# THE EFFECT OF MAGNETIC FIELD ON THE BREAKDOWN OF GASES AT HIGH FREQUENCIES

by

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## ABSTRACT

Two of the phenomena associated with a high frequency gas discharge due to the presence of a magnetic field are energy resonance and reduction of diffusion to the walls of the discharge tube. Both of these effects can be observed simultaneously when the electric and magnetic fields are transverse to each other. The diffusion phenomenon can be studied alone when the two fields are parallel to one another.

The trajectory of a free electron in a magnetic field is a cylindrical helix. The motion parallel to the field is unaffected while that perpendicular is caused to be circular. It is that latter that is responsible for both the diffusion and the energy effects. The radius of curvature of the path of motion, which is small for large values of the field, reduces the diffusion current of electrons in the direction perpendicular to the field. With an electric field perpendicular to the magnetic field, the latter causes the electron to spiral more nearly in phase with the former. At resonance, when the angular frequency of magnetic rotation $\omega_b = \frac{eB}{m}$ is equal to $\omega$, the angular frequency of the electric field, the electron is essentially in a d.c. field. The resonance character of the energy can be best seen by the kinetic theory calculation of $u_e$, the average energy gain of a single

electron between each collision. $u_e$ expressed in volts is given by:

$$u_e = \frac{eE^2}{m(\nu_c^2 + \omega_m^2)} \qquad \omega_m^2 = \frac{\nu_c^2(\omega^2 + \omega_b^2) + (\omega^2 - \omega_b^2)^2}{\nu_c^2 + \omega^2 + \omega_b^2} ,$$

where E is the r.m.s. value of the electric field and $\nu_c$ the frequency of collision between the electron and the gas molecules.

The kinetic theory of a single electron also shows that the diffusion coefficient D and hence the diffusion current in the perpendicular direction, is reduced to a new value $D_m$ by the magnetic field,

$$D_m = \frac{D}{1 + \omega_b^2/\nu_c^2} .$$

When the above is introduced into the solution of the diffusion equation for parallel plates, we get:

$$\frac{\nu_i}{D} = \frac{1}{\Lambda^2(1 + \omega_b^2/\nu_c^2)} = \frac{1}{\Lambda_m^2} \qquad \Lambda = \frac{L}{\pi} ,$$

where $\nu_i$ is the rate of production of electrons per electron, $\Lambda$ the characteristic diffusion length of the parallel plates and L the separation between them. $\Lambda_m$ is an effective diffusion length.

From the random walk theory, we can calculate the number of collisions required by an electron to diffuse to the walls. When this is equated to the number required to raise an electron to ionizing energy, we can obtain a simple breakdown equation for "Heg", helium mixed with the vapor pressure of mercury at room temperature. For this gas $\nu_c \approx$ const x pressure.

$$E_m = E_o \sqrt{\frac{1 + \omega^2/\nu_c^2}{(1 + \omega^2/\nu_c^2)(1 + \omega_b^2/\nu_c^2)}} \quad ,$$

where $E_m$ is the breakdown field for a given value of the transverse magnetic field while E is the breakdown field without the magnetic field. This equation agrees with experiment and shows the resonance character of breakdown at low pressures where the values of $E_m$ become a minimum when $\omega_b \approx \omega$. The diffusion effect is best demonstrated at higher values of $\omega_b$ because the breakdown field levels off to an asymptotic value $E_o/\sqrt{1 + \omega^2/\nu_c^2}$ .

A second method of analyzing the breakdown phenomenon is to solve for the distribution of electrons as a function of velocity from the Boltzmann transport equation. This is done by expanding this equation in spherical harmonics and Fourier series each to the zero and first order components to get a second order differential equation of $f_0$, the distribution function, in terms of the velocity.

In the process of expansion it is possible to show that the diffusion current is not a simple vector but is best described by a diffusion tensor when the magnetic field is present. This tensor is in its simplest form when one of the coordinate axes is in the direction of the magnetic field. If B is taken along the Z axis, then the diffusion tensor $D_{ij}$, (i,j=x,y,z) is given by:

$$D_{ij} = \begin{pmatrix} \dfrac{D}{1 + \omega_b^2/\nu_c^2} & \dfrac{D\ \omega_b/\nu_c}{1 + \omega_b^2/\nu_c^2} & 0 \\ \dfrac{-D\ \omega_b/\nu_c}{1 + \omega_b^2/\nu_c^2} & \dfrac{D}{1 + \omega_b^2/\nu_c^2} & 0 \\ 0 & 0 & D \end{pmatrix}$$

where

$$D = 4\pi \int_0^\infty \frac{\ell v}{3} f_0 v^2 dv = \frac{4\pi}{3\nu_c} \int_0^\infty f_0 v^4 dv \ .$$

D is the ordinary diffusion coefficient and $\ell$ is the mean free path of an electron. The diffusion equation becomes more general and takes the form:

$$D_m\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) + D\frac{\partial^2 n}{\partial z^2} = \nu_i n ,$$

where n is the density of electrons per unit volume. The solution of this equation leads to the following:

$$\frac{\nu_i}{D} = \frac{1}{\Lambda_e^2} = \left\{\left(\frac{\pi}{L_x}\right)^2 + \left(\frac{\pi}{L_y}\right)^2\right\}\frac{1}{1 + \omega_b^2/\nu_c^2} + \left(\frac{\pi}{L_z}\right)^2 .$$

$\Lambda_e$ is an effective diffusion length defined by the above equation. It can be expressed in any coordinates and is particularly simple for a cylindrical cavity

$$\frac{1}{\Lambda_e^2} = \left(\frac{2.405}{R}\right)^2 \frac{1}{1 + \omega_b^2/\nu_c^2} + \left(\frac{\pi}{L}\right)^2 ,$$

where R is the radius of the cylinder and L the height parallel to the magnetic field.

The solution of the diffusion equation is introduced into the differential equation for $f_0$. This enables us to solve for $f_0$ which is then used to obtain a transcendental breakdown equation in terms of the effective parameters $E_e\Lambda_e$, $E_e/p$ and $u_i$.

$$\mathcal{F}(E_e\ {}_e, E_e/p, u_i) = 0 .$$

$E_e$ is an effective d.c. field expressed in terms of the actual r.m.s. value of the high frequency field E as follows:

$$E_e = \frac{E}{\sqrt{1 + \omega_m^2/\nu_c^2}} \qquad \text{for } E \perp B$$

$$E_e = \frac{E}{\sqrt{1 + \omega^2/\nu_c^2}} \qquad \text{for } E \parallel B \text{ or for } B = 0$$

$u_i$ is an effective ionization potential which for "Heg" is defined by the value of the electron energy at which $f_0 = 0$. The way in which $u_i$ is obtained is by extrapolating the solution of $f_0$ in the inelastic region by its tangent at $u = u_x$, the first excitation potential of helium. This is also expressed in terms of $E_e\Lambda_e$ and $E_e/p$. Consequently the final breakdown equation is still a transcendental one:

$$\mathcal{F}'(E_e\Lambda_e, E_e/p) = 0 \quad .$$

This equation is plotted as a single graph of $E_e\Lambda_e$ as a function of $E_e/p$. The resultant curve represents all the breakdown voltages of "Heg" in a uniform electric field with and without a magnetic field, at all frequencies and pressures within the limits of the theory, and for all shaped cavities. The value of $\Lambda_e$ for the latter is obtained from the solution of the diffusion equation.

The theory can be extended to non-uniform fields for rectangular and cylindrical cavities. It has been done for the latter and the solution was used to study the diffusion effect of the magnetic field alone when parallel to the electric field.

The transcendental breakdown equation can be reduced to the ones obtained by the kinetic theory of a single electron by making the approximation that the elastic collision terms at low pressures can be neglected and $u_i$ is assumed to be independent of the magnetic field. This simplified theory agrees well with experiment for a flat cylindrical cavity despite the fact $u_i$ does depend on the magnetic field and is reduced by it for a given pressure. The answer is that the flat cavity is only an approximation to parallel plates with the magnetic field present. $\Lambda_e$ must then be corrected for magnetic distortion. The two corrections for $u_i$ and $\Lambda_e$ very nearly cancel each other and hence account for the agreement of the theory which corrected for neither.

The thesis shows all the theoretical results in form of graphs which are compared with the results obtained experimentally. A diagram of the experimental apparatus together with a complete description and procedure for obtaining measurements are also included.

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# I. INTRODUCTION

## Historical Background

The problem of breakdown of gases by a high frequency electric field in the presence of a magnetic field originated with the investigation of ionospheric propagation. The first laboratory experiment was done by E. W. B. Gill[1] who produced breakdown of air at very low pressures in a glass tube suspended between two plates. He produced a discharge with and without a magnetic field and then compared the relative values of electric field required to breakdown the gas. His magnetic field was 18 gauss which corresponded to the equality of the cyclotron or gyromagnetic frequency of the electron and the radio frequency at 6 meters wavelength. Up to 1/10 mm of pressure no effect was observed, at 1/50 mm the breakdown field was reduced by a factor of 5 and at 1/100 mm by a factor of 40.

Later in 1938, Townsend and Gill[2] performed a more complete set of experiments on air. They produced a discharge in a pyrex sphere 13 cm in diameter located between the plates of a condenser which was part of a high frequency circuit. Measurements of current through the discharge were made in a secondary circuit. The measurements were made at two frequencies of about 30 and 50 mc and at pressures ranging from 120 microns down to 2.5.

A Helmholtz coil supplied a variable magnetic field perpendicular to the electric field. Measurements of breakdown field and the least maintaining field after breakdown as a function of the magnetic field were made. Breakdown was determined by the appearance of a faint glow. The value of the field was read from the current meter which was calibrated by an independent determination of the field between the plates. The absolute accuracy of this calibration was admittedly questionable. A second set of experiments on pure nitrogen and helium was made by A. E. Brown[3] on the same apparatus.

From a qualitative point of view the experiments were successful, since they observed the fundamental resonance phenomenon expected. From a quantitative view, this was not the case, since Townsend did not draw satisfactory theoretical curves to match the experimental ones.

There are a number of criticisms to be made of Townsend's investigations. On the experimental side, the observation of breakdown by the appearance of a faint glow, the use of a glass container, and the means used to measure the value of the electric field all contributed to poor accuracy in determining the breakdown field. The choice of a spherical geometry for the container was an unhappy one, since the diffusion equation in a magnetic field is unmanageable. Theoretically, the single electron approach was good except for the fact that no attempt was made to solve the diffusion equation although Townsend recognized the importance of diffusion. He merely made calculations from energy considerations alone and supplemented the diffusion correction. Comparison of breakdown between the magnetic and non-magnetic case was made only at resonance. The complete range of values of the magnetic field was not analyzed. The final criticism of the theory was that it was not related to the physics of the particular gas except for the collision frequency. The excitation and ionization potentials and probabilities were not mentioned. Nevertheless the theory did predict the results qualitatively.

Present Treatment of the Problem

In the intervening years from 1938 to the present, both the experimental and theoretical tools necessary to solve high frequency problems were improved. The microwave equipment and techniques developed during the war provided more accurate means of measuring fields in a cavity.[4] The use of helium mixed with mercury vapor[5] provided the simplest possible gas for theoretical analysis. With the introduction of the diffusion equation into the breakdown theory by Herlin[6] the single electron analysis to the magnetic problem was possible. This theory will be developed in this thesis.

A second and a more satisfactory theoretical approach to the high frequency breakdown problem is the distribution theory of electrons. The original work of expanding the Boltzmann transport equation into spherical harmonic components was done by Lorentz.[7] This was then extended by Morse, Allis and Lamar[8] properly including the collision terms. The magnetic term was later introduced by Allis and Allen.[9] The application to high frequencies was not made until Margenau[10] extended the theory by the use of Fourier harmonics in the expansion. The development for the solution of the breakdown problem was not complete until the introduction of the diffusion equation. All these elements were combined for the non-magnetic case to obtain a solution for the breakdown of helium by microwaves.[5] The distribution theory is extended in this thesis to include the magnetic term and is applied to the calculation of the breakdown field in helium as a function of the magnetic field.

## Outline

The treatment of breakdown in a magnetic field is first approached from the simplest considerations, namely the behavior of a single electron. This is the topic of Section II. The dynamics of a single electron are analyzed. By averaging simple distributions of the electrons in space, phase with the electric field and time between collision the statistical behavior of the electron is studied. From this we obtain the energy gain of the electrons between collisions and the diffusion properties of an assembly. A diffusion tensor is justified and applied in deriving a diffusion equation with the magnetic term included. These results are used to calculate breakdown fields which are compared with experiment.

In Section III we expand the Boltzmann transport equation in spherical harmonics and Fourier components neglecting all but the zero and first order terms in the expansion. A detailed development of the diffusion tensor is carried out and the nature of the diffusion current is demonstrated. A differential equation for $f_o$ the distribution of electrons as a function of velocity is derived. This is used in obtaining the equation of breakdown of helium. It is a transcendental equation relating breakdown electric field, pressure, frequency and the magnetic field. By a simple approximation this equation is reduced to the one derived by the single electron theory. The differential equation is extended to include inelastic collisions and is used to calculate an effective ionization potential. A single curve of effective quantities is drawn for helium representing the breakdown voltage of both magnetic and non-magnetic high frequency discharge within all appropriate ranges of pressure, frequency and magnetic field.

The theory is made more general in Section IV by solving the breakdown problem for the non uniform electric field. This is used in calculating the diffusion effect of the magnetic field alone when it is parallel to the electric field. The calculations make extensive use of effective diffusion lengths and the general breakdown curve of Section III.

Section V is devoted to the calculation of the breakdown field in transverse electric and magnetic fields from the distribution theory. The single effective voltage curve is again used, but it is corrected for diffusion distortion by the magnetic field. The correlation between these results, those of the simple theory of the single electron and the experimental curves are justified.

The experimental apparatus and the procedure used for making breakdown measurements are described in detail in Section VI.

Finally in Section VII, the conclusion, a survey of the accomplishments of this investigation is made. Evaluation of the results is discussed and new problems along lines similar to those used in this thesis or an extension of them are suggested.

## II. THEORY OF THE AVERAGE FREE ELECTRON

In a gas discharge which contains a large number of electrons, we can choose a particular group of them which is representative of all on the average. To do this, consider any electron during the interval between two successive collisions while in motion under the simultaneous action of an electric and a magnetic field. At the start of the collision, the electron of a particular velocity has an arbitrary direction with the electric field, equally probable over the complete sphere of solid angles. Its phase with the electric field at the start may be anywhere from 0 to $360^o$. Its free path which is curvi-linear may have any length from 0 to $\infty$. Instead of the free path, it is more convenient here to speak about the collision interval t. The longer intervals are less probable and the probability factor is given by $e^{-\nu_c t}$, where $\nu_c$ is the collision frequency or the inverse of the average time of collision.

In general, the above quantities related to the motion of an electron are randomly distributed as described above. Hence any quantity which measures the characteristic behavior of the electron must be a statistical average over the four quantities: <u>direction</u>, <u>phase</u>, <u>collision time</u> and <u>velocity</u>. The first two are random quantities while the velocity and the collision time are distributed. Not all of these quantities need be averaged for some applications; one or two may be ignored depending on the situation. Without going into any more details, we shall apply the above averaging process to a number of specific cases and amplify upon the method as we proceed.

## Energy Gain between Collisions

### Transverse Fields

The most difficult situation for our investigations is the one in which the electric and magnetic fields are perpendicular to each other. The parallel case can be treated as a simple superposition of the electric and magnetic fields acting independently.

First it is necessary to set up the equation of motion of the electron in the given fields. To do this, let us take the electric field in the direction of the X-axis and the magnetic field along the Z-axis as shown in Fig. II-1. Both fields are assumed to be uniform.

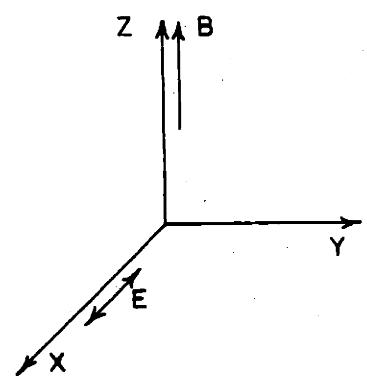



Fig. II-1. Orientation of transverse electric and magnetic fields.

From the general vector equation of the force on an electron, i.e.

$$\vec{F} = m\frac{d\vec{v}}{dt} = -e\vec{E} - e\vec{v}\times\vec{B} \,, \tag{2-1}$$

we get the component equations

$$m\frac{dv_x}{dt} = -\sqrt{2}\, eE_o \cos \omega t - eBv_y \,,$$

$$m\frac{dv_y}{dt} = eBv_x \,, \tag{2-2}$$

$$m\frac{dv_z}{dt} = 0 \,,$$

where $\vec{E} = \vec{i}_x E_x = \vec{i}_x \sqrt{2}\, E_o \cos \omega t$ , $E_o$ being the r.m.s. value of the electric field.

Eliminating $v_y$ from the first two equations of (2-2), we obtain a single second degree equation in $v_x$.

$$\frac{d^2 v_x}{dt^2} + \frac{e^2B^2}{m^2} v_x = \frac{\sqrt{2}\, eE_o}{m}\, \omega \sin \omega t \,. \tag{2-3}$$

The solution to this equation is:

$$v_x = A \cos \omega_b t + B \sin \omega_b t - \frac{\sqrt{2}\, eE_o \omega}{m(\omega^2 - \omega_b^2)} \sin \omega t \; . \qquad (2\text{-}4)$$

A and B are arbitrary constants and $\omega_b = \frac{eB}{m}$ represents the angular gyromagnetic or "cyclotron" frequency.

By substituting the value of $v_x$ in the first of the equations (2-2), a similar solution for $v_y$ is obtained:

$$v_y = A \sin \omega_b t - B \cos \omega_b t + \frac{\sqrt{2}\, eE_o \omega_b}{m(\omega^2 - \omega_b^2)} \cos \omega t \; . \qquad (2\text{-}5)$$

The solution for $v_z$ is simply

$$v_z = \text{constant} \; . \qquad (2\text{-}6)$$

The constants A and B are evaluated by setting $v_x = v_{ox}$, $v_y = v_{oy}$ and $v_z = v_{oz}$ at the initial time $t_o$. The complete solution is then expressed in terms of t, the time at the end of the collision.

$$v_x = v_{ox} \cos \omega_b t - v_{oy} \sin \omega_b t + \frac{\sqrt{2}\, eE_o \omega_b}{m(\omega^2 - \omega_b^2)} \cos \omega t_o \sin \omega_b t$$

$$+ \frac{2\, eE_o \omega}{m(\omega^2 - \omega_b^2)} \left[ \sin \omega t_o \cos \omega_b t - \sin \omega(t_o + t) \right]$$

$$v_y = v_{ox} \sin \omega_b t + v_{oy} \cos \omega_b t + \frac{\sqrt{2}\, eE_o \omega}{m(\omega^2 - \omega_b^2)} \sin \omega t_o \sin \omega_b t$$

$$+ \frac{\sqrt{2}\, eE_o \omega_b}{m(\omega^2 - \omega_b^2)} \left[\cos \omega(t_o + t) - \cos \omega t_o \cos \omega_b t\right] \qquad (2\text{-}7)$$

$$v_z = v_{oz} .$$

$v_{ox}$, $v_{oy}$ and $v_{oz}$ represent the components of a particular value of the electron velocity $v_o$, t the time interval between collisions and $t_o$ the random time between cycles of the alternating electric field. These are the quantities that we must average.

In order to compute the average energy gained by an electron from the electric field between collisions, we shall calculate the average energy at the end of the collision and compare it with its initial value. The average energy in volts is given by:

$$\bar{u} = \frac{m\overline{v^2}}{2e} = \frac{m}{2e}\, \overline{(v_x^2 + v_y^2 + v_z^2)} . \qquad (2\text{-}8)$$

The expression for $\overline{v^2}$ after simplification has the form:

$$\overline{v^2} = v_o^2 + \frac{2\sqrt{2}\, eE_o}{m(\omega^2 - \omega_b^2)} \left[\left\{ v_{ox}(\omega_b - \omega) \cos \omega_b t \cos \omega t - \omega_b v_{oy}\right.\right.$$

$$\left. + v_{oy} \sin \omega_b t(\omega_b \cos \omega t + \omega \sin \omega t)\right\} \overline{\cos \omega t_o}$$

$$+ \left\{ \omega v_{ox} - v_{ox}(\omega \cos \omega_b t \cos \omega t + \omega_b \sin \omega_b t \sin \omega t) \right.$$

$$\left. + v_{oy}(\omega \sin \omega_b t \cos \omega t - \cos \omega_b t \sin \omega t) \right\} \overline{\sin \omega t_o} \Big]$$

$$+ \frac{2e^2E_o^2}{m^2(\omega^2 - \omega_b^2)^2} \Big[ \Big\{ 1 - 2 \cos \omega_b t \cos \omega t \Big\} \Big\{ \omega_b^2 \overline{\cos^2 \omega t_o} + \omega^2 \overline{\sin^2 \omega t_o} \Big\}$$

$$+ \Big\{ \omega_b^2 \overline{\cos^2 \omega(t_o + t)} + \omega^2 \overline{\sin^2 \omega(t_o + t)} \Big\}$$

$$+ (\omega^2 - \omega_b^2) \cos \omega_b t \sin \omega t \overline{\sin 2\omega t_o} - 2\omega\omega_b \sin \omega_b t \sin \omega t \Big] \tag{2-9}$$

Thus, we see that Eq. (2-9) has to be averaged only over $t_o$ and t. For the present, we shall only consider $t_o$. This is to be averaged equally over a complete cycle of the electric field or $\omega t_o$ over an angle of $2\pi$ radians. The values are listed below:

$$\overline{\cos \omega t_o} = \frac{1}{2\pi} \int_0^{2\pi} \cos \omega t_o d(\omega t_o) = 0$$

$$\overline{\sin \omega t_o} = \frac{1}{2\pi} \int_0^{2\pi} \sin \omega t_o d(\omega t_o) = 0 \tag{2-10}$$

$$\overline{\cos^2 \omega t_o} = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \omega t_o d(\omega t_o) = \frac{1}{2}$$

$$\overline{\sin^2 \omega t_o} = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 \omega t_o d(\omega t_o) = \frac{1}{2} .$$

Similarly

$$\overline{\cos^2\omega(t_o + t)} = \overline{\sin^2\omega(t_o + t)} = \frac{1}{2} \; .$$

Finally

$$\overline{\sin 2\omega t_o} = 0 \; .$$

When we substitute the values of (2-10) in Eq. (2-9), we get:

$$\overline{v^2} = v_o^2 + \frac{2e^2E_o^2(\omega^2 + \omega_b^2)}{m^2(\omega^2 - \omega_b^2)^2}\left[1 - \overline{\cos \omega_b t \cos \omega t}\right]$$

$$- \frac{4e^2E_o^2\omega\omega_b}{m^2(\omega^2 - \omega_b^2)^2}\,\overline{\sin \omega_b t \sin \omega t} \; . \qquad (2\text{-}11)$$

The bar above the terms containing t indicate that they are to be averaged over this variable. Previously we said that the distribution in t was exponential and had the form $e^{-\nu_c t}$, where $\nu_c$ was the collision frequency corresponding to a given value of $v_o$ of the velocity. To average any function f(t) we must do the following:

$$\overline{f(t)} = \frac{\int_0^\infty e^{-\nu_c t} f(t)\,dt}{\int_0^\infty e^{-\nu_c t}\,dt} = \nu_c \int_0^\infty e^{-\nu_c t} f(t)\,dt \,. \qquad (2\text{-}12)$$

Before we proceed with averaging the terms in (2-11), we shall first simplify them as follows:

$$\cos \omega_b t \cos \omega t = \frac{1}{2}\left[\cos(\omega - \omega_b)t + \cos(\omega + \omega_b)t\right]$$

$$\sin \omega_b t \sin \omega t = \frac{1}{2}\left[\cos(\omega - \omega_b)t - \cos(\omega + \omega_b)t\right] \,. \qquad (2\text{-}13)$$

Thus, we see that we only have to evaluate the average of a cosine function. The integral has the form:

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \left[e^{-ax}\,\frac{(-\,a \cos bx + b \sin bx)}{a^2 + b^2}\right]_0^\infty$$

$$= \frac{a}{a^2 + b^2} \,. \qquad (2\text{-}14)$$

Then

$$\int_0^\infty e^{-\nu_c t}\cos(\omega \pm \omega_b)t\,dt = \frac{\nu_c}{\nu_c^2 + (\omega \pm \omega_b)^2} \,. \qquad (2\text{-}15)$$

Substituting this expression in (2-12) and subsequently in (2-11), we obtain the desired average value of $\overline{v^2}$. The value for the average energy $\overline{u} = m\overline{v^2}/2e$ then becomes:

$$\overline{u} = \frac{mv_o^2}{2e} + \frac{eE_o^2}{m} \frac{\nu_c^2 + \omega^2 + \omega_b^2}{\left[\nu_c^2 + (\omega + \omega_b)^2\right]\left[\nu_c^2 + (\omega - \omega_b)^2\right]} \qquad (2\text{-}16)$$

The energy gain between collisions $u_e$ is evidently the second term on the right-hand side of Eq. (2-16). This expression can be simplified by expanding the denominator and dividing by the numerator. We then obtain

$$u_e = \frac{eE_o^2}{m} \frac{1}{\nu_c^2 + \omega_m^2} \qquad (2\text{-}17)$$

$$\omega_m^2 = \frac{\nu_c^2(\omega^2 + \omega_b^2) + (\omega^2 - \omega_b^2)^2}{\nu_c^2 + \omega^2 + \omega_b^2} .$$

The above value for $u_e$ has been obtained for a given value of the velocity and no averaging over the velocity was performed. In general, $\nu_c$ is a function of the velocity and $u_e$ must be averaged. Here we shall take the velocity as that of the average electron and take the corresponding value of $\nu_c$.

Physical Interpretation

Before we proceed to plot $u_e$ graphically in terms of the proper variables, let us discuss the physical picture suggested by the mathematics. In making our plot, we shall compare $u_e$ with the steady state energy $u_o$ in an alternating field of angular frequency $\omega$ when no collisions occur and no magnetic field is present ($\nu_c = 0$; $\omega_b = 0$):

$$u_o = \frac{eE_o^2}{m\omega^2} . \qquad (2\text{-}18)$$

If we compare this to Eq. (2-17), we see that the two have the same form if we write:

$$u_e = \frac{eE_o^2}{m\omega_e^2} , \qquad \omega_e^2 = \nu_c^2 + \omega_m^2 , \qquad (2\text{-}19)$$

where $\omega_e$, is called an effective frequency. The physical meaning of this frequency becomes apparent when we consider some simple examples such as a d.c. field, for which $\omega_e = \nu_c$. The effect of collisions is then to produce a phase change of the electron relative to the electric field just as if there were an alternating field present. Similarly, when no magnetic field is applied, but only the electric field, $\omega_e = \nu_c^2 + \omega^2$ represents a higher frequency on the average due to collisions. However, when the magnetic field is also applied transverse to the electric field the situation becomes a little more complicated.

To analyze the last case, let us first consider the motion of an electron in a uniform magnetic field alone. The motion is along a cylindrical helix whose axis is parallel to the magnetic field. In a plane perpendicular, the component of motion is circular with an angular frequency of $\omega_b$. If we now superimpose an alternating electric field parallel to the plane of the circular motion we have our transverse field. The motion of the electron is no longer a simple circle in the reference plane, but nevertheless it still spirals thru $360^\circ$ at the average rate determined by the magnetic field. The result of this spiraling is to throw the electron more nearly in phase with the electric field on the average. When the magnetic field is increased, so that $\omega_b = \omega$, the electron is effectively in a d.c. field except for the collisions. For $\omega_b > \omega$, the electron is again thrown out of phase with the electric field. For extremely large values of the magnetic field, the electron spins very rapidly such that the effective phase variation, hence $\omega_e \approx \omega_b \gg \omega$. Another effective quantity is $\omega_m$, which is essentially the equivalent alternating frequency corresponding to $\omega$ without the magnetic field.

Figure II-2 shows $u_e$ as a function of the magnetic field, expressed by $\omega_b$, in terms of $u_o$. The family of curves is for

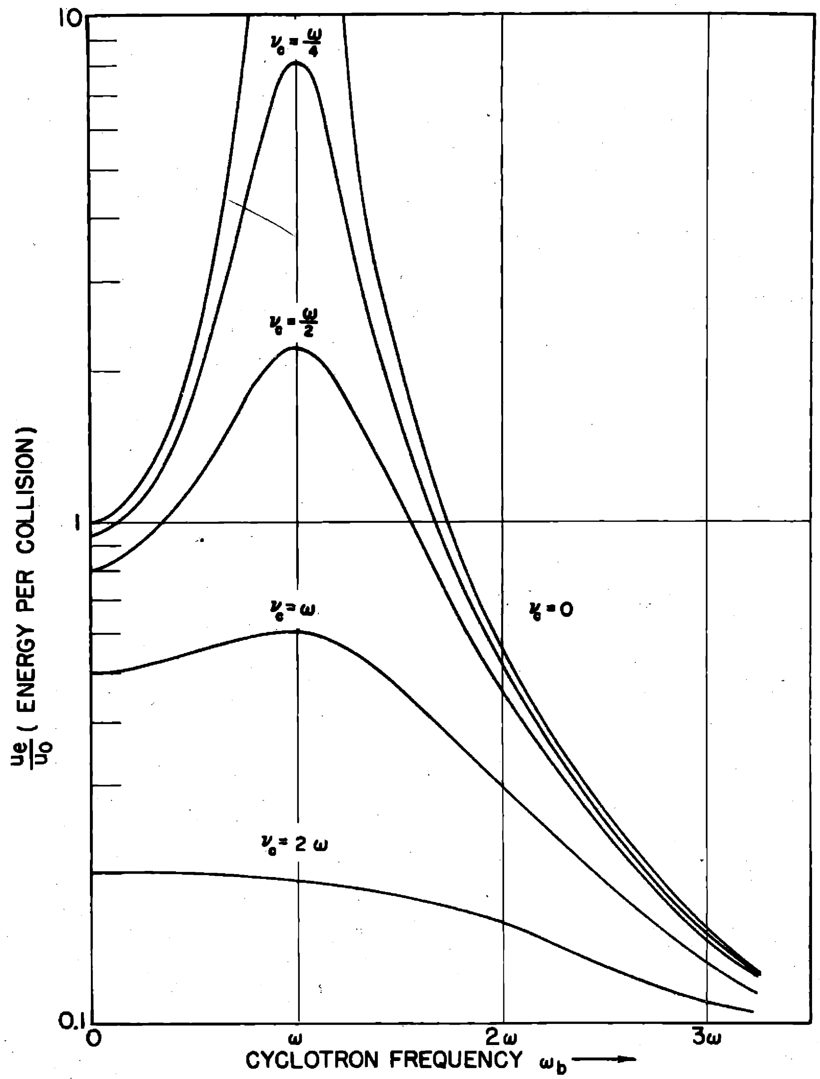



Fig. II-2. The energy gain per collision of an electron in uniform transverse electric and magnetic fields.

different values of $\nu_c$. Since $u_e$ is proportional to the inverse square of $w_e$, the curves shown are in accord with the physical reasoning of the preceding paragraph. The energy gain increases with magnetic field to a maximum near resonance, i.e., where $\omega_b = \omega$. Beyond this point, there is a reduction in $u_e$ which for large values of $\omega_b$ falls below the initial value at $\omega_b = 0$. The resonance effect is more pronounced for smaller values of $\nu_c$, or lower pressures for a given gas. At larger values of $\nu_c$, the rise in $u_e$ becomes less significant until such a value is reached that no maximum exists. This corresponds to $\omega = \nu_c/\sqrt{3}$ . Thus, when $\nu_c > \sqrt{3}\,\omega$ there is no maximum at all, but a steady decline in $u_e$. This is a relatively high pressure region, where the collision frequency is sufficiently large to produce, on the average, a greater number of phase reversals than that which the magnetic field can overcome.

The graph of $u_e$ has been given for a fixed value of the random velocity $v_o$, for which $\nu_c$ is constant. In general, for an assembly of electrons in a gas, the velocity is distributed over all values and hence $\nu_c$ is a function of velocity. To determine the average value of $u_e$, we must integrate over all velocities. This requires that we know both $\nu_c$ and the distribution of electrons as functions of the velocity. Certain exceptional cases are those gases for which the collision cross-section is inversely proportional to velocity and hence $\nu_c$ = constant, at a given pressure, independently of the velocity. Under such conditions, $u_e$ is similarly independent. Helium and hydrogen are gases for which this holds reasonably well and quantitative applications of the preceding formulas are possible. For other gases, a suitable value of $\nu_c$ may be chosen and assumed to be fixed for a given pressure. Here, however, the results are merely good for a qualitative, or at least, a semi-quantitative analysis.

## Breakdown

### Qualitative Description

The reason behind the previous mathematical development was to illustrate the manner in which energy considerations affect the breakdown of gases. If we had an infinite medium in which a uniform electric field produced a discharge in the presence of a transverse uniform magnetic field, we would have the ideal situation for the problem at hand. Under such conditions, there would be no diffusion loss of electrons with the attendant loss of energy. Practically all the energy loss of electrons would be to inelastic collisions. The removal process for electrons would be due to attachment and recombination. These, of course, are low energy phenomena and would represent only a small loss of energy.

The process of the above discharge would be one in which an electron would gain, on the average, the energy $u_e$ during each collision, until its kinetic energy was equal to or greater than an excitation or ionizing energy of the gas molecules. Eventually the electron would lose its energy during an inelastic collision. If we considered the statistical history of a large number of electrons, we would arrive at the conclusion that there is an average energy at which the electrons lose substantially all their energy by excitation and ionization. Also at a fixed pressure, the average collision rate would be constant. With the ionizing and excitation probabilities being also invariant, the logical assumption is that $u_e$ remains constant, independent of the magnetic and electric fields, at the point of breakdown.

$$u_{em} = u_{eo} = \text{constant} ,$$

or

$$\frac{eE_m^2}{m(\nu_c^2 + \omega_m^2)} = \frac{eE_o^2}{m(\nu_c^2 + \omega^2)} = \text{constant} . \qquad (2\text{-}20)$$

The subscripts m and o are used to denote the appropriate quantities when the magnetic field is included or is non-existent. Thus we can express the breakdown field $E_m$ relative to the field $E_o$

for the non-magnetic case as a function of the magnetic field when the pressure and the alternating frequency $\omega$ are fixed.

$$E_m = E_o \sqrt{\frac{1 + \omega_m^2/\nu_c^2}{1 + \omega^2/\nu_c^2}} \sim \sqrt{1 + \frac{\omega_m^2}{\nu_c^2}} . \tag{2-21}$$

We shall make a quantitative use of Eq. (2-21) in the next section. For the present, it is sufficient to note that this equation implies that $E_m$ is proportional to the inverse square of the quantity $u_e$ plotted in Fig. II-2. This permits a qualitative analysis of the experimental results obtained by producing breakdown in a relatively large cylindrical cavity, approximately 3 inches in diameter and 2 inches high. Although the removal process is still by diffusion and not by recombination, the energy considerations permit a fair comparison between the magnetic and the non-magnetic cases because the relative energy loss due to diffusion as a function of the magnetic field plays an increasingly smaller role. The curves of Fig. II-3 show the breakdown field as a function of the magnetic field for air.

As expected, the reduction in breakdown field is greater at lower pressures. At higher pressures, there is no reduction but only an increase which becomes less and less apparent as the pressure increases. Thus at very high pressures, the curve is flat because the large value of the collision frequency completely masks

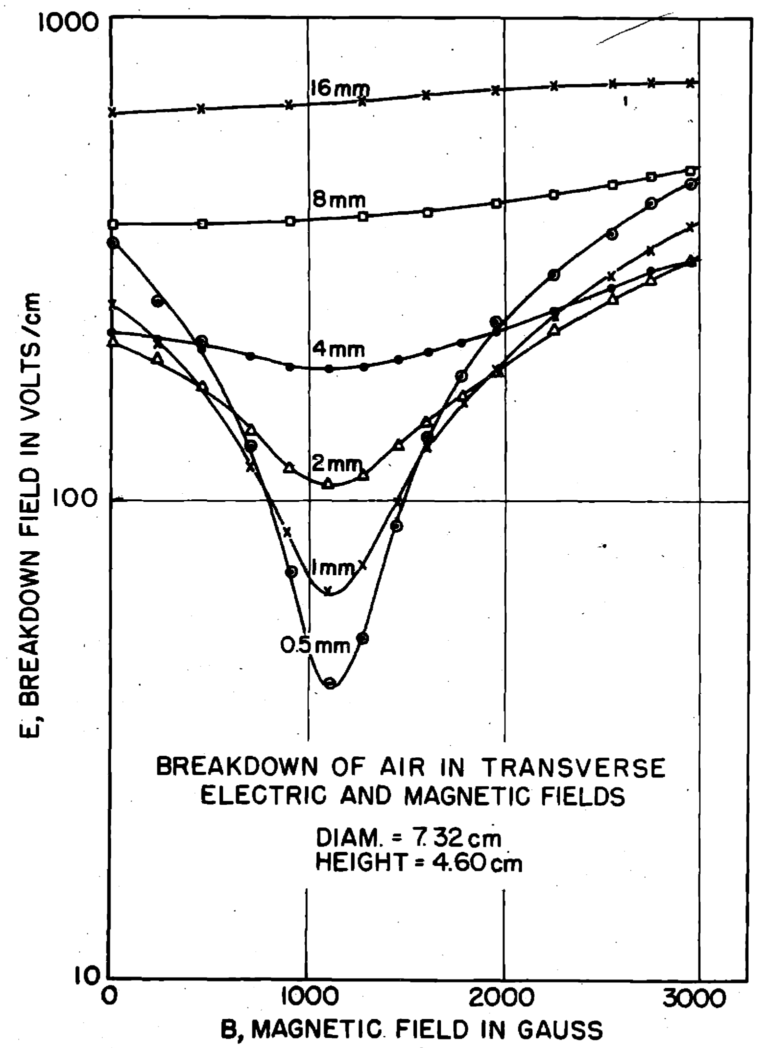



Fig. II-3. Breakdown of air in transverse electric and magnetic fields in a cylindrical cavity (L = 4.60 cm, R = 3.66 cm).

the effect of the magnetic field. The first dip shown in Fig. II-3 occurs at 4 mm, which for air corresponds to about $20 \times 10^9$ collisions/sec or the order of magnitude as $\omega \approx 20 \times 10^9$. This is a general criterion which holds for all gases, i.e., the resonance phenomenon begins to appear when $\nu_c \approx \omega$. For $\nu_c << \omega$, the effect is large as can be seen when p = 0.5 mm. The reduction in the breakdown field is from 350 volts/cm to about 40 volts/cm, almost a factor of 10. The minimum of each curve occurs nearly at B = 1100 gauss, corresponding to the value calculated when $\omega = \omega_b$. Beyond the resonance value, the curves rise steeply as expected, in some cases beyond the initial value $E_o$. The rise, nevertheless, does not seem as steep as one would have anticipated from the previous plot of $u_e$.

The experiment was performed with helium in a 3" x 2" cavity. The curves which are shown in Fig. II-4 show the same general character as those of air except that the reduction in breakdown field is greater for the same pressure, i.e., for air at 1 mm it is a factor of 4 and for helium a factor of 12. This is primarily due to the smaller collision cross-section for helium.

Semi-Quantitative Analysis

In the preceding section, we hinted that the relative rise in the breakdown field at large values of the magnetic field was not as large as one might have expected. This was even more

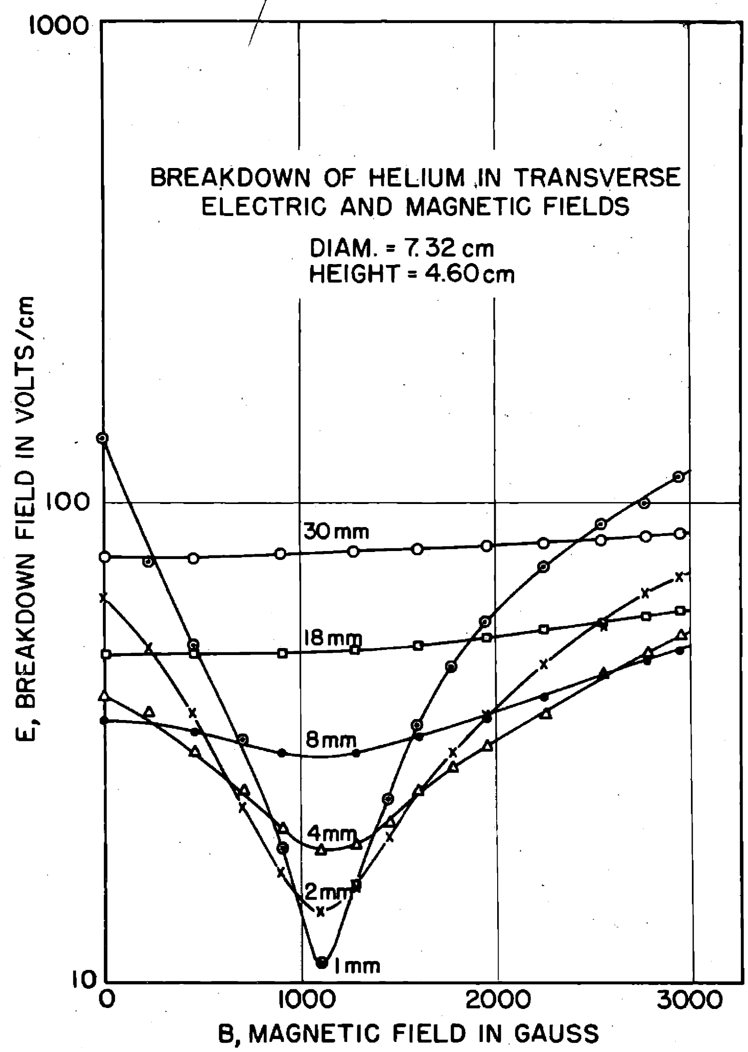



Fig. II-4. Breakdown of helium in transverse electric and magnetic fields in a cylindrical cavity (L = 4.60 cm, R = 3.66 cm).

apparent for helium than for air. Perhaps the best way in which to see this is to use Eq. (2-21) to calculate the breakdown field in terms of the magnetic field and compare the result with the experimental curves. For helium this is quite simple since we are not far from the truth when we say that $\nu_c = cp$ where c is a constant. This approximation has been successfully used by MacDonald and Brown[5] in their studies of helium. The situation for air is more complex since the above approximation is not true. The best one can do is to make a guess at the value of c and recognize the limitations of its application.

One of the simplest methods is to use the curves of Fig. II-3 and the condition that $\nu_c^2 = 3\omega^2$ when there is no maximum in the resonance curves. It appears that this occurs when $p \approx 7$ mm. Upon substituting the value of $\omega = 20 \times 10^9$ we find that $\nu_c \approx 5 \times 10^9 p$. This value agrees fairly well by a second estimate which makes use of the collision probability curves of oxygen and nitrogen.[11] The average value of $P_c \approx 38$ for air. Assuming an average energy of the electron to be $\bar{u} \approx u_i/3$, we use the value of 5 volts. The result is that the mean value of $\nu_c$ is that given above. With this information we obtained two sets of curves at 0.5 mm and 1.0 mm. These are plotted in Fig. II-5 and compared with the corresponding experimental curves.

The significant facts to be gained from this comparison

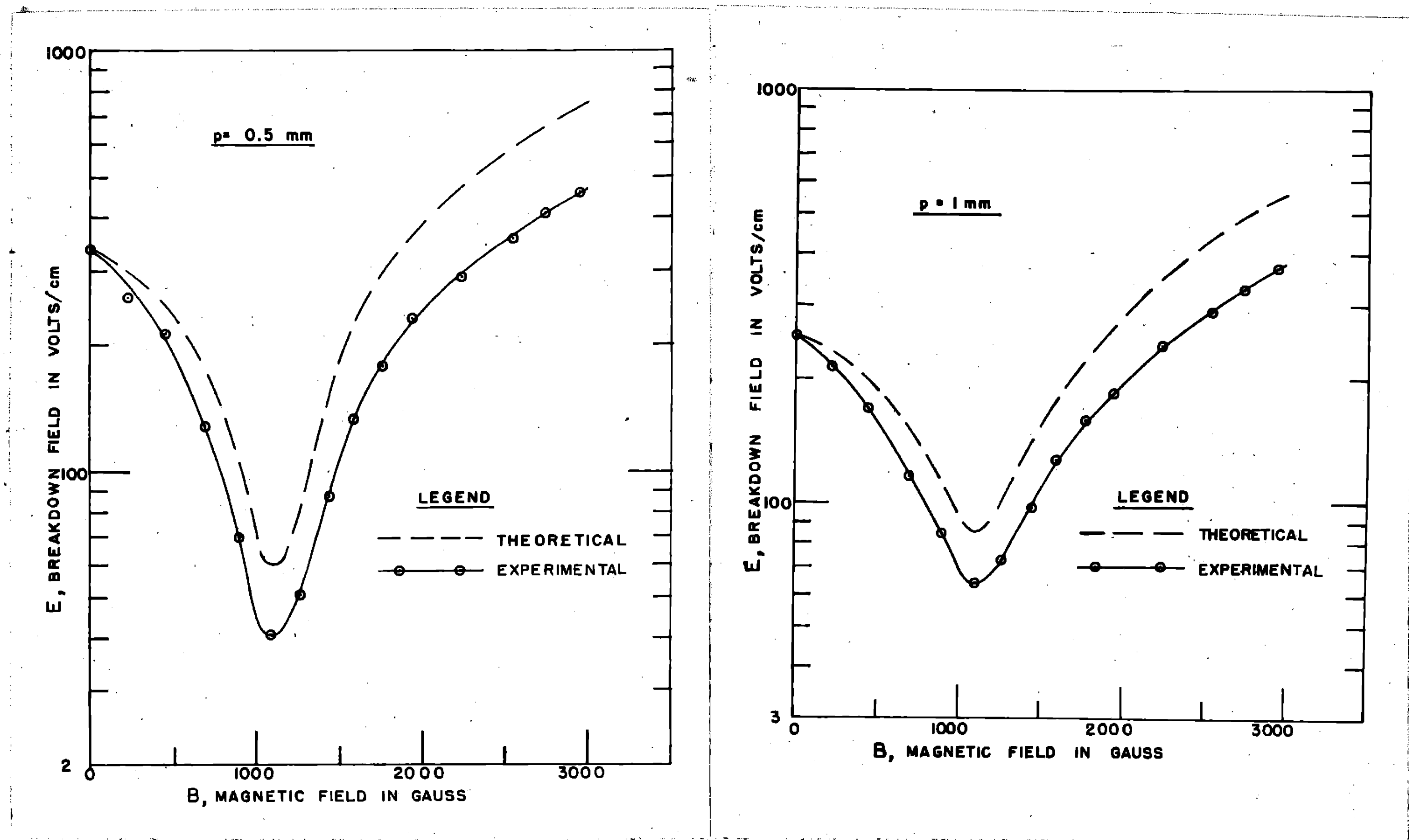



Fig. II-5. Breakdown of air in transverse $\vec{E}$ and $\vec{B}$ in a cylindrical cavity (L = 4.60 cm, R = 3.66 cm). The theoretical curves are those for an infinite medium.

is that although the agreement is far from satisfactory quantitatively the fundamental physical phenomena of the energy resonance is well substantiated by our rough analysis. Furthermore, since air is such a complex gas, the theory is fairly good because the orders of magnitude are reasonable. The second observation to be made is that the separation between the two curves is substantial for large values of B. This effect becomes even more noticeable for helium as illustrated in Fig. II-6. These curves were obtained by using the value of $\nu_c = 2.37 \times 10^9 p$ calculated from Brode's curve[12] of $P_c$ for helium.

The expected agreement for He should have been better except that we have neglected an important factor in our calculations, namely diffusion. The experiment was conducted in as large a cavity as possible, but still small relative to the ideal infinite medium. Hence the loss of energy due to diffusion cannot be neglected particularly for helium where this loss is comparable to the inelastic losses. We shall look further into this problem and account for the apparent discrepancy when we discuss diffusion.

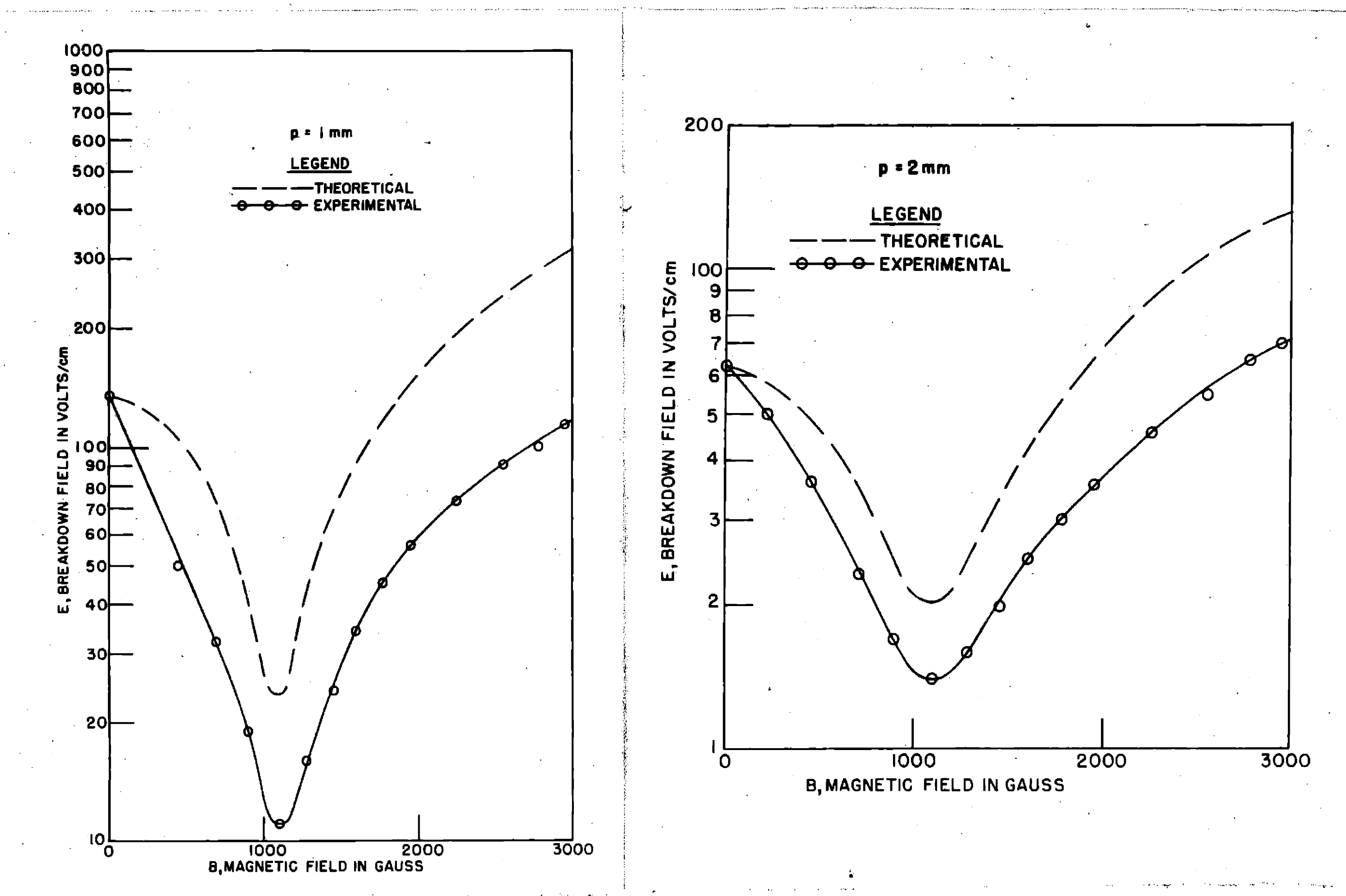



Fig. II-6. Breakdown of helium in transverse $\vec{E}$ and $\vec{B}$ in a cylindrical cavity (L = 4.60 cm, R = 3.66 cm). The theoretical curves are those for an infinite medium.

## Diffusion

### Diffusion Tensor

Diffusion can be defined as a net flow of particles from a region of high to one of lower concentration. Thus diffusion always accompanies a density gradient in an assembly of particles. The existence of a concentration gradient of charged particles occurs because the electrons and the positive ions are absorbed at the walls, reducing the concentration there essentially to zero.

One can derive an expression for the particle current density vector $\vec{\Gamma}$ in terms of the concentration gradient $\nabla n$. The following equation is the result.

$$\vec{\Gamma} = -D\nabla n \; . \tag{2-22}$$

The above equation states that $\vec{\Gamma}$ is proportional to the gradient of the density n and is in the same direction. The proportionality constant D, called the coefficient of diffusion, is defined by the above equation. The implication here is that D is a scalar quantity. However, when there is a magnetic field present, the flow can be anisotropic, i.e., a component of flow can arise due to the presence of a gradient in a direction perpendicular to it. The coefficient D is no longer a simple quantity, but in general a tensor. Equation (2-22) then becomes:

$$\vec{\Gamma} = \vec{i}_x(D_{xx}\frac{\partial n}{\partial x} + D_{xy}\frac{\partial n}{\partial y} + D_{xz}\frac{\partial n}{\partial z})$$

$$+ \vec{i}_y(D_{yx}\frac{\partial n}{\partial x} + D_{yy}\frac{\partial n}{\partial y} + D_{yz}\frac{\partial n}{\partial z}) \qquad (2\text{-}23)$$

$$+ \vec{i}_z(D_{zx}\frac{\partial n}{\partial x} + D_{zy}\frac{\partial n}{\partial y} + D_{zz}\frac{\partial n}{\partial z}) .$$

Later in the development of the theory, we shall study in greater detail the character of the diffusion tensor. For the present, we shall merely demonstrate by elementary means that a component of flow does exist perpendicular to the magnetic field and the gradient. Let us consider the motion of an electron in a magnetic field alone. The direction of the magnetic field is taken along the z axis as before. Equation (2-7) the reduces to a simple form since E, the electric field is taken to be zero.

$$v_x = v_{ox} \cos \omega_b t - v_{oy} \sin \omega_b t$$

$$v_y = v_{ox} \sin \omega_b t - v_{oy} \cos \omega_b t \qquad (2\text{-}24)$$

$$v_z = v_{oz} .$$

We shall assume that concentration gradients exist along the direction of the axes. If we evaluate the averages of $v_x$, $v_y$

and $v_z$ as before, it is necessary to retain the averages of the velocity components $v_{ox}$, $v_{oy}$ and $v_{oz}$, since they now represent the drift components. Previously in deriving the average energy gain we were able to neglect $\bar{v}_{ox}$, $\bar{v}_{oy}$ and $\bar{v}_{oz}$ because they appeared together with the mean square values $\overline{v_{ox}^2}$ etc. which are an order of magnitude larger. If, in addition, we average over the collision time t, we arrive at the result:

$$\bar{v}_x = \bar{v}_{ox} \frac{1}{1 + \omega_b^2/\nu_c^2} + \bar{v}_{oy} \frac{\omega_b/\nu_c}{1 + \omega_b^2/\nu_c^2}$$

$$\bar{v}_y = \bar{v}_{oy} \frac{1}{1 + \omega_b^2/\nu_c^2} - \bar{v}_{ox} \frac{\omega_b/\nu_c}{1 + \omega_b^2/\nu_c^2} \qquad (2\text{-}25)$$

$$\bar{v}_z = \bar{v}_{oz} \ .$$

These quantities represent the drift velocities created by the diffusion current, given by:

$$\Gamma_x = nv_x \qquad \Gamma_y = nv_y \qquad \Gamma_z = nv_z \ . \qquad (2\text{-}26)$$

If no magnetic field is applied, $\omega_b = 0$ and $v_x = v_{ox}$, $v_y = v_{oy}$. These are the net components of the random velocity which exist by virtue of the gradient alone. However, when $\omega_b \neq 0$, these direct diffusion velocity components are reduced by the factor $1 + \omega_b^2/\nu_c^2$ and in addition, transverse components are set up due to the gradient velocity in the perpendicular direction, and these are prefixed by the factor $\omega_b/\nu_c/(1 + \omega_b^2/\nu_c^2)$. The current in the direction of the magnetic field is unaffected. These results are derived by Chapman and Cowling[13] on a more elaborate basis, using a simple statistical approach similar to the one given here.

By taking one of the axes along the direction of the magnetic field, we have simplified our tensor. The comparison of Eqs. (2-25) and (2-26) to (2-23) leads to the result that the diffusion tensor $D_{ij}(i,j = x,y,z)$ is given by:

$$D_{ij} = \begin{pmatrix} \frac{D}{1 + \omega_b^2/\nu_c^2} & \frac{D\,\omega_b/\nu_c}{1 + \omega_b^2/\nu_c^2} & 0 \\ -\frac{D\,\omega_b/\nu_c}{1 + \omega_b^2/\nu_c^2} & \frac{D}{1 + \omega_b^2/\nu_c^2} & 0 \\ 0 & 0 & D \end{pmatrix}. \qquad (2\text{-}27)$$

This result will be derived more rigorously later. The above result shows that $D_{xx} = D_{yy}$ and $D_{xy} = -D_{yx}$. Hence our flow equation now becomes:

$$\vec{\Gamma} = - \vec{i}_x (D_{xx} \frac{\partial n}{\partial x} + D_{xy} \frac{\partial n}{\partial y}) - \vec{i}_y (D_{yx} \frac{\partial n}{\partial x} + D_{yy} \frac{\partial n}{\partial y}) - \vec{i}_z D_{zz} \frac{\partial n}{\partial z} . \tag{2-28}$$

Let us substitute this in the continuity equation which states that:

$$\frac{\partial n}{\partial t} = - \operatorname{div} \vec{\Gamma} + P , \tag{2-29}$$

where P is the production rate. Further simplifications can be introduced by considering only the steady state problem so that $\frac{\partial n}{\partial t} = 0$ and by setting $P = \nu_i n$, $\nu_i$ being the rate of production per particle. With these simplifications and by (2-28), Eq. (2-29) becomes:

$$D_{xx} \frac{\partial^2 n}{\partial x^2} + D_{xy} \frac{\partial^2 n}{\partial x \partial y} + D_{yx} \frac{\partial^2 n}{\partial y \partial x} + D_{yy} \frac{\partial^2 n}{\partial y^2} + D_{zz} \frac{\partial^2 n}{\partial z^2} + \nu_i n = 0 . \tag{2-30}$$

Since $D_{xy} = - D_{yx}$ and $D_{xx} = D_{yy} = D_m$ where

$D_m = D/(1 + \omega_b^2/\nu_c^2)$:

$$D_m\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) + D\frac{\partial^2 n}{\partial z^2} + \nu_i n = 0 \ . \qquad (2\text{-}31)$$

This is the diffusion equation when the coordinate axes are taken along the direction of magnetic field.

Random Walk Theory

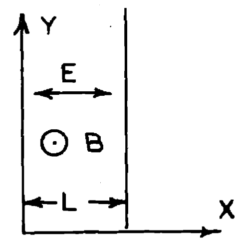
Before applying the diffusion equation to our problem, let us consider an alternative approach to the diffusion phenomenon, namely, that of the random walk. This is essentially the problem of the Brownian motion in which we consider a particle through many successive collisions. The usual procedure is to evaluate the mean square distance travelled in a given direction and relate it to the diffusion coefficient. The details of this development are given by Kennard.[14] The relationship that interests us is that

$$\overline{R^2} = kDt \ . \qquad (2\text{-}32)$$

R is the root mean square distance travelled by the particle, t the time elapsed. The quantity k = 2, 4,6 depending on whether R is a one, two or three dimensional distance. At any rate the above equation can be used to define D.

If instead of studying one electron through many collisions, we considered a large number of equal velocities randomly distributed in space and in the time between collisions, mathematically the problems would be identical. Since the time can be shown to be any value, we shall take it to be $t_c$, the mean time between collisions of the electrons with the gas molecules. For convenience, we shall choose the simplest physical situation, namely, an infinite parallel plate with the electric field perpendicular and the magnetic field transverse to both. The axes are chosen as shown in Fig. II-7. Since the

Fig. II-7. Orientation of $\vec{E}$ and $\vec{B}$ between two parallel plates.




gradient exists only along the X direction, Eq. (2-32) becomes:

$$D = \frac{\overline{x^2}}{2t_c} = \frac{\overline{x^2}}{2} \nu_c \, . \tag{2-33}$$

The procedure now is to solve for the distance x at any time t after the initial collision, assuming that x = 0 when t = 0, and then average $x^2$ over all values of $v_{ox}$, $v_{oy}$ from $-v_o$ to $v_o$ and t from 0 to $\infty$ as before. To do this, we integrate Eq. (2-7) for $v_x$ between the limits indicated. We omit the terms involving the electric fields since we are interested in the diffusion effects of the magnetic field alone. The result of the integration yields:

$$x = \frac{v_{ox}}{\omega_b} \sin \omega_b t + \frac{v_{ox}}{\omega_b} (\cos \omega_b t - 1) \; . \tag{2-34}$$

Upon squaring this expression, we obtain

$$x^2 = \frac{v_{ox}^2}{\omega_b^2} \sin^2 \omega_b t + \frac{v_{oy}^2}{\omega_b^2} (\cos^2 \omega_b t - 2 \cos \omega_b t + 1)$$

$$+ \frac{2 v_{ox} v_{oy}}{\omega_b^2} \sin \omega_b t (\cos \omega_b t - 1) \; . \tag{2-35}$$

Assuming that the average drift velocities $\overline{v}_{ox}$, $\overline{v}_{oy}$ are small compared to the mean square values $\overline{v_{ox}^2}$ and $\overline{v_{oy}^2}$, the term involving the cross product $\overline{v}_{ox} \times \overline{v}_{oy}$ can be neglected when we average over velocity components. Also since $\overline{v_{ox}^2} = \overline{v_{ox}^2} = \overline{v_{oz}^2} = \frac{\overline{v_o^2}}{3}$, Eq. (2-35) simplifies as follows:

$$\overline{x^2} = \frac{2\overline{v_o^2}}{3\omega_b^2} (1 - \cos \omega_b t) \ . \tag{2-36}$$

If we average $\cos \omega_b t$ by integrating, as in Eq. (2-12), we obtain:

$$\overline{x^2} = \frac{2\overline{v_o^2}}{3\omega_b^2} \left(1 - \frac{\nu_c^2}{\nu_c^2 + \omega_b^2}\right) = \frac{2\overline{v_o^2}}{3(\nu_c^2 + \omega_b^2)} \ . \tag{2-37}$$

This represents the mean square distance travelled in the direction perpendicular to the magnetic field. We could have solved the problem in two dimensions in the plane at right angles and would have arrived at the same result. Substituting the above value in Eq. (2-33)

$$D_m = \frac{\overline{v_o^2}}{3\nu_c} \frac{1}{(1 + \omega_b^2/\nu_c^2)} = D \frac{1}{1 + \omega_b^2/\nu_c^2} \ . \tag{2-38}$$

The interpretation of this last expression is that the diffusion coefficient and hence the direct diffusion current is reduced by the magnetic field. Physically, this means that the electrons are spiraling more and more tightly around the initial position as the magnetic field is increased. The lower the pressure, the greater the length of the spiral and the reduction is therefore greater at lower pressures. At very high pressures when $\nu_c \gg \omega_b$, the effect is negligible, since the mean free path is small and nearly a straight line. The alternative consideration of the random walk is that the number of steps required to go a root mean square distance is increased by the magnetic field by the factor $(1 + \omega_b^2/\nu_c^2)$.

Let us now return to the diffusion equation (2-31) and apply it to the one dimensional problem of the parallel plates. The equation becomes:

$$D_m \frac{d^2 n}{dx^2} + \nu_{im} n = 0 \; . \qquad (2\text{-}39)$$

The subscript m is added to production rate $\nu_i$ to indicate that it depends on the magnetic field. When the equation is solved and the boundary conditions, n = 0 at the walls, is applied, we get:

$$n = n_o \sin \frac{\pi x}{L} \; . \qquad (2\text{-}40)$$

The higher order solutions are ignored since they imply negative values of the density, which are physically impossible.

Upon putting (2-40) into the previous equation, we obtain:

$$\frac{\nu_{im}}{D_m} = \frac{\pi^2}{L^2} = \frac{1}{\Lambda^2} \ . \qquad (2\text{-}41)$$

$\Lambda$ is called the characteristic diffusion length of the parallel plates and is equal to $L/\pi$ where L is the distance between the plates. This means that the ratio of the production rate and the diffusion coefficient is invariant, or solving for $\nu_{im}$:

$$\nu_{im} = \frac{D_m}{\Lambda^2} = \frac{D}{\Lambda^2(1 + \omega_b^2/\nu_c^2)} = \frac{D}{\Lambda_m^2} \ . \qquad (2\text{-}42)$$

The above states that the production rate is reduced by the same factors as the diffusion. Since the loss of electrons to the walls is less with a magnetic field than without it, the number required to replace them is reduced. This equation is the one to be satisfied for breakdown, which is defined as the equilibrium condition between the production and removal of electrons.

$\Lambda_m$ is an effective diffusion length which corresponds to an expanded physical distance if the diffusion coefficient remains unaltered. Thus if the mean free path were a straight line, the new length $\Lambda_m$ would be a measure of the actual number of random steps required by an electron to diffuse to the walls. From kinetic theory[15] N, the number of steps is:

$$N = \frac{3\Lambda^2}{\ell^2} \quad \text{or} \quad N_m = \frac{3\Lambda_m^2}{\ell^2}$$

(2-43)

$$\frac{N_m}{N} = 1 + \omega_b^2/\nu_c^2 \ .$$

But this is the same number of steps necessary to raise an electron from zero energy to the ionizing potential. Perhaps a better way of stating this is that while the electron diffuses to the wall another electron is suffering the same number of collisions and gaining an average energy $u_e$ at each collision. But the energy needed to create an electron and to replace the one lost to diffusion is $u_i$ the ionizing potential. This energy must be supplied during the diffusion period. Hence

$$N = \frac{u_i}{u_{eo}} \quad \text{and} \quad N_m = \frac{u_i}{u_{em}}$$

(2-44)

$$\frac{N_m}{N} = \frac{u_{eo}}{u_{em}} \ .$$

In Eq. (2-20) we showed the value of this ratio. Using that result and combining (2-44) and (2-43), we get:

$$\frac{E_o^2}{1 + \omega^2/\nu_c^2} \; \frac{1 + \omega_m^2/\nu_c^2}{E_m^2} = 1 + \omega_b^2/\nu_c^2 \; . \qquad (2\text{-}45)$$

Solving for $E_m$:

$$E_m = \frac{E_o}{\sqrt{1 + \omega_b^2/\nu_c^2}} \sqrt{\frac{1 + \omega_m^2/\nu_c^2}{1 + \omega_b^2/\nu_c^2}} \; . \qquad (2\text{-}46)$$

This is the breakdown equation in which the breakdown field $E_m$ is expressed as a function of the magnetic field when referred to the non-magnetic breakdown field $E_o$. This equation is very similar to (2-21) except that the diffusion factor is included.

Before we plot the results of this equation, let us examine some of the assumptions made in deriving it. The first assumption was that $u_e$ or $u_{em}$ was constant at all velocities. This is only true if $\nu_c$ is independent of the velocity. The second assumption was that the only inelastic collisions were those at the ionizing potential $u_i$. In general, this is not true. However, if helium is mixed with the vapor pressure of mercury, we obtain a mixture which has been called "Heg" and very nearly satisfies this condition. In the region of about 20 volts, there are several metastable energy levels close together, whose excitation probabilities are high compared to the first resonance level in that region. As a consequence, the preponderant number of collisions in this region produce metastable atoms whose lifetimes are large. These eventually ionize the mercury atom by a collision of the second kind to produce electrons. In this collision, the matastable gives up its energy and resumes its normal ground state. The distribution of electrons falls very rapidly, exponentially, beyond these values. Hence it is reasonable to say that for Heg nearly all collisions at an effective potential $u_i$ result in ionization. For our simple picture, we shall assume that $u_i$ is not affected by the magnetic field but is a function of the pressure.

Figure II-8 shows a set of curves obtained by producing breakdown of helium in a cavity 2 7/8" diameter and 1/8" high. This is essentially a parallel plate system in which the discharge is confined to the center of the cavity where the field is nearly uniform and the diffusion loss occurs almost exclusively in the direction of the parallel walls. These curves should be compared to a similar set for helium shown in Fig. II-6 for a larger cavity in which the magnetic field did not control diffusion as completely. The result is that these curves droop more than the previous set for high magnetic field. In addition, the diffusion reduction also manifests itself at higher pressures where no rise is observed. Finally, in the region where the resonance dip is observed, at intermediate pressures there is a

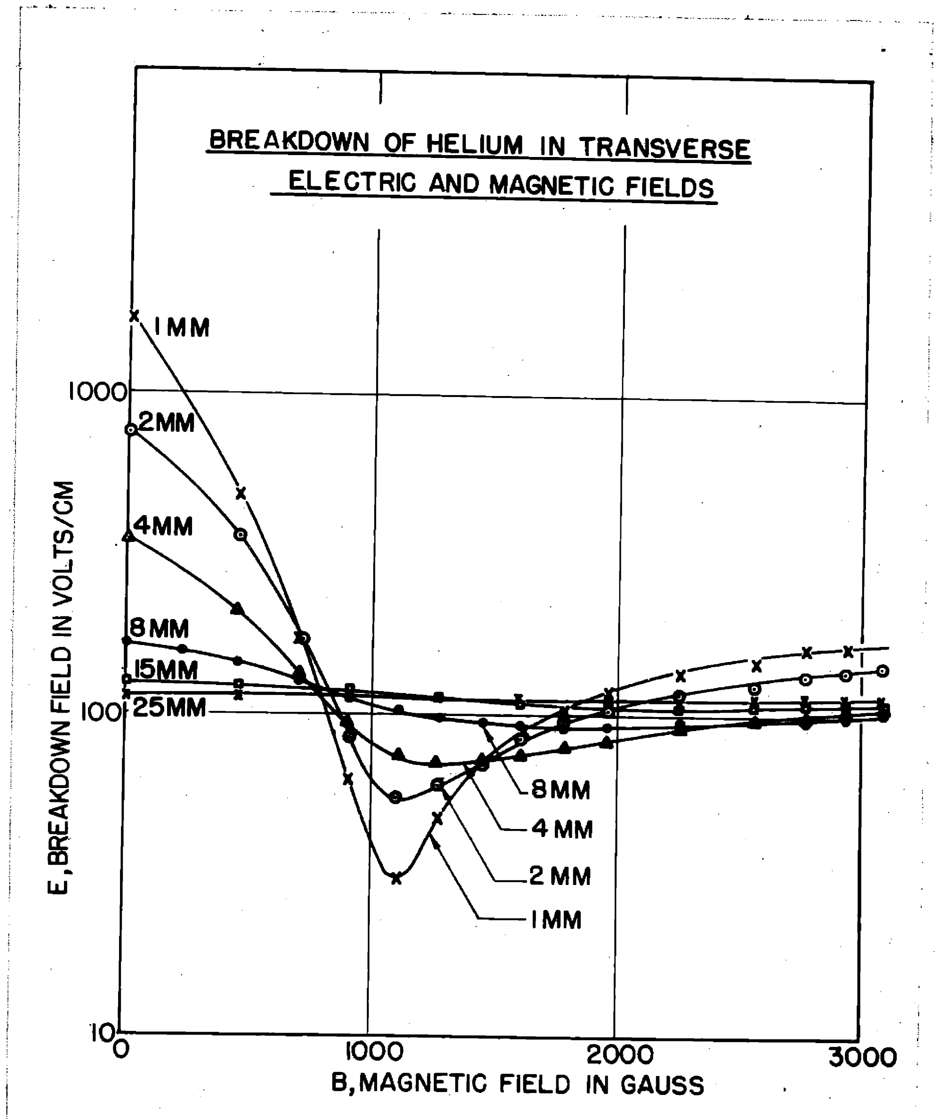



Fig. II-8. Breakdown of helium in transverse electric and magnetic fields in a flat cylindrical cavity (L = 0.318 cm, R = 3.66 cm).

shift in the minimum from $\omega_{b\,min} = \omega$ to the right. This can be easily calculated from Eq. (2-46) by minimizing $E_m$ with respect to $\omega_b^2$. When this is done, the result is that:

$$\omega_{b\,min} = \sqrt{\frac{(\nu_c^2 + \omega^2) + 2\sqrt{(4\nu_c^2 + \omega^2)(\nu_c^2 + \omega^2)}}{3}} \quad . \qquad (2\text{-}47)$$

In the region where the minima are observable, $\nu_c < \omega$ and we can expand this expression by the binomial theorem. Neglecting terms higher order than $\nu_c^2/\omega^2$, we arrive at the result:

$$\omega_{b\,min} \approx \omega + \frac{5}{4}\frac{\nu_c^2}{\omega} \quad . \qquad (2\text{-}48)$$

This expression agrees very well with the experimental results for values of pressure less than 4 mm.

Another consequence of our simple theory is that for very large values of $\omega_b/\nu_c$ the breakdown field approaches a limiting value. This can be seen readily from the graphs. However, if we calculate this limiting value from (2-46), we get the expression:

$$E_{m\infty} = \frac{E_o}{\sqrt{1 + \omega^2/\nu_c^2}} \quad . \qquad (2\text{-}49)$$

For p = 1 mm, this asymptotic value is 200 v/cm and for p = 2 mm is about 185 v/cm as calculated from (2-49).

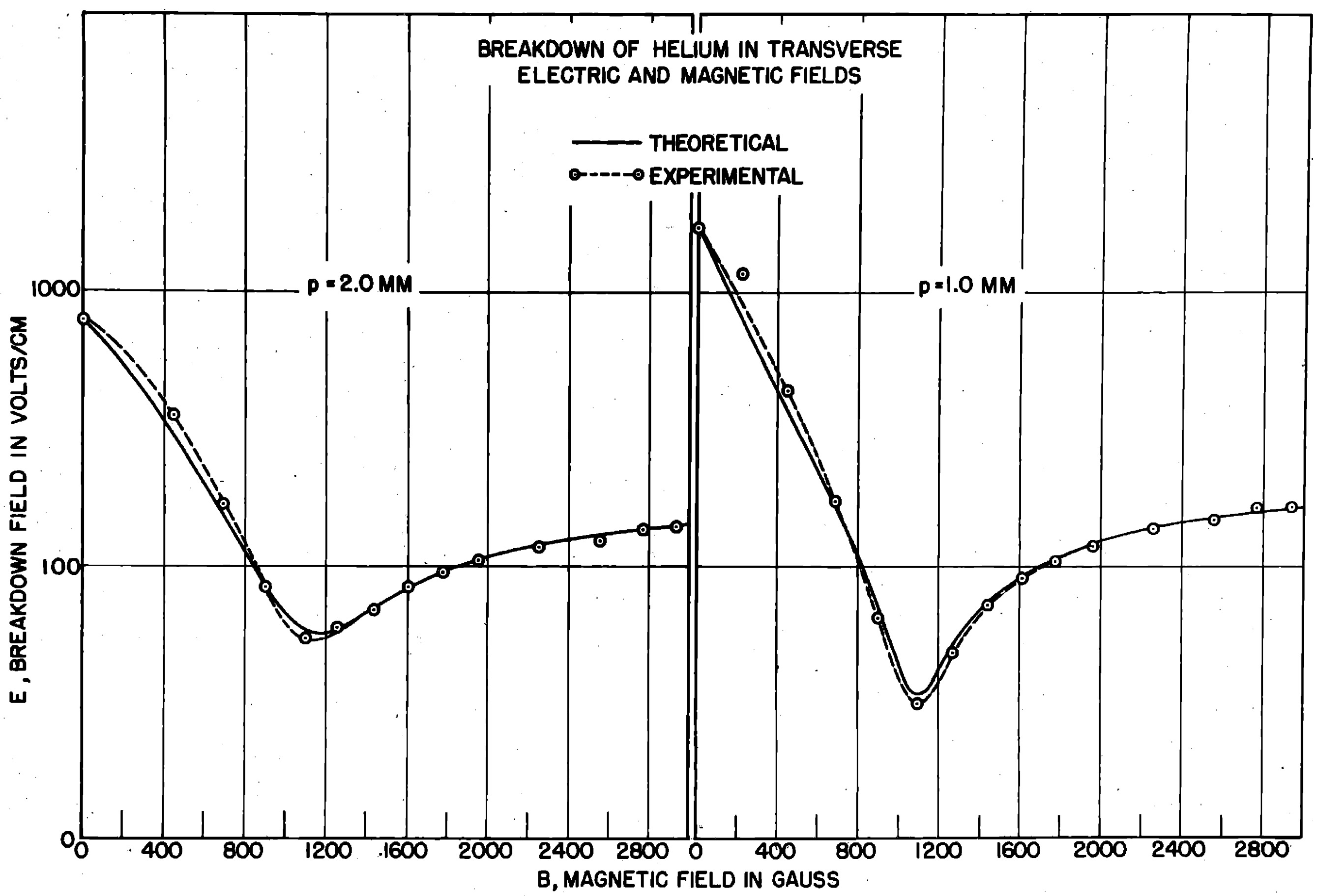



Fig. II-9. Breakdown of helium in transverse $\vec{E}$ and $\vec{B}$ in a flat cylindrical cavity (L = 0.318 cm, R = 3.66 cm). The theoretical curves are those for two infinite parallel plates.

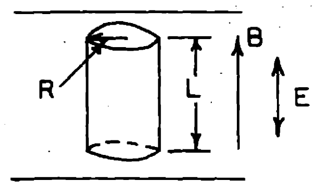
Taking the last two sets of curves we have redrawn them in Fig. II-9 together with the theoretical curves obtained from Eq. (2-46). The solid lines represent the theory and the points are experimental.

Parallel Magnetic and Electric Fields

In order to study the diffusion effects of the magnetic field without considering any other phenomena, the simplest arrangement of the electric and magnetic fields is a parallel one. The motion of a free electron in the parallel direction is dependent only on the electric field and in the perpendicular direction only on the magnetic field. If we considered breakdown between infinite parallel plates, no effect would be observed, since there is no gradient perpendicular to the magnetic field. However, if a glass container of finite dimensions was placed between the parallel plates there would be a reduction in the breakdown field as B is increased. Such an arrangement is shown in Fig. II-10 where B and E are taken in the Z direction and a cylindrical container with its axis also in this direction. L and R are the height and radius, respectively, and represent comparable quantities.

The first step in the solution is to take the diffusion Eq. (2-31) and transform it into cylindrical coordinates. This is done by taking the Laplacian in x and y and substituting its value in r and $\theta$, the polar coordinates. Since the problem is

Fig. II-10. Cylindrical flask between parallel plates with $\vec{E}$ and $\vec{B}$ parallel.




symmetrical in θ, there is no need to include terms involving it. The expression then becomes:

$$D_m \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr}\right) + D \frac{d^2 n}{dz^2} + \nu_i n = 0 \; . \qquad (2\text{-}50)$$

The solution of this equation is obtained by the usual separation of variables and the application of the boundary conditions n = 0 at the walls. As before, we do not admit any but the first modes of electron distributions because higher modes imply negative densities, which is physically impossible. The final result is the following:

$$n = n_o J_o\left(\frac{2.405\, r}{R}\right) \sin \frac{\pi z}{L} \; . \qquad (2\text{-}51)$$

Substituting this in the preceding equation and solving for $\nu_i/D$, we get:

$$\frac{\nu_i}{D} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{2.405}{R}\right)^2 \frac{1}{1 + \omega_b^2/\nu_c^2} = \frac{1}{\Lambda_m^2} . \qquad (2\text{-}52)$$

$\Lambda_m$ is now defined as the new effective diffusion length and includes the mathematical dilation of the radial direction by the magnetic field. For large values of $\omega_b/\nu_c$ the discharge is effectively between parallel plates because diffusion sideways is negligible and $\Lambda_e = L/\pi$.

Repeating the arguments used in the calculation of breakdown by the random walk theory, we can show that:

$$\frac{N_m}{N} = \frac{\Lambda_m^2}{\Lambda_o^2} = \frac{E_o^2}{E_m^2} . \qquad (2\text{-}53)$$

$\Lambda_o$ represents the diffusion length when $\omega_b = 0$ and $E_o$ is the corresponding electric field. Solving for $E_m$ and substituting its value from (2-52) the result is:

$$E_m = E_o \frac{\Lambda_o}{\Lambda_m} . \qquad (2\text{-}54)$$

The plot of this equation is shown for helium at 3000 mc in Fig. II-11. The family of curves are for different values of the pressure. The cavity dimensions were taken as L = 4R. The breakdown field is reduced by the magnetic field, more steeply at lower pressures. For large magnetic fields, the value of $E_m$ approaches a limiting value which is given by

$$E_{\infty} = E_o \frac{\Lambda_o}{\Lambda_z} \qquad \frac{1}{\Lambda_o} = \sqrt{\left(\frac{\pi}{L}\right)^2 + \left(\frac{2.405}{R}\right)^2} \qquad \Lambda_z = \frac{L}{\pi} \; . \qquad (2\text{-}55)$$

Actually we cannot perform this experiment in the laboratory as simply as indicated because breakdown measurements in a glass bottle are not accurate. What was done was to take a cylindrical metal cavity whose dimensions were comparable and perform this experiment with it. The only drawback was that the electric field inside the cavity was not uniform. The analysis developed so far in its present form, therefore, is not applicable. Consequently, we shall delay the discussion until the theory of non-uniform fields is developed.

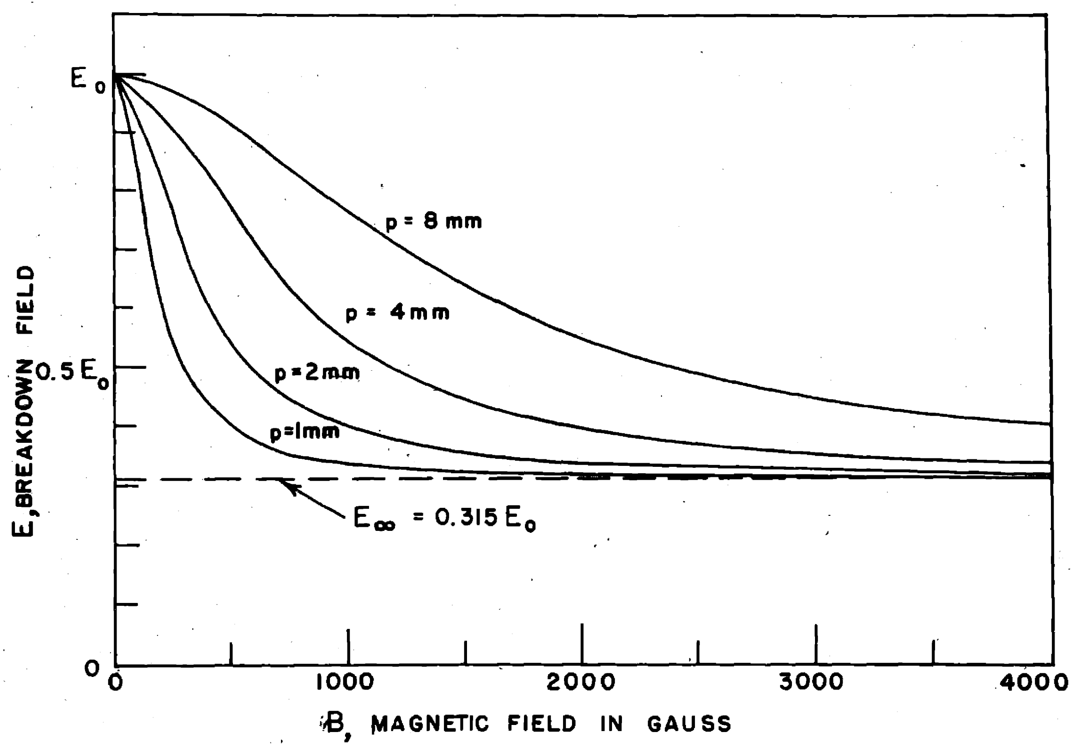



Fig. II-11. Theoretical curves of breakdown of helium in a cylindrical glass flask for which L = 4R, with $\vec{E}$ parallel to $\vec{B}$. $E_o$ is the non-magnetic breakdown field and $E_\infty$ is the parallel plate breakdown field.

## III. DISTRIBUTION THEORY OF ELECTRONS

In the previous section, the behavior of electrons was treated in terms of a single electron which was a statistical representative of the whole assembly. The distribution of electrons in terms of velocity was not computed, although the spatial variation in density was developed in the discussion of diffusion. This section will be devoted to the development of a distribution function which will describe the relative concentration of electrons in velocity, space and time. The latter dependence will be treated only for high frequencies. The inclusion of the magnetic field in the derivation will make it possible to generalize the physical concepts of the preceding section and to put them on a more rigorous basis.

The development is along the general lines of Morse, Allis and Lamar[8] with the time factor as included by Margenau.[10] The fundamental starting point is the Boltzmann transport equation, which is a continuity equation in phase space. The distribution function which satisfies this equation is expanded in spherical harmonics in velocity space. Only the zero and the first order terms are retained on the assumption that the perturbation of the spherical distribution is small and adequately represented by the first order term. The Boltzmann equation is similarly expanded.

The time dependence is taken into account by a Fourier expansion of the distribution function, retaining terms to the zero and first order. Finally, the space dependence is accounted for by the solution of the diffusion equation. Eventually the distribution function is reduced to the solution of a second order differential equation in terms of the velocity. The solution of this equation is used in evaluating the ionization rate $\nu_i$ and the diffusion coefficient D (or $D_m$), whose ratio is then equated to $1/\Lambda^2$ to yield a transcendental equation for the breakdown field.

Spherical Harmonic Expansion

The distribution function $F(x,y,z,v_x,v_y,v_z,t)$ is defined as the density of electrons per unit volume of phase space at any given time.

$$F = \frac{d^6N}{d^3s\ d^3v} \qquad \begin{aligned} d^3s &= dx\ dy\ dz \\ d^3v &= dv_x\ dv_y\ dv_z\ . \end{aligned} \tag{3-1}$$

The spherical harmonic expansion of F in velocity space is done by transforming the Cartesian coordinates $v_x$, $v_y$, $v_z$ to the spherical v, θ, ∅.

Then

$$F = \sum F_{nm}(x,y,z,t)Y_n^m(\theta,\phi) \quad , \tag{3-2}$$

where

$$Y_n^{\pm m}(\theta,\phi) = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\phi \, P_n^m(\cos\,\theta) \quad . \tag{3-3}$$

Since we are neglecting all but the zero and first order harmonics, we should give their values, which are:

$$\begin{aligned} Y_0^0 &= 1 & Y_1^{-1} &= \cos\,\phi\,\sin\,\theta = \frac{v_x}{v} \\ Y_1^0 &= \cos\,\theta = \frac{v_z}{v} & Y_1^1 &= \sin\,\phi\,\sin\,\theta = \frac{v_y}{v} \; . \end{aligned} \tag{3-4}$$

When these are substituted in (3-2),

$$F = F_0 + F_{1-1}\frac{v_x}{v} + F_{11}\frac{v_x}{v} + F_{10}\frac{v_z}{v} + \ldots$$

$$= F_0 + \vec{F}_1 \cdot \frac{\vec{v}}{v} + \ldots \quad , \tag{3-5}$$

where $\vec{F}_1$ represents a vector quantity whose components are $F_{1-1}$, $F_{11}$ and $F_{10}$ and $\vec{v}$ is the velocity vector.

The physical significance to these two terms lies in the following equations:

$$n = \int_0^{\infty} F_0 4\pi v^2 dv \tag{3-6}$$

$$\vec{\Gamma} = \int_0^{\infty} \vec{F}_1 \frac{4\pi}{3} v^3 dv \ . \tag{3-7}$$

Thus the zero order term in the distribution function yields n, density per unit volume, while $\vec{F}_1$ term gives the current vector $\vec{\Gamma}$ in ordinary space.

The continuity equation in six dimensional phase space can be written as:

$$P = \frac{\partial F}{\partial t} + \nabla \cdot (\vec{v}F) + \nabla_v \cdot (\vec{a}F) \ . \tag{3-8}$$

This means that the total production rate per unit volume in phase space is the sum of the time variation in F, the density in phase space, plus the divergence of the current in ordinary or configuration space and the current in velocity space. This is the Boltzmann transport equation. It can be rewritten in simpler form in our case by replacing the expression for $\vec{a}$, the acceleration, by

$$\vec{a} = -\frac{e}{m}(\vec{E} + \vec{v} \times \vec{B}) \qquad (3\text{-}9)$$

setting

$$\nabla \cdot \vec{v} = 0 \quad \text{and} \quad \nabla_v \cdot \vec{E} = 0 \,, \qquad (3\text{-}10)$$

because $\vec{v}$ is independent of x,y,z and $\vec{E}$ is independent of $v_x$, $v_y$, $v_z$. Also

$$\nabla_v \cdot \vec{v} \times \vec{B} = 0 \,. \qquad (3\text{-}11)$$

This can be easily shown by expansion, remembering that $\vec{B}$ is independent of velocity coordinates. When all this is done, the Boltzmann equation becomes:

$$P = \frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F + \vec{a} \cdot \nabla_v F \,. \qquad (3\text{-}12)$$

The value of F from Eq. (3-5) is substituted above and then each term is expanded separately in spherical harmonics. This procedure is a rather involved one and has been carried out by Allis[16] and Herlin with the magnetic term included. The result in terms of the electric and magnetic fields is:

$$P = \left[\frac{\partial F_0}{\partial t} + \frac{v}{3}\nabla\cdot\vec{F}_1 - \frac{e\vec{E}}{3mv^2}\cdot\frac{\partial(v^2\vec{F}_1)}{\partial v}\right] + \frac{\vec{v}}{v}\cdot\left[\frac{\partial\vec{F}_1}{\partial t} + v\nabla F_0 - \frac{e}{m}\vec{E}\frac{\partial F_0}{\partial v} - \frac{e}{m}\vec{B}\times\vec{F}_1\right] \tag{3-13}$$

Expanding P, the production term, in spherical harmonics, we have:

$$P = P_0 + \frac{\vec{v}}{v}\cdot\vec{P}_1 + \dots \quad . \tag{3-14}$$

Substituting this in (3-13) and equating the zero and first order terms:

$$P_0 = \frac{\partial F_0}{\partial t} + \frac{v}{3}\nabla\cdot\vec{F}_1 - \frac{e\vec{E}}{3mv^2}\cdot\frac{\partial(v^2\vec{F}_1)}{\partial v} \tag{3-15}$$

$$\vec{P}_1 = \frac{\partial\vec{F}_1}{\partial t} + v\nabla F_0 - \frac{e}{m}\vec{E}\frac{\partial F_0}{\partial v} - \frac{e}{m}\vec{B}\times\vec{F}_1 \; . \tag{3-16}$$

The expansion thus reduces to a zero order scalar equation and to a first order vector equation.

A further division of the production rate P will be made, namely:

$$P = P_{el} + P_{in} \; . \tag{3-17}$$

$P_{el}$ represents the production term due to elastic collisions. This term appears because the electrons collide elastically with the gas molecules and result in a shift of the number of electrons from one elementary volume of phase space into another. Morse, Allis and Lamar[8] have computed the zero and first order components of this term and obtained the following:

$$P_{0,el} = \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^4 F_0}{\ell}\right) \qquad (3\text{-}18)$$

$$P_{1,el} = -\frac{v}{\ell}F_1 \, , \qquad (3\text{-}19)$$

where M is the mass of the gas molecule and $\ell$ the mean free path of the electron.

The treatment of the inelastic term $P_{in}$ is not so simple, since it involves such processes as exciting and ionizing collisions, electron attachment and recombination. Consequently $P_{in}$ will be expanded in spherical harmonics and written in the scalar and vector equations together with the elastic terms as:

$$P_{0,in} + \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^4 F_0}{\ell}\right) = \frac{\partial F_0}{\partial t} + \frac{v}{3}\nabla\cdot\vec{F}_1 - \frac{e\vec{E}}{3mv^2}\cdot\frac{\partial(v^2\vec{F}_1)}{\partial v} \qquad (3\text{-}20)$$

$$\vec{P}_{1,in} - \frac{v}{\ell}\vec{F}_1 = \frac{\partial \vec{F}_1}{\partial t} + v\nabla F_0 - \frac{e}{m}\vec{E}\frac{\partial F_0}{\partial v} - \frac{e}{m}\vec{B}\times\vec{F}_1 \, . \qquad (3\text{-}21)$$

On physical grounds, it turns out that the $P_{1,in}$ term in the last equation is small compared to the other terms because ionization and excitation collisions occur at high velocities where the density is low. At the low velocities where attachment and recombination occur, the contribution is small because the probabilities of these processes are small. Hence we shall set $P_{1,in} = 0$. Another simplification will be to set $v/\ell = \nu_c$. The equations are then rewritten as:

$$[0] \quad P_{o,in} + \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}(\nu_c v^3 F_0) = \frac{\partial F_0}{\partial t} + \frac{v}{3}\nabla\cdot\vec{F}_1 - \frac{e}{3mv^2}\frac{\partial(v^2\vec{E}\cdot\vec{F}_1)}{\partial v} \qquad (3\text{-}22)$$

$$[1] \quad 0 = \nu_c\vec{F}_1 + \frac{\partial\vec{F}_1}{\partial t} + v\nabla F_0 - \frac{e}{m}\vec{E}\frac{\partial F_0}{\partial v} - \frac{e}{m}\vec{B}\times\vec{F}_1 \quad . \qquad (3\text{-}23)$$

Fourier Expansion

Let the applied electric field be a high frequency one which will result in the breakdown of the gas. The characteristic of breakdown is that it is an equilibrium condition. Consequently, we shall assume that the density is essentially independent of time but the current shall have a component oscillating at the same frequency as the electric field. All harmonics of higher order shall be neglected in the following expansions.

$$\vec{E} = \vec{E} \cos \omega t \qquad F_0 = F_0^0$$

$$\vec{F_1} = \vec{F_1^0} + \vec{F_1^1} \cos \omega t + \vec{G_1^1} \sin \omega t \ . \tag{3-24}$$

In accordance with the assumptions, $F_0$, which is a measure of the density, has no time component. $\vec{F_1}$ which is a measure of the current flow has a d.c. component together with an in phase $\vec{F_1^1}$ and a reactive $\vec{G_1^1}$ first harmonic components. The superscripts indicate the order of the Fourier components. Substituting these values in the preceding equations, we see that we have to evaluate $\vec{E} \cdot \vec{F_1}$. The product results in d.c. terms, a first harmonic and a second harmonic. The latter we shall ignore, since they equate to zero eventually. Then,

$$\vec{E} \cdot \vec{F_1} = \frac{1}{2} \vec{E} \cdot \vec{F_1^1} + \vec{E} \cdot \vec{F_1^0} \cos \omega t \ . \tag{3-25}$$

Let us substitute this value in Eqs. (3-22) and (3-23) together with those of (3-24), then we shall have two sets of equations with d.c. terms and terms in cos ωt and sin ωt. Equating these we should have six equations, three from the scalar and three from the vector equation. These are:

$$[0,0] \quad P_{0,in} + \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}(\nu_c v^3 F_0) = \frac{v}{3}\nabla\cdot \vec{F}_1^0 - \frac{e}{m}\frac{1}{3v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{2}\vec{E}\cdot\vec{F}_1^1\right) \qquad (3\text{-}26)$$

$$[0,1] \quad 0 = \frac{v}{3}\nabla\cdot \vec{F}_1^1 - \frac{e}{m}\frac{1}{3v^2}\frac{\partial}{\partial v}(v^2\vec{E}\cdot\vec{F}_1^0) \qquad (3\text{-}27)$$

$$0 = \frac{v}{3}\nabla\cdot \vec{G}_1^1 \qquad (3\text{-}28)$$

$$[1,0] \quad 0 = \nu_c\vec{F}_1^0 + v\nabla F_0 - \frac{e}{m}\vec{B}\times\vec{F}_1^0 \qquad (3\text{-}29)$$

$$[1,1] \quad 0 = \nu_c\vec{F}_1^1 + \omega\vec{G}_1^1 - \frac{e}{m}\vec{E}\frac{\partial F_0}{\partial v} - \frac{e}{m}\vec{B}\times\vec{F}_1^1 \qquad (3\text{-}30)$$

$$0 = \nu_c\vec{G}_1^1 - \omega\vec{F}_1^1 - \frac{e}{m}\vec{B}\times\vec{G}_1^1 \quad . \qquad (3\text{-}31)$$

It turns out that Eq. (3-27) and (3-28) are redundant and not needed for the solution of the problem. Actually we have four unknown quantities $F_0$, $\vec{F}_1^0$, $\vec{F}_1^1$ and $\vec{G}_1^1$. It is only necessary to eliminate the last three from Eq. (3-26), (3-29), (3-30) and (3-31) to obtain a single equation in $F_0$. This is the desired differential equation. We shall not do this for the general case, although it is possible by eliminating $\vec{G}_1^1$ from the last two equations and then substituting the value of $\vec{F}_1^1$ and $\vec{F}_1^0$ from the preceding equation in the $[0,0]$ equation to arrive at the result. This is too involved and unnecessary for our purposes. The only reason for retaining the general form up to this stage was to show the character of the diffusion tensor when E and B were arbitrarily disposed.

Diffusion Tensor

The manner in which we shall attack the problem of diffusion is to first integrate Eq. (3-26) over velocity space. Let us do this term by term as follows:

$$-\int_0^{\infty} P_{0,in} 4\pi v^2 dv = \nu_i n \ . \qquad (3\text{-}32)$$

The left hand side of the above equation represents the net total of electrons produced at a given point in space. This must then equal $\nu_i$ the net production rate per electron times n the electron density in configuration space.

$$\frac{m}{M} \int_0^{\infty} \frac{1}{v^2} \frac{\partial}{\partial v} (\nu_c v^3 F_0) 4\pi v^2 dv = \frac{4\pi m}{M} \Big[ \nu_c v^3 F_0 \Big]_0^{\infty}$$

$$= 0 \ . \qquad (3\text{-}33)$$

This is equal to zero, since $F_0 \rightarrow 0$ when $v \rightarrow \infty$ and $F_0$ is finite or at least $v^2F_0$ must be finite at v = 0. For similar arguments the last term in Eq. (3-26) vanishes.

$$\frac{e}{3m}\int_0^\infty \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{2}\vec{E}\cdot\vec{F}_1^1\right)4\pi v^2 dv = \frac{4\pi e}{6m}\left[v^2\vec{E}\cdot\vec{F}_1^1\right]_0^\infty = 0 \,. \tag{3-34}$$

Finally, we integrate the remaining term in the equation. This gives us an expression which is not zero, but

$$\int_0^\infty \frac{v}{3}\,\nabla\cdot\vec{F}_1^0 4\pi v^2 dv = \nabla\cdot\vec{\Gamma} \,. \tag{3-35}$$

This must be the case since by the continuity equation for the steady state $\nu_i n = \mathrm{div}\,\vec{\Gamma}$ and in Eq. (3-35) the divergence can be taken outside the integral.

$$\vec{\Gamma} = \int_0^\infty \frac{4\pi}{3}\vec{F}_1^0 v^3 dv \,. \tag{3-36}$$

From Eq. (2-23), this is equal to a vector involving the diffusion tensor $D_{ij}$. Hence it is now possible to define this tensor, i.e., its components in terms of the distribution function. To do this it is necessary to solve $\vec{F}_1^0$ in terms of $F_0$ by the use of Eq. (3-29). For simplification, we shall rewrite the vector:

$$\frac{e\vec{B}}{m} = \vec{\omega}_b \quad \text{or} \quad \frac{eB_x}{m} = \omega_{bx} \quad \frac{eB_y}{m} = \omega_{by} \quad \frac{eB_z}{m} = \omega_{bz} \,. \tag{3-37}$$

$$\nu_c \vec{F}_1^0 - \vec{\omega}_b \times \vec{F}_1^0 = - v \nabla F_0 \,. \tag{3-38}$$

Multiplying this expression vectorially by $\vec{\omega}_b$ and writing the expansion for the triple vector product and substituting for $\vec{\omega}_b \times \vec{F}_1^0$ from (3-38):

$$\nu_c \vec{\omega}_b \times \vec{F}_1^0 - \vec{\omega}_b \times \vec{\omega}_b \times \vec{F}_1^0 = \nu_c(\nu_c \vec{F}_1^0 + v\nabla F_0) + \omega_b^2 \vec{F}_1^0 - (\vec{\omega}_b \cdot \vec{F}_1^0)\vec{\omega}_b = - v\vec{\omega}_b \times \nabla F_0 \tag{3-39}$$

If we multiply Eq. (3-38) scalarly by $\vec{\omega}_b$, we find:

$$\vec{\omega}_b \cdot \vec{F}_1^0 = - \frac{v}{\nu_c} \vec{\omega}_b \cdot \nabla F_0 \tag{3-40}$$

since $\quad \vec{\omega}_b \cdot \vec{\omega}_b \cdot \vec{F}_1^0 = 0 \,.$

Substituting the above value (3-40) in (3-39) and solving for $\vec{F}_1^0$ we get:

$$\vec{F}_1^0 = - \frac{v\left[\nu_c + \frac{\vec{\omega}_b}{\nu_c}\vec{\omega}_b \cdot + \vec{\omega}_b \times\right]}{\nu_c^2 + \omega_b^2} \nabla F_0 \,. \tag{3-41}$$

The above equation provides a great deal of physical information. It states that if the magnetic field and the concentration gradient make an arbitrary angle with each other there are effectively three components of current set up due to the gradient. The first is in the direction of the gradient and represents the diffusion current. This component is reduced from the non-magnetic case by the factor $1 + \omega_b^2/\nu_c^2$. The second component is in the direction of the magnetic field and this is reduced also but the factor is not so simply expressed. Finally, there is the transverse magnetic current which is perpendicular to both the magnetic vector and the concentration gradient. In general this expression is a fairly complicated one and should be considered by expanding (3-41). First, however, it is interesting to consider some special cases.

The simplest case is the one in which the magnetic field and the gradient are parallel to each other. Physically this would be the situation for two infinite parallel planes with the magnetic field perpendicular. Under such conditions, the vector product would vanish and the scalar product would simply be the product of the magnitudes. Since $\vec{\omega}_b$ is parallel to $\nabla F_0$ then:

$$\vec{F}_1^0 = - \frac{v\left(\nu_c + \frac{\omega_b^2}{\nu_c}\right)}{\nu_c^2 + \omega_b^2} \nabla F_0 = - \frac{v}{\nu_c} \nabla F_0 \quad . \tag{3-42}$$

This means that the diffusion current is unaffected by the magnetic field.

If $\omega_b$ is perpendicular to $\nabla F_0$, the scalar product of the two vanishes. In this case, the current is composed of the direct and transverse diffusion currents both of which are perpendicular to the magnetic field. The direct current is reduced by the factor $1 + \omega_b^2/\nu_c^2$ while the transverse current is reduced by $\omega_b/\nu_c/(1 + \omega_b^2/\nu_c^2)$ as compared to the non-magnetic diffusion current.

At this point, it is logical to introduce a new notation. Heretofore we implied that $F_0$ represented a function of velocity and space in which the functions were separable. We can then write:

$$F_0 = n(x,y,z)f_0(v) \ . \tag{3-43}$$

$f_0(v)$ is a normalized function since

$$n = \int_0^\infty F_0 4\pi v^2 dv = n \int_0^\infty f_0(v) 4\pi v^2 dv$$

or

$$\int_0^\infty f_0(v) 4\pi v^2 dv = 1 \ . \tag{3-44}$$

Expanding Eq. (3-41) in Cartesian coordinates we get

$$\vec{F}_1^0 = -\frac{\frac{v}{\nu_c} f_0}{\nu_c^2 + \omega_b^2}\left[\vec{i}_x\left\{(1 + \frac{\omega_{bx}^2}{\nu_c^2})\frac{\partial n}{\partial x} + (\frac{\omega_{bx}\omega_{by}}{\nu_c^2} - \frac{\omega_{bz}}{\nu_c})\frac{\partial n}{\partial y}\right.\right.$$

$$\left. + (\frac{\omega_{bx}\omega_{bz}}{\nu_c^2} + \frac{\omega_{by}}{\nu_c})\frac{\partial n}{\partial z}\right\}$$

$$+ \vec{i}_y\left\{(\frac{\omega_{by}\omega_{bx}}{\nu_c^2} + \frac{\omega_{bz}}{\nu_c})\frac{\partial n}{\partial x} + (1 + \frac{\omega_{by}^2}{\nu_c^2})\frac{\partial n}{\partial y} + (\frac{\omega_{by}\omega_{bz}}{\nu_c^2} - \frac{\omega_{bx}}{\nu_c})\frac{\partial n}{\partial z}\right\}$$

$$\left. + \vec{i}_z\left\{(\frac{\omega_{bx}\omega_{bz}}{\nu_c^2} - \frac{\omega_{by}}{\nu_c})\frac{\partial n}{\partial x} + (\frac{\omega_{by}\omega_{bz}}{\nu_c^2} + \frac{\omega_{bx}}{\nu_c})\frac{\partial n}{\partial y} + (1 + \frac{\omega_{bz}^2}{\nu_c^2})\frac{\partial n}{\partial z}\right\}\right] \quad (3\text{-}45)$$

Using this together with Eqs. (3-36) and (2-23) which define $\vec{\Gamma}$ in terms of $\vec{F}_1^0$ and the diffusion tensor, respectively, we can write the expression for the diffusion tensor. We shall simplify our notation a little more by writing $b_x = \omega_{bx}/\nu_c$, $b_y = \omega_{by}/\nu_c$, $b_z = \omega_{bz}/\nu_c$ and $b^2 = b_x^2 + b_y^2 + b_z^2$. Also setting $v/\nu_c = \ell$ , we get

$$D_{ij} = \int_0^\infty \frac{\ell\, v}{3(1 + b^2)} \begin{pmatrix} (1 + b_x^2) & (b_x b_y - b_z) & (b_x b_z + b_y) \\ (b_x b_y + b_z) & (1 + b_y^2) & (b_y b_z - b_x) \\ (b_x b_z - b_y) & (b_y b_z + b_x) & (1 + b_z^2) \end{pmatrix} 4\pi v^2 f_0 dv \quad (3\text{-}46)$$

Thus if there is no magnetic field $b_x = b_y = b_z = 0$ and we have an isotropic medium as far as diffusion is concerned, i.e., all the diagonal terms are equal and the others are zero. Then

$$D = \frac{\overline{\ell v}}{3} = \int_0^\infty \frac{\ell v}{3} \times 4\pi v^2 f_0 dv \ . \qquad (3\text{-}47)$$

This is the ordinary diffusion coefficient encountered in the literature.

Ordinarily one simplifies the diffusion tensor by selecting the coordinates with one axis in the direction of the magnetic field. Let $b_x = b_y = 0$ and $b_z = b$.

$$D_{ij} = \int_0^\infty \frac{\ell v}{3(1 + b^2)} \begin{pmatrix} 1 & -b & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 + b^2 \end{pmatrix} 4\pi v^2 f_0 dv \ . \qquad (3\text{-}48)$$

Then $D_{xx} = D_{yy} = D_m$ and $D_{zz} = D$ as previously defined. Also $D_{yx} = -D_{xy}$ to give the diffusion equation (2-31) derived in Section II.

## Differential Equation for $f_0$

Previously we indicated that we would solve for the differential equation of $F_0$ by eliminating $\vec{F}_1^0$, $\vec{F}_1^1$ and $\vec{G}_1^1$ from

Eqs. (3-26), (3-29), (3-30) and (3-31). However, since the general case is complicated and not required for our investigations we shall limit the solution to uniform electric and magnetic fields and shall only consider these when they are perpendicular and parallel to each other. Also we shall let $\nu_c$ = constant, since we shall apply the results to helium only.

$\vec{E}$ perpendicular to $\vec{B}$

Let us select $\vec{E}$ along the Z axis and $\vec{B}$ along the X axis. This is for infinite plates perpendicular to Z axis. Hence, there is no variation in x and y. If we write out the equations in component form using the notation by Allis,[16] we get:

$$P_{0,in} + \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}(\nu_c v^3 F_0) = \frac{v}{3}\frac{\partial}{\partial z}F^0_{10} - \frac{e}{m}\frac{1}{3v^2}\frac{\partial}{\partial v}(\frac{v^2}{2}EF^1_{10}) \qquad (3\text{-}49)$$

$$0 = \nu_c F^0_{10} + v\frac{\partial}{\partial z}F_0 - \frac{e}{m}BF^0_{1-1} \qquad \text{(z-comp)} \qquad (3\text{-}50)$$

$$0 = \nu_c F^0_{1-1} + \frac{e}{m}BF^0_{10} \qquad \text{(y-comp)} \qquad (3\text{-}51)$$

$$0 = \nu_c F^1_{10} + \omega G^1_{10} - \frac{e}{m}BF^1_{1-1} - \frac{e}{m}E\frac{\partial F_0}{\partial v} \qquad \text{(z-comp)} \qquad (3\text{-}52)$$

$$0 = \nu_c F^1_{11} + \omega G^1_{11} \qquad \text{(x-comp)} \qquad (3\text{-}53)$$

$$0 = \nu_c F^1_{1-1} + \omega G^1_{1-1} + \frac{e}{m}BF^1_{10} \qquad \text{(y-comp)} \qquad (3\text{-}54)$$

$$0 = \nu_c G^1_{10} - \omega F^1_{10} - \frac{e}{m} B G^1_{1-1} \qquad \text{(z-comp)} \qquad (3\text{-}55)$$

$$0 = \nu_c G^1_{11} - \omega F^1_{11} \qquad \text{(x-comp)} \qquad (3\text{-}56)$$

$$0 = \nu_c G^1_{1-1} - \omega F^1_{1-1} + \frac{e}{m} B G^1_{10} \qquad \text{(y-comp)} \qquad (3\text{-}57)$$

The first equation above is the scalar equation from which we shall eliminate $F^0_{10}$ and $F^1_{10}$ by solving for them in terms of $F_0$ in the rest of the equations. The first step is to see that $F^1_{11}$ and $G^1_{11}$ appear alone, (3-53) and (3-56) and are not needed. To solve for $F^0_{10}$ we only need Eqs. (3-50) and (3-51) to get

$$F^0_{10} = - \frac{\frac{v}{\nu_c} \frac{\partial F_0}{\partial z}}{1 + \omega_b^2/\nu_c^2} \quad . \qquad (3\text{-}58)$$

To solve for $F^1_{10}$, we shall express $G^1_{10}$ and $G^1_{1-1}$ in terms of the F's from (3-52) and (3-54)

$$G^1_{10} = \frac{\frac{\omega_b}{\nu_c} F^1_{1-1} - F^1_{10} + \frac{e}{m} \frac{E}{\nu_c} \frac{\partial F_0}{\partial v}}{\omega/\ \ c} \qquad (3\text{-}59)$$

$$- G^1_{1-1} = \frac{\frac{\omega_b}{\nu_c} F^1_{10} + F^1_{1-1}}{\omega/\nu_c} \qquad (3\text{-}60)$$

Substituting these in (3-55) and (3-57), we get:

$$-\frac{2\omega_b}{\nu_c}F^1_{1-1} + \left(1 + \frac{\omega^2}{\nu_c^2} - \frac{\omega_b^2}{\nu_c^2}\right)F^1_{10} = \frac{eE}{m\nu_c}\frac{\partial F_0}{\partial v} \qquad (3\text{-}61)$$

$$\left(1 + \frac{\omega^2}{\nu_c^2} - \frac{\omega_b^2}{\nu_c^2}\right)F^1_{1-1} + \frac{2\omega_b}{\nu_c}F^1_{10} = \frac{\omega_b}{\nu_c}\frac{eE}{m\nu_c}\frac{\partial F_0}{\partial v} . \qquad (3\text{-}62)$$

Solving this for $F^1_{10}$ we have:

$$F^1_{10} = \frac{\left(1 + \frac{\omega^2}{\nu_c^2} + \frac{\omega_b^2}{\nu_c^2}\right)\frac{eE}{m\nu_c}\frac{\partial F_0}{\partial v}}{\left(1 + \frac{\omega^2}{\nu_c^2} - \frac{\omega_b^2}{\nu_c^2}\right) + \frac{4\omega_b^2}{\nu_c^2}} . \qquad (3\text{-}63)$$

If we substitute $F^0_{10}$ from (3-58) and $F^1_{10}$ from (3-63) in (3-49) we get the differential equation

$$P_{0,in} + \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}(\nu_c v^3 F_0) + \frac{\frac{v^2}{3\nu_c}\frac{\partial^2 F_0}{\partial z^2}}{1 + \omega_b^2/\nu_c^2}$$

$$+ \frac{\frac{1}{6}\frac{e^2E^2}{m^2\nu_c}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial F_0}{\partial v}\right)}{\dfrac{1 + \frac{2\omega^2}{\nu_c^2} + \frac{2\omega_b^2}{\nu_c^2} + \left(\frac{\omega^2 - \omega_b^2}{\nu_c^2}\right)^2}{1 + \frac{\omega^2}{\nu_c^2} + \frac{\omega_b^2}{\nu_c^2}}} = 0 \qquad (3\text{-}64)$$

At this point we shall introduce a number of simplifications such as substituting:

$$F_0 = n(x,y,z)f_0(v) \qquad P_{0,in} = p_{0,in} \times n(x,y,z)$$

$$\frac{\partial^2 F_0}{\partial z^2} = f_0 \frac{\partial^2 n}{\partial z^2} = - f_0 \frac{n}{\Lambda^2} \,, \qquad \Lambda = \frac{L}{\pi}$$

$$\Lambda_e^2 = \Lambda^2(1 + \omega_b^2/\nu_c^2) \tag{3-65}$$

$$\frac{E^2}{2} = E_r^2(\text{rms}) \quad \text{and} \quad \frac{E_r^2}{1 + \omega_m^2/\nu_c^2} = E_e^2$$

$$\omega_m^2 = \frac{\nu_c^2(\omega^2 + \omega_b^2) + (\omega^2 - \omega_b^2)^2}{\nu_c^2 + \omega^2 + \omega_b^2} \,.$$

$E_e$ is an effective d.c. field and $\Lambda_e$ is an effective diffusion length. Substituting these expressions in the preceding equation and multiplying by $\nu_c/n$, we get the simple form

$$\nu_c p_{0,in} + \frac{m}{M}\frac{\nu_c}{v^2}\frac{\partial}{\partial v}(v^3 \nu_c f_0) - \frac{v^2 f_0}{3\Lambda_e^2} + \frac{e^2 E_e^2}{3m^2}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 \frac{\partial f_0}{\partial v}\right) = 0 \tag{3-66}$$

If we let the new variable be the energy $u = mv^2/2e$ in volts, we get:

$$\nu_c p_{0,in} + \frac{2m}{M}\frac{\nu_c^2}{u^{1/2}}\frac{\partial}{\partial u}(u^{3/2} f_0) - \frac{2e}{3m}\frac{u f_0}{\Lambda_e^2} + \frac{2eE_e^2}{3mu^{1/2}}\frac{\partial}{\partial u}\left(u^{3/2}\frac{\partial f_0}{\partial u}\right) = 0 \tag{3-67}$$

This is the differential equation which we have been seeking. Its solution gives us $f_0$, the distribution of electrons as a function of the velocity or in this case u, the energy.

$\vec{E}$ parallel to $\vec{B}$

We can show by a similar derivation that the differential equation for $\vec{E}$ parallel to $\vec{B}$ is identical to the above except that $\Lambda_e$ and $E_e$ take on a different form. The reason is fairly evident from what has gone before in that we shall consider the size of the vessel to be finite in all dimensions. This means that for a rectangular cavity along the principal directions

$$\frac{1}{\Lambda_e} = \sqrt{\frac{1}{\Lambda_z^2} + \left(\frac{1}{\Lambda_x^2} + \frac{1}{\Lambda_y^2}\right) \frac{1}{1 + \omega_b^2/\nu_c^2}} \qquad (3\text{-}68)$$

$$\Lambda_x = \frac{L_x}{\pi}, \ \Lambda_y = \frac{L_y}{\pi} \text{ and } \Lambda_z = \frac{L_z}{\pi} .$$

Similarly for a cylindrical cavity oriented with its axis parallel to $\vec{B}$:

$$\frac{1}{\Lambda_e} = \sqrt{\frac{1}{\Lambda_z^2} + \frac{1}{\Lambda_R^2(1 + \omega_b^2/\nu_c^2)}} \qquad (3\text{-}69)$$

$$\Lambda_z = \frac{L}{\pi} \qquad \Lambda_R = \frac{R}{2.405} .$$

The above diffusion lengths $\Lambda_e$ are obtained from the solution of the diffusion equation.

The effective electric field $E_e$ no longer contains any magnetic terms since the motion of the electron in the direction of the magnetic field depends only upon E. Hence the energy, proportional to $E_e^2$, will not contain $\omega_b$. Therefore

$$E_e^2 = \frac{E_r^2}{1 + \omega^2/\nu_c^2}, \qquad (3\text{-}70)$$

where $E_r$ is the r.m.s. value of the electric field.

It might be mentioned at this point that all our problems can be solved by a single differential equation in which the effective quantities appear. The situation merely requires the proper mathematical interpretation of the quantities in terms of the experimental parameters.

Solution of the Differential Equation

The solution of the differential equation is going to follow a method similar to that used by MacDonald and Brown[5] in their analysis of helium in a high frequency field. The reason for this is that by the use of the effective field and the effective diffusion length, the differential equation (3-67) has been made identical to the one derived by M and B.

It is shown that the collision frequency $\nu_c$ is proportional to the pressure and is given by the expression $\nu_c = kp$ where $k = 2.37 \times 10^9$ and p is in mm. As mentioned previously, helium has a metastable level at 19.8 volts which together with mercury atoms provides an effective ionization potential near this value. An electron near this energy will produce a metastable atom on collision. Thus the effective ionization potential $u_i$ represents this first excitation potential plus a small overshoot energy beyond this value. Actually $u_i$ is selected as a mean value at which all electrons are supposed to ionize. This is an approximation which implies that $p_{0,in}$ the production rate, occurs within a very small interval near $u_i$ and is zero above and below this value. Mathematically this is expressed as:

$$\nu_i = -\int_0^{\infty} p_{0,in} 4\pi v^2 dv = \lim_{\delta \to 0} \int_{u_i - \delta/2}^{u_i + \delta/2} p_{0,in} 4\pi \left(\frac{e}{m}\right)^{3/2} \sqrt{2}\, u^{1/2} du \;. \quad (3\text{-}71)$$

The physical interpretation of this is that no electrons exist beyond the value $u_i$ or that $f_0 = 0$ at $u = u_i$. This is the same assumption made in the single electron theory. Later we shall show how the effective value of $u_i$ is calculated. For the present we shall simply set $p_{0,in} = 0$ for $u < u_i$ and solve the differential equation, which now has the form:

$$\frac{\partial}{\partial u}\left[u^{3/2}(a\frac{\partial f_0}{\partial u} + bf_0)\right] = cu^{3/2}f_0 \ , \qquad (3\text{-}72)$$

where

$$a = \frac{2eE_e^2}{3m} \qquad b = \frac{2m}{M}\nu_c^2 = \frac{2m}{M}k^2p^2$$

$$c = \frac{2e}{3m}\frac{1}{\Lambda_e^2} \ . \qquad (3\text{-}73)$$

Differentiating and dividing by a, Eq. (3-73) becomes:

$$\frac{d^2f_0}{du^2} + (\frac{3}{2u} + A)\frac{df_0}{du} + (\frac{3}{2u}A - B)f_0 = 0 \ , \qquad (3\text{-}74)$$

where

$$A = \frac{b}{a} = \frac{3k^2m^2}{eME_e^2/p^2} \ , \qquad B = \frac{c}{a} = \frac{1}{E_e^2\Lambda_e^2} \ . \qquad (3\text{-}75)$$

We can reduce (3-74) to the normal form[17] by setting:

$$f_0 = g\,\frac{e^{-\frac{A}{2}u}}{u^{3/4}} \ . \qquad (3\text{-}76)$$

Upon substituting this into (3-74) and putting the result into the conventional form, we get:

$$\frac{d^2g}{du^2} - \frac{(4B + A^2)u^2 - 3Au - 3/4}{4u^2} g = 0 . \qquad (3\text{-}77)$$

The solution to this is[17]:

$$g = (u\sqrt{4B + A^2})^{\frac{1}{2}(1 \pm \frac{1}{2})} e^{-\frac{\sqrt{4B + A^2}\,u}{2}} M(\frac{-3A}{4\sqrt{4B + A^2}} + \frac{1}{2}(1 \pm \frac{1}{2}),\ 1 \pm \frac{1}{2},\ u\sqrt{4B + A^2}\ ) . \qquad (3\text{-}78)$$

Let

$$z = u\sqrt{4B + A^2} \qquad \beta = \frac{1}{\sqrt{1 + \frac{4B}{A^2}}} . \qquad (3\text{-}79)$$

Using these expressions and the relation in (3-76), we can express $f_0$ as the solution of the two independent functions in (3-78)

$$f_0 = e^{-\frac{\beta+1}{2}z}\left[M(\tfrac{3}{4}(1-\beta),\tfrac{3}{2},z) - Cz^{-\frac{1}{2}}M(\tfrac{1}{4} - \tfrac{3\beta}{4},\tfrac{1}{2},z)\right] . \qquad (3\text{-}80)$$

In the more standard notation of the confluent hypergeometric function $M(\alpha,\gamma,z)$

$$f_0 = e^{-(1 - \frac{2}{3}\alpha)z}\left[M(\alpha,\tfrac{3}{2},z) - Cz^{-\frac{1}{2}}M(\alpha - \tfrac{1}{2},\tfrac{1}{2},z)\right] , \qquad (3\text{-}81)$$

where

$$\alpha = \frac{3}{4}(1-\beta) \quad \text{and} \quad C = z_i^{1/2} \frac{M(\alpha,\frac{3}{2},z_i)}{M(\alpha - \frac{1}{2},\frac{1}{2},z_i)} \;. \tag{3-82}$$

C is the arbitrary constant which is determined by the condition $f_0 = 0$ at $u = u_i$. The second arbitrary constant should multiply the whole solution of $f_0$ and theoretically can be determined by the "normalization" of $f_0$. However, since we do not make use of this constant, we shall omit it.

Breakdown

To determine the equation for breakdown, we shall evaluate $\nu_i$ and D by the use of the solution for $f_0$. We shall then equate the ratio of these two quantities to $1/\Lambda_e^2$, since this is the result of solving the general form of the diffusion equation. D of course here is then defined as the isotropic coefficient of diffusion computed from the distribution function obtained with the magnetic field present.

To get $\nu_i$ we integrate (3-66) term by term and obtain:

$$\nu_i = -4\pi \int_{v_i}^{\infty} v^2 p_{0,in} dv = 4\pi \frac{m}{M} \int_{v_i}^{\infty} \frac{1}{v^2} \frac{\partial}{\partial v}(v^3 \nu_c f_0) dv$$
$$-4\pi \int_{v_i}^{\infty} \frac{v^4 f_0}{3\nu_c \Lambda_e^2} dv + \frac{4\pi}{3\nu_c} \frac{e^2 E_e^2}{m^2} \int_{v_i}^{\infty} \frac{1}{v^2} \frac{\partial}{\partial v}(v^2 \frac{\partial f_0}{\partial v}) dv \;. \tag{3-83}$$

Since $f_0 = 0$ for $v \geq v_i$ the second integral on the right hand side of the equation is zero.

$$\nu_i = 4\pi \frac{m}{M}\Big[v^3 \nu_c f_0\Big]_{v_i}^{\infty} + \frac{4\pi e^2 E_e^2}{3m^2 \nu_c}\Big[v^2 \frac{\partial f_0}{\partial v}\Big]_{v_i}^{\infty} \quad . \qquad (3\text{-}84)$$

Again the first term on the right vanishes for the same reason. Also $(\frac{\partial f_0}{\partial v})_{\infty} = 0$ since the function and hence its slope do not exist beyond $v = v_i$. Consequently

$$\nu_i = - \frac{4\pi e^2 E_e^2}{3m^2 \nu_c} v_i^2 (\frac{\partial f_0}{\partial v})_{v=v_i} \quad . \qquad (3\text{-}85)$$

In order to find $(\partial f_0/\partial v) v_i$ we transform to the variable z and find $(\partial f_0/\partial z) v_i$. To do this we express the solution of $f_0$ as follows

$$f_0 = f_1 - Cf_2 = f_1 - (\frac{f_1}{f_2})_{z_i} f_2 \qquad (3\text{-}86)$$

$$(\frac{\partial f_0}{\partial z})_{z_i} = (\frac{f_1^1 f_2 - f_2^1 f_1}{f_2})_{z_i} = \frac{W_i}{f_{2_{z_i}}} \quad , \qquad (3\text{-}87)$$

where $W_i$ is the Wronskian of the original equation. By the use of the theory of Wronskians the above expression is evaluated. The answer comes out to be:

$$\left(\frac{\partial f_0}{\partial z}\right)_{z_i} = \frac{1}{2z_i e^{-2az_i} M(a - \frac{1}{2}, \frac{1}{2}, z_i)} . \tag{3-88}$$

From the definition of the coefficient of diffusion D from Eq. (3-47), we have

$$D = 4\pi \int_0^{\infty} \frac{\ell_v}{3} v^2 f_0 dv = \frac{4\pi}{3\nu_c} \int_0^{v_i} v^4 f_0 dv . \tag{3-89}$$

Let

$$v^2 = rz \ , \quad v\,dv = \frac{r}{2}\,dz \ , \quad r = \frac{2e}{m\sqrt{4B + A^2}} \tag{3-90}$$

$$D = \frac{4\pi}{3\nu_c} r^{5/2} \int_0^{z_i} z^{3/2} f_0 dz \quad . \tag{3-91}$$

With the help of the integrals listed by MacDonald[17] this comes out to be:

$$\int_0^{z_i} z^{3/2} f_0 dz = \frac{(1 - \frac{2a}{3}) \frac{4a}{3} e^{-\frac{2az_i}{3}} M(a - \frac{1}{2}, \frac{1}{2}, z_i)}{z_i^{1/2}\left[1 - \frac{M(a, \frac{3}{2}, z_i)}{e^{\frac{2}{3}az_i}}\right]} . \tag{3-92}$$

Using this value and that in (3-88), we get the following ratio first:

$$\frac{\left(\frac{\partial f_0}{\partial z}\right)_{z_i}}{\int_0^{z_i} z^{3/2} f_0 dz} = \frac{\left(1 - \frac{2\alpha}{3}\right)\frac{2\alpha}{3}}{z_i^{3/2}\left[1 - \frac{M(\alpha,\frac{3}{2},z_i)}{e^{\frac{2}{3}\alpha z_i}}\right]} \quad . \tag{3-93}$$

Then for $\nu_i/D$ we get:

$$\frac{\nu_i}{D} = -\frac{\frac{4\pi}{3}\frac{e^2E_e^2}{m^2\nu_c}2r^{1/2}z_i^{3/2}\left(1 - \frac{2\alpha}{3}\right)\frac{2\alpha}{3}}{\frac{4\pi}{3\nu_c}\frac{r^{5/2}}{z}z_i^{3/2}\left[1 - \frac{M(\alpha,\frac{3}{2},z_i)}{e^{\frac{2}{3}\alpha z_i}}\right]} \quad . \tag{3-94}$$

When the algebra is appropriately simplified, we obtain the following:

$$\frac{\nu_i}{D} = \frac{1}{\Lambda_e^2} = \frac{1}{\Lambda_e^2}\frac{1}{\left[\frac{M(\alpha,\frac{3}{2},z_i)}{e^{\frac{2}{3}\alpha z_i}} - 1\right]} \quad . \tag{3-95}$$

When this is solved, the final equation becomes:

$$\frac{M(\alpha,\frac{3}{2},z_i)}{e^{\frac{2}{3}\alpha z_i}} = 2 \tag{3-96}$$

This is the equation for breakdown and is a transcendental equation involving the proper variables[18] $E_e\Lambda_e$, effective voltage, $E_e/p$ and $u_i$ the effective ionization potential. This can be readily shown since:

$$z_i = u_i\sqrt{4B + A^2} \qquad \alpha = \frac{3}{4} - \frac{3}{4\sqrt{1 + \frac{4B}{A^2}}} , \tag{3-97}$$

where

$$B = \frac{1}{E_e^2\Lambda_e^2} \quad \text{and} \quad A = \frac{3m^2}{Me} k^2 \frac{p^2}{E_e^2} = \text{const} \frac{p^2}{E_e^2} \quad .$$

The implicit equation (3-96) can then be written symbolically as:

$$\mathcal{F}(E_e\Lambda_e, \frac{E_e}{p}, u_i) = 0 \quad . \tag{3-98}$$

For the present we shall not pursue this analysis any further until we evaluate $u_i$ in terms of the proper variables above when we calculate the overshoot.

Equivalence of Distribution and Single Electron Theories

Previously we indicated that it was possible to obtain the breakdown equation (2-46) from the distribution theory. In order to do this we must first make a simple approximation in our development of this theory. It was observed experimentally that the phenomena produced by the magnetic field occurred at pressures below 8 mm when the alternating frequency was 3000 mc. At such low pressures, the energy lost to elastic collisions is small compared to that gained from the electric field and that lost by diffusion. The term involving the ratio m/M in the differential equation (3-72) represents this energy loss. If we neglect this term as in the single electron picture, we set b = 0 in (3-73) hence also A = 0 in (3-75) and (3-97). This means that

$$\alpha = \frac{3}{4} \quad \text{and} \quad z_i = \frac{2u_i}{E_e\Lambda_e} \quad . \tag{3-99}$$

The breakdown equation (3-96) takes the simple form

$$\frac{M(\frac{3}{4},\frac{3}{2},z_i)}{e^{z_i/2}} = 2 \quad . \tag{3-100}$$

By trial and error or graphically it is possible to evaluate $z_i$. We find that only one value of $z_i$ satisfies the equation and from the tables[17]

$$z_i = \frac{2u_i}{E_e\Lambda_e} = 4$$

or (3-101)

$$E_e\Lambda_e = \frac{u_i}{2} \quad .$$

In form, this equation is identical to that obtained in (2-46), because if we make the same assumption that $u_i$ is a constant for a given pressure and independent of the magnetic field:

$$E_e \Lambda_e = E_{oe} \Lambda_o = \text{constant}$$

$$\text{or} \qquad E_e = E_{oe} \frac{\Lambda_o}{\Lambda_e} \quad . \tag{3-102}$$

For the parallel electric and magnetic fields this is identical to (2-54) if the subscript e is replaced by m and the same as Eq. (2-46) for the transverse E and B when $E_e = \dfrac{E_m}{\sqrt{1 + \omega_m^2/\nu_c^2}}$ and $\Lambda_e = \Lambda\sqrt{1 + \omega_b^2/\nu_c^2}$ and $E_{oe} = E_o/\sqrt{1 + \omega^2/\nu_c^2}$ and $\Lambda_o = \Lambda$.

Further equivalence can be shown by obtaining Eq. (3-101) from the random walk theory. This is done by calculating the average energy from the distribution function. This can be done analytically. However, it turns out to be simpler by graphical means. Figure III-1 shows the plot of the distribution function $f_0$ with the approximations for low values of pressure:

$$f_0 = e^{-z}\left[M(\tfrac{3}{4},\tfrac{3}{2},z) - 1.443z^{-1/2}M(\tfrac{1}{4},\tfrac{1}{2},z)\right] \quad . \tag{3-103}$$

This is actually similar in character to the distribution function without the approximations. The plot given here is more universal since it is in terms of the ratios of u to the ionization potential.

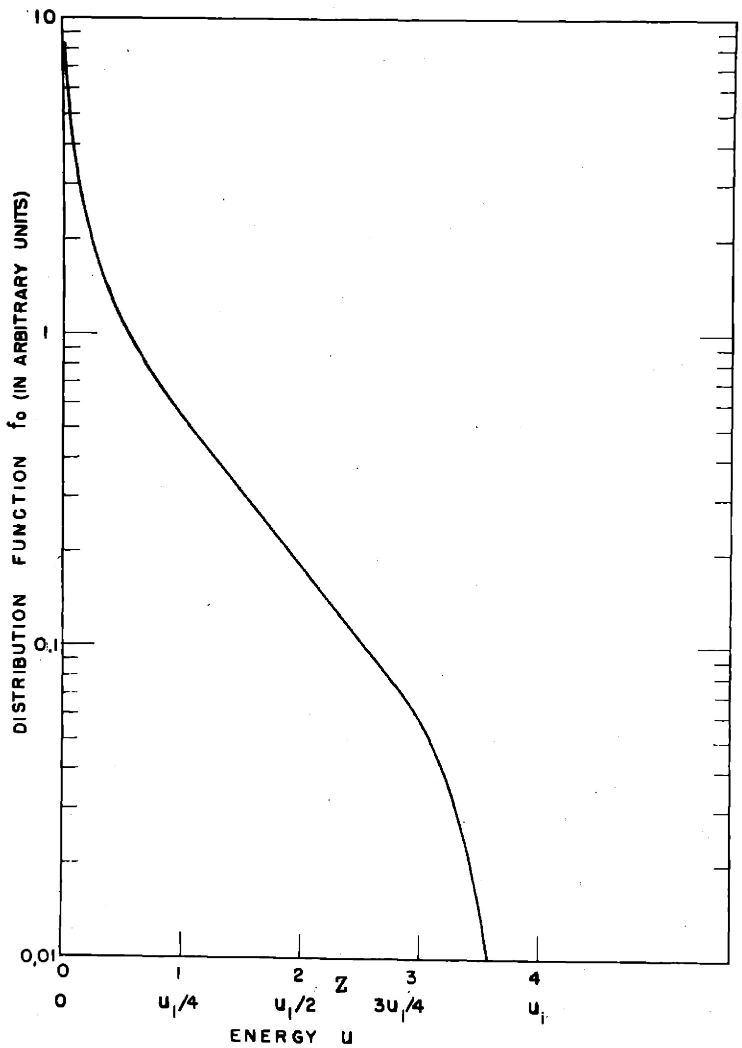



Fig. III-1. Distribution function $f_0$ in arbitrary units. $f_0$ goes to zero at $z = 4$ or $u = u_i$.

Here it is assumed that $E_e\Lambda_e$ = constant = $u_i/2$, but this is only approximate. We shall correct for this when we calculate the overshoot.

The graphical averaging is done by plotting on a linear paper the functions $z^{3/2}f_0$ and $z^{1/2}f_0$ and taking the ratio of the areas under curves. This gives $\bar{z}$ since

$$\bar{z} = \frac{\int_0^{u_i} z^{3/2} f_0 dz}{\int_0^{u_i} z^{3/2} f_0 dz} \quad . \tag{3-104}$$

Although $f_0$ has a weak singularity at the origin the above functions which enter properly into the averaging do not, as can be seen in Fig. III-2. The result of the averaging is that

$$\bar{z} = 1.12 \quad \text{and} \quad \bar{u} = \frac{1.12}{4} u_i = .28\, u_i \quad . \tag{3-105}$$

Using this value we can calculate from Eqs.(2-43) and (2-44) the following:

$$\frac{3\Lambda_e^2}{\bar{\ell}^2} = \frac{u_i}{u_e} \quad \text{or} \quad E_e^2\Lambda_e^2 = \frac{2u_i\bar{u}}{3} \quad . \tag{3-106}$$

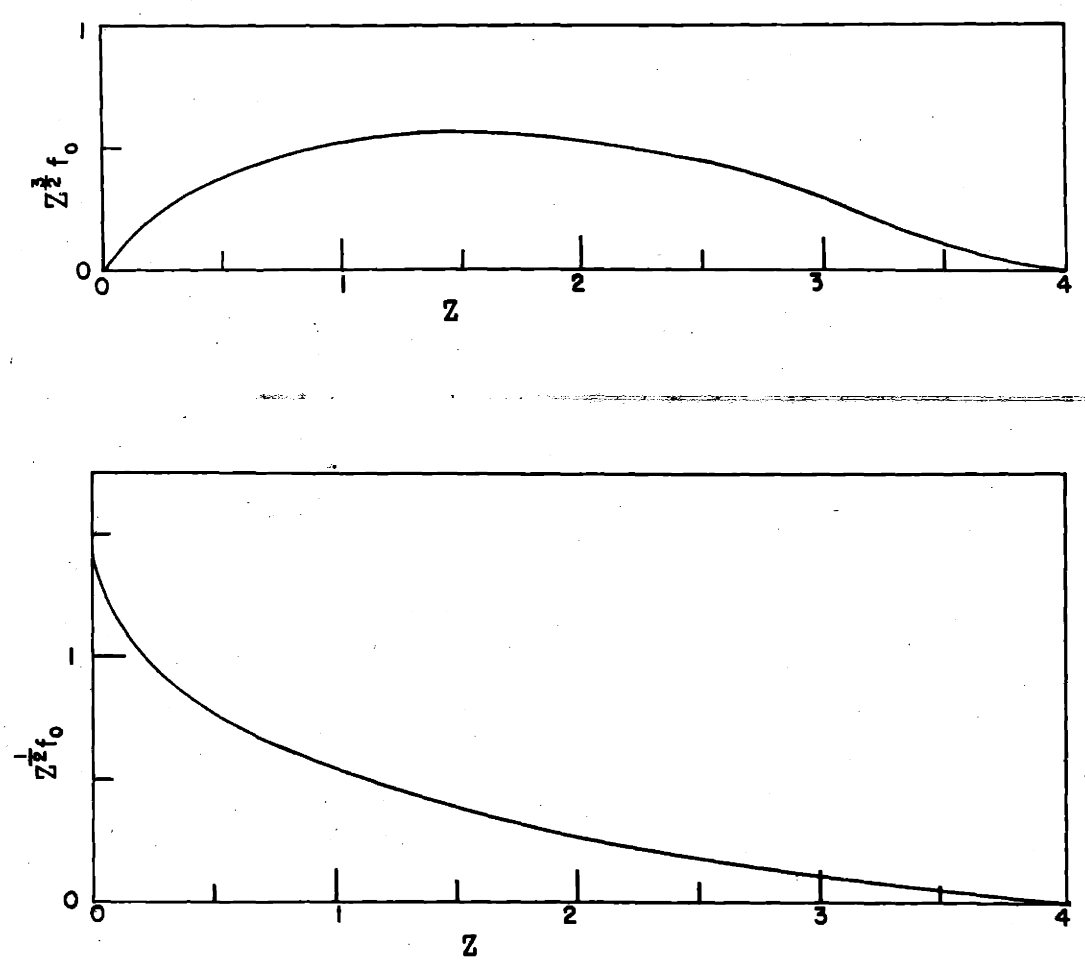



Fig. III-2. Graphs of $z^{3/2}f_0$ and $z^{1/2}f_0$. The ratio of the area under the two curves yields $\bar{z} = 1.12$ and hence $\bar{u} = 0.28\, u_i$.

Since $\overline{\ell^2} = \overline{v^2}/\nu_c^2$ , $\frac{m\overline{v^2}}{2e} = \overline{u}$ and $u_e = \frac{eE_e^2}{m\nu_c^2}$ , then

$$E_e\Lambda_e = .45\ u_i\ . \qquad (3\text{-}107)$$

This agrees very well with (3-101) which says that $E_e\Lambda_e = 0.5\ u_i$. Thus the agreement is within 10°/o. This justifies the random walk theory on a mathematical basis.

Overshoot-Effective Ionization Potential

In the previous analysis we assumed that the distribution function went to zero at an effective potential $u_i$. Physically this potential represented the average ionization potential of the electrons making inelastic collisions beyond the actual ionization value. For helium, the latter was taken as $u_x$ the first excitation potential. The overshoot energy is defined as the difference between $u_i$ and $u_x$.

To evaluate the overshoot energy, we shall solve for the distribution function $f_0$ below $u_x$ setting $p_{0,in} = 0$. For $u > u_x$, we shall use the experimental value of excitation probability obtained by Maier Leibnitz[19] and approximate it by a straight line analytically. Using this we have $p_{0,in}$ and a new function $f_0$ for $u > u_x$. In principle, we can match value and slope of the two functions at $u = u_x$. Actually, however, this will not be necessary as we shall show later. By finding the slope of either function at the matching point and extrapolating the slope until it intersects the axis, we can calculate the effective ionization potential, assuming that the straight line is a good approximation to the distribution function in the inelastic region.

The loss of electrons due to excitation per electron beyond $u_x$ is equal to the number of total collisions $\nu_c$ and the relative probability of excitation $h_x$, defined by the ratio of $P_x$ to $P_c$ the conventional probabilities of excitation and collision. This function is approximated from experimental data quite well as:

$$h_x = h_1(u - u_x) \quad , \tag{3-108}$$

where $h_1 = \frac{1}{135}$ volts$^{-1}$ and $u_x$ = 19.8 volts for helium.

If we substitute this into Eq. (3-67), setting $p_{0,in} = -\nu_c h_x f_0$ we get

$$\nu_c^2 h_1(u - u_x) f_0 = \frac{1}{u^{1/2}} \frac{\partial}{\partial u}\left[u^{3/2} \left(a \frac{\partial f_0}{\partial u} + bf_0\right)\right] - Cuf_0 \quad . \tag{3-109}$$

a, b and c are defined by (3-73).

Equation (3-109) can be rewritten as:

$$\frac{d^2 f_0}{du^2} + \left(\frac{3}{2u} + \frac{b}{a}\right)\frac{df_0}{du} + \left(\frac{3}{2u}\frac{b}{a} - \frac{c}{a} - \frac{\nu_c^2 h_1}{a} + \frac{\nu_c^2 h_1 u_x}{au}\right) f_0 = 0 \tag{3-110}$$

$$\frac{d^2 f_0}{du^2} + \left(\frac{3}{2u} + A\right)\frac{df_0}{du} + \left(\frac{3}{2u} A - B - C + \frac{D}{u}\right) f_0 = 0 , \qquad (3\text{-}111)$$

where $A = \frac{b}{a}$ , $B = \frac{c}{a}$ as in (3-75) and

$$C = \frac{\nu_c^2 h_1}{a} = \frac{3mk^2 h_1}{2e} \frac{p^2}{E_e^2}$$

$$D = \frac{\nu_c^2 h_1 u_x}{a} = \frac{3mk^2 h_1 u_x}{2e} \frac{p^2}{E_e^2} . \qquad (3\text{-}112)$$

Putting (3-111) in the normal form by setting

$$f_0 = g \frac{e^{-\frac{A}{2}u}}{u^{3/4}} \qquad (3\text{-}113)$$

$$\frac{d^2 g}{du^2} + \left[\frac{3}{16u^2} + \frac{3A}{u} + \frac{D}{u} - \frac{A^2}{4} - B - C\right] g = 0 . \qquad (3\text{-}114)$$

The solution to this equation is a confluent hypergeometric function. We shall not proceed with this solution except to state that the solution is expressed in terms of $E_e \Lambda_e$ and $E_e/p$ since $B = \frac{1}{E_e^2 \Lambda_e^2}$ and A, C, D are proportional to $p^2/E_e^2$. We match the two functions $f_0(C = D = 0)$ $u < u_x$ and $f_0$ from (3-113) $u > u_x$ at $u_x$ by equating them and their slopes at $u = u_x$. The energy $u_i$ is then the intersection of the tangent with the axis.

This gives us:

$$u_i = u_x + \mathcal{G}(E_e\Lambda_e, E_e/p) \; . \qquad (3\text{-}115)$$

Previously we showed that for breakdown we obtained a transcendental function $\mathcal{F}(E_e\Lambda_e, E_e/p, u_i) = 0$. In principle, we can eliminate $E_e\Lambda_e$ from these two equations and say that

$$u_i = u_x + \mathcal{H}(E_e/p) \; . \qquad (3\text{-}116)$$

MacDonald and Brown[5] used an approximate method for calculating their value of the effective ionizing potential. In the region of low pressure or high $E_e/p$ it can be seen from their curves that

$$u_i \approx u_x + \text{const} \times E_e/p \; . \qquad (3\text{-}117)$$

This can be shown to be a good approximation theoretically by taking a mean value of $E_e\Lambda_e$ and u and calculating the value of the constant in (3-117). This constant turns out to have the value of about .07.

If we had eliminated $u_i$ instead of $E_e\Lambda_e$ from the two previous transcendental equations, we would get a new implicit function which states that

$$\mathcal{J}(E_e\Lambda_e, E_e/p) = 0 \ . \qquad (3\text{-}118)$$

Or if this could be solved for $E_e\Lambda_e$ as a function of $E_e/p$

$$E_e\Lambda_e = \mathcal{L}(E_e/p) \ . \qquad (3\text{-}119)$$

Using the values obtained by MacDonald and Brown[5] for breakdown of helium without a magnetic field, we plotted this function for a large number of values which agreed with the theoretical values. Figure III-3 shows this plot. According to our analysis it should represent breakdown values for the magnetic field also when $E_e$ and $\Lambda_e$ are properly interpreted. In the next two sections we shall use this curve and analyze its limitations.

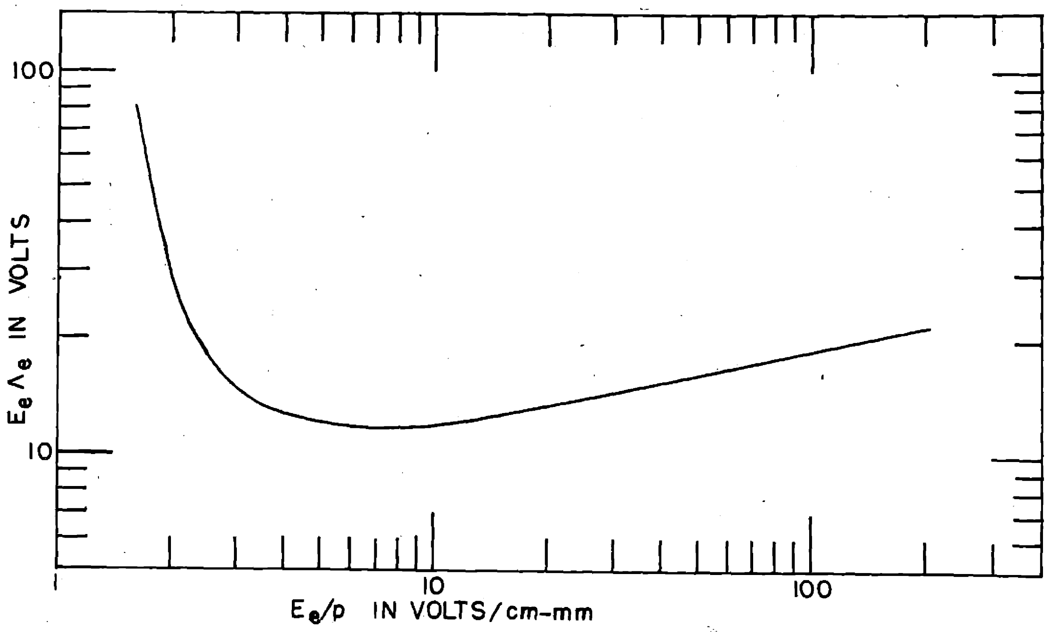



Fig. III-3. The effective breakdown curve of helium.

## IV. BREAKDOWN IN NON-UNIFORM FIELD

In the discussion of the diffusion effect of a magnetic field in Section II we stated that the experimental verification of this phenomenon was to be performed in a large cavity. We had suggested that the ideal experiment should have been made between parallel plates in a cylindrical glass container whose lateral height was nearly the same size as the diameter. Since it was not possible to obtain reproducible data with a glass bottle a cylindrical copper cavity, approximately 3 inches in diameter and 2 inches high, was used. By making the magnetic field parallel to the axis of the cylinder and hence to the electric field, we could study the effect of the magnetic field on direct diffusion. However, since the electric field inside the cavity is no longer uniform, the theory which we have developed is no longer applicable in its present form. Nevertheless the problem can be treated by extending the method used by Herlin and Brown[20] for handling breakdown in cylindrical cavities.

The theory reduces the non-uniform field breakdown to an equivalent parallel plate or uniform field whose magnitude is the same as that of the breakdown field at the center of the cavity. As a consequence of this development, an equivalent parallel plate separation and hence a new diffusion length is computed. This diffusion length is then used in the uniform field theory.

The first assumption made is that the ionization coefficient $\zeta$ defined by:

$$\zeta = \frac{\nu_i}{DE^2} \tag{4-1}$$

is a simple function of the field E as follows:

$$\zeta = \zeta_o\left(\frac{E}{E_o}\right)^{\beta-2} = \left(\frac{k}{E_o}\right)^2\left(\frac{E}{E_o}\right)^{\beta-2}, \tag{4-2}$$

where $\zeta_o$ and $E_o$ are the values of the ionization coefficient and the electric field at the center of the cavity. It is a natural step to extend this equation to include equivalent fields and hence define an equivalent ionization coefficient.

$$\zeta_e = \text{const}(E_e)^{\beta-2} . \tag{4-3}$$

This quantity is put into the diffusion equation which is then solved approximately. The approximation shows that $\nu_i/D$ goes from a maximum at the center to zero at some intermediate radial distance b, which is then defined by the equation:

$$b = \frac{0.831R}{\beta} . \tag{4-4}$$

R is the radius of the cavity and β is the exponent in Eq. (4-2) or (4-3).

The approximations involved in solving the diffusion equation limit the validity of the solution to values of $\frac{L}{R} < 0.5$, where L is the height of the cavity. When the boundary conditions are applied to the solution a transcendental equation results which relates the quantities $(\frac{kL}{\pi})^2$ to b/L. We have not yet defined the quantity k, which we introduced in Eq. (4-2) where,

$$\zeta_o = \frac{k^2}{E_o^2} . \qquad (4\text{-}5)$$

For an equivalent parallel plate system referred to the values at the center of the cavity, from the diffusion equation for such a system, the quantity

$$\left(\frac{\nu_i}{D}\right)_o = \text{constant} = \frac{1}{\Lambda_o^2} , \qquad (4\text{-}6)$$

where $\Lambda_o$ is the equivalent diffusion distance.

But

$$\zeta_o = \left(\frac{\nu_i}{D}\right)_o \frac{1}{E_o^2} = \frac{1}{E_o^2 \Lambda_o^2} . \qquad (4\text{-}7)$$

Hence

$$k = \frac{1}{\Lambda_o} = \frac{1}{\Lambda_e} . \qquad (4\text{-}8)$$

The graph of Figure IV-1 taken from Herlin and Brown[20] shows $(\frac{kL}{\pi})^2$ as a function of b/L. The way in which this curve is used in finding k or $\Lambda_e$ is as follows: First assume that the field is uniform and compute $\Lambda_e = \frac{L}{\pi}$. With this value calculate $E_e$ from Fig. III-3. Knowing this value we find β from Fig. IV-2, which is a plot of $\zeta_e$ for helium as a function of $E_e/p$, by drawing the slope and setting equal to β-2. With this value of β, b can be calculated from Eq. (4-4). Going to the graph of Fig. IV-1 we then calculate the value of k or a new $\Lambda_e$. This will not be the same as the original valüe assumed. The process is then repeated unitl the assumed $\Lambda_e$ checks with the one calculated from Fig. IV-1. Thus $E_e$ and hence $E_o = \sqrt{1 + \frac{\omega^2}{\nu_c^2}}\, E_e$ are known.

The way in which this is applied to the magnetic problem is by a simple device which assumes that the radial distance R, which is perpendicular to the magnetic field, is dilated by the factor $\sqrt{1 + \omega_b^2/\nu_c^2}$ . This is a logical consequence of our hypothesis of the diffusion effect transverse to the magnetic field. In Eq. (4-4) we now use

$$R_b = R\sqrt{1 + \omega_b^2/\nu_c^2} \qquad (4\text{-}9)$$

instead of R, for the particular values of the magnetic field and the pressure. The procedure of finding $\Lambda_e$ and $E_e$ are then identical as above. This scheme was applied to the 3" x 2" cavity

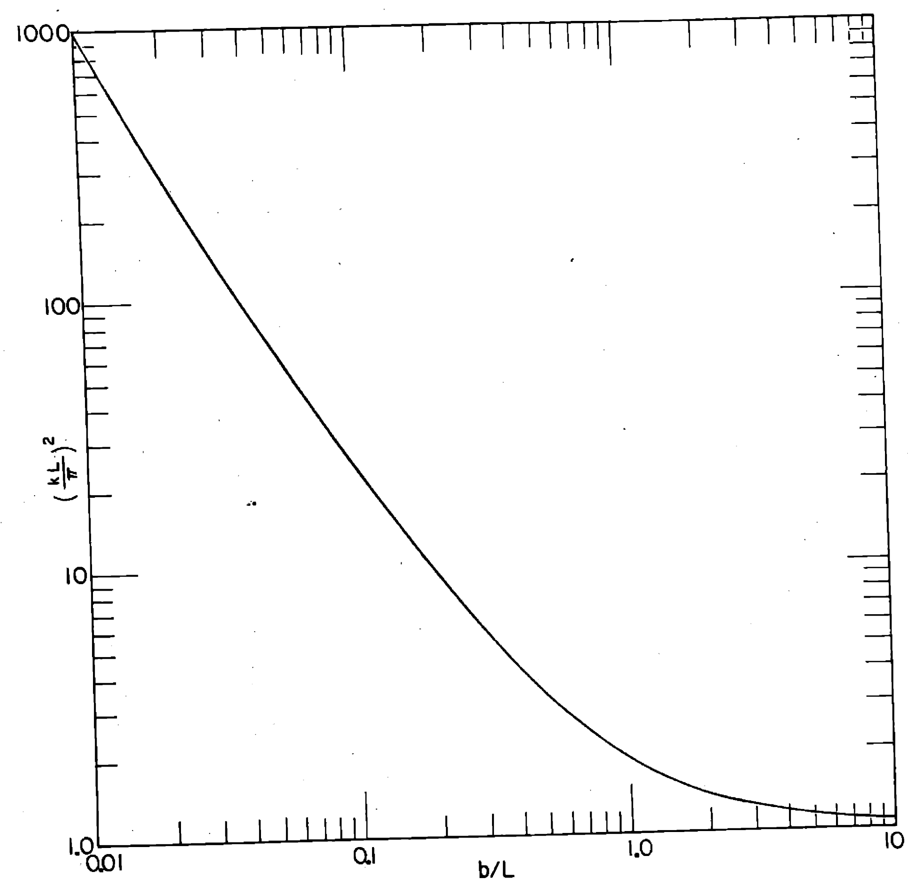



Fig. IV-1. Transformation curve for the non-uniform field diffusion length of a cylindrical cavity. The ordinate is the square of the ratio of the real cavity height to the equivalent infinite parallel plate separation.

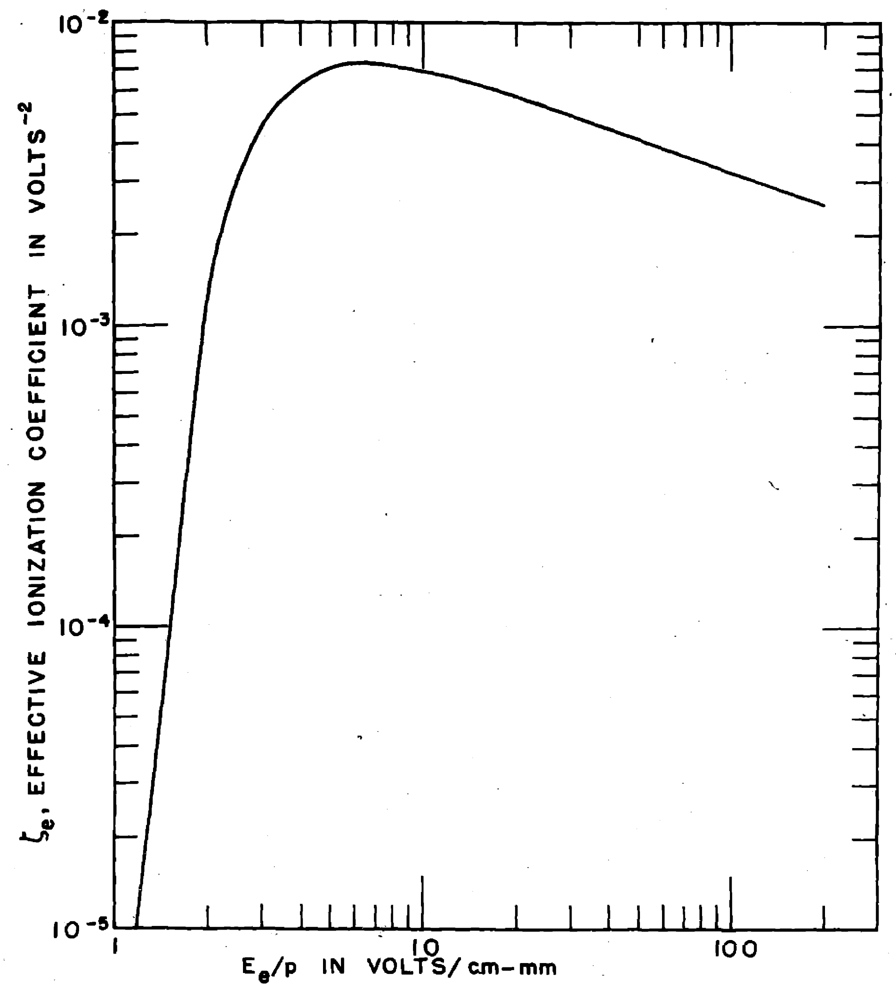



Fig. IV-2. Effective ionization coefficient.

and values of the breakdown field were obtained as a function of the magnetic field theoretically. The solid lines of Fig. IV-3 show the results of the theory. The points on the graphs are experimental data.

The agreement between experiment and theory is quite good. This then verifies the direct diffusion effect of the magnetic field. It also justifies the concept of the mathematical dilation of a diffusion length. In addition, the experiment shows that the value of the breakdown field E is reduced asymptotically to a limiting value shown by the horizontal dotted line. This value corresponds to a parallel plate discharge between plates of 2" separation. It may be also noticed that the reduction proceeds more rapidly as a function of B for smaller pressures.

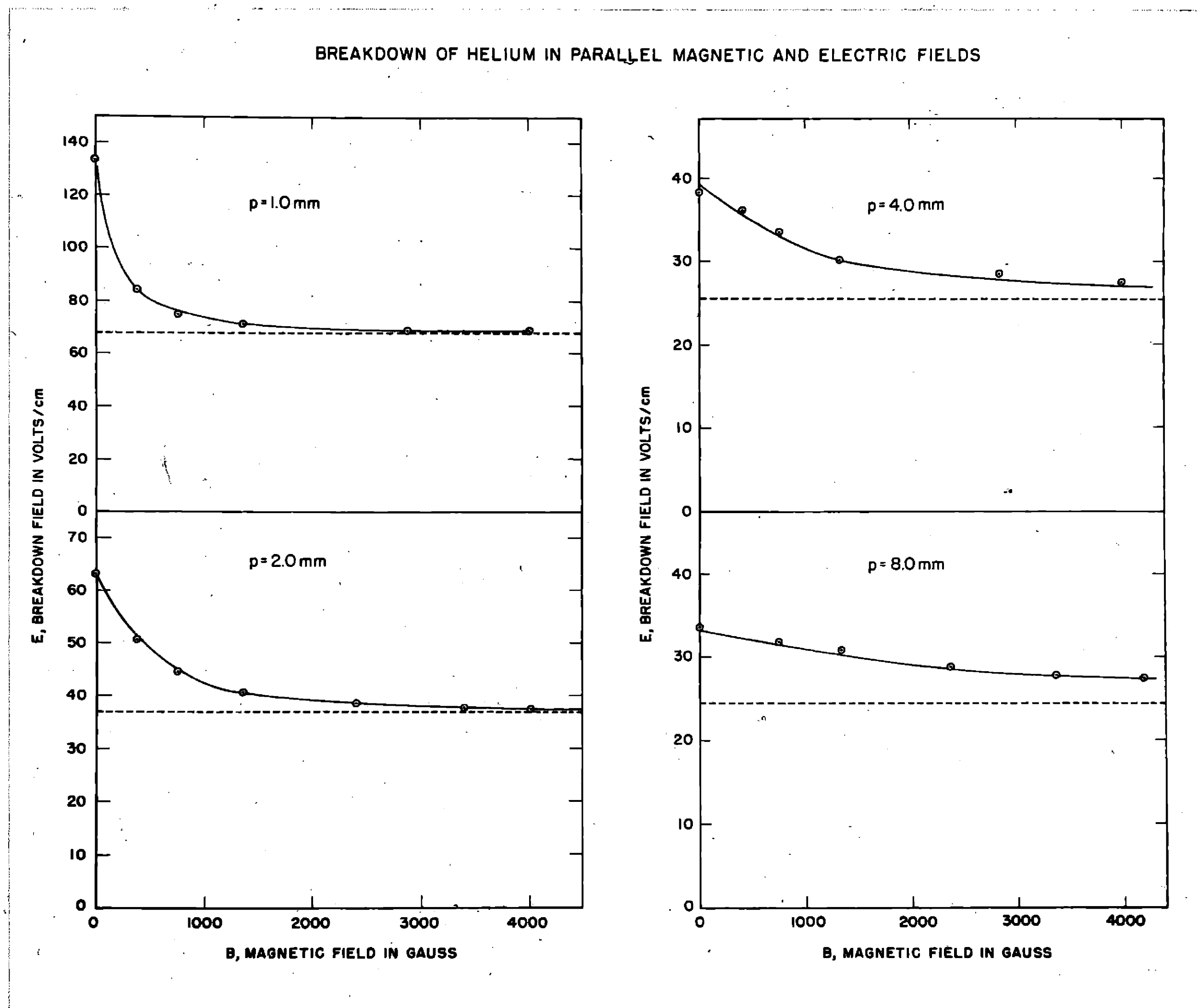



Fig. IV-3. Breakdown of helium in parallel electric and magnetic fields in a cylindrical cavity (L = 4.60 cm, R = 3.66 cm). The solid curves are the theoretical results and the points are experimental data.

## V. BREAKDOWN CALCULATION FROM DISTRIBUTION THEORY

At the end of Section III we drew a single curve shown in Fig. III-3 representing the effective breakdown voltage for helium in a uniform electric field. If we were to use this curve in calculating the breakdown field for the parallel plate cavity of 1/8" separation and 2 7/8" diameter, the first attempt would be to ignore the radial dimension and treat the cavity as two ideal parallel plates of infinite extent. This is what we did in Section II with considerable success.

The way in which we compute the electric field as a function of the magnetic field is to select a given value of the pressure and hence $\nu_c$, then for a given set of values of B calculate $\Lambda_e = \Lambda \sqrt{1 + \omega_b^2/\nu_c^2}$ , $\Lambda = L/\pi$. With this value we enter the curve of Fig. III-3 and by trial and error calculate $E_e$ until $E_e\Lambda_e$ and $E_e/p$ fit the curve. From this value, the electric field given by $E = E_e\sqrt{1 + \omega_m^2/\nu_c^2}$ is computed. This was done for p = 2.0mm and 1 mm. The results are shown in Fig. V-1 and Fig. V-2, respectively, by the dash-dot curves.

The discrepancy between these curves and the experimental ones increases with the magnetic field to approximately 50°/o for 2 mm and 80°/o for 1 mm at 3000 gauss. This deviation seems inconsistent with the results of the random walk theory shown in

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pgs. 103 + 104 do not exist within the original-bound thesis. This is likely a page mis-numbering error by the author.

Fig. II-9 where the agreement was within 10°/o. The difference in the two theories is that the overshoot correction as a function of the magnetic field was not taken into account originally. It was assumed that $u_i$ remained constant. Of course this is not true since we have shown that the overshoot $u_i - u_x$ is nearly proportional to $E_e/p$. But $E_e/p$ decreases with increasing magnetic field and hence the overshoot decreases also. This means that the breakdown field is reduced relative to that calculated before with $u_i$ fixed. The implication is then that the previous theory is incorrect despite its agreement with experiment. The second possibility is that there should be no overshoot correction and that $u_i$ is constant. Actually neither is quite true. We shall show presently that the random walk theory worked as well as it did because we neglected two factors which balanced each other as functions of B, namely, the overshoot correction and the diffusion distortion from an ideal parallel plate system of the actually finite cavity by the magnetic field.

Let us consider the dilated dimensions of the cavity, with respect to diffusion, when B = 3000 gauss for p = 2 mm and 1 mm. Since the magnetic field is perpendicular to the parallel faces of the cavity it will distort the height L by the factor $\sqrt{1 + \omega_b^2/\nu_c^2}$. Similarly, the diameter perpendicular to the field will also be increased by this factor. However, the diameter parallel to the field will be undisturbed. Consequently, as far as diffusion is concerned the cavity appears roughly ellipsoidal. For 2 mm, $L_m = 3.7$ cm, $d_\perp = 41.5$ cm, and $d_\parallel = 3.66$ cm; for 1 mm, $L_m = 7.35$ cm, $d_\perp = 82.5$ cm, and $d_\parallel = 3.66$ cm where $L_m$ is the dilated height. We can neglect the contribution of the perpendicular diameter $d_\perp$ because it is large and hence diffusion in that direction is negligible. In the parallel direction $d_\parallel$ is comparable to $L_m$. This indicates that diffusion in this direction is important and we can no longer treat the cavity as two parallel plates. In addition, since the effective dimensions are of the same order of magnitude, the uniform field theory no longer applies.

Any correction we make at the present will be an approximation because the general diffusion equation has not been developed for cylindrical coordinates except for the magnetic field along the axis of the cylinder. For the transverse case, the equation is complicated and cannot be solved in terms of known functions. Furthermore if this difficulty were overcome, the application of the non-uniform theory developed by Herlin and Brown would also be approximate since they did it for a cylindrical cavity. Our distorted cavity is more nearly elliptical. Despite these difficulties it is possible to make a satisfactory correction to the theory as follows:

Assume that the general diffusion equation in rectangular coordinates is applicable to this problem The effective rectangular diffusion lengths of the cavity will be the following when B is along the X-axis.

$$\Lambda_z = \frac{L}{\pi}\sqrt{1 + \omega_b^2/\nu_c^2} \qquad \Lambda_y = \frac{\sqrt{2}\,R}{2.405}\sqrt{1 + \omega_b^2/\nu_c^2}$$

$$\Lambda_x = \frac{\sqrt{2}\,R}{2.405} \quad . \tag{5-1}$$

Evidently this reduces to the correct value for the non-magnetic case $\omega_b = 0$ since:

$$\frac{1}{\Lambda^2} = \frac{1}{\Lambda_z^2} + \frac{1}{\Lambda_x^2} + \frac{1}{\Lambda_y^2} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{2.405}{R}\right)^2 \quad . \tag{5-2}$$

In the particular case under consideration $\Lambda_y$ can be neglected since it is negligible in comparison to $\Lambda_z$ and $\Lambda_x$.

To apply the non-uniform field theory, we shall interpret $R' = \sqrt{2}\,R$ as the new effective radius and calculate $b' = \frac{.813R'}{\beta}$ as in Section IV. The remainder of the procedure for calculating $E_e$ proceeds exactly as before with $b'/L_m$ as the ordinate in Fig. IV-1. Now, however, the effective diffusion distance $\Lambda_e$ is given by:

$$\Lambda_e = \frac{1}{k} = \frac{L_m}{\pi}\,\frac{1}{\sqrt{\text{ordinate of Fig. Iv-1}}}\ , \quad L_m = L\sqrt{1 + \omega_b^2/\nu_c^2} \quad . \tag{5-3}$$

Thus by successive approximations $E_e$ is found and consequently E, the actual field at the center of the cavity. This was done for helium at pressures of 2 mm and 1 mm. The curves as shown by the solid lines in Fig. V-1 and Fig. V-2. These are the corrected curves for the distribution theory and include both the overshoot and the diffusion corrections. As can be seen, they compensate each other since the theoretical curves are now reasonably close to the experimental ones. The maximum error is about 15°/o and occurs for the 2 mm curve in Fig. V-1. The curve for 1 mm in Fig. V-2 is a little better with about 10°/o error.

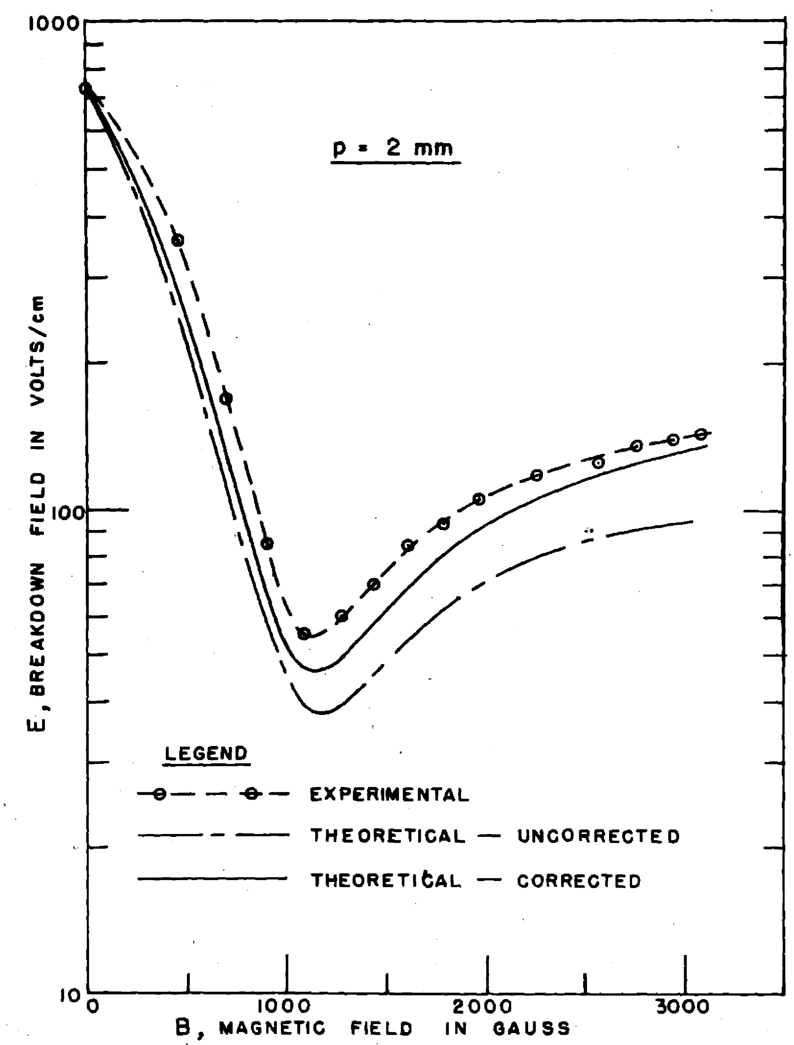


Fig. V-1. Breakdown of helium calculated from the distribution theory. The dot-dash lines are those for infinite plates with the separation of L = 0.318 cm. The solid lines are corrected theoretically for a flat cylindrical cavity (L = 0.318 cm, R = 3.66 cm).

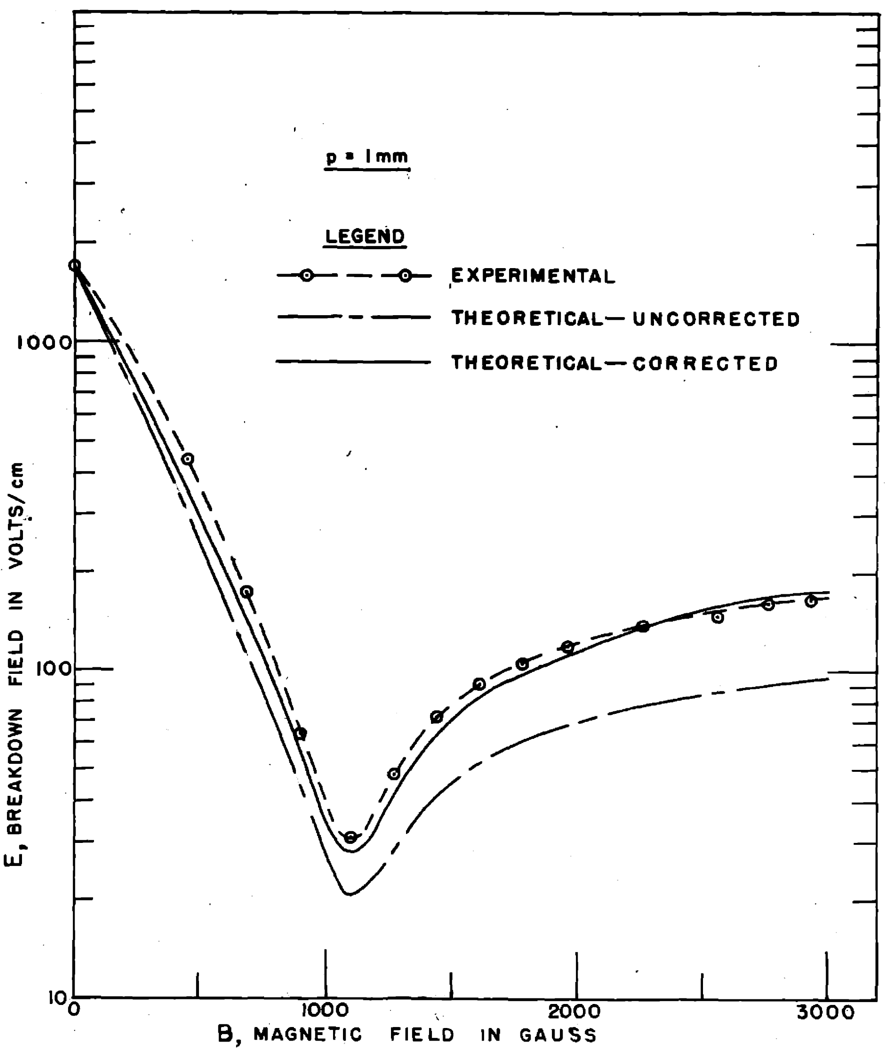


Fig. V-2. Breakdown of helium calculated from the distribution theory. The dot-dash lines are those for infinite plates with the separation of L = 0.318 cm. The solid lines are corrected theoretically for a flat cylindrical cavity (L = 0.318 cm, R = 3.66 cm).

# VI. EXPERIMENTAL APPARATUS AND PROCEDURE

## Apparatus

The equipment used in making the measurements of breakdown is shown in the block diagram of Fig. VI-1. The source of microwave energy was a 10-cm tunable CW magnetron capable of 50 to 100 watts of output power, adjustable by the high voltage power supply. The frequency of the magnetron was monitored by an "echo box" which is a calibrated tunable high Q transmission type cavity ($Q \approx 30{,}000$) with a crystal and meter on the output end. The power fed into the line was varied by a balanced power divider. This was monitored through a 30 db directional coupler by a thermistor and a balanced power bridge, which comprised a bolometer. The slotted section together with the spectrum analyzer, which was used as a sensitive detector, measured the SWR and the phase of the microwave signal. This information was used to determine the Q of the cavity in accordance with the procedure outlined in Technical Report No. 66[21] by S. C. Brown, etc. at M.I.T.

The cavity in which the breakdown was produced was a cylindrical one with glass sealed coaxial loops diametrically opposite on either side of the cavity. The output loop was attached to a reactive calibrated attenuator by a lossless r.f. cable. A crystal detector and a sensitive microammeter were placed in series

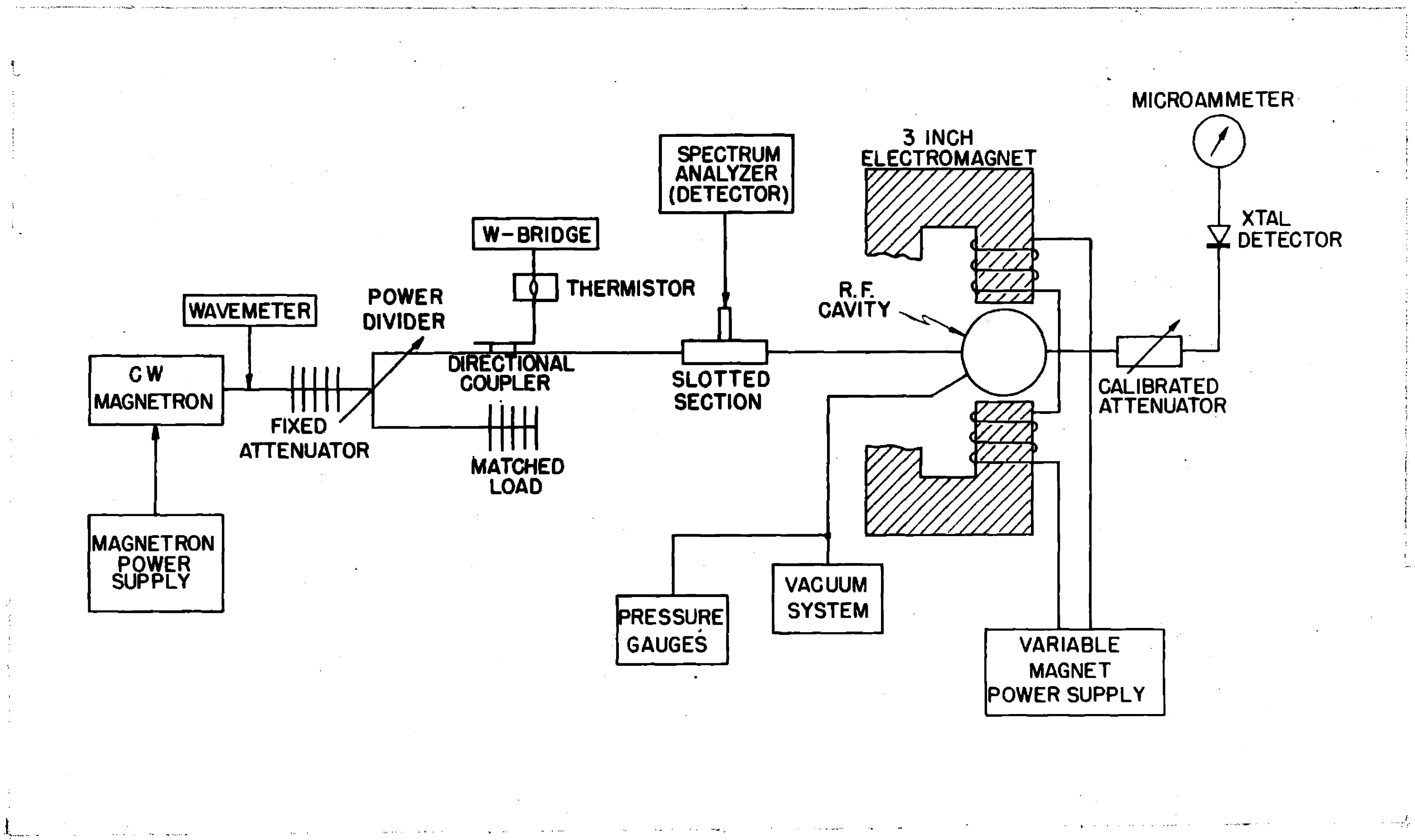


Fig. VI-1. Block diagram of the apparatus.

with the attenuator as shown. The attenuator and the microammeter were calibrated against the bolometer to read the power fed into the cavity at the resonant frequency.

The magnetic field across the cavity was applied by an electromagnet whose pole pieces were 3 inches in diameter. The separation between the pole pieces was adjustable. For a given distance between pole faces, the magnetic field was measured at the center, with the cavity removed, by means of a fixed coil and a ballistic galvanometer up to 2000 gauss and by a compensated torque type fluxmeter from 2000 gauss and higher. The measurements were made for successive settings of an ammeter on the magnet power supply. The magnetic field was determined as a function of the current through the coils for a given gap size.

A vacuum system with suitable pressure gauges was used to evacuate the sealed cavity through a pumping lead. The particular gas to be measured was introduced into the system after evacuation at the desired pressure. This was read by a McCleod gauge or a manometer.

<u>Procedure</u>

The breakdown experiment was performed at fixed values of the pressure while the current through the coils and hence the magnetic field was set at different values. For a given setting the power fed into the cavity was gradually increased from nearly zero by the power divider. The needle on the microammeter was made to rise to a maximum somewhere near mid-scale by the proper adjustment of the calibrated attenuator until it suddenly dropped. This occurred at breakdown when the electron density suddenly increased and detuned the cavity, producing a mismatch in the line and a consequent reduction in power as indicated by the meter. The maximum reading of the microammeter and the setting of the calibrated attenuator measured the power fed into the cavity at breakdown by virtue of the previous calibration.

Having determined the power in the cavity, measured the Q of the cavity, and knowing the geometrical shape and dimensions of the cavity it was a simple process to calculate the electric field at the center of the cavity at breakdown.

# VII. CONCLUSION

The objectives of this investigation of the effects of a magnetic field upon the breakdown of gases at high frequencies, have been successfully accomplished. The two principal phenomena, energy resonance with transverse fields and the reduction of diffusion loss with increasing magnetic field, have been demonstrated experimentally and explained theoretically. The diffusion effect was shown to exist by itself when the electric and magnetic fields were parallel. The resonance phenomena could not be separated because we had to contend with diffusion at all times. Nevertheless this effect was brought into major prominence in the breakdown of air by reducing the diffusion loss in a large cavity.

Although the general theory developed here was applied only to "Heg", helium with mercury vapor, it can be extended to other gases. Pure helium and hydrogen can be treated quite well by the distribution theory derived here. The mathematics and subsequently the numerical work becomes more formidable. In principle, however, the differential equations for the elastic and inelastic region are the same for the two gases since $\nu_c \approx$ constant. The values of the parameters are of course different. The breakdown equation is no longer as simple as for Heg.

The theory can be applied to neon where the mean free path is taken to be constant. The differential equations can no longer be solved for the whole range of frequencies and values of the magnetic field. Since $\nu_c$ is a function of the velocity, the equations are more complex and in order to get analytical solutions the equations have to be solved for selected regions where $\nu_c \ll \omega, \omega_b, \omega_m$ and where $\nu_c \gg \omega, \omega_b, \omega_m$. Nevertheless, by other approximations, neon can be adequately handled. Possibilities for other gases exist, but only at the sacrifice of quantitative accuracy of the present theory. The main obstacle to attempting the analysis of any but "Heg" is of course the complexity, particularly when the magnetic term is included. Fundamentally no more information about the principal phenomena of the magnetic field is gained by using other gases as can be seen by the similarity of the curves for air and helium. Finally, the elegance of the transformation of effective quantities cannot be applied to the other gases.

One of the ideas introduced in the theory was the concept of the diffusion tensor. This was applied in a limited form to the magnetic problem to yield a more general diffusion equation than used heretofore. The natural consequence of this development was the concept of the effective diffusion length, which together with the effective field, extend here to include the magnetic term, served to generalize the theory. In addition, the use of effective quantities reduced numerical work tremendously.

By employing the same approximations and simplifications it was possible to show the equivalence of the distribution theory and the random walk theory of a single electron. It was this simple theory which checked most successfully with the experimental results. The close agreement was fortuitous since we neglected two effects which cancelled each other. More exact theory when corrected for overshoot and diffusion distortion showed a 15°/o difference. For the present this can be regarded as the limit of accuracy of the theory because the correction was approximate. It would be possible to carry out the transverse field experiment in a rectangular cavity whose height was small compared to the other dimensions. Then the diffusion equation could be applied and solved exactly for uniform field. The correction for the non-uniform field could be derived mathematically in a manner similar to that for the cylindrical cavity. We would then obtain a curve very much like that of Fig. IV-1. The possibility of better agreement under such conditions exists.

At this point we should discuss the possible use of the second order terms in both the spherical harmonic and Fourier expansion. These terms could be included in the development together with the zero and first order terms. This was actually done by Allis.[16] A single differential equation for $f_0$, if it were obtained, would be a fourth order. Assuming that it could be solved, the resultant breakdown equation would be extremely complicated. In view of our results, it seems hardly worthwhile to make this attempt at microwave frequencies.

In addition to the two primary effects of energy resonance and diffusion reduction, the magnetic field also reduced the ionization rate within the cavity and decreased the overshoot of electrons above the excitation potential at a given pressure. The diminished production rate of ions was a consequence of the reduction in diffusion while the smaller overshoot resulted from the decreased effective field as a function of B. The reduced overshoot means a smaller value of $u_i$, where the distribution function $f_0$ goes to zero. This means that $f_0$ of Fig. III-1 shifts to the left as the magnetic field increases. Also $\bar{u} \approx .28\, u_i$, the average energy of the electrons is reduced. Chapman and Cowling[22] arrived at this particular conclusion also.

In the discussion of the diffusion problem we showed theoretically that the magnetic field affected not only the direct diffusion current due to the concentration gradient but that it also produced a transverse diffusion current. The existence of this current has not been shown by our breakdown experiments which are unaffected by it. This effect remains yet to be verified experimentally. The manner of handling the problem is an open question.

Another question that merits discussion is the effect of the magnetic field on the limits of the diffusion theory.[23,24]

The one of most importance to us is the mean free path limit. This is obtained by equating the mean free path $\ell$ to the diffusion length $\Lambda$. Since the magnetic field alters the effective value, a new limit can be obtained. Thus for parallel plates:

$$\ell = \frac{1}{pP_c} = \Lambda_e = \Lambda\sqrt{1 + \omega_b^2/\nu_c^2} \qquad (7\text{-}1)$$

or

$$p\Lambda = \frac{1}{P_c\sqrt{(1 + \omega_b^2/\nu_c^2)}} \; . \qquad (7\text{-}2)$$

The last equation states that the mean free path limit, expressed by $P\Lambda =$ constant, for a given value of the magnetic is reduced by the factor $\sqrt{(1 + \omega_b^2/\nu_c^2)}$. This means that the range of applicability of the theory is extended to lower pressures for a given $\Lambda$. The other limits which are of lesser importance for the magnetic case can be investigated along the lines indicated by S. C. Brown and A. D. MacDonald.[24]

We have limited our investigations to the behavior of a breakdown discharge in a magnetic field and said nothing about the steady state high current discharge. The phenomena as observed by Townsend and his coworkers[3] showed characteristics which were similar to those of breakdown. Since the status of the theory treating high density discharge is still undecided, it was not considered. This problem, together with the determination of steady state densities as a function of the magnetic field, are yet to be investigated.

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## BIOGRAPHICAL NOTE

The author was born on December 29, 1915, in Miskolz, Hungary. He came to the United States in August, 1926. He graduated from Boys' High School, Brooklyn, New York, in 1936 with awards in mathematics, science and languages. During the academic year 1936-37, he attended Brooklyn College, where he received two first prizes in mathematics competitions. In 1937 he entered Cooper Union Institute of Technology, New York City, where he held the Schweinburg Scholarship for four years. He graduated cum laude in 1941 with honors in mathematics and received a bachelor's degree in mechanical engineering.

During the summer of 1941, he worked at Curtiss-Wright Corp., Buffalo, New York. From September, 1941 to June, 1942, he was employed by the U.S. Engineer Office, New York City and attended Brooklyn Polytechnic Institute in the evening. He was inducted into the United States Army in August, 1942 and later attended Officers Candidate School at Fort Monmouth, New Jersey. When commissioned in January, 1943, he was sent through preradar and radar schools at Harvard and M.I.T., respectively. At the completion of the courses in August, 1943, he was assigned to the M.I.T. Radiation Laboratory where he worked as a project officer on ground radar until September, 1945 when he was assigned to the

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In October, 1946, the author entered the M.I.T. graduate school to study physics. His thesis work was done in the Microwave Gas Discharge Laboratory under Professor Sanborn C. Brown. He was elected a member of the Sigma Xi society at M.I.T. At present, he is employed by the Geophysical Research Directorate, Cambridge Field Station, U.S.A.F.

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He presented a paper at the meeting of the American Physical Society in Cambridge, June, 1949 on High Frequency Breakdown in Magnetic Field.