SEPARABLE STÄCKEL SYSTEMS

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I. INTRODUCTION

Stäckel concerned himself with the conditions for the separability of the Hamilton-Jacobi equation

$$\sum_i \frac{1}{H_i^2} \left( \frac{\partial \phi}{\partial x_i} \right)^2 + \kappa^2 (E - V) \phi = 0$$

and a condition was that in the quadratic differential form $\sum H_i^2 dx_i^2$ the metric coefficients $H_i^2$ be of the form $H_i^2 = \frac{S}{S^i}$, where $S$ is the Stäckel determinant in which the elements of the $i$th row are dependent upon the $i$th variable at most, and $S^i$ is the cofactor of the $i$th element in the first column. Robertson demonstrated that this was also a condition for the separability of the Schrödinger equation

$$\sum H \frac{\partial}{\partial x_i} \left( \frac{H}{H_i^2} \frac{\partial \phi}{\partial x_i} \right) + \kappa^2 (E - V) \phi = 0$$

There was the further condition that

$$S = \prod \frac{H_i}{f_i}$$

in which $f_i$ is a function of $x_i$ at most.

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In both equations separated solutions of the form $\prod x_i$, where $x_1$ is a function of $x_1$ alone were sought. No great amount was known concerning which coordinate systems permitted the separability of these equations until Weinacht proved the following theorem, which will be discussed in detail in this paper:

The only triply orthogonal Stäckel systems in euclidean space are the confocal $F_2$ and their degenerations.

This statement of the theorem is due to Blaschke, whose proof was geometric rather than analytic. Blaschke limited his discussion to three dimensions. Finally, Eisenhart proved the theorem independently, and determined the Stäckel determinants for euclidean 3-space. We shall follow the proof given by Blaschke, and in particular discuss the case of four dimensions, due to the physical importance of this case in connection with the wave equation.

Eisenhart shows that condition (3) on equation (2) is equivalent to $R_{ij} = 0 \ (i \neq j)$, where $R_{ij}$ are the exponents of the Ricci tensor, so that condition (3) is satisfied by any Stäckel form of a euclidean space or of a space of constant Riemannian curvature. Thus if we confine ourselves to euclidean spaces, the only condition for separability is that the quadratic differential form be of the type given by Stäckel.
II. BLASCHKE'S PROOF OF THE THEOREM
EXTENDED TO FOUR DIMENSIONS

A. Ivory's Property Leads to Stäckel's Quadratic Form.
Consider the three coordinate surfaces $u, v, w$, and the fourth coordinate, time. Then we define Ivory's diagonal property\(^5\) as follows:

In any domain $G$ in which $t_0 < t < t$, $u_0 < u < u$, $v_0 < v < v$, $w_0 < w < w$, the apses of eight pairs of opposing points are equal.

By apsis we mean the geodetic apsis; we define distance in the usual algebraic sense. Thus if the coordinate surfaces are $t, u, v, w$, the point pairs are those given in Table I. We shall show that all spaces which satisfy this condition have the element of length given by Stäckel, that is,

\[
\begin{array}{cccc}
(0000)(1111) & (1000)(0111) \\
(1100)(0011) & (1110)(0001) \\
(0100)(1011) & (0010)(1101) \\
(1010)(0101) & (1001)(0110)
\end{array}
\]

\[
(4) \quad ds^2 = \sum \left[ \frac{dt^2}{S^2} + \frac{du^2}{S^{2u}} + \frac{dv^2}{S^{2v}} + \frac{dw^2}{S^{2w}} \right]
\]

There is no necessity for assuming that the metric is positive definite; the assumption is made for simplicity.
only.

It follows immediately from the Ivory condition that the coordinates \( u, v, w \) are orthogonal. For considering only the three dimensional space coordinates, for sufficiently small \( u_0 < u < u_1, \ v_0 < v < v_1, \ w_0 < w < w_1 \), the relations between the geodetic distance and angle in the parallelepiped (Fig. 1) approach those of a euclidean parallelepiped, and so the spatial diagonals can be considered equivalent to rectangular ones. Therefore in the quadratic expression for arc length the cross product terms will drop out, and if we include time as the fourth dimension we have

\[(5) \quad ds^2 = c dt^2 + f du^2 + g dv^2 + h dw^2\]

Now let us consider the variation in the distance between two points labeled, for example, \((0000)\) and \((1111)\), as the end point \((1111)\) is displaced by \(\delta t, \delta u, \delta v, \delta w\). We have

\[(6) \quad \delta D = c \frac{dt}{ds} \delta t + f \frac{du}{ds} \delta u + g \frac{dv}{ds} \delta v + h \frac{dw}{ds} \delta w\]
The values of \( e, f, g, \) and \( h \) are taken at the end point (1111), and the increments \( \delta t, \delta u, \delta v, \delta w \) refer to this point also.

If we change only \( t \), by the amount \( \delta t \), we see that according to (6) and the assumption of the Ivory property, equal changes of the equal diagonals \((0000)(1111)\) and \((0011)(1100)\) gives

\[
(7a) \quad \frac{\delta D}{\delta t} = \left[ e \frac{dt}{ds} \right]_{0oo} = \left[ e \frac{dt}{ds} \right]_{1oo}
\]

Similarly,

\[
(7b) \quad \frac{\delta D}{\delta u} = \left[ f \frac{du}{ds} \right]_{0oo} = \left[ f \frac{du}{ds} \right]_{1oo}
\]

\[
(7c) \quad \frac{\delta D}{\delta v} = \left[ g \frac{dv}{ds} \right]_{0oo} = \left[ g \frac{dv}{ds} \right]_{1oo}
\]

Let us denote as covariant rectangular coordinates the quantities

\[
(8) \quad e \frac{dt}{ds} = \phi \quad f \frac{du}{ds} = \psi \quad g \frac{dv}{ds} = \omega \quad h \frac{dw}{ds} = \Theta
\]

and as contravariant rectangular coordinates the quantities

\[
(9) \quad \frac{dt}{ds}, \quad \frac{du}{ds}, \quad \frac{dv}{ds}, \quad \frac{dw}{ds}
\]
From (5) we have

\[(10) \quad \frac{\phi^2}{c} + \frac{\psi^2}{f} + \frac{\omega^2}{g} + \frac{\Theta^2}{h} = 1 \]

According to (7a), (7b), and (7c), the diagonals terminating at (1111) and (1100) have the same covariant coordinates \( \phi, \psi, \) and \( \omega \). If we assume, without loss of generality, that the diagonal comes from the interior of \( G \), we have that \( \Theta_{\...} > 0 \), and \( \Theta_{\...} < 0 \), since \( w < w_i \).

Then

\[(11) \quad \frac{\phi_{\mu}^2}{c} + \frac{\psi_{\mu}^2}{f} + \frac{\omega_{\mu}^2}{g} < 1 \]

Due to continuity, this relation holds along the boundary of \( G \) from (1111) to (1100). So in order to obtain the diagonal direction at (1100) from that at (1111) we displace the diagonal continuously from the first point to the latter. The coordinates \( \phi, \psi, \) and \( \omega \) remain fixed, as shown above, while \( \Theta \) changes continuously according to (9), and in addition between the two endpoints the sign of \( \Theta \) reverses.

By similar displacements of the end point of the diagonal, consecutively varying \( \omega \) only, then \( \psi \) only,
and finally $\Phi$ only, we arrive at the point (0000) with
the diagonal directed toward the interior of $G$.

This allows us to consider the movement of the point
(11111) along the diagonal toward the point (0000); we
see that this geodetic diagonal will satisfy a system
of equations

\[ \begin{align*}
\phi &= \phi(t), \\
\psi &= \psi(u), \\
\omega &= \omega(v), \\
\Theta &= \Theta(w)
\end{align*} \]

But by (8) we may write this as

\[ \begin{align*}
\frac{d\phi}{ds} &= \phi(t), \\
\frac{d\psi}{ds} &= \psi(u), \\
\frac{d\omega}{ds} &= \omega(v), \\
\frac{d\Theta}{ds} &= \Theta(w)
\end{align*} \]

However, the point (0000) is not distinct. Therefore,
in the Riemannian space considered there must be a three
parameter family of extremals satisfying equations (13).

The functions $\phi$, $\psi$, $\omega$, and $\Theta$ will depend on
three constants $c_1$, $c_2$, and $c_3$, or more explicitly, upon
the initial direction $\phi_{0000}$, $\psi_{0000}$, $\omega_{0000}$, $\Theta_{0000}$.

Hence equation (9) takes the form

\[ \frac{T}{e} + \frac{U}{f} + \frac{V}{g} + \frac{W}{h} = 1 \]

where we have replaced $\phi^2$ by $T = T(t, c_1, c_2, c_3)$; similarly
\( \psi^2 = U = U(u,c_1,c_2,c_3), \quad \omega^2 = V = V(v,c_1,c_2,c_3), \quad \text{and} \quad \Theta^2 = W = W(w,c_1,c_2,c_3). \) Differentiating (14) with respect to \( c_1, c_2, \) and \( c_3, \) we obtain

\[
\frac{T_1}{e} + \frac{U_1}{f} + \frac{V_1}{g} + \frac{W_1}{h} = 0
\]

\[
\frac{T_2}{e} + \frac{U_2}{f} + \frac{V_2}{g} + \frac{W_2}{h} = 0
\]

\[
\frac{T_3}{e} + \frac{U_3}{f} + \frac{V_3}{g} + \frac{W_3}{h} = 0
\]

in which, for example, \( U_2 = \frac{\partial}{\partial c_2} U(u,c_1,c_2,c_3). \)

In (14) and (15) we have found a linear system of equations in reciprocal values of \( e, f, g, \) and \( h. \) At (0000) the determinant

\[
\begin{vmatrix}
T_1 & T_2 & T_3 & T_4 \\
U_1 & U_2 & U_3 & U_4 \\
V_1 & V_2 & V_3 & V_4 \\
W_1 & W_2 & W_3 & W_4
\end{vmatrix}
\]

(16)

of the system is not identically zero, since at (0000) the direction \( \phi: \psi: \omega: \Theta \) can be chosen arbitrarily. Therefore we can solve equations (13) and (14) for \( e, f, g, \) and \( h \) and obtain the metric form (4) given by Stäckel. Thus we have proved:

THEOREM I. If the parametric surfaces \( t, u, v, w \) of a Riemannian region have the Ivory property, then the element of length of this region must have the form (4) of Stäckel.
B. Converse: A Region with Stäckel's Element of Length Has the Ivory Property.

Starting with a positive definite metric form (4), we seek to determine a family of curves in Riemannian space which satisfies a system of equations of the form (13). First, we insert the values for $e$, $f$, $g$, and $h$ given by (4) into equation (10), obtaining

$$\frac{\phi^2}{S} S'' + \frac{\psi^2}{S} S^{12} + \frac{\omega^2}{S} S^{13} + \frac{\Theta^2}{S} S^{14} = 1$$

which is equivalent to

$$\begin{vmatrix} \phi^2 & T_1 & T_2 & T_3 \\ \psi^2 & U_1 & U_2 & U_3 \\ \omega^2 & V_1 & V_2 & V_3 \\ \Theta^2 & W_1 & W_2 & W_3 \end{vmatrix} = S = \begin{vmatrix} T & T_1 & T_2 & T_3 \\ U & U_1 & U_2 & U_3 \\ V & V_1 & V_2 & V_3 \\ W & W_1 & W_2 & W_3 \end{vmatrix}$$

(17)

The most general solution of this equation contains three arbitrary constants $C_1$, $C_2$, and $C_3$:

$$\begin{align*}
\phi &= \sqrt{T + C_1 T_1 + C_2 T_2 - C_3 T_3} \\
\psi &= \sqrt{U + C_1 U_1 + C_2 U_2 + C_3 U_3} \\
\omega &= \sqrt{V + C_1 V_1 + C_2 V_2 + C_3 V_3} \\
\Theta &= \sqrt{W + C_1 W_1 + C_2 W_2 + C_3 W_3}
\end{align*}$$

(18)

According to the first three equations, $C_1$, $C_2$, and $C_3$ are independent of $W$; similar argument shows them to be independent of $T$, $U$, and $V$. Therefore $C_1$, $C_2$, and
$C_3$ are constants. We take these constants so that the expressions under the radicals are real, and we extract from the radicals any four roots, say the positive ones. Then we assert: In the neighborhood of $(0000)$ the integral curves corresponding to the differential equations (13) form a geodetic line field which intersects orthogonally a family of parallel hypersurfaces. To prove this, we first form the integral curve

(19) \[ p = \int (\phi \delta t + \psi \delta u + \omega \delta v + \Theta \delta w) \]

Here $\phi$, $\psi$, $\omega$, and $\Theta$ are the covariant direction coordinates of the field curve, and $\delta t: \delta s$, $\delta u: \delta s$, $\delta v: \delta s$, $\delta w: \delta s$ are the contravariant direction coordinates of the line of integration.

From the right side of (19) it follows that for the case of fixed end conditions the integral is independent of the path, since, for example, $\phi$ is dependent only upon $t$. Moreover, we know that the cosine of the angle between the field direction and the line of integration is given by

\[ \cos \alpha = \frac{edt \delta t + fdu \delta u + gdy \delta v + hdw \delta w}{ds \delta s} \]
or, in our present notation,

\[
(20) \quad \cos \alpha = \phi \frac{\delta t}{\delta s} + \psi \frac{\delta u}{\delta s} + \omega \frac{\delta v}{\delta s} + \Theta \frac{\delta \psi}{\delta \delta}
\]

From this last expression it is evident that the integral (19) can be written as

\[
(21) \quad p = \int \cos \alpha \, \delta s
\]

Therefore if we extend integral (19) between two points of a field curve, we have from (21) that

\[
(22) \quad p = \int ds
\]

is the length of the field curve. For any other curve connecting the points, we have further from (21) that

\[
(23) \quad p \leq \int \delta s
\]

Consequently the field lines are shortest, and so are geodesics, as was asserted:

\[
(24) \quad \int ds \leq \int \delta s
\]
The hypersurfaces \( p = \text{const.} \) are the orthogonal hypersurfaces of the field.

The geodetic lines, which are orthogonal to the hypersurfaces \( p = \text{const.} \), where \( p \) is given by (18) and (19), satisfy the equations

\[
(25) \quad \frac{\partial p}{\partial C_1} = \text{const.}, \quad \frac{\partial p}{\partial C_2} = \text{const.}, \quad \frac{\partial p}{\partial C_3} = \text{const.}
\]

By taking derivatives of (10) with respect to \( C_1 \) it follows that

\[
(26) \quad \frac{\phi}{e} \frac{\partial \phi}{\partial C_1} + \frac{\psi}{f} \frac{\partial \psi}{\partial C_1} + \frac{\omega}{g} \frac{\partial \omega}{\partial C_1} + \frac{\theta}{h} \frac{\partial \theta}{\partial C_1}
\]

By inserting, for example,

\[
(27) \quad e \frac{dt}{ds} = \phi = \frac{\partial p}{\partial t}
\]

the relation (26) becomes

\[
(28) \quad \frac{\partial^2 p}{\partial t \partial C_1} \frac{dt}{ds} + \frac{\partial^2 p}{\partial u \partial C_1} \frac{du}{ds} + \frac{\partial^2 p}{\partial v \partial C_1} \frac{dv}{ds} + \frac{\partial^2 p}{\partial w \partial C_1} \frac{dw}{ds} = 0
\]

This equation states that along the geodetic lines the field \( \delta p; \delta C_1 \) is constant, in agreement with the first of equations (25).
Thus the field equations (25) appear in detail as

\[ T^* + U^* + V^* + W^* = \text{const.} \]

(29)

\[ T^{**} + U^{**} + V^{**} + W^{**} = \text{const.} \]

\[ T^{***} + U^{***} + V^{***} + W^{***} = \text{const.} \]

where, for example,

\begin{align*}
U^* & = \int \frac{U_1 \, du}{\sqrt{U + C_1 U_1 + C_2 U_2 + C_3 U_3}}, & U^{**} & = \int \frac{U_2 \, du}{\sqrt{U + C_1 U_1 + C_2 U_2 + C_3 U_3}}
\end{align*}

We now consider, in addition to the integral \( p \), those geodetic line fields which correspond to the integrals

\( q_i = e_{ij} \int \phi \, dt + e_{ij} \int \psi \, du + e_{ij} \int \omega \, dv + e_{ij} \int \theta \, dw \)

in which \( e_{ij} = \pm 1 \), the signs given in Table II, being chosen so that

\( p + \sum_{i=1}^{7} q_i = 0 \)

Then the geodesics of the field which intersects orthogonally the surfaces \( q_5 = \text{const.} \), say, satisfy equations corresponding to (29):

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\[-T^* + U^* + V^* + W^* = \text{const.}\]

(33) \[-T^{**} + U^{**} + V^{**} + W^{**} = \text{const.}\]

\[-T^{***} + U^{***} + V^{***} + W^{***} = \text{const.}\]

We now assert that if one takes a geodetically convex domain \(t_0 < t < t_1, u_0 < u < u_1, v_0 < v < v_1, w_0 < w < w_1\), such that the geodetic diagonal \((1111)(0000)\) corresponds to the field (29), then, for example, the diagonal \((0011)\) corresponds to the field (33).

It follows from (29) that

\[T^*_0 + U^*_0 + V^*_0 + W^*_0 = T^*_1 + U^*_1 + V^*_1 + W^*_1\]

(34) \[T^{**}_0 + U^{**}_0 + V^{**}_0 + W^{**}_0 = T^{**}_1 + U^{**}_1 + V^{**}_1 + W^{**}_1\]

\[T^{***}_0 + U^{***}_0 + V^{***}_0 + W^{***}_0 = T^{***}_1 + U^{***}_1 + V^{***}_1 + W^{***}_1\]

where, for example, \(T^*_0 = T^*(t_0)\). It follows by inversion that

\[-T^*_i - U^*_i + V^*_i + W^*_i = -T^*_0 - U^*_0 + V^*_0 + W^*_0\]

\[-T^{**}_i - U^{**}_i + V^{**}_i + W^{**}_i = -T^{**}_0 - U^{**}_0 + V^{**}_0 + W^{**}_0\]

\[-T^{***}_i - U^{***}_i + V^{***}_i + W^{***}_i = -T^{***}_0 - U^{***}_0 + V^{***}_0 + W^{***}_0\]

and these agree with (33), as was asserted.

We are now in a position to compute the lengths of the diagonals. For the diagonal \((0000)(1111)\) we have from (19)

\[p = + \int_{t_0}^{t_1} \phi dt + \int_{u_0}^{u_1} \psi du + \int_{v_0}^{v_1} \omega dv + \int_{w_0}^{w_1} \theta dw\]
and for the diagonal \((1100)(0011)\), for example, we have from (29)

\[
q_5 = -\int_{t_0}^{t_1} \phi dt - \int_{u_0}^{u_1} \psi du + \int_{v_0}^{v_1} \omega dv + \int_{w_0}^{w_1} \Theta dw
\]

According to (35) and (36), \(p = q_5\); that is, the diagonals are of equal length. Thus we have proved:

**THEOREM II.** In every geodetically convex domain of a Stäckel region \(t_0 < t < t_1\), \(u_0 < u < u_1\), \(v_0 < v < v_1\), \(w_0 < w < w_1\), the eight geodetic diagonals are equal.

C. The Theorem of Weinacht.

The theorem proved by Weinacht and Blasche may be stated:

**THEOREM III.** The only triply orthogonal Stäckel systems in euclidean space are the confocal quadrics and their degenerations.

Because of Theorems I and II, the proof of Theorem III reduces to the task of finding all three-surface systems in euclidean space having the Ivory property. We shall prove the theorem for three dimensions before discussing the four dimensional case.

Let \((x_1, y_1, z_1) (u_1, v_1, w)\) and \((x_1', y_1', z_1') (u_1, v_1, w)\) be two sets of four points in euclidean space, \(x, y, z\) in cartesian coordinates and \(u, v, w\) in curvilinear coordinates,
with the Ivory property, and hence Stäckel coordinates. Then because of the equality of the diagonals, for two points \((x, y, z)\) \((u, v, w)\) and \((x^*, y^*, z^*)\) \((u, v, w^*)\) we have

\[
\sum (x - x_i^*)^2 - (x^* - x_i)^2 = 0 \quad (i = 1, 2, 3, 4)
\]

By subtracting one of these equations from the other three, we obtain three linear equations in \(x, y, z; x^*, y^*, z^*\). These can be solved for \(x^*, y^*, z^*\), and so we can eliminate these coordinates and obtain a linear relation between \(x, y,\) and \(z\). If we assume that the surface \(w = \text{const}\). is not a plane, this linear relation-ship can be excluded. Thus we can consider separately the three-surfaced orthogonal system which consists of the planes \(x, y, z = \text{const}\).

Under the assumption that this linear relationship does not exist, we can solve the three equations and obtain \(x^*, y^*,\) and \(z^*\) as a linear expression in \(x, y,\) and \(z\) with determinant different from zero. Then the sur-faces \(w\) and \(w^*\) are transformed affinely into each other by the relation \((u, v, w) \rightarrow (u, v, w^*)\). Finally, if we place the known values for \(x^*, y^*, z^*\) in one of the equa-tions (38), we obtain either an identity in \(x, y,\) and \(z\) or a quadratic or a linear equation in \(x, y,\) and \(z\) as the equation of the surface \(w = \text{const}\). in cartesian
coordinates. What we have shown for the surface \( w = \text{const.} \) naturally applies for the surfaces \( u, v = \text{const.} \). The parametric surfaces are then in general either quadric or plane.

Since we have already considered the case \( x, y, z = \text{const.} \) for which only planes occur, we can assume that the surface \( w = \text{const.} \) is an uneven quadric. Then the surface \( u = \text{const.} \) must also be a quadric, which according to Dupin's Theorem on triply orthogonal surfaces intersects \( w = \text{const.} \) in a line of curvature. If \( w = \text{const.} \) is not a cone or cylinder of revolution, or a sphere, then in general the line of curvature is a curve of the fourth order which is formed by the intersection of the surface with its confocal quadric surface. Through one such curve, say \( u = \text{const.}, v = \text{const.} \), there can pass no quadric surface other than \( u = \text{const.} \) which intersects the surface \( w = \text{const.} \) orthogonally along the curve \( u = \text{const.}, v = \text{const.} \); for if another such surface did exist, the surface \( u = \text{const.} \) would be tangent to it along a curve of the fourth order. Consequently, in the general case the orthogonal system consists only of confocal quadrics.

We now must consider the special cases in which the system contains only planes, spheres, and cones and cylinders of revolution.
If the surface \( w = \text{const.} \) is a cone or cylinder of revolution, then the argument we have just used shows that the plane through the axis of rotation of the surface \( w = \text{const.} \) belongs to the triply orthogonal family of surfaces, and consequently that the system is transformed into itself by rotations about the axis. As we have shown, the surface \( w = \text{const.} \) is affine to the other surfaces of this family with the same axis of rotation, and so these other surfaces are also cones (or cylinders) of revolution. Therefore the intersection of the system of surfaces \( w = \text{const.} \) with a plane through the axis of rotation forms a family of straight lines whose orthogonal trajectories are either straight lines or conic sections. However, due to the condition of orthogonality with the concentric cones of rotation, this conic section must be circular, and so the straight line family must be either a cluster of straight lines with circular orthogonal trajectories or a parallel cluster. Therefore we obtain as a triply orthogonal system either coaxial concentric cones of revolution, the planes through the axis of rotation, and the concentric spheres; or coaxial cylinders of revolution with their planes of symmetry.

If the surface \( w = \text{const.} \) is either a cylinder
(such as a parabolic cylinder) or a sphere, then the orthogonal surfaces may be determined quite simply. For the cylinders the orthogonal surfaces are the confocal cylinders and the planes orthogonal to the axes of these cylinders, since the orthogonal surfaces must intersect along lines of curvature; similarly, in the case of the sphere, the orthogonal surfaces must be either planes or cones, or both.

Finally, there remains for consideration only the case for which with the insertion of the known values of $x^*, y^*, z^*$ in (37) all the equations are satisfied identically in $x, y, z$. According to (37), the previously considered affine transformation $(x, y, z) \rightarrow (x^*, y^*, z^*)$ permits the points $x_i^*, y_i^*, z_i^*$ to be transformed back into the points $x_i, y_i, z_i$, and has the additional property of transforming each point pair $(x, y, z) (x_i, y_i, z_i)$ the same distance from the point pair $(x^*, y^*, z^*) (x_i^*, y_i^*, z_i^*)$. Consequently it preserves all distances affinely; that is, it is a congruent transformation. In addition, by this transformation the points $x_i^*, y_i^*, z_i^*$ are taken into the points $x_i, y_i, z_i$, and so this congruent transformation is involutory; that is, it is merely a reflection in a plane or in a straight line. If we move $w^*$ with
respect to \( w \), it is evident that the four points 
\( x_i, y_i, z_i \) lie in a plane. Thus we again have the case 
in which the triply orthogonal system consists only 
of planes.

This proof of the theorem could be generalized to 
four dimensions, but the geometric reasoning involved 
would be rather difficult. However, all the considera-
tions of this proof generalize directly to higher dimen-
sions. For example, Dupin’s Theorem on triply orthog-
onal surfaces has been generalized to \( n \) dimensions.\(^7\) 
For our purposes, however, a complete generalization 
is unnecessary; for if we limit our considerations to 
spaces in which the fourth variable is time, then the 
metric coefficient of this variable can be taken as 
unity. We will then have a quadruply orthogonal system 
of surfaces, and the theorem just proved applies with-
out change to the three dimensional subspace of ordinary 
space coordinates. In other words, in any time-space 
coordinate system the three dimensional space coor-
dinates must be confocal quadrics or their degenerations 
if the four dimensional coordinate system is to have 
the Stäckel form. We may expect that application of 
the generalization of Dupin’s Theorem to the proof of 
Theorem III for spaces of dimension higher than three
would, in more general cases than we shall consider, lead to generalized surfaces of revolution. The reasoning would certainly be analogous to that of the proof given for euclidean three-space.
III. EISENHART'S RESULTS

A. The Stäckel Form.

Eisenhart obtained the same results as those given in Theorem I and Theorem II, but in a manner which leads directly to the formation of the Stäckel determinants. One of his principal results is stated here without proof.

Writing (4) in a more general form we have

\[ ds^2 = \sum_i e_i H_i^2 \, dx_i^2 \]

in which \( e_i \) is plus or minus one, as the case may be. Then Eisenhart states:

A necessary and sufficient condition that (38) be of the Stäckel form is that

\[ \frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} + \frac{\partial \log H_i^2}{\partial x_j} + \frac{\partial \log H_j^2}{\partial x_i} = 0 \]

and

\[ \frac{\partial^2 \log H_i^2}{\partial x_i \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_j^2}{\partial x_j} \frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_j^2}{\partial x_k} \frac{\partial \log H_k^2}{\partial x_j} = 0 \]
Thus in a sense, in the Eisenhart discussion these equations correspond to the Ivory property presented by Blaschke. The equations are arrived at through consideration of the conditions for integrability of geodesics. He further shows that expressions satisfying these equations are given by

\begin{equation}
H_i^2 = X_i \prod_{j \neq i} (\sigma_{ij} + \sigma_{ji}) \quad (j \neq i)
\end{equation}

in which, for example, \( \sigma_{ij} \) is a function of \( x_i \) at most.

These results can be used to generalize those of the preceding section, being particularly useful in the cases for which with \( n = 4 \), \( H_1^2 \) is not constant, or for which \( n > 4 \). With the previous method either of these cases would have involved difficult geometrical considerations.

We shall consider, as before, the simplest case, namely when by suitable choice of \( x_1 \) (previously the time coordinate) we have \( H_1^2 = 1 \). It follows immediately from (41) that \( \sigma_{j1} = 0 \), for \( j > 1 \), since \( \sigma_{j1} \) is a function of \( x_j \) at most. Thus the factor \( \sigma_{1j} \) may be removed from the product (41), and we have

\begin{equation}
H_j^2 = X_j \sigma_{1j} \prod_{k \neq j} (\sigma_{jk} + \sigma_{kj}) \quad (j, k = 2, 3, \ldots n, j \neq k)
\end{equation}
We now introduce the Riemannian symbol for the quadratic form (38), with all the e's being equal to plus one. This tensor is given by

\[ R_{ijkl} = H_i^2 \left( \frac{\partial^2 \log H_i}{\partial x_i^2} + \frac{\partial \log H_i}{\partial x_j} \frac{\partial}{\partial x_j} \log \frac{H_i}{H_j} \right) \]

\[ + H_j^2 \left( \frac{\partial^2 \log H_i}{\partial x_i^2} + \frac{\partial \log H_i}{\partial x_i} \frac{\partial \log H_i}{\partial x_j} \right) + \sum_k \left( \frac{H_i H_j}{H_k} \right)^2 \frac{\partial \log H_i}{\partial x_k} \frac{\partial \log H_i}{\partial x_k} \]

(43)

In the region under consideration this must equal zero. Taking \( i = 1 \), and \( j > 1 \), we have that since \( H_1^2 = 1 \) the only non-zero term in (43) is the center term. If the factor in \( x_1 \) of \( H_j \) is given by \( f(x_1) = (a_j x_1 + b_j) \), \( a \) and \( b \) being constants, then the center term gives

\[ \frac{\partial}{\partial x_i} \left[ \frac{a_j}{a_j x_i + b_j} \right] + \left[ \frac{\partial}{\partial x_i} (a_j x_i + b_j) \right]^2 = 0 \]

We have then from (42) that \( \sigma_{1j} = (a_j x_1 + b_j)^2 \). By suitable choice of \( x_1 \) we have that either \( \sigma_{1j} = x_1^2 \) or \( \sigma_{1j} = c^2 \), \( c \) being a constant. Having obtained this, we substitute (42) into (43) for \( i, j > 1 \), and find that the hypersurfaces are of constant curvature \( \frac{1}{x_1^2} \) for the case \( \sigma_{1j} = x_1^2 \), and euclidean for the case \( \sigma_{1j} = c^2 \). Since \( H_1^2 = 1 \), these hypersurfaces are geodesically parallel. Thus we have either concentric hyperspheres or parallel hyperplanes.
These cases for which $H_1^2 = 1$ are of particular interest with regard to the solution of the wave equation, which may be reduced to the solution of the 4-dimensional Laplace equation. In this case, $H_1^2 = 1$ may be considered the metric coefficient of the time variable, and the separable systems are then just those of the three dimensional case in which $H_2^2 dx_2^2 + H_3^2 dx_3^2 + H_4^2 dx_4^2$ takes on the various forms discussed in detail in the previous sections, and also discussed in even greater detail by Eisenhart.

Generalizations of the various three dimensional cases may be obtained in this way. Thus the parallel hyperplanes and concentric hyperspheres above correspond to the three dimensional cases in which $H_1^2 = 1$. As mentioned previously, we may expect that certain non-constant values of $H_1^2$ in spaces for which $n > 3$ correspond to generalized surfaces of revolution. This is indeed the case, as is pointed out by Eisenhart. However, the only cases of physical importance for spaces in which $n = 4$ are those for which $H_1^2 = 1$; this includes of course the spaces of constant curvature.

B. Conditions for Separability.

As was pointed out in the introduction, the Hamilton-
Jacobi equation and the Laplace equation will separate in any coordinate system in which the fundamental quadratic form is that given by Stäckel, and only in these systems.

The Schrödinger equation will also separate in these systems, provided also that

$$S = \prod \frac{H_i}{f_i}$$

In his paper Eisenhart proved that this latter condition is satisfied by any Stäckel form of a euclidean space or of a space of constant Riemannian curvature. He proceeded as follows:

'From the relation $E_i^2 = \frac{S}{S^n}$ and the definition $\phi_i = \frac{S^{i\alpha}}{S^n}$ we have that

$$\frac{\partial \log S}{\partial x_j} = \frac{1}{H_j} \left( S_j' + \phi_j^2 S_{j2} + \cdots + \phi^n_j S_{j'n} \right)$$

We again use the notation that $S^{i\alpha}$ is the cofactor of the element $S_{i\alpha}$ in the determinant $S$, understanding that $S$ is such that $\phi_1$ is independent of $x_1$, and $(\phi_1 - \phi_j H_1^2)$ is independent of $x_j$. Expressed otherwise, this gives

$$\frac{\partial \phi_i}{\partial x_i} = 0 \ , \ \frac{\partial \phi_i}{\partial x_j} = (\phi_i - \phi_j) \frac{\partial \log H_i^2}{\partial x_j}$$
Therefore, using also the fact that $S_{j\kappa}$ is a function of $x_j$ at most,

$$\frac{\partial^2 \log S}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_k} \left( \frac{1}{H_j^2} \right) \left[ S'_{j1} + \epsilon_j^2 S'_{j2} + \cdots + \epsilon_j^{n-1} S'_{jn} \right] +$$

$$\frac{1}{H_j^2} \left[ S'_{j2} (\epsilon_j^2 - \epsilon_k^2) \frac{\partial \log H_j^2}{\partial x_j} + \cdots + S'_{jn} (\epsilon_j^{n-1} - \epsilon_k^{n-1}) \frac{\partial \log H_j^2}{\partial x_j} \right]$$

And since

$$\frac{\partial}{\partial x_k} \left( \frac{1}{H_j} \right) = - \frac{1}{H_j^2} \frac{\partial \log H_j^2}{\partial x_k}$$

We can combine the above to obtain

$$(46) \quad \frac{\partial^2 \log S}{\partial x_j \partial x_k} = - \frac{1}{H_j^2} \frac{\partial \log H_j^2}{\partial x_k} \left[ S'_{j1} + \sum_{\alpha=2}^{n} \epsilon_k^\alpha S'_{j\alpha} \right]$$

From the elementary properties of determinants we have the identity

$$S_{j1} + \sum_{\alpha} \epsilon_k^\alpha S_{j\alpha} = 0$$

Differentiating this with respect to $x_j$ we obtain

$$(47) \quad S'_{j1} + \sum_{\alpha} \epsilon_k^\alpha S'_{j\alpha} = \sum_{\alpha} S_{j\alpha} (\epsilon_j^\alpha - \epsilon_k^\alpha) \frac{\partial \log H_j^2}{\partial x_j}$$

where we have again used the relations (45). But since

$$S_{j\alpha} \epsilon_k^\alpha = S_{j\alpha} \frac{S_{\alpha k}}{S_k^2} = 0 \quad (j \neq k)$$
we may write this as

\[ S_{j_1} + \sum_{k} \Phi_k S_{j_k} = \left[ \sum_{k} S_{j_k} \Phi_k \right] \frac{\partial \log H_k^2}{\partial x_j} = \frac{S}{S'} \frac{\partial \log H_k^2}{\partial x_j} = H_j^2 \frac{\partial \log H_k^2}{\partial x_j} \]

Therefore (46) becomes

\[ \frac{\partial^2 \log S}{\partial x_j \partial x_k} = - \frac{\partial \log H_i^2}{\partial x_k} \frac{\partial \log H_k^2}{\partial x_j} \]

In view of (39), this gives

\[ \frac{\partial^2 \log S}{\partial x_j \partial x_k} = \frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} = \frac{\partial^2 \log H_k^2}{\partial x_j \partial x_k} \]

Now condition (3) is equivalent to the relation

\[ \frac{\partial^2 \log}{\partial x_j \partial x_k} \frac{\Pi H_i}{S} = 0 \quad (j, k = 1, \ldots, n, j \neq k) \]

since each term in the product \( \frac{\Pi H_i}{S} \) is required to be a function of only one variable. Therefore, making use of (49), condition (3) is equivalent to

\[ \frac{\partial^2 \log}{\partial x_j \partial x_k} \frac{\Pi' H_i}{S} + \frac{\partial^2 \log H_i}{\partial x_j \partial x_k} + \frac{\partial^2 \log H_i}{\partial x_j \partial x_k} - \frac{\partial^2 \log S}{\partial x_j \partial x_k} = \frac{\partial^2 \log}{\partial x_j \partial x_k} \frac{\Pi' H_i}{S} = 0 \]

in which the \( \Pi' \) indicates the product of all the \( H_i's \) except \( H_1 \) and \( H_j \).
We now make use of the Riemannian symbol for $i, j, k$ different\(^7\)

\[ R_{jik} = \frac{e_i H_i^2}{4} \left[ 2 \frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} + \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_i^2}{\partial x_k} \right. \]

\[ - \left. \frac{\partial \log H_j^2}{\partial x_i} \frac{\partial \log H_k^2}{\partial x_i} - \frac{\partial \log H_j^2}{\partial x_k} \frac{\partial \log H_k^2}{\partial x_j} \right] \]

(51)

Adding (40) to this we obtain

\[ R_{jik} = \frac{3 e_i H_i^2}{4} \frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} \]

Now\(^10\) \(g_{ij} = \frac{e_i}{H_i^2}\), and \(g^{ij} = 0\) \((1 \neq j)\)

We also know that\(^11\)

\[ R^{i}_{ik} = R^{i}_{jki} = g^{i}_{ij} R_{jik} \]

Therefore we have

\[ R_{jk} = \frac{3}{4} \frac{\partial^2 \log}{\partial x_j \partial x_k} H_i^2 \]

And so according to (50),

(52) \(R_{jk} = 0\)
We therefore conclude that condition (3) is satisfied if the fundamental metric is of the Stäckel form in a euclidean space or in a space of constant Riemannian curvature.

Thus the necessary and sufficient condition that Laplace's equation be solvable by separation of variables, with a solution of the form $\Pi x_1$ is that the metric be of the Stäckel form; for the separation of the Schrödinger equation there is the additional condition that the space be euclidean or of constant curvature.

We have shown that in euclidean 3-space the only coordinate systems satisfying these conditions are the confocal quadrics and their degenerations. We have shown further that these conditions are also satisfied in euclidean 4-space, and it is obvious from the proof in the first section that these conditions are satisfied in a euclidean space of any dimension, or in any n-dimensional Riemannian space of constant curvature, if in these spaces the Ivory property (generalized to the higher dimensions) holds.

For the particular problem of the wave equation we have seen that the separable system consists of parallel hypersurfaces.
It should be noted parenthetically that the Laplace equation can be solved by a form of separation of variables when the metric is not of the Stäckel type. The metric coefficients may take the more general form

\[ H_i^2 = \frac{S}{S_{ii}} u \]

in which \( u \) is an arbitrary function of the several variables. In this case a modification of condition (3) becomes a necessary condition for the separation of the Laplace equation, namely

\[ S = \prod H_i \frac{R^2}{f_i} u \]

in which \( R \) is another arbitrary function of the \( n \) variables. The separated solution function can no longer be expressed in the form \( \prod x_i \), but assumes the more general form

\[ \frac{\prod x_i}{R} \]

For example, Laplace's equation will separate in toroidal coordinates, which satisfy (53) and (54), but not in bipolar coordinates, which do not satisfy these condi-
tions jointly. It appears that no results have been obtained concerning what coordinate systems permit the separability of Laplace's equation so that the solution takes the form (55).\textsuperscript{13}
IV. THE WAVE EQUATION

A. The Four Dimensional Stäckel Determinant.

The equation to be separated is

\begin{equation}
\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}
\end{equation}

in which \( \nabla^2 \phi \) is to be expressed in any of the usual forms for three dimensional space coordinate systems. By making the change of variable

\[ \tau = \text{i} c t \]

we obtain Laplace's equation in four dimensions. If we choose \( x_1 = \tau \) as the time variable to insure that \( H_1^2 = 1 \), then (56) becomes

\begin{equation}
\frac{\partial^2 \phi}{\partial \tau^2} + \nabla^2 \phi = 0
\end{equation}

To separate this equation we first find the Stäckel determinant. This is very easy, since the Stäckel determinants for three dimensional space are known. Since \( H_1^2 = \frac{S}{S_{11}} = 1 \), we have that \( S_{11} = S \). Furthermore, the cofactors \( S_{11} \), for \( i = 2, 3, 4 \) in the four dimensional case will equal, respectively, the cofactors
for $i = 1, 2, 3$ in the three dimensional case if the four dimensional determinant has the form

\[
S_{(4)} = \begin{vmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & S_{(3)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{vmatrix}
\]

where $S_{(3)}$ is the Stäckel determinant for the proper system of coordinates in the three dimensional space, such as one of those listed by Eisenhart. That is, the form (58) of the Stäckel determinant applies to the wave equation (57) in which $\nabla^2 \phi$ is expressed in any of the confocal quadric coordinate systems, degenerate or otherwise.

B. Example: Parabolic Coordinates.

As a particular case, let us examine the separation of the wave equation expressed in parabolic coordinates. The space coordinate surfaces are the confocal paraboloids of revolution $x_2, x_3 = \text{const.}$ and the planes $x_4 = \text{const.}$ through the axis of rotation. As in the general case, $x_3 = \tau$. We then have

\[
\tau = x_1, \\
x = x_2 x_3 \cos x_4 \\
y = x_2 x_3 \sin x_4 \\
z = \frac{1}{2}(x_2^2 - x_3^2)
\]
The Stäckel determinant (58) takes the form

$$S = \begin{vmatrix}
1 & -1 & 0 & 0 \\
0 & x_2^2 & 1 & \frac{1}{x_2^2} \\
0 & x_3^2 & -1 & \frac{1}{x_3^2} \\
0 & 0 & 0 & -1
\end{vmatrix}$$

(60)

This determinant is, of course, not the unique solution to the problem, since the separated equations contain the usual arbitrary constants. The metric coefficients are given by

$$H_i^2 = 1, \quad H_2^2 = H_3^2 = x_2^2 + x_3^2, \quad H_4 = x_2^2 x_3^2$$

The separated equations have the form

$$\frac{1}{f_i} \frac{d}{dx_i} \left( f_i \frac{dX_i}{dx_i} \right) + X_i \sum_{j=1}^{k} S_{ij} \alpha_i = 0$$

(61)

in which the $\alpha_i$ are separation constants, and the $f_i$, according to equation (3), satisfy

$$\prod_{i} H_i = \prod f_i = x_2 x_3$$

from which we obtain

$$f_1 = f_4 = 1, \quad f_2 = x_2, \quad f_3 = x_3$$
The separated equations then become

\[ \frac{d^2 X_1}{d x_1^2} - \alpha_2 X_1 = 0 \]

\[ \frac{d^2 X_2}{d x_2^2} + \frac{1}{x_2} \frac{d X_2}{d x_2} + \left( \alpha_2 x_2^2 + \alpha_3 + \frac{\alpha_4}{x_2^2} \right) = 0 \]  

(62)

\[ \frac{d^2 X_3}{d x_3^2} + \frac{1}{x_3} \frac{d X_3}{d x_3} + \left( \alpha_2 x_3^2 - \alpha_3 + \frac{\alpha_4}{x_3^2} \right) = 0 \]

\[ \frac{d^2 X_4}{d x_4^2} - \alpha_4 X_4 = 0 \]

Here \( \alpha_1 \) has been taken as zero in conformance with the derivation of equation (61). \( ^{14} \)


8) L. F. Eisenhart, Riemannian Geometry, p. 119.


10) Ibid, p. 15, p. 43.


12) P. M. Morse and H. Feshbach, Methods of Theoretical Physics, M.I.T. Notes, Chap. I.

13) It is pointed out in reference (11) that certain coordinate systems satisfying equations (53) and (54) were given by M. Böcher in his dissertation "Über die Reihenentwickelungen der Potentialtheorie" (Leipzig, 1894). Böcher treats cyclides which are generalizations of Dupin cyclides.

14) P. Moon, Engineering Applications of Static Field Theory, M.I.T. Notes, Chap. 7.