AN EXACT SOLUTION FOR A FINITE MOVING DISLOCATION
IN AN ELASTIC HALF-SPACE, WITH APPLICATION TO
THE SAN FERNANDO EARTHQUAKE OF 1971

by

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A.B., University of California at Berkeley
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ABSTRACT

Seismic displacements caused by a shear dislocation point source in two dimensions are obtained for the unit-step time function, by using Cagniard's method. The solution is checked for all the known properties of the direct body waves, surface P and S waves, Rayleigh waves and the static displacement. The method was then modified to calculate the elastic radiation from a finite two-dimensional fault propagating with constant velocity, constant dislocation-slip along a straight line and it was checked against the known infinite medium solution. The method was then applied to the half-space case with parameters chosen to model the Pacoima Dam record of the San Fernando earthquake, of February, 1971. Synthetic seismograms are produced and checked for their first motion, the static limit of the solution and the far field Rayleigh waves. Comparison between the model and the observed seismogram puts constraints on the geometry of the faulting with respect to the station, the dislocation offset near the hypocenter, and the range of permissible rupture velocity. It also sheds light on the contribution of the free surface to the portion of the Pacoima Dam seismogram that can not be explained by the infinite medium solution.

Guided by the available data on the faulting process, a series of models approximating the known constraints were developed. Displacements and accelerations are compared to observations, and show improvements in the results as our models approximate better the known constraints. The major
features in the observed displacements and acceleration records are success fully compared with the theoretical results. The highest accelerations on the Pacoima Dam record are shown to be caused by the surface P waves described for the line source by Lapwood (1949), followed closely by Rayleigh and shear waves generated from the upper tip of the fault.

Thesis Supervisor: Keiiti Aki
Title: Professor of Geophysics
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Immesurable gratitude is also properly due to my parents for their continuing moral support.

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CHAPTER 1

Introduction and Review of Near-field Studies

1.1 Introduction

The studies of seismic sources and seismic waves within distances shorter than the wavelengths has lagged behind other branches of seismology because of several complications. In the far field, seismic waves may be decomposed in various rays and modes arriving separately in time and allowing the investigator to use the appropriate approximation required to gain insight on the properties of the source and the medium in the period range of interest. It is for this reason that first motion seismology and long period normal mode seismology has contributed most of the available knowledge about the earth. In the near field studies of seismic waves and sources, one faces a seismic motion in which all waves arrive almost simultaneously. Proximity to the source may cause severe interferences between waves arriving from different portions of the source. The need for improving near field predictive theory is well put in the words of Professor Keiti Aki (1971\textsuperscript{19}): "On the other hand, one must use 'EXACT THEORY' if the records are obtained close to the source", the meaning of the word "EXACT" must be taken in its proper perspective. I will expand on this point further in discussing the Pacoima Dam record of the San Fernando earthquake which is under study here.
1.2 Review of application of static and dynamic dislocation theory to near field studies of earthquakes

Most of the theoretical work on dynamic and static elastic fields generated by earthquake sources have been carried out within the framework of the elastic dislocation theory. In a pioneering work, Steketee (1958) calculated the static field due to a shear dislocation in which a slip (discontinuity in displacement) occurs in the direction parallel to the fault plane. Maruyama (1963) obtained an expression for elastic displacements due to a dynamic shear dislocation in an unbounded medium, and showed that the dislocation source is equivalent to a distribution of double couples. Burridge and Knopoff (1964) also obtained the equivalent body force distributions for general discontinuities in displacements and stress. Their equivalence relation is applicable to a general inhomogeneous, anisotropic, bounded medium. Maruyama (1964) has calculated the Green's function for static displacements in an infinite and semi-infinite medium. The corresponding two dimensional solutions were given also by Maruyama (1966). Successful applications of static dislocation theory are quite numerous, and the reader is referred to an extensive review by Savage (1972). I will review here only the work relevant to the San Fernando earthquake of February, 1971 and other relevant near field studies.

The conventional dislocation theory and its applications
to earthquake seismology, contains some arbitrariness in that one has to specify the dislocation function \( \mathbf{\Delta u}(r,t) \) on the fault plane in order to predict the elastic field. It is desirable that \( \mathbf{\Delta u} \) be determined from the initial conditions on the stress state of the medium, the fracture criterion and material properties. Unfortunately, the mathematical difficulties involved in these crack problems are quite formidable even in the simplest of cases, as discussed by Kostrov (1966), Burridge (1969) and others. The formula obtained by Brune (1970) is more practical and has been enthusiastically applied to many earthquakes by his followers. He assumed an instantaneous stress drop over the whole fault plane (implying an infinite rupture velocity) to a relation between the effective stress and the initial particle velocity. He also computed the corresponding far-field seismic spectra, which in my opinion is useful only for an order of magnitude estimate of the physical parameters of the seismic source because of the lack of rigorousness in the derivation. Iida and Aki (1972) have solved a problem of longitudinal shear crack (anti-plane stress drop in two dimensions), and shown that the source time function of Brune (1970) is applicable only to supersonic rupture if a suitable correction is applied. They concluded that the nature of source time function is quite different for subsonic rupture. For example, the front of the rupture
has a singularity in the stress components similar to the static ones. The most promising approach toward obtaining a realistic and reliable source function seems to be the numerical methods such as the finite element and finite difference methods. Some progress has been made in this direction by Burridge (1969), Hanson et al. (1971), Dietrich (1972) and others.

The first successful application of dislocation theory to the near field displacement of an earthquake was made by Aki (1968). He integrated Maruyama's (1963) Green's function over the fault surface to obtain the seismic displacement near a propagating strike-slip fault. The synthetic seismogram was obtained by using step-like source time function with a finite rise-time and compared with the twice integrated accelerogram recorded at station No. 2 of the Parkfield earthquake of (1966) (Cloud, 1966). The synthetic wave-form agreed well with the observed one, for the displacement in the direction perpendicular to the fault plane. The observed displacement required a slip of about 50 cm., which was 10 times greater than the slip actually left on the ground surface as reported by Allen and Smith (1966). The discrepancy was attributed to a decoupling layer at a depth of 100 meters. Haskell (1969) made a similar calculation using a different technique, assuming a ramp function for the source time dependence and found a similar result, that the fault slip
required to explain the observed large impulsive displacement, is an order of magnitude greater than that detected on the surface. Boore et al. (1971) gave an analytic solution for an infinite, uniformly moving two dimensional dislocation model. Their solution was restricted to rupture velocities smaller than the shear wave velocity of the medium because they have assumed a steady state fault, moving with uniform rupture velocity since \( t = -\infty \).

Since the medium is assumed to be infinite in the work of Aki (1968) and Haskell (1969), they can not explain the surface waves which often are the main part of a seismogram. The method may be applicable to the beginning of body waves before the effects of free surface become significant on the observed record. It is for this reason that the following workers have attempted successfully to explain the initial P and S motion on several well studied earthquakes.

Kanamori (1971) was able to explain the initial P and S motion from the Sanriku earthquake of 1933, at a distance comparable to the fault dimension. The agreement between the observed and the synthetic records was rather remarkable. It gave much credibility to the dislocation theory. Kanamori assumed in his work a rupture velocity of 3.5 km/sec, a rise time of 10 sec., and a dislocation of 2.5 to 5 meters which is in good agreement with estimates from the surface wave amplitudes. In all the previous work and much of what will
be reviewed next, it was assumed that the free surface approximately doubles the amplitudes of the infinite medium solution. This assumption is known to be valid for SH-type motion and up to now there has not been any quantitative and theoretical justification for it. The results that will be presented here, will give the first check on the validity of that intuitive assumption in the near-field of a finite source. The rupture velocity (3.5 km/sec) chosen by Kanamori is still below the shear wave velocity (4.6 km/sec) for the medium. He gave an estimate of the stress drop to be about 39 bar. Kanamori (1972) made another successful comparison between a dislocation model and the observed seismogram of the Tottori earthquake of 1943 at Abuyama station located at a distance of approximately 4-5 times the fault length. He concluded that the rupture velocity was 2.3 km/sec in the medium with shear wave velocity 3.5 km/sec and an average fault slip of 2.5 m, which was in an approximate agreement with the slip determined from the triangulation data. He estimated the stress drop to be 83 bar and an effective stress of 30 to 100 bars, based on the maximum particle velocity on the fault inferred from the rise time T = 3 sec. that was determined by comparing observed and synthetic seismograms. He required bilateral faulting to minimize the disagreement between the observed and the theoretical records. Still the agreement was less satisfactory than the case of Sanriku earthquake,
possibly due to the greater distance between the station and the source. Abe (1974a) made a similar study on the observed displacements of the Saitama earthquake of 1931 as recorded at Hongo station. He concluded that the rupture was bilateral at a velocity of 2.3 km/sec., with rise time of 2 sec. and an average fault slip of 1 meter. The station was at a distance approximately 3 times the fault length. The fit of the initial P and S motion was remarkable. The shear wave velocity of the medium was 3.5 km/sec., the effective stress was estimated between 30 and 60 bars and the stress at 43 bars. The obtained fault slip was consistent with the geodetic observations. In another study, Abe (1974b) determined the parameters of the Wakasa Bay earthquake of 1963. He obtained a rupture velocity of 2.3 km/sec (shear wave velocity 3.5 km/sec), rise time of two seconds, effective stress of 40 bars and a stress drop of 32 bars. He used the initial motion on the seismogram at Maizuru located at about twice the fault length from the epicenter. The agreement between the synthetic and observed wave forms was very well at Maizuru and to a lesser at Abyama located about five times the fault length away from the epicenter. The faulting was bilateral with a dislocation amplitude of .65 m. The initial half-cycle P and S motion at several other far stations was successfully compared with the model prediction while disagreement in the latter portions were attributed to the free surface effects.
1.3 Review of the source mechanism study on the San Fernando earthquake of February, 1971

Let us now turn to the works done on the San Fernando earthquake of February, 1971, which is the subject of the present thesis. The San Fernando earthquake is one of the best studied earthquakes in the history of seismology. A great deal of work was done to estimate the source parameters, and explain the observed physical quantities of that earthquake. Trifunac (1974) has summarized the results of various estimates of the source parameters made by independent observers, and his summary with some modification will be reproduced here, in table 1. The fault plane solutions obtained by Whitcomb (1971), Wesson et al. (1971), Dillinger and Espinosa (1971) and Canitez and Toksöz (1972) are all consistent within the observational error and are consistent with the observed surface faulting (Burford, et al., 1971; Kamb et al., 1971; Lahr et al., 1971). The Pacoima Dam accelerograms (Trifunac and Hudson, 1971) have provided the opportunity and the challenge to study and explain the good quality strong motion record in the vicinity of a thrust fault breaking the earth surface, which is the main goal of this thesis. The distribution of the aftershocks outlining the fault boundary (Allen et al., 1971), and the estimates of seismic moment from surface waves by Canitez and Toksöz (1972) indicate the two dimensionality assumption is a reasonable approximation. This assumption has been made by Boore and Zoback (1974),
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for the San Fernando earthquake

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Litchiser (1974) in studying the San Fernando earthquake. Hypocentral depth can be reasonably constrained at a depth of 14 km. within a few kilometers (Canitez and Toksöz, 1972; Mikumo, 1973; Hanks, 1974). A change in the epicentral location as determined by Allen et al. (1973), by 4 km to the north was proposed by Hanks (1974). According to Boore and Zoback (1974) that would make the fit between the theoretical wave form and the observed one at the Pacoima Dam poor. The average ($\bar{\Delta u}$) of the fault slip $\Delta u$ on the fault plane is related to the seismic moment with the well-known relation:

$$M_o = \mu \bar{\Delta u} S$$

where $M_o$ is the seismic moment, $\mu$ is the rigidity, $S$ is the fault area. $M_o$ is reasonably well constrained for a dynamic model from the spectra of long period surface waves, within a factor of 2 or less as was demonstrated by Tsai and Aki (1969) and pointed out again by Aki (1972a).

Let us now look at table 1, which is adapted from Trifunac (1974) with more data and more explanations added to it. Three determinations of the seismic moment from the spectra of surface waves are given. Aki (1971b) determined the moment from long period surface waves to be $0.77 \times 10^{26}$ dyne·cm. He expressed confidence that his surface wave determination is accurate within a factor of 2 (personal
communication (1974), in view of the good quality long period data available. Canitez and Toksöz (1972) determined it to be \(0.75 \times 10^{26}\) dyne-cm. They matched the observed spectra at six different stations having a wide azimuthal coverage about 80° around the source. The quality of the fit was quite good. Wyss (1971) determined the moment to be \(1.9 \times 10^{26}\) dyne-cm. He used the spectra at periods of 50 sec. at two stations only. The azimuthal coverage was in the neighborhood of 5° or 10°. One station was at Stuttgart, West Germany; it yielded a seismic moment of \(0.8 \times 10^{26}\) dyne-cm. The other was at Uppsala, Sweden, and it gave a moment of \(3 \times 10^{26}\) dyne-cm. The reported moment has the average of the two values. All the previous determinations, approximated the source by a point-double couple. Alewine (1974), using several point sources on the San Fernando fault plane, have showed that the point source estimates of the moment tend to underestimate the moment by approximately a factor of 2. This will bring the moment estimates of Aki (1971b) and Canitez and Toksöz (1972) closer to other estimates of the moment from surface waves and static-work. Body wave determinations of the moment by different observers, show more scattering as shown in table 1. Near field studies cannot provide a good constraint on the moment because the dislocation field is strongly dependent on the distance between the source and the receiver. The amplitude in the near field is more dependent on the fault slip at the portion
of the fault that is closest to the station rather than the total moment. Thus it seems that to obtain a useful dynamic dislocation model, one should constrain the seismic moment from surface waves data. The fault area that ruptured during an earthquake is not very well constrained because of the uncertainty in determining the fault boundaries, specially the deep part of it. It is usually presumed that the fault is outlined by the aftershocks that occur due to the stress concentration at the dislocation surface boundary. This is perhaps the best way to constrain the fault area. Any attempt to constrain the fault area by a dislocation model cannot yield a better estimate because the area is dependent on both the seismic moment, the average fault slip \( \bar{\Delta}u \) as well as the rigidity. Other methods of estimating the area were reviewed by Aki (1972a). He concluded that for shallow earthquakes, the fault area is probably accurate within a factor of 2. A glance at table 1, shows that the fault area estimate, obtained from aftershocks by three independent observers is the most consistent estimate. This area is likely to be larger than the actual area of the dynamic rupture as was pointed out by Aki (1968, 1972b). Thus a fault area which is approximately bounded by: \( 220 \text{ km}^2 \leq S \leq 460 \text{ km}^2 \) seems to be a reasonable constraint on the dynamic rupture area during the San Fernando earthquake.

Let us look now at some of the quantitative studies
related to the San Fernando to get a feeling for other important parameters needed for describing a dislocation model, namely the rupture velocity, the rise time and still the connection between $\Delta u$ and $S$. Based on a qualitative study of the Pacoima Dam record, Bolt and Gopalakrishnan (1972), identified a phase on the accelogram as shear radiation from the point of the closest approach to the fault, proposed a rupture velocity of 2.5 km/sec.

They also pointed out, that because of the proximity of the ruptured fault trace to the Pacoima Dam, Rayleigh waves may have not been large enough to be observable on the accelogram. Niazy (1973), based on an exact solution for a finite two dimensional dislocation model in an infinite medium, excluded high rupture velocity close to the assumed shear wave velocity of the medium (3.3 km/sec) on the basis of the sharp wave-form associated with such a high rupture velocity which was not observed on the Pacoima Dam record. He also excluded low rupture velocity (1 km/sec) again on the basis of inconsistency with the observation. Using a fault slip of about 1 meter, $\mu = 3 \times 10^{11}$ dyne/cm$^2$, $M_o = 0.75 \times 10^{26}$ dyne cm (Canitez and Toksöz, 1972), fault area $S=270$ km$^2$, rupture velocity of 2.5 km/sec, he was able to explain the observed amplitude and major features of the displacement record by assuming that the effect of the free surface is approximately to double the amplitude. Tsai and
Patton (1972) were able to explain the major features on the velocity record using Haskell's (1969) method for a three dimensional infinite medium solution. They used a fault area of 231 km$^2$, $\mu = 3 \times 10^{11}$ dyne cm$^{-2}$, $\Delta u = 1.5$ meter, rupture velocity $v = 3$ km/sec, rise time $T = .6$ sec. This implied a seismic moment $M_o = 1.01 \times 10^{26}$, still within a factor of 2 of the surface wave determination for the moment. Boore and Zoback (1974), did a similar work to that of Tsai and Patton (1972). They used a two dimensional infinite medium solution for the velocity field, and concluded that $v = 2.5$ km/sec, $T = .6$ sec, $\Delta u = 1$ meter, satisfy the observations. Bolt (1972), by studying the frequency content of the accelerogram record, and by assuming that the low frequency portion of the record ends at the time when the shear radiation from the closest approach distance of the dislocation reaches the station, he revised the rupture velocity estimate given by Bolt and Gopalakrishnan (1972) from 2.5 km/sec to 3 km/sec. He proposed that the latter portion of the record is caused by random bursts of high frequency energy from the far side of the rupture front. Mikumo (1973) using a three dimensional infinite medium dislocation model obtained the following parameter for the San Fernando earthquake:

\[
V = 2.5 \text{ km/sec}, \ M_o = 1.1 \times 10^{26}, \ S = 266 \text{ km}^2, \ \Delta u = 1.4 \text{ m},
\]

stress drop 40-65 bars, $T = 1$ sec
He tried to compensate the effects of the free surface by resolving the displacements from each elementary dislocation into P, SV and SH components along each ray path and then multiplied by the reflection coefficient at the surface appropriate to these components. After doing this, the compensated displacements were again corrected for the XYZ-coordinate system, and all contributions from the entire fault surface were superposed. The approach presumes that the waves are plane and hence, at best it is only approximate. Mikumo stated that the approach has failed to yield reasonable results. Most of the previous quantitative results seem to yield a consistent picture about the rupture velocity, the rise time and the average slip $\bar{\Delta u}$, and reasonable fault area. Trifunac (1974) carried out an inversion, assuming a certain fault shape, constant rupture velocity $v = 2$ km/sec., for the distribution of $\Delta u$ over various fault elements. In addition to the Pacoima Dam record, he used four additional records, at stations ranging in distance from the epicenter between 1 to 3 times the focal depth. He took the upper frequency limit to be 1 Hz. (p. 152), and the lower limit at .1 Hz. (p. 160).

The rise time was .5-1 sec. The fault area was arbitrarily chosen to be $= 130$ km$^2$. The inversion scheme resulted in slip displacements as large as 11.9 meters near the hypocentral region, decreasing near the middle of the
fault to about 1 meter and increasing to 5-7 meters close to the surface. This yielded a seismic moment of $1.53 \times 10^{26}$ dyne cm, $\Delta u = 3.93$ meters.

I believe that these high slip values are a consequence of the following factors.

1. Total area used in this model was chosen arbitrarily and considerably smaller than the area constrained by the aftershocks, and the area estimates used by all other dislocation models for the San Fernando earthquake.

2. The very high slip near the hypocentral region is again a consequence of choosing the area of the fault there, considerably narrower than that outlined by the aftershocks.

3. The fault dip was constrained by 6 independent observers to be in the neighborhood of 52° (see table 1), using fault plane solution and surface wave data. Whitecomb et al. (1973) give error bound on it of $\pm 3°$. Still Trifunac (1974) used an arbitrary value of 40°.

4. By adding records other than that at the Pacoima Dam to the inversion data, Trifunac did not improve the inversion but probably degraded it. The other records, contained a considerable surface wave contribution which the model cannot include, because it had no free surface in it (Haskell, 1969). Indeed the comparison between the theoretical and observed in his model, for stations other than the Pacoima Dam, is quite poor in my opinion (see Trifunac, 1974, Fig. 5);
the records speak for themselves. Hanks (1974) has studied the Pacoima Dam qualitatively using Brune's (1970) model. He proposed that the epicenter be moved northward by about 4 km, and I have already mentioned the objection of Boore and Zoback (1974) to that. Indeed Boore and Zoback (1974) argue even against making the hypocenter shallower (as proposed by Allen et al. (1973)) because that also causes the agreement between the theoretical and observed velocity wave forms to deteriorate. Thus if we reject the proposal of moving the epicenter northward, the proposed increase in rupture velocity to 2.8 km/sec (Hanks (1974), p. 1217) is no longer needed. On the basis of the pulse in the velocity record at the time of the shear wave arrival at the Pacoima Dam, Hanks proposes an unusually energetic initial rupture. Using Brune's model, he concludes that the initial rupture had a dimension of 6-3 km, an average slip of 4.6-9.2 meters and a stress drop 350-1400 bars. Now the velocity pulse on the Pacoima Dam record was explained successfully by Tsai and Patton (1972) using a uniform dislocation offset of 1.5 meters, a rupture velocity of 3 km/sec, seismic moment of $7.5 \times 10^{25}$ dyne cm and a fault area of 231 km$^2$ by using a three dimensional dislocation model. It was also explained successfully by Boore and Zoback (1974) using a two dimensional dislocation model with uniform $\Delta u = 1$ meter, and a uniform rupture velocity of 2.5 km/sec. There is no need to require
such an energetic initial rupture near the hypocentral region. Thus all the features of the Pacoima Dam record that require a dramatically energetic initial rupture, according to Brune's model, can be very easily explained in the light of simple and quantitative uniform dislocation models with uniform dislocation slip of the order of one meter, and there is no need to require a highly non-uniform stress distribution.

I want to emphasize here that all the previously described dislocation models can not explain the latter portion of the Pacoima Dam record which is 6 seconds after the triggering of the instrument. That is the most interesting portion which contains the highest acceleration ever recorded during an earthquake (1.25 g).

It is very rich in high frequency motion. Such high accelerations are of great importance in earthquake engineering especially to determine if they could have relatively long periods to pose any threat to engineering structures. Another conflict between proposed dislocation models and the observations concerns the initial motions on the vertical and on the S15°W horizontal displacements. The latter was erroneously reported as S15°E in Trifunac and Hudson (1971) (see Trifunac, 1974, for this correction). The question is whether they were real or only a result of the filtering, instrument response and the numerical integration involved in obtaining the observed displacement. There is no way in which a reasonable dislocation
model in an infinite medium model could explain the senses of those initial motions. I will discuss this question in the light of a dislocation model in half-space near the end of this dissertation.

Other interesting studies relevant to the San Fernando earthquakes are the work by Jungels and Frazier (1973) and by Mal (1972). Jungels and Frazier (1973), used a finite element procedure to generate a two dimensional static dislocation model, that includes geological structure. Their best model was obtained by separating the fault into two distinct parts, having offsets near the surface, a factor of 5 larger than the average slip. Their seismic moment was only $0.62 \times 10^{26}$ and an average stress drop of 24 bars. One permissible solution that gives results consistent with the observations allows as much as five meters slip in the hypocentral region. However, I do not consider these results as constraining the dynamic slip to have such a large initial value, because they may contain aseismic slip contribution. Mal (1972) computed far-field Rayleigh waves from a two dimensional moving thrust fault, that reaches the free surface. He assumed a uniform rupture velocity and step function for the source time dependence. His results (Fig. 9) indicate that the amplitudes of the Rayleigh waves are infinite at the time of arrival from the point of ground breakage. His results are of limited use for two reasons. The two dimensionality assumption is usually only valid in the near-field of seismic source (e.g. Pacoima Dam station
for the San Fernando earthquake, or station no. 2 of the
Parkfield earthquake) but there, the body waves are quite
significant and inseparable from the surface waves. It is
there that an exact solution is needed. Another weakness
is that the field was only calculated for a fault dip angle
\( \theta = 45^\circ \); it will be shown in this work that the solution
for a different angle will involve another linearly independent
set of eigenfunctions and not a simple rotation of the 45°
solution. Lastly, he showed the behavior of the solution at
station at the epicenter and at a station approximating the
location of the Pacoima Dam, both stations are in the near-
field and hence the behavior of the solution is not very
meaningful there. The far-field behavior of his solution was
not exhibited, where the solution is mathematically valid but
physically not realizable because of the two dimensionality
assumption.

1.4 Significance of the free surface effect and the
unexplained portion of the Pacoima Dam record

From the results of all the previous works done on near-
field of dislocation models, and the work relevant to the
San Fernando earthquake, we can safely conclude:
1. Dislocation models in an infinite medium adequately
describe the initial body wave displacement in the
vicinity of earthquake sources.
2. Dislocation models in an infinite medium adequately describes the first portion (6 sec.) of the Pacoima Dam record with the exception of the initial motion, which may have been caused by filtering or may be real at least in part.

3. Plane-wave approach of Mikumo (1973) to correct for the free surface for P and SV waves was not satisfactory.

4. Initial motion of the Pacoima Dam record of displacement cannot be explained for the source models in the infinite medium.

5. The latter portion of the Pacoima Dam record that contains the highest recorded acceleration (1.25 g) and large high frequency motion can not be explained in terms of dislocation models in an infinite medium. It is important in earthquake engineering to understand this portion to see what causes these high accelerations.

1.5 Importance of an exact solution

It is clear now that we have to formulate a more realistic model if we are to explain the latter portion of Pacoima Dam record. It may still be debatable as to whether or not it would be easier and more flexible to follow a numerical scheme like finite element to gain insight in the record of interest. Jungels and Fraziers (1973) mentioned that the finite element technique could serve to simulate the San
Fernando dynamic record, but they added that it is not well suited to producing high frequency motion, higher than 1 Hz, which is exactly the portion of interest in the Paccima Dam record. According to Smith (1974), the dynamic finite element analysis requires at least 8 elements per wave length. Hence the error could become a problem at high frequency. For all these reasons, I have decided to find exact solutions for seismic motion due to finite dislocation models in a halfspace and compare them with the Paccima record.
CHAPTER 2

Theory

2.1 Boundary conditions for the line source solution

Let us consider the elastic field generated by a line source equivalent to a shear dislocation buried in an elastic half space. The equivalent source is a double couple with vanishing total moment but the moment of the component couple is given by $M_\circ = \mu \Delta u \, d\Sigma$ as was shown by Maruyama (1963) and Burridge and Knopoff (1964) where $\mu$ is the rigidity and $\Delta u$ is the slip on the fault plane $d\Sigma$. We require a free surface without traction and the Sommerfeld-radiation condition that no waves come from infinity. Since the field due to a double couple can be obtained from that of a single force by superposition, we shall first consider the field due to a single force. We will always assume here linear elasticity, homogeneity and isotropy of the medium.

2.2 Steady state solution for the line source

We take a coordinate system in which the origin is at the free surface, the x-axis is parallel to the free surface and the z-axis is perpendicular to it downward. The concentrated force $(F_x, F_z) S(x)e^{i\omega t}$ where $S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dk$, is at depth $h$. Let us denote quantities in the region $z < h$ with a "-" superscript, and quantities in the region $z > h$ with a "+" superscript. The x-component displacement $u$ and
the z-component displacement $w$ can be written as:

$$
u = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial z}$$

(1)

$$w = \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x}$$

where

$$\varphi = \begin{cases} 
\psi^- & z \leq h \\
\psi^+ & z > h 
\end{cases}$$

$$\psi = \begin{cases} 
\psi^- & z \leq h \\
\psi^+ & z > h 
\end{cases}$$

They satisfy the wave equations

$$\frac{\partial^2 \varphi}{\partial t^2} = \alpha^2 \nabla^2 \varphi$$

(2)

$$\frac{\partial^2 \psi}{\partial t^2} = \beta^2 \nabla^2 \psi$$

where $\alpha$ is the P-wave velocity of the medium and $\beta$ is the S-wave velocity of the same medium.

Now $F(x) e^{i\omega t}$ is equivalent to a discontinuity in the stress component $\tau_{zz}$ at $z = h$ as was shown by Burridge and Knopoff (1964). Similarly, $F(x) e^{i\omega t}$ is equivalent to a discontinuity in $\tau_{xz}$ at $z = h$. The displacement components $u$ and $w$ are continuous at $z = h$. The free surface condition

$$\tau_{xz} = \tau_{zz} = 0 \quad \text{at} \quad z = 0$$
Thus we have the following equations:

\[
\begin{align*}
\left[ \tau_\beta^- - \tau_\beta^+ \right]_{z=h} &= -F_\beta e^{i\omega t} \quad (a) \\
\left[ u^+ - u^- \right]_{z=h} &= 0 \quad (b) \\
\left[ \tau_{x\beta}^+ - \tau_{x\beta}^- \right]_{z=h} &= -F_x e^{i\omega t} \quad (c) \\
\left[ W^+ - W^- \right]_{z=h} &= 0 \quad (d) \\
\left[ \tau_{z\beta}^- \right]_{z=0} &= 0 \quad (e) \\
\left[ \tau_{x\beta}^- \right]_{z=0} &= 0 \quad (f)
\end{align*}
\]

The following potentials satisfy the wave equation (2) and the radiation condition:

\[
\begin{align*}
\psi^- &= \Phi^- e^{i\omega t - ikx + \gamma_4 z} + \Phi^- e^{i\omega t - ikx - \gamma_4 z} \\
\psi^+ &= \Phi^+ e^{i\omega t - ikx + \gamma_4 z} + \Phi^+ e^{i\omega t - ikx - \gamma_4 z} \quad \left( z < h \right) \\
\psi^- &= \Phi^- e^{i\omega t - ikx + \gamma_\beta z} + \Phi^- e^{i\omega t - ikx - \gamma_\beta z} \\
\psi^+ &= \Phi^+ e^{i\omega t - ikx + \gamma_\beta z} + \Phi^+ e^{i\omega t - ikx - \gamma_\beta z} \quad \left( z > h \right)
\end{align*}
\]

\[\gamma_4 = i\sqrt{\frac{\omega^2}{\alpha^2} - k^2} \quad , \quad \alpha^2 = (\lambda + 2\mu)/\rho \quad , \quad \operatorname{Re}(\gamma_4) > 0\]

\[\gamma_\beta = i\sqrt{\frac{\omega^2}{\beta^2} - k^2} \quad , \quad \beta^2 = \mu/\rho \quad , \quad \operatorname{Re}(\gamma_\beta) > 0\]

\(\rho = \text{density} \), \(\lambda, \mu\) are the Lamé's constants.
The stresses $\tau_{xz}$, $\tau_{zz}$ are given in terms of the potentials by:

$$
\tau_{xz} = \mu \left( \frac{3}{2} \frac{\partial \psi}{\partial z} + \frac{3}{2} \frac{\partial \psi}{\partial x} - \frac{3}{2} \frac{\partial \psi}{\partial y} \right)
$$

$$
\tau_{zz} = \lambda \nabla^2 \psi + \gamma \mu \left( \frac{3}{2} \frac{\partial \psi}{\partial z} - \frac{3}{2} \frac{\partial \psi}{\partial x} \right)
\tag{5}
$$

Thus making use of (1), (5) and (4), equations (3) give us

$$(2\mu k^2 - \rho\omega^2) \Phi_+ e^{2\phi h} + 2i\mu k \gamma \Phi_+ e^{2\phi h} - (2\mu k^2 - \rho\omega^2) \Phi^- e^{2\phi h}$$

$$- (2\mu k^2 - \rho\omega^2) \Phi^- e^{2\phi h} - 2i\mu k \gamma \Phi^- e^{2\phi h} + 2i\mu k \gamma \Phi_+ e^{2\phi h}$$

$$= - F_x$$

$$-ik \Phi_+ e^{2\phi h} \gamma \Phi_+ e^{2\phi h} + ik \Phi^- e^{2\phi h} + ik \Phi_+ e^{2\phi h} \gamma \Phi_+ e^{2\phi h} + 2i \gamma \Phi_- e^{2\phi h} = 0$$

$$\mu [2i k \gamma \phi^+ e^{2\phi h} + (2k^2 - \omega^2) e^{-\phi h} \Phi_+ + 2i k \gamma \phi^- e^{2\phi h} \Phi^- - 2i k \gamma \phi^- e^{2\phi h} \Phi_+]$$

$$= - F_x$$

$$-2i k \phi^+ e^{2\phi h} + ik \phi^+ e^{2\phi h} \gamma \Phi_+ e^{2\phi h} + 2i \gamma \phi^- e^{2\phi h} - ik \phi^- e^{2\phi h} \gamma \phi^- e^{2\phi h} = 0$$

$$(2\mu k^2 - \rho\omega^2) \Phi_+ + (2\mu k^2 - \rho\omega^2) \Phi^- + 2i \mu k \Phi^- \gamma \Phi_+ - 2i \mu k \gamma \Phi_+ \Phi^- = 0$$

$$-2ik \gamma \Phi_+ + 2ik \gamma \Phi^- + (2k^2 - \omega^2) \Phi_+^- + (2k^2 - \omega^2) \Phi_-^+ = 0$$

$$= 0$$

$$(3, a)$$

$$(3, b)$$

$$(3, c)$$

$$(3, d)$$

$$(3, e)$$

$$(3, f)$$
Thus we have six equations in the six unknowns \( \Phi^-, \Phi^-, \Psi^-, \Psi^-, \Phi^+ \) and \( \Psi^+ \). Since we will be mainly concerned with the field observed on the free surface, we give solutions for \( \Phi^- \) and \( \Psi^- \).

\[
\begin{align*}
\Phi^- & = \frac{e^{-2\beta h}}{2\rho \omega^2} \left[ \frac{ik}{\omega} F_x - F_z \right] \\
\Phi^- & = \frac{1}{\rho \omega^2} \left[ \frac{-2ik(2k^2 - \omega^2)e^{-2\beta h}(F_x+i\omega F_z)}{F_-} - \frac{F_+}{2F_-} \frac{2\omega F_z - i\omega F_x}{\omega^2} \right] \\
\Psi^- & = \frac{-1}{\rho \omega^2} e^{-2\beta h} \left[ F_x + \frac{ik}{\omega} F_z \right] \\
\Psi^- & = \frac{1}{\rho \omega^2} \left[ \frac{2ik(2k^2 - \omega^2)e^{-2\beta h}(2\omega F_x - i\omega F_z)}{F_-} - \frac{F_+}{2F_-} \frac{2\omega F_x + i\omega F_z}{\omega^2} \right]
\end{align*}
\] (6)

where \( F_- \), \( F_+ \) are defined by:

\[
\begin{align*}
F_- & = 4k^2 \omega^2 \omega^2 + (2k^2 - \omega^2)^2 \\
F_+ & = 4k^2 \omega^2 \omega^2 + (2k^2 - \omega^2)^2
\end{align*}
\] (7)

\( F_- = 0 \) gives the well-known period equation for Rayleigh waves.
From now on, we will drop the "-" superscript, keeping in mind that we are referring to the field at the free surface. The displacement potentials due to \((F_x', F_z') e^{i\omega t}\) will thus be given by (4) and (6) in the form:

\[
\frac{e^{i\omega t}}{\kappa} \int_{-\infty}^{\infty} \varphi(k, \omega) dk, \quad \frac{e^{i\omega t}}{\kappa} \int_{-\infty}^{\infty} \psi(k, \omega) dk
\]

for the steady state.

The displacements due to a double-couple are obtained by superposition of those due to the single force. If we denote the potentials caused by \(F_x\) by an x-subscript and those by \(F_z\) by a z-subscript, the displacements due to a double couple become:

\[
\begin{align*}
\varphi &= z M_0 n_x f_x \left( \varphi_{x,xx} + \varphi_{x,xz} + \varphi_{z,xh} + \varphi_{z,zh} \right) - M_0 \left( n_x f_z + n_z f_x \right) \\
\psi &= z M_0 n_x f_x \left( \varphi_{x,xz} - \varphi_{x,xx} + \varphi_{z,xh} - \varphi_{z,zh} \right) - M_0 \left( n_x f_z + n_z f_x \right) \\
\end{align*}
\]

(8)

where \(M_0\) is the moment of the component couple, related to the fault slip of the infinitesimal dislocation by the
previously mentioned relation: \( M_0 = \mu \Delta u d \Sigma \), \( \hat{n} = (n_x, n_z) \) is the unit normal to the fault plane, \( \hat{f} = (f_x, f_z) \) is the unit normal to the auxiliary plane, so that \( n_x f_x + n_z f_z = 0 \).

If we take \( \hat{n} \) or \( \hat{f} \) parallel to the z-axis then clearly the first term of the right hand side of (8) becomes zero, if \( \hat{n} \) (or \( \hat{f} \)) makes 45° with the x-axis the second term becomes zero. Thus the solution for a double couple of arbitrary orientation is a linear combination of the above two independent solutions.

2.3 Step function solution for the line source, using Cagniard's method

We shall now obtain the displacements \( u, w \) when the source time function is a unit, step-function by the use of Cagniard's method. The method is discussed in detail by Cagniard (1962). We note that each term in equation (8) consists of the sum of upgoing waves and downgoing waves.

For example,

\[
\psi_x = \Phi_{lx}^- e^{i\omega t - ikx + \gamma_d z} + \Phi_{zx}^- e^{i\omega t - ikx - \gamma_d z}
\]

where \( \Phi_{lx}^- \), \( \Phi_{zx}^- \) are obtained by putting \( F_z = 0 \) in equation (6). The solution corresponding to the "1" subscript is the infinite medium solution obtained by Niazy (1973). The other terms with the subscript "2" involve the Rayleigh denominator and hence the Rayleigh
roots. The steady state solution is a sum of terms of the form:

\[ I = e^{i\omega t} \sum_{-\infty}^{\infty} F(\omega, k)e^{-ikx-h2t} \]

where \( c = \alpha \) or \( \beta \), where \( F \) is either even or odd in \( k \).

To apply Cagniard's method, we obtain first the response to \( e^{pt} \) by letting \( p = i\omega \), and look for a transformation \( \tau(k) \) so that we can write \( I = e^{pt}\int_{0}^{\infty} g(\tau) p^n e^{-pt} d\tau \)

and identify \( g(\tau) \) as the impulse response or one of its derivatives or indefinite integrals depending on the integer value \( n \). We shall assume that \( p \) is positive, real and rely on Cagniard's assertion that the result will be unique and valid for complex \( p \) as long as the impulse response has integrable singularities if any (Cagniard, 1962).

Now depending on whether \( F(\omega, k) \) is an odd or even function of \( k \), we can rewrite:

\[ I = \begin{cases} 2e^{pt} Re. \int_{0}^{\infty} F(p,k)e^{-ikx-h2t} dk & \text{for even } F \\
2e^{pt} Im. \int_{0}^{\infty} F(p,k)e^{-ikx-h2t} dk & \text{for odd } F \end{cases} \]

The singularities of the integrand are shown in Fig. 1 in the \( k \)-plane. We take the branch cut along the imaginary axis connecting pairs of branch points. We choose our Riemann
Fig. 1 Branch cut and singularities in the complex $k$-plane. The branch cut is along the imaginary axis and the branches are determined by eq. 9. The integration path is along the real axis.
sheet by requiring that \( \text{Re}(\mathcal{U}_c) \geq 0 \) for \( \text{Re}(k) \leq 0 \). This will assure the radiation condition along the path of integration \( \alpha \leq k \leq \infty \). On this Riemann sheet, we find that \( \text{Im}(\mathcal{U}_c) \leq 0 \) for the quadrant I and \( \text{Im}(\mathcal{U}_c) > 0 \) for the quadrant IV. Our choice of branch cut, then determines the signs of \( \mathcal{U}_c \) for \( k \leq 0 \).

Summarizing, we have

\[
\text{Re}(\mathcal{U}_c) > 0 \quad \text{in I} \quad \text{Re}(\mathcal{U}_c) < 0 \quad \text{in II} \\
\text{Re}(\mathcal{U}_c) < 0 \quad \text{in III} \quad \text{Re}(\mathcal{U}_c) > 0 \quad \text{in IV}
\]

where

\[
\mathcal{U}_c = \sqrt{k^2 + \frac{p^2}{c^2}}
\]

The situation is similar to that discussed by Garvin (1956). We now put \( p^2 = ikx + h\mathcal{U}_c \), and solving for \( k \), we get:

\[
k = \frac{p}{r^2} \left[ \frac{h}{\sqrt{r^2 - \frac{r^2}{c^2}}} - i\sqrt{r^2 - \frac{r^2}{c^2}} \right], \quad r^2 = x^2 + h^2
\]

and

\[
dk = \frac{p}{r^2} \left[ \frac{h}{\sqrt{r^2 - \frac{r^2}{c^2}}} - i\sqrt{r^2 - \frac{r^2}{c^2}} \right] d\tau
\]

\[
\mathcal{U}_c = \frac{p}{h} \left[ \tau \left( 1 - \frac{x^2}{r^2} \right) - \frac{ixh}{r^2} \sqrt{r^2 - \frac{r^2}{c^2}} \right]
\]
Fig. 2 $\tau(k)$ for $x < 0$: The image of the positive real axis is denoted by $\tau(k_r)$, and the negative part of it by $\tau(-k_r)$. Similarly, the image of the positive imaginary axis by $\tau(ik_i)$ and the negative imaginary axis by $\tau(-ik_i)$. 
Fig. 3 Behavior of $\tau(k)$ for $x > 0$ with the same notations as in Fig. 2.
The behavior of this conformal mapping for $x > 0$, $x < 0$ is shown in figures 2 and 3, together with the singularities in the $\zeta$-plane. Clearly we can deform the integration path which is the image of the positive real $k$-axis, to the real axis of the $\zeta$-domain, without crossing any singularities.
Since $\tau(k = 0) = h/c$, we can write using the unit-step function $H(\tau - h/c)$:

$$I = e^{pt} \int_{\text{Im.}}^{\text{Re.}} \left[ H(\tau - \frac{h}{c}) \frac{d}{d\tau} F(\tau, p) \right] e^{-p\tau} d\tau$$

$$= e^{pt} \int_{\text{Im.}}^{\text{Re.}} \left[ H(\tau - \frac{h}{c}) g(\tau) \right] e^{-p\tau} d\tau$$

Where $g(\tau)$ does not include $p$. Thus we recognize $H(\tau - h/c)g(\tau)$ as the integral of the impulse response or the response to the unit step function. Because of $H(\tau-h/c)$, it vanishes for $\tau < h/c$. Then for $\tau > h/c$, the step function response is given by $g(\tau)$. The derivatives of the potentials that are necessary to compute $u, w$ as shown in equation (8) are given in Appendix II.

We notice that the potential terms corresponding to the infinite medium (that is $\phi_1, \Psi_1$ terms) vanishes until the direct P or S waves arrive at $\tau = r/c$. For the free surface terms ($\phi_2, \Psi_2$ terms) the same is true if $c = \alpha$.

For $c = \beta$, the situation is slightly more complicated. For the sake of definiteness, let $x < 0$. Then the image of the positive imaginary k-axis start getting folded at $\tau = \frac{r}{\beta}$ as shown in figure 3. This corresponds to a point

$$\text{Im}(k) = -x \text{ in the k-plane. If } -x < \frac{r}{\beta}$$

of $\Psi$ in the $\tau$-plane is on the upper side of the fold and
\( u_a \) has exactly the same behavior as \( u_b \) over the real \( \tau \)-axis. In that case there is no contribution from \( \phi_z \), \( \Psi_z \) terms until \( \tau > \frac{r}{\beta} \). However, if \( \frac{-x}{\beta r} p > \frac{p}{\alpha} \)
then the zero of \( \frac{u}{\alpha} \) in the \( \tau \)-plane is on the lower side of the fold. For \( \tau \) in the range:

\[
-\frac{x}{\alpha} + h\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} < \tau < \frac{r}{\beta}
\]

We have contribution from \( \phi_z \) and \( \Psi_z \)-terms. The time \( \tau = -\frac{x}{\alpha} + h\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} \) is the arrival time of a surface P-wave that started from the source as a shear-wave, arrived to the free surface at the critical angle, and travelled as a P-wave along the surface to the station. The case of \( x > 0 \) is analogous.

2.4 Exact solution for the finite source

We will look now at the field caused by a finite propagating two dimensional fault with a constant rupture velocity and uniform slip along a fault plane. For the line source, the field was composed of integrals of the form:

\[
I = e^{pt} \int_0^\infty F(-ip,k) e^{-i\kappa x - \frac{\kappa^2 c}{2\kappa}} dk
\]

\[
= 2 e^{pt} \left\{ Re. \int_0^\infty F(-ip,k) e^{-i\kappa x - \frac{\kappa^2 c}{2\kappa}} dk \quad \text{for } F \text{ even in } k
\right. \\
\left. Im. \int_0^\infty F(-ip,k) e^{-i\kappa x - \frac{\kappa^2 c}{2\kappa}} dk \quad \text{for } F \text{ odd in } k
\right\}
\]

\[\kappa = \alpha \text{ or } \beta\]
Suppose now we allow the line source to move along a straight line as shown in figure 4. The line intersects the z-axis at a depth \( h_0 \), and it makes an angle \( \theta \) with the x-axis. The observation point is on the free surface at a distance \( x_0 \). Let \( \xi \) be the distance along the straight line measured from its intersection with the z-axis. We require that the dislocation source at \( \xi = 0 \) to start acting at \( t = 0 \), and a dislocation source at \( \xi \) to start acting at \( t = \xi / v \), where \( v \) is the rupture velocity. Let \( a = \cos \theta \), \( b = \sin \theta \), then the contribution from the fault segment between \( \xi \) and \( \xi + d\xi \) will be

\[
 d\xi \quad I = 2e^{ip(\xi - v)} \Re \int_0^\infty F(-ip, k) e^{-ik(x_0 - a\xi)} - 2i(h_0 - b\xi) \, dk \, d\xi
\]

where for the sake of definiteness, we chose \( F \) to be even in \( k \). Then the solution for the moving fault which starts from \( \xi = 0 \) and ends at \( \xi = \delta \) will be given by:

\[
 J = \int_0^\delta I \, d\xi
\]

Assuming the interchangeability of the order of integration above, we carry out the integration with respect to \( \xi \) and we get:

\[
 J = K(\delta) - K(0)
\]
Fig. 4 Geometry under consideration: the cosine of the dip angle is denoted by "a" and the sine by "b". A line source dislocation is denoted by a double couple D. The critical length $L_{\text{critical}}$ is explained in the text. The field generated by the sources $D_1$ and $D_2$ just below and above the critical length are shown in Fig. 8.
where
\[ K(\zeta) = \exp(\text{Re} \int_0^\infty \frac{F(-ip,k) e^{-ikx-h_0^2 - p \zeta} \zeta}{aik + b \zeta^2 - \frac{p}{\nu}} \, dk) , \]

\[ \chi = x_0 - a \zeta , \quad h = h_0 - b \zeta \]

Like the line source solution, we make the substitution
\[ p \zeta' = ikx + h_0 \zeta + \frac{p \zeta}{\nu} \]

Letting \( \tau = \tau' - \frac{\zeta}{\nu} \) the behavior of the transformation \( \tau(k) \) is identical to that for the line source, as shown in figure 2 and figure 3. The only potential problem is that the integration path are the roots introduced by the integration, namely the roots of the equation:

\[ aik + b \zeta^2 - \frac{p}{\nu} = 0 \]

The behavior of these roots is different for the supersonic propagation \( v > c \) and the subsonic propagation \( v < c \). Both cases will be discussed separately below:

**Case I:**
\[ v < c , \quad k_1 = \frac{-ip}{\nu} + \frac{p}{\nu} \left(1 - \frac{v^2}{c^2}\right)^{1/2} \]

\( k_1 \) does not satisfy the radiation condition, because

\[ \mathcal{A}(k_1) = \left[ \frac{p}{\nu} - i k_1 a \right] \frac{1}{b} \]
has a positive imaginary part in the lower half of the k-plane, hence it is not on the Riemann sheet of interest, as was specified in the discussion for the line source, by equation (9). On the other hand, \( k_i^+ \) is on our Riemann sheet but clearly in the fourth quadrant in the k-plane. Thus, for \( x < 0 \), it is clear from figure 3 that there is no interference with the integration path. For \( x > 0 \), the situation is slightly more complicated, and we have

\[
\tau(k_i^+) = \frac{ax + bh}{V} + \frac{i}{V} \left( bx_0 - ah_0 \right) \left( 1 - \frac{v^2}{c^2} \right)^{1/2}
\]

From figure 4 we find that \( ah_0 - bx_0 \) is the distance between the fault plane and the station. Thus \( \text{Im}(\tau(k_i^+)) < 0 \) and the pole \( k_i^+ \) again does not interfere with deforming the integration path in figure 2, to the real axis.

**Case II:** \( v > c \) We have

\[
k_i^+ = \frac{iP}{\sqrt{a + b}} \left[ -q + b \left( \frac{v^2}{c^2} - 1 \right)^{1/2} \right]
\]

\[
\tau(k_i^+) = \frac{ax + bh}{V} + \frac{i}{V} \left( \frac{v^2}{c^2} - 1 \right)^{1/2} (ah_0 - bx_0) \equiv \tau_i^+ 
\]

clearly if \( \tau_i^+ > \frac{h}{c} \) then we have to add or subtract \( \pi i \) times the residue of the function at \( \tau_i^+ \) depending on whether \( x < 0 \) or \( x > 0 \) when we deform the integration path to the real axis. That will not cause any difficulty in the interpretation, because this contribution will appear
as a Dirac-delta function $\delta(\tau - \tau_i)$ in the time domain with an appropriate constant factor.

The other root gives us:

$$\tau_i = \tau(k_i) = \frac{ax + bh}{v} - \frac{1}{\nu} \left( \frac{V_i}{c_i} - 1 \right)^{1/2} \left( a h_0 - b x_0 \right)$$

and if $\tau_i > \frac{h}{c}$ we have a similar $\delta$-function contribution $\delta(\tau - \tau_i)$

Since, as discussed in the review of the literatures in chapter I, in most earthquakes (including the San Fernando earthquake), the rupture velocity is lower than the shear wave velocity, let us focus our attention to the case $\nu < \beta$. Because of the factor $\frac{1}{aik + b^2 - \frac{p}{v}}$, the response obtained by Cagniard's method will have the form:

$$I = \sum_{Re.} e^{pt} \int_0^\omega G(\tau) e^{\nu t} d\tau$$

in contrast to the line source case where it was of the form

$$I = \sum_{Im.} e^{pt} \int_0^\omega f(\tau) e^{-\nu t} d\tau$$

Thus we identify the integrand of the finite source solution as the impulse response rather than the step-function response. This response is still singular because of the factor $dk/d\tau$ which tends to $\omega$ as $\tau \rightarrow V/c$. This singularity,
together with the determination of the upper limit of integration, will be discussed in the next paragraph. In essence, what will be done next, can be thought of as a simultaneous integration in space and time that will yield the step-function displacements for a finite, moving two dimensional fault.

I will discuss here, in details the method of integration for the simple case of the infinite medium terms. The results for the free surface terms are similar, but the algebra is considerably more cumbersome. Thus all but the most basic considerations of the free surface terms will be left to Appendix III. As an example, let us focus our attention on the term

$$2\pi \rho \varphi_{121, xx} = \text{Re. } \frac{k^2}{\alpha k^2} \frac{1}{\alpha i k + b n^2 \frac{1}{\alpha}} \left| \begin{array}{c} \xi = \infty \\ \xi = 0 \end{array} \right. \quad (11)$$

where Laplace transformation parameter $p$ will be dropped because it cancels out from the numerator and the denominator of the above expression. Thus we define:

$$k' = \frac{1}{r^2} \left[ h \frac{r^2 - r^2}{q^2} - i x \right]$$

$$\frac{d k'}{d \tau} = \frac{1}{r^2} \left[ \frac{h \tau}{\sqrt{r^2 - r^2}} - i x \right]$$
where

$$\tau = t - \frac{E}{v}, \quad h = h_0 - b\xi, \quad x = x_0 - a\xi, \quad r^2 = x^2 + h^2$$

$$\zeta = \frac{1}{h} \left[ \tau \left(1 - \frac{x^2}{r^2}\right) - \frac{i x h}{r^2} \sqrt{\frac{\tau^2 - r^2}{a^2}} \right]$$

Since we are taking the real part of the right hand side of equation (11), the upper limit of integration corresponds to the value from which the P-wave have just arrived at the time $t$. That is to say it must be a root of the equation $\tau^2 - \frac{r_1^2}{a^2} = 0$. We must choose the smallest root $\zeta_1$ that satisfies the condition: $0 \leq \zeta, \leq L$ where $L$ is the final length of the fault. If the smallest root is still larger than $L$, then we take the upper limit $\zeta = L$. Unfortunately this will always make the right hand side of equation (11) infinite as long as $0 \leq \zeta_1 \leq L$, $\zeta = \zeta_1$, because $$\frac{d\zeta}{dr} \bigg|_{\xi = \xi_1} = \infty$$

The same is true for other derivatives of the potentials. Thus it seems that our method will always yield an infinite solution in the dynamic range: $0 \leq \zeta_1 \leq L$. This may seem surprising, and disheartening, but let us look at the horizontal displacement caused by the initial P-motion as obtained by Niazy (1973). To examine the nature of the singularity, it suffices to look at $U_H$ as given in Niazy (1973), with $\Theta = 45^\circ$, so that
\( n_x f_x = 0 \). Before the shear waves arrive, the step function, line source response is given by:

\[
U_H = \frac{-M_0}{\pi \rho r^6} \text{Re} \left[ t \sqrt{t^2 - \frac{r_1^2}{a^2}} \left( h^3 - 2x^3h \right) - \frac{x^3ht^3}{\sqrt{t^2 - \frac{r_1^2}{a^2}}} \right]
\]

Then the impulse response is:

\[
\dot{U}_H = \frac{-M_0}{\pi \rho r^6} \text{Re} \left[ (h^3 - 2x^3h) \left( \sqrt{\frac{t^2 - \frac{r_1^2}{a^2}}{t^2 - \frac{r_1^2}{d^2}}} + \frac{t^2}{t^2 - \frac{r_1^2}{d^2}} \right) - x^3h \left( 3\frac{t^2}{t^2 - \frac{r_1^2}{a^2}} - \frac{t^4}{(t^2 - \frac{r_1^2}{d^2})^{3/2}} \right) \right]
\]

We then let \( t = t' / \sqrt{v}, x = x_0 - a \bar{q} \), \( h = h_0 - b \bar{q} \) and try to integrate \( \dot{U}_H \) with respect to \( \bar{q} \), where the upper limit of integration is \( \bar{q}_1 \), as defined previously.

All terms will give us a finite result except the term \( \frac{t^4}{(t^2 - \frac{r_1^2}{d^2})^{3/2}} \), because it has a singularity of order larger than 1, its integral will clearly be infinite at that singularity. Thus our result is not too surprising. Suppose that rather than looking at this form of the impulse response for the finite source, we look at the step-function response defined by

\[
\varphi_{1z, xx} = \frac{1}{2 \pi \rho} \text{Re} \left\{ \int \frac{k^2}{\omega} \frac{d\omega}{d\tau} \frac{d\tau}{\omega + b_2q^2} \right\} \bar{q} = \delta \left| \bar{q} = 0 \right.
\]

(11)

To carry out the integration with respect to \( \tau \), we make the substitutions:

\[
q = \frac{\bar{q}}{a}, \quad \tau = \text{Csc}(\theta), \quad \text{so that}
\]

\[
\sqrt{\tau^2 - q^2} = q \text{Cot}(\theta), \quad d\tau = -q \text{Csc}(\theta) \text{Cot}(\theta) d\theta
\]
We then write the integrand in (12), as a function of $e^{i\theta}, e^{-i\theta}$ and make the substitution $z = e^{i\theta}, \, d\theta = -i \frac{dz}{z}$

This set of transformations, change the integrand into a rational function of $z$, in particular, we have:

$$k' = i a_4 \frac{z^2 + 2 a_3 z + i a_1}{z^2 - 1} = i a_4 \frac{(z - q_5)(z - q_6)}{z^2 - 1}$$

$$\frac{dk'}{dz} = -i a_4 \frac{z^2 + 2 a_3 z - i a_4}{z^2 + 1} = -i a_4 \frac{(z - q_5)(z - q_6)}{z^2 + 1}$$

$$\gamma' = \frac{a_2 z^2 + 2 i a_1 z + a_4}{z^2 - 1} = a_2 \frac{(z - q_5)(z - q_6)}{z^2 - 1}$$

$$a_1 = \frac{h}{r a'}, \quad a_2 = \frac{x}{r a'}, \quad a_3 = \frac{h}{r z}, \quad a_4 = \frac{x}{r z}$$

$$q_1 = \frac{i}{a_1} \left( a_2 + \frac{1}{a_1} \right), \quad q_2 = \frac{i}{a_1} \left( a_2 - \frac{1}{a_1} \right)$$

$$q_3 = -\frac{i}{a_2} \left( a_1 - \frac{1}{a_1} \right), \quad q_4 = -\frac{i}{a_2} \left( a_1 + \frac{1}{a_1} \right)$$

$$a_i k' + \frac{b a_2'}{V} \frac{1}{V} = \frac{z^2 (-a q_i V + b a_2 V - 1) + 2 i V (a q_i + b a_2) - a_q V + b a_2 V + 1}{V (z^2 - 1)}$$

$$= \frac{(-a q_i V + b a_2 V - 1)}{V (z^2 - 1)} (z - q_5)(z - q_6)$$

$$q_5 = \sqrt{a^2 - v^2 + i V (a x + b h)}$$

$$q_6 = -\frac{r \sqrt{a^2 - v^2 + i V (a x + b h)}}{V (a h_0 - b x_0) + r a}$$
Now the factors coming up from the consecutive change of variables are:

\[
\frac{dk'}{dr} \frac{d\theta}{dz} = -2a_4 q \frac{(z - q_4)(z - q_2)}{(z^2 - 1)^2}
\]

and

\[
\frac{1}{aik + b_2} - \frac{i}{V} \frac{dk'}{dr} \frac{d\theta}{dz} = -2V a_4 q \frac{(z - q_3)(z - q_4)}{(z^2 - 1)^2 (z - q_5)(z - q_6)}
\]

with \( \eta_2 = -aq_v y + ba_2 y - l \)

Thus equation (12) becomes:

\[
\psi_{12,xx} = -\frac{a_4^2 q a_4 y}{2\pi \rho \eta_2} \left[ \frac{(z - q_3)^2 (z - q_5)^2 (z - q_6)(z - q_3)}{(z^2 - 1)^2 (z - q_5)(z - q_6)} \right]_{\xi=0}^{\xi=\delta} (13)
\]

Now

\[
\xi = \sqrt{1 - \frac{r^2}{a^2 t^2}} + \frac{ir}{a t}
\]

Thus the upper limit of integration will always be \( z = i \), because \( \delta \) is a root of \( r^2 a^2 = r^2 \) as long as \( \delta < L \).

If \( \delta = L \), then it will move on the unit circle according to the equation:

\[
z = \sqrt{1 - \frac{(x_0 - aL)^2 + (y_0 - bL)^2}{a^2 (t - \frac{L}{V})^2}} + \frac{i \sqrt{(x_0 - aL)^2 + (y_0 - bL)^2}}{a (t - L/V)}
\]

for \( \delta = 0 \):

\[
z = \sqrt{1 - \frac{r_0^2}{a^2 t^2}} + \frac{ir_0}{a t}, \quad r_0^2 = x_0^2 + y_0^2
\]
As a check on the method, the displacements were synthesized at a point close to the center and checked against the known solution (Niazy, 1973).

We note if the integrand in (12) was multiplied by \( \tau^n \) this will contribute a factor of \( \frac{(ziq)^n \tau^n}{(\tau^2 - 1)^n} \) to the integrand in (13). This means we can convolve with any polynomial in time, and still get an exact solution, thus the problem is solved in principle for any source time function which can be approximated by a polynomial. This will be discussed further in the next chapter.

The free-surface terms are of the form:

\[
I = \text{Re.} \int \frac{P(k)}{(ai\xi + br\eta - \frac{1}{\nu}k)} \frac{dk}{d\tau} d\tau \bigg|_0^\delta
\]

where \( F_- \) is the Rayleigh function, with \( P(k) \) being odd or an even function in \( k \), which will determine whether we take the real or the imaginary part of the integrand. The first step toward utilizing the previously discussed transformations here, is to separate the integrand above into a part that involves

\[
\xi' (\xi' = \sqrt{k^2 + \frac{1}{\beta^2}} \quad \text{if } c = \alpha \quad \text{and} \quad \xi' = \sqrt{k^2 + \frac{1}{\alpha^2}} \quad \text{if } c = \beta)
\]

a part that does not involve it. This is accomplished by
multiplying the integrand by \[ \frac{4k^2 \gamma \gamma' + (2k^2 + 1/\beta^2)^2}{4k^2 \gamma \gamma' + (2k^2 + 1/\beta^2)^2} \]

This will make the denominator in the integrand a polynomial function in \( k \) and \( \gamma \). Thus we can write \( I \) as

\[ I = J_1 + J_2 \]

\[ \int \frac{ \gamma \gamma' }{ \gamma \gamma' + (2\gamma^2 + \frac{1}{\beta^2})^2 } \, \, d\gamma \bigg|_{\gamma = 0} \]

\[ J_1 = \int R_1(k, \gamma) \frac{d\gamma}{d\eta} \, d\eta \bigg|_{\xi = 0} \]
\[ J_2 = \int R_2(k, \gamma) \frac{d\gamma}{d\eta} \, d\eta \bigg|_{\xi = 0} \]

\( R_1, R_2 \) are rational functions of \( k \).

If we utilize the previously discussed transformations, the integrand in \( J_1 \) will become a rational function of \( z \). The only potential problem will be in determining \( \delta \) when \( c = \beta \); that will be discussed later. The poles of \( R_1, R_2 \) clearly contain the roots of the Rayleigh function and will be discussed in Appendix III. For \( J_2 \), the integrand has the added complication of \( \gamma = (k^2 + \frac{1}{c^2})^{1/2} \). Using the previous transformations, we get:

\[ \gamma = \frac{2i}{\xi^2 - 1} \left[ b_j \xi^j \right]^{1/2} \quad , \quad 0 \leq j \leq 4 \]

where we assumed that the summation convention of repeated subscripts holds. We find that \( b_j = b_{4-j} \), and

\[ b_4 = \frac{1}{4} \left( q_i^2 - \frac{1}{c^2} \right) \quad , \quad b_3 = -iq_i q_j \quad , \quad b_2 = -q_i^2 + \frac{q_j^2}{2} + \frac{1}{2} \frac{1}{c^2} \]
with \( a_1, a_2 \) as defined previously.

Then \( J_2 \) becomes:

\[
J_2 = \int_{R_3(z)} \frac{d^2}{\sqrt{(z-q_\alpha)(z-q_\beta)(z-q_\gamma)(z-q_\delta)}} \bigg|_0^\delta, \quad R_3 \text{ is a rational function}
\]

where \( q_\alpha, q_\beta, q_\gamma, q_\delta \) are roots of \( b_jz^j \) which are discussed in Appendix III. The branch of the square root must be determined from the radiation condition on \( \psi'_c \), which is just like that of \( \psi_c \) given by equation (9). \( J_2 \) can now be evaluated in terms of the elliptic integrals. The method and the branch, of the square root are again outlined in Appendix III.

If \( c = a, \zeta \) is determined in exactly the same way as the previously discussed, infinite medium case. For \( c = \beta \), determining \( \delta \) is slightly more complicated. Let \( \delta_i \) be the smallest root of \( \tau^2 - \frac{r^2}{\beta^2} \) such that \( 0 < \delta_i < \ell \).

Consider now the critical length \( L_c \) shown in figure 4. It is the length at which a shear wave generated from a point source located at \( L_c \) on the fault plane, will hit the surface at the critical angle and reach the station as a P-wave; it was discussed in § 2.3. \( L_c \) is determined by the equation

\[
\frac{|x|}{r} = \frac{\beta}{a}
\]

\[
L_c = \left\{ \begin{array}{ll}
\frac{dh_0 + x_0}{a + bd} & \text{for } x < 0 \\
\frac{dh_0 - x_0}{-a + bd} & \text{for } x > 0 \\
\end{array} \right., \quad d = \frac{\beta}{\sqrt{a^2 - \beta^2}}
\]
If \( \xi_1 < L_c \) then \( \gamma = \gamma_1 \), however if \( \xi_1 > L_c \) then \( \gamma \) is determined by solving the equation:

\[
\gamma = \frac{|x|}{\alpha} + h \left( \frac{1}{\beta^2} - \frac{1}{\alpha^2} \right)^{1/2} \quad \text{for} \quad \gamma
\]

The last value for \( \gamma \), is a root of \( \gamma^2 \). It is worthwhile here to point out that in principle, we still can compute the field for any time function which is a polynomial, in an exact way even for our half-space solution, just like what was discussed for the infinite medium case.

I will now summarize what has been accomplished in this chapter. The steady state solution for a line dislocation source in an elastic half space was obtained. We then applied Cagniard's method to obtain the corresponding step function displacements. We then computed the impulse response due to a finite source, and it was found to be infinite and of use only to compute the corresponding step function displacements. Appropriate transformations were then found to compute the desired step function displacements, as well as for any source time function that can be approximated by a polynomial in the time domain. The method yields an exact solution for any polynomial source time function. A check on the method was then proposed and will be carried out in the next chapter.
CHAPTER 3

Interaction Between the Free Surface and the Elastic Radiation of a Moving Dislocation

3.1 Properties of the line source solution

I will first discuss the properties of the line source solution and exhibit computer plots of the displacements versus time, for different locations of the observation point on the free surface. Three checks were made on our line source solution. The first one was on the known radiation pattern associated with our double couple solution. The second one is to check the known properties of Rayleigh waves in two dimensions. Figure 5 shows the far-field Rayleigh waves from a double couple source located at \((x_s, z_s) = (0, h)\). The x-axis is on the free surface, positive to the right. The origin is on the free surface, and the positive z-axis is directed downward. The orientation of the double couple is shown in Figure 4, for a dip angle of 52°. The displacements are shown for a station located at \((100, 0)\) km, the depth of the source \(h = .1\) km, for dip angles \(\theta = 51°, 45°, 0°\). It was assumed that \(\alpha = 6\) km/sec and \(\beta = 3.5\) km/sec. Changing the location of the station to \((1000,0)\) km and to \((10,000,0)\) km the amplitude and the waveform remained the same within the line thickness of those shown in Figure 5. All the features shown, conform to the known expected behavior of the Rayleigh
Fig. 5a Behavior of the horizontal displacement caused by the Rayleigh wave:
The arrival time |x|/c is mislocated by .025 sec. to the left on the drawing, it is .025 sec. to the right of its indicated location. The time dependence of the source is a step-function. Theta, is the dip angle of the double couple, x is the directed horizontal distance and H is the depth of the source.
Fig. 5b Vertical displacement due to Rayleigh wave with the same nomenclature as those on Fig. 5a. The positive direction is downward.
waves in two dimensions.

Since the time dependence of the source was a step function, we expect the limit of the solution as the time tends to infinity, to reduce to the static limit given by Maruyama (1966). Unfortunately, the analytic expression for the displacements as given by eq. 8, and Appendix II are too complicated to take their limit as the time tends to infinity analytically. Alternatively, we can check the algebra as well as the logic of the computer program by sampling the displacements at sufficiently large values in time, for several values of the space parameters *x* and *h*. This, however, is still quite problematical, because the derivatives of the potentials as given in Appendix II are sums of terms of the form: \( f(t) + g(t) \), where

\[
\lim_{t \to \infty} f(t) \to at^h + bt^2 + c \quad \text{and} \quad \lim_{t \to \infty} g(t) \to -at^h + dt^2 + e
\]

Thus when we sample the function for sufficiently large *t*, we have cancellation of the most significant bits, in evaluating those derivatives. The same is true in evaluating the Rayleigh functions \( F_- , F'_- \), in order to compute those derivatives. Also the displacements are the sum of derivatives, which are of the form:

\( x(t) + y(t) \) where \( \lim_{t \to \infty} x(t) \to pt^2 + q \), \( \lim_{t \to \infty} y(t) \to -pt^2 + r \)

Lastly, in evaluating the static solution itself as given by Maruyama (1966), we can get further cancellation depending on the source orientation and the space coordinates
involved, implying still more cancellations in taking our static limit.

Because of the loss of accuracy described above, it was found that the double precision arithmetic on the IBM 370/M165 (16 figures carried in the computation), was not adequate to perform satisfactory checks on the static limit. Indeed a similar problem occurs in computing the Rayleigh waves when the depth of the source is small. This is due to the fact that the Rayleigh pole is mapped into:

$$\tau = \frac{|\chi|}{c_R} - i \text{sign}(\chi) \sqrt{\frac{1}{c_R^2} - \frac{1}{c^2}}$$

(in the $\tau$-plane)

where $c$ is $\alpha$ or $\beta$ and $c_R$ is the Rayleigh velocity.

Thus, as $h \to 0, \tau \to \frac{|\chi|}{c_R}$ becomes a zero of the Rayleigh functions $F_-, F'_-$ as given in Appendix II. This will cause a precision problem, in looking at the Rayleigh waveforms, when $h$ is small.

To resolve the above problems, a multiple precision program was developed, which carried fifty figures in the computations and made adequate, satisfactory checks on Rayleigh waveforms as shown in Fig. 5. The computation time on the IBM 370/M165 was less than one minute to generate two hundred values in time, for the two components of displacement.

Let us look now at some details of seismograms at various stations. In Figures 6 and 7, we show displacements for stations at $x=7.1$ km, $x=7.5$ km, for the source at depth
Fig. 6 Development of surface waves in the horizontal component as the horizontal distance is changed. P denotes the P-wave arrival, S the shear wave, R the Rayleigh wave. Note the P and the S radiation pattern.
Fig. 7 Behavior of the vertical component corresponding to Fig. 6. Positive motion here is downward on the surface of the half space.
h = .1 km with a dip angle of 51°. The time denoted by P is the arrival of the P-waves. The displacement has a square root infinity there. In other words, \( u \rightarrow \frac{A}{\sqrt{t - \frac{r}{a}}} \) as \( t \rightarrow (\frac{r}{a}) \), and the same is true for \( w \). The shear wave arrival has also a similar singularity and is denoted by S. We note that for \( |x| = .1 \) km, we see only the P and the S-arrivals. The SP arrival is the surface P waves that result from the conversion of the S-wave incident at the free surface to P-waves. The arrival time is \( \frac{|x|}{a} + h \sqrt{\frac{1}{\beta^2} - \frac{1}{\alpha^2}} \) and it appears only when the incidence angle is equal to or greater than the critical angle \( \text{Sin}^{-1} \left( \frac{\beta}{\alpha} \right) \) as discussed in chapter II. These waves have been described by Lapwood (1949) for the case \( \frac{|x|}{h} \gg 1 \). To emphasize further the significance of shallow sources in the near-field of earthquakes, seismograms were generated from dislocation sources indicated by \( D_1 \) and \( D_2 \) in Fig. 4, and they are shown in Fig. 8.

Finally the Rayleigh waves are denoted by R. They are not quite well developed at \( |x| = .5 \) km, and seem to be of lower frequency than the body phases for the source depth considered here. The sense of first motion of body waves clearly conform to the well known far field pattern, although in the vicinity of a nodal plane where the sense of motion is less apparent, finer sampling time is needed to confirm it.
Fig. 8 Significance of the critical length $L_c$ and the development of the surface. 
P-waves: $\alpha = 5.6$ km/sec and $\beta = 3.18$ km/sec for this figure. As we changed $\delta$ to 
3.17 km/sec the changes in the surface are minimal within this figure.
3.2 Properties of the finite source solution

Now, we shall test the validity of our method, described in Chapter II, used for integrating the line source solution to obtain the finite source solution. For this purpose, we shall calculate the infinite medium term, and compare with the known exact solution given by Niazy (1973) by using a different technique. To compare with the known solution, we computed both components of displacement at a station located at 10 meters from the middle of a fault whose length is $4\sqrt{2}$ km in a medium with the following parameters:

$$\alpha = 6 \text{ km/sec}, \beta = 3.5 \text{ km/sec}, v = 2.5 \text{ km/sec}; \Delta U = 76.2 \text{ cm}$$

The solution will be generated for two source time functions, and it is exact for both of them. The first source time function is a step function. Since the station is very close to the middle of the fault, the parallel component of displacement should approximate the boundary condition by exhibiting a step like motion with height equal $\Delta U/2$. The component perpendicular to the fault should be similar to an impulse as discussed by Aki (1966) and Niazy (1973).

The second source time function is given by:
The function \( g(t) \) gives a particle velocity which is continuous in time on the fault plane and is characterized by the rise time \( t_o \). It reduces to a step function as \( t_o \to 0 \). The details of the convolution technique used for the source function \( g(t) \) is given in Appendix IV.

Let us look now at Fig. 9. The plots to the left show the parallel component of displacement as a function of time. The upper traces are the step function solution and the lower traces are those for \( g(t) \) as the source time function with \( t_o = .6 \) sec. The major difference between the two is the smoothing of the waveform and the decrease in the amplitude of the perpendicular component for the finite rise-time source.

It appeared that our ability to convolve exactly with any source time function that is a polynomial in time (e.g. see Appendix IV) might be a desirable feature to keep for the half-space terms because the step function solution implies an infinite seismic energy release. However the computation time for exact convolution is considerable and
Fig. 9 The finite source, infinite medium solution: The location of the station and the wave velocities are discussed in the text. The top traces are for for a step-function source time dependence. The bottom traces are for $g(t)$ as the source time function (eq.14) with $t_0 = .6$ sec. Time is in seconds and the vertical scale is in centimeters.
does not justify its extensive use. For example, generating two hundred points of the step function for the infinite medium terms takes approximately .55 minutes, while generating eighty points for the finite rise-time source function \( g(t) \) takes about 1.9 minutes on the IBM 370/M165. We note also that we encountered a precision problem in calculating the finite rise-time solution when we sample the solution beyond 10 second after the P-arrival. We used the double precision arithmetic on the IBM 370/M165 for that computation. A glance on Appendices I, II and III show that computing the free surface terms using the exact convolution method should be at least an order of magnitude more time consuming because of the great increase in complexity of the terms to be evaluated. This will add to the precision problem which will require a lengthy program written using multiple precision arithmetics. Both the time for developing and for running such a program will be prohibitive. For these reasons, we will compute only the step function solution exactly, and the convolution, if needed, will be done numerically.

3.3 Properties of the finite source in a half-space

We consider now the motion of a half-space caused by a moving step function dislocation. Unfortunately the half-space solution is considerably more complicated than the corresponding infinite medium solution studied by Niazy
(1973). Fifty-two double precision FORTPAN IV subroutines and a main program were needed to generate the solution for a reasonable amount of computer time, computer programming and core space requirements. As it happens often in computer programming, one can cut the computation time at the expense of increasing the core space and vice versa. A typical time for computing the seismic displacement at 50 time points is about 4 minutes on the IBM 370/M165. As a result of the great amount of computation involved, the round-off error accumulates and the final result is accurate to 4 or 5 figures during the arrivals of various waves. The precision rapidly deteriorates when we sample the displacement for larger values of time than say twice the arrival time Rayleigh waves from the upper tip of the fault. This precision problem also occurs when the station is close to the fault and to a lesser extent in evaluating the displacement at the arrival of Rayleigh waves from the upper tip of the fault when that tip gets too close to the free surface.

Thus, our method is restricted by the available computer time and by the required precision. Since we are mainly interested in the San Fernando earthquake of Feb. 1971, and the accelerogram recorded at the Pacoima Dam, we shall choose the geometry and the wave medium to be applicable to that earthquake.
Consider now the geometry shown in Figure 4, we shall make a few checks on our computer program. The first one is that the initial motion of the half-space solution should be similar to that of the infinite medium case, because we expect that the free surface does not reverse the polarity of the first motion of incident waves. The second check is on the Rayleigh waves mostly generated from the upper tip of the fault. It should have the same waveform and the same amplitude in the far-field, independent of the horizontal distance. Figure 10 exhibits these features very clearly.

The following parameters were used for the wave medium and dislocation model parameters,

\[ \beta = 3.5 \text{ km/sec}, \quad \alpha = \sqrt{3} \beta \approx 6 \text{ km/sec}, \quad V_T \approx 7.2 \beta \approx C_R, \quad \Delta u = 76.2 \text{ cm}, \delta = 0.99 \]

where \( L \) is the length of a fault that extends from the free-surface to the hypocenter making an angle \( \theta = 52^\circ \) with the free surface as shown in Figure 4, \( \delta \) is the length at which the fault is stopped and \( C_R \) is the Rayleigh velocity of the medium. The most important check is the static limit. Because of the precision problem and the expense of the program, it was done at a few points near the source. The result agreed perfectly with the result given by Maruyama (1966).

Lastly the surface S-P waves should attenuate as \( x^{-3/2} \) in the far-field as shown by Lapwood (1949).

Let us look now at Figure 11. It compares the infinite medium solution with the half-space solution at a distance
Fig. 10 Independence of Rayleigh wave motion in the far field of the horizontal distance: $x_h$ is the location of the station, $|x|$ is the separation between the upper tip of the fault and the station and $c_h$ is the Rayleigh velocity. The sampling interval is .05 sec.
\( x_0 = -5h_0 \) where \( h_0 \) as shown in Figure 4 is equal to 14 km. The following parameters were chosen for the model:

\( \alpha = 5.6 \text{ km/sec}, \beta = 3.2 \text{ km/sec}, V = 7.5 \text{ km/sec}, b = .99, \Delta u = 1 \text{ meter}, \theta = 52^\circ \)

The sampling interval is chosen as .21 sec, so that the high frequency Rayleigh waves shown in Fig. 10 is not included. The amplitude of the shown S-P phase was checked in the far field to have the proper dependence on distance. The difference between the infinite medium and the half-space solutions is striking.

We turn next to a comparison of the near-field of a dislocation source between the infinite medium and half-space solutions. The parameters are identical to the previous case except for the location of the station which is taken to be at \( x = 8 \) km. This approximates the location of the Pacoima Dam station with respect to the San Fernando earthquake as shown in Fig. 4. We show in Fig. 12, the horizontal displacement for the half-space in comparison with those for the infinite medium solution for two different rupture velocities. The lower velocity is closer to the values adopted in previous studies of the Pacoima Dam records as discussed in Chapter 1. A similar comparison for the vertical displacement is shown in Fig. 13.

There are a few important features to observe on these figures. First, our half-space solution is about twice as large and has approximately the same waveform as the infinite
Fig. 11 Horizontal and vertical displacements at a station $(x,z)=(-5h,0)$, with the geometry as shown in Fig. 4. Parameters are discussed in the text. The sampling interval is .21 sec. The top traces are the half-space solution and the bottom ones are the corresponding infinite medium solution. Time is in seconds and the vertical is in centimeters.
Fig. 12 Dependence of the horizontal displacement on the rupture velocity: top traces are the half-space solution and the bottom ones are the corresponding infinite medium solution. Time is in seconds from the P-wave arrival and the vertical scale is in centimeters.
Fig. 13 Vertical displacement corresponding to Fig. 12.
medium solutions, as far as long period motions on the record are concerned. The same is true for both rupture velocities. As reviewed in Chapter 1, all the previous workers, who computed synthetic seismograms using the infinite medium solution, took the free surface effects into account by doubling the amplitude of the infinite medium solution. The assumption is valid for SH type motion. Our solution here demonstrates the first confirmation on that assumption for long period P and SV motion.

Another important feature to observe in the record, is that the effect of increasing the rupture velocity is similar for both solutions. It squeezes the waveforms in time and increases the amplitude.

Consider now the features of short period motion in our half space solution. Two main short period wavelets can be recognized as the surface P discussed in section 3.1 (see also Lapwood, 1949) and the stopping phase (Savage, 1966). The surface P waves are generated when the fault surpasses the critical length $L_c$ as shown in Fig. 4. The later high frequency motion corresponds to the shear wave and (or) the Rayleigh wave from the upper tip of the fault.

From comparison of Fig. 12 and Fig. 13, we find that the general behavior is common between the two components of displacement. Namely, we see that the half-space solution is about twice the infinite medium solution for long periods of both components. We also see the high frequency surface
P caused by the fault propagating beyond the critical length $L_c$ and then the stopping phases on both components. Increasing the rupture velocity has also the effect of increasing the amplitude and squeezing the waveform in time for both components.

Before concluding this chapter, let us discuss an alternative to the analytic approach we have followed here. In principle, we could have numerically superposed the contributions from many dislocation line sources distributed along the fault plane, by introducing the delay time appropriate for a given rupture velocity. There are mainly two reasons for not following that approach. The first reason is that because of the singularities in the integrand the error may become too large specially when sampling the Rayleigh waves from the upper tip of the fault as it approaches the free surface and in sampling the field when the fault passes close to the station. Unfortunately both cases are of great interest in studying the Pacoima Dam record of the San Fernando earthquake. The second reason is that the numerical solution is expected to cost as much as our exact finite source solution to generate. Because of the precision problem encountered in evaluating and checking the line source solution, it is necessary to use the multiple precision program to generate the line source solution for any numerical scheme. In other words we are
likely to need to carry more than sixteen figures in the computation which is more than the IBM370/M165 double precision arithmetics permits. If we were to use 50 line sources to approximate the finite source solution, and we want to generate 50 points in time, this will mean that we have to generate 2500 points for the line source solution. This is expected to cost considerably more than our exact solution requires. So, not only does there not seem to be an advantage in any numerical integration approach, but also there is no control over the error involved in the computation, which may be complicated by precision problems and/or slow convergence of the scheme specially in sampling the near-field high frequency motion.

We now summarize the results shown in this chapter. We developed a multiple precision program to check on the algebra and the logic of the computer program. A satisfactory check of the line source solution was made by comparison with Maruyama's (1966) static solution. The far field displacements due to Rayleigh waves were computed for various orientations of the double-couple source and shown to have their known properties. We synthesized the displacements at different stations and then demonstrated the importance of the surface P waves in the near-field of a shallow dislocation source, by putting the source above and below the critical point. The validity of the finite
source solution was tested for the infinite medium terms using the exact solution given by Niazy (1973), obtained by a different technique. The final computer program for a finite-source, half-space solution, consisted of 52 FORTRAN IV subroutines and a main program using the IBM370/M165 double precision arithmetics. The program was used to generate the displacements at the far field, at a distance five times the depth to the hypocenter and at a near-field point corresponding to the Pacoima Dam station with respect to the San Fernando fault. For long waves computed at the near-field point, the effect of the free surface is approximately doubling the infinite medium solution. For short waves, however, the effect generates two distinct arrivals, one is the surface P-waves which results from S to P conversion at the free surface and the other is Rayleigh and the shear wave arrival from the upper tip of the fault.
CHAPTER 4

Interpretation of the Pacoima Dam Record of the San Fernando Earthquake

4.1 Modeling the San Fernando earthquake

Let us first consider the choice of compressional and shear wave velocities $\alpha$ and $\beta$ for the medium in which our earthquake model will be placed. From Appendix III, it seems that our solution may be critically dependent on the ratio $\xi = \alpha^2 / \beta^2$. Using the notation of Appendix III, some singularities are critically dependent on whether $\xi$ is greater or less than $\xi_1$, where $\xi_1$ is given in Appendix III. Those singularities are, however, removable singularities and brought to our integration domain from the lower Riemann sheet when we rationalized the denominators in the various free-surface terms.

Thus, that singularity is not fundamental. Indeed computer results for the line source, half-space, solution, show that the seismic motion is not sensitive to the relative magnitude of $\xi$ to $\xi_1$. As shown in Fig. 8 no observable change was seen as we changed $\xi$ from $\xi < \xi_1$ to $\xi > \xi_1$. This confirms our intuition that since the poles are not on the Riemann sheet of interest, changing their location slightly should not influence the solution significantly.

Let us now choose appropriate values for $\alpha$ and $\beta$. Bolt and Gopalakrishnan (1972) and Bolt (1972) used
\( \alpha = 5.5 \text{ km/sec}, \beta = 3.3 \text{ km/sec} \). Boore and Zoback (1974) used \( \alpha = 5.7 \text{ km/sec}, \beta = 3.1 \text{ km/sec} \), in their study of the Pacoima Dam record of the San Fernando earthquake. Hanks (1974) suggested that \( \alpha = 5.6 \text{ km/sec}, \beta = 3.2 \text{ km/sec} \), which happened to be the average of the other two values. Other observers used values that are close to these values. Slight differences in the medium wave velocities, do not affect our results on the Pacoima Dam seismograms. Thus, we shall use Hanks' values for \( \alpha \) and \( \beta \) in this study.

We saw in Chapter 1, that estimates of the seismic moment from surface waves ranged from \( 0.75 \times 10^{26} \) to \( 1.9 \times 10^{26} \) dyne-cm. We also discussed what Alewine (1974) pointed out that the single point source estimate for the moment of the San Fernando earthquake, tend to underestimate the moment by approximately a factor of 2. The moment estimate \( M_0 = 1.1 \times 10^{26} \) dyne-cm chosen by Mikumo (1973), seems to be a reasonable median value. We shall choose a fault area \( S = 400 \text{ km}^2 \) which is slightly less than the \( 440 \text{ km}^2 \) obtained from the first few months of aftershocks data as shown in table 1. Thus using the rigidity \( \mu \) for the crust \( \sim 3 \times 10^{11} \) dyne/cm² we get a fault slip

\[
\Delta u = \frac{M_0}{\mu S} = 1 \text{ meter.}
\]

In all the models considered in the following, we shall use this fault slip estimate.

From the discussion given in chapter 1 of the data and the results that were presented by various workers on the
San Fernando earthquake, we may choose the following appropriate source parameters. The dip angle of the deep part of the fault will be taken to be 52°. The hypocentral depth will be 14 km. Then the location of the epicenter puts the Pacoima Dam station at \( x_o = 8 \) km, as shown in Fig. 4.

The first set of models we consider have the parameters that we just described. We will look at the two components of displacements as the fault moving with constant dip is stopped at an upper limit \( \theta = 90 L, 94 L, 8 L \) where \( L \) is the length that connects the free surface with the hypocenter, along the fault plane. The displacements are convolved with the instrument response and low-cut-filtered for frequencies lower than 0.06 c/s. The result is then convolved with with \( g \) as given by eq. 14 in Chapter 3. For this case, we chose a rise time close to what most other researchers have used, which is \( t_o = 0.6 \) second. All the convolutions are carried out numerically in the time domain.

It is well known, however, that the fault dip as directly observed at the surface has a smaller dip angle than that determined from the fault plane solution. Kamb et al. (1971) report on the Tujunga fault segment that the fault dips 20-25° to the north and is a thrust. Proctor et al. (1972) give two cross sections revealing a near-surface dip of 20-22° for the same segment. Also, the projection of the fault surface from the hypocentral area
using the 52° dip would put the fault trace further to the north than the observed trace. Thus as a reasonable second simple model, we approximate the San Fernando fault with two segments. The geometry is shown in fig. 14. The lower portion of the fault $L_1$ has the same dip determined from the fault plane solution. The upper segment $L_2$ has a smaller dip of 40° similar to what Boore and Zoback (1974) have used. The fault trace is at a distance of 5 km from the station, corresponding to the distance between the observed San Fernando fault trace and the Pacoima Dam station. Several maps are reproduced in the literatures showing the San Fernando valley map with the location of the main shock, the Pacoima Dam station and the fault traces on the ground. The reader is referred, for example, to fig. 1a of Hanks (1974) for a relevant map. In fig. 4 and fig. 14, we are clearly looking from the northwest side at a cross section perpendicular to the fault trace and passing through the station.

We also consider another model in which the station is at a distance $x_0 = 12$ km with the geometry as shown in fig. 4. In other words it is on the down thrown side of the fault. Allen et al. (1973), revised the hypocentral location of the main shock as given by Allen et al. (1971). They displaced the epicenter by about 1 km to the north and they assigned a depth of 8.4 km. That depth was about
Fig. 14 Geometry of the second set of models: $L_1 = 11$ km, the horizontal distance between the upper end of $L_1$ and the station is constrained by the observed distance between the Pacoima Dam and the fault trace. Distances in the drawing are in kilometers. The rupture velocity $v_1$ on $L_1$ is kept constant at 2.5 km/sec, while two values are considered for $v_2$ on $L_2$. 
6 km shallower than the previous depth determination. Hanks (1974) pointed out that the surface wave spectral studies of Canitez and Toksöz (1972) as well as tentative identification of pP at teleseismic distances suggest a hypocentral depth of 12-14 km. This depth with the epicentral location of Allen et al. (1973) would put the main shock well below the fault plane determined by well located aftershocks (Whitcomb et al., 1973). To resolve this inconsistency, Hanks (1974) proposed moving the epicenter northward by about 4 km. As far as the initial portion of the Pacoima Dam record is concerned, this is equivalent to putting the station at $x_0 = 12$ km in Fig. 4. The latter portion of the record is more affected by the near surface, low angle segment of the fault. Although only the initial part of the motion is relevant for checking the earthquake focus proposed by Hanks, we show the entire displacement for the sake of interesting comparison between motions on either sides of a thrust fault.

4.2 Towards a better understanding of the Pacoima Dam record

Consider now the first model we discussed in the last section. Figs. 15 & 16 show the displacements convolved with $g'(t)$, the instrument response and filtered to eliminate periods longer than 16 seconds. The instrument response is that of a simple damped pendulum with natural frequency of 19 Hz. and 60% critical damping as given by Trifunac and
Fig. 15 Behavior of the horizontal displacement for the first set of models. Time is in seconds after the p-arrival and the displacement is in centimeters; these will be the units for all the rest of the figures. The observed record is the bottom trace. The half-space solution is on the left and the infinite medium solution is on the right. All theoretical results are convolved as described in the text.
Fig. 16 Behavior of the vertical displacement corresponding to Fig. 15.
and Hudson (1971). The function \( g(t) \) is given by eq. 14 in Chapter 3, with \( t = 0.6 \text{ sec} \). All the convolution was done by using numerical linear convolution in the time domain. On the basis of comparing the computed results shown in Figs. 12 & 13, with the observed record reported by Trifunac and Hudson (1971) and shown in Figs. 15 & 16, we chose the lower rupture velocity of 2.5 km/sec for the model under consideration here. This value is consistent and close to rupture velocity values used by other observers using dislocation models as we reviewed in Chapter 1.

The infinite medium solution is shown on the right hand side of both Figures for the purpose of comparison. There are some interesting things to observe on these figures. The first one is the development of the surface P motion as well as the later high frequency motion as the fault is stopped closer and closer to the free surface. Those latter phases are as we discussed in chapter 3 caused by the shear and Rayleigh waves from the upper tip of the fault. It is clear from those figures that the half-space solution with the fault reaching its closest distance to the free surface is the best model that fits the observation. Still this model has two basic problems. The first one is that the theoretical vertical displacement is larger in amplitude than the observed one while the horizontal one is smaller. The second one is that the agreement between the horizontal
component and the observed one is rather poor after four seconds. We expect the first problem to definitely become less serious if we make the fault plane dips less steeply close to the free surface, in agreement with the observed small dip at the surface. The second problem may also improve if we make the upper portion of the fault dips less steeply and decrease the rupture velocity on that portion. The reason for this conjecture is that we expect more contribution from the surface P waves but this can be proven or disproven by getting concrete results from our model. We will verify this conjecture by an appropriate model, later on.

Another interesting feature to observe on figures 15 & 16 is the initial motion. The vertical component motion is initially upward as one expects for an infinite medium. Then, about 1.5 seconds later it goes downward for a duration of one second. The amplitude of the very first motion is smaller than the downward motion, apparently explaining the puzzling downward motion on the observed record. Filtering and numerical processing of the observed motion probably amplified the downward motion, as was seen in the results obtained by numerical filters of Niazy (1973), Mikumo (1973) and Trifunac (1974). Our convolution filter in the time domain minimizes the distortion at the beginning of the record. At the time when the theoretical vertical
displacement starts its initial downward motion, the horizontal component also exhibits a small high frequency motion to the negative x-direction. If this motion is magnified by the topography, for example and again amplified by the numerical processing, it is possible to explain the observed behavior of initial motion by our simple model.

Consider next the geometry of fig. 4 with $x_0 = 12$ km. This is the last case that we discussed at the end of the first section in this chapter. Figure 17 shows the behavior of the convolved displacements for the half-space and the infinite medium solution. The convolution was performed just like the previous case. For this geometry, the first arrival after the P-motion in the half-space case, is the surface P-wave from the lower tip of the fault. We observe that because of the location of the station, the initial negative motion in the horizontal displacement is quite small and then we get a large positive motion to the right. That positive motion is rather surprising considering that the station is on the fault side that is thrown down and to the left. The motion continues until the shear wave or the surface P (for the half-space) arrives to the station and then $u$ starts changing sign. Thus for the first two and a half seconds, the waveform for $u$ is quite similar to that shown in fig. 15, with the station at $x_0 = 8$ km which is on the upthrown side of the fault.
Fig. 17 Convolved displacements at a station $x_0 = 12$ km in Fig. 4. Top traces are the half space solution while the bottom ones are the corresponding infinite medium solution. Note the early arrival of the SP in the half space solution preceding the S arrival as indicated on the infinite medium solution.
similarity is apparent except for the small, high frequency kink shown in fig. 15 (half-space case) at about 1 second. Thus, the model we are considering now can not explain the observed initial motion of the horizontal component of displacement.

The vertical component has a clear, initial downward motion in agreement with the observed motion. The only problem is that at two and a half to three seconds it shows no sign of recovery to the upward direction contradictory to the observed displacement. It does seem however that while the observation on the initial motion from this model are not quite consistent with the observed initial motion at the Pacoima Dam, but they can not be taken as positive proof against Hanks (1974) proposal of moving the epicenter northward by about four kilometers. The reason is that the two-segmented fault affects significantly the waveforms two and a half seconds after the origin time. Figure 18, shows the results for the exact step-function solution.

The next model we shall consider, is the one shown in fig. 14. We shall start by keeping the rupture velocity over $L_1$ constant at 2.5 km/sec. Over $L_2$, we start by choosing a rupture velocity of 1.85 km/sec as proposed by Bodre and Zoback (1974). Figure 19 shows the horizontal displacement from $L_1$, $L_2$ and $L_1 + L_2$ for the infinite
Fig. 18 Exact step-function solution corresponding to Fig. 17.
Fig. 19 Step-function horizontal displacement: right hand side is the infinite medium solution and the left hand side is the half-space solution. Bottom traces are the contribution from $L_1$, middle traces are from $L_2$ and top ones are from $L_1 + L_2$. The rupture velocity $v_2=1.85$ km/sec.
Fig. 20 Corresponding vertical displacement.
medium and the half-space solutions. Figure 20 shows the corresponding vertical component of displacement. The most striking difference between the infinite medium and the half-space results is the significant increase in the high frequency content of the record. This difference will be dramatized when we look at the acceleration records. The shear wave arrivals from the ends of the various segments is more pronounced in the half-space solution than in the infinite medium solution. We also have rather high frequency surface P waves in the half-space case, which are generated when the fault surpasses the critical length \( L_c \) that we discussed in chapter 2. The surface P waves seem to cause a rather complicated high frequency motion. There may also be surface S waves as discussed by Lapwood (1949), whose arrival time \( |x| / \beta \) for shallow sources may make it quite difficult to identify from the direct shear wave arrival or the Rayleigh arrival. Our line source solution indicates that it is negligible compared to the surface P, the direct S or the Rayleigh phases.

For the purpose of comparison with the observations, we convolve our step-function response in the usual way. We show in fig. 21 the horizontal displacement generated from \( L_1, L_2 \) and \( L_1 + L_2 \) for the convolved infinite medium and half-space solutions. The rise time \( t_o = .6 \) second and the sampling interval is \( \Delta t = .1 \) second. The observed
Fig. 21 Convolved horizontal displacements of Fig. 19, compared with the observations (bottom trace) with $t_0 = 0.6$ sec.
Fig. 22 Vertical displacements corresponding to Fig. 21.
horizontal component is also included for comparison. The corresponding vertical component of displacement is shown in fig. 22. Changing the rise time \( t_0 \) to 0.2 second increases the high frequency content in the record and the results for the horizontal and vertical displacements are shown in fig. 23. There is an apparent decrease in the amplitude even for the low frequency motion in the latter case as compared with the previous one. The reason is that we are convolving with \( g'(t) \). \( g(t) \) is defined by eq. 14 which is zero outside the range \( 0 \leq t \leq t_0 \). Since our sampling interval is fixed at 0.1 second which is only half our rise time \( t_0 = 0.2 \) sec, we are clearly undersampling \( g'(t) \) and the result is an artificially attenuated amplitude.

However, since the frequency content of the larger rise time \( t_0 = 0.6 \) is apparently in better agreement with the observed record as far as the absolute amplitude is concerned. We will not pay too much attention to the amplitude for the case of \( t_0 = 0.2 \).

Let us pause briefly here and discuss the results shown in figures 21 and 22. In particular, we focus our attention on the \( L_1 + L_2 \) traces as compared with the observations. The theoretical horizontal component compares much better with the observed one than any model that we have discussed so far. The theoretical record is starting to reproduce some of the finer details of the
observed motion. For example the small higher frequency motion in the positive x-direction at about 4 seconds is clearly apparent at the correct time on the theoretical record. The second large, long-period positive swing on the horizontal observed record is now closely matched by a similar motion in the theoretical model. However, the corresponding swing has a shorter period and occurs about a second too soon. This suggests a lower rupture velocity on L₂ and this will be considered later.

Consider now the vertical component of displacement shown in fig. 22. Again the fit here is better than the earlier models that we have considered up to now. There is a small upward initial motion on the theoretical record, followed by a longer period downward motion at about the same time that the observed initial motion reaches its maximum. There is still however a discrepancy in the amplitude of the theoretical versus the observed motion. The theoretical is still considerably larger than the observed amplitude while it was vice versa for the horizontal component. This may still indicate a smaller dip for the upper portion of the fault, than what we used in the model.

We show next a comparison between the Pacoima Dam accelerogram and the theoretical acceleration obtained from our model. We obtained the acceleration by numerically differentiating the displacement twice and
then convolving as we did for the displacement. The sampling interval was $\Delta t = .1$ second. This will introduce aliasing error. We will be able to look at frequencies less than the Nyquist frequency $1/(2\Delta t) = 5$ Hz. Also for the extreme rise time value $t_0 = .2$ sec, the computed acceleration will still be smaller in amplitude than the actual one because of under sampling the function $g'(t)$ as we discussed previously. Furthermore, the amplitudes of the higher frequency will be smaller than they should be because of the numerical differentiation. The problems are less serious for the case of $t_0 = .6$ second.

With these numerical problems in mind, let us look at fig. 24 and 25. The top two traces show the acceleration for the current model for the two rise times we considered. The third trace is the Pacoima Dam record digitized from Hanks (1974). Since no numerical processing will be done on this record, the digitization errors are not important. The bottom trace is for the last model that we will discuss later. Let us focus our attention on the top three traces. The top trace with $t_0 = .2$ has higher frequency content than the other theoretical trace. It also has a much larger amplitude than the observed. The amplitude of the motion show that we can not increase the frequency content of our theoretical solution by simply decreasing $t_0$ and $\Delta t$, because that will make the amplitude of the record much larger than the observed amplitude.
Fig. 24 Display of horizontal accelerations for different models. The right hand side show the infinite medium accelerations. The left hand side is the half-space solution. The accelerations unit is cm/sec².
Fig. 25 Vertical accelerations corresponding to Fig. 24.
Comparison between the theoretical and the observed, show that if we delay the Rayleigh phase on the theoretical (that is the last major motion) by about one second, then it will correspond to the last major motion on the observed record. This should bring the largest amplitudes on our theoretical record closer to the largest accelerations on the observed record. Those large accelerations in the theoretical solution are caused by the surface P arrival when the fault surpasses the critical length $L_c$. The mentioned delay suggests still a slower rupture velocity on the upper portion of the fault. This is the model that we will discuss next.

If we require the mentioned delay time, for the latter portion of the record, then we have to lower the rupture velocity on the upper portion of the fault. A rupture velocity of 1.5 km/sec on $L_2$ should be adequate. This is the last model we shall consider in this thesis. The acceleration records from this model are shown in the lower tr. as on figs. 24 and 25, for $t_o = .6$ second. It seems now reasonably clear that the highest acceleration ever recorded during an earthquake up to now, was caused by near field surface P waves followed closely by the so-called breakout phases. Those breakout phases for the Pacoima Dam record seem to be a mixture of shear and Rayleigh motion arriving close to each other. In both fig. 24 and 25 the
acceleration from the infinite medium solution are included
to dramatize the effect of the free surface on the accelerations.

For the sake of completeness and comparison we include
both the exact displacement and the convolved one for the
last model. The contribution from L₁ is kept the same while
on L₂ we chose a rupture velocity of 1.5 km/sec. Fig. 26
shows the horizontal displacement from L₂, L₁ + L₂ for the
infinite medium and the half-space solutions. Fig. 27 shows
the corresponding vertical component of displacement. To
compare with the observed displacements, we convolve in the
usual way with \( t_o = 0.6 \) second. Fig. 28 shows the convolved
final model for the horizontal displacement. The fit is
again an improvement over the previous models. The same
is true for the vertical component of displacement shown in
fig. 29. Thus the assumption of a lower rupture velocity is
consistent with the observed displacements and accelerations.
Clearly the observed accelerograms were considerably more
complicated than theoretical ones. However, we were still
able to present a model that explains the general feature
of displacement and acceleration seismograms. The
discrepancies between our theoretical model and the
observations will be discussed in the next section.
Fig. 26 Horizontal displacements corresponding to Fig. 19 but with $v_2 = 1.5 \text{ km/sec}$. 
Fig. 27. Vertical displacements corresponding to Fig. 26.
Fig. 28 Convolved horizontal displacements with $v_z = 1.5$ km/sec, as compared with observation on the bottom trace.
Fig. 29 Vertical displacement corresponding to Fig. 28.
4.3 Explained and unexplained features of the Pacoima Dam record; discussion and conclusions

We have introduced a series of models to approximate the faulting process of the San Fernando earthquake, using the Pacoima Dam records. We have assumed that the medium is homogeneous, isotropic and perfectly elastic, and we assumed two dimensionality. We have computed the displacements and from it the accelerations and compared with the observations. In spite of all the simplifying assumptions that we made and in spite of the fact that we only chose a few reasonable models out of the denumerable infinity of models, the major features and some of the finer details on the displacement records were well explained. Including the free surface, and approximating the fault by two segments to be consistent with the available data, have in my opinion introduced a rather dramatic improvement of the previous infinite medium solutions.

By comparing the theoretical and observed accelerograms, we were able to choose a rupture velocity on the upper side of the fault that explained the highest accelerations on the Pacoima Dam records as due to surface P waves generated by the conversion, at the free surface, of the S waves radiated from the shallow part of the fault, to P waves. The latter high accelerations are easily identified with the break out phases in the model. Those are Rayleigh and shear waves radiated from the upper tip of the fault.
However, the observed accelerogram was much more complicated than the theoretical one. The record, for example, shows considerable high frequency motion following the initial P arrival, while the predicted one showed no such motion. Clearly the reason for this discrepancy must be due to complexity in the source and the medium that our simplified model can not account for. It is well known that homogeneity assumption breaks down for very short wavelengths. Those waves would usually be severely scattered by the small scale heterogeneity in the earth. The short wavelengths are expected to be also sensitive, to the complicated topography at the Pacoima Dam as was demonstrated by Bouchon (1973). In addition to all that, the three dimensionality may also account for some of the complexity in the observed record. Indeed we expect the high frequency surface P and Rayleigh waves to be very sensitive to the surface ground conditions and topography. There is no known quantitative account of how the effects of the layering, topography and heterogeneity along the wave path can be corrected for. The linearity assumption may have also been somewhat violated for the Pacoima Dam, because there was a crack in the concrete of the instrument house (Trifunac and Hudson, 1971) caused by the earthquake.

Considering all those facts, it is only natural that we can not match all the details on the accelerogram. One
may ask however about the possibility of inverting our solution to get useful constraint on some of the parameters involved. For example, one may constrain the fault plane from some of the static results. We can then divide the fault plane in small portions and try to solve for appropriate sets of rupture velocities and dislocation slip that would minimize the differences between the observed and calculated records. This however, is a formidable and an expensive project.

One may also ask if the method we developed here can be extended to more realistic models. For example if we can solve the problem in a layered medium. Experience with the half-space solution indicate that although this may be possible but the results are likely to be too cumbersome. Even in the simple case of the half-space, we had to develop fifty two FORTRAN IV subroutines and a main program to carry out the computation. Any significant increase in complexity of the computation may make the problem untractable.

Thus, it seems that more complicated models should be formulated and tackled by numerical methods like finite element for example. Those methods are limited however by the computer time available. Some hybrid methods combining the analytical and the numerical approaches may provide an optimum compromise to tackle and solve more realistic problems in the study of earthquake source mechanism.
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ADDENDUM

APPENDIX I

Elliptic Integrals

We are faced in our problem with the need to evaluate three kinds of elliptic integrals with complex argument for all three types, with modulus $k$ that can be real or imaginary, less than 1 or greater than 1 in magnitude, and with complex parameter $\nu$ for the third type. The need for efficiency in the computation can not be over-emphasized. In the following, I will discuss the methods that were found to be most appropriate.

The complete elliptic integral of the first kind is defined by

$$K(k^2) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad (A1.1)$$

If $k^2 < 0$, we reduce it to the case of $k^2 > 0$ by using the equation:

$$K(k^2) = \frac{1}{|k|} K\left(\frac{-k^2}{1-k^2}\right), \quad k' = \sqrt{1-k^2}$$

the above definition for $k'$ will be used consistently throughout this Appendix. The above equation is eq. 17.4.17 of Thompson (1964), with $m = -k^2$, and $\varphi = \pi/2$.

If $k^2 > 1$, we reduce it to the case of $k^2 < 1$ by using the
equation:

\[ K(k^2) = \frac{1}{k} \left[ K\left(\frac{1}{k^2}\right) + iK'\left(\frac{1}{k^2}\right) \right] \]

where \( K'(k^2) = K(k^2), \quad i = \sqrt{-1} \)

The above equation can be found on p. 319 of Erdelyi et al. (1953), or eq. 162.02 of Byrd and Friedman (1971). Thus we have to worry only about evaluating \( K(k^2) \) where \( 0 < k^2 < 1 \). To this end, we write \( K(k^2) = \frac{\pi}{k} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) \) [eq. 17.3.9 of Thompson (1964)] where \( F \) is the hypergeometric function defined by the series:

\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}
\]

\[
K = \frac{1}{\pi} \sum_{n=0}^{\infty} \left( \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{n!} \right)^2 \left( \frac{\Gamma\left(n + \frac{1}{2}\right)}{n!} \right)^2 \left( \frac{\Gamma\left(n + 1\right)}{n!} \right)^2
\]

where \( \Gamma(x) \) is the gamma (factorial) function, \( \Gamma(x + 1) = x \Gamma(x) \). The above series is sufficient to calculate \( K \) efficiently when \( k^2 < 1/2 \), but the convergence deteriorates rapidly when \( k^2 > 1/2 \). Thus for \( k^2 > 1/2 \), we use the equation

\[
F(a, b; a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(1-2n)\Gamma(1-2)}{\Gamma(a+b+n)\Gamma(1-2)}
\]

\[(A1.2)\]
where \((x)_h = \frac{\Gamma(x+n)}{\Gamma(x)}\), where \(\psi(x)\) is the digamma function

\[
\frac{\Gamma'(x)}{\Gamma(x)} \quad \text{defined by:}
\]

\[
\begin{align*}
\psi\left(\frac{1}{2}\right) &= -\gamma - 2 \ln(2) \\
\psi(x+1) &= \psi(x) + \frac{1}{x} \\
\psi(1) &= -\gamma
\end{align*}
\]  \hspace{1cm} (A1.3)

where \(\ln(z)\) is the natural logarithm and \(\gamma\) is the Euler constant given by:

\[
\gamma = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right] \approx 0.5772156649\ldots \quad (A1.4)
\]

where eq. (2) is given by Oberhittinger (1964), eq. 15.3.10, equs. (3) and (4) are given by Davis (1964), eqs. 6.3.5, 6.1.3 respectively (for a 24 digit decimal value of \(\gamma\) see Liepman (1964)).

Now we can calculate \(K(k^2)\) for \(-\infty < k^2 < \infty\), by using the previous procedures.

The complete elliptic integral of the second kind, is defined by:
\[
E(k^2) = \int_0^1 \frac{1-k^2s^2}{1-s^2} \, ds = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}} \, d\theta
\]

Define the nome \( q \) by:

\[
q = e^{\pi K'/K}
\]

with \( K' \) and \( K \) as defined previously, then \( E \) will be given by:

\[
E = \frac{1}{3}(1+k^2)K + \frac{\pi^2}{K} \left[ \frac{12}{12} - 2 \sum_{s=1}^{\infty} q^{2s} (1-q^{2s})^{-2} \right]
\]

The above is eq. 17.3.23 of Thomson (1964). It is adequate for evaluating \( E \) once \( K \) is determined.

The complete elliptic integral of the third kind is defined by:

\[
\Pi(k^2, \nu) = \int_0^1 \frac{1}{(1+\nu s^2)(1-k^2s^2)(1-s^2)} \, ds = \int_0^{\pi/2} \frac{1}{(1+\nu \sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \, d\theta
\]

We have no need to evaluate initially, but it can be evaluated by the method for the incomplete kind. The incomplete elliptic integral of the first kind, is defined by:
\[ F(z, k^2) = \int_0^3 \frac{d\xi}{\sqrt{(1-k^2\xi)(1-k^2\xi^2)}} = \int_0^{\sin^{-1}(z)} \frac{d\theta}{\sqrt{1-k^2\sin^2 \theta}} \]

If \( k^2 < 0 \), we reduce it to the case \( k^2 > 0 \) by using:

\[ F(z, k^2) = k' \cdot F(\frac{z \cdot k'}{\sqrt{1-k'^2z^2}}, k_1) \quad k_1 = \frac{|k|}{k} \]

The above is eq. 160.02 of Byrd and Friedman (1971).

If \( k^2 > 1 \), we reduce that case to one with \( k^2 < 1 \) by:

\[ F(z, k^2) = \frac{1}{k} \cdot F(kz, k^{-2}) \]

(eq. 162.02 of Byrd and Friedman (1971))

When \( z \) is complex, we define

\[ w = \psi + i\psi = \sin^{-1}(z) \]

where \( \sin^{-1}(z) \) is taken as a conformal mapping from the \( z \)-plane (see Kober (1957)) to the \( W \)-plane. We then write

\[ F(\phi + i\psi, k^2) = F(\gamma, k^2) + i F(\mu, k^2) \]

where \( \cot^2 \chi \) is the positive root of the equation

\[ x^2 - \left[ \cot \phi + k^2 \sin^2 \psi \cdot \csc^2 \phi - k^2 \right] x - k^2 \cot^2 \phi = 0 \quad (A1.5) \]
and \[ k \tan^2 \mu = \tan^2 \varphi \cot^2 \lambda - 1 \]

Proper limiting process must be taken in the above equation when \(|\varphi|\) is close to \(\pi/2\) or \(\pi\). Equation 5 is given by eq. 17.4.11 of Thomson (1964). If \(\varphi = \pi/2\) (i.e. \(z \gg 1\)), we have two cases:

\[ 1 < z < 1/k \]

then we write:

\[ F(z, k^2) = K(k^2) + iF\left(\frac{\sqrt{z^2 - 1}}{k^2}, k^2\right) \]

for \(\frac{1}{k} < z < \infty\), we have:

\[ F(z, k^2) = F\left(\frac{1}{k^2}, k^2\right) + iK' \]

Note that if \(G(z)\) is an elliptic integral, then \(G(-z) = -G(z)\), which enables us to consider the cases for \(\varphi = -\frac{\pi}{2}\) above. The last two equations are given by Byrd and Friedman (1971), eqs. 115.02 and 115.03.

The incomplete elliptic integral of the second kind, is defined by:

\[ E(z, k^2) = \int_0^z \frac{1-k^2 \xi^2}{\sqrt{1-\xi^2}} d\xi = \int_0^{\sin^{-1}(z)} \sqrt{1-k^2 \sin^2 \theta} d\theta \]

It satisfies the equation:

\[ Z(u, k^2) = E(u, k^2) - uE/K \]
where \( u = F(z, k^2) \)

We will drop the \( k^2 \) argument in what follows, the dependence, on \( k^2 \) of the functions involved is understood implicitly. In the last equation, \( \zeta \) is the Jacobi's Zeta function (see Copson (1970), p. 405) defined by

\[
\zeta(u) = \frac{\theta'(u)}{\theta(u)}
\]

\[
\theta(u) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{in\pi u / \kappa}
\]

with \( q \) as defined previously, and \( |q| < 1 \) thus \( \theta \) is an analytic function of \( u \) and it is most suitable for numerical computation because it is rapidly convergent, so is its derivative:

\[
\theta'(u) = \frac{i\pi}{\kappa} \sum_{n=-\infty}^{\infty} (-1)^n n q^{n^2} e^{in\pi u / \kappa}
\]

Thus having calculated \( E, K, \zeta, u \) we calculate \( E(u) \) as:

\[
E(u) = \zeta(u) + \frac{uE}{K}
\]

The incomplete elliptic integral of the third kind is defined by:

\[
\Pi(z, u, k^2) = \int_0^z \frac{1}{(1+uv^2)^{(1-k^2v^2)}} dv
\]
To evaluate it we define

$$a = F\left(\sqrt{-\frac{u}{k}}, k^2\right)$$

$$\therefore \Pi = u + \sqrt{-\frac{u}{k(1 + u)(u + k)}} \left[ \frac{1}{2} \ln \frac{\Theta(u - a)}{\Theta(u + a)} + u \Omega(u) \right]$$

(See Whittaker and Watson (1969) or Copson (1970) for a detailed discussion of $\Theta$, $\Omega$ and other functions related to elliptic integrals and elliptic functions.)

To determine the proper branch of the logarithmic function as $z$ changes, we first consider the following.

Let \( u^- = u - a \)

\( u^+ = u + a \)

$$\ln \frac{\Theta(u^-)}{\Theta(u^+)} = \sum_{m=1}^{\infty} \frac{q^m}{m(1 - q^{2m})} \left[ e^{i\pi u^+/k} + e^{-i\pi u^+/k} \right]$$

$$- \sum_{m=1}^{\infty} \frac{q^m}{(1 - q^{2m})m} \left[ e^{i\pi u^-/k} + e^{-i\pi u^-/k} \right]$$

The above is obtained from eq. 1050.2 of Byrd and Friedman (1971). The region of convergence of the series:
\[ \sum_{m=1}^{q} \frac{q^m}{m(1-q^m)} \left[ e^{im\pi u/K} + e^{-im\pi u/K} \right] \]

is the strip \( |\text{Im}(u)| < \text{Im}(iK') \)

Now the transformation \( F(z) \) maps the \( z \) plane into a rectangle in the \( u \)-plane defined by:

\[
\begin{align*}
|\text{Im}(u)| & \leq K' \\
|\text{Re}(u)| & \leq K
\end{align*}
\]

for \( 0 < k^2 < 1 \)

(see Kober (1957))

Thus although \( u \), a lie in the region of convergence of the above series but \( u^+ \) and \( u^- \) may not be. To fix the situation, we make use of the equation:

\[ \Theta(u + z_2 K') = -q^{-1} e^{-\pi i u/K} \Theta(u) \quad \text{(Copson (1970), p. 406)} \]

where it remains only to determine the branch of \( \ln(-1) \).

It is suggested that an initial branch is fixed using the trapezoidal rule and then changing the branch if \( z \) traces a curve that encloses \( \mp i n^{-1/2} \) according to Cauchy integral theorem (residue theorem).

The only problem that is left now is to evaluate \( F(x, k^2) \) where \( |x| < 1 \), \( 0 < k^2 < 1 \).
If \( k^2 \ll 1/2 \), we use the series expansion put in a form suitable for computation by DiDonato and Hershey (1959):

Let

\[
\begin{align*}
\phi &= \sin^{-1} x \\
\mathcal{H}_0 &= 1, \quad A_0 = \phi, \quad n = 0, \quad \Sigma = 0
\end{align*}
\]

Then \( n \to n+1 \)

\[
\begin{align*}
\mathcal{H}_{2n} &= \frac{2n-1}{2n} k^2 \mathcal{H}_{2n-2} \\
A_{2n} &= \frac{2n-1}{2n} A_{2n-2} - \frac{\sin \phi \cos \phi}{2n} \\
\Delta_1 &= \mathcal{H}_{2n} \cdot A_{2n} \\
\Sigma &= \Sigma + \Delta_1 \\
F &= \Sigma + \phi
\end{align*}
\]

For \( k^2 \gg 1/2 \), we modify algorithm 2 of Bulirsch (1965) which is essentially a Gauss' transformation (Byrd and Friedman (1971), p. 39). The algorithm is essentially as follows:

\(|z| < 1 \quad \text{and} \quad z \text{ is real}

we want \( F(z, k^2) \), let

\[
\begin{align*}
\mu_{i+1} &= \frac{1}{2} \left( \mu_i + z_i \right) \\
\nu_{i+1} &= \sqrt{\mu_i z_i} \\
k_{i+1} &= (\mu_i^2 k_i^2) \left/ \left( 4 \mu_{i+1}^2 \right) \right. \\
\alpha_{i+1} &= (b_i / \mu_i) + a_i \\
b_{i+1} &= a_i \nu_i + b_i \\
\sigma_{i+1} &= \left( \nu_i + \mu_i \sigma_i \right) / \left( \mu_{i+1} (1 + \sigma_i) \right)^{1/2} \\
\tilde{z}_{i+1} &= \sigma_{i+1} \tilde{z}_i \\
F(z, k^2) &= -\frac{\alpha_{i+1}}{\mu_i \tilde{z}_{i+1}} \ln \left( \frac{1 - \tilde{z}_{i+1} \mu_{i+1}}{1 + \tilde{z}_{i+1} \mu_i} \right)
\end{align*}
\]

\[
\begin{align*}
\mu_0 &= 1 \\
\nu_0 &= k \\
k_0 &= 1 - k^2 \\
\alpha_0 &= 1 \\
b_0 &= 1 \\
\sigma_0 &= \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} \\
\tilde{z}_0 &= z
\end{align*}
\]
APPENDIX II

Derivatives of the Potentials

I shall describe here the derivatives of the potentials that are needed to compute the displacements. The infinite medium terms of the potentials will be denoted by a "1" subscript, and the free-surface terms by a "2" subscript. Potential caused by a force \( F_x \) in the \( x \)-direction, will have \( x \) as the second subscript. Those caused by \( F_z \), will have \( z \) as the second subscript. Differentiation with respect to any variable \( \xi \), will be denoted by:

\[
\frac{\partial ( )}{\partial \xi} ( ) \xi
\]

We use the following notations consistent with chapter 2.

Let

\[
k = \frac{1}{r^2} \left[ h \frac{\tau^2 - r^2}{\beta^2} - i \kappa \right], \quad \frac{dk}{d\tau} = \frac{1}{r^2} \left[ \frac{h \kappa}{\tau^2 - r^2} - i \kappa \right]
\]

\[
z_\beta = \sqrt{k^2 + \frac{1}{\beta^2}} = \frac{1}{h} \left[ \tau \left( 1 - \frac{x^2}{r^2} \right) - i \frac{x}{r^2} \sqrt{\tau^2 - \frac{r^2}{\beta^2}} \right]
\]

\[
z_\alpha = \sqrt{k^2 + \frac{1}{\alpha^2}}
\]

\[
F = 4k^2 z_\beta^2 - (2k^2 + \frac{1}{\beta^2})^2, \quad F_+ = 4k^2 z_\beta^2 + (2k^2 + \frac{1}{\beta^2})^2, \quad G = 2k^2 + \frac{1}{\beta^2}
\]

\[
k' = \frac{1}{r^2} \left[ h \frac{\tau^2 - r^2}{\alpha^2} - i \kappa \right], \quad \frac{dk'}{d\tau} = \frac{1}{r^2} \left[ \frac{h \kappa}{\tau^2 - r^2} - i \kappa \right]
\]

\[
z_\alpha' = \sqrt{k^2 + \frac{1}{\alpha^2}} = \frac{1}{h} \left[ \tau \left( 1 - \frac{x^2}{r^2} \right) - i \frac{x}{r^2} \sqrt{\tau^2 - \frac{r^2}{\alpha^2}} \right], \quad z_\beta' = \sqrt{k^2 + \frac{1}{\beta^2}}
\]

\[
F' = 4k^2 z_\beta z_\beta' - (2k^2 + \frac{1}{\beta^2})^2, \quad F'_+ = 4k^2 z_\beta z_\beta' + (2k^2 + \frac{1}{\beta^2})^2, \quad G' = 2k^2 + \frac{1}{\beta^2}
\]
The displacements due to a double couple are given by equation (8) in chapter 2. The derivatives of the potentials needed for the calculation are:

\[
\varphi_{i_2,x} = \text{Re.} \left( \frac{1}{2\pi^s} \left[ k' \frac{d k'}{d \tau} \right] \right)
\]

\[
\varphi_{i_2,xx} = \text{Im.} \left( \frac{1}{2\pi^s} \left[ k' \frac{d k'}{d \tau} \right] \right)
\]

\[
\varphi_{i_2,x h} = -\varphi_{i_2,x x} = -\frac{1}{2\pi^s} \text{Im.} \left[ k' \frac{d k'}{d \tau} \right]
\]

\[
\varphi_{i_2,x h} = -\varphi_{i_2,x x} = \frac{1}{2\pi^s} \text{Re.} \left[ k' \frac{d k'}{d \tau} \right]
\]

\[
\varphi_{i_2,x x} = -\frac{1}{2\pi^s} \text{Im.} \left[ \frac{k'}{2\pi^s} \frac{d k'}{d \tau} \right]
\]

\[
\varphi_{i_2,x x} = \frac{1}{2\pi^s} \text{Re.} \left[ \frac{k'}{2\pi^s} \frac{d k'}{d \tau} \right]
\]

\[
\varphi_{i_2,x h} = -\varphi_{i_2,x x} = -\frac{1}{2\pi^s} \text{Im.} \left[ k' \frac{d k'}{d \tau} \right]
\]

\[
\varphi_{i_2,x h} = -\varphi_{i_2,x x} = \frac{1}{2\pi^s} \text{Re.} \left[ k' \frac{d k'}{d \tau} \right]
\]

\[
\varphi_{i_2,x x} = \frac{1}{\pi^s} \text{Re.} \left[ \frac{-F' \frac{k'}{2 \frac{d k'}{d \tau}} + \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x x} = \frac{1}{\pi^s} \text{Im.} \left[ \frac{F' \frac{k'}{2 \frac{d k'}{d \tau}} + \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x h} = \frac{1}{\pi^s} \text{Re.} \left[ \frac{k' \frac{d k'}{d \tau}}{2 F'} - \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x h} = \frac{1}{\pi^s} \text{Re.} \left[ \frac{k' \frac{d k'}{d \tau}}{2 F'} + \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x x} = \frac{1}{\pi^s} \text{Re.} \left[ \frac{-F' \frac{k'}{2 \frac{d k'}{d \tau}} + \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x x} = \frac{1}{\pi^s} \text{Re.} \left[ \frac{k' \frac{d k'}{d \tau}}{2 F'} - \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x h} = \frac{1}{\pi^s} \text{Re.} \left[ \frac{2 k' \frac{d k}{d \tau}}{2 F'} - \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]

\[
\varphi_{i_2,x h} = \frac{1}{\pi^s} \text{Im.} \left[ \frac{k' \frac{d k}{d \tau}}{2 F'} - \frac{2 k' \frac{d k}{d \tau}}{2 \varphi} \right]
\]
For the finite source, we separate the primed""and unprimed quantities above. For example, for the P-terms, we multiply the primed quantities by \( \frac{1}{aik + b\frac{\nu^2}{v}} \). The unprimed, shear terms are multiplied by \( \frac{1}{aik + b\frac{\nu^2}{v}} \). Then the integration can be performed as was outlined in chapter 2, and described in detail in Appendix III.
APPENDIX III

Singularities and Integration of the Free Surface Terms

It was mentioned near the end of Chapter II that the free surface terms can be written in the form:

\[ I = J_1 + J_2 \]

\[ J_1 = \int R_1(k, \zeta) \frac{dk}{d\tau} d\tau \bigg|_0^\beta \]

\[ J_2 = \int R_2(k, \zeta) \frac{1}{\zeta'} \frac{dk}{d\tau} d\tau \bigg|_0^\beta \]

where \( R_1, R_2 \) are rational functions of \( k, \zeta \), given by:

\[ R_1 = \frac{P_1(k, \zeta)}{Q(k, \zeta)} \]

\[ R_2 = \frac{P_2(k, \zeta)}{Q(k, \zeta)} \]

\[ \zeta = \sqrt{k^2 + 1/c^2} \]

Here, \( P_1, P_2, Q \) are polynomial functions, \( Q \) is defined as

\[ Q = (aik + b\zeta - 1/v) F_+ F_- \]

\[ F_+ = 4k^2 \zeta \zeta' - (2k^2 + \frac{1}{\beta^2})^2 \]

\[ F_- = 4k^2 \zeta \zeta' + (2k^2 + \frac{1}{\beta^2})^2 \]

Now \( F_+ F_- = 16k^4 - (2k^2 + \frac{1}{\beta^2})^4 = 0 \) give the roots of the Rayleigh equation which are dependent on the material properties of the medium, where in consistency with previous convention we dropped the Laplace transform parameter \( p \).

Let us review briefly the dependence of those roots on the material properties of the medium. The equation is
a cubic equation in \( k^2 \) and can be solved by standard methods, given for example by Vilhelm (1969). To examine the properties of the solution, we define some local variables (local for this Appendix) \( \xi = \alpha^2/\beta^2 \) where \( \alpha \) and \( \beta \) are the P-wave and S-wave velocities of the medium. Also, we define \( F \) by:

\[
F(\xi) = \frac{\left[ \frac{\xi^3}{32} - \frac{5}{48} \xi^2 + \frac{11}{96} \xi - \frac{1}{27} \right]}{\left[ \frac{\xi^2}{12} - \frac{\xi}{16} + \frac{1}{9} \right]^{3/2}}
\]

\( F \) has the following properties:

\[
F(1) < 1 \quad \text{and} \quad \lim_{\xi \to 0} F(\xi) = \frac{3\sqrt{3}}{4} > 1
\]

and \( F(\bar{\xi}) = 1 \) implies \( \bar{\xi} = \bar{\xi}_{12} \approx 3.1104353 \)

so that \( F(\xi) \leq 1 \) for \( 1 < \xi \leq \bar{\xi}_1 \)

\( F(\xi) > 1 \) for \( \xi > \bar{\xi}_1 \)

For \( \xi \in (1, \bar{\xi}_1) \), we define \( \varphi = \cos^{-1}(F(\xi)) \)

then the three roots for \( k^2 \) are \(-x_1^2, -x_2^2, -x_3^2\) where

\[
x_1^2 = \frac{\beta^{-2}}{\bar{\xi}^{-1}} \left[ -2\left( \frac{\xi^2}{12} - \frac{\xi}{6} + \frac{1}{9} \right)^{1/2} \cos \left( \frac{\varphi}{3} - \frac{\pi}{3} \right) + \frac{(3\xi-2)}{6} \right] = 1/C_R
\]

\[
x_2^2 = \frac{\beta^{-2}}{\bar{\xi}^{-1}} \left[ -2\left( \frac{\xi^2}{12} - \frac{\xi}{6} + \frac{1}{9} \right)^{1/2} \cos \left( \frac{\varphi}{3} - \frac{\pi}{3} \right) + \frac{(3\xi-2)}{6} \right]
\]
\[ x_3^2 = \frac{\beta^{-2}}{\xi - 1} \left[ -2 \left( \frac{\xi^2}{12} - \frac{\xi}{6} + \frac{1}{9} \right)^{1/2} \cos \left( \frac{\varphi}{3} + \frac{\pi}{3} \right) + \frac{(3\xi - 2)}{6} \right] \]

where \( C_R \) is the Rayleigh wave velocity. For a Poisson's solid, \( \xi = 3 \) and we have:

\[ x_1^2 = \frac{3 + \sqrt{3}}{4} \beta^{-2} \quad x_2^2 = \frac{1}{4} \beta^{-2} \quad x_3^2 = \frac{3 - \sqrt{3}}{4} \beta^{-2} \]

For \( \xi > \xi_1 \), we define \( \varphi = \text{Cosh}^{-1} \left( F(\xi) \right) \) and we have:

\[ x_1^2 = \frac{\beta^{-2}}{\xi - 1} \left[ -2 \left( \frac{\xi^2}{12} - \frac{\xi}{6} + \frac{1}{9} \right)^{1/2} \text{Cosh} \left( \frac{\varphi}{3} \right) + \frac{(3\xi - 2)}{6} \right] = 1/c_R^2 \]

\[ x_2^2 = \frac{\beta^{-2}}{\xi - 1} \left[ -2 \left( \frac{\xi^2}{12} - \frac{\xi}{6} + \frac{1}{9} \right)^{1/2} \left( \text{Cosh} \frac{\varphi}{3} + i\frac{\sqrt{3}}{2} \text{Sinh} \frac{\varphi}{3} \right) + \frac{(3\xi - 2)}{6} \right] \]

\[ x_3^2 = \frac{\beta^{-2}}{\xi - 1} \left[ -2 \left( \frac{\xi^2}{12} - \frac{\xi}{6} + \frac{1}{9} \right)^{1/2} \left( \text{Cosh} \frac{\varphi}{3} - i\frac{\sqrt{3}}{2} \text{Sinh} \left( \frac{\varphi}{3} \right) \right) + \frac{(3\xi - 2)}{6} \right] \]

These roots are mapped by the transformation that was discussed in Chapter II into corresponding roots in the
z-plane. The transformation is given by:

\[ k = \frac{ig_1 z^2 + 2ag_2 z + a_1}{z^2 - \frac{1}{c^2}} , \quad a_1 = \frac{h}{rc} , \quad a_2 = \frac{x}{rc} , \quad r^2 = x^2 + \frac{1}{c^2} \]

The roots in the z-plane are given by:

\[ r_3 = \frac{-\sqrt{x_1^2 - \frac{1}{c^2}} + i a_2}{a_1 + x_1} \]
\[ r_4 = \frac{-\sqrt{x_1^2 - \frac{1}{c^2}} + i a_2}{a_1 + x_1} \]
\[ r_5 = \frac{-\sqrt{x_1^2 - \frac{1}{c^2}} + i a_2}{a_1 - x_1} \]
\[ r_6 = \frac{-\sqrt{x_1^2 - \frac{1}{c^2}} + i a_2}{a_1 - x_1} \]
\[ r_7 = i \left[ \frac{a_2 + \sqrt{\frac{1}{c^2} - x_2^2}}{a_1 + x_2} \right] \]
\[ r_8 = i \left[ \frac{a_2 - \sqrt{\frac{1}{c^2} - x_2^2}}{a_1 + x_2} \right] \]
\[ r_9 = i \left[ \frac{a_2 + \sqrt{\frac{1}{c^2} - x_2^2}}{a_1 - x_2} \right] \]
\[ r_{10} = i \left[ \frac{a_2 - \sqrt{\frac{1}{c^2} - x_2^2}}{a_1 - x_2} \right] \]
\[ r_{11} = i \left[ \frac{a_2 + \sqrt{\frac{1}{c^2} - x_3^2}}{a_1 + x_3} \right] \]
\[ r_{12} = i \left[ \frac{a_2 - \sqrt{\frac{1}{c^2} - x_3^2}}{a_1 + x_3} \right] \]
\[ r_{13} = i \left[ \frac{a_2 + \sqrt{\frac{1}{c^2} - x_3^2}}{a_1 - x_3} \right] \]
\[ r_{14} = i \left[ \frac{a_2 - \sqrt{\frac{1}{c^2} - x_3^2}}{a_1 - x_3} \right] \]

where \( c = \alpha \) or \( \beta \)
The roots of the factor \( qik + b\gamma - \frac{1}{v} \) were discussed in Chapter II, and are given by:

\[
\begin{align*}
    r_1 &= -\frac{\sqrt{c^2 - v^2} + iv(ax + bh)}{v(ah_0 - bx_0) + rc} \\
    r_2 &= \frac{\sqrt{c^2 - v^2} + iv(ax + bh)}{v(ah_0 - bx_0) + rc}
\end{align*}
\]

where \( v \) is the rupture velocity, \( x = x_0 - a\gamma, h = h_0 - b\gamma, r^2 = x^2 + h^2 \)

\( \gamma \) is the upper limit of integration as we discussed in Chapter II, \( x_0, h_0, a \) and \( b \) are defined as shown in Fig. 4.

Two more roots of the integrand in the \( z \)-plane corresponding to the time infinity in the \( \tau \)-plane, are given by

\[
\begin{align*}
    r_{15} &= 1 \\
    r_{16} &= -1
\end{align*}
\]

We note that \( r_1 \) through \( r_{14} \) are all poles of multiplicity equal to 1, \( r_{15}, r_{16} \) are poles of multiplicity equal to 5 for the free surface terms and equal to 3 for the infinite medium terms. We now look at the branch points associated with \( \zeta' \). We saw in Chapter II that the factor

\( \zeta' = \sqrt{\frac{c^2 - v^2}{k + \frac{1}{c^2}}} \)

in the denominator in the integrand of \( J_2 \), will give us:

\( \zeta' = \frac{2i}{\zeta_{\text{min}}} \left[ b_j \zeta^j \right]^{1/2}, \quad 0 \leq j \leq 4 \)

where \( b_j = b_{4-j} \), and we assumed that the summation convention holds over repeated indices.
\[ b_4 = \frac{1}{4} \left( a_1^2 - \frac{1}{c^2} \right), \quad b_3 = -i a_1 a_2, \quad b_2 = -a_2^2 + \frac{a_1^2}{2} + \frac{1}{2} \frac{1}{c' z} \]

with \( a_1 \) and \( a_2 \) as defined previously. Because of the symmetry in the coefficients \( b_j \)'s, we can easily find the roots of \( b_j z^j \) by using the transformation \( y = z + \frac{1}{z} \) so \( b_j z^j \) reduces to:

\[ y^2 + \frac{b_3}{b_4} y + \frac{b_2}{b_4} - 2 = 0 \]

with roots \( y = q_1, q_2 \) and then solving the two quadratics:

\[ q_1 = z + \frac{1}{z}, \quad q_2 = z + \frac{1}{z} \]

Carrying out the algebra, we find the roots:

\[ q_a = \frac{1}{c'^2 h^2 - c^2 r^2} \left[ -r \sqrt{c^2 - c'^2} (c'h - cr) + i x c' (c'h - cr) \right] \]

\[ q_b = \frac{1}{c'^2 h^2 - c^2 r^2} \left[ r \sqrt{c^2 - c'^2} (c'h + cr) + i x c' (c'h + cr) \right] \]

\[ q_8 = \frac{1}{c'^2 h^2 - c^2 r^2} \left[ -r \sqrt{c^2 - c'^2} (c'h + cr) + i x c' (c'h + cr) \right] \]

\[ q_6 = \frac{1}{c'^2 h^2 - c^2 r^2} \left[ r \sqrt{c^2 - c'^2} (c'h - cr) + i x c' (c'h - cr) \right] \]
I will now discuss in detail how to integrate the free surface terms. Let us first look at the rational terms. They are of the form:

\[ I = \left. \frac{P_3(z)}{Q_3(z)} \right|_{z(t, \xi=0)}^{z(t, \xi=\xi)} \]

\( \xi \) - here is measured along the fault,

\[ Q_3 = (z^3-1)^5 \prod_{j=1}^{14} (z-r_j) \]

and \( P_3 \) is a polynomial of degree = 22. To carry out the integration, we have to write the integrand in the form:

\[ \frac{P_3(z)}{Q_3(z)} = \sum_{j=1}^{14} \frac{B_j}{z-r_j} + \sum_{k=1}^{5} \frac{A_k^+}{(z-1)^k} + \frac{A_k^-}{(z+1)^k} \]

\( B_j \) is easily computed by the equation:

\[ B_j = \frac{P_3(r_j)}{Q'_3(r_j)} \quad , \quad Q'_3(z) = \frac{d}{dz} Q_3(z) \]

To compute \( A_k^\pm \), it was found that the easiest method that can be easily programmed in a high level computer language like FORTRAN IV, is to throw the poles \( z=\mp 1 \) to infinity. The standard method is discussed by Ahlfors (1966). To compute say \( A_k^+ \) we write \( \eta = \frac{1}{z-1} \) so that
\[ z = \frac{1}{\eta} + 1 \]

We replace \( z \) by \( \frac{1}{\eta} + 1 \) in the expression

\[ \frac{P_3(\eta)}{Q_3(\eta)} \]

and then simplify to get:

\[ \frac{P_3(\eta)}{Q_3(\eta)} = \frac{P_4(\eta)}{Q_4(\eta)} = A_5 \eta^5 + A_4 \eta^4 + A_3 \eta^3 + A_2 \eta^2 + A_1 \eta + \frac{P_5(\eta)}{Q_5(\eta)} \]

where degree \((P_5) = degree \((Q_5)\). We do the same for \( A_K \).

There are still a few computational problems which I will discuss next for the cases \( C = \alpha \) and \( C = \beta \) separately.

**Case I: \( C = \alpha \)**

The upper limit of integration was shown (Chapter II) to be:

\[ \delta = \begin{cases} \delta_1 & \text{if } 0 \leq \delta_1 \leq L \\ L & \text{if } \delta_1 > L \end{cases} \]

\( \delta_1 \) is the smallest root of the equation

\[ \tau = \frac{r^2}{a^2}, \quad \tau = \tau - \frac{\delta}{V}, \quad x = x_0 - a \delta, \quad h = h_0 - b \delta, \quad r = x^2 + h^2 \]

For \( \delta = \delta_1 \), we have \( z = i \). If \( \xi < \xi_1 \), then there is the potential problem that \( r_j = i \) for some \( j = j_0 \) and for some time value \( t \). This will give us a logarithmic infinity in the term

\[ \int \frac{R_{j_0} \, dz}{z - r_{j_0}} \bigg|_{z = z_0}^{z = i} = R_{j_0} \log \frac{i - r_{j_0}}{z_0 - r_{j_0}} \]

Indeed it was found that for the geometry of Fig. 4
\( r_9 \) and \( r_{13} \) are equal to \( i \) at the values of

\[
\gamma_j^+ = \frac{x_0 h_0 \Xi_j^+}{a + b \Xi_j^+}, \quad \Xi_j^+ = \frac{1 - x_j^2 \alpha^2}{x_j^2 \alpha^2}, \quad j = 2 \text{ or } 3
\]

which corresponds to time values

\[
t_j = \frac{\gamma_j}{v} + \sqrt{\frac{(x_0 - a \gamma_j)^2 + (h_0 - b \gamma_j)^2}{\alpha^2}}
\]

Mathematically this is a removable singularity. It comes from rationalizing the denominator, and hence bringing the poles on the other Riemann sheets to our integration path. Computationally, it does not pose a serious problem because the logarithmic singularity grows very slowly, and we can sample the function at time values very close to \( t_j \) without any serious precision problem.

A more serious problem is caused by the fact that at \( \gamma = \gamma_j^- \) we have \( a_r = x_j^- \) and thus \( |r_{10}| \) and \( |r_{14}| \) tend to \( \infty \) there. This causes no problem mathematically because we simply have poles tending to infinity there. Computationally, if \( \gamma \) is close to \( \gamma_j^- \), we have a severe precision problem caused by cancellation in computing \( r_{13}, r_{14} \) and all other decomposition constants \( B_j, A_k^+ \).

Thus if for a given rupture velocity our sampling interval in time is very small, then \( \gamma \) may come too close to \( \gamma_j^- \) and we may have a precision problem. Alternatively, if for a given sampling interval in time, the rupture velocity is made sufficiently small (which means that we have finer
sampling intervals on the fault plane), we may run into a similar problem.

Both of the previous problems would not occur if \( \xi > \xi_p \) because \( x_2, x_3 \) will both be complex rather than real. In that case, none of the roots \( \eta_k, 1 \leq k \leq 16 \) is purely imaginary. This will improve our computational ability here, but will cause a problem in integrating the irrational part and that will be discussed later.

Case II: \( C = \beta \)

The upper limit of integration was discussed in Chapter II. It is given by:

\[
\vartheta = \begin{cases} 
\vartheta_1 & \text{for } 0 \leq \vartheta_1 \leq L_c \\
\vartheta_2 & L_c < \vartheta_2 < L, \vartheta_1 < \vartheta_2 \\
L & L < \vartheta_2 
\end{cases}
\]

\( \vartheta_1 \) is the smallest root of the equation \( \tau^2 = r^2/\beta^2 \)

\[
L_c = \begin{cases} 
\frac{x_0 + \frac{dh_0}{\alpha + bd}}{a + bd} & \text{for } x < 0 \\
\frac{dh_0 - x_0}{d \beta - a} & \text{for } x > 0 
\end{cases}, \quad d = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}
\]

\( \vartheta_2 \) is the root of the first degree equation:
\[ \tau = \frac{|x|}{a} + h \left( \frac{1}{\beta^2} - \frac{1}{\alpha^2} \right)^{1/2} \]

If \( \theta = \theta_2 > \theta_1 \), then \( \left| \frac{r^2}{\beta^2 \tau^2} \right| \) and it follows that the upper limit of integration in the \( z \)-plane is no longer \( \tau = i \). Recalling that our integration path in the \( z \)-plane was given by:

\[ z = \sqrt{1 - \frac{r^2}{\beta^2 \tau^2}} + \frac{ir}{\beta \tau} \]

We see that \( z = iy \), and \( y \) may be greater than or smaller than \( 1 \) depending on the choice of the sign of the square root above.

Let us now look at the irrational terms assuming that \( c = \alpha \) for the moment. They are in the form:

\[ J_2 = \int \frac{P_5(z)}{Q_3(z)} \frac{dz}{\sqrt{b_4 \sqrt{(z-q_\alpha)(z-q_\beta)(z-q_8)(z-q_{8'})}}} \bigg|_{z(t, \delta = \delta)}^{z(t, \delta = 0)} \]

with the notations as explained previously in this Appendix.

We have here degree \( (P_5(z)) = 24 = \) degree \( (Q_3(z)) \). We write:

\[ \frac{P_5(z)}{Q_3(z)} = \frac{a_{24}}{\sqrt{b_4}} + \sum_{j=1}^{14} \frac{C_j}{z - r_j} + \frac{5}{2} \sum_{k=1}^5 \left( \frac{D_k^+}{(z-1)^k} + \frac{D_k^-}{(z+1)^k} \right) \frac{1}{\sqrt{b_4}} \]
In the above equation, \( a_{24} \) is the coefficient of \( z^{24} \) in \( P_5 \), \( C_j \), \( D_k^\tau \) are decomposition constants obtained by the previously discussed method. We have to evaluate terms of the form:

\[
\frac{1}{\sqrt{b_4}} \int \frac{dz}{(z-q)^m \sqrt{(z-q_\alpha)(z-q_\beta)(z-q_\delta)(z-q_\varepsilon)}} , \quad 0 \leq m \leq 5
\]

Let \( \gamma^2 = \frac{(q_\beta - q_\delta)}{(q_\alpha - q_\delta)} \frac{z - q_\alpha}{z - q_\beta} \)

(Copson, 1970, p. 400)

\[
\therefore \frac{1}{\sqrt{b_4}} \int \frac{dz}{(z-q)^m \sqrt{(z-q_\alpha)(z-q_\beta)(z-q_\delta)(z-q_\varepsilon)}} = \frac{2}{\sqrt{b_4(q_\alpha-q_\beta)(q_\delta-q_\varepsilon)}} z
\]

\[
\int \left( \frac{1 - A^2 y^2}{1 - A^2 x^2} \right)^m \frac{dz}{(1 - y^2)(1 - x^2)}
\]
where we have:

\[ A_1^2 = \frac{q_\alpha - q_\delta}{q_\beta - q_\delta}, \quad A_2^2 = A_1^2 \frac{q_\beta - q}{q_\alpha - q}, \quad k^2 = \frac{(q_\alpha - q_\delta)(q_\beta - q_\delta)}{(q_\alpha - q_\delta)(q_\beta - q_\delta)} \]

In spite of the fact that \( q_\alpha, q_\beta, q_\gamma, q_\delta, q \) and \( b_4 \) all depend on the geometry of the problem and the upper limit of integration, we notice that:

\[ k^2 = \frac{\alpha^2 - \beta^2}{\alpha^2}, \quad b_4 (q_\alpha - q_\delta)(q_\beta - q_\delta) = -\frac{1}{\beta^2} \]

Similarly, \( A^2 \) is independent of the geometry of the problem and the upper limit of integration; in other words, it depends only on the elastic constants of the medium.

The integration path in the \( \xi \)-plane is the image of the portion of the unit circle in the \( z \)-plane that is given by:

\[ z = \sqrt{1 - \frac{r^2}{\alpha^2 \tau^2}} + \frac{ir}{\alpha \tau} \]

with \( \tau = t - \frac{\xi}{v}, \quad \xi = \tau \omega - \eta \), \( \eta = h_0 - b \delta \), and \( 0 \leq \delta \leq \gamma \).

In terms of \( t, \xi \) the curve in the \( \xi \)-plane is specified by:

\[ \xi = \frac{\alpha}{\alpha^2 - \beta^2} \left[ \alpha + \frac{\chi \beta}{\alpha \tau} + \frac{\beta^2}{\alpha^2 \tau^2} \left( 1 - \frac{r^2}{\alpha^2 \tau^2} \right) \right] + \frac{i \eta \beta}{\alpha \tau} \]
Thus in the dynamic range $\theta < \varpi$, so that the upper limit of integration in the $z$-plane is given by $z = i$, we have in the $\zeta$-plane:

$$\zeta^2(z=i,t) = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \left[ \varpi \sqrt{\alpha^2 \beta^2 + i \beta \varpi} \right]$$

The behavior of the above curves is exhibited in Figure (A3-1) together with the relevant singularities in the $\zeta$-plane. In the Figure (A3-1), we exhibit $\zeta$ sometimes as a function of $z$ and sometimes as a function of $\theta$ and $t$, for the sake of clarity and compactness. We have assumed in Figure (A3-1) that $\alpha = 6$ km/sec, $\beta = 3.5$ km/sec, so that $\zeta = \frac{\varpi^2}{3\varpi} < \zeta_i$. The roots $\zeta(r_2), \zeta(r_3)$ lie on the curve $\zeta(\varpi, t)$, so that the integration curve is equivalent to a straight line connecting the upper limit of integration with the lower one, because it lies in a region of analyticity of the integrand. Thus by using the Cauchy integral theorem, we can evaluate the integral, by evaluating it along two straight lines connecting the upper and lower limits to the origin, as shown in Fig. (A3-1).

I will now discuss how to evaluate the integral along a straight line. Let

$$Z_m = \int_0^{\varpi} \left( \frac{1 - A_i \zeta^2}{1 - A_i \zeta^2} \right)^m \frac{d\zeta}{(1 - \zeta^2)(1 - k^2 \zeta^2)}$$
\( z_m \) was evaluated by Byrd and Friedman (1971) and it is given by their equations 340.04 and 336.00 through 336.03. For the sake of completeness, we reproduce the results here.

\[
z_m = \left( \frac{A^1}{A} \right)^2 m \sum_{j=0}^{m} \frac{(A^2 - A^4)^j}{A^4_j} \binom{m}{j} V_j
\]

where \( \binom{m}{j} \) are the binomial coefficients and \( V_j \) are given by:

\[
V_0 = F(s_1, k^2)
\]

\[
V_1 = \prod (s_1, -A^2, k^2)
\]

\[
V_2 = \frac{1}{2(A^2-1)(k^2-A^2)} \left[ A^2 E(s_1, k^2) + (k^2-A^2) F(s_1, k^2) + (2A^2k+2A^2-A^4-3k^2) \prod (s_1, -A^2, k^2) - \frac{A^4 s_1 \sqrt{(1-s_1^2)(1-k^2s_1^2)}}{1-A^4 s_1^2} \right]
\]

\[
V_{m+3} = \frac{1}{2(m+2)(1-A^2)(1^2-A^2)} \left[ (2m+1) k^2 V_m + 2(m+1) (A^2 k^2 + A^2 - 3k^2) V_{m+1} + (2m+3) (A^4 - 2A^2 k^2 - 2A^2 + 3k^2) V_{m+2} + \frac{A^4 s_1 \sqrt{(1-s_1^2)(1-k^2s_1^2)}}{(1-A^4 s_1^2)^{m+2}} \right]
\]
In the above, $F$, $E$ and $\Pi$ are the incomplete elliptic integrals of the First, Second and Third kind respectively. Also, $A^2$, $s_1$, are general complex numbers, and evaluating those elliptic integrals is discussed in details in Appendix I.

If $\xi > \xi_1$, then $r_9$, $r_{13}$ are no longer purely imaginary, $\gamma(r_9)$ can no longer interfere with the integration path, but $\gamma(r_{13})$ enters inside the curve $\gamma(z=i\tau)$ as shown in Figure (A3-1). In this case the integration curve is no longer always equivalent to a straight line connecting the upper and the lower limits of integrations but may differ from it by $2\pi i$ times the residue at $\gamma(r_{13})$. This makes the computer programming somewhat more difficult, than for the case $\xi < \xi_1$.

Let us discuss now the branch of the radical

$$\sqrt{(1-x^2)(1-k^2x^2)}$$

We mentioned that it must be determined from the radiation condition on $\gamma_0$ (see Chapter II). From Figures 2 and 3, we see that for real time values we have:

$$\text{Re}(\gamma_0) > 0$$

$$\text{Im}(\gamma_0) = \begin{cases} 0 & \text{when } \tau = r/\alpha \quad (i.e. \tau = i) \\ > 0 & \text{for } x < 0 \quad , \tau > r/\alpha \\ < 0 & \text{for } x > 0 \quad , \tau > r/\alpha \end{cases}$$

$$\gamma_0 = \frac{z_i}{z^2 - 1} \left[ \frac{b_4(z - q_0)(z - q_8)(z - q_0)(z - q_8)}{(z - q_0)(z - q_8)} \right]^{1/2}$$

$$= \frac{z_i}{z^2 - 1} \frac{d^2}{dz^2} \sqrt{\frac{b_4(q_0 - q_8)(q_8 - q_0)}{(1 - s^2)(1 - k^2s^2)}}$$
\[ z_\beta = i \sqrt{-1} \frac{1}{\beta^2} \frac{d}{dy} \sqrt{(1-\xi^2)(1-k^2\xi^2)} \]
\[ = \frac{i}{\beta^2} (1 + \left( \frac{1}{r} \int \frac{1}{\sqrt{1-\xi^2}} \right) \frac{\xi (q_\beta - q_\delta)(q_\omega - q_\delta)}{q_\omega - q_\delta} \frac{\xi \sqrt{(1-\xi^2)(1-k^2\xi^2)}}{[\xi^2 - \xi_1^2]^2} \]

From the last equation, it was found that the branch cut that satisfies the radiation condition is the one shown in Figure (A3-2). To examine it, we let:

\[ f = (1-\xi^2)(1-k^2\xi^2) = \Re e^{i\theta}, \quad \theta = \phi_1 + \phi_2 + \phi_3 + \phi_4 \]

Assuming \[ \sqrt{(q_\omega - q_\delta)(q_\beta - q_\delta)} b_4 = \frac{i}{\beta} \]
we write:

\[ q = \sqrt{f} = -R^{1/2} e^{i\theta/2} \]

I will consider next the case when \( c = \beta \). This is quite analogous to the case when \( c = \alpha \), except when \( \xi > L_c \).

In that case, we compute \( q_\alpha, q_\beta, q_\omega \) and \( q_\delta \) by letting \( c = \beta \), \( c' = \alpha \) and \[ \sqrt{c^2 - c'^2} = -i \sqrt{\alpha^2 - \beta^2} \]
Since \( z = iy \) at the upper limit which makes \( \nu_d(iy) = 0 \), we ordered \( q_\alpha, q_\beta, q_\omega \) and \( q_\delta \) (i.e. rename when necessary) so that the integration path in the \( \zeta \)-plane is in a bounded region. In particular, at \( z = iy \), we will have \( \zeta = 0 \).

The integration curve is determined by:
Fig. A3-1 Integration path ($\zeta(\nu,t)$) and singularities in the $\xi$-plane for $c=\alpha$. 
Fig. A3-2 Branch cut in the $\xi$-plane for $c=\alpha$. 

$0 \leq \phi_1 < 2\pi$

$0 \leq \phi_2 < 2\pi$

$0 \leq \phi_3 < 2\pi$

$0 \leq \phi_4 < 2\pi$
\[ s^2 = \left[ \beta^2 \left( \frac{\alpha^2 - \beta^2}{\beta^2 \tau^2} \right) - \beta \left( \beta + \frac{\gamma}{\beta \tau} \right) \left( \frac{1}{\alpha^2 - \beta^2} + \frac{\gamma}{\beta \tau} \right) \right] \]

\[ = i \beta \sqrt{1 - \frac{\beta^2}{\beta^2 \tau^2}} \left\{ \frac{\alpha^2 - \beta^2}{\beta^2 \tau^2} \left( \frac{\gamma}{\beta \tau} \right) \right\} / R \]

where \( R = (\alpha^2 - \beta^2)^{3/2} \frac{\hbar^2}{\beta^2 \tau^2} \left( \frac{\alpha^2 - \beta^2}{\beta^2 \tau^2} + \frac{2 \gamma}{\beta \tau} \right) \left( \frac{1}{\alpha^2 - \beta^2} + \frac{\gamma}{\beta \tau} \right) \left( \frac{\beta^2 - r^2 \gamma}{\beta^2 \tau^2} \right) \)

The behavior of the integration curve and the locations of the relevant singularities are shown in Figure A3-3, for \( \alpha = 6 \text{ km/sec}, \beta = 3.5 \text{ km/sec} \) (\( \xi < \xi_1 \)). Again as we make \( \xi > \xi_1 \), all the poles on the imaginary axis move out of the integration curve except one which moves to the right and makes the computer programming difficult just like the case \( c = \alpha \).

The branch cut for the factor \( \sqrt{(1 - \xi^2)(1 + k^2 \xi^2)} \) is given by Fig. A3-4. Where \( k \) satisfies the equation

\[ -k^2 = \frac{\beta^2 - \alpha^2}{\beta^2 \tau^2}, \quad |k| > 1 \]

\[ \xi = (1 - \xi^2)(1 + k^2 \xi^2) = Re^{i \theta}, \quad \theta = \pi + \frac{4}{j \alpha}, \phi_j. \]

where it was assumed that:

\[ \sqrt{b_4 (q_\alpha - q_\beta)(q_\beta - q_\alpha)} = i / \alpha \]
Fig. A3-3 Integration path \( \xi(t) \) and singularities in the \( \xi \)-plane for \( \alpha = \beta \).
Fig. A3-4 Branch cut in the $\zeta$-plane for $\epsilon = \beta$. 

\begin{align*}
0 < \phi_1 & \leq 2\pi \\
-\frac{3\pi}{2} < \phi_2 & \leq \frac{\pi}{2} \\
-\frac{3\pi}{2} < \phi_3 & \leq \frac{\pi}{2} \\
-\pi < \phi_4 & \leq \pi
\end{align*}
\[ g = R^{1/2} e^{i\theta/2} \sqrt{(1 + k^2 \xi^2)(1 - \xi^2)} \]

This completes the discussion of the singularities of the integrands and the method of integration.
Appendix IV

Analytic Convolution of the Infinite Medium Solution

Let \( f_0 \) denote the impulse response of a derivative of one of the potentials as given in Appendix II. Consider now the source time function \( g(t) \) as given by eq. 14 in Chapter III. Then the response to \( g(t) \) will be given by:

\[
f_g = \int_0^{t - r_0/c} f_0(t - \tau) q(\tau) d\tau
\]

where \( f_0(t) = 0 \) for \( \tau < r_0/c \); \( r_0 \) is the distance between the hypocenter and the station and \( c \) is the wave velocity which is equal to \( \alpha \) or \( \beta \) for the medium. From eq. 14, we then have the following cases:

\[
f_g = \begin{cases} 
\int_0^{t_0/2} \frac{(t - \tau)^2}{t_0} f_0(t - \tau) d\tau + \int_{t_0/2}^{t_0} q_j \gamma \int_0^{t - r_0/c} f_0(t - \tau) d\tau + \int_0^{t - r_0/c} f_0(t - \tau) d\tau & \text{(I)} \\
\int_0^{t_0/2} \frac{(t - \tau)^2}{t_0} f_0(t - \tau) d\tau + \int_{t_0/2}^{t_0} q_j \gamma \int_0^{t - r_0/c} f_0(t - \tau) d\tau & \text{(II)} \quad \text{(for } t_0 = t - r_0/c) \\
\int_0^{t - r_0/c} \frac{(t - \tau)^2}{t_0} f_0(t - \tau) d\tau + \int_{t - r_0/c}^{t_0} q_j \gamma \int_0^{t - r_0/c} f_0(t - \tau) d\tau & \text{(III, a)} \quad \text{(for } \frac{t_0}{2} > t - r_0/c) \\
\int_0^{t - r_0/c} \frac{(t - \tau)^2}{t_0} f_0(t - \tau) d\tau + \int_{t - r_0/c}^{t_0} q_j \gamma \int_0^{t - r_0/c} f_0(t - \tau) d\tau & \text{(III, b)} \quad \text{(for } \frac{t_0}{2} < t - r_0/c) 
\end{cases}
\]
We then make first the substitution:

\[ \eta = \tau \quad \text{so that} \quad d\tau = -d\eta \quad \text{and} \quad f_6(t-\tau) = f_6(t+\eta) \]

and then \( t' = t + \eta \) so that \( \eta = t' - t \), \( d\eta = dt' \)

This gives us the following results:

Case I \((t_0 < t - r_0/c)\):

\[ f_9 = \sum_{j=0}^{2} I_{1j} + \sum_{j=0}^{2} I_{2j} + I_3 \]

\[ I_{10} = \frac{r^2}{c^2} \int_{t - t_0/2}^{t} f_6(t') dt' \]

\[ I_{11} = -\frac{2t}{c^2} \int_{t - t_0/2}^{t} \frac{d}{dt'} f_6(t') dt' \]

\[ I_{12} = \frac{1}{t^2} \int_{t - t_0/2}^{t} \frac{d}{dt'} f_6(t') dt' \]

\[ I_{20} = a_j t^j \int_{t - t_0}^{t - t_0/2} f_6(t') dt' \quad 0 \leq j \leq 3 \quad \text{(See eq. 14 for } a_j) \]

\[ I_{21} = (-3t^2a_3 - 2a_2t - a_1) \int_{t - t_0}^{t - t_0/2} f_6(t') dt' \]

\[ I_{22} = (3ta_3 + a_2) \int_{t - t_0}^{t - t_0/2} t^{12} f_6(t') dt' \]

\[ I_3 = \int_{t_0/c}^{t} f_6(t') dt' \]
Case II \((t_b = t - t_o/c)\): This is a limiting case of the previous case, and hence can be treated in a similar way. Since for a given \(t_o, r_o/c\) this case has a zero probability of occurrence for random sampling of \(f_g(t)\), it is not of practical significance and there is no need to consider it further.

\[
\text{Case (II,a): } f_g = \frac{1}{t_o^2} \int_{t_o/c}^{t} (t'^2 - 2tt' + t^2) f_s(t') dt'
\]

\[
\text{Case (II,b): } f_g = \frac{1}{t_o^2} \int_{t - t_o/2}^{t} (t'^2 - 2tt' + t^2) f_s(t') dt' + \int_{t_o/c}^{t - t_o/2} a_j(t-t') f_s(t') dt'
\]

Thus we have only to evaluate integrals of the form:

\[
\int_{t'}^n f_s(t') dt'
\]

which is discussed in Chap. II and Appendix III.