ANALYTICAL AND EXPERIMENTAL STUDIES
OF TRANSIENT STRESSES IN AIRPLANE
WINGS DURING LANDING (DROP TESTS)

by

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Signature of Author

Department of Aeronautical Engineering
September 12, 1947

Certified by

Thesis Supervisor

Chairman, Departmental Committee on Graduate Students
Cambridge, Massachusetts
September 12, 1947

Professor Joseph S. Newell
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

Dear Sir:

Enclosed herewith is my thesis entitled "Analytical and Experimental Studies of Transient Stresses in Airplane Wings during Landing (Drop Tests)" which is submitted in partial fulfillment of the requirements for the degree of Master of Science in Aeronautical Engineering.

Respectfully submitted

[Signature]

Oswaldo do Nascimento Leal
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I-INTRODUCTION

Transient wing stresses which cannot be predetermined by the concept that an airplane is a rigid body are encountered during landings of airplanes.

In many cases the transient oscillations, excited by the landing impact force, produce critical design conditions and the practical importance of their analysis has been emphasized by several investigators, M. A. Biot and R. L. Bisplinghoff (Ref. 2), E. G. Keller (Ref. 6), A. J. Yorgiadis (Ref. 7) and others.

The National Bureau of Standards (Ref. 4) has carried out an experimental investigation of M. A. Biot and R. L. Bisplinghoff's analysis of landing impact (Ref. 4).

The wing was a tapered box beam of rectangular cross section having a distribution of mass and a flexural rigidity along the wing, approximately proportional to that for the B-24 airplane.

Four engine masses were mounted symmetrically in the wings so as to excite flexural vibrations without torsion when the model was dropped vertically to receive a load impact force below the center of gravity.

This thesis follows the approach recommended by M. A. Biot and R. L. Bisplinghoff (Ref. 2) for the investigation of transient stresses in airplane wings during landing.

The scope of the present work is an investigation of transient stresses in a model, straight tapered wing, with two parallel magnesium alloy spars, having a distribution of mass and a flexural and torsional rigidity along the wing so as to provide dynamic characteristics similar to some airplanes.

Two unbalanced engine masses were mounted on the wings so as to excite coupling between the flexural and torsional vibrations, when the model was dropped vertically, to receive a load impact force below the center of gravity.

A study of the determination of natural principal oscillations of a free wing with rigid fuselage is also carried out.
II - SUMMARY

In this investigation, the following assumptions of the work of M. A. Biot and R. L. Bisplinghoff (Ref. 2) were adopted:

a) The stresses in the structure are considered to be caused by a landing impact force of known time history applied directly to the structure.

b) The time history of the impact force may be investigated independently of the elastic properties of the structure.

c) The damping and aerodynamic forces may be neglected.

d) The response of the structure is represented as a superposition of natural modes excited by the landing impact.

For the drop tests, a model was built having a wind tunnel flutter test model wing (Figs. 2 and 12), which was designed to be dynamically similar to some airplanes.

The mass distribution and the flexural rigidity (Fig. 4) and the torsional rigidity (Fig. 5) along the span, were respectively provided in design and determined experimentally.
Two unbalanced engines were mounted on the wings so as to excite coupling between the flexural and torsional vibrations, when the model was dropped vertically to receive a load impact below the center of gravity. (Figs. 14 and 15).

The airplane structure was replaced by a dynamic model for the analytical computations. (Fig. 13)

The flexibility influence coefficients $C_{mn}$'s for the wing were computed. (Table 3)

The natural frequencies and mode shapes of the free model were computed using a matrix iteration procedure. (Table 4)

Drop tests of the model were made from one, two and three inches and the time history of the landing impact force, bending moments in the right and left wings at station 5.5 in. from the center line, and torsional moment in the left wing were recorded. (Figs. 27, 28, and 29)

Using approximations for the actual landing impact force, analytical computations of the bending and torsional moments at the same station were made. (Tables 5, 6, 7)

The recorded and computed bending and torsional moments were compared. (Table 8)
III - DESCRIPTION OF THE MODEL

Wing.

The wing model was designed to approximate dynamic characteristics of several aircraft for wind-tunnel flutter tests.

The wing model is a 60 in. straight tapered wing pictured in Figs. 2 and 12.

It has the following general properties:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span</td>
<td>60 in.</td>
</tr>
<tr>
<td>Tip chord</td>
<td>6 in.</td>
</tr>
<tr>
<td>Root chord</td>
<td>11 in.</td>
</tr>
<tr>
<td>Area</td>
<td>510 sq. in.</td>
</tr>
<tr>
<td>Aspect Ratio</td>
<td>7.06</td>
</tr>
<tr>
<td>Airfoil</td>
<td>N75</td>
</tr>
</tbody>
</table>

33.3% chord perpendicular to fuselage center line.

It has two parallel magnesium alloy spars located as shown in Fig. 12.

The dimensions of the spars are:

<table>
<thead>
<tr>
<th>Station</th>
<th>Front Spar depth (in.)</th>
<th>Rear Spar depth (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.356</td>
<td>.300</td>
</tr>
<tr>
<td>6</td>
<td>.308</td>
<td>.260</td>
</tr>
<tr>
<td>12</td>
<td>.260</td>
<td>.220</td>
</tr>
<tr>
<td>18</td>
<td>.213</td>
<td>.180</td>
</tr>
<tr>
<td>24</td>
<td>.165</td>
<td>.140</td>
</tr>
<tr>
<td>30</td>
<td>.118</td>
<td>.100</td>
</tr>
</tbody>
</table>

Thickness of front spar = .064 in.

Thickness of rear spar = .032 in.
3 1/4 in. duralumin torque tubes are riveted perpendicular to the spars at stations indicated in Fig. 2.

All tube wall thicknesses are .013 in. and the outside diameters are:

<table>
<thead>
<tr>
<th>Stat.</th>
<th>2.5, 4.5, 6.5</th>
<th>8.5, 11.5, 14.5</th>
<th>18.5, 20.5, 23.5, 27.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out. Diam.</td>
<td>1/4 in.</td>
<td>3/16 in.</td>
<td>1/8 in.</td>
</tr>
</tbody>
</table>

The airfoil shape and chord length are fixed by wooden ribs located at one inch intervals spanwise, beginning at station 3.

Balsa nose-blocks are attached forward to add weight and maintain the shape of the leading edge, but these blocks are split between ribs so as not to furnish torsional or bending stiffness.

A wood trailing edge strip is glued across the after ends of the ribs.

Model cement was used for joints, except for the riveted torque-tube spar attachment and for the screwed engine-support spar attachment.

Model weight and inertial properties were adjusted to conform to the specifications. Elastic properties were fixed during construction, and have been checked by static tests. The mass distribution and the elastic properties are summarized in Figs. 4 and 5.

Radius of gyration of sections at 30% chord.

Center of gravity of sections at 33.3% chord.
Engines

Two engines weighing 584 grs. and having a moment of inertia equal to 5410 grs. in.\(^2\) were connected to the wing spars at station 9.5 in.

The center of gravity of the engine is 3.375 in. forward of the Elastic Axis. (Figs. 14 and 15)

Fuselage

The fuselage is represented by a steel bar 1" x 0.5" x 33 1/4", having on one end a block 2" x 2" x 1" where two ball bearings are tightly fitted, through which passes the support axle for the swinging of the model. On the other end it has connected the fixing support for the wing and the landing gear. (Figs. 16 and 22)

The apparatus is similar to a pendulum designed to have an effective mass equal to the desired fuselage mass.

Landing Gear

The landing gear is made of a hollow cylinder having inside a piston which supports a spring. It is fastened on the top to the fuselage bar and has on the bottom a guide for the piston axle which also supports the wheels. (Figs. 16 and 22)

The piston travels in oil so as to provide damping, which could be varied by changing the oil.

The spring could be changed.

The landing wheels were model airplane rubber wheels, 1 3/8" x 5/8".
IV - PROCEDURE AND TESTS

Preliminaries

The determination of the bending rigidity along the span had been made assuming that the spars provided all bending strength.

Symmetrical loads were applied at different stations of the Elastic Axis, and the deflections were read to the nearest .001 in.

The bending stiffness $E I$ was computed by application of the formula

$$B.M. = EI \frac{d^2 z}{dy^2}$$

and the results were averaged for the two wings. (Fig. 4)

The torsional rigidity had been computed by applying anti-symmetrical couples along the span, measuring the twisting, and applying the approximate formula

$$T.M. = GJ \frac{d\theta}{dy}$$

which neglects the contribution of differential bending of the spars. The results were averaged for the two wings. (Fig. 5)

The distribution of mass (Fig. 4) the center of gravity of the sections (33.3% chord) and the radius of gyration (30% chord) had been provided in the design and checked experimentally.
Using this data, the moment of inertia of the wing along the span was computed. (Fig. 6)

The fuselage with landing gear was designed so as to have an effective mass, at the location of the elastic Axis of the wing, in the same proportion to the mass of the wing as the average medium bomber (weighing about 25,000 lbs.).

The fuselage (Fig. 16, 12) has already been described. Schematically it is a bar pinned at one end, having the landing gear and the wing in the other, so it could be raised and dropped vertically in order to receive a vertical load when the landing gear strikes the ground. The bar was made long enough to approximate a vertical drop in the non pinned end.

To compute the effective mass of the fuselage, the following reasoning was used:

\[ F \text{ vertical force on the landing gear.} \]
\[ r \text{ distance between the point of application of the force and the pin.} \]
\[ m_b \text{ mass of the bar} \]
\[ i \text{ radius of gyration of the bar.} \]
\[ m_f \text{ effective mass (fuselage mass)} \]
\[ \theta \text{ angle of swinging of the bar.} \]
\[ a \text{ acceleration produced by the force } F \]

For a mass \( m \) acted by a vertical force \( F \), causing...
an acceleration a

\[ F = m_f a \quad \text{then} \quad \frac{F}{a} = m_f \]

For a pinned bar acted by a vertical force \( F \) causing a swinging at a distance \( r \) from the pin \( \theta \) of the bar.

\[ Fr = m_b i^2 \ddot{\theta} \]

The acceleration \( a \) of the point of application of the force is

\[ a = r \ddot{\theta} \quad \text{then} \quad \ddot{\theta} = \frac{a}{r} \]

Substituting and transposing

\[ \frac{F}{a} = m_b \frac{i^2}{r^2} \]

Then, by definition of effective mass

\[ m_f = m_b \frac{i^2}{r^2} \]

To get the desired effective mass \( m_f \) for the fuselage, the mass \( m_b \) and the radius of gyration \( i \) of the bar were varied by adding weights to the bar at different distances from the pinned end, until the desired \( m_f \) at \( r \) was obtained.

The radius of gyration of each combination was
obtained by swinging the bar as a pendulum.

**Dynamic Model**

With these data, the dynamic model (Fig. 13) was computed, assuming the fuselage infinitely rigid, and by design, its effective mass was located in the elastic axis of the wing, and according to the following steps:

The wing was divided into four sections:

- 2" — 7.5", 7.5" — 12", 12" — 19", 19" — 30" (Fig. 3)

The centers of gravity of the sections were determined and the stations for the dynamic model taken at these centers of gravity. (Fig. 3)

The section from 0 to 2" was assumed pertaining to the fuselage, since it consisted only of the two spars that were clamped to the fuselage bar, and consequently the semi-span of the dynamic model was decreased by two inches.

The mass of each section was concentrated in two equal point masses connected by an infinitely rigid bar so as to conserve the same center of gravity and the same moment of inertia.

At the station 9.5 (section 2 of Fig. 3) where the engine (with a mass equal to 584 grs. and a moment of inertia about the center of gravity equal to 5410 gr. in.²) was located so that its center of gravity was 4.375 inches
forward of the center of the gravity of the section, the resulting center of gravity, mass and moment of inertia was computed and the same procedure already described was followed.

Influence Coefficients

The bending influence coefficients \( \tilde{C}_{mn} \) defined as the deflection of the Elastic Axis at station \( m \) due to a unit load at station \( n \), \( \tilde{C}_{mn} \) defined as the slope of the Elastic Axis at station \( m \) due to a unit load at station \( n \), were computed from the flexure formula

\[
\frac{d^2z}{dyz} = \frac{M}{EI}
\]

on the assumption that the root was clamped, so that the Elastic Axis behaves as a cantilever beam.

Consequently, the slope and deflection at a distance \( l \) from the root, due to a unit load at \( l \) are respectively

\[
\frac{dz}{dy} = \int_0^l \frac{1-y}{EI} \, dy
\]

\[
z = \int_0^l \frac{(1-y)^2}{EI} \, dy
\]
Noticing (if \( m \) is nearer the root than \( n \)) that \( \tilde{C}_{nm} = \tilde{C}_{mn} \) and using Maxwell's Theorem, all the \( \tilde{C}_{mn} \) can be determined by means of the first formula.

For a cantilever beam, when \( m \) is nearer the root than \( n \), the deflection at \( n \) due to the load at \( m \) is equal to the deflection at \( m \) due to the same load at \( m \), plus the distance between \( m \) and \( n \), times the slope at \( m \) when the load is at \( m \).

Using this and Maxwell's Theorem, all the \( \tilde{C}_{mn} \) can be determined by the use of the two formulas above.

Figures 7, 8, 9 and 10 show the computations of these coefficients for the four stations and Table 1 gives the results for all \( \tilde{C}_{mn} \).

The Torsional Influence Coefficients \( \tilde{C}_{mn}^T \) defined as the angle of twist of the Elastic Axis at station \( m \) due to a unit torque at station \( n \), were computed from the formula

\[
\frac{d\theta}{dy} = \frac{T}{GJ}
\]

Consequently, the angle of twist \( \theta \) at a distance \( l \) from the root due to a unit torque at \( l \) or at a distance greater than \( l \) is

\[
\theta = \int_0^l \frac{dy}{GJ}
\]

The angle of twist at \( l \) is the same as the angle of twist at \( b \), when the torque is applied at a distance \( b \) smaller than \( l \).
Figure 9 shows the computations of these coefficients and Table 2 gives the results for all $C_{mn}$.

The Flexibility Influence Coefficients of the wing, $C_{mn}$ defined as the deflection of the mass $m$ due to a unit load on the mass $n$, were computed treating the dynamic model as a structure where the Elastic Axis can bend and twist, but the transverse bars are infinitely rigid, using the Bending and Torsional Influence Coefficients already discussed.

Table 3 gives the results for all $C_{mn}$.

**Coupled Frequencies and Mode Shapes**

The coupled frequencies and mode shapes for a free wing with rigid fuselage were computed solving the equation determined in the Analysis of this thesis by matrix iteration, following the steps recommended in References 8 and 10.

The results of the computed frequencies and mode shapes for the first three modes are tabulated in Table 4.

**Releasing Set-up**

The releasing set-up shown in figure 17 was made so as to drop the model in an almost strain-free condition, which was desirable to prevent the setting up of vibrations during the free fall, excited by the sudden removal of the dead weight forces upon release of the model.

The model was supported by seven strings connected
to a wooden bar by means of regulating screws.

The bar was connected by five strings, having regulating screws on the bar, to a central ring connected by string to another ring which was connected by a thin supporting rope to a horse mounted over the apparatus. (Fig. 13)

The releasing was made by cutting the string connecting the two rings. (Fig. 17)

The wooden bar was caught in the drop by two arms fixed on the horse, so as to prevent the bar from falling on the model, and at the same time allowing the wing to vibrate without restraint. (Fig. 18)

Before the drops, the model was raised so that the wheels would clear the ground; the regulating screws on the bar were adjusted so as to strain-free the model; the desired height of drop was adjusted by varying the length of the supporting rope to the horse.

The bottom ring was prevented from dropping on the model after the connecting string to the upper ring was cut, by a string tied to the horse so that it was loose when the model was raised and it would not interfere with the free fall. (Fig. 17)
**Instrumentation**

The instrumentation of the model included strain gage pick-ups for measuring bending and torsional moments at station 5.5 of the right wing and bending moment at the same station of the left wing, in order to check the symmetry of the drop.

In order to measure the impact force, the model was made to fall on a steel beam rigidly supported at both ends and the bending moment at the center was measured by a strain gage pick-up. (Fig. 17)

The signals of these strain gage pick-ups were amplified with a 4 channel consolidated equipment and recorded in a 5 channel recorder. (Figure 20)

Pure bending moment and pure torsional moment on the wing were measured by attaching strain gages to the top and bottom of the front and rear spars and the wiring diagram used is shown in Figure 21.

A static calibration was obtained before the tests by applying known forces and moments, and recording the output.

Figure 19 shows the set-up used for calibration of pure bending (load applied on the elastic axis) and pure torsion (torque applied to the elastic axis).

Figures 24, 25 and 26 show the calibration curves for impact load, bending and torsional moment respectively.
Dynamic Response

The Dynamic Response was computed following the method outlined in the Analysis of this work, and a sample of the computations for the different landing impact forces obtained for one, two and three inch drops is shown in the Appendix of this thesis.

The landing impact forces obtained in the tests (Figs. 27, 28 and 29) were approximated by analytical functions and the analytical computations carried out.

The numerical values were substituted in the final formulas and the results were tabulated in Tables 5, 6 and 7.

The maximum bending and maximum torsional moments at station 5.5 on the wing, corresponding to the station 3.5 on the dynamic model, were computed by loading the dynamic model with forces \( m_n \phi_n^{(r)} \omega_r^2 \xi_r \) where

- \( m_n \) Mass of particle \( n \).
- \( \phi_n^{(r)} \) Relation of wing tip deflection and particle \( n \) deflection in the mode \( r \) relative to the equilibrium position.
- \( \omega_r \) Natural frequency of the wing in mode \( r \).
- \( \xi_r \) Wing tip deflection in mode \( r \) relative to the equilibrium position.
Calling

\[ M^{(r)}_R = K^{(r)}_M \xi_r \]  
Bending moment in mode \( r \).

\[ T^{(r)}_R = K^{(r)}_T \xi_r \]  
Torsional moment in mode \( r \).

\( b_n \)  
Distance from the station of particle \( n \) to the station where the bending moment is being computed.

\( d_n \)  
Distance from the particle \( n \) to the Elastic Axis.

\( M_R \)  
Total bending moment.

\( T_R \)  
Total torsional moment.

Then

\[ M^{(r)}_R = K^{(r)}_M \xi_r = \sum_n b_{mn} \phi^{(r)}_m \omega^2 r \xi_r \]

\[ M_R = \sum_r M^{(r)}_R = \sum_r K^{(r)}_M \xi_r = \sum_r \sum_n b_{mn} \phi^{(r)}_n \omega^2 r \xi_r = \]

\[ = \sum_r \xi_r \omega^2 r \sum_n b_{mn} \phi^{(r)}_n \]

and consequently

\[ K^{(r)}_M = \omega^2 r \sum_n b_{mn} \phi^{(r)}_n \]

Similarly

\[ T_R = \sum_r T^{(r)}_R = \sum_r K^{(r)}_T \xi_r \]

\[ K^{(r)}_T = \omega^2 r \sum_n d_{mn} \phi^{(r)}_n \]
For the computations of the contributions of each mode, the times which gave the maximum $M_r$ and $T_r$ were used without regard to the fact that the contribution of each mode was maximum or not.

Tests

In the first place the landing impact force was studied by substituting the wing with an equivalent mass and recording the impact force for different heights of drops.

The spring and oil of the shock absorber were varied and four different wheel tires were tried. The impact forces for each combination were recorded for different heights of drops.

After the study of the landing impact force, one combination of spring and oil of the shock absorber and wheel tires was chosen for the final tests.

Calibration curves for the impact load, bending moment and torsional moment were made.

The wing was then mounted on the apparatus and three drop tests for one, two and three inches were repeated three times, so that the average values could be taken.

Figures 27, 28 and 29 are the records for drops from one, two and three inches respectively, showing the measured bending and torsional moments on the right wing at station 5.5 inches, bending moment on the left wing at the same station, and the landing impact force.
The bending moments on the left wing were measured in order to check the symmetry of the drops.
V - RESULTS AND DISCUSSION

The records showing the results of the drop tests from the heights of one, two, and three inches are shown in figures 27, 28 and 29.

The maximum observed bending and torsional moment in the wings at station 5.5 inches from the root are given in table 8, where a comparison between the observed and computed values is made.

The computed bending moments are smaller than the observed, and the difference varies from 17.3 percent for a three inch drop, to 23.1 percent for a one inch drop.

The computed torsional moments are greater than the observed, and the difference varies from 129.2 percent for a two inch drop, to 159.0 percent for a three inch drop.

The time for the computed maximum bending moments is always smaller than the time for those observed, while the time for the computed maximum torsional moments is always greater than the time for those observed.

The coupled symmetrical free wing frequencies were measured experimentally and found:

First mode 4.0 c.p.s.
Second mode 7.5 c.p.s.
Third mode 12.2 c.p.s.

A comparison between these results with the computed values of table 5, shows that the accuracy of the
computation of the first mode is excellent, while for the second mode, the error is about 30 percent and for the third mode about 10 percent.

Studying the mode shapes (Table 5), the following conclusions were reached:

a) The first coupled mode is essentially made up of the first restrained bending mode with very small torsion.

b) The second coupled mode is essentially made up of the first restrained torsional mode with some bending.

c) The third coupled mode is essentially made up of the second restrained torsional mode with some bending.

This study of the natural coupled frequencies and mode shapes leads to the conclusion that the method employed to determine the bending stiffness is accurate enough for these computations, but the method employed to determine the torsional stiffness, disregarding the differential bending of the spars, is not satisfactory.

The first mode predominates in the determination of the bending moment, while the second and third have a very small influence, as can be seen in tables 5, 6 and 7. Therefore the accuracy of the first mode explains by itself the satisfactory results of the bending moment computations.
The second mode predominates in the determination of the torsional moment, while the first and third have a negligible influence, as can be seen in tables 5, 6 and 7. The inaccuracy of the computed second mode, and the fact that the determination of the moment depends on the square of the frequency, explain the poor results of the torsional moment computations.

During the investigation the problem of getting a shock absorber for the model dynamically similar to the actual one, since a dynamically similar wing was being used, came up when the relation between the frequency of the shock absorber and the frequency of the wing, was thought to be kept the same as for actual airplanes.

In first approximation, disregarding damping, the square of the natural frequency of the shock absorber is proportional to the static deflection, so if the same relation of frequencies is essential, a model shock absorber having the same static deflection as the actual airplane is required.
The wing is assumed to be completely free and is attached to a rigid fuselage.

The plane EFGH representing the undeformed plane of the wing is parallel to the Newtonian axis $x' - y'$.

The plane defined by the lines AB and CD always remains tangent to the elastic plate at the center line of the airplane.

The coordinate $\hat{q}$, $\hat{c}$ and $\hat{\chi}$ define the position in space of the plane ABCD with respect to the equilibrium position EFGH, as shown in the figure.

The vector $q_m$ defines the position of the mass $m_m$ normal to the plane EFGH.

The vector $a_m$ defines the position of the mass $m_m$ normal to the plane defined by the lines AB and CD.

Using the fundamental formulas

$$\sum m_m \frac{d^2\mathbf{r}_{cg}}{dt^2} = \sum m_m \frac{d^2\mathbf{r}_{cg}}{dt^2} = 0$$

$$\frac{d}{dt} \sum m_m \left( \mathbf{p}_m + \mathbf{a}_m \right) \times \ddot{\mathbf{v}}_m = 0$$

$$\mathbf{F}_m^* = \mathbf{F}_m - m_m \frac{d^2 \mathbf{r}_m}{dt^2}$$
\[ \bar{r}_m = \sum_{n} \phi_{m,n}^{-1} \ddot{q}_n \]

\( \phi_{m,n}^{-1} \) Dyadic of stiffness influence coefficients.

Observing that

\[ \bar{r}_m = \bar{r}_{cg} + \ddot{\bar{p}}_m + \ddot{q}_m \]

\[ \frac{d^2F_m}{dt^2} = \frac{d^2q_m}{dt^2} \quad \text{since } \ddot{\bar{p}}_m \text{ is constant and } \frac{d^2\bar{r}_{cg}}{dt^2} = 0 \]

\[ \ddot{\bar{p}}_m = x_m \bar{I} + y_m \bar{J} \]

Then

\[ \ddot{q}_m = \ddot{q} + x_m \dddot{x} + y_m \dddot{y} + \dddot{q}_m \]

and

\[ \dddot{q}_m = \dddot{q}_m - x_m \dddot{x} - y_m \dddot{y} - \dddot{q} \]

Hence we obtain the first fundamental formula:

\[ \sum_{m} \frac{d^2q_m}{dt^2} = 0 \]
Consequently, the second fundamental equation becomes

\[ \sum \overline{\rho}_m x \frac{d^2 \vec{q}_m}{dt^2} = 0 \]

or

\[ \sum\overline{\rho}_m (x_m \mathbf{i} + y_m \mathbf{j}) x \frac{d^2 \vec{q}_m}{dt^2} \mathbf{k} = 0 \]

\[ \sum\overline{\rho}_m \left( -x_m \frac{d^2 \vec{q}_m}{dt^2} \mathbf{j} + y_m \frac{d^2 \vec{q}_m}{dt^2} \mathbf{i} \right) = 0 \quad \text{II} \]

The third fundamental equation reduces to:

\[ m \frac{d^2 \vec{q}_m}{dt^2} + \sum \phi (\vec{q}_m - x_n \mathbf{x} \mathbf{k} - y_n \mathbf{y} \mathbf{k} - \vec{q}) = 0 \quad \text{III} \]

We can reduce these equations to the scalar forms:

\[ \sum x_m \ddot{a}_m = 0 \quad \text{I} \]

\[ \sum y_m x_m \ddot{a}_m = 0 \quad \text{II} \]

\[ \sum y_m x_m \ddot{a}_m = 0 \quad \text{II} \]
\[ m_i \ddot{x}_m + \sum_{k} k_{mn} \left( q_n - x_n X - y_n \Theta - q \right) = 0 \]  

Substituting as solutions:

\[ q = A \sin (\omega t + \psi) \]

\[ q_m = A_m \sin (\omega t + \psi) \]

\[ \chi = \chi \sin (\omega t + \psi) \]

\[ \theta = \Theta \sin (\omega t + \psi) \]

we obtain:

\[
\begin{align*}
\sum_{i} m_i A_i &= 0 \\
\sum_{i} x_i A_i &= 0 \\
\sum_{i} y_i A_i &= 0 \\
\end{align*}
\]

\[ m_i A_i \omega^2 + \sum_{k} k_{mn} \left( k_n - x_n X - y_n \Theta - A \right) = 0 \]

Rewriting the last equation in terms of the flexibility influence coefficients \( C_{mn} \)'s, we get:

\[
\sum_{i} C_{mn} x_i A_i = \frac{1}{\omega^2} \left( A_m - x_m X - y_m \Theta - A \right)
\]

We want to determine \( \frac{A}{\omega^2} \), \( \frac{X}{\omega^2} \) and \( \frac{\Theta}{\omega^2} \).
Multiplying both sides by $m_n$ and summing up on $m$:

$$\sum_n \sum_m m_n C_{mn} m_n A_n =$$

$$= \frac{-1}{\omega^2} \sum_m \left( m_m A_m - m_m x_m X - m_m y_m Y - m_m A \right)$$

As

$$\sum m_m A_m = 0$$

and

$$\sum m_m x_m X = \sum m_m y_m Y = 0$$

Since the axes are through the C.G.

we have

$$\alpha = -\frac{A}{\omega^2} = \frac{1}{M} \sum_n \sum_m m_n C_{mn} m_n A_n$$

Multiplying both sides by $m_n y_m$ and summing up on $m$:

$$\sum_n \sum_m m_n y_m C_{mn} m_n A_n =$$

$$= \frac{-1}{\omega^2} \sum_m \left( m_m y_m A_m - m_m y_m y_m X - m_m y_m^2 Y - m_m y_m A \right)$$
Assuming that the airplane is symmetrical and the plane of symmetry is through the center of gravity, and since the axis pass through the C.G., we have:

\[ \sum m_m y_m A_m = \sum m_m y_m x_m = \sum m_m y_m = 0 \quad \text{and} \]

\[ \sum m_m y_m^2 = I_x \]

We have:

\[ \beta = -\frac{\dot{\omega}}{\omega^2} = \frac{1}{I_x} \sum m_m y_m C_{mn} m_n A_n \]

Multiplying both sides by \( m_m x_m \), and summing up, similarly we get:

\[ \gamma = -\frac{\dot{\gamma}}{\omega^2} = \frac{1}{I_y} \sum m_m x_m C_{mn} m_n A_n \]

Introducing \( \alpha, \beta \) and \( \gamma \), we obtain

\[ \sum C_{mn} m_n A_n = \frac{A_m}{\omega^2} + y_m \beta + x_m \gamma + \alpha \]

This can be written as

\[ \sum C_{mn} m_n A_n - y_m \beta - x_m \gamma - \alpha = \frac{A_m}{\omega^2} \]
Applying the iteration procedure to this set of equations, observing that for this model

a) If \( m \) or \( n \) is equal to zero or one, then \( C_{mn} = 0 \)

b) If \( m \) is on one side of the center line and \( n \) is on the other, then also \( C_{mn} = 0 \)

\[
\begin{bmatrix}
C_{22} & C_{23} & C_{24} & C_{25} & 0 & 0 & 0 & 0 \\
C_{32} & C_{33} & C_{34} & C_{35} & 0 & 0 & 0 & 0 \\
C_{42} & C_{43} & C_{44} & C_{45} & 0 & 0 & 0 & 0 \\
C_{52} & C_{53} & C_{54} & C_{55} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66} & C_{67} & C_{68} & C_{69} \\
0 & 0 & 0 & 0 & C_{76} & C_{77} & C_{78} & C_{79} \\
0 & 0 & 0 & 0 & C_{86} & C_{87} & C_{88} & C_{89} \\
0 & 0 & 0 & 0 & C_{96} & C_{97} & C_{98} & C_{99}
\end{bmatrix}
\begin{bmatrix}
m_2 \\
m_3 \\
m_4 \\
m_5 \\
m_6 \\
m_7 \\
m_8 \\
m_9
\end{bmatrix}
\begin{bmatrix}
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6 \\
A_7 \\
A_8 \\
A_9
\end{bmatrix}
\]

\[
\frac{-\beta}{\sqrt{\gamma}}
\begin{bmatrix}
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6 \\
y_7 \\
y_8 \\
y_9
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_9
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
= \frac{1}{\omega^2}
\begin{bmatrix}
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6 \\
A_7 \\
A_8 \\
A_9
\end{bmatrix}
\]
Applying this iteration to the entire system leads to both symmetrical and anti-symmetrical modes.

However, we may break the problem into two parts: a system of equations for the symmetrical modes and another for the anti-symmetrical ones.

**Symmetrical Oscillations**

For symmetrical oscillations $\beta = \gamma = 0$, $A_2 = A_6$, $A_3 = A_7$, $A_4 = A_8$, $A_5 = A_9$

Then

\[
\begin{bmatrix}
C_{22} & C_{23} & C_{24} & C_{25} \\
C_{32} & C_{33} & C_{34} & C_{35} \\
C_{42} & C_{43} & C_{44} & C_{45} \\
C_{52} & C_{53} & C_{54} & C_{55}
\end{bmatrix}
\begin{bmatrix}
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5
\end{bmatrix}
= \frac{1}{\omega^2}
\begin{bmatrix}
A_2 \\
A_3 \\
A_4 \\
A_5
\end{bmatrix}
\]

where

\[
\alpha = \sum_{m=2}^{5} \sum_{n=2}^{5} \frac{m_m C_{mn} m_n A_n}{M/2}
\]
Applying to the test model, (Fig. 13) we have:

\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & C_{18} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} & C_{28} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} & C_{57} & C_{58} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} & C_{67} & C_{68} \\
C_{71} & C_{72} & C_{73} & C_{74} & C_{75} & C_{76} & C_{77} & C_{78} \\
C_{81} & C_{82} & C_{83} & C_{84} & C_{85} & C_{86} & C_{87} & C_{88}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6 \\
A_7 \\
A_8
\end{bmatrix}
\]

\[
\sum_{m=1}^{l} \sum_{n=1}^{r} \frac{m \cdot C_{mn} \cdot n \cdot A_n}{M/2}
\]

Using the symbols: \( \lfloor \rfloor \) for row and \( \lfloor \rfloor \) for column, and noticing that

\[
\left\{ \sum_{m=1}^{l} \sum_{n=1}^{r} \frac{m \cdot C_{mn} \cdot n \cdot A_n}{M/2} \right\} = \left[ \begin{array}{c}
\frac{\sum_{m=1}^{l} m \cdot C_{m1}}{M/2} \\
\frac{\sum_{m=1}^{l} m \cdot C_{m2}}{M/2} \\
\vdots \\
\frac{\sum_{m=1}^{l} m \cdot C_{m8}}{M/2}
\end{array} \right]
\begin{bmatrix}
m_1 A_1 \\
m_2 A_2 \\
m_3 A_3 \\
m_4 A_4 \\
m_5 A_5 \\
m_6 A_6 \\
m_7 A_7 \\
m_8 A_8
\end{bmatrix}
\]
Taking \( n = 1, 2, 3, \ldots, 8 \), we can write

\[
\begin{bmatrix}
    \{C_{1n}\} & \{m_n A_n\} \\
    \vdots & \vdots \\
    \{C_{8n}\} & \{m_n A_n\}
\end{bmatrix}
\begin{bmatrix}
    \sum_{m=1}^{8} \frac{m m_n C_{mn}}{M/2} \\
    \vdots \\
    \sum_{m=1}^{8} \frac{m m_n C_{mn}}{M/2}
\end{bmatrix}
\begin{bmatrix}
    \{m_n A_n\} \\
    \vdots \\
    \{m_n A_n\}
\end{bmatrix} = \begin{bmatrix}
    A_1 \\
    \vdots \\
    A_8
\end{bmatrix}
\]

Calling \( E_{mn} = \frac{\sum_{m=1}^{8} m m_n C_{mn}}{M/2} \), we can write

\[
\begin{bmatrix}
    \{C_{1n} - E_{mn}\} & \{m_n A_n\} \\
    \vdots & \vdots \\
    \{C_{8n} - E_{mn}\} & \{m_n A_n\}
\end{bmatrix}
\begin{bmatrix}
    \{A_1\} \\
    \vdots \\
    \{A_8\}
\end{bmatrix} = \frac{i}{\omega^4}
\]

Calling \( C_{mn} - E_{mn} = D_{mn} \), in order to facilitate the computations, we can write

\[
\begin{bmatrix}
    m_1 \\
    0 \quad m_2 \\
    \vdots \\
    \vdots \\
    \vdots \\
    0 \quad 0 \quad \ldots \quad m_n
\end{bmatrix}
\begin{bmatrix}
    \{A_n\}
\end{bmatrix} = \frac{i}{\omega^4}
\begin{bmatrix}
    \{A_n\}
\end{bmatrix}
\]
The same equations are obtained if we write down the Kinetic and Potential energy of the system and apply Lagrange's Equation.
Stresses in the Undamped Structure during Transient Oscillations.

In terms of the normal co-ordinates $\xi_r$ (amplitudes of the wing tip relative to the C.G. axis in the mode r), the differential equation is (Ref. 2)

$$M_r \ddot{\xi}_r + N_r \dot{\xi}_r \omega_r = Q_r(t)$$

Where:

$$M_r = \sum_{m=0}^{\infty} m \left( \phi_m^{(r)} \right)^2$$ Generalized mass in mode r

$$\phi_m^{(r)} \mid_{\theta_m} = 1$$

$\omega_r$ Circular frequency in mode r.

$$Q_r(t) = \phi_{1f}^{(r)} F(t)$$ Generalized force in mode r as a function of time.

The subscript f refers to the station at which the landing impact force $F(t)$ is applied.

Hence, the displacements of the principal oscillations obey the same laws as that of a simple oscillator of mass $M_r$ and spring $k_r = M_r \omega_r^2$. 
The expression giving the displacement \( \xi_r \) under the action of the generalized force \( Q_r(t) \) of arbitrary time history is well known and usually designated as Duhamel's integral. (Ref. 2 and 3)

\[
\xi_r = \frac{l}{M_r \omega_r} \int_0^t Q_r(\tau) \sin \omega_r (t - \tau) \, d\tau
\]

where \( \tau \) is the variable of integration.

Hence, the computation of the displacement \( \xi_r \) at any time \( t \), requires the evaluation of a definite integral between the limits 0 and \( t \).

The time history of \( \xi_r \) is obtained by repeating this process for every value of \( t \).

Fortunately, in the practical problems we are interested only in the maximum value of \( \xi_r \) instead of its complete time history.

The displacement under the landing impact force may be described as a superposition of \( \xi_0 \), a rigid body translation (the motion of the center of gravity) and deformations of the structure in all of its natural vibratory modes relative to the center of gravity.

We are interested only in the deformations of the structure.

Having computed the displacements \( \xi_1, \xi_2 \ldots \) we can compute the stresses.

The bending and torsional moment at any station in each mode \( r \) can be computed by considering each mass
of the model loaded by forces.

\[ F_n(r) = m_n \times \phi_n(r) = m_n \phi_n(r)^2 \omega_r^2 \xi_r \]

and superimpose the effect of each mode.

The justification for this procedure rests on the fact that since we neglect damping, the modes are uncoupled under the landing impact force. (Ref. 2 and 4)
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Sample of Computations of Dynamic Response

In order to facilitate the typewriting, we are going to introduce here the following notations:

\[ \gamma_r = x_r \]
\[ \omega_r = \omega_r \]
\[ \gamma = \gamma \]
\[ \frac{\phi_0^{(r)}}{M_r \omega_r} = g_r \]

Then,

\[ \gamma_r = \frac{\phi_0^{(r)}}{M_r \omega_r} \int_0^t f(\tau) \sin \omega_r (t - \tau) d\tau \]

can be written as

\[ x_r = g_r \int_0^t f(T) \sin \omega_r (t - T) dT \]
According to the record (Fig. ??) the landing impact force for one inch drop can be approximated by

\[ F(t) = \begin{cases} \frac{a}{t_1} t & \text{for } 0 < t < t_1 \\ \frac{at_2}{t_2-t_1} - \frac{a}{t_2-t_1} t & \text{for } t_1 < t < t_2 \end{cases} \]

\[ a = 21.8 \text{ lbs.} \]
\[ t_1 = 0.0074 \text{ sec.} \]
\[ t_2 = 0.3398 \text{ sec.} \]
Using the new notations:

For \( t < t_1 \)

\[ F(T) = \frac{a}{t_1} T \quad \text{for} \quad T < t_1 \]

Then

\[ x_r = G_r \int_{0}^{t} \frac{a}{t_1} T \sin \omega_r (t-T) \, dT = \]

\[ = G_r \frac{a}{t_1} \frac{1}{\omega_r} \left( t - \frac{1}{\omega_r} \sin \omega_r t \right) \quad (a) \]

For \( t > t_1 \)

\[ F(T) = \frac{a}{t_1} T \quad \text{for} \quad 0 < T < t_1 \]

\[ F(T) = F(T) \quad \text{for} \quad t_1 < T \]
Then

\[ x_r = G_r \int_0^t F(T) \sin \omega_r (t-T) \, dT = \]

\[ = G_r \int_0^{t_1} \frac{a}{t_1} T \sin \omega_r (t-T) \, dT + \]

\[ + G_r \int_{t_1}^t F(T) \sin \omega_r (t-T) \, dT \]

Solving and substituting the limits:

\[ x_r = G_r \frac{a}{t_1 \omega_r} \left[ \frac{1}{\omega_r} \sin \omega_r (t-t_1) + t_1 \cos \omega_r (t-t_1) - \right. \]

\[ - \frac{1}{\omega_r} \sin \omega_r t \right] + \]

\[ + G_r \int_{t_1}^t F(T) \sin \omega_r (t-T) \, dT \]  \hspace{1cm} (b)
For \( t_1 < t < t_2 \)

\[
F(T) = \frac{a}{t_1} T \quad \text{for} \quad 0 < t < t_1
\]

\[
F(T) = \frac{a t_2}{t_2-t_1} - \frac{a}{t_2-t_1} T \quad \text{for} \quad t_1 < T < t_2
\]

Then

\[
x_r = G_r \int_0^{t_1} \frac{a}{t_1} T \sin \omega_r (t-T) \, dT +
\]

\[
+ G_r \int_{t_1}^{t} \left[ \frac{a t_2}{t_2-t_1} - \frac{a}{t_2-t_1} T \right] \sin \omega_r (t-T) \, dT
\]

\[
\int_{t_1}^{t} \sin \omega_r (t-T) \, dT = \frac{1}{\omega_r} \left[ 1 - \cos \omega_r (t-t_1) \right]
\]

\[
- \frac{1}{\omega_r} \sin \omega_r (t-t_1)
\]

Substituting and simplifying:

\[
x_r = G_r \frac{a}{\omega_r} \left[ \frac{t_2}{t_1(t_2-t_1)} \frac{1}{\omega_r} \sin \omega_r (t-t_1) +
\]

\[
+ \frac{t_2 - t}{t_2-t_1} -
\]

\[
- \frac{1}{t_1 \omega_r} \sin \omega_r t \right] \quad (c)
\]
According to the records (Figs. 28 & 29) the landing impact force for two and three inch drops can be approximated by

\[ F(t) \]

For 2" drop
\[ b = 31.2 \text{ lbs.} \]
\[ t_1 = 0.026 \text{ sec.} \]
\[ t_2 = 0.210 \text{ sec.} \]
\[ p = 0.036 \text{ sec.} \]

For 3" drop
\[ b = 40.4 \text{ lbs.} \]
\[ t_1 = 0.031 \text{ sec.} \]
\[ t_2 = 0.209 \text{ sec.} \]
\[ p = 0.040 \text{ sec.} \]

Calling \( c = \frac{\pi}{p} \) and \( d = b \sin \frac{\pi}{p} t_1 \)

\[ F(t) = b \sin ct \quad \text{for } 0 < t < t_1 \]

\[ F(t) = d \frac{t_2 - t}{t_2 - t_1} = \frac{dt_2}{t_2 - t_1} - \frac{d}{t_2 - t_1} t \quad \text{for } t_1 < t < t_2 \]
For $0 < t < t_1$

$$F(T) = b \sin cT$$

for $0 < T < t_1$

$$x_r = G_r \int_0^t F(T) \sin \omega_r (t-T) \,dT =$$

$$= G_r \int_0^t b \sin cT \sin \omega_r (t-T) \,dT$$

Solving and simplifying

$$x_r = G_r \frac{b}{w_r^2 - c^2} \left[ w_r \sin ct - c \sin w_r t \right]$$

(d)

For $t_1 < t < t_2$

$$F(T) = b \sin cT$$

for $0 < T < t_1$

$$F(T) = \frac{dt_2}{t_2-t_1} - \frac{d}{t_2-t_1} T$$

for $t_1 < T < t_2$

$$x_r = G_r \int_0^t F(T) \sin \omega_r (t-T) \,dT =$$

$$= G_r \int_0^t b \sin cT \sin \omega_r (t-T) \,dT +$$

$$+ G_r \int_{t_1}^t \left[ \frac{dt_2}{t_2-t_1} - \frac{d}{t_2-t_1} T \right] \sin \omega_r (t-T) \,dT$$
Calling \( m = w_r - c \)
\[ n = w_r + c \]

\[
\int_0^t \sin cT \sin w_r (t-T) \,dT =
\]
\[
= \frac{1}{2} \left[ \frac{1}{m} \sin (w_r t - m t_1) - \frac{1}{n} \sin (w_r t - n t_1) - \frac{2c}{w_r^2 - c^2} \sin w_r t \right]
\]

\[
\int_{t_1}^t \sin w_r (t-T) \,dT = \frac{1}{w_r} \left[ 1 - \cos w_r (t-t_1) \right]
\]

\[
\int_{t_1}^t T \sin w_r (t-T) \,dT = \frac{1}{w_r} t - \frac{1}{w_r} \sin w_r (t_t_1) - \frac{1}{w_r} t_1 \cos w_r (t-t_1)
\]

Substituting and simplifying:

\[
x_r = \alpha_r \left\{ \frac{d}{2} \left[ \frac{1}{m} \sin (w_r t - m t_1) - \frac{1}{n} \sin (w_r t - n t_1) - \frac{2c}{w_r^2 - c^2} \sin w_r t \right] +
\right.
\]

\[
+ \frac{d}{(t_2 - t_1) w_r} \left[ (t_2 - t) + \frac{1}{w_r} \sin w_r (t-t_1) - (t_2 - t_1) \cos w_r (t-t_1) \right] \right\}
\]

(e)
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Table 1 - Matrix of the Flexibility Influence

Coefficients $c_{mn}^{sf}$ of the Elastic Axis.
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**Table 2 - Matrix of the Torsional Influence**

-\( \Theta T \)

Coefficients \( c_{mn} \) of the Elastic Axis.
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<td>38.41</td>
<td>41.57</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3.390</td>
<td>1.663</td>
<td>28.75</td>
<td>19.08</td>
<td>59.43</td>
<td>36.84</td>
<td>111.63</td>
<td>92.69</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.421</td>
<td>6.630</td>
<td>-2.686</td>
<td>21.84</td>
<td>36.84</td>
<td>88.76</td>
<td>95.06</td>
<td>138.60</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5.844</td>
<td>4.181</td>
<td>46.24</td>
<td>38.41</td>
<td>111.63</td>
<td>95.06</td>
<td>265.55</td>
<td>246.03</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3.741</td>
<td>8.346</td>
<td>19.88</td>
<td>41.57</td>
<td>92.69</td>
<td>138.60</td>
<td>246.03</td>
<td>355.51</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 - Matrix of the Flexibility Influence Coefficients

C<sub>mn</sub>'s of the Wing.

(Elements in the table have been multiplied by 10<sup>2</sup>)
<table>
<thead>
<tr>
<th>m</th>
<th>$\phi_1^{(1)}/\phi_{1m}$</th>
<th>$\phi_2^{(2)}/\phi_{2m}$</th>
<th>$\phi_3^{(3)}/\phi_{3m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-.0913746</td>
<td>.164947</td>
<td>.0156114</td>
</tr>
<tr>
<td>1</td>
<td>-.0703343</td>
<td>.0348814</td>
<td>.0239043</td>
</tr>
<tr>
<td>2</td>
<td>-.0681937</td>
<td>.355313</td>
<td>-.0585303</td>
</tr>
<tr>
<td>3</td>
<td>.0518603</td>
<td>-1.24455</td>
<td>.158059</td>
</tr>
<tr>
<td>4</td>
<td>.0629429</td>
<td>.228866</td>
<td>-.164853</td>
</tr>
<tr>
<td>5</td>
<td>.300882</td>
<td>-.369947</td>
<td>-.602444</td>
</tr>
<tr>
<td>6</td>
<td>.355815</td>
<td>.967733</td>
<td>.190426</td>
</tr>
<tr>
<td>7</td>
<td>.848989</td>
<td>-.340666</td>
<td>-.924423</td>
</tr>
<tr>
<td>8</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
</tbody>
</table>

Table 4 - Coupled Symmetrical Wing Modes and Frequencies.
Max. Impact Force 21.8 lbs.
Time for Max. Torsional Moment .090 sec.

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_r )</td>
<td>25.4189</td>
<td>34.7540</td>
<td>83.0260</td>
</tr>
<tr>
<td>( M_r \times 10^2 )</td>
<td>.116232</td>
<td>.817758</td>
<td>.137237</td>
</tr>
<tr>
<td>( \xi_r ) at .1163 sec.</td>
<td>-4.44121</td>
<td>.486192</td>
<td>.059916</td>
</tr>
<tr>
<td>( K(r) )</td>
<td>6.95</td>
<td>-4.14</td>
<td>-6.74</td>
</tr>
<tr>
<td>( M(r) )</td>
<td>-30.866</td>
<td>-2.013</td>
<td>-4.04</td>
</tr>
<tr>
<td>( T(r) )</td>
<td>-.467</td>
<td>13.995</td>
<td>-.026</td>
</tr>
<tr>
<td>( \xi_r ) at .090 sec.</td>
<td>-3.80987</td>
<td>.641367</td>
<td>.005174</td>
</tr>
</tbody>
</table>

\( \omega_r \) Natural symmetrical frequencies of the wing (rad/sec.)

\( M(r) \) Resultant Bending Moment in mode r (in.lb.)

\( T(r) \) Resultant Torsional Moment in mode r (in.lb)

\( M_r \) Generalized mass in mode r (lb. sec.²/in.)

\( \xi_r \) Wing tip deflection in mode r relative to the C.G. of the model (in.).

Table 5 - Computation of Dynamic Response for One Inch Drop.
Max. Impact Force 31.2 lbs.
Time for Max. Bending Moment .110 sec.
Time for Max. Torsional Moment .085 sec.

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_r$</td>
<td>25.4189</td>
<td>34.7540</td>
<td>83.0260</td>
</tr>
<tr>
<td>$M_{1r} \times 10^2$</td>
<td>.116232</td>
<td>.817758</td>
<td>.137237</td>
</tr>
<tr>
<td>$\xi_{jr}$ at .110 sec</td>
<td>-4.87505</td>
<td>.57372</td>
<td>.03774</td>
</tr>
<tr>
<td>$X_{M}$</td>
<td>6.95</td>
<td>-4.14</td>
<td>-6.74</td>
</tr>
<tr>
<td>$M_{R}$</td>
<td>-33.881</td>
<td>-2.375</td>
<td>-0.254</td>
</tr>
<tr>
<td>$\xi_{jr}$ at .085 sec</td>
<td>-3.23567</td>
<td>.73263</td>
<td>-.01792</td>
</tr>
<tr>
<td>$K_{T}$</td>
<td>.1228</td>
<td>21.82</td>
<td>-5.12</td>
</tr>
<tr>
<td>$T_{R}$</td>
<td>-.397</td>
<td>15.986</td>
<td>.092</td>
</tr>
</tbody>
</table>

The symbols used are the same as in Table 5.

Table 6 - Computations of Dynamic Response for two Inch Drop.
Max. Impact Force 40.4 lbs.
Time for Max. Torsional Moment .085 sec.

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_r$</td>
<td>25.4189</td>
<td>34.7540</td>
<td>83.0260</td>
</tr>
<tr>
<td>$M_r \times 10^2$</td>
<td>.116232</td>
<td>.817758</td>
<td>.137237</td>
</tr>
<tr>
<td>$E_{\gamma}$ at .108 sec.</td>
<td>-5.74609</td>
<td>.58076</td>
<td>.07599</td>
</tr>
<tr>
<td>$K_M^{(r)}$</td>
<td>6.95</td>
<td>-4.14</td>
<td>-6.74</td>
</tr>
<tr>
<td>$M^{(r)}_R$</td>
<td>-39.935</td>
<td>-2.818</td>
<td>-.512</td>
</tr>
<tr>
<td>$E_{\gamma}$ at .085 sec.</td>
<td>-5.01506</td>
<td>.87486</td>
<td>-.031417</td>
</tr>
<tr>
<td>$K_T^{(r)}$</td>
<td>.1228</td>
<td>21.82</td>
<td>-5.12</td>
</tr>
<tr>
<td>$T_{RH}^{(r)}$</td>
<td>-.616</td>
<td>19.089</td>
<td>.161</td>
</tr>
</tbody>
</table>

The symbols used are the same as in Table 5.

Table 7 - Computations of Dynamic Response for
Three Inch Drop.
<table>
<thead>
<tr>
<th>ITEM</th>
<th>Height of Drop</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1&quot;</td>
<td>2&quot;</td>
<td>3&quot;</td>
<td></td>
</tr>
<tr>
<td>Max. Bend. Mom. (in. lb.)</td>
<td>Observed</td>
<td>-43.30</td>
<td>-45.50</td>
<td>-52.30</td>
</tr>
<tr>
<td></td>
<td>Computed</td>
<td>-33.28</td>
<td>-36.51</td>
<td>-43.27</td>
</tr>
<tr>
<td></td>
<td>Percent</td>
<td>23.1</td>
<td>19.8</td>
<td>17.3</td>
</tr>
<tr>
<td>at Station 5.5 in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Tors. Mom. (in. lb.)</td>
<td>Observed</td>
<td>5.80</td>
<td>6.84</td>
<td>7.20</td>
</tr>
<tr>
<td></td>
<td>Computed</td>
<td>13.50</td>
<td>15.68</td>
<td>18.63</td>
</tr>
<tr>
<td></td>
<td>Percent</td>
<td>132.8</td>
<td>129.2</td>
<td>159.0</td>
</tr>
<tr>
<td>at Station 5.5 in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time for Max. Bend. Mom. (sec.)</td>
<td>Observed</td>
<td>.1363</td>
<td>.129</td>
<td>.127</td>
</tr>
<tr>
<td></td>
<td>Computed</td>
<td>.1163</td>
<td>.110</td>
<td>.108</td>
</tr>
<tr>
<td>Time for Max. Tors. Mom. (sec.)</td>
<td>Observed</td>
<td>.071</td>
<td>.065</td>
<td>.059</td>
</tr>
<tr>
<td></td>
<td>Computed</td>
<td>.090</td>
<td>.085</td>
<td>.085</td>
</tr>
</tbody>
</table>

Table 8 - Comparison between the Observed and Computed Values.
SECTIONS AND STATIONS TAKEN ALONG THE WING FOR THE DYNAMIC MODEL.

ENGINE STATION

35 1/3\% C - LINE OF CENTERS OF GRAVITY.

ELASTIC AXIS Z

FIG. 3
MOMENT OF INERTIA PER INCH

Y3.

SPAN

\[
\begin{align*}
1 & \int_0^{25} \lambda (y) dy = 1543 \text{ gms in}^2 \\
2 & \int_{7.5}^{12} \lambda (y) dy = 874 \text{ gms in}^2 \\
3 & \int_7^{12} \lambda (y) dy = 720 \text{ gms in}^2 \\
4 & \int_7^{20} \lambda (y) dy = 385 \text{ gms in}^2
\end{align*}
\]
EVALUATION OF $C_{ff}$ and $C_{ff}$

$b_1 = 3$ inches.

\[
\int_0^b \frac{(b_1 - y)^2}{EI} \, dy = 26.46 \times 10^{-4} \text{ in lb}
\]

\[
\int_0^b \frac{(b_1 - y)^4}{EI} \, dy = 52.32 \times 10^{-4} \text{ in lb}
\]
EVALUATION OF $\bar{C}_{66}$ and $\bar{C}_{88}$

$D = 7.5$ inches

\[
\begin{align*}
\int_0^{b_a} \frac{(b_a - y)}{E_1} \, dy &= 187.84 \times 10^{-3} \text{ in}^{-1} \\
\int_0^{b_a} \frac{(b_a - y)^2}{E_1} \, dy &= 888.96 \times 10^{-2} \text{ in}^{-1}
\end{align*}
\]
EVALUATION OF $\bar{C}_{66}^e$ AND $\bar{C}_{88}^e$

$h = 13$ inches

\[
\int_0^{b_3} \frac{(b_3-y)}{EI} dy = 675.32 \times 10^4 \text{ in}.
\]

\[
\int_0^{b_3} \frac{(b_3-y)^2}{EI} dy = 525.80 \times 10^3 \text{ in/lb}.
\]

FIG. 9

O.N. LEAL
SEP 5-47
EVALUATION OF $C_{44}$ and $C_{44}$

$b_4 = 21$ inches

$$\int_0^{b_4} \frac{(b_4-y)}{EI} \, dy = 235.44 \times 10^3 \text{ in.lb.}$$

$$\int_0^{b_4} \frac{(b_4-y)^2}{EI} \, dy = 275.07 \times 10^5 \text{ in.lb.}$$

FIG. 10
EVALUATION OF $\bar{c}_m$ 

$10^3/G_1$ (1/16 in.$^{-2}$) 

$\int_0^{b_1} \frac{dy}{G_1} = 2.42 \times 10^{-3}$ /lb.in. 
$\int_0^{b_2} \frac{dy}{G_1} = 10.16 \times 10^{-3}$ /lb.in. 
$\int_0^{b_3} \frac{dy}{G_1} = 28.87 \times 10^{-3}$ /lb.in. 
$\int_0^{b_4} \frac{dy}{G_1} = 82.11 \times 10^{-3}$ /lb.in. 

SPANWISE STATION (IN.) 

FIG. 11
Fig. 12 - Wing Model without Engine.

Fig. 13 - Dynamic Model
Fig. 14 - Engine Model.
Fig. 15 - Wing Model with Engines.
Fig. 16 - Details of Apparatus.
Fig. 18 - Release Mounting.
Fig. 19 - Calibration Set-up for Bending and Torsion.
Fig. 20 - Measuring Instruments.
(Set-up for one spar. Same wiring used for both spars).

(Signal from front spar)

- Bending

(Signal from rear spar)

- To Amplifier
- Torsion

(Detail of transformer wiring).

Fig. 21 - Wiring Set-up for Tests.
FIG 22. SCHEMATIC DIAGRAM OF APPARATUS.
CALIBRATION CURVE

PLOT OF LANDING IMPACT FORCE VS.
RECORD DISPLACEMENT

FIG. 24

O.N. LEAL
SEP 7-97
CALIBRATION CURVE

PLOT OF BENDING MOMENT

VS.

RECORD DISPLACEMENT

FIG. 25

D.Y. LEAL
SEP 7 47
CALIBRATION CURVE

PLOT OF TORSIONAL MOMENT

vs.

RECORD DISPLACEMENT

FIG. 26
BENDING - RIGHT WING

50 lb.in.

TORSION - RIGHT WING

5 lb.in.

BENDING - LEFT WING

50 lb.in.

IMPACT FORCE

2g

.01 sec.

FIG. 27 - ONE INCH DROP RECORD.
Fig. 28 - Two inch drop record.
FIG. 29 - THREE INCH DROP RECORD.