MATHEMATICAL THEORY OF LUBRICATION-TYPE FLOW

by

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1940

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY 1947

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PREFACE:

The purpose of the present work is to extend the boundaries of the mathematical theory of lubrication by giving consideration to second order corrections, which are normally neglected. The end product of this analysis is a set of partial differential equations in two dimensions, the solutions to which represent corrections to be applied to results obtainable from the conventional theory.

The author wishes to express his appreciation to Professor E. Reissner for his many suggestions and references to literature pertaining to this analysis. In addition he wishes to thank the members of his staff of the Computing Section of the Instrumentation Laboratory for their assistance in performing the numerical computations incident to this work, and in particular to thank Mr. Frank Sander for his aid in checking the algebraic correctness of many of the formulae derived. Finally, the author extends his appreciation to Mr. L. E. Payne and his associates of the Technical Publications Section of Jackson and Moreland for their help in typing and reproducing this thesis.
Summary:

The mathematical theory of lubrication-type flow, interpreted in a general sense as the theory of the motion of a viscous fluid between two arbitrary neighboring surfaces, has previously been treated, by many investigators, on the basis of a number of simplifying assumptions. These assumptions consider as negligible:

(a) Body forces and acceleration stresses acting on the fluid.
(b) Compressibility.
(c) Variations in viscosity of the fluid.
(d) Geometrical effects associated with curvature of the surfaces.
(e) Thermal effects.

Although variable viscosity, geometrical effects, and thermal effects have been treated in isolated instances, no previous work has been found by the author in which these various phenomena have been unified into a single theory.

The starting point of the analysis of Part A of this work is a set of differential equations representing the motion and thermal behavior of an isotropic, linear, viscous fluid. The density, coefficient of viscosity, specific heat, and coefficient of thermal conductivity of the fluid are allowed to be variable functions of position and time in these equations. The fluid is assumed to be bounded by two neighboring surfaces which are in an arbitrary state of motion and which are such that their separation is small in comparison with their other dimensions. The problem is to determine the fluid motion and temperature distribution between the surfaces on the basis of the general
differential equations and given boundary conditions.

In order to give a general treatment of all the geometrical problems involved, tensor methods are employed in this analysis. A permanently fixed reference surface which approximates the two boundary surfaces is one of the coordinate surfaces of a system of curvilinear coordinates. By means of relations between tensors in three dimensional space and tensors in the two dimensional reference surface, it is found possible to express the general equations of motion and heat flow in a dimensionless form which involves only two dimensional tensor quantities. In this procedure eight auxiliary dimensionless parameters are introduced. In a relatively general class of problems each of these parameters is small in comparison with unity. Pressure, velocity, temperature and other related quantities are expanded in multiple power series of these dimensionless parameters and the general differential equations are transformed into separate sets which are studied individually. The integration of these equations in the direction perpendicular to the reference surface results in the reduction of each of these sets to a single two-dimensional partial differential equation involving a dimensionless pressure as the dependent variable and the curvilinear coordinates in the reference surface as independent variables. The first of these equations is the conventional lubrication equation:

\[
\text{div} \left( h^3 \text{grad} \, p \right) = f
\]

in which \( h \) is the separation between the surfaces, \( p \) is the basic approximation to the pressure, and \( f \) is a quantity involving the velocities of the boundary surfaces. The divergence
and gradient are two-dimensional operators associated with the reference surface. Subsequent equations have a form identical with that above, but with \( p \) in the role of a correction term to be applied to the basic approximation and with \( f \) as a more general function derived from preceding terms in the series. These equations are derived for first order correction terms representing geometrical effects, body forces, variations in viscosity and density, acceleration stresses, and heat conduction, and for second order terms representing changes in viscosity and density arising from heat generated by the motion. Because of the algebraic complexity of the analysis, the calculation of further corrections does not appear feasible in a general treatment.

In Part B the general theory is illustrated by applications to lubrication analysis. After referring the tensor equations of Part A to plane and cylindrical coordinate systems, an analysis is given of thermal effects and geometrical corrections in flooded journal bearings. It is found that the heat generated by the motion tends to increase the eccentricity of the shaft, for a fixed shaft load, whereas the friction coefficient is decreased. Geometrical corrections to the lift force and frictional moment on the shaft are found to be negligibly small.

Non-linear thermal problems associated with coupling between viscosity, velocity, and temperature are also studied.
Comparative data, determined by the non-linear theory and by the more approximate series-expansion theory of Part A, are presented for typical cases of parallel plane flow to illustrate the degree of accuracy attainable by the latter theory.
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INTRODUCTION

The conventional mathematical theory of the type of fluid flow encountered in lubrication problems is based, with a large number of simplifying assumptions, upon an analysis of the Navier-Stokes equations for the motion of a viscous fluid. The original form of the theory, as developed by Reynolds, and extended by Sommerfeld, relates to the steady-state flow of a fluid between two smooth and nearly parallel surfaces in a state of relative motion. The bearing surfaces are assumed to be of infinite width (that is, to extend without limit in the direction normal to that of their relative velocity), and to be of such a nature that the state of flow is independent of position along the normal. As is customary in viscous flow problems, the fluid is regarded as adhering to the bearing surfaces. The angle of inclination between the surfaces is assumed to be very small, so that the flow is essentially unidirectional in character. Since the fluid film is generally very thin, the pressure is taken to be constant over the film thickness and, for journal bearings, the effect of curvature of the surfaces is neglected. The fluid is assumed to be incompressible, to possess a constant coefficient of viscosity, and to be weightless; for this last reason, the effect of body forces and stresses arising from acceleration of the fluid particles can be neglected.

1 References corresponding to the numbered superscripts are found in the Bibliography
In consideration of the fundamental importance of lubrication in modern industrial applications and the fact that Reynolds' original paper was published over sixty years ago, it is surprising to note that only within the last decade or two has any concerted effort been made to remove the many restrictions from Reynolds' theory. The success of this theory in practical lubrication design has clearly substantiated the essential validity of the concepts which it involves; however, the necessity of extending its scope has become increasingly apparent with the accumulation of a large mass of experimental data. Recent theoretical developments have included considerable work relating to the two-dimensional partial differential equation associated with bearings of finite width, as contrasted with the ordinary (one-dimensional) differential equation of Reynolds' theory. In addition, partial investigations have been made of the effect of viscosity changes introduced by local variations in the temperature and pressure of the fluid. Among those investigators who have made prominent contributions to the theory of bearings of finite width may be mentioned Michell, who first considered this problem for a slider bearing, and more recently Muskat and his collaborators, who have obtained considerable theoretical and experimental data for journal and slider bearings. The effects of pressure and temperature have been studied theoretically, among others, by Needs, Muskat and Evinger, and Christopherson. These effects, which have been examined only briefly by these investigators, are of importance in estimating
the temperature rise due to the operation of bearings, and may
prove to be of interest in connection with the property of
"oiliness" and other peculiarities of thin-film lubrication.
For a more detailed list of references on general lubrication
problems, see Norton.\textsuperscript{16}

This investigation was initiated in connection with a
lubrication problem of a classified nature, in which interest
centered about the accurate theoretical prediction of various
small torques acting on a lubricated shaft. These included
torques arising from slight departures from uniformity or sym-
metry in the shaft and supporting bearing, such as those that
might arise from a non-uniform temperature distribution within
the bearing or that might be produced by the asymmetric fluid
flow pattern set up around a shaft of a slightly elliptical
shape. In many of the cases, a rather delicate balance between
relatively large opposing forces seemed to be involved; hence,
the degree of accuracy in the basic theory used in prediction
was considered to be of fundamental importance. In addition,
estimates of behavior under extremely wide ranges of operating
conditions were necessary, requiring a theory as free from
basic limitations as possible. The conventional lubrication
theory certainly has adequate accuracy for many investigations
in the lubrication field, and indeed has been found directly
applicable to some of the problems arising in the present in-
vestigation. However, it seemed desirable to conduct a sys-
tematic treatment of the general lubrication-type fluid flow in
such a form that, in principle, the theory could be extended to an arbitrarily high degree of accuracy; not only to increase the accuracy obtainable, but also to establish the essential limitations of the simplified theory. The material presented herein consists of those mathematical tools which have been developed in an effort to meet these requirements.

Many sciences, including that of lubrication, for convenience may be divided into four somewhat overlapping classes of investigation. The first of these is the development of the physical concepts and a study of the basic physical phenomena governing the process. For example, in this region of study we may include such work as that relating to the physical and chemical character of boundary lubrication. Once the fundamental concepts have been formulated, the second step is the reduction of these to mathematical terms - usually by the derivation of differential equations governing the behavior. The third class of investigations is that which seeks solution to these differential equations in various practical problems, providing theoretical data for use in applications. Much of the work of the applied mathematics falls into this third classification. In the fourth and final step, the results of theory are subjected to experimental test. These steps seldom proceed in the order described; more often than not a large proportion of the experimental data is accumulated before the basic physical concepts are clearly defined.
The present analysis falls essentially into the second category; that is, the reduction of the fundamental concepts to mathematical terms. The basic physical properties of viscous fluids are assumed from the beginning to be satisfactorily represented by a general form of the Navier-Stokes differential equations, together with a heat flow equation. However, these equations are three-dimensional in character, and in only a very few instances have they been found integrable in an exact form. Generally, approximations must be introduced which reduce the mathematical calculations to the solution of problems in one or two dimensions before any real progress can be achieved. The central problem of the present development is therefore the derivation of a set of two-dimensional partial differential equations in terms of which the three-dimensional fluid flow can be satisfactorily described.

In the conventional theory of lubrication, the relative thinness of the lubricant film, in comparison with the other dimensions of the system, provides the basic key to the approximation. Assume, for simplicity, that the fluid is flowing between two surfaces which are stationary and lie very close together, and that the fluid velocity and viscosity are such that the flow is predominantly viscous in character. Then, by isolating a portion of the system for which the dimensions are small in comparison with the overall system dimensions but which, nevertheless, is large in comparison with the linear separation of the surfaces, the local flow may be approximated by a unidirectional flow between two parallel planes. This situation is
depicted in Figure 1. The velocity distribution will be approximately parabolic, as indicated, and the pressure \( p \) will be substantially constant across the fluid layer between the surfaces.

![Figure 1](image)

**FIGURE 1**

Let the vector \( \bar{v} \) denote the fluid velocity averaged over the distance between the surfaces. Then, if \( h \) is the distance of separation and \( \mu \) is the coefficient of viscosity of the fluid, the well-known formulae for parallel plane flow give

\[
\bar{v} = -\frac{h^2}{12\mu} \text{grad } p
\]

The quantity \( h \) will, in general, be a variable function of position on the bearing surfaces. If the fluid is assumed to be incompressible, the equation of continuity for the flow may be written

\[
\text{div}(h\bar{v}) = 0
\]

since \( h\bar{v} \) is a vector giving the total volume rate of flow over a cross-sectional area of height \( h \) and unit length. Substituting the previous expression for \( \bar{v} \), and assuming \( \mu \) to be a constant,

\[
\text{div}(h^3 \text{grad } p) = 0
\]

(1)

* See, for example, Lamb, (Ref. 19), pp 561-584.
Equation (1) provides the basic partial differential equation for determining the pressure, assuming \( h \) to be a known function. The divergence and gradient operators employed are to be interpreted as the two-dimensional operators associated with some mean surface lying between the two given surfaces. Thus, if the surfaces are approximately plane, we may write

\[
h = h(x, y) \quad p = p(x, y)
\]

and

\[
\frac{\partial}{\partial x} \left[ h^3 \frac{\partial p}{\partial x} \right] + \frac{\partial}{\partial y} \left[ h^3 \frac{\partial p}{\partial y} \right] = 0
\]

where \( x \) and \( y \) represent rectangular coordinates in a reference plane between the two surfaces. In the event that the surfaces are in a state of relative motion, a simple shear type of flow may be superimposed on that above - giving in place of (1) a corresponding equation with a non-zero right-hand member, determined from the surface velocities.

Part A of this thesis also employs the relative thinness of the lubricant film as the basis for approximation, with the important difference that techniques are included for taking a finite film thickness into account. A generalized form of the equations of motion and heat flow for a viscous fluid is selected as the starting point, in which the physical "constants" of the fluid, such as the coefficient of viscosity, density, specific heat, etc., are allowed to be variable functions. All terms in the equations, including those relating to body forces and inertia effects, are retained in the analysis. By an appropriate
selection of dimensionless variables, these equations are written so that their various terms may be readily classified, according to order of magnitude, in terms of a set of auxiliary dimensionless parameters. In most applications many, if not all, of these dimensionless parameters are small in comparison with unity. A series expansion of pressure, velocity, etc. in terms of powers of these parameters is therefore assumed, leading to a sequence of separate sets of three-dimensional partial differential equations. By the use of the boundary conditions on the two surfaces, these sets may then be reduced individually to the form of two-dimensional equations that are analogous to equation (1).

In this analysis it is found possible to take account of the following:

(a) Geometrical properties, such as deviations of $h$ from a constant value or curvature of the boundary surfaces.

(b) General body forces acting on the fluid.

(c) Stresses related to acceleration of the fluid.

(d) Arbitrary motion of the boundary surfaces.

(e) Changes in the fluid temperature resulting from the dissipation of mechanical energy by the action of viscosity or by heat imposed from an external source.

(f) Variations in the density, coefficient of viscosity, specific heat, and thermal conductivity of the fluid with temperature and pressure.

The theory it is proposed to develop will still be an approximate one. The final results provide, primarily, a basic partial differential equation which, for stationary surfaces, is identical with equation (1), followed by a sequence of similar
non-homogeneous partial differential equations. These yield first-order corrections for each of the various types of perturbing effects to be considered. Because of the complexity of the work required, it is found relatively impracticable to investigate higher order corrections, although in principle this could be done.

It is therefore necessary to impose certain assumptions limiting the extent of various departures from uniformity. Specifically, it is immediately pointed out that the work is essentially restricted to liquids in its applications; the orders of magnitude usually involved with gaseous media as the lubricant are such as to require an entirely different type of mathematical treatment. However, it is shown in Section 6 that, within the limitations to be imposed, it is still possible to treat a wide class of problems in a relatively exact manner. Furthermore, the analysis clearly distinguishes between various orders of approximation, so that a reasonable idea can be formed as to the nature and rough magnitude of the residual errors. A detailed discussion of these matters is reserved for Section 6, when specific equations and parameters can be examined.

In order to provide a degree of generality, and in order to avoid the duplication of a considerable mass of algebraic detail in applying the results to different types of coordinate systems, tensor notation and concepts are employed throughout Part A. The results of this work may then be applied directly to problems involving plane, cylindrical,
spherical, and conical bearings without the necessity for a separate detailed derivation in each instance. No more knowledge of tensor methods is required than may be found in the first four chapters of Weatherburn\textsuperscript{20}. A very condensed derivation of the principal equations for plane bearings is given in Part B, Section 7, in rectangular Cartesian coordinates.

The main results of the theory presented herein are based upon the assumption that all of a certain set of dimensionless parameters are small. Within fairly broad limits, which are discussed in detail in Section 6, this is the case in practice. The theory is not, however, equipped in its present form to treat the non-linear type of problem encountered when significant temperature changes take place within a bearing. The resistance of the lubricant film itself to the flow of heat causes the presence of a higher relative temperature at the center of the film than that found at the boundary surfaces, since heat is continually being generated within the film. This higher temperature in turn reduces the viscosity at the center, causing a departure from the simple parabolic velocity distribution of the fluid, and therefore causing an unevenness in the rate at which heat is generated, as well. The net result is that the temperature and velocity distributions across the film are related by a pair of non-linear differential equations, the integration of which is necessary in order to set up the basic two-dimensional partial differential equations describing the flow.

In Section 1, the physical foundations of the theory of viscous flow are discussed, with particular reference to those fluid properties of interest in lubrication. In Section 2 the
first step in the general mathematical plan of attack is carried out, which involves the derivation in tensor form of a suitably general form of the equations of motion, continuity, and heat flow for an ideal viscous fluid. In Section 3 these equations are made dimensionless and are referred to a coordinate system involving a fixed reference surface approximating the boundary surfaces, and the associated family of parallel surfaces. The introduction of a specific distinction between directions parallel and perpendicular to the reference surface makes it possible to express the covariant derivatives occurring in the general tensor equations of Section 2 in terms of combinations of derivatives normal to the reference surface, and in various surface tensors and operators. The reduction of the three-dimensional differential equations to a set of related two-dimensional equations on the reference surface is thereby simplified.

The boundary conditions to be applied to the differential equations, both on the boundary surfaces and on the edge of the lubricated region, are discussed in Section 4. By assuming multiple series expansions of the velocity vector, pressure, and other parameters in powers of the several small dimensionless parameters, and by employing the boundary conditions on the two neighboring surfaces, it is possible to derive in a formal manner the approximating two-dimensional partial differential equations which are the objective of this portion of the analysis. This derivation is the subject matter of Section 5. The formal results of Section 5 are in Section 6 subjected
to an examination of their practical convergence and general range of validity. In this section, attention is directed primarily toward the validity of the mathematical approximations used, since the fundamental physical foundations of the analysis are examined in Section 1.

Part B is concerned with translating the results of Part A into a form suitable for analysis of lubrication problems. In Section 7, a direct derivation of the principal equations for plane bearings is presented in a condensed form, without the use of tensors. The tensor results of Part A are translated into a form suitable for the analysis of journal bearing problems in Section 8, by referring the general equations to a cylindrical coordinate system. An illustration of the application of the general equations to practical lubrication analysis is presented in Section 9, in which corrections to the theory of flooded journal bearings which arise from geometrical and thermal effects are examined.

In Section 10, comparisons are made between results predicted by the non-linear theory of large thermal effects and results obtained from the less exact linear theory of Part A, for a simple problem of plane parallel flow in which the former theory can be used numerically. A reasonable estimate of the range of validity of the linear theory is thereby obtained. Section 11 contains a brief analytical treatment of the problem of large thermal effects in slider bearings of infinite width. Techniques for numerical application of the non-linear theory to this specialized problem are presented.
PART A. THEORY OF LUBRICATION-TYPE FLOW, IN TENSOR FORM

Section 1. Basic Assumptions:

The final evaluation of the ability of this theory of lubrication-type flow to predict experimental data must, of course, come from experiment itself. However, we may examine with profit the basic physical assumptions chosen as the starting point for the analysis and the mathematical conclusions drawn from these assumptions, in order to form a reasonable estimate of the class of problems to which the theory may be expected to apply. Although it is rather difficult to draw a unique dividing line between the two, an attempt will be made in this section, and in Section 6, to distinguish between assumptions that are primarily of a physical nature, in the sense of providing a starting point for mathematical reasoning, and those that are necessary only to justify the mathematical approximations used in the course of the development. For example, it will be found during the investigation that certain of the series-expansion procedures employed cannot be regarded as valid in cases involving relatively large temperature changes in a system. However, a significant part of the analysis is unaffected by this difficulty and, therefore, is as valid as the original physical assumptions. By a careful distinction between the types of assumptions introduced, a satisfactory starting point will be provided for future analysis in certain cases where our present hypotheses fail. This section is therefore concerned with the physical foundations of the theory to be developed; discussion
of the mathematical approximations introduced in the course of
the analysis is properly withheld until Section 6, where the
various parameters and equations are considered in detail.

The first assumption to be introduced is that the lubricant
is an ideal, linear, isotropic, viscous fluid. To formulate this
hypothesis in precise terms, let \( \sigma^{ij} \) denote the stress tensor,
referred for convenience to an arbitrary rectangular Cartesian
coordinate system \( x^h = (x^1, x^2, x^3) \). We shall define the pressure
p as the negative of the mean value of the principal stresses;

\[
\rho = - \frac{1}{3} \left( \sigma^{11} + \sigma^{22} + \sigma^{33} \right) 
\]

(1.1)

We now postulate that the value of \( p \) at any point is definable
entirely in terms of the density and temperature of the fluid at
that point; that is to say, for given values of density and
temperature, the pressure is independent of the instantaneous
manner in which the fluid is being deformed, or the past history
of the deformation.

We specify, in addition, that the components of the stress
tensor shall consist of homogeneous linear combinations of the
quantity \( p \) and the various components of the rate-of-strain
tensor. The coefficients involved in these linear combinations
can be functions of position in the fluid; however, the require-
ment that the fluid be isotropic imposes the rather strong
restriction that, at any fixed point, the coefficients shall
not depend upon the particular orientation chosen for the
coordinate axes.
The ideal fluid defined by these properties will possess a dissipation function; that is, a function expressing a rate at which mechanical energy is converted into thermal energy per unit volume of the fluid. We shall extend the properties of the ideal fluid to include its thermal behavior by postulating first of all the conservation of energy. In addition, we shall require that the fluid be a linear and isotropic conductor of heat. This implies that the vector defining the rate of heat conduction per unit cross-sectional area shall be proportional and parallel to the negative of the vector representing the temperature gradient. The assumption of isotropy carries the implication that the coefficient of proportionality in this relation shall be independent of direction at any fixed point, although it may be a scalar function of position. Finally, we shall assume that the relations between the temperature, the pressure, and the internal energy of the fluid (total energy minus kinetic energy) shall be governed by the usual laws of thermodynamics as applied to the fluid in an equilibrium condition, and shall not depend upon the state of fluid motion.

The fluid possessing these properties is by no means the most general that could be considered. In the first place, the pressure $p$ could itself depend on the rate-of-strain tensor. Also, the dependence of stress on strain rate might very well include non-linear features. A very general fluid possessing properties of this type, although still falling within the definition of a viscous fluid, has been discussed.
by Reiner. Finally, a physical fluid might be anisotropic in behavior. Since the fluid flow in a bearing is largely composed of a shear in a single direction, it would not be surprising to find molecular orientation effects occurring which tended to produce anisotropy.

With respect to hydrocarbon oils of the type normally employed as lubricants, it is fairly safe to state that these various complicating effects, if they exist, are sufficiently small to be neglected for most purposes. Although no attempt has been made to do so, properties of these types occurring in a moderate degree could readily be incorporated in the structure of the present analysis by use of perturbation procedures of the type which we have employed in treating variable viscosity, density, etc. The behavior of such materials as greases, however, appears to follow the more complicated laws of plastic solids (see, for example, Norton, pp. 14-15), and does not fall within the scope of this investigation.

It has been stated that the pressure is prescribed to be a function of the fluid temperature and density. In practice we shall prefer to adopt pressure and temperature as the basic thermodynamic quantities, regarding the density as a function of these variables instead. It will be assumed that the other physical properties of the fluid are similarly defined; namely, that the coefficient of viscosity, the thermal conductivity, the internal energy, and the specific heat are functions of pressure and temperature.
The second major assumption to be introduced is that the fluid flow is essentially stable in character; that is to say, non-turbulent. Although the laws of viscous motion may be assumed to apply to turbulent flow on a local scale, the type of mathematical analysis required for investigation of turbulent motion calls for an entirely different approach from that used in this work. The application of our results is essentially limited to liquids by this factor, since the orders of magnitude involved in the use of gaseous lubricants are generally such as to place the flow definitely in the turbulent region.

Because the fluid flow with which we are concerned occurs between surfaces that are generally in very close proximity to each other, the physical behavior of fluids in intimate contact with solids is of considerable interest. It is assumed, in the first place, that a viscous fluid adheres to a surface with which it is in contact. This property is expressed mathematically by setting the (vector) velocity of the fluid at each point of a surface equal to the velocity of the surface itself at that point. The exact physical status of this hypothesis, although it is widely used in practice, is perhaps still open to slight question. It has been pointed out (see, for example, Bateman\textsuperscript{22} et al., pp 89-111) that the assumption that a fluid may slip at a surface, with a definite coefficient of sliding friction, still permits the explanation of many important cases of viscous flow. Goldstein\textsuperscript{23} (pp 676-680) presents a summary of the empirical data supporting the hypothesis that a fluid adheres to a surface, pointing out
that if this assumption were false a new parameter involving a scale factor could be expected to enter into the dimensional analyses involving fluid motion about solid objects. From the fact that no such parameter has been found necessary, he concludes that slipping of a viscous fluid past a solid surface probably does not occur. It is not clear that this argument can be extended to problems in the lubrication field, however, since the importance of even a very small effect of this type could conceivably be greatly magnified by the relative thinness of the lubricant film, whereas it might easily pass unnoticed in problems involving larger dimensions.

In our further considerations we neglect any effects arising from surface tension. In addition, it is assumed that proximity to a surface per se exerts no influence on the physical properties of a fluid. With respect to these two assumptions and to that regarding possible slipping at a surface, the situation can be summarized in the following way. If these effects are of any significance, it can be expected that they should show up in the lubrication problem to a far greater extent than in most others, since the proximity of the surfaces implies that a greater relative portion of the fluid will be affected. In the case of boundary lubrication, in which the separation between surfaces may be only a few hundred-thousandths of an inch, phenomena do occur which have not as yet been satisfactorily explained on the basis of a purely hydrodynamical lubrication theory; however, the mathematical aspects of lubrication theory have not advanced
sufficiently to date to enable any positive conclusion to be
drawn from this fact. From the fact that the phenomena of the
so-called thick film lubrication have been at least partially
explained on the basis of a hydrodynamical theory, a very
tentative conclusion may be suggested that in thick film
lubrication these various complicating factors are at most
of minor importance. It is to be hoped that the results of
the present investigation, by providing techniques for more
accurate mathematical analysis, may be of some assistance in
further studies of these surface effects.

As a practical expedient it is assumed that the position
and velocity of the boundary surfaces are exactly known at all
times. Such effects as elastic deformation arising from the
action of the fluid stresses are to be neglected. By this
means, our attention is concentrated solely upon the hydro-
dynamical portion of the problem in this development. If it
is necessary in practice to take elastic effects into account,
it is probable that an iterative procedure involving separate
and alternate consideration of the fluid flow problem and the
elastic deformation problem would be found rapidly convergent.

In problems typified by that of the vibrations of a high-speed
rotating shaft, the shaft position and velocity may be specified
in terms of a minimum number of arbitrary functions of time,
and the fluid forces acting on the shaft then calculated in
terms of these unknown functions and their derivatives. Such
results may then be used in writing the necessary differential
equations for the vibration.
With a similar aim it is assumed in the analysis that the temperatures of the boundary surfaces are known functions. In many problems of heat transmission between solids and fluids the effect of the thermal resistance of a relatively thin film or surface coating of impurity must be taken into account. In boilers, for example, the thermal resistance of a deposit of scale may be a significant portion of the total resistance to heat flow through the boiler wall. This phenomenon is normally treated analytically by assuming the existence of a surface heat transfer coefficient, defined as the heat transferred across a unit area per unit time, corresponding to a unit temperature differential between the portion of the fluid immediately in contact with the surface and the temperature of the wall immediately below the surface. Although it is doubtful that this type of effect is directly significant in the lubrication process, the existence of such surface heat transfer coefficients is, nevertheless, assumed. The purpose of this assumption is only secondarily to secure an added degree of generality in our results; the primary aim is to provide at least an approximate technique for taking into account the overall thermal resistance of the bearing. In a mathematically exact analysis of the problem of the generation and transmission of heat in a bearing it is, of course, necessary to take into account the exact temperature distribution set up in the bearing material, as well as that set up in the fluid. The use of heat transfer coefficients, however, permits approximate consideration of
the effect of the bearing without the necessity for an 
elaborate analysis of this heat conduction problem. In 
assuming that the temperatures of the boundary surfaces 
are known, we shall henceforth have reference to the 
temperature of the solid boundary at points immediately 
beneath the surface. In practical analysis, however, this 
may be interpreted to be the temperature of the atmosphere 
surrounding the bearing.

Conventional practice in the analysis of stresses in 
thin elastic shells considers the forces acting on the edges 
only in an integrated sense; that is to say, the exact 
distribution of stress across the thickness of the shell, 
at its edge, is disregarded and only the net force and 
moment is taken into account. The effect of this assumption 
is to lead to a single mathematically possible state of stress 
within the shell, which yields the required force and moment 
on the boundary. Other such possible states of stress exist, 
with the same force and moment; since the difference between 
two such states corresponds to a zero net force and moment, 
however, it is assumed, because of the thinness of the shell, 
that this difference is of significant magnitude only at 
points immediately adjacent to the edge. By a similar 
reasoning process we are led to neglect analogous edge 
effects in the lubrication problem. We assume in this 
analysis only that the value of the fluid pressure or 
velocity averaged between the two surfaces is specified 
at points about the edge of a region under consideration,
and that the exact pressure or velocity distribution is unimportant. In this manner we obtain one possible state of fluid flow, and suppose that the true state of flow differs from that calculated at most in a small region about the edge.

Finally, in order to perform all necessary mathematical operations without attention to rigor, it is specifically assumed that all functions with which we shall deal possess as many continuous derivatives as are necessary in the analysis, and that such series expansions as we may employ are absolutely and uniformly convergent within the range in which they are used.
Section 2. Tensor Equations of Motion, Continuity, and Heat Flow for an Ideal Viscous Fluid:

In this section we derive the fundamental partial differential equations of fluid motion and heat flow, taking account of possible variations in all mechanical and thermal properties of the fluid such as density, viscosity, and thermal conductivity. A brief vector derivation of such equations has been given recently by Cope\(^24\); we shall prefer, however, to re-derive his results, using tensor notation and concepts throughout*. Previous tensor treatments of the equations of motion and continuity by Levi-Civita\(^25\) and Syage\(^26\) lack the desired degree of generality; however, the underlying ideas of these papers have been of considerable aid in our development. These earlier results are included as special cases in those which follow. Except in notation, our final equations agree with those derived by Cope.

Two fundamental laws** of tensor analysis serve conveniently to bridge the gap between physical equations in a particular Euclidean coordinate system and the general tensor statement of these equations. The first of these laws is

(a) If a set of quantities
\[
\dot{A}^{i\ldots j}_{k\ldots l} \quad (i, \ldots j, k, \ldots l = 1, 2, 3)
\]

is defined with respect to a particular system of coordinates \(X^1\), we may consider these quantities as components of a tensor,

*The reader unfamiliar with tensor methods is referred to Weatherburn\(^20\). Only material to be found in the first four chapters of this book is employed in the present analysis.

**Weatherburn (loc. cit.), p. 20 and pp. 26-27.
defining the corresponding components in any other coordinates \( \tilde{X}^i \) by the fundamental law of transformation:

\[
\tilde{A}^i_{\ldots l} = A^i_{\ldots l} \left( \frac{\partial \tilde{X}^i}{\partial X^r} \frac{\partial X^r}{\partial x^k} \frac{\partial x^k}{\partial x^l} \right)
\]

(2.1)

The second of these laws is

(b) If the components of a tensor all vanish, in one particular coordinate system, then they also vanish in all other coordinate systems. This statement follows at once from the basic transformation law (2.1).

The joint implication of these two statements is that it is possible to write a physical law in a particular set of Euclidean coordinates \( X^i \), in such manner that it satisfies all requirements of tensor notation, we may be assured that the formal equation thus arrived at is actually a true statement of the law in tensor form. The only precaution that must be observed is to make sure that the tensor quantities thus defined coincide in meaning with the interpretations usually assigned to the symbols involved. The basic tensor character of such quantities as stress, rate of strain, force, velocity, etc., may be regarded as well known, however, so that we shall proceed without further attention to this point.

*Here, as in the remainder of our development, we employ the summation convention according to which the repetition of an index as a subscript and superscript within a single term indicates that a summation is to be performed on that index over the values 1, 2, 3. See, for example, Weatherburn (loc. cit.), p. 2.

**That is, coordinates in which the element of distance \( ds \) is given by

\[
(ds)^2 = (dx')^2 + (dx^2)^2 + (dx^3)^2
\]
By way of illustration, let us consider the equations of motion for an arbitrary continuous medium. We first employ Euclidean coordinates \((x_1^i, x_2^i, x_3^i)\), letting the stress components be denoted by \((\sigma_1^{11}, \sigma_1^{12}, \ldots, \sigma_3^{32}, \sigma_3^{33})\), the vector body force per unit mass by \((f_1^i, f_2^i, f_3^i)\), the acceleration vector of a particle of the medium by \((a_1^i, a_2^i, a_3^i)\), and the density by \(\rho\). Then the equations of motion may be written

\[
\frac{\partial \sigma_1^{1i}}{\partial x_1^i} + \frac{\partial \sigma_2^{11}}{\partial x_2^i} + \frac{\partial \sigma_3^{13}}{\partial x_3^i} + \rho f_1^i = \rho a_1^i
\]

\[
\frac{\partial \sigma_1^{2i}}{\partial x_1^i} + \frac{\partial \sigma_2^{22}}{\partial x_2^i} + \frac{\partial \sigma_3^{23}}{\partial x_3^i} + \rho f_2^i = \rho a_2^i
\]

\[
\frac{\partial \sigma_1^{3i}}{\partial x_1^i} + \frac{\partial \sigma_2^{32}}{\partial x_2^i} + \frac{\partial \sigma_3^{33}}{\partial x_3^i} + \rho f_3^i = \rho a_3^i
\]

or by employing the summation convention in the condensed form

\[
\frac{\partial \sigma_1^{ki}}{\partial x_k^i} + \rho f_1^i = \rho a_1^i \quad (i = 1, 2, 3)
\]

In seeking to express the corresponding equation in a general coordinate system, we first observe that the quantities \(f_i^i\) and \(a_i^i\) may be regarded as contravariant vectors. In another coordinate system \(\bar{x}_i^i\) we may then define

\[
\bar{f}_i^i = f_i^j \frac{\partial \bar{x}_j^i}{\partial x_1^i} \quad \bar{a}_i^i = a_i^j \frac{\partial \bar{x}_j^i}{\partial x_1^i}
\]

*See, for example, Goldstein\textsuperscript{23}, equation (19), p. 96.
A tensor $\overline{\sigma}^{ij}$ which reduces to $\sigma^{ij}$ in the $X^1$ coordinates may also be defined by the relation

$$\overline{\sigma}^{ij} = \sigma^{lm} \frac{\partial X^i}{\partial x^l} \frac{\partial X^j}{\partial x^m}$$

Note that if the $\overline{X}^1$ coordinates are another set of Euclidean coordinates with a different orientation, the partial derivatives $\frac{\partial \overline{X}^1}{\partial X^1}$ are simply direction cosines expressing the relative orientation of the axes, and the above formulae are the usual ones for transformation of axes. Therefore, with respect to Euclidean coordinates, $\overline{X}^1$, $\overline{a}^i$, and $\overline{\sigma}^{ij}$ have their usual meanings, and we shall regard the above equations of transformation simply as extending the definition of these quantities to more general coordinate systems.

A tensor quantity reducing to $\frac{\partial \sigma^{ij}}{\partial X^1}$ in the $X^1$ coordinates is the covariant derivative $\overline{\sigma}^{ij}_{,j}$. We therefore write tentatively as the required tensor equation

$$\overline{\sigma}^{ij}_{,j} + \rho \overline{f}^i = \rho \overline{a}^i$$

To prove that this equation is actually the correct one, let

$$\overline{A}^i = \overline{\sigma}^{ij}_{,j} + \rho \overline{f}^i - \rho \overline{a}^i$$

Each of the quantities on the right is a contravariant vector; hence, so is $\overline{A}^i$. However, in the particular Euclidean system of coordinates $X^1$, we have $A^i = 0$. Hence $\overline{A}^i = 0$ in any coordinates, and our equation is proved valid. Dropping the
bars we may therefore write as the basic equation of motion

\[ \sigma^{ij}_{,j} + \rho f^i = \rho a^i \]  

(2.2)

where it is understood that (2.2) is to hold with respect to all coordinate systems. A similar sequence of arguments may be used to establish the validity of each of the fundamental equations to be derived below. The relations used as the starting point are stated without proof, however, and the reader can readily verify that they constitute a simple tensor generalization of the corresponding equations in Euclidean coordinates.

Let \( \mu_i(x^1, x^2, x^3, t) \) denote the covariant components of the vector velocity of the fluid. A suitable definition for the rate of strain tensor \( \varepsilon_{ij} \) reducing to the conventional definition* in Euclidean coordinates, is

\[ \varepsilon_{ij} = \frac{1}{2} (\mu_{i,j} + \mu_{j,i}) \]  

(2.3)

In order to express the equations of motion (2.2) in terms of the velocity components, it is necessary to obtain an explicit expression for the stress tensor \( \sigma^{ij} \) in terms of \( \varepsilon_{ij} \).

Using an argument that depends only upon

(a) the isotropy of the fluid

*Goldstein (loc. cit.), p. 91. Note that the rate of strain components used by Goldstein are exactly twice those given above.
(b) the symmetry of the stress and strain-rate tensors

c) the assumption that the components of the stress
tensor are to be linear functions of the components
of the strain-rate tensor, Jeffreys\textsuperscript{27} (pp. 83-86),
has shown that, in Euclidean coordinates,

$$\sigma^{i\hat{d}} = -\delta^{i\hat{d}} \rho - \frac{2}{3} \mu \delta^{i\hat{d}} \delta^{k\ell} \varepsilon_{k\ell} + 2\mu \delta^{i\hat{k}} \delta^{j\ell} \varepsilon_{k\ell}$$

Here \( \rho \) and \( \mu \) are scalar quantities which may, of course,
be identified with the pressure and coefficient of viscosity,
respectively, and \( \delta^{ij} \) is the so-called Kronecker delta defined
by*

$$\delta^{i\hat{j}} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In order to obtain the tensor generalization of the stress
strain-rate equation, we require a contravariant tensor reducing
to \( \delta^{ij} \) in Euclidean coordinates. For this purpose let the
fundamental Riemannian metric defining the infinitesimal distance
ds for an arbitrary coordinate system be

$$(ds)^2 = g_{ij} dx^i dx^j$$

and, in the usual notation, let the tensor \( q^{ij} \) be reciprocal
to \( g_{ij} \)

$$q^{ij} q_{jk} = \delta^i_k$$

*We shall also write the Kronecker delta in the form \( \delta^i_j \), and it can be
shown that it is a mixed tensor having the same components in any coordinate
system. See Weatherburn (loc. cit.), p. 25.
The tensor $g^{ij}$ is that which is sought, since in Euclidean coordinates

$$g^{ij} = \delta^{ij}$$

Hence we find as our general stress strain-rate relation

$$\sigma^{ij} = -g^{ij}\mu - \frac{2}{3} \mu g^{ij} g^{kl} \varepsilon_{kl} + 2 \mu g^{ik} g^{jl} \varepsilon_{kl}$$

From (2.4), since the tensor $g^{ab}$ may be treated as a constant in covariant differentiation,

$$\sigma^{ij}_{,j} = -g^{ij}\frac{\partial}{\partial x^j} + \left[ -\frac{2}{3} g^{ij} g^{kl} + 2 g^{ik} g^{jl} \right] \left[ \mu \varepsilon_{kl} + \mu \varepsilon_{kl}_{,j} \right]$$

We may now make use of (2.3) in expressing the last member of this equation in terms of the velocity components. Simplifying the resulting expression by making use of the symmetry of $g^{ab}$, by permitting the interchange of order in covariant differentiations*, and by employing the contravariant velocity components

$$\mu^a = g^{ab} \mu_b$$

we obtain

$$\sigma^{ij}_{,j} = -g^{ij}\frac{\partial}{\partial x^j} + \mu g^{ik} \mu^j_{,k} + \frac{1}{3} \mu g^{ij} \mu^k_{,k}$$

$$\sigma^{ij}_{,j} = -\frac{2}{3} g^{ij} \mu_j \mu^k_{,k} + g^{ik} \mu_j \mu^i_{,k} + g^{ik} \mu_j \mu^i_{,k}$$

(2.5)

*This is not allowable in a general Riemannian space; however, in the ordinary "flat" space it is permitted, as may be seen by temporarily referring tensors to an Euclidean coordinate system.
In Euclidean coordinates the acceleration vector $a^i$ may be expressed in terms of the velocity components by the relation*

$$a^i = \frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j}$$

The general tensor form of this equation is evidently

$$a^i = \frac{\partial u^i}{\partial t} + u^j \mu_{ji}^i$$

(2.6)

Substituting (2.5) and (2.6) into (2.2), with some rearrangement of terms, we obtain the fundamental equations of motion expressed in terms of the contravariant velocity components:

$$\mu \left\{ g^{ik} u_{ik} + \frac{1}{3} g^{ij} u_{ij} \right\} + \mu_{ij} \left\{ g^{ik} u_{ij}^k + g^{kij} - \frac{2}{3} g^{ij} u_{k} \right\} = -g^{ij} \rho_{ij} =$$

$$-\rho f^i + \rho \frac{\partial u^i}{\partial t} + \rho u^j \mu_{ij}^i$$

(2.7)

The continuity equation in Euclidean coordinates may be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i} (\rho u^i) = 0$$

In a general coordinate system we therefore have the tensor

*Goldstein (loc. cit.), p. 96.
equation

\[ \frac{\partial \rho}{\partial t} + (\rho u^i)_{,i} = 0 \]  \hspace{1cm} (2.8)

or in expanded form

\[ \frac{\partial \rho}{\partial t} + \rho u^i_{,i} + \rho u^i u^i = 0 \]  \hspace{1cm} (2.9)

We wish finally to derive the energy transfer equation for our system, taking into account thermal conductivity. If we let \( E(\rho, T) \) denote the total energy of the fluid per unit mass, at rest, then the total energy per unit mass in motion is \( E + \frac{1}{2} u^2 \), and the total energy per unit volume is \( \rho[E + \frac{1}{2} u^2] \). Here \( u^2 \) is the square of the velocity magnitude, given by

\[ u^2 = u^i u_i = g_{ij} u^i u^j \]

If we consider an arbitrary moving volume of fluid \( V \), composed at all times of the same fluid particles, the rate of increase of energy within \( V \) will then be

\[ \frac{d}{dt} \int_V \rho[E + \frac{1}{2} u^2] \, dV \]

(2.10)

The rate of increase of energy in \( V \) must equal the rate at which work is done on the fluid within \( V \) by forces acting across the surface, plus the rate of work done by body forces, plus the rate of heat conduction into \( V \). To evaluate the first of these, let \( S \) be the surface bounding \( V \), with the unit outward normal vector \( n_j \). The vector force acting across an element
of surface area \( ds \) will be \( \sigma^{ij} n_j ds \), and the required rate of work is

\[
\int_S \sigma^{ij} n_i u_j dS
\]  

(2.11)

The rate of work from body forces is evidently

\[
\int_V \rho u_i f^i dV
\]  

(2.12)

If \( k \) denotes the thermal conductivity of the fluid, the vector \( q_i \) representing the rate of heat flow per unit area is given by

\[
q_i = -k T_{,i}
\]

where \( T \) is the temperature. Then the total rate of heat conduction into \( V \) is

\[
\int_S q_i (-n^i) dS = \int_S k T_{,i} n_i dS
\]

\[
= \int_S k q_i T_{,i} n_i dS
\]

(2.13)

The desired energy balance may now be obtained in integral form by equating (2.10) to the sum of (2.11), (2.12), and (2.13):

\[
\frac{d}{dt} \int_V \rho [E + \frac{1}{2} u^2] dV = \int_S \sigma^{ij} u_i n_j dS
\]

\[
+ \int_V \rho u_i f^i dV + \int_S k q_i T_{,i} n_i dS
\]

(2.14)
In order to put (2.14) into a differential form, we must first bring the differentiation in the left-hand member under the integral sign. Let \( dm = \rho dV \) be an infinitesimal element of mass, then, if we regard the integral as a limit of a sum of the form

\[
\sum \left[ E + \frac{1}{2} u^2 \right] \Delta m
\]

we see that in carrying out the differentiation the quantities \( \Delta m \) should be treated as constant, and

\[
\frac{d}{dt} \int_V \rho \left[ E + \frac{1}{2} u^2 \right] dV = \frac{d}{dt} \int_{\text{mass in } V} \left[ E + \frac{1}{2} u^2 \right] dV \\
= \int_{\text{mass in } V} \left\{ \frac{d}{dt} \left[ E + \frac{1}{2} u^2 \right] \right\} dV \\
= \int_V \rho \frac{d}{dt} \left[ E + \frac{1}{2} u^2 \right] dV
\]

Here and elsewhere in the analysis the operator \( \frac{d}{dt} \) is to be interpreted as providing the time rate of change with respect to a given moving fluid particle. Thus, for any scalar or tensor quantity \( A \),

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + u^i A_{,i} \tag{2.15}
\]

To transform the surface integrals of (2.14) into volume integrals, we employ Green's theorem, according to which

\[
\int_s F^i n_i dS = \int_V F^i_{,i} dV
\]

33
for any vector $F^i$. Then

$$\int_S \sigma^{ij} u_i \eta_j \, ds = \int_V (\sigma^{ij} u_{i,j}) \, dV$$

and

$$\int_S \kappa \theta^{ij} T_{ij} \eta_j \, ds = \int_V (\kappa \theta^{ij} T_{ij},_j) \, dV$$

Combining these results we find from (2.14)

$$\int_V \left\{ \rho \frac{d}{dt} [E + \frac{1}{2} u^2] - (\sigma^{ij} u_{i,j})_j - \rho u_i f^i - (\kappa \theta^{ij} T_{ij},_j) \right\} \, dV = 0$$

Since the volume $V$ is entirely arbitrary, we thus must have

$$\rho \frac{d}{dt} [E + \frac{1}{2} u^2] = (\sigma^{ij} u_{i,j})_j + \rho u_i f^i + (\kappa \theta^{ij} T_{ij},_j) \quad \text{(2.16)}$$

By means of the equations of motion in the form (2.2) we may transform (2.16) into a slightly simpler form. We find

$$(\sigma^{ij} u_{i,j})_j + \rho u_i f^i = \sigma^{ij} u_{i,j} + \sigma^{ij} u_i a^i + \rho u_i f^i$$

$$= \sigma^{ij} u_{i,j} + \rho u_i a^i$$

$$= \sigma^{ij} u_{i,j} + \frac{\rho}{2} \frac{d}{dt} (u^2)$$

Equation (2.16) thus becomes

$$\rho \frac{dE}{dt} = \sigma^{ij} u_{i,j} + (\kappa \theta^{ij} T_{ij},_j) \quad \text{(2.17)}$$
To express the quantity $\sigma_{ij} u_{i,j}$ in terms of $p$ and the velocity components, we make use of (2.3) and (2.4). Then (2.17) may be written

$$\rho \frac{\partial E}{\partial t} = -\rho \dot{u}_i \dot{u}_i + \Phi + (k \rho \dot{u}_i T_{,i})_{,j}$$

(2.18)

where

$$\Phi = \mu \left\{ \gamma_{ij} \gamma^{lm} \dot{u}_l \dot{u}_m + \dot{u}_i \dot{u}_j - \frac{2}{3} (\dot{u}_i \dot{u}_i)^2 \right\}$$

(2.19)

The quantity $(-p \dot{u}_i, i)$ in (2.18) represents the rate at which work is done in compressing the fluid, whereas $\Phi$ represents the rate of conversion of mechanical energy into thermal energy by the action of viscosity.

Finally, if $E$ is regarded as a function of $p$ and $T$, we may write (2.18) in the expanded form

$$\rho \frac{\partial E}{\partial p} \left( \frac{\partial p}{\partial t} + u_i \rho_{,i} \right) + \rho \frac{\partial E}{\partial T} \left( \frac{\partial T}{\partial t} + u_i T_{,i} \right) =$$

$$-\rho \dot{u}_i \dot{u}_i + \Phi + (k \rho \dot{u}_i T_{,i})_{,j}$$

(2.20)

Equations (2.7), (2.9), and (2.20) constitute the basic tensor equations of motion, continuity, and heat flow which formed the objective of this section.
Section 3. Fluid and Heat Flow Equations Expressed in an
Intrinsic Coordinate System:

If two neighboring surfaces lie sufficiently close to each other, it is clear that, physically, the essential features of the flow of a viscous fluid between them should be describable in terms of two dimensional phenomena. If we assume the validity of the simplified type of theory discussed in the Introduction, as represented by equation (1), or its non-homogeneous counterpart for the case of moving surfaces, the specification of the pressure \( p \) is sufficient to determine the average velocity vector \( \bar{v} \), and hence, by use of the parabolic distribution law, the complete velocity distribution. The pressure \( p \), it will be noted, arises as the solution to a two dimensional partial differential equation. Furthermore, it is reasonable to suppose that any corrections to this approximate theory will possess a somewhat similar character.

The theory we propose to develop is based essentially upon an examination of the exact equations of fluid motion and heat flow of Section 2, rather than upon physical intuition. However, we shall allow intuition to guide us to the extent of suggesting the seemingly most profitable form in which to put these equations. As the first step it would appear logical to employ a system of curvilinear coordinates \((x^1, x^2, x^3)\) in which the two surfaces correspond approximately to a parametric surface, say \( X^3 = 0 \). We might, of course, so choose the coordinates that each of the surfaces was exactly expressible in the form \( X^3 = \text{constant} \); however, if the
surfaces were moving this choice would have the disadvantage of continually changing our coordinate system. Instead we prefer to employ coordinates based on a permanently fixed system of parallel surfaces, one of which (S) is selected as the reference surface $X^3 = 0$. We let the coordinate $X^3$ be a constant multiple of the perpendicular distance of a variable point from S, and let $X^1$ and $X^2$ be an arbitrary system of curvilinear coordinates in S.

It is further evident from the approximate theory that our equations should be written in such form as to single out the $X^3$ direction for special consideration. Since we are seeking differential equations involving functions of $X^1$ and $X^2$ only, it is found advantageous to express the basic equations (2.7), (2.9), and (2.20) in terms of two dimensional vectors and tensors and the corresponding covariant differentiations associated with the reference surface, treating $X^3$ vector components as scalars and $X^3$ differentiations as operations of a scalar type. It is then possible, in Section 5, to eliminate the $X^3$ direction from consideration. As a part of the developments of the present section, a natural classification of the terms of our equations according to order of magnitude appears; to clarify this situation the equations are written in a dimensionless form.

Let the two surfaces bounding the fluid be denoted by $S_1$ and $S_2$, and let S be the fixed reference surface approximating $S_1$ and $S_2$. Let $h_o$ and $L_o$ be fixed constants representing typical values of the distance
separating $S_1$ and $S_2$ and the overall dimensions of the system, respectively. For example, in the case of a cylindrical shaft and bearing, $h_0$ might be chosen to be the average radial clearance between shaft and bearing, and $L_0$ the shaft radius. We write

$$\frac{h_0}{L_0} = \varepsilon$$  \hspace{1cm} (3.1)

and assume $\varepsilon \ll 1$.

Let $(X^1, X^2)$ denote an arbitrary curvilinear coordinate system in the surface $S$, such that $X^1$ and $X^2$ have the dimensions of length and are measured in units in which* $L_0 = O(1)$. As the third coordinate $X^3$ we select the perpendicular distance of a variable point to $S$, multiplied by the dimensionless ratio $\frac{L_0}{h_0} = \frac{1}{\varepsilon}$, and with a sign so chosen that $X^3$ is increasing from $S_1$ toward $S_2$. In these coordinates the total volume of fluid under consideration may be regarded as occupying a somewhat distorted cube, since the surfaces $S_1$ and $S_2$ correspond to values of $X^3$ of the order of $L_0$.

We let the equations defining $S_1$ and $S_2$ be

$$S_1: \hspace{1cm} X^3 = L_0 H_{10}(x^1, x^2; t)$$

$$S_2: \hspace{1cm} X^3 = L_0 H_{20}(x^1, x^2; t)$$  \hspace{1cm} (3.2)

*Throughout the next few sections the symbol $O(\cdot)$ is used rather loosely, to indicate a quantity of the general order of magnitude of the quantity within the parentheses. Thus $L_0 = O(1)$ is to read, "the numerical value of $L_0$ is of the order of magnitude of unity."
and also write

\[ H(x'_j x^j; t) = H_{(2)}(x'_j x^j; t) - H_{(1)}(x'_j x^j; t) \]  

Then, in accordance with our choice of sign for \( x^3 \),

\[ H(x'_j x^j; t) > 0 \]

and the quantity \( h_0 H(x'_j x^j; t) \) is the physical distance between \( S_1 \) and \( S_2 \). The quantities \( H_{(1)}, H_{(2)}, \) and \( H \) are evidently dimensionless, and are of the order of magnitude of unity.

In order to provide a notation which singles out the \( x^3 \) direction for special consideration, we henceforth use Latin letters to denote tensor indices having the range (1, 2, 3), and Greek letters for indices having the range (1, 2). We then may write, for example,

\[
(d\alpha)^2 = g_{ij} dx^i dx^j \\
= g_{\alpha\beta} dx^\alpha dx^\beta + g_{\alpha 3} dx^\alpha dx^3 \\
+ g_{3\alpha} dx^3 dx^\alpha + g_{33} (dx^3)^2
\]

Because of the orthogonality of the \( x^3 \) coordinate to the other two,

\[ g_{\alpha 3} = g_{3\alpha} = 0 \quad (\alpha = 1, 2) \]  

(3.4)
whence the above expression for \((ds)^2\) reduces simply to
\[
(ds)^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta + g_{33} \, (dx^3)^2
\]

We do not specifically assume that \(X^1\) and \(X^2\) are mutually orthogonal, although this will be the case in all contemplated applications.

Before proceeding further we may profitably examine the relations between two and three dimensional tensors that we have occasion to use later. Let us consider coordinate transformations of the type

\[
\bar{x}' = \bar{x}'(x', x^2) \quad \bar{x}^2 = \bar{x}^2(x', x^2) \quad \bar{x}^3 = \bar{x}^3
\]  \hspace{1cm} (3.5)

We call a set of quantities \(A^\alpha_{\gamma...\delta} \) \((\alpha, \beta, \gamma, ...\delta = 1, 2)\) a two dimensional tensor if it obeys the tensor law (2.1) with respect to transformations of the special type above, or, in other words, if

\[
\bar{A}^\alpha_{\gamma...\delta} = A^\gamma_{\lambda...\epsilon} \frac{\partial \bar{x}^\alpha}{\partial x^\lambda} \cdots \frac{\partial \bar{x}^\delta}{\partial x^\epsilon}
\]  \hspace{1cm} (3.6)

Such a tensor may of course be a function of \(X^3\). In this connection we may readily demonstrate the following assertion:

Let \(A^\alpha_{\gamma...\delta}\) be an arbitrary two dimensional tensor, in the sense understood above. Then \(\frac{\partial}{\partial x^3} A^\alpha_{\gamma...\delta}\) is also a two dimensional tensor of the same character. For proof, we observe
that the right hand member of (3.6) consists of a sum of terms, each of which contains an A multiplied by a factor independent of $X^3$ by (3.5). Thus, by direct differentiation of (3.6),

\[
\left( \frac{\partial}{\partial X^3} A^{\alpha \ldots \beta}_{\gamma \ldots \zeta} \right) = \left( \frac{\partial}{\partial X^3} A^{\alpha \ldots \beta}_{\lambda \ldots \rho} \right) \frac{\partial X^\alpha}{\partial X^n} \ldots \frac{\partial X^\alpha}{\partial X^r} \frac{\partial X^\beta}{\partial X^s} \ldots \frac{\partial X^\beta}{\partial X^s}
\]

showing that the derivative also satisfies (3.6).

We may also demonstrate the following: Let $T^{i \ldots j}_{k \ldots l}$ be an arbitrary three dimensional tensor. Then the set of quantities obtained by assigning the value 3 to certain of the indices $(i, \ldots, j, k, \ldots, l)$ and allowing the others to range independently over the values 1 and 2 is a two dimensional tensor.

By way of illustration, we consider the simple example $T^{i j}_{k l}$. From (2.1),

\[
\overline{T}^{i j}_{k l} = T^{l m}_{k l} \frac{\partial X^i}{\partial X^l} \frac{\partial X^j}{\partial X^m} \frac{\partial X^k}{\partial X^l}
\]

If we set $i = \alpha = 1, 2; j = 3; k = \beta = 1, 2$, for example, then

\[
\overline{T}^{\alpha 3}_{\beta} = T^{l m}_{\alpha \beta} \frac{\partial X^\alpha}{\partial X^l} \frac{\partial X^3}{\partial X^m} \frac{\partial X^\beta}{\partial X^l}
\]

Now, with respect to transformations of the restricted type (3.5), the only non-vanishing terms on the right of the above
equation are those for which \( m = 3 \) and \( l \) and \( p \) have the values 1 or 2. Hence

\[
\overline{T}_{\beta}^{\alpha 3} = T_{\delta}^{\gamma 3} \frac{\partial X^\alpha}{\partial X^\gamma} \frac{\partial X^\delta}{\partial X^\beta}
\]

which is a transformation of the type (3.6). \( T_{\delta}^{\alpha 3} \) is thus a two dimensional tensor, as asserted, and it is evident that the above reasoning is applicable to tensors of any rank.

As an example of the application of this proposition, \( T_{\delta}^{\alpha 3} \) will be a two dimensional tensor. Specifically, when \( X^3 = 0 \), \( T_{\delta}^{\alpha 3} \) will be the fundamental covariant tensor for the reference surface \( S \). For the contravariant velocity vector \( u^i \) we find the following behavior with respect to two dimensional transformations:

- \( u^\alpha = \text{contravariant vector} \)
- \( u^3 = \text{scalar invariant} \)
- \( u^\alpha_\beta = \text{mixed tensor} \)
- \( u^\alpha_3 = \text{contravariant vector} \)
- \( \frac{\partial u^\alpha}{\partial X^3} = \text{contravariant vector} \)
- \( u^3_\alpha = \text{covariant vector} \)
- \( \frac{\partial u^3}{\partial X^\alpha} = \text{covariant vector} \)
- \( u^3_3 = \text{scalar invariant} \)
We note that \( u^3_{ij} \) and \( \frac{\partial u^i}{\partial x^3} \) are not, in general, equal, for the former quantity is to be interpreted as the result of setting \( i = \alpha \) and \( j = 3 \) in the equation

\[
\mu^i_{ij} = \frac{\partial u^i}{\partial x^j} + \mu^k \{ i, j \}
\]

A similar statement evidently applies to \( u^3_\alpha \) and \( \frac{\partial u^3}{\partial x^\alpha} \).

The basic equations of motion, continuity, and heat flow developed in Section 2 are, of course, directly applicable without change to our present coordinate system, because of their tensor character. However, we wish to put these equations into a form involving only dimensionless variables, on the one hand, and which is written in terms of two dimensional tensors, on the other. As a preliminary step, we write

\[
\eta^i = \frac{x^i}{L_o} \quad U^i = \frac{u^i}{u_o}
\]

where \( u_o \) is a representative magnitude of the velocity vector. The velocity components \( U^1 \) and \( U^2 \) are evidently \( O(1) \) by the definition of \( u_o \); we assume (and later show) that \( U^3 \) is \( O(1) \) also*.

*Note that the contravariant velocity component \( u^3 \) is actually \( \frac{1}{L} \) times the ordinary physical component of velocity in the direction normal to \( S \), by our choice of coordinates. Hence \( U^3 = O(1) \), implies that the physical velocity component in the \( X^3 \) direction is \( O(L u_o) \).
To investigate the character of the covariant derivatives of the velocity vector occurring in the equations of motion, we first must study the properties of the fundamental tensor and the associated Christoffel symbols*. We observe that since the \( x^i \) coordinates have been chosen to have the dimensions of length, the associated tensors \( g_{ij} \) and \( g^{ij} \) are dimensionless.

Now, expanding \( g_{\alpha \beta} \) and \( g^{\alpha \beta} \) in power series in the distance from \( S \), we may write**

\[
\nabla_{\alpha \beta}(x', x^2, x^3) = g_{\alpha \beta}(x', x^2) + \left( \frac{E x^3}{L_0} \right) g_{\alpha \kappa \beta}(x', x^2)
\]

\[
+ \left( \frac{E x^3}{L_0} \right)^2 g_{\alpha \kappa \lambda \beta}(x', x^2)
\]

and

\[
\nabla^{\alpha \beta}(x', x^2, x^3) = g^{\alpha \beta}(x', x^2) + \left( \frac{E x^3}{L_0} \right) g^{\alpha \kappa \beta}(x', x^2)
\]

\[
+ \left( \frac{E x^3}{L_0} \right)^2 g^{\alpha \kappa \lambda \beta}(x', x^2) + \left( \frac{E x^3}{L_0} \right)^3 g^{\alpha \kappa \lambda \nu \beta}(x', x^2) + \ldots
\]

where the quantities \( g_{\alpha \kappa \beta} \) and \( g^{\alpha \kappa \beta} \) are dimensionless tensors associated with the reference surface \( S \), and are all

* See Weatherburn (loc. cit.), Chapter IV, for the definition of these symbols.

** The series for \( g_{\alpha \beta} \) terminates with the quadratic term; however, that for \( g^{\alpha \beta} \) does not terminate in general.
of the order of magnitude of unity. We also note that

$$Q_{33}(x', x^2, x^3) = \varepsilon^2 \quad Q_{33}^{\alpha\beta}(x', x^2, x^3) = \frac{1}{\varepsilon^2}$$

(3.10)

In the $y$ coordinates we write

$$G_{\alpha\beta}(y', y^2, y^3) \equiv q_{\alpha\beta}(x', x^2, x^3) \quad G_{\alpha\beta}^{\alpha\beta}(y', y^2) \equiv q_{\alpha\beta}^{\alpha\beta}(x', x^2)$$

$$G_{\alpha\beta}^{\gamma\delta}(y', y^2, y^3) \equiv q_{\alpha\beta}^{\gamma\delta}(x', x^2, x^3) \quad G_{\alpha\beta}^{\gamma\delta}(y', y^2) \equiv q_{\alpha\beta}^{\gamma\delta}(x', x^2)$$

(3.11)

The $G$'s differ by factors of $L_0^2$ from the values which the fundamental tensors would have when referred to the $y$ coordinates; however, these quantities have the advantage of being dimensionless. From (3.8)

$$G_{\alpha\beta}(y', y^2, y^3) = G_{\alpha\beta}^{\gamma\delta}(y', y^2) + \varepsilon(y^3) G_{\gamma\delta}^{\alpha\beta}(y', y^2)$$

$$+ \varepsilon^2(y^3)^2 G_{\gamma\delta}^{\alpha\beta}(y', y^2)$$

(3.12)

and from (3.9)

$$G_{\alpha\beta}^{\gamma\delta}(y', y^2, y^3) = G_{\alpha\beta}^{\gamma\delta}(y', y^2) + \varepsilon(y^2) G_{\gamma\delta}^{\alpha\beta}(y', y^2)$$

$$+ \varepsilon^2(y^3)^2 G_{\gamma\delta}^{\alpha\beta}(y', y^2) + \varepsilon^3(y^3)^3 G_{\gamma\delta}^{\alpha\beta}(y', y^2) + \ldots$$

(3.13)
For later use, we require certain simple algebraic relations between the tensors $G_{\alpha \beta \gamma \delta}$ and $G^{\alpha \beta \gamma \delta}$. From equations (3.8) to (3.13), and the relation

$$g_{ij} g^{jk} = \delta^k_i$$

we observe that

$$\left\{ G_{\alpha \beta \gamma \delta} + \varepsilon (q^3) G_{\alpha \beta \gamma \delta} + \ldots \right\} \left\{ G^{\beta \gamma \delta} + \varepsilon (q^3) G^{\beta \gamma \delta} + \ldots \right\} = \delta^\gamma_\alpha$$

identically in $Y^3$. Expanding this equation and equating coefficients of like powers of $Y^3$ on the left and right hand sides, we readily obtain

$$G_{\alpha \beta \gamma \delta} G^{\beta \gamma \delta} = \delta^\gamma_\alpha \quad (3.13.1)$$

$$G_{\alpha \beta \gamma \delta} G^{\beta \gamma \delta} + G_{\alpha \beta \gamma \delta} G^{\alpha \beta \gamma \delta} = 0 \quad (3.13.2)$$

with analogous relations derivable from higher order terms.

For the Christoffel symbols of the first kind in the $X$ coordinates, in which only the indices 1 and 2 are involved, we have

$$[\alpha, \beta \gamma] = \frac{1}{L_0} [\alpha, \beta \gamma]_0 + \frac{\varepsilon (q^3)}{L_0} [\alpha, \beta \gamma], \quad + \frac{\varepsilon^2 (q^3)^2}{L_0^2} [\alpha, \beta \gamma]_2$$

$$= \frac{1}{L_0} \left\{ [\alpha, \beta \gamma]_0 + \varepsilon (q^3)[\alpha, \beta \gamma], \quad + \varepsilon^2 (q^3)^2 [\alpha, \beta \gamma]_2 \right\}$$
where \([\alpha, \beta \gamma]_n\) are dimensionless quantities given by

\[
[\alpha, \beta \gamma]_n = \frac{L_o}{2} \left( \frac{\partial G_{\omega \alpha \beta}}{\partial x^\gamma} + \frac{\partial G_{\omega \beta \alpha}}{\partial x^\gamma} - \frac{\partial G_{\omega \gamma \alpha}}{\partial x^\beta} \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial G_{\omega \alpha \beta}}{\partial y^\gamma} + \frac{\partial G_{\omega \beta \alpha}}{\partial y^\gamma} - \frac{\partial G_{\omega \gamma \alpha}}{\partial y^\beta} \right) \quad (n = 0, 1, 2)
\]

Also

\[
[3, \alpha \beta] = -\frac{1}{2} \frac{\partial G_{\omega \alpha \beta}}{\partial x^3}
\]

\[
= -\frac{1}{2L_o} \left\{ \varepsilon G_{\omega \alpha \beta} + 2\varepsilon^2(y^3) G_{\omega 3 \alpha \beta} \right\}
\]

\[
[\alpha, 3 \beta] = [\alpha, \beta 3] = +\frac{1}{2L_o} \left\{ \varepsilon G_{\omega \alpha \beta} + 2\varepsilon^2(y^3) G_{\omega 3 \alpha \beta} \right\}
\]

and

\[
[\alpha, 33] = [3, \alpha 3] = [3, 3 \alpha] = [3, 33] = 0
\]

Using these relations, we find for the Christoffel symbols of the second kind in the X coordinates the values

\[
\{ \alpha^\gamma \} = g^{\alpha \lambda} [\lambda, \beta \gamma] + g^{\alpha 3} [3, \beta \gamma]
\]

\[
= g^{\alpha \lambda} [\lambda, \beta \gamma]
\]

\[
= \frac{1}{L_o} \left\{ \{ \alpha^\gamma \}_o + \sum_{n=1}^{\infty} \varepsilon^n \left( y^3 \right)^n \left( \frac{1}{n!} \delta^\alpha \gamma \right) \right\}
\]

(3.14)
where

\[ \{ \alpha \beta \} = G_{\alpha \beta}^{\alpha \beta} \{ \lambda, \beta \gamma \} \]  \hspace{1cm} (3.15) 

and

\[
\begin{align*}
\mathcal{A}_{\beta \gamma} & = G_{\alpha \beta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \gamma}^{\alpha \lambda} \{ \lambda, \beta \gamma \}, \\
\mathcal{B}_{\beta \gamma} & = G_{\alpha \beta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \gamma}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \zeta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \zeta}^{\alpha \lambda} \{ \lambda, \beta \gamma \}, \\
\mathcal{C}_{\beta \gamma} & = G_{\alpha \beta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \gamma}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \zeta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \zeta}^{\alpha \lambda} \{ \lambda, \beta \gamma \}, \\
\vdots & \quad \vdots \\
\mathcal{D}_{\beta \gamma} & = G_{\alpha \beta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \gamma}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \zeta}^{\alpha \lambda} \{ \lambda, \beta \gamma \} + G_{\alpha \zeta}^{\alpha \lambda} \{ \lambda, \beta \gamma \}.
\end{align*}
\]  \hspace{1cm} (3.16)

Also

\[
\begin{align*}
\{ 3 \beta \} & = g^{33} [3, \alpha \beta] = -\frac{1}{2 L_0} \left\{ \frac{1}{E} G_{01 \alpha \beta} + 2 (q^{3}) G_{02 \alpha \beta} \right\} \\
\{ \alpha \gamma \} & = \{ \alpha \beta \} = g^{\alpha \lambda} \{ \lambda, \beta \gamma \} \\
\{ 3 \beta \} & = \{ \alpha \beta \} = g^{\alpha \lambda} \{ \lambda, \beta \gamma \} \\
& = \frac{1}{L_0} \sum_{n=1}^{\infty} \varepsilon^n [q^{3}]_{n-1} [\beta \lambda \zeta \phi].
\end{align*}
\]  \hspace{1cm} (3.17)
where
\[
\begin{align*}
\sqrt{\Gamma^\kappa_{\alpha\beta\gamma}} &= \frac{1}{2} G^{\alpha\lambda}_{\gamma\mu} G_{\mu\lambda\kappa} \\
\sqrt{\Gamma^\kappa_{\beta\lambda\gamma}} &= \frac{1}{2} G^{\alpha\lambda}_{\beta\mu} G_{\mu\lambda\kappa} + G^{\alpha\lambda}_{\gamma\mu} G_{\mu\kappa\lambda} \\
\sqrt{\Gamma^\kappa_{\gamma\lambda\beta}} &= \frac{1}{2} G^{\alpha\lambda}_{\gamma\mu} G_{\mu\lambda\kappa} + G^{\alpha\lambda}_{\gamma\mu} G_{\mu\kappa\lambda}
\end{align*}
\]

(3.19)

Finally
\[
\begin{align*}
\{3\} &= \{\alpha\} = \{\alpha\} = \{\alpha\} = \{\alpha\} = 0 \quad (3.20)
\end{align*}
\]

We may readily show that the $\sqrt{\Gamma^\kappa_{\alpha\beta\gamma}}$ are two dimensional tensors. The quantities $\sqrt{\Gamma^\kappa_{\alpha\beta\gamma}}$ are seen directly to be tensors from (3.19) since they are composed of the sums of inner products of tensors. To prove that the quantities $\sqrt{\Gamma^\kappa_{\alpha\beta\gamma}}$ are tensors, we note first that the fundamental transformation law for the Christoffel symbols $\{i_{j}k\}$ is*
\[
\begin{align*}
\{i_{j}k\} &= \{a_{b}c\} \frac{\partial x^{b}}{\partial \bar{x}^{d}} \frac{\partial x^{c}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{i}}{\partial x^{d}} + \frac{\partial^{2} x^{m}}{\partial x^{d} \partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{i}}
\end{align*}
\]

where \(\{i_{j}k\}\) denotes the set of quantities formed from a new set of coordinates $\bar{x}^{i}$ and the corresponding $\bar{\Gamma}^{i}_{j}k$.

*Weatherburn (loc. cit.), p. 58.
Now if, in particular, we consider transformations of the special type (3.5), and restrict i, j, and k to the values 1 and 2, then

\[ \{ \alpha \} = \{ \lambda \} \left\{ \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\rho}{\partial x^\gamma} \frac{\partial y^\kappa}{\partial x^\lambda} \right\} + \frac{\partial^2 x^\lambda}{\partial x^\alpha \partial x^\gamma} \frac{\partial y^\kappa}{\partial x^\lambda} \]

Also at \( x^3 = 0 \)

\[ \left\{ \alpha \right\}_{x^3=0} = \left\{ \lambda \right\}_{x^3=0} \left\{ \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\rho}{\partial x^\gamma} \frac{\partial y^\kappa}{\partial x^\lambda} \right\} + \frac{\partial^2 x^\lambda}{\partial x^\alpha \partial x^\gamma} \frac{\partial y^\kappa}{\partial x^\lambda} \]

A comparison of these equations with (3.6) shows, of course, that the Christoffel symbols themselves are not tensors. We note, however, that \( \frac{\partial x^\mu}{\partial x^\gamma} \frac{\partial x^\rho}{\partial x^\lambda} \frac{\partial y^\kappa}{\partial x^\lambda} \) and \( \frac{\partial^2 x^\lambda}{\partial x^\alpha \partial x^\gamma} \frac{\partial y^\kappa}{\partial x^\lambda} \) are independent of \( x^3 \) by (3.5). Hence we may subtract the second of these equations from the first to obtain

\[ \left[ \left\{ \alpha \right\} - \left\{ \alpha \right\}_{x^3=0} \right] = \left[ \left\{ \lambda \right\} - \left\{ \lambda \right\}_{x^3=0} \right] \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\rho}{\partial x^\gamma} \frac{\partial y^\kappa}{\partial x^\lambda} \]

showing that the quantities \( \left\{ \alpha \right\} - \left\{ \alpha \right\}_{x^3=0} \) form a two-dimensional tensor. By equation (3.14) these quantities are scalar multiples of

\[ \mathcal{E}(y^3) \tilde{\Gamma}^{\alpha}_{\beta \gamma} + \mathcal{E}^2(y^3)^2 \tilde{\Gamma}^{\alpha}_{\beta \gamma} + \cdots \]

and hence the latter must also constitute a tensor, identically in \( y^3 \). Since the terms of this series are independent, each \( \tilde{\Gamma} \) must be a tensor, which was to be proved.
We now turn to the evaluation of such covariant derivatives as shall be required, in terms of the variables $Y_i$ and the two dimensional tensor quantities introduced above. A vertical bar $(\mu^{\alpha}_{i\beta}, A^\alpha_{i\beta}, \text{etc.})$ is used to denote the operation of dimensionless covariant differentiation with respect to the reference surface $S$. Specifically, we introduce the following definitions, which may be at once generalized:

$$\phi_{1\theta} = \frac{\partial \phi}{\partial Y^\theta}$$

$$\mu^\alpha_{i\beta} = \frac{\partial \mu^\alpha}{\partial Y^\theta} + \mu^\lambda \{^\alpha_\lambda \},$$

$$\mu_{i\alpha\beta} = \frac{\partial \mu^\alpha}{\partial Y^\theta} - \mu^\lambda \{^\alpha_\lambda \},$$

$$A^\alpha_{i\beta} = \frac{\partial A^\alpha}{\partial Y^\theta} + A^\lambda \{^\alpha_\lambda \} - A^\alpha \{^\lambda_\beta \}.$$  

(3.21)

We must here distinguish between such quantities as $\mu^\alpha_{i,\beta}$ and $\mu^\alpha_{i\beta}$. Both are two dimensional tensors; the former is to be interpreted as the result of forming first the three dimensional tensor $\mu^i_{j}$ and subsequently restricting the indices to the values 1 and 2, whereas the latter is the result of a two dimensional operation on a two dimensional vector.

For future reference, certain formulae for the two dimensional covariant derivatives of the tensors $G_{01}^{i\alpha\beta}$, $G^{i\alpha\beta}$, and $\sqrt{\cdot}^{i\alpha\beta}$ may conveniently be derived at this point.
any three dimensional covariant tensor \( A_{ij} \) we have by definition

\[
A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - A_{im} \{^m_\cdot jk\} - A_{mj} \{^m_\cdot ik\}
\]

Letting \( i, j, k = \alpha, \beta, \gamma \) \((= 1, 2)\) and noting that \( m \) still has the range 1, 2, 3, we find

\[
A_{\alpha\beta,\gamma} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} - A_{\alpha\lambda} \{^\lambda_\beta\gamma\} - A_{\lambda3} \{^3_\beta\gamma\} - A_{\lambda\varepsilon} \{^\varepsilon_\alpha\gamma\} - A_{3\varepsilon} \{^3_\alpha\gamma\}
\]

If we now make use of (3.14) to express the quantities \( \{^\lambda_\beta\gamma\} \) and \( \{^\lambda_\alpha\gamma\} \) in series form, we obtain upon collecting terms in like powers of \( \varepsilon \)

\[
A_{\alpha\beta,\gamma} = \frac{1}{L_o} \left\{ \frac{\partial A_{\alpha\beta}}{\partial y^\gamma} - A_{\alpha\lambda} \{^\lambda_\beta\gamma\} - A_{\lambda\varepsilon} \{^\varepsilon_\alpha\gamma\} - \sum_{n=1}^{\infty} \varepsilon^n (y^3)_n \left[ A_{\alpha\lambda} \{^\lambda_0\beta\gamma\} + A_{\lambda\varepsilon} \{^\varepsilon_0\alpha\gamma\} \right] \right\}
\]

\[
- A_{\alpha3} \{^3_\beta\gamma\} - A_{3\varepsilon} \{^3_\alpha\gamma\}
\]

\[
= \frac{1}{L_o} \left\{ A_{\alpha\beta,\gamma} - \sum_{n=1}^{\infty} \varepsilon^n (y^3)_n \left[ A_{\alpha\lambda} \{^\lambda_0\beta\gamma\} + A_{\lambda\varepsilon} \{^\varepsilon_0\alpha\gamma\} \right] \right\}
\]

\[
- A_{\alpha3} \{^3_\beta\gamma\} - A_{3\varepsilon} \{^3_\alpha\gamma\}
\]

(3.21.1)
In a similar manner it may be shown that

\[ B^\alpha{}_{\gamma, \lambda \tau} = \frac{1}{L_0} \left\{ B^\alpha{}_{\gamma, \lambda \tau} + \sum_{n=1}^{\infty} \epsilon^n(q^3)^n \left[ B^\alpha{}_{\nu \lambda \gamma} + B^\lambda{}_{\nu \lambda \gamma} \right] \right\} + B^\alpha{}_{\gamma, \lambda \tau} \{ \beta \} + B^\alpha{}_{\gamma, \lambda \tau} \{ \lambda \} \]

(3.21.2)

With respect to three dimensional covariant differentiation*,

\[ \Phi_{i j, \kappa} = \Phi_{i j, \kappa} = 0 \]

From (3.11) we therefore find

\[ G_{i j, \kappa} = G_{i j, \kappa} = 0 \]

(3.21.3)

Now let \( A_{i j} = G_{i j} \) and \( B_{i j} = G_{i j} \) in (3.21.1) and (3.21.2). Since

\[ G^{\alpha \beta} = G_{3 \beta} = G^{\alpha \beta} = G^{\alpha \beta} = 0 \]

we obtain by use of (3.21.3) the relations

\[ G^{\alpha \beta}_{\gamma, \lambda} - \sum_{n=1}^{\infty} \epsilon^n(q^3)^n \left[ G^{\alpha \beta}_{\nu \lambda, \gamma} + G^{\alpha \beta}_{\lambda \gamma, \nu} \right] = 0 \]

(3.21.4)

\[ G^{\alpha \beta}_{1 \gamma} + \sum_{n=1}^{\infty} \epsilon^n(q^3)^n \left[ G^{\alpha \beta}_{\nu \lambda, \gamma} + G^{\alpha \beta}_{\lambda \gamma, \nu} \right] = 0 \]

(3.21.5)

*Weatherburn (loc. cit.), p. 63.
Equations (3.21.4) and (3.21.5) are true identically in \( Y^3 \); however, the tensors \( G_{\alpha \beta} \) and \( G^{\alpha \beta} \) themselves depend upon \( Y^3 \), in the manner given by the series (3.12) and (3.13). Substituting from these series and collecting terms in like powers of \( Y^3 \), we find from the terms which are independent of \( Y^3 \)

\[
G_{\alpha \beta \gamma} = G^{\alpha \beta} = 0 \quad (3.21.6)
\]

and from the terms involving \( Y^3 \) to the first power,

\[
G_{\alpha \beta \gamma} = G_{\alpha \beta \gamma} \Gamma^{\lambda}_{\beta \gamma} + G_{\alpha \beta \gamma} \Gamma^{\lambda}_{\alpha \gamma} \quad (3.21.7)
\]

\[
G^{\alpha \beta} = -G^{\alpha \lambda} \Gamma^{\lambda}_{\alpha \gamma} - G^{\lambda \beta} \Gamma^{\lambda}_{\gamma \gamma} \quad (3.21.8)
\]

By means of these results we may also calculate the quantity \( \Gamma^{\alpha}_{\beta \gamma} \). From the definition of the tensor \( \Gamma^{\alpha}_{\beta \gamma} \) as given by (3.19), we obtain by successive application of (3.21.6), (3.21.7), and (3.13.1)

\[
\Gamma^{\alpha}_{\beta \gamma} = \left[ \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} \Gamma^{\rho \eta}_{\beta \gamma} \right]_{\gamma}
\]

\[
= \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} G^{\rho \eta}_{\beta \gamma}
\]

\[
= \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} \left[ G^{\rho \eta}_{\gamma \lambda} \Gamma^{\lambda}_{\beta \gamma} + G^{\rho \eta}_{\lambda \sigma} \Gamma^{\lambda}_{\gamma \gamma} \right]
\]

\[
= \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} \left[ \Gamma^{\lambda}_{\beta \gamma} \Gamma^{\lambda}_{\gamma \lambda} + \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} \Gamma^{\rho \sigma}_{\beta \gamma} \right]
\]

\[
= \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} \left[ \Gamma^{\rho \sigma}_{\beta \gamma} + \frac{1}{2} G^{\alpha \eta}_{\rho \sigma} \Gamma^{\rho \sigma}_{\beta \gamma} \right]
\]

\[
\quad (3.21.9)
\]
We now wish to derive certain auxiliary formulae to be used in expressing the equations of Section 2 in a dimensionless form and in terms of two dimensional tensors. Let \( \phi \) be an arbitrary scalar quantity. Then

\[
\phi; \alpha = \frac{\partial \phi}{\partial x^\alpha} = \frac{1}{L_0} \frac{\partial \phi}{\partial y^\alpha} = \frac{1}{L_0} \phi_{,\alpha}
\]  

(3.22)

\[
\phi; \beta = \frac{1}{L_0} \frac{\partial \phi}{\partial y^\beta}
\]

(3.23)

Also we find

\[
\phi; \alpha \beta = (\phi; \alpha)_\beta = \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} - \phi_{,\lambda} \{^\alpha^\beta^\lambda\} - \phi_{,\beta} \{^\alpha^\beta^3\}
\]

\[
= \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} - \phi_{,\lambda} \{^\alpha^\beta^\lambda\} - \phi_{,\beta} \{^\alpha^\beta^3\}
\]

(3.24)
Here we have used the definition

\[ \phi_{\lambda\rho} = \frac{\partial^2 \phi}{\partial y^\lambda \partial y^\rho} - \phi_{\lambda\rho} \left\{ \frac{\partial}{\partial y^\rho} \right\} \]

which follows from the first and third of equations (3.21), as well as the formulae (3.14) to (3.20) for the Christoffel symbols. According to the previous discussion of two-dimensional tensors, the quantity \( \frac{\partial \phi}{\partial y^3} \) in (3.24) is to be treated as a scalar invariant, as well as \( y^3 \) and its powers. Using similar methods we also find

\[ \phi_{\lambda 3} = \phi_{3\lambda} = \frac{1}{L^3_0} \left\{ \frac{\partial \phi}{\partial y^3} - \sum_{n=1}^{\infty} \epsilon^n \left[ (y_3)^{n-1} \phi_{\lambda 3} \right] \right\} \]  

(3.25)

\[ \phi_{33} = \frac{1}{L^3_0} \left\{ \frac{\partial^2 \phi}{(\partial y^3)^2} \right\} \]  

(3.26)

For the vector \( u^\lambda \) and its first covariant derivatives we obtain

\[ u^\lambda = u_0 U^\lambda \]  

(3.27)

\[ u_{\lambda \rho} = \frac{u_0}{L_0} \left\{ U^\lambda_{\rho} + \sum_{n=1}^{\infty} \epsilon^n \left[ (y_3)^{n-1} U^\lambda \right] \right\} \]  

(3.28)

\[ u_{\lambda 3} = \frac{u_0}{L_0} \left\{ \frac{\partial U^\lambda}{\partial y^3} + \sum_{n=1}^{\infty} \epsilon^n \left[ (y_3)^{n-1} U^\lambda \right] \right\} \]  

(3.29)

\[ u_{3 \rho} = \frac{u_0}{L_0} \left\{ \frac{1}{\epsilon} \left[ -\frac{1}{2} U^3 G_{\rho \lambda \varnothing} \right] + U_{\rho \lambda \varnothing} \right\} \]  

(3.30)
\( u_{j3}^3 = \frac{M_0}{L_0} \left\{ \frac{\partial U}{\partial \eta_j} \right\} \) \hspace{1cm} (3.31)

In order to calculate the second derivatives of the type \( u_{j3}^3 \) we first note that for an arbitrary mixed tensor \( A_j^\alpha \)

\[
A_\sigma^\alpha = \frac{1}{L_0} \left\{ \frac{1}{\varepsilon} \left[ \frac{1}{2} A_3^\alpha \delta_{0\sigma} \right] + 1. \left[ U_{\sigma1} - \left( \psi^3 \right) A_3^\alpha \delta_{01} \right] \right\} + \sum_{n=1}^{\infty} \varepsilon^n \left[ \left( \psi^3 \right)^n A_\lambda^\alpha \delta_{\sigma\lambda} \right] + \left( \psi^3 \right)^n A_\sigma^\alpha \delta_{\lambda3} \right\}
\]

Taking \( A_j^\alpha = u_j^\alpha \) we may then substitute the values (3.28) to (3.30) into the above expression. For the term \( A_\sigma^\alpha \) we may take the two dimensional covariant derivative of (3.28), obtaining

\[
A_\sigma^\alpha = \frac{M_0}{L_0} \left\{ U_{\sigma1}^\alpha + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{\partial \eta_n} \right\}
\]

Making the required substitutions, and arranging the results in ascending powers of \( \varepsilon \), we arrive at the result

\[
u_{\sigma3}^3 = \frac{M_0}{L_0} \left\{ \frac{1}{\varepsilon} \left[ \frac{1}{2} \frac{\partial U}{\partial \eta^3} \delta_{0\sigma} \right] + 1. \left[ U_{\sigma1} - \left( \psi^3 \right)^3 \frac{\partial U}{\partial \eta^3} \delta_{01} \right] \right\} + \sum_{n=1}^{\infty} \varepsilon^n \left[ \left( \psi^3 \right)^n U_{\sigma1}^\lambda \delta_{\lambda3} \right] + \left( \psi^3 \right)^n U_{\sigma1}^\lambda \delta_{\lambda3} \right\}
\]

(continued)
\[ \begin{align*}
+ \left( \gamma^3 \right)^{n-1} U_{13}^\alpha \left( \eta^{\alpha} \gamma_{13} \right) + \left( \gamma^3 \right)^{n-1} U_{13}^\alpha \left( \gamma^{\alpha} \right) \left( \gamma_{03} \gamma^3 \right) - \left( \gamma^3 \right)^{n} U_{13}^\alpha \left( \gamma^{\alpha} \right) \left( \gamma_{03} \gamma^3 \right) \\
+ \left( \gamma^3 \right)^{n} U_{13}^\alpha A_{0\gamma 13} + \left( \gamma^3 \right)^{n-1} U_{13}^\alpha B_{0\gamma 13} \end{align*} \]

where \( A_{0\gamma 13} \) and \( B_{0\gamma 13} \) are tensors given by

\[ A_{0\gamma 13} = \frac{1}{2} \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{013\beta} - \frac{1}{2} \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{01}\beta \gamma \]  \hspace{1cm} (3.33)

\[ A_{0\gamma 13} = \frac{1}{2} \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{013\beta} - \frac{1}{2} \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{01}\beta \gamma + \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{01}\beta \gamma \]  \hspace{1cm} (3.34)

\[ A_{0\gamma 13} = \frac{1}{2} \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{013\beta} - \frac{1}{2} \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{01}\beta \gamma + \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{01}\beta \gamma \]  \hspace{1cm} (3.35)

\[ B_{0\gamma 13} = \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{013\beta} \]  \hspace{1cm} (3.36)

\[ B_{0\gamma 13} = \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) G_{013\beta} + \sum_{r=1}^{n} \left( \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) \left( \gamma_{03} \gamma^3 \right) - \left( \gamma^{\alpha} \gamma_{03} \gamma^3 \right) \left( \gamma_{03} \gamma^3 \right) \right) \]  \hspace{1cm} (n = 2, 3, ...)

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In a similar fashion we may calculate the derivatives found on the following pages.

\[ \mu, \sigma = \frac{\eta_0}{L_0} \left\{ \frac{1}{2} \delta \gamma_{\lambda} \gamma_{\gamma \lambda} - \frac{1}{2} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} - \frac{1}{2} \gamma_{\lambda} \gamma_{\gamma \lambda \beta \gamma} \right. \\
+ \frac{1}{2} \frac{\partial g^3}{\partial \gamma_{\lambda} \gamma_{\gamma \lambda \beta}} + 1 \cdot \left[ U^3_{\lambda \gamma} - (\eta^3) U^3_{\lambda \gamma} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} - (\eta^3) U^3_{\lambda \gamma} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} \right] \\
+ (\eta^3) U^3_{\lambda \gamma} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} + (\eta^3) \frac{\partial g^3}{\partial \gamma_{\lambda} \gamma_{\gamma \lambda \beta}} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} - \frac{1}{2} \gamma^3_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} \right\} \\
+ \sum_{n=1}^{\infty} \frac{\eta^n}{n!} \left[ - (\eta^3)^n U^3_{\lambda \gamma} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} + (\eta^3)^{n+1} U^3_{\lambda \gamma} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} \right] \\
- (\eta^3)^n U^3 \left( \frac{1}{2} \gamma_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} + \gamma_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} \right) \right\} \\
(3.38) \]

with

\[ \gamma_{\lambda} \gamma_{\gamma \lambda \beta} = \frac{1}{2} \gamma_{\lambda \gamma \lambda \beta} - \frac{1}{2} \gamma_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} - \gamma_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} \]  
(3.39)

\[ \gamma_{\lambda} \gamma_{\gamma \lambda \beta} = \frac{1}{2} \gamma_{\lambda \gamma \lambda \beta} - \frac{1}{2} \gamma_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} + \gamma_{\lambda \gamma \lambda \beta} \gamma_{\lambda} \gamma_{\gamma \lambda \beta} \]  
(3.40)
\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]

\[ \epsilon^n \left[ (\theta Y_3) \right. \left. \right] \]

\[ \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \]

\[ U_{\gamma \beta}^{\alpha} = - \frac{M_0}{L_0^2} \left\{ \left[ \frac{2}{\partial (\theta Y_3)^2} U_1^{\alpha} \right] + \sum_{n=1}^{\infty} \epsilon^n \left[ 2(\theta Y_3)^{n-1} \frac{\partial U_1^{\lambda}}{\partial Y_3} \right] \right\} \]
\[ E_{\kappa \lambda}^{\alpha} = (n-1) \sum_{\alpha \beta \gamma} \frac{\partial \alpha \beta \gamma}{\partial \kappa \lambda \mu} + \sum_{\alpha \beta \gamma} \frac{\partial \alpha \beta \gamma}{\partial \kappa \lambda \mu} \sum_{\alpha \beta \gamma} \frac{\partial \alpha \beta \gamma}{\partial \kappa \lambda \mu} \] 

\[ (n=2, 3, \ldots) \] 

\[ \mu_3, \kappa_3 = \mu_0, \lambda_3 = \frac{\mu_0}{L_0} \left\{ \frac{1}{\varepsilon} \left[ -\frac{1}{2} \frac{2\nu^3}{\gamma} G_{\lambda \alpha \lambda \alpha} \right] + \varepsilon \frac{3}{2\gamma} U_3 \lambda_3 + \left( \frac{\nu}{\gamma} \right)^{3/2} G_{\kappa \lambda \kappa \lambda} \right\} + \sum_{n=1}^{\infty} \varepsilon^n \left[ \left( \frac{\nu}{\gamma} \right)^{n-1} U_3 \lambda_3 \sum_{\alpha \beta \gamma} \frac{\partial \alpha \beta \gamma}{\partial \kappa \lambda \mu} \right] + \left( \frac{\nu}{\gamma} \right)^n U_3 \left( \frac{1}{2} \sum_{\alpha \beta \gamma} \frac{\partial \alpha \beta \gamma}{\partial \kappa \lambda \mu} + \sum_{\alpha \beta \gamma} \frac{\partial \alpha \beta \gamma}{\partial \kappa \lambda \mu} \right) \] 

\[ \mu_3, \alpha_3 = \frac{\mu_0}{L_0} \left\{ \frac{\partial^2 U_3}{(2\gamma^3)^2} \right\} \] 

The above results may now be used in writing the equations of motion (2.7) in a dimensionless form. We let \( \mu_0, \rho_0, \) and \( f_0 \) be representative values of viscosity, density, and body force per unit mass, respectively, and introduce the dimensionless quantities

\[ \tau = \frac{\mu_0}{L_0} \frac{u}{l}, \quad \rho = \frac{h_0^2}{\mu_0 u_0 l_0}, \quad R = \frac{\rho_0 u_0 h_0^2}{\mu_0 L_0} \] 

\[ F^\alpha = \frac{f^\alpha}{f_0}, \quad F^3 = \frac{f^3}{f_0}, \quad \phi = \frac{h_0^2 \rho_0 f_0}{\mu_0 u_0} \] 

\[ \tau = \mu_0 \left[ 1 + K \mu \bar{u} \right], \quad \rho = \rho_0 \left[ 1 + K \rho \bar{\rho} \right] \] 

In addition, let
where \( \bar{\mu} \) and \( \bar{\rho} \) are dimensionless functions of pressure and temperature and where \( K_\mu \) and \( K_\rho \) are dimensionless constants. There is evidently a freedom of choice available for \( K_\mu \) and \( K_\rho \), since only the combinations \( K_\mu \bar{\mu} \) and \( K_\rho \bar{\rho} \) are directly defined by (3.50). We assume in the following that these constants have been so adjusted in any particular problem that the functions \( \bar{\mu} \) and \( \bar{\rho} \) are approximately of unit magnitude at the point of their largest value. If \( \mu \) and \( \rho \) remain relatively close to the constant values \( \mu_0 \) and \( \rho_0 \), as is the case in many practical problems, the constants \( K_\mu \) and \( K_\rho \) are then small in comparison with unity. It will become clear at a later stage of the investigation that the degree of arbitrariness in the choice of \( K_\mu \) and \( K_\rho \) has no effect on the analytical or numerical results derived by their use. The introduction of these auxiliary parameters is for the sole purpose of providing a convenient technique for setting up iterative procedures to take account of the effect of variable viscosity and density.

Finally, we shall write

\[
F^\alpha = \sum_{n=0}^{\infty} \varepsilon^n \left[ (\psi^3)^n F_{0\alpha}^\alpha \right] \quad F^3 = \sum_{n=0}^{\infty} \varepsilon^n [ (\psi^3)^n F_{0\alpha}^3 ] \quad (3.51)
\]

where the quantities \( F_{0\alpha}^i \) are functions of \( \psi^1 \) and \( \psi^2 \) only.

Considering first the equations of motion for the \( \psi^1 \) and \( \psi^2 \) directions, with \( i = \alpha \) in (2.7), we make the substitutions (3.49) to (3.51) and use our series expressions for the various covariant derivatives which occur in this equation. Performing
a few simplifications by means of the formulae for the various surface tensors, and arranging the results according to powers of the parameters $\varepsilon, \phi, K_\mu, K_\rho, \text{ and } R$, we then obtain the following equation:

\[
\frac{\partial^2 U^\alpha}{(2\eta^2)^2} - G^\alpha_{\omega\rho} P_{\rho\beta} = \varepsilon \left\{ - \frac{2U^\kappa}{2\eta^3} \Gamma^\kappa_{\mu\rho} - 2 \frac{2U^\lambda}{2\eta^3} \Gamma^\lambda_{\mu\rho} \lambda + (\eta^2) G^\alpha_{\omega\rho} P_{\rho\beta} \right\}
\]

\[+ K_\mu \left\{ - \frac{2U^\kappa}{(2\eta^2)^2} - \frac{2U^\lambda}{2\eta^3} \right\} + \phi \left\{ - F^\alpha_{\omega} \right\} + R \left\{ \frac{\partial V^\alpha}{\partial \zeta} \right\}
\]

\[- 2(\eta^2) \frac{2U^\lambda}{2\eta^3} \Gamma^\lambda_{\mu\rho} - U^\lambda \left( \Gamma^\kappa_{\mu\rho} \Gamma^\kappa_{\lambda\rho} \right) + \frac{1}{3} \Gamma^\kappa_{\mu\rho} \Gamma^\kappa_{\lambda\rho} - \frac{1}{3} G_{\alpha\beta} G_{\omega\rho}(\eta^2)
\]

\[+ (\eta^3) G^\alpha_{\omega\beta} P_{\rho\beta} \right\} + \varepsilon K_\mu \left\{ - \frac{L}{2} \frac{2U^\kappa}{2\eta^3} G^\beta_{\omega\rho} G_{\omega\rho\sigma}
\]

\[+ 2 \frac{2U^\lambda}{2\eta^3} \left( \Gamma^\lambda_{\mu\rho} \right) \right\} + \varepsilon F^\alpha_{\omega} \left\{ - (\eta^3) F^\alpha_{\omega} \right\} + \varepsilon R \left\{ (\eta^3) U^\beta U^\tau \Gamma^\tau_{\omega\rho} \right\}
\]

\[+ K_\rho \phi \left\{ - \phi \right\} + K_\rho R \left\{ \frac{\partial V^\alpha}{\partial \zeta} \right\} + O(\varepsilon^3, \varepsilon^2 K_\mu, \varepsilon^2 \phi, \varepsilon^2 R, \varepsilon K_\rho \phi, \varepsilon K_\rho R)
\]

(3.52)
Here we have employed the abbreviated notation

\[ \frac{dU^\alpha}{d\tau} = \frac{\partial U^\alpha}{\partial \tau} + U^\beta U^\alpha_{\beta} + U^3 \frac{\partial U^\alpha}{\partial \eta^3} \]  
(3.53)

We note that the operator \( \frac{d}{d\tau} \) is nonlinear.

In a similar manner the equation of motion in the \( \eta^3 \) direction becomes simply

\[ \frac{\partial P}{\partial \eta^3} = \varepsilon^2 \left\{ \frac{2}{3} \frac{U^3}{(\partial \eta^3)^2} + \frac{1}{3} \frac{\partial^2}{\partial \eta^3} U^\beta_{\eta^3} \right\} + \varepsilon \phi \left\{ \frac{\partial^3 \phi}{\partial \eta^3} \right\} + \varepsilon R \left\{ \frac{1}{2} U^\sigma U^\tau \sigma_{\eta^3} \right\} + O(\varepsilon^3, \varepsilon^2 \kappa, \varepsilon^2 \Phi, \varepsilon^2 R, \varepsilon \phi \kappa, \varepsilon^2 R \kappa) \]  
(3.54)

and the continuity equation (2.9) assumes the form

\[ U^\alpha_{\eta^3} + \frac{\partial U^3}{\partial \eta^3} = \varepsilon \left\{ -\frac{1}{4} U^3 \frac{\partial}{\partial \eta^3} U^\eta_{\eta^3} \right\} - U^3 \frac{\partial U^\eta_{\eta^3}}{\partial \eta^3} \} + \kappa \left\{ \left\{ -\frac{\partial^2 P}{\partial \eta^3} \right\} - \rho \left\{ U^\alpha_{\eta^3} + \frac{\partial U^3}{\partial \eta^3} \right\} \right\} + \varepsilon \left\{ -\frac{1}{4} U^3 \frac{\partial}{\partial \eta^3} U^\eta_{\eta^3} \right\} - \varepsilon^2 \frac{\partial U^3}{\partial \eta^3} \]  
(3.55)

where

\[ \frac{\partial \rho}{\partial \tau} = \frac{\partial \rho}{\partial \eta^3} + U^\alpha \frac{\partial \rho}{\partial \eta^3} + U^3 \frac{\rho}{\partial \eta^3} \]  
(3.56)
Equation (3.55) may be somewhat simplified by eliminating the quantity \(-\varepsilon (U_{1Q} \varepsilon + 2U_{3})\) from the term in \(K_{p}\) on the right. We thereby obtain

\[
U_{1Q} \varepsilon + \frac{2U_{3}}{\varepsilon} = \varepsilon \left\{ (q^{3} U_{1} \varepsilon)_{1Q} + U_{3} \varepsilon \right\} + K_{p} \left\{ \frac{d\varepsilon}{dt} \right\} \\
+ \varepsilon^{2} \left\{ - \frac{(q^{3})^{2} U_{1} \varepsilon}{(q^{3})^{2}} + \frac{(q^{3}) U_{3} \varepsilon}{(q^{3})^{2}} \right\} + K_{p}^{2} \left\{ \frac{d\varepsilon}{dt} \right\} \\
+ O(\varepsilon^{3}, \varepsilon^{2} K_{p}, \varepsilon K_{p}^{2}, K_{p}^{3})
\]

(3.57)

Slight simplifications may also be made in the \(\varepsilon^{2}\) terms of (3.52) and (3.54), since

\[
\varepsilon^{2} \left\{ - \frac{1}{3} G_{\alpha \beta} (U_{1Q} \varepsilon + \frac{2U_{3}}{\varepsilon}) \right\} = \varepsilon^{2} \left\{ - \frac{1}{3} G_{\alpha \beta} (U_{1Q} \varepsilon + \frac{2U_{3}}{\varepsilon}) \right\} \varepsilon \beta \\
= O(\varepsilon^{3}, \varepsilon^{2} K_{p})
\]

(3.58)

and

\[
\varepsilon^{2} \left\{ \frac{4}{3} \frac{U_{3}^{2}}{(q^{3})^{2}} + \frac{1}{3} \frac{2}{(q^{3})^{2}} U_{1Q} \varepsilon \right\} = \varepsilon^{2} \left\{ \frac{4}{3} \frac{U_{3}^{2}}{(q^{3})^{2}} \right\} + \varepsilon^{2} \frac{2}{3} \frac{2}{(q^{3})^{2}} (U_{1Q} \varepsilon + \frac{2U_{3}}{\varepsilon}) \\
= \varepsilon^{2} \left\{ \frac{2^{2} U_{3}^{2}}{(q^{3})^{2}} \right\} + O(\varepsilon^{3}, \varepsilon^{2} K_{p})
\]

(3.59)
Before discussing the heat flow equation (2.20) we wish to obtain expressions for the quantities \( \rho \frac{\partial E}{\partial \rho} \) and \( \rho \frac{\partial E}{\partial T} \) occurring in this equation. The temperature \( T \) and pressure \( p \) will generally be measured from some reference levels \( T_r \) and \( p_r \), such as room temperature and atmospheric pressure. The absolute temperature and pressure will then be \( (T + T_r) \) and \( (p + p_r) \), respectively. From thermodynamic relations we then have

\[
\left( \frac{\partial E}{\partial \rho} \right)_T = -\left( T + T_r \right) \left( \frac{\partial V}{\partial T} \right)_\rho - \left( \rho + \rho_r \right) \left( \frac{\partial V}{\partial \rho} \right)_T \tag{3.60}
\]

\[
\left( \frac{\partial E}{\partial T} \right)_\rho = -\left( \rho + \rho_r \right) \left( \frac{\partial V}{\partial T} \right)_\rho + C \tag{3.61}
\]

where \( V \) is the specific volume and \( C \) the specific heat at constant pressure.

Let \( k_0 \) and \( C_{p_0} \) be representative values of thermal conductivity and specific heat, and let \( T_0 \) be a representative value of the change or variation in temperature. We then define the dimensionless quantities

\[
\Theta = \frac{T}{T_0} \quad \Theta_r = \frac{T_r}{T_0} \quad \rho_r = \frac{k_0}{\rho_0 \mu_0 \omega_0} \frac{h_r}{\mu_0 \omega_0 L_0} \tag{3.62}
\]

\[
\bar{C}_\rho = \frac{C_{\rho}}{C_{\rho_0}} \quad k = k_0 \left[ 1 + k_0 \frac{T_r}{T_0} \right]
\]

\[
\lambda = \frac{\rho_0 \mu_0 h_0^2 C_{\rho_0}}{k_0 L_0} \quad \eta = \frac{\mu_0 \omega^2}{k_0 T_0}
\]
As in the definition (3.50) for $\bar{\rho}$ and $\bar{\rho}$, the constant $K_k$ in the above equation is so adjusted that $K = O(1)$. Employing the relation $V = \frac{1}{\bar{\rho}}$ together with our previous expression (3.50) for $\rho$, we find from (3.60) and (3.61),

$$
\rho \left( \frac{\partial E}{\partial t} \right)_T = \frac{K \rho}{1 + K \rho \bar{\rho}} \left\{ (\theta + \theta_r) \frac{\partial \rho}{\partial \theta} + (p + p_r) \frac{\partial \rho}{\partial p} \right\} 
$$

(3.63)

$$
\rho \left( \frac{\partial E}{\partial t} \right)_p = \frac{\mu \mu_{\infty} \rho \theta_0}{h \theta_0 \theta_0} \left\{ \frac{\lambda}{\eta} C \bar{\rho} + K \rho \left[ \frac{\lambda}{\eta} \bar{\rho} \bar{C} \rho + \frac{p + p_r}{1 + K \rho \bar{\rho}} \frac{\partial \rho}{\partial \theta} \right] \right\}
$$

(3.64)

Introducing the further definitions

$$
\begin{align*}
\frac{d \theta}{d t} &= \frac{\theta}{\eta} + U \theta \theta_{1 \alpha} + U^3 \frac{\partial \theta}{\partial \eta^3} \\
\frac{d \rho}{d \eta} &= \frac{\rho}{\eta} + U \rho \rho_{1 \alpha} + U^3 \frac{\partial \rho}{\partial \eta^3}
\end{align*}
$$

(3.65)

we may now express the heat flow equation (2.20) in the following form:

$$
\frac{\partial^2 \theta}{\partial \eta^3} \theta^2 = \varepsilon \left\{ - \frac{\partial \theta}{\partial \eta^3} \rho_{1 \alpha} \right\} + \eta \left\{ - C \theta \theta_{1 \alpha} \frac{\partial \theta}{\partial \eta^3} \right\}
$$

$$
+ \lambda \left\{ C \frac{d \theta}{d \eta} \right\} + K_k \left\{ - \frac{\partial \theta}{\partial \eta^3} \theta^2 - \frac{2 \theta}{\partial \eta^3} \frac{\partial \theta}{\partial \eta^3} \right\}
$$

(continued)
\[ + \varepsilon^2 \left\{ - G_{\alpha\beta} \Theta_{\alpha\beta} - (\varepsilon^3) \frac{\partial \Theta}{\partial y^3} \right\} + \varepsilon \eta \left\{ (\varepsilon^3) \frac{\partial U}{\partial y^3} \right\} \]

\[ + \varepsilon K_k \left\{ - k \frac{\partial \Theta}{\partial y^3} \right\} + \eta K^n \left\{ - \frac{\partial U}{\partial y^3} \right\} \]

\[ + \eta K_p \left\{ (\gamma + \xi) \frac{\partial \rho}{\partial \xi} + \frac{D}{\mu} \frac{d \Theta}{d \xi} \right\} + \lambda K_p \left\{ \frac{\partial C}{\partial t} \right\} \]

\[ + \mathcal{O}(\varepsilon^3, \varepsilon^2 \eta, \varepsilon \eta K_k, \varepsilon \eta K^n, \eta K_p^2) \] (3.66)

In concluding this section it may be remarked that many of the developments contained herein are simply special applications of the Theory of Induction, which is well known in tensor analysis but which was not known to the author at the time these derivations were undertaken.
Section 4, Boundary Conditions:

In preparation for the analysis of Section 5, in which the solution of the three dimensional partial differential equations (3.52), (3.54), (3.57), and (3.66) is represented in terms of two dimensional quantities, it is necessary to examine in detail the boundary conditions which must be imposed on these equations. As a first step, we define the region in space which is to be isolated for study. The boundary surfaces $S_1$ and $S_2$, and the reference surface $S$, may be supposed given. The surfaces $S_1$ and $S_2$ may move in an arbitrary manner; however, they are not allowed to come into contact with each other. The reference surface $S$ is assumed to be permanently fixed in space.

![Figure 2a](image-url)
Figure 2b

We now draw a boundary curve $B$ lying in $S$, as indicated in Figure 2a, and construct the normals to $S$ at all points of $B$. We select as the volume of fluid to be studied that which is bounded by the surfaces $S_1$ and $S_2$, and about the edges by the normals to $S$ on $B$. This volume is indicated schematically by the shaded area in Figure 2b.

We first consider the boundary conditions which are to be satisfied on $S_1$ and $S_2$. The motion of $S_1$ and $S_2$ may be expressed by velocity vectors having contravariant components $N_{(1)}^i$ and $N_{(2)}^i$, respectively, in the $X^i$ coordinates. We shall write

$$V_{(1)}^i = \frac{N_{(1)}^i}{\mu_0} \quad V_{(2)}^i = \frac{N_{(2)}^i}{\mu_0} \quad \quad (4.1)$$

and may consider $V_{(1)}^i$ and $V_{(2)}^i$ to be prescribed functions of $\Psi'$, $\Psi^\wedge$, and $\tau$. Since the fluid is assumed to adhere to the surfaces, the boundary conditions to be satisfied by $U^\kappa$ and $U^3$ are then
\[ U^\alpha = U^\alpha_{1}, \quad (\Psi^3 = H_{11}) \]
\[ U^\alpha = U^\alpha_{2}, \quad (\Psi^3 = H_{12}) \]  
\[ U^3 = U^3_{1}, \quad (\Psi^3 = H_{11}) \]
\[ U^3 = U^3_{2}, \quad (\Psi^3 = H_{12}) \]  

where \( \Psi^3 = H_{11} \) and \( \Psi^3 = H_{12} \) are the equations defining \( S_1 \) and \( S_2 \), as given by (3.2) and (3.7).

To provide boundary conditions on \( \Theta \), it will be assumed that the temperature of the walls immediately beneath a microscopically thin surface layer is prescribed as a function \( \Theta^1 \), \( \Theta^2 \), and \( \Theta \) on each surface. As indicated in Section 1, heat conduction problems in the surrounding material will be neglected. We shall, however, assume that the surfaces \( S_1 \) and \( S_2 \) possess surface heat transfer coefficients \( k_{S_1} \) and \( k_{S_2} \) such as might be associated with a thin surface film of metallic oxide or other impurity of a relatively low thermal conductivity. The quantities \( k_{S_1} \) and \( k_{S_2} \) will be defined as the rate of flow of heat per unit surface area, per unit temperature differential between the fluid immediately adjacent to the surface and the material of the boundary immediately below the surface film. In the applications, \( k_{S_1} \) and \( k_{S_2} \) may be zero, infinite, or may possess some intermediate values.

Let \( \mathbf{n}_{1i} \) and \( \mathbf{n}_{2j} \) be the covariant components of the unit normal vectors \( \mathbf{n}_{1} \) and \( \mathbf{n}_{2} \) to the surfaces \( S_1 \) and \( S_2 \). These vectors are chosen to be directed away from the region occupied by the fluid, as shown in Figure 2b. Then in the \( X^1 \) coordinates
and corresponding notation the rate at which heat is lost from the fluid across an area $dS_1$ of $S_1$ is

$$-k \varrho^{ij} T_{ij} n_{1i} \, dS_1,$$

where $k$, $\varrho^{ij}$, and $T$ are to be evaluated for the portion of the fluid at the surface. However, the heat conducted across the area $dS_1$ of the surface film is also given by

$$k_{s1} (T - T_{in}) \, dS_1,$$

where $T_{in}$ is the prescribed temperature beneath the film. Thus, by equating these two expressions we find as the required boundary condition to be satisfied by the temperature $T$

$$-k \varrho^{ij} T_{ij} n_{1i} = k_{s1} (T - T_{in}) \quad (4.4)$$

and at $S_2$ we have a similar equation

$$-k \varrho^{ij} T_{ij} n_{2i} = k_{s2} (T - T_{in}) \quad (4.5)$$

From the equations (3.2) defining the surfaces, we may readily derive the components of $n_{1i}$ and $n_{2i}$, obtaining

$$n_{1i\alpha} = \frac{\varepsilon L \alpha \varrho H_{1i,\alpha}}{\sqrt{1 + \varepsilon^2 L^2 \varrho^{ij} \varrho H_{1i,}\beta H_{1i,\beta}}}$$

$$n_{1i3} = \frac{-\varepsilon}{\sqrt{1 + \varepsilon^2 L^2 \varrho^{ij} \varrho H_{1i,}\beta H_{1i,\beta}}}$$

$$n_{2i3} = \frac{-\varepsilon}{\sqrt{1 + \varepsilon^2 L^2 \varrho^{ij} \varrho H_{2i,}\beta H_{2i,\beta}}}$$

$$n_{2i\alpha} = \frac{\varepsilon L \alpha \varrho H_{2i,\alpha}}{\sqrt{1 + \varepsilon^2 L^2 \varrho^{ij} \varrho H_{2i,}\beta H_{2i,\beta}}}$$
\[ n_{(2)} \alpha = \frac{-\mathcal{E} L_0 H_{(2)}, \alpha}{\sqrt{1 + \mathcal{E}^2 L_0^2 \mathcal{G} \mathcal{R} H_{(2)}, \beta H_{(2)}, \gamma}} \]
\[ n_{(2)} \gamma = \frac{\mathcal{E}}{\sqrt{1 + \mathcal{E}^2 L_0^2 \mathcal{G} \mathcal{R} H_{(2)}, \beta H_{(2)}, \gamma}} \]  

(4.6)

Here the tensor \( \mathcal{G} \mathcal{R} \) is to be evaluated at the surface in question.

Finally, we define the dimensionless quantities

\[ \bar{k}_{s_1} = \frac{k_{s_1} h_o}{k} \]
\[ \bar{k}_{s_2} = \frac{k_{s_2} h_o}{k} \]  

(4.7)

and note that these are simply ratios of the surface heat transfer coefficients \( k_{s_1} \) and \( k_{s_2} \) to the overall heat transfer coefficient for a fluid film of thickness \( h_o \). The \( k \) of the denominator in each case is to be evaluated for fluid at the temperature and pressure corresponding to that prevailing at the appropriate surface; in view of the uncertainty as to the nature and amount of surface film in any given practical case, together with a marked lack of experimental data regarding the behavior of the resulting surface heat transfer coefficient with temperature and pressure, it would appear quite reasonable to write \( k = k_o \) in (4.7).

Transforming (4.4) into the \( Y^1 \) coordinates and the associated dimensionless notation, we find by the aid of (4.6) and (4.7) for \( Y^3 = H_{(1)} \)
\[ \frac{\partial \theta}{\partial \eta^2} - k_{s1} \theta = - k_{s1} \theta_{(1)} \]

\[ \quad + \varepsilon^2 \left[ G_{(2)}^{\eta^2 \eta^2} \left( \theta_{(\eta)} + \frac{1}{2} \frac{\partial \theta}{\partial \eta^3} H_{(1)}(\eta) \right) \right] + O(\varepsilon^3) \] (4.8)

and similarly for \( Y^3 = H_{(2)} \)

\[ \frac{\partial \theta}{\partial \eta^3} + k_{s2} \theta = - k_{s2} \theta_{(2)} \]

\[ \quad + \varepsilon^2 \left[ G_{(2)}^{\eta^2 \eta^3} \left( \theta_{(\eta)} + \frac{1}{2} \frac{\partial \theta}{\partial \eta^3} H_{(2)}(\eta) \right) \right] + O(\varepsilon^3) \] (4.9)

where

\[ \theta_{(1)} = \frac{T_{(1)}}{T_0} \quad \theta_{(2)} = \frac{T_{(2)}}{T_0} \] (4.10)

The quantities \( k_{s1}, k_{s2}, \theta_{(1)}, \theta_{(2)}, H_{(1)}, \) and \( H_{(2)} \) are here to be considered as prescribed functions of \( Y^1, Y^2, \) and \( \tau \).

In accordance with the discussion of Section 1, edge effects will be neglected, and only average values of pressure or velocity will be specified along the normals to the boundary curve \( B \). In order to formulate the required conditions analytically, let \( \Phi \) denote the arc length of \( \mathbb{G} \), measured from an arbitrary fixed point on this curve. We shall designate by \( P(\Phi, Y^3, \tau) \) the value of \( P \) at points lying on the normal to \( S \) drawn through the point of \( \Phi \) of \( \mathbb{G} \). Then the average pressure along this normal
between \( S_1 \) and \( S_2 \) is

\[
\frac{1}{H} \int_{H_{i1}}^{H_{i2}} P(\sigma_0, \psi^3, \tau) \, d\psi^3
\]

The required boundary condition is therefore

\[
\frac{1}{H} \int_{H_{i1}}^{H_{i2}} P(\sigma_0, \psi^3, \tau) \, d\psi^3 = P_0(\sigma_0, \tau)
\]  \hspace{1cm} (4.11)

where \( P_0(\sigma_0, \tau) \) is a specified function of its arguments.

Although the type of boundary condition represented by (4.11) is the most common in practice, two others may be considered here for future reference. The first of these specifies that the rate at which fluid leaves the region interior to \( \mathcal{B} \) is to be a given function of the arc \( \mathcal{A}_0 \). The second, arising in connection with the intersection of two regions such as are shown in Figure 4, specifies the continuity of pressure and fluid flow at the common boundary.

To formulate the boundary condition of the former type, let \( \overline{\mathbf{n}}(\mathcal{A}) \) be the unit vector which lies in \( S \), which is

![Diagram](image)

**Figure 3**

75
normal to $\mathcal{B}$, and which is directed away from the interior of $\mathcal{B}$ as indicated in Figure 3. Let the (dimensionless) covariant components of this vector be denoted by $\eta_{(\mathcal{B})\alpha}$. The rate of flow of fluid mass across the element of area defined by the normals to $S$ based upon $dA_{\mathcal{B}}$ is

$$
dA_{\mathcal{B}} \int_{H_{ho}}^{H_{ho}} (\rho u \alpha) \eta_{(\mathcal{B})\alpha} (h_0 d\ell^3)
$$

The rate of fluid flow per unit length of arc is obtained by dividing this expression by $dA_{\mathcal{B}}$.

Noting that $\eta_{(\mathcal{B})\alpha}$ does not depend upon $\ell^3$, we may then write the required boundary condition in the dimensionless form

$$
\int_{H_{ho}}^{H_{ho}} (1 + K \eta) U^3 d\ell^3 = \frac{m(4_{\mathcal{B}}, \ell)}{\rho \bar{u}_0 h_0}
\tag{4.12}
$$

where $m(4_{\mathcal{B}}, \ell)$ is the rate of flow of mass outward across $\mathcal{B}$ per unit arc length, and is a specified function.

The boundary conditions appropriate to the line of intersection between two connecting regions, as shown in Figure 4, may be readily obtained from (4.11) and (4.12).

Figure 4
Distinguishing the quantities pertaining to the two domains by single and double primes, respectively, we let the common boundary curve $\mathcal{B}$ be the intersection between the two reference surfaces $S'$ and $S''$, and construct the normals $\vec{n}_{(H',H''},\vec{n}_{(H')}$ as shown.

As the first boundary condition we write

$$\frac{1}{H'} \int_{H''}^{H'} \! \! \! \! P' d\gamma^3' = \frac{1}{H''} \int_{H''}^{H''} \! \! \! \! P'' d\gamma^3'' \quad (4.13)$$

A comparison with (4.11) shows that (4.13) is an expression of the statement that the pressure, as calculated from formulae pertaining to $S'$, has the same average value along the line $l'$ of Figure 4 as does the pressure pertaining to $S''$ averaged along $l''$. The second boundary condition, expressing the continuity of fluid flow across $\mathcal{B}$, may be written by analogy with (4.12) in the form

$$n_{(P') \alpha} \int_{H''}^{H'} \! \! \! \! (1 + K_P P') U^\alpha d\gamma^3' + n_{(P') \alpha} \int_{H''}^{H''} \! \! \! \! (1 + K_P P'') U^\alpha d\gamma^3'' = 0 \quad (4.14)$$

Neither (4.13) nor (4.14) is related to the actual pressure or velocity distribution in the neighborhood of $\mathcal{B}$. The quantities $P'$, $P''$, $U^\alpha'$, and $U^\alpha''$ are fictitious pressures and velocities calculated from simplified formulae which approximate the true values of these quantities only at points removed from $\mathcal{B}$, in accordance with our discussion of edge effects. The true nature of these conditions may be clarified by considering (4.13) and (4.14) as applying to the limiting configuration
of two separate boundary curves $\mathcal{B}'$ and $\mathcal{B}''$ as shown in Figure 4, which are initially removed a sufficient distance from $\mathcal{B}$ so that edge effects may be neglected, and which are then allowed to approach each other. From this point of view, it is reasonable to suppose that these equations are essentially correct. A more refined analysis might replace (4.13) by an equation which made allowances for a small drop in pressure arising from the flow around the corner; however, a consideration of this phenomenon is beyond the scope of the present analysis.

We note at this point that no boundary conditions for the temperature $\Theta$ are given on $\mathcal{B}$. If the quantities $\varepsilon, \phi, \ldots, K_r$ are all small, as is assumed in the major part of this analysis, it is clear from (3.66) that $\Theta$ will be very nearly linear in $y^3$ and hence determined almost entirely by the local temperatures $\Theta_{10}$ and $\Theta_{20}$. The physical significance of this statement is that even though the fluid may be a relatively poor conductor of heat, the thickness of the fluid film is so small that under normal circumstances any differential in temperature carried into the system by the fluid crossing $\mathcal{B}$ is very rapidly adjusted by conduction to or from the boundary surfaces. The principal exceptions to this statement occur when the surfaces act as perfect insulators ($k_s = 0$), or when the heat convection ($\lambda$) term of (3.66) is not small. In general, any adjustment between the temperature of the fluid upon entry into the system and that of the walls may be regarded as an edge effect, at least as long as we retain
our present position that the temperature of the walls is to be considered a **specified** function of position and time.

We conclude this Section with the derivation of a convenient auxiliary relation which arises from the boundary conditions on $S_1$ and $S_2$. For any function $f(x^i, t)$, the rate of change of $f$ for a specified moving particle of fluid is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial t} + u^\alpha \frac{\partial f}{\partial x^\alpha} + u^3 \frac{\partial f}{\partial x^3}$$

In particular let $f$ be the distance of an arbitrary fluid particle from $S_1$ along a line perpendicular to $S$. Then

$$f = h_0 (\eta^3 - H_{\eta^3})$$

and we have

$$\frac{\partial f}{\partial t} = -h_0 \frac{\partial H_{\eta^3}}{\partial t} = -\varepsilon u_0 \frac{\partial H_{\eta^3}}{\partial \eta^3}$$

$$f_{,\alpha} = -h_0 H_{\eta^3, \alpha} = -\varepsilon H_{\eta^3, \alpha}$$

$$f_{,3} = \frac{1}{\ell_0} \frac{\partial f}{\partial \eta^3} = \varepsilon$$

$$u^\alpha = u_0 U^\alpha \quad u^3 = u_0 U^3$$
Combining these results,

$$
\frac{df}{dt} = \varepsilon \mu \left\{ - \frac{\partial H_{\alpha}}{\partial \zeta} - U^\alpha H_{\alpha i i} + U^3 \right\}
$$

For those fluid particles on \( S_1 \), \( f = 0 \) and \( \frac{df}{dt} = 0 \) by our assumption that the fluid adheres to the surface. Since for these particles \( U^\alpha = V_{(i)}^\alpha \) and \( U^3 = V_{(i)}^3 \), we find

$$
\frac{\partial H_{\alpha}}{\partial \zeta} = - V_{(i)}^\alpha H_{\alpha i i} + V_{(i)}^3
$$  \tag{4.15}

and similarly

$$
\frac{\partial H_{\alpha}}{\partial \zeta} = - V_{(i)}^\alpha H_{\alpha i i} + V_{(i)}^3
$$  \tag{4.16}
Section 5. Derivation of the General Two Dimensional Equations:

The development of Section 3 is quite general in nature and rests essentially upon the sole assumption that the fluid with which we are concerned obeys the basic tensor equations (2.7), (2.9), and (2.20). The convergence of the infinite series represented by the derived equations (3.52), (3.54), (3.57), and (3.66) depends only upon the convergence of the series (3.13) for $g^\alpha\beta$, which is well assured as long as the reference surface is analytic.

In the following, no attempt at rigor is made, and the validity of our results rests upon physical intuition. In the major part of the present section it is assumed that the parameters $\epsilon, \phi, k_\mu, k_\rho, R, \eta, \lambda,$ and $k_k$ are all small compared with unity, and that $U^\alpha, U^3, P,$ and $\Theta$ can be formally expanded in multiple power series in these parameters in such manner as to satisfy identically the necessary boundary conditions. Section 6 is then devoted to an examination of the physical order of magnitude of these parameters and the other dimensionless variables, with the aim of establishing bounds within which this series treatment can be expected to yield accurate practical results. The ultimate validity of this theory is measured by the degree to which it proves self-consistent, together with its ability to predict experimental results.

Before considering the results obtainable by this series technique, we wish to examine the nature of the main problem.
which has not yet been satisfactorily treated by this means. In the discussion of Section 6 it will be found that the parameters \( \varepsilon, \phi, K_\rho, R, \) and \( K_k \) may be regarded as small in almost all problems in the lubrication field. Since the viscosity of a lubricating fluid is highly dependent upon temperature, however, the rapid generation of heat within the lubricant film may be responsible for significant relative changes in viscosity that are large enough to invalidate our later approximations. The variation of the viscosity over the film thickness is a basic factor in determining the distribution of velocity, whereas the velocity distribution and viscosity themselves determine the rate of generation of heat. The velocity, temperature, and viscosity are therefore closely coupled to each other, and the solution of a system of non-linear differential equations is found to be necessary even to determine the distribution of velocity and temperature across the film thickness.

To examine this problem in some detail, let us set

\[
\varepsilon = \phi = K_\rho = R = K_k = 0
\]

in equations (3.52), (3.54), (3.57), and (3.66). Since \( K_\mu \) is no longer to be regarded as small, we shall simplify our notation in these equations by writing

\[
\tilde{\mu} = 1 + K_\mu \mu = \frac{\mu}{\mu_0} \quad (5.1)
\]
We then obtain

\[ \frac{\partial}{\partial \eta^3} \left\{ \tilde{\mu} \frac{\partial u^\alpha}{\partial \eta^3} \right\} = G^{\kappa \sigma}_{\alpha \beta} p, \beta \]  

(5.2)

\[ \frac{\partial p}{\partial \eta^3} = 0 \]  

(5.3)

\[ U^\alpha_{,\alpha} + \frac{\partial u^3}{\partial \eta^3} = 0 \]  

(5.4)

and

\[ \frac{\partial^2 \theta}{\partial \eta^3 \partial \eta^3} = -\eta \tilde{\mu} G^{\kappa \sigma \nu \mu}_{\alpha \beta \lambda \eta} \frac{\partial u^\alpha}{\partial \eta^3} \frac{\partial u^\nu}{\partial \eta^3} + \lambda \zeta \frac{\partial \theta}{\partial \eta^3} \]  

(5.5)

as the basic set of equations to be studied.

Henceforth, the term in \( \lambda \) will be neglected. Although there exist practical cases in which \( \lambda \) is small at the same time that \( \eta \) is large, the real reason for omitting the \( \lambda \) term is that the following analysis does not appear to be possible if it is included. The artificiality of this procedure may, perhaps, be excused on the ground that the neglect of this term enables us to gain an insight into the structure of the simplified problem which may later be employed to advantage in solving the more general case, by iterative techniques or otherwise.

From equation (5.3) it is seen that \( P \) is a function of \( Y^1 \) and \( Y^2 \) only, say

\[ P = \psi(y^1, y^2) \]  

(5.6)
The right hand member of (5.2) is thus independent of \( Y^3 \), and we may integrate this equation a first time to obtain

\[ \mu \frac{\partial \eta^\alpha}{\partial Y^3} = A^\alpha + (Y^3) G^{\alpha \beta}_{\omega \gamma} \psi_{\beta \gamma} \]  

(5.7)

where \( A^\alpha \) is an unknown vector function of \((Y^1, Y^2)\). Setting \( \lambda = 0 \), we may then write (5.5) in the form

\[ \frac{\partial^2 \Theta}{\partial Y^3 \partial Y^3} = -\frac{n}{\mu} \frac{\partial}{\partial Y^3} \left[ \mu \frac{\partial \eta^\alpha}{\partial Y^3} \right] \left[ \mu \frac{\partial \eta^\beta}{\partial Y^3} \right] \]

\[ = -\frac{n}{\mu} \frac{\partial}{\partial Y^3} \left[ A^\alpha \eta^\alpha + (Y^3) G^{\alpha \beta}_{\omega \gamma} \psi_{\beta \gamma} \right] \left[ A^\beta + (Y^3) G^{\beta \gamma}_{\omega \delta} \psi_{\gamma \delta} \right] \]

\[ = -\frac{n}{\mu} \left\{ A^\alpha A^\alpha + 2(Y^3) A^\alpha \psi_{\alpha \gamma} + (Y^3)^2 G^{\alpha \beta}_{\omega \gamma} \psi_{\beta \gamma} \psi_{\gamma \delta} \right\} \]

(5.8)

where the relation

\[ G^{\alpha \beta}_{\omega \gamma} G^{\beta \gamma}_{\omega \alpha} = \delta^\omega_\alpha \]

has been used, together with the definition

\[ A^\alpha = G^{\alpha \beta}_{\omega \gamma} A^\beta \]

With attention confined to a fixed point \((Y^1, Y^2)\), we may now consider (5.8) as an ordinary differential equation with \( \Theta \) as the dependent variable and \( Y^3 \) the independent variable.
The quantity $\hat{\kappa}$ is, in general, a function of $\Theta$ and $P$; however, since $P$ does not depend upon $\Psi^3$, $\hat{\kappa}$ may here be regarded as a function of $\Theta$ only, with $P (= \Psi)$ in the role of a parameter. The quantities $\alpha^\kappa A_\kappa$, $\alpha^\kappa \Psi_\kappa$, and $G_{\alpha\beta} \Psi_{\alpha} \Psi_{\beta}$ may also be considered as parameters, although their values (as well as that of $\Psi$ itself) are unknown.

Equation (5.8) is therefore a second order non-linear ordinary differential equation, and is to be solved under the boundary conditions

$$\frac{2\Theta}{2\Psi^3} - \bar{k}_s, \Theta = - \bar{k}_s, \Theta, (\Psi^3 = H_{\alpha 1})$$

$$\frac{2\Theta}{2\Psi^3} + \bar{k}_s, \Theta = \bar{k}_s, \Theta, (\Psi^3 = H_{\alpha 2})$$

which are derived from (4.8) and (4.9) by setting $\varepsilon = 0$. The solution to this differential system may be regarded as known, in principle, and may be written in the form

$$\Theta = \Theta (\Psi^3, A^\kappa A_\kappa, A^\kappa \Psi_\kappa, G_{\alpha\beta} \Psi_{\alpha} \Psi_{\beta}; \Psi)$$

where the dependence of $\Theta$ on $\Psi$ arises from the dependence of $\hat{\kappa}$ on the pressure, and where the dependence of $\Theta$ on the known quantities $\eta, \bar{k}_s, \bar{k}_s, \Theta_{(1)}, \Theta_{(2)}, H_{(1)}$ and $H_{(2)}$ has not been shown explicitly.

Since $\hat{\kappa} = \hat{\kappa}(\Theta, \Psi)$, the quantity $\frac{1}{\hat{\kappa}}$ is a function of...
of the same type as $\Theta$ itself; that is,

$$
\frac{1}{\mu} = \Phi (\eta^3; \mathbf{A}^\alpha \mathbf{A}_\alpha, \mathbf{A}^\alpha \mathbf{A}_\alpha, G_{\omega \beta} \mathbf{\Psi}^{\alpha} \mathbf{\Psi}^{\beta}, \Psi)
$$  \hspace{1cm} (5.9)

where $\Phi$ may be regarded as a known function of its (unknown) arguments. We then find from (5.7) and the boundary conditions (4.2)

$$
V_{\alpha} - V_{\alpha} = \int_{H_{\alpha}}^{H_{\alpha}} \frac{\partial V^\alpha}{\partial y^3} \, dy^3
$$

$$
= \int_{H_{\alpha}}^{H_{\alpha}} \left[ A^\alpha + (y^3) G_{\alpha \beta} \mathbf{\Psi}^{\beta} \right] \Phi \, dy^3
$$  \hspace{1cm} (5.10)

In (5.10) there are four unknown quantities; namely, $A^1$, $A^2$, $\Psi_{11}$, and $\Psi_{12}$. Two functional relations (for $\alpha = 1,2$) are provided between these quantities by this equation. It is thus possible, in principle, to solve (5.10) for $A^1$ and $A^2$ in terms of $(V_{\alpha} - V_{\alpha})$, $(V_{\alpha} - V_{\alpha})$, $\Psi_{11}$, $\Psi_{12}$, $\Psi$, and the known quantities $\eta$, $k_3$, $\ldots$ $H_{\alpha}$, and $H_{\alpha}$ listed previously. By an argument based on the tensor character of equation (5.10), it may be shown that the desired solution can be written in the form

$$
A^\alpha = V^\alpha F_1 (\mathbf{V}, \mathbf{V}, \mathbf{\Phi}, \mathbf{\Phi}', \Phi', \Psi)
$$

$$
+ G_{\omega \beta} \mathbf{\Psi}^{\alpha} F_2 (\mathbf{V}, \mathbf{V}, \mathbf{\Phi}, \mathbf{\Phi}', \Phi', \Psi)
$$  \hspace{1cm} (5.11)
where we have introduced the notation

\[ \nabla^\alpha = \nabla_{(12}^\alpha - \nabla_{(11}^\alpha \]

\[ \nabla \cdot \nabla = G_{\alpha \beta} \nabla^\alpha \nabla^\beta \]

\[ \nabla \cdot \nabla' = \nabla^\kappa \psi_{1 \alpha} \]

\[ \nabla' \cdot \nabla' = G_{\alpha \beta} \psi_{1 \alpha} \psi_{1 \beta} \]

The quantity \( \Phi \) may be expressed in terms of the same variables, with an additional dependence on \( y^3 \).

Equations (5.7), (5.9), and (5.11) now yield

\[ \frac{\partial u^\kappa}{\partial y^3} = \nabla^\kappa \vec{F}, \Phi + G^{\alpha \beta}_{\alpha \beta} \psi_{1 \alpha} \vec{F}_2 \Phi + (y^3) G_{\alpha \beta} \psi_{1 \beta} \Phi \]

Writing

\[ C_0(y^3) = \left\{ \int_{U_{10}} \Phi \, dy^3 \right\} \]

\[ C_1(y^3) = \left\{ \int_{U_{10}} \Phi \, dy^3 \right\} \]

\[ (5.12) \]
we then find

\[ U^\alpha = V^\alpha_{\lambda\mu} + (V^\alpha F_1 + G^{\alpha\beta}_{\lambda\mu} \gamma_{1\beta} F_2) C_0 (\psi^3) \]

\[ + G_{\alpha\beta}^{\lambda\mu} \gamma_{1\beta} C, (\psi^3) \]

Also

\[ \int_{H_{1\alpha}}^{H_{1\beta}} U^\alpha d\psi^3 = H V^\alpha_{\lambda\mu} + \left[ V^\alpha F_1 + G^{\alpha\beta}_{\lambda\mu} \gamma_{1\beta} F_2 \right] \int_{H_{1\alpha}}^{H_{1\beta}} C_0 (\psi^3) d\psi^3 \]

\[ + G_{\alpha\beta}^{\lambda\mu} \gamma_{1\beta} \int_{H_{1\alpha}}^{H_{1\beta}} C, (\psi^3) d\psi^3 \]

(5.13)

We note that as in the case of differentiation, the operation of integration with respect to \( \psi^3 \) leaves the two dimensional tensor character of a set of quantities unaltered.

Now, by the definition of two dimensional covariant differentiation,

\[ \left\{ \int_{H_{1\alpha}}^{H_{1\beta}} U^\alpha d\psi^3 \right\}_{\lambda\mu} = \frac{\partial}{\partial \psi^3} \left\{ \int_{H_{1\alpha}}^{H_{1\beta}} U^\alpha d\psi^3 \right\} + \left\{ \frac{\alpha}{\lambda\mu} \right\}_{\alpha} \int_{H_{1\alpha}}^{H_{1\beta}} U^\alpha d\psi^3 \]

Using the boundary conditions (4.2), the fact that \( \left\{ \frac{\alpha}{\lambda\mu} \right\}_{\alpha} \) does not depend on \( \psi^3 \), and the equations (4.15) and (4.16) for \( \frac{\partial}{\partial \psi^3} \) and \( \frac{\partial H_{1\beta}}{\partial \psi^3} \), this relation becomes

\[ \left\{ \int_{H_{1\alpha}}^{H_{1\beta}} U^\alpha d\psi^3 \right\}_{\lambda\mu} = H_{1\beta} \left[ U^\alpha \right]_{\psi^3 = H_{1\beta}} - H_{1\alpha} \left[ U^\alpha \right]_{\psi^3 = H_{1\alpha}} \]

(continued)
\[ + \int_{H_{i,j}}^{H_{i,j}^{H(2)}} \frac{\partial U}{\partial y^3} \, d\eta^3 + \int_{H_{i,j}}^{H_{i,j}^{H(2)}} \left\{ U^{\alpha}_{\lambda \alpha} \right\} \, U^2 \, d\eta^3 \]
\[ = H_{i,j}^{H(2)} - H_{i,j}^{H(1)} \alpha \, V_{\alpha} \, + \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha}_{\lambda \alpha} \, d\eta^3 \]
\[ = (V_{\alpha}^3 - V_{\alpha}^{H(1)}) - \frac{\partial H}{\partial \tau} + \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha}_{\lambda \alpha} \, d\eta^3 \]

We then obtain

\[ \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha}_{\lambda \alpha} \, d\eta^3 = \frac{\partial H}{\partial \tau} - (V_{\alpha}^3 - V_{\alpha}^{H(1)}) + \left\{ \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha} \, d\eta^3 \right\}_{\lambda \alpha} \quad (5.14) \]

Equation (5.4) may now be integrated between the limits
\[ \eta^3 = H_{i,j}^{H(1)} \quad \text{and} \quad \eta^3 = H_{i,j}^{H(2)} \]

\[ \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha}_{\lambda \alpha} \, d\eta^3 + \int_{H_{i,j}}^{H_{i,j}^{H(2)}} \frac{\partial U^3}{\partial \eta^3} \, d\eta^3 = \]
\[ \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha}_{\lambda \alpha} \, d\eta^3 + (V_{\alpha}^3 - V_{\alpha}^{H(1)}) = \]
\[ \frac{\partial H}{\partial \tau} + \left\{ \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha} \, d\eta^3 \right\}_{\lambda \alpha} = 0 \]

whence

\[ \left\{ \int_{H_{i,j}}^{H_{i,j}^{H(2)}} U^{\alpha} \, d\eta^3 \right\}_{\lambda \alpha} = - \frac{\partial H}{\partial \tau} \]

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A comparison of this equation with (5.13) then gives

$$\left\{ H V_{\alpha}^{ \kappa} \right\}_{\alpha} + \left\{ (V^{\kappa} F_1 + G_{\kappa\beta} \psi_{\kappa} F_2) \int_{H_{\beta\gamma}}^{H_{\beta\gamma}} C_0 (\theta^\gamma) \, d\theta^\gamma \right\}_{\lambda\kappa}$$

$$+ \left\{ G_{\kappa\beta} \psi_{\kappa} \int_{H_{\beta\gamma}}^{H_{\beta\gamma}} C_1 (\theta^\gamma) \, d\theta^\gamma \right\}_{\lambda\kappa} = - \frac{\partial H}{\partial \gamma}$$

(5.15)

Examination of previous formulae reveals that the only unknown quantities in (5.15) are $\psi$ and its various derivatives, which occur implicitly in $F_1$, $F_2$, $C_1$, and $C_2$, as well as explicitly. This equation therefore constitutes a partial differential equation for determining $\psi(Y^1, Y^2)$.

The functions used in (5.15) involve a large number of parameters, which may be constant or which may be variable functions of $Y^1$ and $Y^2$. Furthermore, many of the quantities required in (5.15) are obtainable only by the solution of implicit functional relations. Finally, the partial differential equation ultimately derived is of a complicated non-linear type. It is therefore not to be expected that this procedure, in its present general form, will be directly useful in practice. However, the application of this technique to the solution of a suitably simplified problem in one dimension is shown, in a later section, to be quite feasible.

In the remainder of this section it is assumed that the parameters $\epsilon, \phi, K_\mu, K_\rho, R, \eta, \lambda$, and $K_\kappa$ are all small in comparison with unity.
We begin by assuming the following series expressions:

\[ U^\alpha = U_{(0)}^\alpha + \varepsilon U_{(\varepsilon)}^\alpha + \phi U_{(\phi)}^\alpha + K_\mu U_{(\mu)}^\alpha + K_\rho U_{(\rho)}^\alpha + R U_{(R)}^\alpha + \eta U_{(\eta)}^\alpha + \lambda U_{(\lambda)}^\alpha + K_k U_{(k)}^\alpha + \varepsilon^2 U_{(EE)}^\alpha + \varepsilon \phi U_{(E\phi)}^\alpha + \varepsilon K_\mu U_{(E\mu)}^\alpha + \cdots \]

\[ U^3 = U_{(0)}^3 + \varepsilon U_{(\varepsilon)}^3 + \phi U_{(\phi)}^3 + \cdots + K_k U_{(k)}^3 + \varepsilon^2 U_{(EE)}^3 + \cdots \]

\[ \rho = P_{(0)} + \varepsilon P_{(\varepsilon)} + \phi P_{(\phi)} + \cdots + K_k P_{(k)} + \varepsilon^2 P_{(EE)} + \cdots \]

\[ \Theta = \Theta_{(0)} + \varepsilon \Theta_{(\varepsilon)} + \phi \Theta_{(\phi)} + \cdots + K_k \Theta_{(k)} + \varepsilon^2 \Theta_{(EE)} + \cdots \]

(5.16)

Examination of the basic equations shows that the variables \( \bar{\mu}, \bar{\rho}, \bar{C}_\phi, \) and \( \bar{k} \) may also be expected to depend upon \( \varepsilon, \phi, \ldots, \lambda, K_k \). We therefore write in addition

\[ \bar{\mu} = \bar{\mu}_{(0)} + \varepsilon \bar{\mu}_{(\varepsilon)} + \phi \bar{\mu}_{(\phi)} + \cdots \]

\[ \bar{k} = \bar{k}_{(0)} + \varepsilon \bar{k}_{(\varepsilon)} + \phi \bar{k}_{(\phi)} + \cdots \]

(5.17)
Since $\mu$, $\rho$, $C_\mu$, and $k$ are functions of temperature and pressure, the derived quantities $\tilde{\mu}, \ldots, \tilde{k}$ may be expressed in terms of $P$ and $\Theta$. If it is assumed that these functions are analytic, the expansions of (5.17) may be conveniently related to those of (5.16). We have, for example,

$$
\tilde{\mu} = \tilde{\mu}(P, \Theta)
$$

$$
= \tilde{\mu}(P_{(0)} + \epsilon P_{(\epsilon)} + \cdots + K_k P_{(k)} + \cdots \Theta_{(0)} + \epsilon \Theta_{(\epsilon)} + \cdots + K_k \Theta_{(k)} + \cdots)
$$

$$
= \tilde{\mu}_{(0)} + \epsilon \tilde{\mu}_{(\epsilon)} + \cdots + K_k \tilde{\mu}_{(k)} + \cdots
$$

identically in $\epsilon, \phi, \ldots, K_k$. Here the functions $P_{(0)}$, $P_{(\epsilon)}$, $\tilde{\mu}_{(0)}$, $\tilde{\mu}_{(\epsilon)}$, $\Theta_{(0)}$, $\Theta_{(\epsilon)}$ etc. are independent of the parameters.

Setting $\epsilon = \phi = \cdots = K_k = 0$ in this relation, we therefore find

$$
\tilde{\mu}_{(0)} = \tilde{\mu}(P_{(0)}, \Theta_{(0)})
$$

(5.18)

Also differentiating the expression for $\tilde{\mu}$ with respect to $\epsilon$, and setting $\epsilon = \phi = \cdots = K_k = 0$,

$$
\tilde{\mu}(\epsilon) = \left\{ \frac{\partial \tilde{\mu}(P, \Theta)}{\partial \epsilon} \frac{\partial P}{\partial \epsilon} + \frac{\partial \tilde{\mu}(P, \Theta)}{\partial \epsilon} \frac{\partial \Theta}{\partial \epsilon} \right\}_{\epsilon = \phi = \cdots = K_k = 0}
$$

$$
= \frac{\partial \tilde{\mu}(P_{(0)}, \Theta_{(0)})}{\partial P} P_{(\epsilon)} + \frac{\partial \tilde{\mu}(P_{(0)}, \Theta_{(0)})}{\partial \Theta} \Theta_{(\epsilon)}
$$

(5.19)
In this manner the coefficients in (5.17) may be expressed in terms of $P_{(\alpha)}$, $P_{(\epsilon)}$, $\ldots$, $\Theta_{(\alpha)}$, $\Theta_{(\epsilon)}$, $\ldots$ etc. In practice we shall find occasion to employ only those quantities typified by (5.18) and (5.19); however, higher order terms may be similarly treated.

We now substitute the series (5.16) and (5.17) into (3.52), (3.54), (3.57), and (3.66), collecting terms in like powers of $\epsilon$, $\phi$, $\ldots$, $K_k$. For those terms independent of the parameters, we find

$$\frac{\partial^2 U_{(\alpha)}^\alpha}{(\partial \eta^3)^2} - G_{(\alpha)}^{\alpha \beta} P_{(\alpha)}^\beta = 0$$  \hspace{1cm} (5.20)

$$\frac{\partial P_{(\alpha)}}{\partial \eta^3} = 0$$  \hspace{1cm} (5.21)

$$U_{(\alpha)}^\alpha + \frac{\partial U_{(\alpha)}^3}{\partial \eta^3} = 0$$  \hspace{1cm} (5.22)

$$\frac{\partial^2 \Theta_{(\alpha)}}{(\partial \eta^3)^2} = 0$$  \hspace{1cm} (5.23)

It will be instructive to analyze this case in detail before considering higher order terms.

Applying the expansions of (5.16) to the general boundary conditions (4.2), (4.3), (4.8), and (4.9), it is clear that we must require

$$U_{(\alpha)}^\alpha = V_{(1)}^\alpha \hspace{0.5cm} (\eta^3 = H_{(1)}) \hspace{1cm} U_{(\alpha)}^\alpha = V_{(2)}^\alpha \hspace{0.5cm} (\eta^3 = H_{(2)})$$  \hspace{1cm} (5.24)
\[ U_{(\omega)}^3 = V_{(1)}^3 \ (y^3 = H_{(1)}) \quad U_{(\omega)}^3 = V_{(2)}^3 \ (y^3 = H_{(2)}) \] (5.25)

and

\[
\begin{align*}
\frac{\partial \Theta_{(\omega)}}{\partial y^3} - \bar{K}_5 \Theta_{(\omega)} &= -\bar{F}_5 \Theta_{(n)} \quad (y^3 = H_{(1)}) \\
\frac{\partial \Theta_{(\omega)}}{\partial y^3} + \bar{K}_5 \Theta_{(\omega)} &= \bar{K}_5 \Theta_{(b)} \quad (y^3 = H_{(2)})
\end{align*}
\] (5.26)

From (5.21), \( P_{(\omega)} \) is independent of \( y^3 \). We may therefore write

\[ P_{(\omega)} = \Psi_{(\omega)} (y', y^2) \] (5.27)

where \( \Psi_{(\omega)} \) is a function to be determined. Substitution of this function into (5.20) then gives

\[
\frac{\partial^2 U_{(\omega)}^\alpha}{\partial (y^3)^2} = G_{(\omega)}^{\alpha\beta} \Psi_{(\omega)1\beta}
\]

Noting that the right hand member of this equation is independent of \( y^3 \), we see that the general solution for \( U_{(\omega)}^\alpha \) is

\[ U_{(\omega)}^\alpha = a_{(\omega)}^\alpha (y', y^2) + (y^3) b_{(\omega)}^\alpha (y', y^2) + \frac{1}{2} (y^3)^2 c_{(\omega)}^{\alpha\beta} \Psi_{(\omega)1\beta} \] (5.28)

where \( a_{(\omega)}^\alpha \) and \( b_{(\omega)}^\alpha \) are undetermined vector functions of
The boundary conditions (5.24) then give

\[ \alpha_{(0)} = \frac{V_{11}^\alpha H_{11} - V_{12}^\alpha H_{11}}{H_{11} - H_{11}} + \frac{1}{2} H_{11} H_{12} G^{\alpha \beta}_{(0)} \psi_{(11)} \psi_{(12)} \]

\[ \beta_{(0)} = \frac{V_{12}^\alpha - V_{11}^\alpha}{H_{12} - H_{11}} - \frac{1}{2} \left[ H_{11} + H_{12} \right] G^{\alpha \beta}_{(0)} \psi_{(11)} \psi_{(12)} \]  

(5.29)

Substitution of these values into (5.28) yields the following expression for \( U^{\alpha}_{(0)} \):

\[ U^{\alpha}_{(0)} = \left\{ \frac{V_{11}^\alpha H_{12} - V_{12}^\alpha H_{11}}{H} + \frac{1}{2} H_{11} H_{12} G^{\alpha \beta}_{(0)} \psi_{(11)} \psi_{(12)} \right\} 

+ (\psi^3)^3 \left\{ \frac{V_{11}^\alpha H_{12} - V_{12}^\alpha H_{11}}{H} - \frac{1}{2} \left[ H_{11} + H_{12} \right] G^{\alpha \beta}_{(0)} \psi_{(11)} \psi_{(12)} \right\} 

+ \frac{1}{2} (\psi^3)^2 \left\{ G^{\alpha \beta}_{(0)} \psi_{(11)} \psi_{(12)} \right\} \]

This equation may be simplified in form by use of a new variable \( \xi \), defined by

\[ \xi = \psi^3 - \frac{1}{2} \left( H_{11} + H_{12} \right) \]  

(5.30)

The quantity \( \xi \) represents distance in the direction perpendicular to \( S \), as does \( \psi^3 \), but is measured from the middle surface between \( S_1 \) and \( S_2 \) rather than from \( S \). In differentiation of quantities containing this variable, it should be noted that \( \xi \) depends,
in general, on \( y^1, y^2, \) and \( \tau \), and hence cannot be treated as a constant. However,

\[
\frac{\partial \xi^3}{\partial y^3} = 1
\]

so that in differentiations or integrations with respect to \( y^3 \) the quantities \( y^3 \) and \( \xi \) may be used interchangeably.

By means of (5.30), \( U_{10}^{\alpha} \) may be written

\[
U_{10}^{\alpha} = \frac{1}{2} (V_{11}^{\alpha} + V_{12}^{\alpha}) + \xi \left( \frac{V_{12}^{\alpha} - V_{11}^{\alpha}}{H} \right) - \frac{1}{2} \left( \frac{H^2}{4} - \xi^3 \right) G_{12}^{\alpha\beta} \psi_{10110}^{\beta}
\]

(5.31)

The quantity \( U_{10}^{3} \) may now be found from (5.22). Expressing this equation in the form

\[
\frac{\partial U_{10}^{3}}{\partial \xi} = -U_{10}^{\alpha} \Gamma_{\alpha}^{\beta} \xi_{\beta}
\]

the differentiation of (5.31) provides the necessary value for \( U_{10}^{\alpha} \). Integration of the resulting expression with respect to \( \xi \) then yields

\[
U_{10}^{3} = C + \xi \left[ -\frac{1}{2} A_{10}^{\alpha} + \left( \frac{H^2}{8} G_{12}^{\alpha\beta} \psi_{10110}^{\beta} \right)_{10} - \frac{B_{10}^{\alpha}}{H} \xi_{10} \right]
\]

\[
+ \frac{\xi^2}{2} \left[ -\left( \frac{B_{10}^{\alpha}}{H} \right)_{10} - G_{12}^{\alpha\beta} \psi_{10110}^{\beta} \xi_{10} \right] + \frac{\xi^3}{3} \left[ -\frac{1}{2} G_{12}^{\alpha\beta} \psi_{10110}^{\beta} \right]
\]

(5.32)

*Throughout the rest of this section we employ the abbreviations

\[
A^{\alpha} = V_{11}^{\alpha} + V_{12}^{\alpha} \quad B^{\alpha} = V_{12}^{\alpha} - V_{11}^{\alpha}
\]
where $C$ is a constant of integration. The equation

$$\left[ U_{10}^3 \right]_{\xi = -\frac{H}{2}} + \left[ U_{10}^3 \right]_{\xi = \frac{H}{2}} = V_{11}^3 + V_{12}^3$$

obtained from the boundary conditions (5.25)* then yields

$$C = \frac{1}{2} (V_{11}^3 + V_{12}^3) + \frac{H^2}{8} \left[ \left( \frac{\partial}{\partial H} \right)_{1\alpha} + C_{1\alpha}^{\alpha\beta} \frac{V_{1\alpha}}{1\beta} \right]_{1\xi} \right]$$

(5.33)

The single boundary condition used in deriving (5.33) is not equivalent to the two boundary conditions given by (5.25); thus, another independent condition may still be applied to $U_{10}^3$.

This second condition will require that a certain relation exist between the derivatives of $\psi_{10}$ and the quantities $V_{11}^\alpha$, $V_{12}^\alpha$, $H_{11}$, and $H_{12}$; since the latter set are known functions; however, this relation constitutes a partial differential equation for $\psi_{10}$.

Instead of proceeding directly on this basis, we employ the equivalent and simpler technique developed previously in connection with (5.14) and the equations which followed. In this manner we obtain

$$\left\{ \int_{H_{10}}^{H_{13}} \frac{U_{10}^\alpha}{H} \, d\zeta \right\}_{1\alpha} = \left\{ \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{U_{10}^\alpha}{H} \, d\zeta \right\}_{1\alpha} = - \frac{2H}{\partial T}$$

*Note that $Y^3 = H_{11}$ corresponds to $\zeta = -\frac{H}{2}$, and $Y^3 = H_{10}$ to $\zeta = +\frac{H}{2}$.
Upon substituting the value given for \( U_{(o)}^{\alpha} \) by (5.31), carrying out the indicated operations, and rearranging the results, we have

\[
G_{(\alpha)}^{(B)} \left[ H^3 \psi_{(o)1\alpha} \right]_{1\beta} = 6 \left( H A^{\alpha} \right)_{1\alpha} + 12 \frac{\partial H}{\partial \xi}
\]

(5.34)

as the required two dimensional partial differential equation for \( \psi_{(o)}(1^1, 1^2) \). Equation (5.34) provides the fundamental first approximation which forms the basis for most of the standard theory of lubrication. If the surfaces \( S_1 \) and \( S_2 \) are stationary, (5.34) reduces at once to the more simply derived equation (1) in the Introduction.

To obtain \( \Theta_{(o)} \), we observe from (5.23) that \( \Theta_{(o)} \) is linear in \( Y^3 \), and a simple calculation using the boundary conditions (5.26) gives

\[
\Theta_{(o)} = \frac{\bar{k}_{51} \Theta_{(o)} + \bar{k}_{52} \Theta_{(2)} + \bar{k}_{51} \bar{k}_{52} \left[ \Theta_{(o)} (H_{(o)} - Y^3) + \Theta_{(2)} (Y^3 - H_{(o)}) \right]}{\bar{k}_{51} + \bar{k}_{52} + \bar{k}_{51} \bar{k}_{52} H}
\]

(5.35)

Here \( \Theta_{(1)} \), \( \Theta_{(2)} \), \( \bar{k}_{51} \), and \( \bar{k}_{52} \) are in the most general case arbitrary functions of \( 1^1, 1^2, \) and \( \xi \). In terms of the variable \( \xi \) we may write

\[
\Theta_{(o)} = \Theta_{(o)(av)} + \xi \Theta_{(o)}'
\]

(5.36)

where \( \Theta_{(o)(av)} \) is the value of \( \Theta_{(o)} \) averaged between \( S_1 \) and \( S_2 \),
given by the formula

\[
\Theta_{\omega}(\omega) = \frac{\bar{r}_{s1} \Theta_{\omega}^{(1)} + \bar{r}_{s2} \Theta_{\omega}^{(2)} + \frac{1}{2} \bar{r}_{s1} \bar{r}_{s2} H (\Theta_{\omega}^{(1)} + \Theta_{\omega}^{(2)})}{\bar{r}_{s1} + \bar{r}_{s2} + \bar{r}_{s1} \bar{r}_{s2} H}
\]

(5.37)

and where

\[
\Theta_{\omega}^{(1)}' = \frac{\bar{r}_{s1} \bar{r}_{s2} (\Theta_{\omega}^{(2)} - \Theta_{\omega}^{(1)})}{\bar{r}_{s1} + \bar{r}_{s2} + \bar{r}_{s1} \bar{r}_{s2} H}
\]

(5.38)

We now proceed to the calculation of the first order correction terms \(U^{\alpha}_{(e)}\), \(U^{\alpha}_{(\phi)}\), etc. Substituting the series (5.16) and (5.17) into (3.52), (3.54), (3.57), and (3.66), and employing equations of the form (5.18), we obtain the following sets of equations from those terms linear in the parameters:

\[
\begin{align*}
\frac{\partial^2 U^{\alpha}_{(e)}}{(\partial \eta^3)^2} - G_{\omega\beta} P_{(e)\beta} & = - \frac{\partial U^{\alpha}_{(e)}}{\partial \eta^3} \eta^{\beta} \\
- \frac{3}{2} \frac{\partial U^{\beta}_{(e)}}{\partial \eta^3} \eta^{\alpha} + (\eta^3) G_{\alpha\beta} P_{(e)\beta} & = 0 \\
\frac{\partial P_{(e)}}{\partial \eta^3} & = 0 \\
U^{\alpha}_{(e) \eta} + \frac{\partial U^{3}_{(e)}}{\partial \eta^3} & = -(\eta^3) U^{\beta}_{(e)} \eta^{\alpha} - \frac{3}{2} U^{3}_{(e)} \eta^{\alpha} \\
\frac{\partial^2 \Theta^{(e)}}{(\partial \eta^3)^2} & = - \frac{\partial \Theta^{(e)}}{\partial \eta^3} \eta^{\alpha}
\end{align*}
\]

(5.39)
\[
\frac{\partial^2 U^{\alpha}_{(\phi)}}{(\partial \psi^3)^2} - G^{\alpha \beta}_{(\phi)} P_{(\phi)1\beta} = - F^{\alpha}_{(\phi)}
\]

\[
\frac{\partial P_{(\phi)}}{\partial \psi^3} = 0
\]

\[
U^{\alpha}_{(\phi)1\alpha} + \frac{\partial U^3_{(\phi)}}{\partial \psi^3} = 0
\]

\[
\frac{\partial \Theta_{(\phi)}}{(\partial \psi^3)^2} = 0
\]

(5.40)

\[
\frac{\partial^2 U^{\alpha}_{(\mu)}}{(\partial \psi^3)^2} - G^{\alpha \beta}_{(\mu)} P_{(\mu)1\beta} = - \bar{\mu} (P_{(\mu)}, \Theta_{(\mu)}) \frac{\partial^2 U^{\alpha}_{(\mu)}}{(\partial \psi^3)^2}
\]

\[- \frac{\partial \bar{\mu} (P_{(\mu)}, \Theta_{(\mu)})}{\partial \psi^3} \frac{\partial U^{\alpha}_{(\mu)}}{\partial \psi^3} \]

\[
\frac{\partial P_{(\mu)}}{\partial \psi^3} = 0
\]

\[
U^{\alpha}_{(\mu)1\alpha} + \frac{\partial U^3_{(\mu)}}{\partial \psi^3} = 0
\]

\[
\frac{\partial \Theta_{(\mu)}}{(\partial \psi^3)^2} = 0
\]

(5.41)
\[
\begin{align*}
\frac{\partial^2 U^\alpha_{(p)}}{(\partial \eta^3)^2} - G_{(c)}^{\alpha(\beta} P^{(p)}_{(\beta)} = 0
\end{align*}
\]

\[
\frac{\partial P^{(p)}}{\partial \eta^3} = 0
\]

\[
\begin{align*}
U^\alpha_{(p)} + \frac{\partial U^3_{(c)}}{\partial \eta^3} = & - \frac{\partial \rho(P_{(c)}, \Theta_{(c)})}{\partial \tau} \\
- U^\alpha_{(c)} \rho(P_{(c)}, \Theta_{(c)})_{(c)} - U^3_{(c)} \frac{\partial \rho(P_{(c)}, \Theta_{(c)})}{\partial \eta^3} = & 0
\end{align*}
\]

\[
\frac{\partial^2 \Theta_{(c)}}{(\partial \eta^3)^2} = 0
\]

(5.42)

\[
\begin{align*}
\frac{\partial^2 U^\alpha_{(r)}}{(\partial \eta^3)^2} - G^{\alpha(\beta} P_{(r)}^{(\beta)} = & \frac{\partial U^\alpha_{(r)}}{\partial \Theta_{(c)}} + U^\beta_{(c)} U^\alpha_{(r)} 13 \\
+ & U^3_{(c)} \frac{\partial U_{(r)}}{\partial \eta^3} \\
\frac{\partial P^{(r)}}{\partial \eta^3} = & 0 \\
U^\alpha_{(r)} + \frac{\partial U^3_{(r)}}{\partial \eta^3} = & 0 \\
\frac{\partial^2 \Theta_{(r)}}{(\partial \eta^3)^2} = & 0
\end{align*}
\]

(5.43)
\[
\frac{\partial^2 U^\alpha_{(\eta)}}{(\partial \eta^3)^2} - G_{\alpha\beta}^{\lambda(\eta)} P_{(\eta)\beta} = 0
\]
\[
\frac{\partial P_{(\eta)}}{\partial \eta^3} = 0
\]
\[
U^\alpha_{(\eta)\alpha} + \frac{\partial U^3_{(\eta)}}{\partial \eta^3} = 0
\]
\[
\frac{\partial^2 \Theta_{(\eta)}}{(\partial \eta^3)^2} = -G_{\alpha\beta}^\lambda \frac{\partial U^\alpha_{(\eta)}}{\partial \eta^3} \frac{\partial U^\beta_{(\eta)}}{\partial \eta^3}
\]
(5.44)

\[
\frac{\partial^2 U^\alpha_{(\eta)}}{(\partial \eta^3)^2} - G_{\alpha\beta}^\lambda P_{(\eta)\beta} = 0
\]
\[
\frac{\partial P_{(\eta)}}{\partial \eta^3} = 0
\]
\[
U^\alpha_{(\eta)\alpha} + \frac{\partial U^3_{(\eta)}}{\partial \eta^3} = 0
\]
\[
\frac{\partial^2 \Theta_{(\eta)}}{(\partial \eta^3)^2} = C \left( P_{(\eta)}, \Theta_{(\eta)} \right) \left\{ \frac{\partial \Theta_{(\eta)}}{\partial \eta^3} \right\}
\]
\[
+ U^\alpha_{(\eta)} \Theta_{(\eta)\alpha} + U^3_{(\eta)} \frac{\partial \Theta_{(\eta)}}{\partial \eta^3}
\]
(5.45)
and finally

\[
\frac{\partial^2 U_{(k)}^\alpha}{\partial \eta^3} - G_{\alpha \beta} \frac{\partial P_{(k)}}{\partial \eta^3} = 0
\]

\[
\frac{\partial P_{(k)}}{\partial \eta^3} = 0
\]

\[
U_{(k)\alpha} + \frac{\partial U_{(k)}^3}{\partial \eta^3} = 0
\]

\[
\frac{\partial^2 \theta_{(k)}}{\partial \eta^3} = - \overline{K}(P_{(w)}, \theta_{(w)}) \frac{\partial^2 \theta_{(w)}}{\partial \eta^3}
\]

\[
- \frac{\partial \overline{K}(P_{(w)}, \theta_{(w)})}{\partial \eta^3} \frac{\partial \theta_{(w)}}{\partial \eta^3}
\]

\[\text{(5.46)}\]

The quantities \( V_{(1)}^i, V_{(2)}^i, \theta_{(1)}, \) and \( \theta_{(2)} \) expressing the boundary conditions on \( S_1 \) and \( S_2 \) are independent of the parameters \( \epsilon, \phi, \ldots, K_k \). Examination of (4.2), (4.3), (4.8), and (4.9) thus shows that the boundary conditions for the differential systems (5.39) to (5.46) are all homogeneous; that is, of the form

\[
U_{(1)}^\alpha = U_{(1)}^3 = 0 \quad (\eta^3 = H_{(1)}, H_{(2)})
\]

\[
\frac{\partial \theta_{(1)}}{\partial \eta^3} - \overline{K}_{(1)} \theta_{(1)} = 0 \quad (\eta^3 = H_{(1)})
\]

\[
\frac{\partial \theta_{(1)}}{\partial \eta^3} + \overline{K}_{(1)} \theta_{(1)} = 0 \quad (\eta^3 = H_{(12)})
\]

\[\text{(5.47)}\]
where any of the subscripts $\xi, \phi, \ldots, k$ may be attached.

We also observe that in each of (5.39) to (5.46) we have

$$\frac{\partial P_{ij}}{\partial \Psi^3} = 0; \text{ thus we may write}$$

$$P_{ij} = \Psi_i^j (\Psi', \Psi^2)$$

(5.48)

The remainder of our work will be greatly facilitated by first solving the general system:

$$\begin{align*}
\frac{\partial^2 U^\alpha}{\partial \Psi^3} &= G_{\alpha\beta} \Psi_{\beta} + L^\alpha (\Psi', \Psi^2, \Psi^3) \\
\frac{\partial U^3}{\partial \Psi^3} &= - U^\alpha_{,\alpha} + M (\Psi', \Psi^2, \Psi^3) \\
\frac{\partial^2 \Theta}{\partial (\Psi^3)^2} &= N (\Psi', \Psi^2, \Psi^3)
\end{align*}$$

(5.49)

under the boundary conditions (5.47), where $\Psi = \Psi (\Psi', \Psi^2)$ and where $L^\alpha, M, \text{ and } N$ are arbitrary vector or scalar functions of $(\Psi^1, \Psi^2, \Psi^3)$. Explicit formulae may then be obtained for each of the systems (5.39) to (5.46) by proper selection of $L^\alpha, M, \text{ and } N$.

The function $U^\alpha$ satisfying the first of equations (5.49) and vanishing on the boundaries may be verified to be

$$U^\alpha = - \frac{1}{2} (\Psi^3 - H_{1i}) (H_{1i} - \Psi^3) G_{\alpha\beta} \Psi_{\beta} - \frac{1}{H} \int_{H_{1i}}^{H_{1i}} (\Psi^3 - \Psi^3 - H_{1i}) L^\alpha (\Psi) d\Psi$$

$$- \frac{1}{H} \int_{H_{1i}}^{H_{1i}} (\Psi^3 - H_{1i}) (H_{1i} - \Psi) L^\alpha (\Psi) d\Psi$$

(5.50)
where \( L^\alpha(\eta) \) stands for \( L^\alpha(\eta', \eta^3, \eta) \).

We now integrate the second of equations (5.49) between \( H_(1) \) and \( H_(2) \). Since \( U^3 = 0 \) at the limits of integration, by equations (5.47), the term \( \frac{2U^3}{\eta^3} \) drops out, and we find

\[
\int_{H_(u)}^{H_(1)} U^\alpha_{\xi \alpha} d\eta^3 = \int_{H_(u)}^{H_(2)} M d\eta^3
\]

(5.51)

Upon substitution of an expression for \( U^\alpha_{\xi \alpha} \) derived from (5.50), the above relation will provide a partial differential equation for \( \psi' (\eta^1, \eta^2) \).

Instead of performing this substitution directly, we first use previously developed techniques to obtain

\[
\int_{H_(u)}^{H_(2)} U^\alpha_{\xi \alpha} d\eta^3 = \left\{ \int_{H_(u)}^{H_(2)} U^\alpha_{\xi \alpha} d\eta^3 \right\}_{\xi \alpha} + H_m (\xi \alpha) \left[ U^\alpha \right]_{\eta^3 = H_m}^H
\]

\[
- H_(1)(\xi \alpha) \left[ U^\alpha \right]_{\eta^3 = H_(1)}^\eta
\]

\[
= \left\{ \int_{H_(u)}^{H_(2)} U^\alpha_{\xi \alpha} d\eta^3 \right\}_{\xi \alpha}
\]

(5.52)

since, in the present instance, \( U^\alpha \) vanishes on the boundaries, by (5.47). Integrating (5.50) between \( H_(1) \) and \( H_(2) \), and
carrying out two integrations by parts, we find

\[
\int_{H_{ii}}^{H_{zz}} U^\alpha dy^3 = -\frac{1}{2} \gamma_{\alpha\beta} \int_{H_{ii}}^{H_{zz}} (y^3 - H_{ii}) (H_{zz} - y^3) dy^3 \\
+ \frac{1}{H} \int_{H_{ii}}^{H_{zz}} dy^3 \left\{ (H_{zz} - y^3) \left[ (H_{ii} - y^3) L^\alpha (y) dy + (H_{ii} - y^3) L^\alpha (y) dy \right] \right\}
\]

\[
= -\frac{H^3}{12} \gamma_{\alpha\beta} \int_{H_{ii}}^{H_{zz}} (y^3 - H_{ii}) (H_{zz} - y^3) L^\alpha (y, y^3, y^3) dy^3
\]

(5.53)

Combining (5.52) and (5.53), with minor rearrangements, the basic equation (5.51) becomes

\[
\gamma_{\alpha\beta} \left[ H^3 \psi_{1\alpha} \right] = -\frac{1}{2} \int_{H_{ii}}^{H_{zz}} W(y, y^3, y^3) dy^3 \\
- 6 \left\{ \int_{H_{ii}}^{H_{zz}} (y^3 - H_{ii}) (H_{zz} - y^3) L^\alpha (y, y^3, y^3) dy^3 \right\}_{1\alpha}
\]

(5.54)

To calculate $\Theta$ we integrate the last of equations (5.49) twice, employing (5.47) to determine the integration constants. The final result of this operation may be expressed in the
following symmetric form:

\[
\Theta = -\frac{1}{H(k_3 + k_{32} + k_3 k_{32} H)} \int_{H_{\text{lin}}}^{H_{\text{lin}}} \left\{ H + \bar{k}_{31} (\psi^2 - H_{\text{lin}})(\psi - H_{\text{lin}}) \\
+ \bar{k}_{32} (H_{\text{lin}} - \psi^3)(H_{\text{lin}} - \psi) \right\} N(\psi, \psi^2, \psi) \, d\psi \\
- \frac{1}{H} \left\{ \int_{H_{\text{lin}}}^{\psi^3} (\psi - H_{\text{lin}})(H_{\text{lin}} - \psi^3) N(\psi, \psi^2, \psi) \, d\psi \\
+ \int_{\psi^3}^{H_{\text{lin}}} (\psi^2 - H_{\text{lin}})(H_{\text{lin}} - \psi) N(\psi, \psi^2, \psi) \, d\psi \right\}
\]

(5.55)

For the first order correction in $E$, the right hand side of equation (5.39) gives

\[
L^{(\epsilon)} = - \frac{\partial U_{\alpha \beta}}{\partial \psi^3} \int_{\epsilon_{\alpha \beta}}^{\psi^3} - 2 \frac{\partial U_{\alpha \beta}}{\partial \psi^3} \int_{\epsilon_{\alpha \beta}}^{\psi^3} + (\psi^3) G_{\alpha \beta} \left( \psi_{\alpha \beta} \right)
\]

(5.56)

\[
M^{(\epsilon)} = - (\psi^3) U_{\alpha \beta} \int_{\epsilon_{\alpha \beta}}^{\psi^3} - U_{\alpha \beta} \int_{\epsilon_{\alpha \beta}}^{\psi^3}
\]

(5.57)

\[
N^{(\epsilon)} = - \frac{\partial U_{\alpha \beta}}{\partial \psi^3} \int_{\epsilon_{\alpha \beta}}^{\psi^3}
\]

(5.58)
To evaluate the second term of (5.54), we first use (5.28) and (5.29) to calculate

$$
\int_{H_{\ell 0}}^{H_{r 2}} (y^3 - H_{\ell 0})(H_{r 2} - y^3) \frac{\partial U_{10}}{\partial y^3} dy^3 = \frac{1}{6} H^2 B^\sigma
$$

Also

$$
\int_{H_{\ell 0}}^{H_{r 2}} (y^3 - H_{\ell 0})(H_{r 2} - y^3) \left[ (y^3) G_{\alpha \beta \psi_{10} \psi_{10}} \right] dy^3 = \frac{1}{12} H^3 (H_{10} + H_{r 2}) G_{\alpha \beta \psi_{10} \psi_{10}}
$$

Substituting these results into (5.56), we obtain

$$
\int_{H_{\ell 0}}^{H_{r 2}} (y^3 - H_{\ell 0})(H_{r 2} - y^3) L_{(\alpha)} (y^1, y^2, y^3) dy^3 =

- \frac{H^2}{6} \left\{ \sum_{\ell \geq 0} \gamma B^\alpha + 2 \sum_{\ell \geq 0} B^\sigma \right\}

+ \frac{H^3}{12} (H_{10} + H_{r 2}) G_{\alpha \beta \psi_{10} \psi_{10}}
$$
Thus

\[-6 \left\{ \int_{H_{11}}^{H_{12}} (\hat{\mathcal{F}}^2 - H_{11})(H_{12} - \hat{\mathcal{F}}^3) \mathcal{L}^{\alpha}_{(1\varepsilon)} (\hat{\mathcal{F}}, \hat{\mathcal{F}}^3, \hat{\mathcal{F}}^3) \, d\mathcal{F}^3 \right\}_{1\varepsilon} =\]

\[\left\{ \int_{H_{11}}^{H_{12}} (\mathcal{F}^2 + 2 \mathcal{F}^3) B^{\alpha} + 2 \mathcal{F}^3 \mathcal{B}^{B} - \frac{\mathcal{F}^3}{2} (H_{11} + H_{12}) G^{\alpha \beta}_{(H_{11} \varepsilon \varepsilon)} \right\}_{1\varepsilon} (5.59)\]

To evaluate the term \(\left\{ -12 \int_{H_{11}}^{H_{12}} M_{(\varepsilon)} \, d\mathcal{F}^3 \right\}\) of (5.54), we proceed indirectly. Let

\[I^{(n)}(f) = \int_{H_{11}}^{H_{12}} (\mathcal{F}^3)^n f(\mathcal{F}, \mathcal{F}^3, \mathcal{F}^3) \, d\mathcal{F}^3\]

for any scalar or tensor function \(f\). Then, by previous methods, we may readily show that

\[[I^{(n)}(f)]_{1\beta} = I^{(n)}(f)_{1\beta} + H^{(n)}_{12} H^{(n)}_{12} \beta [f] \mathcal{F}^3 = H_{12}\]

\[\quad - H^{(n)}_{11} H^{(n)}_{11} \beta [f] \mathcal{F}^3 = H_{11}\]

From equation (3.21.9), giving the general covariant
\[
\Gamma^\alpha_{\beta\gamma\delta} \equiv \Gamma_{\beta\gamma\delta}^\alpha
\]

Then

\[
\left\{ \Gamma^\alpha_{\beta\gamma\delta} I^{(\gamma)}(U_{(\alpha)}^\beta) \right\}_{\beta} = \Gamma^\alpha_{\beta\gamma\delta} I^{(\gamma)}(U_{(\alpha)}^\beta) + \Gamma^\alpha_{\beta\gamma\delta} \left\{ I^{(\gamma)}(U_{(\alpha)}^\beta) \right\} + H_{(3)} H_{(2)} (U_{(\alpha)}^\beta)_{A^3 = H_{(3)}} - H_{(1)} H_{(2)} (U_{(\alpha)}^\beta)_{A^3 = H_{(1)}}
\]

(5.60)

However, from (5.22) and an integration by parts,

\[
I^{(\gamma)}(U_{(\alpha)}^\beta) = - I^{(\gamma)} \left( \frac{\partial U_{(\alpha)}^3}{\partial A^3} \right)
\]

\[
= H_{(3)} (U_{(\alpha)}^3)_{A^3 = H_{(3)}} - H_{(2)} (U_{(\alpha)}^3)_{A^3 = H_{(2)}} + I^{(\gamma)}(U_{(\alpha)}^3)
\]

Substituting this expression into (5.60) and employing the boundary conditions (5.24) and (5.25),

\[
\left\{ \Gamma^\alpha_{\beta\gamma\delta} I^{(\gamma)}(U_{(\alpha)}^\beta) \right\}_{\beta} = \left\{ \Gamma^\alpha_{\beta\gamma\delta} I^{(\gamma)}(U_{(\alpha)}^\beta) + \Gamma^\alpha_{\beta\gamma\delta} I^{(\gamma)}(U_{(\alpha)}^3) \right\} + H_{(3)} V_{(3)} - H_{(2)} V_{(2)}^3 + H_{(2)} V_{(2)}^3 - H_{(3)} H_{(3)} V_{(3)}^\beta - H_{(1)} H_{(1)} V_{(1)}^\beta
\]

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From (5.57) we then find

\[
\int_{H_{11}}^{H_{12}} M_e \, d\mathcal{A}^3 = -\int_{03\beta\lambda}^{\alpha} \Gamma^{(11)}(U_{10}^\beta) - \int_{03\beta\lambda}^{\alpha} \Gamma^{(1)}(U_{10}^\beta)
\]

\[
= \int_{03\beta\lambda}^{\alpha} \left\{ H_{11} V_{11}^\beta - H_{12} V_{12}^\beta + H_{13} H_{12} \right\} V_{12}^\beta
\]

\[
- H_{11} H_{12} \frac{\partial V_{11}^\beta}{\partial \xi} \right\} - \left\{ \int_{03\beta\lambda}^{\alpha} \Gamma^{(11)}(U_{10}^\beta) \right\}_{1\beta}
\]

\[
= \int_{03\beta\lambda}^{\alpha} \left\{ H_{11} \frac{\partial H_{11}}{\partial \xi} - H_{12} \frac{\partial H_{12}}{\partial \xi} \right\}
\]

\[
- \left\{ \int_{03\beta\lambda}^{\alpha} \Gamma^{(11)}(U_{10}^\beta) \right\}_{1\beta}
\]

By direct calculation from (5.28) and (5.29) we find

\[
\Gamma^{(11)}(U_{10}^\beta) = \frac{H}{c} \left\{ (H_{12} + 2H_{11}) V_{11}^\beta + (H_{11} + 2H_{12}) V_{12}^\beta \right\}
\]

\[-\frac{H^3}{24} (H_{11} + H_{12}) G_{03}^\beta \psi_{10}^1 \right\}
\]

Thus (5.61) becomes

\[
\int_{H_{11}}^{H_{12}} M_e \, d\mathcal{A}^3 = \int_{03\beta\lambda}^{\alpha} \left\{ H_{11} \frac{\partial H_{11}}{\partial \xi} - H_{12} \frac{\partial H_{12}}{\partial \xi} \right\}
\]

\[-\frac{H^3}{24} (H_{11} + H_{12}) G_{03}^\beta \psi_{10}^1 \right\}
\]

\[
+ \frac{H^3}{24} (H_{11} + H_{12}) G_{03}^\beta \psi_{10}^1 \right\}
\]
Combining this equation with (5.59) and substituting the result into (5.54), we obtain the following partial differential equation for $\psi_{(e)}$:

\[
G^{\alpha\beta}_{(e)} \left[ H^3 \psi_{(e)1\alpha} \right]_{1\beta} = -12 \sum_{\alpha\beta} \left\{ H_{i\alpha} \frac{\partial H_{i\beta}}{\partial \epsilon} - H_{i\beta} \frac{\partial H_{i\alpha}}{\partial \epsilon} \right\}
\]

\[
+ \left\{ 2 \sum_{\alpha\beta} H \left[ (H_{i\alpha} + 2H_{i\beta}) \psi_{i\beta} + (H_{i\beta} + 2H_{i\alpha}) \psi_{i\alpha} \right] \right\}_{1\beta}
\]

\[
+ H^2 \left[ \sum_{\alpha\beta} \mathcal{B}^{\beta} + 2 \sum_{\alpha\beta} \mathcal{B}^{\alpha} \mathcal{B}^{\beta} \right] \right\}_{1\beta}
\]

\[
- \frac{1}{2} \left\{ H^3 (H_{i\alpha} + H_{i\beta}) \left( \sum_{\alpha\beta} G^{\beta\gamma}_{(e)} + G^{\beta\gamma}_{(e)} \psi_{i\alpha1\gamma} \right) \right\}_{1\beta}
\]

(5.62)

For the temperature correction $\Theta_{(e)}$, we find at once from (5.35) and (5.58):

\[
N_{(e)} = - \sum_{\alpha\beta} \left\{ \frac{\kappa_{s1} \kappa_{s2} (\Theta_{s2} - \Theta_{s1})}{\kappa_{s1} + \kappa_{s2} + \kappa_{s1} \kappa_{s2} H} \right\}
\]
Thus, from (5.55), noting that $N_{(e)}$ is independent of $Y^3$, we have

$$
\Theta_{(e)} = \frac{\Gamma_{\alpha\beta\gamma} \Gamma_{\alpha\beta\gamma}}{2 \left[ \kappa_{15} + \kappa_{15} + \kappa_{15} \kappa_{15} H \right]^{2}} \left\{ \begin{array}{l}
2 \kappa_{15} H \left( Y^3 - H_{11} \right) \\
+ \kappa_{15} H (H_{11} - Y^3) + (\kappa_{15} + \kappa_{15} + \kappa_{15} \kappa_{15}) (H_{11} - Y^3)(Y^3 - H_{11}) \end{array} \right\}
$$

(5.63)

For the first order body force $(\phi)$ correction we observe from (5.40) that

$$
L_{(\phi)}^\alpha = - F_{\alpha}^{\alpha}
$$

$$
M_{(\phi)} = N_{(\phi)} = 0
$$

Since $F_{\alpha}^{\alpha}$ is independent of $Y^3$, (5.54) gives at once

$$
G_{[\alpha\beta]} \left[ H^3 \psi_{(\phi)} \right]_{\alpha=\beta} = \left[ H^3 F_{\alpha}^{\alpha} \right]_{\alpha=\beta}
$$

(5.64)

whereas from (5.55)

$$
\Theta_{(\phi)} = 0
$$

(5.65)
For the $\mu$ correction, (5.41) yields

$$L_{(\mu)}^\alpha = -\frac{2}{\alpha} \left[ \bar{\mu}_{(0)} \frac{\partial U_{(0)}}{\partial Y^3} \right]$$

$$M_{(\mu)} = N_{(\mu)} = 0$$

We observe at once that

$$\Theta_{(\mu)} = 0 \quad (5.66)$$

To find the differential equation for $Y_{(\mu)}$ we must evaluate

$$\left. -6 \left\{ \int_{H_{11}}^{H_{12}} (Y^3 - H_{11})(H_{12} - Y^3) L_{(\mu)}^\alpha dY^3 \right\}_\alpha = \right.$$

$$\left. 12 \left\{ \frac{B^\alpha}{H} \int_{H_{11}}^{H_{12}} \bar{\mu}_{(0)} \left[ Y^3 - \frac{1}{2} (H_{11} + H_{12}) \right] dY^3 \right\}_\alpha \right.$$  

$$\left. + 12 \left\{ C^\alpha_{\beta} \psi_{(0)} \int_{H_{11}}^{H_{12}} \bar{\mu}_{(0)} \left[ Y^3 - \frac{1}{2} (H_{11} + H_{12}) \right]^2 dY^3 \right\}_\alpha \right. \quad (5.67)$$

where the right hand member of this equation is obtained through an integration by parts and use of (5.28) and (5.29).

The dependence of $\bar{\mu}_{(0)}$ on $Y^3$ is relatively simple, since
\[ \bar{\mu}_{(o)} = \bar{\mu}(P_{(o)}, \Theta_{(o)}) \] and since \( P_{(o)} \) is independent of \( Y^3 \) and \( \Theta_{(o)} \) is linear in \( Y^3 \).

Using the expression for \( \Theta_{(o)} \) as given by (5.36), the integrals of (5.67) then assume the form

\[
\int_{-H/2}^{H/2} \bar{\mu}_{(o)} \left[ Y^3 - \frac{1}{2} (H_{11} + H_{12}) \right] dY^3 = \int_{-H/2}^{H/2} \bar{\mu}_{(o)} (P_{(o)}, \Theta_{(o)(aw)} + 5 \Theta_{(o)}) dY^3
\]

\[
\int_{-H/2}^{H/2} \bar{\mu}_{(o)} \left[ Y^3 - \frac{1}{2} (H_{11} + H_{12}) \right]^2 dY^3 = \int_{-H/2}^{H/2} \bar{\mu}_{(o)} (P_{(o)}, \Theta_{(o)(aw)} + 5 \Theta_{(o)}) dY^3
\]

(5.68)

If \( \bar{\mu}(P, \Theta) \) is known as a relatively simple function of its arguments, the integrals (5.68) may be evaluated analytically or numerically for any particular problem, in terms of the quantities \( \Theta_{(o)(aw)} \) and \( \Theta_{(o)} \) given by (5.37) and (5.38). For the present, however, we shall confine our attention to the special case wherein \( \bar{\mu} \) is a linear function of \( P \) and \( \Theta \).

Specifically, we write

\[
\bar{\mu}(P, \Theta) = P \frac{\partial \bar{\mu}}{\partial P} + \Theta \frac{\partial \bar{\mu}}{\partial \Theta}
\]

(5.69)

where \( \frac{\partial \bar{\mu}}{\partial P} \) and \( \frac{\partial \bar{\mu}}{\partial \Theta} \) are assumed to be fixed constants. In assuming that \( K_{\bar{\mu}} \) is small, we are implicitly agreeing to consider only small relative changes in \( \bar{\mu} \), so that the assumption of linearity would appear consistent with the remainder of our approximations. The fact that \( \bar{\mu} \) as defined by (5.69) vanishes
with $P$ and $\Theta$ amounts to nothing more than a definition of the constant $\mu_0$ of (3.50) in terms of the reference levels chosen for pressure and temperature measurements.

Substitution of (5.69) into (5.68) yields

$$\int_{H_0}^{H(z)} \bar{\mu}(z) \left[ \psi^3 - \frac{1}{2} (H_0 + H(z)) \right] \, \psi \, \, d\psi^3 = \frac{H_0^3}{12} \frac{\partial \psi}{\partial \psi^3} \frac{\partial \mu}{\partial \psi}$$

$$\int_{H_0}^{H(z)} \bar{\mu}(z) \left[ \psi^3 - \frac{1}{2} (H_0 + H(z)) \right]^2 \, \psi \, \, d\psi^3 = \frac{H_0^3}{12} \left[ \psi^3 \frac{\partial \mu}{\partial \psi} + \Theta \frac{\partial \psi}{\partial \psi^3} \right]$$

(5.70)

The combination of (5.54), (5.67), and (5.70) then provides the basic differential equation for $\psi_{(\mu)}$; namely

$$G_{(\mu)}^{\alpha \beta} \left[ H^3 \psi_{(\mu)} \right]_{\alpha \beta} = \frac{\partial \bar{\mu}}{\partial \psi} \left\{ H^3 \psi_{(\mu)} \right\}_1 \left[ G_{(\mu)}^{\alpha \beta} \psi_{(\mu)} \right]_{\alpha \beta}$$

$$+ \frac{\partial \bar{\mu}}{\partial \psi} \left\{ H^2 \psi_{(\mu)} B^\alpha + H^3 \psi_{(\mu)} \right\}_1 \left[ G_{(\mu)}^{\alpha \beta} \psi_{(\mu)} \right]_{\alpha \beta}$$

(5.71)

To obtain the $\rho$ correction, we note from (5.42) that

$$L_{(\rho)}^\alpha = 0$$

$$M_{(\rho)} = - \frac{\partial \bar{\psi}_{(\rho)}}{\partial \psi} - U_{(\rho)}^\alpha \frac{\partial \psi}{\partial \psi^3} - U_{(\rho)}^3 \frac{\partial \bar{\psi}_{(\rho)}}{\partial \psi^3}$$

$$N_{(\rho)} = 0$$
From the third of these equations

\[ \Theta_{(p)} = 0 \]  \hspace{1cm} (5.72)

Also we have

\[ -12 \int_{H_{11}}^{H_{12}} M_{(p)} \, d\gamma^3 = 12 \int_{H_{11}}^{H_{12}} \frac{\partial \rho_{(o)}}{\partial \gamma} \, d\gamma^3 \]

\[ + 12 \int_{H_{11}}^{H_{12}} U_{(0) \lambda} \rho_{(o)} \, d\gamma^3 + 12 \int_{H_{11}}^{H_{12}} U_{(o) \gamma} \frac{\partial \rho_{(o)}}{\partial \gamma} \, d\gamma^3 \]  \hspace{1cm} (5.73)

We first observe that

\[ \int_{H_{11}}^{H_{12}} \frac{\partial \rho_{(o)}}{\partial \gamma} \, d\gamma^3 = \frac{2}{H_{11}} \int_{H_{11}}^{H_{12}} \rho_{(o)} \, d\gamma^3 + \frac{2H_{11}}{\partial \gamma} \left( \rho_{(o)} \right)_{\gamma^3 = H_{11}} \]

\[ - \frac{\partial H_{12}}{\partial \gamma} \left( \rho_{(o)} \right)_{\gamma^3 = H_{12}} \]

Using the formulae (4.15) and (4.16) for \( \frac{\partial H_{11}}{\partial \gamma} \) and \( \frac{\partial H_{12}}{\partial \gamma} \), this relation becomes

\[ \int_{H_{11}}^{H_{12}} \frac{\partial \rho_{(o)}}{\partial \gamma} \, d\gamma^3 = \frac{2}{H_{11}} \int_{H_{11}}^{H_{12}} \rho_{(o)} \, d\gamma^3 + \left[ -V_{\alpha}^{\alpha} H_{11} \lambda + V_{\alpha}^{\gamma} \right] \left( \rho_{(o)} \right)_{\gamma^3 = H_{11}} \]

\[ - \left[ -V_{\alpha}^{\alpha} H_{12} \lambda + V_{\alpha}^{\gamma} \right] \left( \rho_{(o)} \right)_{\gamma^3 = H_{12}} \]  \hspace{1cm} (5.74)
We now integrate the third term of (5.73) by parts, and by means of (5.22) obtain

\[
\int_{H_{1i}}^{H_{12}} U_{10}^3 \frac{\partial \overline{\rho}_{10}}{\partial \eta^3} \ d\eta^3 = V_{12}^3 \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{12}} - V_{1i}^3 \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{1i}} \\
- \int_{H_{1i}}^{H_{12}} \overline{\rho}_{10} \frac{\partial U_{10}^3}{\partial \eta^3} \ d\eta^3 \\
= V_{12}^3 \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{12}} - V_{1i}^3 \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{1i}} \\
+ \int_{H_{1i}}^{H_{12}} \overline{\rho}_{10} U_{10}^\alpha \ d\eta^3
\]

From this result

\[
\int_{H_{1i}}^{H_{12}} U_{10}^\alpha \overline{\rho}_{10} \ d\eta^3 + \int_{H_{1i}}^{H_{12}} U_{10}^3 \frac{\partial \overline{\rho}_{10}}{\partial \eta^3} \ d\eta^3 = \\
\int_{H_{1i}}^{H_{12}} \left[ \overline{\rho}_{10} U_{10}^\alpha \right]_{\eta^3 = H_{12}} \ d\eta^3 + V_{12}^3 \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{12}} - V_{1i}^3 \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{1i}} = \\
\left\{ \int_{H_{1i}}^{H_{12}} \overline{\rho}_{10} U_{10}^\alpha \ d\eta^3 \right\}_{\eta^3 = H_{12}} + \left[ V_{12}^3 - H_{12} U_{12i}^\alpha \right] \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{12}} \\
- \left[ V_{1i}^3 - H_{1i} U_{1i}^\alpha \right] \left[ \overline{\rho}_{10} \right]_{\eta^3 = H_{1i}}
\]

(5.75)
Combining (5.73), (5.74), and (5.75) we then obtain for \( \psi(p) \) the equation

\[
G_{\alpha\beta}^{[H^3 \psi(p)_{1\alpha}]} \left[ \begin{array}{c} H_{11} \\ H_{12} \end{array} \right]_{1\alpha} = 12 \frac{\partial}{\partial \omega} \int_{H_{11}}^H r_{1\alpha} \, d\Psi^3 + 12 \left\{ \int_{H_{11}}^H r_{1\alpha} U_{1\alpha} \, d\Psi^3 \right\}_{1\alpha}
\]

(5.76)

The integrals of (5.76) may be treated in an analogous manner to those of (4.68). Considering in detail only the special case in which \( \rho \) is linear in \( P \) and \( \Theta \), we may write

\[
\rho(p, \Theta) = P \frac{\partial \rho}{\partial P} + \Theta \frac{\partial \rho}{\partial \Theta}
\]

(5.77)

where \( \frac{\partial \rho}{\partial P} \) and \( \frac{\partial \rho}{\partial \Theta} \) are regarded as constants. Using this relation in (5.76) we find

\[
G_{\alpha\beta}^{[H^3 \psi(p)_{1\alpha}]} \left[ \begin{array}{c} H_{11} \\ H_{12} \end{array} \right]_{1\alpha} = 12 \frac{\partial}{\partial P} \frac{\partial}{\partial \omega} (H \psi_{1\alpha})

+ 12 \frac{\partial}{\partial \Theta} \frac{\partial}{\partial \omega} (H \Theta_{1\alpha}(m)) + \left\{ \left[ C_{\alpha \beta} - H^3 C_{\alpha \beta} \right] \right\}_{1\alpha}

\[
+ \frac{\partial}{\partial \Theta} \left\{ \Theta_{1\alpha} H^2 B_{1\alpha} \right\}_{1\alpha}
\]

(5.78)
From (5.43),

\[
L^\alpha_{(R)} = \frac{\partial U^\alpha_{(0)}}{\partial \xi^3} + U_{(0)} \frac{\partial U^\alpha_{(0)}}{\partial \xi^3} + U_{(0)}^3 \frac{\partial U^\alpha_{(0)}}{\partial \xi^3}
\]

\[
M_{(R)} = N_{(R)} = 0
\]

To calculate \( \int_{H_{11}}^{H_{12}} (Q^3 - H_{11})(Q^3 - H^3) L^\alpha_{(R)} d\xi^3 \) we proceed directly; however, the calculation is somewhat simplified by use of \( \xi \), rather than \( \xi^3 \), as the variable of integration.

From (5.31), noting that in general \( \xi \) depends upon \( \xi^3 \), we find for \( \frac{\partial U^\alpha_{(0)}}{\partial \xi^3} \) the expression

\[
\frac{\partial U^\alpha_{(0)}}{\partial \xi^3} = \left[ \frac{1}{2} \frac{\partial A^\alpha}{\partial \xi^3} - \frac{\partial}{\partial \xi^3} \left( \frac{H^2}{2} G_{(0)}^{\alpha \lambda \mu} \psi_{(0) \mu} \right) + \frac{B^\alpha}{H} \frac{\partial \xi}{\partial \xi^3} \right]
\]

\[
+ 5 \left[ \frac{\partial}{\partial \xi^3} \left( \frac{\dot{B}^\alpha}{H} \right) + G_{(0)}^{\alpha \lambda \mu} \psi_{(0) \mu} \frac{\partial \xi}{\partial \xi^3} \right] + 5 \left[ \frac{1}{2} G_{(0)}^{\alpha \lambda \mu} \frac{\partial \psi_{(0) \mu}}{\partial \xi^3} \right]
\]

Thus, by direct integration,

\[
\int_{H_{11}}^{H_{12}} (Q^3 - H_{11})(Q^3 - H^3) \frac{\partial U^\alpha_{(0)}}{\partial \xi^3} d\xi^3 =
\]

\[
\int_{-\frac{H}{2}}^{\frac{H}{2}} \left( \frac{H^2}{4} - \xi^2 \right) \frac{\partial U^\alpha_{(0)}}{\partial \xi^3} d\xi =
\]

(continued)
\[ \begin{align*}
&= \frac{H^3}{12} \frac{\partial A^\alpha}{\partial \xi} - \frac{H^5}{60} \tilde{G}^{\alpha \lambda} \frac{\partial \psi_{\alpha\lambda}}{\partial \xi}
&\quad - \frac{H^4}{24} \frac{\partial H}{\partial \xi} \tilde{G}^{\alpha \lambda} \psi^{\lambda(\alpha)\beta} + \frac{H^2}{6} \frac{\partial B^\alpha}{\partial \xi} B^\alpha \tag{5.79}
\end{align*} \]

The covariant derivative \( U^\alpha_{\lambda(\alpha)\beta} \) is found from (5.31) to be

\[ \begin{align*}
U^\alpha_{\lambda(\alpha)\beta} &= \left[ \frac{1}{2} A^\alpha_{\lambda(\alpha)\beta} - \left( \frac{H^2}{8} \tilde{G}^{\alpha \lambda} \psi^{\lambda(\alpha)\beta} \right)_{\lambda(\alpha)\beta} + \frac{B^\alpha}{H} \xi_{\lambda(\alpha)\beta} \right] \\
&\quad + \frac{5}{8} \left[ \left( \frac{B^\alpha}{H} \right)_{\lambda(\alpha)\beta} + \tilde{G}^{\alpha \lambda} \psi^{\lambda(\alpha)\beta} \xi_{\lambda(\alpha)\beta} \right] + \frac{5}{8} \left[ \frac{1}{2} G_{\lambda(\alpha)\beta} \psi^{\lambda(\alpha)\beta} \right] \tag{5.80}
\end{align*} \]

Carrying out the required integrations, and collecting terms where possible, equations (5.31) and (5.80) yield

\[ \int \frac{H}{4} \left( \frac{H^2 - 5^2}{2} \right) U^\beta_{\alpha(\alpha)\beta} \psi^{\alpha(\alpha)\beta} \psi^{\beta(\beta)\beta} = \frac{H^3}{24} A^\beta A^\alpha + \frac{H^3}{12} \xi_{\lambda(\alpha)\beta} A^\beta B^\alpha \]

\[- \frac{H^5}{120} A^\beta \tilde{G}^{\alpha \lambda} \psi^{\lambda(\alpha)\beta} - \frac{H^5}{120} A^\alpha_{\lambda(\alpha)\beta} \tilde{G}^{\alpha \lambda} \psi^{\lambda(\alpha)\beta} \]

\[- \frac{H^4}{48} H_{\lambda(\alpha)\beta} A^\beta \tilde{G}^{\alpha \lambda} \psi^{\lambda(\alpha)\beta} + \frac{H^4}{120} B^\beta \left( \frac{B^\alpha}{H} \right)_{\lambda(\alpha)\beta} \]

(continued)
\[- \frac{H^4}{60} \xi_{1\beta} B^\alpha G_{\alpha\beta}^{\beta\pi} \psi_{(111)}^\pi + \frac{H^4}{120} \xi_{1\beta} B^\beta G_{\alpha\beta}^{\alpha\lambda} \psi_{(111)}^\lambda + \frac{H^7}{560} G_{\alpha\beta}^{\alpha\lambda} G_{\alpha\beta}^{\beta\pi} \psi_{(111),\pi}^\pi \psi_{(111)}^\pi \]

\[+ \frac{H^6}{240} H_{1\beta} G_{\alpha\beta}^{\alpha\lambda} G_{\alpha\beta}^{\beta\pi} \psi_{(111)}^\lambda \psi_{(111)}^\pi \]

\[(5.31)\]

In order to calculate the necessary integral for the last term of \( L^\alpha_{(R)} \), we use equations (5.32) and (5.33), giving \( U_{(10)}^3 \) in terms of \( \xi^\mu \). We then obtain

\[
\int \frac{H^4}{4} \left( \frac{H^2}{4} - \xi^2 \right) U_{(10)}^3 \frac{\partial U_{(10)}^{\alpha}}{\partial \xi^\beta} d \xi = \left\{ \frac{H^2}{12} (V_{(1)}^3 + V_{(2)}^3) + \frac{H^4}{60} \left( \frac{B^\beta}{H} \right)_{1\beta} \right. \\
+ \frac{H^6}{60} G_{\alpha\beta}^{\beta\pi} \psi_{(111)}^\pi \xi_{1\beta}^\pi \right\} B^\alpha + \left\{ - \frac{H^5}{240} A_{1\beta}^\alpha + \frac{H^7}{120} G_{\alpha\beta}^{\alpha\lambda} \psi_{(111),\pi}^\lambda \psi_{(111)}^\pi \right. \\
+ \frac{H^6}{480} H_{1\beta} G_{\alpha\beta}^{\beta\pi} \psi_{(111)}^\pi \psi_{(111)}^\lambda G_{\alpha\beta}^{\alpha\lambda} \psi_{(111)}^\lambda \left\} \right.
\]

\[(5.32)\]
The desired integral for $L^{\alpha}_{(R)}$ may now be found by the combination of (5.79), (5.81), and (5.82). Using the auxiliary formulae

$$\frac{\partial \xi}{\partial \nu} = -\frac{1}{2} \left( \frac{\partial H_{(1)}}{\partial \nu} + \frac{\partial H_{(2)}}{\partial \nu} \right)$$

$$\xi_{1\beta} = -\frac{1}{2} \left( H_{(1)} \beta + H_{(2)} \beta \right)$$

which are obtained from (5.30), we find

$$\int_{H_{(1)}}^{H_{(2)}} (q^{3} - H_{(1)} - q^{3}) L^{\alpha}_{(R)} \, dq^{3} = \int_{H_{(1)}}^{H_{(2)}} \left( \frac{H^{3}}{3} - \xi^{2} \right) L^{\alpha}_{(R)} \, d\xi^{3} =$$

$$\frac{H^{3}}{12} \frac{\partial A^{\alpha}}{\partial \nu} - \frac{H^{5}}{60} G_{\alpha \beta} \frac{\partial Y_{(0)1\beta}}{\partial \nu} - \frac{H^{4}}{24} \frac{\partial H_{(1)}}{\partial \nu} G_{\alpha \beta} \psi_{(0)1\beta}$$

$$+ \frac{H^{3}}{24} A^{\gamma} A_{1\beta}^{\alpha} - \frac{H^{5}}{120} A^{\beta} G_{\alpha \beta} \psi_{(0)1\beta} - \frac{H^{5}}{120} A_{1\beta}^{\alpha} G_{\alpha \beta}^{\gamma} \psi_{(0)1\beta}$$

$$- \frac{H^{4}}{48} H_{1\beta} A^{\beta} G_{\alpha \beta}^{\gamma} \psi_{(0)1\beta} - \frac{H^{5}}{240} A_{1\beta}^{\alpha} G_{\alpha \beta}^{\gamma} \psi_{(0)1\beta}$$

$$+ \left[ \frac{H^{7}}{120} G_{\alpha \beta}^{\gamma} G_{\alpha \beta}^{\eta} \psi_{(0)1\beta} \psi_{(0)1\beta} \right]_{1\beta} + \frac{H^{7}}{120} G_{\alpha \beta}^{\gamma} G_{\gamma \beta}^{\eta} \psi_{(0)1\beta} \psi_{(0)1\beta}$$

$$+ \frac{H^{3}}{120} B^{\gamma} B_{1\beta}^{\alpha} + \frac{H^{2}}{60} H_{1\beta} B^{\beta} B^{\alpha} + \frac{H^{3}}{60} B_{1\beta}^{\beta} B^{\alpha}$$
Noting that $M_{(R)} = 0$, the basic partial differential equation for $\psi_{(R)}$ may now be found by substitution of the above expression into (5.54). We thus have

\[
G_{\omega}^{\alpha\beta} \left[ H^3 \psi_{(R)\lambda\alpha} \right]_{\beta} = -\frac{1}{2} \left[ H^3 \frac{\partial A^\alpha}{\partial \tau} \right]_{\lambda\alpha} \\
+ \frac{1}{10} \left[ G_{\omega}^{\alpha\lambda} H^5 \frac{\partial \psi_{(0)\lambda\beta}}{\partial \tau} \right]_{\lambda\alpha} + \frac{1}{4} G_{\omega}^{\alpha\lambda} \left[ H^4 \frac{\partial H}{\partial \tau} \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} \\
- \frac{1}{4} \left[ H^3 A^\beta A^\alpha_{\lambda\beta} \right]_{\lambda\alpha} + \frac{1}{20} G_{\omega}^{\alpha\lambda} \left[ H^5 A^\beta \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} \\
+ \frac{1}{20} G_{\omega}^{\beta\lambda} \left[ H^5 A^\alpha_{\lambda\beta} \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} + \frac{1}{5} G_{\omega}^{\alpha\lambda} \left[ H^4 H_{\lambda\beta} A^\beta \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} \\
+ \frac{1}{40} G_{\omega}^{\alpha\lambda} \left[ H^5 A^\beta \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} - \frac{3}{560} G_{\omega}^{\alpha\lambda} G_{\omega}^{\beta\pi} \left[ H^7 \psi_{(0)\lambda\beta} \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} \\
- \frac{3}{560} G_{\omega}^{\alpha\lambda} G_{\omega}^{\beta\pi} \left[ H^7 \psi_{(0)\lambda\beta} \psi_{(0)\lambda\beta} \right]_{\lambda\alpha} - \frac{1}{20} \left[ H^3 B^\beta B^\alpha_{\lambda\beta} \right]_{\lambda\alpha} \\
- \frac{1}{10} \left[ H^2 H_{\lambda\beta} B^\beta B^\alpha_{\lambda\beta} \right]_{\lambda\alpha} - \frac{1}{10} \left[ H^3 B^\beta B^\alpha_{\lambda\beta} \right]_{\lambda\alpha} \quad (5.84)
\]
From the fact that \( N^{(R)} = 0 \) we have at once from (5.55)

\[
\Theta^{(R)} = 0
\]

(5.85)

For the \( \lambda \), \( \eta \), and \( k \) correction terms, equations (5.44) to (5.46) give

\[
\begin{align*}
L^\alpha_{(\eta)} &= L^\alpha_{(\lambda)} = L^\alpha_{(k)} = 0 \\
M_{(\eta)} &= M_{(\lambda)} = M_{(k)} = 0
\end{align*}
\]

(5.86)

and

\[
N_{(\eta)} = -G_{(0)\alpha\beta} \frac{\partial U^\alpha_{(0)}}{\partial \eta^3} \frac{\partial U^\beta_{(0)}}{\partial \eta^3}
\]

(5.87)

\[
N_{(\lambda)} = \Gamma_{(0) \alpha \beta} \left\{ \frac{\partial \Theta_{(0)}}{\partial \xi} + U^\alpha_{(0)} \Theta_{(0) \alpha \beta} + U^\beta_{(0)} \frac{\partial \Theta_{(0)}}{\partial \eta^3} \right\}
\]

(5.88)
\[ N_{(k)} = -\frac{\partial}{\partial \varphi} \left\{ k(p_{(\omega)}, \Theta_{(\omega)}) \frac{\partial\Theta_{(\omega)}}{\partial \varphi} \right\} \]  

(5.89)

In the present investigation our interest in the temperature \( \Theta \) arises primarily from our interest in calculating fluid velocities and stresses. Now temperature variations within the system can arise in two ways; namely, by imposition of non-uniform temperatures externally (i.e. on the walls) and by the generation of heat within the fluid through the action of viscosity. The first type of variation is represented by \( \Theta_{(\omega)} \), and the second by \( \Theta_{(\eta)} \). The \( \lambda \) and \( k \) terms then occur only as corrections to \( \Theta_{(\omega)} \), and it will be observed from (5.88) and (5.89) that \( \Theta_{(\lambda)} = \Theta_{(k)} = 0 \) whenever \( \Theta_{(\omega)} = 0 \). Also (5.86), together with a subsequent examination of the boundary conditions to be satisfied on \( \Omega \), shows that

\[
\begin{align*}
\{V_{(\eta)}^\alpha &= U_{(\omega)}^\alpha = U_{(k)}^\alpha = 0 \\
P_{(\eta)} &= P_{(\omega)} = P_{(k)} = 0
\}
\]  

(5.90)

If the walls are at a non-uniform temperature, then \( \Theta_{(\omega)} \neq 0 \) and temperature corrections for the fluid stresses will be found from \( U_{(\omega)}^\alpha \), \( P_{(\omega)} \), \( U_{(\rho)}^\alpha \), and \( P_{(\rho)} \), which have already been calculated. If, on the other hand, the walls are at a constant temperature, which may be chosen to be zero by an appropriate selection of the reference level, then \( \Theta_{(\omega)} = 0 \) and the lowest
order temperature corrections in the stresses will be those arising from $U_{\eta}^{\kappa}$, $P_{\mu \eta}$, $U_{\rho \eta}^{\kappa}$, and $P_{\rho \eta}$ If $\Theta_{(\omega)} \neq 0$ then the $\lambda$ and $\kappa$ terms may be neglected in comparison with those arising from $\Theta_{(\omega)}$ itself; if $\Theta_{(\omega)} = 0$ the first order $\lambda$ and $\kappa$ terms vanish identically. Thus, in either event, since we wish to derive only the first non-vanishing corrections in each physical situation, the quantities $\Theta_{(\lambda)}$ and $\Theta_{(\kappa)}$ may be neglected.

In the remainder of this section, we therefore calculate $\Theta_{(\eta)}$, $P_{\mu \eta}$, and $P_{\rho \eta}$, concluding with a derivation of formulæ for the stresses at the boundary surfaces in terms of velocity and pressure.

Equation (5.31) for $U_{(\omega)}^{\kappa}$ in terms of $\xi$, together with the definition (5.87) of $N_{(\eta)}$ gives

$$N_{(\eta)} = - C_{\omega \lambda \nu \beta} \left[ \frac{B^\alpha}{H} + \frac{G_{\omega \lambda}^{\eta \alpha} Y_{\nu \lambda \omega}}{H} \right] \left[ \frac{B^\beta}{H} + \frac{G_{\omega \beta}^{\eta \beta} Y_{\nu \lambda \omega}}{H} \right]$$

(5.91)

By a straightforward, but quite tedious, application of the general formula (5.55) we obtain $\Theta_{(\eta)}$ in the form

$$\Theta_{(\eta)} = \frac{1}{48 H (k_{s1} + k_{s2} + k_{s1} k_{s2} H)} \left\{ 12 G_{\omega \lambda \nu \beta} B^\alpha B^\beta + H \gamma_{(1)} C_{\omega \lambda}^{\alpha \beta} Y_{\nu \lambda \omega} \right\} \times \left[ 4 + (k_{s1} + k_{s2}) H - 2 (k_{s2} - k_{s1}) + \xi \right]$$

(continued)
\[-4H^2 B^\alpha \psi_{101 \alpha} \left[ (\kappa s_2 - \kappa s_1) H - 2(\kappa s_1 + \kappa s_2) \frac{s}{3} \right] \] \\
\[+ \left( \frac{H^2}{4} - \frac{s^2}{3} \right) \left\{ \frac{1}{2H^2} G_{\alpha \beta} B^\alpha B^\beta + \frac{5}{3H} B^\alpha \psi_{101 \alpha} \right\} \]
\[+ \frac{1}{12} \left( \frac{H^2}{4} + \frac{s^2}{3} \right) G_{\alpha \beta} \psi_{101 \alpha} \psi_{101 \beta} \]

(5.92)

Substitution of the series expressions (5.16) and (5.17) into (3.52), (3.54), and (3.57) gives the following system of equations for $U^\alpha_{(\mu \eta)}, U^3_{(\mu \eta)}, P_{(\mu \eta)}$:

\[\frac{\partial^2 U^\alpha_{(\mu \eta)}}{(\partial \eta^3)^2} - G_{\alpha \beta} P_{(\mu \eta) 1 \beta} = - \frac{2}{2\eta^3} \left[ \frac{\partial U^\alpha_{(\mu \eta)}}{\partial \eta^3} \right] \]
\[\frac{\partial P_{(\mu \eta)}}{\partial \eta^3} = 0 \]
\[U^\alpha_{(\mu \eta) 1 \alpha} + \frac{\partial U^3_{(\mu \eta)}}{\partial \eta^3} = 0 \]

(5.93)

where, in writing these expressions, use has been made of the fact that $U^\alpha_{(\eta)} = 0$. Thus, from the second of these equations,
we may write

$$P_{(\omega \eta)} = \Psi_{(\omega \eta)}(4', 4''')$$

(5.94)

and the analysis leading to equation (5.54) is applicable.

Equations (5.93) yield

$$L_{(\omega \eta)}^\alpha = -\frac{2}{\partial^3} \left[ \bar{\mu}_{(\omega \eta)} \frac{\partial u_{(\omega \eta)}^\alpha}{\partial x^3} \right]$$

and

$$M_{(\omega \eta)} = 0$$

By an integration by parts

$$\int_{H_{ii}}^{H_{ii}} (4'^3 - H_{ii}) (4''^3 - 4'^3) L_{(\omega \eta)}^\alpha \, d4'^3 = \int_{-H_{ii}}^{H_{ii}} \left( \frac{1}{2} 4^{'2} - 4'^3 \right) L_{(\omega \eta)}^\alpha \, d4'^3$$

$$= -\int_{-H_{ii}}^{H_{ii}} \left( \frac{1}{4} (H_{ii}^2 - 4'^2) \frac{\partial}{\partial x^3} \left[ \bar{\mu}_{(\omega \eta)} \frac{\partial u_{(\omega \eta)}^\alpha}{\partial x^3} \right] \right) \, d4'^3$$

$$= -2 \int_{-H_{ii}}^{H_{ii}} \bar{\mu}_{(\omega \eta)} \frac{\partial u_{(\omega \eta)}^\alpha}{\partial x^3} \, d4'^3$$

(5.95)
The methods used in obtaining (5.19) give

\[ \bar{\mu}(\eta) = \frac{\partial \mu(P_{\omega}, \Theta_{\omega})}{\partial P} P_{\eta} + \frac{\partial \mu(P_{\omega}, \Theta_{\omega})}{\partial \Theta} \Theta_{\eta} \]

\[ = \frac{\partial \bar{\mu}(P_{\omega}, \Theta_{\omega})}{\partial \Theta} \Theta_{\eta} \]

(5.96)

since \( P_{\eta} = 0 \) by (5.90). The problem of evaluating the integral (5.95) is thus analogous to that previously encountered in the integrals (5.68) for the \( \mu \) corrections. If, as before, we assume \( \bar{\mu} \) to be linear in \( P \) and \( \Theta \), then \( \frac{\partial \bar{\mu}}{\partial \Theta} \) reduces to a constant and the calculation is greatly simplified.

With this assumption and the methods employed previously, we therefore obtain the following differential equation for \( \psi(\mu, \eta) \):

\[ G_{\alpha \beta}^{3} \left[ H^{2} \psi(\mu, \eta) \right] \mid \beta = \frac{\partial \bar{\mu}}{\partial \Theta} \left\{ \frac{H^{2}}{30} B^{\alpha} B^{\beta} \psi_{\omega 1} \right\}

+ G_{\alpha \beta}^{\lambda \lambda} \psi_{\omega 1} \left[ \frac{H^{2}}{20} G_{\omega 1 \beta \gamma} B^{\beta} B^{\gamma} + \frac{H^{2}}{336} G_{\omega 1 \beta \gamma} \psi_{\omega 1 3} \psi_{\omega 1 1 3} \right]

+ \frac{1}{F_{\omega 1} + F_{\omega 2} + F_{\omega 1} F_{\omega 2} H} \left[ \frac{H^{4}}{4} G_{\omega 1 \beta \gamma} B^{\beta} B^{\gamma} \left[ H(4 + (F_{\omega 1} + F_{\omega 2}) H) G_{\omega 1 \beta \gamma} \psi_{\omega 1} \right]

- 2(F_{\omega 1} + F_{\omega 2}) B^{\alpha} \right] \quad (continued)
\[ + \frac{H^3}{12} B^\alpha \Psi_{(10)\beta} \left[ 2(k_{1s} + k_{2s}) B^\alpha - H^2(k_{2s} - k_{1s}) G_{(03)}^\alpha \Psi_{(10)\beta} \right] \]

\[ + \frac{H^3}{48} G_{(03)}^{\alpha \beta} \Psi_{(10)\beta} \Psi_{(10)\gamma} \left[ H(4 + (k_{1s} + k_{2s}) H) G_{(03)}^\alpha \Psi_{(10)\beta} \right] \]

\[-3(k_{2s} - k_{1s}) B^\alpha \right) \right] \right) \right]_{\alpha} \]

Finally, the \((\rho \eta)\) corrections are to be calculated from the following equations:

\[ \frac{\partial^2 U_{(\rho \eta)}^\alpha}{(\partial \Psi^3)^2} - G_{(03)}^{\alpha \beta} \left[ P_{(\rho \eta)\beta} \right] = 0 \]

\[ \frac{\partial P_{(\rho \eta)}}{\partial \Psi^3} = 0 \]

\[ U_{(\rho \eta)\alpha} + \frac{\partial U_{(\rho \eta)}^3}{\partial \Psi^3} = - \frac{\partial \tilde{P}_{(\rho \eta)}}{\partial \tilde{L}} - U_{(\rho \eta)\alpha} \tilde{P}_{(\rho \eta)\alpha} \]

\[ - U_{(\rho \eta)}^3 \frac{\partial \tilde{P}_{(\rho \eta)}}{\partial \Psi^3} \]
These equations may be treated in exactly the same manner as those for $U_{(\rho)}^\gamma$, $P_{(\rho)}$, $U_{(\rho)}^3$, giving, by comparison with (5.76),

$$G_{\alpha \beta} \left[ H^{\alpha} \eta^{(\eta) \beta} \right]_{13} = 12 \frac{\partial}{\partial \eta} \int_{H_{\eta \gamma}}^{H_{\eta \gamma}} \bar{\rho}(\eta) \, d\eta \, d^3 \eta$$

$$+ 12 \left\{ \int_{H_{\eta \gamma}}^{H_{\eta \gamma}} \bar{\rho}(\eta) U_{(\rho)}^\alpha \, d\eta \, d^3 \eta \right\} \bar{\eta}$$

(5.98)

In the special case that $\bar{\rho}$ is a linear function of $\rho$ and $\Theta$, we have

$$\bar{\rho}(\eta) = \frac{\partial \bar{\rho}}{\partial \eta} \Theta(\eta)$$

(5.99)

where $\frac{\partial \bar{\rho}}{\partial \eta}$ is to be considered a constant.

Because of the small magnitude of this correction, in comparison with the effect of temperature on viscosity as represented by the $(\mu \eta)$ term, we consider only the simplified case of (5.98) for which

$$k_{31} = k_{32} = \infty$$

(5.100)

With this simplification the substitution of (5.99) into (5.98), together with equations (5.31) and (5.92) for $U_{(\rho)}^\alpha$ and $\Theta(\eta)$,
then yields the following differential equation for \( \psi_{(\rho \eta)} \):

\[
G_{\alpha \beta} \left[ H^3 \psi_{(\rho \eta)1\beta} \right]_{1\alpha} = \frac{2}{\Theta} \frac{\partial}{\partial t} \left\{ H G_{\alpha \beta} B^\beta B^\gamma \right. \\
+ \frac{H^5}{20} G_{\alpha \beta} \psi_{1\alpha 1\beta} \psi_{1011\gamma} \left\} + \frac{2}{\Theta} \left\{ A^\alpha \left[ \frac{H}{2} G_{\alpha \beta} B^\beta B^\gamma \right. \\
+ \frac{H^5}{40} G_{\alpha \beta} \psi_{1011\beta} \psi_{1011\gamma} \right] + B^\alpha \left[ \frac{H^3}{30} B^\beta \psi_{1011\beta} \right] \\
- G_{\alpha \beta} \psi_{1011\beta} \left[ \frac{H^3}{10} G_{\alpha \beta} B^\beta B^\gamma + \frac{H^5}{50} G_{\alpha \beta} \psi_{1011\beta} \psi_{1011\gamma} \right] \left\} \right. \\
- G_{\alpha \beta} \psi_{1011\beta} \left[ \frac{H^3}{10} G_{\alpha \beta} B^\beta B^\gamma + \frac{H^5}{50} G_{\alpha \beta} \psi_{1011\beta} \psi_{1011\gamma} \right] \left\} \right. \\
- G_{\alpha \beta} \psi_{1011\beta} \left[ \frac{H^3}{10} G_{\alpha \beta} B^\beta B^\gamma + \frac{H^5}{50} G_{\alpha \beta} \psi_{1011\beta} \psi_{1011\gamma} \right] \left. \right. \\
(5.101)
\]

The differential equations (5.34), (5.62), (5.64), (5.71), (5.73), (5.84), (5.97), and (5.101) for \( \psi_1 \), \( \psi_2 \), \( \psi_3 \), \( \psi_4 \), \( \psi_5 \), \( \psi_6 \), \( \psi_7 \), \( \psi_8 \), \( \psi_9 \), \( \psi_{10} \), \( \psi_{11} \), \( \psi_{12} \), \( \psi_{13} \), \( \psi_{14} \), \( \psi_{15} \), and \( \psi_{16} \), respectively, form the basic
system which has been the objective of this analysis. Aside from the assumption of the convergence of our series procedure, the additional assumptions or approximations which have been made in deriving these equations are the following:

(a) that \( \mu \) and \( \bar{\rho} \) are linear functions of \( P \) and \( \theta \) (made in deriving the equations for \( \Psi_{(\mu)} \), \( \Psi_{(p)} \), \( \Psi_{(\mu\eta)} \), and \( \Psi_{(p\eta)} \))

(b) that \( k_{s1} = k_{s2} = 00 \) (made in deriving the equation for \( \Psi_{(p\eta)} \))

Neither of these approximations is essential to the success of the derivation. The integrals of the type of (5.68) involving \( \mu \) and \( \bar{\rho} \) can be evaluated analytically, as long as \( \mu \) and \( \bar{\rho} \) are known explicitly as functions of \( P \) and \( \theta \); if finite heat transfer coefficients are assumed, no more serious obstacle than added algebraic complexity of the equation for \( \Psi_{(p\eta)} \) will result. It is also clear that, in principle, this analysis may be extended to include correction terms of any desired order without introducing essentially new techniques. It should be equally clear at this point that the algebraic labor involved in calculating many more terms would be prohibitive, at least as long as the present degree of generality is retained. For specific problems of a more restricted scope, however, it is possible that the derivation of higher order corrections would be quite feasible.

The partial differential equations developed in this section are to be solved over a region of the reference surface \( S \) enclosed within a boundary curve \( \mathcal{B} \), or else over two or more such regions
connected. The boundary condition of the type (4.11), applicable when the average pressure is a prescribed function of arc \( \Theta \) on \( \mathcal{B} \), is to be satisfied identically in the parameters \( \epsilon, \ldots, \kappa \).
For the correction terms of the order considered, \( P \) is independent of \( Y^3 \), and equation (4.11) gives simply

\[
\psi_{(1)} + \epsilon \psi_{(\epsilon)} + \phi \psi_{(\phi)} + K_m \psi_{(m)} + K_\rho \psi_{(\rho)}
\]

\[+ R \psi_{(R)} + \eta K_m \psi_{(m\eta)} + \eta K_\rho \psi_{(\rho\eta)} = P_{(\mathcal{B})}(\Theta, \tau)\]

where the various \( \psi \)'s are to be evaluated for points \( (Y^1, Y^2) \) on \( \mathcal{B} \). Since \( P_{(\mathcal{B})}(\Theta, \tau) \) does not depend on the parameters, we must have

\[
\psi_{(1)} = P_{(\mathcal{B})}(\Theta, \tau) \quad (5.102)
\]

and

\[
\psi_{(\epsilon)} = \psi_{(m)} = \psi_{(\phi)} = \psi_{(R)} = \psi_{(m\eta)} = \psi_{(\rho\eta)} = 0 \quad (5.103)
\]
as the boundary conditions to be satisfied by the individual \( \psi \)'s on \( \mathcal{B} \). It should be noted, however, that equations of the type of (5.103) will not hold for all higher order corrections.
To formulate the boundary conditions of the type (4.12), in which the rate of fluid flow across \( \mathcal{B} \) is prescribed, we first note that

\[
(1 + K_\rho \bar{\rho}) U' = U'^{(\rho)} + \varepsilon [U'^{i(\varepsilon)}] + \phi [U'^{i(\phi)}] + K_\mu [U'^{(\mu)}] + K_\rho [U'^{(\rho)} + \bar{\rho}(\rho) U'^{(\rho)}] + R [U'^{(R)}] \\
+ \eta [U'^{(\eta)}] + \lambda [U'^{(\lambda)}] + K_k [U'^{(k)}] \\
+ \eta K_\mu [U'^{(\mu \eta)}] + \eta K_\rho [U'^{(\rho \eta)} + \bar{\rho}(\eta) U'^{(\eta)}] \\
+ \frac{H^2}{2} \Lambda - \frac{H^3}{12} G_{i(\text{rot})} \Psi_{i(\text{rot})} \lambda
\]  

(5.104)

where only those terms of the order considered in this section have been included. Evaluation of (4.12) then requires the computation of the integral of this expression between \( S_1 \) and \( S_2 \).

The integral of \( U'^{(\rho)} \) may be calculated directly from (5.31), giving

\[
\int_{H_{i(\lambda)}}^{H_{i(\lambda)}} U'^{(\rho)} \, d\gamma^3 = \frac{H}{2} \Lambda - \frac{H^3}{12} G_{i(\text{rot})} \Psi_{i(\text{rot})} \lambda
\]

while from (4.46), for the various correction terms,

\[
\int_{H_{i(\lambda)}}^{H_{i(\lambda)}} U'^{(\rho)} \, d\gamma^3 = -\frac{H^3}{12} G_{i(\text{rot})} \Psi_{i(\text{rot})} \lambda
\]

\[
- \frac{1}{2} \int_{H_{i(\lambda)}}^{H_{i(\lambda)}} (\gamma^3 - H_{i(\lambda)}) (H_{i(\lambda)} - \gamma^3) L_{i(\lambda)} \, d\gamma^3
\]
The quantities \( \int_{\Sigma_{12}} \mathcal{R}_{\alpha} \mathcal{U}_{\mu} \, d\gamma \) and \( \int_{\Sigma_{12}} \mathcal{P}_{\eta} \mathcal{U}_{\mu} \, d\gamma \) have been calculated in the course of deriving the \( \rho \) correction equation (5.78), and that for \( \rho \eta \), (5.101). Thus all of the integrals involved in the substitution of (5.104) into the boundary condition (4.12) have already been evaluated.

Performing this substitution, and collecting like terms, we readily find the following set of boundary conditions for the individual \( \Psi \)'s:

\[
H^3 n_{1\beta\alpha} G_{\alpha\lambda} \Psi_{\eta\lambda\lambda} = 6 H n_{\beta\lambda\lambda} A^\lambda
\]

\[-12 \frac{m(\beta, T)}{\rho u_0 n_0} \]  

\[
H^3 n_{\beta\alpha} G_{\alpha\lambda} \Psi_{\eta\lambda\lambda} = H^2 n_{1\beta\alpha} \left[ \gamma_{\lambda\beta} B^\lambda + 2 \gamma_{\lambda\beta} B^\beta \right]
\]

\[ - \frac{H^3}{2} (H_{11} + H_{12}) n_{1\beta\alpha} G_{\alpha\lambda} \Psi_{\eta\lambda\lambda} \]  

(5.105)

(5.106)

\[
H^3 n_{1\beta\alpha} G_{\alpha\lambda} \Psi_{\eta\lambda\lambda} = H^3 n_{1\beta\alpha} F_{\eta\alpha}
\]  

(5.107)
\[ H^3 \eta_{(\beta)\alpha} G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} = \frac{2\mu}{\Delta P} \psi_{(\omega)} H^3 \eta_{(\beta)\alpha} G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} \]

\[ + \frac{2\mu}{\Delta \Theta} \eta_{(\beta)\alpha} \left[ H^2 \Theta_{(\omega)} B^\alpha + H^3 \Theta_{(\omega)(\mu)(\lambda)} G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} \right] \]  \hspace{1cm} (5.108)

\[ H^3 \eta_{(\beta)\alpha} G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} = \frac{2\phi}{\Delta \Theta} H^2 \Theta_{(\omega)} \eta_{(\beta)\alpha} B^\alpha \]

\[ + \eta_{(\beta)\alpha} \left[ C \mu A^\lambda - H^3 G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} \right] \left[ \frac{2\phi}{\Delta P} \psi_{(\omega)} + \frac{2\phi}{\Delta \Theta} \Theta_{(\omega)(\mu)(\lambda)} \right] \]  \hspace{1cm} (5.109)

\[ H^3 \eta_{(\beta)\alpha} G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} = \eta_{(\beta)\alpha} \left\{ - \frac{H^3}{2} \frac{2A^\lambda}{\Delta \Theta} \right\} \]

\[ + \frac{H^5}{10} G^{\alpha^\lambda}_{(\omega)} \frac{\partial \psi_{(\omega)\lambda}}{\partial \Theta} + \frac{H^4}{4} \frac{\partial A}{\partial \Theta} G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} \]

\[ - \frac{H^3}{4} A^\beta A^\alpha_{13} + \frac{H^5}{20} A^\beta G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} B \]

\[ + \frac{H^5}{20} A^\alpha_{13} G^{(3\pi)}_{(\omega)} \psi_{(\mu)\lambda} \pi + \frac{H^4}{8} H_{13} A^\beta G^{\alpha^\lambda}_{(\omega)} \psi_{(\mu)\lambda} \]

\[ \text{(continued)} \]
\[ + \frac{H^5}{40} A_\beta^\alpha \psi_{(1)\lambda} - \frac{3}{560} G_{\alpha \beta} G_{\alpha \beta} \left[ H^3 \psi_{(1)\lambda} \psi_{(1)\mu} \right]_{\beta} \]

\[ - \frac{3H^7}{560} G_{\alpha \beta} G_{\alpha \beta} \psi_{(1)\lambda} \psi_{(1)\mu} - \frac{H^3}{20} B^\beta B^\alpha \]

\[ - \frac{H^2}{10} H_{1\beta} B^\beta B^\alpha - \frac{H^3}{10} B^\beta B^\alpha \}

(5.110)

Also, we find that

\[ H^3 \eta_{(1)\alpha} G_{\alpha \beta}^{\lambda \gamma} \psi_{(1)\mu} = 0 \]

\[ H^3 \eta_{(1)\alpha} G_{\alpha \beta}^{\lambda \gamma} \psi_{(1)\mu} = 0 \]

Equations (5.103) and (5.110.1) show that the boundary conditions of both the first and second kinds for \( \psi_{(1)} \), \( \psi_{(2)} \), and \( \psi_{(3)} \).
are homogeneous; equations (5.86) show that the differential
equations for these functions are also homogeneous. Therefore,

\[ \psi(\eta) = \psi(\lambda) = \psi(\kappa) = 0 \quad (5.110.2) \]

For \( \psi(\mu\eta) \) we have

\[
H^3 \eta_{\alpha\beta\alpha} G_{\alpha\beta}^{\alpha\gamma} \psi(\mu\eta)_{\gamma\lambda} = \frac{2}{\delta^2} H \eta_{\alpha\beta} \left\{ \frac{H^3}{36} B^\alpha B^\beta \psi(\omega)_{\omega\beta} \\
+ G_{\gamma\delta} \psi(\omega)_{\gamma\delta} \left[ \frac{H^3}{20} G_{\omega\delta} \beta \gamma B^\beta B^\gamma + \frac{H^2}{336} G_{\omega\delta} \psi(\omega)_{\omega\beta} \psi(\omega)_{\omega\gamma} \right] \right\}
\]

\[
+ \frac{2}{\delta^2} H \eta_{\alpha\beta} \left\{ \frac{H}{4} G_{\gamma\delta} \beta \gamma B^\beta B^\gamma \left[ H \left( 4 + (\bar{\kappa}_1 + \bar{\kappa}_2) H \right) G_{\gamma\delta}^{\alpha\lambda} \psi(\omega)_{\omega\lambda} \\
- 2 (\bar{\kappa}_1 + \bar{\kappa}_2) B^\alpha \right] + \frac{H^3}{12} B^\beta \psi(\omega)_{\omega\beta} \left[ 2 (\bar{\kappa}_1 + \bar{\kappa}_2) B^\alpha \\
- H^2 (\bar{\kappa}_2 - \bar{\kappa}_1) G_{\gamma\delta}^{\alpha\lambda} \psi(\omega)_{\omega\lambda} \right] \right\}
\]

\[
+ \frac{H^5}{48} G_{\gamma\delta}^{\beta\gamma} \psi(\omega)_{\omega\beta} \psi(\omega)_{\omega\gamma} \left[ H \left( 4 + (\bar{\kappa}_1 + \bar{\kappa}_2) H \right) G_{\gamma\delta}^{\alpha\lambda} \psi(\omega)_{\omega\lambda} - 3 (\bar{\kappa}_2 - \bar{\kappa}_1) B^\alpha \right]\}
\]

\( (5.111) \)
Finally, the boundary condition for $\psi_{(\eta)}$ is

$$H^3 n_{(\beta)\alpha} G_{(\alpha)}^{\alpha \lambda} \psi_{(\eta)\lambda} = \frac{\partial}{\partial \eta} n_{(\beta)\alpha} \left\{ A^K \left[ \frac{H}{2} G_{(\alpha \beta)} B^K B^\gamma \right] + \frac{H^5}{40} G_{(\alpha \beta)} \psi_{(\gamma)\beta} \psi_{(\gamma)\beta} \right\} + B^K \left[ \frac{H^3}{30} B^K \psi_{(\gamma)\beta} \right]$$

$$- G_{(\alpha)}^{\alpha \lambda} \psi_{(\eta)\lambda} \left[ \frac{H^3}{10} G_{(\alpha \beta)} B^K B^\gamma \right] + \frac{H^5}{210} G_{(\alpha \beta)} \psi_{(\gamma)\beta} \psi_{(\gamma)\beta} \right\}$$

(5.112)

In summary, if the region of $S$ under consideration is enclosed by a single boundary curve $\partial S$, and the average pressure is specified at each point of $\partial S$, the boundary conditions (5.102) and (5.103) should be employed. If, on the other hand, the rate of fluid flow across $\partial S$ is prescribed, the appropriate boundary conditions are those given by (5.105) to (5.112). The boundary conditions arising in the problem of the intersection of two
regions, as given in general form by (4.13) and (4.14), may
be written in terms of \( \Psi_{(1)} \), \( \Psi_{(2)} \), etc. by a simple and
obvious modification of (5.102), (5.103), and (5.105) to
(5.112).

Since one of the primary objectives of this analysis is the
calculation of the net forces and moments exerted by the lubricating fluid on the surfaces with which it is in contact, we
conclude this section with a derivation of formulae for the
fluid stresses at \( S_1 \) and \( S_2 \). In particular, we seek the
"x" and the "y" components of the force vector exerted by
the fluid on elements of surface area \( ds_1 \) and \( ds_2 \).

As before, \( n_{(1)} \) and \( n_{(2)} \) denote the unit vectors normal
to \( S_1 \) and \( S_2 \), directed away from the fluid mass. In the
dimensionless notation, the components of these vectors are
found from (4.6) to be

\[
\begin{align*}
\eta_{(1)\alpha} &= \frac{\varepsilon H_{(1)\alpha}}{D_{(1)}} \\
\eta_{(2)\alpha} &= -\frac{\varepsilon H_{(2)\alpha}}{D_{(2)}} \\
\eta_{(1)3} &= -\frac{\varepsilon}{D_{(1)}} \\
\eta_{(2)3} &= \frac{\varepsilon}{D_{(2)}}
\end{align*}
\]

(5.113)

where

\[
\begin{align*}
D_{(1)} &= \sqrt{1 + \varepsilon^2 \left[ G^{\beta \gamma}_{\alpha \gamma} + \varepsilon H_{(1)} G^{\beta \gamma}_{\alpha \gamma} + \ldots \right] H_{(1)\beta} H_{(1)\gamma}} \\
D_{(2)} &= \sqrt{1 + \varepsilon^2 \left[ G^{\beta \gamma}_{\alpha \gamma} + \varepsilon H_{(2)} G^{\beta \gamma}_{\alpha \gamma} + \ldots \right] H_{(2)\beta} H_{(2)\gamma}}
\end{align*}
\]

(5.114)
Let $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ be the values of the stress tensor, evaluated for points on $S_1$ and $S_2$. The force exerted by the fluid on the area $dS_1$ will then be

$$-\sigma_{ij}^{(1)} n_{ij} dS_1,$$

and that exerted on $dS_2$

$$-\sigma_{ij}^{(2)} n_{ij} dS_2.$$

In evaluating net forces and moments it is convenient to base calculations upon elements of area in the reference surface $S$, since these are related in a natural way to the differentials of the coordinates $\nu^\alpha$, and also are independent of time. If we let $d\mathbf{s}$ denote the projection of $dS_1$ upon $S$, then

$$dS_1 = D_{i(i)} d\mathbf{s}$$

and similarly

$$dS_2 = D_{i(j)} d\mathbf{s}.$$

We then may write

$$\left\{ \begin{align*}
\mathbf{T}_{(1)}^i d\mathbf{s} &= -\sigma_{ij}^{(1)} n_{ij} dS_1, \\
\mathbf{T}_{(2)}^i d\mathbf{s} &= -\sigma_{ij}^{(2)} n_{ij} dS_2
\end{align*} \right\}$$

(5.115)
where $\mathbf{F}_{(m)}^i$ and $\mathbf{F}_{(n)}^i$ represent the stress vectors acting on $d\mathbf{s}_1$ and $d\mathbf{s}_2$, referred, however, to $d\mathbf{s}$ as the element of area. Application of the previous equations for $d\mathbf{s}_1$ and $d\mathbf{s}_2$ then gives

$$
\begin{align*}
\mathbf{F}_{(m)}^i &= -\mathbf{F}_{(n)}^i \cdot D_{mn} \cdot n_{mn}^j \\
\mathbf{F}_{(n)}^i &= -\mathbf{F}_{(m)}^i \cdot D_{nm} \cdot n_{nm}^j
\end{align*}
$$

(5.116)

The substitution of the equations (5.113) for the normal vectors into the above relations yields

$$
\begin{align*}
\mathbf{F}_{(m)}^\lambda &= \mathbf{A} \left[ \mathbf{R}_{(m)}^\lambda \cdot \mathbf{A} - \mathbf{R}_{(n)}^\lambda \cdot \mathbf{A} \cdot H_{(m)1}\beta \right] \\
\mathbf{F}_{(n)}^3 &= \mathbf{A} \left[ \mathbf{R}_{(m)}^3 \cdot \mathbf{A} - \mathbf{R}_{(n)}^3 \cdot \mathbf{A} \cdot H_{(n)1}\beta \right] \\
\mathbf{F}_{(n)}^\lambda &= -\mathbf{A} \left[ \mathbf{R}_{(m)}^\lambda \cdot \mathbf{A} - \mathbf{R}_{(n)}^\lambda \cdot \mathbf{A} \cdot H_{(n)1}\beta \right] \\
\mathbf{F}_{(n)}^3 &= -\mathbf{A} \left[ \mathbf{R}_{(m)}^3 \cdot \mathbf{A} - \mathbf{R}_{(n)}^3 \cdot \mathbf{A} \cdot H_{(n)1}\beta \right]
\end{align*}
$$

(5.117)

From (2.3) and (2.4)

$$
\mathbf{R}^i = -\mathbf{q}^i / \mathbf{A} - \frac{n}{3} \mu \mathbf{q}^i \mathbf{u}^k,^k + \mu \mathbf{q}^i \mathbf{u}^k,^k + \mu \mathbf{q}^i \mathbf{u}^j,^k
$$

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\[ \Gamma^{\alpha\beta} = -\frac{g^{\alpha\beta}}{\rho} - \frac{2}{3} \mu g^{\alpha\beta}(u^r_{,r} + u^3_{,3}) \]

\[ + \mu g^{\beta\gamma} u^\gamma_{,r} + \mu g^{\alpha\gamma} u^\gamma_{,r} \]

\[ \Gamma^{3\beta} = \mu \frac{u^\beta_{,3}}{\varepsilon^2} + \mu g^{\beta\gamma} u^3_{,r} \]

\[ \Gamma^{33} = \frac{1}{\varepsilon^2} \left\{ \frac{1}{\rho} - \frac{2}{3} \mu u^r_{,r} + \frac{4}{3} \mu u^3_{,3} \right\} \]

Expressing these results in the dimensionless notation of Section 3, we then derive by means of (3.12), (3.13), (3.28) to (3.31), (3.49), and (3.50) the following expressions:

\[ \Gamma^{\alpha\beta} = \frac{\mu u^\beta_{,3}}{H_0^2} \left\{ \left[ -G^{\alpha\beta}_{\omega} \right] + \varepsilon \left[ -\left(\frac{4}{3}\right) G^{\alpha\beta}_{\omega} P \right] \right. \]

\[ + O(\varepsilon^2, \varepsilon^3 K_{\mu}) \right\} \]

\[ \Gamma^{3\beta} = \frac{\mu u^\beta_{,3}}{H_0^2} \frac{1}{\varepsilon} \left\{ \varepsilon \left[ \frac{2 u^\alpha}{\partial y^3} \right] + \varepsilon K_{\mu} \left[ \mu \left[ \frac{2 u^\alpha}{\partial y^3} \right] \right] \right. \]

\[ + O(\varepsilon^3, \varepsilon^3 K_{\mu}) \right\} \]

\[ \Gamma^{33} = \frac{\mu u^\beta_{,3}}{H_0^2} \frac{1}{\varepsilon^2} \left\{ \left[ -P \right] + \varepsilon^2 \left[ -\frac{2}{3} u^r_{,r} + \frac{4}{3} \mu \frac{2 u^3}{\partial y^3} \right] \right. \]

\[ + O(\varepsilon^3, \varepsilon^2 K_{\mu}) \right\} \]
The substitution of these results into (5.117) yields

\[
\frac{\alpha^\infty}{\alpha^{(1)}} = \frac{\mu u_0 L_0}{\hbar_0} \left\{ \mathcal{E} \left[ \frac{\partial u^\infty}{\partial y^3} + P G^{\alpha\beta}_{1(0)} H_{1111} \right] + \mathcal{E} \mathcal{K}_\mu \left[ -\frac{\partial u^\infty}{\partial y^3} \right] + O(\mathcal{E}^3, \mathcal{E}^3 \mathcal{K}_\mu) \right\} \\
(5.118)
\]

\[
\frac{\alpha^3}{\alpha^{(1)}} = \frac{\mu u_0 L_0}{\hbar_0} \frac{1}{\mathcal{E}} \left\{ \left[ -P \right] + O(\mathcal{E}^2) \right\} \\\n(5.119)
\]

\[
\frac{\alpha^\infty}{\alpha^{(2)}} = \frac{\mu u_0 L_0}{\hbar_0} \left\{ \mathcal{E} \left[ \frac{\partial u^\infty}{\partial y^3} - P G^{\alpha\beta}_{1(0)} H_{1111} \right] + \mathcal{E} \mathcal{K}_\mu \left[ -\frac{\partial u^\infty}{\partial y^3} \right] + O(\mathcal{E}^3, \mathcal{E}^3 \mathcal{K}_\mu) \right\} \\
(5.120)
\]

\[
\frac{\alpha^3}{\alpha^{(2)}} = \frac{\mu u_0 L_0}{\hbar_0} \frac{1}{\mathcal{E}} \left\{ \left[ P \right] + O(\mathcal{E}^2) \right\} \\\n(5.121)
\]
In these expressions the quantities \( P, \xi, \mu, \) etc are to be evaluated for \( Y^3 = H^{(1)2} \) or \( H^{(2)} \).

Recalling that \( \overrightarrow{\xi}^i \) and \( \overrightarrow{\xi}^i \) are contravariant vectors, and hence that the numerical magnitude of their \( Y^3 \) components is larger by a factor of \( \frac{1}{\varepsilon} \) than the corresponding components in Euclidean coordinates, we see that the effect of factoring \( \frac{1}{\varepsilon} \) from (5.119) and (5.121) is to give terms in like powers of the parameters within the brackets in (5.118) to (5.121) the same physical order of magnitude. Thus we may note at once that the \( Y^3 \) components of stress are \( O(\varepsilon) \) compared with the \( Y^3 \) component. It is also of interest to observe that in the leading term of (5.118),

\[
\mathcal{E} \left[ \frac{\partial \xi^\alpha}{\partial Y^3} + P \, G^{\alpha \beta}_{\nu} H_{\mu \nu \beta} \right]
\]

and the corresponding term of (5.120), the quantity \( \frac{\partial \xi^\alpha}{\partial Y^3} \) may be interpreted directly as the contribution to the component of the stress vector tangent to \( S \) arising from viscous drag, while the quantity \( P G^{\alpha \beta}_{\nu} H_{\mu \nu \beta} \) corresponds to the action of a normal pressure \( P \) against a surface which is not quite parallel to the reference surface. The latter term is, of course, \( O(\varepsilon) \) compared with the main effect of \( P \).

In substituting the basic series expressions of (5.16) and (5.17) into the equations for \( \overrightarrow{\xi}^3 \) and \( \overrightarrow{\xi}^{-3} \), we retain only the linear terms together with those in \( P_{(\mu \nu)} \) and \( P_{(\rho \nu)} \). The inclusion of the \( \mathcal{E}^2 \) terms of (5.119) and (5.121) would
be inconsistent with our previous work, since the term involving $P_{(EE)}$ has not been evaluated. We readily obtain

$$
\Phi_{(1w)}^{(3)} = \frac{\mu_0}{\hbar_0} \frac{L_0}{\hbar} \left\{ \left[-\chi_{(w)} + \epsilon [\chi_{(E)}] + \phi [\chi_{(\phi)}] \right.ight.
\left. + K_\mu [\chi_{(\mu)}] + K_\rho [\chi_{(\rho)}] + R [\chi_{(R)}] \right.
\left. + \eta K_\mu [\chi_{(\mu\eta)}] + \eta K_\rho [\chi_{(\rho\eta)}] \right\} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \righthand side of equation (5.118) and equation (5.120) are 0(\epsilon) to begin with. Performing the required series substitutions, we find from (5.118)
\[ \frac{\partial \phi}{\partial t} = \frac{\mu_0 u_0 L_0}{h_0} \left\{ \frac{1}{\epsilon} \left[ (\frac{\partial U^{(\phi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] \\
+ \epsilon^2 \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} + \psi(t \phi) H_{\psi^3 \psi^3} G_{\psi^3} H_{\psi^3 \psi^3} \right] \\
+ \epsilon \phi \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] + \epsilon K_{\psi} \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] \\
+ (\bar{\psi}(t \phi))_{\psi^3} \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] + \epsilon K_{\psi} \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] \\
+ \epsilon \phi \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] + \epsilon K_{\psi} \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] \\
+ \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} + (\bar{\psi}(t \phi))_{\psi^3} \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] + \epsilon K_{\psi} \left[ (\frac{\partial U^{(\psi)}}{\partial \psi^3})_{\psi^3} + \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right] \\
+ \psi(t \phi) G_{\psi^3} H_{\psi^3 \psi^3} \right\} \right] \\
\right) \\
\text{(5.124)} \]

where \( \frac{\partial U^{(\phi)}}{\partial \psi^3} \), \( \frac{\partial U^{(\psi)}}{\partial \psi^3} \), \( \bar{\psi}(t \phi) \), \( \bar{\phi}(t \phi) \), etc. indicate the values of the quantities in parentheses at \( Y^3 = H(1) \).
Similarly,

\[ \hat{G}_{\alpha \beta}^{12} = -\frac{M_0 m_0 L_0}{\hbar_0^2} \left\{ \epsilon \left[ \left( \frac{\partial U_{10}}{\partial y^1} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} \right] \\
+ \epsilon^2 \left[ \left( \frac{\partial U_{10}}{\partial y^2} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} + \psi_{10} H_{1211\beta} G^{\alpha \beta}_{0\alpha} H_{1211\beta} \right] \\
+ \epsilon \phi \left[ \left( \frac{\partial U_{10}}{\partial y^3} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} \right] \\
+ \epsilon \phi \left[ \left( \frac{\partial U_{10}}{\partial y^3} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} + (\bar{u}_{10})_{12} \left( \frac{\partial U_{10}}{\partial y^3} \right)_{12} \right] \\
+ \epsilon \phi \left[ \left( \frac{\partial U_{10}}{\partial y^3} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} \right] \\
+ \epsilon \phi \left[ \left( \frac{\partial U_{10}}{\partial y^3} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} \right] \right\} \\
+ \epsilon \phi \left[ \left( \frac{\partial U_{10}}{\partial y^3} \right)_{12} + \psi_{10} G^{\alpha \beta}_{0\alpha} H_{1211\beta} \right] \}

(5.125)
where the factor \((-1)\) which arises in each term has been taken outside of the brackets for convenience.

By means of the various techniques used previously in this section, we derive the following expressions for the quantities \((\frac{\partial U_{(1)}}{\partial \eta^3})_{(11)}\), etc., required in the evaluation of (5.124) and (5.125):

\[
(\frac{\partial U^\kappa_{(1)}}{\partial \eta^3})_{(11)} = \frac{B}{H} - \frac{H}{2} G_{\alpha \omega \lambda} \psi_{(\omega)1\lambda}
\]  

(5.126)

\[
(\frac{\partial U^\kappa_{(1)}}{\partial \eta^3})_{(12)} = \frac{B}{H} + \frac{H}{2} G_{\alpha \omega \lambda} \psi_{(\omega)1\lambda}
\]  

(5.127)

\[
(\frac{\partial U^\kappa_{(1)}}{\partial \eta^3})_{(11)} = -\frac{H}{2} G_{\alpha \omega \lambda} \psi_{(\omega)1\lambda} + \frac{B}{2} \Gamma_{\omega \lambda}^\gamma \gamma + B\beta \Gamma_{\alpha \lambda}^\gamma \gamma
\]

\[-\frac{H}{4} (H_{11} + H_{12}) G_{\alpha \omega \lambda} \psi_{(\omega)1\lambda} - \frac{H}{2} \Gamma_{\omega \lambda}^\gamma \gamma G_{\alpha \omega \lambda} \psi_{(\omega)1\lambda}
\]

\[+ \frac{H}{6} G_{\alpha \omega \lambda} \psi_{(\omega)1\lambda}
\]  

(5.128)
\[
\left( \frac{\partial U_0^\kappa}{\partial \phi} \right)_{\lambda} = \frac{H}{2} G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} - \frac{B^\kappa}{2} \Gamma_0^{\gamma} \rho_{3\gamma} - B^\beta \Gamma_0^{\gamma} \rho_{3\beta}
\]

\[+ \frac{H}{4} (H_{\lambda\lambda} + H_{\lambda\gamma}) G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} - \frac{H^2}{12} \Gamma_0^{\gamma} \rho_{3\gamma} G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} \]

\[+ \frac{H^2}{6} G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} \]  (5.129)

\[
\left( \frac{\partial U_0^\kappa}{\partial \theta} \right)_{\lambda} = -\frac{H}{2} G_{\omega\omega}^{\kappa\lambda} \psi_{(\mathbf{1})\lambda} + \frac{H}{2} F_{\omega\omega}^{-\kappa} \]  (5.130)

\[
\left( \frac{\partial U_0^\kappa}{\partial \theta} \right)_{\lambda} = \frac{H}{2} G_{\omega\omega}^{\kappa\lambda} \psi_{(\mathbf{1})\lambda} - \frac{H}{2} F_{\omega\omega}^{-\kappa} \]  (5.131)

\[
\left( \frac{\partial U_0^\kappa}{\partial \phi} \right)_{\lambda} = -\frac{H}{2} G_{\omega\omega}^{\kappa\lambda} \psi_{(\mu)\lambda} + \frac{H}{2} \psi_{(\phi)} G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} \]

\[+ \frac{\partial U_0}{\partial \theta} \left[ \frac{H}{2} \psi_{(\phi)} G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} + \frac{1}{2} \theta_{(\omega)} B^\kappa \right. \]

\[\left. - \frac{H^2}{6} \theta_{(\omega)} G_{\omega\omega}^{\kappa\lambda} \psi_{(\omega)\lambda} \right] \]  (5.132)
\[
\left( \frac{\partial U^{x}_{(w)}}{\partial y^3} \right)_{(1)} = \frac{H}{2} G^{\alpha \lambda}_{(x)} \Psi_{(x)1\lambda} + \frac{2 \mu}{3} \left[ -\frac{H}{2} \Theta_{(o)\lambda} G^{\alpha \lambda}_{(o)} \Psi_{(o)1\lambda} \right]
\]

\[
+ \frac{2 \mu}{3} \left[ -\frac{H}{2} \Theta_{(o)\lambda} G^{\alpha \lambda}_{(o)} \Psi_{(o)1\lambda} + \frac{1}{2} \Theta_{(o)'} B^\alpha \right]
\]

\[
- \frac{H^2}{6} \Theta_{(o)'} G^{\alpha \lambda}_{(o)} \Psi_{(o)1\lambda} \right] \] (5.133)

\[
(\bar{\mu}_{(w)})_{(1)} = \frac{2 \mu}{3} \bar{\Psi}_{(o)} + \frac{2 \mu}{3} \left[ \Theta_{(o)\lambda} - \frac{H}{2} \Theta_{(o)'} \right] (5.134)
\]

\[
(\bar{\mu}_{(w)})_{(2)} = \frac{2 \mu}{3} \bar{\Psi}_{(o)} + \frac{2 \mu}{3} \left[ \Theta_{(o)\lambda} + \frac{H}{2} \Theta_{(o)'} \right] (5.135)
\]

\[
- \left( \frac{\partial U^{x}_{(o)}}{\partial y^3} \right)_{(1)} = \left( \frac{\partial U^{x}_{(o)}}{\partial y^3} \right)_{(2)} = \frac{H}{2} G^{\alpha \lambda}_{(x)} \Psi_{(x)1\lambda} \] (5.136)

\[
\left( \frac{\partial U^{x}_{(o)}}{\partial y^3} \right)_{(o)} = -\frac{H}{2} G^{\alpha \lambda}_{(o)} \Psi_{(o)1\lambda} - \frac{2}{3} \left[ \frac{H}{4} A^\alpha \right]
\]

\[
+ \frac{2}{3} \left[ \frac{H^3}{24} G^{\alpha \lambda}_{(o)} \Psi_{(o)1\lambda} \right] - \left[ \frac{H}{8} A^\alpha A^\beta \right]_{1\beta}
\]

(continued)
\[ \begin{align*}
+ \left\{ \frac{H^3}{48} \left[ A^\alpha G^{\beta\gamma}_{\omega\omega} \Psi_{\omega\omega 1\lambda} + A^\beta G^{\alpha\lambda}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \right] \right\}_{1\beta} \\
- \left[ \frac{H}{24} B^\alpha B^\beta \right]_{1\beta} - \left[ \frac{H^5}{240} G^{\alpha\lambda}_{\omega\omega} G^{\beta\gamma}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \right]_{1\beta} \\
+ \frac{H}{24} B^\beta A^\alpha_{1\beta} + \frac{2}{3} \left[ \frac{H}{12} B^\alpha \right] + \left[ \frac{H}{24} A^\beta B^\alpha \right]_{1\beta} \\
- \left[ \frac{H^3}{240} B^\alpha G^{\beta\gamma}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \right]_{1\beta} - B^\alpha \left[ \frac{H^3}{240} G^{\beta\gamma}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \right]_{1\beta} \\
- \frac{H^2}{240} \left[ H B^\beta G^{\alpha\lambda}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \right]_{1\beta} + \frac{H^3}{120} B^\beta G^{\alpha\lambda}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \\
\end{align*} \]

(5.137)

\[ \left( \frac{\partial \psi_{(\omega)}^{\alpha}}{\partial \psi^3} \right)_{(2)} = \frac{H}{2} G^{\alpha\lambda}_{\omega\omega} \Psi_{(\omega) 1\lambda} + \frac{2}{3} \left[ \frac{H}{4} A^\alpha \right] \\
- \frac{2}{3} \left[ \frac{H^3}{24} G^{\alpha\lambda}_{\omega\omega} \Psi_{(\omega) 1\lambda} \right] + \left[ \frac{H}{8} A^\alpha A^\beta \right]_{1\beta} \\
- \left\{ \frac{H^3}{48} \left[ A^\alpha G^{\beta\gamma}_{\omega\omega} \Psi_{\omega\omega 1\pi} + A^\beta G^{\alpha\lambda}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \right] \right\}_{1\beta} \\
+ \left[ \frac{H}{24} B^\alpha B^\beta \right]_{1\beta} + \left[ \frac{H^5}{240} G^{\alpha\lambda}_{\omega\omega} G^{\beta\gamma}_{\omega\omega} \Psi_{\omega 1\omega 1\lambda} \Psi_{\omega 1\omega 1\lambda} \right]_{1\beta} \\
+ \frac{H}{24} B^\beta A^\alpha_{1\beta} + \frac{2}{3} \left[ \frac{H}{12} B^\alpha \right] + \left[ \frac{H}{24} A^\beta B^\alpha \right]_{1\beta} \]

(continued)
\[- \left[ \frac{H^3}{240} B^\alpha G^{\beta \gamma \delta} \pi_1 \pi_2 \right]_{1\beta} - B^\alpha \left[ \frac{H^3}{120} G^{\beta \gamma \delta} \pi_1 \pi_2 \right]_{1\beta} \]

\[- \frac{H^2}{240} \left[ H B^\beta G^{\alpha \gamma \delta} \pi_1 \pi_2 \right]_{1\beta} + \frac{H^3}{120} B^\beta \left[ B^\gamma \pi_1 \pi_2 \right] \]

\[(\frac{\partial U^{(\text{free})}}{\partial \pi^3})_{0\beta} = - \frac{\mu}{2} G^{\alpha \gamma \delta} \pi_{0\beta} \pi_{1\gamma} \pi_{2\delta} + \frac{\mu}{\delta \theta} \left\{ B^\alpha \left[ \frac{1}{12} H G^{\beta \gamma \delta} B^\alpha B^\beta \right] 

+ \frac{H^3}{240} G^{\beta \gamma \delta} \pi_{0\beta} \pi_{1\gamma} \pi_{2\delta} \right\} + \frac{H^3}{360} G^{\alpha \gamma \delta} \pi_{0\alpha} \pi_{1\gamma} \pi_{2\delta} \left[ B^\beta \pi_{0\beta} \pi_{1\beta} \right] \]

\[+ \frac{H^2}{240} \frac{G^{\alpha \gamma \delta} \pi_{0\beta} \pi_{1\alpha} \pi_{2\gamma} \pi_{3\delta}}{48H (\bar{k}_{n1} \bar{k}_{n2} + \bar{k}_{n2} \bar{k}_{n1} H)} \left\{ \left[ 12 \frac{G^{\beta \gamma \delta}}{\bar{k}_{n1} \bar{k}_{n2} \bar{k}_{n1} H} \right] \right\}

\[+ H^4 G^{\beta \gamma \delta} \pi_{0\beta} \pi_{1\gamma} \pi_{2\delta} \pi_{3\alpha} \left[ 4 \left( \bar{k}_{n1} \bar{k}_{n2} \bar{k}_{n1} H \right) \right] \]

\[- 4 H^3 B^\beta \pi_{0\beta} \pi_{1\beta} \left( \bar{k}_{n2} - \bar{k}_{n1} \right) \]

\[+ \frac{\mu}{\delta \theta} \left[ \frac{B^\alpha}{2} - \frac{H^2}{6} G^{\alpha \gamma \delta} \pi_{1\gamma} \pi_{2\delta} \right] \left\{ -2 \left( \bar{k}_{n2} - \bar{k}_{n1} \right) \left[ 12 \frac{G^{\beta \gamma \delta}}{\bar{k}_{n1} \bar{k}_{n2} \bar{k}_{n1} H} \right] \right\}

+ H^4 G^{\beta \gamma \delta} \pi_{0\beta} \pi_{1\gamma} \pi_{2\delta} \pi_{3\alpha} \left[ 8 H^2 \left( \bar{k}_{n1} \bar{k}_{n2} \bar{k}_{n1} \bar{k}_{n2} \right) B^\beta \pi_{0\beta} \pi_{1\beta} \right] \]

\[\text{(5.138)} \]

\[\text{(5.139)} \]
\[
\begin{align*}
\left( \frac{\partial \psi}{\partial y} \right)_{(y)} &= \frac{H}{2} G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} + \frac{2 \mu}{\xi} \left\{ B^\mu \left[ \frac{1}{12} H G_{\alpha \beta} B^\alpha B^\beta \right. \\
&\left. + \frac{H^2}{240} G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \right] + \frac{H^3}{360} G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \left[ B^\alpha B^\beta \psi_{(\alpha \beta) \gamma} \right] \right\} \\
- &\frac{H^2 \mu}{2} G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \left\{ \left[ 12 G_{\alpha \beta} B^\alpha B^\beta \right. \\
&\left. + H^4 G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \right] \cdot \left[ 4 + (\bar{k}_{s1} + \bar{k}_{s2}) H \right] \right\} \\
- &4 H^3 B^\gamma \psi_{(\alpha \beta) \gamma} (\bar{k}_{s2} - \bar{k}_{s1}) \right\} \\
- &\frac{\left[ \frac{B^\mu}{2} + \frac{H^2}{6} G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \right]}{48 H (\bar{k}_{s1} + \bar{k}_{s2} + \bar{k}_{s1} \bar{k}_{s2} H)} \left\{ -2 (\bar{k}_{s2} - \bar{k}_{s1}) \left[ 12 G_{\alpha \beta} B^\alpha B^\beta \right. \\
&\left. + H^4 G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \right] + 8 H^2 (\bar{k}_{s1} + \bar{k}_{s2}) B^\beta \psi_{(\alpha \beta) \gamma} \right\} \\
&\left( \bar{\mu}(y) \right)_{(y)} = \frac{\partial \bar{\mu}}{\partial y} \left\{ (4 + 2 \bar{k}_{s2} H) \left[ 12 G_{\alpha \beta} B^\alpha B^\beta \right. \\
&\left. + H^4 G_{\alpha \beta} \psi_{(\alpha \beta) \gamma} \right] - 8 H^3 \bar{k}_{s2} B^\beta \psi_{(\alpha \beta) \gamma} \right\} \\
\end{align*}
\]
\[
(\tilde{\mu}_\eta)_{(2)} = \frac{\partial \tilde{\mu}}{\partial \theta} \frac{1}{\eta \delta H(k_{s1} + k_{s2} + k_{s1} + k_{s2})} \left\{ \left[ \eta^2 + 2k_{s1} H \right] \left[ 12 G^{(\eta)}_{(2)} B^{B} B^{\gamma} + H^4 G^{(\eta)}_{(0)} \psi_{(\eta)} \right] \right\} + 8 H^3 k_{s1} B^{B} \psi_{(\eta)} \right\}
\]

(5.142)

\[
\left( \frac{\partial U^{(\eta)}}{\partial y^3} \right)_{(1)} = - \frac{H}{2} G^{(\eta)}_{(0)} \psi_{(\eta)} 1 \gamma
\]

(5.143)

\[
\left( \frac{\partial U^{(\eta)}}{\partial y^3} \right)_{(2)} = \frac{H}{2} G^{(\eta)}_{(0)} \psi_{(\eta)} 1 \gamma
\]

(5.144)
Section 6. Range of Applicability of Theory, and Practical Convergence, of series Expansions:

The discussion of section 1 is concerned with the physical assumptions underlying the foregoing analysis. On the basis of the hypotheses introduced, the formulae derived in Sections 2, 3, and 4 may be strictly justified. In particular, the validity of the series operations of section 3 rests solely upon the convergence of the basic series (3.9) for $\zeta^f$. Hence, if the reference surface $S$ is analytic and if the fluid obeys the assumed laws of viscous motion, the fundamental partial differential equations (3.52), (3.54), (3.57), and (3.66) are justified.

The series expansions of pressure, velocity, etc., in terms of the various parameters, however, are not mathematically rigorous. It is by no means certain, without further examination, that series of the type introduced are convergent. Furthermore, since it is practical to calculate only a very limited number of terms of these series, it is necessary not only that convergence occur, but also that it takes place quite rapidly. No attempt is made in this analysis to prove rigorous convergence of our results; instead it is preferred to indicate the class of physical problems wherein it may be expected that a reasonable degree of convergence occurs. With this aim, in view we consider in this section the order of magnitude of the parameters $\varepsilon, \phi, \kappa_p, \kappa_r, \gamma, \lambda$ and $K_w$ and the other dimensionless variables which have been employed.
The basic parameters depend on the quantities \( h_0, L_o, u_0, \rho_o, \mu_0, K_o, C_p, T_o \) and \( f_o \). The quantities \( K_\rho, K_p \) and \( K_\kappa \) are defined only implicitly; the remaining parameters are defined only implicitly; the remaining parameters are defined as follows:

\[
\begin{align*}
\varepsilon &= \frac{h_o}{L_o} \\
\phi &= \frac{h_o^2 \rho_o f_o}{\mu_0 u_o} \\
R &= \frac{\rho_o u_0 h_o^2}{\mu_0 L_o} \\
\eta &= \frac{\mu_0 u_o}{K_o T_o} \\
\lambda &= \frac{\rho_o u_0 h_o^2 C_p}{K_o L_o}
\end{align*}
\]  

(6.1)

Some of the physical constants of the problem are fluid properties which are approximately the same under all conditions of temperature and pressure met in practice. The remainder may vary considerably with the type of problem involved. In the following the discussion is confined to lubricating oils for which the values

\[
\begin{align*}
\rho_o &= 0.9 \text{ gm/cm}^3 \\
K_o &= 1.5 \times 10^4 \text{ erg/sec-cm-}^\circ C \\
C_p &= 2 \times 10^7 \text{ erg/gm-}^\circ C
\end{align*}
\]  

(6.2)

may be assumed independent of the problem under discussion. In addition, the body force exerted by gravity is

\[
f_o = 980 \text{ cm/sec}^2
\]  

(6.3)

The coefficient of viscosity \( \mu_o \) is subject to considerable variation; however a reasonable range is

\[
\mu_o = 1 - 10 \text{ gm/cm-sec} = 1 - 10 \text{ poise}
\]
(a) **Magnitude of ε:**

It is difficult to specify an exact magnitude for the parameter $\varepsilon$, without reference to a particular practical problem. This parameter is always small in problems of the lubrication type; a reasonable range might be

$$0.001 < \varepsilon < 0.1$$

It is expected that at most the linear terms in $\varepsilon$ need be employed in calculations to within ordinary engineering accuracy; in many important cases even these may safely be neglected.

(b) **Magnitude of $\phi$:**

It is to be noted that $p_0 f_0$ represents the gradient of the hydrostatic pressure induced by a body force $f_0$. Then the quantity $\frac{h_0^2}{12 \mu_0} p_0 f_0$ is the average velocity which would be set up in a fluid subjected to this body force and confined between two stationary parallel planes at a distance $h_o$ apart. From the relation

$$\phi = 12 \frac{h_0^2}{12 \mu_0} \frac{p_0 f_0}{u_0}$$

it is thus clear that $\phi$ may be regarded as proportional to the ratio of this hypothetical velocity, under the action of body forces only, to the velocity $u_0$. If the true flow is primarily dominated by a pressure gradient of magnitude $|\nabla p|$ then

$$\phi = 12 \frac{p_0 f_0}{|\nabla p|}$$

In practice $\phi$ is generally quite small. For example, letting $h_o = 0.005 \text{ cm.}$, $\mu_0 = 1 \text{ gm/cm-sec,}$
\( u_0 = 10 \text{ cm/sec} \), and employing the values (6.2) and (6.3) for \( \rho_0 \) and \( f_0 \), we find \( \phi = 0.0022 \).

(c) **Magnitude of** \( R \):

Assuming a minimum viscosity of one poise, letting \( h_o = 0.01 \text{ cm} \), and writing \( R \) in the form we find

\[
R = \frac{\rho_0 h_o}{\mu_o} (u_0 \epsilon) \quad \text{we find} \quad R = 0.009 u_0 \epsilon
\]

where \( u_0 \) is to be expressed in \( \text{cm/sec} \). Since \( \epsilon \) is a small number, it is clear that only at very high speeds is \( R \) of importance. In addition, we obtain from (6.1) and (6.2)

\[
\lambda = 1.33 \times 10^3 \mu_o R
\]

where \( \mu_o \) is measured in poise. It is thus evident from the magnitude of \( \lambda \) that the series procedure for \( \lambda \) is invalidated at much lower speeds than \( R \).

(d) **Magnitude of** \( K_\rho, K_\kappa \):

Because of the implicit manner in which these parameters have been defined, we must examine the quantities

\[
K_\rho \bar{\rho} = \frac{\rho}{\rho_0} - 1 \quad \quad K_\kappa \bar{k} = \frac{k}{k_0} - 1
\]

rather than the parameters themselves. The compressibility of oil is represented by an increase in density of the order of magnitude of one part in ten thousand per atmosphere pressure. A ten percent change in density would therefore require approximately 15,000 pounds per square inch applied pressure. The coefficient of
thermal expansion of oil is of the order of magnitude of 0.001 per degree Centigrade; thus a temperature variation of 100°C would be responsible for a change of about 10% in $\overline{\rho}$. From these considerations, a maximum value for $K\overline{\rho}$ would appear to be 0.1, with a normal range of values in the vicinity of 0.01. Since $\overline{\rho}$ is assumed to be of the order of magnitude of unity, similar values for $K\rho$ prevail.

Specific data relating to the variation of the thermal conductivity of lubricating oils has not been found by the author. However, the thermal conductivity of organic liquids in general appears to be relatively insensitive to changes in temperature and pressure (see, for example, McAdams\textsuperscript{23}, pp. 17-18, 321-322). Since the influence of variable $K$ plays a distinctly secondary role in our analysis, we are probably safe in neglecting terms in $K_k$.

(e) **Magnitude of $\lambda$:**

Assuming $h_0 = 0.01$ cm,

$$\lambda = 12 \cdot u_0 \varepsilon$$

where $u_0$ is expressed in cm/sec. This condition limits $u_0$ to the order of magnitude of a few cm/sec., if $\lambda$ is to be small. Under more favorable conditions, with $h_0 = 0.025$ cm, $\varepsilon = 0.002$,

$$\lambda = 0.006 \cdot u_0$$

and a maximum permissible $u_0$ is of the order of 50 cm/sec. We must therefore conclude that $\lambda$ has a definite tendency toward finite values, indicating that in general lubrication analysis the effect of thermal convection is important in determining
temperature distributions.

(f) Magnitudes of $\eta$ and $K \mu$:

Since there is a very close relation between temperature and viscosity, it is appropriate to discuss $\eta$ and $K \mu$ together. In the first place, a maximum change of perhaps 10% in the viscosity may be expected to arise from changes in pressure; this may generally be neglected in comparison with changes arising from temperature variation.

A 10% change in viscosity may arise from a change of as little as $2^\circ C$. When variations in viscosity are restricted to this magnitude, the quantity $T_o$ in the formula for $\eta$ may be assigned the value $2^\circ C$, whence

$$\eta = \frac{\mu_o u_o^2}{3 \times 10^4}$$

with $\mu_o$ and $u_o$ in c.g.s. units. It is therefore clear that velocities of 100 cm/sec. or larger will invalidate the assumption that $\eta$ is small in comparison with unity.

By increasing the permissible variation in $\mu$, somewhat larger velocities may be tolerated. In general the important parameter is not $\eta$ or $K \mu$ by itself, but the combination $\eta K \mu \frac{\partial \mu}{\partial T}$. Assuming a linear relation between viscosity and temperature, we find

$$\eta K \mu \frac{\partial \mu}{\partial T} = \frac{u_o^2}{k_o} \frac{\partial \mu}{\partial T}$$

When $\frac{\partial \mu}{\partial T}$ is restricted to a range of 0.1 poise $/^\circ C$ to 0.5 poise $/^\circ C$ and $k_o = 1.5 \times 10^4$ erg/sec-cm-$^\circ C$, it is seen that $\eta K \mu \frac{\partial \mu}{\partial T}$ in general lies between $\frac{u_o^2}{1.5 \times 10^5}$ and $\frac{u_o^2}{3 \times 10^4}$, showing that, necessarily, $u_o < 100$ cm/sec, if the approximate analysis of the thermal problem is to be used.
We conclude this section with a discussion of the order of magnitude of some of the other dimensionless variables employed in the analysis. By the definition of \( u_0 \) as a representative magnitude of velocity, it is clear that \( U^i \) is \( O(1) \); also the dimensionless boundary velocities \( V_{(1)}^i \) and \( V_{(2)}^i \) are \( O(1) \). Similarly, by the definition of \( h_a \), it follows that \( H = O(1) \).

If the surfaces \( S, S_1, \text{ and } S_2 \), are smooth, it is to be expected that the differentiation with respect to \( y^a \), or \( y^3 \), will not produce an essential change in the order of magnitude of a quantity; this can be indicated symbolically by

\[
\frac{d}{dy^i} = O(1)
\]

A similar statement is not always true with respect to the operation \( \frac{d}{d\tau} \); however, noting that one unit of \( \tau \) corresponds to the physical time required for a particle moving with a speed \( u_0 \) to traverse a distance \( L_0 \), it is clear that only in problems involving externally imposed motions will the equality \( \frac{d}{d\tau} = O(1) \) fail to hold.

Finally, the dimensionless pressure \( \Phi \) can be expressed as

\[
\Phi = 12 \frac{h_{a_0}^2}{12 \mu_0 L_0} \frac{\Phi}{u_0}
\]

It is seen that \( \Phi \) is expressed as the ratio of two velocities. If the fluid flow is dominated by the pressure, then this ratio is of the order of magnitude of unity. As the importance of surface velocities in determining the flow increases, however, this ratio decreases. Thus in all cases \( \Phi = O(1) \).
PART B. APPLICATION OF THE GENERAL THEORY IN THE ANALYSIS OF JOURNAL AND SLIDER BEARINGS

Section 7. Development of General Equations for Plane Coordinates:

In this section we present a greatly condensed derivation of the principal mathematical results of Part A, as applied to the class of problems wherein the lubricant film is approximately plane. The development commences with the general equations of fluid motion and heat flow, in rectangular Cartesian coordinates, and reproduces the essential steps of Sections 3, 4, and 5 without the use of tensor concepts or notation. Most of the equations developed are also applicable, with changes in notation only, to an approximately cylindrical film; the revisions necessary for the cylindrical case are developed in Section 8 by means of the tensor equations of Part A.

Let \((x_1, y_1, z_1)\) denote a set of rectangular Cartesian coordinates*, and let two surfaces, \(S_1\) and \(S_2\), be defined by

\[
\begin{align*}
S_1: \quad z &= h_0(x_1, y_1, t) \\
S_2: \quad z &= h_0(x_1, y_1, t)
\end{align*}
\] (7.1)

where \(t\) represents time, and where \(h_{(2)} - h_{(1)} = h > 0\). As the reference surface \(S\) we select the \((x_1, y_1)\) plane. Let \(h_0\) denote a constant representative (e.g., average) value of \(h(x_1, y_1, t)\), and let \(L_0\) denote a fixed distance which is representative of

* The subscript "1" is attached to these coordinates in order to reserve \((x, y, z)\) without subscripts for a corresponding set of dimensionless coordinates to be used in the major part of the analysis.
the dimensions of the system in the \( x_1 \) and \( y_1 \) directions. In the following we assume \( \frac{h_0}{D_0} \ll 1 \).

To study the relations between the components of fluid velocity \((u,v,w)\), the pressure \((p)\), the viscosity \((\mu)\), the density \((\rho)\), the vector body force per unit mass \((f^x, f^y, f^z)\), the temperature \((T)\), the internal energy per unit mass \((E)\), and the thermal conductivity \((k)\), we employ the following generalized form of the equations of viscous flow and thermal transfer:

\[
\mu \left[ \nabla^2 u + \frac{1}{3} \frac{\partial \Delta}{\partial x_1} \right] + \frac{2\mu}{\partial x_1} \left[ 2 \frac{\partial u}{\partial x_1} - \frac{\partial \Delta}{\partial x_1} \right] + \frac{2\mu}{\partial x_1} \left[ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} \right] - \frac{\partial \Delta}{\partial x_1} = -\rho f^x + \rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + v \frac{\partial u}{\partial y_1} + w \frac{\partial u}{\partial z_1} \right] \\
\mu \left[ \nabla^2 v + \frac{1}{3} \frac{\partial \Delta}{\partial y_1} \right] + \frac{2\mu}{\partial y_1} \left[ \frac{\partial v}{\partial x_1} + \frac{\partial v}{\partial y_1} \right] + \frac{2\mu}{\partial y_1} \left[ \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial y_1^2} \right] - \frac{\partial \Delta}{\partial y_1} = -\rho f^y + \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + v \frac{\partial v}{\partial y_1} + w \frac{\partial v}{\partial z_1} \right] \\
\mu \left[ \nabla^2 w + \frac{1}{3} \frac{\partial \Delta}{\partial z_1} \right] + \frac{2\mu}{\partial z_1} \left[ \frac{\partial w}{\partial x_1} + \frac{\partial w}{\partial y_1} \right] + \frac{2\mu}{\partial z_1} \left[ \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial y_1^2} \right] - \frac{\partial \Delta}{\partial z_1} = -\rho f^z + \rho \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x_1} + v \frac{\partial w}{\partial y_1} + w \frac{\partial w}{\partial z_1} \right] 
\] (7.2)
\[
\mu \left[ \nabla^2 \omega + \frac{1}{3} \frac{\partial \Phi}{\partial \omega} \right] + \frac{2 \mu}{\partial x,} \left[ \frac{\partial \omega}{\partial x} + \frac{2 \mu r}{\partial y,} \right] \\
+ \frac{2 \mu}{\partial y,} \left[ \frac{\partial \omega}{\partial y} + \frac{2 \mu r}{\partial z,} \right] + \frac{2 \mu}{\partial z,} \left[ \frac{\partial \omega}{\partial z} - \frac{2}{3} \Delta \right] - \frac{\partial \Phi}{\partial z,} = \\
- \rho f^2 + \rho \left[ \frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} + w \frac{\partial \omega}{\partial z} \right] \\
\frac{\partial \Phi}{\partial t} + \rho \Delta + u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} + w \frac{\partial \Phi}{\partial z}, = 0 \tag{7.4}
\]

and
\[
\rho \frac{\partial \Phi}{\partial t} + \rho \frac{\partial \Phi}{\partial T} \frac{dT}{dt} = -\rho \Delta + \Phi \\
+ \frac{2}{\partial x,} \left( \frac{k \partial T}{\partial x} \right) + \frac{2}{\partial y,} \left( \frac{k \partial T}{\partial y} \right) + \frac{2}{\partial z,} \left( \frac{k \partial T}{\partial z} \right) \\
\tag{7.6}
\]

where \( \Phi \) is the dissipation function given by
\[
\Phi = \mu \left[ 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right. \\
+ \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 - \frac{2}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 \left] \right. \\
\tag{7.7}
\]

Here we have used the auxiliary definitions
\[
\Delta = \frac{\partial u}{\partial x} + \frac{2 \mu r}{\partial y} + \frac{2 \mu r}{\partial z,} \tag{7.8}
\]
\[
\n\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z^2}
\]

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial u_1}{\partial x_1} + \frac{\partial v}{\partial y_1} + \frac{\partial w}{\partial z}
\]

In order to express equations (7.2) to (7.6) in a convenient dimensionless form, we introduce the following definitions:

\[ u_0 \] a representative magnitude of the velocity vector

\[ f_0 \] a representative magnitude of the body force vector

\[ \mu_0, \rho_0, k_0, c_{p0} \] representative values of the coefficient of viscosity, the density, the thermal conductivity, and the specific heat at constant pressure, respectively

\[ p_r, T_r \] absolute pressure and temperature selected as the reference levels from which \( p \) and \( T \) are measured (i.e., \( p+p_r \) and \( T+T_r \) are the absolute pressure and temperature, respectively)

\[ T_0 \] a fixed temperature, representative of the temperature differential occurring in the system.

In terms of these quantities, we define the following set of dimensionless parameters and variables:

\[
\begin{align*}
\varepsilon &= \frac{h_0}{L_0} \\
\phi &= \frac{h_0^2 \rho_0 f_0}{\mu_0 u_0} \\
R &= \frac{\rho_0 u_0 h_0^2}{\mu_0 L_0} \\
\lambda &= \frac{\rho_0 c_{p0} h_0^2}{k_0 L_0} \\
\eta &= \frac{\mu_0 u_0^2}{k_0 T_0}
\end{align*}
\]

(continued)
\[ \tau = \frac{u_0}{t_0}, \quad x = \frac{x_1}{t_0}, \quad \eta = \frac{\eta_1}{t_0}, \quad z = \frac{z_1}{h_0} = \frac{z_1}{\eta_0} \]

\[ P = \frac{h_0^2}{\mu_0 u_0 l_0} F \quad U = \frac{u}{u_0} \quad V = \frac{v}{u_0} \quad W = \frac{w}{u_0} \]

\[ F^x = \frac{f^x}{f_0} \quad F^\eta = \frac{f^\eta}{f_0} \quad F^z = \frac{f^z}{f_0} \]

\[ \Theta = \frac{\Theta}{\Theta_0} \quad \Theta_r = \frac{\Theta_r}{\Theta_0} \quad P_r = \frac{h_0^2}{\mu_0 u_0 l_0} P_r \quad \overline{C}_h = \frac{C_h}{C_{ho}} \]

(7.11)

We assume that the vector \((\mathbf{F}^x, \mathbf{F}^\eta, \mathbf{F}^z)\) may be expanded in a power series in the dimensionless variable \(\frac{z_1}{t_0} = \varepsilon Z\),

\[ F^x(x, \eta, z) = \sum_{n=0}^{\infty} \varepsilon^n Z^n F_{(n)}^x (x, \eta) \]  

(7.12)

with two similar equations for \(\mathbf{F}^\eta\) and \(\mathbf{F}^z\). In addition, we define dimensionless parameters \(k_\mu\), \(k_\rho\), and \(k_k\), and corresponding dimensionless functions \(\overline{\mu}(p, \theta)\), \(\overline{\rho}(p, \theta)\) and \(\overline{K}(p, \theta)\) by the relations

\[ \mu = \mu_0 [1 + k_\mu \overline{\mu}]; \quad \rho = \rho_0 [1 + k_\rho \overline{\rho}]; \quad k = k_0 [1 + k_k \overline{K}] \]  

(7.13)

It is assumed that \(\overline{\mu}, \overline{\rho}, \overline{K} = O(1)\) and that \(k_\mu, k_\rho, k_k \ll 1\).

In the following we assume \(\varepsilon, \phi, K_\mu, K_\rho, R, \eta, \lambda\), and \(k_k\) to be small in comparison with unity. Retaining terms through the second degree only in these parameters, the substitution of (7.11) into (7.2) and (7.3) gives

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\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial P}{\partial x} = \kappa \mu \left[ -\mu \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} \right] + \phi \left[ -F_{0x} \right] \\
+ R \left[ \frac{\partial \phi}{\partial \xi} \right] + \varepsilon^2 \left[ -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right] + \phi \left[ -2 F_{0x} \right] \\
+ K \phi \left[ -\rho F_{0x} \right] + K \rho \left[ \frac{\partial \phi}{\partial \xi} \right]
\]

(7.14)

\[
\frac{\partial^2 v}{\partial z^2} - \frac{\partial P}{\partial y} = \kappa \mu \left[ -\mu \frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} \right] + \phi \left[ -F_{0y} \right] \\
+ R \left[ \frac{\partial \phi}{\partial \xi} \right] + \varepsilon^2 \left[ -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} \right] + \phi \left[ -2 F_{0y} \right] \\
+ K \phi \left[ -\rho F_{0y} \right] + K \rho \left[ \frac{\partial \phi}{\partial \xi} \right]
\]

(7.15)

Similarly (7.4) yields

\[
\frac{\partial \rho}{\partial x} = \varepsilon^2 \left[ \frac{\partial \rho}{\partial x} \right] + \phi \left[ \rho F_{0x} \right]
\]

(7.16)

and the continuity equation (7.5) may be written in the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \kappa \rho \left[ \frac{\partial \rho}{\partial \xi} \right] + K \rho \left[ \frac{\partial \rho}{\partial \xi} \right]
\]

(7.17)

The operator \( \frac{\partial}{\partial \xi} \) in these equations is defined by

\[
\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} + W \frac{\partial}{\partial z}
\]

(7.18)

In writing the dimensionless form of the thermal transfer
equation (7.6), we employ the thermodynamic formulae

\[
\left( \frac{\partial E}{\partial \mu} \right)_T = -(T + T_r) \left( \frac{\partial}{\partial T} \left( \frac{1}{\rho} \right) \right)_\mu - (\mu + \mu_r) \left( \frac{\partial}{\partial \mu} \left( \frac{1}{\rho} \right) \right)_T \quad (7.19)
\]

\[
\left( \frac{\partial E}{\partial \rho} \right)_\mu = -(\mu + \mu_r) \left( \frac{\partial}{\partial \rho} \frac{1}{\rho} \right)_\mu + C_p \quad (7.20)
\]

We then obtain

\[
\frac{\partial^2 \theta}{\partial z^2} = -\eta \left[ -\left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] + \lambda \left[ \overline{C_p} \frac{\partial \theta}{\partial z} \right] 
+ K_k \left[ -\overline{K} \frac{\partial^2 \theta}{\partial z^2} - \frac{\overline{K}}{\rho} \frac{\partial \theta}{\partial z} \right] + \varepsilon^2 \left[ -\left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \right]
+ \eta K_m \left[ -\overline{m} \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] + \eta K_{m_r} \left[ (\theta + \theta_r) \frac{\partial \theta}{\partial z} \frac{d \rho}{d z} \right]
+ \varepsilon \left[ \overline{C_m} \frac{d \theta}{d z} \right] + \lambda K_{\rho} \left[ \overline{C_p} \frac{\partial \theta}{\partial z} \right] 
\]

(7.21)

To formulate the boundary conditions on these equations, we let \((u(1), v(1), w(1))\) be the vector velocity of \(S_1\), and \((u(2), v(2), w(2))\) that of \(S_2\). Introducing the definitions

\[
\begin{align*}
U_{\|} &= \frac{u}{u_0} & V_{\|} &= \frac{v}{u_0} & W_{\|} &= \frac{w}{u_0} \\
U_{\perp} &= \frac{u_{\perp}}{u_0} & V_{\perp} &= \frac{V_{\perp}}{u_0} & W_{\perp} &= \frac{W_{\perp}}{u_0}
\end{align*}
\]

(7.22)

and
\[ H_{ij}(x, y, \tau) = \frac{h_{ij}(x, y, \tau)}{h_0}; \quad H_{uu}(x, y, \tau) = \frac{h_{uu}(x, y, \tau)}{h_0} \]
\[ H(x, y, \tau) = H_{uu}(x, y, \tau) - H_{ij}(x, y, \tau) \quad (7.23) \]

we require that

\[ U, V, W = \begin{cases} 
U_{ij}, V_{ij}, W_{ij} & (z = H_{ii}) \\
U_{ij}, V_{ij}, W_{ij} & (z = H_{ii}) 
\end{cases} \quad (7.24) \]

To provide boundary conditions for the temperature \( \Theta \), we let \( k_{s1} \) and \( k_{s2} \) be surface heat transfer coefficients at \( S_1 \) and \( S_2 \) respectively. Then in terms of dimensionless quantities,

\[
\frac{\partial \Theta}{\partial z} - \bar{k}_{s1} \Theta = -\bar{k}_{s1} \Theta_{ij} + \varepsilon^2 \left[ \frac{\partial H_{ij}}{\partial x} \left( \frac{\partial \Theta}{\partial x} + \frac{1}{2} \frac{\partial \Theta}{\partial z} \frac{\partial H_{ij}}{\partial y} \right) \right. \quad (z = H_{ii}) \quad (7.25)
\]

\[
\frac{\partial \Theta}{\partial z} + \bar{k}_{s2} \Theta = \bar{k}_{s2} \Theta_{ij} + \varepsilon^2 \left[ \frac{\partial H_{ij}}{\partial x} \left( \frac{\partial \Theta}{\partial x} + \frac{1}{2} \frac{\partial \Theta}{\partial z} \frac{\partial H_{ij}}{\partial y} \right) \right. \quad (z = H_{ii}) \quad (7.26)
\]

to within terms of the second degree in the parameters, where \( T(1) \) and \( T(2) \) denote the temperatures of the walls \( S_1 \) and \( S_2 \) immediately beneath the surface, and where
\[
\begin{align*}
\Theta_1 & = \frac{T_1}{T_0} \\
\Theta_2 & = \frac{T_2}{T_0} \\
\bar{k}_{s1} & = \frac{k_{s1} h_0}{k_0} \\
\bar{k}_{s2} & = \frac{k_{s2} h_0}{k_0}
\end{align*}
\]

(7.27)

No boundary conditions on \( S_1 \) and \( S_2 \) are provided for the pressure \( P \); it is assumed instead that a boundary curve \( \mathcal{C} \) is given in the \( (x, y) \) plane, and that the average value of \( P \) between \( S_1 \) and \( S_2 \) is specified as a function of the arc \( \mathcal{L} \) of this curve, and time:

\[
\frac{1}{H} \int_{H_0}^{H_2} P(\mathcal{L}, z, \tau) \, dz = P(\mathcal{L}, \tau)
\]

(7.28)

where \( P(\mathcal{L}, \tau) \) is a preassigned function. Alternately, it may be assumed that the rate of flow of fluid mass per unit length of arc outward across \( \mathcal{C} \), denoted by \( m(\mathcal{L}, \tau) \), is specified. In terms of dimensionless variables this condition requires that

\[
\begin{align*}
\eta_{(\mathcal{L})x} & \int_{H_0}^{H_2} (1 + K_p \bar{P}) U \, dz + \eta_{(\mathcal{L})y} \int_{H_0}^{H_2} (1 + K_p \bar{P}) V \, dz = \frac{m(\mathcal{L}, \tau)}{\rho_0 U_0 h_0}
\end{align*}
\]

(7.29)

where \( \eta_{(\mathcal{L})x} \) and \( \eta_{(\mathcal{L})y} \) are the components of the unit outer normal vector to \( \mathcal{L} \).

We now assume that \( U, V, W, P, \Theta, \mu, \rho, \bar{C}, \bar{k} \), and \( \bar{k} \) may be expanded in multiple power series in the parameters \( \varepsilon, \phi, K_\mu, K_p, R, \eta, \lambda \), and \( K_k \).
\[ \begin{array}{l}
U = U_{(0)} + \varepsilon U_{(\varepsilon)} + \phi U_{(\phi)} + K_{\mu} U_{(\mu)} + K_{\rho} U_{(\rho)} + \eta U_{(\eta)} + \lambda U_{(\lambda)} + K_{K} U_{(K)} + \varepsilon^{2} U_{(\varepsilon \varepsilon)} + \cdots \\
+ \varepsilon \phi U_{(\varepsilon \phi)} + \varepsilon K_{\mu} U_{(\varepsilon \mu)} + \cdots \\

V = V_{(0)} + \varepsilon V_{(\varepsilon)} + \cdots + K_{K} V_{(K)} + \varepsilon^{2} V_{(\varepsilon \varepsilon)} + \cdots \\

W = W_{(0)} + \varepsilon W_{(\varepsilon)} + \cdots + K_{K} W_{(K)} + \varepsilon^{2} W_{(\varepsilon \varepsilon)} + \cdots \\
P = P_{(0)} + \varepsilon P_{(\varepsilon)} + \cdots + K_{K} P_{(K)} + \varepsilon^{2} P_{(\varepsilon \varepsilon)} + \cdots \\
\Theta = \Theta_{(0)} + \varepsilon \Theta_{(\varepsilon)} + \cdots + K_{K} \Theta_{(K)} + \varepsilon^{2} \Theta_{(\varepsilon \varepsilon)} + \cdots \\
\end{array} \] (7.30)

and
\[ \begin{array}{l}
\bar{\mu} = \bar{\mu}_{(0)} + \varepsilon \bar{\mu}_{(\varepsilon)} + \cdots + K_{K} \bar{\mu}_{(K)} + \varepsilon^{2} \bar{\mu}_{(\varepsilon \varepsilon)} + \cdots \\
\bar{K} = \bar{K}_{(0)} + \varepsilon \bar{K}_{(\varepsilon)} + \cdots + K_{K} \bar{K}_{(K)} + \varepsilon^{2} \bar{K}_{(\varepsilon \varepsilon)} + \cdots \\
\end{array} \] (7.31)

Since \( \bar{\mu} = \bar{\mu}(P, \Theta) \), \( \bar{p} = \bar{p}(P, \Theta) \), \( \bar{c}_{a} = \bar{c}_{a}(P, \Theta) \), and \( \bar{k} = \bar{k}(F, \Theta) \), there will exist identities relating the quantities (7.30) and (7.31), typified by the following

\[ \bar{\mu}_{(\varepsilon)} = \bar{\mu}(P_{(\varepsilon)}, \Theta_{(\varepsilon)}) \] (7.32)

\[ \bar{\mu}_{(\varepsilon)} = \frac{\partial \bar{\mu}(P_{(\varepsilon)}, \Theta_{(\varepsilon)})}{\partial P} P_{(\varepsilon)} + \frac{\partial \bar{\mu}(P_{(\varepsilon)}, \Theta_{(\varepsilon)})}{\partial \Theta} \Theta_{(\varepsilon)} \] (7.33)
Substituting the series (7.30) and (7.31) into the differential equations (7.14), (7.15), (7.16), (7.17), (7.21) and the boundary conditions (7.24), (7.25), and (7.26), and further assuming $U(1)$, $V(1)$, $W(1)$, $\Theta(1)$, $U(2)$, $V(2)$, $W(2)$, and $\Theta(2)$ to be independent of the parameters, we may collect like terms to obtain the differential systems which follow. The first of these is obtained from those terms which do not involve the parameters, whereas the remainder correspond to the linear terms.

\[
\frac{\partial^2 U(\omega)}{\partial z^2} - \frac{\partial P(\omega)}{\partial x} = 0
\]  
(7.34)

\[
\frac{\partial^2 V(\omega)}{\partial z^2} - \frac{\partial P(\omega)}{\partial y} = 0
\]  
(7.35)

\[
\frac{\partial P(\omega)}{\partial z} = 0
\]  
(7.36)

\[
\frac{\partial U(\omega)}{\partial x} + \frac{\partial V(\omega)}{\partial y} + \frac{\partial W(\omega)}{\partial z} = 0
\]  
(7.37)

\[
\frac{\partial^2 \Theta(\omega)}{\partial z^2} = 0
\]  
(7.38)

\[
U(\omega), V(\omega), W(\omega) = U(1), V(1), W(1) \quad (z = H_{11})
\]  
(7.39)

\[
U(\omega), V(\omega), W(\omega) = U(2), V(2), W(2) \quad (z = H_{12})
\]  
(7.40)

\[
\frac{\partial \Theta(\omega)}{\partial z} - \kappa_s \Theta(\omega) = -\kappa_s \Theta(1) \quad (z = H_{11})
\]  
(7.41)
\[
\frac{\partial \Theta_{\psi}}{\partial z} + \frac{K_{s2} \Theta_{\psi}}{z} = \frac{K_{s2} \Theta_{\psi}}{z} \quad (z = H_{12})
\]

\[
\frac{\partial^2 U_{(\psi)}}{\partial z^2} - \frac{\partial P_{(\psi)}}{\partial x} = 0
\]

\[
\frac{\partial^2 V_{(\psi)}}{\partial z^2} - \frac{\partial P_{(\psi)}}{\partial y} = 0
\]

\[
\frac{\partial P_{(\psi)}}{\partial z} = 0
\]

\[
\frac{\partial U_{(\psi)}}{\partial x} + \frac{\partial V_{(\psi)}}{\partial y} + \frac{\partial W_{(\psi)}}{\partial z} = 0
\]

\[
\frac{\partial^2 \Theta_{(\psi)}}{\partial z^2} = 0
\]

\[
\frac{\partial^2 U_{(\psi)}}{\partial z^2} - \frac{\partial P_{(\psi)}}{\partial x} = - \frac{F_{\psi x}}{x}
\]

\[
\frac{\partial^2 V_{(\psi)}}{\partial z^2} - \frac{\partial P_{(\psi)}}{\partial y} = - \frac{F_{\psi y}}{y}
\]

\[
\frac{\partial P_{(\psi)}}{\partial z} = 0
\]

\[
\frac{\partial U_{(\psi)}}{\partial x} + \frac{\partial V_{(\psi)}}{\partial y} + \frac{\partial W_{(\psi)}}{\partial z} = 0
\]

\[
\frac{\partial^2 \Theta_{(\psi)}}{\partial z^2} = 0
\]
\[
\frac{\partial^2 U_{(m)}}{\partial z^2} - \frac{\partial P_{(m)}}{\partial x} = -\frac{\partial}{\partial z} \left[ \mu_{(m)} \frac{\partial U_{(m)}}{\partial z} \right]
\]
\[
\frac{\partial^2 V_{(m)}}{\partial z^2} - \frac{\partial P_{(m)}}{\partial y} = -\frac{\partial}{\partial z} \left[ \mu_{(m)} \frac{\partial V_{(m)}}{\partial z} \right]
\]
\[
\frac{\partial P_{(m)}}{\partial z} = 0
\]
\[
\frac{\partial U_{(m)}}{\partial x} + \frac{\partial V_{(m)}}{\partial y} + \frac{\partial W_{(m)}}{\partial z} = 0
\]
\[
\frac{\partial^2 \Theta_{(m)}}{\partial z^2} = 0
\]

\[
\frac{\partial^2 U_{(p)}}{\partial z^2} - \frac{\partial P_{(p)}}{\partial x} = 0
\]
\[
\frac{\partial^2 V_{(p)}}{\partial z^2} - \frac{\partial P_{(p)}}{\partial y} = 0
\]
\[
\frac{\partial P_{(p)}}{\partial z} = 0
\]
\[
\frac{\partial U_{(p)}}{\partial x} + \frac{\partial V_{(p)}}{\partial y} + \frac{\partial W_{(p)}}{\partial z} = -\frac{\partial \tilde{P}_{(p)}}{\partial z} - U_{(p)} \frac{\partial \tilde{P}_{(p)}}{\partial x} - \left( V_{(p)} \frac{\partial \tilde{P}_{(p)}}{\partial y} - W_{(p)} \frac{\partial \tilde{P}_{(p)}}{\partial z} \right)
\]
\[
\frac{\partial^2 \Theta_{(p)}}{\partial z^2} = 0
\]
\[
\frac{\partial^2 U_{RI}}{\partial x^2} - \frac{\partial P_{RI}}{\partial x} = \frac{\partial U_{W}}{\partial x} + U_{10} \frac{\partial U_{W}}{\partial x} + V_{10} \frac{\partial U_{W}}{\partial y} + W_{10} \frac{\partial U_{W}}{\partial z}
\]
\[
\frac{\partial^2 V_{RI}}{\partial y^2} - \frac{\partial P_{RI}}{\partial y} = \frac{\partial V_{W}}{\partial x} + U_{10} \frac{\partial V_{W}}{\partial x} + V_{10} \frac{\partial V_{W}}{\partial y} + W_{10} \frac{\partial V_{W}}{\partial z}
\]
\[
\frac{\partial^2 P_{RI}}{\partial z^2} = 0
\]
\[
\frac{\partial U_{RI}}{\partial x} + \frac{\partial V_{RI}}{\partial y} + \frac{\partial W_{RI}}{\partial z} = 0
\]
\[
\frac{\partial^2 \Theta_{RI}}{\partial z^2} = 0
\]
\[
\frac{\partial^2 U_{WI}}{\partial x^2} - \frac{\partial P_{W}}{\partial x} = 0
\]
\[
\frac{\partial^2 V_{WI}}{\partial y^2} - \frac{\partial P_{W}}{\partial y} = 0
\]
\[
\frac{\partial^2 P_{W}}{\partial z^2} = 0
\]
\[
\frac{\partial U_{WI}}{\partial x} + \frac{\partial V_{WI}}{\partial y} + \frac{\partial W_{WI}}{\partial z} = 0
\]
\[
\frac{\partial^2 \Theta_{WI}}{\partial z^2} = -\left[\left(\frac{\partial U_{W}}{\partial z}\right)^2 + \left(\frac{\partial V_{W}}{\partial z}\right)^2\right]
\]
\[
\frac{\partial^2 U_{(2)}}{\partial z^2} - \frac{\partial P_{(2)}}{\partial x} = 0
\]

\[
\frac{\partial^2 V_{(2)}}{\partial z^2} - \frac{\partial P_{(2)}}{\partial y} = 0
\]

\[
\frac{\partial P_{(2)}}{\partial z} = 0
\]

\[
\frac{\partial U_{(2)}}{\partial x} + \frac{\partial V_{(2)}}{\partial y} + \frac{\partial W_{(2)}}{\partial z} = 0
\]

\[
\frac{\partial^2 \Theta_{(2)}}{\partial z^2} = \bar{C}_{k(2)} \left[ \frac{\partial \Theta_{(1)}}{\partial t} + U_{(1)} \frac{\partial \Theta_{(1)}}{\partial x} + V_{(1)} \frac{\partial \Theta_{(1)}}{\partial y} + W_{(1)} \frac{\partial \Theta_{(1)}}{\partial z} \right]
\]

\[
\frac{\partial^2 U_{(1)}}{\partial z^2} - \frac{\partial P_{(1)}}{\partial x} = 0
\]

\[
\frac{\partial^2 V_{(1)}}{\partial z^2} - \frac{\partial P_{(1)}}{\partial y} = 0
\]

\[
\frac{\partial P_{(1)}}{\partial z} = 0
\]

\[
\frac{\partial U_{(1)}}{\partial x} + \frac{\partial V_{(1)}}{\partial y} + \frac{\partial W_{(1)}}{\partial z} = 0
\]

\[
\frac{\partial^2 \Theta_{(1)}}{\partial z^2} = -2 \bar{B} \left[ \bar{k}_{(1)} \frac{\partial \Theta_{(1)}}{\partial z} \right]
\]
The boundary conditions have been given only for \( U(0), V(0), \ldots \Theta(0) \); those appropriate to the systems (7.43) to (7.50) are

\[
U(0) = V(0) = W(0) = 0 \quad (z = H_{11}, H_{12})
\]  
(7.51)

\[
\frac{\partial \Theta_{1}}{\partial z} = K_{1} \Theta_{1} = 0 \quad (z = H_{11})
\]  
(7.52)

\[
\frac{\partial \Theta_{0}}{\partial z} + K_{2} \Theta_{0} = 0 \quad (z = H_{12})
\]  
(7.53)

where any of the subscripts \( \xi, \phi, \mu, \rho, R, \eta, \lambda, \) or \( k \) may be inserted in the parentheses.

To solve the differential system (7.34) to (7.42) for \( U(0), V(0), \) and \( \Theta(0) \), we first note from (7.36) that \( \pi(0) \) is independent of \( z \). Writing

\[
\pi(0) = \psi(0)(x, \eta)
\]  
(7.54)

we may integrate (7.34) and (7.35) subject to the boundary conditions (7.39) and (7.40) to obtain \( U(0) \) and \( V(0) \). The results may be expressed in simplified form by use of the variables

\[
\xi = z - \frac{1}{2} (H_{11} + H_{12})
\]  
(7.55)

and
\[
\begin{align*}
A^{(\alpha)} &= U_{(1)} + U_{(2)} \\
B^{(\alpha)} &= U_{(2)} - U_{(1)} \\
A^{(\psi)} &= V_{(1)} + V_{(2)} \\
B^{(\psi)} &= V_{(2)} - V_{(1)}
\end{align*}
\] (7.56)

In terms of these quantities

\[
U_{(\omega)} = \frac{1}{2} A^{(\alpha)} + \frac{5}{H} B^{(\alpha)} - \frac{1}{2} \left( \frac{H^2}{4} - \xi^2 \right) \frac{\partial U_{(\omega)}}{\partial \xi}
\] (7.57)

\[
V_{(\omega)} = \frac{1}{2} A^{(\psi)} + \frac{5}{H} B^{(\psi)} - \frac{1}{2} \left( \frac{H^2}{4} - \xi^2 \right) \frac{\partial V_{(\omega)}}{\partial \eta}
\] (7.58)

The third velocity component \( \hat{w}_{(0)} \) may now be found from (7.37) and one of the two boundary conditions (7.39) or (7.40). The remaining boundary condition then provides a partial differential equation for \( V_{(\omega)}(x,y) \), since all other quantities in the resulting equation are known. To derive this differential equation in a relatively simpler manner, however, we may integrate (7.37) to obtain
\[ \int_{H_{ii}}^{H_{ii+1}} \frac{\partial W_{ii}}{\partial z} \, dz = W_{i+1} - W_{ii} = - \int_{H_{ii}}^{H_{ii+1}} \left( \frac{\partial U_{ii}}{\partial x} + \frac{\partial V_{ii}}{\partial y} \right) \, dz \]

\[ = - \frac{\partial}{\partial x} \int_{H_{ii}}^{H_{ii+1}} U_{ii} \, dz - \frac{\partial}{\partial y} \int_{H_{ii}}^{H_{ii+1}} V_{ii} \, dz \]

\[ + \frac{\partial H_{ii}}{\partial x} U_{ii+1} + \frac{\partial H_{ii}}{\partial y} V_{ii+1} - \frac{\partial H_{ii}}{\partial x} U_{ii} - \frac{\partial H_{ii}}{\partial y} V_{ii} \]

(7.59)

From kinematical relations,

\[ \begin{align*}
\frac{\partial H_{ii}}{\partial t} &= W_{ii} - \frac{\partial H_{ii}}{\partial x} U_{ii} - \frac{\partial H_{ii}}{\partial y} V_{ii} \\
\frac{\partial H_{i+1}}{\partial t} &= W_{i+1} - \frac{\partial H_{i+1}}{\partial x} U_{i+1} - \frac{\partial H_{i+1}}{\partial y} V_{i+1}
\end{align*} \]

(7.60)

whence (7.59) yields

\[ \frac{\partial}{\partial x} \int_{H_{ii}}^{H_{ii+1}} U_{ii} \, dz + \frac{\partial}{\partial y} \int_{H_{ii}}^{H_{ii+1}} V_{ii} \, dz = - \frac{\partial H}{\partial t} \]

or, in terms of \( \xi \) as the variable of integration,

\[ \frac{\partial}{\partial x} \int_{-\frac{H}{2}}^{\frac{H}{2}} U_{\xi i} \, d\xi + \frac{\partial}{\partial y} \int_{-\frac{H}{2}}^{\frac{H}{2}} V_{\xi i} \, d\xi = - \frac{\partial H}{\partial t} \]

(7.61)
Upon substituting from (7.57) and (7.58), and performing the operations indicated in (7.61), we find

\[
\frac{2}{\delta X} \left( H^3 \frac{\partial^2 \psi_{11}}{\partial X^2} \right) + \frac{2}{\delta X} \left( H^3 \frac{\partial^2 \psi_{12}}{\partial X^2} \right) = 6 \frac{2}{\delta X} \left( HA' \chi \right) \\
+ 6 \frac{2}{\delta X} \left( HA' \chi' \right) + 12 \frac{2H}{\delta X} 
\]

(7.62)

as the required partial differential equation. A simple calculation from (7.38), (7.41), and (7.42) yields

\[
\Theta_{(0)} = \Theta_{(0)(av)} + \frac{5}{4} \Theta_{1} \tag{7.63}
\]

where \( \Theta_{(0)(av)} \) is the value of \( \Theta_{(0)} \) averaged between \( S_1 \) and \( S_2 \), given by the formula

\[
\Theta_{(0)(av)} = \frac{K_{s1} \Theta_{11} + K_{s2} \Theta_{12} + \frac{1}{2} K_{s1} K_{s2} H (\Theta_{11} + \Theta_{12})}{K_{s1} + K_{s2} + K_{s1} K_{s2} H} 
\]

(7.64)

and where

\[
\Theta_{1} = \frac{K_{s1} K_{s2} (\Theta_{11} - \Theta_{12})}{K_{s1} + K_{s2} + K_{s1} K_{s2} H} 
\]

(7.65)

Each of the remaining systems (7.43) to (7.50) is a special case of the more general system
\[ \frac{\partial^2 u}{\partial z^2} - \frac{\partial p}{\partial x} = L^0(x, \psi, z, \tau) \quad (7.66) \]

\[ \frac{\partial^2 v}{\partial z^2} - \frac{\partial p}{\partial y} = L^1(x, \psi, z, \tau) \quad (7.67) \]

\[ \frac{\partial p}{\partial z} = 0 \quad (7.68) \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = M(x, \psi, z, \tau) \quad (7.69) \]

\[ \frac{\partial^2 \psi}{\partial z^2} = N(x, \psi, z, \tau) \quad (7.70) \]

in which \( L^0(x), L^1(y), M, \) and \( N \) are arbitrary functions of the indicated variables. To solve (7.66) to (7.70), subject to boundary conditions of the type of (7.51), (7.52), and (7.53), we write

\[ p = \psi(x, y) \quad (7.71) \]

Equations (7.51), (7.57), and (7.58) then yield

\[ u = -\frac{1}{2} \left( \frac{H^2}{4} - \xi^2 \right) \frac{\partial \psi}{\partial x} - \frac{1}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} (\frac{H}{2} + \xi)(\frac{H}{2} - \xi) L^0(x) \, d\xi, \]

\[ - \frac{1}{H} \int_{\frac{H}{2}}^{\frac{H}{2}} (\frac{H}{2} + \xi)(\frac{H}{2} - \xi) L^1(x) \, d\xi, \quad (7.72) \]
\[ V = -\frac{1}{2} \left( \frac{H^2}{4} - \xi^2 \right) \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} \left( \frac{H}{2} + \xi \right) \left( \frac{H}{2} - \xi \right) L^{(\psi)} d\xi, \]

\[-\frac{1}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} \left( \frac{H}{2} + \xi \right) \left( \frac{H}{2} - \xi \right) L^{(\psi)} d\xi. \tag{7.73}\]

By means of (7.51), (7.59), and the technique used in deriving (7.61), we obtain

\[ \frac{\partial}{\partial x} \int_{-\frac{H}{2}}^{\frac{H}{2}} V d\xi + \frac{\partial}{\partial y} \int_{-\frac{H}{2}}^{\frac{H}{2}} V d\xi = \int_{-\frac{H}{2}}^{\frac{H}{2}} M d\xi. \tag{7.74}\]

The substitution of (7.72) and (7.73) into this relation then gives, by suitable integrations by parts and minor rearrangements,

\[ \frac{2}{\partial x} \left( H^{3} \frac{\partial \psi}{\partial x} \right) + \frac{2}{\partial y} \left( H^{3} \frac{\partial \psi}{\partial y} \right) = -12 \int_{-\frac{H}{2}}^{\frac{H}{2}} M(x, y, \xi, \tau) d\xi \]

\[-6 \frac{\partial}{\partial x} \int_{-\frac{H}{2}}^{\frac{H}{2}} \left( \frac{H^2}{4} - \xi^2 \right) L^{(\psi)}(x, y, \xi, \tau) d\xi \]

\[-6 \frac{\partial}{\partial y} \int_{-\frac{H}{2}}^{\frac{H}{2}} \left( \frac{H^2}{4} - \xi^2 \right) L^{(\psi)}(x, y, \xi, \tau) d\xi. \tag{7.75}\]

as the required partial differential equation for \( \psi(x, y) \).

For \( \theta \), we find from (7.52), (7.53), and (7.70)
\[ \Theta = \frac{1}{H(k_{s1} + k_{s2} + k_{s1}k_{s2}H)} \left\{ \int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{ H + k_{s1}(\frac{H}{2} + \xi)(\frac{H}{2} + \xi) \right. \right. \\
+ k_{s2}(\frac{H}{2} - \xi)(\frac{H}{2} - \xi) \right\} \left. \right. \\
N(x, \psi, \xi, \tau) d\xi, \\
\left. \right. \\
- \frac{1}{H} \left\{ \int_{-\frac{H}{2}}^{\frac{H}{2}} (\frac{H}{2} + \xi)(\frac{H}{2} - \xi)N(x, \psi, \xi, \tau) d\xi, \\
\left. \right. \\
+ \int_{-\frac{H}{2}}^{\frac{H}{2}} (\frac{H}{2} + \xi)(\frac{H}{2} - \xi)N(x, \psi, \xi, \tau) d\xi \right\} \right\} \] (7.76)

We now apply (7.75) and (7.76) to the systems (7.43) to (7.50), obtaining the following results:

\[ \frac{2}{\partial x} \left( H^3 \frac{\partial \psi}{\partial x} \right) + \frac{2}{\partial y} \left( H^3 \frac{\partial \psi}{\partial y} \right) = 0 \] (7.77)

\[ \Theta(\psi) = 0 \] (7.78)

\[ \frac{2}{\partial x} \left( H^3 \frac{\partial \psi}{\partial x} \right) + \frac{2}{\partial y} \left( H^3 \frac{\partial \psi}{\partial y} \right) = \frac{2}{\partial x} \left( H^3 F_{\alpha x} \right) + \frac{2}{\partial y} \left( H^3 F_{\alpha y} \right) \] (7.79)

\[ \Theta(\phi) = 0 \] (7.80)
\[
\frac{d}{dx} \left( H^3 \frac{\partial \psi^{(0)}}{\partial x} \right) + \frac{d}{dy} \left( H^3 \frac{\partial \psi^{(0)}}{\partial y} \right) = 12 \frac{\partial}{\partial x} \left\{ \frac{B^{(x)}}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} \xi \mu^{(0)} d\xi \right\} \\
+ 12 \frac{\partial}{\partial y} \left\{ \frac{B^{(y)}}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} \xi \mu^{(0)} d\xi \right\} + 12 \frac{\partial}{\partial x} \left\{ \frac{\partial \psi^{(0)}}{\partial x} \int_{-\frac{H}{2}}^{\frac{H}{2}} \xi \mu^{(0)} d\xi \right\} \\
+ 12 \frac{\partial}{\partial y} \left\{ \frac{\partial \psi^{(0)}}{\partial y} \int_{-\frac{H}{2}}^{\frac{H}{2}} \xi \mu^{(0)} d\xi \right\} \\
(7.81)
\]

\[
\Theta^{(0)} = 0 \\
(7.82)
\]

Noting that
\[
\mu^{(0)} = \tilde{\mu} (P^{(0)}, E^{(0)}) = \tilde{\mu} (\psi^{(0)}, E^{(0)(av)} + \xi \tilde{\psi}^{(0)}) \\
(7.83)
\]

where \(\psi^{(0)}\) is independent of \(\xi\), it is clear that the integrals of (7.81) may be evaluated analytically, provided that \(\tilde{\mu} (P, E)\) is a sufficiently simple function. In the remainder of our calculations we assume
\[
\tilde{\mu} (P, E) = \frac{\partial \tilde{\mu}}{\partial P} P + \frac{\partial \tilde{\mu}}{\partial E} E \\
(7.84)
\]

where \(\frac{\partial \tilde{\mu}}{\partial P}\) and \(\frac{\partial \tilde{\mu}}{\partial E}\) are constants. Equation (7.81) then yields
\[
\frac{d}{dx} \left( H^3 \frac{\partial \psi^{(0)}}{\partial x} \right) + \frac{d}{dy} \left( H^3 \frac{\partial \psi^{(0)}}{\partial y} \right) = \frac{\partial \mu}{\partial P} \left\{ \frac{\partial}{\partial x} \left( H^3 \psi^{(0)} \frac{\partial \psi^{(0)}}{\partial x} \right) + \frac{\partial}{\partial y} \left( H^3 \psi^{(0)} \frac{\partial \psi^{(0)}}{\partial y} \right) \right\} \\
+ \frac{\partial \mu}{\partial E} \left\{ \frac{\partial}{\partial x} \left[ H^2 \Theta^{(0)} B^{(x)} + H^3 \Theta^{(0)(av)} \frac{\partial \psi^{(0)}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ H^2 \Theta^{(0)} B^{(y)} + H^3 \Theta^{(0)(av)} \frac{\partial \psi^{(0)}}{\partial y} \right] \right\} \\
(7.85)
\]

For the calculation of \(\psi^{(0)}\) we find the differential equation
\[
\frac{\partial}{\partial x} \left( \frac{H^3 \partial^2 \psi_{(R)}}{\partial x^2} \right) + \frac{\partial}{\partial y} \left( \frac{H^3 \partial^2 \psi_{(R)}}{\partial y^2} \right) = 12 \frac{\partial}{\partial z} \int_{-\frac{H}{2}}^{\frac{H}{2}} \tilde{\rho}(\omega) \, d\omega \\
+ 12 \frac{\partial}{\partial x} \left\{ \int_{-\frac{H}{2}}^{\frac{H}{2}} \tilde{\rho}(\omega) U_{\omega} \, d\omega \right\} + 12 \frac{\partial}{\partial y} \left\{ \int_{-\frac{H}{2}}^{\frac{H}{2}} \tilde{\rho}(\omega) V_{\omega} \, d\omega \right\} 
\] (7.86)

Assuming
\[
\tilde{\rho}(P,\Theta) = \frac{\partial \tilde{\rho}}{\partial P} P + \frac{\partial \tilde{\rho}}{\partial \Theta} \Theta 
\] (7.87)

with \(\frac{\partial \tilde{\rho}}{\partial P}\) and \(\frac{\partial \tilde{\rho}}{\partial \Theta}\) as constants, we find
\[
\left( \frac{\partial}{\partial x} \left( \frac{H^3 \partial^2 \psi_{(R)}}{\partial x^2} \right) + \frac{\partial}{\partial y} \left( \frac{H^3 \partial^2 \psi_{(R)}}{\partial y^2} \right) = 12 \frac{\partial}{\partial z} \left[ \frac{\partial \tilde{\rho}}{\partial P} \psi_{(R)} + \frac{\partial \tilde{\rho}}{\partial \Theta} H \Theta (\omega) \omega \right] \\
+ \frac{\partial}{\partial x} \left\{ [6 HA^3(\omega) - H^3 \partial^2 \psi_{(R)}] \left[ \frac{\partial \tilde{\rho}}{\partial P} \psi_{(R)} + \frac{\partial \tilde{\rho}}{\partial \Theta} H \Theta (\omega) \omega \right] + \frac{\partial \tilde{\rho}}{\partial \Theta} H^2 \Theta (\omega) B^3(\omega) \right\} \\
+ \frac{\partial}{\partial y} \left\{ [6 HA^3(\omega) - H^3 \partial^2 \psi_{(R)}] \left[ \frac{\partial \tilde{\rho}}{\partial P} \psi_{(R)} + \frac{\partial \tilde{\rho}}{\partial \Theta} H \Theta (\omega) \omega \right] + \frac{\partial \tilde{\rho}}{\partial \Theta} H^2 \Theta (\omega) B^3(\omega) \right\} 
\] (7.88)

Also
\[
\Theta(\rho) = 0 
\] (7.89)

After extensive calculation, we obtain the following differential equation for \(\psi_{(R)}\):

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Also
\[ \Theta_{(R)} = 0 \] (7.91)

In the present investigation we calculate only the first of the quantities \( \Theta_{(\eta)}, \Theta_{(x)}, \) and \( \Theta_{(k)} \). The last two of these are of relatively minor importance, since they may be regarded simply as corrections to \( \Theta_{(\omega)} \), which vanish when \( \Theta_{(\omega)} = 0 \). We find

\[
\begin{align*}
\frac{2}{\partial x} \left( H^3 \frac{\partial \psi_{(\eta)}}{\partial x} \right) + \frac{2}{\partial y} \left( H^3 \frac{\partial \psi_{(\eta)}}{\partial y} \right) &= 0 \\
\frac{2}{\partial x} \left( H^3 \frac{\partial \psi_{(x)}}{\partial x} \right) + \frac{2}{\partial y} \left( H^3 \frac{\partial \psi_{(x)}}{\partial y} \right) &= 0 \\
\frac{2}{\partial x} \left( H^3 \frac{\partial \psi_{(k)}}{\partial x} \right) + \frac{2}{\partial y} \left( H^3 \frac{\partial \psi_{(k)}}{\partial y} \right) &= 0
\end{align*}
\] (7.92)

\[ \Theta_{(\eta)} = \frac{1}{48 H (k_{s1} + k_{s2} + k_{s1} k_{s2})} \left\{ \left[ 12 \left( B^{(\eta)} \right)^2 + B^{(\psi)} \right] \\
+ H^4 \left( \frac{\partial ^2 \psi_{(\eta)}}{\partial x^2} + \left( \frac{\partial \psi_{(\eta)}}{\partial y} \right)^2 \right) \right\} \left[ 4 + (k_{s1} + k_{s2}) H - 2(k_{s2} - k_{s1}) \right] \\
- 4 H^2 \left[ B^{(\eta)} \frac{\partial \psi_{(\eta)}}{\partial x} + B^{(\psi)} \frac{\partial \psi_{(\eta)}}{\partial y} \right] \left[ (k_{s2} - k_{s1}) H - 2(k_{s1} + k_{s2}) \right] \right\} \]

(continued)
\[ + \left( \frac{H^2}{4} - \xi \right) \left\{ \frac{1}{2H^2} \left[ B^{(2)}_{\eta} B^{(3)}_{\eta} \right] + \frac{5}{3H} \left[ B^{(2)}_{\eta} \frac{\partial V_{\eta}}{\partial x} + B^{(3)}_{\eta} \frac{\partial V_{\eta}}{\partial y} \right] \right\} + \frac{1}{12} \left( \frac{H^2}{4} + \xi \right) \left[ \left( \frac{\partial H}{\partial x} \right)^2 + \left( \frac{\partial H}{\partial y} \right)^2 \right] \] (7.93)

In continuing this analysis to the correction terms which are quadratic in the parameters \( \xi, \phi, \ldots K_k \), we consider only the terms in \( K_\eta \) and \( K_\eta \), and derive only the two dimensional partial differential equation for the pressure. For the former terms we find

\[ \begin{align*}
\frac{\partial^2 U_{\eta \eta}}{\partial z^2} - \frac{\partial P_{\eta \eta}}{\partial x} &= -\frac{\partial}{\partial z} \left[ \bar{\mu}_{\eta \eta} \frac{\partial V_{\eta \eta}}{\partial z} \right] \\
\frac{\partial^2 V_{\eta \eta}}{\partial z^2} - \frac{\partial P_{\eta \eta}}{\partial y} &= -\frac{\partial}{\partial z} \left[ \bar{\mu}_{\eta \eta} \frac{\partial V_{\eta \eta}}{\partial z} \right] \\
\frac{\partial P_{\eta \eta}}{\partial z} &= 0 \\
\frac{\partial U_{\eta \eta}}{\partial x} + \frac{\partial V_{\eta \eta}}{\partial y} + \frac{\partial W_{\eta \eta}}{\partial z} &= 0
\end{align*} \] (7.94)

subject to the boundary conditions (7.51).

Subsequent examination of the boundary conditions for curve \( \xi \) shows that \( \Psi_\eta = 0 \). Then, assuming \( \frac{\partial \mu}{\partial \theta} \) to be a constant,

\[ \bar{\mu}_{\eta \eta} = \frac{\partial \mu}{\partial \theta} \Theta_\eta \] (7.95)
The combination of (7.75), (7.93), (7.94), and (7.95) then yields

\[
\frac{\partial}{\partial x} \left( H \frac{\partial \psi_{10}}{\partial x} \right) + \frac{\partial}{\partial y} \left( H \frac{\partial \psi_{10}}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{H^3}{3c} B' \left( \frac{\partial \psi_{10}}{\partial x} + B' \frac{\partial \psi_{10}}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left[ \frac{H^3}{20} \left( B^{(x)}_1 + B' \frac{\partial \psi_{10}}{\partial x} \right) \right] + \frac{H^3}{36} \left( \left( \frac{\partial \psi_{10}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{10}}{\partial y} \right)^2 \right) \right]
\]

\[
+ \frac{1}{\tilde{R}_{31} + \tilde{R}_{32} + \tilde{R}_{51} + \tilde{R}_{52}} \left\{ \frac{H}{4} \left[ B^{(x)}_1 + B' \frac{\partial \psi_{10}}{\partial x} \right] \right\} \left[ H \left( 4 + \left( \tilde{R}_{51} + \tilde{R}_{52} \right) H \right) \frac{\partial \psi_{10}}{\partial x} \right] \right] \right]
\]

\[
-2 \left( \tilde{R}_{31} + \tilde{R}_{32} \right) B' \right] + \frac{H^3}{12} \left[ B^{(x)}_1 \frac{\partial \psi_{10}}{\partial x} + B' \frac{\partial \psi_{10}}{\partial y} \right] \left[ 2 \left( \tilde{R}_{31} + \tilde{R}_{32} \right) B' \right] \right]
\]

\[
- \frac{H^2}{48} \left[ \left( \frac{\partial \psi_{10}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{10}}{\partial y} \right)^2 \right] \left[ H \left( 4 + \left( \tilde{R}_{51} + \tilde{R}_{52} \right) H \right) \frac{\partial \psi_{10}}{\partial x} \right] \right]
\]

\[
+ \frac{1}{\tilde{R}_{31} + \tilde{R}_{32} + \tilde{R}_{51} + \tilde{R}_{52}} \left\{ \frac{H}{4} \left[ B^{(x)}_1 + B' \frac{\partial \psi_{10}}{\partial x} \right] \right\} \left[ H \left( 4 + \left( \tilde{R}_{51} + \tilde{R}_{52} \right) H \right) \frac{\partial \psi_{10}}{\partial y} \right] \right]
\]

\[
-2 \left( \tilde{R}_{31} + \tilde{R}_{32} \right) B' \right] + \frac{H^3}{12} \left[ B^{(x)}_1 \frac{\partial \psi_{10}}{\partial x} + B' \frac{\partial \psi_{10}}{\partial y} \right] \left[ 2 \left( \tilde{R}_{31} + \tilde{R}_{32} \right) B' \right] \right]
\]

\[
- \frac{H^2}{48} \left[ \left( \frac{\partial \psi_{10}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{10}}{\partial y} \right)^2 \right] \left[ H \left( 4 + \left( \tilde{R}_{51} + \tilde{R}_{52} \right) H \right) \frac{\partial \psi_{10}}{\partial y} \right] \right]
\]

\[
-3 \left( \tilde{R}_{32} - \tilde{R}_{31} \right) \left\{ \right. \left. \right\} \right)
\]

(7.96)
Since the effect of $\Phi_{(\mu\eta)}$ is generally quite small in comparison with that of $\Phi_{(\mu\eta)}$, we present only a simplified form of the $\Phi_{(\mu\eta)}$ equation in which the terms involving $\frac{F_{s1}}{2x}$ and $\frac{F_{s2}}{2x}$ are omitted (corresponding to the limiting case $\frac{F_{s1}}{2x} = \frac{F_{s2}}{2x} = \infty$). We obtain as the result

\[
\frac{2}{3x} \left( \alpha^3 \frac{\partial \Phi_{(\mu\eta)}}{\partial x} \right) + \frac{2}{3y} \left( \alpha^3 \frac{\partial \Phi_{(\mu\eta)}}{\partial y} \right) = \frac{2\alpha}{3x} \frac{2}{3y} \left\{ H \left[ 2B_{1}^{(x)} + 2B_{1}^{(y)} \right] \right. \\
\left. + \frac{H^5}{40} \left[ \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial x} \right)^2 + \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial y} \right)^2 \right] \right\} + \frac{2\alpha}{3x} \frac{2}{3y} \left\{ A_{(x)} \left[ \frac{H}{2} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \\
+ \frac{H^5}{40} \left( \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial x} \right)^2 + \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial y} \right)^2 \right) \right\} \right. \\
\left. + B_{(x)} \left[ \frac{H^3}{20} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \right\} \\
+ \frac{2\alpha}{3x} \frac{2}{3y} \left\{ A_{(y)} \left[ \frac{H}{2} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \\
+ \frac{H^5}{40} \left( \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial x} \right)^2 + \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial y} \right)^2 \right) \right\} \right. \\
\left. + B_{(y)} \left[ \frac{H^3}{20} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \right\} \\
- \frac{2\alpha}{3x} \frac{2}{3y} \left\{ A_{(x)} \left[ \frac{H}{2} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \\
+ \frac{H^5}{40} \left( \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial x} \right)^2 + \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial y} \right)^2 \right) \right\} \\
+ B_{(x)} \left[ \frac{H^3}{20} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \\
\left. - \frac{2\alpha}{3x} \frac{2}{3y} \left\{ A_{(y)} \left[ \frac{H}{2} \left( 2B_{1}^{(x)} + 2B_{1}^{(y)} \right) \right] \\
+ \frac{H^5}{40} \left( \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial x} \right)^2 + \left( \frac{\partial \Phi_{(\mu\eta)}}{\partial y} \right)^2 \right) \right\} \right\} \\
(7.97)
\]

Since, for terms of the order calculated, the pressure is independent of $z$, the boundary condition (7.28) on the curve $\delta$ is simply
\[ \Psi_{(r)} + \varepsilon \Psi_{(e)} + \phi \Psi_{(h)} + K_{\mu} \Psi_{(\mu)} + K_{\rho} \Psi_{(\rho)} + R \Psi_{(r)} + \eta \Psi_{(\eta)} \]

\[ + \lambda \Psi_{(\lambda)} + K_{\kappa} \Psi_{(\kappa)} + \eta K_{\kappa} \Psi_{(\kappa\eta)} + \eta K_{\rho} \Psi_{(\rho\eta)} = P_\theta (A, \tau) \]  

(7.98)

Since \( P_\theta \) is independent of the parameters, we have

\[ \Psi_{(r)} = P_\theta (A, \tau) \]  

(7.99)

\[ \Psi_{(e)} = \Psi_{(h)} = \cdots = \Psi_{(\kappa)} = \Psi_{(\kappa\eta)} = \Psi_{(\rho\eta)} = 0 \]  

(7.100)

Satisfaction of the boundary condition (7.29) imposes the following conditions on the individual \( \Psi_\lambda \):

\[ H^3 \left[ \eta_{(\theta)} x \frac{\partial \Psi_{(\lambda)}}{\partial x} + \eta_{(\theta)} y \frac{\partial \Psi_{(\lambda)}}{\partial y} \right] = 6 H \left[ \eta_{(\theta)} x A'' + \eta_{(\theta)} y A' \right] \]

\[ - 12 \frac{m (A, \tau)}{\rho_0 u_0 h_0} \]  

(7.101)

\[ H^3 \left[ \eta_{(\theta)} x \frac{\partial \Psi_{(\lambda)}}{\partial x} + \eta_{(\theta)} y \frac{\partial \Psi_{(\lambda)}}{\partial y} \right] = 0 \]  

(7.102)

\[ H^3 \left[ \eta_{(\theta)} x \frac{\partial \Psi_{(\lambda)}}{\partial x} + \eta_{(\theta)} y \frac{\partial \Psi_{(\lambda)}}{\partial y} \right] = H^3 \left[ \eta_{(\theta)} x F_{(\theta)} x + \eta_{(\theta)} y F_{(\theta)} y \right] \]  

(7.103)
\[
H^3 \left[ \eta(\beta)_x \frac{\partial \psi^{(\mu)}}{\partial x} + \eta(\beta)_y \frac{\partial \psi^{(\mu)}}{\partial y} \right] = \frac{\partial}{\partial \beta} \eta_{i(\alpha)} H^3 \left[ \eta(\beta)_x \frac{\partial \psi^{(\mu)}}{\partial x} + \eta(\beta)_y \frac{\partial \psi^{(\mu)}}{\partial y} \right] \\
+ \frac{2}{\partial \beta} \eta_{i(\alpha)} H^2 \left[ \eta(\beta)_x B^{(x)} + \eta(\beta)_y B^{(y)} \right] \\
+ \frac{2}{\partial \theta} \Theta_{\alpha(\omega)} H^3 \left[ \eta(\beta)_x \frac{\partial \psi^{(\mu)}}{\partial x} + \eta(\beta)_y \frac{\partial \psi^{(\mu)}}{\partial y} \right] \\
(7.104)
\]

\[
H^3 \left[ \eta(\beta)_x \frac{\partial \psi^{(\mu)}}{\partial x} + \eta(\beta)_y \frac{\partial \psi^{(\mu)}}{\partial y} \right] = \frac{\partial}{\partial \beta} \eta_{i(\alpha)} H^3 \left[ \eta(\beta)_x B^{(x)} + \eta(\beta)_y B^{(y)} \right] \\
+ \frac{2}{\partial \beta} \eta_{i(\alpha)} \left[ \frac{1}{\partial x} \psi^{(\mu)} + \frac{1}{\partial y} \Theta_{\alpha(\omega)} \right] \left[ \partial H \left( \eta(\beta)_x A^{(x)} \right) + \eta(\beta)_y A^{(y)} \right] \\
+ \frac{2}{\partial \beta} \eta_{i(\alpha)} \left( -H^3 \left( \eta(\beta)_x \frac{\partial \psi^{(\mu)}}{\partial x} + \eta(\beta)_y \frac{\partial \psi^{(\mu)}}{\partial y} \right) \right) \\
(7.105)
\]

\[
H^3 \left[ \eta(\beta)_x \frac{\partial \psi^{(\mu)}}{\partial x} + \eta(\beta)_y \frac{\partial \psi^{(\mu)}}{\partial y} \right] = \eta(\beta)_x \left\{ \frac{H^3}{2} \frac{2A^{(x)}}{\partial x} + \frac{H^5}{10} \frac{2\psi^{(\mu)}}{\partial x} \right. \\
+ \frac{H^5}{4} \frac{2\psi^{(\mu)}}{\partial x} - \frac{H^3}{4} \left( A^{(x)} \frac{2A^{(x)}}{\partial x} + A^{(y)} \frac{2A^{(y)}}{\partial x} \right) \\
+ \frac{H^5}{20} \left( \frac{2A^{(x)}}{\partial x} + \frac{2A^{(y)}}{\partial x} \right) + \frac{H^5}{20} \left( \frac{2A^{(x)}}{\partial x} + \frac{2A^{(y)}}{\partial x} \right) \\
+ \frac{H^5}{8} \left( \frac{2A^{(x)}}{\partial x} + \frac{2A^{(y)}}{\partial x} \right) \frac{2\psi^{(\mu)}}{\partial x} \\
+ \frac{H^5}{40} \left( \frac{2A^{(x)}}{\partial x} + \frac{2A^{(y)}}{\partial x} \right) \frac{2\psi^{(\mu)}}{\partial x} \\
- \frac{3}{560} \frac{\partial}{\partial x} \left[ \frac{H^7}{2} \left( \frac{2\psi^{(\mu)}}{\partial x} \right)^2 \right] - \frac{3}{560} \frac{\partial}{\partial y} \left[ \frac{H^7}{2} \left( \frac{2\psi^{(\mu)}}{\partial y} \right)^2 \right] \\
(continued)
\]
\[-\frac{3}{560} H^7 \left[ \frac{\partial^2 \psi_{\omega}}{\partial x^2} \frac{\partial \psi_{\omega}}{\partial x} + \frac{\partial^2 \psi_{\omega}}{\partial x \partial y} \frac{\partial \psi_{\omega}}{\partial y} \right] - \frac{H^3}{20} \left( B^{(x_x)} \frac{\partial B^{(x_x)}}{\partial x} + B^{(y_y)} \frac{\partial B^{(y_y)}}{\partial y} \right) \]

\[-\frac{H^2}{10} B^{(x)} \left( \frac{\partial^2 H}{\partial x^2} B^{(x)} + \frac{\partial^2 H}{\partial y^2} B^{(y)} \right) - \frac{H^3}{10} B^{(x)} \left( \frac{\partial B^{(x)}}{\partial x} + \frac{\partial B^{(y)}}{\partial y} \right) \right\} \]

\[+ \eta_{(\psi \omega)} \left\{ -\frac{H^3}{2} \frac{\partial A^{(\psi \omega)}}{\partial x} + \frac{H^5}{10} \frac{\partial^2 A^{(\psi \omega)}}{\partial x \partial y} + \frac{H^4}{4} \frac{\partial H \frac{\partial \psi_{\omega}}{\partial y}}{\partial y} \right. \]

\[-\frac{H^3}{4} \left( \frac{\partial A^{(\psi \omega)}}{\partial x} + A^{(\psi \omega)} \frac{\partial A^{(\psi \omega)}}{\partial y} \right) + \frac{H^5}{20} \left( \frac{\partial A^{(x_x)}}{\partial x} \frac{\partial^2 \psi_{\omega}}{\partial y^2} + A^{(y_y)} \frac{\partial^2 \psi_{\omega}}{\partial y^2} \right) \]

\[+ \frac{H^5}{20} \left( \frac{\partial A^{(x_x)}}{\partial x} + A^{(x_x)} \frac{\partial A^{(x_x)}}{\partial y} \right) + \frac{H^4}{8} \frac{\partial \psi_{\omega}}{\partial y} \left( \frac{\partial H \frac{\partial A^{(x_x)}}{\partial x}}{\partial x} + \frac{\partial H A^{(y_y)}}{\partial y} \right) \]

\[+ \frac{H^5}{40} \frac{\partial \psi_{\omega}}{\partial y} \left( \frac{\partial A^{(x_x)}}{\partial x} + A^{(x_x)} \frac{\partial A^{(x_x)}}{\partial y} \right) - \frac{3}{560} \frac{\partial}{\partial x} \left[ H^7 \frac{\partial^2 \psi_{\omega}}{\partial x^2} \frac{\partial \psi_{\omega}}{\partial y} \right] - \frac{3}{560} \frac{\partial}{\partial y} \left[ H^7 \frac{\partial \psi_{\omega}}{\partial y} \right]^2 \]

\[-\frac{3}{560} H^7 \left( \frac{\partial^2 \psi_{\omega}}{\partial x^2} \frac{\partial^2 \psi_{\omega}}{\partial y^2} + \frac{\partial \psi_{\omega}}{\partial y} \frac{\partial^2 \psi_{\omega}}{\partial y^2} \right) - \frac{H^3}{20} \left( B^{(x_x)} \frac{\partial B^{(x_x)}}{\partial x} + B^{(y_y)} \frac{\partial B^{(y_y)}}{\partial y} \right) \]

\[-\frac{H^2}{10} B^{(x_y)} \left( \frac{\partial^2 H}{\partial x^2} B^{(x)} + \frac{\partial^2 H}{\partial y^2} B^{(y)} \right) - \frac{H^3}{10} B^{(x_y)} \left( \frac{\partial B^{(x)}}{\partial x} + \frac{\partial B^{(y)}}{\partial y} \right) \right\} \]

(7.106)

\[H^3 \left[ \eta_{(\psi \omega)} \frac{\partial \psi_{\omega}}{\partial x} + \eta_{(\psi \omega)} \frac{\partial \psi_{\omega}}{\partial y} \right] = 0 \]

(7.107)
\[ H^3 \left[ \eta_{18} \frac{\partial \psi_{18}}{\partial x} + \eta_{181} \frac{\partial \psi_{18}}{\partial y} \right] = 0 \] (7.108)

\[ H^3 \left[ \eta_{18} \frac{\partial \psi_{18}}{\partial x} + \eta_{181} \frac{\partial \psi_{18}}{\partial y} \right] = 0 \] (7.109)

\[ H^3 \left[ \eta_{18} \frac{\partial \psi_{18}}{\partial x} + \eta_{181} \frac{\partial \psi_{18}}{\partial y} \right] = \]

\[ \frac{2\mu}{3e} \eta_{18} \left\{ \frac{H^3}{30} \left[ (B^{18})^2 + (B^{18})^2 \right] \frac{\partial \psi_{18}}{\partial x} + \right. \]

\[ + \frac{H^7}{336} \left( \left( \frac{\partial \psi_{18}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{18}}{\partial y} \right)^2 \right) \} \right\} + \frac{2\mu}{3e} \eta_{181} \left\{ \frac{H^3}{30} \left[ (B^{18})^2 + (B^{18})^2 \right] \frac{\partial \psi_{18}}{\partial x} + \right. \]

\[ + \frac{H^7}{336} \left( \left( \frac{\partial \psi_{18}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{18}}{\partial y} \right)^2 \right) \} \right\} + \]

\[ \frac{2\mu}{3e} \eta_{18} \left\{ \frac{H^3}{30} \left[ (B^{18})^2 + (B^{18})^2 \right] \frac{\partial \psi_{18}}{\partial x} + \right. \]

\[ + \frac{H^7}{336} \left( \left( \frac{\partial \psi_{18}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{18}}{\partial y} \right)^2 \right) \} \right\} + \]

\[ \frac{2\mu}{3e} \eta_{181} \left\{ \frac{H^3}{30} \left[ (B^{18})^2 + (B^{18})^2 \right] \frac{\partial \psi_{18}}{\partial x} + \right. \]

\[ + \frac{H^7}{336} \left( \left( \frac{\partial \psi_{18}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{18}}{\partial y} \right)^2 \right) \} \right\} + \]

\[ \frac{2\mu}{3e} \eta_{18} \left\{ \frac{H^3}{30} \left[ (B^{18})^2 + (B^{18})^2 \right] \frac{\partial \psi_{18}}{\partial x} + \right. \]

\[ + \frac{H^7}{336} \left( \left( \frac{\partial \psi_{18}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{18}}{\partial y} \right)^2 \right) \} \right\} + (\text{continued}) \]
\[ + \frac{2\alpha}{\sigma} \frac{\partial}{\partial \eta_{(1)\eta_1}} \left\{ \frac{H}{4} \left[ B(x)^2 + B_1(Y)^2 \right] \left[ H \left( 4 + (\bar{\kappa}_1 + \bar{\kappa}_2) H \right) + \frac{2\gamma_{(0)}}{2\eta_1} \right] \right\} \]

\[ - 2 \left[ (\bar{\kappa}_1 + \bar{\kappa}_2) B_1(Y) \right] + \frac{H^3}{12} \left[ B(x) \frac{\partial \psi_{(0)}}{\partial x} + B_1(Y) \frac{\partial \psi_{(0)}}{\partial y} \right] \left[ 2 (\bar{\kappa}_1 + \bar{\kappa}_2) B_1(Y) \right] \]

\[ - H^2 (\bar{\kappa}_2 - \bar{\kappa}_1) \frac{\partial \psi_{(0)}}{\partial x} \left[ \frac{\partial \psi_{(0)}}{\partial x} + \left( \frac{\partial \psi_{(0)}}{\partial y} \right)^2 \right] \left[ H \left( 4 + (\bar{\kappa}_1 + \bar{\kappa}_2) H \right) \right] \frac{2\gamma_{(0)}}{2\eta_1} \]

\[ - 3 (\bar{\kappa}_2 - \bar{\kappa}_1) B_1(Y) \right\} \right\} \] (7.110)

\[ H^3 \left[ n_{(1)1} \frac{\partial \psi_{(0)}}{\partial x} + n_{(1)1} \frac{\partial \psi_{(0)}}{\partial y} \right] = \frac{2\alpha}{\sigma} n_{(1)1} \left\{ A(x) \left[ \frac{H}{2} \left( B(x)^2 + B_1(Y)^2 \right) \right] \right. \]

\[ + \frac{H^5}{40} \left( \left( \frac{\partial \psi_{(0)}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{(0)}}{\partial y} \right)^2 \right) \right] + B(x) \left[ \frac{H^3}{30} \left( B(x) \frac{\partial \psi_{(0)}}{\partial x} + B_1(Y) \frac{\partial \psi_{(0)}}{\partial y} \right) \right] \]

\[ - \frac{2\gamma_{(0)}}{\partial x} \left[ \frac{H^3}{10} \left( B(x)^2 + B_1(Y)^2 \right) + \frac{H^7}{210} \left( \left( \frac{\partial \psi_{(0)}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{(0)}}{\partial y} \right)^2 \right) \right] \right\} \]

\[ + \frac{2\alpha}{\sigma} n_{(1)1} \left\{ A(x) \left[ \frac{H}{2} \left( B(x)^2 + B_1(Y)^2 \right) + \frac{H^5}{40} \left( \left( \frac{\partial \psi_{(0)}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{(0)}}{\partial y} \right)^2 \right) \right] \right. \]

\[ + B(x) \left[ \frac{H^3}{30} \left( B(x) \frac{\partial \psi_{(0)}}{\partial x} + B_1(Y) \frac{\partial \psi_{(0)}}{\partial y} \right) \right] \]

\[ - \frac{2\gamma_{(0)}}{\partial x} \left[ \frac{H^3}{10} \left( B(x)^2 + B_1(Y)^2 \right) + \frac{H^7}{210} \left( \left( \frac{\partial \psi_{(0)}}{\partial x} \right)^2 + \left( \frac{\partial \psi_{(0)}}{\partial y} \right)^2 \right) \right] \right\} \] (7.111)
It is to be noted that the boundary conditions of the first kind, (7.100), or of the second kind, (7.102), (7.107) (7.108), and (7.109) for \( \psi_{(e)}, \psi_{(\eta)}, \psi_{(\lambda)}, \) and \( \psi_{(k)} \) are all homogenous. This fact, together with the homogeneity of the corresponding differential equations (7.77) and (7.92) shows that

\[
\psi_{(e)} = \psi_{(\eta)} = \psi_{(\lambda)} = \psi_{(k)} = 0
\]  \tag{7.112}

The quantities \( \psi_{(\eta)}, \psi_{(\lambda)}, \) and \( \psi_{(k)} \) are also zero in the general case of a curved reference surface; however the vanishing of \( \psi_{(e)} \) is associated with the fact that the reference surface in the present instance is a plane. The presence of \( \varepsilon^2 \) terms in the basic equations (7.14), (7.15), and (7.16) indicates that corrections of the second order in \( \varepsilon \) occur even in the plane case.

In order to evaluate net forces and moments acting on specified portions of the boundary surfaces, stress formulae are required. For convenience, our calculations are based upon elements of area of the reference surface (i.e., the \((x_1, y_1)\) plane). Letting \( dS_1 \) be an element of area of \( S_1 \) and \( dS = dx_1 \, dy_1 \) the projection of \( dS_1 \) upon the \((x_1, y_1)\) plane, we denote by \( \Gamma_{(11)}^X \, dx_1 \, dy_1, \Gamma_{(11)}^Y \, dx_1 \, dy_1, \) and \( \Gamma_{(11)}^Z \, dx_1 \, dy_1 \) the \( x, \) \( y, \) and \( z \) components of the force exerted by the fluid on \( dS_1 \), and \( \Gamma_{(21)}^X \, dx_1 \, dy_1, \Gamma_{(21)}^Y \, dx_1 \, dy_1, \) and \( \Gamma_{(21)}^Z \, dx_1 \, dy_1 \) the components of force exerted on a similar element of area \( dS_2 \) of \( S_2 \).

Using the stress formulae
\[
\begin{align*}
\sigma_{xx} &= -\mu - \frac{2}{3} \mu \Delta + 2 \mu \frac{\partial u}{\partial x}, \\
\sigma_{yy} &= -\mu - \frac{2}{3} \mu \Delta + 2 \mu \frac{\partial u}{\partial y}, \\
\sigma_{zz} &= -\mu - \frac{2}{3} \mu \Delta + 2 \mu \frac{\partial u}{\partial z}, \\
\sigma_{xy} &= \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \\
\sigma_{yz} &= \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \\
\sigma_{xz} &= \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right),
\end{align*}
\] (7.113)
we find, in dimensionless notation, the following equations for \( \bar{q}^{x,y,z}_{11} \) and \( \bar{q}^{x,y,z}_{12} \):

\[
\bar{q}^{x}_{11} = \frac{\mu_0 u_0 L_0}{h_0} \left\{ E \left[ \frac{\partial^2 u}{\partial z^2} + P \frac{\partial H_0}{\partial x} \right] + \epsilon K_{uu} \left[ \mu_0 \frac{\partial^2 u}{\partial z^2} \right] + O(\varepsilon^2) \right\}
\] (7.114)

\[
\bar{q}^{y}_{11} = \frac{\mu_0 u_0 L_0}{h_0} \left\{ E \left[ \frac{\partial^2 v}{\partial x^2} + P \frac{\partial H_0}{\partial y} \right] + \epsilon K_{uv} \left[ \mu_0 \frac{\partial^2 v}{\partial x^2} \right] + O(\varepsilon^2) \right\}
\] (7.115)

\[
\bar{q}^{z}_{11} = \frac{\mu_0 u_0 L_0}{h_0} \left\{ \left[ -P \right] + O(\varepsilon^2) \right\}
\] (7.116)

\[
\bar{q}^{x}_{12} = -\frac{\mu_0 u_0 L_0}{h_0} \left\{ E \left[ \frac{\partial^2 u}{\partial z^2} + P \frac{\partial H_0}{\partial x} \right] + \epsilon K_{uu} \left[ \mu_0 \frac{\partial^2 u}{\partial z^2} \right] + O(\varepsilon^2) \right\}
\] (7.117)

\[
\bar{q}^{y}_{12} = -\frac{\mu_0 u_0 L_0}{h_0} \left\{ E \left[ \frac{\partial^2 v}{\partial x^2} + P \frac{\partial H_0}{\partial y} \right] + \epsilon K_{uv} \left[ \mu_0 \frac{\partial^2 v}{\partial x^2} \right] + O(\varepsilon^2) \right\}
\] (7.118)
$$Q_{12}^x = \frac{\text{MouL} \omega}{h_0^2} \left\{ [P] + O(\epsilon^3) \right\}$$  \hspace{1cm} (7.119)

The quantities $\frac{2V}{\partial z}$, $\frac{2V}{\partial z}$, $P$, and $\mathbf{u}$ of these formulæ are to be evaluated at the appropriate surface, $S_1$ or $S_2$.

Introducing the series expansions (7.30) and (7.31), the above relations become

$$Q_{12}^x = \frac{\text{MouL} \omega}{h_0^2} \left\{ \epsilon \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial x} \right] + \epsilon \phi \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial x} \right] ight. $$

$$+ \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} \psi_{\nu} \frac{\partial H}{\partial x} \right] + \epsilon K_{\nu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial x} \right] $$

$$+ \epsilon \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial x} \right] + \epsilon \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial x} \right]$$

$$\left. \left. + \epsilon \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial x} \right] \right\} \right\}$$  \hspace{1cm} (7.120)

$$Q_{12}^y = \frac{\text{MouL} \omega}{h_0^2} \left\{ \epsilon \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial y} \right] + \epsilon \phi \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial y} \right] ight. $$

$$+ \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} \psi_{\nu} \frac{\partial H}{\partial y} \right] + \epsilon K_{\nu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial y} \right] $$

$$+ \epsilon \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial y} \right] + \epsilon \epsilon K_{\mu} \left[ \left( \frac{\partial V}{\partial z} \right)_{n1} + \psi_{\nu} \frac{\partial H}{\partial y} \right]$$

(continued)
\[ F_{in} = - \frac{m_0 u_0 L_0}{h_0} \left\{ \left[ \phi \left( \psi_{in} \right) \right] + K_m \left[ \psi_{in} \right] + K_p \left[ \psi_{ir} \right] + R \left[ \psi_{ir} \right] + \eta K_m \left[ \psi_{in} \eta \right] + \eta K_p \left[ \psi_{ir} \eta \right] \right\} \]  

(7.122)

\[ F_{(2)} = - \frac{m_0 u_0 L_0}{h_0} \left\{ \left[ \left( \frac{\partial(\psi_{(2)})}{\partial z} \right)_{(2)} + \psi_{(2)} \frac{\partial H_{(2)}}{\partial x} \right] + E \phi \left[ \left( \frac{\partial(\psi_{(1)})}{\partial z} \right)_{(1)} + \psi \frac{\partial H_{(1)}}{\partial x} \right] + \eta K_m \left[ \left( \frac{\partial(\psi_{(1)})}{\partial z} \right)_{(1)} + \psi \frac{\partial H_{(1)}}{\partial x} \right] + \eta K_p \left[ \left( \frac{\partial(\psi_{(2)})}{\partial z} \right)_{(2)} + \psi_{(2)} \frac{\partial H_{(2)}}{\partial x} \right] \right\} \]  

(7.123)
\[ \mathbf{T}_{(\Delta_2)} = -\frac{\mu_0 u_{\Delta_2} L_0}{h_0} \left\{ \varepsilon \left[ \left( \frac{\partial V_{(\Delta_2)}}{\partial z} \right)_{(\Delta_2)} + Y_{(\Delta_2)} \frac{\partial H_{(\Delta_2)}}{\partial y} \right] + \varepsilon \left[ \left( \frac{\partial V_{(\Delta_2)}}{\partial z} \right)_{(\Delta_2)} + Y_{(\Delta_2)} \frac{\partial H_{(\Delta_2)}}{\partial y} \right] \right\} \\
+ \varepsilon K_m \left[ \left( \frac{\partial v_{(\Delta_2)}}{\partial z} \right)_{(\Delta_2)} + Y_{(\Delta_2)} \frac{\partial H_{(\Delta_2)}}{\partial y} \right] + \varepsilon K_0 \left[ \left( \frac{\partial v_{(\Delta_2)}}{\partial z} \right)_{(\Delta_2)} + Y_{(\Delta_2)} \frac{\partial H_{(\Delta_2)}}{\partial y} \right] \\
+ \varepsilon H_{(\Delta_2)} \left[ \left( \frac{\partial v_{(\Delta_2)}}{\partial z} \right)_{(\Delta_2)} + Y_{(\Delta_2)} \frac{\partial H_{(\Delta_2)}}{\partial y} \right] + \varepsilon K_m \left[ \left( \frac{\partial v_{(\Delta_2)}}{\partial z} \right)_{(\Delta_2)} + Y_{(\Delta_2)} \frac{\partial H_{(\Delta_2)}}{\partial y} \right] \right\} \\
(7.124) \\

\[ \mathbf{T}_{(\Delta_2)} = \frac{\mu_0 u_{\Delta_2} L_0}{h_0} \left\{ [Y_{(\Delta_2)}] + \varepsilon [Y_{(\Delta_2)}] + K_m [Y_{(\Delta_2)}] + K_0 [Y_{(\Delta_2)}] \right\} \\
+ R [Y_{(\Delta_2)}] + \eta K_m [Y_{(\Delta_2)}] \right\} + \eta K_0 [Y_{(\Delta_2)}] \right\} \\
(7.125) \\

The quantities \( \frac{\partial V_{(1)}}{\partial z} \), \( \frac{\partial V_{(2)}}{\partial z} \), \( \frac{\partial \mu_{(1)}}{\partial z} \), etc. denote the values of the expression within the parentheses at the surface \( S_1 \) or \( S_2 \) indicated by the subscript. As the final step of this section we tabulate these values. Since the quantities in question differ at \( S_1 \) and \( S_2 \) only by changes of sign in some of the terms, to save space we tabulate values pertaining to \( S_1 \), encircling those signs that are to be changed at \( S_2 \).
\[
\begin{align*}
\left( \frac{\partial V_{\omega}}{\partial x} \right)_{\eta} &= \frac{B^{(x)}}{H} \Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial x} \\
\left( \frac{\partial V_{\omega}}{\partial y} \right)_{\eta} &= \frac{B^{(y)}}{H} \Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial y} \\
\left( \frac{\partial V_{\omega}}{\partial z} \right)_{\eta} &= -\Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial x} + H \frac{\partial F_{\omega}}{\partial x} \\
\left( \frac{\partial V_{\omega}}{\partial z} \right)_{\eta} &= -\Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial y} + H \frac{\partial F_{\omega}}{\partial y} \\
\left( \frac{\partial V_{\omega}}{\partial z} \right)_{\eta} &= -\Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial z} + \frac{H}{2} \frac{\partial F_{\omega}}{\partial z} \\
\left( \frac{\partial V_{\omega}}{\partial z} \right)_{\eta} &= -\Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial z} + \frac{H}{2} \frac{\partial F_{\omega}}{\partial z} \\
\left( \frac{\partial V_{\omega}}{\partial z} \right)_{\eta} &= -\Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial z} + \frac{H}{2} \frac{\partial F_{\omega}}{\partial z} \\
\left( \frac{\partial V_{\omega}}{\partial z} \right)_{\eta} &= -\Theta \frac{H}{2} \frac{\partial \psi_{\omega}}{\partial z} + \frac{H}{2} \frac{\partial F_{\omega}}{\partial z} \\
\end{align*}
\]
\[
\frac{\partial U_{(1)}}{\partial z} = \Theta \frac{H}{2} \frac{\partial \psi}{\partial x} \\
\frac{\partial U_{(2)}}{\partial z} = \Theta \frac{H}{2} \frac{\partial \psi}{\partial x}
\]

\[
\frac{\partial U_{(1)}}{\partial z} = \Theta \frac{H}{2} \frac{\partial \psi_{(1)}}{\partial x} \Theta \frac{\partial}{\partial z} \left( \frac{H}{4} A^{(w)} \right) + \Theta \frac{\partial}{\partial z} \left( \frac{H^3}{24} \frac{\partial \psi}{\partial x} \right) \\
\Theta \frac{\partial}{\partial x} \left[ \frac{H}{8} A^{(w)^2} \right] \Theta \frac{\partial}{\partial z} \left[ \frac{H}{8} A^{(w)} A^{(w)} \right] + \Theta \frac{\partial}{\partial z} \left\{ \Theta \frac{\partial}{\partial x} \left[ \frac{H^3}{24} (2A^{(w)} \frac{\partial \psi}{\partial x}) \right] \\
+ \Theta \frac{\partial}{\partial x} \left[ \frac{H^3}{240} (\frac{\partial \psi_{(1)}}{\partial x})^2 \right] \right\}
\]

\[
\Theta \frac{\partial}{\partial z} \left[ \frac{H}{240} \left( \frac{\partial \psi_{(1)}}{\partial x} \right)^2 \right] \right\} \Theta \frac{\partial}{\partial z} \left[ \frac{H^5}{240} \frac{\partial \psi_{(1)}}{\partial x} \right] + \Theta \frac{\partial}{\partial z} \left[ \frac{H}{24} B^{(w)^2} \right] \Theta \frac{\partial}{\partial z} \left[ \frac{H}{24} B^{(w)} B^{(w)} \right] \\
\Theta \frac{\partial}{\partial x} \left[ \frac{H}{240} \left( \frac{\partial \psi_{(1)}}{\partial x} \right)^2 \right] \right\} \Theta \frac{\partial}{\partial z} \left[ \frac{H^5}{240} \frac{\partial \psi_{(1)}}{\partial x} \right] + \Theta \frac{\partial}{\partial z} \left[ \frac{H}{24} B^{(w)^2} \right] \Theta \frac{\partial}{\partial z} \left[ \frac{H}{24} B^{(w)} B^{(w)} \right] \\
+ \Theta \frac{\partial}{\partial z} \left[ \frac{H^5}{240} \frac{\partial \psi_{(1)}}{\partial x} \right] + \Theta \frac{\partial}{\partial z} \left[ \frac{H}{24} B^{(w)^2} \right] + \Theta \frac{\partial}{\partial z} \left[ \frac{H}{24} B^{(w)} B^{(w)} \right] - \Theta \frac{\partial}{\partial z} \left[ \frac{H^3}{240} \frac{\partial \psi_{(1)}}{\partial x} \right]
\]

\[
- \frac{H^2}{240} \frac{\partial}{\partial x} \left[ H B^{(w)} \frac{\partial \psi_{(1)}}{\partial x} \right] - \frac{H^2}{240} \frac{\partial}{\partial z} \left[ H B^{(w)} \frac{\partial \psi_{(1)}}{\partial x} \right] \\
- \frac{H^3}{120} \frac{\partial}{\partial x} \left( \frac{\partial B^{(w)}}{\partial x} + \frac{\partial B^{(w)}}{\partial y} \right)
\]

(7.135)
\[
\left( \frac{\partial^2 V(r)}{\partial T^2} \right)_{T} = \Theta \frac{\partial}{\partial T} \left[ \frac{H^4}{24} A'(\psi) \right] + \Theta \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{24} A' \psi \right] + \Theta \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{24} \frac{\partial^2 \psi}{\partial T^2} \right] \\

\Theta \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{48} A'(\psi)^2 \right] + \Theta \left( \frac{H^3}{48} A'(\psi)^2 \right) \Theta \frac{\partial}{\partial T} \left[ \frac{H^3}{24} B'(\psi) \right] + \Theta \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{24} B'(\psi)^2 \right] \\

\Theta \frac{\partial}{\partial T} \left[ \frac{H^5}{240} \frac{\partial^2 \psi}{\partial T^2} \right] + \Theta \frac{\partial}{\partial T} \left[ \frac{H^5}{240} \left( \frac{\partial^2 \psi}{\partial T^2} \right)^2 \right] + \frac{H}{24} \left( \frac{\partial^2 A'(\psi)}{\partial T^2} \right) + \frac{H}{24} \left( \frac{\partial^2 B'(\psi)}{\partial T^2} \right) \\

+ \frac{2}{3} \left( \frac{H}{12} B'(\psi) \right) + \frac{2}{3} \left( \frac{H}{24} A'(\psi)\right) + \frac{2}{3} \left( \frac{H}{24} A'(\psi) \right) - \frac{2}{3} \left( \frac{H^3}{240} B'(\psi) \right) \\

- \frac{2}{3} \left( \frac{H^3}{240} B'(\psi) \right) - \frac{H}{240} \left[ \frac{H^3}{240} \frac{\partial \psi}{\partial T} \right] - \frac{H}{240} \left[ \frac{H^3}{240} \frac{\partial \psi}{\partial T} \right] \\

- \frac{H^2}{240} \left( \frac{\partial B'(\psi)}{\partial T} \right) + \frac{H^2}{240} \left( \frac{\partial B'(\psi)}{\partial T} \right) + \frac{H^3}{120} \left( \frac{\partial^2 \psi}{\partial T^2} \right) + \frac{H^3}{120} \left( \frac{\partial^2 \psi}{\partial T^2} \right)
\]

(7.136)

\[
\left( \frac{\partial^2 V(r)}{\partial T^2} \right)_{T} = \Theta \frac{\partial}{\partial T} \left[ \frac{H^4}{24} \psi \right] + \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{24} \psi \right] + \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{24} \frac{\partial^2 \psi}{\partial T^2} \right] \\

+ \Theta \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{48} \psi \right] + \Theta \frac{\partial}{\partial T} \left( \frac{H^3}{48} \psi \right) \Theta \frac{\partial}{\partial T} \left[ \frac{H^3}{24} B'(\psi) \right] + \Theta \frac{\partial^2}{\partial T^2} \left[ \frac{H^3}{24} B'(\psi)^2 \right] \\

+ \Theta \frac{\partial}{\partial T} \left[ \frac{H^5}{240} \frac{\partial^2 \psi}{\partial T^2} \right] + \Theta \frac{\partial}{\partial T} \left[ \frac{H^5}{240} \left( \frac{\partial^2 \psi}{\partial T^2} \right)^2 \right] + \frac{H}{24} \left( \frac{\partial^2 A'(\psi)}{\partial T^2} \right) + \frac{H}{24} \left( \frac{\partial^2 B'(\psi)}{\partial T^2} \right) \\
\frac{H^2}{240} \left( \frac{\partial B'(\psi)}{\partial T} \right) + \frac{H^2}{240} \left( \frac{\partial B'(\psi)}{\partial T} \right) + \frac{H^3}{120} \left( \frac{\partial^2 \psi}{\partial T^2} \right) + \frac{H^3}{120} \left( \frac{\partial^2 \psi}{\partial T^2} \right)
\]

(continued)
\[ \left( \frac{\partial V_{(m)}}{\partial z} \right)_{\nu} = -\frac{
abla \cdot \bar{\phi}_{(m)}}{2} + \frac{2\mu}{\delta \theta} \left\{ B^{(4)} \left[ \frac{1}{12\bar{H}} (B^{\omega 2} + B^{\psi 2}) \right] + \frac{H^3}{240} \left( \frac{\partial \bar{\phi}_{(m)}}{\partial x} \right)^2 \right\} + \frac{1}{350} \frac{2\mu}{\delta \theta} \left[ B^{(4)} \left( \frac{\partial \bar{\phi}_{(m)}}{\partial x} \right)^2 \right] \]

\[ + \frac{H^2}{48H(K_{s1} + K_{s2} + K_{s1}, K_{s2})} \left\{ -2(K_{s2} - K_{s1}) \left[ 12(B^{\omega 2} + B^{\psi 2}) \right] 
+ 8H^3(K_{s1} + K_{s2}) \left[ B^{(4)} \frac{\partial \bar{\phi}_{(m)}}{\partial x} + B^{\psi 4} \frac{\partial \bar{\phi}_{(m)}}{\partial y} \right] \right\} \] (7.137)

\[ \frac{\partial \bar{\phi}_{(m)}}{\partial z} = \frac{2\mu}{\delta \theta} \left\{ B^{(4)} \left[ \frac{1}{12\bar{H}} (B^{\omega 2} + B^{\psi 2}) \right] + \frac{H^3}{360} \left( \frac{\partial \bar{\phi}_{(m)}}{\partial x} \right)^2 \right\} + \frac{H^2}{48H(K_{s1} + K_{s2} + K_{s1}, K_{s2})} \left\{ -2(K_{s2} - K_{s1}) \left[ 12(B^{\omega 2} + B^{\psi 2}) \right] 
+ 8H^3(K_{s1} + K_{s2}) \left[ B^{(4)} \frac{\partial \bar{\phi}_{(m)}}{\partial x} + B^{\psi 4} \frac{\partial \bar{\phi}_{(m)}}{\partial y} \right] \right\} \]

\[ + \frac{H^2}{48H(K_{s1} + K_{s2} + K_{s1}, K_{s2})} \left\{ -2(K_{s2} - K_{s1}) \left[ 12(B^{\omega 2} + B^{\psi 2}) \right] 
+ 8H^3(K_{s1} + K_{s2}) \left[ B^{(4)} \frac{\partial \bar{\phi}_{(m)}}{\partial x} + B^{\psi 4} \frac{\partial \bar{\phi}_{(m)}}{\partial y} \right] \right\} \] (7.138)

\[ (\bar{\mu}_{(m)})_{(1)} = \frac{2\mu}{\delta \theta} \left\{ 4 + 2K_{s2} + H^3(K_{s1} + K_{s2}, K_{s1}, K_{s2}) \left[ B^{(4)} \frac{\partial \bar{\phi}_{(m)}}{\partial x} + B^{\psi 4} \frac{\partial \bar{\phi}_{(m)}}{\partial y} \right] \right\} \] (7.139)
\[
(\vec{n}_{12}) = \frac{2u}{2\phi} \frac{2\phi}{4H(\bar{K}_{12} + \bar{K}_{52} + \bar{K}_{31} + \bar{K}_{32} + \bar{H})} \left\{ (4 + 2\bar{K}_{31} + \bar{H}) \left[ 12 \left( B^{\omega_2} + B^{\phi_1} \right) \right. \right.
\]
\[+ H (\frac{2\phi}{2x} + \frac{2\phi}{2y}) \left. \right] + 8 \bar{K}_{31} H^2 \left[ B^{\omega_2} \frac{2\phi}{2x} + B^{\phi_1} \frac{2\phi}{2y} \right] \right\} \]
\[= (7.140)\]

\[
(\frac{\partial V}{\partial x}) = 0 \frac{H}{2} \frac{\partial^2 \phi}{\partial x^2} \]
\[= (7.141)\]

\[
(\frac{\partial V}{\partial y}) = 0 \frac{H}{2} \frac{\partial^2 \phi}{\partial y^2} \]
\[= (7.142)\]
Section 8. General Equations Referred to Cylindrical Coordinates:

To provide a convenient starting point for subsequent analysis of the lubrication of journal bearings, in this section we refer the general tensor equations of Part A to a system of coordinates in which the reference surface S is a circular cylinder. In this task our attention is restricted to the final results of the previous analysis; that is, to the basic partial differential equations for \( \psi_0, \psi_1, \ldots, \psi_{(p)} \) the boundary conditions on the curve \( B \), and the formulae relating to the stresses exerted on \( S_1 \) and \( S_2 \) by the fluid. Many of the resulting relations are found to be identical, except for minor changes in notation, with those developed in Section 7 for the case of a plane reference surface. To save space, only those equations requiring essential modification are written here.

Let \((r, \phi, z_1)\)* denote a system of cylindrical coordinates, in which the reference surface is a circular cylinder defined by

\[
S: \quad r = r_0
\]  
(8.1)

Also, let the equations defining \( S_1 \) and \( S_2 \) be

\[
S_1: \quad r = r_0 + h(0)(\phi, z_1, t) \\
S_2: \quad r = r_0 + h(1)(\phi, z_1, t)
\]  
(8.2)

where

\[
h = h(2) - h(1) > 0.
\]

* As in Section 7, the subscript "1" is employed for the Z coordinate, reserving Z without subscripts for a corresponding dimensionless variable.
As in the preceding, we let \( h_0 \) be a representative value of \( h(\varphi, Z_1, t) \), and let \( L_0 = r_0 \). For the \( x^1 \) coordinate system we choose
\[
 x^1 = r_0 \varphi \quad x^2 = Z_1 \quad x^3 = \frac{r-r_0}{\varepsilon} \tag{8.3}
\]
Since, in cylindrical coordinates,
\[
 (ds)^2 = (dr)^2 + r^2 (d\varphi)^2 + (dz_1)^2
\]
we find
\[
 (ds)^2 = \left(1 + \frac{\varepsilon x_3}{r_0}\right)^2 (dx')^2 + (dx^2)^2 + \varepsilon^2 (dx^3)^2 \tag{8.4}
\]
whence
\[
 \begin{align*}
 g'' &= \left(1 + \frac{\varepsilon x_3}{r_0}\right)^2 \quad g_{12} = 1 \quad g_{33} = \varepsilon^2 \\
 g'' &= \left(1 + \frac{\varepsilon x_3}{r_0}\right)^2 \quad g_{22} = 1 \quad g_{33} = \frac{1}{\varepsilon^2} \\
 g_{12} &= g_{23} = g_{31} = g_{12} = g_{23} = g_{31} = 0
\end{align*} \tag{8.5}
\]
Expanding equations (8.5) in power series in \( \frac{\varepsilon x_3}{r_0} \),
and comparing the results with equations (3.3) to (3.13)
inclusive we find
\[
 \begin{align*}
 G_{\alpha\beta\gamma} &= 1 \quad G_{\alpha\beta\gamma} = 2 \quad G_{\alpha\beta\gamma} = 1 \\
 G_{[\alpha][1]22} &= 1 \quad G_{[\alpha][1]22} = 0 \quad G_{[1]22} = 0 \\
 G_{[\alpha]} &= 1 \quad G_{[\alpha]} = -2 \quad G_{[\alpha]} = 3 \\
 G_{[22]} &= 1 \quad G_{[22]} = 0 \quad G_{[22]} = 0
\end{align*} \tag{8.6}
\]
Since each of the quantities \( G_{[\alpha][\beta]}^{\gamma} \) is a constant,
the Christoffel symbols of the first and second kinds in
\( (\alpha, \beta, \gamma) \) vanish identically, whence, from (3.14),
\[
 \left\{ \begin{array}{c}
 \alpha \\
 \beta \gamma \\
 \end{array} \right\}_0 = \sum_{\alpha}^{\gamma} \beta \gamma = 0 \tag{8.7}
\]
The operation of two-dimensional covariant differentiation thus reduces to that of simple partial differentiation, as it does in the plane case. For the two dimensional tensors \( \partial_{[\alpha}^{\beta} \) we obtain by means of (3.19) the values

\[
\begin{align*}
\partial_{[1}^{2} &= 1 \\
\partial_{[2}^{1} &= 1 \\
\partial_{[\alpha}^{2} &= 0 \\
\partial_{[\alpha}^{\beta} &= 0 \quad (\alpha \neq \beta)
\end{align*}
\]  

(8.8)

Let \((u^r, u^\theta, u^z)\) denote the components of the fluid velocity. We then define*

\[
\begin{align*}
\mathbf{U} &= \frac{u^\theta}{u_o} \\
\mathbf{V} &= \frac{u^z}{u_o} \\
\mathbf{W} &= \frac{u^r}{\epsilon u_o}
\end{align*}
\]  

(8.9)

Letting \((u_{(1)}^r, u_{(1)}^\theta, u_{(1)}^z)\) and \((u_{(2)}^r, u_{(2)}^\theta, u_{(2)}^z)\) denote the velocity components of \(S_1\) and \(S_2\) respectively, we also define

\[
\begin{align*}
U_{(1)} &= \frac{u_{(1)}^\theta}{u_o} \\
V_{(1)} &= \frac{u_{(1)}^z}{u_o} \\
W_{(1)} &= \frac{u_{(1)}^r}{u_o}
\end{align*}
\]  

(8.10)

\[
\begin{align*}
U_{(2)} &= \frac{u_{(2)}^\theta}{u_o} \\
V_{(2)} &= \frac{u_{(2)}^z}{u_o} \\
W_{(2)} &= \frac{u_{(2)}^r}{u_o}
\end{align*}
\]

and

\[
\begin{align*}
A^{(\theta)} &= U_{(1)} + U_{(2)} \\
A^{(z)} &= V_{(1)} + V_{(2)} \\
B^{(\theta)} &= U_{(2)} - U_{(1)} \\
B^{(z)} &= V_{(2)} - V_{(1)}
\end{align*}
\]  

(8.11)

* This choice, which is not entirely in accord with usual practice, is made in order to increase the degree of correspondence between the results of this section and those pertaining to the case of a plane reference surface.
In addition, we let \((f^r, f^\theta, f^z)\) denote the components of the vector body force, and

\[
F^\phi = \frac{f^\phi}{f_0}, \quad F^z = \frac{f^z}{f_0}, \quad F^r = \frac{f^r}{\epsilon f_0}
\]  

Finally, we note that the dimensionless coordinates

\((y^1, y^2)\) as defined by (3.7) are

\[
y^1 = \phi \quad \quad y^2 = \frac{z}{r_0}
\]  

(8.13)

We write

\[
z = \frac{z}{r_0}
\]  

(8.14)

and henceforth employ the simpler and more readily visualized notation \((\phi, z)\) rather than \((y^1, y^2)\). We retain the notation \(y^3\), however, in those cases where this quantity is needed.

Transcribing our general tensor formulae into the present notation, we find, from (5.34),

\[
\frac{\partial}{\partial \phi} \left( H^3 \frac{\partial \psi_{(0)}}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( H^3 \frac{\partial \psi_{(0)}}{\partial z} \right) = 6 \frac{\partial}{\partial \phi} \left( H A^{(\phi)} \right) + 6 \frac{\partial}{\partial z} \left( H A^{(z)} \right) + 12 \frac{\partial H}{\partial \epsilon}
\]

(8.15)

as the partial differential equation for \(\psi_{(0)}\). We observe that this equation is identical with the previously derived equation (7.62) for the plane case, under the transformation

\[
(x, y, z) \quad \quad \rightarrow \quad \quad (\phi, z, y^3)
\]

(8.16)
We may, in fact, state as a general proposition that those equations of Part A which do not involve the tensors \( \Gamma^{\alpha}_{\beta \gamma} \), \( \mathbf{G}_{[\alpha \beta]} \), etc., may be correctly written in cylindrical coordinate form by applying the transformation (8.16) to the corresponding equations of Section 7.

Using equations (8.6), (8.8), and (8.11), the differential equation (5.62) for \( \psi(\epsilon) \) becomes:

\[
\frac{d}{d\phi} \left( H^3 \frac{d\psi(\epsilon)}{d\phi} \right) + \frac{d}{d\zeta} \left( H^3 \frac{d\psi(\epsilon)}{d\zeta} \right) = 12 \left( H_{(2)} \frac{dH_{(2)}}{d\zeta} - H_{(1)} \frac{dH_{(1)}}{d\zeta} \right) \\
+ \frac{d}{d\phi} \left\{ 3H (H_{(1)} + H_{(2)}) A^{(\phi)} + 4H^2 B^{(\phi)} \right\} \\
+ \frac{d}{d\zeta} \left\{ 3H (H_{(1)} + H_{(2)}) A^{(\zeta)} + 2H^2 B^{(\zeta)} \right\} \\
+ \frac{1}{2} \frac{d}{d\phi} \left\{ H^3 (H_{(1)} + H_{(2)}) \frac{d\psi(\epsilon)}{d\phi} \right\} - \frac{1}{2} \frac{d}{d\zeta} \left\{ H^3 (H_{(1)} + H_{(2)}) \frac{d\psi(\epsilon)}{d\zeta} \right\} \\
(8.17)
\]

whereas, from (5.63),

\[
\Theta(\epsilon) = -\frac{\bar{K}_{s1} \bar{K}_{s2} (\Theta_{(2)} - \Theta_{(1)})}{2 \left[ \bar{K}_{s1} + \bar{K}_{s2} + \bar{K}_{s1} \bar{K}_{s2} H \right]^2} \left\{ 2H + \bar{K}_{s1} \bar{K}_{s2} H \left( \frac{H}{2} + \frac{3}{2} \right) \\
+ \bar{K}_{s2} H \left( \frac{H}{2} - \frac{3}{2} \right) + \left( \bar{K}_{s1} + \bar{K}_{s2} + \bar{K}_{s1} \bar{K}_{s2} H \right) \left( \frac{H^2}{4} - \frac{3}{2} \right) \right\} \\
(8.18)
\]
Examination of equations (5.64), (5.65), (5.66), (5.71),
(5.72), (5.78), (5.84), (5.85), (5.90), (5.92), (5.97), and
(5.101) shows that the differential equations for \( \Psi_\eta \), \( \Psi_\mu \),
\( \Psi_\rho \), \( \Psi_\kappa \), \( \Psi_\lambda \), \( \Psi_\kappa \), \( \Psi_\eta \) and \( \Psi_\eta \)
together with the values of \( \Theta_\eta \), \( \Theta_\kappa \), \( \Theta_\rho \), \( \Theta_\kappa \) and \( \Theta_\eta \)
are those given in Section 7, transformed according to (8.16).

With respect to boundary conditions of the first kind
to be satisfied on \( \mathcal{B} \), it is clear that equations (7.99)
and (7.100) hold without change in cylindrical coordinates.
From (5.105) to (5.112) it follows that the boundary con-
ditions of the second kind are also obtainable from equations
(7.101) to (7.111) by the transformation (8.16), with the
exception of equation (7.102) for \( \Psi_\epsilon \). The appropriate form of
this equation, obtained from (5.106), is

\[
H^3 \left[ n_{(\kappa)} \frac{\partial \Psi_\epsilon}{\partial \varphi} + n_{(\rho)} \frac{\partial \Psi_\epsilon}{\partial \varphi} \right] = H^2 \left( 3n_{(\kappa)} B_{(\kappa)}^\eta + n_{(\rho)} B_{(\rho)}^\epsilon \right) \\
+ H^3 \left( H_\epsilon - H_\rho \right) n_{(\kappa)} \frac{\partial \Psi_\epsilon}{\partial \varphi}
\]

(8.19)

Considering, finally, the stresses exerted on \( S_1 \) and
\( S_2 \) by the fluid, we find that the components of stress in
the direction perpendicular to \( S \) are correctly given by
(7.122) and (7.125) with the addition of a term in \( \Psi_\epsilon \).
For the $\Phi$ and $Z$ components we have, from (5.124) and (5.125),

$$
\Phi_{(n)} \phi \equiv \mu_0 \frac{u_0 c_0}{h_0^2} \left\{ \varepsilon \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + \varepsilon^2 \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + 
\psi_{(\varepsilon)} \frac{\partial H_{(n)}}{\partial \phi} - 2 H_{(n)} \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + \varepsilon \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + 
\varepsilon K_{\mu} \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + \varepsilon R \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + 
\varepsilon \eta K_{\mu} \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] + \varepsilon \eta K_{\rho} \left[ (\frac{\partial U_{(s)}}{\partial y^3})_{(n)} + \psi_{(\phi)} \frac{\partial H_{(n)}}{\partial \phi} \right] \right\}
$$

(8.20)
\[ \mathcal{F}_{(1)} = \frac{\mu_0 \mu_0 \sigma_0}{h^2} \left\{ \epsilon \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(1)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] + \epsilon \left[ \left( \frac{\partial V_{(t)}}{\partial y^3} \right)_{(1)} + \psi_{(t)} \frac{\partial H_{(t)}}{\partial \varphi} \right] \right. \\
+ \epsilon \phi \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(s)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] + \epsilon \kappa_\mu \left[ \left( \frac{\partial V_{(t)}}{\partial y^3} \right)_{(t)} + \psi_{(t)} \frac{\partial H_{(t)}}{\partial \varphi} \right] \\
+ \epsilon \kappa_\rho \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(s)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] \\
+ \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(t)}}{\partial y^3} \right)_{(t)} + \psi_{(t)} \frac{\partial H_{(t)}}{\partial \varphi} \right] \right\} \\
+ \epsilon R \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(s)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] + \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(t)}}{\partial y^3} \right)_{(t)} + \psi_{(t)} \frac{\partial H_{(t)}}{\partial \varphi} \right] \\
+ \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(s)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] \right\} \\
+ \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(t)}}{\partial y^3} \right)_{(t)} + \psi_{(t)} \frac{\partial H_{(t)}}{\partial \varphi} \right] \right\} \\
+ \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(s)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] \right\} \right\} \\
+ \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(t)}}{\partial y^3} \right)_{(t)} + \psi_{(t)} \frac{\partial H_{(t)}}{\partial \varphi} \right] \right\} \right\} \right\} \\
+ \epsilon \eta \kappa_\rho \left[ \left( \frac{\partial V_{(s)}}{\partial y^3} \right)_{(s)} + \psi_{(s)} \frac{\partial H_{(s)}}{\partial \varphi} \right] \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right}
\[ \frac{\mathbf{F}_{(2)}}{2} = -\frac{\mu_0\mu_0\epsilon_0}{h_0^2} \left\{ \epsilon \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(0)} + \psi_{(0)} \frac{\partial H_{(0)}}{\partial z} \right] + \epsilon^2 \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(2)} + \psi_{(0)} \frac{\partial H_{(0)}}{\partial z} \right] \\
+ \psi_{(1)} \frac{\partial H_{(0)}}{\partial z} \right\} + \epsilon \phi \left[ (\frac{\partial V_{(1)}}{\partial y^3})_{(0)} + \psi_{(0)} \frac{\partial H_{(1)}}{\partial z} \right] \\
+ \epsilon K_\mu \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(1)} + \psi_{(0)} \frac{\partial H_{(0)}}{\partial z} + (\overline{\mu}_{(0)})_{(2)} (\frac{\partial V_{(0)}}{\partial y^3})_{(2)} \right] \\
+ \epsilon K_\mu \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(2)} + \psi_{(0)} \frac{\partial H_{(2)}}{\partial z} \right] + \epsilon R \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(0)} + \psi_{(0)} \frac{\partial H_{(0)}}{\partial z} \right] \\
+ \epsilon \eta K_\mu \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(1)} + \psi_{(0)} \frac{\partial H_{(0)}}{\partial z} + (\overline{\mu}_{(0)})_{(2)} (\frac{\partial V_{(0)}}{\partial y^3})_{(2)} \right] \\
+ \epsilon \eta K_\mu \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(2)} + \psi_{(0)} \frac{\partial H_{(2)}}{\partial z} \right] \right\} \\
+ \epsilon \eta K_\mu \left[ (\frac{\partial V_{(0)}}{\partial y^3})_{(2)} + \psi_{(0)} \frac{\partial H_{(2)}}{\partial z} \right] \right\} \right\} \\
(3.23) \]

The values of the \( y^3 \) derivatives of \( U_{(0)} \), \( V_{(0)} \), \( U_{(0)} \), \( V_{(0)} \), \( U_{(1)} \), \( V_{(1)} \), etc. may be found from equations (7.126) to (7.138) and the transformation (8.16). In addition, the following quantities, determined from (5.128) and (5.129), are required for the \( \epsilon \) terms:
\[
\left( \frac{\partial U_{(e)}}{\partial y^3} \right)_{(0)} = -\frac{H}{2} \frac{\partial \psi_{(e)}}{\partial \phi} + \frac{3}{2} B^{(p)} + \left[ \frac{H}{2} (H_0 + H_{(e)}) - \frac{5H^2}{12} \right] \frac{\partial \psi_{(e)}}{\partial \phi} \\
(8.24)
\]

\[
\left( \frac{\partial V_{(e)}}{\partial y^3} \right)_{(0)} = -\frac{H}{2} \frac{\partial \psi_{(e)}}{\partial \phi} + \frac{1}{2} B^{(z)} - \frac{H}{12} \frac{\partial \psi_{(e)}}{\partial \phi} \\
(8.25)
\]

\[
\left( \frac{\partial U_{(e)}}{\partial y^3} \right)_{(z)} = \frac{H}{2} \frac{\partial \psi_{(e)}}{\partial \phi} - \frac{3}{2} B^{(p)} - \left[ \frac{H}{2} (H_0 + H_{(e)}) + \frac{5H^2}{12} \right] \frac{\partial \psi_{(e)}}{\partial \phi} \\
(8.26)
\]

\[
\left( \frac{\partial V_{(e)}}{\partial y^3} \right)_{(z)} = \frac{H}{2} \frac{\partial \psi_{(e)}}{\partial \phi} - \frac{1}{2} B^{(z)} - \frac{H}{12} \frac{\partial \psi_{(e)}}{\partial \phi} \\
(8.27)
\]
Section 9. Analysis of Geometrical and Thermal Corrections for Flooded Journal Bearings

The preceding general theory is applied in this section to investigate corrections to the theory of flooded journal bearings arising from curvature of the boundary surfaces and from heat generated by distortion of the fluid. The lifting force and frictional moment for such bearings have previously been calculated by Muskat and Morgan, using the first approximation lubrication theory represented by equation (8.15) together with an expansion of the pressure in a power series in the ratio of shaft displacement to radial clearance. Similar series are employed in the present analysis; however, we

Figure 5
retain fewer terms in this expansion. Our results are therefore mutually compared, rather than with the curves calculated by Muskat and Morgan.

The bearing to be analyzed and the system of coordinates employed are indicated schematically in Figure 5. The \( z \) axis is assumed to lie along the center line of the shaft, with the \((x_1, y_1)\) plane bisecting the bearing. The shaft radius is denoted by \( r_0 \), and that of the bearing surface by \( r_0 + h_0 \). The center line of the shaft is assumed to be displaced by a distance \( d \) in a horizontal direction with respect to the center line of the bearing. The polar angle \( \phi \) is measured from the \( x_1 \) axis in the counter-clockwise direction, and the direction of shaft rotation is positive, as shown. The bearing is assumed to be immersed in a bath of lubricant, so that at the edges \((z_1 = \pm \infty)\) the pressure \( p \) has a constant value \( \bar{p} \). By symmetry, \( p \) must be an even function of \( z_1 \).

As the reference distance \( L_0 \) we select the shaft radius \( r_0 \), writing as the dimensionless coordinate in the \( z_1 \) direction

\[
\bar{z} = \frac{z_1}{r_0}
\]  
(9.1)

The quantity \( h_0 \) has already been defined as the difference between bearing and shaft radii; in accordance with previous notation we let

\[
\varepsilon = \frac{h_0}{r_0}
\]  
(9.2)
Also, we define a dimensionless parameter $\delta$ by the relation

$$\delta = \frac{d}{h_0} \quad (9.3)$$

It is clear from Figure 5 that the equality $\delta = 0$ corresponds to a situation in which the shaft is perfectly centered within the bearing, and that the equality $\delta = 1$ corresponds to a situation of contact between the shaft and bearing.

With $\omega$ denoting the angular velocity of the shaft, the velocity of the shaft surface, given by

$$u_0 = r_0 \omega \quad (9.4)$$

is selected as the reference velocity $u_0$. In terms of the number of shaft revolutions per unit time, denoted by $N$,

$$u_0 = 2\pi r_0 N \quad (9.5)$$

The surface $S_1$ is that of the shaft, and $S_2$ is the bearing surface. The reference surface $S$ is assumed to coincide with $S_1$, whence

$$H(\phi, \tau) = 0 \quad (9.6)$$

Also, since the surface $S_2$ is stationary, and since the velocity of $S_1$ is in the $\phi$ direction only, equations (8.10) give

$$U(0) = 1 \quad V(0) = 0 \quad W(0) = 0 \quad (9.7)$$

$$U(\omega) = 0 \quad V(\omega) = 0 \quad W(\omega) = 0$$
Therefore, from (8.11),

\[
\begin{align*}
A^{(q)} &= 1 \\
B^{(q)} &= -1 \\
A^{(a)} &= B^{(a)} = 0
\end{align*}
\]  

(9.8)

By a simple trigonometric calculation

\[
H(q, z, \tau) = \frac{1}{h_o} \left\{ \delta^2 \cos \phi + \sqrt{(\theta_0 + h_0)^2 - \delta^2 \sin \phi} - r_0 \right\}
\]  

(9.9)

In terms of dimensionless variables this expression becomes

\[
H = (1 + \delta \cos \phi) - \varepsilon \left[ \frac{\delta^2}{4^2} (1 - \cos 2\phi) \right] + O(\varepsilon)
\]  

(9.10)

In the following work we wish to evaluate the quantities \(\psi_{(\phi)}, \psi_{(\ell)},\) and \(\psi_{(\mu, \eta)}\) defined in the previous analysis, and in terms of these functions to calculate expressions for the lifting force and frictional moment exerted on the shaft by the lubricant. Forces contributed by terms of the type \(\psi_{(\phi)}, \psi_{(\eta)}, \psi_{(\tau)},\) are neglected. Considering first \(\psi_{(\phi)},\) we require a solution to the differential equation (8.15), which may be written by means of (9.8) in the form

\[
\frac{\partial}{\partial \phi} \left( H^3 \frac{\partial \psi_{(\phi)}}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( H^3 \frac{\partial \psi_{(\phi)}}{\partial z} \right) = 6 \frac{\partial H}{\partial \phi}
\]  

(9.11)
The boundary conditions to be imposed on this equation are

$$
\psi_{(0)} = \overline{P}, \quad (z = \pm z_0)
$$  \hspace{1cm} (9.12)

where

$$
\overline{z} = \frac{\overline{z}}{r_0} \quad \overline{P} = \frac{\mu_0 \nu_0 \overline{r}_0}{\mu_0 \nu_0 r_0} \overline{P}
$$  \hspace{1cm} (9.13)

We first note from (9.10) that \( H(\phi) \), and therefore \( \psi_{(0)} \), depends upon \( \epsilon \). We therefore introduce the series

$$
\psi_{(0)} = \psi_{(00)} + \epsilon \psi_{(0\epsilon)} + \epsilon^2 \psi_{(0\epsilon\epsilon)} \cdots \hspace{1cm} (9.14)
$$

and expand both sides of (9.11) in power series in \( \epsilon \).

Assuming \( \overline{P} \) to be independent of \( \epsilon \), we find for \( \psi_{(00)} \) and \( \psi_{(0\epsilon)} \) the equations

$$
\frac{\partial}{\partial \phi} \left[ (1 + \delta \cos \phi)^3 \frac{\partial \psi_{(00)}}{\partial \phi} \right] + \frac{\partial}{\partial z} \left[ (1 + \delta \cos \phi)^3 \frac{\partial \psi_{(00)}}{\partial z} \right] = -6 \delta \cos \phi
$$  \hspace{1cm} (9.15)

$$
\frac{\partial}{\partial \phi} \left[ (1 + \delta \cos \phi)^3 \frac{\partial \psi_{(0\epsilon)}}{\partial \phi} \right] + \frac{\partial}{\partial z} \left[ (1 + \delta \cos \phi)^3 \frac{\partial \psi_{(0\epsilon)}}{\partial z} \right] = -3 \delta^2 \sin 2\phi
$$

$$
+ \frac{\partial}{\partial \phi} \left[ \frac{3 \delta^2}{4} (1 + \delta \cos \phi) (1 - \cos 2\phi) \frac{\partial \psi_{(00)}}{\partial \phi} \right]
$$

$$
+ \frac{\partial}{\partial z} \left[ \frac{3 \delta^2}{4} (1 + \delta \cos \phi) (1 - \cos 2\phi) \frac{\partial \psi_{(00)}}{\partial z} \right]
$$  \hspace{1cm} (9.16)

to be solved under the boundary conditions

$$
\begin{align*}
\psi_{(00)} &= \overline{P} \\
\psi_{(0\epsilon)} &= 0 \\
(z = \pm z_0)
\end{align*}
$$  \hspace{1cm} (9.17)
To calculate the function \( \psi_{(00)} \) from (9.15) and the first of equations (9.17), we let

\[
\psi_{(00)} = \psi_{(00)0} + \delta \psi_{(00)1} + \delta^2 \psi_{(00)2} \quad \ldots \ldots \quad (9.18)
\]

and expand both sides of (9.15) in a power series in \( \delta \).

In this manner we obtain a sequence of Poisson equations for the individual quantities \( \psi_{(00)0} \), \( \psi_{(00)1} \), etc. These equations, treated by Muskat and Morgan, readily yield the solution

\[
\psi_{(00)} = \overline{P} + 6 \delta \sin \phi \left[ 1 - \frac{\cosh \frac{\pi}{\lambda}}{\cosh \frac{\pi}{\lambda_0}} \right]
+ 3 \delta^2 \sin 2 \phi \left[ -\frac{3}{2} + \frac{\cosh \frac{\pi}{\lambda}}{\cosh \frac{\pi}{\lambda_0}} + \frac{1}{2} \frac{\cosh \frac{2\pi}{\lambda}}{\cosh \frac{2\pi}{\lambda_0}} \right] + \ldots \ldots (9.19)
\]

which may be seen directly to satisfy the boundary condition (9.17). In this analysis we carry terms through the second power in \( \delta \) only. Higher order terms for the function \( \psi_{(00)} \) have been given by Muskat and Morgan; however, the retention of these would greatly complicate our subsequent calculations.
To calculate $\psi_{(\omega)}$ we proceed in a similar manner, writing

$$\psi_{(\omega)} = \psi_{(\omega)0} + \delta \psi_{(\omega)1} + \delta^2 \psi_{(\omega)2} + \ldots \ldots$$ \hfill (9.20)

Noting from (9.19) that

$$\frac{\partial \psi_{(\omega)}}{\partial \phi} = O(\delta) \quad \frac{\partial \psi_{(\omega)}}{\partial z} = O(\delta)$$ \hfill (9.21)

we see that (9.16) may be written

$$\frac{\partial}{\partial \phi} \left[ (1 + \delta \cos \phi) \frac{\partial \psi_{(\omega)}}{\partial \phi} \right] + \frac{\partial}{\partial z} \left[ (1 + \delta \cos \phi)^3 \frac{\partial \psi_{(\omega)}}{\partial z} \right] =$$

$$- 3 \delta^2 \sin 2\phi + O(\delta^3)$$ \hfill (9.22)

Since the boundary conditions (9.17) for $\psi_{(\omega)}$ are homogeneous, the substitution of (9.20) into (9.22) yields

$$\psi_{(\omega)0} = \psi_{(\omega)1} = 0$$ \hfill (9.23)

By means of this relation, the substitution of the series (9.20) into (9.22) yields

$$\frac{\partial^2 \psi_{(\omega)2}}{\partial \phi^2} + \frac{\partial^2 \psi_{(\omega)2}}{\partial z^2} = - 3 \sin 2\phi$$ \hfill (9.24)

as the differential equation for $\psi_{(\omega)2}$. The solution to (9.24) which vanishes at $z = z_0$ is

$$\psi_{(\omega)2} = \frac{3}{4} \sin 2\phi \left[ 1 - \frac{\cosh 2z}{\cosh 2z_0} \right]$$ \hfill (9.25)
whence

\[ \psi_{(\epsilon)} = \frac{3\delta^2}{4} \sin 2\varphi \left[ 1 - \frac{\cosh \frac{2\epsilon}{2}}{\cosh \frac{2z}{2z_0}} \right] + O(\delta^3) \]  

(9.26)

The function \( \psi_{(\epsilon)} \) is thus found from (9.14), (9.19), and (9.26) to be

\[ \psi_{(\epsilon)} = \bar{F} + 6\delta \sin \varphi \left[ 1 - \frac{\cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} \right] \]

\[ + 3\delta^2 \sin 2\varphi \left[ -\frac{3}{2} + \frac{\cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} + \frac{1}{2} \frac{\cosh \frac{2z}{2z_0}}{\cosh \frac{2z_0}{2z_0}} \right] \]

\[ + \frac{3\delta^2}{4} \sin 2\varphi \left[ 1 - \frac{\cosh \frac{2z}{2z_0}}{\cosh \frac{2z_0}{2z_0}} \right] + O(\delta^3, \epsilon^2) \]  

(9.27)

To calculate \( \psi_{(\epsilon)} \) we employ the differential equation (8.17), together with the simplifications

\[ H_0 = 0, \quad H_\omega = H, \quad \frac{\partial H_\omega}{\partial \tau} = \frac{\partial H_{(\epsilon)}}{\partial \tau} = 0 \]  

(9.28)

and the relations (9.8). We thereby obtain

\[ \frac{\partial}{\partial \varphi} \left[ H^3 \frac{\partial \psi_{(\epsilon)}}{\partial \varphi} \right] + \frac{\partial}{\partial z} \left[ H^3 \frac{\partial \psi_{(\epsilon)}}{\partial z} \right] = -\frac{\partial}{\partial \varphi} \left[ H^2 \right] \]

\[ + \frac{1}{2} \frac{\partial}{\partial \varphi} \left[ H^4 \frac{\partial \psi_{(\epsilon)}}{\partial \varphi} \right] + \frac{\partial}{\partial z} \left[ \frac{1}{2} \frac{\partial}{\partial z} \left[ H^4 \frac{\partial \psi_{(\epsilon)}}{\partial z} \right] \right] \]  

(9.29)

to be solved under the boundary conditions

\[ \psi_{(\epsilon)} = 0 \quad (z = \pm z_0) \]  

(9.30)
Since, in the final expression for the pressure, the quantity $\psi_1(\varepsilon)$ is to be multiplied by $\varepsilon$, we may neglect terms of order $\varepsilon$ in $H$ and $\psi_1(0)$ in eq. (9.30). These terms would contribute effects of the order $\varepsilon^2$ in the final result, and could not be included legitimately without the additional inclusion of $\varepsilon^2$ terms in $\psi_1(0)$. The quantity $H$ in (9.29) is therefore to be identified with $1 + \delta \cos \phi$, and $\psi_1(0)$ with $\psi_1(100)$ as given by (9.19). We note from (9.21) and (9.29) that $\psi_1(0) = O(\delta)$.

With these simplifications we write

$$\psi_1(\varepsilon) = \varepsilon \psi_1(\varepsilon)_1 + \varepsilon^2 \psi_1(\varepsilon)_2 + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ ld
Substituting this expression into (9.33) we obtain for \( \psi(z) \) the value

\[
\psi(z) = \sin 2\phi \left\{ -\frac{1}{4} \left( \frac{1 - \cosh 2z}{\cosh 2z_0} \right) - \frac{3}{2} \left( \frac{z \sinh 2z}{\cosh 2z_0} \right) \right. \\
- \frac{z_0 \tanh z_0 \cosh 2z}{\cosh 2z_0} \right) \left( \frac{\cosh 2z - \cosh 2z_0}{\cosh 2z_0} \right) \\
\left. - \frac{3}{2} \left( \frac{z \sinh 2z}{\cosh 2z_0} - \frac{z_0 \tanh 2z_0 \cosh 2z}{\cosh 2z_0} \right) \right\}
\]

(9.35)

Thus, combining these expressions,

\[
\psi(z) = \delta \sin \phi \left\{ 1 - \frac{\cosh 2z}{\cosh 2z_0} + \frac{3z \sinh 2z - 3z_0 \tanh z_0 \cosh 2z}{\cosh 2z_0} \right\} \\
+ \delta^2 \sin 2\phi \left\{ -\frac{1}{4} \left( \frac{1 - \cosh 2z}{\cosh 2z_0} \right) - \frac{3}{2} \left( \frac{z \sinh 2z - z_0 \tanh 2z_0 \cosh 2z}{\cosh 2z_0} \right) \right. \\
- \left( \frac{z_0 \tanh z_0 \cosh 2z}{\cosh 2z_0} \right) \left( \frac{\cosh 2z - \cosh 2z_0}{\cosh 2z_0} \right) \\
\left. - \frac{3}{2} \left( \frac{z \sinh 2z}{\cosh 2z_0} - \frac{z_0 \tanh 2z_0 \cosh 2z}{\cosh 2z_0} \right) \right\}
\]

(9.36)

To calculate \( \psi(\eta) \) we employ the differential equation (7.96), substituting \( \phi \) for \( x \) and \( z \) for \( y \). For simplicity we assume a single constant heat transfer coefficient

\[
\bar{K}_s = \bar{K}_{s_2} = \bar{K}_s
\]

(9.37)
Using the relations (9.8) we then find

\[
\frac{\partial}{\partial \phi} \left( H^3 \frac{\partial \psi^{(m)}}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( H^3 \frac{\partial \psi^{(m)}}{\partial z} \right) = \frac{\partial \mu}{\partial \theta} \frac{\partial}{\partial \phi} \left( \frac{H^3 \partial \psi^{(0)}}{\partial \phi} \right) \\
+ \frac{H^7}{336} \frac{\partial \psi^{(0)}}{\partial \phi} \left[ \left( \frac{\partial \psi^{(0)}}{\partial \phi} \right)^2 + \left( \frac{\partial \psi^{(0)}}{\partial z} \right)^2 \right] + \frac{H^2}{2K_s} \frac{\partial \psi^{(0)}}{\partial \phi} + \frac{H}{2 + K_s H} \\
+ \frac{H^3}{3(2 + K_s H)} \frac{\partial \psi^{(0)}}{\partial \phi} + \frac{H^6}{24K_s} \frac{\partial \psi^{(0)}}{\partial \phi} \left[ \left( \frac{\partial \psi^{(0)}}{\partial \phi} \right)^2 + \left( \frac{\partial \psi^{(0)}}{\partial z} \right)^2 \right] \\
+ \frac{\partial \mu}{\partial \theta} \frac{\partial}{\partial z} \left\{ \frac{H^3}{20} \frac{\partial \psi^{(0)}}{\partial z} + \frac{H^7}{336} \frac{\partial \psi^{(0)}}{\partial \phi} \left[ \left( \frac{\partial \psi^{(0)}}{\partial \phi} \right)^2 + \left( \frac{\partial \psi^{(0)}}{\partial z} \right)^2 \right] \right\} \\
+ \frac{H^2}{2K_s} \frac{\partial \psi^{(0)}}{\partial z} + \frac{H^6}{24K_s} \frac{\partial \psi^{(0)}}{\partial z} \left[ \left( \frac{\partial \psi^{(0)}}{\partial \phi} \right)^2 + \left( \frac{\partial \psi^{(0)}}{\partial z} \right)^2 \right] \right\} \\
(9.38)
\]

as the required differential equation.

The right-hand side of equation (9.38) is expanded in a power series in \( \delta \), after collecting terms to derive the following simplified equation:

\[
\frac{\partial}{\partial \phi} \left( H^3 \frac{\partial \psi^{(m)}}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( H^3 \frac{\partial \psi^{(m)}}{\partial z} \right) = \delta \sin \phi \left( \alpha_0 + \alpha, \frac{\cosh z}{\cosh z_0} \right) \\
\frac{\partial}{\partial \phi} \left( H^2 \frac{\partial \psi^{(0)}}{\partial \phi} \right) + \frac{\partial}{\partial \phi} \left( H^6 \frac{\partial \psi^{(0)}}{\partial \phi} \right) + \delta^2 \sin 2 \phi \left( \beta_0 + \beta, \frac{\cosh z}{\cosh z_0} + \beta, \frac{\cosh z}{\cosh z_0} \right) \right)
\]

(9.39)

where the relations \( H = 1 + \delta \cos \phi \) and \( \psi^{(0)} = \psi^{(0)} \) have been used, and where
\[ \alpha_0 = -\frac{\partial \mu}{\partial \theta} \left[ \frac{24 + 40\overline{K}_s + 14\overline{K}_s^2 + \overline{K}_s^3}{2\overline{K}_s (2 + \overline{K}_s)^2} \right] \]

\[ \alpha_1 = \frac{\partial \mu}{\partial \theta} \left[ \frac{12 + \overline{K}_s}{5(2 + \overline{K}_s)} \right] \]

\[ \beta_0 = \frac{\partial \mu}{\partial \theta} \left[ \frac{24 + 36\overline{K}_s + 24\overline{K}_s^2 + 5\overline{K}_s^3}{\overline{K}_s (2 + \overline{K}_s)^3} \right] \]

\[ \beta_1 = -\frac{\partial \mu}{\partial \theta} \left[ \frac{60 + 12\overline{K}_s + 7\overline{K}_s^2 - 2\overline{K}_s^3}{10\overline{K}_s (2 + \overline{K}_s)^2} \right] \]

\[ \beta_2 = -\frac{\partial \mu}{\partial \theta} \left[ \frac{12 + \overline{K}_s}{5(2 + \overline{K}_s)} \right] \]

As in previous calculations, we write

\[ \psi_{(\mu \eta)} = \delta \psi_{(\mu \eta)} + \delta^2 \psi_{(\mu \eta)}^2 + \cdots \]  

(9.41)

obtaining

\[ \frac{\partial^2 \psi_{(\mu \eta)}}{\partial \varphi^2} + \frac{\partial^2 \psi_{(\mu \eta)}}{\partial z^2} = \sin \varphi \left( \alpha_0 + \alpha, \cosh \frac{z}{\cosh \varphi} \right) \]

(9.42)

and

\[ \frac{\partial^2 \psi_{(\mu \eta)}^2}{\partial \varphi^2} + \frac{\partial^2 \psi_{(\mu \eta)}^2}{\partial z^2} = \sin 2 \varphi \left( \beta_0 + \beta_1, \cosh \frac{z}{\cosh \varphi} + \beta_2, \cosh \frac{2z}{\cosh \varphi} \right) \]

\[ -3 \frac{\partial}{\partial \varphi} \left( \cos \varphi \frac{\partial \psi_{(\mu \eta)}}{\partial \varphi} \right) - 3 \cos \varphi \frac{\partial^2 \psi_{(\mu \eta)}}{\partial z^2} \]

(9.43)
The solutions to these equations, which vanish at \( z = \pm z_0 \), are found to be

\[
\Psi_1(\mu_1) = \sin \phi \left[ -a_0 \left( 1 - \frac{\cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} \right) + \frac{a_1}{2} \left( \frac{z \sinh \frac{z}{2} - z_0 \tanh \frac{z_0}{2} \cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} \right) \right]
\]

(9.44)

and

\[
\Psi_2(\mu_2) = \sin 2 \phi \left[ \lambda_0 + \frac{\cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} \lambda_1 + \frac{z \sinh \frac{z}{2}}{\cosh \frac{z_0}{2}} \lambda_2 + \frac{z \sinh \frac{z}{2} - \tanh \frac{z_0}{2} \cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} \lambda_3 + \frac{\cosh \frac{z}{2}}{\cosh \frac{z_0}{2}} \lambda_4 \right]
\]

(9.45)

where

\[
\begin{align*}
\lambda_0 &= \frac{3a_0 - \beta_0}{4} \\
\lambda_1 &= \frac{2a_1 - 2\beta_1 - 3a_0 + a_1 \frac{z_0 \tanh \frac{z_0}{2}}{4}}{6} \\
\lambda_2 &= -\frac{a_1}{4} \\
\lambda_3 &= \frac{\beta_1}{4} \\
\lambda_4 &= \frac{\beta_0 - a_0 + \frac{\beta_1 - a_1 \beta_2}{3} - \frac{\beta_2 z_0 \tanh \frac{z_0}{2}}{4}}{4}
\end{align*}
\]

(9.46)
In terms of the calculated quantities $\psi_{(\phi)}$, $\psi_{(\phi\phi)}$, $\psi_{(\phi\phi\phi)}$, and $\psi_{(\phi\phi\phi\phi)}$, we now wish to evaluate the net lift force and moment exerted on the shaft by the fluid. By symmetry, we may neglect the $z$ components of force, since these integrate to give a zero resultant. For the stress component $F_{(\phi)}$ we find from (8.20) and the relation $H_{(1)} = 0$

$$F_{(\phi)} = \frac{\mu_0 u_0 c_0}{h_0^2} \left\{ \varepsilon \left[ \left( \frac{\partial U_{(c)}}{\partial y^3} \right)_{(i)} \right] + \varepsilon^2 \left[ \left( \frac{\partial U_{(c)}}{\partial y^3} \right)_{(ii)} \right] 
+ \varepsilon \eta \kappa \mu \left[ \left( \frac{\partial U_{(\phi\phi)}}{\partial y^3} \right)_{(i)} + \left( \mu_{(\phi)} \right)_{(ii)} \left( \frac{\partial U_{(c)}}{\partial y^3} \right)_{(i)} \right] \right\}$$

(9.47)

and from (7.122)

$$F_{(c)} = -\frac{\mu_0 u_0 c_0}{h_0^2} \left\{ \psi_{(c)} + \varepsilon \psi_{(\phi\phi)} + \eta \kappa \mu \psi_{(\phi\phi\phi)} \right\}$$

(9.48)

where only terms of the type considered in this section have been included.

The derivatives used in (9.47) are found, from (7.126), (7.137), and (8.24), to be

$$\left( \frac{\partial U_{(c)}}{\partial y^3} \right)_{(i)} = -\frac{1}{H} - \frac{H}{2} \frac{\partial \psi_{(c)}}{\partial \phi}$$

(9.49)

$$\left( \frac{\partial U_{(\phi\phi)}}{\partial y^3} \right)_{(i)} = -\frac{3}{2} - \frac{H}{2} \frac{\partial \psi_{(\phi\phi)}}{\partial \phi} + \frac{H^2}{12} \frac{\partial \psi_{(c)}}{\partial \phi}$$

(9.50)
\[
\left( \frac{\partial U_1(u)}{\partial y^2} \right)_0 = -\frac{H}{2} \frac{\partial \psi_0}{\partial \phi} + \frac{\partial \mu}{\partial \theta} \left\{ -\frac{1}{12} \frac{H^3}{144} \left( \frac{\partial \psi_1}{\partial \phi} \right)^2 \right. \\
\left. - \frac{H^3}{240} \left( \frac{\partial \psi_1}{\partial \phi} \right)^2 + \frac{6 + 5K_s H}{12 K_s (2 + K_s H)} \frac{\partial \psi_0}{\partial \phi} \right. \\
\left. + \frac{H^3}{18 (2 + K_s H)} \left( \frac{\partial \psi_1}{\partial \phi} \right)^2 + \frac{H^4}{48 K_s} \frac{\partial \psi_0}{\partial \phi} \left[ \left( \frac{\partial \psi_1}{\partial \phi} \right)^2 + \left( \frac{\partial \psi_0}{\partial \phi} \right)^2 \right] \right\}
\]

(9.51)

Also

\[
\left( \bar{\mu} \right)_0 = \frac{\partial \mu}{\partial \theta} \left\{ \frac{1}{2K_s H} \frac{H^2}{\epsilon (2 + K_s H)} \frac{\partial \psi_1}{\partial \phi} + \frac{H^3}{24 K_s} \left[ \left( \frac{\partial \psi_1}{\partial \phi} \right)^2 + \left( \frac{\partial \psi_0}{\partial \phi} \right)^2 \right] \right\}
\]

(9.52)

In evaluating (9.49) we first expand \( H \) and \( \psi_1 \) in series in \( \epsilon \), as given by (9.10) and (9.14), subsequently combining the \( \epsilon \) terms thus obtained with those arising from (9.50). In evaluating (9.50), (9.51), and (9.52) we let \( H = 1 + \delta \cos \phi \) and \( \psi_1 = \psi_0 \) as before.

The net vertical lift force \( F \) and moment \( M \) are given by the integrals

\[
F = -c_0^2 \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \left( F_{u_0} \sin \phi + F_{u_0}^\theta \cos \phi \right) d\phi \\
M = -c_0^3 \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} F_{u_0}^\theta d\phi
\]

(9.53)

(9.54)

The horizontal component of force on the shaft is found to be
identically zero. After considerable algebraic work we obtain the following expressions for these integrals:

\[
F = \frac{2\pi \mu_0 u_0 r_0}{h_0^2} \left\{ \delta \left[ F_{(0)} + \varepsilon F_{(e)} + \eta \kappa \mu \frac{\partial \mu}{\partial \varepsilon} F_{(\mu \eta)} \right] + O(\delta^3) \right\}
\]

\[
M = \frac{2\pi \mu_0 u_0 r_0}{h_0} \left\{ m_{(0)} + \varepsilon m_{(e)} + \eta \kappa \mu \frac{\partial \mu}{\partial \varepsilon} M_{(0 \mu \eta)} \right\}
\]

\[
+ \delta^2 \left[ m_{(20)} + \varepsilon m_{(2e)} + \eta \kappa \mu \frac{\partial \mu}{\partial \varepsilon} M_{(2 \mu \eta)} \right] + O(\delta^4) \right\}
\]

(9.55)

(9.56)

where

\[
F_{(0)} = 6 (z_0 - \tanh z_0)
\]

\[
F_{(e)} = 6 z_0 - 7 \tanh z_0 - 3 z_0 \tanh^2 z_0
\]

\[
F_{(\mu \eta)} = c_0 (z_0 - \tanh z_0) + c_1 (z_0 - \tanh z_0)
\]

\[
+ c_1 (z_0 \tanh z_0)
\]

(9.57)
\[ M_{(0\sigma)} = 2z_0 \quad M_{(2\sigma)} = 4z_0 - 3\tanh z_0 \]
\[ M_{(0k)} = 3z_0 \quad M_{(1\sigma)} = \frac{3}{2}z_0 \text{sech}^2z_0 - \tanh z_0 \]
\[ M_{(0\mu)} = d_0z_0 \]
\[ M_{(2\mu)} = d_1z_0 + d_2\tanh z_0 + d_3\frac{z_0^2}{2}\tanh^2 z_0 + d_4 z_0 \text{sech}^2 z_0 \]

and

\[ c_0 = \frac{24 + 40\overline{K}_s + 14\overline{K}_s^2 + \overline{K}_s^3}{2\overline{K}_s (2 + \overline{K}_s)^2} \]
\[ c_1 = \frac{12 + \overline{K}_s}{10(2 + \overline{K}_s)} \]
\[ d_0 = \frac{6 + \overline{K}_s}{6\overline{K}_s} \]
\[ d_1 = \frac{540 + 796\overline{K}_s + 286\overline{K}_s^2 + 19\overline{K}_s^3}{30\overline{K}_s (2 + \overline{K}_s)^2} \]
\[ d_2 = -\frac{60 + 107\overline{K}_s + 42\overline{K}_s^2 + 3\overline{K}_s^3}{5\overline{K}_s (2 + \overline{K}_s)^2} \]
\[ d_3 = -\frac{12 + \overline{K}_s}{20(2 + \overline{K}_s)} \]
\[ d_4 = -\frac{12 + \overline{K}_s}{20(2 + \overline{K}_s)} \]

(9.58)

(9.59)

(9.60)
In the remainder of this work it is convenient to introduce a parameter \( m \), defined by

\[
    m^2 = - \eta K_0 \frac{\partial \mu}{\partial \theta}
\]

(9.61)

From the definitions of \( \mu \) and \( \eta \) given by (3.50) and (3.62) it is found that

\[
    m = \frac{\mu_0}{\sqrt{K_0 | \frac{\partial \mu}{\partial \theta} |}}
\]

(9.62)

where \( K_0 \) is the standard value of thermal conductivity of the fluid, and \( | \frac{\partial \mu}{\partial \theta} | \) is the slope of the viscosity-temperature curve at the reference temperature. Since these latter quantities are fixed properties of the fluid, it is seen that \( m \) is a parameter which is equivalent to velocity, but which is dimensionless.

Solving equation (9.62) for \( \mu_0 \), and substituting the result into (9.55) and (9.56), we find

\[
    F = C m \delta \left[ F_{(0)} + \varepsilon F_{(e)} - m^2 F_{(\mu \eta)} \right]
\]

(9.63)

\[
    M = h_0 C m \left\{ \left[ M_{(0)} + \varepsilon M_{(e)} - m^2 M_{(0 \mu \eta)} \right] + \delta^2 \left[ M_{(20)} + \varepsilon M_{(2e)} - m^2 M_{(2 \mu \eta)} \right] \right\}
\]

(9.64)

where
\[ C = \frac{2\pi \mu_0 r_0^3}{\frac{1}{k_0} \left. \frac{\partial \mu}{\partial T} \right|_0} \]  

(9.65)

Henceforth we shall write

\[ F' = \frac{F}{Z_0 C} \quad M' = \frac{M}{Z_0 h_0 C} \]  

(9.66)

The factor \( z_0 \) has been removed from \( F \) and \( M \) since, for the case of very long bearings, \( F \) and \( M \) are proportional to \( z_0 \). The quantities \( F' \) and \( M' \), therefore, represent in dimensionless form the lifting force and moment per unit length of the bearing, and \( (M'/F') \) is a representation of a coefficient of friction.

We first wish to examine the order of magnitude of the \( \varepsilon - \) corrections. Letting

\[
\begin{align*}
\bar{M}(\varepsilon) &= \bar{M}(0\varepsilon) + \delta^2 \bar{M}(2\varepsilon) \\
\bar{M}(0) &= \bar{M}(00) + \delta^2 \bar{M}(20)
\end{align*}
\]

(9.67)

Figure 6 and 7 show plots of the ratios \( (F(\varepsilon)/F(0)) \) and \( (M(\varepsilon)/M(0)) \) against \( z_0 \), for \( \delta = 0.0, 0.2, 0.4 \). To within terms of the order considered, \( (F(\varepsilon)/F(0)) \) is independent of \( \delta \); hence only one curve of this ratio has been drawn.

Since in general the parameter \( \varepsilon \) is of the order of magnitude of 0.01 or less, it is clear from these curves that \( \varepsilon F(\varepsilon) \) and \( \varepsilon M(\varepsilon) \)
FIGURE 7

$\frac{M_\infty}{M_0}$ vs. $Z_0$
may be neglected in comparison with \( F_0 \) and \( M_0 \) respectively, except perhaps in extremely narrow bearings. Henceforth we shall omit the terms in \( \varepsilon \) from our calculations.

An important role in conventional lubrication analysis is played by the so-called Sommerfeld variable \( S \), defined in our notation by

\[
S = \frac{1}{\varepsilon^2} \frac{\mu_0 N}{F/4r_0^2z_0}
\]

From (9.5), (9.62), (9.65), and (9.66) we find

\[
S = \frac{m}{\pi^2 F'}
\]

or an equivalent form (neglecting \( F' \))

\[
S = \frac{z_0}{\pi^2 \delta (F_{10} - m^2 F_{10})}
\]

A further useful relation, obtained from (9.63) and (9.66), is

\[
S = \frac{F'z_0}{m (F_{10} - m^2 F_{10})}
\]

Figure 8 to 12 inclusive depict the shaft displacement \( \delta \) and friction coefficient \( (M'/F') \) as functions of \( S \), for various combinations of shaft load \( (F') \), heat-transfer coefficient \( (K_1) \), and bearing length \( (z_0) \). Although in the analysis \( K_1 \) has been treated as a heat transfer coefficient associated with surface phenomena, for practical purposes it may be regarded
as an overall coefficient between the fluid and the external surroundings of the bearing.

The curve $F' = 0$ in each case represents that determined from conventional theory. It is clear from (9.69) that if $F' = 0$ then $m = 0$ also, if $S$ is to remain finite. From assigned values of $\delta$, the quantity $S$ is found from (9.70) with $m = 0$; also

$$\frac{M'}{F'} = \frac{M_{(o)}}{\delta F_{(o)}} \quad (F' = 0)$$  \hspace{1cm} (9.72)

For $F' \neq 0$, $m$ may be selected as the independent parameter, and $S$ found from (9.69). Using a value for $\delta$ determined from (9.71), the ratio $(M'/F')$ is then obtained by the relation

$$\frac{M'}{F'} = \frac{\left[ M_{(o)} + \delta^2 M_{(2o)} \right] - m^2 \left[ M_{(o,\mu,\eta)} + \delta^2 M_{(2o,\mu,\eta)} \right]}{\delta \left[ F_{(o)} - m^2 F_{(\mu,\eta)} \right]}$$  \hspace{1cm} (9.73)

Since both $m^2$ and $\delta$ are assumed small in this analysis, curves have been constructed only over the range for which

$$\delta \leq 0.3 \quad m \leq 0.5$$  \hspace{1cm} (9.74)

Figure 8 illustrates the behavior of $\delta$ as a function of $S$ for a relatively small heat transfer coefficient ($K_3 = 0.5$). It is seen from these curves that thermal effects tend to increase $\delta$, and that the effect is greater for shorter bearings. Comparison of Figures 8 and 9 shows that an increase in $K_3$ decreases this effect; since a larger $K_3$ means a smaller increase in temperature, this is to be expected. It should be
mentioned that a small residual effect will remain even at \( \bar{K}_s = \infty \), however, since the non-zero thermal resistance of the fluid film has been taken into account.

Figure 10 illustrates the effect of bearing length on friction. To the scale chosen this effect is quite small; hence in Figures 11 and 12 the single value \( z_o = 1.0 \) has been used. These curves depict the effect of shaft load (\( F' \)) on friction. It is of interest to note that the decrease in friction arising from decrease in viscosity more than counteracts the increase arising from a larger \( \delta \). As in the case of \( \delta \), it may be observed that an increase in \( \bar{K}_s \) results in a smaller spread between the curves.
Section 10. Exact Analysis Of Thermal Effects In Combined Pressure
And Shear Flow Between Parallel Planes:

The analysis of thermal effects for the flooded journal
bearing is based on the supposition that the quantity $\eta K \frac{d\varepsilon}{d\theta}$
is small in comparison with unity. In order to investigate the
limits within which this assumption leads to essentially accurate
results a problem is selected in which exact techniques may be
employed without unduly great computational labor, and the results
obtained by means of exact techniques are compared with those
derived from the linearized theory. The problem to be analyzed
is that of the fluid and heat flow between two infinite parallel
planes under the combined action of a shear induced by motion of
one of the planes in a direction parallel to itself together with
a constant pressure gradient in the same direction.

In the dimensionless notation of Section 7, $S_1$ and $S_2$ are
the planes $Z = 0$ and $Z = 1$, respectively. The plane $S_1$ is
assumed to be stationary; the plane $S_2$ is assumed to move with
a unit (dimensionless) velocity in the positive $X$ direction.
From elementary considerations

\[
\begin{align*}
U & = U(z) & V & = W = 0 \\
\frac{dP}{dx} & = P' \text{ constant} \\
\Theta & = \Theta(z)
\end{align*}
\]

(10.1)
As in the beginning of Section 5, we write

$$\frac{\mu}{\mu_0} = 1 + K_\mu \tilde{\mu} = \tilde{\mu}$$  \hspace{1cm} (10.2)

and assume

$$\tilde{\mu} = \tilde{\mu}(\theta)$$  \hspace{1cm} (10.3)

Neglecting terms in all parameters but $K_\mu$ and $\theta$, equations (7.14) and (7.21) become

$$\frac{d}{dz} \left\{ \tilde{\mu}(\theta) \frac{dU}{dz} \right\} = P'$$  \hspace{1cm} (10.4)

$$\frac{d^2 \theta}{dz^2} = -\eta \tilde{\mu}(\theta) \left( \frac{dU}{dz} \right)^2$$  \hspace{1cm} (10.5)

The remaining equations of motion (7.15) and (7.16), and the continuity equation (7.17), are seen by (10.1) to be satisfied identically.

Assuming for simplicity that the surface temperatures $\Theta_y$ and $\Theta_z$ are zero, the boundary conditions to be satisfied by $\Theta$ are found from (7.25) and (7.26) to be

$$\frac{d\theta}{dz} - \kappa_s \theta = 0 \quad z=0$$

$$\frac{d\theta}{dz} + \kappa_s \theta = 0 \quad z=H=1$$  \hspace{1cm} (10.6)
The boundary conditions for $U(z)$ are

$$U(0) = 0 \quad U(1) = 1$$  \hspace{1cm} (10.7)

In general the differential system (10.4) to (10.7) is
non-linear in character. We may, however, reduce this system
to one involving only linear differential equations by assuming
a specialized relation between viscosity and temperature; namely,

$$\tilde{\mu}(\theta) = \frac{1}{1 + \alpha \theta}$$  \hspace{1cm} (10.8)

where $\alpha$ is a suitably chosen positive constant. Although
equation (10.8) is not satisfactory for the accurate representa-
tion of viscosity over a wide temperature range, it is far
more accurate than a linear relation of the form

$$\tilde{\mu} = 1 - (\text{constant}) \times \theta$$

and reproduces the essential features found in practice. It
is therefore well suited to the main purpose of this section,
which is to study the range of validity of the linearized theory.

The integration of equation (10.4), with respect to $z$, gives

$$\tilde{\mu}(\theta) \frac{dU}{dz} = \mathcal{P} + A$$  \hspace{1cm} (10.9)
where $A$ is a constant to be determined. From equations (10.5), (10.8), and (10.9),

$$\frac{d^2 \theta}{dz^2} = -\frac{\eta}{\tilde{\mu}(\theta)} \left[ \tilde{\mu}(\theta) \frac{dV}{dz} \right]^2$$

$$= -\frac{\eta}{\tilde{\mu}(\theta)} \left[ p'z + A \right]^2$$

$$= -\eta (p'z + A)^2 (1 + \alpha \theta)$$

(10.10)

Since $\eta$, $p'$, $A$, and $\alpha$ are constants, and $\Theta = \Theta(z)$, equation (10.10) is a linear second order differential equation for $\Theta$.

It is convenient at this point to introduce a new dependent variable $\phi$, defined by

$$\phi = 1 + \alpha \theta$$

(10.11)

and a new parameter

$$m = \sqrt{\alpha \eta}$$

(10.12)

In terms of these quantities, the differential equation (10.10) becomes

$$\frac{d^2 \phi}{dz^2} + m^2 (p'z + A)^2 \phi = 0$$

(10.13)
whereas the boundary conditions (10.6) may be written

$$
\begin{align*}
\frac{d\phi}{dz} - \bar{K}_s \phi &= -\bar{K}_s, \\
\frac{d\phi}{dz} + \bar{K}_{s_2} \phi &= \bar{K}_{s_2},
\end{align*}
$$
\[ z = 0 \]
\[ z = 1 \]

(10.14)

To simplify the calculation, we henceforth assume

$$
\bar{K}_{s_1} = \bar{K}_{s_2} = \bar{K}_s
$$

(10.15)

It is clear from (10.13) that the quantities \( P^1 \) and \( A \) influence \( \phi \) only through the combinations \( mp^1 \) and \( m \alpha \); that is, to say, we may write

$$
\phi = \phi (z; mp', mA, \bar{K}_s)
$$

(10.16)

Assuming \( \phi \) to be known as a function of the indicated variables, (10.8), (10.9), and (10.11) are combined to obtain

$$
\frac{dU}{dz} = \frac{P'z + A}{\bar{\kappa} (\theta)}
$$

\[ = (P'z + A)(1 + \alpha \theta) \]

\[ = (P'z + A) \phi (z; mp', mA, \bar{K}_s) \]

\[ = \frac{1}{m} \left\{ (m P'z + mA) \phi (z; mp', mA, \bar{K}_s) \right\} \]

(10.17)
Integrating this equation between the limits \( z = 0 \) and \( z = 1 \), and employing the boundary conditions (10.7),

\[
m \int \frac{dU}{dz} \, dz = m = \int \left\{ (mP' + mA) \phi(z, mP', mA, \overline{K}_s) \right\} \, dz
\]

(10.18)

or in other words

\[
m = \overline{F} \left( mP', mA, \overline{K}_s \right)
\]

(10.19)

In practice, quantities \( m \), \( P' \), and \( \overline{K}_s \) are usually assigned. The quantity \( A \) is then determined by equation (10.19). For present purposes, however, since the object is to obtain typical results rather than a solution to a preassigned problem, it is convenient to specify values of \( mP' \), \( mA \), and \( \overline{K}_s \). The quantity \( m \) is then obtainable from (10.19), and hence the values of \( P' \) and \( A \) themselves may be found.

In carrying out the required operations numerically, power series solutions of the system (10.13) and (10.14) may be
employed*. The quantity m, temperature, and velocity distributions are then quite simply obtained in terms of series operations.

In addition to the preceding relations, derived upon the basis of the exact theory, corresponding results from the linearized theory are required. For the calculation of $\Theta(\eta)$, equation (7.93) is used, setting

$$
\begin{align*}
\mathcal{B}_1^{(\nu)} &= 1 & \mathcal{B}_2^{(\eta)} &= 0 & H &= 1 \\
\frac{\partial \psi_{10}}{\partial x} &= \rho' & \frac{\partial \psi_{10}}{\partial y} &= 0 \\
\bar{K}_s &= \bar{K}_{s_1} = \bar{K}_s
\end{align*}
$$

(10.20)

The result is

$$
\Theta(\eta) = \frac{1}{24K_s} \left[ 12 + (\rho')^2 \right] + \frac{\rho' \bar{\zeta}}{3(2 + \bar{\zeta})} \\
+ \left( \frac{1}{4} - \bar{\zeta}^2 \right) \left[ \frac{1}{2} + \frac{\rho' \bar{\zeta}}{3} + \frac{1}{12} \left( \frac{1}{4} + \bar{\zeta}^2 \right) (\rho')^2 \right]
$$

(10.21)

*Solutions to (10.13) may be represented in terms of Bessel functions of the orders $-\frac{1}{2}$ and $+\frac{1}{2}$; however, the author has been unable to obtain tables of all of the quantities needed in the analysis.
with

\[ \mathcal{f} = z^{-\frac{1}{2}} \]  

(10.22)

In Figure 13, values of \( \alpha \Theta \) are plotted in four typical cases, for which \( mA = 1 \) and \( m^p = -1.0, -0.5, 0.0, \) and \( +0.3, \) with \( \mathcal{K}_s = 1. \) The corresponding values of \( A, P^1, \) and \( m^2 \) are indicated in the following table:

<table>
<thead>
<tr>
<th>( m^p )</th>
<th>( -1 )</th>
<th>-0.5</th>
<th>0</th>
<th>+0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>1.591</td>
<td>0.875</td>
<td>0.415</td>
<td>0.191</td>
</tr>
<tr>
<td>( P^1 )</td>
<td>-1.591</td>
<td>-0.437</td>
<td>0.000</td>
<td>+0.057</td>
</tr>
<tr>
<td>( m^2 )</td>
<td>0.395</td>
<td>1.306</td>
<td>5.800</td>
<td>27.293</td>
</tr>
</tbody>
</table>

The full lines in Figure 13 represent temperatures determined from the exact theory; the dotted curves are found from (10.21) and the relation \( \alpha \Theta = m^2 \Theta(\eta) \). The approximate temperatures are, in each case, larger than the true temperatures; the approximate curve for \( m^p = +0.3 \) does not fall on the plot. Since the approximate curves are in effect determined on the basis of a viscosity which is too large (i.e., \( \mathcal{K}_s = 1 \)) this behavior is to be expected. Good agreement between the approximate and exact theories is obtained only for the smallest value of \( m^2 \); however, even this value \( (0.395) \) is larger than the maximum of \( m^2 = 0.25 \) employed in the journal-bearing calculations of Section 9.
FIGURE 14

$\theta$ vs. $Z$

$P^2 = 0$

By Linear & Exact Theories

--- Linear Theory
--- Exact Theory

$m^2 = 0.067$
(LINEAR & EXACT)

$m^2 = 0.343$

$m^2 = 1.248$

$m^2 = 5.800$

$\theta$

$Z$

0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6
Figure 14 presents similar curves for the case in which \( P^l = 0 \). For this special case equations (10.13) and (10.14) may be solved in a simple closed form. For \( \varphi (z) \) we find

\[
\varphi (z) = C_1 \cos m A z + C_2 \sin m A z
\]

where

\[
C_1 = \frac{K_s m A \cos m A + K_s^2 \sin m A + K_s m A}{2K_s m A \cos m A + (K_s^2 - m^2 A^2) \sin m A}
\]

\[
C_2 = \frac{K_s m A \sin m A + K_s^2 (1 - \cos m A)}{2K_s m A \cos m A + (K_s^2 - m^2 A^2) \sin m A}
\]

The quantity \( m \) is given in simple form by

\[
m = 2 C_2
\]

The curves of Figure 14, for \( \overline{K_s} = 1 \); \( m_A = 0.25, 0.50, 0.75, \) and \( 1.00 \) display a behavior similar to that shown in Figure 13.

In Figure 15, the special case \( P^l = 0 \) is considered in slightly greater detail, by plotting the maximum value of \( \Delta \theta \) (i.e. that for \( z = 0.5 \)) against \( m \), for \( \overline{K_s} = 0.5, 1.0, \) and \( 2.0 \).

*This relation is valid only when \( \overline{K_s} = K_{s2} \).
It is seen from this figure that, although, as predicted by the linearized theory, the temperature increases approximately as the square of $m$ for small values of this parameter, the curves tend to become linear as $m$ increases. As expected, smaller temperatures are found to be associated with the larger values of $K_s$. 
Section 11, Analytical Procedure For Investigation Of Thermal Effects In Slider Bearings Of Infinite Width:

The problem to be studied in the present section is that of determining non-linear thermal effects, such as those analyzed in Section 10, for the case of a slider bearing of infinite width and finite length. The analysis is somewhat similar to that of Section 10; however, in the present case, the problem is complicated by the fact that the pressure gradient is an unknown function which must be determined.

The essential physical features of the problem are depicted, in dimensionless form, in Figure 16. We let the lower surface \( z = 0 \) move with unit velocity in the positive \( X \) direction. The upper surface, \( z = H(x) \), is assumed to be stationary. Letting \( L_0 \) be the (physical) length of the bearing, in dimensionless notation we may assign the coordinates \( X = 0 \) and \( X = 1 \) to the ends.

![Figure 16](image-url)
The pressure $P$ is assumed to be zero at $X = 0$ and $X = 1$; the quantity $H$ is a linear function of $X$, which may be written by a proper choice of $h_0$ in the form

$$H(x) = 1 - AX \quad (0 < A < 1)$$

(11.1)

Since the bearing is of infinite width, we may write

$$\begin{align*}
U &= U(x, z) \\
V &= 0 \\
W &= W(x, z) \\
\Theta &= \Theta(x, z)
\end{align*}$$

(11.2)

Also, as in the previous case,

$$P = P(x)$$

(11.3)

Neglecting terms in $E$, $\phi$, $K_p$, $R$, and $K_k$, equations (7.14), (7.17), and (7.21) become, respectively

$$\frac{2}{\partial z} \left\{ \tilde{u} \frac{2u}{\partial z} \right\} = \frac{dP}{dx}$$

(11.4)

$$\frac{2u}{\partial x} + \frac{2w}{\partial z} = 0$$

(11.5)
\[ \frac{\partial^2 \Phi}{\partial z^2} = -\eta \tilde{\mu} \left( \frac{\partial \Phi}{\partial z} \right)^2 + \lambda \tilde{C}_\mu \frac{\partial \Phi}{\partial z} \]  

(11.6)

where, as before

\[ \tilde{\mu} = \tilde{\mu}(\theta) = \frac{\mu}{\mu_0} = 1 + K_\mu \tilde{\mu} \]

(11.7)

These equations are to be solved under the boundary conditions

\[ \begin{aligned}
U(x, \theta) &= 1 & U(x, H(x)) &= 0 \\
W(x, 0) &= W(x, H(x)) = 0
\end{aligned} \]  

(11.8)

and

\[ \begin{aligned}
\frac{\partial \Theta}{\partial z} - \bar{K}_{s1} \Theta &= 0 & (z = 0) \\
\frac{\partial \Theta}{\partial z} + \bar{K}_{s2} \Theta &= 0 & (z = H(x))
\end{aligned} \]  

(11.10)

(11.11)
In accordance with the general discussion of this type of problem given at the beginning of Section 5, we omit the \( \lambda \) term of (11.6) in the subsequent analysis, even though it may not be negligible in practice. It is hoped that the techniques developed by study of the resulting simplified problem may later be extended to take this term into account.

Following the pattern of Section 10, we integrate equation (11.4) to obtain

\[
\hat{\mu}(\theta) \frac{\partial U}{\partial z} = z \frac{dP}{dx} + A(x)
\]  
(11.12)

where \( A(x) \) is an unknown function. Writing

\[
\hat{\mu}(\theta) = \frac{1}{1 + \lambda \theta}
\]  
(11.13)

we then find from (11.12), (11.13), and the simplified form of (11.6) the equation

\[
\frac{\partial^2 \varphi}{\partial z^2} = - \eta [p' z + A]^{2 (1 + \lambda \theta)}
\]  
(11.14)

or

\[
\frac{\partial^2 \varphi}{\partial z^2} + m^2 [p' z + A]^2 \varphi = 0
\]  
(11.15)
where
\[ \varphi = 1 + \kappa \theta \]  \hspace{1cm} (11.16)
and
\[ m^2 = \alpha \eta \]  \hspace{1cm} (11.17)

Although the quantities \( \varphi, P^1, \) and \( A \) depend on \( X \), for any fixed \( X \) we may regard (11.15) as an ordinary differential equation with \( Z \) as the independent variable. Writing, as in Section 10,
\[ \bar{k}_{s_1} = \bar{k}_{s_2} = \bar{k}_s \]  \hspace{1cm} (11.18)
the boundary conditions for (11.15) are
\[ \begin{align*}
\frac{\partial \varphi}{\partial z} - \bar{k}_s \varphi &= -\bar{k}_s \quad (z = 0) \\
\frac{\partial \varphi}{\partial \bar{z}} + \bar{k}_s \varphi &= \bar{k}_s \quad (z = H(x))
\end{align*} \]  \hspace{1cm} (11.19)

It is convenient at this point to introduce a new independent variable, \( Z' \), defined by
\[ Z' = \frac{Z}{H} \]  \hspace{1cm} (11.20)
In terms of \( z' \), equations (11.15) and (11.19) become

\[
\frac{\partial^2 \varphi}{\partial (z')^2} + \left[ (mH^2p') \right] z' + (mA) \bigg[ z \bigg] \varphi = 0 \tag{11.21}
\]

\[
\begin{aligned}
\frac{\partial \varphi}{\partial z}, - (H\bar{k}_s) \varphi &= -(H\bar{k}_s) \quad (z' = 0) \\
\frac{\partial \varphi}{\partial z}, + (H\bar{k}_s) \varphi &= (H\bar{k}_s) \quad (z' = 1)
\end{aligned}
\tag{11.22}
\]

These equations show that the various parameters entering into the determination of \( \varphi \) so combine that we may write

\[
\varphi = \varphi(z', mH^2P', mA, H\bar{k}_s) \tag{11.23}
\]

For any fixed value of \( X \), the quantities \( P^1 \) and \( A \) may be regarded as unknown parameters. A functional relation between these parameters may be obtained by use of (11.12) and the boundary conditions (11.8) for \( U(x, z) \). Employing (11.13) and (11.16) we find

\[
\frac{\partial U}{\partial z} = (zP' + A) \varphi \tag{11.24}
\]
or, in terms of \( \mathbf{z}' \),

\[
\frac{2U}{\rho} = \frac{1}{m} \left[ (mH^2p')z' + (mHA) \right] \mathcal{Q}(z', mH^2p', mHA, H\bar{k}_s) \tag{11.25}
\]

Integrating this relation between \( z' = 0 \) and \( z' = 1 \), we find, by (11.8),

\[
(mH^2p')I_1(x) + (mHA)I_0(x) = -m \tag{11.26}
\]

where

\[
I_0(x) = \int_0^1 \mathcal{Q}(z', mH^2p', mHA, H\bar{k}_s) \, dz' \tag{11.27}
\]

and

\[
I_1(x) = \int_0^1 z' \mathcal{Q}(z', mH^2p', mHA, H\bar{k}_s) \, dz' \tag{11.28}
\]

Since \( I_0 \) and \( I_1 \) may be considered as functions of the variables \( mH^2p^l \), \( mHA \), and \( H\bar{k}_s \), equation (11.26) provides an implicit functional relation between these parameters and \( m \), which may be solved for \( mHA \). We thereby obtain an equation of the form

\[
mHA = f(mH^2p^l, H\bar{k}_s, m) \tag{11.29}
\]
To carry out this operation in practice, we may specify values of \( mh^2 p^1 \), \( mHA \), and \( Hk_s \), determining \( m \), from (11.26), as a function of these three parameters. By a sequence of cross-plots taken from curves of \( m \) versus \( mHA \) for constant values of \( mh^2 p^1 \) and \( Hk_s \), we may then obtain curves representing the functional relation (11.29) for a preassigned constant value of \( m \).

By use of this relation we may eliminate \( mHA \) from (11.27) and (11.28), obtaining

\[
I_0 = I_0(m, H(\alpha)^2 p'(\alpha), H(\alpha)k_s, m)
\]

and

\[
I_1 = I_1(m, H(\alpha)^2 p'(\alpha), H(\alpha)k_s, m)
\]  \hspace{1cm} (11.30)

Since for any particular problem the quantity \( m \) will be a constant, \( I_0 \) and \( I_1 \) are essentially functions of two variables only, and can be readily plotted. These quantities, together with a similarly defined function

\[
I_2 = \int (z')^2 \varphi(z', mh^2 p', mHA, Hk_s) dz'
\]  \hspace{1cm} (11.31)

may now be used in the derivation of a differential equation for \( P(\alpha) \).
For this purpose we first integrate (11.24) to obtain

$$U(x, z) = \int_0^z (z', P' + A) \varphi(x, z') \, dz', \quad (11.32)$$

where $z'$ is a variable of integration, unrelated to $z$'. Then

$$\frac{\partial U}{\partial x} = \frac{d^2 P}{dx^2} \int_0^z z \varphi(x, z) \, dz' + \frac{dA}{dx} \int_0^z \varphi(x, z) \, dz' + \frac{dP}{dx} \int_0^z z \frac{\partial \varphi(x, z)}{\partial x} \, dz' + A \int_0^z \frac{\partial \varphi(x, z)}{\partial x} \, dz', \quad (11.33)$$

We now integrate equation (11.5) between the limits $z = 0$ and $z = H(x)$, and apply the boundary conditions (11.8) for $W(x, z)$. We thereby find

$$\int_0^{H(x)} \frac{\partial U(x, z)}{\partial x} \, dz = 0 \quad (11.34)$$

Upon substitution from (11.33) this relation becomes

$$\frac{d^2 P}{dx^2} \int_0^{H(x)} [H(x) - z] z \varphi(x, z) \, dz' + \frac{dA}{dx} \int_0^{H(x)} [H(x) - z] \varphi(x, z) \, dz' + \frac{dP}{dx} \int_0^{H(x)} [H(x) - z] z \frac{\partial \varphi(x, z)}{\partial x} \, dz' + A \int_0^{H(x)} [H(x) - z] \frac{\partial \varphi(x, z)}{\partial x} \, dz' = 0 \quad (11.35)$$
or in an equivalent form

\[
H \frac{d^2 P}{d x^2} \int_0^H z \phi(x, z) d z - \frac{d^2 P}{d x^2} \int_0^H z^2 \phi(x, z) d z \\
+ H \frac{d A}{d x} \int_0^H \phi(x, z) d z - \frac{d A}{d x} \int_0^H z \phi(x, z) d z \\
+ H \frac{d P}{d x} \int_0^H z^2 \frac{2 \phi}{d x} d z - \frac{d P}{d x} \int_0^H z^2 \frac{2 \phi}{d x} d z \\
+ H A \int_0^H \frac{\phi}{d x} d z - A \int_0^H z \frac{2 \phi}{d x} d z = 0
\]

(11.36)

However, from (11.20), (11.27), (11.28), and (11.31),

\[
\begin{align*}
\int_0^H \phi(x, z) d z &= H I_0(x) \\
\int_0^H z \phi(x, z) d z &= H^2 I_1(x) \\
\int_0^H z^2 \phi(x, z) d z &= H^3 I_2(x)
\end{align*}
\]

(11.37)

Also, by differentiating the first of these relations with respect to \( X \), we have

\[
\frac{d}{d x} (H I_0) = \frac{d H}{d x} \phi(x, H) + \int_0^H \frac{\phi}{d x} d z
\]

(11.38)
whence

\[ \int_0^H \frac{\partial \varphi}{\partial x} \, dz = \frac{d}{dx} (H I_1) - \frac{d}{dx} \frac{d}{dx} \varphi(x, H) \tag{11.39} \]

In a similar manner we derive the equations

\[ \int_0^H z \frac{\partial \varphi}{\partial x} \, dz = \frac{d}{dx} (H^3 I_2) - H \frac{d}{dx} \frac{d}{dx} \varphi(x, H) \tag{11.40} \]

\[ \int_0^H z^2 \frac{\partial \varphi}{\partial x} \, dz = \frac{d}{dx} (H^3 I_2) - H^2 \frac{d}{dx} \frac{d}{dx} \varphi(x, H) \tag{11.41} \]

Substituting (11.37), (11.39), (11.40), and (11.41) into equation (11.36), after minor rearrangement of terms we find the simplified relation

\[ H \frac{d}{dx} \left( H^2 I_1, \frac{d}{dx} \right) - \frac{d}{dx} \left( H^3 I_2 \frac{d}{dx} \right) \]

\[ + H \frac{d}{dx} (H I_0 A) - \frac{d}{dx} (H^2 I_1 A) = 0 \tag{11.42} \]

By differentiation of (11.26), however,

\[ \frac{d}{dx} \left( H^2 I_1, \frac{d}{dx} \right) + \frac{d}{dx} (H I_0 A) = 0 \tag{11.43} \]
We therefore obtain

$$\frac{d}{dx} (H^2 I_2 \frac{dP}{dx}) + \frac{d}{dx} (H^2 I, A) = 0$$  \hspace{1cm} (11.44)

Integration of this equation once, and the elimination of A by means of equation (11.26), then gives

$$H^2 \frac{dP}{dx} = \frac{C}{H} \left( \frac{I_0}{I_0 I_2 - I_1^2} \right) + \left( \frac{I_1}{I_0 I_2 - I_1^2} \right)$$  \hspace{1cm} (11.45)

where C is a constant of integration. From (11.30) and a similar equation for I_2 we may write

$$\frac{I_0}{I_0 I_2 - I_1^2} = F \left( m H^2 P', H \bar{k}_s, \mu \right)$$

$$\frac{m I_1}{I_0 I_2 - I_1^2} = G \left( m H^2 P', H \bar{k}_s, \mu \right)$$  \hspace{1cm} (11.46)

Equation (11.45) then yields

$$m H^2 P' = \left( \frac{m C}{H} \right) F \left( m H^2 P', H \bar{k}_s, \mu \right)$$

$$+ G \left( m H^2 P', H \bar{k}_s, \mu \right)$$  \hspace{1cm} (11.47)
Solving this equation for $mH^2p^1$, we obtain a functional relation of the form

$$mH^2p^1 = J \left( \frac{mc}{H}, H \kappa_1, m \right)$$  \hspace{1cm} (11.48)

We therefore find

$$P(x) = \int_0^x \frac{1}{mH(x)^2} J \left( \frac{mc}{H(x)}, H(x), \kappa_1, m \right) \, dx$$  \hspace{1cm} (11.49)

where the boundary condition (11.9) at $x = 0$ has been satisfied automatically, and where $C$ must be so adjusted that the boundary condition at $x = 1$ is also satisfied.

This adjustment of $C$, however, is unnecessary if one is interested in obtaining a general family of typical solutions, rather than the solution to a preassigned problem. The dimension $L_0$ does not enter into any of the parameters which we have used; neither has the boundary condition $P(1) = 0$ been previously employed in this analysis. It is therefore sufficient to fix $C$ and carry out the integration of (11.49) to such point that $P(X_2) = 0$, assuming such a point to exist for an arbitrary $C$. The distance from 0 to $X_2$ may then be taken as the length of the bearing. This procedure has the disadvantage that the ratio of maximum to minimum $H$ cannot be specified a priori; however, the solution to a family of
similar problems can be carried out by this means in a relatively simple manner.

No attempt has been made in the present investigation to carry out a numerical analysis based upon this technique; however, the essential steps for such a calculation may be summarized as follows:

(a) For a set of preassigned values of \(m^2 P^1\), mHA, and \(H_k^8\) solutions to the differential system (11.21) and (11.22) are obtained, and the integrals \(I_0\), \(I_1\), and \(I_2\) are calculated.

(b) For one or more values of \(m\), equation (11.26) is solved for mHA in terms of \(m H^2 P^1\), \(H_k^8\), and \(m\).

(c) From the results of step (b), \(I_0\), \(I_1\), and \(I_2\) are expressed as functions of \(m H^2 P^1\), \(H_k^8\), and \(m\). The quantities \(F\) and \(G\) of (11.46) are also expressed in a similar fashion.

(d) Equation (11.47) is solved for \(m H^2 P^1\) in terms of \(m C\), \(H_k^8\), and \(m\) as parameters.

(e) \(P\) is determined as a function of \(X\) from (11.49), where \(C\) may be adjusted by trial and error in such manner that \(P(1) = 0\).

The solution to the differential equation (11.21) may be expressed in the form

\[
\varphi = J_{-\frac{1}{4}} \left[ D_1 J_{-\frac{1}{4}} (J) + D_2 J_{\frac{1}{4}} (J) \right] \tag{11.50}
\]

where \(J_{-\frac{1}{4}}\) and \(J_{\frac{1}{4}}\) are Bessel functions, where \(D_1\) and \(D_2\) are arbitrary constants, and where

\[
J = (\text{constant}) \times \left[ (m H^2 P^1) z' + (m HA) \right]^2 \tag{11.51}
\]
Tables of these functions over the range $0 \leq \delta < \infty$, together with related integrals, would prove quite useful in carrying out a numerical analysis of the preceding. In view of the original neglect of the term in $\lambda$ of equation (11.6), however, it is not clear that an extended numerical analysis based on the results of this section would be justified, except perhaps for the purpose of verifying the validity of simpler and more approximate theoretical techniques. It is unfortunately true that almost as much work would be required for the solution of a single problem by this method as for the solution of a large number of problems.
Appendix I: Bibliography


Appendix II:

Biographical Note:

The author was born in Kansas City, Missouri, on February 14, 1920, and received his elementary and secondary education in that city. From September, 1936, to June, 1938, he was enrolled as a student in the field of Chemical Engineering, at the Kansas City Junior College. In September, 1938, he transferred to the Massachusetts Institute of Technology as a student in the same field, receiving the degree of Bachelor of Science in October, 1940.

Feeling that his interests lay in the field of theoretical science, he thereupon enrolled as a student in the Department of Mathematics at M.I.T. for the academic year 1940 - 41. During the summer of 1941 he studied Applied Mathematics under the program of Advanced Instruction and Research in Mechanics inaugurated at Brown University. In September, 1941, he was the recipient of a fellowship which enabled him to continue his studies in the Applied Mathematics field at Brown University until September, 1942. During the summer of 1942 he also held a part time position as Instructor in the Department of Engineering.

Between September, 1942, and November, 1945, the author served in civilian capacity as a member of the staff of the Applied Mechanics Section of the Research Laboratory
at the Watertown Arsenal, Watertown, Massachusetts. In this position he was engaged in mathematical research in the Ordnance field. The author returned to M.I.T. in November, 1945, receiving an appointment as Research Associate in the Department of Aeronautical Engineering, which he held until June, 1947. In this position his attention was divided between the pursuit of his doctorate studies in the Department of Mathematics and a program of mathematical research performed in the Computing Section of the Instrumentation Laboratory.

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