqrt{\sigma_m^2-\lambda^2}} \frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}} d\lambda$$

$$+ \int_{\sqrt{\sigma_m^2-\delta^2}}^{\sigma_m} \hat{\rho}(\lambda)\Phi_{k,\lambda}^{0,+}\overline{\tilde{S}_{mk}(\lambda)\phi_m} \frac{1}{\log x}\frac{d}{d\lambda} x^{\sqrt{\sigma_m^2-\lambda^2}} \frac{\sqrt{\sigma_m^2-\lambda^2}}{\sqrt{\lambda^2-\sigma_k^2}} d\lambda$$

$$= O(x^\delta) + \frac{1}{\log x}\left(\hat{\rho}(\lambda)\Phi_{k,\lambda}^{0,+}\overline{\tilde{S}_{mk}(\lambda)\phi_m} x^{\sqrt{\sigma_m^2-\lambda^2}} \frac{\sqrt{\sigma_m^2-\lambda^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)\Big|_{\sqrt{\sigma_m^2-\delta^2}}^{\sigma_m}$$

$$- \frac{1}{\log x}\int_{\sqrt{\sigma_m^2-\delta^2}}^{\sigma_m} \frac{d}{d\lambda}(\hat{\rho}(\lambda)\Phi_{k,\lambda}^{0,+}\overline{\tilde{S}_{mk}(\lambda)} \frac{\sqrt{\sigma_m^2-\lambda^2}}{\sqrt{\lambda^2-\sigma_k^2}}) x^{\sqrt{\sigma_m^2-\lambda^2}}\overline{\phi_m} d\lambda$$

The fourth summand is easiest, since it is clearly bounded by $x^{\frac{1}{2}}$ near $x = 0$.

The harder part to prove is the case $i \neq j$. If $\hat{\rho}(\lambda)$, and thus $\hat{\eta}(\lambda)$, are 0 in a neighborhood of $\lambda = 0$, the above proof works essentially without modification. The case of $\hat{\rho}$ supported in $[-\frac{\sigma_1}{2}, \frac{\sigma_1}{2}]$ is slightly trickier, since the kernel of $U_0(\rho)m_{12}$ is given by

$$\frac{1}{\pi}\sum_{\sigma_k \in spec\Delta_{\partial X}} \int_{\sigma_k}^{\infty} \hat{\rho}(\lambda)(\Phi_{k,\lambda}^{0,+}\overline{\Phi_{k,\lambda}^{+}} - \Phi_{k,\lambda}^{0,-}\overline{\Phi_{k,\lambda}^{-}}) \frac{1}{\sqrt{\lambda^2-\sigma_k^2}} d\lambda. \tag{4.6}$$

The problem arises when $\sigma_k = 0$, which only happens when $\sigma_k = 0$, because then we are dividing by λ and integrating down to 0. Note that by choosing $\mathrm{supp}\hat{\rho} \subset [-\frac{\sigma_1}{2}, \frac{\sigma_1}{2}]$, we have eliminated the terms which behave like those in the previous proof and thus have simplified our calculations.

First we work on getting a bound on $U_0(\rho)m_{12}$ near the boundary, say $x' = 0$:

$$\int_0^{\infty} \hat{\rho}(\lambda)\frac{1}{\lambda}(\Phi_{k,\lambda}^{0,+}\overline{\Phi_{k,\lambda}^{+}} - \Phi_{k,\lambda}^{0,-}\overline{\Phi_{k,\lambda}^{-}}) d\lambda. \tag{4.7}$$

But, at $\lambda = 0$, $\Phi_{k,\lambda}^{0,+} = \text{const} = \Phi_{k,\lambda}^{0,-}$ and $\Phi_{k,\lambda}^{+} = \text{const} = \Phi_{k,\lambda}^{-}$. Therefore, the integrand in (4.7) is continuous. We can use a Taylor expansion of the integrand to see that near $x' = 0$, (4.7) will be bounded by a constant times $\log x'$.

Then consider the behavior, near the boundary $x' = 0$, of $m_{21}^* U_0(\eta)$, where $\hat{\eta}$ is supported in $[-\frac{3\sigma_I}{4}, \frac{3\sigma_I}{4}]$:

$$\begin{aligned}
\pi m_{21}^* U_0(\eta) &= \int_0^\infty \hat{\rho}(\lambda)\lambda(\Phi_{k,\lambda}^+ \overline{\Phi_{k,\lambda}^{0,+}} - \Phi_{k,\lambda}^- \overline{\Phi_{k,\lambda}^{0,-}})d\lambda \\
&= \frac{1}{2}\phi_k \int_0^\infty \hat{\rho}(\lambda)\lambda(\Phi_{k,\lambda}^+(x')^{-i\lambda} - \Phi_{k,\lambda}^-(x')^{i\lambda})d\lambda \\
&= \frac{1}{2} i\phi_k(\log x')^{-1}\hat{\rho}(0)[(\lambda\Phi_{k,\lambda}^+)_{|\lambda=0} + (\lambda\Phi_{k,\lambda}^-)_{|\lambda=0}] \\
&+ \frac{1}{2}(\log x')^{-1}\phi_k \int_0^\infty \{D_\lambda(\hat{\rho}(\lambda)\lambda\Phi_{k,\lambda}^+)(x')^{-i\lambda} + D_\lambda(\hat{\rho}(\lambda)\lambda\Phi_{k,\lambda}^-)(x')^{i\lambda}\}d\lambda \\
&= (\log x')^{-2}\frac{1}{2}\phi_k \int_0^\infty \{D_\lambda^2(\hat{\rho}(\lambda)\lambda\Phi_{k,\lambda+})(x')^{-i\lambda} - D_\lambda^2(\hat{\rho}(\lambda)\lambda\Phi_{k,\lambda}^-)(x')^{i\lambda}\}d\lambda
\end{aligned}$$

Properties of the Mellin (Fourier) transform in turn allows us to bound

$$\left|\int_0^\infty \{D_\lambda^2(\hat{\rho}(\lambda)\lambda\Phi_{k,\lambda+})(x')^{-i\lambda} - D_\lambda^2(\hat{\rho}(\lambda)\lambda\Phi_{k,\lambda}^-)(x')^{i\lambda}\}d\lambda\right| \leq |f(x')| \tag{4.8}$$

where $f(x') \in L_b^p$ for $p > 2$, for (x, x') in a compact set away from the corners. One can get similar decay at the boundary $x = 0$, and away from the corners. Therefore, the product of $m_{21}^* U_0(\eta)$, $U_0(\rho)m_{12}$, and $(1 - \chi)$ is integrable.

Q.E.D.

Lemma 4.4.2 *Let $\chi_0, \chi_\infty \in C_c^\infty(X \times \tilde{X})$, with $\chi_0 = 1$ in a neighborhood of the corner $\partial X \times \{0\} \times \partial X$ and 0 outside a product neighborhood of the corner, and χ_∞ satisfy the same conditions for $\partial X \times \{\infty\} \times \partial X$. Then*

$$\begin{aligned}
&b\text{-}Tr[\tilde{M}_+ U_0(\eta), U_0(\rho)M_+] \\
&= \lim_{\epsilon \downarrow 0} \\
&\Big(Tr\int_{\partial X}\int_{\partial X}\int_0^\infty\int_\epsilon^{\frac{1}{\epsilon}} \chi_0(x,y,xs,y')(\tilde{M}_+U_0(\eta))(x,y,xs,y')(U_0(\rho)M_+)(xs,y',x,y)\frac{dx}{x}\frac{ds}{s} \\
&- Tr\int_{\partial X}\int_{\partial X}\int_0^\infty\int_\epsilon^{\frac{1}{\epsilon}} \chi_\infty(x,y,\frac{1}{xs},y')(\tilde{M}_+U_0(\eta))(x,y,\frac{1}{xs},y')(U_0(\rho)M_+)(\frac{1}{xs},y',x,y)\frac{dx}{x}\frac{ds}{s} \\
&- \gamma\log\epsilon\Big)
\end{aligned}$$

Proof : Essentially by definition we have

$$\begin{aligned}
&b\text{-}\mathrm{Tr}[\tilde{M}_+U_0(\eta), U_0(\rho)M_+] \\
&= \lim_{\epsilon\downarrow 0}\Big(\mathrm{Tr}\int_{x>\epsilon}\int_{\tilde{X}}(\tilde{M}_+U_0(\eta))(p',p)(U_0(\rho)M_+)(p,p') \\
&\quad - \mathrm{Tr}\int_{\epsilon<x'<1/\epsilon}\int_X(\tilde{M}_+U_0(\eta))(p',p)(U_0(\rho)M_+)(p,p') - \gamma\log\epsilon\Big)
\end{aligned} \tag{4.9}$$

where γ is the constant which makes the limit exist. Then rewrite

$$\tilde{M}_+U_0(\eta)(p',p) = (\chi_0+\chi_\infty)\tilde{M}_+U_0(\eta)(p',p) + (1-\chi_0-\chi_\infty)\tilde{M}_+U_0(\eta)(p',p).$$

Substituting this into (4.9), we get two pairs of terms. In the limit as ϵ goes to 0, the pair with a $(1-\chi_0-\chi_\infty)$ adds to give 0, since the integrands are integrable over the whole of $X\times\tilde{X}$ by Lemma 4.4.1, and thus we can switch the order of integration. We are left with

$$\begin{aligned}
&b\text{-}\mathrm{Tr}[\tilde{M}_+U_0(\eta), U_0(\rho)M_+] \\
&= \lim_{\epsilon\downarrow 0}\Big(\mathrm{Tr}\iint_{\{x>\epsilon\}\times\tilde{X}}(\chi_0+\chi_\infty)(\tilde{M}_+U_0(\eta))(p,p')(U_0(\rho)M_+)(p',p) \\
&\quad - \mathrm{Tr}\iint_{\{1/\epsilon>x'>\epsilon\}\times X}(\chi_0+\chi_\infty)(\tilde{M}_+U_0(\eta))(p,p')(U_0(\rho)M_+)(p',p) \\
&\quad -\gamma\log\epsilon\Big).
\end{aligned} \tag{4.10}$$

We have managed to reduce the b-trace to integrals near the corners. In this region we can introduce local coordinates $(x,y)\in X$ and $(x',y')\in\tilde{X}=[0,\infty)\times\partial X$.

We can lift the kernels of $\tilde{M}_+U_0(\eta)$ and $U_0(\rho)M_+$ to $(X\times\tilde{X})_b$, which is $(X\times\tilde{X})$ with the corners blown up (in the same manner as we blew up the diagonal corners of X^2 to get X_b^2 in Chapter 2). Let's work just with the integral near the corner $\partial X\times\{0\}\times\partial X$; the calculation for the other corner is essentially the same, using $1/x'$ instead of x'. On the support of χ_0 on $(X\times\tilde{X})_b$ we can introduce coordinates

$$r = x+x', \tau = \frac{x-x'}{x+x'}, y, y'$$

with $y, y' \in \partial X$. For $\delta > 0$, let $\zeta_\delta \in C^\infty((X \times \partial X)_b)$ be such that $0 \le \zeta_\delta \le 1$ and near $r = 0$,

$$\zeta_\delta(r, \tau) = \begin{cases} 1 \text{ if } 1 - |\tau| > \delta/2 \\ 0 \text{ if } 1 - |\tau| < \delta/4 \end{cases}$$

Multiplying by ζ_δ has the effect of cutting off an amount proportional to δ near $lb \sqcup rb$ (that is, the boundary faces of $(X \times \tilde{X})_b$ which do not meet the lifted diagonal).

Let α and β denote the lifts, respectively, of $\tilde{M}_+ U_0(\eta)$ and $U_0(\rho) M_+$ to $(X \times \tilde{X})_b$. Rewriting, we have

$$\chi_0 \alpha\beta = \chi_0 \zeta_\delta \alpha\beta + \chi_0(1 - \zeta_\delta)\alpha\beta.$$

For the first term, we can use projective coordinates $s = x'/x$, x (and, of course, the usual tangential coordinates y and y'). Then

$$\int_{x>\epsilon} \int_{\tilde{X}} \chi_0 \zeta_\delta \alpha\beta - \int_{x'>\epsilon} \int_X \chi_0 \zeta_\delta \alpha\beta = \int_{\partial X} \int_{\partial X} \int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} \chi_0 \zeta_\delta \alpha\beta.$$

Then

$$\begin{aligned} &\mathrm{Tr}\Big(\int_{x>\epsilon} \int_{\tilde{X}} \chi_0(\tilde{M}_+ U_0(\eta))(p, p')(U_0(\rho) M_+)(p', p) \\ &- \int_{x'>\epsilon} \int X(\chi_0)(\tilde{M}_+ U_0(\eta))(p, p')(U_0(\rho) M_+)(p', p)\Big) \\ &= \mathrm{Tr} \int_{\partial X} \int_{\partial X} \int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} \chi_0 \zeta_\delta \alpha\beta + \mathrm{Tr}\left[\iint_{r(1+\tau)>2\epsilon} \chi_0(1 - \zeta_\delta)\alpha\beta - \iint_{r(1-\tau)>2\epsilon} \chi_0(1 - \zeta_\delta)\alpha\beta\right] \end{aligned}$$

The left hand side is independent of δ, and the second and third terms of the right hand side go to 0 as δ goes to 0 by an application of Lemma 4.4.1. The first summand goes to

$$\mathrm{Tr} \int_{\partial X} \int_{\partial X} \int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} \chi_0 \alpha\beta,$$

which, translated to the usual language, is, in the limit as $\epsilon \downarrow 0$

$$\mathrm{Tr} \int_{\partial X} \int_{\partial X} \int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} \chi_0(x, y, xs, y')(\tilde{M}_+ U_0(\eta))(x, y, xs, y')(U_0(\rho) M_+)(xs, y', x, y) \frac{dx}{x} \frac{ds}{s}.$$

Q.E.D.

4.5 Dependence Only on Leading Behavior

This section shows that the b-trace depends only on the leading behavior of $\Phi^{\pm}_{k,\lambda}$ at the boundary. Recall that near the boundary

$$2\Phi^{+}_{k\lambda} = x^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k + \sum_{0\leq\sigma_m\leq\lambda} S_{mk}(\lambda)x^{-i\sqrt{\lambda^2-\sigma_m^2}}\phi_m + \sum_{\sigma_n>\lambda} \tilde{S}_{nk}(\lambda)x^{\sqrt{\sigma_n^2-\lambda^2}}\phi_n + O(x\log x).$$

By the leading behavior, $\Phi^{+}_{k,\lambda,L}$ of $\Phi^{+}_{k,\lambda}$ we mean the first two summands in the expansion above, and we have an analogous definition for $\Phi^{-}_{k,\lambda,L}$. Since the kernel of M_+ is given in terms of the $\Phi^{\pm}_{k,\lambda}$, for example

$$m_{ii} = \frac{1}{\pi}\sum_{\sigma_k\geq 0}\int_{\sigma_k}^{\infty}[\Phi^{0,+}_{k,\lambda}\overline{\Phi^{0,+}_{k,\lambda}} + \Phi^{0,-}_{k,\lambda}\overline{\Phi^{0,-}_{k,\lambda}}]\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda$$

it makes sense to ask if the limit in (4.10) or in Lemma 4.4.2, which is the limit of integrals over the corners of $X \times \partial X$, depends on the lower order terms in the expansion of $\Phi^{\pm}_{k,\lambda}$. The answer is

Lemma 4.5.1 *The integrals in (4.10) and in Lemma 4.4.2 depend only on the leading behavior of $\Phi^{\pm}_{k\lambda}$ at the boundary.*

Proof: Again, we give the proof only in case ∂X is connected, for notational ease.

The reason is that only the leading expansion can give something which is not L^1 on $X \times \tilde{X}$. We prove this for the corner near $x' = 0$ and for one term in the expansion of $U_0(\rho)m_{ii}$, but it is true for the other terms as well.

By the argument of Lemma 4.4.1, we can bound

$$|m^{*}_{ii}U_0(\eta)| \leq c\left|\frac{\log x}{\log x'}\right|.$$

Then consider

$$\int_{\sigma_k}^{\infty}\Phi^{0,+}_{k,\lambda}\overline{\Phi^{+}_{k,\lambda}}\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda$$

$$= \int_{\sigma_k}^{\infty}(x')^{i\sqrt{\lambda^2-\sigma_k^2}}\phi_k[\overline{\Phi^{+}_{k,\lambda,L}} + \sum_{\sigma_m>\lambda}\overline{\tilde{S}_{mk}}(\lambda)x^{\sqrt{\sigma_m^2-\lambda^2}}\overline{\phi_m} + O(x\log x)]\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda.$$

The term with $O(x \log x)$ can be bounded by $|c(\log x')^{-1}(\log x)^2 x|$, which, multiplied by $\chi_0 m^*_{ii} U_0(\eta)$, clearly results in something integrable, and thus the limits in question are 0.

Consider, then, the other term. For any small $\delta > 0$,

$$\left| \int_{\sigma_k}^{\sigma_m - \delta} (x')^{i\sqrt{\lambda^2 - \sigma_k^2}} \phi_k \overline{\tilde{S}_{mk}}(\lambda) x^{\sqrt{\sigma_m^2 - \lambda^2}} \overline{\phi_m} \hat{\rho}(\lambda) \frac{\lambda}{\sqrt{\lambda^2 - \sigma_k^2}} d\lambda \right| \leq c \left| \frac{\log x}{\log x'} x^{\delta/4} \right| .$$

Multiplied by $\chi_0 m^*_{ii} U_0(\eta)$, this results in something integrable for any $\delta > 0$; and thus the limits in question are 0. Since this is true for all small $\delta > 0$, and since the point $\lambda = \sigma_m$ is included in the leading part of $\Phi^+_{k,\lambda}$, we are done.

Q.E.D.

Chapter 5

Final Calculations

Finally, we will calculate $b\text{-Tr}(\tilde{M}_+U_0(\rho)M_+)$ by calculating the integrals over the corners of $\tilde{M}_+U_0(\eta)U_0(\rho)M_+$, to which we reduced the problem in the previous chapter. The result of this chapter is

Theorem 5.0.1 *The part of the b-trace corresponding to the continuous spectrum is given by*

$$b\text{-}Tr(\tilde{M}_+U_0(\rho)M_+) = \frac{1}{i2\pi}\int \hat{\rho}(\lambda)\frac{d}{d\lambda}\log\det\Psi(\lambda)d\lambda + \frac{1}{4}\sum_{\sigma_k^2\in\operatorname{spec}\Delta_{\partial X}}\hat{\rho}(\sigma_k) + \frac{c}{4}\hat{\rho}(0)$$

where c is the number of connected components of ∂X.

Note that this and the results of Sections 4.1 and 4.2 are enough to show Theorem 0.2.1.

We use Lemma 4.4.2 or the intermediate result, equation 4.10, which reduce the left hand side of the above equation to a limit of integrals at the corners. Lemma 4.5.1 shows that the limit depends only on the leading behavior of $\tilde{M}_+U_0(\eta)$ and $U_0(\rho)M_+$ at the corners. A similar argument shows that on the support of χ_0 or χ_∞, we may use the density dv_0dv_0, where dv_0 is the product density (on $\tilde{X} = [0,\infty)\times\partial X$), since the difference between this and the density on $X\times\partial X$ is integrable.

We give the proof for ∂X connected, although the general case is only marginally more difficult (for notational reasons).

5.1 Notation

In an attempt to make calculations clearer, we fix some notation. Recall that M_+ is a 2×2 matrix with entries m_{ij}. The leading parts of $m_{11}^* U_0(\eta)$ and $U_0(\rho) m_{11}$ are given by

$$m_{11}^* U_0(\eta) \sim \tfrac{1}{4\pi} \textstyle\sum_{\sigma_k \geq 0} [\alpha_k(x,x')\phi_k(y)\overline{\phi_k(y')} + \sum_{\sigma_m \geq 0} \alpha_{mk}(x,x')\phi_m(y)\overline{\phi_k(y')}]$$
$$U_0(\rho) m_{11} \sim \tfrac{1}{4\pi} \textstyle\sum_{\sigma_l \geq 0} [\beta_l(x',x)\phi_l(y')\overline{\phi_l(y)} + \sum_{\sigma_n \geq 0} \beta_{nl}(x',x)\phi_k(y')\overline{\phi_n(y)}]$$

where

$$\alpha_k(x,x') = \int_{\sigma_k}^{\infty} [(\frac{x'}{x})^{-i\sqrt{\tau^2-\sigma_k^2}} + (\frac{x'}{x})^{i\sqrt{\tau^2-\sigma_k^2}}]\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau$$

$$\alpha_{mk}(x,x') = \int_{\max(\sigma_m,\sigma_k)}^{\infty} \Big[x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}}S_{mk}(\tau) + x^{i\sqrt{\tau^2-\sigma_m^2}}(x')^{i\sqrt{\tau^2-\sigma_k^2}}T_{mk}(\tau)\Big]\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau$$

$$\beta_l(x',x) = \int_{\sigma_l}^{\infty} [(\frac{x'}{x})^{i\sqrt{\lambda^2-\sigma_l^2}} + (\frac{x'}{x})^{-i\sqrt{\lambda^2-\sigma_l^2}}]\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_l^2}}d\lambda$$

$$\beta_{nl}(x,x') = \int_{\max(\sigma_n,\sigma_l)}^{\infty} \Big[x^{i\sqrt{\lambda^2-\sigma_n^2}}(x')^{i\sqrt{\lambda^2-\sigma_l^2}}\overline{S_{nl}(\lambda)} + x^{-i\sqrt{\lambda^2-\sigma_n^2}}(x')^{-i\sqrt{\lambda^2-\sigma_l^2}}\overline{T_{nl}(\lambda)}\Big]\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_l^2}}d\lambda.$$

Integrating the leading terms of $\chi m_{11}^* U_0(\eta)(x,y,x',y')U_0(\rho)m_{11}(x',y',x,y)$ over $y \in \partial X$, $y' \in \partial X$, gives

$$\frac{1}{16\pi^2}\gamma(x,x') = \frac{1}{16\pi^2}\sum_k [\alpha_k(x,x')\beta_k(x',x) + \alpha_k(x,x')\beta_{kk}(x',x) + \alpha_{kk}(x,x')\beta_k(x',x) + \sum_m \alpha_{mk}(x,x')\beta_{mk}(x',x)]. \quad (5.1)$$

We need to define the analogous leading terms for a pair of off-diagonal terms; ie, $m_{ij}^* U_0(\eta) U_0(\rho) m_{ji}$, where $i \neq j$. We will do only the case $i = 1$, $j = 2$, since the proofs for the other off-diagonal contribution is the same. The leading parts of $m_{12}^* U_0(\eta)$ and $U_0(\rho) m_{21}$ are given, respectively, by

$$m_{12}^* U_0(\eta) \sim \frac{1}{4\pi} \sum_{\sigma_k \geq 0} [\alpha'_k(x,x')\phi_k(y)\overline{\phi_k(y')} + \sum_{\sigma_m \geq 0} \alpha'_{mk}(x,x')\phi_m(y)\overline{\phi_k(y')}]$$

$$U_0(\rho) m_{21} \sim \frac{1}{4\pi} \sum_{\sigma_l \geq 0} [\beta'_l(x',x)\phi_l(y')\overline{\phi_l(y)} + \sum_{\sigma_n \geq 0} \beta'_{nl}(x',x)\phi_k(y')\overline{\phi_n(y)}]$$

where

$$\alpha'_k(x,x') = \int_{\sigma_k}^{\infty} [(\frac{x'}{x})^{-i\sqrt{\tau^2-\sigma_k^2}} - (\frac{x'}{x})^{i\sqrt{\tau^2-\sigma_k^2}}]\hat{\eta}(\tau)\frac{1}{\sqrt{\tau^2-\sigma_k^2}} d\tau$$

$$\alpha'_{mk}(x,x') = \int_{\max(\sigma_m,\sigma_k)}^{\infty} \Big[x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}} S_{mk}(\tau) - x^{i\sqrt{\tau^2-\sigma_m^2}}(x')^{i\sqrt{\tau^2-\sigma_k^2}} T_{mk}(\tau)\Big] \hat{\eta}(\tau)\frac{1}{\sqrt{\tau^2-\sigma_k^2}} d\tau$$

$$\beta'_l(x',x) = \int_{\sigma_l}^{\infty} [(\frac{x'}{x})^{i\sqrt{\lambda^2-\sigma_l^2}} - (\frac{x'}{x})^{-i\sqrt{\lambda^2-\sigma_l^2}}]\hat{\rho}(\lambda)\frac{\lambda^2}{\sqrt{\lambda^2-\sigma_l^2}} d\lambda$$

$$\beta'_{nl}(x,x') = \int_{\max(\sigma_n,\sigma_l)}^{\infty} \Big[x^{i\sqrt{\lambda^2-\sigma_n^2}}(x')^{i\sqrt{\lambda^2-\sigma_l^2}} \overline{S_{nl}(\lambda)} - x^{-i\sqrt{\lambda^2-\sigma_n^2}}(x')^{-i\sqrt{\lambda^2-\sigma_l^2}} \overline{T_{nl}(\lambda)}\Big] \hat{\rho}(\lambda)\frac{\lambda^2}{\sqrt{\lambda^2-\sigma_l^2}} d\lambda.$$

Integrating the leading terms of $m_{12}^* U_0(\eta)$ multiplied by those of $U_0(\rho) m_{21}$ over $y \in \partial X$, $y' \in \partial X$ results in

$$\frac{1}{16\pi^2}\gamma'(x,x') = \frac{1}{16\pi^2}\sum_k [\alpha'_k(x,x')\beta'_k(x',x) + \alpha'_k(x,x')\beta'_{kk}(x',x) + \alpha'_{kk}(x,x')\beta'_k(x',x) + \sum_m \alpha'_{mk}(x,x')\beta'_{mk}(x',x)] \qquad (5.2)$$

5.2 Contribution from the Corner $x = 0$, $x' = 0$

The goal of this section is to prove

Proposition 5.2.1 *The contribution of the integral near the corner $x = 0$, $x' = 0$ is*

$$\lim_{\epsilon \downarrow 0} Tr\left(\iint_{\{x>\epsilon\}\times\tilde{X}} \chi_0(\tilde{M}_+ U_0(\eta))(p,p')(U_0(\rho)M_+)(p',p) - \iint_{\{x'>\epsilon\}\times X} \chi_0(\tilde{M}_+ U_0(\eta))(p,p')(U_0(\rho)M_+)(p',p)\right) = \frac{1}{8}\sum_{\sigma_k^2 \in \operatorname{spec}\Delta_{\partial X}} \hat{\rho}(\sigma_k)\Psi_{kk}(\sigma_k) + \frac{1}{8}\sum_{\sigma_k=0} \hat{\rho}(0)\Psi_{kk}0$$

where χ_0 is a smooth function on $X \times \tilde{X}$, with its support contained in a product neighborhood of the corner $\partial X \times \{0\} \times \partial X$, and equal to 1 in a smaller neighborhood of the corner.

We prove this lemma in two subsections below. The first subsection shows that the contribution of the diagonal entries ($m_{ii}^* U_0(\eta)$ and $U_0(\rho) m_{ii}$) is 0. The second subsection calculates the contribution of the off-diagonal entries.

It will be helpful to make use of some symmetries involved in the problem. To do this, first choose χ_0 to depend only on $x + x'$, and have support in $x + x' < a$, where a is chosen small enough that $x < a$ is contained in a product neighborhood of the boundary.

Next, fix some notation. Let $R^+_{0,\epsilon}$ denote the region $\{x > \epsilon\} \cap \{x < a, x' < a\}$ and $R^-_{0,\epsilon}$ denote the region $\{x' > \epsilon\} \cap \{x < a, x' < a\}$. Let

$$r_0 : \{x < a, x' < a\} \to \{x < a, x' < a\} \tag{5.3}$$

be given by $r_0((x, x')) = (x', x)$. Note that r_0 maps $R^+_{0,\epsilon}$ diffeomorphically to $R^-_{0,\epsilon}$, $r_0 \circ r_0 = Id$, and $\chi_0 \circ r_0 = \chi_0$.

Consider any distributions, smooth enough that their product is defined, on $[0, a] \times [0, a]$, $\alpha = \alpha(x, x')$ and $\beta = \beta(x', x)$. With the notation fixed above,

$$\int_{x>\epsilon}\int \chi_0\alpha\beta - \int_{x'>\epsilon}\int \chi_0\alpha\beta = \int_{R^+_{0,\epsilon}} \chi_0\alpha\beta - \int_{R^-_{0,\epsilon}} \chi_0\alpha\beta = \int_{R^+_{0,\epsilon}} \chi_0[\alpha\beta - (\alpha \circ r_0)(\beta \circ r_0)]. \tag{5.4}$$

Thus, if $\alpha\beta$ is even under r_0, (5.4) is 0.

5.2.1 Contribution of the Diagonal Entries

Lemma 5.2.1 *The contribution of the diagonal entries at the corner $x = 0$, $x' = 0$ is 0; i.e.,*

$$\lim_{\epsilon\downarrow 0} Tr\left(\iint_{\{x>\epsilon\}\times \tilde{X}} \chi_0(m_{ii}^* U_0(\eta))(p,p')(U_0(\rho)m_{ii})(p',p) - \iint_{\{x'>\epsilon\}\times X} \chi_0(m_{ii}^* U_0(\eta))(p,p')(U_0(\rho)m_{ii})(p',p)\right) = 0.$$

Proof: As previously noted, we need only consider the leading parts of $m_{11}^* U_0(\eta)$ and $U_0(\rho)m_{11}$, which means showing that

$$\lim_{\epsilon\downarrow 0}\left(\int_{x>\epsilon}\int \chi_0\gamma(x,x')\frac{dx}{x}\frac{dx'}{x'} - \int_{x'>\epsilon}\int \chi_0\gamma(x,x')\frac{dx}{x}\frac{dx'}{x'}\right) = 0 \tag{5.5}$$

by (5.1).

First, we make use of the symmetries involved. A simple calculation shows that $\alpha_k \circ r_0 = \alpha_k$ and $\beta_k \circ r_0 = \beta_k$. In addition, for any m for which $\sigma_m = \sigma_k$ (in particular for $m = k$), $\alpha_{mk} \circ r_0 = \alpha_{mk}$ and $\beta_{mk} \circ r_0 = \beta_{mk}$. This, along with the definition of γ, immediately gives us that the left hand side of (5.5) is equal to

$$\lim_{\epsilon\downarrow 0}\left(\int_{R^+_{0,\epsilon}} \sum_{\substack{\sigma_k,\sigma_m\geq 0\\ \sigma_k\neq\sigma_m}} \alpha_{mk}(x,x')\beta_{mk}(x',x) - \int_{R^-_{0,\epsilon}} \sum_{\substack{\sigma_k,\sigma_m\geq 0\\ \sigma_k\neq\sigma_m}} \alpha_{mk}(x,x')\beta_{mk}(x',x)\right). \tag{5.6}$$

Finally, since $\chi_0\alpha_{mk}\beta_{mk}$ is in $L^1_b([0,a]\times[0,a])$ by the lemma below, (5.6) is 0.

Q.E.D.

Lemma 5.2.2 *For $\sigma_m \neq \sigma_k$, $\chi_0\alpha_{mk}\beta_{mk}$ is in $L^1_b([0,a]\times[0,a])$.*

Proof of Lemma:

We have

$$\alpha_{mk}(x,x') = \int_{\max(\sigma_m,\sigma_k)}^{\infty}\left[x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}}S_{mk}(\tau) + x^{i\sqrt{\tau^2-\sigma_m^2}}(x')^{i\sqrt{\tau^2-\sigma_k^2}}T_{mk}(\tau)\right]\hat{\eta}(\tau)\frac{\tau d\tau}{\sqrt{\tau^2-\sigma_k^2}}.$$

We will work only with the first summand in the integrand, since the other one can be treated independently in the same way.

$$\int_{\max(\sigma_m,\sigma_k)}^{\infty} x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}} S_{mk}(\tau)\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau$$
$$= \int_{\max(\sigma_m,\sigma_k)}^{\infty}[-\frac{\tau\log x}{\sqrt{\tau^2-\sigma_m^2}} - \frac{\tau\log x'}{\sqrt{\tau^2-\sigma_k^2}}]^{-1} D_\tau(x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}})S_{mk}(\tau)\frac{\hat{\eta}(\tau)\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau \tag{5.7}$$

If $\sigma_k > \sigma_m$, rewrite this as

$$\int_{\sigma_k}^{\infty}[-\frac{\sqrt{\tau^2-\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x - \log x']^{-1} D_\tau(x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}})S_{mk}(\tau)\hat{\eta}(\tau)d\tau$$
$$= \int_{\sigma_k}^{\infty}(x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}})D_\tau\left([\frac{\sqrt{\tau^2+\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x - \log x']^{-1}S_{mk}(\tau)\hat{\eta}(\tau)\right)d\tau \tag{5.8}$$

since $S_{mk}(\sigma_k) = 0$. Consider

$$D_\tau\left([-\frac{\sqrt{\tau^2-\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x - \log x']^{-1}S_{mk}(\tau)\hat{\eta}(\tau)\right).$$

Since $D_\tau S_{mk}$ is integrable, and behaves like $c(\sqrt{\tau^2-\sigma_k^2})^{-1}$ near $\tau = \sigma_k$, we can bound

$$\left|\frac{D_\tau(S_{mk}(\tau)\hat{\eta}(\tau))}{\frac{\sqrt{\tau^2-\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x + \log x'}\right|$$
$$= \left|\frac{D_\tau(S_{mk}(\tau)\hat{\eta}(\tau))}{(\sqrt{\tau^2-\sigma_k^2})^{\frac{1}{2}}(\frac{\sqrt{\tau^2-\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x + \log x')^{\frac{1}{2}}(\frac{\log x}{\sqrt{\tau^2-\sigma_m^2}} + \frac{\log x'}{\sqrt{\tau^2-\sigma_k^2}})^{\frac{1}{2}}}\right|$$
$$\leq (\log x)^{-\frac{1}{2}}(\log x')^{\frac{1}{2}}|g(\tau)|$$

where $g(\tau) \in L^{p'}(\mathbf{R})$ for any $p' < 4/3$. Similarly, for the term with the derivative falling on the denominator, we have

$$\left|\frac{\log x\,(S_{mk}(\tau)\hat{\eta}(\tau)\tau(\sigma_m^2 - \sigma_k^2))}{(\sqrt{\tau^2-\sigma_m^2})^{3/2}\sqrt{\tau^2-\sigma_k^2}(\frac{\sqrt{\tau^2-\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x + \log x')^2}\right|$$

$$\leq \left|\frac{\log x\,(c\hat{\eta}(\tau)\tau(\sigma_m^2 - \sigma_k^2))}{(\sqrt{\tau^2-\sigma_k^2})^{\frac{3}{2}}(\frac{\log x}{\sqrt{\tau^2-\sigma_m^2}} + \frac{\log x'}{\sqrt{\tau^2-\sigma_k^2}})^{\frac{1}{2}}(\frac{\sqrt{\tau^2-\sigma_k^2}}{\sqrt{\tau^2-\sigma_m^2}}\log x + \log x')^{\frac{1}{2}}(\frac{1}{\sqrt{\tau^2-\sigma_m^2}} + \frac{\log x'}{\log x\sqrt{\tau^2-\sigma_k^2}})}\right|$$

$$\leq \left|(\log x)^{-\frac{1}{2}}(\log x')^{\frac{1}{2}}g'(\tau)\right|.$$

with $g'(\tau) \in L^{p'}(\mathbf{R})$ for any $p' < 4/3$. Thus, using the mapping properties of the Fourier transform (or here, the inverse Mellin transform), we get

$$|\alpha_{mk}| \leq |(\log x)^{-\frac{1}{2}}(\log x')^{-\frac{1}{2}}f(x,x')|$$

with $f(x,x') \in L^p_b([0,a]_x)$ uniformly in x' and $f(x,x') \in L^p_b([0,a]_{x'})$ uniformly in x, for any $p > 4$.

If $\sigma_m > \sigma_k$, we merely rewrite (5.7) as

$$\int_{\sigma_m}^{\infty}\frac{\sqrt{\tau^2-\sigma_m^2}}{\sqrt{\tau^2-\sigma_k^2}}[\log x + \frac{\sqrt{\tau^2-\sigma_m^2}}{\sqrt{\tau^2-\sigma_k^2}}\log x']^{-1}D_\tau(x^{-i\sqrt{\tau^2-\sigma_m^2}}(x')^{-i\sqrt{\tau^2-\sigma_k^2}})S_{mk}(\tau)\hat{\eta}(\tau)d\tau$$

and proceed as before.

We can bound β_{mk} in exactly the same way. The product, then, is bounded by

$$|\alpha_{mk}\beta_{mk}| \leq |(\log x)^{-1}(\log x')^{-1}fg|,$$

where $f \in L^p_b([0,a)_{x'})$ and $g \in L^p_b([0,a)_x)$ for any $p > 4$. Therefore, $\alpha_{mk}\beta_{mk}$ is L^1_b on the region in question.

Q.E.D.

5.2.2 Contribution of the Off-diagonal Entries

The main result of this section is

Proposition 5.2.2 *For $i \neq j$, the contribution of $(m_{ij}^* U_0(\eta))(U_0(\rho) m_{ji})$ near the corner $x = 0$, $x' = 0$ is*

$$\begin{aligned} \lim_{\epsilon\downarrow 0} Tr\Big(&\textstyle\iint_{\{x>\epsilon\}\times \tilde{X}} \chi_0(m_{ij}^* U_0(\eta))(p,p')(U_0(\rho)m_{ji})(p',p) \\ &- \textstyle\iint_{\{x'>\epsilon\}\times X} \chi_0(m_{ij}^* U_0(\eta))(p,p')(U_0(\rho)m_{ji})(p',p)\Big) \\ &= \tfrac{1}{16}\textstyle\sum_{\sigma_k\geq 0}\hat{\rho}(\sigma_k)(S_{kk}(\sigma_k)+\overline{S_{kk}}(\sigma_k)). \end{aligned}$$

We limit ourselves to proving the case $i = 1, j = 2$. As before, only the leading terms at the boundary can give a non-zero contribution. Therefore, by (5.2), we need to calculate

$$\frac{1}{16\pi^2}\lim_{\epsilon\downarrow 0}\Big[\int_{R^+_{0,\epsilon}} \gamma'(x,x')\frac{dx}{x}\frac{dx'}{x'} - \int_{R^-_{0,\epsilon}} \gamma'(x,x')\frac{dx}{x}\frac{dx'}{x'}\Big]$$

where

$$\begin{aligned} \gamma'_k(x,x') = \sum_k [&\alpha'_k(x,x')\beta'_k(x',x) + \alpha'_k(x,x')\beta'_{kk}(x',x) + \\ &\alpha'_{kk}(x,x')\beta'_k(x',x) + \sum_m \alpha'_{mk}(x,x')\beta'_{mk}(x',x)]. \end{aligned}$$

We break the proof of the proposition down into three lemmas.

Lemma 5.2.3 *For all k, the limit*

$$\lim_{\epsilon\downarrow 0}\Big[\int_{R^+_{0,\epsilon}} \alpha'_k\beta'_k + \sum_m \alpha'_{mk}\beta'_{mk} - \int_{R^-_{0,\epsilon}} \alpha'_k\beta'_k + \sum_m \alpha'_{mk}\beta'_{mk}\Big] = 0.$$

Proof: As for the diagonal contribution, we can make use of some symmetries to get a few results without much work. Simple calculations show that

$$\alpha'_k \circ r_0 = -\alpha'_k$$

$$\beta'_k \circ r_0 = -\beta'_k$$

and if $\sigma_m = \sigma_k$, then

$$\alpha'_{mk} \circ r_0 = \alpha'_{mk}$$
$$\beta'_{mk} \circ r_0 = \beta'_{mk}.$$

This shows that the pairings $\alpha'_k\beta'_k$ and $\alpha'_{mk}\beta'_{mk}$, for $\sigma_m = \sigma_k$, contribute nothing to the b-Trace. For $\sigma_m \neq \sigma_k$, the argument of Lemma 5.2.2 shows that $\chi_0\alpha'_{mk}\beta'_{mk}$ is in L^1_b, and thus

$$\lim_{\epsilon \downarrow 0}\left(\int_{R^+_{0,\epsilon}} \chi_0\alpha'_{mk}\beta'_{mk} - \int_{R^-_{0,\epsilon}} \chi_0\alpha'_{mk}\beta'_{mk}\right) = 0. \tag{5.9}$$

Q.E.D.

It remains to calculate the contributions of α'_k multiplied by β'_{kk} and α'_{kk} multiplied by β'_k.

Lemma 5.2.4 *For the terms with* $\sigma_k = 0$.

$$\lim_{\epsilon\downarrow 0}[\int_{R^+_{0,\epsilon}} \alpha'_k\beta'_{kk} + \alpha'_{kk}\beta'_k) - \int_{R^-_{0,\epsilon}} (\alpha'_k\beta'_{kk} + \alpha'_{kk}\beta'_k)] = \pi^2\hat{\rho}(0)(S_{kk}(0) + \overline{S}_{kk}(0))$$

Proof: First we show that the product of $\alpha'_k\beta'_{kk}$ is in L^1_b near the blown up corner $x = 0, x' = 0$ and thus contributes nothing.

Here we return to the convention that either $\hat{\rho}$ is 0 in a neighborhood of the origin (and $\hat{\eta}$ is too) or that the support of $\hat{\rho}$ is contained in $[-\sigma_J/2, \sigma_J/2]$ (and then the support of $\hat{\eta}$ is contained in $[-3\sigma_J/4, 3\sigma_J/4]$) that we discussed in section 4.3. If $\hat{\rho}$ is 0 in a neighborhood of the origin, then it is pretty obvious that the product $\alpha'_k\beta'_{kk}$ is L^1_b in a neighborhood of the corner. If not, we have

$$|\alpha'_k(x, sx)| = |\int_k^{\infty}(s^{-i\tau} - s^{i\tau})\hat{\eta}(\tau)\frac{1}{\tau}d\tau|$$
$$\leq C$$

and

$$|\beta'_{kk}(x,x')| = |\int_0^\infty [(xx')^{i\lambda}\overline{S_{00}(\lambda)} - (xx')^{-i\lambda}\overline{T_{00}(\lambda)}]\hat{\rho}(\lambda)\lambda d\lambda|$$
$$\leq |(\log xx')^{-2} f|$$

where $f \in L^p_b$, in s say, for $p > 2$. This is enough to ensure the the product in question is in L^1_b.

Now turn to the pair α'_{kk} and β'_k. We have

$$\lim_{x,x'\downarrow 0} \alpha'_{kk}(x,x') = \lim_{x,x'\downarrow 0}(\int_0^\infty [(xx')^{-i\tau}S_{kk}(\tau) - (xx')^{i\tau}T_{kk}(\tau)]\frac{\hat{\eta}(\tau)}{\tau}d\tau$$
$$= \frac{i\pi}{2}(S_{kk}(0) + \overline{S_{kk}}(0))\hat{\eta}(0)$$

and it approaches this limit rapidly. Additionally,

$$\lim_{\epsilon\downarrow 0}\int_0^\infty \int_\epsilon^{\epsilon/s} \beta'_k(x,xs)\frac{dx}{x}\frac{ds}{s} = -\log s\int_0^\infty \int_0^\infty (s^{i\lambda} - s^{-i\lambda})\lambda\hat{\rho}(\lambda)d\lambda$$
$$= -i\int_0^\infty \int_{-\infty}^\infty s^{i\lambda}\frac{d}{d\lambda}(\lambda\hat{\rho}(\lambda))d\lambda\frac{ds}{s}$$
$$= -2\pi i\hat{\rho}(0).$$

Q.E.D.

Lemma 5.2.5 *If* $\sigma_k > 0$, *then*

$$\lim_{\epsilon\downarrow 0}\int_0^\infty \int_\epsilon^{\epsilon/s} \alpha'_k\beta'_{kk} + \alpha'_{kk}\beta'_k\frac{dx}{x}\frac{ds}{s} = \pi^2\hat{\rho}(\sigma_k)(S_{kk}(\sigma_k) + \overline{S_{kk}(\sigma_k)}).$$

Proof: We will calculate only the term with α'_k and β'_{kk}, since the other calculation is similar.

We know that

$$\beta'_{kk}(x,xs) = \int_{\sigma_k}^\infty [(x^2s)^{i\sqrt{\lambda^2-\sigma_k^2}}\overline{S_{kk}(\lambda)} - (x^2s)^{-i\sqrt{\lambda^2-\sigma_k^2}}\overline{T_{kk}(\lambda)}]\hat{\rho}(\lambda)\frac{\lambda^2}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda$$
$$= -i\sigma_k\hat{\rho}(\sigma_k)(\log(x^2s))^{-1}(\overline{S_{kk}(\sigma_k)} + S_{kk}(\sigma_k))$$
$$- (\log(x^2s))^{-1}\int_{\sigma_k}^\infty \Big((x^2s)^{i\sqrt{\lambda^2-\sigma_k^2}}D_\lambda[\lambda\overline{S_{kk}(\lambda)}\hat{\rho}(\lambda)]$$
$$+(x^2s)^{-i\sqrt{\lambda^2-\sigma_k^2}}D_\lambda[\lambda\overline{T_{kk}}(\lambda)\hat{\rho}(\lambda)]\Big)\,d\lambda.$$

$\alpha'(x, xs)$ is bounded, and

$$\alpha_k'(x, sx) = -\frac{2}{i\sigma_k}\hat{\eta}(\sigma_k)(\log s)^{-1} + \int_{\sigma_k}^{\infty} (\log s)^{-1}[s^{-i\sqrt{\tau^2-\sigma_k^2}} + s^{i\sqrt{\tau^2-\sigma_k^2}}]D_\tau \frac{\hat{\eta}(\tau)}{\tau} d\tau.$$

First, note that the second integrand in the expression for β_{kk}' is small enough that the product of it and α_k' is in L_b^1, and thus the limit in question is 0 for that summand. Thus, we need only consider, for the β_{kk}' term, the leading part, $d_1/\log(x^2 s)$, where $d_1 = -i\sigma_k\hat{\rho}(\sigma_k)(S_{kk}(\sigma_k) + \overline{S_{kk}(\sigma_{kk})})$. Now we show that the limit of the integral over a compact set in s away from $s = 0$ is 0. Here, $(\log(x^2 s))^{-1}$ behaves like $(\log(x^2))^{-1}$. Then, for $0 < c < d < \infty$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \left| \int_c^d \int_\epsilon^{\epsilon/s} \chi_0 \alpha_k'(x, xs) d_1 (\log(x^2 s))^{-1} \frac{dx}{x}\frac{ds}{s} \right| &\le \lim_{\epsilon\downarrow 0} C' \int_c^d \log|\log x|\Big|_\epsilon^{\epsilon/s} \frac{ds}{s} \\
&\le \lim_{\epsilon\downarrow 0} C' \int_c^d \log\left|1 - \frac{\log s}{\log \epsilon}\right| \frac{ds}{s} \\
&= 0.
\end{aligned}$$

The antisymmetry of $\alpha_k'\beta_{kk}'$ under r_0, combined with the results from above, means that we have reduced the problem to

$$\begin{aligned}
&2d_1 \lim_{\epsilon\downarrow 0} \int_0^c \int_\epsilon^{\epsilon/s} \chi_0 \alpha_k'(x, xs)(\log x^2 s)^{-1} \frac{dx}{x}\frac{ds}{s} \\
&= 2d_1 \lim_{\epsilon\downarrow 0} \left[\int_{\frac{\epsilon}{a-\epsilon}}^c \int_\epsilon^{\epsilon/s} \alpha_k'(x, xs)(\log x^2 s)^{-1} \frac{dx}{x}\frac{ds}{s} + \int_0^{\frac{\epsilon}{a-\epsilon}} \int_\epsilon^{\frac{a}{1+s}} \alpha_k'(x, xs)(\log x^2 s)^{-1} \frac{dx}{x}\frac{ds}{s} \right]
\end{aligned} \tag{5.10}$$

where the second line is obtained by approximating χ_0 by a function which is 1 when $x + xs < a < 1$ and 0 when $x + sx > a$; since the difference of χ_0 and this approximation has support in a compact set away from the corners, it disappears in the limit.

Then calculate the two terms separately. With $d_2 = -2\hat{\eta}(\sigma_k)/i\sigma_k$, we have

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_{\frac{\epsilon}{a-\epsilon}}^{c}\int_{\epsilon}^{\epsilon/s}\alpha_k(x,xs)(\log x^2 s)^{-1}\frac{dx}{x}\frac{ds}{s}\\
&=\frac{1}{2}\lim_{\epsilon\downarrow 0}\int_{\frac{\epsilon}{a-\epsilon}}^{c}\alpha_k(x,xs)\log\left|\frac{2\log\epsilon-\log s}{2\log\epsilon+\log s}\right|\frac{ds}{s}\\
&=\frac{d_2}{2}\lim_{\epsilon\downarrow 0}\int_{\frac{\epsilon}{a-\epsilon}}^{c}(\log s)^{-1}\log\left|\frac{2\log\epsilon-\log s}{2\log\epsilon+\log s}\right|\frac{ds}{s}
\end{aligned}$$

since the rest of α_k is in L^1_b, which makes its contribution go to 0. Setting

$$t=\frac{\log s}{\log\epsilon}.$$

we get, with $c(\epsilon)\to 0$ and $d(\epsilon)\to 1$.

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\int_{\frac{\epsilon}{a-\epsilon}}^{c}\int_{\epsilon}^{\epsilon/s}\alpha'_k(x,xs)(\log x^2 s)^{-1}\frac{dx}{x}\frac{ds}{s} &= -\frac{d_2}{2}\lim_{\epsilon\downarrow 0}\int_{c(\epsilon)}^{d(\epsilon)}\log\left(\frac{2-t}{2+t}\right)\frac{dt}{t}\\
&=-\frac{d_2}{2}\int_0^{1/2}\log\left(\frac{1-t'}{1+t'}\right)\frac{dt'}{t'}.
\end{aligned}$$

For the second term in (5.10), we get, similarly,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_0^{\frac{\epsilon}{a-\epsilon}}\int_{\epsilon}^{\frac{a}{1+s}}\alpha'_k(x,xs)(\log x^2 s)^{-1}\frac{dx}{x}\frac{ds}{s}\\
&=d_2\lim_{\epsilon\downarrow 0}\int_0^{\frac{\epsilon}{a-\epsilon}}\int_{\epsilon}^{\frac{a}{1+s}}(\log s)^{-1}(\log x^2 s)^{-1}\frac{dx}{x}\frac{ds}{s}\\
&=\frac{d_2}{2}\lim_{\epsilon\downarrow 0}\int_0^{\frac{\epsilon}{a-\epsilon}}(\log s)^{-1}\log\left|\frac{2\log a-2\log(1+s)+\log s}{2\log\epsilon+\log s}\right|\frac{ds}{s}
\end{aligned}$$

With the substitution $t=\log s/\log\epsilon$, and $f(\epsilon)\to 0$ this becomes

$$\begin{aligned}
-\frac{d_2}{2}\lim_{\epsilon\downarrow 0}\int_{1-f(\epsilon)}^{\infty}\log\left|\frac{\frac{2\log a-2\log(1+s(t))}{\log\epsilon}+t}{2+t}\right|\frac{dt}{t} &= -\frac{d_2}{2}\int_1^{\infty}\log\left(\frac{t}{2+t}\right)\frac{dt}{t}\\
&=\frac{d_2}{2}\int_0^2\log(1+\xi)\frac{d\xi}{\xi}
\end{aligned}$$

Putting it all together, we get that

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_0^\infty\int_\epsilon^{\epsilon/s}\alpha_k'(x,xs)\beta_{kk}'(x,sx)\frac{dx}{x}\frac{ds}{s}\\
&= -d_1d_2[\int_0^{1/2}\log\left(\frac{1-t'}{1+t'}\right)\frac{dt'}{t'}-\int_0^2\log(1+t)\frac{dt}{t}]\\
&= -d_1d_2[-\operatorname{Li}_2(\frac{1}{2})+\operatorname{Li}_2(-\frac{1}{2})+\operatorname{Li}_2(-2)]\\
&= 2\hat{\rho}(\sigma_k)(S_{kk}(\sigma_k)+\overline{S_{kk}(\sigma_k)})\frac{\pi^2}{4}
\end{aligned}$$

where Li_2 is the dilogarithm and the last equality is obtained by consulting **[Lewin]**.

Q.E.D.

5.3 Contribution from the Corner $x = 0,\ x' = \infty$

Finally we are going to calculate the last part of the b-Trace.

The main result of this section is

Proposition 5.3.1 *The contribution to the b-trace of the integral over the corner $x = 0,\ x' = \infty$ is given by*

$$\frac{1}{2\pi i}\sum_{\sigma_m,\sigma_k\geq 0}\int_{|\lambda|>\max(\sigma_k,\sigma_m)}\overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}\Psi_{mk}(\lambda)d\lambda+\frac{1}{8}\sum_{\sigma_k^2\in\operatorname{spec}\Delta_{\partial X}}\hat{\rho}(\sigma_k)\Psi_{kk}(\sigma_k)$$
$$+\frac{1}{8}\sum_{\sigma_k=0}\hat{\rho}(0)\Psi_{kk}(0).$$

It will be helpful if we fix a little notation, reminiscent of that for the other corner, before we actually start calculating. We choose χ_∞ to be a function only of $x+1/x'$, with support in the region $\{x<a,(1/x')<a\}$. In analogy with the previous section, we define $R^+_{\infty,\epsilon}$ to be the region $\{x>\epsilon\}\cap\{x<a,(1/x')<a\}$ and $R^-_{\infty,\epsilon}$ to be the region $\{(1/x')>\epsilon\}\cap\{x<a,(1/x')<a\}$. Let

$$r_\infty:\{x<a,(1/x')<a\}\to\{x<a,(1/x')<a\}$$

be given by $r_\infty((x,x')) = (1/x', 1/x)$. As before, r_∞ maps $R^+_{\infty,\epsilon}$ diffeomorphically to $R^-_{\infty,\epsilon}$, and $r_\infty \circ r_\infty = Id$, and $\chi_\infty \circ r_\infty = \chi_\infty$. We use the fact that if $\alpha\beta \circ r_\infty = \alpha\beta$, then $\int_{R^+_{\infty,\epsilon}} \alpha\beta - \int_{R^-_{\infty,\epsilon}} \alpha\beta = 0$.

As in the case for the calculation at the other corner, we break this down into two subsections: the first for the proof of the contribution of a pair of diagonal terms, and the second for the contribution of the off-diagonal terms.

5.3.1 Contribution of the Diagonal Entries

The main result of this section is

Proposition 5.3.2 *The contribution of the $m^*_{ii}U_0(\eta)$ and $U_0(\rho)m_{ii}$ at the corner $x = 0$, $x' = \infty$ is*

$$\frac{1}{8\pi i} \sum_{\sigma_m,\sigma_k \geq 0} \int_{|\lambda| > \max(\sigma_k,\sigma_m)} \overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}\Psi_{mk}(\lambda)d\lambda$$

This is rather harder to prove than Lemma 5.2.1, so the proof will be broken into a number of lemmas. We continue to use the α_k's, α_{mk}'s, etc. defined in the first section of this chapter. As before, the Proposition immediately reduces to showing that

$$\lim_{\epsilon \downarrow 0}\left[\int_{R^+_{0,\epsilon}} \chi_\infty \gamma(x,x')\frac{dx}{x}\frac{ds}{s} - \int_{R^-_{0,\epsilon}} \chi_\infty \gamma(x,x')\frac{dx}{x}\frac{ds}{s}\right]$$
$$= \frac{2\pi}{i} \sum_{\sigma_m,\sigma_k \geq 0} \int_{|\lambda| > \max(\sigma_k,\sigma_m)} \overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}\Psi_{mk}(\lambda)d\lambda$$

Lemma 5.3.1 *For all k,*

$$\lim_{\epsilon \downarrow 0}\left(\int_{x>\epsilon}\int \chi_\infty(\alpha_k\beta_k + \alpha_k\beta_{kk} + \alpha_{kk}\beta_k) - \int_{1/x'>\epsilon}\int \chi_\infty(\alpha_k\beta_k + \alpha_k\beta_{kk} + \alpha_{kk}\beta_k)\right) = 0.$$

Proof: We show that each summand in the integrands above is L^1_b, which is sufficient.

Consider first

$$
\begin{aligned}
|\alpha_k(x,x')| &= \left|\int_{\sigma_k}^{\infty}[(\tfrac{x'}{x})^{-i\sqrt{\tau^2-\sigma_k^2}} + (\tfrac{x'}{x})^{i\sqrt{\tau^2-\sigma_k^2}}]\hat{\eta}(\tau)\tfrac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau\right| \\
&= \left|\int_{\sigma_k}^{\infty}[\log x - \log x']^{-1}[D_\tau(\tfrac{x'}{x})^{-i\sqrt{\tau^2-\sigma_k^2}} - D_\tau(\tfrac{x'}{x})^{i\sqrt{\tau^2-\sigma_k^2}}]\hat{\eta}(\tau)d\tau\right| \\
&= \left|[\log x - \log x']^{-1}\int_{\sigma_k}^{\infty}[(\tfrac{x'}{x})^{-i\sqrt{\tau^2-\sigma_k^2}} - (\tfrac{x'}{x})^{i\sqrt{\tau^2-\sigma_k^2}}]D_\tau\hat{\eta}(\tau)d\tau\right| \\
&\leq |[\log x - \log x']^{-1}f(x,x')|
\end{aligned}
\tag{5.11}
$$

where $f(x,x')$ is L_b^p, for any $p \geq 2$ in x uniformly in x' on the support of χ_∞, or L_b^p in x' uniformly in x on the same region. The same bound can be made on β_k.

For the next piece we will use the projective coordinates $s = (xx')^{-1}$, x. Consider

$$
\begin{aligned}
&\alpha_{kk}(x,(sx)^{-1}) \\
&= \int_{\sigma_k}^{\infty}[s^{-i\sqrt{\tau^2-\sigma_k^2}}S_{kk}(\tau) + s^{i\sqrt{\tau^2-\sigma_m^2}}T_{kk}(\tau)]\hat{\eta}(\tau)\tfrac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau \\
&= i(\log s)^{-1}[S_{kk}(\sigma_k) - T_{kk}(\sigma_k)]\hat{\eta}(\sigma_k) \\
&\quad + (\log s)^{-1}\int_{\sigma_k}^{\infty}[s^{-i\sqrt{\tau^2-\sigma_k^2}}D_\tau(S_{kk}(\tau)\hat{\eta}(\tau)) - s^{i\sqrt{\tau^2-\sigma_m^2}}D_\tau(T_{kk}(\tau)\hat{\eta}(\tau))]d\tau.
\end{aligned}
\tag{5.12}
$$

We can bound β_{kk} in the same way. The inequalities in (5.11) and (5.12) and their analogues for the β's are enough to show that the products which appear in Lemma 5.3.1 are integrable on the support of χ_∞.

Q.E.D.

Lemma 5.3.2 *If $\sigma_m = \sigma_k$, then*

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_{R^+_{\infty,\epsilon}}\chi_\infty\alpha_{mk}\beta_{mk} - \int_{R^-_{\infty,\epsilon}}\chi_\infty\alpha_{mk}\beta_{mk} \\
&= -2\pi\int_{\sigma_k}^{\infty}\{S_{mk}(\lambda)D_\lambda[\overline{S_{mk}}(\lambda)\hat{\rho}(\lambda)] - T_{mk}(\lambda)D_\lambda[\overline{T_{mk}}(\lambda)\hat{\rho}(\lambda)]\}d\lambda
\end{aligned}
$$

Proof: Let

$$
\begin{aligned}
\alpha_{mk}^o &= \frac{1}{2}[\alpha_{mk} - \alpha_{mk}\circ r_\infty] \\
\alpha_{mk}^e &= \frac{1}{2}[\alpha_{mk} + \alpha_{mk}\circ r_\infty]
\end{aligned}
$$

be, respectively, the odd and even (under r_∞) parts of α_{mk}, and similarly for β_{mk}. Clearly, $\alpha_{mk} = \alpha^e_{mk} + \alpha^o_{mk}$. We need to calculate the contribution of $\alpha_{mk}\beta_{mk}$; only the pairing of an even and an odd can contribute something nonzero.

Consider first the pairing α^e_{mk} and β^o_{mk}. We will use the more refined version of the integral at the corners, Lemma 4.4.2. We write the kernels in terms of the projective coordinates $s = 1/xx'$, x:

$$\begin{aligned}\alpha^e_{mk}(x, 1/xs) &= \tfrac{1}{2}\int_{\sigma_k}^{\infty}(s^{i\sqrt{\tau^2-\sigma_k^2}} + s^{-i\sqrt{\tau^2-\sigma_k^2}})(S_{mk}(\tau) + T_{mk}(\tau))\hat{\eta}(\tau)\tfrac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} s^{-it}(S_{mk}(\sqrt{t^2+\sigma_k^2}) + T_{mk}(\sqrt{t^2+\sigma_k^2}))\hat{\eta}(\sqrt{t^2+\sigma_k^2})dt\end{aligned}$$

and

$$\beta^o_{mk} = \frac{1}{2}\int_{\sigma_k}^{\infty}(s^{-i\sqrt{\lambda^2-\sigma_k^2}} - s^{i\sqrt{\lambda^2-\sigma_k^2}})(\overline{S_{mk}}(\lambda) - \overline{T_{mk}}(\lambda))\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda.$$

We want to know

$$\lim_{\epsilon\downarrow 0}\int_0^\infty\int_\epsilon^{\frac{\epsilon}{s}} \alpha^e_{mk}\beta^o_{mk}\frac{dx}{x}\frac{ds}{s}. \tag{5.13}$$

Since $\alpha^e_{mk}(x, 1/xs)\beta^o_{mk}$ is independent of x, pointwise

$$\int_\epsilon^{\frac{\epsilon}{s}} \chi_\infty\alpha^e_{mk}\beta^o_{mk}\frac{dx}{x} \to -(\log s)\alpha^e_{mk}\beta^o_{mk}.$$

This is integrable, since $\beta^o_{mk} \sim c/\log s$ and $\alpha^e_{mk} \sim 1/(\log s)f$, where $f \in L^p_b$ for $p > 2$. Additionally,

$$|\int_\epsilon^{\frac{\epsilon}{s}} \chi_\infty\alpha^e_{mk}\beta^o_{mk}\frac{dx}{x}| \leq |(\log s + c)(\alpha^e_{mk}\beta^o_{mk})| \tag{5.14}$$

for small ϵ. Therefore, by the dominated convergence theorem, (5.13) is given by

$$\begin{aligned}&-\int_0^\infty \log s\,\alpha^e_{mk}\beta^o_{mk}\tfrac{ds}{s} \\ &= \tfrac{1}{2}\int_0^\infty \alpha^e_{mk}\int_{\sigma_k}^\infty D_\lambda(s^{-i\sqrt{\lambda^2-\sigma_k^2}} + s^{i\sqrt{\lambda^2-\sigma_k^2}})(\overline{S_{mk}}(\lambda) - \overline{T_{mk}}(\lambda))\hat{\rho}(\lambda)d\lambda\tfrac{ds}{s} \\ &= -\tfrac{1}{2}\int_0^\infty \alpha^e_{mk}\int_{\sigma_k}^\infty(s^{-i\sqrt{\lambda^2-\sigma_k^2}} + s^{i\sqrt{\lambda^2-\sigma_k^2}})D_\lambda[(\overline{S_{mk}}(\lambda) - \overline{T_{mk}}(\lambda))\hat{\rho}(\lambda)]d\lambda\tfrac{ds}{s}.\end{aligned}$$

Substituting in the value of α^{ϵ}_{mk} we get

$$
\begin{aligned}
&-\int_0^\infty \log s\alpha^e_{mk}\beta^o_{mk}\tfrac{ds}{s}\\
&= -\tfrac{1}{4}\int_0^\infty \int_{-\infty}^\infty s^{-it}[S_{mk}(\tau(t)) + T_{mk}(\tau(t))]\hat{\eta}(\tau(t))\\
&\qquad \times \int_{\sigma_k}^\infty (s^{-i\sqrt{\lambda^2-\sigma_k^2}} + s^{i\sqrt{\lambda^2-\sigma_k^2}})D_\lambda[(\overline{S_{mk}}(\lambda) - \overline{T_{mk}}(\lambda))\hat{\rho}(\lambda)]d\lambda dt\tfrac{ds}{s}\\
&= -\tfrac{\pi}{2}\int_{-\infty}^\infty \int_{\sigma_k}^\infty (\delta(t+\sqrt{\lambda^2-\sigma_k^2}) + \delta(t-\sqrt{\lambda^2-\sigma_k^2}))[S_{mk}(\tau(t)) + T_{mk}(\tau(t))]\hat{\eta}(\tau(t))\\
&\qquad \times D_\lambda[(\overline{S_{mk}}(\lambda) - \overline{T_{mk}}(\lambda))\hat{\rho}(\lambda)]d\lambda dt\\
&= -\pi\int_{\sigma_k}^\infty [S_{mk}(\lambda) + T_{mk}(\lambda)]\hat{\eta}(\lambda)D_\lambda[(\overline{S_{mk}}(\lambda) - \overline{T_{mk}}(\lambda))\hat{\rho}(\lambda)]d\lambda.
\end{aligned}
$$

For the term with $\alpha^o_{mk}\beta^e_{mk}$ we get

$$
\begin{aligned}
&\pi\int_{\sigma_k}^\infty \hat{\rho}(\lambda)(\overline{S_{mk}}(\lambda) + \overline{T_{mk}}(\lambda))D_\lambda[S_{mk}(\lambda) - T_{mk}(\lambda)]d\lambda\\
&= -\pi\int_{\sigma_k}^\infty [S_{mk}(\lambda) - T_{mk}(\lambda)]D_\lambda[\hat{\rho}(\lambda)(\overline{S_{mk}}(\lambda) + \overline{T_{mk}}(\lambda))]d\lambda
\end{aligned}
$$

Summing, the entire contribution of $\alpha_{mk}\beta_{mk}$, for $\sigma_m = \sigma_k$, is given by

$$
-2\pi\int_{\sigma_k}^\infty \{S_{mk}(\lambda)D_\lambda[\overline{S_{mk}}(\lambda)\hat{\rho}(\lambda)] - T_{mk}(\lambda)D_\lambda[\overline{T_{mk}}(\lambda)\hat{\rho}(\lambda)]\}d\lambda
$$

Q.E.D.

Finally, we need to calculate the messiest contribution: that of α_{mk}, β_{mk} when $\sigma_m \neq \sigma_k$.

Lemma 5.3.3 *As* $\epsilon \to 0$,

$$
\begin{aligned}
&\int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} \chi_\infty\alpha_{mk}(x, 1/sx)\beta_{mk}(x, 1/xs)\frac{dx}{x}\frac{ds}{s}\\
&+ 2\pi\log\epsilon\int_{\max(\sigma_m,\sigma_k)}^\infty [\overline{S_{mk}(\lambda)}S_{mk}(\lambda) + \overline{T_{mk}(\lambda)}T_{mk}(\lambda)]\hat{\rho}(\lambda)\lambda\frac{\sqrt{\lambda^2-\sigma_k^2} - \sqrt{\lambda^2-\sigma_m^2}}{\lambda^2-\sigma_k^2}d\lambda\\
&\to \frac{2\pi}{i}\int_{\sigma_k}^\infty \overline{S_{mk}(\lambda)}\hat{\rho}(\lambda)\left(\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)^{\frac{1}{2}}\frac{d}{d\lambda}\left(S_{mk}(\lambda)\left(\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)^{\frac{1}{2}}\right)d\lambda\\
&- \frac{2\pi}{i}\int_{\sigma_k}^\infty \overline{T_{mk}(\lambda)}\hat{\rho}(\lambda)\left(\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)^{\frac{1}{2}}\frac{d}{d\lambda}\left(T_{mk}(\lambda)\left(\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)^{\frac{1}{2}}\right)d\lambda
\end{aligned}
$$

Proof: Recall that

$$\alpha_{mk}(x,1/xs) = \int_{\max(\sigma_m,\sigma_k)}^{\infty} \Big[x^{-i\sqrt{\tau^2-\sigma_m^2}+i\sqrt{\tau^2-\sigma_k^2}}s^{i\sqrt{\tau^2-\sigma_k^2}}S_{mk}(\tau) \\ + x^{i\sqrt{\tau^2-\sigma_m^2}-i\sqrt{\tau^2-\sigma_k^2}}s^{-i\sqrt{\tau^2-\sigma_k^2}}T_{mk}(\tau)\Big]\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau$$

and

$$\beta_{mk}(x,1/xs) = \int_{\max(\sigma_m,\sigma_k)}^{\infty} \Big[x^{i\sqrt{\lambda^2-\sigma_m^2}-i\sqrt{\lambda^2-\sigma_k^2}}s^{-i\sqrt{\lambda^2-\sigma_k^2}}\overline{S_{mk}(\lambda)} \\ + x^{-i\sqrt{\lambda^2-\sigma_m^2}+i\sqrt{\lambda^2-\sigma_k^2}}s^{i\sqrt{\lambda^2-\sigma_k^2}}\overline{T_{mk}(\lambda)}\Big]\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda.$$

We will work first with the product arising from the first summand in each integrand; that is, A_{mk} and B_{mk}, where

$$A_{mk}(x,1/xs) = \int_{\max(\sigma_m,\sigma_k)}^{\infty} x^{-i\sqrt{\tau^2-\sigma_m^2}+i\sqrt{\tau^2-\sigma_k^2}}s^{i\sqrt{\tau^2-\sigma_k^2}}S_{mk}(\tau)\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\tau$$

$$B_{mk}(x,1/xs) = \int_{\max(\sigma_m,\sigma_k)}^{\infty} x^{i\sqrt{\lambda^2-\sigma_m^2}-i\sqrt{\lambda^2-\sigma_k^2}}s^{-i\sqrt{\lambda^2-\sigma_k^2}}\overline{S_{mk}(\lambda)}\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\lambda.$$

Since

$$\int_{\epsilon}^{\frac{\epsilon}{s}} \chi_\infty A_{mk}B_{mk} = \int_{\epsilon}^{\frac{\epsilon}{s}} A_{mk}B_{mk} - \int_{\epsilon}^{\frac{\epsilon}{s}}(1-\chi_\infty)A_{mk}B_{mk}, \tag{5.15}$$

and Lemma 5.3.4 below shows that the integral over s of the second term goes to 0 as ϵ goes to 0, we need only concern ourselves with the first term on the right hand side of (5.15).

For the sake of sanity-preservation let's assume that $\sigma_k > \sigma_m$. It makes almost no difference in the calculations. We have

$$\int_{\epsilon}^{\frac{\epsilon}{s}} A_{mk}(x,1/xs)B_{mk}(x,1/xs)\frac{dx}{x} \\ = \int_{\sigma_k}^{\infty}\int_{\sigma_k}^{\infty} \epsilon^{D(\tau,\lambda)}\frac{s^{i\sqrt{\tau^2-\sigma_m^2}-\sqrt{\lambda^2-\sigma_m^2}} - s^{i\sqrt{\tau^2-\sigma_k^2}-\sqrt{\lambda^2-\sigma_k^2}}}{D(\tau,\lambda)} \\ \times S_{mk}(\tau)\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}\overline{S_{mk}(\lambda)}\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}d\tau d\lambda \tag{5.16}$$

where

$$D(\tau,\lambda) = i(\sqrt{\tau^2-\sigma_k^2} - \sqrt{\lambda^2-\sigma_k^2} - \sqrt{\tau^2-\sigma_m^2} + \sqrt{\lambda^2-\sigma_m^2}).$$

Note that $D(\tau,\lambda)$ is 0 if and only if $\tau = \lambda$, but despite this, the integrand is continuous. A change of coordinates to $r = \sqrt{\lambda^2-\sigma_k^2}$ and $\xi = \sqrt{\tau^2-\sigma_m^2} - \sqrt{\lambda^2-\sigma_m^2}$ for the first summand and $\xi = \sqrt{\tau^2-\sigma_k^2} - \sqrt{\lambda^2-\sigma_k^2}$ for the second summand gives us that (5.16) is equal to

$$\int_{-\infty}^{\infty}\int_0^{\infty} s^{i\xi}\left(\frac{N_1(\xi,r)}{D_1(\xi,r)}x - \frac{N_2(\xi,r)}{D_2(\xi,r)}\right) dr d\xi \tag{5.17}$$

where

$$\lambda(r) = \sqrt{r^2+\sigma_k^2}$$

and

$$\begin{aligned}
\tau_1(\xi,r) &= \sqrt{(\xi+\sqrt{r^2+\sigma_k^2-\sigma_m^2})^2+\sigma_m^2}\\
\tau_2(\xi,r) &= \sqrt{(\xi+r)^2+\sigma_k^2}\\
D_1(\xi,r) &= i\left(\frac{\sqrt{\tau_2^2(\xi,r)-\sigma_k^2}-r-\xi}{\xi+\sqrt{r^2+\sigma_k^2-\sigma_m^2}}\right)\\
D_2(\xi,r) &= i(\xi-\sqrt{\tau_1^2(\xi,r)-\sigma_m^2}+\sqrt{r^2+\sigma_k^2-\sigma_m^2})\\
N_1(\xi,r) &= H(\xi+\sqrt{r^2+\sigma_k^2-\sigma_m^2}-\sqrt{\sigma_k^2-\sigma_m^2})S_{mk}(\tau_1(\xi,r))\overline{S_{mk}}(\lambda(r))\hat{\rho}(\lambda(r))\\
&\quad\times\hat{\eta}(\tau_1(\xi,R))[\tau_1^2(\xi,r)-\sigma_k^2]^{-\frac{1}{2}}e^{(\xi+\sqrt{r^2+\sigma_k^2-\sigma_m^2})D_1(\xi,r)}\\
N_2(\xi,r) &= H(\xi+r)S_{mk}(\tau_2(\xi,r))\overline{S_{mk}}(\lambda(r))\hat{\rho}(\lambda(r))\hat{\eta}(\tau_2(\xi,r))e^{D_2(\xi,r)}.
\end{aligned}$$

The change of coordinates is justified since the integrand remains continuous.

If (5.17) is L_b^1 in s, then we can use the Fourier inversion formula to say that the integral of (5.17) over s is equal to

$$2\pi\int_0^{\infty}\left(\frac{N_1(\xi,r)}{D_1(\xi,r)} - \frac{N_2(\xi,r)}{D_2(\xi,r)}\right)_{|\xi=0} dr. \tag{5.18}$$

A brief sketch of how to show that (5.17) is in L_b^1 in s follows.

If

$$\int_0^\infty \left(\frac{N_1(\xi,r)}{D_1(\xi,r)} - \frac{N_2(\xi,r)}{D_2(\xi,r)}\right) dr \tag{5.19}$$

is in L^1 and its first ξ derivative is in L^p for some $p > 1$, then (5.17) is L^1_b in s. Clearly the problem is near $\xi = 0$. Let $D_1 = \xi d_1$ and $D_2 = \xi d_2$. Then $d_i \neq 0$, and we may rewrite (5.19) as

$$\frac{1}{\xi}\int_0^\infty \left(\frac{N_1(\xi,r)}{d_1(\xi,r)} - \frac{N_2(\xi,r)}{d_2(\xi,r)}\right) dr = \int_0^1 \frac{d}{dz}\int_0^\infty \left(\frac{N_1(z,r)}{d_1(z,r)} - \frac{N_2(z,r)}{d_2(z,r)}\right)_{|z=t\xi} drdt \tag{5.20}$$

which is clearly continuous and compactly supported in ξ, and thus L^1. Differentiating the right hand side of (5.20) with respect to ξ gives us something which is L^p in ξ, for $p < 2$. This is because of some nice properties of S_{mk}. First, near $\lambda = \sigma_k$, $S_{mk}(\lambda) \sim c\sqrt{\lambda^2 - \sigma_k^2}$. Secondly, although $S_{mk}(\lambda)$ may not be smooth, it is continuous, and near an element σ_l of the point spectrum of the Laplacian on the boundary, it is smooth in $\sqrt{\lambda^2 - \sigma_l^2}$, so a z derivative falling on $S_{mk}(\tau_l(z,r))$ gives something continuous when integrated over r, and then the ξ derivative of the integral does not behave worse than $1/\sqrt{\xi^2 - \sigma_l^2}$.

Now we return to (5.18). Evaluating the integrand at $\xi = 0$ and returning to the coordinate λ, we have

$$\begin{aligned}
&\int_0^\infty \int_\epsilon^{\frac{\xi}{s}} A_{mk}B_{mk}\frac{dx}{x}\frac{ds}{s} \\
&= \frac{2\pi}{i}\int_{\sigma_k}^\infty \overline{S_{mk}(\lambda)}\hat{\rho}(\lambda)\left(\frac{d}{d\lambda}(S_{mk}(\lambda))\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}} + \frac{1}{2}S_{mk}(\lambda)\frac{d}{d\lambda}\left(\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)\right)d\lambda \\
&- 2\pi\log\epsilon\int_{\sigma_k}^\infty \overline{S_{mk}(\lambda)}\hat{\rho}(\lambda)S_{mk}(\lambda)\lambda\frac{\sqrt{\lambda^2-\sigma_k^2}-\sqrt{\lambda^2-\sigma_m^2}}{\lambda^2-\sigma_k^2}d\lambda
\end{aligned}$$

Similarly, for $(\alpha_{mk} - A_{mk})$ multiplied by $\beta_{mk} - B_{mk}$, we get a contribution of

$$\int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} \chi_\infty(\alpha_{mk} - A_{mk})(\beta_{mk} - B_{mk})$$
$$= -\frac{2\pi}{i}\int_{\sigma_k}^\infty \overline{T_{mk}(\lambda)}\hat{\rho}(\lambda)\left(\frac{d}{d\lambda}(T_{mk}(\lambda))\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}} + \frac{1}{2}T_{mk}(\lambda)\frac{d}{d\lambda}\left(\frac{\sqrt{\lambda^2-\sigma_m^2}}{\sqrt{\lambda^2-\sigma_k^2}}\right)\right)d\lambda$$
$$-2\pi\log\epsilon\int_{\sigma_k}^\infty \overline{T_{mk}(\lambda)}\hat{\rho}(\lambda)T_{mk}(\lambda)\lambda\frac{\sqrt{\lambda^2-\sigma_k^2}-\sqrt{\lambda^2-\sigma_m^2}}{\lambda^2-\sigma_k^2}d\lambda + f_\epsilon$$

where f_ϵ goes to 0 as ϵ goes to 0 by Lemma 5.3.4 below.

The other two products arising from this division of α_{mk} and β_{mk} contribute nothing in the limit. Consider the pairing of $(\alpha_{mk} - A_{mk})$ and B_{mk}. By Lemma 5.3.4 we need only consider

$$\int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} (\alpha_{mk} - A_{mk})B_{mk}\frac{dx}{x}\frac{ds}{s}$$
$$= \int_0^\infty \int_{\sigma_k}^\infty \int_{\sigma_k}^\infty \left(\frac{s^{-i\sqrt{\tau^2-\sigma_m^2}-i\sqrt{\lambda^2-\sigma_m^2}} - s^{-i\sqrt{\tau^2-\sigma_k^2}-i\sqrt{\lambda^2-\sigma_k^2}}}{D(\lambda,\tau)}\right)$$
$$\times T_{mk}(\tau)\overline{S_{mk}}(\lambda)\hat{\eta}(\tau)\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\lambda d\tau\frac{ds}{s}$$

where

$$D(\lambda,\tau) = i\sqrt{\tau^2-\sigma_m^2} + i\sqrt{\lambda^2-\sigma_m^2} + i\sqrt{\tau^2-\sigma_k^2} + i\sqrt{\lambda^2-\sigma_k^2}$$

and we have assumed that $\sigma_k > \sigma_m$, although it makes no difference except in the limits of integration. Notice that the denominator is non-zero, and we may write, for example,

$$\int_0^\infty \int_{\sigma_k}^\infty \int_{\sigma_k}^\infty \left(\frac{s^{-i\sqrt{\tau^2-\sigma_k^2}-i\sqrt{\lambda^2-\sigma_k^2}}}{D(\lambda,\tau)}\right) T_{mk}(\tau)\overline{S_{mk}}(\lambda)\frac{\hat{\rho}(\lambda)\lambda}{\sqrt{\lambda^2-\sigma_k^2}}\frac{\hat{\eta}(\tau)\tau}{\sqrt{\tau^2-\sigma_k^2}}d\lambda d\tau\frac{ds}{s}$$
$$= \int_0^\infty \int_{\sigma_k}^\infty \int_{\sigma_k}^\infty \left(\frac{sD_s s^{-i\sqrt{\tau^2-\sigma_k^2}-i\sqrt{\lambda^2-\sigma_k^2}}}{D(\lambda,\tau)(-\sqrt{\tau^2-\sigma_k^2}-\sqrt{\lambda^2-\sigma_k^2})}\right)$$
$$\times T_{mk}(\tau)\overline{S_{mk}}(\lambda)\hat{\eta}(\tau)\hat{\rho}(\lambda)\frac{\lambda}{\sqrt{\lambda^2-\sigma_k^2}}\frac{\tau}{\sqrt{\tau^2-\sigma_k^2}}d\lambda d\tau\frac{ds}{s}$$
$$= 0$$

since $S_{mk}(\lambda) \sim c\sqrt{\lambda^2 - \sigma_k^2}$ near $\lambda = \sigma_k$.

Q.E.D.

Lemma 5.3.4 *For $\sigma_m \neq \sigma_k$,*

$$\lim_{\epsilon \downarrow 0} \int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} (1 - \chi_\infty)\alpha_{mk}(x, 1/xs)\beta_{mk}(x, 1/xs)\frac{dx}{x}\frac{ds}{s} = 0$$

Proof: Suppose that $\chi_\infty(x + sx) \equiv 1$ when $x + sx < b$. Then we can estimate

$$\begin{aligned} &\int_0^\infty \int_\epsilon^{\frac{\epsilon}{s}} |(1 - \chi_\infty)\alpha_{mk}(x, 1/xs)\beta_{mk}(x, 1/xs)|\frac{dx}{x}\frac{ds}{s} \\ &\leq \int_0^{\epsilon/b} \int_{b/2}^{\epsilon/s} |\alpha_{mk}(x, 1/xs)\beta_{mk}(x, 1/xs)|\frac{dx}{x}\frac{ds}{s} \\ &\quad + \int_{b/\epsilon}^\infty \int_\epsilon^{b/2s} |\alpha_{mk}(x, 1/xs)\beta_{mk}(x, 1/xs)|\frac{dx}{x}\frac{ds}{s}. \end{aligned} \tag{5.21}$$

Now we would like to estimate the size of $\alpha_{mk}(x, 1/xs)$. Consider, for example, the first summand in the expression for α_{mk}:

$$\begin{aligned} &\int_{\max(\sigma_k,\sigma_m)}^\infty x^{-i\sqrt{\tau^2-\sigma_m^2}}(xs)^{i\sqrt{\tau^2-\sigma_k^2}} S_{mk}(\tau)\hat{\eta}(\tau)\frac{\tau}{\sqrt{\tau^2 - \sigma_k^2}}d\tau \\ &= \int_{\max(\sigma_k,\sigma_m)}^\infty x^{-i\sqrt{\tau^2-\sigma_m^2}}(xs)^{i\sqrt{\tau^2-\sigma_k^2}} D_\tau \left[\left(\frac{-\sqrt{\tau^2 - \sigma_k^2}}{\sqrt{\tau^2 - \sigma_m^2}}\log x + \log(xs)\right)^{-1} S_{mk}(\tau)\hat{\eta}(\tau)\right] d\tau. \end{aligned}$$

Consider the region where $s < 1$. Then, in the region of interest (that which appears on the right hand side of (5.21))

$$\log(xs) \leq \log \epsilon$$

so that

$$\frac{-\sqrt{\tau^2 - \sigma_k^2}}{\sqrt{\tau^2 - \sigma_m^2}}\log x + \log(xs) \neq 0.$$

Then, in this region we can use the same methods as in the proof of Lemma 5.2.2 to bound the right hand side of (5.22) by

$$|(\log x)^{-\frac{1}{2}}(\log xs)^{-\frac{1}{2}}f(s)|$$

where $f \in L^p_b$ for $p > 2$. We can do the same for the region with $s > 1$, $b/2s < x < \epsilon$, and for the other summand in the expression for α_{mk}, and for β_{mk}. We have, then, that

$$\int_0^\infty \int_\epsilon^{\epsilon/s} (1-\chi_\infty)\alpha_{mk}(x, 1/xs)\beta_{mk}(x, 1/xs)$$
$$\leq \int_0^{\epsilon/b} \int_{b/2}^{\epsilon/s} |(\log(xs)\log x)^{-1} fg| \frac{dx}{x}\frac{ds}{s} + \int_{b/2\epsilon}^{\epsilon/s} |(\log(xs)\log x)^{-1} fg| \frac{dx}{x}\frac{ds}{s}$$

with f and g in L^p_b in s, for $p > 2$. Consider just the integral for $s < 1$, which is indicative of what happens.

$$\int_0^{\epsilon/b} \int_{b/2}^{\epsilon/s} |(\log(xs)\log x)^{-1} fg| \frac{dx}{x}\frac{ds}{s}$$
$$= \int_0^{\epsilon/b} \left|(\log s)^{-1} fg\,(\log|\log x| - \log|\log xs)|_{b/2}^{\epsilon/s}\right| \frac{dx}{x}\frac{ds}{s}$$
$$= \int_0^{\epsilon/b} \left|(\log s)^{-1} fg\right| \left|\log|\log \epsilon/s| - \log|\log b/2| + \log\left|\frac{\log b/2 + \log s}{\log \epsilon}\right|\right| \frac{dx}{x}\frac{ds}{s}.$$

The integrand on the right side is L^1_b, independent of ϵ, and so

$$\lim_{\epsilon \downarrow 0} \int_0^1 \int_\epsilon^{\epsilon/s} (1-\chi_\infty)\alpha_{mk}(x, 1/xs)\beta_{mk}(x, 1/xs) \to 0. \tag{5.22}$$

Since we can say the same for the integral over $s > 1$, we are finished.

Q.E.D.

Proof of Proposition 5.3.2 Since by Lemma 5.3.1 the only non-zero contributions come from a pairing of α_{mk} and β_{mk}, we need only sum over all indices m and k to get the contribution of the diagonal terms near the corner $x = 0$, $x' = \infty$. Note that since

$$T_{mk}(\lambda) = \frac{\sqrt{\lambda^2 - \sigma_k^2}}{\sqrt{\lambda^2 - \sigma_m^2}} \overline{S_{km}}(\lambda), \tag{5.23}$$

summing over m and k gets rid of the $\log \epsilon$ terms which appear in Lemma 5.3.3. This and the fact that

$$\Psi_{mk}(\lambda) = \begin{cases} \left(\frac{\sqrt{\lambda^2-\sigma_k^2}}{\sqrt{\lambda^2-\sigma_m^2}}\right)^{1/2} S_{mk}(\lambda) \text{ if } \lambda > \max(|\sigma_k|, |\sigma_m|) \\ \left(\frac{\sqrt{\lambda^2-\sigma_k^2}}{\sqrt{\lambda^2-\sigma_m^2}}\right)^{1/2} T_{mk}(\lambda) \text{ if } \lambda < -\max(|\sigma_k|, |\sigma_m|) \end{cases}$$

give the desired result.

Q.E.D.

5.3.2 Contribution of the Off-diagonal Entries

Proposition 5.3.3 *The sum of the contributions of the integrals of the off-diagonal entries at the corner $x = 0, x' = \infty$ is*

$$\frac{1}{4\pi i}\sum_{m,k}\int_{|\lambda|>\max(\sigma_k,\sigma_m)}\overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}\Psi_{mk}(\lambda)d\lambda+\frac{1}{8}\sum\hat{\rho}(\sigma_k)\Psi_{kk}(\sigma_k)+\frac{1}{8}\sum_{\sigma_k=0}\hat{\rho}(0)\Psi_{kk}(0).$$

Fortunately, by now we have done most of the work for this one. Below we outline a proof for the contribution of $m_{12}^{*}U_0(\eta)$ multiplied by $U_0(\rho)m_{21}$.

Lemma 5.3.5 *With α_k', β_k' defined as in section 1,*

$$\lim_{\epsilon\downarrow 0}\left[\int_{R^{+}_{\infty,\epsilon}}\chi_\infty\alpha_k'\beta_k' - \int_{R^{-}_{\infty,\epsilon}}\chi_\infty\alpha_k'\beta_k'\right] = 0$$

Proof: Since

$$\alpha_k \circ r_\infty = \alpha_k,\ \beta_k \circ r_\infty = \beta_k,$$

this follows from our usual symmetry arguments.

Q.E.D.

Lemma 5.3.6 *The sum of the contributions of the pairings of α_k' and β_{kk}' and α_{kk}' paired with β_k' at the corner $x = 0$, $x' = \infty$ is*

$$\pi^2\hat{\rho}(\sigma_k)(S_{kk}(\sigma_k) + \overline{S_{kk}(\sigma_k)}).$$

Proof: The proof is just a combination of the proofs of Lemmas 5.2.4 and 5.2.5.

Q.E.D.

The above two lemmas are true for the analogous terms from the other off-diagonal pair, $m_{21}^{*}U_0(\eta)$ and $U_0(\rho)m_{12}$.

Lemma 5.3.7

$$\lim_{\epsilon\downarrow 0}\sum_{mk}[\int_{R_{0,\epsilon}^{+}}\alpha'_{mk}\beta'_{mk}-\int_{R_{0,\epsilon}^{-}}\alpha'_{mk}\beta'_{mk}]=\frac{2\pi}{i}\sum_{m,k}\int_{|\lambda|>\max(\sigma_k,\sigma_m)}\lambda\overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}(\frac{1}{\lambda}\Psi_{mk})d\lambda.$$

Proof: The proofs of Lemma 5.3.2 and Lemma 5.3.3 work equally well here, with the only real difference being the changes in the factors of λ.

Q.E.D.

Proof of Proposition: We have calculated the contribution of one off-diagonal pair of terms here. The other off-diagonal pair gives the same delta functions at the spectrum of the boundary Laplacian, but for the continuous part, which would be the analog of Lemma 5.3.7, we have

$$\frac{2\pi}{i}\sum_{m,k}\int_{|\lambda|>\max(\sigma_k,\sigma_m)}\frac{1}{\lambda}\overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}(\lambda\Psi_{mk})d\lambda.$$

Summing over the two off-diagonal pairs, then, gives us the result claimed in the proposition.

Q.E.D.

5.4 Proof of Theorem 5.0.1

Propositions 5.2.1 and 5.3.1 show that

$$b\text{-Tr}(\tilde{M}_{+}U_0(\rho)M_{+})$$
$$=\frac{1}{2\pi i}\sum_{\sigma_m,\sigma_k\geq 0}\int_{|\lambda|>\max(\sigma_k,\sigma_m)}\overline{\Psi_{km}}(\lambda)\hat{\rho}(\lambda)\frac{d}{d\lambda}\Psi_{mk}(\lambda)d\lambda+\frac{1}{4}\sum_{\sigma_k^2\in\operatorname{spec}\Delta_{\partial X}}\hat{\rho}(\sigma_k)\Psi_{kk}(\sigma_k)$$
$$+\frac{1}{4}\sum_{\sigma_k=0}\hat{\rho}(0)\Psi_{kk}(0).$$

Then, using the fact that for $A(t)$ an invertible matrix with continuous dependence on $t \in \mathbf{R}$, $A^{-1}(t)\frac{d}{dt}A(t) = \frac{d}{dt}\log\det A(t)$, the first term on the right hand side gives $1/(2\pi i)\frac{d}{d\lambda}\log\det\Psi(\lambda)$ except at points where $\log\det\Psi(\lambda)$ might jump ($\lambda = \sigma_i$). Recalling the discussion of the jumps in the argument of the determinant of the scattering matrix (Chapter 2, Section 2), we are done.

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