A MATHEMATICAL SURVEY OF COMPUTING DEVICES
WITH AN APPENDIX ON AN ERROR ANALYSIS OF DIFFERENTIAL
ANALYZERS

by

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PREFACE

The purpose of this Thesis is stated in the Introduction, Chapter I. The main body of the work is compiled largely from the literature, more with a view to usefulness than originality. The Appendix is the author's development of some ideas communicated to her by Dr. S.H. Caldwell of the Electrical Engineering Department, and confirms the experience of workers with such machines.
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I

INTRODUCTION

It is hoped that the body of this thesis will serve in either one of two ways: as an introduction to the subject of computing devices for those who are newly entering the field, or as a brief guide to workers in other fields who would have an overall picture of the subject with a view to its possible bearing on their own work.

The approach is mathematical in the sense that the emphasis is on the mathematical forms which can be treated by the methods. However, sufficient description has been included to give an idea of the working principles of the machines involved. No particular attempt has been made to outline the applications of the devices in the various scientific and engineering fields. The literature on the various problems which have been treated by the methods is most extensive, and only a few applications are cited in each section as examples.
It is felt that the value of making the approach mathematical is to reveal the changing trend in the philosophy of computational methods.

We might begin by quoting an interesting passage from S. Lilley's article on Mathematical Machines (239):

"One of the reasons why mathematical machines were developed in the period after the Industrial Revolution rather than in the earlier merchant capital period was that the industrial period represented to the mathematician a much greater variety of problems which could only be solved within the given limits of space and time by the use of every available instrument. Another reason was that the Industrial Revolution had developed the necessary mechanical technique. To these must be added another reason, namely, the way in which men's minds had been altered by the circumstances in which they lived. . . . .

Such men (as Lord Kelvin) were used to thinking of the machine as typical of man's attack on the problem of living, and of a differential equation as representing the motion of a machine; and it was not difficult for them to reverse the process and make the machine represent and solve the equation. But to men of Newton's time, a differential equation represented the motion of the moon or of a projectile, neither of which could be used conversely to solve the equation. . . . . A similar ideological effect appears in the recent development of the Integral Equation and of machines for solving them. . . . . They are the mathematical method of taking into account the history of a system."
The most recent trend towards large scale digital computing, as away from the less flexible continuous devices, has a direct bearing on the researches of the pure mathematician. He must now consider to what extent his work can be furthered by these machines, and to what extent these machines need his aid if they are to produce meaningful and accurate results. A short chapter is devoted to this topic later on in the thesis.

Finally, one must consider the attitude of what might be called, for lack of a better expression, the practical mathematician. He is presented with the problem with all its requirements sufficient to define a solution, and is required to produce that solution.

A man of considerable experience in this respect is J.L. Comrie working in England. On the style of the Scientific Computing Service, Limited, London, England, he suggests the establishment of a central organization supported by public funds, equipped with all types of machines, and ready to solve any problems sent in. It is now realized that large machines in such a central computational laboratory could have teletype connections to leading research organisations, so that they could be operated from those organisations on a time allotment basis. Such proposals are already being definitely considered in this country as well as in England. See article on the Admiralty Computing Service (245).

Out of Comrie's experience in handling the hosts of
problems presented to him, he has come to the conclusion that "the full exploitation of the capabilities of the commercial calculating machines (including the National Accounting Machine and the Hollerith) is usually the most efficient way of dealing with problems, and specially designed calculating machines and instruments are necessary only for large-scale investigations of infrequent occurrence." Comrie's idea of a serviceable Differential Analyser is a mechanical one with the usual Kelvin integrating units costing about £1000 and giving say 1/2% accuracy. See the various references under his name.

But C.W.Hansel in his article on Graphical Computation (2) has gone so far as to advocate graphical methods in science and engineering when high accuracy is required. He shows by illustration how charts comply with highly accurate data more adequately than numerical methods.

Certainly this entire matter is one for careful consideration in these days of such tremendous expansion. Suffice it to say that there remain two sides to the question:

The occurrence of problems requiring a numerical or graphical solution will encourage the building of computing devices; the building of computing devices will encourage the attempts to find numerical or graphical solutions to existing unsolved problems.
A Note on the Existing Literature:

There are very few comprehensive works in our general field. By far the most comprehensive thing is a book recently written by F.J. Murray on "The Theory of Mathematical Machines" (246). Murray has very carefully examined and classified, according to #2 of page 8, the "possible mechanisms which provide information concerning the relationships between a specified set of mathematical concepts". He has not made any attempt to point out how these mechanisms have been employed to date in actual machines, but his book will be invaluable to designers of new machines. Hazen's Doctoral Thesis of 1931 (237) is particularly useful up to that date because it shows how the mathematical forms arise in practice as well as what machines have been built to handle them. His applications are largely in the field of Electrical Engineering, and his classification according to #1 of page 8 of this thesis. Seminars covering the following groups of machines: planimeters, integrators, integraphs, harmonic analysers, and differentiators, were given at M.I.T. by K.Y. Wang, N. Howitt, and J. Russell in 1929 (76).

There are several more specialized articles. "The Handbook of the Napier Tercentenary Exhibition" 1914, edited by E.M. Horsburgh (10) contains many machines of historical interest. F.S. Dellenbaugh wrote a thesis in 1921 in which he surveyed the field of Harmonic Analysis very thoroughly (88). J.S. Frame did the same for Algebraic Equation solvers in
1945 (29). D. Baxandall, Deputy Keeper of the Science Division of the Science Museum, South Kensington, England, has written many interesting articles, largely on the early digital types and on slide rules. They are to be found in the 14th. edition of the Encyclopaedia Britannica, 1929, under the heading of Calculating Machines, and of Mathematical Instruments (9, 53); and in the Catalogue of Collections in the Science Museum, South Kensington (207). A brief survey concentrating on the early digital types and on harmonic analysers was given by S. Lilley in 1942 (239). A good history of the early digital machines is included in the 1946 "Manual of Operation for a Sequence Controlled Calculator" of the Harvard University Computation Laboratory (205). F. A. Willers has written two little books on "Mathematische Instrumente" and "Graphische Integration" respectively (7) and more up to date articles in the current German literature. E. W. Crew wrote about early digital types and about harmonic analysers in 1941 (234).

L. J. Comrie, already referred to in the introduction, has written numerous articles on the subject of small-scale digital computation, as, for example, (212) and (213). See also Turck (221).

A very good understanding of mechanical differential analysers may be obtained by reading the paper of Bush and Caldwell on the latest machine at M.I.T. (137). A very interesting thesis on the applications of Differential Analyser technique is the one written by Caldwell in 1933 (138).
In order to keep up to date on the latest developments one may refer to "Mathematical Reviews" under the heading of Numerical and Graphical Methods (246); to "Mathematical Tables and other Aids to Computation" usually under the heading of Mechanical Aids to Computation (247); and to "Physics Abstracts" (248) known prior to 1941 as "Science Abstracts" Section A Physics, under several possible headings.

We have drawn our material freely from these articles, trying to concentrate their essence into one whole, bringing the information up to date and supplementing it as far as possible by references to the original literature.

It is believed that a perusal of these articles, along with references to any particular machines which may excite the reader's curiosity, will serve as an adequate complement to this thesis.
II

PROPOSED ATTACK ON SURVEY

In making any sort of overall survey of such a broad field, it is necessary to have the subject matter well organized and weighted. As one looks into the literature, one finds more and more references to specialised machines, and realises that any sort of complete catalogue of them would be most extensive and of no particular value other than historical. Our aim is therefore to give a judicious selection of examples at each stage, pointing out the salient features of each.

Any attempt at classifying existing type machines is clouded by the number of variable factors. Some possible breakdowns are:

1. From the mechanical point of view, operational (i.e. performing, as required, \( \pm, \times, \div, \int \), or \( \frac{d}{dx} \)) and constraint (i.e. equation solvers).

2. From the theoretical point of view, digital (both direct computers and adjusters) and continuous variable. (including analogue models).
3. From the historical point of view, there is much overlapping, as the more successful machines have been refined over the years.

4. From the practical point of view, according to the type of problem solved, and with what form and accuracy.

5. From the financial point of view, according to cost and therefore to availability.

We propose to follow through the continuous variable types, grouped according to type of problem, each group being listed chronologically; and then through digital types, listed chronologically. Data on existing machines will be given largely in tabular form, with reference as far as possible to the original accounts of the machines. The mathematical capabilities and implications of each group of machines will be considered in connection with the appropriate tables.

**Discrete Variable Types:**

Every field of endeavour handling great quantities of data is looking, now more than ever, to machine aids. Any such data representable in a systematic manner by letters, digits (with any radix, in particular 2, 8, or 10) by a yes-or-no code, having to be analyzed, stored, or operated upon by any mathematical process reducible to a sequence of the fundamental operations of arithmetic, is adaptable to digital computing methods. See Chapter IV for further details.
Advantages of this type of machine are:

1. The accuracy, limited only by that of the given data.
2. The flexibility, in particular, the adaptability to all algebraic, differential, integral, and transcendental equations for which a solution is determined by initial, boundary, or periodicity conditions.
3. The capacity to store data.
4. The reliability, because of readily designed checking routines.
5. The possibility of introducing sub-sequence controlled codes for the expansion of determinants, solution of systems of algebraic equations, and so on.

Disadvantages are:

1. The high cost, anywhere form $50,000 to $500,000 for the large machines now in operation or construction. However, teletype connections to the outlying laboratories would make them more generally available.
2. The need for skilled mathematicians to break down each type of problem into the form best adapted to the capabilities of the machine.
3. The labour involved in coding these sequences of operations so that they can be sent in as orders to the machine. Dr. Rubinoff of the Harvard Computation Laboratory is considering the design of a coding machine, however, which would reduce this aspect of the work to a mechanical routine.
4. The uncertainty at present existing as to the extent of the errors of truncation and rounding-off, and the reliability of certain checking routines.

5. The fact that the failure of one small element may lead to completely erroneous results, the entire machine being rendered inactive until that element is located and repaired.

Continuous Variable Types:

Less sweeping statements can be made for these machines. Their accuracy is always limited to that of a physical measurement, and subject to change with the variations of that measurement with physical conditions. Their speed varies greatly with the type of mechanism used, and can, as in the case of electronic computers, be practically instantaneous. Advantages are therefore seen to be, in general, the absence of the disadvantages outlined above for digital machines, with the possible exception of high cost.

More positive Advantages are:

1. The ease with which the parameters can be varied.
2. The variety of possible forms of input or output.
3. The ability to perform directly other than the arithmetical operations.

Disadvantages are:

1. The limited accuracy. It is usually sufficient for engineering purposes, however, and note that the new M.I.T.
Differential Analyzer usually gives an accuracy of one part in 10,000.

2. The absence of checking techniques, other than those of repetition on the same or on a different machine.

3. The inflexibility, each machine being restricted to one class of problem.

4. The limitations being imposed by the fact that each machine is essentially an assembly of like units. The order, degree or extensiveness of the problem being solved is limited by the number of these units.

5. The difficulties arising in handling more than one independent variable.

Conclusions:

From the above discussion, it is apparent that both types of devices will continue to have their uses, and no one device will be able to supplant all the rest. Of course, we have made general statements which will be modified in the next two chapters.

However, it is interesting to note that the trend of the advances in digital types, spurred on by the requirements of modern warfare, is now towards an optimum model, not so swift that the answers cannot be recorded as quickly as they are produced, not so vast that the potentialities are never fully utilized, not so intricate that the repair times become too large a fraction of the total operating time. With continuous types, the trend is toward low cost, low accuracy, high speed electronic devices.
SPECIFICATIONS

It is of some value to make a fairly complete list of the items which could possibly pertain to any one machine:

Model
Date of Completion
Inventor or designer
Location
Availability
Initial Cost
Operating Cost
Number and skill of operators required

Time efficiency ratio = \(\frac{\text{time machine is in operation}}{\text{total time operators on duty}}\)

Speed (overall, access to stored data, etc.)

Principle of operation
Generation of Functions
Input and output
Storage capacities (both internal and external)

Flexibility
Accuracy
Checking system
Error analysis
Existing Literature
III

CONTINUOUS VARIABLE COMPUTING DEVICES

The earliest comprehensive theory of continuous type machines was given by Hele Shaw in 1885. (236). He points out that any mechanism by means of which the velocity ratio between two rotating pieces can be determined at any instant or made to vary in any required manner enables two variable quantities to be dealt with numerically. He discusses many of the possibilities for obtaining $dy/dx$ and $\int y \, dx$, and gives applications. For example:

1. The disc and roller mechanism is used for
   
   i. Integration in the Moseley Constant Steam Engine Indicator.
      
   ii. Differentiation in the Ship Speed Indicator of Hele Shaw.
   
   iii. $x, +, \log x^2, x^3, x^4, \sqrt{x}, 3\sqrt{x}$ etc. as shown by the Rev. J.
       Moseley in Phil. Mag. XXX p.171. of 1847.

2. The disc, globe, and cylinder used for
   
   i. $\int f(x).g(x) \, dx$ by Lord Kelvin.
   
   ii. $\int F_1(x).F_2(x)\ldots F_n(x) \, dx$ and $d^n y/dx^n$ by Hele Shaw.

3. Models using cones instead of discs for $\iint \phi (x,y) \, dx \, dy$. 

One of the earliest comprehensive articles on graphical methods was given also by Hele Shaw in 1892 (3). He makes the following classification:

1. Addition of parallel segments - as employed in the slide-rule, shearing force, and load diagrams.
2. The addition of non-parallel segments. The principle of reciprocal figures.
3. Graphical multiplication. The rectification of areas.

He points out that the Forth Bridge, Eiffel Tower and St. Gothard tunnel were built by using graphical methods in spite of the fact that the necessary theory for such structures hadn't yet been evolved or was too tedious to apply.

Later articles under this chapter heading have been rather more specialized, and have been referred to in the note on the literature in Chapter I.

A series of interesting articles is given by M. Fry in Machine Design (235). He states that the greatest refinements in continuous mechanisms in recent years have been made in connection with gun fire-control apparatus. He describes the following mechanisms there developed, and points out that these could readily be employed as the basic elements in computational machines:

I. & II. Basic designs for addition, multiplication by a constant and by two variables, and division.

III. Cam mechanisms, for using functions mechanically
impracticable or known only as graphs. These are both two and three dimensional.

IV. (i) the two-disc integrator, invented by Hermann and Lammle in 1814 and is friction geared.

(ii) the ball integrator, on the same principle as the above, but able to handle greater torque.

(iii) Varigear, or straight gearing, where the integration is performed in a large number of finite steps, as in using the trapezoidal rule.

(iv) Integration with respect to time, achieved with a variable speed drive.

V. Differential equations and differentiation, the latter being achieved with a ball integrator and differential or, if just with respect to time, with a tachometer.

VI. Servomechanisms and their applications to continuous variable machines. This includes the Ward-Leonard speed control, and various electric and hydraulic devices.

VII. Stepping follow-ups, i.e. servos in which the drive unit is either fully energized in one direction or the other, or is cut off altogether. Such a follow-up tends to follow the controlling quantity by periodic steps.

A similar and somewhat more general treatment is given by Murray (240) in Part II of his book. An important book on "Computing Mechanisms and Linkages" by A. Svoboda (243) will soon appear.
SECTION A

ARITHMETIC AND TABULATION OF FUNCTIONS

Here, it is usual to use digital machines or books of tabulated values. However, there are a few continuous devices, some of which are of considerable practical value.

The two chief methods are graphical with the straight edge as the essential principle; and scalar, with the measured scale as the essential principle. Nomograms are, in a sense, a combination of these two principles. There are a few devices for the direct handling of the arithmetic operations.

RULERS AND SLIDE RULES

Rulers of great precision and marked with almost any scale are available.

Slide rules will always remain a tool of immense practical value. There is no need to repeat the excellent survey given in the catalogue of Collections in the Science Museum (207). There are many specialized slide rules, as for the solution of cubics and catenaries. Developments are not yet at an end. Spielrein (13) describes a slide rule to convert \( x+iy \) to \( re^{i\theta} \) and the reverse, and to perform multiplication and division in complex numbers. G.A. Hay (12) shows how to increase the accuracy in reading and setting the slide rule by getting vernier action out of the variable spacing of the logarithmic graduation.
NOMOGRAMS (4, 8, 246)

Nomographic or alignment charts are still of considerable use to engineers, and have developed out of the usual methods of solving equations by graphical representation. These charts are designed to use as few lines as possible, to interpolate along a scale rather than between curves, to avoid all unnecessary construction lines, and to show up instantly the change in one of the variables due to changes in the others.

The principle of nomographic charts consists in the representation of an equation connecting three variables, \( f(u,v,w) = 0 \), by means of three scales along three curves (or straight lines) in such a manner that a straight line (the isopleth or index line) cuts the three scales in values of \( u, v, \) and \( w \) satisfying the equation.

Lipka (4) makes a survey of some of the equations that can be so represented, showing the representation in each case, and the extension in most cases to \( n \) variables. Just as an example, we give

\[
f_1(u) = f_2(v) f_3(w)
\]
or the equivalent logarithmic form

\[
f_1(u) = f_2(v) f_3(w),
\]
both of which are representable by a Z chart.

Constructions found more recently, especially by French mathematicians, are listed in "Mathematical Reviews" under the
heading of Numerical and Graphical Methods. (240). D'Ocagne was the first to give the theory in his book entitled "Traité
de Nomographie". (11). A recent and useful text on the
subject is the one by Allcock and Jones (8).

ARITHMETICAL OPERATIONS

The best survey here is given by Murray (240), Part II.
Addition can be achieved by the superposition of electrical
potential energy differences, scalar and vector, mechanical
forces and torques, or light fluxes, linear displacements,
rotations by means of a differential, resistances, etc.

Multiplication can be realised through some physical
equivalent of one of the following forms:

1. Similar triangles.
2. \( xy = (x+a)(y+b) - ay - bx - ab \).
3. \( xy = 1/4 \left[ (x+y)^2 - (x-y)^2 \right] \).
4. \( z = xy = f(x,y) \). The planes through the line \( x=y=0 \)
   intersect the surface \( f(x,y) = \) constant in a parabola.
5. \( \sin A \cdot \sin B = 1/2 \left[ \cos(A-B) - \cos(A+B) \right] \).

Multiplication by a constant is more easily achieved, a
rotation or displacement being readily so multiplied.
There are many possible electrical set-ups, starting with
the simple wattmeter, and including the one of P.A. Borden (20).

Subtraction and division can generally be achieved by
inverting the operations for addition and multiplication,
respectively. H.D. Green has designed a square root extractor.

(30)"
CURVE SMOOTHING

A special rather interesting device called a Trend Analyzer has been built by Sreedharan Pillai of the Indian Academy of Science (14). It mechanizes the general graduation formula for smoothing of time series of T.E.W. Schumann. It consists of a wooden frame, 50 rods suspended by springs, with a ring on each of the 50 rods, bars and so on. Observed values are plotted, and the graph placed so that the rods coincide with the 50 plotted points. The rods are then adjusted through the springs so that the rings coincide with the points. A wire is passed through the rings and given a tension. It puts the rings into the smoothed position required by Schumann's Formula. The degree of smoothness depends on the thickness and elastic properties of the wires.
SECTION B

ALGEBRA

ALGEBRAIC EQUATIONS

Most of the equations in this section are of one of two types:

(A) the system of $n$ equations $\sum_{j=1}^{n} a_{ij} x_j + D_i = 0$,

(B) the polynomial equation $\sum_{j=0}^{n} c_j z^j = 0$,

distinctions being made according as the coefficients are real or complex, and according as to whether the complex roots or just the real roots can be found. The evaluation of certain algebraic forms is also considered later on.

 Upon reading through the original literature, one is amazed by the ingenuity displayed in the designs of some of the machines, and especially of some of the dynamic and hydrostatic balances.

The solving of algebraic equations has been of widespread interest for many centuries, both as a classical and as a practical problem. It is therefore not surprising that so many attempts have been made at building continuous machines to avoid the tedious numerical methods. In spite of this, even to-day, most of the practical men follow the prescriptions of the methods of Newton, Horner, or Doolittle, using desk calculators, and always in some fear of making mistakes and having to repeat great quantities of the calculations. The possible numerical methods are carefully outlined
by Lacock in his thesis (34), to which we might add the following: Doolittle's method for equations (A) as detailed by Mills (176) which has the advantage of being self-checking at each stage of the work; also P.D.Crut's method (185) for equations (A), which has the advantage of being readily extended to the case of complex coefficients.

It is interesting to note that in solving equations (A) the accuracy can always be readily improved with any method for obtaining an approximate solution. The $x_1$ are first found to the accuracy permitted by the method, say $X_1$. Then

$$\sum a_{1j}x_j + D_1 = d_1$$

say, are computed to as many figures as desired with a desk computer, or by a supplementary part of the machine, as with Mallock (37). If these values $d_1$ do not approximate to zero sufficiently well, they may be set as new constants in a set of equations with the same coefficients $a_{1j}$, of which the solutions are the errors of the first approximations. This is because

$$\sum a_{1j}X_j - \sum a_{1j}x_j = d_1$$

ie.

$$\sum a_{1j}(x_j - X_j) + d_1 = 0$$

By repeating this procedure, as high accuracy as desired may be obtained. The errors at each stage are also found to the accuracy permitted by the machine. This method is always used with the Wilbur Machine (46). Prof. Hitchcock has extended the process to include the case of complex roots, as detailed by Lacock in his aforementioned thesis, p. 31.
Quite an extensive survey of this class of machines has recently been made by J.S. Frame (29). We have, according to him, listed the machines chronologically, but in the six following natural groups:

1. Graphical and Visual
2. Kinematic Linkages
3. Dynamic Balances
4. Hydrostatic Balances
5. Electric and Electromagnetic
6. Harmonic analysis

Of the machines listed by Frame, we give only the ones which could be considered reasonably successful. The others in our list are either quite recent or else appearing only in thesis form. The many elaborate principles of operation of the earlier machines are outlined in the article by Frame, and references to the original literature there given. We shall therefore give supplementary notes only on the more modern and practical machines.

As early as 1770, Rowing considered the possibility of drawing the graph of a polynomial continuously by local motion. Mechanical difficulties ruled out the possibility of building such a machine in those days. It was not until the twentieth century that machine design was sufficiently refined to enable Wilbur to build his kinematically linked model at M.I.T.; and not until the late nineteenth century that electric circuit
theory and technique sufficiently developed to suggest to Felix Lucas the possibility of using electronic devices. In fact, it is very interesting to trace through his six papers and see how his approach gradually passed from mechanical principles, through a generalization of Rolle's Theorem, to electrical principles.

Recent developments have chiefly occurred amongst the last two classes of the six. They are essentially electrical, because of their speed, ability to handle complex numbers if required, and the fact that tremendous accuracy is not required in view of the several methods available for improving it once an approximate solution is obtained. They can be divided into the two groups:

Group I: Network Equation Solvers.

For solving type (A) equations, a very compact instrument has been made available commercially by Clifford and others, of the Consolidated Engineering Corporation of Pasadena, California (17). It uses the Gauss-Seidel or Classical Iterative Method, the cycle being repeated until the variables remain the same over two successive cycles. The conditions for convergence are discussed from a practical standpoint. To solve 12 simultaneous equations takes from one quarter to one seventh the time on a desk calculator.

For solving type (B) equations, there are machines built on the principle outlined by Hazen in his thesis pp.169-175 (237).
We quote his summary of his idea:

"... Use n phase-shifting transformers plus one static transformer for the constant term, from each of which is derived an alternating potential difference representing the angle and magnitude of one term of the right hand side of the equation

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n. \]

These potential differences are added by connecting the units in series to give \( f(x) \). A high impedance vibration galvanometer or other indicating instrument is connected to read the sum \( |f(x)| \) or, if desired, the vector \( f(x) \) can be read off by a simple null method. The roots then correspond to \( f(x) = 0 \).

From these ideas, put into practice as outlined on the following page, Hazen went on with his work in network analysis, eventually designing and constructing the Network Analyser. This machine ( ) properly belongs in the section on Analogue Models, since, in practice, the equations actually being solved are of no explicit importance. However, it is interesting to note that the Network Analyser actually solves the particular set of simultaneous linear algebraic equations

\[ \sum_{j=1}^{n} Z_{ij} I_j = E_i \]

where \( i = 1, 2, \ldots, n \).

and \( Z_{ij} = Z_{ji} \),

the \( Z, I, \) and \( E \) having their usual electrical engineering significance of impedance, current strength, and potential difference, respectively.
The outstanding example is the Hart and Travis instrument at the University of Pennsylvania. (31). Two compromise machines have been built at M.I.T. by Cabaniss (26) and Lacock (34) respectively. They are cheaper but not as accurate as the Moore School model, using Selsyn motors to represent phase shift, the output voltage of these motors not being quite regular. Cabaniss gets 2% accuracy, which he can refine by a semi-mechanical method to 0.15%, or by a numerical method of successive approximation which is unlimited in its accuracy. He also gives the error theory in detail. Lacock gets a comparable accuracy. Both machines are open to improvement.

Group II: Harmonic Analysis Equation Solvers.

So far, these will only handle type (R) equations.

The Isograph, built by T.C.Fry of the Bell Telephone Company, (27) and the Harmonic Synthesizer-Analyser built by S.L.Brown (22) are notable. Both solve \( f(re^{i\theta}) = 0 \) by summing separately the sin and cos terms in \( f(re^{i\theta}) - c_0 \) and mapping this function for fixed "r" as a curve in the complex plane.

In passing, we note that the S.L.Brown instrument has been used by its inventor and Wheeler to treat a more extensive class of problems (23). This includes the solution of type (B) equations with complex coefficients as well as real ones, and also the following:
The drawing of Lissajou's figures and their application (24), the solving of equations like
\[ x + \sin x = 3 , \]
\[ \log \left( 2x^3 + 5x^2 + 10x + 15 \right) = 5/x , \]
\[ 3 \tan^2 x + 2 \cos^2 x + \sin x + \cosec x = 5 , \]
and the drawing of the contour map for the surface
\[ A(x,y) = \sum_{k=0}^{n} \sum_{h=0}^{m} a_{hk} \cos 2\pi hx \cos 2\pi ky . \quad (25) \]

Another, somewhat older, but fairly flexible machine in this group of algebraic equation solvers is the Russell and Wright machine (42). This device traces electrically, by means of a mirror galvanometer spot, the curves of \( y=\sum a_i x^i \)
and of \( y=\int_0^x f(x) \, dx \). Hence it can also be used for Harmonic Analysis. Russell and Wright's circuits can be extended to solve equations in one variable with various functions \( F(X) \) if \( F \) and \( X \) are tabulated, and to solve some transcendental equations like \( \sum a_i e^{b_i x} = 0 \) where the \( a_i, e_i, b_i \) are given constants.
ALGEBRAIC EQUATION SOLVERS

Notes on the tables to follow:

Under (A) is listed the number of unknowns in type (A) equations, viz. \( \sum_{j=1}^{n} a_{ij} x_j + D_i = 0 \).

Under (B) is listed the degree of the polynomial equation of type (B), viz. \( \sum_{j=0}^{n} c_{i} z^j = 0 \).

A fixed number means that any number up to and including that number can be handled by the machine.

The general integer \( n \) means that the apparatus can readily be extended to handle any reasonable number of unknowns or terms as the case may be.

R is an abbreviation for real.

C is an abbreviation for complex.
<table>
<thead>
<tr>
<th>PRINCIPLE</th>
<th>NAME</th>
<th>REF.#</th>
<th>LOCATION</th>
<th>YEAR</th>
<th>ACCURACY</th>
<th>ROOTS</th>
<th>COEFFTS. (A)</th>
<th>COEFFTS. (B)</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphic and</td>
<td>Lagrange</td>
<td>29</td>
<td>France</td>
<td>1795</td>
<td>Graphic</td>
<td>R</td>
<td>R</td>
<td></td>
<td>n Many construction lines</td>
</tr>
<tr>
<td>Visual</td>
<td>Lill</td>
<td>29</td>
<td>France</td>
<td>1867</td>
<td></td>
<td>R</td>
<td>R</td>
<td></td>
<td>n Fewer construction lines</td>
</tr>
<tr>
<td></td>
<td>Cunynghame</td>
<td>29</td>
<td>England</td>
<td>1886</td>
<td></td>
<td>C</td>
<td>R</td>
<td></td>
<td>n For $x^n+ax+b=0$ only</td>
</tr>
<tr>
<td></td>
<td>Mehmke</td>
<td>29</td>
<td>Germany</td>
<td>1893</td>
<td></td>
<td>R</td>
<td>R</td>
<td></td>
<td>4 If 4 curved scales in space are cut by a plane, the 4 readings satisfy a functional relationship.</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>5 In fact, only solves $x^2+ax^2+bx^2+cx+d=0$</td>
</tr>
<tr>
<td>Kinematic</td>
<td>Blashforth</td>
<td>16</td>
<td>England</td>
<td>1892</td>
<td>Low</td>
<td>R</td>
<td>R</td>
<td></td>
<td>n</td>
</tr>
<tr>
<td>Linkages</td>
<td>NaeBauer</td>
<td>29</td>
<td>Germany</td>
<td>1910</td>
<td></td>
<td>R</td>
<td>R</td>
<td></td>
<td>n Same troubles of constraint and approximation to gear ratios as in the early differential analyzers.</td>
</tr>
<tr>
<td></td>
<td>Muirhead</td>
<td>29</td>
<td>Scotland</td>
<td>1912</td>
<td></td>
<td>R</td>
<td>R</td>
<td></td>
<td>n Same principle as Lagrange.</td>
</tr>
<tr>
<td></td>
<td>Wilbur</td>
<td>46</td>
<td>M.I.T. (U.S.A.)</td>
<td>1936</td>
<td>Usually 1% of largest unknown</td>
<td>R</td>
<td>R</td>
<td>9</td>
<td>Gets 3 significant figures in 1 to 3 hours, where desk calculator takes 8 hours, now in disuse.</td>
</tr>
<tr>
<td></td>
<td>Vidal</td>
<td>45</td>
<td>France</td>
<td>1936</td>
<td>Low</td>
<td>R</td>
<td>R</td>
<td></td>
<td>n</td>
</tr>
<tr>
<td>PRINCIPLE</td>
<td>NAME</td>
<td>REF.#</td>
<td>LOCATION</td>
<td>YEAR</td>
<td>ACCURACY</td>
<td>ROOTS</td>
<td>COEFFTS.</td>
<td>REAL or COMPLEX</td>
<td></td>
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<td>-----------------</td>
<td></td>
</tr>
<tr>
<td>Dynamic Balances</td>
<td>Lalanne</td>
<td>29</td>
<td>France</td>
<td>1840</td>
<td>Low</td>
<td>R</td>
<td>R</td>
<td>n Reproduces product moments whose sum a ( x^n ) is in equilibrium.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Kelvin</td>
<td>29</td>
<td>England</td>
<td>1878</td>
<td></td>
<td>R</td>
<td>R</td>
<td>Merely proposed.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Exner</td>
<td>39</td>
<td>Germany</td>
<td>1881</td>
<td></td>
<td>C</td>
<td>R</td>
<td>n Weights placed on spiral curves to represent the complex numbers.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Boys</td>
<td>29</td>
<td>England</td>
<td>1886</td>
<td></td>
<td>R</td>
<td>R</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Grant</td>
<td>29</td>
<td>U.S.A.</td>
<td>1896</td>
<td></td>
<td>R</td>
<td>R</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Skutsch</td>
<td>28</td>
<td>Germany</td>
<td>1902</td>
<td></td>
<td>R</td>
<td>R</td>
<td>n Kinematic circular balance.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Peddie</td>
<td>29</td>
<td>England</td>
<td>1912</td>
<td></td>
<td>R</td>
<td>R</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Many others</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>n</td>
<td></td>
</tr>
<tr>
<td>Hydrostatic Balances</td>
<td>Delmanet</td>
<td>29</td>
<td>France</td>
<td>1898</td>
<td>Low</td>
<td>R</td>
<td>R</td>
<td>3 Only solves ( x^3 + x = c ) 0 or ( x^3 - x = c ) 0. Upwards forces proportional to volumes displaced by suitably chosen solids.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Meslin</td>
<td>29</td>
<td>France</td>
<td>1900</td>
<td></td>
<td>R</td>
<td>R</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td>PRINCIPLE</td>
<td>NAME</td>
<td>REF. #</td>
<td>LOCATION</td>
<td>YEAR</td>
<td>ACCURACY</td>
<td>ROOTS</td>
<td>COEFFTS.</td>
<td>REAL or COMPLEX</td>
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</tr>
<tr>
<td>Hydrostatic balances cont'd.</td>
<td>Skutsch</td>
<td>29</td>
<td>Germany</td>
<td>1902</td>
<td>Low</td>
<td>R</td>
<td>R</td>
<td>$\sum a_n (p+a_n q)^n=0$ where $q=p/c$ and the roots are found as functions of $q$.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fuchs</td>
<td>29</td>
<td>Germany</td>
<td>1914</td>
<td>&quot;</td>
<td>R</td>
<td>R</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Schumann</td>
<td>43</td>
<td>England</td>
<td>1940</td>
<td>Fair</td>
<td>R</td>
<td>R</td>
<td>n Also used in computing multiple correlation coefficients.</td>
<td></td>
</tr>
<tr>
<td>Electric and electromagnetic</td>
<td>Lucas</td>
<td>35</td>
<td>France</td>
<td>1888</td>
<td>Fair</td>
<td>C</td>
<td>C</td>
<td>n See text.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Kann</td>
<td>29</td>
<td>Germany</td>
<td>1902</td>
<td>&quot;</td>
<td>R</td>
<td>R</td>
<td>n Electric counterpart of Lalanne. Variable resistances give moments of weights.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Russell</td>
<td>42</td>
<td>England</td>
<td>1909</td>
<td>1%</td>
<td>C</td>
<td>R</td>
<td>n Multiplication by electric slide-rule i.e., by logarithmic slide resistances.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wright</td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>Russell</td>
<td>41</td>
<td>England</td>
<td>1909</td>
<td>1%</td>
<td>C</td>
<td>R</td>
<td>n Lucas' idea, but using magnetism also. n-l ammeters, n-l rheostats.</td>
<td></td>
</tr>
<tr>
<td>PRINCIPLE</td>
<td>NAME</td>
<td>REF.#</td>
<td>LOCATION</td>
<td>YEAR</td>
<td>ACCURACY ROOTS</td>
<td>REAL or COMPLEX COEFFTS. (A) (B)</td>
<td>REMARKS</td>
<td></td>
<td></td>
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<td>------------------------------------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Electric and Electromagnetic cont'd.</td>
<td>After Mallock</td>
<td>37</td>
<td>England</td>
<td>1933</td>
<td>0.1% of largest root</td>
<td>R R 10</td>
<td>Built by Cambridge Inst. Co. Transformers wound partially for each term involved. Also considers n eqs. in 3 unknowns, n&gt;3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bode</td>
<td>19</td>
<td>Germany</td>
<td>1937</td>
<td>2.5% of largest unknown</td>
<td>R R n</td>
<td>An electrical network.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lacock</td>
<td>34</td>
<td>M.I.T.</td>
<td>1938</td>
<td>2 %</td>
<td>C R</td>
<td>n 30 min. to locate each pair of roots. 30 to 60 min. to refine them by Hitchcock's method. See p. 26.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Berry, Clifford &amp; others</td>
<td>17</td>
<td>Calif.-ornia (U.S.A.)</td>
<td>1946</td>
<td>0.04% of sum of unknowns</td>
<td>R R 12</td>
<td>Also evaluates symmetric matrices. Uses iterative methods and electric circuits. Handles 3 or 4 figure coefficients. See p. 24.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hughes &amp; Wilson</td>
<td>33</td>
<td>Harvard Univ. (U.S.A.)</td>
<td>1946</td>
<td>1 %</td>
<td>R R n</td>
<td>Built to solve secular equations, but could in principle be used to solve systems of alg. eqs. See p. 35.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PRINCIPLE</td>
<td>NAME</td>
<td>REF.#</td>
<td>LOCATION</td>
<td>YEAR</td>
<td>ACCURACY</td>
<td>ROOTS</td>
<td>COEFFTS.</td>
<td>(A)</td>
<td>(B)</td>
</tr>
<tr>
<td>-------------</td>
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</tr>
<tr>
<td>Analysis</td>
<td></td>
<td>190</td>
<td>N.Y., U.S.A.</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>22</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wheeler</td>
<td></td>
<td></td>
<td>(U.of Texas)</td>
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</tbody>
</table>
EVALUATION OF ALGEBRAIC FORMS

Some of the machines already listed can readily be adapted to evaluate certain algebraic forms. For example, the machine of Schumann (43) was designed to evaluate \( k_2, k_3, \Sigma xy, \) and \( \Sigma xz \) directly, where \( k_2 \) and \( k_3 \) are the partial regression coefficients of \( x \) on \( y \) and \( z \); the small letters indicating the departures of the independent variables \( Y \) and \( Z \) and of the dependent variable \( X \) from their respective average values.

For completeness, we note that the equations of linear regression are

\[
    x_i = k_2 y_i + k_3 z_i + d_i, \quad i = 1, 2, 3, \ldots n,
\]

where the variables take on each of the corresponding \( n \) values.

The condition that \( \sum_{i=1}^{n} d_i^2 \) is minimum determines \( k_2 \) and \( k_3 \).

The multiple correlation coefficient \( R \) can then be found from the formula derived by Schumann, viz.

\[
    R^2 = (k_2 \Sigma xy + k_3 \Sigma xz)/\Sigma x^2.
\]

Poggi (39) has built a graphical-mechanical machine for evaluating \( \sum_{1}^{n} a_m x^m + a_0 \) by evaluating \( \sum_{1}^{n} A_m \cos ma + A_0 \).

The apparatus gets very cumbersome for \( n \) much more than 4 or 5.

Evans and Feiser (28) have mechanized the form

\[
    2f \cos(hx-ky) \quad \text{which occurs as a structure factor, \( h \) and \( k \) being integers. The average error is \( f/75 \). \( hx \) and \( ky \) are given by a pair of disc and roller multipliers; the difference by a differential; and the sinusodial motion by a Scotch cross-head.}
\]

Merril (38) has built a slide-disc calculator to compute root mean square values, and therefore, in particular, standard deviations which are of the form \( (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} / n^{1/2} \).
MacDougall and Wilson (36) built a mechanical analyzer with an application in molecular physics. A set of coupled harmonic oscillators solve the numerical secular equations occurring in the theory of vibrations of polyatomic molecules and elsewhere. Each oscillator consists of a shaft with cross-arms carrying weights to increase the moment of inertia and springs to provide a restoring force. The unit is free to rotate about the axis of the shaft, except for the restoring force of the springs. The various units are connected to one another by additional springs. In operation, the springs and weights are adjusted to values which represent the coefficients of the secular equations to be solved; the whole device is driven by an eccentric of variable speed, and the natural frequencies, equal to the roots of the equation, are measured.

Hughes and Wilson (33) have recently built a special electrical network for solving secular equations, particularly the ones occurring in Quantum Mechanics. The basic circuit for solving

\[
\begin{vmatrix}
-xB_{1k} & B_{12} \\
B_{21} & -xB_{2k}
\end{vmatrix} = 0
\]

is as alongside. The current is varied at P or at Q. The circuit is readily extended for solving higher order equations.
SECTION C

DIFFERENTIATION

From the mathematical form, it is generally easy to find and evaluate the mathematical form of the derivative. From graphical or tabular data, the derivatives may be found mechanically, or by using one of the many formulae of finite difference theory. Hence, there is no great need to devise differentiating machines, and few have been devised. From a practical point of view, it is difficult to build good differentiators because a slight error in following the curve leads to large errors in the derivative. The opposite is true in the case of integration.

We now give a brief outline of the history:

1890 (circa) Hele Shaw
(a) Using disc, screw, two wheels.
(76)
(b) Using Kelvin's disc, ball and cylinder inversely. Not too effective.

1904 (76) Murray Using system of rails, rulers on drawing board.

1916 (48) Elmendorf The same principle as Murray, but using a mirror instead of the celluloid strip, viz. the tangency of a slope-taking element is judged by viewing the curve continued back on itself in a mirror held normal to the plane of the paper. The mirror is continually adjusted as the curve is traversed, so that the curve shows no break as it enters the mirror.
In his book (240) Murray outlines several differentiating principles, but points out that there are serious practical difficulties involved in using them in devices.

Dauphine outlines the following possibilities in his Thesis of 1936 (47) for drawing the derivative curve of a graphical function: Condenser, optical, bending of beams, inverse of differential analyzer arrangement, transformer, perpendicular telescope, and Prof. N. Searé's maximum density of a photographic line method. He goes on to propose and recommend for construction designs for the optical and condenser methods. The other methods would require more refined treatment before being built.

In 1942, Pekeris and White of M.I.T. (49) noted that the usual processes of numerical differentiation involve the summation of neighbouring ordinates to which weighting factors have been applied. They therefore extend the method to the computation of continuous numerical derivatives. The polynomial approximation of the function to be differentiated is replaced by a development into a finite series of a set of orthogonal functions. By suitable choice of functions, the integrals required for evaluation of coefficients are put into the form

\[ R(x) = \int_a^b P(x z) Q(z) dz. \]

This integral is of the general form that can be evaluated with the cinema integragraph. (See Section D). With the M.I.T. cinema integragraph, error is 1/2 to 1% of the integral of the absolute values of the functions to be integrated.
Three electronic differentiating circuits were perfected by Schmitt and Tolles in 1942 (57). Over a wide range of frequency and wave form they each generate output voltages accurately proportional at every instant to the derivative of their input potentials.
SECTION D

INTEGRATION

This is the largest section, and falls into two main groups — evaluation of integral forms, and the solution of Integral and Differential Equations. First, we shall outline the evaluation of particular integrals, then the evaluation of general integrals, which is the transition to the solving of integral and differential equations.

Instruments which perform integrations and operate on graphs are usually classified as Mathematical Instruments. There is no clear-cut distinction between these graphico-mechanical devices and continuous devices in general. They are very extensively treated in the literature; descriptively by Morin (68), Galle (58), Horsburgh (10), and Baxandall (207), and theoretically by Murray (240) Part IV. There are many other books and papers on the subject, which includes Planimeters, Integrometers, Integraphs, and Harmonic Analyzers.

Machines which solve integral equations are generally called integraphs in a more general sense, since they usually operate on some other principle than the graphico-mechanical.

Machines which solve differential equations are usually called differential analyzers, after the first one built and so named by Vannevar Bush at M.I.T.

In order to illustrate the fact that the classifications
are by no means absolute, we show that the following group of integral equations can be solved by means of a differential analyzer. Our reference is pp. 4 & 5 of Lovitt (1903):

Consider this group of Volterra integral equations of the second kind:

\[ u(x) + \int_0^x \left[ a_1(x) + a_2(x)(x-t) + \ldots + a_n(x) \frac{(x-t)^{n-1}}{(n-1)!} \right] u(t) dt = f(x). \]  

Take a set of constants \( c_i \), \( i=1,2,\ldots,n \), all \( =0 \).

Then in Lovitt's notation, \( f(x) = \emptyset(x) \).

We can now solve with a differential analyzer

\[ \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n(x) y = \emptyset(x), \]  

with the initial settings \( y=dy/dx=d^2y/dx^2=\ldots=d^{n-1}y/dx^{n-1}=0 \),

to get, say, a table of values of \( y \) against \( x \).

One can then find a table of values of \( u(x) = \frac{d^n y}{dx^n} \) by using one of the finite difference formulae, or feed in \( y \) from its graph and use the differential analyzer to solve

\[ u(x) = \frac{d^n y}{dx^n} \]  

provided \( n \) is not too large.

Hence all equations of form (i) can be solved by a differential analyzer.

Conversely, whenever we solve (ii) with a given set of initial conditions, we solve (i), where \( f(x) \) is determined as in Lovitt (1903) by the initial conditions. See example p. 8.

A mathematical classification of this group of devices is therefore not too rigorous, and we shall simply point out the possible extensions to the applications of each device as it occurs.
PLANI METERS

These are instruments for measuring area. They give the mathematical form \( \int_a^b f(x)dx \).

Operating directly on the area are the following:

1. Glass ruled into squares.
2. The "harp" planimeter, consisting of many threads strung parallel to one another on a frame, to form \( \sum_{i=1}^{n} y_i \Delta x \).
3. Planimeters on the roller wheel principle to form \( \sum_{i=1}^{n} y_i \Delta x \).
4. Conversion planimeters, which convert a polygon into a triangle of the same area.
5. Weighing methods, in which a replica of the area is simply cut out of some material of uniform density and thickness and weighed.
6. Optical methods, in which the total illumination passing through a mask of the area is measured by a photo-tube. This is extended in the cinema integrator to enable the solution of more complex problems.

Operating directly on the perimeter are instruments with an integrating wheel (polar—if one end of the tracing arm is constrained to move in a circle; linear—if in a straight line) and instruments without an integrating wheel. We list the outstanding examples in historical order.
<table>
<thead>
<tr>
<th>Date</th>
<th>Name</th>
<th>Location</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1814</td>
<td>Hermann and</td>
<td>Munich</td>
<td>In these first three instruments, the area is measured as the revolutions of a roller working by frictional contact on a disc or cone. The distance of the roller from the centre of the disc or apex of the cone being always proportional to the breadth of the figure, the revolution of the cone or disc depending also on the length of the figure.</td>
</tr>
<tr>
<td></td>
<td>Oppikofer</td>
<td>Berne</td>
<td>(76)</td>
</tr>
<tr>
<td>1836</td>
<td>Ernst</td>
<td>Zürich</td>
<td>Commercial model developed out of the above and working on the same principle.</td>
</tr>
<tr>
<td></td>
<td>(76)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1849</td>
<td>Welti</td>
<td>Zürich</td>
<td>Disc planimeter, a modification of the above disc and roller types.</td>
</tr>
<tr>
<td></td>
<td>(76)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1856</td>
<td>Amsler</td>
<td>Königsberg</td>
<td>The most successful of the earlier instruments working on the following principle: a roller, moved obliquely across a smooth surface, turns through an amount corresponding to the distance it has moved perpendicular to the axis. Hence the breadth of a figure being measured by the inclination of a bar carrying the roller, and the length being measured by the distance which the wheel moved, the actual turning of the roller gives the area.</td>
</tr>
<tr>
<td></td>
<td>(76)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1882</td>
<td>Hohmann</td>
<td>Bamberg</td>
<td>Precision polar planimeters, like the Amsler, but not depending on the smoothness of the paper.</td>
</tr>
<tr>
<td></td>
<td>(76)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1887</td>
<td>Prytz</td>
<td>Copenhagen</td>
<td>Hatchet planimeter. Very simple and easy to use. The theory gives the area approximately, but to the same order as in practice. See also next page.</td>
</tr>
<tr>
<td></td>
<td>(76)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1919</td>
<td>Hulka</td>
<td>Germany</td>
<td>For the mechanical evaluation, in Cartesian coordinates, of $\int y , dx$, $1/2 \int y^2 , dx$, and $1/3 \int y^3 , dx$.</td>
</tr>
<tr>
<td></td>
<td>(62)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1919</td>
<td>Naatz</td>
<td>Germany</td>
<td>A rectangular frame, with two straight edges pivoted at two adjacent corners AB of the frame. A transparent celluloid strip is arranged to slide along the frame parallel to the side AE, and has a central line with two cross-lines.</td>
</tr>
<tr>
<td></td>
<td>(49)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DATE</td>
<td>REF. #</td>
<td>NAME</td>
<td>LOCATION</td>
</tr>
<tr>
<td>------</td>
<td>--------</td>
<td>---------------</td>
<td>--------------</td>
</tr>
<tr>
<td>1924</td>
<td>73</td>
<td>Beneze-Wolf</td>
<td>Germany</td>
</tr>
<tr>
<td>1933</td>
<td>60</td>
<td>Weber-Kern</td>
<td>Germany</td>
</tr>
<tr>
<td>1934</td>
<td>59</td>
<td>Gradstein</td>
<td></td>
</tr>
</tbody>
</table>

**A Note on the Hatchet Planimeter:**

This is such a simple and useful instrument that we shall describe it further. It consists of a rod bent down at right angles at either end, the rod being about 9" long and the end pieces about 3". One end is pointed, the other is shaped to a knife edge lying parallel to the rod. Call the pointed end P and the knife edge Q. Q can only move freely along the line PQ. When P traces the given curve, Q will describe a curve such that PQ is always tangent to it. Start with P at G, the approximate centre of gravity of the area, move P along the radius vector to the curve, completely
around the curve and back along the same radius vector to G. Then the area is, as shown by Hill (61), approximately \((PQ)^2\varphi\) where \(\varphi\) is the angle between the initial and final positions of \(PQ\). The error is within that of experiment in measuring areas up to 23 square inches for a 10" rod.

The hatchet planimeter can be used to solve differential equations of the type \(d\theta/dt = \left[\varphi(t) - \theta\right]/c\) with \(c\) constant, such as arise in heat conduction problems. Callender (57) gives the following directions: the hatchet is placed on the graph on which \(\varphi(t)\) is plotted, and the pointer is made to follow the \(\varphi\) contour. The tracing hatchet then traces a curve satisfying \(\sin(\tan^{-1} d\theta/dt) = \left[\varphi(t) - \theta(t)\right]/L\).

If \(L \gg \text{var}[\varphi]\) we have approximately \(d\theta/dt = (\varphi - \theta)/L\).

Dora Wehage (163) has shown how polar planimeters can be used to treat the following problems of A. Multiple Integrals (1 to 5) and of B. Partial Differential Equations (6 and 7):

1. Angular moments.
2. Polar moments of inertia for rotating bodies.
3. Degrees of accuracy.
4. Mean geometrical distances.
5. Radiation of heat from a square and from a cylinder.
6. Asymptotes of a pseudosphere.
7. Telegraphic equations with given initial and boundary conditions.
INTEGROMETERS

These are linear planimeters extended to evaluate \( \int_c^{} y^2 dx \), \( \int_c^{} y^3 dx \), and so on.

INTEGRATORS (DEFINITE INTEGRALS)

This is rather a general term which we shall use here to denote instruments giving various other definite integrals.

1. H.L. Hazen, in a memorandum on line integrals of 20th. March, 1931, shows how \( \oint f(z) dz \) can be evaluated by using a special set-up on the differential analyzer, where \( z \) is a complex variable, and \( f(z) \) is a function of a complex variable.

2. Lorenz describes an instrument for the mechanical calculation of surface moments in his paper of 1938. (66).

3. Nyström evaluates the area of a surface \( \iint \sec \theta \, dx \, dy \), where \( \theta \) is the angle between the normal and the vertical, as follows: the surface is assumed to be given graphically by its level curves, with constant differences of elevation between the levels. \( \sec \theta \) is determined graphically by a set of lattice points by means of a transparent piece of celluloid with rectangular rulings. Using these values as new ordinates, the volume of the solid formed is approximated to by the usual methods. This in 1940 (72).

4. In 1941, Berger (54) described a moment planimeter which measures the product of inertia \( \int xy \, dA \) by measuring the moments about the two axes passing through the origin but inclined at angle \( \theta \) to the \( x \) axis. The mathematical
details show how to locate the centre of gravity.

5. Nyström had an instrument for the evaluation of Stieltjes integrals $\int f(x) \, d(g(x))$ in 1936 (7). It is described by Laurila in 1940 (64) who shows that it really consists of an attachment for the Ott analyzer; See page 48. The curve for $g(x)$ is on a drawing board which is supported by and moves with one of the carriages of the analyzer. The curve may be followed either manually or by means of a template. Such integrals as $\int \int f(t) \, dt$, $\int F(y(x)) \, dx$, $\int \int x^n \, dx \, dy$, $\int y^2 \, dx$, $\int y^{1/2} \, dx$ have been obtained.

INTEGRAPHS

These are instruments for drawing the graph of a function for which the derivative is given; or, in other words, for drawing the integral curve of a function. More generally, they give the function $F(x) = \int_a^x f(x) \, dx$ when the function $f(x) = F'(x)$ is given.

<table>
<thead>
<tr>
<th>DATE</th>
<th>REF.#</th>
<th>NAME</th>
<th>REMARKS</th>
</tr>
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<tbody>
<tr>
<td>1836</td>
<td>76</td>
<td>Coriolis</td>
<td>Mechanical, involving cylinder, template and string.</td>
</tr>
<tr>
<td>1861</td>
<td>76</td>
<td>Zmurko</td>
<td>An extension of the Gonnella-Welti disc and wheel planimeter principle.</td>
</tr>
<tr>
<td>1875</td>
<td>76</td>
<td>Thomson (James)</td>
<td>The brother of Lord Kelvin. Devised the sphere, cylinder and disc integrating unit.</td>
</tr>
<tr>
<td>1875</td>
<td>76</td>
<td>Kelvin</td>
<td>Showed how to use his brother's device in the solving of differential equations.</td>
</tr>
<tr>
<td>DATE</td>
<td>REF.#</td>
<td>NAME</td>
<td>REMARKS</td>
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<td>-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1878</td>
<td>52</td>
<td>Abdank Abakanowicz</td>
<td>Developed Coriolis' idea into the model most used in practice to-day. The input for x is a two-wheeled carriage which rolls parallel to the x axis and which carries the entire instrument; the input and output units for the dependent variables are small carriages carrying pointers and pencils, respectively, which roll on tracks attached to the main carriage; the integrator is a friction wheel which constrains the motion of the output pencil.</td>
</tr>
<tr>
<td>1878</td>
<td>74</td>
<td>C.V.Boys</td>
<td>Developed a similar instrument in England.</td>
</tr>
<tr>
<td>1890</td>
<td>76</td>
<td>Frères</td>
<td>Used disc and roller principle.</td>
</tr>
<tr>
<td>1922</td>
<td>63</td>
<td>Karapetoff</td>
<td>A more recent graphico-mechanical device based on the principle of parallel double tongs.</td>
</tr>
<tr>
<td>1926</td>
<td>75</td>
<td>Robb</td>
<td>A sharp, rounded edge pressed against a plane can move freely in a direction tangential to itself, but not freely in a transverse direction. With suitable guidance, such an edge can be made to trace out a curve. Here, this principle is realized as a small sharp-edged wheel turning on an axle replaces the knife-edge.</td>
</tr>
<tr>
<td>1941</td>
<td>74</td>
<td>Pascal</td>
<td>A modification of the Abdank Abakanowicz integrator so as to solve certain types of first and second order differential equations. It is more compact but less flexible than a differential analyzer. The modifications consist of curved tracks, cams, and parallelograms to act as linkages between integrator and input and output units.</td>
</tr>
</tbody>
</table>
INTEGRATORS (INDEFINITE INTEGRALS)

We now list the machines which evaluate the indefinite integral \( \int_a^x f(x) \, dx \). The value is given as a reading on an integrator wheel as \( x \) changes continuously, rather than being given as the graph of a function.

1931 77 Salomon
A mechanical device to find the area under a curve wrapped around a cylinder. 
\( \varphi = px, \cos \theta = qy, \) so that

\[
A-A = - \int \cos \theta \, d\varphi = -pq \int y \, dx,
\]

and the problem is reduced to measuring an angular rotation as the pointer follows the curve.

1939 81 Werkmeister
( Ott Integrator)

See figure alongside. The ends \( E \) and \( D \) move in a straight line in a guide \( EF \), and rollers are fitted at \( A \) and \( C \) to trace the curves.

PRODUCT INTEGRAPHS

Fundamentally, these machines evaluate \( \int_a^b \) or \( \int_a^x \) \( f(x)g(x) \, dx \).

It is evident at once that a great many of the integrals commonly occurring in practice come under this heading, for example: Formulae for moments of inertia, centres of gravity, Fourier Coefficients and so on. As a consequence, most machines have been adapted to handle just one of the special forms, and some of these have already been listed. We now give the general Product Integraphs, followed by a sub-section on Harmonic Analyzers. Extensions for the solution of differential and integral equations are indicated.
NAME and DATE
Kelvin 1876

LOCATION and REF. #
England 79

ERROR PRINCIPLES OF OPERATION REMARKS

Sphere,cylinder and disc mechanism. Disc turns as \( \int_a^x f(x)dx \); displacement of ball from centre of disc =g(x); cylinder turns as \( \int_a^x f(x)g(x)dx \).

1. Plots \( F(x) = \int_a^x f(x)g(x)dx \) against \( x \).
2. Solves differential equations of the second order with variable coefficients.
3. The principle could be extended to solving \( f(x,d^iu/dx^i,d^iu/dx^i-1,\ldots du/dx, u) = 0 \). See page 86 eq.(ix).

Stewart 1925

M.I.T. 78

Potentiometers and servomechanisms transfer functions \( f_1(x) \) and \( f_2(x) \) from input tables to an electric integrating watthour-meter which gives \( F(x) = \int_a^x f_1(x)f_2(x)dx \), which is then plotted on an output table.

Auxiliary linkage mechanisms can be used to plot product of two of the functions (input or output) against \( x \).

1. Plots \( F(x) = \int_a^x f(x)g(x)dx \) against \( x \).
2. \( F(x) = \int_a^x f(x)g(x)dx \)
3. Computes traveling wave transients in A.C. transmission lines which are of form \( \frac{d}{dt} \int_0^t f(t-y)A(y)dy \).
4. Solves \( \frac{1}{dx} \left( \frac{dy}{F(x)} \right) = y(x) \) for \( y(x) \) if \( F(x) \) is known.
5. \( \phi(x) = \int_a^x f(x)\phi(x)dx \) for \( \phi(x) \).
A development of the Stewart machine of 1925. It has all the same potentialities, with the addition of back-coupling, which enables the direct solution of certain integral equations.

Does 1 to 5, inclusive, as for the Stewart machine, and the following:
6. Solves \( \Theta(x) = \int_a^x f_1(\Theta(x) + K)dx \)
for \( \Theta(x) \).
7. Solves \( \Theta(x) - F(x) = \int_a^x f_1(x)\Theta(x)dx \)
for \( \Theta(x) \) where \( f_1 \) and \( F \) are known functions and related by \( F(x) = K \), \( \int_a^x f_1(x)dx \); which is really the equivalent of problem 6.
8. Solves \( \Theta(x) - K = \int_a^x \Theta(x)dx \) for \( \Theta(x) \).

A development of the Bush, Gage & Stewart machine of 1927 to handle more complex problems. There are now 4 input tables and 2 output tables, and 2 sets of integrating units of the Kelvin dics and wheel type. Each function may be put in as a function of one of \( x, y \), or \( z \). Note that this machine is the direct predecessor of the Bush differential analyzer. Page 77.

Using 3 input tables, the first stage integration gives \( y = \int_m^x f_a(f_1(x)f_2(x)dx \)
on the first output table. Using the fourth input table and first output table, the second stage integration gives \( z = \int_n^x (y + f_d)dx \) on the second output table. Putting the first equation in the second, and differentiating, we get \( \frac{dz}{dx} = f_1 f_1 + f_2 f_2 + \frac{df_d}{dx} \)

where \( f_1, f_2, f_a, f_b \) are functions of one of \( x, z \), or \( \frac{dz}{dx} \). Most total second order differential equations occurring in practice can be put in this form.
<table>
<thead>
<tr>
<th>NAME</th>
<th>LOCATION</th>
<th>ERROR</th>
<th>PRINCIPLES OF OPERATION</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nessi</td>
<td>Paris</td>
<td>70</td>
<td>Rulers, travelling, carriage, etc. Usually requires only one operator.</td>
<td>1. One arrangement of the apparatus gives $\int_{0}^{t_1} f(t) , dg(t_1-t) , dt$ where $f$ and $g$ are given functions. Since $\frac{d^n y}{dx^n} = f(x)$ can be reduced to evaluating $\int_{0}^{x} \frac{dg(x-a)}{ds} f(s) , ds$ with $g(x) = x^n/n!$ for the case of $y$, or $g(x) = x/(x-k)!$ for the case of $y(k)$, it can also be solved.</td>
</tr>
<tr>
<td>Van den Akker</td>
<td>Germany</td>
<td>0.2%</td>
<td>Elaborate mechanism, with input board, multiplier and integrator. Linear displacements are transmitted by means of thin steel tapes operating against springs. A spot of light is used to follow the curves.</td>
<td>2. Another arrangement of the apparatus gives $\int_{0}^{t} (f_1(x) - f_2(x)) , dg(x)$ where $f_1, f_2,$ and $g$ are given functions. Very specialized—applied to the computation of colour specifications from spectrophotometric curves.</td>
</tr>
<tr>
<td>Laurila</td>
<td>Germany</td>
<td>65</td>
<td>Linkage mechanism.</td>
<td>Gives $\int_{0}^{x} f(x)g(x) , dx$.</td>
</tr>
</tbody>
</table>
HARMONIC ANALYZERS AND SYNTHESIZERS

These instruments are really a specialized form of product integragraphs which are concerned with the mathematical form
\[ \int_a^b f(x) \cos(nx) \, dx \], the form of the Fourier Coefficients in the expansion of \( f(x) \) into a Fourier Series. This problem has been very extensively treated because of its many practical applications. It covers both the analysis and synthesis of electric, heat, sound and other waves; of vibrations in molecules, machinery, and other objects; and of tides, astronomical observations, and other periodically varying quantities.

Numerical methods are often tedious; and are carefully detailed, for example, by Grover (191); instrumental methods are based on the following principles:

1. Direct reading (using resonance, differential dynamometer, or special circuits).
2. Operating directly from a trace of the curve, giving the results directly.
3. Operating directly from a trace of the curve, giving the results from a secondary curve.
4. Operating on selected ordinates of the curve.

In an article on sound analyzers, Hall (94) has shown that sound analyzers fall into one of five groups:

1. Graphic (Henrici)  
2. Resonance (Hickman, Freystedt)  
3. Heterodyne (Hall)  
4. Stroboscopic (Nemes)  
5. Diffraction (Meyer, Germansky).
Dellenbaugh has written a very exhaustive thesis on the subject of Harmonic Analyzers (88). There are numerous more limited but more up-to-date articles on the subject. Our tables are intended to be representative but not exhaustive. We now give some descriptive details of the more important instruments listed.

The Kelvin Synthesizer (123) uses the pin and slot devices to get the algebraic sum of the amplitudes of the various motions, the motions being combined in sums on a chord which passes around the pulleys on the moving parts.

The Henrici machine (100) is equipped with an integrating wheel like that attached to a planimeter or integrator. The wheel is run around one period of the curve to be analyzed to produce each coefficient. If the planimeter carries two integrating wheels whose axes at each instant make angles $kx$ and $\pi/2 - kx$ with the $y$ axis, and the point of intersection of these axes is capable of moving parallel to the $y$ axis, then as the tracer point passes once completely around the boundary, the wheels give readings proportional to $\int \sin kx \, dy$ and $\int \sin(\pi/2 - kx) \, dy = \int \cos kx \, dy$, from which the Fourier coefficients can be found. See figure over.

The Michelson and Stratton machine is a sequence of eighty of the units illustrated on the next page. Its working principle depends on the addition of forces in spiral springs and is as accurate as Hooke's Law. (108).
As in the figure above, $s$ is one of 80 small springs attached side by side to lever $C$, a hollow cylinder pivoted on knife-edges at its centre. $S$ is the large counter-spring. The harmonic motion produced by the eccentric $A$ is communicated to $x$ by the rod $R$ and lever $B$, the amplitude of the motion at $x$ depending on the adjustable rod $d$. Resultant motion is recorded by the pen at $u$. Under the pen, a slide moves with a speed proportional to the angular motion of a cone $D$.

The eccentrics have periods from 1 to 80 respectively, being geared to wheels whose number of teeth are in the proper ratio. These wheels are all fastened together on the same axis and form the cone $D$. 

See Henrici,

See Michelson and Stratton and below.
The machine designed by Chubb for the Westinghouse Co. (86) was meant for analyzing oscillograms. A turntable, upon which is placed a template of the polar plot of the curve to be analyzed, can turn about its centre and move back and forth with simple harmonic motion. A cross-bar across the centre of the disc bears a small wheel which rests against the template and is forced to follow its contour. A planimeter at the end of the cross-bar evaluates the integral.

Kranz (101) shows that if the number of ordinates used per wave-length of original curve is not less than twice the order of the highest harmonic present in the curve, his process of synthesis contains no approximation, that is to say, the use of additional ordinates or of additional elements on the synthesizer would add nothing to the accuracy of the results. He also calculates the results of the violation of the above condition.

Archer (82) shows that two or more simple harmonic motions with any desired amplitudes and phases may be combined optically, and hence Lissajou's figures or the combinations of a wave with one or more of its harmonics may be demonstrated. Light from a point source, after successive reflections from plane mirrors mounted to rotate or vibrate, is focussed on a screen, but the motion of the mirrors is controlled both as to phase and amplitude by intermittent currents obtained from a rotating interrupter.
In the Brown machine of 1939, analysis is accomplished by setting the amplitudes of the 30 elements corresponding to the values of 30 of 32 equi-spaced ordinates from a wave-length of the curve to be analyzed. Fifteen sin and fifteen cos components of the curve being analyzed are then determined from the values of 30 of 32 equi-spaced ordinates of a wave-length from a single trace produced by the machine. The mechanical synthesizer plots a sum of harmonic terms against the angle. It works on the principle of the Kelvin tide predictor, and draws accurate resultant curves for ratios of harmonic amplitudes as great as 200. Use of the Synthesizer by the method of Kranz is illustrated, and the method extended to the determination of high order harmonics; in fact to 59 harmonics, using 120 equi-spaced ordinates. (21).

Maxwell's harmonic synthesizer (107) containing two elements was built to facilitate electron diffraction studies. An alternating supply voltage is modulated by variometers of special design which are driven at harmonic frequencies by a gear train. The combined outputs are detected by a vacuum tube voltmeter and photographically recorded by a galvanometer mirror. Precision in the location of maxima or minima is 0.03 or 0.04 radians. Several other such devices have been built. See reference.

The Rymer and Butler machine is slow in operation, being scarcely an improvement over numerical methods. However, it works on an interesting electrical principle. (118).
<table>
<thead>
<tr>
<th>NAME</th>
<th>YEAR</th>
<th>REPORT</th>
<th>AN. or SYN</th>
<th># OF ELEMENTS</th>
<th>TYPE</th>
<th>PURPOSE</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kelvin</td>
<td>1878</td>
<td>123</td>
<td>A</td>
<td>5</td>
<td>Disc, sphere &amp; cylinder</td>
<td>Analyse graphs.</td>
<td>Two cranks give simple harmonic angular motion to two of the discs, respectively, so that the cylinders give $\int_0^X y \cos\left(\frac{2x}{c}\right) dx$. Shafts from these to two other integrating units are geared to give $\int_0^X y \sin\left(\frac{2wx}{c}\right) dx$.</td>
</tr>
<tr>
<td>Kelvin</td>
<td>1881</td>
<td>123</td>
<td>A &amp; S</td>
<td>11</td>
<td>&quot;</td>
<td>Tide predictor.</td>
<td>Built by Admiralty. A year's tides in any port can be run off in 4 hours. For other tide predictors, see below &amp; Special Publication #32 of the U.S. Coast and Geodetic Survey.</td>
</tr>
<tr>
<td>Henrici</td>
<td>1894</td>
<td>100</td>
<td>A</td>
<td>5</td>
<td>&quot;</td>
<td>Analyse graphs.</td>
<td>Gives 10 components in 15 min. Refined in 1910 by Conradi (see below). Basic wavelength 40 cm. Accuracy 0.003 to 0.03%. Best example of Kelvin principle.</td>
</tr>
<tr>
<td>Sharp</td>
<td>1894</td>
<td>120</td>
<td>&quot;</td>
<td>&quot;</td>
<td>Mechanical at a time.</td>
<td>&quot;</td>
<td>Uses rolling wheel and tracer. Gives amplitude and epoch directly.</td>
</tr>
<tr>
<td>Yule</td>
<td>1895</td>
<td>128</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>Uses planimeter, ruler, toothed wheels.</td>
</tr>
<tr>
<td>Michelson</td>
<td>1898</td>
<td>108</td>
<td>A</td>
<td>80</td>
<td>&quot;</td>
<td>&quot;</td>
<td>Number could readily be increased to several hundred. 1 or 2% accuracy. See text.</td>
</tr>
<tr>
<td>and Stratton</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td>&quot;</td>
<td></td>
</tr>
<tr>
<td>Milne</td>
<td>1906</td>
<td>111</td>
<td>S</td>
<td>3</td>
<td>Like Kelvin</td>
<td>Synthesize graphs!</td>
<td>Provides for an incommensurable ratio of speeds of two components by means of parallel cones connected by a shifting leather belt. Draws resultant curve. Amplitude or period of constituent components can be varied during motion.</td>
</tr>
<tr>
<td>NAME</td>
<td>YEAR</td>
<td>REF.#</td>
<td>AN. or SYN.</td>
<td># OF ELEMENTS</td>
<td>TYPE</td>
<td>PURPOSE</td>
<td>REMARKS</td>
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<td>----------------------------------------------</td>
</tr>
<tr>
<td>Cady</td>
<td>1906</td>
<td>85</td>
<td>S</td>
<td>4</td>
<td>Mechanical Synthesize size graphs</td>
<td>Compact and portable for drawing the sum of sine waves; fundamental, 2nd. 3rd. or 5th harmonic.</td>
<td></td>
</tr>
<tr>
<td>Henrici</td>
<td>1910</td>
<td>110</td>
<td>A</td>
<td>30</td>
<td>Like Kelvin graphs</td>
<td>Acoustic A very popular machine for analysing sounds. 0.003 to 0.03% accuracy. Very fine workmanship. Gives 30 harmonics in 5 or 6 hrs. Design by Conradi of Zurich.</td>
<td></td>
</tr>
<tr>
<td>Mader</td>
<td>1910</td>
<td>119</td>
<td>A</td>
<td>One at a time</td>
<td>Mechanical Analyses graphs</td>
<td>Involves planimeter, rolling wheel and tracer.</td>
<td></td>
</tr>
<tr>
<td>Boucherot</td>
<td>1913</td>
<td>68</td>
<td>A</td>
<td>One at a time</td>
<td>Mechanical Analyses graphs</td>
<td>Involves planimeter, rolling wheel and tracer.</td>
<td></td>
</tr>
<tr>
<td>Lütsche</td>
<td>1915</td>
<td>104</td>
<td>A &amp; S</td>
<td>75 in, 9 cm</td>
<td>Mechanical Analysis graphs</td>
<td>Involves planimeter, rolling wheel and tracer.</td>
<td></td>
</tr>
<tr>
<td>U.S. Gov't.</td>
<td>1916</td>
<td>129</td>
<td>A &amp; S</td>
<td>38</td>
<td>Mechanical Tide predictor</td>
<td>Built for U.S. Coast and Geodetic Survey.</td>
<td></td>
</tr>
<tr>
<td>Raymond</td>
<td>1918</td>
<td>114</td>
<td>S</td>
<td>3</td>
<td>Electric Graphs</td>
<td>At the U.of California. Draws the resultant curve of components of incommensurate period and any desired decrement.</td>
<td></td>
</tr>
<tr>
<td>See Remarks</td>
<td>1919</td>
<td>130</td>
<td>A &amp; S</td>
<td>16</td>
<td>Like Kelvin Tide predictor</td>
<td>Built in England for the Argentine Naval Commission. One year's tides in 150 min. Others supplied to France, Brazil, Japan.</td>
<td></td>
</tr>
<tr>
<td>NAME</td>
<td>YEAR</td>
<td>REF. NO.</td>
<td>AN. or SYN.</td>
<td>NO. OF ELEMENTS</td>
<td>TYPE</td>
<td>PURPOSE</td>
<td>REMARKS</td>
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<td>-------------------------------------------------------------------------</td>
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</tr>
<tr>
<td>Bush</td>
<td>1920</td>
<td>84</td>
<td>A</td>
<td>One at a time</td>
<td>MechanicalCurrent</td>
<td>Mechanism built out of transparent discs, planimeter, fish line. Analyses oscillograph curves.</td>
<td></td>
</tr>
<tr>
<td>Woodbury</td>
<td>1921</td>
<td>127</td>
<td>A</td>
<td>One at a time</td>
<td>Electric network</td>
<td>Has two frequency ranges: 20-1250 cycles and 80-6000 cycles. Works on the principle of tuned circuits. Can perform one analysis in 5 minutes.</td>
<td>Accuracy to date exceeded only by Henried.</td>
</tr>
<tr>
<td>Mason</td>
<td>1921</td>
<td>106</td>
<td>A</td>
<td>One at a time</td>
<td>Electric waves</td>
<td>Electric Current or voltage waves. Amplitude correct to within 1/20 of fundamental.</td>
<td></td>
</tr>
<tr>
<td>Wegel and</td>
<td>1924</td>
<td>126</td>
<td>A</td>
<td>One at a time</td>
<td>Mechanical Seismograms</td>
<td>Powered by a three phase motor. Phase and amplitude of components and period of the motion variable. Resultant motion recorded on drum. To study seismograms, vibrations of machinery, traffic and so on.</td>
<td></td>
</tr>
<tr>
<td>Moore</td>
<td>1926</td>
<td>87</td>
<td>A</td>
<td>Electric waves</td>
<td>Mechanical Graphs</td>
<td>The analyzer is mechanical, the synthesizer is built on the Kelvin Tide Predictor principle. Dimensions of whole are 9 1/2&quot; x 2&quot; x 9&quot;. At the Riverbank Labs. Geneva, Ill.</td>
<td></td>
</tr>
<tr>
<td>Rohé and</td>
<td>1926</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rémy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kranz</td>
<td>1927</td>
<td>101</td>
<td>A and S</td>
<td></td>
<td>Mechanical Graphs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Harvey</td>
<td>1930</td>
<td>95</td>
<td>A</td>
<td>One at a time</td>
<td>Mechanical Graphs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NAME</td>
<td>YEAR</td>
<td>REF. NO.</td>
<td>AN. or SYN.</td>
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<td>-------------------------------------------------------------------------</td>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Robertson</td>
<td>1932</td>
<td>/16</td>
<td>A</td>
<td>2</td>
<td>Kelvin</td>
<td>Mechanical Graphs</td>
<td>A generalization of the Kelvin Tide Predictor and equation solving machine. The extension to several components is shown.</td>
</tr>
<tr>
<td>Harvey</td>
<td>1934</td>
<td>96</td>
<td>A</td>
<td>One at a time.</td>
<td>Grapho-</td>
<td>mechanical Graphs</td>
<td>Represents a rediscovery of Yule's principle. Can also be used to give areas and first and second moments about an axis.</td>
</tr>
<tr>
<td>Archer</td>
<td>1937</td>
<td>82</td>
<td>S</td>
<td>2 or more</td>
<td>Optical</td>
<td>See remarks.</td>
<td>Used to combine a wave with one or more of its harmonics. See text.</td>
</tr>
<tr>
<td>Montgomery</td>
<td>1938</td>
<td>113</td>
<td>A and S</td>
<td>Gets first 30 harmonics</td>
<td>Optical</td>
<td>Motion picture sound track.</td>
<td>Function to be analyzed is supplied in the form of a variation in the density or width of a film. Each term obtained independently of the others, 30 terms in 1/3 min. Agrees well with an analysis on the Henrici machine.</td>
</tr>
<tr>
<td>Brown</td>
<td>1939</td>
<td>21</td>
<td>A and S</td>
<td>15 sin 15 cos.</td>
<td>Kelvin</td>
<td>Graphs</td>
<td>At U. of Texas. Analysis accomplished by setting the amplitudes of the 30 elements to correspond to the values of 30 of 32 equi-spaced ordinates from a wave-length of the curve to be analyzed. See text for further details.</td>
</tr>
<tr>
<td>Walther, Dreyer,</td>
<td>1939</td>
<td>124</td>
<td>S</td>
<td>4</td>
<td>Kelvin</td>
<td>Mechanical Graphs</td>
<td>A small model, built from standard Meccano parts.</td>
</tr>
<tr>
<td>Estenfeld</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maxwell</td>
<td>1940</td>
<td>107</td>
<td>S</td>
<td>2</td>
<td>Electrical</td>
<td>Electron diffraction.</td>
<td>See text.</td>
</tr>
<tr>
<td>NAME</td>
<td>YEAR</td>
<td>REF NO.</td>
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<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Straiton and Terhune</td>
<td>1943</td>
<td>/22</td>
<td>A</td>
<td>n</td>
<td>Photographic.</td>
<td>Graphs. At U. of Texas. Fourier coefficients are determined from area measurements on photographs of the function to be analyzed, with the graph sheet formed into a series of semi-cylinders.</td>
<td></td>
</tr>
<tr>
<td>Shilton</td>
<td>1944</td>
<td>/21</td>
<td>S</td>
<td>3</td>
<td>Hydraulic Synthesis</td>
<td>Acoustic synthesis of complex waves. It utilizes hydraulic means for summing the motion of three pistons which are driven by a system of gears and cams from a main shaft. There is some control over the harmonics.</td>
<td></td>
</tr>
<tr>
<td>Rymer and Butler</td>
<td>1944</td>
<td>/18</td>
<td>A</td>
<td>18</td>
<td>Electric</td>
<td>A D.C. Network involving standard radio parts + a mirror galvanometer and potentiometer. See text.</td>
<td></td>
</tr>
<tr>
<td>Farth and Pringle</td>
<td>1944</td>
<td>/99</td>
<td>A and S</td>
<td>9 sin, 9 cos.</td>
<td>Photo-electric.</td>
<td>Only a demonstration model has been built. It is of low accuracy, and the principle does not seem too adequate. But see p. 67; and (90).</td>
<td></td>
</tr>
</tbody>
</table>
PARAMETRIC PRODUCT INTEGRALS

In the study of the transient responses of networks, there arises a parametric product integral in the following way:— Indicial admittance $A(t)$ is defined as the response of a network to the sudden application of a unit force. To find the response to an arbitrary form $e(t)$, assume the force is applied as the limiting case of a series of infinitesimal step functions. Consider the differential force increment taking place at time $t=x$. It is of magnitude $(de)_{t=x}$ or $(\frac{de}{dx})_{t=x}$. Response of network is $A(t-x)(de)_{t=x}$. Note that this is valid only for $t\geq x$. It = $A(0)$ at $t=x$. Response to the entire function is found by summing and taking the limit:

$$i(t) = e(0)A(t) + \int_{0}^{t} A(t-x)e'(x)dx = \frac{d}{dt} \int_{0}^{t} A(t-x)e(x)dx.$$ 

This is the superposition theorem in two of its many forms.

In order to solve the type of equation suggested by this formula, the principle of the cinema integrator was proposed by Dr. Norbert Wiener and developed at M.I.T. by Gould and by Gray. The principle is sketched below.

![Diagram of integrator circuit](image)

With everything stationary as in the above diagram, the flux would be proportional to $\int_{a}^{b} f(x)g(x)dx$, as given by a simple product integrator. By using a separate film for each
of a set of values of \( y \), \( P(y) = \int_a^b f(x,y)g(x)\,dx \) can be evaluated. More conveniently, by moving the \( f(x) \) film continuously along the \( x \) axis so that at each instant it is displaced a distance \( y \), one can obtain the continuous evaluation of the parametric product integral \( P(y) = \int_a^b f(x-y)g(x)\,dx \).

See p.188 of Hazen and Brown (97) for conditions under which the kernel \( f(x,y) \) can be converted to the form \( F(X+Y) \) so that a continuous solution is possible.

Cinema integrals have been used extensively to solve many other integral equations of practical interest. To amplify the remarks on the Gray machine, we note that (i) the Pearson coefficient of correlation may be given in integral form which further involves a parameter when the quantities vary with time. It is then of the form

\[
\lim_{A \to a} \int_0^A x(z)y(z+t)\,dz \\
\sqrt[3]{\int_0^A x^2(z)\,dz \left[ \int_0^A y^2(z)\,dz \right]} 
\]

(ii) Periodogram analysis is to find the hidden periodicities in irregularly varying quantities, and has been applied, for example, to the flow of the Ohio River. Let \( f(t) \) be the variation of the quantity with time. Define

\[
\varphi(u) = \lim_{k \to a} \frac{1}{k-u} \int_0^{k-u} f(t)f(t+u)\,dt \\
\theta(b) = \frac{2}{\pi} \int_0^a \varphi(u) \frac{\sin bu}{u} \,du .
\]

\( \varphi(u) \) contains the elements of the correlation of a function against itself. \( \theta(b) \) is similar to the Fourier transform.
of $\Theta(u)$. The curve of $\Theta(b)$ will show a sudden vertical rise for each value of $b$ corresponding to the frequency of a periodic component of $f(t)$.

W.R. Redeman Jr. (98) has treated very thoroughly the problem of classifying and solving linear integral equations, pointing out which ones are best adapted to solution by the cinema integrator. In (99) he gives a further application to inter-reflection problems.

Warren (125) has written a similar thesis, showing which equations could, to that date (1945) be solved, practically, only by cinema integrator. He concentrates on the homogeneous equation

$$u(x) = \int_a^b G(x,s)u(s)ds$$

with $G(x,s)$ symmetrical. Characteristic values of $\lambda$ and the corresponding characteristic functions have to be obtained. He also gives considerable attention to the equation of cantilever beam theory

$$u(x) = \int_0^1 G(x,s)u(s)ds.$$  

As a few final remarks, we note that the convolution integral of the functions $f(t)$ and $g(t)$ is

$$\int_0^t f(t-x)g(x)dx$$

and so is of our general form.

The continuous solutions, when and if they exist, for Fredholm and Volterra equations as found by the possible analytical methods, 1. Successive substitution, 2. Fredholm theory, 3. Hilbert-Schmidt theory (for symmetric kernel only), are given in detail by Lovitt in his text (103). They can usually be evaluated numerically, although the process is very tedious, because they are in the form of infinite series.
<table>
<thead>
<tr>
<th>NAME and DATE</th>
<th>LOCATION</th>
<th>ERROR</th>
<th>PRINCIPLES OF OPERATION</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gould 1927</td>
<td>M.I.T.</td>
<td>2%</td>
<td>Uses infra-red radiation, which is measured by thermopiles. There are many practical difficulties. Setting up the functions takes 1/2 to 3 hours, machine operation takes 1 minute.</td>
<td>Principally used only for the integral $\int_a^b g(x)f(y^2x)dx$. Has been abandoned in favour of Gray's machine (below).</td>
</tr>
<tr>
<td>Gray 1931</td>
<td>M.I.T.</td>
<td>2 to 5%</td>
<td>Photocells were by this time sufficiently improved for visible light to be used.</td>
<td>1. Solves superposition theorem in form $1(t) = \frac{d}{dt} \int_0^t A(t-x)e(x)dx$.</td>
</tr>
<tr>
<td>Hazen and Brown 1940</td>
<td>M.I.T.</td>
<td>1%</td>
<td>f(x) and g(x) may be in analytic, tabular, or graphical form, and are plotted on photo-film to be used by the machine. Automatic controls make the machine capable of evaluating the integrals 1 &amp; 2. Solutions are graphical or tabular.</td>
<td>Conversion from f(x,y) to F(Y±X) is possible under certain conditions. See page 188 of reference.</td>
</tr>
</tbody>
</table>

1. Fourier transforms.  
2. Coefficient of correlation.  
3. Periodogram analysis.  
4. Fredholm type integral equations,  
5. Poisson type integral equations of form $s(t) = -S'(t) - \int_0^t (s(T)-S'(tT)) dt$.  

(See text).
FOURIER TRANSFORM COMPUTERS

In the form \( F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos(yx) dx \), these are a particular case of parametric product integrals just as Fourier coefficients are of product integrals. They may also be considered as the limiting case of Fourier coefficients as the range of integration tends to \((-\infty, \infty)\). For this reason, any harmonic analyzer which gives a large number of terms will give at the same time a corresponding number of equally spaced values of the Fourier transform.

Devices designed principally for evaluating Fourier transforms work as a limiting case of this principle.

As pointed out by Redheffer (115), the usual methods are point by point. They evaluate the integral at several values of \( y \) and plot the curve through the points so obtained. To do this by punched card methods (both \( x \) and \( y \) discrete) costs about \$50 for each curve. To do this by differential analyzer (\( x \) continuous, \( y \) discrete) would cost about \$500 for each point. To do this by Redheffer's new machine (\( x \) discrete, \( y \) continuous), the cost of operation is very low and the Fourier integral is really approximated to by the sum of a Fourier series.

An electronic computer for the generalized transform \( F(x) = \int_{a}^{b} f(t)K(x, t)dt \) is now under consideration at M.I.T. A design has been proposed by D.H. Wallman.
<table>
<thead>
<tr>
<th>NAME and DATE</th>
<th>LOCATION</th>
<th>REF.</th>
<th>ERROR</th>
<th>PRINCIPLES OF OPERATION</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laurila 1940</td>
<td>Germany</td>
<td>102</td>
<td></td>
<td>Adaptation of the Mader-Ott harmonic analyzer.</td>
<td>Evaluates the Fourier transform of a given real function $f(x)$. $F(y)=\int_a^b f(x)\cos(yx+\phi),dx$ is produced on a cathode ray oscillograph screen if $f(x)$ is given as a mask cut out of black paper or as a record in density variation on a film. From two such graphs, the Fourier transform is obtained.</td>
</tr>
<tr>
<td>Born, Fürth &amp; Pringle 1945</td>
<td>Scotland</td>
<td>83</td>
<td></td>
<td>Adaptation of Montgomery's photoelectric harmonic analyzer. It allows the resolution of a complicated oscillation into a continuous frequency spectrum of harmonic oscillation.</td>
<td></td>
</tr>
<tr>
<td>Also; Fürth &amp; Pringle 1946</td>
<td></td>
<td>90</td>
<td></td>
<td>The product and the integral are taken by linear potentiometers, while the complex exponential is generated by mechanical linkages. Machine gives direct readings of phase and absolute value, or real and imaginary parts.</td>
<td></td>
</tr>
<tr>
<td>Redheffer M.I.T. 1946</td>
<td></td>
<td>115</td>
<td></td>
<td></td>
<td>1. Obtains Fourier transform in the form $\int f(x)\exp(i(xy-\phi(x))),dx$. 2. Obtains sum of Fourier Series $a_n\sin(ny+b_n)$. 3. Gives the Faltung convolution of $f$ and $g$, $\int_0^y f(x)g(x-t),dx$ as a function of $t$ and $y$. 4. Solves simultaneous linear eqs. 5. Evaluates certain parametric integrals such as Laplace transforms. 6. Could sum complex power series. 7. Could compute the integral of antenna theory $\int f(x)\exp\left[i(\phi(x)-kp(x)\cos(x-y))\right]dx$.</td>
</tr>
</tbody>
</table>
DIFFERENTIAL EQUATION SOLVERS

A. DESCRIPTIVE

ORDINARY DIFFERENTIAL EQUATIONS

The approaches range from simple graphical methods (1), through specialized small machines for low order equations, to Differential Analyzers with many variations in the basic units. The smaller machines are readily built, fairly inexpensive, and with a high time-efficiency ratio. Differential Analyzers are much more difficult to build, many refinements being necessary to ensure accuracy of better than 2%. For example, the new Differential Analyzer at M.I.T. which is the largest of its kind, cost about $200,000 to build, not counting the $200,000 or $300,000 telephone exchange system which was a gift from the Bell Laboratories and is an integral part of the machine. In peacetime, it operates at about $20 an hour for a 9 or 10 hour day. Two or more operators are required, depending on the number of input tables to be manned.

In (1) graphical methods are adapted to equations of the first degree and n th, order; of the n th, degree and first order, and to the set of equations \( P_i \frac{d^n y_i}{dx^n} = Q_i \) where \( i = 1, 2, \ldots, k \) and \( P_i \) and \( Q_i \) all functions of \( x \), and \( y_i \).

Myers' Machine (155) is of the mechanical rolling type, in which all the variables of the equation are represented by the linear displacements of mechanical links. It gives the graphical solution of \( ay + by + cy + d = 0 \), where \( a \) and \( d \) are functions of \( x, y, \) and \( dy/dx \), and \( b \) and \( c \) are constants.
Bullard and Moon (134) built a machine in 1930 to solve \( \frac{d^2y}{dx^2} + f(x)y = 0 \). The method depends on the equations of motion of a current carrying coil, suspended in a magnetic field, being of the form given by this equation. The current is varied with time as \( f(t) \), and the motion of the system then represents the solution of the equation. Motion is usually realized as the motion of a spot of light. Many physical sources of error assure an accuracy of only 2 to 5%.

In 1936, Henshaw built a somewhat similar machine to solve second order differential equations, and we quote from his thesis (147) "by means of a dynamic analogy proposed by Dr. N. Minorski. A torsion pendulum is so arranged that its restoring torque can be regulated according to the desired function of time and displacement. Provision is made for tracing the graph of the oscillation. In this manner, any differential equation of the general form \( \ddot{\theta} + f(t, \theta) = 0 \) may be solved.

"The main part of the apparatus consists of two coils mounted on a vertical shaft which is suspended from a fine steel wire. Each coil is surrounded by a horizontal solenoid rigidly mounted upon the base of the apparatus. When electric currents flow through the coils and solenoids, the moving coils tend to orient themselves with their axes parallel to the axes of the stationary solenoids. If displaced from this position and released, the moving coil assembly will oscillate according to the equation of the torsion pendulum. A
recording arm attached to the vertical shaft traces the
graph of the motion upon a sheet of paper mounted upon a
drum rotating at a uniform rate."

The apparatus to solve several equations of the Mathieu
type $\ddot{\theta} + (a + b \cos wt) = 0$ was actually constructed and
tested. Graphs were compared with numerical results obtained
by the method of successive approximation. The total error
was found to be about 5%, with the occasional error rising
as high as 12%. Various refinements are proposed on his p. 62.

The chief disadvantage of this machine would seem to
be that a complete revision of the wiring is necessary for
each $f(\theta, t)$ if more than just the parameters are varied.
However, once the apparatus is set up, the graphs are soon
obtained.

Varney (62) points out that the four basic devices
required for the solution of differential equations are
1. Integrating units
2. Multiplying units
3. Coupling devices
4. Generators of functional coefficients. He proposes an all
electric machine costing about $5000, and makes a prelimin-
ary model. His integrators are induction-type A.C. Watthour
meters, his couplings are electro-magnetic torque amplifiers,
and his follow-up and control are through a variable auto-
transformer, such as the General Radio Variac. This machine
has a flexible scale to relate the Variac voltages to num-
erical values of the variables.
Electronic Differential Analyzers.

A very promising type of differential analyzer is now being developed by many experimentalists both in this country (eg. J. Ham, working under Prof. S. Caldwell and R. E. Scott and A. B. Macnee working under Prof. H. Wallman at M.I.T.) and elsewhere, (eg. I. S. Brück in Russia (/33)). This type of machine is of a continuous electronic nature, all magnitudes being represented by voltages, the output being plotted against the input on the screen of a cathode ray oscillograph. The simplest input, representing the independent variable, which can very readily be introduced is time. Machines using this would solve differential equations with time as the independent variable. However, it is much more desirable to be able to solve general differential equations, and efforts are now being concentrated upon designing generalized units.

This class of machines has the advantage that the solution is presented graphically in 1/120 of a second after the connections and adjustments are completed. These machines will be relatively inexpensive ($200 to $500 depending on the number of units desired) and readily constructed. They cannot hope to achieve more than 1% accuracy, but it is hoped sufficient for most practical problems.

Many technical difficulties have still to be surmounted by clever circuit design.

Notice that \( u \cdot v = \int u \, dv + \int v \, du \)

and also that \( \int u \, dv = \int \frac{u \, dv}{dt} \)
so that a feed-back integrator which integrates with respect to time only could be made into a general integrator if combined with a differentiator and a multiplier. But both differentiators and multipliers are hard to design. Theoretically, feed-back amplifiers can also be used as differentiators, but an investigation has shown that the errors are excessive unless the basic amplifier has an infinite bandwidth. There are various possibilities for multipliers.

Multipliers based on the form \((x+y)^2 - (x-y)^2 = 4xy\) are unsatisfactory because it is difficult to obtain two square law devices with identical characteristics and covering a sufficient range to handle the input voltages. Logarithmic Amplifiers have been constructed, in which the logarithms and antilogarithms of voltages are taken to obtain functional products. See S. Wingate's Thesis (164). The most promising principle is that of pulse techniques which is as follows: The lengths of two sets of pulses represent the two quantities to be multiplied. The repetition frequencies for the two sets of pulses are slightly different, so that the second pulse occurs at a succession of positions along the first. The time during which the pulses coincide will be proportional to the lengths of each pulse and hence to their product. This quantity can be measured by the direct current output of a detector biased to respond only to pulses of an amplitude greater than either pulse alone. The construction of a successful multiplier based on this principle is to be the subject of a Master's Thesis presented at M.I.T. by J. Ham.
Scott and Macnee are experimenting with an electron beam deflection multiplier. Four plates are mounted in place of the normal cathode ray screen and charged according to a scheme which deflects the electron beam by an amount proportional to the product of the two variables.

Scott has further suggested using the input \( \varphi = t \) \( \varphi = (2-t) \)

and so on in the time ranges \((0,1), (1,2), (2,3)\) and so on, respectively, so that \( d\varphi/dt = \pm 1 \), and multiplication may be performed by switching a sign inverter in and out of the circuit. This avoids the need for a multiplier altogether.

Circuits for addition, inversion, and multiplication by a constant are easily designed. It remains to discuss the introduction of functional coefficients and coupling.

Functional coefficients will be generated as in the mechanical differential analyzers if their mathematical form is known. In fact, all the block diagram coupling schemes worked out for the mechanical machines will carry over to these electronic ones. However, the question of introducing functions from graphical or tabular data has yet to be investigated.

The coupling of a system of integrating units is being studied by Scott and Macnee and by Bruk. It presents the difficulties of synchronization. This, mathematically, leads to errors in the initial settings, which would come in in a similar way at the beginning of each sweep.
In a recent paper, Bruk (1933) proposes using a sine wave as the independent variable, thus achieving a periodic solution to which initial conditions could be easily applied. However, he still requires a differentiator and a multiplier, with all the attendant difficulties.

From the above discussion it should be apparent that the field of electronic differential analyzers is still very much in its infancy, but it promises to yield some very practical machines in the field of low-cost, high-speed, one-to-five-percent-accuracy computing.
All the other machines listed are Differential Analyzers, built on the same principle as the original Bush machine of 1930. The principle of operation is as follows, and we quote Dr. V. Bush from p. 459 of (135): "A bus shaft is assigned to each significant quantity appearing in the equation. The several relations existing between these are then set up by means of connections to the operating units ... a functional relation by connecting the two corresponding shafts to an input table" or by arranging connections so as to solve the differential equation generating the function, most of the commonly occurring functions being treated by Guerrieri in (140) "a sum by placing an adder in position, an integral relationship by an integrator, and so on. When all the relationships involved have been thus represented, a final connection is made which represents the equality expressed in the equation."

Complicated situations arising in the solution of many problems, such as discontinuities in coefficients, have to be treated individually, and many techniques have been developed through practice. These are well known to operators of the machines, but have yet to be written about systematically.

In connection with the large machine at the University of Manchester (143), we may point out the following features; the special input table, allowing for solutions with a time lag, useful in problems with mixed difference and differential equations; the frontlash units throughout.
In three years of operation, a direct multiplier has never been needed, and the machine never supplied with one. The average problem requires one day to be set up in the machine, and a quarter of an hour of operation to run it through. Note that this has been the great source of improvement in the new M.I.T. machine, where the automatic tape feed cuts the time of setting up, a new problem to a matter of an hour or so. High speed automatic cameras were fitted up to record the results by acting as photographic revolution counters.

The Beard Machine (13) is interesting because of its applications. All the common actuarial functions, \( v^n, (1+i)^n, \int f(t)v^t dt \), and so on, have been generated by it. It is proving useful in the production of contingent functions, where two integrators have to be connected up"in cascade." The machine requires the functions to be continuous. This is not really an objection in actuarial work, since one can use all the usual approximations made in obtaining continuous functions from curtate functions. Furthermore, functions can be readily plotted from select tables. The differential analyser can employ directly the observed function \( \mu_x \). The errors introduced by backlash are discussed by Beard. He points out that most actuarial functions are monotonic, so that backlash is kept at a minimum. Notable exception is \( \mu_x l_x \), which changes at age 76.
<table>
<thead>
<tr>
<th>NAME</th>
<th>YEAR</th>
<th>REF, NO.</th>
<th>LOCATION</th>
<th>ACCURACY</th>
<th>UNITS</th>
<th>INPUT</th>
<th>OUTPUT</th>
<th>INTERCON-AMPLIFICATION</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thomson and Tait</td>
<td>1876</td>
<td>79,149</td>
<td>England</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Only a laboratory model, with disc, ball and cylinder as the unit.</td>
</tr>
<tr>
<td>Bush</td>
<td>1930</td>
<td>135</td>
<td>U.S.A. (M.I.T.)</td>
<td>0.1%</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>Mechanical Torque</td>
<td>One input table may be polar. Has multiplying units which are still used considerably. Has 18 bus shafts.</td>
</tr>
<tr>
<td>Bullard and Moon</td>
<td>1930</td>
<td>134</td>
<td>England</td>
<td>2 to 5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>See text.</td>
</tr>
<tr>
<td>Hartree and Porter</td>
<td>1935</td>
<td>145</td>
<td>England (Univ. of Manchester)</td>
<td>2%</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>Mechanical Torque</td>
<td>Small scale model, mainly standard Meccano parts.</td>
</tr>
<tr>
<td>Hartree</td>
<td>1935</td>
<td>143</td>
<td>England (Univ. of Manchester)</td>
<td>0.1%</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>1935</td>
<td>161</td>
<td>U.S.A. (Moore Sch. of E.E., U.of Penn.)</td>
<td></td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>1935</td>
<td>161</td>
<td>U.S.A. (Aberdeen Proving Centre, Md.)</td>
<td></td>
<td>10</td>
<td></td>
<td></td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>NAME</td>
<td>YEAR</td>
<td>REF. NO.</td>
<td>LOCATION</td>
<td>ACCURACY</td>
<td>NO. OF TABLES</td>
<td>NO. OF INPUT UNITS</td>
<td>NO. OF OUTPUT UNITS</td>
<td>INTERCONNECTIONS</td>
<td>AMPLIFICATION</td>
</tr>
<tr>
<td>--------------</td>
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<tr>
<td>Massey</td>
<td>1938</td>
<td>153</td>
<td>Ireland (Queen's Univ., Belfast)</td>
<td>1 to 2%</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>Mechanical</td>
<td>Two stage torque amplifiers</td>
</tr>
<tr>
<td>Lennard-Jones</td>
<td>1939</td>
<td>150</td>
<td>England (Math'l. Lab., U.of Cambridge)</td>
<td>2%</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>Myers</td>
<td>1939</td>
<td>155</td>
<td>Australia (Univ. of ink line Sydney)</td>
<td>0.01&quot;</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1939</td>
<td></td>
<td>England (Univ. of Cambridge)</td>
<td>0.1%</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Before</td>
<td>1940</td>
<td>?</td>
<td>Leningrad, Russia</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Before</td>
<td>156</td>
<td>Norway (Inst. of Astrophysics, Oslo.)</td>
<td></td>
<td>12</td>
<td>7</td>
<td>1</td>
<td>Mechanical</td>
<td>Torque amplifiers</td>
</tr>
<tr>
<td>Varney</td>
<td>1941</td>
<td>162</td>
<td>U.S.A. (U. of Washington, St. Louis, Mo.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beard</td>
<td>1942</td>
<td>131</td>
<td>England (Pearl Assurance Co., London)</td>
<td>1%</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>Mechanical</td>
<td>Torque amplifiers</td>
</tr>
<tr>
<td>NAME</td>
<td>YEAR</td>
<td>REF. NO.</td>
<td>LOCATION</td>
<td>ACCURACY</td>
<td>NO. OF TABLES</td>
<td>NO. OF UNITS</td>
<td>INTERCON-AMPLIFI-NECTIONS</td>
<td>I-CATION</td>
<td>REMARKS</td>
</tr>
<tr>
<td>-------------</td>
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</tr>
<tr>
<td>Bush and Caldwell</td>
<td>1942</td>
<td>/37</td>
<td>U.S.A. (M.I.T.)</td>
<td>0.01%</td>
<td>18</td>
<td>3</td>
<td>1</td>
<td></td>
<td>Has speed control; switching mechanisms to set connections, gear ratios.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Three independent sets of 6 each. Provision for 12 more.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sauer and Pösch</td>
<td>1943</td>
<td>/57</td>
<td>Germany</td>
<td>0.03%</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>Has differential gear-type adders.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>per cycle in circle test.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Meyer zur Capellen</td>
<td>1943</td>
<td>/49-1</td>
<td>Germany</td>
<td></td>
<td></td>
<td></td>
<td>Mechanical Electric Functions graphed on follow-up drums rather than on drawing boards, based on a design made in 1937.</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bruk</td>
<td>1946</td>
<td>/33</td>
<td>U.S.S.R.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>See text.</td>
</tr>
</tbody>
</table>


PARTIAL DIFFERENTIAL EQUATIONS

Continuous devices for solving these are of a rather artificial nature if not indirect.

D. Wehage (/63) has described a planimeter method of integrating \( \partial^2 u / \partial x \partial y + f(u) = 0 \) based on an analytic approximation consisting of a series of explicit integrals.

H. Füttinger (/39) has developed the vector integrator for the mechanical solution of potential and vortex problems. These are graphical-numerical methods for attacking some particular boundary value problems. The author has explored the Rankine Method in connection with vortices and surface and volume displacements, with a view to its application to the approximate solution of the integral equation. The determination of hollow space forms from previously chosen physical relationships has been achieved.

Bernstein (/32) has used the idea of Calton's Board, which was originally designed to demonstrate Gauss' Theory of Errors. The original experiment consisted of spheres falling between nails arranged in the form of Pascal's Triangle. Bernstein's modification is to use two boxes whose bases are arranged to slide, and which contain a number of cells. Each cell is graduated along its depth. Each cell of one box is filled with sand to a height which will represent the ordinates of the function. This box is placed over the other, but rotated in a horizontal plane through 180° relative to it. By removing the sliding base, the sand falls through into the cells of the other box, and by this operation the funct-
ion has been transformed. This can be used to solve $\Delta U = 0$ and $\Delta U = \text{Constant}$. Bernstein has applied it to solve:

1. $\Delta U = 0$ for various boundary conditions.
2. Calculation of a turning moment due to torsion of a girder of T-Section.

Other methods come under the more specialized heading of analogue type models, although the term here is very loosely applied.

Schmidt (/58) built a high precision electrolytic tank for the solution of the two dimensional potential problem and used it to solve for 1. The ideal fluid flow around a cylinder and for 2. The ideal fluid flow around a plate with free rays.

Kron (/48) has recently built electric circuits to represent the Schrödinger equation for one, two, and three independent space variables in orthogonal curvilinear coordinate systems, with provision for an arbitrary potential function.

Gutenmacher in Russia, has been interested in electric circuit representations of various physical problems for the past six or seven years. In 1940 (/41) he solved some partial differential equations of the second order with 2% accuracy. More recently, he has treated systems of equations and integral equations (/42) and the equations of various physical phenomena (/42) in a similar manner.
B. THEORETICAL

The most remarkable feature of the machines in this section is that they will solve non-linear equations just as readily as linear equations, this being their greatest mathematical feature. Their greatest mathematical limitation lies in the fact that they will essentially only solve ordinary differential equations. The extent to which they can be used to treat problems with more than one independent variable is discussed on page 145 of this thesis. See also (146).

These machines always give trouble at all singular points, whether of the variable, differential coefficients, or functional coefficients. Of course, this is generally true for most computing devices, being the failure of a physical system to comply with the mathematical idealization. It is in discussing the behaviour of solutions in the neighbourhood of singular points that the methods of analysis are still supreme.

These machines are especially useful in two-boundary problems, i.e. those with boundary conditions to be satisfied at the starting and finishing ends of the range. One has to satisfy those at the start and then run through the solution using various boundary values of the derivatives until the solution satisfies the boundary conditions at the far end. In fact, the machines are most useful when a large number of equations of the same form have to be solved.
The mathematical capacities of a machine consisting of an interconnected system of integrators and adders are fully discussed in a paper by Claude Shannon (1959). He arrives at some very interesting theorems which we now cite:

**Theorem I**:

A necessary and sufficient condition that a system of differential equations can be solved using only integrators and adders is that they can be written in the form

$$\frac{dy_k}{dy_1} = \sum_{i,j=0}^{n} a_{ijk} y_i \frac{dy_j}{dy_1}$$  \hspace{1cm} (i)

where $k=2,3, \ldots, n$; $y_1 = 1$; $y_0$ is the independent variable and $y_k$ are the dependent variables, amongst which are the dependent variables of the original system.

In his recent book on the mathematical theory of machines, (p. 79 of 240) Murray has given these equations in a somewhat different form, and from these he has derived some less general possible forms which are at once more familiar. In each case he shows how the system of equations can be set up in the machine, and indicates the maximum number of integrating units required. In practice, of course, many less units are generally required, since advantage is taken of the special features of the system at hand. For example, the solution of

$$(\frac{dy}{dx})^2 = x \sin y$$

requires 20 integrating units in the general form of Murray's Theory. Actually, we now show that it can be
solved with only 6 units as in the following set-up:

\[
\begin{align*}
\frac{dy}{dx} & \quad (dy/dx)^2 \\
2 \frac{dy}{dx} & \quad x \sin y \\
\Sigma & \quad \int x \, ds \sin y \\
\Sigma & \quad \int \sin y \\
\Sigma & \quad \int \cos y \\
\Sigma & \quad \int \\
\end{align*}
\]

\[
sin y = -\frac{d^2 z}{dy^2} \quad x \sin y \quad y = \int \frac{dy}{dx} \, dx
\]

\[
\cos y = \frac{d}{dy} \quad -\int \sin y \, dx \quad \left(\frac{dy}{dx}\right)^2 = \int 2 \frac{dy}{dx} \, d\left(\frac{dy}{dx}\right)
\]

\[
\cdot \cdot \cdot \sin y
\]

The equations of constraint:

\[
\left(\frac{dy}{dx}\right)^2 = x \sin y.
\]

From a practical point of view, Murray's approach is of no great value. However, in the event that some particularly involved system of equations were encountered, it would serve as a good guide towards a first solution. It must always be borne in mind that economy in the number of units is desirable, the maximum number yet available in any one machine being 18, at M.I.T. (137).
Out of general mathematical interest, we give Murray's results:

He considers \( n \) integrators with the following notation:

- \( W_i = \text{output} \) for the \( i \)th integrator, \( i = 1, 2, \ldots, n \).
- \( U_i = \text{linear input} \)
- \( V_i = \text{angle input} \)

Then equivalent to system \((i)\) of Shannon are the following:

\[
\frac{dW_i}{dx} = U_i \left( \frac{dV_i}{dx} \right) \quad \text{(ii)}
\]

Since all other operational units are adders or gear boxes, all \( U_i, V_i \) are linear combinations of \( U_i, V_i, W_i, x \). For them to be uniquely determined, the following equations must be unique:

\[
U_i = a_i, 0 + a_i x + \sum_{j=1}^{n} a_{i,j} W_j
\]
\[
V_i = b_i, 0 + b_i x + \sum_{j=1}^{n} b_{i,j} W_j
\]
\[
\text{(iii)}
\]

this being a third equivalent system.

As a fourth equivalent system, Murray gives

\[
\frac{dW_{ki}}{dx} = w_k \left( \frac{dW_i}{dx} \right)
\]
\[
w_k = A_k + A_k \sum_{p,q} A_{k,p,q} W_p, q \quad k = 1, 2, \ldots, n
\]
\[
\text{(iv)}
\]

which shows how to connect up the machine directly, if not in the most practical manner.

Murray then goes on to consider some expanded systems, which can be obtained from the original system by introducing \( p \) new unknowns with \( p \) appropriate new equations and their appropriate initial conditions. By appropriate is meant that by eliminating
the new unknowns, the original system is obtained. Hence if the expanded system can be solved by our machine, so can the original system.

Some such solvable original systems are shown by Murray to be:

\[ \frac{dw_k}{dx} = p_{k,0} + \sum_{q=1}^{n} p_{k,q}(dw_q/dx) \quad (v) \]

where \( p_{k,0}, p_{k,q} \) are arbitrary linear combinations of the \( w_k \) and \( x \).

\[ \frac{dw_i}{dx} = \sum_{j=0}^{n} p_{i,j}(dw_j/dx) \quad (vi) \]

where \( j = 0, 1, 2, \ldots, n \), and \( w_0 = x \).

\[ \frac{dw_i}{dx} = \sum_{j=0}^{n} r_{i,j}(dw_j/dx) \quad (vii) \]

where \( r_{i,j} = p_{i,j}/q_{i,j} \) = rational functions of \( w_k \) and \( x \).

\[ \frac{dw_i}{dx} = r_i \quad (viii) \]

where \( i = 1, 2, \ldots, n \).

Note that higher order systems can be reduced to set (viii).

For example, if

\[ \frac{d^k y}{dx^k} = f(x, y, dy/dx, d^2y/dx^2, \ldots, d^{k-1}y/dx^{k-1}) \quad (ix) \]

where \( f \) is a rational function, we can let \( w_i = d^{i-1}y/dx^{i-1} \) and so derive the system of type (viii), namely:

\[ \frac{dw_0}{dx} = 1, \quad \frac{dw_1}{dx} = w_2, \ldots, \quad \frac{dw_{k-1}}{dx} = w_k, \]

and \[ \frac{dw_k}{dx} = f(w_0, w_1, \ldots, w_k) \].
Two more original systems which can be expanded to one of the general solvable forms (i to iv) and which are therefore also solvable, are:

\[ f_i = 0 \]  \hspace{1cm} (x)

where \( i = 1, 2, \ldots, n \), and the \( f_i \) are polynomials in \( w_k \) and \( dw_k/dx \).

\[ f_i = 0 \]  \hspace{1cm} (xi)

where \( i = 1, 2, \ldots, n \), and the \( f_i \) are constructed by rational operations on the \( w_k \) and \( dw_k/dx \) and on functions of one variable, which are themselves solutions of algebraic differential equations.

After this lengthy digression, we return to Shannon's paper (159) and cite his general theorem on the generation of functions using only a system of integrators and adders.

**Theorem II:**

A function of one variable can be generated if and only if it is not hypertranscendental.

It is assumed that the usual classifications of functions hold, there being two types of transcendental (or non-algebraic) functions distinguishable as follows:

If \[ \sum_j A_j x^{n_j} y^{n_{0j}} (y')^{n_{1j}} (y'')^{n_{2j}} \ldots (y^{(m)})^{n_{mj}} = 0, \]

where the \( n_i \)'s are integers and \( y^{(k)} = d^k y/dx^k \),
then \( y \) is an algebraic transcendental function of \( x \).

If no such relation exists, then \( y \) is a hypertranscendental function of \( x \). Two outstanding examples of hypertranscendental functions are

\[
\Gamma(t) = \int_0^\infty t^{x-1} e^{-x} \, dx, \quad \text{the Gamma Function, and}
\]

\[
\mathcal{J}(t) = \sum_{k=0}^{\infty} \frac{1}{k^t} \quad \text{with } t < 1, \quad \text{the Riemann Zeta Function.}
\]

**Theorem III:**

If \( y = f(x) \) can be generated, so can \( f'(x) \); \( \int_a^x f(x) \, dx \); and \( x = f^{-1}(y) \).

**Theorem IV:**

If \( f(x) \) and \( g(x) \) can be generated, so can \( f[g(x)] \).

**Theorem V:**

Any function \( f(x) \) which is continuous in a closed interval can be generated in this interval to within a preassigned error \( \epsilon > 0 \) using only a finite number of integrators.

Proof of this follows from Weierstrass' Theorem on the approximation to continuous functions by polynomials.

To generate a continuous functions with a finite number of discontinuities, one need only stop the machine at each discontinuity and turn the cranks by hand.

Shannon makes various extensions of the above theorems to a hypothetical machine consisting only of integrators and adders, with more than one independent variable source.
of drive. He also shows how to approximate to constant multiplying factors with gear ratios, in the event that an integrating unit cannot be spared to perform the multiplication by the usual method of \( \int_0^v k \, dv = kv \). He also works out an estimate of the time saved by using an Automatic Speed Control.

The theory up to this point applies to any interconnection of integrating units and adders. Whether these be Kelvin mechanical units, feed-back amplifiers, watthour meters, or anything else is of no significance except insofar as they reproduce the operation exactly. However, most practical machines are also equipped with some sort of devices for introducing functions. If the data is empirical or tabular, or if an economy of integrating units has to be effected, these devices are used. They are generally not as accurate as internal generation methods, but adequate for most of the problems encountered in practice. In the case of mechanical Differential Analysers, input tables are used and the graph followed with a pointer by hand, or as in the case of the University of Pennsylvania machine, by optical following. A function unit for use with the M.I.T. Machines has been designed by R. Taylor, but it has never been utilised because the error theory required is too complicated to apply for all but very smoothly varying functions. This unit entails tabulating fourth differences of the function to be introduced, and then using four integrators to solve \( \frac{d^4y}{dx^4} = \Delta^4y/\Delta x^4 \). (160.)
APPLICATIONS

Because of the vastness of the field of application of ordinary differential equations, we think it not inappropriate to list some of the more outstanding examples which have been solved, if not for the first time, at least in greater detail, by differential analyzers.

<table>
<thead>
<tr>
<th>YEAR</th>
<th>NAME</th>
<th>REF.#</th>
<th>PROBLEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1931</td>
<td>Bush and</td>
<td>136</td>
<td>In the Fermi-Thomas model of complex atoms there arises the equation</td>
</tr>
<tr>
<td></td>
<td>Caldwell</td>
<td></td>
<td>$y^{1/2} \cdot \frac{d^2 x}{dy^2} = x^{3/2}$ subject to $x(0) = 1$ and $x(\infty) = 0$.</td>
</tr>
<tr>
<td>1932</td>
<td>Morse</td>
<td>154</td>
<td>Solutions of the equations arising in the quantum mechanics of collision processes.</td>
</tr>
<tr>
<td>1933</td>
<td>Caldwell</td>
<td>138</td>
<td>A thesis discussing the solution of problems arising in a great many branches of science and engineering, many of them of classical interest.</td>
</tr>
<tr>
<td>1936</td>
<td>Manning and</td>
<td>152</td>
<td>Calculation of the self-consistent field for tungsten.</td>
</tr>
<tr>
<td></td>
<td>Millman</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1945</td>
<td>Maginniss</td>
<td>151</td>
<td>Review of some problems solved by the General Electric Machine in the various fields of Applied Mathematics. The reference contains a good bibliography of other problems solved by this and other machines.</td>
</tr>
</tbody>
</table>
## SECTION E

### GENERATION OF FUNCTIONS

<table>
<thead>
<tr>
<th>YEAR</th>
<th>NAME</th>
<th>REF.#</th>
<th>FUNCTION GENERATED</th>
<th>PRINCIPLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1920</td>
<td>Barbillion and Dugit</td>
<td>16</td>
<td>$f(u,v)$</td>
<td>Uses specialized Watt-meter with two curved needles turning about a common axis.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1. Finding vel. of aircraft relative to air.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2. Control of carburation in explosion motors.</td>
</tr>
<tr>
<td>1929</td>
<td>Borden</td>
<td>20</td>
<td>$f(u,v)$</td>
<td>Mutual reaction between two fixed associated electrical conductors is proportional to the product of the strengths of the magnetic fields produced by the currents in the conductors.</td>
</tr>
<tr>
<td>1935</td>
<td>Sewig</td>
<td>44</td>
<td>$f(u,v)$ and $f(u/v)$</td>
<td>Uses electric circuits.</td>
</tr>
<tr>
<td>1945</td>
<td>Fry</td>
<td>235</td>
<td>Functions not readily formulated or known empirically only.</td>
<td>Two and three dimensional cams.</td>
</tr>
<tr>
<td>1946</td>
<td>Murray</td>
<td>240</td>
<td>Fourier Series.</td>
<td>Electrical system of amplifiers, condensers and resistors.</td>
</tr>
</tbody>
</table>

We have only listed some specific examples. Many of the other machines discussed can be used to generate functions, as indicated. In particular, we refer to the theory of Claude Shannon discussed under the section of Differential Equations. Mechanical differential analyzers are useful for numerical, electronic differential analyzers for graphical generation of functions.
SECTION F

SOLUTION BY ANALOGY

In the physical world, there are groups of phenomena superficially different but capable of the same form of mathematical expression. The phenomena in a group are termed analogues of one another. If the difference is just one of scale, then they are termed models. The solution to a problem in one representation is therefore known if the behaviour of the phenomenon in an analogous representation is known. In such cases, the problem can be solved by analogy, without any particular recourse to the mathematical form.

However, one has to be careful about boundary values and ranges over which the equations are valid. In fact, the limitations of analogue representation have to be carefully studied. But for engineering purposes, the method is a powerful one, and quite sufficiently accurate.

Considerable work has been done at M.I.T. in this direction, but the results are only available in Thesis or Internal Contribution form.

Ideally, the experiments of the analogy method could be carried out in either direction, but we list the practical situation in each case.

The examples are grouped according to the class of problem solved.
Supplementary notes on the tables:-

The noted English Professor, Hele-Shaw, realized that the laws governing the distribution of magnetic flux in two dimensional space are those of streamline motion of a perfect incompressible fluid. He found difficulty in:

1. Mapping the stream-lines of the fluid.
2. Not being able to find a "perfect" fluid.
3. The representation of magnetic permeability.

These difficulties he treated, respectively, as follows:

1. He used thin colour bands consisting of some liquid with a dye like potassium permanganate.
2. He used Sir G.G.Stokes' Theorem that the stream-lines of any viscous fluid flowing in a very thin layer between parallel bounding walls are identical with those of a "perfect" fluid. He found glycerine very satisfactory.
3. For a viscous liquid flowing in a thin parallel layer, the quantity crossing per second across a plane of unit length normal to the flow is proportional to the cube of the thickness of the layer. Hence he could use the thickness of the layer to represent permeability. In this way, he could vary it between 0 and 1000.

He managed to obtain many very good photographs of quite a variety of distributions.

Meuller, working on a Thesis at M.I.T. considers the mechanical - electrical analogy in great detail.
Mechanical stress involves six components at each point, and an integral over the surface gives the total force. The analogous electric potential has only one component at each point, and there is no integral.

Mechanical force is proportional to the displacement. In liquids and gases, the forces are normal to the surface. On the other hand, the Coulomb potential is given by \( \frac{\vec{r} \cdot \vec{n}}{r_n} \), and \( E - E_0 = q/c \).

Shearing forces are proportional to velocity gradients. But Ohm's Law is \( I = E/R \).

Force = mass x acceleration.

But electrically, \( E = L \frac{dI}{dt} \).

Mueller is principally concerned with the problem of longitudinal stability of the aeroplane. His electrical machine will solve a set of linear, total, differential equations with constant coefficients, and the dependent variables occurring with their first and second derivatives only. He is concerned with coupling terms, those describing the effect of one variable upon the other.

He shows that the electrical analogue of a mechanical system in which the stress - strain relation is linear is the classical passive network, for which the coefficient matrix of the governing equations is symmetric about the main diagonal. The longitudinal stability equations do not fall into this classification, however.

He achieves 10% accuracy, as good as the data on which the calculations are based.
<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>ANALOGUE</th>
<th>YEAR</th>
<th>NAME</th>
<th>REF.#</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid Flow</td>
<td>Electrolytic</td>
<td>1943</td>
<td>Schmidt</td>
<td>158</td>
<td>A high precision electrolytic tank for the solution of two dimensional problems. Applied to 1. Ideal fluid flow around a cylinder. 2. Ideal fluid flow around a plate with free rays. The principle also used in aeronautical research.</td>
</tr>
<tr>
<td>Fluid Flow</td>
<td>Electric Circuits</td>
<td></td>
<td>Katzoff S. and Finn R.S.</td>
<td></td>
<td>With reference to jet-propulsion. (M.I.T.)</td>
</tr>
<tr>
<td>Fluid Flow</td>
<td>Windtunnel</td>
<td></td>
<td></td>
<td></td>
<td>Used extensively in aeronautical research.</td>
</tr>
<tr>
<td>Supersonic Flow</td>
<td>Hydrodynamic</td>
<td>1946</td>
<td>Goldmann K. and Meerbaum S.</td>
<td></td>
<td>To study the supersonic flow of incompressible fluids, with reference to aerodynamical research. (M.I.T.)</td>
</tr>
<tr>
<td>Hydraulic Flow in Electrical Networks</td>
<td>1940</td>
<td>Strandrud H.T.</td>
<td>177</td>
<td>This problem has been attacked from both directions by many workers at M.I.T., and is one of considerable practical interest. We give the names of other contributors: A.M. Alexander, T.R. Camp, H.W. Englund, H.L. Hazen, J.D. Collins, E.L. Jones, A.C. Cook, G.J. Laurent, H.W. Lob, M.S. McIlroy, J.S. Quill. Strandrud outlines the status of the general problem. It is to find the hydraulic head at various points for expected maximum flow. The pipe, head, junctions, and quantity of flow are represented respectively by electric resistance, E.M.F., electric junctions and current strength. If the pipes are of variable cross-section, in the electric analogue, ( R=Kx^{-1} ), where ( x ) is 1.85. Strandrud develops non-linear resistances to follow this equation directly, and avoid cut and try methods. He proposes tungsten lamps.</td>
<td></td>
</tr>
<tr>
<td>PROBLEM</td>
<td>ANALOGUE</td>
<td>YEAR</td>
<td>NAME</td>
<td>REF.#</td>
<td>REMARKS</td>
</tr>
<tr>
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<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Vibrating Systems</td>
<td>Electric Circuits</td>
<td>1943</td>
<td>Olson</td>
<td>74</td>
<td>A text book to show the equivalence of electric and mechanical (both rectilineal and rotational) and acoustic systems. Electrical vibrations occur in a network of meshes with resistances, inductances and capacitances as the elements. The number of degrees of freedom = number of independent, closed meshes. Mechanical vibrations occur in a network of meshes with masses, springs and brakes as the elements. The number of degrees of freedom = the number of independent variables required to specify the motion.</td>
</tr>
<tr>
<td>Soil Mechanics</td>
<td>Mechanical</td>
<td></td>
<td></td>
<td></td>
<td>Contributions at M.I.T. by Bunker, Rohde and Jennings.</td>
</tr>
<tr>
<td>Torsion Problem</td>
<td>Membrane</td>
<td></td>
<td>Timoshenko</td>
<td>78</td>
<td>Section 76. See also Sokolnikoff (76).</td>
</tr>
<tr>
<td>Torsion Problem of Elasticity surfaces</td>
<td>Other</td>
<td>Timoshenko</td>
<td></td>
<td>Section 82. Soap films, paraffin surfaces, interfaces between two immiscible liquids.</td>
<td></td>
</tr>
<tr>
<td>Rotating Cylinder of Fluid</td>
<td></td>
<td></td>
<td>Timoshenko</td>
<td></td>
<td>All the problems may be formulated in the Dirichlet form: $\Delta \phi = -2$ with $\phi$ on the boundary = 0; or in the equivalent Neumann form: $\Delta \phi = 0$ with $\partial \phi / \partial n$ on the boundary equal to a preassigned function.</td>
</tr>
</tbody>
</table>
First attempts to solve this problem were static (no adjustment of phase angle possible) and used direct current. For several generating stations of different capacities, situated at different points in the network, operating at different voltages, and connected to the network through transformers of appreciable reactance, one assumes all the impedances are pure reactances, and reduces them all to a percentage basis. Mr. H.R. Wilson showed how this could be achieved in the 1916 issue of the General Electric Review, and W.W. Lewis developed a machine of commercial value by 1920. Short circuit studies are here possible, since phase angle can be neglected.

A static three-phase A.C. machine to investigate performances with unbalanced short circuits.

A three-phase A.C. machine with three-phase generators, motors, static loads and lumped three-phase artificial lines. Many later models have been built. Those with larger ratings facilitate stability studies.

Since rotating systems to represent generating stations are costly and impractical, M.I.T. built a static type miniature system in which the stations are represented by static phase-shifting transformers. Voltage, current, power, reactive power and phase angle can be measured at any point of the network. This machine has 8 generating stations, 60 lines and cables, 40 loads & 4 ratio-changing transformers. It operates at 200 volts and 60 cycles, using 0.5 amperes. 2% on the voltages, 3 to 5% on the power measurements is the order of error. Portable indicating instruments, whose presence need not be corrected for, give the measurements to 1% accuracy. A large problem requires 50 man hours, 20 of which use the machine.
<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>ANALOGUE</th>
<th>YEAR</th>
<th>NAME</th>
<th>REF.#</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network Analysis</td>
<td>Network Analyzer (Concl.)</td>
<td>1930</td>
<td>Travers and Parker</td>
<td>180</td>
<td>A machine on the same principle as the preceding one, but built for the Westinghouse Electric Company. 1.5% accuracy is obtained. Problems solved are in one of four classes: 1. Load division, 2. System stability, 3. Mutual induction, 4. Short-circuit analysis. General It is to be noted that network analyzers have been largely developed at M.I.T. and at the General Electric Co., Schenectady N.Y. However, most large electrical companies are now equipped with them. See also 175.</td>
</tr>
<tr>
<td>Unsteady State Diffusion Problems</td>
<td>Hydrodynamic</td>
<td>1936</td>
<td>More A.D.</td>
<td>172</td>
<td>Describes the &quot;Hydrocal&quot;, a hydrodynamic calculating machine for studying heat transfer and other diffusion problems. The medium in which the flow is to be studied is divided into a number of sections, and each is dealt with separately. Water is supplied from a tank to a number of tubes, each one of which represents a sectional element. Small-bore flow tubes are arranged to have a resistance to the flow proportional to the known thermal (or other) resistance. The volume of water represents the quantity of heat, and the height it rises in the standpipe the rise in temperature. Five, and 18 section models are described.</td>
</tr>
<tr>
<td>Magnetic Problems</td>
<td>Hydrodynamic</td>
<td>1904</td>
<td>Hele-Shaw</td>
<td>169, 170</td>
<td>See text.</td>
</tr>
<tr>
<td>Electric Discharge</td>
<td>Hydrodynamic</td>
<td>1927</td>
<td>Cramp W.</td>
<td>167</td>
<td>Current, self-induction and capacity are represented respectively, by fluid velocity, mass and a surge-pipe. The model is fed by a continuous stream of fluid, and equipped with valves to govern the pressure in the surge-pipe.</td>
</tr>
</tbody>
</table>
IV

DISCRETE VARIABLE COMPUTING DEVICES

NUMERICAL METHODS

Numerical methods of solving almost every type of mathematical problem arising out of physical considerations have been developed, and are described in standard books such as Boole's "Finite Differences", Steffensen's "Interpolation", Freeman's "Mathematics for Actuarial Students " Vol.II, and Milne-Thomson's "Calculus of Finite Differences", ( 184, 199, 189, 197.)

The purpose of digital-type computing devices is merely to facilitate the evaluation of these numerical forms. Now, the basic numerical process is that of addition. Once that is achieved, subtraction can be effected by the use of complementary numbers, multiplication by repeated addition, division by repeated subtraction, and all the other numerical processes by combinations of these four processes. Digital machines, large or small, are therefore essentially only required to be capable of the basic process of addition. The main trend of experimental research in this field has been to make as swift and automatic as possible the mechanization
of evaluating a sequence of numerical processes; of mathematical research to develop numerical processes best suited to the capabilities of the machine at hand.

The experimental development has just been outlined in a very interesting article "On High-speed Calculating Machines" by the American Naval Mathematician H.E. Goheen (236). Reference to this paper is strongly recommended. This subject will be more fully discussed under Section D of the present chapter.

The mathematical development is now to be discussed. For anyone interested in a bibliography of numerical methods, we refer to pp. 338-396 of the Manual of Operation of Harvard's Mark I machine (204). The list is not exhaustive, but gives the works found useful in one and a half years of operation of the machine, and covers the following subjects:

1. Historical Background of Automatic Calculating Machinery.
4. Linear Algebraic Equations, Determinants, and Matrices.
5. Least Squares Methods.
6. Square Roots and Higher Roots of Numbers.
7. The Location and Separation of the Zeros of a Polynomial.
8. The Calculation Of Zeros of a Polynomial.
10. Implicit Functions.
11. Harmonic Analysis.
13. Finite Differences.
15. Direct Interpolation. (Functions of one and of Several Variables.)
17. Interpolation Tables.
18. Asymptotic Expansions.
20. Numerical Integration of Definite Integrals. (Functions of one and of Several Variables.)
23. Integral Equations.

We draw attention to the following supplementary items:—

Marchant Methods (195) published by the Marchant Calculating Machine Co. which are of great value to anyone equipped with a desk computer. Many short-cuts are indicated.

L.J. Comrie's article (2/3) in which he gives some practical pointers on the usual methods used with desk computers. He also points out the usefulness of multi-register adding and subtracting machines for checking tables by differencing.

The problems of evaluating matrices, inverting matrices, and hence of solving simultaneous linear algebraic equations, are made very precise in a series of articles by E.S. Dwyer (186, 187, 188.)
in the interesting new Journal, Psychometrica, devoted to a mathematical study of the human faculties of thought, will, and so on. L.R.Tucker (200) there also gives a method for finding the inverse of a matrix. See also F.Waugh and P.Dwyer (203).

P.D.Crout (185) has devised a short method for evaluating determinants and solving systems of linear algebraic equations with real or complex coefficients, which is very well adapted to desk calculator methods.

H.Lipson and C.A.Beevers (194) give an improved numerical method for two-dimensional Fourier Synthesis for crystals.

The Relaxation Methods developed in the last few years by R.V.Southwell (198, with further applications as listed in the Harvard Manual of Operation,204) are proving to be so useful that we give a brief description of them. In developing the working formulae for the computation of stresses in braced frameworks, Southwell realized that the methods he was using of relaxing the constraints systematically avoided extensive systems of algebraic equations. Furthermore, it contained a fundamental principle in statics, and was of such a mathematical form as to be applicable to many other fields such as electrical networks; gyrostatic systems; vibrating systems; continuous systems involving problems of equilibriu, elastic stability, and vibrations; and non-linear systems. Relaxation methods therefore provide a means of breaking down many problems of the above types into numerical forms readily utilized by digital computers.
THE ALGEBRA OF MACHINE OPERATIONS

There are really two chief approaches to the problem of numerical solution by mechanical means. One is that of direct calculation, the other is that of solution by adjustment.

In direct calculation, the procedure is prescribed and followed through from the beginning, using as many terms as are required to give the desired accuracy. We give an example:

To solve the system of differential equations

\[ \frac{dy}{dx} = f(x, y, z) \]
\[ \frac{dz}{dx} = g(x, y, z) \]

where \( f \) and \( g \) can be evaluated by tables and calculating machines. Then

\[ y = y_0 + \int_{x_0}^{x} f(x, y, z) \, dx \]
\[ z = z_0 + \int_{x_0}^{x} g(x, y, z) \, dx \, . \]

Next, \( y \) and \( z \) are calculated for equally spaced values of \( x \) by replacing the integrals by approximations. Hence if \( x_i, y_i, z_i \) are known for \( i=1, 2, \ldots, (n-1) \), we can get

\[ y_n = y_{n-p} + \int_{x_{n-p}}^{x_n} f(x, y, z) \, dx \]
\[ z_n = z_{n-p} + \int_{x_{n-p}}^{x_n} g(x, y, z) \, dx \, , \]

where \( p \) is an integer depending on the accuracy desired. We need \( p \) starting values, which may be got, for example, by Taylor Series.

In the method of solution by adjustment, the procedure is prescribed as a closed cycle, and it is followed through repeatedly until the desired accuracy is attained. The mathematical formulation of this method generally used is the Gauss-Seidel Method applied to systems of linear algebraic equations. For the details see Murray p. 85 (240).
Experience with the new large-scale digital computers has shown that there is a need to develop:

1. Self-terminating formulae for all functions. A troublesome function to generate, for example, is $\text{arc sinh } x$.
2. Finite Difference Formulae which shall converge more rapidly.
3. Iteration formulae.

Considerable work is being done along these lines at the present time by R. Courant in New York. See listings in the Harvard Manual of Operation (204) under the heading of Partial Differential Equations. See also (182.)

ERRORS AND FALLACIES

The need for very careful analysis of the mathematics used in the high speed machines has become evident through practice. The ever important question of convergence becomes exceedingly important when the mesh length becomes very small and the number of net points correspondingly greater. This point is discussed in greater detail in the next chapter. Errors which would be negligible in a less extensive computation may now accumulate sufficiently to produce a result which is very far from the truth. It is beyond the scope of this thesis to discuss these matters. We only wish to emphasize their importance and to give a few illustrations. Reference may be made to the lectures delivered at the Harvard Symposium in January of this year and now being

Hans Rademacher, Professor of Mathematics at the University of Pennsylvania, is currently investigating the accumulation of round-off errors, which may become serious even in a 23 digit number when the rounding off is performed some few million times. Truncation errors, arising out of replacing the continuum by a mesh of points of finite spacing, are also under investigation. See also (201).

Checking routines, which are inserted at every possible stage of the computation, may easily be fallacious. A trivial example illustrates this point. Suppose the checking formula is \( (2-x) 10^{-10} = 0 \). Check the point \( x=1 \). Left hand side is then \( 10^{-10} \), which is very small, yet \( x \) is in error by 50%. Again, suppose the checking formula is \( (2.01-x) 10^{10} = 0 \). Check the point \( x=2 \). Left hand side is then \( 10^8 \), which is very great, yet \( x \) is only in error by 0.5%.

It is hoped that this, and the discussion in the next chapter, is sufficient to indicate the trend of mathematical thought stimulated by the needs and capabilities of our new large-scale machines.
SECTION A

PRIMITIVE METHODS

These are referred to briefly in a great many articles. We would refer to the article on Calculating Machines in the Encyclopaedia Britannica (206) and to a quaint little pamphlet which may be obtained from the Marchant Calculating Machine Company entitled "From Og to Googol". (224). We mention a few of the methods.

The use of the 10 fingers for counting is a very natural way. Other pairs of hands may be called into use to represent the tens and the hundreds and so on. Sea shells of various colours and sizes may be used similarly. Knotted strings were used by the Chinese, and developed independently by the Peruvians under the name of Quupus. The Romans developed a checkerboard arrangement that was very useful for business transactions. These were all in use Before Christ.

About the most successful of the devices of the pre-Christian era was a system of wires and beads known in Ancient Rome as the Abacus, in China and Japan as the Suan Pan, and apparently developed independently in Russia and in India, also. They are still used extensively by Orientals, even by those living in the world's leading cities.

In 1614, John Napier, a Scottish nobleman, devised a unique mechanical means of calculating by a system of numbered rods called "Napier's Bones" in which multiplication is performed by addition.
SECTION B

DESK CALCULATORS

A great many articles deal with these machines and their applications. E.W.Crew (234) gives a good outline of the working principles. Murray (240) gives considerable detail of the mathematical theory of the working parts. Quite an extensive listing of the machines is given in the article on Calculating Machines given in the Encyclopaedia Britannica (206). We shall merely list the ones of historical or practical interest. See also p. 6.

Time Magazine, November 25, 1946, page 35, records a contest which took place in Tokyo between a Japanese working with an Abacus and an American working with a modern $700.00 Friden Calculating machine. It is interesting to note that the Japanese had the greater speed in addition, subtraction, and division, although he lost out in multiplication. The Japanese also came out with greater overall accuracy. Considering the fact that a pre-war Abacus cost about 25 cents, all that can be said for the modern electrical machine is that it eliminates the necessity for thought.

Blaise Pascal built the first calculator in the modern sense in 1642. His machine and all the others of this section contain three basic mechanisms:

1. Counting mechanism.
2. Digit-adding mechanism. (Leibniz wheel, rocking segment, or Odner wheel)
3. Carrying mechanism.
It should be pointed out that the only basic numerical process is that of addition. Once that is achieved, subtraction can be effected by the use of complementary numbers; multiplication by repeated addition; and division by repeated subtraction. The taking of square, cube and higher roots can be reduced to combinations of the above basic operations, and so, in fact, can all numerical processes. All desk calculators are therefore capable of addition. And, to-day, most of them are capable of automatic subtraction and division if not of automatic multiplication.

We give some average figures on the capabilities of modern desk calculators:-

Capacity: 10 columns of keyboard keys, 10 lower dial digits, 20 upper dial digits, thus enabling, for example, the multiplication of two ten digit figures to give an accuracy of 20 significant figures. See p. 110.

Speed: Varies from 400 to 1300 counts per minute. Compare with the speeds to be attained with the modern large scale digital computers of one count every one or two micro-seconds.

Features: Automatic carriage shift, electrical clearance of dials, decimal point indicators, fully automatic division.

Extras: Features which may be added as extras are fully automatic multiplication, a product-accumulating register, a split keyboard, and negative multiplication and division.

Limitations: Design imperfections can be shown up in certain artificial problems. For example, the 9's do not carry over to the extreme left in the Fridén machine. So that, if one multiplies on a number, and subtracts off the same number, one will obtain the result all zeros. But, if one subtracts off a number, and then multiplies on the same number, one obtains 01010101.... which is obviously not 00000.....
<table>
<thead>
<tr>
<th>NAME</th>
<th>DATE</th>
<th>NATIONALITY</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pascal</td>
<td>1642</td>
<td>French</td>
<td>Capable of addition and subtraction. Built as an aid in his Father's business.</td>
</tr>
<tr>
<td>Morland</td>
<td>1666</td>
<td>English</td>
<td>Capable of +, -, and x. Low-priced machines modelled after this one still used.</td>
</tr>
<tr>
<td>Leibniz</td>
<td>1695</td>
<td>German</td>
<td>Capable of +, -, ÷, and x by addition. Employed the Leibniz stepped wheel.</td>
</tr>
<tr>
<td>Thomas</td>
<td>1820</td>
<td>French</td>
<td>Charles Xavier Thomas employed the stepped drum principle in the first successful commercial model. It was capable of +, -, x, and ÷.</td>
</tr>
<tr>
<td>Babbage</td>
<td>1832</td>
<td>English</td>
<td>A difference or computing engine. Although capable of +, -, x, and ÷, it was primarily designed to tabulate values of a function for equidistant values of the argument by using Finite Difference methods. This is really the original punched-card calculating machine. One set of cards carried the variable, the other set carried the sequence of operations.</td>
</tr>
<tr>
<td>Odhner</td>
<td>1880</td>
<td>Russian</td>
<td>Used pinwheel mechanism. The next five machines are based on the same principles.</td>
</tr>
<tr>
<td>or so.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brunsviga</td>
<td>1892</td>
<td>German</td>
<td></td>
</tr>
<tr>
<td>Dactyle</td>
<td>1905</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Marchant</td>
<td>1911</td>
<td>U.S.A.</td>
<td></td>
</tr>
<tr>
<td>Facit</td>
<td>1918</td>
<td>U.S.A.</td>
<td></td>
</tr>
<tr>
<td>Gauss</td>
<td>1923</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Millionaire</td>
<td>1899</td>
<td></td>
<td>The first successful machine to perform multiplication directly, using mechanical multiplying tables up to 9x9.</td>
</tr>
<tr>
<td>NAME</td>
<td>DATE</td>
<td>REMARKS</td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>-------</td>
<td>----------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Madas</td>
<td>1908</td>
<td>Some still in use.</td>
<td></td>
</tr>
<tr>
<td>Mercedes-Euklid</td>
<td>1910</td>
<td>Some still in use.</td>
<td></td>
</tr>
<tr>
<td>Monroe</td>
<td>1911</td>
<td>Revised since that date and currently very popular.</td>
<td></td>
</tr>
<tr>
<td>Comptometer</td>
<td>1927</td>
<td>Capable of automatic addition.</td>
<td></td>
</tr>
<tr>
<td>Burroughs</td>
<td></td>
<td>These are the most common adding and listing machines used in business in this country today. They all occur as one or more of the following forms:—</td>
<td></td>
</tr>
<tr>
<td>Corona</td>
<td></td>
<td>a) Single counter adding machines, with subtraction by the complementary method.</td>
<td></td>
</tr>
<tr>
<td>Smith-Corona</td>
<td></td>
<td>b) Same but with direct subtraction.</td>
<td></td>
</tr>
<tr>
<td>Federal</td>
<td></td>
<td>c) Duplex and multiple counter adding machines without direct subtraction.</td>
<td></td>
</tr>
<tr>
<td>National</td>
<td></td>
<td>d) Same but with direct subtraction.</td>
<td></td>
</tr>
<tr>
<td>Sundstrand</td>
<td></td>
<td>See Encyclopaedia for further details. Prices range from about $80.00 up.</td>
<td></td>
</tr>
<tr>
<td>Underwood</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Victor</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>National</td>
<td></td>
<td>Cash registers, capable of addition. These are of enormous importance in small business.</td>
<td></td>
</tr>
</tbody>
</table>

F. Emde (2/5) has considered the accuracy of which the modern machines are capable. The common arrangement of $n$ figure capacity in two of the registers and of $2n$ in the products register is impractical, and it is better to have all three registers of the same capacity. However, each register should have sufficient capacity for dealing with two or three more figures than are required in the final result. We quote these considerations for what they may be worth.
SECTION C

Punched Card Machines

In the early nineteenth century, it was realized that it would be of great advantage to design a machine in which the arithmetical processes, from statement to answer, should be both mechanical and automatic. This is really the principle of sequence control, now being exploited to such a great extent in our modern, large-scale digital computers.

Babbage, taking his cue from the Jacquard punched cards used in pattern loom weaving, invented his analytical engine. Scheutz, a Swedish printer, later refined it. It remained for Dr. H. Hollerith, spurred by the needs of the American Government in taking its census, to introduce the electromagnetic principle, and so develop the whole family of Hollerith Machines. These are now entirely under the control of the International Business Machines Corporation.

Another family of punched card machines also developed is the Powers' Family, now under the control of the Remington-Rand Company. The Powers machines are not as rapid or flexible as the I.B.M. Machines, their cards being punched with round holes and sorted mechanically.

Hollerith Machines - Descriptive

In the Hollerith method of computation, the numbers are recorded by means of rectangular holes punched in cards, and are read into the machines by means of electrical contacts through the holes. In all the machines, the principle
is the same. The tabulating card, acting as an insulator, passes between a wire brush and a brass roller. A hole punched in the card permits the brush and the roller to make contact, thus completing an electric circuit which contains an electromagnet. The position of the hole in the card determines the time when the magnet is operated, and all parts of the machine which are synchronized with the passage of the card will therefore perform different functions depending upon the position of the hole in the card. In the sorting machine, this magnet opens a chute along which the card slides until it reaches the proper receptacle. In the adding machines, the digits on the card are added. In the multiplying machine, the multiplicand and multiplier are read from the card. To give an idea of the speeds, sorting is done at the rate of 20,000 cards per hour; the summary punch operates at the rate of 10 columns per second, and 1.5 seconds are required to feed a new card.

There are various other specialized machines. The tabulator is a high-speed adding machine of large capacity which reads the numbers to be added from holes punched in the cards. An attached printing mechanism will print information from each card or the accumulated totals only. The summary punch punches on a new card the totals accumulated in the counters of the tabulator. The high-speed reproducer transfers information from one card to another. The interpreter prints in type at the top of the card the information contained on the card.
An automatic plugboard reduces the setting up of a battery of machines to work through any one problem to a simple matter. Literal as well as numerical data can be handled by machines designed for the purpose. Of course, a suitable coding system makes the handling of any type of data possible with the straight numerical machines.

**Hollerith Machines - Applications**

The question is not so much, "Can the problem be solved by the machines?" but, "Have I enough operations of this type to justify such powerful equipment?"

The machines were originally developed for business purposes. The book printed in 1935 under the direction of G.W. Bachne contains many contributions from persons working with Hollerith Machines in the following and related fields:

- Registration of Personnel.
- Business and Business Administration.
- Scoring of Psychological, educational and other tests.
- Medical, Legal, and other Research Projects.
- Statistical Problems.

This text (/3/) also contains a brief, well-illustrated account of the various types of machines.

The text by W.J. Eckert (2/4) is more recent (1940). Eckert has been connected with the Thos. J. Watson Astronomical Computing Bureau attached to Columbia University in New York City, and shows how various scientific problems
can be formulated for solution by Hollerith Machines, so that any skilled operator of the machines could solve them. He also includes a few pages of descriptive material. The problems discussed are:

- Interpolation, Mechanical Quadrature etc.
- Numerical Harmonic Analysis and Synthesis.
- The Multiplication of Series.
- The Numerical Solution of Differential Equations.

He gives applications to various fields of science and engineering, with particular emphasis on astronomy.

W.L. Comrie's paper of 1944 (2/3) outlines some outstanding examples which we quote out of general interest.

1. The synthesis of the harmonic terms in the Moon's motion from 1935 to 2000 required 7 months; this was adding figures at the rate of 20 a second. It used half a million cards in which were punched twenty million holes. Some hundred million figures were added.

2. A Fourier Analysis involving 140,000 multiplications was performed and checked by two operators in four days.

S. Bergman of Harvard University (208) formulates the torsion problem of elasticity in the Dirichlet form, that is, Laplace Equation with the function specified on the boundary. He then works out the details of a punched card routine for the solution, using his previously published method of solution of orthogonal polynomials.
L. Fienstein and M. Schwarzschild (2/6) have treated the equation \[ \frac{d^2y}{dx^2} = b(x) \cdot y + c(x) \].

M. Kormes (2/7) has prescribed and done some exercises on a punched card routine for the solution of the Dirichlet Problem, which is \[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \] with \( u = f(x, y) \), a prescribed function on the boundary. We now indicate the procedure as an illustration of the method:- Replace the plane continuum for \( x \) and \( y \) by a fine net of mesh-length \( h \).

Using the definition \( \partial u/\partial x = \lim_{h \to 0} \frac{u(x+h, y) - u(x-h/2, y)}{h} \), we have the difference equation corresponding to Laplace equation

\[ \frac{1}{h^2}\left[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)\right] = 0. \]

Therefore \( u \) at an interior point of the net \( P(i, k) \) is given by

\[ u_{i,k} = \frac{[u_{i-1,k} + u_{i+1,k} + u_{i,k-1} + u_{i,k+1}]}{4}. \]

Assume a reasonable set of values for the interior points of the net as a first approximation. Apply this last equation to get a second approximation. Repeat until the values stop changing in the last significant figure desired. This procedure converges to the solution of the difference equation.

And the solution of the difference equation converges to the solution of the Dirichlet problem as \( h \to 0 \). To give an idea of the speed, Kormes used 145 net points and required 30 minutes of machine time for each approximation. Some 10 approximations were required to give a two figure accuracy.

M. and J. P. Kormes (2/8) have done similar work on some initial value problems.

G. W. King (192) and Thomas G. B. and King G. W. (192) have prepared tables of exponential and logarithmic functions.
SECTION D

AUTOMATIC SEQUENCE CONTROLLED CALCULATORS

As pointed out in the first page of this chapter, the main trend of experimental research in the field of digital computers has been to link up and speed up the evaluation of any required sequence of numerical operations. Apart from the two families of machines just described which operate entirely on punched cards, we shall list the machines in chronological order, and the trend of development will become obvious.

At this point, reference may be made to the article on High Speed Calculating Machines by H.E. Goheen (236); to a humorous, philosophical article in the New Republic by J.R. Newman in which he rumours that the new Princeton machine may be called MANIAC (Mechanical And Numerical Integrator And Calculator.) (244); and to the Seminar Papers given this Spring at M.I.T. with reference to the proposed Project Whirlwind (226).

The first of these gives a very imaginative overall picture, the second a brief, amusing one, the third a highly technical discussion of the structure of the modern giants.

When once the numerical method for the problem has been established by trained mathematicians, the process for these machines is briefly as follows:

Initial settings are made. Values of the independent
variables and the main sequence and sub-sequences of the operations to be performed on them are coded and prepared for input on punched cards or whatever it may be. Everything is put into position and the machine is set into operation.

In the course of the computation, some numbers have to be stored to await further use, and this is achieved by some internal storage technique. Choices are made between one of two numbers or between one of two procedures depending on the condition of some outside factor to appear as the computation progresses. Interpolation processes may go into operation as required, perhaps running through some pre-prepared table of values. Sub-sequence routines to evaluate determinants and so on may go on independently, the results being stored until required in the main sequence pattern. Checking circuits carry out the pre-coded checking routines at each stage of the game, and stop the entire machine if the error tolerance is exceeded. Only the end results desired are put out and recorded in usable, usually typewritten, form.

For obvious physical reasons, the most economic digital system to use is the binary system. Therefore many of the latest machines are using binary representation throughout, converting from and to the decimal system only at the beginning and end of the computation, respectively. Conversion can be achieved by a further sequence of arithmetical processes, and a part of the machine will simply be employed for this purpose. Designers, but fortunately not operators of the
machines, will have to be familiar with binary arithmetic. Although it looks odd, it is not very difficult, since the basic rules for addition are simply

\[0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0\text{ carry } 1.\]

And the only multiplication tables are

\[0 \times 0 = 0, \quad 0 \times 1 = 1 \times 0 = 0, \quad 1 \times 1 = 1.\]

One process for converting a decimal integer to a binary is

\[
\begin{align*}
109 & \div 2 = 54 + 1 \\
54 & \div 2 = 27 + 0 \\
27 & \div 2 = 13 + 1 \\
13 & \div 2 = 6 + 1 \\
6 & \div 2 = 3 + 0 \\
3 & \div 2 = 1 + 1 \\
1 & \div 2 = 0 + 1 \\
\end{align*}
\]

So that \(109 \equiv 1101101\)

The reverse process for converting a binary integer to a decimal is

\[
\begin{align*}
2 \times 0 + 1 & = 1 \\
2 \times 1 + 1 & = 3 \\
2 \times 3 + 0 & = 6 \\
2 \times 6 + 1 & = 13 \\
2 \times 13 + 1 & = 27 \\
2 \times 27 + 0 & = 54 \\
2 \times 54 + 1 & = 109 \\
\end{align*}
\]

So that \(1101101 \equiv 109\)

Fractional numbers may be treated as follows

\[
\begin{align*}
0.142 \times 2 & = 0.284 & \text{So that } 0.142 & \equiv 0.0010010--- \\
.284 \times 2 & = 0.568 & \text{And then } \pi & \equiv 3.142 \equiv 11.00100--- \\
.568 \times 2 & = 1.136 \\
.136 \times 2 & = 0.272 \\
.272 \times 2 & = 0.544 \\
.544 \times 2 & = 1.088 \\
.088 \times 2 & = 0.176 \\
\end{align*}
\]

Incidentally, note that the Harvard Mark II machine uses a special 4 digit system as described by Goheen p.10 (236).

Further details on number representation may be found in books on the Theory of Numbers.
As a simple example of the sort of sequence of orders that is coded into the machine at the outset is the approximation to \( \exp(x) \) by \( 1 + x + x^2/2 \) for \( x \ll 1 \):

- \( x \) and computation orders introduced on external storage to internal storage.
- Add 1 and \( x \).
- Store \( (1+x) \) on internal storage.
- Compute \( x^2/2 \) from stored value of \( x \).
- Add \( x^2/2 \) and \( (1+x) \), to get the final result \( \exp(x) \).
- Store \( \exp(x) \) internally, or transmit to output for external storage.

It is to be noted that all the larger machines listed are all equally flexible as regards solving a variety of problems. In the tables we give only the problems which they were originally intended to solve. These were principally of such a nature as to aid in the war effort. In fact, the E.N.I.A.C. at the University of Pennsylvania was built under such severe war pressure as to contain many serious faults of design and workmanship, and is now almost considered obsolete. However, this is a slight exaggeration. The Mark II machine of Harvard has many special design features such as four-relay number system, and round off to the lowest digit rather than the nearest one, which have marked disadvantages, but make the machine better adapted to the special Naval Problems it is to solve.
Notes on the tables to follow:

The data on the American machines is largely taken from a table recently compiled by Mr. Jos. O. Harrison, Jr., who is working with the relay Government machines at the Aberdeen Proving Ground, Maryland. These figures are as reliable as possible. However, it must be realized that they are tentative as the machines are constantly being re-designed, improved, and so on. For example, the design of the Princeton machine very much depends on the development of the selectron tube, which is proving to be quite a difficult engineering problem. If this tube is not soon ready for use, then the entire machine may be re-designed to use magnetic storage.

Data on the Bell Telephone Machines is average, over the various machines of that type.

In addition to those machines mentioned in the tables, computers are being considered by the Eastman Kodak Company, Rochester, N.Y.; the Prudential Life Insurance Company; and the National Bureau of Standards. The latter has extended contracts to both the Raytheon Manufacturing Co. and the Electronic Control Co., the former to build a machine for Navy and other security problems, the latter to build a machine to take care of census problems. In fact, the National Bureau of Standards eventually hopes to develop an optimum model for general scientific and engineering problems, this to be made commercially available.
<table>
<thead>
<tr>
<th>NAME</th>
<th>REF.#</th>
<th>PURPOSE</th>
<th>DATE</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duplex</td>
<td>223</td>
<td>Survey problems.</td>
<td>1914</td>
<td>Two desk calculators in tandem. Sequence only semi-automatic.</td>
</tr>
<tr>
<td>Comptometer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I.B.M. Difference Tabulator</td>
<td>230</td>
<td>Interpolation and calculation of correlation coefficients.</td>
<td>1930</td>
<td>Developed under the I.B.M. research project at Columbia University, New York.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weygant</td>
<td>222</td>
<td>Matrix Multiplier and algebraic equation solver.</td>
<td>1933</td>
<td>An electrically powered mechanical machine. The multiplying action is achieved with telephone components and relays operated by two keys. Sequence of multiplications and additions is automatic.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tucker</td>
<td>220</td>
<td>Matrix Multiplication (15 rows, any number of columns)</td>
<td>1940</td>
<td>Operating in Dr. L. Thurstone's lab. at the Univ. of Chicago. A modification of the I.B.M. Test Scoring Machine. The values of the product elements are indicated one at a time by a milliammeter. Gives 2 digit accuracy. 15 min. for 60x12 det.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Twin Marchant</td>
<td>211</td>
<td>Various formulae.</td>
<td>1942</td>
<td>Two desk calculators in tandem.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alt</td>
<td>205</td>
<td>Matrix Multiplication</td>
<td>1946</td>
<td>A special new I.B.M. general multiplier. Has been used on pairs of 9 row matrices with elements of not more than 6 significant figures which are done in 9 minutes. Cards with the elements and the row and column numbers punched in are inserted, and the product of the two matrices is produced without further attention.</td>
</tr>
<tr>
<td>LOCATION</td>
<td>INTERNAL STORAGE</td>
<td>MANUFACTURER PROBLEMS FOR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------------------------</td>
<td>------------------</td>
<td>---------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ballistic Research Lab. Aberdeen Proving Ground, Maryland.</td>
<td>Relay</td>
<td>I.B.M.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Government Ordnance Department. Two such machines.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Army Ground Forces # 4, Fort Bliss, Texas.</td>
<td>Relay</td>
<td>Bell Telephone Labs.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Army Ground Forces</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Naval Research Lab. Bellevue, Maryland.</td>
<td>Relay</td>
<td>Bell Telephone.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Navy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Naval Proving Ground, Dahlgren, Va.</td>
<td>Relay</td>
<td>I.B.M.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Naval Ordnance.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>National Advisory Committee for Aeronautics.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All the above machines are under Government contract and have been recently built or are about to be completed. They are all small, automatic sequence machines, and are listed in the Symposium Notes (23/).
<table>
<thead>
<tr>
<th>Calculator</th>
<th>Harvard Mark I</th>
<th>I.R.M.</th>
<th>Bell Telephone</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>Relay</td>
<td>Relay</td>
<td>Relay</td>
</tr>
<tr>
<td>Circuits</td>
<td>Decimal</td>
<td>Decimal</td>
<td>Decimal</td>
</tr>
<tr>
<td>Number System</td>
<td>Electro-mechanical Counters</td>
<td>Relays</td>
<td>Relays</td>
</tr>
<tr>
<td>Storage Equipment</td>
<td>One digit for sign; 23 decimal digits with fixed decimal point.</td>
<td>One digit for sign; 23 decimal digits with fixed decimal point.</td>
<td>Sign, 7 decimal digits, &amp; an exponent (base 10) from -19 to +19.</td>
</tr>
<tr>
<td>Internal Storage Capacity</td>
<td>72 numbers</td>
<td>27 numbers</td>
<td>30 numbers</td>
</tr>
<tr>
<td>Program Control</td>
<td>Punched Paper tapes</td>
<td>Hand Plugboard</td>
<td>Punched Paper tapes</td>
</tr>
<tr>
<td>Automatic Operations</td>
<td>$+, -, \times, \div, \sin, \log_{10}, \exp_{10}$, interpolation.</td>
<td>$+, -, \times, \div, \sqrt{}$, $\sin, \cos$, arc tan, $\log_{10}$, $\exp_{10}$</td>
<td></td>
</tr>
<tr>
<td>Add Time</td>
<td>1/3 sec.</td>
<td>0.025 sec</td>
<td>0.3 sec</td>
</tr>
<tr>
<td>Multiply Time</td>
<td>6 sec.</td>
<td>0.2 sec</td>
<td>1 sec</td>
</tr>
<tr>
<td>Input, Output</td>
<td>Card feeds and card feeds</td>
<td>Card feeds and card punch</td>
<td>Paper tape readers and punches, Typewriters</td>
</tr>
<tr>
<td>External Storage Devices</td>
<td>Card feeds and card punch.</td>
<td>Paper tape readers and punch.</td>
<td>Two typewriters</td>
</tr>
<tr>
<td>Normal Checking Procedure</td>
<td>Programmed checks</td>
<td>Separate runs on two different checking</td>
<td>Self-calculators</td>
</tr>
<tr>
<td>Location</td>
<td>Harvard Computation Laboratory, U.S.A.</td>
<td>Aberdeen Proving Ground, Md., U.S.A.</td>
<td>See page 122</td>
</tr>
<tr>
<td>Date of Completion</td>
<td>August 1944</td>
<td>Recent</td>
<td>Recent</td>
</tr>
<tr>
<td>Cost of size and</td>
<td>$400,000</td>
<td>51' x 8'</td>
<td></td>
</tr>
<tr>
<td>Problems Used for</td>
<td>Navy Ordnance, Hankel &amp; Bes- sel functions</td>
<td>Government</td>
<td>Government</td>
</tr>
<tr>
<td>Sub-Sequence mechanism</td>
<td>10x10 determinant; choice of the better of two series</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ref. Numbers</td>
<td>225, 205, 230</td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td>-----------</td>
<td>------------</td>
<td>----------------</td>
</tr>
<tr>
<td>Mark II</td>
<td>Electronic</td>
<td>Electronic</td>
<td>Princeton</td>
</tr>
<tr>
<td>Relay</td>
<td>Decimal</td>
<td>Binary</td>
<td>Electronic</td>
</tr>
<tr>
<td>Decimal</td>
<td>Decimal</td>
<td>Binary</td>
<td>Binary</td>
</tr>
<tr>
<td>Relays</td>
<td>Vacuum Tubes</td>
<td>Mercury</td>
<td>Selectrons</td>
</tr>
<tr>
<td>(Coded</td>
<td>(18,000)</td>
<td>Tanks</td>
<td></td>
</tr>
<tr>
<td>decimal)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sign, 10 decimal</td>
<td>Sign, 10 digits, &amp; an</td>
<td>44 binary digits with</td>
<td>40 binary digits with</td>
</tr>
<tr>
<td></td>
<td>decimal digits</td>
<td>fixed binary</td>
<td>fixed binary</td>
</tr>
<tr>
<td>exponent(base 10) with fixed</td>
<td>point.</td>
<td>point.</td>
<td></td>
</tr>
<tr>
<td>from -15 to +15</td>
<td>decimal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>point.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 numbers</td>
<td>20 numbers</td>
<td>1024 words</td>
<td>4096 words</td>
</tr>
<tr>
<td>Paper</td>
<td>Plugging and</td>
<td>Orders from</td>
<td>Orders from</td>
</tr>
<tr>
<td>tapes</td>
<td>switch setting</td>
<td>internal</td>
<td>internal</td>
</tr>
<tr>
<td></td>
<td></td>
<td>storage</td>
<td>storage</td>
</tr>
<tr>
<td>+, -, x, 1/√,</td>
<td>+, -, x, ¹,</td>
<td>+, -, x, ¹</td>
<td>+, -, x, ¹</td>
</tr>
<tr>
<td>reciprocal,</td>
<td>2¹</td>
<td></td>
<td></td>
</tr>
<tr>
<td>cos, arc tan,</td>
<td>log₁₀, exp₁₀</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200 micro-sec.</td>
<td>50 micro-sec</td>
<td>10 micro-sec.</td>
<td></td>
</tr>
<tr>
<td>0.7 sec.</td>
<td>2800 micro-sec.</td>
<td>2000 micro-sec.</td>
<td>100 micro-sec.</td>
</tr>
<tr>
<td>Card Feeds</td>
<td>Magnetic tape</td>
<td>Magnetic tape</td>
<td></td>
</tr>
<tr>
<td>and card</td>
<td>readers and</td>
<td>readers and</td>
<td></td>
</tr>
<tr>
<td>punch.</td>
<td>magnetizers.</td>
<td>magnetizers</td>
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<td>Four typewriters</td>
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<td>Programmed</td>
<td>Alternate test</td>
<td>Self-checking</td>
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<td>checks</td>
<td>runs with production runs</td>
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<tr>
<td>Cambridge Mass.</td>
<td>Built at Moore</td>
<td>Moore Sch.</td>
<td>Institute for</td>
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<td>To be moved to</td>
<td>Sch. U. of Penn.</td>
<td>U. of Penn.</td>
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<td>Estimate</td>
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<tr>
<td>$400,000</td>
<td>$500,000</td>
<td>end of 1947</td>
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<td>Navy</td>
<td>Army</td>
<td>War. Dept.</td>
<td>Non-linear</td>
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<tr>
<td>Ordnance</td>
<td>Ordnance</td>
<td>Ordnance</td>
<td>Appl. Maths.</td>
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</tbody>
</table>

227 228, 229
| Whirlwind  | N.P.L.     | Univ.of | Ceska  |
| I & II    | England    | Cambridge | Technika  |
| Electronic | Electronic | Electronic | Relay |
| Binary    | Binary     |          |        |
| Electrostatic | Mercury |          | Magnetic |
| dielectric | delay      |          |        |
| plates     | line       |          |        |
| 40 binary digits. | Semilogarithmic | to take care of binary point. | 40 binary digits |
| 195,000 words |          |          | 10,000 words |
| Orders from internal storage |            |          | 1 micro-sec. |
| +,-,x,÷ |          |          | Film or magnetic tape |
|          | Electronic | computing circuits |
| N.I.T. | National   | England | Technologishes |
| U.S.A. | Physical Lab. England | Institut,Praha, Tchechoslovakia |
| 1948, 1950 |          | No time limit | 1-200,000 crowns (≈20,000) |
| Very large | Large Pilot model first | General Applied Mathematics Problems |
| Guided-missile flight (Navy) simulator. | (To be Called Electronic Delay Storage Automatic) |
| Determinants etc. | Dr. Womersley | A. Svoboda |
SECTION E

ELECTRICAL NETWORK REPRESENTATION

Although somewhat similar in practice, these methods must be distinguished from the continuous variable techniques used in network analysis. The numerical solution to any required degree of accuracy of many partial differential equations and of some integral equations by considering the solution at the points of a net in mathematical space can be achieved by making measurements at the corresponding junctions of an equivalent electrical network. The mathematical potentialities of such methods are now being extensively investigated.

It has long been known (see for example, Emmonds (234-) or Vazsonyi (202)) that the two dimensional Laplace Equation \( \Delta \phi = 0 \) in rectangular coordinates, may be solved to any degree of accuracy by drawing a net of squares each with area \( \Delta x \Delta y \), and associating with line segment the number unity. If, with each junction representing a point of space, the correct value of \( \phi \) is associated, then each \( \phi \) must be the average of the four neighbouring \( \phi \)'s. See p. 115.

In his paper of 1945 (193) Kron shows how to extend this principle to certain linear and non-linear partial differential equations, This involves permitting several lines in the net to meet at a junction at arbitrary angles, associating with each line an arbitrary complex number, and associating each junction with not one but several dependent
variables. Kron gives the details of the physical interpretation of the mathematical nets associated with certain initial value, boundary value, and characteristic value problems. The details have to be studied carefully.

For initial value problems, the networks may be looked upon as supplying a schedule of operations that could be put into a digital calculating machine. For time varying problems, time can be introduced as an extra spatial variable. The example of transient heat flow is given.

For boundary value problems, there are four possible methods of solution, the first three being cut-and-try methods. They are: 1. The method of weighted averages. 2. The method of unbalanced currents and voltages. 3. Relaxation methods. 4. The diffusion method, in which the boundary value problem is changed into an initial value problem by adding to the original partial differential equation a time variable of the form $A\partial \Theta/\partial t$, allowing the currents to "diffuse in time".

For characteristic value problems, the methods are similar to those for boundary value problems. An additional method of unbalanced admittances is also indicated. By calculating the power in the network, the characteristic value of the assumed function is found. The example of the linear harmonic oscillator is given.

It is noted that the unbalanced currents at the junctions are easily calculated to give a measure of the deviation from the correct solution. Direct network analyzer sol-
utions may be checked and refined by these methods of Kron.

Sample calculations have been done for many problems
by research engineers at the General Electric Company at
(193) gives several references at the end of his article.
Notable examples are on:-

Elastic Field Problems
Compressible and Incompressible Flow Field Problems
The Schrödinger Equation of Atomic Physics
Maxwell's Field Equations
Molecular Vibrations (J. Chem. Phys. 1946.)

The behaviour of many phenomena may also be represented
by integral equations or by finite difference equations.
Equivalent circuit methods of solving such equations can also
be worked out. This has been done for the equations of
Elasticity by Carter (270) in a companion paper to that of
Kron (279).

It is interesting to note the proposal for a mech-
anical analogue of the simplest of these electrical networks
recently made by I.S.Bruk (209). It solves the Poisson
Difference Equation by using several parallel racks, placed
between guides and connected by gears mounted in the reces-
es of the racks.
RELATIONSHIP WITH PURE MATHEMATICS

BEARING ON ANALYSIS

From the point of view of pure mathematics, computing devices can only be regarded as the tools of analysis—sometimes easing the labour of numerical computation, sometimes avoiding it altogether by giving an approximate solution by the principle of analogy. Thus it is apparent that such devices can never replace the methods of analysis. However, they can help to guide the calculations of the pure mathematician by giving an approximate form of the solution which is sought. They can also challenge his skill at establishing existence theorems with proper regard to initial and boundary conditions. We give a few relevant remarks from the Symposium lecture delivered by R. Courant on the Method of Finite Differences for the Solution of Partial Differential Equations.

Any numerical method replaces an analytical problem $P$ with solution $S$, which we will suppose exists, by an approx-
imate problem $P_h$ with solution $S_h$. When $h \to 0$, $P_h \to P$. Does $S_h \to S$? Answer is no, unless $P_h$ is properly chosen. It can be shown that $P_h$ is properly chosen if one introduces difference quotients of any order in place of the corresponding differential coefficients. The convergence of $S_h$ upon $S$ increases in rapidity with the order of the differential equation. Hence, if $L$ is a differential operator, why not solve $LL(u) = 0$ instead of $L(u) = 0$?

Courant also points out that the domain of dependence of the functions upon the independent variable is equivalent to the range of influence of a set of the initial and boundary data. All values used in finite difference methods must be kept inside this range to ensure convergence. John Von Neumann of the Institute for Advanced Study at Princeton is interested in computational experiments on convergence, as an aid in developing the theory.

Since the transition from a tabular solution to a graphical one is essentially just a mechanical process, the great limitation of machine results is not the form in which they are obtained, but the fact that they cannot be expressed in terms of parameters of the system or of known constants or functions. Electronic continuous variable computers designed to display the results graphically are about the most that can be hoped for in showing readily the dependence of the solution on parameters. In this respect they will be invaluable to the mathematician and to the teacher of mathematics.
It is also interesting to note that analysis is primarily concerned with studying the behaviour of mathematical forms in the neighbourhood of singular points, or, conversely, of building up the mathematical form from a desired behaviour in the neighbourhood of singular points. On the other hand, computational devices are primarily capable of describing the behaviour of mathematical forms over the regions in which they are well-behaved only, and must leave untouched the building up of mathematical forms. Such, then, is the complementary nature of the relationship between analysis and computational methods.

The following is an interesting example of interplay in this relationship, however. The solutions of differential equations can't be obtained in a closed form in the neighbourhood of singular points in many cases. There is then no elegant way of joining the solutions together at the boundaries, as is necessary in physical problems. To avoid such troubles, solutions of differential equations of the form
\[ p(z) \frac{d^2 \phi}{dz^2} + f(z) \frac{d\phi}{dz} + g(z) \phi = 0 \]
can be obtained as an integral of the form
\[ \phi = \int_C K(z,t) \nu(t) \, dt . \]
Such integrals can often be handled conveniently with cinema integrals.
The following table shows the position of computing devices in the Scientific Method:

<table>
<thead>
<tr>
<th>Scientist</th>
<th>Contribution</th>
<th>Errors</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimentalist</td>
<td>Observation</td>
<td>Physical Measurement, &amp; Methodological.</td>
<td></td>
</tr>
<tr>
<td>Applied Mathematician</td>
<td>Formulation</td>
<td>Idealization</td>
<td>Error may be reduced by using empirical data.</td>
</tr>
<tr>
<td>Pure Mathematician</td>
<td>Discovery of Existence Theorems.</td>
<td>None</td>
<td>More extensive theorems now required.</td>
</tr>
<tr>
<td></td>
<td>Numericalization</td>
<td>Truncation</td>
<td>Error decreases with h.</td>
</tr>
<tr>
<td>Practical Mathematician</td>
<td>Finding of Model Representation.</td>
<td>Varies</td>
<td>eg. no error in Kelvin integrator.</td>
</tr>
<tr>
<td></td>
<td>Selection of Numericalization.</td>
<td></td>
<td>Experience invaluable here.</td>
</tr>
<tr>
<td>Operator</td>
<td>Machine Operation</td>
<td>Physical (slip, no round-off backlash, etc.) with continuous or round-off. It increases as $1/\sqrt{h}$.</td>
<td></td>
</tr>
<tr>
<td>Mathematician</td>
<td>Analysis of Results</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>Engineer</td>
<td>Application</td>
<td>According to requirements.</td>
<td></td>
</tr>
</tbody>
</table>
QUESTIONS OF SOLVABILITY

The fabulous speeds and unlimited digital accuracies now becoming a reality in the realm of discrete variable computation can be considered as dividing the question of solvability into five parts, which we now list and illustrate.

1. Same solutions attainable more quickly. eg. Hartree's 15 year calculations on atomic structure could now be done in a few days.

2. Solutions attainable more accurately, since the complexity imposed by using more stringent numerical forms no longer a barrier. eg. various hydrodynamic problems.

3. Solutions which can now be undertaken, formerly considered too extensive. eg. the numerical integration of the equations of motion of Mercury, the Earth, and the Moon for the next 50 years.

4. Solutions which cannot be attained even now owing to unsuitable formulation of the problem. eg. approximation to log n through the inequality

\[ \frac{1}{2} + \frac{1}{3} + - - - + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \frac{1}{3} - - - + \frac{1}{(n-1)} \]

It can be shown that \( 4 \times 10^6 \) terms are required to make \( 1 + \left( 1/2 \right) + \left( 1/3 \right) - - - + \left( 1/n \right) \geq 20 \). A machine that does \( 10^6 \) operations per second would take at least 400 seconds to evaluate this. But to compute \( \log n \) of the order \( 10^6 \) would take at least \( 10^{(10^5 - 6)} \) seconds or about \( 10^{10^5} \) centuries, which is clearly out of the question.

5. Solutions still beyond computational reach, eg. the three body problem in all generality. But see Caldwell(138)pp.25-36.
LOGIC AND PHILOSOPHY

It is not surprising that the advent of such powerful computational tools as our new high-speed, digital machines should require a consideration of the basic postulates of mathematical logic in their design. For, in essence, the machines have to reproduce the basic thought processes in ordered sequence and at high speed. The network of wires to conduct these thought processes have been likened by Dr. N. Wiener to the nervous system in the higher animals. By a study of the nervous system one can therefore hope to gain fresh ideas to apply to the machine designs. H. E. Goheen (236) considers that the four aspects of thinking involved in a routine computation are memory, choice, sequence discrimination, and reflex patterns. The mathematical logic pertaining to these four faculties as detailed, as, for example, in the classic work "Principia Mathematica" by Whitehead and Russell (244) has to be carefully studied.

The converse question of how far the new machines can be used to speed the study of logic has yet to be seen. For considerations such as are detailed in "Psychometrica" (p. 12) for example, deal with the more subtle of the human faculties of adaptability, creative ability, invention, and so on, in a very cold and numerical form. In fact, fears that the brilliancy of our new machines will detract from Man's dignity are apt to be taken as more than a joke. But Pascal, himself the inventor of a computing machine, allayed these
fears once and for all when he said:

"Man is but a reed — the weakest thing in Nature — but he is a reed that thinks. It is not necessary that the whole Universe should arm itself to crush him. A vapour, a drop of water, is enough to kill him. But if the Universe should crush him, Man should still be nobler than that which slays him, for he knows that he dies; but of the advantage which it has over him the Universe knows nothing. Our dignity consists then wholly in thought. Our elevation must come from this, not from space and time which we cannot fill."
APPENDIX

Analysis of Errors
in the Mechanical Differential Analyzer.
OUTLINE

Analysis of errors in the Mechanical Differential Analyzer.

Practical Approach:-

Mechanical Sources of error:
1. Variation of dimensions of moving parts.
2. Slip between integrator wheel and disc.

Practical reduction of these errors:
1. By accurate machining, operating at thermal equilibrium.
2. Torque amplification or polaroid follow-up.
3. Lashlocks and frontlash units.

Operational tests for overall error consisting of comparison of actual results with calculated values.
1. V. Bush's Circle test.
2. Caldwell's linearizing process.

Theoretical Approach:-

Reasons for only treating the case of one independent variable.
Mechanical equivalence of solving differential equations and generating functions.

Derivation of the Error Function=Actual Function-Accurate Function
(i) For direct integrators
1. For one integrating unit  a.) Because of backlash
   b.) Because of backlash and slip.
2. For n-1 integrating units. a.) Because of backlash and slip.
3. Illustration from example of Prof. Caldwell.

(ii) For inverse integrators

General Conclusions.
ANALYSIS OF ERRORS IN THE MECHANICAL DIFFERENTIAL ANALYZER

PRACTICAL APPROACH

Mechanical Sources of Error

From an operational point of view, errors, as distinct from mistakes, can arise in three main ways. 1. As the result of variation of dimensions of the moving parts (this includes gear ratios, scaling, length of shafts, pitches of screw, and so on) 2. As the result of slip between integrator wheel and disc, and as a result of backlash at all connecting points.

In practice, to evaluate $W = \int U dV$, the following factors are defined:

$k$ = the integrator constant, the same for each integrating unit of given design, and therefore usually the same for each unit of a given machine.

$s$ = the scale factor, for each shaft, being determined by the values of the variable to be represented.

So that, $U = s_1 y$ where $y$ = distance of integrator wheel out from zero setting

$V = s_2 x$ where $x$ = number of revolutions of disc.

and $W = ks_1 s_2 \int y dx$.

Practical Reduction of Errors

1. In practice, the dimensions can be kept quite constant by use of suitable materials, by very accurate machining of the parts, and by running the machine "hot", that is, taking readings only when the machine has been running long
Turntable
Drive Shaft
(dx)

Integrator
lead screw
(y)

Output shaft(\int y \, dx)

Fig. 1.

Symbolic Representation of a
Single Integrating Unit

actual y

Fig. 2.

Interconnections of
the Basic Differential Analyzer
Elements.

Fig. 3.

M - motor
F - follow-up system
O - output table
I - input table
T - integrator turntable
L - integrator lead screw
enough with the appropriate cooling system in operation so as to be at thermal equilibrium. Thermal adjustments of the lead screw are made in the new M.I.T. Machine of 1943, as follows:-- the end of the lead screw from which all measurements are made is fixed in pre-loaded ball bearings. The other end is free to expand thermally as the machine warms up, finally attaining a limiting position at thermal equilibrium.

2. To estimate slip, the integrator constant is measured directly. Readings are taken for various carriage settings and at various angular velocities of the disc over some given range, usually 0 to 1000 revolutions per minute. Tests show that departures from the correct constant vary continuously with the torque, and are nearly proportional to it at low speeds. This was the great mechanical difficulty which prevented Lord Kelvin from developing his principles. V. Bush was the first to overcome this difficulty by introducing torque amplification. The principle of the torque amplifier is essentially that of the winch. The winch is kept rotating. A rope is wound several times around it, and a small pull on one end of the rope is found to produce a much stronger pull on the other. The best technical description of one and two-stage torque amplification as applied to differential analyzers is given on p. 57 of vol. 160 of "The Engineer" of 1935. (165). A polaroid interconnection, or rather, follow-up device, is used in the General Electric Machine at Schenech- tady, and provides another solution. Here, the position of
the output shaft is controlled by the position of the integrator wheel with only a beam of polarized light as the interconnection. The integrator constant also depends very slightly on the speed after a certain point. In practice, this point is determined by the error tolerance. A speed control, such as one of the Ward-Leonard type, is usually used to make the independent variable turn at such a speed that the integrator which is turning fastest turns at this top speed. An estimate of the time saved in this way is made by Shannon p.352 (159) by assuming a random distribution of the maximum speeds.

3. Backlash, and as a particular case, errors in the initial settings of the moving parts, can become serious through accumulation if the way in which the shafts are geared together is continually reversed. Originally, in the American machines, lashlocks were used to eliminate backlash within the drives of the integrating units themselves. Nowadays, drives are equipped with frontlash units. So-called by Prof. N.Wiener, these are units with negative backlash to compensate for the positive backlash present in the drives. They are described in detail in (135). Even the polaroid follow-up device suffers from a sort of optical backlash, and, furthermore, introduces problems of instability, as discussed on p.223 of (149).

To check on overall performance, there are several operational tests. Bush's Circle test p.469 of (135) is quite
stringent and widely used. In it, the machine is set up to solve \( d^2y/dx^2 = -y/(K^2G^2) \), where \( K \) is the integrating constant and \( G \) is the gear ratio. This involves two integrators, and therefore provides a method of checking the integrators in pairs. The solution is \( y = A \sin(x+B) \) so that \( (dy/dx)^2 + y^2 = \text{constant} \). Hence, if the output table is connected to record \( dy/dx \) as a function of \( y \), the plotted result should be a circle. Errors would accumulate so that the machine could be left running, and the stability of the solution could be checked over a period of time. Backlash errors make the output pencil spiral outwards. Other errors affect the shape of the circle, introducing a slight eccentricity. Just out of general interest, we show the set-up for solving this problem in the usual notation:

\[
\begin{align*}
\frac{d^2y}{dx^2} &= -\frac{1}{K^2G^2} y \\
\frac{dy}{dx} &= \int_{K^2G^2} y \, dx.
\end{align*}
\]
Tests with the General Electric Machine (149) show that backlash as so determined increases with:

1. Increase in speed of the output shaft.
2. Increase in the directness of the drive.
3. Decrease in the integration constant.

Growths ranging from 0.1 to 1.05% per cycle were observed with the General Electric Machine running up to 970 revolutions per minute.

In the generation of functions with inverses, the following is a practical way of determining overall errors. It was devised by Dr. S. H. Caldwell of M.I.T. and consists of linearizing the function and making a comparison with experiment.

Let \( f(y) = \) inverse function of \( f(y) \). (eg. \( \text{Ie}^y = \log y \)). Hence, \( \text{II}f(y) = f(y) \), by the definition of inverse function. We give a slight generalization of Prof. Caldwell's Theory.

<table>
<thead>
<tr>
<th>( y )</th>
<th>( Y=x )</th>
<th>( Y'=a+bx )</th>
<th>( y' )</th>
<th>Error = ( \frac{y-y'}{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>( \log y )</td>
<td>( \log y' = a+bx )</td>
<td>( e^x )</td>
<td>( a+bx )</td>
</tr>
<tr>
<td>( \log x )</td>
<td>( e^y )</td>
<td>( e^y' = a+bx )</td>
<td>( \log(a+bx) )</td>
<td>( \log x - \log(a+bx) )</td>
</tr>
</tbody>
</table>

and therefore generally

\[ f(x) \text{ If}(y) \text{ If}(y') = a+bx \quad \text{II}f(a+bx) = f(a+bx) \quad f(x) = f(a+bx) \]

So that percentage error in generating \( f(x) = \frac{f(x) - f(a+bx)}{f(x)} \times 100 \)

where \( a \) and \( b \) are constants which can be determined by
plotting the empirical points to get a straight line differing slightly from the linearized function as below:

\[ Y = a + bx \] (empirical)

\[ Y = x \]

Note that

\[ a \neq 0 \]
\[ b \neq 1 \]
THEORETICAL APPROACH

Reasons for treating the case of one independent variable:

Any sort of theoretical analysis would have to develop out of a precise formulation of the classes of functions and equations being considered.

We shall disregard the case of more than one independent variable as not being handled directly by any practical continuous variable machine. Shannon (59) considers this case as a purely abstract one because it is useful in proving some of the theorems for one independent variable. Furthermore, while it is true that differential analyzers have been used on partial differential equations with two independent variables, in each case the problem has been reduced mathematically to a long sequence of ordinary differential equations all of the same form. An account of one such method is given by Hartree and Womersley (46). This method consists of approximating by use of finite intervals in one of the variables, and integrating exactly in the other, later correcting for the leading term of the error introduced by the use of finite intervals. The method depends on the form of the boundary conditions, which must be specified on an open set of boundary points. The method is justified by direct computation for the types of boundary conditions most likely to occur in physical applications. Ideally, such problems are best adapted to solution by the new large-scale digital machines — the mathematical analysis into finite intervals.
being extended to both independent variables, and the operations being carried on at the points of a net.

We shall therefore consider one independent variable \( y_1 = x \) and \( n-1 \) dependent variables \( y_k \) where \( k = 2, 3, 4, \ldots, n \).

**Solving equations and generating functions:**

By Theorem I of Shannon's paper (159), \( f(x) = y \) can be generated if and only if there exists a set of equations

\[
\frac{dy_k}{dy_1} = \sum_{i,j=0}^{n} a_{ijk} \frac{dy_j}{dy_1}, \quad \text{where } k = 2, 3, 4, \ldots, n,
\]

such that one of the \( y_k = f(x) \).

In this respect, the generation of functions and the solving of differential equations are equivalent machine processes. We note here for completeness that Theorem I states that a necessary and sufficient condition that a system of differential equations can be solved using only integrators and adders is that they can be written in the above form.

In practice, integrating units are often connected inversely, so that instead of obtaining \( y = \int (dy/dx) dx \), \( x = \int dy/(dy/dx) \) is obtained. This essentially amounts to integrating the reciprocal of a function rather than the function itself. Both the error function and the exact theory are somewhat different for the inverse wiring.

**Derivation of error function:**

**Direct Integration - One Integrating Unit:**

Refer to Fig. 239 which is taken from page 222 of (149).

The machine is set in motion. When \( x \) is accurately at the
initial point of the range \( x_0 \), then, owing to backlash and initial setting errors, \( y \) is at \( y+q \), where \( q \) may be positive or negative. Suppose \( x \) increases monotonically. Then, as long as \( y \) is monotonic, and hence \( dy/dx \) is of constant sign, only errors of slip and dimension enter in the form of a factor \((1+p)\). Hence, the actual equation of constraint is
\[
Y(x) = (1+p) \int_{x_0}^{x} (y+q) \, dx
\]
as opposed to the ideal constraint
\[
y(x) = \int_{x_0}^{x} y \, dx.
\]
Now consider these two quantities \( p \) and \( q \).

\( p \) is a function of \( l \). The relative speed and acceleration between the integrator wheel and disc above a certain point; 2. Temperature; 3. The coefficients of thermal expansion of the various shafts and so on. Therefore, for a machine running "hot" and with speed control, \( p \) can be taken as a constant.

\( q \) is initially a constant, and changes only when \( dy/dx \) changes sign. In any range over which \( dy/dx \) is of constant sign, \( q \) is therefore a constant.

**Direct Integration — (n-1) Integrating Units:**

**Backlash Errors:**

We take \( y_0 = 1 \), \( y_1 = x \), and the dependent variables \( y_k = y_2, y_3, \ldots, y_n \). Consider \((n-1)\) integrators, including adders in the general sense. Suppose that backlash in the \( k \) th. integrator occurs at points \( x_{ki} \) and is \( q_{ki} \) at the \( i \) th. such point. Note that \( q_{k0} \) includes the initial setting.
error. We assume, as suggested by experiment, that $q_{ki}$ is
1. A direct function of the speed of the output shaft
2. An inverse function of the integrator constant $K$
3. $K$ is positive for direct drive and negative for indirect
drive.

Then the error in $y_k$ at the point $x_{km} = \sum_{i=0}^{m} q_{ki}$.
It is evident that this error is monotonic increasing if the
$y_k$ shaft is always subject to direct drive. However, this
depends on the particular equation being solved. In the
special case of $y = \sin x$, both the $y$ and $dy/dx$ drives
would remain direct, and there would be a steady growth of
error due to backlash. Furthermore, the interdependence
of the $q_{ki}$ is a second order effect, which we here neglect.

Special Case:

We give, as an illustration, the generation of $y = \sin x$.
This requires two integrating units, the exact constraint
being

$$\frac{-1}{KG^2} \int y \, dx = \frac{dy}{dx}$$

where $K =$ integrator
constant.
$G =$ gear ratio.

Now refer to Fig. 2. There are similar curves for $dy/dx$.
At $x = 0$ \quad $\pi/2$ \quad $3\pi/2$ \quad $2\pi$ \quad ---
Backlash error in $y = q_o + q_1 + q_o + q_1 + q_o + q_1 + q_2 + q_o + q_1 + q_2$ ---
Backlash error in $\frac{dy}{dx} = r_o + r_1 + r_o + r_1 + r_o + r_1 + r_2$ ---

Note that $q_o$ and $r_o$ include initial setting errors.

Over the range $0$ to $\pi/2$ the actual constraint is
\[- \frac{1}{KG^2} \int_0^{\pi/2} (y + q_0 \oslash) dx = dy/dx + r_0 ,\]

and over the range \(\pi/2\) to \(\pi\) the actual constraint is

\[- \frac{1}{KG^2} \int_{\pi/2}^{\pi} (y + q_0 + q_1) dx = dy/dx + r_0 ,\]

and so on. So that over the general ranges it is, respectively,

\[- \frac{1}{KG^2} \int_{\pi/2}^{\pi} (y + \sum_{i=0}^{n} q_i) dx = dy/dx + \sum_{i=0}^{n-1} r_i,\]

and

\[- \frac{1}{KG^2} \int_{\pi/2}^{\pi} (y + \sum_{i=0}^{n} q_i) dx = dy/dx + \sum_{i=0}^{n} r_i.\]

**Direct Integration — (n-1) Integrating Units**

**All Errors** (Backlash, Slip, etc.)

Assuming that the \(p_i\) are independent of the \(q_{ik}\) to

the first order of small quantities, the actual constraint is

\[Y_k = (1+p_k) \int_{x_0}^{x} \left( \sum_{i=0}^{n} b_{ik} Y_i + \sum_{j=0}^{m} q_{jk} \right) d(\sum_{l=0}^{n} c_{lk} Y_k) + Y(x_0) + r_k,\]

where, in our notation, \(m\) is chosen so that \(x_m = x = Y_1\),

and \(r_k\) is the initial setting error in \(Y_k\).

The ideal constraint is

\[y_k = \int_{x_0}^{x} \left( \sum_{i=0}^{n} b_{ik} Y_i \right) d(\sum_{l=0}^{n} c_{lk} Y_k) + y(x_0).\]

Note that \(p_k\), \(Y_k\), and \(y_k\) are functions of \(x\).

The error in generating the function \(y_k(x) = Y_k - y_k = E_k\) is

\[E_k = (1+p_k) \int_{x_0}^{x} \left\{ \sum_{i=0}^{n} b_{ik} (Y_i - Y_i) + \sum_{j=0}^{m} q_{jk} \right\} d(\sum_{l=0}^{n} c_{lk} Y_1) + (1+p_k) \int_{x_0}^{x} \sum_{i=0}^{n} b_{ik} Y_i d(\sum_{l=0}^{n} c_{lk} Y_1) - \int_{x_0}^{x} \sum_{i=0}^{n} b_{ik} Y_i d(\sum_{l=0}^{n} c_{lk} Y_1) + Y_k(x_0) - y_k(x_0) + r_k.\]
Assuming that \( d\mathbf{Y}_i = dy_i \) to the first order, for all \( i \),

\[
\mathbf{E}_k = (1+p_k) \int_{x_0}^{x} \left( \sum \_i \, b_{ik} \mathbf{E}_i + \sum \_j \, q_{jk} \right) \, d(\sum \_l \, c_{lk} \mathbf{Y}_l) + \mathbf{E}_k(x_0) + r_k \\
+ p_k \int_{x_0}^{x} \sum \_i \, b_{ik} \mathbf{Y}_i \, d(\sum \_l \, c_{lk} \mathbf{Y}_l)
\]

Note that except for this last term, \( \mathbf{E}_k \) satisfies the same constraint as does \( \mathbf{Y}_k \). The last term, however, may easily be as important as some of the error correction terms in the constraint equation for \( \mathbf{Y}_k \). However, ignoring all the error correction terms altogether, \( \mathbf{E}_k \) is seen to satisfy the same constraint as does \( \mathbf{Y}_k \). This last statement is in general agreement with the results that have been observed with the new M.I.T. Differential Analyzer operating over a period of some three or four years.

The above analysis was suggested by the following actual problem, as worked out by Dr. S. H. Caldwell of M.I.T. We reproduce it as an illustration of the general theory.

To solve

\[
d^2y/dx^2 = y f(x)
\]  

(i)

where it is assumed that \( f(x) \) is some function introduced without error, and that there is no further backlash after the initial. In our notation, \( p_1 \) and \( q_1 \) are the respective errors for the \( i \)th, integrator, and are here constants.

\[
\mathbf{Y}(x) = y(x) + \mathbf{E}(x)
\]  

(ii)

The actual equations satisfied are:

For integrator \# 1

\[
\mathbf{Y} = (1+p_1) \int (y+\mathbf{E}+q_1) \, dx
\]  

(iii)

For integrator \# 2

\[
\frac{d\mathbf{Y}}{dx} = (1+p_2) \int (f(x) + q_2) \, d\mathbf{Y}
\]  

(iv)
For integrator \# 3

\[ Y = (1+p_3) \int (dY/dx + q_3) \, dx \quad (v) \]

The interconnections are as follows:

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<table>
<thead>
<tr>
<th>generator of f(x)</th>
<th>Theoretical</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>f(x)</td>
<td>f(x)</td>
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<tr>
<td></td>
<td>y</td>
<td>y + E</td>
</tr>
<tr>
<td></td>
<td>ydx</td>
<td>((l+p_1)(y+E+q_1))dx</td>
</tr>
<tr>
<td></td>
<td>dy/dx</td>
<td>dY/dx</td>
</tr>
</tbody>
</table>

---

(1) (2) (3)

Substitute for dY from (iii) into (iv). Then substitute for dY/dx from (iv) in (v), to get:

\[ Y = (1+p_3) \int \left\{ (1+p_2) \int \left[ f(x) + q_2 \right] (1+p_1)(y+E+q_1) dx + q_3 \right\} dx. \]

By differentiating through twice, we get; retaining only the first order terms in the p's, q's and E,

\[ d^2y/dx^2 = (1+p_1)(1+p_2)(1+p_3) \left[ (f+q_2)(y+E+q_1) + dq_3/dx \right], \]

or

\[ d^2y/dx^2 + d^2E/dx^2 = (1+p_1+p_2+p_3)(fy+fE+fq_1+yq_2+ dq_3/dx) \].

Using (i), we have:

\[ d^2E/dx^2 = f(x) \, \xi(x) + fq_1+yq_2 + dq_3/dx + (p_1+p_2+p_3)(fy) \]

or

\[ d^2E/dx^2 = f(x) \, \xi + \left\{ (p_1+p_2+p_3)y + q_2 \right\} f(x) + q_3 y + dq_3/dx \]

so that E(x) is seen to satisfy the same differential equation which y(x) satisfies, except for the correction terms involving the p's and q's.
Inverse Integration

The basic differential relation for integrators is \( \frac{dY}{dx} = y \). When suitably wired, an integrator can therefore be made to give \( x = \int \frac{dY}{y} \). In the M.I.T. of 1945, one of the integrators is so wired, and can readily be switched into the circuit.

Suppose that initial and backlash errors in \( y \) are equal to \( \sum_{j=0}^{m} q_j \), where \( m \) is defined as before. A servo-mechanism following an increment in \( Y \) produces an increment in \( x \). But the servo loop contains \( 1/y \). For a given \( dY \), the required \( dx \) is proportional to \( 1/y \). Therefore we would expect the ratio error to vary as \((1+p/q)\). The actual constraint imposed on one integrator is then

\[
x = (1+p/q) \int \frac{dY}{(y+ \sum_{j} q_{j})}.
\]

This is the form originally proposed by Dr. S. H. Caldwell.

General Conclusions

The error function of a function generated by a mechanical system of integrators and adders satisfies a differential equation of the same form as the one determining the function generated, except for small correction terms in the equation of the same order of magnitude as the individual errors in each integrator or adder. The same statement can be made for the error function of the output of the \( k \) th. integrator in solving a system of simultaneous ordinary differential equations.

It follows that errors in the neighbourhood of sing-
ular points may rise above the error tolerance. Usually, mechanical difficulties are encountered in running the machine as singular points are approached, so that the machine is incapable of producing good values at such points.

But, if the individual errors for each unit are kept at a minimum by careful design and maintenance of the machine, the overall error will only become appreciable over regions in which any one of the functions represented by a shaft displacement is badly behaved.

Error theory carried over to electronic differential analyzers would have to consider that:

1. The initial setting error, a constant for each sweep of the oscillograph screen, may be relatively large.
2. The constraint imposed by each integrating unit is not exact, even if everything is in good working order. This would now take the place of the previous p error.

However, owing to the difficulties involved in keeping the electrical elements constant, better than 1% accuracy could hardly be hoped for. An error survey is as yet hardly worthwhile, since even the basic designs have still to be agreed upon. Some work is being done along these lines by A.B. Macnee at M.I.T. But the most satisfactory test of error is still empirical, to photograph the curves and to compare them with the known accurate results.
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Note:

Most of the abbreviations are standard, and may be found in full on pages 397 - 404 of the Manual of Operation Harvard Calculator, Mark I. (204).

We now give those which might cause trouble:

E.E.-Electrical Engineering (Department).


M.I.T.-Massachusetts Institute of Technology.

M.T.A.C.-Mathematical Tables and Other Aids to Computation.

Encyclopaedia Br.-Encyclopaedia Britannica.

Jour.or Trans.A.I.E.E.-Journal or Transactions of the American Institute of Electrical Engineers.

I.E.E.-Institute of Electrical Engineers (British).
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p. 161 missing from original


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ERRATA

PAGE LINE DETAILS

11 5  y on completely.
26 2 up i in coefficients
37 1  The reference number is 240.
63 6  In the formula for Θ(b) it is sin bu
66 1  last Put semi-colon after dy/dx.
74 4,5 Correct words are "a multiplier".
79 1  Remove stars in headings
79 1  In reference 149 read Ward.
80 (i) Σ sign in the formula.
84 1  diag. Σ sign on the top member of the symbol for the adder.
85 1  Comma after Then, and after Shannon,
85 (iv) ' on the second A_k in the formula.
86 1  a in obtained.
87 2 up Should be n^j and not nlj.

120 6 Comma after "tentative".
123 IBM Should read 6 and not 23 decimal digits.
125 U. of Cambridge. End bracket after bottom entry.
146 Σ sign in equation
151 diag. ∫ sign in ∫ ydx and in (1+q_1)∫(y+E+q_1)dx

69 Enter new sheet in place of old.

PAGE REF. # DETAILS

154 2 Read "to" for "ot" in Br.Ass'n.Report.
155 1 One l in Hansel.
156 1 Put ; after III.
157 33 Read 18, 103-107.

158 Enter new sheets in place of old.

162 104 Pages 453-456.
164 124 Omit n in Überlagerung
165 141 i in Elektrische
166 149 Ref. is to "Electrical Engineering."
153 Insert "Scale" after the word "Small".
168 174 Year is (1943).
176 Publisher is McGraw-Hill. Same in (179).
169 185 Vol. 60.
193 And
170 Ref. 204 (1946) Ref. 208 Punch-Card.
171 Ref. 215 (1936) Ref. 224 (1946) Ref. 225 to Electrical
172 232 Publisher is J. Wiley and Sons, New York.
173 242 Year is (1942).