Essays on the Theory of Financial Innovation

by

Kazuhiko ŌHASHI

Submitted to the Alfred P. Sloan School of Management in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Management

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

This thesis conducts theoretical investigation on financial innovation i.e., innovation of securities. It consists of three chapters where chapter 1 and 2 analyze the innovation of futures contracts by futures exchanges, and chapter 3 analyzes the innovation of securities in general, including that of futures contracts.

Chapter 1 studies the innovation of several futures contracts by futures exchanges in a model similar to Duffie-Jackson (1989), and investigates the conditions that induce the exchanges to create futures efficiently. We show that, in an economy where investors' preferences can be represented by mean-variance utility functions, if the investors have symmetric information and transaction fees are small, a centralized futures exchange with a quadratic transaction fee schedule attains efficient futures innovation. Since decentralized exchanges can be stuck in inefficient innovation, centralization is more desirable than decentralization. In contrast, if investors have differential information, centralization can be harmful to the investors' welfare. We take this result as a caveat against centralization. Optimal industry structure and the number of contracts to be created may depend crucially on the distribution of information.

Chapter 2 studies the determination of the number of futures contracts created by an exchange. We show that even though the exchange can create the futures contracts costlessly, if the investors have differential information, there may be an upper bound on the number of the contracts that the exchange is willing to create. That is, under differential information, the number of contracts is endogenously determined through the innovation by the exchange even without the cost of innovation. We investigate, in a stylized economy, what type of contracts and how many of them the exchange is willing to create, whether such innovation is desirable to the investors, and when underproduction and overproduction of the contracts can occur.

Chapter 3 studies the innovation of securities in general. When securities are created by private security-innovators, it is reasonable to expect that the created securities are traded. In this chapter, we call a set of securities possibly created if some
of them are traded with positive probability, and characterize such possibly created securities. This characterization enables us to predict the type of securities that are unlikely to be created as long as the security-innovator creates securities for their own private benefits. Since this condition is necessary for any reasonable criterion of security-innovation, this paper complements the results by many other papers that assume some specific objectives of the innovators, including chapter 1 and 2. We discuss the relation between the characterization of possibly created securities and no-trade theorem of Holmstrom–Myerson (1983), the number of created securities given the distribution of information, and possible inefficiency of the security innovation by private innovators.

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# 3 Characterization of Security-Innovation

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Chapter 1

Efficient Futures Innovation with Small Transaction Fee: Centralization vs Decentralization

1.1 Introduction

In traditional models of financial economics, it is usually assumed at the starting point that available securities are given exogenously. Nonetheless, during the past two decades of financial innovation, we have witnessed an explosion in the variety of financial instruments. Many types of securities have been created. Among them, financial futures are the most significant and successful (Miller (1990)), and futures exchanges are the workhorses of such innovation.

Futures exchanges are non-profit organizations governed by membership associations. Members, typically acting as brokers, profit from providing transaction services; either directly through brokerages or indirectly through market making. Exchanges are operated for the benefit of these members. Thus, in order to understand futures innovation as an economic activity, it is appropriate to formalize futures exchanges as business organizations that maximize their members' benefit. This view has been suggested in several papers; e.g. Silber (1981), Black (1986), and Miller (1986). The-
oretical investigation along this line was also done by Duffie–Jackson (1989), Cuny (1993), and Hara (1992) in an economy where the investors' utilities have mean-variance representations.

With this perspective in mind, one natural question arises: "If the exchanges create futures for their own members' sake, do they yield any benefit to society?" For the special case where only one futures contract is created in the whole economy, Duffie–Jackson (1989) answer this question in the affirmative. They show that if the exchange designs the single contract to maximize its transaction volume, the resulting innovation is efficient in a certain sense. For the more realistic case where more than one futures contracts are created, however, they merely demonstrated the possibility of inefficient innovation of the contracts that maximize trading volume.

This paper investigates the efficiency of the innovation of more than one futures contracts. We call futures innovation efficient or (constrained) optimal if the corresponding equilibrium allocation is not Pareto dominated by the equilibrium allocation under any other set of futures for which the same number of contracts can be created.

To accommodate differential information, we consider an economy with CARA utilities and Gaussian random shocks where several futures contracts are created by the exchanges and where transaction fee revenue is the major source of the member brokers' benefit. Since the exchanges create futures to maximize the members' fee revenue, and since more than one contracts are created, the choice of the fee schedule and the industry structure – i.e., whether it is centralized (a single exchange creates all futures) or decentralized (each exchange creates a single unique contract) – is a natural focal point of the analysis. We also consider the effect of differential information on the efficiency of futures innovation.

Assuming the futures' payoffs are taken to be mutually uncorrelated, our main results are summarized as follows:
1. When investors have symmetric information,

(a) A centralized futures exchange endowed with a quadratic fee schedule yields efficient futures innovation when fees become small. This is because, if the transaction fee is quadratic and close to zero, a change of futures innovation increases the centralized exchange’s fee revenue whenever it Pareto improves the investors’ welfare, since, without transaction fee, each investor’s utility level is monotone in the sum of the square trading volume of futures.

(b) Decentralized exchanges may be stuck in inefficient creation of futures. This is because those exchanges may fail to coordinate to internalize the surplus of creating the efficient futures, since each exchange designs futures to maximize its own fee revenue, not the total fee revenue of the futures industry.

(c) For efficient futures innovation, centralization is always more desirable than decentralization.

2. When investors have differential information,

(a) Centralization can be harmful to efficient futures innovation because of the underproduction of futures contracts. The reason is as follows: When information is differentially distributed, creating additional futures may transmit more information. Better information, however, does not necessarily imply more trade (Pfleiderer (1984) and Wang (1992)). Hence, it may be more profitable for the centralized exchange not to create new futures. This can result in underproduction of futures.

(b) Decentralization can be more efficient than centralization. The reason is as follows: The decentralized exchanges create futures to maximize their own fee revenue. Hence, even in the case that the total fee revenue of the futures industry decreases as the result of the innovation of an additional futures contract,
the decentralized exchanges do not coordinate and will still create the contracts so long as their fee revenue is positive. In this way, the underproduction of futures can be avoided.

(c) Given the distribution of information, the number of futures created in an equilibrium can be different between different industry structures.

Therefore, whether centralization is more desirable than decentralization depends on the distribution of information. Under symmetric information, inefficiency due to the decentralized (and segmented) innovators is also observed in other types of innovation (e.g. Hart (1980), Makowski (1980), and Pesendorfer (1991)). Inefficiency occur because the decentralized innovators fail to internalize the surplus from innovating the different but complementary goods. Thus, a simple remedy is to let one innovator participate in all the possible innovation and internalize all the surplus. In this paper, this corresponds to centralization. In this sense, centralization for efficient innovation is quite appealing. However, under differential information, this mechanism need not work, and what is worse, inefficient innovation can occur because of centralization. Since differential information is one important motive to trade futures in reality, we take this result as a caveat against thoughtless centralization.

Our analysis is different from previous literature in the following sense. First, previous papers assume that each exchange can create only one contract so that the number of futures created depends on the number of exchanges. Here, the same number of contracts can be created under both the centralized (a single exchange) and the decentralized (multiple exchanges) industry structures. This allows the comparison of allocational efficiency between different industry structures when the same number of contracts can be created. (This is the reason why the terms centralized and decentralized are used here instead of monopoly and oligopoly, as used in the previous literature.) Second, unlike Duffie–Jackson (1989) or Hara (1992) who start with an a priori specified (proportional) fee schedule and analyze its equilibrium char-
acterization and welfare consequences, we consider a class of well-behaved feasible fee schedules and then look for the schedule desirable for efficiency. Third, we investigate how differential information among investors affects futures innovation.

The remainder of the paper is organized as follows: In section 1.2, we discuss the basic framework of the model with a very simple example. In section 1.3, we formulate an economy with futures innovation under symmetric information. We define equilibrium and efficiency concepts of futures innovation. Section 1.4 presents the conditions for optimality of the innovation of several futures contracts under symmetric information. We show that centralization is always more efficient than decentralization. Section 1.5 investigates how differential information affects the conditions for efficient futures innovation. We show that decentralization can be more efficient than centralization. In section 1.6, we offer some concluding remarks.

Finally, there are many other important discussions related to financial innovation. In particular, see Allen–Gale (1988, 1991) for the relationship between asset creation and equilibrium prices, Anderson–Harris (1986) for the timing of innovation, Madan–Sobrera (1991) and Chen (1992) for the pricing and marketing of derivative securities, Ross (1989) and Pesendorfer (1991) for the effect of marketing cost on innovation, Demange–Laroque (1992) and Rahi (1992) for the security-innovation under asymmetric information, Tufano (1989) for empirical evidence in asset creation by investment banks, and Van Horne (1985) for a general introduction.
1.2 Basic Framework

1.2.1 Timing

We consider an economy where uncertainty, resolved at the terminal date, is described by a finite number of random variables called risk factors. There are three types of decision makers: a regulator, futures exchanges, and investors. Each economy is identified with the collection of the investors' initial endowments of the units of the risk factors, preferences, and private signals about the risk factors. The timing is as follows.

- Distribution of possible economies is determined
- An economy is drawn
- Futures are traded
- Regulations are set
- Futures are created
- Uncertainty is resolved
- Before ex-ante
- Ex-ante
- Interim
- Ex-post

< Fig.1.2.1. >

Before the distribution of the possible economies (i.e. that of the possible initial endowments, preferences, and private signals) is determined, the regulator sets the
regulations (Before ex-ante). After the distribution is determined and is publicly known, futures exchanges create futures under these regulations to maximize their expected transaction fee revenue, given the investors' optimal trading of the created contracts (Ex-ante). Then, an economy is drawn (i.e. the endowments and signals are realized), and the investors trade the innovated futures (Interim). Finally, uncertainty is resolved (i.e. the risk factors are realized) (Ex-post).

We are interested in the ex-ante efficiency of futures innovation at the time when the exchanges create contracts. We look for the regulations that induce the exchanges to create futures efficiently. In doing so, we imply by symmetric information that the distribution of the possible economies is degenerate to one point. On the other hand, under differential information, this distribution is not degenerate.

1.2.2 A Simple Example

In order to illustrate the points of the discussion, consider a simple economy under symmetric information. Since the distribution of the possible economies is degenerate to one point under symmetric information, no private signal is informative. Thus, without loss of generality, we treat the model as if no investor received a private signal.

Suppose that there are two jointly normally distributed risk factors \( \tilde{Z} = (\tilde{z}_1, \tilde{z}_2)^T \sim N(0, I_2) \) where \( I_2 \) is the \( 2 \times 2 \)-identity matrix. There are two investors \((i = 1, 2)\) with the identical CARA utility \( E[-exp\{(−1/r)\tilde{W}\}] \) over the distribution of wealth \( \tilde{W} \) at the terminal date. Note that if \( \tilde{W} \) is normally distributed, a monotone transformation of this utility has a mean-variance representation \( E[\tilde{W}] - (1/(2r))Var[\tilde{W}] \) The initial endowments of the units of the risk factors are given by \( \epsilon_1 = -\epsilon_2 = (1, 1)^T \). Hence, the investors are endowed with the stochastic wealth at the terminal date \( \tilde{\epsilon}_1 = -\tilde{\epsilon}_2 = \tilde{z}_1 + \tilde{z}_2 \), where \( \tilde{\epsilon}_i = \epsilon_i^T \tilde{Z} \) and \( \epsilon_i \in R^2 \ (i = 1, 2) \).
Look at <Fig.1.2.2>. Originally, there is no means to share the risk before the terminal date (1). Now, assume that a futures contract with the terminal date payoff \(a\tilde{z}_1 + b\tilde{z}_2 (a^2 + b^2 = 1)\) appears spontaneously. It is easy to check that investor \(i\)'s equilibrium position \(y_k\) of this contract is just the projection \(y_k = -(a,b)e_k\) of \(-e_i\) on \((a,b)^T\) i.e. \(-y_1 = y_2 = a + b\) (2). Obviously, in this example, if \((a,b) = \pm(1/\sqrt{2},1/\sqrt{2})\), the investors can attain full risk sharing and no other contract design \((a',b') \neq \pm(1/\sqrt{2},1/\sqrt{2})\) can Pareto improve the investors' welfare (3). In this sense, both futures contracts \(\pm(1/\sqrt{2},1/\sqrt{2})\) are efficient. (In ex-ante sense. See <Fig.2.1>. Recall that, under symmetric information, the distribution of the possible economies is degenerate at one point.)

\[<Fig.1.2.2>\]

Now, instead of assuming that a futures contract appears spontaneously, suppose that an exchange creates the contract costlessly. Note that, in the absence of transaction fees, the exchange is indifferent as to the design of the futures contract. In order to give the exchange an appropriate incentive, suppose that the regulator can control the transaction fee schedule and that, in particular, she chooses a quadratic fee \(c_m(y) = (1/(2m))y^2\), where \(y\) is the futures position taken by an investor.
Look at <Fig.1.2.3.>. For a contract \((a, b)^T\), the investors now take a smaller futures position than the mere projection as in (2) of <Fig.1.2.2.> because of the transaction fee, and the exchange earns the fee revenue \(\frac{2}{m}(\frac{m}{m+r})^2(a + b)^2\) (1). In order to maximize the fee revenue, the exchange innovates either the contract \((1/\sqrt{2}, 1/\sqrt{2})^T\) or \(-(1/\sqrt{2}, 1/\sqrt{2})^T\), which generates the fee revenue \(\frac{4}{m}(\frac{m}{m+r})^2\) (2). As we have seen in (3) of <Fig.1.2.2.>, both contracts \(\pm(1/\sqrt{2}, 1/\sqrt{2})^T\) are desirable. But, the investors' welfare is now lower than that in (3) of <Fig.1.2.2.> because of the distortion caused by the transaction fee \((1/(2m))y^2\). (Observe that the arrows do not reach the origin.) Taking the exchanges' status quo as zero fee revenue, the regulator thus tries to set the fee schedule as close to zero as possible (i.e. \(m \not\rightarrow \infty\)) so that it can reduce the distortion while giving the exchange an appropriate incentive for innovating a desirable futures contract. In the limit of decreasing transaction fees, the efficient allocation can be attained (3).

\(<Fig.1.2.3>\)

In what follows, we study whether and how the efficiency result in this example can be extended to a more general setting.\(^1\) In particular, we investigate the efficiency

\(^1\)Setting a proportional, instead of quadratic, fee schedule \(c(y) = T \cdot |y| (T > 0)\) yields a result similar to this example. In fact, Duffle–Jackson (1989) showed that, under a proportional fee
of creating more than one futures contracts. We will see that the choice of the transaction fee schedule and of the industry structure becomes an important issue for efficient innovation. We also study the effect of differential information on innovation.

1.3 Innovation with Symmetric Information

Under symmetric information, the distribution of the possible initial endowments of the risk factors, preferences, and private signals degenerates to one point. Hence, when the exchanges create futures, they know the exact initial endowments and preferences that the investors will receive. Also, since no private signal is informative, we can treat the model as if no investor received a private signal. We are interested in symmetric information for the following two reasons: First, it avoids unnecessary complexity and hence serves as a benchmark. Second, it is this case that all previous papers have considered. Though we consider an economy with CARA utility functions and Gaussian random shocks to accomodate differential information later, we can obtain the same results in the economy with mean-variance utilities and general random shocks considered by Duffie-Jackson (1989) and Hara (1992).

1.3.1 An Economy and Futures Markets

We consider an economy whose timing is given by < Fig.1.2.1 >. Uncertainty, resolved at the terminal date, is described by an $N$-dimensional random vector $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_N)^T$. It can be interpreted as state variables or risk factors. We assume that $\tilde{Z} \sim N(0, I_N)$ where $I_N$ is the $N \times N$-identity matrix. A single consumption commodity, acting as numeraire, is available only at the terminal date. We take $L \equiv \{\tilde{x} : (\exists e \in R)(\exists x \in R^N) \tilde{x} \equiv e + x^T\tilde{Z}\}$ as the commodity space.

schedule, as long as only one contract is created in the whole economy, a similar efficiency result holds for a more general distribution of the initial endowments and preferences. For the case of creating more than one futures, however, they consider the case of a zero transaction fee and show the possibility of inefficient innovation by the decentralized exchanges of the that maximize the transaction volume.
There are $K$ investors. Each $k \in \{1, \ldots, K\}$ has an initial endowment $\tilde{e}_k = e_k^T \tilde{Z}$, $e_k \in \mathbb{R}^N$. Note that because of the orthogonality of $\tilde{z}_k$'s, $\tilde{e}_k \in L$ and $e_k \in \mathbb{R}^N$ can be identified. Investor $k$'s preferences are represented by a CARA utility function $U_k : L \rightarrow \mathbb{R}$, $U_k(\tilde{x}) = E[-\exp\{(-1/r_k)\tilde{z}\}]$ ($r_k > 0$). An economy is thus described as a collection $(e_k, r_k)_{k=1,\ldots,K}$.

A futures contract is a contingent claim that pays its holder $\tilde{f} = \tilde{f}^T \tilde{Z}$ ($f \in \mathbb{R}^N$) units of numeraire in exchange of the futures price $p \in \mathbb{R}$ units of numeraire at the terminal date when state $Z = (z_1, \ldots, z_N)^T$ occurs. $p$ is determined before the terminal date when the contracts are agreed. In short, a holder of one unit of a futures receives $\tilde{f} - p$ at the terminal date.

The investors pay transaction fees to the brokers according to a given transaction fee schedule $c: \mathbb{R} \rightarrow \mathbb{R}_+$. We denote by $C$ the set of feasible fee schedules. There is a continuum of homogeneous potential brokers. Each of them has zero weight and consumes only brokerage fee revenue. A futures exchange is a non-profit organization governed by its member brokers. The population of members in each exchange is taken to be one. Given a transaction fee schedule, an exchange designs futures contracts to maximize the members' fee revenue.

We assume that, in the economy as a whole, only up to $T$ ($\leq N$) futures can be created by the exchanges without cost. We assume that $T$ is exogenously given, but we do not assume that the exchanges are obliged to create all $T$ contracts. The question we ask is, thus, under what circumstances the fee-revenue-maximizing exchanges design futures efficiently, when they can create a certain number of futures very cheaply. In this paper, we do not address questions related to set-up costs of the contracts.
Since the investors have symmetric information, and there is no set-up costs, W.L.O.G. we assume that all \( I \) contracts are created\(^2\). Each contract is indexed by \( i \in \{1, \ldots, I\} \). Payoffs and futures prices are denoted by \( \tilde{f}_i = f_i^T \tilde{Z} \quad (f_i \in \mathbb{R}^N) \) and \( p_i \in \mathbb{R} \), respectively. We denote the \( I \)-dimensional random vector of futures payoffs by \( \tilde{F} \equiv [\tilde{f}_1, \ldots, \tilde{f}_I]^T \), the corresponding \( N \times I \) matrix \([f_1, \ldots, f_I] \in \mathbb{R}^{N \times I}\) where \((\forall i) \quad \tilde{f}_i = f_i^T \tilde{Z} \) by \( F \in \mathbb{R}^{N \times I} \), and the \( I \)-dimensional futures price vector by \( P \equiv [p_1, \ldots, p_I]^T \in \mathbb{R}^I \). Similarly, we identify \( \tilde{F} \in \Pi_I^T L \) and \( F \in \mathbb{R}^{N \times I} \).

### 1.3.2 Regulations on Futures Markets

The regulator sets and enforces regulations on futures markets to attain efficient innovation. We assume, however, that her discretionary power is limited. She does not have the precise information about the investors' characteristics when setting the regulations. When enforcing the regulations, the regulator can verify only the variables that the exchanges can. Specifically, the futures payoffs, the positions in futures taken by the investors, and the fee charges are verifiable, but the investors' risk tolerance and initial endowments are not.\(^3\) Our purpose is to find a set of easily enforceable regulations that induce efficient futures innovation.

This formulation is justified for the following reasons: First, we investigate the characterization of a fee schedule and industry structure that lead to efficient futures innovation. We do not ask how they are chosen. Second, the presence of the regulator captures the fact that the futures industry is well regulated (Anderson (1984)). Third, this formulation makes the model tractable. We discuss each regulation in more detail.

\(^2\)See section 5.

\(^3\)For example, these assumptions imply that all the regulations should be neither individual investor specific nor individual contract specific.
Contracts' Payoffs

The regulator restricts futures contracts to have mutually orthogonal payoffs i.e. \( \text{Cov}(\bar{f}_i, \bar{f}_i') = 0 \) for any \( i \neq i' \).\(^4\)

Industry Structure

We call an industry structure centralized if all contracts are created by a single futures exchange. We call it decentralized if there are \( T \) exchanges and each of them creates a single unique contract. The regulator chooses one of these two industry structures.

Transaction Fee Schedule

The regulator sets and enforces a transaction fee schedule to give the exchanges incentives to create the futures and to operate the markets.\(^5\)

To select a fee schedule, she determines the unit of a contract and a fee schedule over it. As in Duffie–Jackson (1989), we define the unit of a contract by the standard deviation of its payoff unspanned by other contracts. Each contract has one unit. Because of the mutual orthogonality, this restriction is simply expressed as \( \text{Var}(\bar{f}_i) = 1 \) for all \( i \). A transaction fee schedule is given by a function of position \( c: \mathbb{R} \to \mathbb{R}_+ \). It satisfies the following properties: \( c \) is positive, convex, attaining its minimum at the

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\(^4\)This is a nontrivial assumption, which may exclude several possible exchanges' behavior such as creating similar contracts to compete for large trading demand or to force investors' unnecessary trade. However, this assumption in turn enables us to avoid the inefficiency problems caused by the overproduction of too similar futures contracts. Thus, we can do clear analysis on the other important issues in efficient innovation such as the effect of the fee schedule and the industry structure. Furthermore, since we are interested in the limiting case where the transaction fees are negligible, an additional contract is meaningful only for its payoff unspanned by the other futures, and hence it makes sense to assume that the regulator restricts each futures payoff to have only non-redundant part. Finally, this assumption makes our analysis tractable.

\(^5\)Since the cost to work as a broker is assumed negligible, however small the fee is, futures markets are operated. A regulator takes brokers' status quo as zero fee revenue and set a fee schedule as small as possible to reduce the distortion due to the transaction fees. We will investigate the welfare property of futures innovation in this limit.
origin by $c(0) = 0$, symmetric i.e. $c(y) = c(-y)$, and continuously differentiable except possibly at origin, but always has right and left derivatives $(c_+, c_-)$ at the origin. We define by $C$ the set of feasible transaction fee schedules.

**Summary**

Hereafter, let us denote the futures contracts $\tilde{f}_i$ and $\tilde{F}$ and the initial endowments $\tilde{e}_k$ in terms of their coordinates $f_i$ and $F$ and $e_k$ where $\tilde{f}_i = f^T \tilde{Z}$, $\tilde{F} = F^T \tilde{Z}$, and $\tilde{e}_k = e^T \tilde{Z}$. Define $O^{N,T} \equiv \{ F \in R^{N \times T} | F^T F = I_T \}$ where $I_T$ is the $T \times T$ identity matrix. Regulations are summarized as follows: Feasible futures contracts should satisfy $F \in O^{N,T}$. Each $f_i$ in $F \in O^{N,T}$ is counted as one unit. Transaction fee $c(y_i)$ is paid for each contract $i$ according to the units $y_i$ traded. If all $T$ contracts can be created by a single exchange, the industry is called centralized. If each futures is created by a distinct exchange, it is called decentralized.

### 1.3.3 Equilibrium for a Given Fee Schedule

Given futures contracts $F \in O^{N,T}$, transaction fee schedule $c \in C$, and futures prices $P \in R^T$, an investor $k$ chooses the optimal futures position $y_k$ so that

$$y_k \in \text{ArgMax}_{y \in R^T} \ U_k [e_k^T \tilde{Z} + (F^T \tilde{Z} - P)^T y - \sum_1^T c(y_i)]$$

(1.1)

**Definition 1.3.1.**

A market equilibrium given $(F,c) \in O^{N,T} \times C$ is an element $(P,(y_1,...,y_K))$ $\in R^T \times R^{T \times K}$ such that

(a) Given $(P,(F,c))$, $y_k$ is optimal, i.e. solves (1.1) for all $k$

(b) Markets clear, i.e. $\sum_1^K y_k = 0$
Under the assumed regulations, a market equilibrium given \((F, c)\) exists, and the set of equilibrium futures positions is unique. (See Lemma 1.A.1. in the appendix.) Futures exchanges understand the relation between \((F, c)\) and a market equilibrium. Taking other exchanges' contract design as given, each exchange innovates contract \(f_i\)'s simultaneously to maximize its own transaction fee revenue. Now, for any contract design \(F' \in O^{N,T}\), denote the corresponding market equilibrium futures positions by \((y'_1, \ldots, y'_K)\).

**Definition 1.3.2.**

An innovation equilibrium given \(c \in C\) under centralized industry structure is an element \(\{(P, (y_1, \ldots, y_K)), F\} \in R^T \times R^{T \times K} \times O^{N,T}\) such that

(a) \((P, (y_1, \ldots, y_K))\) is a market equilibrium, given \((F, c) \in O^{N,T} \times C\)

(b) \(F\) maximizes transaction fee revenue, given \(c \in C\), i.e.

\[
F \in \text{ArgMax}_{F' \in O^{N,T}} \sum_{i=1}^{T} \sum_{k=1}^{K} c(y'_{ki})
\]

An innovation equilibrium under decentralized industry structure is a Nash equilibrium \(\{(P, (y_1, \ldots, y_K)), F\} \in R^T \times R^{T \times K} \times O^{N,T}\) that satisfies (a) and

(c) For exchange \(i\), given other exchanges' contracts \(f_{-i}\), contract \(f_i\) maximizes its own transaction fee revenue i.e.

\[(\forall i), \text{given } f_{-i}, f_i \in \text{ArgMax}_{F' \subseteq F' \in [f', f_{-i}]} \text{ArgMax}_{F' \in O^{N,T}} \sum_{k=1}^{K} c(y'_{ki})\]

where \(f_{-i} \equiv [f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_T]\) and \([f', f_{-i}] \equiv [f_1, \ldots, f_{i-1}, f', f_{i+1}, \ldots, f_T]\).
1.3.4 Small-Fee Equilibria

The regulator chooses a fee schedule that gives the exchanges appropriate incentives to create futures. Since the contracts can be costlessly created, the futures markets are operated however small the fee revenue is. Thus, as we saw in the example of section 2, the regulator tries to set the transaction fee schedule as close to zero as possible so that she can reduce the distortion caused by the transaction fees. To approximate the equilibrium in the limit, we take a sequence of innovation equilibria with decreasing fee schedules \( \{c_m\}_m (c_m \searrow 0) \).\(^6\) In particular, we set \( c_1 = c \in C \) and consider the scaling down i.e. \( c_m \equiv (1/m)c \). Since innovation equilibria for a given \( c_m \) may not be unique, the approximation can be defined up to subsequences of equilibrium selections. This leads to the following definition:

**Definition 1.3.3.**

A small-fee equilibrium given \( c \in C \) is an element \( \{(P^*, (y^*_1, ..., y^*_K)), (F^*)\} \in R^T \times R^{T \times K} \times O^{N,T} \) such that,

\[
P^* = \lim_{m(l)} P^{m(l)}, \quad y^*_k = \lim_{m(l)} y^{m(l)}_k, \quad \text{and} \quad F^* = \lim_{m(l)} F^{m(l)}
\]

for some subsequence \( \{(P^{m(l)}, (y^{m(l)}_1, ..., y^{m(l)}_K)), (F^{m(l)})\} \) of some sequence of a selection of innovation equilibria \( \{(P^m, (y^m_1, ..., y^m_K)), F^m\} \) with decreasing fee schedule \( c_m \equiv (1/m)c \).

As fees become close to zero, an innovation equilibrium will be close to one of the small-fee equilibria. In this sense, small-fee equilibria approximate innovation equilibria with zero fee. Now, for given \( F \in O^{N,T} \) and \( (r_k, \epsilon_k)_{k=1, ..., K} \), define

\[
y_k(F) \equiv [y_{k1}(F), ..., y_{kI}(F)]^T \quad \text{where} \quad y_{ki}(F) \equiv r_k \left[ \sum_{k'} r_{k'} \frac{\epsilon_{k'}}{r_k} - \frac{\epsilon_k}{r_k} \right] f_i
\]

\[(1.2)\]

\(^6\)We cannot simply take \( c = 0 \) because, then, the exchanges's revenue maximization is not well-defined.
$y_k(F)$ is investor $k$'s market equilibrium futures position when $F$ is created and the transaction fee is zero. Since a small-fee equilibrium is the limit of decreasing fee schedule, it is natural to expect that it looks as if it were an innovation equilibrium without transaction fee. The following lemma confirms this conjecture.

**Lemma 1.3.2.**

If $\{(P^*, (y_1^*, ..., y_K^*)), F^*\}$ is a small-fee equilibrium, given $c \in C$, then it looks as if it were an innovation equilibrium with zero fee charge. That is, $y_k^* = y_k(F^*)$, and $F^*$ satisfies, for a centralized industry structure,

$$F^* \in \text{ArgMax}_{F^* \in O^N, T} \sum_{i=1}^{T} \sum_{k=1}^{K} c(y_{ki}(F'))$$

and for a decentralized industry structure, for any $i = 1, ..., T$,

$$f_i^* \in \text{ArgMax}_{f^* \in (f^* \in [f^*, f^*]) \in O^N, T} \sum_{i=1}^{T} \sum_{k=1}^{K} c(y_{ki}(F')) \text{ for given } f_{-i}^*$$

Thus, a small-fee equilibrium looks like a hypothetical innovation equilibrium with zero fee charge. The converse need not be true. This lemma allows us to focus on the allocation associated with $\{(y_1(F^*), ..., y_K(F^*)), F^*\}$ to analyze the welfare of a small-fee equilibrium where $F^*$ satisfies the conditions in Lemma 1.3.1.

**1.3.5 Futures Innovation (FI-) Optimality**

We are interested in the efficiency of small-fee equilibria. By Lemma 1.3.1, we only have to analyse the welfare of a hypothetical innovation equilibrium with zero transaction fee. We call a set of futures contracts $F' \in O^{N, T}$ Pareto-dominates $F \in O^{N, T}$ at zero transaction fees, if some investors attain strictly higher and all attain not lower utility level in a market equilibrium given $F'$ and $c \equiv 0$ than in a market equilibrium given $F$ and $c \equiv 0$. 

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Definition 1.3.4.

$F \in O^{N,T}$ is Futures Innovation (FI-) optimal, if there are no other sets of the contracts $F' \in O^{N,T}$ that Pareto-dominate $F$ at zero transaction fee.

FI-optimality is called "Pareto optimality" in Duffie–Jackson (1989). This is a weak optimality concept because it compares only allocations implemented as market equilibria. It is important to observe that because CARA utility functions combined with Gaussian random shocks has the equivalent mean-variance representation, when there is no transaction fee, each investor’s equilibrium utility level is monotone in his sum of the square trading volume. Thus, FI-optimality is characterized in terms of the square trading volume.

Lemma 1.3.2.

$F \in O^{N,T}$ is FI-optimal, if and only if there exists no $F' \in O^{N,T}$ whose corresponding sum of square transaction volumes is at least as large as that of $F$ for all investors, and is strictly greater than that of $F'$ for some investors.

That is, $F \in O^{N,T}$ is FI-optimal, if and only if ($\exists F' \in O^{N,T}$) such that

$$\forall k = 1, \ldots, K \quad y_k(F')^T y_k(F') \geq y_k(F)^T y_k(F)$$

$$\exists k = 1, \ldots, K \quad y_k(F')^T y_k(F') > y_k(F)^T y_k(F)$$

(1.3)
1.4 Conditions for FI-Optimality

1.4.1 Centralization Efficiency Result

To attain efficient innovation, the regulator should set a fee schedule under which the exchanges will not design FI-inefficient futures. Our first proposition shows that, under a centralized industry structure, such a fee schedule exists and has a quadratic form.

We denote by \( y_{k_i}^m(F) \) investor \( k \)'s market equilibrium position in the \( i \)-th contract given a set of futures \( F \in O^{N,T} \) and a fee schedule \( c_m = (1/m)c \). Also, recall that \( y_{k_i}(F) \) denotes \( k \)'s market equilibrium position in \( i \), given \( F \in O^{N,T} \) and zero transaction fee \( c \equiv 0 \).

Proposition 1.4.1.

When multiple contracts are innovated and the fee is sufficiently small, any Pareto-dominating contracts generates a higher fee revenue than the Pareto-dominated contracts for a centralized exchange in all economies, if and only if a transaction fee schedule \( c \) is quadratic.

The intuition for this result is clear. As in Lemma 1.3.2, when there is no transaction fee, each investor's market equilibrium utility level is monotone in the sum of her square trading volume. If a fee schedule is quadratic, by Lemma 1.3.1, a centralized exchange's fee revenue is (roughly, when the fee is small,) monotone in the sum of square trading volume of all investors. Now, suppose \( \hat{F} \) Pareto-dominates \( F \). Then, by Lemma 1.3.1, for all investors, the sum of square trading volume under \( \hat{F} \) is greater than that under \( F \) when the fee is zero. However, this implies the exchange's revenue is also larger under \( \hat{F} \) than under \( F \). That is, when the fee is sufficiently small, a quadratic fee schedule reflects the investors' preferences correctly in the direction of Pareto improvement.
Given a quadratic transacion fee schedule, we can show the existence of a small-fee equilibrium under a centralized industry structure. We can also expect its FI-optimality from the above discussion. The next proposition shows that this is true. We call it the Centralization Efficiency Result.

**Proposition 1.4.2.**

Under a centralized industry structure with a quadratic transaction fee schedule, a small-fee equilibrium exists and is FI-optimal.

The monopolistic industry structure considered in the previous literature can be described as a degenerated case with a single futures innovation (i.e. $I = 1$). In this case, we can weaken the condition on a fee schedule needed for optimality. The following result includes proposition 6 in Duffie–Jackson (1989) (with a proportional fee schedule) as a special case.

**Proposition 1.4.3.**

Suppose that there exists one exchange and that only one contract is created in the whole economy. Then, any small-fee equilibrium allocation associated with any fee schedule $c \in C$ is FI-constrained optimal.

The reason for this result is simple. With a single contract traded, there is a monotonic relation between the square trading volume and the fee charge for each investor, so long as $c \in C$. Thus, if a change of futures design Pareto-improves the welfare so that it increases all investors' square trading volume, it must increase also the exchanges fee revenue. In an innovation equilibrium, of course, this never occurs.
1.4.2 Inefficiency Examples

Example 1.4.1: Inefficiency due to Non-quadratic Fee Schedule

There are three risk factors \( N = 3 \) and three investors with identical risk tolerance \( r_k = r \) \( (k = 1, 2, 3) \) and initial endowments \( e_1 = (-1/2, -1/2, 0)^T \), \( e_2 = (-1/2, -3/4, 0)^T \), \( e_3 = (1, 1, 0)^T \). Transaction fee is proportional to trading volume i.e. \( c(y) = T|y| \) \( (T > 0) \). There is a centralized exchange creating two futures \( (T = 2) \).

One can show that each of the following sets of contracts, \( F = [f_1, f_2] \) with \( f_1 = (1, 0, 0)^T \), \( f_2 = (0, 1, 0)^T \) and \( F' = [f'_1, f'_2] \) with \( f'_1 = (1/2, 1/2, -1/\sqrt{2})^T \), \( f'_2 = (1/2, 1/2, 1/\sqrt{2})^T \), constitutes a small-fee equilibrium with the same limiting trading volume equal to 4. (See appendix.) However, \( F \) Pareto dominates \( F' \) at zero transaction fee. Indeed,

\[
\begin{align*}
y_1(F)^Ty_1(F) &= 5/16 \geq 9/32 = y_1(F')^Ty_1(F') \\
y_2(F)^Ty_2(F) &= 13/16 \geq 25/32 = y_2(F')^Ty_2(F') \\
y_3(F)^Ty_3(F) &= 2 = 2 = y_3(F')^Ty_3(F')
\end{align*}
\]

By Lemma 1.3.2, this shows the inefficient futures innovation under a centralized industry structure with a proportional fee schedule. \( \Box \)

Recall that, by Lemma 1.3.2, the investors' welfare improves monotonically to the sum of their square trading volume. On the other hand, with a proportional fee schedule, the exchange tries to maximize the sum of the trading volume. Because of this discrepancy of interests, the exchange fails to design contracts that reflect investors' preferences correctly. Consequently, inefficient innovation results.

This result clarifies how multiplicity of futures, in contrast to the case with a single futures, affects the conditions to attain efficiency. When only one futures is created in the whole economy, Duffie–Jackson (1989) shows that designing the contract to
maximize the volume of trade yields efficient innovation. The results above show that this should be modified to maximize the sum of square trading volume, when multiple futures are innovated. There is no contradiction between them. As in Proposition 1.4.3, in the case of one contract, there is a monotonic relation between the square trading volume and the trading volume for each investor. Hence, Pareto improvement, which is equivalent to the increase of square trading volume of each investor, implies the increase of trading volume of each investor. This degree of freedom allowed Duffie–Jackson to characterize efficient futures innovation in terms of trading volume for a single contract case. In the case of multiple futures, we lose this degree of freedom and are required to use square volume of trade.

Example 1.4.2: Inefficiency due to Decentralized Industry Structure

There are three risk factors i.e. $N = 3$. There are four investors with identical risk tolerance $r_k = r$ ($k = 1, 2, 3, 4$), and initial endowments given by $-e_1 = e_3 = (1, 0, 0)^T$, $-e_2 = e_4 = (0, 2, 2)^T$. There are two exchanges, each of which creates one contract simultaneously under a quadratic fee schedule.

Consider the following sets of futures contracts, $F \equiv [f_1, f_2]$ with $f_1 = (1, 0, 0)^T$, $f_2 = (0, 1/\sqrt{2}, 1/\sqrt{2})^T$ and $F' \equiv [f'_1, f'_2]^T$ with $f'_1 = (0, 1, 0)^T$, $f'_2 = (0, 0, 1)^T$. One can show that each of them constitutes a small-fee equilibrium with $c(y) = y^2$. (See appendix.) But,

$$y_1(F)^T y_1(F) = y_3(F)^T y_3(F) = 1$$
$$y_2(F)^T y_2(F) = y_4(F)^T y_4(F) = 8$$

$$y_1(F')^T y_1(F') = y_3(F')^T y_3(F') = 0$$
$$y_2(F')^T y_2(F') = y_4(F')^T y_4(F') = 8$$

Hence, $F$ Pareto-dominates $F'$. This shows the possibility of inefficient small-fee
equilibrium allocation\(^7\) under decentralization. \(\Box\).

Observe that, in this example, if the exchanges can coordinate to create the contracts \(F\), the fee revenue of the industry increases and an FI-efficient small-fee equilibrium is attained. However, each exchange cares only for its own fee revenue, and there is no mechanism to induce such coordination. Because of this lack of coordination mechanism, the decentralized exchanges may fail to internalize the surplus from creating more efficient futures and can get stuck in an FI-inefficient innovation.

1.4.3 Centralization vs Decentralization

Example 1.4.2. shares similarities with Duffie–Jackson’s (1989) Example 4 in which futures are innovated inefficiently by the transaction-volume-maximizing exchanges. This suggests that inefficiency due to decentralized industry structure be prevalent regardless the fee schedule. The following lemma shows that this conjecture is true.

**Lemma 1.4.1.**

For any fee schedule \(c \in C\), there exists an economy in which inefficient futures innovation can occur under a decentralized industry structure when the fee is small.

Recall that the regulations are set before an economy is drawn. Thus, the regulator has no idea about the economy when she sets the regulations. However, by Proposition 1.4.2, she can guarantee that all possible small-fee equilibria be FI-optimal. We state this claim as the following proposition:

---

\(^7\)One may suspect that this inefficient innovation is caused by the orthogonal restriction among futures payoffs. However, removing it makes the situation even worse. In fact, suppose that there is no restriction on the futures payoffs. Then, for any small fee, both exchanges innovate futures \(f = (0, 1/\sqrt{2}, 1/\sqrt{2})\), which results in apparent inefficiency. Moreover, in this situation, no efficient innovation can be supported in an equilibrium.
Proposition 1.4.4.

Under symmetric information, if the mutual orthogonality restriction on contracts' payoffs and a quadratic fee schedule are enforceable, then the centralized exchange should be chosen over the decentralized exchanges to attain efficient futures innovation.

Notice that this is a weak sense of domination. Decentralization is not desirable because it may not yield efficient innovation, though it may also result in efficient innovation. We interpret this proposition as follows: Since the regulator has little knowledge of the economy and small discretionary power, though she can influence the exchanges' futures innovation, she cannot choose which equilibrium will emerge. Hence, she wants all possible equilibria to be efficient. In our economy with symmetric information where the investors behave in the mean-variance manner, this is possible by selecting a quadratic fee schedule and a centralized exchange.

1.5 Innovation with Differential Information

We introduce differential information among the investors because, in reality, this is one important motive to trade futures. Since, under differential information, there is no simple relation between the transaction volume and the investors' utility level, unlike in the case of symmetric information, we have little hope of identifying a simple mechanism to implement efficient futures innovation. Thus, the purpose of this section is not to find conditions for efficiency but to point out the cause of inefficient innovation intrinsic to differential information. Since it turns out that the problem is the underproduction of futures by a centralized exchange, we take the result as a caveat against thoughtless centralization.
Why, then, does centralization cause underproduction? Why does the number of the futures created, in addition to their design, matter under differential information? The reason is as follows: When the investors have symmetric information, their demands for futures depend solely on their hedging needs. Hence, so long as there is unhedged risk, an exchange is willing to create an additional contract. Consequently, so long as there are enough investors, generically, the maximal number $I$ of contracts are created.\textsuperscript{8} Thus, under symmetric information, only the contract design matters. In contrast, when the investors have differential information, their demand for futures depends on their information as well. Creating an additional futures may transmit more information. However, better information does not necessarily imply more trade (Pfleiderer (1984) and Wang (1992)). Hence, the exchange may be better off by not creating a new contract even though there remains some unhedged risk. The ability to coordinate in creating futures enables the centralized exchange to exploit this situation, which may result in the underproduction of futures contracts. In this way, the number of contracts can be a non-trivial issue under differential information.

Since analyzing the effect of both the design and the number of the innovated contracts simultaneously is a complicated task, we restrict ourselves to a simple model in which the design matters only if underproduction of futures occurs. Specifically, we consider an economy with two risk factors where two contracts can be created and there are two types of investors with private signals about the risk factors. The fee schedule is taken to be quadratic.\textsuperscript{9} As a usual device to construct a noisy rational expectation equilibrium, we assume that noise or liquidity trade exists for any contract created by the exchanges. We then study the effects of differential information on futures innovation and the investors' welfare.

\textsuperscript{8}The maximal number of futures are not created only if all the possible risk sharing opportunity is absorbed by smaller number of contracts. But, in this case, both the design and the number of contracts no longer matters for efficiency.

\textsuperscript{9}Though we analyse only a quadratic fee schedule case for tractability, we conjecture that the similar result holds for general fee schedule $c \in C$. 
1.5.1 An Economy with Differential Information

The timing is given in <Fig.1.2.1.>. There are two risk factors \( \tilde{z}_n \sim N(0, 1) \) (\( n = 1, 2 \)) and four investors (\( k = 1, 2, 3, 4 \)) with identical CARA utility \( E[-exp\{(-1/r)\tilde{W}\}] \). Investors 1 and 3 (resp. 2 and 4) receive private information about risk factor 1 (resp. 2) as signals \( \tilde{s}_k = \tilde{z}_1 + \tilde{\epsilon}_1 \) (\( k = 1, 3 \)) (resp. \( \tilde{s}_k = \tilde{z}_2 + \tilde{\epsilon}_2 \) (\( k = 2, 4 \))) where (\( \forall n = 1, 2 \)) \( \tilde{\epsilon}_n \sim N(0, \sigma^2) \). Investor \( k \) has the initial endowment of the risk factors \( c_k = (c_{k1}, c_{k2})^T \in R^2 \). To construct a noisy rational expectation equilibrium, we assume that, for each contract \( i \), there exists independent noise or liquidity trade \( \tilde{y}_{Li} \sim N(0, \eta^2)^{10} \). We assume that up to two contracts can be created i.e. \( \tilde{T} = 2 \). We write \( \tilde{y}_L \equiv \tilde{y}_{L1} \) if only one contract is created, and that \( \tilde{y}_L = (\tilde{y}_{L1}, \tilde{y}_{L2})^T \) if two contracts are created. We assume that \( (\tilde{z}_1, \tilde{z}_2, \tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{y}_{L1}, \tilde{y}_{L2}) \) are independently distributed for any innovation of futures contracts. The distribution of \( (\tilde{s}_1, \tilde{s}_2, \tilde{y}_L) \) is public information, but their realization is the private information of the receiver. Given futures contracts \( F \), a fee schedule \( c_m \in C \), and equilibrium prices \( \tilde{P} = P(\tilde{s}_1, \tilde{s}_2, \tilde{y}_L, F, c_m) \), investor \( k \) takes positions \( y_k(\tilde{s}_1, \tilde{P}, F, c_m) \) so that

\[
y_k(s_k, F, c_m) \in \text{ArgMax}_y E[\tilde{W}_k(y, F, c_m)|s_k, P] - \frac{1}{(2\tau)} \text{Var}[\tilde{W}_k(y, F, c_m)|s_k, P]
\]

s.t. \( \tilde{W}_k(y, F, c_m) = s_k \tilde{Z} + (F^T \tilde{Z} - \tilde{P})^T y - \sum c_m(y) \)

\( E[\cdot|s_k, P] \) and \( \text{Var}[\cdot|s_k, P] \) are conditional expectation and conditional variance on \( (s_k, P) \), respectively. By definition, \( \sum_{k=1}^2 y_k(s_k, P(s_1, s_2, \tilde{y}_L, F, c_m), F, c_m) = 0 \).

The regulator sets a quadratic fee schedule \( c_m(y) = (1/(2m))y^2 \). Since the distribution of the private signals is not verifiable, the unit of a contract is determined by the unconditional standard deviation of the contract's payoff unspanned by the

\[10\]The existence of noise or liquidity trade has a drawback to obscure the welfare analysis. However, it also has the merit to guarantee that the investors' ex-ante utility levels are well-defined without any further restriction on the parameter values.
others'. Two futures can be created ($\mathcal{I} = 2$). The exchanges know only public information. They innovate futures to maximize their own expected fee revenue. A centralized exchange chooses $F \in O^{2.1} \cup O^{2.2}$ to maximize

$$E[\sum_{k=1}^{2} y_k(\tilde{s}_k, \tilde{x}_k, \tilde{P}, F, c_m)]$$

and decentralized exchanges choose $f_i$ to maximize

$$E[\sum_{k=1}^{2} y_k^2(\tilde{s}_k, \tilde{x}_k, \tilde{P}, F, c_m)]$$

subject to $F = [f_i, f_{i'}] \in O^{2.2}$ given $f_i$ where $\{i, i'\} = \{1, 2\}$.

**Definition 1.5.1.**

$F \in O^{2.1} \cup O^{2.2}$ weakly Futures Innovation (FI-) optimal, if there exists no other set of futures contracts $F' \in O^{2.1} \cup O^{2.2}$ that Pareto-improves the investors ex-ante utilities at zero transaction fee.

We investigate the FI-optimality of small-fee equilibria where each investor's ex-ante utility is given by $\lim_{m \to -\infty} E[-e^{xp(-1/r)}\tilde{W}_k(\tilde{y}_k, F, c_m)]$. Specifically, we compare the welfare consequence of centralization and decentralization and ask which is more desirable to the investors in terms of weak FI-optimality.

### 1.5.2 Underproduction of Futures Contracts with Differential Information

First, consider the innovation by the centralized exchange. When only one contract is created, it is analytically difficult to obtain a noisy rational expectation equilibrium price for the general design of the futures. (We have to solve a fifth-degree equation. See Lemma A.3. in the appendix.) This is because the informativeness of the price
depends on the design of the contract. For tractability, we will focus on a particular index futures \( f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T \), and assume that the centralized exchange is required to select this design when it creates only one contract. Under this assumption, the centralized exchange decides how many futures to create.

When two contracts are created, however, since there are only two risk factors, the information transmitted by the futures' prices does not depend on the design of the contracts. Hence, we can obtain the rational expectation equilibrium for general innovation. Lemma 1.A.4. and 1.A.5. provide the rational expectation equilibrium prices, the normalized expected fee revenue, and the investors' ex-ante utility levels in the small fee equilibria for the innovation of a single index futures \( f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T \) and two futures, respectively.

Note that, since the expected square trading volume of each contract is positive, the decentralized futures exchanges always supply two futures contracts. On the other hand, a centralized exchange creates two futures if and only if they generate higher expected fee revenue than one futures. Thus, under centralization, the number of innovated futures can vary, as economic environments vary. However, unlike the symmetric information case, this need not yield the futures innovation more desirable to the investors than decentralization. We can confirm this claim by constructing such an economy.

**Proposition 1.5.1.**

There exists an economy where the futures innovation by the centralized exchange is not weakly FI-optimal. That is, while the investors, ex-ante, unanimously prefer the innovation of two futures that that of one, a centralized exchange creates only one futures, and decentralized exchanges creates two.
Thus, from the investors' viewpoint, centralization need not yield the futures innovation more desirable than decentralization under differential information.\textsuperscript{11} Now, consider when this inefficiency is likely to occur. Suppose first that the liquidity trade is large (i.e. $\eta$ is large). Then, the exchange is likely to benefit from creating an additional contract by capturing the large liquidity motivated trade. Hence, two contracts are likely to be created. Suppose next that the innovators have inaccurate information (i.e. $\sigma^2$ is large). Then, the situation is similar to symmetric information and the effect of the informativeness of the futures prices on transaction is small. Thus, adding new futures does not significantly affect the demand for the incumbent and is likely to increase the total square trading volume. Thus, two futures are likely to be created. Consequently, in these situations, underproduction of futures is unlikely.

Suppose, on the other hand, that information is accurate (i.e., $\sigma^2$ is small). Consider how investor 1 (or 3) trade the index futures $f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T$. Since investor 1 (or 3) is informed about the first factor but uninformed about the second, he faces two kinds of adverse selection. One is for him: He has better knowledge about the half of the futures payoff than investor 2 (or 4) and the noise (or liquidity) trader. The other is against him: He has worse knowledge about the other half of the futures payoff than investor 2 (or 4).

Now, compare the cases of one futures innovation and two futures innovation. As the private information of the investors gets more accurate, both adverse selection effects become tighter. When only one futures $f_{in}$ is created, though the first induces investor 1 (or 3) to trade more aggressively, the second makes him to trade more cautiously. On the other hand, when two contracts are created, investor 1 (or 3) can trade futures more aggressively on his information, because, with two contracts, he can hedge against the risk factor of which he is uninformed. This difference of

\textsuperscript{11}Though this proposition assumes a quadratic fee schedule, we conjecture that the similar result holds for general small fee schedules $c \in C$. This is because negative correlation between information and futures transaction volume is an independent phenomena of the choice of the fee schedule.
aggressiveness to trade on the private information affects the informativeness of the futures price(s). Consequently, in an equilibrium, the price of the index futures reflects less information of the investors than those of two futures. However, this implies more trade due to the difference of the information with a single index futures than with two futures. Hence, if the reduction of the trade motivated by the informational difference outweighs the increase of trade by capturing new liquidity (or noise) trade, the centralized exchange creates only one contracts rather than two.

Whether this innovation-decision by the exchange fits the investors' benefit is a subtle question. This is because the conflicting effects of the innovation of a new contract: While the creation of a new contract provides the investors the additional means of hedging and speculation, the increase of the informativeness of the prices may decrease the quality of hedging or speculation in ex-ante sense. (Suppose hypothetically that the futures prices is extremely accurate i.e., almost equal to their true payoffs. Then, in ex-ante sense, such futures provide little hedging opportunity to the investors, and the speculation benefit, which is the difference between the payoffs and the prices, is almost nil.) When the exchange creates one futures, but the benefit of an additional contract as a new hedging and speculation tools exceeds the deterioration of the quality of hedging or speculation due to the release of new information, underproduction can occur. We confirm this thought experiment by some numerical examples.

< Fig.1.5.1. > plots\textsuperscript{12} the expected square transaction volume $\Pi^{(1)}$ (i.e., normalized expected fee revenue) when the single index futures $f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T$ is created, against $\sigma$, the standard deviation of the error in signals, and $\eta$, the standard deviation of the noise (or liquidity) trade. As $\sigma$ becomes smaller, i.e., signals become more accurate, $\Pi^{(1)}$ tends to increase in this environment. This is because for each investor the benefit of knowing more about the half of the futures $f$'s payoff exceeds

\textsuperscript{12}In all plots, we take risk tolerance $r$ to be 1. We also assume that the investors does not have initial endowments of the risk factors. Thus, In these plots, the investors gain from the speculation trade. However, for the small endowment of the risk factors, we obtain the essentially same results.
the loss of knowing less about the other half of the payoff. <Fig.1.5.2.> plots $\Pi^{(2)}$, the expected sum of the square trading volume when two futures are created, against $\sigma$ and $\eta$. Introduction of new futures induces the investors to trade more aggressively on their own information. The futures prices becomes more informative, which symmetrizes the information among the investors and reduces the transaction due to the difference of the information. Consequently, the expected sum of equilibrium square transaction volume increase less than $\Pi^{(1)}$ as information becomes very accurate. <Fig.1.5.3.> plots their difference, $\Pi^{(1)} - \Pi^{(2)}$. If it is positive, revenue from one futures is greater than that of two, and the centralized exchange creates only one futures.

<Fig.1.5.4.> and <Fig.1.5.5.> plots $U^{(1)}$, the investors' ex-ante utility when one futures $f = (1/\sqrt{2}, 1/\sqrt{2})^T$ is created, and $U^{(2)}$, the investors' ex-ante utility when two futures are created, respectively. <Fig.1.5.6.> shows their difference $U^{(1)} - U^{(2)}$. It can be seen that, for some environment where one futures is created by the centralized exchange, the investors prefer the innovation of two futures (See the proof of Proposition 1.5.1.).

We conclude that, with differential information, the number of futures innovated by a centralized exchange can vary endogenously and that underproduction of futures by a centralized exchange can occur. When information has a significant effect on the futures trading, centralization need not be a better idea than decentralization.
1.6 Concluding Remarks

(1) In this paper, we considered the efficiency of futures innovation when a certain number of multiple contracts can be created without cost. Our analysis shows that industry structure and distribution of information among investors has a non-trivial effect on the number of innovated futures and its welfare consequence. This result generates the following two questions for immediate future research:

a. "Explicit consideration of set-up cost of futures"

With the set-up cost of a contract borne by member brokers, the fee revenue should be away from zero to cover the cost. The first question we have to answer is whether we can meaningfully define the notion of efficiency in this context. If we can\textsuperscript{13}, then we can investigate the conditions for efficient innovation in this case. It would be interesting to see how the results would change, if the regulator could put a non-discriminatory lump-sum tax on investors. In searching for an efficient combination of such a lump-sum tax and a transaction fee scheme, we may be able to apply some results from optimal taxation theory. Extending the analysis in this direction would allow us to make more practical predictions.

b. "Information distribution and market incompleteness"

We have observed that, under a centralized exchange, the number of futures innovated in equilibrium varies endogenously depending on the distribution of information among the investors. This is because more accurate information does not necessarily imply more equilibrium trading volume. Pfeifferer (1984), with an exogenously given single asset, showed that this

\textsuperscript{13}Duffie–Jackson (1989) proposes a notion of efficiency with fee revenue away from zero, which may be appropriately extended. Taking a fee schedule, a set of futures is called "(constrained) Pareto optimal" with respect to that fee schedule, if there is no set of futures that leads to Pareto superior equilibrium allocation under the same fee scheme whether or not investors are compensated by the amount of fee charges. It is worth remarking that a quadratic fee schedule attains this sense of efficiency for any fee level among the contracts with orthogonal payoffs. Hence, again, a quadratic fee schedule seems good for efficient innovation.
non-monotonic relationship between information and trading volume can exist if there is correlation among the errors in the investors' private signals. This suggests that the non-monotonicity between information and trading volume, and hence the underproduction of futures, may occur quite frequently. The specific information distribution in this paper did not give us a transparent expression. However, by using a different distribution, we may obtain a simpler characterization. Mapping the relationship between the number of futures observed in equilibrium and the information distribution is interesting in its own right.

(2) Inefficiency due to the lack of a coordination mechanism under the decentralized innovators is observed in other contexts of innovation problems, though the precise mechanism that generates coordination problems is different. For the innovation of financial assets such as CMO's, asset-backed securities, and zero-coupon bonds, Pesendorfer (1991) shows that inefficient innovation may occur due to the lack of a coordination mechanism, if there is a fixed marketing cost per customer.

(3) In our model, we restrict the feasible regulations so that the regulator has much less discretionary power compared with a "Social Planner." For example, she cannot set a discriminatory fee schedule depending on the investors' types or the contract designs. She cannot enforce discriminatory lump-sum transfers among investors. This formulation has an advantage in that the enforcement of the regulations requires the regulator to possess little knowledge of the characteristics of the economy. We show that a certain sense of efficiency (FI-optimality) can be attained with these (informationally) easily enforceable regulations. Hara (1991) introduces a "Social Planner" and defines a stronger optimality notion that allows the planner to perturb the allocation with arbitrary lump-sum transfers among the investors. In the model where only one futures contract is innovated and the transaction fee is set to zero, he shows that the futures innovation by the volume-maximizing exchange generically yields an
inefficient allocation with respect to his stronger efficiency criterion.

(4) We assume that even when several contracts are created by the decentralized exchanges, the investors can still submit the demand functions that can depend on the prices of the futures traded in the different exchanges. Though this assumption cannot be any restriction in a continuous time trading model in which the information is continuously resolved, the impact of this type of institutional friction on the futures prices, their informativeness, and the welfare is worth analysing.

(5) The presence of the noise or liquidity trader under differential information is for the technical device to construct a noisy rational expectation equilibrium. While this approach has the disadvantage of obscuring the welfare analysis, it also has the advantage that the investors’ ex-ante utility levels are well defined without any further restriction on the parameter values, unlike the alternative formulation where the liquidity shocks are introduced through the investors’ initial endowments.
1.7 Appendix 1

Lemma 1.A.1.

For a given \((F,c) \in O^{N,T} \times C\), there exists a market equilibrium \((P^*, (y^*_1, \ldots, y^*_K))\) in which the set of equilibrium futures positions \((y^*_1, \ldots, y^*_K)\) is unique. For any \(k = 1, \ldots, K\) and \(i = 1, \ldots, T\), they solve the following

\[
- p^*_i - c'(y^*_k) - (1/r_k)[y^*_ki + e_k^T f_i] = 0 \quad \text{if } y^*_ki \neq 0
\]

\[
c_- \leq - p^*_i - (1/r_k)e_k^T f_i \leq c_+ \quad \text{if } y^*_ki = 0
\]

where, if \(y^*_ki \neq 0\) for some \(k\), then

\[
p^*_i = \frac{-\sum_k 1_{\{x\neq 0\}}(v^*_k) e_k^T f_i}{\sum_k 1_{\{x\neq 0\}}(v^*_k) r_k} - \frac{\sum_k 1_{\{x\neq 0\}}(v^*_k) r_k c'(v^*_ki)}{\sum_k 1_{\{x\neq 0\}}(v^*_k) r_k}
\]

where \(1_{\{x\neq 0\}}(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Proof of Lemma 1.A.1.

Note first that if \(\bar{x} \sim N(m, \sigma^2)\), then \(E[-\exp((-1/r_k)\bar{x})] = -\exp((-1/r_k)E[\bar{x}] - (1/(2r_k))\text{Var}[\bar{x}])\). Hence, (1) is equivalent to

\[
y_k \in \text{ArgMax}_{y \in R^T} -y^T P - (1/(2r_k))(e_k^T e_k + y^T y + 2y^T F^T e_k) - \sum_i c(y_i)
\]

For a given \(P\), first order condition (F.O.C.) (which is sufficient, too) will be,

\[
- p_i - c'(y_k) - (1/r_k)[y_ki + e_k^T f_i] = 0 \quad \text{if } y_ki \neq 0
\]

\[
c_- \leq - p_i - (1/r_k)e_k^T f_i \leq c_+ \quad \text{if } y_ki = 0
\]

Let \(y_ki(p_i)\) be the solution of the above. Since \(c'(x)\) is continuous and increasing in \(x \neq 0\) and \(r_k > 0\), \(y_ki(p_i)\) is a continuous decreasing function of \(p_i\) with properties \(\lim_{p_i \to \infty} y_ki(p_i) = -\infty\), and \(\lim_{p_i \to -\infty} y_ki(p_i) = \infty\), Thus, for each \(i\), there exist some \(p^*_i\) such that \(\sum_k y_ki(p^*_i) = 0\). \(P^* \equiv [p^*_1, \ldots, p^*_K]^T\) constitutes a market-equilibrium price for \((F, c)\). Furthermore, the fact that \(y_ki(p_i)\) is strictly decreasing in \(p_i\) for all \(k\) whenever \(y_ki(p_i) \neq 0\) shows the uniqueness of the market equilibrium futures position given \((F, c)\). Finally, expression (1.4) is from direct computation. \(\square\)
The following technical lemma is useful to prove Lemma 1.3.1.

**Lemma 1.A.2.**

\[
(\forall c \in C)(\forall m \geq 0)(\exists \, D^c_m > 0)(\forall \, F \in O^{N,I}, k, i)
\]

\[
D^c_m \to 0 \ (m \to \infty) \text{ and } | y_{k_i}^m(F) - y_{k_i}(F)| \leq D^c_m
\]

where \( y_{k_i}^m(F) \) is investor \( k \)'s market equilibrium futures position of contract \( i \) given \( (F, c_m) \), and \( y_{k_i}(F) = r_k \left[ \sum_{k' \neq k} \frac{e_{k_i}^T}{r_{k_i}} \right] f_i \) (defined by (1.2)). That is, market equilibrium futures positions \( y_{k_i}^m(F) \) given \( (F, c_m) \) converge to \( y_{k_i}(F) \) uniformly in \( F, k \) and \( i \).

**Proof of Lemma 1.A.2.**

\( y_{k_i}^m(F) \) satisfies

\[
- p_i^m - (1/m)c'(y_{k_i}^m(F)) - (1/r_k)[y_{k_i}^m(F) + x_k^T f_i] = 0 \quad \text{if } y_{k_i}^m(F) \neq 0
\]

\[
(1/m)c_+ \leq - p_i^m - (1/r_k)x_k^T f_i \leq (1/m)c_+ \quad \text{if } y_{k_i}^m(F) = 0
\]

\[
\sum_k y_{k_i}^m(F) = 0 \quad \text{for all } i
\]

By Lemma 1.A.1., for each \( F \in O^{N,I} \) and \( m, \{y_{k_i}^m(F)\}_{k, i} \) exists and is unique.

We first claim that \( (\exists \, M > 0)(\forall F, m, k, i) \ | y_{k_i}^m(F) | \leq M \). Indeed, suppose not, then W.O.L.G. there exists a subsequence \( \{n\} \) of \( \{m\} \) and the corresponding sequence \( \{F^n, k^n, i^n\} \) such that \( y_{k^n_{i^n}}(F^n) \to \infty \) as \( n \to \infty \). Market clearing condition then implies that \( (\exists \, k^n \neq k^n) \ y_{k^n_{i^n}}(F^n) \to - \infty \) as \( n \to \infty \). But, since futures position is strictly decreasing in futures price for any investor, it must be both

\[
p_i^n = - (1/n)c'(y_{k^n_{i^n}}(F^n)) - (1/r_k)[y_{k^n_{i^n}}(F^n) + e_{k^n i}^T f_i] \to - \infty, \text{ and}
\]

\[
p_i^n = - (1/n)c'(y_{k^n_{i^n}}(F^n)) - (1/r_k)[y_{k^n_{i^n}}(F^n) + e_{k^n i}^T f_i] \to \infty
\]

A contradiction. Now, we can write

\[
y_{k_i}^m(F) = (r_k)(-p_i^m) - e_k^T f_i + \delta_{ki}^m
\]

where \( |\delta_{ki}^m| \leq D^c_m \)

\[
\equiv \text{Max}[(1/m)(\text{Max}_k r_k)(c'(M) - c'(-M)), (1/m)(c_+ - c_-)]
\]

\[
\to 0 \ (m \to \infty)
\]

\[
\delta_{ki}^m \equiv (-r_k)(1/m)c'(y_{k_i}^m(F)) \quad \text{if } y_{k_i}^m(F) \neq 0
\]
Using the market clearing condition, we have
\[-p_i^m = \sum_{k \in K} e_i^T f_i + \sum_{k \in K} \delta_k^{e_i^T} \Rightarrow \sum_{k \in K} e_i^T \delta_k^{e_i^T} f_i (m \to \infty)\]

Hence, \(y_{ki}^m(F) \rightarrow r_k \left[ \sum_{k \in K} e_i^T \delta_k^{e_i^T} - \frac{e_i^T}{r_k} \right] f_i \equiv y_{ki}(F)\)

Since \(y_{ki}^m(F) = y_{ki}(F) + \sum_{k'} \delta_k^{e_i^T} + \delta_k^{e_i^T}\)
\[|y_{ki}^m(F) - y_{ki}(F)| = \left| \sum_{k \in K} \delta_k^{e_i^T} + \delta_k^{e_i^T} \right| \leq (K+1) \hat{D}_m \equiv D_m \rightarrow 0 (m \to \infty) \quad \square\]

**Proof of Lemma 1.3.1.**

We prove the decentralized case. (The centralized case is proven in the same way.)

W.L.O.G., we can write \(F^m \to F^* (m \to \infty)\) Suppose that, for some \(i\), \(f_i^*\) does not satisfy (3) i.e. \((\exists \hat{F} \in O^{N,T}) \hat{f}_i = f_i^*\) and \(\sum_{k=1}^K c(y_{ki}(\hat{F})) - \sum_{k=1}^K c(y_{ki}(F^*)) = \epsilon > 0\)

Define \(\hat{F}^m\) by \(\hat{f}_i^m = f_i^m\) \(\hat{f}_i^m = \hat{f}_i\) for all \(m\). By construction, \(\hat{F}^m \to \hat{F}\). Hence, by continuity of \(c\) and \(y_{ki}(\cdot)\), for large \(m\),
\[\sum_{k=1}^K c(y_{ki}(\hat{F}^m)) \geq \sum_{k=1}^K c(y_{ki}(\hat{F})) - \epsilon/4\]

Now, by (Lemma 1.A.2.) and continuity of \(c\), for large \(m\),
\[|\sum_{k=1}^K c(y_{ki}(\hat{F}^m)) - \sum_{k=1}^K c(y_{ki}(\hat{F}))| \leq \epsilon/4\]

With the definition of \(F^m\), they imply for large \(m\),
\[\sum_{k=1}^K c(y_{ki}(F^m)) \geq \sum_{k=1}^K c(y_{ki}(\hat{F}^m)) \geq \sum_{k=1}^K c(y_{ki}(\hat{F})) - \epsilon/2\]

On the other hand, since \(F^m \to F^*\), \(y_{ki}(F^m) \to y_{ki}(F^*)\) and that by Lemma 1.A.1., \(y_{ki}^m(F^m) \to y_{ki}(F^*)\) uniformly in \(k\) and \(i\) as \(m \to \infty\). Therefore, for \(m\) large enough,
\[|\sum_{k=1}^K c(y_{ki}(F^m)) - \sum_{k=1}^K c(y_{ki}(F^*))| \leq \epsilon/4\]

But, then, for large \(m\), \(\sum_{k=1}^K c(y_{ki}(F^m)) - \sum_{k=1}^K c(y_{ki}(F^m)) \geq \epsilon/4\). A contradiction. \(\square\)

**Proof of Lemma 1.3.2.**

When \(c \equiv 0\), the optimization problem (1.1) of investor \(k\) will be
\[y_k \in \text{ArgMax}_{y \in \mathbb{R}^T} -y^T P \cdot (1/r_k) [\epsilon_i^T c_k + y^T y + 2y^T F^\top c_k]\]
(F.O.C.) and equilibrium price will be
\[-p_i^* = (1/r_k)[y_{ki} + \epsilon_i^T f_i] = 0, \quad p_i^* = -\sum_{k \in K} e_i^T \delta_k^{e_i^T} / r_k f_i\]

Hence, equilibrium futures position \(y_{ki}\) is in fact given by
\[y_{ki} = r_k \left[ \sum_{k \in K} e_i^T / r_k \right] f_i = y_{ki}(F)\]
From the above equations, the equilibrium utility level given $F$ will be

$$U_k[F] \equiv U_k[c^T_k \tilde{Z} + (F^T \tilde{Z} - P^*)^T y_k] = (1/(2r_k))[c_k^T c_k + y_k(F)^T y_k(F)]$$

Lemma 1.3.2. is immediate from Definition 1.3.4. \(\Box\)

**Proof of Proposition 1.4.1.**

**Sufficiency**

Assume that $c$ is quadratic and that $\hat{F} \in O^{N,I}$ Pareto dominates $F \in O^{N,I}$ at zero transaction fee. Then, by Lemma 1.3.2., (1.3) holds. But, this implies

$$\sum_{i=1}^{I} \sum_{k=1}^{K} c(y_{ki}(\hat{F})) > \sum_{i=1}^{I} \sum_{k=1}^{K} c(y_{ki}(F))$$

By Lemma 1.4.2., $y_{ki}^m(F') \rightarrow y_{ki}(F')$ ($m \rightarrow \infty$) uniformly in $F' \in O^{N,I}$, $k, i$. Hence, the claim follows from the continuity of $c$.

**Necessity**

By assumption for any $F, \hat{F} \in O^{N,I}$, if

$$(\forall k) \ \sum_{i=1}^{I} [y_{ki}(\hat{F})]^2 \geq \sum_{i=1}^{I} [y_{ki}(F)]^2, \ (\exists k') \ \sum_{i=1}^{I} [y_{k'i}(\hat{F})]^2 > \sum_{i=1}^{I} [y_{k'i}(F)]^2,$$

then $\sum_{i=1}^{I} \sum_{k=1}^{K} c(y_{ki}(\hat{F})) > \sum_{i=1}^{I} \sum_{k=1}^{K} c(y_{ki}(F))$

We claim that $\exists \phi : R_+ \rightarrow R_+$ strict increasing and $\sum_{i=1}^{I} c(y_{ki}(F)) = \phi(\sum_{i=1}^{I} [y_{ki}(F)]^2)$

To see this, consider an economy with four investors with initial endowments $e_1 = -e_2, e_2 = -e_4, e_2 \neq e_1$. By symmetry of initial endowments, one can show $y_{i1}(F) = -y_{3i}(F), y_{2i}(F) = -y_{4i}(F)$ and $c(y_{i1}(F)) = c(y_{3i}(F)), c(y_{2i}(F)) = c(y_{4i}(F))$. Now, suppose that the claim is not true. Then, there exist some $e_1, e_2, F$, and $\hat{F}$ such that

$$(\forall k) \sum_{i=1}^{I} [y_{ki}(\hat{F})]^2 > \sum_{i=1}^{I} [y_{ki}(F)]^2, \text{ and } \sum_{i=1}^{I} \sum_{k=1}^{K} c(y_{ki}(\hat{F})) \leq \sum_{i=1}^{I} \sum_{k=1}^{K} c(y_{ki}(F))$$

But then, $\hat{F}$ is not strictly preferred by the exchange to $F$ though $\hat{F}$ Pareto dominates $F$. A contradiction to the assumption. Hence, $\sum_{i=1}^{I} c(y_{ki}(F)) = \phi(\sum_{i=1}^{I} [y_{ki}(F)]^2)$.

This implies $c(y_{ki}(F)) = \phi([y_{ki}(F)]^2)$ and $\sum_{i=1}^{I} c(y_{ki}(F)) = \sum_{i=1}^{I} \phi([y_{ki}(F)]^2) = \phi(\sum_{i=1}^{I} [y_{ki}(F)]^2)$.

Hence, $\phi$ is linear i.e. $\phi(x) = a + bx$. Finally, since $c(0) = 0, a = 0$. Thus, $c(y) = by^2$. \(\Box\)
Proof of Proposition 1.4.2.

W.L.O.G. set \( c(y) = y^2 \) and \( c_m(y) = (1/(2m))y^2 \). For each \( m \), the F.O.C. of each investor \( k \)'s optimization problem given \( F \) will be

\[-p_i - (1/r_k + 1/m)y_{ki} + (1/r_k)e_k^T f_i = 0\]

It is easy to show that the equilibrium price and futures position will be

\[-p_i = \frac{1}{\sum_{k'=1}^{K} \frac{1}{|r_{k'}/m| + 1}} \left( \sum_{k'=1}^{K} \frac{1}{|r_{k'}/m| + 1} e_{k'}^T f_i \right)\]

\[y_{ki}^m(F) = \frac{r_k}{|r_k/m| + 1} \left( \sum_{k'=1}^{K} \frac{1}{|r_{k'}/m| + 1} e_{k'}^T - \frac{e_k^T}{r_k} \right) f_i\]

For each \( m \), the monopolist chooses \( F^m \) so that

\[F^m \in \text{ArgMax}_{F \in O^{N,I}} \sum_{k=1}^{K} \sum_{i=1}^{T} [y_{ki}^m(F)]^2\]

Note that \( O^{N,I} :\equiv \{ F \in R^{N \times I} | F^T F = I_I \} \) is compact. \( O^{N,I} \) is a Stiefel manifold with normalization and orthogonality restriction. It is these restrictions that make \( O^{N,I} \) compact. Indeed, by identifying \( R^{N \times I} \) with \( NI \)-dimensional Euclid space \( R^{NI} \), \( F^T F \) defines a continuous (inner product-like) function from \( R^{N \times I} \times R^{N \times I} \) to \( R^{I \times I} \). \( O^{N,I} \) is a inverse image of a point \( I_I \) by this map. Hence, it is closed. Boundedness is obvious from its definition. For the use of a Stiefel manifold in an economic model with incomplete markets, see Husseini–Lasry–Magill (1990). Since \( y_{ki}^m(F) \) given by the equation above is continuous in \( F \), the optimization problem above has its solution \( F^m \).

Take a sequence \( \{F^m\}_m \subset O^{N,I} \). W.L.O.G. we can assume \( \{F^m\}_m \) itself converges to some \( F^* \in O^{N,I} \). It is easy to check that along this sequence, the corresponding \((P^m, (y^m_1, ..., y^m_K)) \to (P^*, (y_1^*, ..., y_K^*)) (m \to \infty)\) and hence \( \{(P^*(y_1^*, ..., y_K^*)), F^*\} \) constitutes a small-flee equilibrium.

We only have to show it is FI-constrained optimal. Suppose that some \( F' \in O^{N,I} \) Pareto dominates \( F^* \in O^{N,I} \) at zero transaction fee. But, by Lemma 1.3.2, this contradicts Lemma 1.3.1. Hence, \( \{(P^*(y_1^*, ..., y_K^*)), F^*\} \) is FI-constrained optimal.

\( \square \)

Proof of Proposition 1.4.3.

Consider a transaction fee schedule \( c \in C \). By assumption, especially symmetry of \( c \) i.e. \( c(-y) = c(y) \), for any \( c \in C \) and \( y, y' \in R, y'^2 > (\text{resp.} =) y^2 \) implies \( c(y') \)
> (resp. \(\geq\)) \(c(y)\).

Now, suppose that \(\{(P^*,(y_1^*,\ldots,y_K^*),f^*)\}\) is a small-fee equilibrium but not FI-constrained optimal i.e. \(f^*\) is Pareto dominated by some \(f'\). Then, by Lemma 1.3.2,
\[
(\forall \ k) \ y_k(f')^T y_k(f') \geq y_k(f^*)^T y_k(f^*) \quad (\exists \ k) \ y_k(f')^T y_k(f') > y_k(f^*)^T y_k(f^*).
\]
But, this implies that
\[
(\forall \ k) \ c(y_k(f')) \geq c(y_k(f^*)) \quad (\exists \ k) \ c(y_k(f')) > c(y_k(f^*))
\]
But, this contradicts Lemma 1.3.1. Hence, \(\{(P^*,(y_1^*,\ldots,y_K^*),f^*)\}\) is FI-constrained optimal.

\(\square\)

**Proof of the claim in Example 1.4.1.**

By Lemma 1.3.1, if \(F^*\) is in a small-fee equilibrium, then
\[
F^* \in \text{ArgMax}_{F \in O^{3,2}} \sum_{k=1}^{3}\sum_{i=1}^{2} \ y_{ki}(F) \mid (2)
\]
where \(y_{ki}(F)\) is given by (2). Assume now that the exchange designs contracts so that all investors take non-zero position in all futures. Then, using the fact that \(\sum_{k=1}^{3} c_k = 0, y_{ki}(F) = -e_k f_i, F\) can be taken so that \(y_{1i}(F), y_{2i}(F) \geq 0\). Hence, the optimization problem will be equivalent to
\[
F^* \in \text{ArgMax}_{F \in O^{3,2}} (1/4)(2, 1, 0)^T (f_1 + f_2) + (1/4)(2, 3, 0)^T (f_1 + f_2)
\]
which is simplified as
\[
F^* \in \text{ArgMax}_{F \in O^{3,2}} (1, 1, 0)^T (f_1 + f_2)
\]
Obviously, any \(F \in O^{3,2}\) with \(f_1 + f_2 = (1, 1, 0)^T\) can be a solution. Hence, both \(F\) and \(F'\) are solutions of this maximization problem and they are shown to generates trading volume 4 in this limit. One can also show that any choice of contracts that allow one of the investors to trade only one futures results in trading volume strictly less than 4 at zero transaction fee. Thus, for \(m\) large enough, W.L.O.G. we can assume that the exchange selects contracts that induce the investors to take non-zero position in all futures.

To prove that \(F\) and \(F'\) are small-fee equilibria, we have to show there exist sequences of innovation equilibria that converge to \(F\) and \(F'\). We claim \(\{F^m\}_m\) defined by \(F^m = F\) and \(\{F'^m\}_m\) by \(F'^m = F'\) constitute such sequences. Indeed, given a contract \(\hat{F}\) that induce all investors to take non-zero position in all futures,
the equilibrium positions are given by

\[ y_{11}^m(\hat{F}) = -x_k \hat{f}_1 + (r/2)(T/m)[1/3 - \text{Sign}(y_{11}^m(\hat{F}))] \]

Hence, the brokerage fee revenue given \( c_m(y) = (T/m)|y| \) will be

\[ 2(T/m)[(1, 1, 0)(\hat{f}_1 + \hat{f}_2) - (2/3)(r/2)(T/m)] \]

whose maximum is attained by choosing \( \hat{F} \in O^{3,2} \) so that \( \hat{f}_1 + \hat{f}_2 = (1, 1, 0)^T \). Since \( F \) and \( F' \) satisfy this condition and since, for large \( m \), all investors take non-zero position in all futures given \( F \) or \( F' \) and \( (1/m)c \), \( F^m \) and \( F'^m \) so defined are in fact innovation equilibria for each \( m \).

The rest of the claim in this example is from direct computation. \( \square \)

**Proof of the claim in Example 1.4.2.**

To prove that both \( F \) and \( F' \) are small-fee equilibrium futures contracts given \( c(y) = y^2 \), we have to show they are the limits of the sequences of small-fee equilibrium. Now, consider the \( m \)-th economy with fee schedule \( c_m = (1/(2m))y^2 \). By identical risk tolerance \( \nu = r \) and non-aggregate shock \( \sum_{k=1}^4 e_k = 0 \), it is easy to show that the equilibrium price will be \( p_{i1}^m = 0 \), and equilibrium position will be

\[ y_{11}^m(F) = \frac{m}{2(1 + m)}[ -e_k^T \hat{f}_1 ] \]

Now, suppose \( f_2' = (0, 1, 0)^T \) is given. Exchange 2 chooses

\[ f_2' \in \text{ArgMax2}[ (y_{11}^m(\hat{F}))^2 + (y_{22}^m(\hat{F}))^2 ] \]

\[ \text{s.t. } \hat{f}_1^T \hat{f}_2 = 0 \text{ and } \hat{f}_2^T \hat{f}_2 = 1 \]

Take \( \hat{f}_2 = (a, b, c)^T \), then the constraints are given by \( b = 0 \) and \( a^2 + b^2 + c^2 = 1 \). Exchange 2 hence chooses \( (a, b, c) \) so that

\[ (a, b, c) \in \text{ArgMax}[ \frac{m}{2(1 + m)}^2[a^2 + (b + c)^2] \text{ s.t. } a^2 + c^2 = 1 \]

Obviously, \( (a, b, c) = (0, 0, 1) \) constitutes one of the solutions. On the other hand, given \( f_2' = (0, 0, 1) \),

\[ f_1' = (0, 1, 0)^T \in \text{ArgMax2}[ (y_{11}^m(\hat{F}))^2 + (y_{22}^m(\hat{F}))^2 ] \]

\[ \text{s.t. } \hat{f}_2^T \hat{f}_1 = 0 \text{ and } \hat{f}_1^T \hat{f}_1 = 1 \]

Hence, \( F'^m = F' \) is in a innovation equilibrium given \( c_m \) and \( F' \) is a small-fee equilibrium. Similarly, one can show \( F'^m = F \) is a innovation equilibrium. \( \square \)
Proof of Lemma 1.4.1.

We only have to give an inefficiency example for each \( c \in C \). Consider the same situation as that in Example 1.4.2. That is, \(-c_1 = c_3 = (1, 0, 0)^T\) and \(-c_2 = c_4 = (0, 2, 2)^T\), and \(r_k = r\) for all \( k = 1, 2, 3, 4\). Suppose \( c_m(y) = (1/m)c(y) \) \( c \in C \) is given. Suppose also that \( f_1 = (0, 1, 0)^T\) is innovated. By orthogonality, \( f_2 = (a, 0, b)^T\) and \( a^2 + b^2 = 1\). The market equilibrium futures position will satisfy, for all \( k\),

\[
-p_2 + (1/m) c'(y_{k2}) - (2/r) [y_{k2} + f_2^T e_k] = 0
\]

\[
p_2 = (-1/2m)[c'(y_{12}) + c'(y_{22})]
\]

One can derive

\[
a - y_{12} = (1/2m)(r/2)[c'(y_{12}) - c'(y_{22})], \quad 2b - y_{22} = (1/2m)(r/2)[c'(y_{22}) - c'(y_{12})]
\]

Note, W.L.O.G. equilibrium position satisfies \( |y_{k2}| \leq 2 \) for all \( k \). Define \( M^c \equiv (r/2)c'(2)\), then \( |a - y_{12}| \leq M^c/m \) and \( |2b - y_{22}| \leq M^c/m \). Hence, one can write \( y_{12} = a + \delta_m^1, \ y_{22} = 2b + \delta_m^2 \) where \( |\delta_m^1| \leq M^c/m, \ |\delta_m^2| \leq M^c/m \). The exchange chooses \((a, b)\) so that

\[
\text{Max } c(y_{12}) + c(y_{22}) \text{ s.t. } a^2 + b^2 = 1 \ i.e.
\]

\[
\text{Max } c(a + \delta_m^1) + c(2b + \delta_m^2) \text{ s.t. } a^2 + b^2 = 1
\]

By convexity of \( c \in C \), maximum is attained by setting \( b = 1 \) when \( a + \delta_m^1 < 2b + \delta_m^2 \) can be attained. But, this is possible for sufficiently large \( m \). In the same way, given \( f_2 \), one can show \( f_1 \) maximized an exchange's revenue when the fee is small. Hence, they constitutes a small-fee equilibrium. As is in the Example 1.4.2, the corresponding allocation is FI-inefficient. (Note that the FI-efficient equilibrium is also attainable as a small-fee equilibrium.) \( \square \)

Lemma 1.4.3.

Suppose that the investors have no initial endowments of the risk factors (i.e., \((\forall k)e_k = 0\)), that a single contract \( f = (a, b)^T \ (a^2 + b^2 = 1) \) is created, and that the noisy REE price has the linear form \( \bar{p} = \alpha_1 \bar{\delta}_1 + \alpha_2 \bar{\delta}_2 + K \bar{y}_L \). Then, when there is no transaction fee (i.e., \( m = \infty \)), \( \alpha_1, \alpha_2 \), and \( K \) must satisfy

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\[ K_1 = K/\alpha_1, \quad K_2 = K/\alpha_2 \]

\[
\frac{2aK_1}{1+\sigma^2} = \frac{-2bK_2}{1+\sigma^2+K_2^2\eta^2} - \frac{1}{r}(\frac{b_0^2}{1+\sigma^2} + \frac{b^2(\sigma^2+K_2^2\eta^2)}{1+\sigma^2+K_2^2\eta^2})
\]

\[
\frac{2bK_1}{1+\sigma^2} = \frac{-2aK_1}{1+\sigma^2+K_1^2\eta^2} - \frac{1}{r}(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2})
\]

\[
\begin{align*}
\left(\frac{\sigma^2}{1+\sigma^2} + \frac{a^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2} + \frac{b^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right)\alpha_1 &= \left(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right) \frac{1}{1+\sigma^2} + \\
\left(\frac{a^2\sigma^2}{1+\sigma^2} + \frac{b^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right)\alpha_2 &= \left(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right) \frac{1}{1+\sigma^2+K_2^2\eta^2} + \\
\left(\frac{a_0^2\sigma^2}{1+\sigma^2} + \frac{b_0^2(\sigma^2+K_2^2\eta^2)}{1+\sigma^2+K_2^2\eta^2}\right)\alpha_2 &= \left(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right) \frac{1}{1+\sigma^2+K_2^2\eta^2} + \\
2\left(\frac{\sigma^2}{1+\sigma^2} + \frac{a^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2} + \frac{b^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right)\alpha_2 &= 2\left(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right) \frac{bK_2}{1+\sigma^2+K_2^2\eta^2} + \\
2\left(\frac{\sigma^2}{1+\sigma^2} + \frac{a^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2} + \frac{b^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right)\alpha_2 &= 2\left(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2}\right) \frac{aK_1}{1+\sigma^2+K_2^2\eta^2} - \frac{1}{r}(\frac{b_0^2}{1+\sigma^2} + \frac{a_0^2(\sigma^2+K_1^2\eta^2)}{1+\sigma^2+K_1^2\eta^2})\left(\frac{\sigma^2}{1+\sigma^2} + \frac{b^2(\sigma^2+K_2^2\eta^2)}{1+\sigma^2+K_2^2\eta^2}\right)
\end{align*}
\]

**Proof of Lemma 1.A.3.**

Suppose that there exists no transaction fee and that the REE price is given by 
\( \hat{p} = \alpha_1 \hat{s}_1 + \alpha_2 \hat{s}_2 + K \hat{y}_L \). Define \( \hat{p}_{(1)} = \alpha_1 \hat{s}_1 + K \hat{y}_L \) and \( \hat{p}_{(2)} = \alpha_2 \hat{s}_2 + K \hat{y}_L \). Then,

\[
E[\hat{Z}|s_1,p] = \begin{bmatrix}
\frac{1}{1+\sigma^2} s_1 \\
\frac{1}{1+\sigma^2+K_1^2\eta^2} p(2)
\end{bmatrix}
\]

\[
E[\hat{Z}|s_2,p] = \begin{bmatrix}
\frac{1}{1+\sigma^2+K_2^2\eta^2} p(1) \\
\frac{1}{1+\sigma^2} s_2
\end{bmatrix}
\]

\[
\text{Var}[\hat{Z}|s_1,p] = \begin{bmatrix}
\frac{\sigma^2}{1+\sigma^2} & 0 \\
0 & \frac{\sigma^2+K_1^2\eta^2}{1+\sigma^2+K_1^2\eta^2}
\end{bmatrix}
\]

\[
\text{Var}[\hat{Z}|s_2,p] = \begin{bmatrix}
\frac{\sigma^2+K_2^2\eta^2}{1+\sigma^2+K_2^2\eta^2} & 0 \\
0 & \frac{\sigma^2}{1+\sigma^2}
\end{bmatrix}
\]

Solving the investors' optimization problem by using these conditional expectations and variances, and imposing the equilibrium condition, the claim follows from direct computation. \( \square \)
Lemma 1.A.4.

If the centralized exchange creates a single index futures \( f_{in} = (a, b)^T \), as the transaction fee gets close to zero i.e., \( m \to \infty \),

(1) The noisy REE price of the index contract converges to

\[
\tilde{p}_{in} = \alpha_{in}(\delta_1 + \delta_2 + K_{in}^* \tilde{y}_L) + \bar{p}_{in}
\]

where \( K_{in}^* \) is the unique real solution of

\[
4r \eta^2 K^3 + \sqrt{2}(1 + 2\sigma^2)\eta^2 K^2 + 2\sqrt{2}\sigma^2(1 + \sigma^2) = 0
\]

\[
\alpha_{in} = \frac{1}{2\sqrt{2}}\left\{ \frac{1}{1+\sigma^2} + \frac{1}{1+\sigma^2+K_{in}^* \eta^2} \right\}
\]

\[
\bar{p}_{in} = \frac{-1}{2\sqrt{2}}\left\{ \frac{\sigma^2}{1+\sigma^2}(e_{11} + e_{31} + e_{22} + e_{42}) + \frac{\sigma^2 + K_{in}^* \eta^2}{1+\sigma^2+K_{in}^* \eta^2}(e_{11} + e_{32} + e_{21} + e_{41}) \right\}
\]

(2) The expected fee revenue \( \times(2m) \) converges to

\[
\frac{1}{4}\left[ \frac{2(1+\sigma^2)}{K_{in}^*} + \eta^2 \right] + \sum_{k=1}^{4} \frac{1}{\text{Var}_{n}[f_{in}]} [\text{Cov}_k[f_{in}, \tilde{e}_k] + r \tilde{p}_{in}]^2 + \eta^2
\]

where \( \text{Var}_{n}[f_{in}] = (1/2)\left\{ \frac{\sigma^2}{1+\sigma^2} + \frac{\sigma^2 + K_{in}^* \eta^2}{1+\sigma^2+K_{in}^* \eta^2} \right\} \)

\[
\text{Cov}_k[f_{in}, \tilde{e}_k] = (1/\sqrt{2})\left\{ \frac{\sigma^2 + K_{in}^* \eta^2}{1+\sigma^2+K_{in}^* \eta^2} e_{k1} \right\} \text{ if } k = 1, 3
\]

\[
\text{Cov}_k[f_{in}, \tilde{e}_k] = (1/\sqrt{2})\left\{ \frac{\sigma^2 + K_{in}^* \eta^2}{1+\sigma^2+K_{in}^* \eta^2} e_{k2} \right\} \text{ if } k = 2, 4
\]

(3) Investor \( k \)'s ex-ante utility \( U^{(1)}_k \) converges to

\[
U^{(1)}_k = -\sqrt{1 - \frac{2\eta^2}{10r} K_{in}^* \left\{ \frac{1}{1+\sigma^2} + \frac{1}{1+\sigma^2+K_{in}^* \eta^2} \right\}}
\times \exp\left[ \frac{1}{2\sqrt{2}} e_k^T \tilde{e}_k - \frac{1}{2} \left( (E_{in}+(1/r)) [\text{Cov}_k[f_{in}, \tilde{e}_k] + \text{Cov}[\tilde{e}_k, E_k[f_{in}]-\bar{p}_{in}]^3] \right. \right]
\]

\[
\text{where } \text{Cov}[\tilde{e}_k, E_k[f_{in}] = \frac{1}{2\sqrt{2}} \left\{ \frac{1}{1+\sigma^2} - \frac{1}{1+\sigma^2+K_{in}^* \eta^2} \right\} (e_{k1} - e_{k2}) \text{ if } k = 1, 3
\]

\[
\text{Cov}[\tilde{e}_k, E_k[f_{in}] = \frac{1}{2\sqrt{2}} \left\{ \frac{1}{1+\sigma^2} - \frac{1}{1+\sigma^2+K_{in}^* \eta^2} \right\} (-e_{k1} + e_{k2}) \text{ if } k = 2, 4
\]
Proof of Lemma 1.A.4.

Substituting $f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T$ and $\tilde{p}_{in}$ in the conditional expectations and the conditional variances in the proof of Lemma 1.A.3, and imposing the equilibrium condition, we obtain (1). It is now easy to calculate investor $k$’s REE futures position, which is $\tilde{y}_k(f_{in}) = (-1)^k[(\tilde{s}_1 - \tilde{s}_2)/(4k_{in})] + (1/4)\tilde{y}_L + (1/(Var_k[\tilde{f}_{in}]))[Cov_k[\tilde{f}_{in}, \tilde{z}_k] + r\tilde{p}_{in}]$. Finally, (2) and (3) follows from direct computation. □

Lemma 1.A.5.

If the exchange(s) cerate(s) two futures contracts $f_1 = (a,b)^T$ and $f_2 = (-b,a)^T$, then, as the transaction fee gets close to zero i.e., $m \to \infty$,

(1) The REE prices converge to

$$\tilde{p}_1 = \alpha_{pl}\{a(\tilde{s}_1 + K_{pl}\tilde{y}_L) + b(\tilde{s}_2 + K_{pl}\tilde{y}_L)\} + a\tilde{p}_{pl1} + b\tilde{p}_{pl2}$$

$$\tilde{p}_2 = \alpha_{pl}\{-b(\tilde{s}_1 + K_{pl}\tilde{y}_L) + a(\tilde{s}_2 + K_{pl}\tilde{y}_L)\} - b\tilde{p}_{pl1} + a\tilde{p}_{pl2}$$

where $K_{pl} = 1/\sigma^2/(2r)$, $\alpha_{pl} = 1/\sigma^2 + 1/\sigma^2 + K_{pl}\eta^2/(1 + \sigma^2 + K_{pl}\eta^2)$, $\tilde{p}_{pl1} = -1/2\eta(1 + \sigma^2 + K_{pl}\eta^2)/\sigma^2 + K_{pl}\eta^2 + K_{pl}\eta^2/(1 + \sigma^2 + K_{pl}\eta^2)$, $\tilde{p}_{pl2} = -1/2\eta(1 + \sigma^2 + K_{pl}\eta^2)/\sigma^2 + K_{pl}\eta^2 + \eta^2/(1 + \sigma^2 + K_{pl}\eta^2)$

(2) The expected fee revenue $\times(2m)$ converges to

$$\eta^2[1 - \frac{1}{\sigma^2 + K_{pl}\eta^2} \frac{2(1 + \sigma^2)K_{pl}\eta^2}{\sigma^2(\sigma^2 + K_{pl}\eta^2)^2}] + 2\eta^2$$

$$\frac{\sum_{k=1}^{4} Cov_k[\tilde{f}_{pl1}, \tilde{z}_{pl1}]^2}{Var_k[\tilde{f}_{pl1}]^2} + \sum_{k=1}^{4} \frac{Cov_k[\tilde{f}_{pl2}, \tilde{z}_{pl2}]^2}{Var_k[\tilde{f}_{pl2}]^2}$$

where $Var_k[\tilde{f}_{pl1}] = \sigma^2/1 + \sigma^2$ $(k = 1, 3)$, $\frac{\sigma^2 + K_{pl}\eta^2}{1 + \sigma^2 + K_{pl}\eta^2}$ $(k = 2, 4)$

$Var_k[\tilde{f}_{pl2}] = \frac{\sigma^2 + K_{pl}\eta^2}{1 + \sigma^2 + K_{pl}\eta^2}$ $(k = 1, 3)$, $\frac{\sigma^2}{1 + \sigma^2}$ $(k = 2, 4)$

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\[ \text{Cov}_k[\tilde{f}_{pl1}, \tilde{e}_{k1}] = \frac{\sigma^2_k}{1+\sigma^2} e_{k1} \quad (k = 1, 3), \quad \frac{\sigma^2 + K_{pl}^2 \eta^2}{1+\sigma^2 + K_{pl}^2 \eta^2} e_{k1} \quad (k = 2, 4) \]

\[ \text{Cov}_k[\tilde{f}_{pl2}, \tilde{e}_{k2}] = \frac{\sigma^2 + K_{pl}^2 \eta^2}{1+\sigma^2 + K_{pl}^2 \eta^2} e_{k2} \quad (k = 1, 3), \quad \frac{\sigma^2}{1+\sigma^2} e_{k2} \quad (k = 2, 4) \]

(3) Investor \( k \)'s ex-ante utility \( U_k^{(2)} \) converges to

\[
U_k^{(2)} = - \sqrt{\frac{\sigma^2 (\sigma^2 + K_{pl}^2 \eta^2)}{(1+\sigma^2)(1+\sigma^2 + K_{pl}^2 \eta^2)}} \frac{1}{4+\frac{\sigma^2}{1+\sigma^2 + K_{pl}^2 \eta^2}} \left[ \frac{1}{\sigma^2 + K_{pl}^2 \eta^2} \right]^2 \exp\left[ \frac{1}{2 \sigma^2} \left\{ \frac{1}{2 \sigma^2} \left( \sum_{i=1}^{4} e_{k1}^i - \frac{1}{2} \sum_{i=1}^{4} e_{k2}^i \right)^2 (e_{k1}^1 - (1/2) \sum_{i=1}^{4} e_{k1}^i e_{k2}^i e_{k2}^i)^2 \right\} \right]
\]

Proof of Lemma 1.A.5.

Similar to that of Lemma 1.A.4. \( \square \)

Proof of Proposition 1.5.1.

First of all, comparing \(< \text{Fig.1.5.3.} >\) and \(< \text{Fig.1.5.6.} >\), the claim seems plausible. In fact, take \((r, \sigma, \eta) = (1, 1.2, 0.7)\). Then, \(\Pi^{(1)} - \Pi^{(2)} = 0.128 > 0\), but \(U^{(1)} - U^{(2)} = -0.037 < 0\), proving the claim. \( \square \)
References


18 ——— (1990), "Financial Innovations and Market Volatility"


<Fig.1.5.1: \(\Pi^{(1)}\) plotted against \(\sigma\) and \(\eta\)>
<Fig 1.5.2: $\Pi^{(2)}$ plotted against $\sigma$ and $\eta$>
<Fig.1.5.3: \Pi^{(1)} - \Pi^{(2)} plotted against \sigma and \eta>
<Fig.1.5.4: $U^{(1)}$ plotted against $\sigma$ and $\eta$>
\textit{Fig.1.5.5}: $U^{(2)}$ plotted against $\sigma$ and $\eta$. 
<Fig.1.5.6: $U^{(1)} - U^{(2)}$ plotted against $\sigma$ and $\eta$>
Chapter 2

Endogenous Determination of the Degree of Market-Incompleteness in Futures Innovation

2.1 Introduction

Innovation of numerous types of new securities changed the financial markets dramatically in the past two decades. Futures contracts are among the most successful and important instruments in this financial innovation (Miller (1986)). What type of contracts are likely to be popular, and hence, created? What is the welfare implication of such innovation of futures contracts? A number of researchers have addressed these questions either empirically (Silber (1981), and Black (1988)) or theoretically (Duffie-Jackson (1989), Cuny (1993), Hara (1992), Ohashi (1992), and Rahi (1992)).

All of these papers analyze futures innovation as economic activities of the exchanges, and share the following two basic assumptions: First, an exchange creates futures contracts to maximize its transaction fee revenue (or its proxy), which is a certain function of the investors' futures positions\(^1\). Second, there are too few futures

\(^1\)For example, Duffie-Jackson (1989) and Rahi (1992) take, as the exchanges' objective, the maximization of the sum of the trading volume of the innovated futures, while Hara (1992) takes
for the investors to attain Pareto-optimal risk-sharing, and hence the risk-sharing opportunities available to the investors depend on the innovation of futures contracts.

These papers then ask how the futures contracts are created as the result of the exchanges' maximization behavior, and how the investors' risk-sharing opportunities are affected by such innovation of contracts. To address these questions, these papers assume that the number of contracts is given exogenously, and ask how the design of these fixed number of contracts is determined.

These papers certainly take one step forward from the classical general equilibrium theory with incomplete markets (e.g., Arrow (1964), Radner (1972), Hart (1975), and Duffie-Shafer (1985)) in which both the number and the design of securities are taken as given. However, their analysis is not completely satisfactory. Especially, by assumption, they can explain little about how the number of the contracts is determined. In this paper, we address this question and investigate the determination of the number of contracts in the context of futures innovation.

One immediate answer to this question may be cost of innovation: Since creating futures is costly, there must be an upper bound on the number of contracts to be created. This is the rationale of the approach, as a first approximation, to take the number of the contracts as given. This paper, however, shows that this argument cannot be the whole story of the determination of the number of contracts. In fact, we show that when the investors have differential information, the exchange may decide not to create an additional futures contract, even if such contract can be costlessly created. This finding is important to understand futures innovation because, in reality, the investors are likely to be differently informed about the different sources of uncertainty, and because once an exchange is established, the cost of creating an

the maximization of the positive fee charge proportional to the trading volume, and Ohashi (1992) considers the maximization of a general convex fee charge. One exception is Cuny (1993), which assumes that a futures exchange maximizes the revenue from the sales of its membership to the speculators (market-makers).
additional contract seems not too large.

We also analyze the determination of the design of the contracts, and the efficiency of the futures innovation. As in Duffie-Jackson (1989), Cuny (1993), Ohashi (1992), and Rahi (1992), we call futures innovation efficient or (constrained) optimal, if no feasible change of the innovation can Pareto-improve the investors' ex-ante utility\(^2\).

To deal with differential information in a tractable way, we consider an economy in which uncertainty is described by Gaussian random shocks, two types of investors — investors with CARA utility functions and a liquidity trader — exist, and a single centralized exchange can create possibly several futures contracts\(^3\). We assume that the exchange can create up to a given number of contracts costlessly\(^4\), that the exchange's objective is to maximize the expected sum of the square trading volume of the contracts\(^5\). Our main results are as follows:

1. When the investors have symmetric information,

   (a) The exchange creates as many futures contracts as possible, and the maximal number of feasible contracts is always attained. This is because, under symmetric information with CARA utility and Gaussian shocks, the demand for the futures depends solely on the hedging and liquidity needs, and hence as long as there is unhedged risk, it is always more beneficial for the exchange to create an additional contract.

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\(^2\)For the formal definition, see section 4.

\(^3\)This assumption is consistent with the previous papers all of which assume either mean-variance utility functions or CARA utility functions combined with Gaussian shocks. Also, the results of this chapter will not change even though several exchanges create futures, as long as they are allowed to create the same contracts.

\(^4\)This upper bound is due to the cost of creation, and can be large enough to complete the markets. The difference between most of the previous papers and this paper is that, in the previous cases, an exogenously given number of contracts are required to be created, while here the number of contracts is a endogenously determined choice variable.

\(^5\)For the justification of these assumptions, see section 2. We conjecture, however, that the results about the determination of the number of contracts will not depend much on this particular choice of the exchange's objective, as long as the objective is increasing in the trading volume of the created futures.
(b) Futures innovation to maximize the expected sum of the square trading volume is efficient for both the investors and the liquidity trader.

2. When the innovators have differential information,

(a) The centralized exchange need not create as many contracts as possible, and the maximal number of feasible contracts may not be created. This is for the following reason: With differential information, the transaction of the futures contracts depends on the demand due to the liquidity and hedging needs and the demand due to the difference of information. The innovation of a new contract, while capturing the additional liquidity and hedging transaction, transmits the additional information through its price, which may symmetrize the investors' information, and may decrease the transaction due to the difference of information. If this reduction is too large, the exchange prefers not to create the additional contract\(^6\).

(b) The welfare consequence of the futures innovation is ambiguous. This is because the investors' ex-ante utility depends on the information released through the futures prices as well, and because the better information need not imply the higher ex-ante utility\(^7\).

Thus, although the futures contracts can be costlessly created, if the investors have differential information, there can be an endogenously determined upper bound on the number of the contracts that the exchange is willing to create.

\(^6\)Similar phenomenon that innovation of an additional futures can reduce the transaction is reported by Ohashi (1992) in a simulation example. This paper attempts to analyze this issue systematically.

\(^7\)This is due to Hirschleifer effect. If too much information is publicly known before trading, the investors lose the benefit from the hedging and speculation in ex-ante sense.
The informational role played by the futures prices becomes crucial. However, there is no general closed-form formula of the noisy rational expectation equilibrium prices for the general information distribution among the risk-averse investors. Thus, for further investigation, we have to restrict ourselves to a more specified model. In a stylized economy where up to two contracts can be created, we analyze what design of contracts and how many of them the exchange creates, whether such innovation is desirable to the investors, and when underproduction and overproduction of the contracts can occur. The results are summarized as follows:

(c) Suppose that the volatility of liquidity trade is small relative to the investors' risk tolerances, that hedging demand is small, and that the investors have very accurate information. Then, it is more beneficial for the exchange to create the contracts that transmits less information, and the exchange creates a single contract rather than two contracts. Furthermore, in this situation, such innovation is desirable to the investors, in the sense that the investors, ex-ante, unanimously prefer the innovation of one contract to that of two.

(d) If the volatility of the liquidity trade is large relative to both the accuracy of the private information and the risk tolerances, then the exchange creates two futures in order to capture the large transaction caused by the liquidity trade. In this situation, such innovation is desirable to the investors.

These results have some empirical implication. That is, if the liquidity trade is small, and the difference of information among the investors is severe, we can expect that the smaller number of index-type futures tend to be created. This is because such contracts transmit less information among the investors, and hence there are likely to be more transaction due to difference of information.

The relation between the exchange's innovation decision and the investors' ex-ante welfare is rather subtle. Especially, the exchange may create the number of contracts
that the investors do not want.

(e) For a given volatility of the liquidity trade and the risk tolerances, if neither liquidity nor hedging demand is too large but both are large enough relative to the risk tolerances, the exchange always create two futures. However, if the investors have very accurate information, the investors are better off with only one contract. Thus, such innovation by the exchange is undesirable to the investors. (Over-production)

(f) If the volatility of the liquidity trade is small relative to the accuracy of private information, but is large relative to the risk tolerances, then the exchange creates only one contract, while the investors are better off with two contracts. (Under-production)

The following two points are worth noting: First, the relation between the transaction volume of a security and the information is an important topic in recent financial economics, and is investigated by many authors including Pfleiderer (1984) and Wang (1992). In most of the models, however, a traded security is exogenously given, and the effect of information on the transaction volume is analyzed within this fixed asset structure. Unlike their models, in this paper, the information transmitted by the security prices is determined endogenously through the futures innovation. Second, it is an analogy to Grossman (1977, 1981) that all the feasible securities need not be created under differential information. Grossman argues that, if the acquisition of information is costly, markets are likely to be incomplete because, otherwise, the security prices are so informative that the informed investors cannot benefit from their informational advantage and hence lose the incentive to be informed with cost. In this paper, information is costlessly endowed, but securities (futures) are created by an innovator (exchange). The exchange may create fewer contracts than it can because creating more contracts may symmetrize the investors' information, and reduce the transaction due to the difference of information.
This paper is organized as follows: In section 2.2, we formulate a model of futures innovation. In section 2.3, we investigate the determination of the number of the created contracts. In a stylized economy with differential information, we analyze the design and number of contracts that are likely to be created. In section 2.4, we analyze the ex-ante utility of the investors. We investigate, in the stylized economy, whether the contracts, which we observe the exchange create in section 2.3, are desirable to the innovators. In section 2.5, we investigate the inefficiency due to the choice of the number of contracts i.e., underproduction and overproduction of the contracts. In section 2.6, we offer some concluding remarks.

Finally, there are many other important discussions about financial innovation. In particular, see Allen-Gale (1988, 1991) for the relation between asset creation and equilibrium prices, Anderson-Harris (1986) for the timing of innovation, Madan-Soubra (1991) and Chen (1992) for the pricing and marketing of small derivative securities, Ross (1989) and Pesendorfer (1991) for the effect of marketing cost on innovation, Demange-Laroque (1992) and Rahi (1992) for the security-innovation under asymmetric information, Tufano (1989) for empirical evidence in asset creation by investment banks, and Van Horne (1985) for a general introduction.
2.2 A Model

2.2.1 An Economy and Futures Markets

We consider an economy with three dates \{0, 1, 2\}. Uncertainty, resolved at date 2, is described by a \(N\)-dimensional random vector \(\tilde{Z} = (\tilde{z}_1, ..., \tilde{z}_N)^T\). For tractability, we assume that \(\tilde{Z}\) has \(N\)-dimensional standard normal distribution i.e., \(\tilde{Z} = (\tilde{z}_1, ..., \tilde{z}_N)^T \sim N(0, I_N)\) where \(I_N\) is a \(N \times N\)-identity matrix. A single good is available in each event at date 2. We take \(L \equiv \{\tilde{x} : (\exists r \in R)(\exists x \in R^N)\tilde{x} = r + x^T\tilde{Z}\}\) as the commodity space.

There are three types of agents; a futures exchange, a liquidity trader, and investors. The exchange creates futures contracts, which the liquidity trader and the investors trade. The timing is as follows: At date 0, ex-ante, every agent has the same information about the characteristics of the economy (i.e., the preferences and the joint distribution of \(\tilde{Z}\), the initial endowments, and the private signals), and the exchange creates futures contracts. At date 1, interim, the initial endowments and the private signals are realized, and the liquidity trader and the investors trade the created futures. At date 2, ex-post, all uncertainty is resolved, the futures are liquidated, and consumption occurs.

There are \(K\) investors. Each \(k (= 1, ..., K)\) has the initial endowment of \(e_k \in R^N\) units of the risk factors i.e., \(\tilde{c}_k = e_k^T\tilde{Z} \in L\) of date 2 risky income, and receives the private signal \(\tilde{s}_k\) at date 1. We assume \(e_k\) is publicly known. His preference is given by a CARA VNM utility with the risk tolerance coefficient \(r_k\), \(U_k(\tilde{x}) = E[-exp(-(1/r_k)\tilde{x})] (\forall \tilde{x} \in L)\).

The liquidity trader has a random endowment of the risk factors \(\tilde{\omega} = (\tilde{\omega}_1, ..., \tilde{\omega}_N)^T\) i.e., \(\tilde{\omega}^T\tilde{Z}\) of date 2 risky income, but receives no private signal at date 1. Due to the implicit cost to utilize a sophisticated limit order and to learn the structure of the economy, she submits a market order and may behave in a way of bounded
rationality. More precisely, she submits a market order before observing the futures prices to minimize the unconditional variance of a particular proxy of her risky date 2 income\(^6\). We assume that the endowment shock \(\tilde{\omega}\) has \(N\)-dimensional standard normal distribution i.e., \(\tilde{\omega} = (\tilde{\omega}_1, ..., \tilde{\omega}_N)^T \sim N(0, \eta^2 I_N)\) \((\eta \in R)\). A stochastic economy is hence described by a collection \(\{(e_k, \delta_k, r_k)_{k=1,...,K}, \tilde{\omega}\}\).

A futures contract is a contingent claim that pays its holder \(f = f^T Z\) \((f \in R^n)\) units of the good in exchange of the futures price \(p \in R\) units of the good at date 2 when the state \(Z = (z_1, ..., z_N)\) occurs. \(p\) is determined at date 1 when the contract is agreed. In short, a holder of one unit of a futures receives \(\tilde{f} - p\) units of the good at date 2. When \(I\) futures contracts are created, we denote the array of futures payoffs by a \(I\)-dimensional random vector \(\tilde{F} = [\tilde{f}_1, ..., \tilde{f}_I]^T\). We denote the corresponding array of the coordinates by a \(N \times I\) matrix \(F = [f_1, ..., f_I] \in R^{N \times I}\) where \(\tilde{F} = F^T \tilde{Z}\) i.e., \((\forall i)\) \(\tilde{f}_i = f_i^T \tilde{Z}\). Since there is one-to-one correspondence between \(\tilde{F}\) and \(F\), we identify \(\tilde{F}\) with \(F\). Finally, we denote the \(I\)-dimensional random vector of futures prices by \(\tilde{P} = [\tilde{p}_1, ..., \tilde{p}_I]^T\) and its realization by \(P = [p_1, ..., p_I]^T \in R^I\).

A single centralized futures exchange is responsible for the innovation of futures contracts. The exchange is uninformed. Thus, at date 0, knowing the characteristics of the economy \(\{(e_k, \delta_k, r_k)_{k=1,...,K}, \tilde{\omega}\}\), the exchange creates futures to maximize the unconditionally expected sum of the square trading volume of the created contracts.

Two points are worth noting: First, though we consider the case where the exchange is a monopolistic supplier of the futures, our results will not change even if several exchanges create the futures, as long as they are allowed to create the same

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\(^6\)As usual, the presence of such a liquidity trader is to construct a noisy rational expectation equilibrium. Introducing the shocks through the liquidity trade rather than the endowment shock of the risk-averse investor has the advantage to guarantee that the investors' ex-ante utility exists without any further restriction on the parameters. Besides the obvious interpretation as noise trading, with this formulation, we can interpret the liquidity trader as a proxy of an extremely risk-averse uninformed hedger who faces a high cost of submitting a sophisticated limit order and learning the structure of the economy.
contracts with one another. Second, we take the maximization of the expected sum of the square trading volume as the objective of the exchange for the following two reasons: First, this objective of the exchange has a good welfare property at least in the symmetric information case. (See section 2.4.) Second, under differential information, this is the only analytically tractable criterion that can be interpreted as the approximation of the limit of the exchange's transaction revenue maximization when the transaction fee is scaled down to zero. We conjecture, however, the results about the determination of the number of contracts do not depend much on this particular choice of the innovation-criterion, as long as the exchange's objective is a monotonic function of the transaction volume.

Feasible futures innovation is subject to the physical constraint: the exchange can create only up to $\bar{T} \leq N$ futures. Futures markets are regulated by a regulator (e.g. CFTC), and the feasible innovation is also subject to the legal constraints. In particular, we assume that the payoff of each contract should be normalized to have unit variance, that the contracts are required to be mutually unconditionally uncorrelated, and that certain design of contracts may be prohibited. In short, the feasible set $F$ of contracts is described as follows: $F = \{ F \in \mathbb{R}^{N \times I} \cap A : F^T F = I_I, I \leq \bar{T} \}$ where $A$ is a closed set in $\mathbb{R}^{N \times \bar{T}}$.

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9This is presumably because of the cost of creation. Note that $\bar{T}$ can be large enough to essentially complete the markets (i.e. $\bar{T} = N$). However, we will see that this upper bound need not be attained when the investors have differential information.

10We need this normalization for the unit of a contract to be well-defined. Since there is no transaction cost in the economy, the economic value of a futures contract stems only from its ability of spanning. Thus, it makes sense for the regulator to require the contract's payoff to be non-redundant to the other. Because the regulator cannot verify the quality of private information or the realization of initial endowments, the regulator can impose the criterion only in the unconditional sense. See Ōhashi (1992).
2.2.2 Equilibrium and Futures Innovation

Now, suppose that futures contracts $F$ is created at date 0, and that a date 1 noisy rational expectation equilibrium price (hereafter, abbreviated as NREE-price) w.r.t. $F$ is given by $\tilde{P}$. Investor $k$ takes the futures position plan $\tilde{y}_k = y_k(F \mid \tilde{s}_k, \tilde{P}, \tilde{y}_{-k})$, where $y_k(\cdot)$ is a function of the created futures $F$ and the available information $(\tilde{s}_k, \tilde{P}, \tilde{y}_{-k})$ and $y_{-k} \equiv [y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_K]$, so that

$$y_k(F \mid s_k, P, y_{-k}) \in ArgMax_y E[-\exp(-(1/r_k)\tilde{W}_k) \mid s_k, P, y_{-k}]$$

s.t. $\tilde{W}_k = \tilde{\varepsilon}_k + (\tilde{F} - \tilde{P})^T \tilde{y} = \varepsilon_k^T \tilde{Z} + (F^T \tilde{Z} - \tilde{P})^T \tilde{y}$

(2.1)

where $E[\cdot \mid s_k, P, y_{-k}]$ and $Var[\cdot \mid s_k, P, y_{-k}]$ are conditional expectation and conditional variance given $(s_k, P, y_{-k})$, respectively.

Meanwhile, the liquidity trader receives the endowment shock, $\omega$, of the risk factors, and submits a market order $\tilde{y}_L = y_L(\omega)$ using her proxy $P_L \in R^l$ of the NREE-price $\tilde{P}$\(^{11}\) so that

$$y_L(F \mid \omega) \in ArgMin_y Var[\omega^T \tilde{Z} + (F^T \tilde{Z} - P_L)^T \omega]$$

Hence, the liquidity trader’s futures position is given by

$$y_L(F \mid \omega) = -Cov[\tilde{P}, \omega^T \tilde{Z}] = -F^T \omega$$

(2.2)

\(^{11}\)In this sense, the liquidity trader is of bounded rationality. This behavior is interpreted due to the cost to analyze the structure of the economy and calculate the NREE-price. Note that $P_L$ can be the objective expectation $E[\tilde{P}]$ or any subjective expectation $E_L[\tilde{P}]$ of $\tilde{P}$. Note also that, if investors have symmetric information and $P_L = E[\tilde{P}]$, then this objective of the liquidity trader is fully rational as the variance-minimization of date 2 risky income.
By definition of a NREE, we have

\[
\sum_{k=1}^{K} y_k(F \mid s_k, P, y_{-k}) = -y_L(F \mid \omega) = F^\top \omega \text{ a.s.}
\]

(2.3)

Finally, the futures exchange, knowing the economy \((e_k, \tilde{\omega}_k, r_k)_{k=1,...,K}, \tilde{\omega}\) and the relation between the futures innovation \(F \in F\) and the corresponding NREE-futures position \((\tilde{y}_k(F))_{k=1,...,K}, \tilde{y}_L(F)\)\(^{12}\), creates futures contracts \(F\) so that

\[
F \in \arg\max_{F' \in F} E[\sum_{k=1}^{K} y_k(F')y_k(F') + y_L(F')y_L(F')]
\]

(2.4)

2.3 Determination of the Number of Contracts

2.3.1 General Formula

Throughout the rest of this paper, we assume that, in an equilibrium, \(\{\tilde{Z}, (\tilde{\omega}_k)_{k=1,...,K}, \tilde{\omega}, \tilde{P}\}\) is multivariately normally distributed. This assumption is for the tractability of the model, but is consistent with a large class of noisy rational expectations models\(^{13}\). For random variables \(\tilde{z}, \tilde{z}'\) and a given NREE \((\tilde{y}_k)_{k=1,...,K}, \tilde{y}_L, \tilde{P}\) w.r.t. \(F\), let us denote by \(E_k[\tilde{z}]\) (resp. \(\text{Var}_k[\tilde{z}]\) and \(\text{Cov}_k[\tilde{z}, \tilde{z}']\)) the conditional expectation (resp. variance and covariance) of \(\tilde{z}\) (and \(\tilde{z}'\)) on \((\tilde{\omega}_k, \tilde{P}, \tilde{y}_{-k})\) i.e., \(E_k[\tilde{z}] = E[\tilde{z} \mid \tilde{\omega}_k, \tilde{P}, \tilde{y}_{-k}]\) (resp. \(\text{Var}_k[\tilde{z}] = \text{Var}[\tilde{z} \mid \tilde{\omega}_k, \tilde{P}, \tilde{y}_{-k}]\) and \(\text{Cov}_k[\tilde{z}] = \text{Cov}[\tilde{z}, \tilde{z}' \mid \tilde{\omega}_k, \tilde{P}, \tilde{y}_{-k}]\)). Under this normality assumption, given the contracts \(F\) and the information \((\tilde{\omega}_k, \tilde{P}, \tilde{y}_{-k})\), the investor \(k\)'s optimization problem (2.1) is equivalent to the following mean-variance optimization problem:

\(^{12}\tilde{y}_k(F)\) and \(\tilde{y}_L(F)\) is the abbreviation of \(\tilde{y}_k(F \mid s_k, P, y_{-k})\) and \(\tilde{y}_L(F \mid s_k, P, y_{-k})\), respectively.

\(^{13}\)For justification of this assumption, see Admati-Pfeiderer (1987).
\( y_k(F) \in \text{ArgMax}_y E_k[\hat{W}_k] - (1/(2r_k))\text{Var}_k[\hat{W}_k] \)

s.t. \( \hat{W}_k = \hat{e}_k + (\hat{F} - \hat{P})^\top \hat{y} = e_k^\top \hat{Z} + (F^\top \hat{Z} - \hat{P})^\top \hat{y} \)

From the first-order condition, its solution is

\[
y_k = r_k \text{Var}_k[\hat{F}]^{-1}(E_k[\hat{F}] - P) - \text{Var}_k[\hat{F}]^{-1}\text{Cov}_k[\hat{F}, \hat{e}_k]
\]

(2.5)

Imposing the market clearing condition (2.3), the NREE-price is given by

\[
\hat{P} = [\sum_{k=1}^K r_k \text{Var}_k[\hat{F}]^{-1}]^{-1} \times [\sum_{k=1}^K r_k \text{Var}_k[\hat{F}]^{-1} E_k[\hat{F}] - \sum_{k=1}^K \text{Var}_k[\hat{F}]^{-1}\text{Cov}_k[\hat{F}, \hat{e}_k] - F^\top \omega]
\]

(2.6)

It turns out convenient to decompose the NREE-price \( \hat{P} \) into two components i.e.,

\[
\hat{P} = \hat{P}_S + P_H
\]

s.t. \( \hat{P}_S = [\sum_{k=1}^K r_k \text{Var}_k[\hat{F}]^{-1}]^{-1} [\sum_{k=1}^K r_k \text{Var}_k[\hat{F}]^{-1} E_k[\hat{F}] - F^\top \omega] \)

\[
P_H = -[\sum_{k=1}^K r_k \text{Var}_k[\hat{F}]^{-1}]^{-1} [\sum_{k=1}^K \text{Var}_k[\hat{F}]^{-1}\text{Cov}_k[\hat{F}, \hat{e}_k]]
\]

(2.7)

where \( \hat{P}_S \) represents the random fluctuation of the NREE-price due to the private signals \((\hat{s}_k)_{k=1, \ldots, K}\) and the liquidity shock \(\hat{\omega}\), and \(P_H\) represents the deterministic bias of the NREE-price due to the hedging needs \((e_k)_{k=1, \ldots, K}\).
Using (2.7), k’s futures position (2.5) is rewritten as

\[ y_k = r_k \text{Var}_k[\tilde{F}]^{-1}(E_k[\tilde{F}] - \tilde{P}_S) - r_k \text{Var}_k[\tilde{F}]^{-1}(\text{Cov}_k[\tilde{F}, \varepsilon_k/r_k] - P_H) \]

(2.8)

Hence, in a noisy rational expectation equilibrium, investor k’s demand consists of two components: One is the speculation demand, \( r_k \text{Var}_k[\tilde{F}]^{-1}(E_k[\tilde{F}] - \tilde{P}_S) \), for which the investor bets on the difference between his rational guess of the futures’ true payoff and their prices. The other is the hedging demand, \( -r_k \text{Var}_k[\tilde{F}]^{-1}(\text{Cov}_k[\tilde{F}, \varepsilon_k/r_k] - P_H) \), for which the investor’s position depends on the transaction motivated by the hedging needs.

Now, let us define \( m_k = r_k \text{Var}_k[\tilde{F}]^{-1} \), assume that \( (\forall k)E[E_k[\tilde{F}] - \tilde{P}_S] = 0 \), which is satisfied in most of NREE models, and suppose that I futures are created. Then, we have

\[
E[\sum_{k=1}^{K} y_k^T(F)y_k(F) + y_L^T(F)y_L(F)]
\]

\[
= \sum_{k=1}^{K} E[(E_k[\tilde{F}] - \tilde{P}_S)^T m_k^2(E_k[\tilde{F}] - \tilde{P}_S)]
+ \sum_{k=1}^{K} (\text{Cov}_k[\tilde{F}, \varepsilon_k/r_k] - P_H)^T m_k^2(\text{Cov}_k[\tilde{F}, \varepsilon_k/r_k] - P_H)
+ I\eta^2
\]

(2.9)

where \( \eta^2 \) is the variance of the liquidity shock \( \omega_n \) on each \( n \)-th risk factor. The futures exchange understands the sources of the futures trading — the speculation demand corresponding to \( \sum_{k=1}^{K} E[(E_k[\tilde{F}] - \tilde{P}_S)^T m_k^2(E_k[\tilde{F}] - \tilde{P}_S)] \), the hedging demand corresponding to \( \sum_{k=1}^{K} (\text{Cov}_k[\tilde{F}, \varepsilon_k/r_k] - P_H)^T m_k^2(\text{Cov}_k[\tilde{F}, \varepsilon_k/r_k] - P_H) \), and the liquidity demand corresponding to \( I\eta^2 \) — creates futures \( F \in \mathbb{F} \) to maximize the expected sum of the square trading volume (2.9).
In the following, we analyze such innovation of futures contracts by the exchange. We take cases; one with symmetric information among the investors and the other with differential information. It turns out that, since the demand for the futures can be affected by the information that the created contracts transmit, the exchange's innovation-decision under differential information is much more complicated than that under symmetric information, and for further investigation, we need further specification of the model. As a first step to deal with this problem under differential information, we will focus on a stylized economy where up to two contracts can be created, and analyze the futures innovation in this economy.

2.3.2 Futures Innovation with Symmetric Information

When the investors have symmetric information, there is no difference of information and hence no speculation demand. Thus, the transaction of futures depends only on the hedging demand and the liquidity demand. Since the demand for each contract is separable under the specification above, especially due to the mean-variance representation, we can expect that the more contracts are created, the more transaction occurs, and hence that the exchange creates as many contracts as possible as long as there is large enough hedging or liquidity needs. The following claim shows that this conjecture is correct. Define \( \Delta_k \equiv \frac{e_k}{r_k} - \frac{\sum_{k'=1}^{K} e_{k'}}{\sum_{k'=1}^{K} r_{k'}} \). \( \Delta_k \) represents the portion of \( k \)'s hedging needs that the economy can absorb.

**Proposition 2.3.1.**

Under symmetric information, if there is liquidity demand (i.e. \( \eta^2 > 0 \)), or if there are many enough investors with well-diversed hedging needs relative to the economy (i.e., \( \text{Rank}[\Delta_1, \ldots, \Delta_I] \geq I \)), then the maximal number \( \overline{I} \) of feasible futures contracts are created by the exchange that maximizes the expected sum of the square trading volume.
Thus, when the investors have symmetric information, generically, the exchange creates as many contracts as possible, and the given upper bound on the number of the contracts is always attained. Clearly, similar argument holds for other innovation criteria, e.g. the maximization of the expected sum of the trading volume.

2.3.3 Futures Innovation with Differential Information

An Innovation Problem Specific to Differential Information

When the investors have differential private information about the realization of the risk factors, the equilibrium futures prices (and positions) transmit the information. As long as the rational expectation equilibrium is not fully revealing, two things are different from the symmetric information case: First, there is the transaction due to the difference of information. Second, the informational content of the futures prices can affect also the transaction motivated by the hedging needs. Since the information transmitted by the futures prices depends on both the number and the design of the contracts, the exchange faces more complicated innovation-decision than in the case of symmetric information.

Take, for example, a simple case where only one contract is created. Let us further assume that there is no hedging needs (i.e., (\forall k)c_k = 0). We call this case purely speculative. The exchange creates a contract to maximize

\[
E[\sum_{k=1}^{K} y_k^2(f) + y_k^2(f)] = \sum_{k=1}^{K} \frac{r_k^2 \text{Var}[E_k[\hat{f}] - \hat{P}_S]}{\text{Var}_k[\hat{f}]^2} + \eta^2
\]

\[
= \sum_{k=1}^{K} m_k^2(f) \text{Var}[E_k[\hat{f}] - \hat{P}_S] + \eta^2
\]

We can interpret \( m_k^2(f) \equiv (\frac{r_k}{\text{Var}_k[\hat{f}]})^2 \) as the square of the information-adjusted risk tolerance of investor \( k \). Now, the exchange faces the following trade-off: Creating a contract that transmits more information among the investors tends to decrease
the conditional variance of the futures payoff, \( \text{Var}[E_k[f]] \), and hence increase the investors’ information-adjusted risk tolerances, \( m_k^2(f) \). Being more tolerant toward the risk, the investors tend to trade more futures. On the other hand, the increase of the informativeness of the price decreases the difference of information among the investors, \( \text{Var}[E_k[\hat{f}] - \hat{F}_S] \), and tends to reduce the speculation trade. The exchange has to create the contract that balances these effects to maximize the expected square trading volume.

In the general case, the problem is more complicated. While the innovation of an additional contract captures the additional liquidity (and maybe hedging) transaction, the information released by the price of the new contract may suppress the trading due to the difference of information (and maybe that due to hedging needs). This trade-off can limit the number of futures that the exchange is willing to create. In this sense, under differential information, the choice of the number of the contracts matters to the exchange. In the following, we consider a stylized economy, and investigate when this phenomenon can eventually occur.

A Factor-Informed Economy

Under differential information, the informational role played by the futures prices is crucial to determine the investors’ demands for the futures. Hence, we have to know the exact relationship between the created futures and the informativeness of their prices. However, there is no closed-form formula of the NREE-prices for the general information distribution among the risk-averse investors even for this well-specified economy. This analytical difficulty stems from the fixed-point like nature of the NREE-price: The NREE-price should be the market clearing price for which the investors’ demands should incorporate fully the information obtained from the NREE-price. To determine the explicit relationship between the created futures contracts and the informational content of their prices so that we can tell the relationship between the created contracts and their NREE-demands, we will focus on a stylized
economy with the following specification. We take this approach as a first step to analyze this problem, and will refer this economy as a factor-informed economy.

A Factor-Informed Economy:  

There are two risk factors, $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2)^T \sim N(0, I_2)$. Four investors exist with an identical CARA VNM utility $U(\tilde{x}) = E[-\exp(-(1/r)\tilde{x})]$ and the initial endowments $\tilde{e}_k \equiv e_k \tilde{Z}$ where $e_k \equiv (e_{k1}, e_{k2})^T \in R^2 \ (k = 1, 2, 3, 4)$. Investors 1 and 3 (resp. 2 and 4) receive a private signal $\tilde{s}_1$ and $\tilde{s}_3$ (resp. $\tilde{s}_2$ and $\tilde{s}_4$) about risk factor 1 (resp. 2), but are uninformed about factor 2 (resp. 1), where $\tilde{s}_1 = \tilde{s}_3 = \tilde{z}_1 + \tilde{e}_1$ and $\tilde{s}_2 = \tilde{s}_4 = \tilde{z}_2 + \tilde{e}_2$ where $\tilde{e} \equiv (\tilde{e}_1, \tilde{e}_2)^T \sim N(0, \sigma^2 I_2) \ (\sigma \in R)$. The liquidity trader receives a random shock $\tilde{\omega} \equiv (\tilde{\omega}_1, \tilde{\omega}_2)^T \sim N(0, \eta^2 I_2) \ (\eta \in R)$. $(\tilde{Z}, \tilde{e}, \tilde{\omega})$ are independently distributed.

The futures exchange can create up to two futures contracts. The feasible set $F$ of the contracts is as follows: When one contract is created $F = (1, 0)^T, (0, 1)^T, (1/\sqrt{2}, 1/\sqrt{2})^T, \text{or } (-1/\sqrt{2}, 1/\sqrt{2})^T$. When two contracts are created $F = [(1, 0)^T, (0, 1)^T] \text{ or } [(1/\sqrt{2}, 1/\sqrt{2})^T, (-1/\sqrt{2}, 1/\sqrt{2})^T]$. Note that, if $F = (1, 0)^T$, the futures payoff is $F^T \tilde{Z} = \tilde{z}_1$, and that, if $F = (1/\sqrt{2}, 1/\sqrt{2})^T$, the futures payoff is $F^T \tilde{Z} = (1/\sqrt{2})\tilde{z}_1 + (1/\sqrt{2})\tilde{z}_2$. Thus, the exchange decides whether to create a plain futures $(1, 0)^T, (0, 1)^T$ or an index futures $(1/\sqrt{2}, 1/\sqrt{2})^T, (-1/\sqrt{2}, 1/\sqrt{2})^T$, and how many of them to create (one or two).  

In the following, we take cases and investigate when the exchange prefers to create an index futures rather than a plain futures, and one contract rather than two contracts.

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14 This economy is also considered by Ōhashi (1992). Subrahmanyan (1991) analyzes an economy in which the informed investors have similar information distribution but behave strategically as local insiders with risk-neutral utilities.

15 Since the only difference between innovating the contracts $F$ and $-F$ is the sign (plus or minus) of the futures position, W.L.O.G. we omit the contracts $(-1, 0)^T, (0, -1)^T, (-1/\sqrt{2}, -1/\sqrt{2})^T \text{ or } (1/\sqrt{2}, -1/\sqrt{2})^T$. This restriction on the feasible contracts is basically from the analytical tractability. Outside these contracts, it seems quite difficult to obtain the NREE-price in a tractable form.
A Purely Speculative Case

We start with a purely speculative case, where there is no hedging needs of the investors (i.e., \((\forall k)e_k = 0\)), and hence the transation of the contracts is due to either the speculation or the liquidity. Facing the trade-off, the futures exchange tries to balance the gains from the liquidity trade and the speculation trade by appropriately choosing the design and the number of the contracts. We take the following cases.

Case 1: One contract can be created \((T = 1)\)

Since \(e_k = 0\) for all \(k\), W.L.O.G, it suffices for us to compare only the contracts \(f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T\) and \(f_{pl} = (1, 0)^T\). Throughout this paper, we consider the NREE-price which is linear in the private signals and the liquidity shock. We show in Lemma 2.A.1. in the appendix that if the single index futures \(f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T\) is created, its NREE-price \(\hat{p}_{in}\) is given by \(\hat{p}_{in} = \alpha_{in}(\tilde{\sigma}_1 + \tilde{\sigma}_2 + K_{in}f_{in}^T\hat{\omega})\), and that if the plain futures \(f_{pl} = (1, 0)^T\) is created, its NREE-price \(\hat{p}_{pl}\) is given by \(\hat{p}_{pl} = \alpha_{pl}(\tilde{\sigma}_1 + K_{pl}f_{pl}^T\hat{\omega})\). Comparing the expected sum of the square trading volume for both contracts, we obtain the following claim.

Lemma 2.3.1.

In a purely speculative factor-informed economy, when only one contract can be created, the exchange strictly prefers to create an index futures rather than a plain futures, if and only if

\[
K_{in}^2 < 2(1 + \sigma^2)/ \left[ \eta^2 \left\{ 1 - \frac{1}{1 + \sigma^2 + K_{pl}^2\eta^2} \frac{4(1 + \sigma^2)K_{pl}^2\eta^2}{\sigma^2(\sigma^2 + K_{pl}^2\eta^2)} \right\} \right]
\]

\[(2.10)\]

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Equation (2.10) merely tells us the fact that the decision whether to create an index futures or a plain futures is complicated. It is therefore instructive to analyze some of the limiting cases. We consider two situations; one in which the investors’ information becomes very accurate, and the other in which the liquidity trade becomes very volatile\textsuperscript{16}

**Proposition 2.3.2.**

In a purely speculative factor-informed economy with a given risk tolerance \( r \), when only one contract can be created,

(1) If the investors have very accurate private information (i.e., \( \sigma^2 \) is close to zero), and if the volatility of the liquidity trade is small relatively to the risk tolerance (i.e., \( \eta^2 < 16r^2 \)), then the exchange creates an index futures rather than a plain futures.

(2) For any accuracy of private information, if the volatility, \( \eta^2 \), of the liquidity trade is large enough, then the exchange creates a plain futures rather than an index futures.

The intuition of these results is as follows: First, suppose that a plain futures \( f_{pt} = (1, 0)^T \) is created. Since the payoff of this contract, \( \bar{z}_1 \), is determined only by risk factor 1, investors 1 and 3 have strictly more accurate information about the payoff than investors 2 or 4. Thus, as the information becomes more accurate, the risk that investors 1 and 3 face to trade the futures (with investors 2 and 4) decreases. In the limit, investors 1 and 3 has no risk in trading this plain contract (since they know the realization of its payoff), and hence behave risk-neutrally. Since the information that a competitive risk-neutral trader possesses is fully revealed in the

\textsuperscript{16}We are interested in these cases for the following reason: First, if the information becomes pretty inaccurate i.e., \( \sigma^2 \not\to \infty \), the situation resembles the symmetric information case, which we analyzed in the previous section. Second, if the volatility of the liquidity trade gets very small i.e., \( \eta^2 \not\to 0 \), then since there is no hedging needs, there will be no trade, which is uninteresting.
equilibrium price, with extremely accurate private information, there is almost no difference of information between the informed investors (1 and 3) and the uninformed investors (2 and 4), and therefore there is almost no transaction due to the difference of information.

On the other hand, if an index futures $f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T$ is created, since the payoff of this contract $(1/\sqrt{2})z_1 + (1/\sqrt{2})z_2$ depends on both factors 1 and 2, no investor has superior information to any other investors about the contract’s payoff. Thus, as long as there is some liquidity trade (i.e., $\eta^2 > 0$), however accurate information the investors have, investors 1 and 3 (resp. 2 and 4) always face the risk from risk factor 2 (resp. 1) in trading this index futures. Consequently, even in the limit, the investors remain risk-averse, and the equilibrium price is not fully revealing. Hence, with the index futures, there always remains some transaction due to the difference of information, which may give the index futures higher expected square trading volume than the plain futures when the information is extremely accurate.

The condition $\eta^2 < 16r^2$ in the first statement and the second statement come from the strict convexity of the exchange’s (quadratic) objective function. Because of the strict convexity, given the same volume of trade, the exchange prefers the contract design for which the trading clusters to one type of the investors rather than is devided equally to both types of the investors. For an index futures, each type of investors trade evenly with the liquidity trader, while, for a plain futures, the transaction between the investors and the liquidity trader clusters to the better informed investors. If the liquidity trade, $\eta^2$, is large enough, the benefit from this clustering may outweigh the cost from the smaller transaction due to the informational difference, which may make a plain futures more attractive to create.
**Case 2: Two contracts can be created** \((T = 2)\)

To start with, the following observations are useful: First, in the factor-informed economy, if two futures contracts are created, the information transmitted by their prices is the same between the innovation of \([(1, 0)^T, (0, 1)^T]\) and \([(1/\sqrt{2}, 1/\sqrt{2})^T, (-1/\sqrt{2}, 1/\sqrt{2})^T]\) because there are only two risk factors. Second, given the first observation, the expected sum of the square trading volume under the futures \([(1, 0)^T, (0, 1)^T]\) is equal to that under the futures \([(1/\sqrt{2}, 1/\sqrt{2})^T, (-1/\sqrt{2}, 1/\sqrt{2})^T]\), as long as the equilibrium allocation is the same\(^{17}\).

Moreover, the transaction of a plain futures \(f_{pl} = (1, 0)^T\) will not be affected by the innovation of an additional plain contract \((0, 1)^T\), and vice versa. (See equations in Lemma 2.A.1.) This is because, given the specification of the economy, the futures contract \((0, 1)^T\) with the payoff \(\tilde{z}_2\) neither adds any information nor provides any hedging to the contract \((1, 0)^T\) with the payoff \(\tilde{z}_1\). This independence of the demands between the two plain futures leads us to the following claim.

**Lemma 2.3.2.**

Suppose that a plain contract \((1, 0)^T\) (resp. \((0, 1)^T\)) has been created. Then, creating the additional plain futures \((0, 1)^T\) (resp. \((1, 0)^T\)) increases the expected sum of the square trading volume. Thus, when two futures can be created, the exchange never creates one plain contract instead of two contracts.

With these observations, the relevant question to the exchange turns out to be the decision whether to create a **single index** futures or **two** (index or plain) futures.

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\(^{17}\)This is because \([(1/\sqrt{2}, 1/\sqrt{2})^T, (-1/\sqrt{2}, 1/\sqrt{2})^T]\) is just the rotation of the orthogonal base \([(1, 0)^T, (0, 1)^T]\) (and vice versa), which preserves the distance of a point from the origin measured by the sum of the square of the coordinates.
As is in the previous case, the precise choice-criterion is complex, which we cite as Lemma 2.A.2. in the appendix. We again consider some limiting situations.

**Proposition 2.3.3.**

In a purely speculative factor-informed economy with a given risk tolerance $r$, when two contracts can be created,

1. If the investors have very accurate private information (i.e., $\sigma^2$ is close to zero), and if the volatility of the liquidity trade is small relatively to the risk tolerance (i.e., $\eta^2 < (16/7)r^2$), then the exchange creates a single index futures rather than two futures.

2. For any accuracy of private information, if the volatility, $\eta^2$, of the liquidity trade is large enough, then the exchange creates two contracts rather than one (either index or plain) contract.

Similarly to Proposition 2.3.2, the exchange create futures by taking account of the balance between the informativeness of the prices and the transaction due to the difference of information. However, since the exchange can create two contracts, it faces the following trade-off: Creating the additional contract captures the additional liquidity trade, which increases the square trading volume (Positive effect). On the other hand, new information transmitted by the price of the additional contract may reduce the difference of information among the investors and the transaction due to the informational difference (Negative effect).

Proposition 2.3.3. shows that when the investors have very accurate information, the negative effect exceeds the positive effect, and the exchange creates a single index futures rather than two futures. That is, if the different investors are informed about the different components of the uncertainty, and if their informational difference is large in the sense that their private information is quite accurate, then we tend to observe less complete futures markets even if there is no cost to create the contracts.
This result is, of course, subject to the condition that the liquidity trade is not too large. This is because, if the liquidity demand is so large that the gain from capturing the additional liquidity trade exceeds the loss from the reduction of the trade due to the difference of information, the exchange is always better off by creating two futures. Note that since the gain from the additional liquidity trade makes the innovation of two contracts more attractive to the exchange, the condition on the volatility of the liquidity trade ($\eta^2 < (16/7)r^2$) for which the exchange prefers to create a single index futures becomes much more strict than that in the previous case ($\eta^2 < 16r^2$) where only one contract can be created.

The empirical implication of these propositions, hence, will be as follows: When the difference of information among the investors is severe, an index futures can be more popular than several plain futures. Thus, in this situation, we can expect that the smaller number of index futures are likely to be created rather than the larger number of plain contracts.

Subrahmanyam (1991) shows a similar result about the popularity of an index futures, however, in a different setting where the risk-neutral informed traders (insiders), each of whom is locally informed about one risk factor, trade strategically, and the trading volume — popularity of a contract — is solely determined by the trading decision of the discretionary uninformed liquidity traders, not by that of the informed investors. In this paper, there is no discretionary liquidity trader, and the investors behave competitively. The popularity of a contract, measured by its expected square trading volume, is determined by the informed traders' trading decision.
A General Case

In this case, the investors have some hedging needs (i.e., \( \exists k \epsilon_k \neq 0 \)). Though the effect of the informativeness of the futures on the hedging demand is ambiguous, it seems likely that the hedging needs tend to increase the trading of the futures. This conjecture is true for some limiting cases. (We consider the case where two contracts can be created.)

**Proposition 2.3.4.**

In a general factor-informed economy with risk tolerance \( r \) and initial endowments \( e_k = (e_{k1}, e_{k2})^T (k = 1, ..., 4) \), when two contracts can be created,

1. If the investors have very accurate information, then the exchange creates a single index futures rather than two futures, if and only if,
   \[
   \frac{7}{4} r^2 + \sum_{j=1}^{2} \sum_{k=1}^{4} (e_{kj} - \frac{1}{4} \sum_{k'=1}^{4} e_{k'j})^2
   -2 \max \left[ \sum_{k=1}^{4} \left\{ \frac{e_{k1} + e_{k2}}{2} - \frac{1}{4} \sum_{k'=1}^{4} e_{k'1} \right\}^2, \sum_{k=1}^{4} \left\{ \frac{e_{k1} + e_{k2}}{2} - \frac{1}{4} \sum_{k'=1}^{4} e_{k'2} \right\}^2 \right]< 4 r^2
   \]

2. If the volatility of the liquidity trade is large enough, then the exchange creates two contracts rather than one contract.

3. If the investors initial endowments are diversified (i.e., \( \dim[(e_k - \frac{1}{4} \sum_{k'=1}^{4} e_{k'})_{k=1,2,3,4}] = 2 \)), and if the volatility of the liquidity demand is very small i.e., close to zero, the exchange create two contracts.

When the investors have very accurate information, the transaction due to the hedging needs increases as the number of the contracts increases. This makes the innovation of two futures more attractive to the exchange so that the condition under

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\( ^{18} \)The result for the general parameter values can be calculated easily. However, the expression is just cumbersome, and as Lemma 2.A.1. and 2.A.2, there is little we can learn from it. Hence, we omit citing it.
which it creates a single index futures becomes much more strict. The first claim shows this point. The second claim is an obvious consequence of Proposition 2.3.3. Finally, as the liquidity trade becomes smaller, the NREE-price becomes more revealing. In the limit, it becomes fully revealing, and the transaction depends solely on the hedging trade, which yields the third claim.

2.4 Ex-Ante Efficiency of Innovation

2.4.1 Ex-Ante Utility and Definition of Efficiency

We have seen that, depending on the economic environments, the futures exchange may create several different design and number of contracts. It is now interesting to ask whether and in what sense futures innovation by the exchange is socially desirable. For this purpose, we investigate the investors' ex-ante welfare. In this subsection, we derive a formula of investor $k$'s ex-ante utility for a general economy formulated in section 2, and define (constrained) efficiency or optimality of the innovation.

First, for a NREE, investor $k$'s ex-ante utility, $U_k(\tilde{W}_k) \equiv E[-\exp(-(1/r_k)\tilde{W}_k)]$, is given in terms of either the NREE-prices, $\bar{P}$, or the NREE-futures positions, $\tilde{y}_k$, as follows:

**Proposition 2.4.1.**

$$U_k(\tilde{W}_k) = U_k(\tilde{e}_k) \sqrt{\frac{\text{det}[\text{Var}_k[\tilde{F}]]}{\text{det}[\text{Var}[\bar{F}-\bar{P}]]}}$$

$$\times \exp\left[\frac{-1}{2}\{E[\bar{F}-\bar{P}]-(1/r_k)\text{Cov}_k[\bar{F}, \tilde{e}_k]-(1/r_k)\text{Cov}[\tilde{e}_k, E_k[\bar{F}-\bar{P}]]\}^T \right.$$

$$\left.\times \{\text{Var}_k[\bar{F}] + \text{Var}[E_k[\bar{F}-\bar{P}]} \}^{-1} \right.$$  

$$\times \{E[\bar{F}-\bar{P}]-(1/r_k)\text{Cov}_k[\bar{F}, \tilde{e}_k]-(1/r_k)\text{Cov}[\tilde{e}_k, E_k[\bar{F}-\bar{P}]]\}$$

$^{19}$\text{det}(M) implies the determinant of a matrix $M$. Similar expression, but with some special conditions, appears in Admati-Pfeiderer (1987) and Demange-Laroque (1992).
\[
= U_k(\tilde{\varepsilon}_k) \sqrt{\frac{\text{det}[\text{Var}_k[\tilde{F}]]^{-1}}{\text{det}[\text{Var}_k[\tilde{F}]^{-1} + (1/r_k^2)\text{Var}[\tilde{y}_k(F)]]}} \times \exp\left[\frac{-1}{2}\{(1/r_k)E[\tilde{y}_k(F)] - (1/r_k^2)\text{Cov}_k[\tilde{y}_k(F), \tilde{\varepsilon}_k]\}\right] \\
\times \{(1/r_k)E[\tilde{y}_k(F)] - (1/r_k^2)\text{Cov}_k[\tilde{y}_k(F), \tilde{\varepsilon}_k]\}\}
\]

(2.11)

Observe that investor $k$'s ex-ante utility is determined by the factors such as the interim riskiness, $\text{Var}_k[\tilde{F}]$, of the futures, the informativeness, $\text{Var}[\tilde{F} - \tilde{P}]$, of the NREE-prices, the interim hedging quality, $\text{Cov}_k[\tilde{F}, \tilde{\varepsilon}_k]$, of the futures payoffs, the ex-ante hedging quality, $\text{Cov}_k[E_k[\tilde{F}] - \tilde{P}, \tilde{\varepsilon}_k]$, of the futures payoffs net their prices, and so on. In the following subsections, we analyze this complicated object step by step.

Our next task is thus to define a criterion of efficiency (or optimality) of the innovation. We will employ the notion of constrained efficiency (or optimality) similar to that in Duffie-Jackson (1989), Cuny (1993), and Ohashi (1992). Since the liquidity trader may behave in a boundedly rational manner, we define the optimality with and without the liquidity trader.

**Definition 2.4.1.**

1. The created futures $F \in F$ is weakly futures innovation optimal (weakly FI-optimal), if there is no other feasible innovation $F' \in F$ that Pareto-improves the investors' ex-ante utility.

2. The created futures $F \in F$ is futures innovation optimal (FI-optimal), if there is no other feasible innovation $F' \in F$ that Pareto-improves both the investors' and the liquidity trader's ex-ante utility.
As is discussed in section 2.2.2, under symmetric information, if the liquidity trader uses the objective expectation of the price so that \( P_L = E[\bar{P}] \), her behavior is fully rational as the minimization of the variance of the after-trade date 2 risky income. Hence, with symmetric information, FI-optimality makes sense. Under differential information, the liquidity trader is boundedly rational in the sense that she does not fully utilize the available information. Although the liquidity trader’s (boundedly rational) utility increases as the number of contracts increases, as long as we require full rationality of the agents for welfare analysis, only weak FI-optimality makes sense under differential information.

2.4.2 Efficiency with Symmetric Information

To understand the relationship between the ex-ante utility and the futures innovation, it is more convenient to express the utility level in terms of the futures positions (i.e., the second expression in (2.11)). We have the following claim.

**Proposition 2.4.2.**

When the investors have symmetric information so that no one receives private information, if \( I \) futures are created, investor \( k \)'s ex-ante utility is given by

\[
U_k(\bar{W}_k) = U_k(\bar{\epsilon}_k) \sqrt{ \frac{1}{\left(1 + \frac{\eta}{\sum_{k'=1}^{K} r_{k'}}\right)^2 I} } \\
\times \exp \left[ \frac{-1}{2r_k^2 \left(1 + \frac{\eta}{\sum_{k'=1}^{K} r_{k'}}\right)^2} \left( E[\bar{y}_k(F)^T \bar{y}_k(F)] - \left( \frac{r_k}{\sum_{k'=1}^{K} r_{k'}} \right)^2 I \eta^2 \right) \right]
\]

(2.12)
Furthermore, as long as the liquidity trader uses the objective proxy $P_L = E[\hat{P}]$ of the NREE-price, the exchange that creates the contracts to maximize the expected sum of the square trading volume attains FI-optimal futures innovation\(^{20}\).

This result, in some sense, supports a naive conjecture: The more the futures are tailed, the higher must be the investors' welfare. This argument, however, may hinge crucially upon the symmetry of information.

2.4.3 Efficiency with Differential Information

Under differential information, the futures contracts play two roles; to provide the means of risk sharing and, additionally, to transmit the information. The investor's ex-ante utility hence depends on both the hedging quality and the information that the contracts transmit. To understand the informational effect, we first focus on a purely speculative case.

A Purely Speculative Case

Assuming $E[\tilde{y}_k] = 0$, investor $k$'s ex-ante utility is given by

Lemma 2.4.1.

$$U_k(\hat{W}_k) = -\frac{\det[\text{Var}_k[\hat{F}]]}{\det[\text{Var}[\hat{F} - P]]}$$

$$= -\frac{\det[\text{Var}_k[\hat{F}]^{-1}]}{\sqrt{\det[\text{Var}_k[\hat{F}]^{-1} + (1/\tau_k^2)\text{Var}[\tilde{y}_k(F)]]}}$$

(2.13)

\(^{20}\)The desirability to maximize the sum of the square trading volume over other types of convex objective functions under symmetric information is reported by Hashi (1992) for the case without the liquidity trader.
Thus, there is trade-off: More informative futures tends to increase the investor's ex-ante utility by reducing the payoff risk, \( \text{Var}_k[\hat{F}] \), of the contracts and facilitating the speculation. However, more informative prices also may reduce the ex-ante utility because the profitability of the speculation, measured by \( \text{Var}[\hat{F} - \hat{P}] \), decreases. Thus, the total informational effect of the futures on the ex-ante utility is hard to tell without knowing the explicit relationship between the created futures and their informativeness. For the special case where only one contract is created, however, we have more transparent characterization.

**Proposition 2.4.3.**

In a purely speculative economy, suppose that only one contract can be created and that \( E[\tilde{y}_k] = 0 \). Then, the innovation of a futures \( f \) is weakly FI-optimal, if and only if there is no other feasible contract \( f' \) for which the improvement of its expected trading volume w.r.t. that of \( f \) exceeds the improvement of its interim precision w.r.t. that of \( f \) for all investors, that is,

\[
(\exists f' \in F) \ (\forall k) \ \frac{E[\tilde{y}_k^2(f')]}{E[\tilde{y}_k^2(f)]} > \frac{1}{\text{Var}_k[\hat{f}]} \frac{\text{Var}_k[f]}{\text{Var}_k[\hat{f}]}
\]

(2.14)

Note that when the investors have symmetric information, the interim precision of a contract payoff is equal to 1 for all investors, and hence R.H.S. of the inequality in (2.14) is always 1. Therefore, the expected square trading volume maximization always attains weak FI-optimality under symmetric information. (Recall that it is also FI-optimal.) Clearly, this claim need not hold under differential information, because the interim precision of a futures payoffs, \( \frac{1}{\text{Var}_k[f]} \), can be different for different contracts. Hence, under differential information, the innovation to maximize the expected square trading volume need not be weakly FI-optimal.
This result, although general, has two shortcomings: First, it merely states the general difficulty of attaining efficient innovation, and in order to apply this result, we have to know the explicit relationship between the created futures and their informativeness. Second, since it assumes the innovation of only one contract, the choice of the number of the futures, which is a problem intrinsic to differential information, is out of its sight.

For this reason, we again consider the factor-informed economy. Instead of looking for a innovation criterion that attains efficiency, however, we are interested in whether the futures contracts that the exchange is willing to create are desirable to the investors. In this subsection, we focus on the futures contracts considered in section 2.3. In the next section, we analyze the inefficiency related to the choice of the number of the contracts.

Lemma 2.4.2.\textsuperscript{21}

In a purely speculative factor-informed economy, given risk tolerance \( r \),

(1) When only one contract can be created,

(a) For given volatility, \( \eta^2 > 0 \), of the liquidity trade, if the investors' private information is accurate (i.e., \( \sigma^2 \) is small) enough, then, ex-ante, the investors unanimously prefer the innovation of an index futures to that of a plain futures.

(b) If the volatility, \( \eta^2 \), of the liquidity trade is large enough and the private information is accurate enough (i.e., \( \sigma^2 < (\sqrt{5} - 1)/4 \)), then, ex-ante, the investors unanimously prefer the innovation of an index futures to that of a plain futures.

\textsuperscript{21}Lemma 2.A.3. in the appendix shows the exact, but completely untransparent, condition for the general case.
(2) When two contracts can be created,

(a) For given volatility, $\eta^2 > 0$, of the liquidity trade, if the investors have very accurate private information (i.e., $\sigma^2$ is close to 0), then, ex-ante, the investors unanimously prefer the innovation of an index futures to that of two futures.

(b) For given accuracy, $\sigma^2 > 0$, of the private information, if the volatility, $\eta^2$, of the liquidity trade is large enough, then, ex-ante, the investors prefer the innovation of two futures to that of a single index contract.

The intuition of these results is as follows: As the private information becomes more accurate, the NREE-price indicates the true payoff more accurately, and the price of a plain contract or the prices when two contracts are created are much more accurate than that of an index contract when it is created alone. This is because the investors trade more aggressively according to his private information in the former case than in the latter case\(^{22}\). However, as the price gets closer to the true payoff, the profitability from the speculation (the payoff net the price) decreases. Consequently, when the private information is very accurate, the benefit from the speculation disappears when a plain contract or two contracts are created, while it does not if a single index contract is created. This yields the investors' higher ex-ante utility with a single index futures. On the other hand, when the volatility of the liquidity trade is large, for each trader, the benefit from the speculation on the factor that he is informed of is also large. The presence of two contracts enables the investors to take advantage of this opportunity, and hence is more desirable than a single contract innovation.

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\(^{22}\)For an index contract, each investor has inferior information to some other investors about the half of the payoff. If only one index contract can be traded, no investor can hedge against the risk due to this inferior information (or adverse selection), and hence trades the contract more cautiously.
By this lemma and Proposition 2.3.2, we can now tell whether the futures innovation by the exchange is desirable to the innovators in the limiting cases analyzed in section 2.3.

**Proposition 2.4.4.**

In a purely speculative factor-informed economy with a given risk tolerance \( r \), suppose that the exchange create futures to maximize the expected sum of the square trading volume. Then,

(1) When only one contract can be created,

(a) For the volatility, \( \eta^2 > 0 \), of the liquidity trade small enough w.r.t. the risk tolerance (i.e., \( \eta^2 < 16r^2 \)), if the investors' private information is accurate (i.e., \( \sigma^2 \) is small) enough, then the futures innovation by the exchange is weakly FI-optimal.

(b) If the private information is accurate (i.e., \( \sigma^2 < (\sqrt{5} - 1)/4 \)), then the futures innovation by the exchange is not weakly FI-optimal for large volatility, \( \eta^2 \), of the liquidity trade.

(2) When two contracts can be created,

(a) If the volatility is small enough w.r.t. the risk tolerance (i.e., \( \eta^2 < (16/7)r^2 \)) and the investors have very accurate private information (i.e., \( \sigma^2 \) is close to 0), then the futures innovation by the exchange is weakly FI-optimal.

(b) For given accuracy, \( \sigma^2 > 0 \), of the private information, if the volatility, \( \eta^2 \), of the liquidity trade is large enough, then the futures innovation by the exchange is weakly FI-optimal.
A General Case

We now analyze the efficiency of the innovation in a general factor-informed economy. As is in the previous section, we only consider the case where two contract can be created.

Proposition 2.4.5.

In a general factor-informed economy with a given risk tolerance $\tau$,

(1) For given volatility, $\eta^2 > 0$, of the liquidity trade, if the investors have very accurate information (i.e., $\sigma^2$ is close to zero), then, ex-ante, the investors prefer the innovation of a single index futures to that of two futures. In this situation, the futures innovation to maximize the sum of the expected trading volume is weakly FI-optimal if and only if the inequality in (1) of Proposition 2.3.4. holds.

(2) For given accuracy, $\sigma^2 > 0$, of the private information, if the volatility, $\eta^2$, of the liquidity trade is large enough, then, ex-ante, the investors prefer the innovation of two futures to that of one futures, and the futures innovation to maximize the expected sum of the trading volume is weakly FI-optimal.

(3) For given accuracy, $\sigma^2 > 0$, of the private information, if the investors have diversified initial endowments, and the volatility, $\eta^2$, of the liquidity trade is small enough, then, ex-ante, the investors prefer the innovation of two futures to that of one futures, and the futures innovation to maximize the expected sum of the trading volume is weakly FI-optimal.

The intuition of these results is quite similar to that of Proposition 2.4.4: First, the release of too much information before trading hurts the investors' ex-ante welfare,
not only because the benefit from the speculation decreases, but also because the ex-ante hedging quality of the futures decreases (i.e., Hirschleifer effect). For this reason, when their private information is very accurate, the investors are better off with the less informative futures contracts, and hence they prefer the innovation of a single index futures to that of a single plain or two futures. Second, as the liquidity trading becomes large, the benefit from the speculation becomes large. The presence of two contracts gives each investor more opportunities to take advantage of his own superior information than one contract. (Recall that each investor is informed about only one risk factors.) Hence, for the large liquidity trade, the investors prefer to have two contracts. Finally, as the liquidity trade becomes small, the futures prices becomes more revealing, and the investors' information is more symmetrized. In the limit, the informational content that the futures transmit does not depend on their innovation. Since, for the fixed quality of information, the benefit from the speculation is the same and that from the hedging is higher with more contracts, the investors prefer to have two contracts.

2.5 Underproduction and Overproduction

Recall that, under symmetric information, the exchange creates as many contracts as possible, and this innovation attains FI-optimality. Hence, how the exchange's endogenous choice of the number of contracts affects the investors' welfare matters only when the investors have differential information. It is therefore interesting to ask when the undesirable innovation to the investors can occur due to the exchange's choice of the wrong number of contracts. Some sufficient conditions for such underproduction and overproduction (in terms of weak FI-optimality) in a purely speculative factor-informed economy are as follows\(^{23}\).

\(^{23}\)Lemma 2.4.4. in the appendix shows the general, but completely untransparent, necessary and sufficient conditions.
Proposition 2.5.1.

In a purely speculative factor-informed economy, in terms of weak FI-optimality,

(1) Underproduction of the futures contracts occurs for any given accuracy, $\sigma^2 > 0$, of the private information, if the volatility, $\eta^2 > 0$, of the liquidity trade is small enough w.r.t. $\sigma^2$, and the risk tolerance $r > 0$ is small enough w.r.t. $\eta^2$.

(2) Overproduction of the futures contracts occurs, if the volatility, $\eta^2 > 0$, of the liquidity trade is large enough w.r.t. the risk tolerance $r > 0$ (i.e., $\eta^2 > (16/7)r_2$), and the investors have very accurate private information (i.e., $\sigma^2$ is close to zero).

The intuition of these results is as follows: First, for underproduction, suppose that the exchange has already created a contract, and has to decide whether to create an additional contract. If the volatility of the liquidity trade is small enough, the loss from the decrease of the speculation trade exceeds the gain from capturing the additional liquidity trade. Consequently, the exchange creates only one contract. However, the investors need not be satisfied with this innovation decision. In particular, when the investors' risk tolerance is small, since the liquidity trader submits inelastic demand, the futures prices deviate more from the true payoffs, and the investors gain from the speculation is relatively large. If this gain remains large enough even after the release of the new information from the innovation of the second contract, the investors prefer to have two contracts so that they can enjoy this gain from the speculation with higher degree of freedom. This is how underproduction occurs.

On the other hand, if the volatility of the liquidity trade is large enough, the gain from capturing the additional liquidity trade exceeds the loss from the reduction of the speculation trade. Thus, the exchange creates two futures. However, if the private
information is very accurate so that the futures prices comes to reflect the true payoff of the contracts much more accurately when two contracts are created, the loss of the profitability of speculation is large. Consequently, the investors prefer the innovation of a single index contract. This is how overproduction occurs.

The overproduction result extends to the general factor-informed economy.

**Proposition 2.5.2.**

In a general factor-informed economy, if the volatility of the liquidity trade and the hedging needs are so large that the converse of the inequality in Proposition 2.3.4. (1) holds, then for the private information accurate enough, overproduction of the futures occurs in terms of weak FI-optimality.

The intuition of this results is almost the same as the above: The innovation of an additional contract is beneficial to the exchange by capturing the large liquidity and hedging demand, while the increase of the informativeness of the price due to the innovation of new contract reduces the ex-ante gains from speculation and the hedging quality of the contracts. This discrepancy may lead to the inefficient innovation. Whether or not the presence of hedging needs increases the possibility of this types of overproduction is ambiguous. This is because, other thing being equal, if there are some hedging needs, two contracts are more likely created, but then the investors benefit to have more contracts also increases.

Obtaining a simple expression for the underproduction in a general factor-informed economy is hard. However, we conjecture that this type of underproduction becomes less likely in the presence of the hedging needs. This is because, with some hedging demand, the innovation of two futures is more likely, and the investors' benefit to have two contracts also increases. It should be reminded, though, that there may be other types of underproduction besides this particular type.
2.6 Concluding Remarks

(1) This paper analyzes the determination of the number of futures when an exchange creates the contracts to maximize the expected sum of the square trading volume of the created contracts. Though we choose this specific innovation criterion of the exchange for both economic and analytical reasons, we conjecture that the economically equivalent phenomena occur for any given criterion that depends on the trading volume of the contracts (e.g. trading volume maximization, or proportional fee maximization). While almost all papers on security-innovation by an organized exchange assume some specific innovation criteria (e.g. Duffie-Jackson (1989), Cuny (1993), Hara (1992), and Rahi (1992)), Ohashi (1993) provides a characterization of security-innovation under a condition necessary for any reasonable innovation criterion, which therefore is free from specificity of any other criteria.

(2) In order to analyze the innovation decision and its welfare consequence under differential information, we need to know the explicit relationship between the created contracts and their informativeness. For this reason, we have restricted our analysis under differential information to a stylized economy, which is considered in Ohashi (1992) and similar to the one in Subrahmanyan (1991). We conjecture that other types of specification can generate similar results, as long as the innovation of a new futures contract provides new information about the incumbent contracts.

(3) The problem of futures innovation under differential information is considered also in Ohashi (1992) and Rahi (1992). Assuming the risk-neutrality of the informed investors and trivializing the derivation of rational equilibrium expectations prices, Rahi (1992) investigates the effect of the information transmitted by futures prices on production. Though the exchange is assumed to create the contracts which maximize the expected transaction volume, in Rahi (1992), the relationship between the informativeness and the liquidity of futures markets and its implications for futures innovation decisions is not described.
2.7 Appendix 2

Proof of Proposition 2.3.1.

This claim follows from the mean-variance representation of the optimization problem and the fact that, under the given assumptions, especially the orthogonality of the contracts' payoffs, with symmetric information, the demand for each contract is independent of those of the others. In fact, if $I$ number of futures $F$ are created, we have

$$y_k(F) = r_k[-F^T e_k + \sum_{k'=1}^{K} f_{k'}^T e_k'] = [f_1^T r_k(-e_k + \sum_{k'=1}^{K} r_{k'}), \ldots, f_I^T r_k(-e_k + \sum_{k'=1}^{K} r_{k'})]^T$$

$$\equiv [f_1^T r_k \Delta_1, \ldots, f_I^T r_k \Delta_I]^T$$

Hence, the expected sum of the square trading volume is

$$E[\sum_{k=1}^{K} y_k^T(F)y_k(F) + y_L^T(F)y_L(F)] = \sum_{k=1}^{K} \sum_{i=1}^{I} (f_i^T r_k \Delta_i)^2 + I \eta^2$$

Clearly, its maximum value increases as $I$ increases. □

Lemma 2.A.1.

(1) If the single index futures $f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T$ is created, its NREE price is

given by

$$\bar{p}_{in} = \alpha_{in}(\tilde{s}_1 + \tilde{s}_2 + K_{in} f_{in}^T \tilde{\omega})$$

where $\alpha_{in} = \frac{1}{2\sqrt{2}} \left( \frac{1}{1+\sigma^2} + \frac{1}{1+\sigma^4 + K_{in}^2 \eta^2} \right)$, and $K_{in}$ is the unique real solution for $24 \ 4r \eta^2 K^3 + \sqrt{2}(1 + \sigma^2) \eta^2 K^2 + 2\sqrt{2} \sigma^2 (1 + \sigma^2) = 0$

Investor $k$'s NREE-futures position is given by

$$y_k(f_{in}) = (-1)^k [ (\tilde{s}_1 - \tilde{s}_2)/(4K_{in}) ] + (1/4) f_{in}^T \tilde{\omega}$$

The expected square trading volume of $f_{in}$ is given by

$^{24}$In fact, $K_{in} = (-\sqrt{2}/(12 \eta^2/3 \sigma))(\{(216 \sigma^2 r^2 + 216 \sigma^4 r^2 + \eta^2 + 6 \eta^2 \sigma^2 + 12 \eta^2 \sigma^4 + 8 \eta^2 \sigma^6 - 12 \sqrt{3} \sigma(1 + \sigma^2)^1/2 \eta^2 r^2 + 108 \sigma^4 r^2 + 12 \eta^2 \sigma^4 + 108 \sigma^6 r^2 + 6 \eta^2 \sigma^2 + \eta^2)^1/2 \}^{1/3} + \{(216 \sigma^2 r^2 + 216 \sigma^4 r^2 + \eta^2 + 6 \eta^2 \sigma^2 + 12 \eta^2 \sigma^4 + 8 \eta^2 \sigma^6 + 12 \sqrt{3} \sigma(1 + \sigma^2)^1/2 \eta^2 r^2 + 108 \sigma^4 r^2 + 12 \eta^2 \sigma^4 + 108 \sigma^6 r^2 + 6 \eta^2 \sigma^2 + \eta^2)^1/2 \}^{1/3} + \eta^2/3 + 2 \eta^2/3 \sigma^2 \}$. Note also that $K_{in} < 0$. 

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\[ E[\sum_{k=1}^{K} y_k^2(f_{in}) + y_k^2(f_{in})] = (1 + \sigma^2)/(2K_{in}^2) + \eta^2/4 + \eta^2 \]

(2) If the plain futures \( f_{pl} = (1, 0)^T \) is created, its NREE-price \( \tilde{p}_{pl} \) is given by

\[ \tilde{p}_{pl} = \alpha_{pl}(\tilde{s}_1 + K_{pl} f_{pl}^T \tilde{\omega}) \]

where \( \alpha_{pl} = \left( \frac{1}{\sigma^2} + \frac{1}{\sigma^2 + K_{pl}^2 \eta^2} \right) \left( \frac{1 + \sigma^2}{\sigma^2} + \frac{1 + \sigma^2 + K_{pl}^2 \eta^2}{\sigma^2 + K_{pl}^2 \eta^2} \right) \),

and \( K_{pl} = -\sigma^2/(2r) \)

Investor k's futures position \( y_k(f_{pl}) \) is given by

\[ y_k(f_{pl}) = r \frac{1}{1 + \sigma^2} \left[ \frac{1}{\sigma^2 + K_{pl}^2 \eta^2} \right] \left\{ \frac{K_{pl}^2 \eta^2}{\sigma^2 + K_{pl}^2 \eta^2} - \frac{2\sigma^2 + K_{pl}^2 \eta^2}{\sigma^2 + K_{pl}^2 \eta^2} \tilde{s}_1 \right\} \]

(if \( k = 1 \) or 3)

\[ y_k(f_{pl}) = \frac{1}{1 + \sigma^2 + K_{pl}^2 \eta^2} \left[ \frac{1}{\sigma^2 + K_{pl}^2 \eta^2} \right] \left\{ \frac{-K_{pl}^2 \eta^2}{\sigma^2 + K_{pl}^2 \eta^2} \tilde{s}_1 + K_{pl} f_{pl}^T \tilde{\omega} \right\} \]

(if \( k = 2 \) or 4)

The expected square trading volume of \( f_{pl} \) is given by

\[ E[\sum_{k=1}^{K} y_k^2(f_{pl}) + y_k^2(f_{pl})] = \eta^2/2 \left[ 1 - \frac{1}{(1 + \sigma^2 + K_{pl}^2 \eta^2)^2} \frac{2(1 + \sigma^2)K_{pl}^2 \eta^2}{\sigma^2(1 + \sigma^2 + K_{pl}^2 \eta^2)} \right] + \eta^2 \]

**Proof of Lemma 2.A.1.**

Since there are only two types of private information and the distribution of the investors' initial endowments \( \tilde{\epsilon}_k \) is publicly known, W.L.O.G, it suffices for us to consider only the information transmitted by the prices. With the conjectured linear form of the prices above, since the prices follow the normal distribution, we can apply the standard procedure outlined in section 2.3.1. Hence, we merely cite some key results needed for the derivation. One can prove them by simple computation.

For the index futures \( f_{in} = (1/\sqrt{2}, 1/\sqrt{2})^T \), with the price form above, the conditional expectation and variance of \( \tilde{Z} \) on \( (\tilde{s}_1, \tilde{P}_{in}) \) will be
\[ E[\tilde{Z} \mid s_1, P_{in}] = \left[ \begin{array}{c} \frac{s_1}{1+\sigma^2} \\ \frac{s_2 + K_{in} f_{1n}^T \omega}{1+\sigma^2 + K_{in}^2 \eta^2} \end{array} \right] \]

\[ \text{Var}[\tilde{Z} \mid s_1, P_{in}] = \left[ \begin{array}{cc} \frac{\sigma^2}{1+\sigma^2} & 0 \\ 0 & \frac{\sigma^2 + K_{in}^2 \eta^2}{1+\sigma^2 + K_{in}^2 \eta^2} \end{array} \right] \]

Note that \( E[f_{in} \mid s_1, P_S] = (1/\sqrt{2}, 1/\sqrt{2})^T E[\tilde{Z} \mid s_1, P_S] \) and \( \text{Var}[f_{in} \mid s_1, P_S] = (1/\sqrt{2}, 1/\sqrt{2})^T \text{Var}[\tilde{Z} \mid s_1, P_S](1/\sqrt{2}, 1/\sqrt{2}) \) Similar to the investors 2, 3, and 4. Imposing the market clearing condition (2.3), and comparing the coefficients of the NREE-price, we obtain equations that \( \alpha_{in} \) and \( K_{in} \) should satisfy. Derivation of the other equations is direct.

In the same way, we can obtain the equations for the plain contract \( f_{pl} = (1, 0)^T \). The conditional expectation and variance of \( \tilde{Z} \) on \((\tilde{s}_1, \tilde{P}_S)\) will be

\[ E[\tilde{Z} \mid s_1, P_{pl}] = \left[ \begin{array}{c} \frac{s_1}{1+\sigma^2} \\ \frac{s_1 + K_{pl} f_{pl}^T \omega}{1+\sigma^2 + K_{pl}^2 \eta^2} \end{array} \right] \]

\[ \text{Var}[\tilde{Z} \mid s_1, P_{pl}] = \left[ \begin{array}{cc} \frac{\sigma^2}{1+\sigma^2} & 0 \\ 0 & \frac{\sigma^2 + K_{pl}^2 \eta^2}{1+\sigma^2 + K_{pl}^2 \eta^2} \end{array} \right] \]

where, by calculating the equilibrium condition, we get \( \alpha_{pl} \) and \( K_{pl} \) as above. The other equations are from direct computation. \( \square \)

**Proof of Lemma 2.3.1.**

Direct from comparing the equations in Lemma 2.A.1. \( \square \)

**Proof of Proposition 2.3.2.**

Note first that \( K_{in} < 0 \) is the unique real solution of the equation 

\[ 4r\eta^2K^3 + \sqrt{2}(1 + \sigma^2)\eta^2K^2 + 2\sqrt{2}\sigma^2(1 + \sigma^2) = 0. \]

We can show that as \( \sigma^2 \searrow 0, K_{in} \to (-1)/(2\sqrt{2}r) \), and hence \( K_{in}^2 \to 1/(8r^2) \). On the other hand, the R.H.S. of equation (2.10) converges to \( 2/\eta^2 \) as \( \sigma^2 \searrow 0 \). Hence, as long as \( 1/(8r^2) < 2/\eta^2 \), for \( \sigma^2 \) small enough, equation (2.10) is satisfied. This proves the first claim.

The second claim follows from the fact that as \( \eta^2 \nearrow \infty, K_{in} \to -(1+2\sigma^2)/(2\sqrt{2}r) \) or \( K_{in}^2 \to (1 + 2\sigma^2)^2/(8r^4) > 0 \), while the R.H.S. of (2.10) converges to 0. \( \square \)
Proof of Lemma 2.3.2.

We can calculate the NREE-futures position for the plain contract $(0, 1)^T$ for all investors. It turns out that investor $k$'s equilibrium position of the contract $(1, 0)^T$ depends only on $\hat{s}_1$ and $(1, 0)^T \tilde{\omega}$, while $k$'s position of $(0, 1)^T$ depends only on $\hat{s}_2$ and $(0, 1)^T \tilde{\omega}$. Thus, $k$'s futures positions of these contracts are, as random variables, uncorrelated. Hence, the expectation of the sum of the square of these positions is equal to the sum of the expectation of the squares. The claim is now immediate. □

Lemma 2.A.2.

In a purely speculative factor-informed economy, when two contracts can be created, the exchange strictly prefers to create a single index futures rather than two futures, if and only if

$$K_{in}^2 < 2(1 + \sigma^2)/[\eta^2 \{ 7 - \frac{1}{\left( \frac{1 + \sigma^2}{\sigma^2} + \frac{K_{mn}^2 \eta^2}{\sigma^2 \eta^2} \right)^2} \frac{8(1 + \eta^2)K_{mn}^2 \eta^2}{\sigma^2 \eta^2 (\sigma^2 + K_{mn}^2 \eta^2)} \}]$$

Proof of Lemma 2.A.2.

By the observations made in the section, the innovation of a contract $(1, 0)^T$ does not affect the transaction of the contract $(0, 1)^T$ (and vice versa), and the expected sum of the square trading volume of the contracts $[(1, 0)^T, (1, 0)^T]$ is equal to that of $[(1/\sqrt{2}, 1/\sqrt{2})^T, (-1/\sqrt{2}, 1/\sqrt{2})^T]$. Also, because $(\forall k)e_k = 0)$, by symmetry, the expected square trading volume of the contract $(1, 0)^T$ is equal to that of $(0, 1)^T$. Therefore, the expected sum of the square trading volume when two contracts are created is just twice as much as that of the contract $(1, 0)^T$. The rest of the claim follows by simple calculation. □

Proof of Propositions 2.3.3. and 2.3.4.

Similar to the proof of Proposition 2.3.2. □

The following technical lemma is useful to prove the claim in Proposition 2.4.1.
Technical Lemma

(1) If $\tilde{\epsilon}$ is $I$-dimensionally normally distributed, then
\[
E[\exp(-\tilde{\epsilon}^T\tilde{\epsilon})] = \frac{1}{\sqrt{\det(I_I + 2\text{Var}[\tilde{\epsilon}])}} \exp\left[-E[\tilde{\epsilon}]^T(I_I + 2\text{Var}[\tilde{\epsilon}])^{-1}E[\tilde{\epsilon}]\right]
\]

(2) If $\tilde{u}$ is 1-dimensionally and $\tilde{v}$ is $I$-dimensionally normally distributed, then
\[
E[\exp(\tilde{u} - \tilde{v}^2)] = \frac{1}{\sqrt{\det(I_I + 2\text{Var}[\tilde{v}])}} \exp\left[E[\tilde{u}] - (1/2)\text{Var}[\tilde{u}] - (E[\tilde{v}] + C\text{ov}[	ilde{u}, \tilde{v}])^T(I_I + 2\text{Var}[\tilde{v}])^{-1}(E[\tilde{v}] + C\text{ov}[	ilde{u}, \tilde{v}])\right]
\]

Proof of Technical Lemma

(1) is from a direct computation. It is the $I$-dimensional version of Rao's Law.

Now, observe that
\[
\tilde{u} - \tilde{v}^2 = \tilde{u} - E[\tilde{u} | \tilde{v}] + E[\tilde{u}] - C\text{ov}[	ilde{u}, \tilde{v}]^T\text{Var}[	ilde{v}]^{-1}E[\tilde{v}] - \{\tilde{v} - (1/2)C\text{ov}[	ilde{u}, \tilde{v}]^T\text{Var}[	ilde{v}]^{-1}\}^T
\]
\[
\{\tilde{v} - (1/2)C\text{ov}[	ilde{u}, \tilde{v}]^T\text{Var}[	ilde{v}]^{-1}\} + \{(1/2)C\text{ov}[	ilde{u}, \tilde{v}]^T\text{Var}[	ilde{v}]^{-1}\}^T \{\tilde{v} - (1/2)C\text{ov}[	ilde{u}, \tilde{v}]^T\text{Var}[	ilde{v}]^{-1}\}
\]

Now, noticing that $\tilde{u} - E[\tilde{u} | \tilde{v}]$ is independent of $\tilde{v}$, apply (1). Then, we get the desirable result. □

Proof of Proposition 2.4.1.

Substitute the first-order condition of the interim utility maximization i.e., the interim optimal futures demand (2.5), into the budget constraint in (2.1). Take the unconditional expectation of the interim utility level measured w.r.t. this interim optimal wealth distribution. We can define $\tilde{u} = -(1/r_k)E_k[\tilde{\epsilon}_k] - (1/(2r_k^2))\text{Var}_k[\tilde{\epsilon}_k]$ and $\tilde{v} = (1/\sqrt{2})\text{Var}[\tilde{F}]^{-1/2}[E_k[\tilde{F}] - \tilde{P} - (1/r_k)C\text{ov}_k[\tilde{F}, \tilde{\epsilon}_k]]$, and apply the technical lemma. We can prove the second claim in the same way. □

Proof of Proposition 2.4.2.

Recall that $E[\tilde{F}] = 0$ and $\text{Var}[\tilde{F}] = I_I$. Notice that, in equilibrium, $y_k = r_k F^T[\sum_{k=1}^K \frac{r_k}{r_k^2} - \frac{r_k}{r_k^2} + \frac{\tilde{\epsilon}_k}{\sum_{k=1}^K r_k}]$. Then, the derivation of equation (2.12) is direct.

Now, we prove the second claim. Observe first that, for the fixed number $I$ of the innovated futures, if the exchange creates futures so that $E[\sum_{k=1}^K y_k(F)^T y_k(F)] =
\[ \sum_{k=1}^{K} E[y_k(F)^\top y_k(F)] \] is maximized among the sets of I contracts in F, then, by definition, (\forall F' \in F) F' contains I contracts, and (\forall k) E[y_k(F')^\top y_k(F')] \geq E[y_k(F)^\top y_k(F)] and (\exists k) E[y_k(F')^\top y_k(F')] > E[y_k(F)^\top y_k(F)]. But, this implies that the futures F cannot be Pareto-dominated by any other feasible set of futures with I contracts.

Next, by comparing the ex-ante utility, we can show that investor k prefers the innovation of the I contracts to the innovation of the maximal number \( \tilde{I} \), if and only if \( E[y_k(F(\tilde{I}))^\top y_k(F(\tilde{I}))] - E[y_k(F(I))^\top y_k(F(I))] < (\tilde{I} - I)r_k^2(\frac{n}{\sum_{i=1}^{K} r_k'})^2 \{ 1 + (\frac{n}{\sum_{i=1}^{K} r_k'})^2 \} \log \{ 1 + (\frac{n}{\sum_{i=1}^{K} r_k'})^2 \} \}. \) However, we can show \( \frac{n}{\sum_{i=1}^{K} r_k'}^2 \{ 1 + (\frac{n}{\sum_{i=1}^{K} r_k'})^2 \} \log \{ 1 + (\frac{n}{\sum_{i=1}^{K} r_k'})^2 \} < 0 \). Hence, investors always prefer the innovation of the maximal number of \( \tilde{I} \) contracts to that of \( I < \tilde{I} \).

Finally, observe that the liquidity trader, who behaves rationally, also prefers to have as many contracts as possible. Since the exchange creates \( \tilde{I} \) contracts under symmetric information by Proposition 2.3.1, with the first observation, we obtain the second claim. \( \square \)

**Proof of Lemma 2.4.1.**

Immediate from Proposition 2.4.1. \( \square \)

**Proof of Proposition 2.4.3.**

Under the given assumption, the ex-ante utility is given by lemma 2.4.1. Since there is only one contract and \( E[\hat{y}_k(f)] = 0, \) \( Var[\hat{y}_k(f)] = E[\hat{y}_k^2(f)] \). The claim follows from a direct computation. \( \square \)

The following lemma is useful to prove Propositions 2.4.4. Its proof is direct from applying Lemma 2.4.1. to the factor-informed economy and comparing the ex-ante utilities.
Lemma 2.A.3.

In a purely speculative factor-informed economy,

(1) Ex-ante, the investors unanimously prefer the innovation of an index (resp. a plain) contract to that of a plain (resp. an index) contract, if and only if

$$-K_{in}\left[\frac{1}{1+\sigma^2} + \frac{1}{1+\sigma^2+K_{in}^2\eta^2}\right] > (\text{resp. } <)$$

$$\frac{2\sqrt{2}}{r\left[\frac{1+\sigma^2+K_{in}^2\eta^2}{\sigma^2+K_{in}^2\eta^2}\right]^2} \max (\text{resp. } \min) \left[\frac{\sigma^2 K_{pl}^2 \eta^2}{(\sigma^2+K_{pl}^2\eta^2)^2} + \frac{(1+\sigma^2)(2\sigma^2+K_{pl}^2\eta^2)^2}{\sigma^2(\sigma^2+K_{pl}^2\eta^2)^2}, \frac{K_{pl}^2 \eta^2}{\sigma^2+K_{pl}^2\eta^2}\right]$$

(2) Ex-ante, the investors unanimously prefer the innovation of an index contract to that of two contracts, if and only if

$$1 - \frac{\sqrt{2}K_{in}\eta^2}{16r}\left[\frac{1}{1+\sigma^2} + \frac{1}{1+\sigma^2+K_{in}^2\eta^2}\right] > \frac{(3\sigma^2+K_{pl}^2\eta^2)^2}{\sigma^4(\sigma^2+K_{pl}^2\eta^2)^2}\left[\frac{1+\sigma^2}{\sigma^2+K_{pl}^2\eta^2}\right]^4\left[\frac{1+\sigma^2}{\sigma^2+K_{pl}^2\eta^2}\right]^4\left[\frac{1+\sigma^2+K_{in}^2\eta^2}{\sigma^2+K_{in}^2\eta^2}\right]^4\left[\frac{1+\sigma^2+K_{pl}^2\eta^2}{\sigma^2+K_{pl}^2\eta^2}\right]^4$$

Proof of Lemma 2.4.2.

Note that as $$\sigma^2 \searrow 0$$, $$K_{in} \rightarrow -1/(2\sqrt{2}r)$$ and that as $$\eta^2 \nearrow \infty$$, $$K_{in} \rightarrow -(1+2\sigma^2)/(2\sqrt{2}r)$$. The claim follows from taking the limits of the inequalities in Lemma 2.A.3. □

Proof of Proposition 2.4.4.

By Lemma 2.4.2. and Propositions 2.3.2. and 2.3.3. □

Proof of Proposition 2.4.5.

Apply Propositions 2.4.1. and 2.3.4. to the factor-informed economy, and take limits. When we take the limit of $$\eta^2 \searrow 0$$, use $$K_{in}\eta \to 0$$ ($$\eta^2 \searrow 0$$). □

The following characterization of the underproduction and overproduction (in terms of weak FI-optimality) of the futures contracts in a purely speculative factor-informed economy is immediate from Lemma 2.3.3. and Lemma 2.A.3.
Lemma 2.4.

In a purely speculative factor-informed economy, from the investors’ viewpoint, the underproduction (resp. overproduction) of the futures contracts occurs, if and only if

\[ K_{in}^2 < (\text{resp.} >) 2(1 + \sigma^2)/[\eta^2 \{7 - \frac{1}{\sigma^2 + \frac{1}{\eta^2} + \frac{8(1 + \sigma^2)K_{pl}^2\eta^2}{\sigma^2 + K_{pl}^2\eta^2}\}] \],

and

\[ 1 - \frac{\sqrt{2}K_{in}\eta^2}{16r} \left[ \frac{1}{1 + \sigma^2} + \frac{1}{1 + \sigma^2 + K_{in}^2\eta^2} \right] < (\text{resp.} >) \frac{\frac{1}{\sigma^2 + \frac{1}{\eta^2} + \frac{(2\sigma^2 + K_{pl}^2\eta^2)^2}{\sigma^2 + 2(2\sigma^2 + K_{pl}^2\eta^2)}}}{\{1 + \frac{1}{\sigma^2 + \frac{1}{\eta^2} + \frac{(1 + \sigma^2)^2}{\eta^2}\}\}} ^r \]

Proof of Propositions 2.5.1. and 2.5.2.

Overproduction results are immediate from Lemma 2.4.2, and Propositions 2.3.3, 2.3.4, and 2.4.5. Underproduction result in Proposition 2.4.6. is proven as follows. First, notice that the first inequality in Lemma 2.4. is satisfied if \( K_{in}^2 < (2(1 + \sigma^2)/(7\eta^2) \). Since \( K_{in}^2 \eta^2 \sim 0 \) as \( \eta^2 \sim 0 \), for \( \eta^2 \) small enough, the first inequality in Lemma 2.4. is satisfied. Next, set \( q \equiv \eta^2/(4r^2) \). Then, we can show that \( rK_{in} \to -1/\{2\sqrt{2}(1 + 2\sigma^2)\} \) as \( q \to \infty \) (or equivalently as \( r \to 0 \) for a fixed \( \eta^2 \)). Finally, using this relation, we can show that, for large \( q \equiv \eta^2/(4r^2) \), the second equation in Lemma 2.4. is satisfied. Thus, underproduction occurs for \( \eta^2 \) small enough w.r.t. \( \sigma^2 \) and for \( r \) small enough w.r.t. \( \eta^2 \). \( \square \)
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Chapter 3

Characterization of Security-Innovation

3.1 Introduction

During the past two decades, the successive innovation of new financial instruments has changed the working of financial markets, and now, creating new financial securities is one of the most important part of the business for many financial institutions. To understand the consequence of this economic activity, many researchers have investigated many different aspects of financial innovation.

One important branch of them, either commercially or academically, is the innovation of the securities traded in the organized exchanges such as futures and options. In particular, innovation of futures contracts has attracted a lot of academic (and business) interests, and many papers have been written on this issue (e.g. Silber (1981), Miller (1986), Black (1986), Duffie-Jackson (1989), Cuny (1993), Hara (1992), Ōhashi (1992, 1993), and Rahi (1992)).

All papers, however, start with some specific innovation-criteria of the security-innovators. They are, for example, the maximization of membership-fee (Cuny (1993)), trading volume (Silber (1981), Black (1986), Duffie-Jackson (1989), and Rahi (1992)),

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proportional transaction fee revenue (Hara (1992)), quadratic transaction fee revenue (Ohashi (1993)), and general convex transaction fee revenue (Ohashi (1992)). Each paper, then, characterizes the futures innovation under these specific objectives of the security-innovators, and investigates the welfare consequence of such innovation. Each result is, hence, subject to the specific choice of the innovation-criterion of the innovators.

In this paper, we take a different approach. That is, we do not look for the characterization of the innovation for the specific objectives. Instead, we are interested in the general properties that the created securities should satisfy for any reasonable innovation-criterion. The purpose of this paper is, hence, to characterize the general security-innovation by any type of private security-innovators. This analysis is important because this characterization enables us to obtain some restrictions on the securities created by private innovators. Especially, it allows us to predict what type of securities will, in general, never be created by the private innovators that create the securities for their own private benefits. In other words, this characterization provides us the limit of the securities supplied by the private innovation mechanism. In this way, the results in this paper complement many other papers whose results depend on the specific choice of security-innovators’ objectives.

For this purpose, our first task is to find an appropriate necessary condition that the created securities satisfy for any reasonable objective of the innovation. We propose the following condition: A set of securities is possibly created by security-innovators if at least some of them are traded with positive probability in an equilibrium.

Clearly, this condition is necessary for all innovation-criteria used in the papers above that depend on the transactions of the securities such as the maximization of the trading volume, and that of the general transaction fee revenue. Note that this condition is also necessary for the innovation under any lump-sum participation
fee charge, as long as the investors’ participation in the markets is verified by their transactions of the securities. Other objectives of the innovation that require the trade of created securities, such as the maximization of the sales-value of the assets (Allen–Gale (1988)), and the maximization of the benefit from the trade (Demange–Laroque (1992)), satisfy this condition, too. In this sense, this condition is necessary for any reasonable innovation-criterion.

One may wonder if this condition is too weak, and would argue that the appropriate criterion should be that a set of securities is possibly created if each of them are traded with positive probability in an equilibrium. Though seemingly intuitive, this conjecture is not correct. We show that, if the investors have differential information, this stronger condition of the security-innovation need not be necessary even for the maximization of the expected trading volume. This is because, under differential information, a security transmits information as well, and even the security that is not traded in the equilibrium can affect the transactions of the other securities through the information that its price transmits.

This subtlety suggests the importance of the information transmitted by the securities in determining whether or not securities are possibly created. Our second task is, thus, to characterize this necessary condition of the innovation in the way that we can accomodate the informational role played by the securities. Note that the securities are traded in an equilibrium only when the investors benefit from such transactions given the information available in the equilibrium. Hence, supposing that there is an equilibrium for the created securities, we obtain the following characterization: A set of securities is possibly created by some private innovators only if some transactions of these securities can Pareto-improve the investors' welfare given the information available in the equilibrium. In other words, the securities are possibly created only if the investors' initial endowments are not constrained optimal w.r.t. the transactions of these securities subject to the available information.
This characterization is closely related to the result by Holmstrom-Myerson (1983) about common knowledge and no-trade theorem at the interim stage i.e., after the investors receive their private and public information. In fact, it is one of the main points of this paper to show how this seemingly different topic, security-innovation by private innovators, can be understood in a unified way from the viewpoint of no-trade theorem. To understand the importance of the role of information, we also investigate the relation between the security-innovation and no-trade theorem at the ex-ante stage i.e., of Milgrom-Stoky (1982).

Our third task is to apply this characterization to obtain some restrictions on the structure of securities that we can expect to be created. Since we have a necessary condition of the innovation, we can tell what kind of securities are, in general, unlikely to be created. We are especially interested in the number of securities that can be created, and show that if the aggregated information fully reveals the true state of the nature, the securities that attain the complete markets cannot be created by the private security-innovators that seek their own private benefits.

This result is an analogy to Grossman (1977, 1980) in the context of the security-innovation by the innovators. Unlike Grossman's case, in this paper, information is costlessly endowed to the investors, but the release of new public information due to the innovation of a new security and the resulting reduction of the trade imposes a bound on the number of the securities created by the innovators. To illustrate this claim, we provide an example in which the introduction of a new security makes the whole securities untraded.

Our final task is to investigate the effect of the security-innovation on the investors' welfare in the ex-ante sense. We first show that no innovation of possibly created securities will put the investors' welfare down below their status quo i.e., the welfare level without the securities. Note that we know, by famous Hart's (1975) example, that when the investors have symmetric information, if there are more than two
goods or more than two trading dates, the innovation of an additional security can make all the investors worse off. With differential information, this result can be strengthened. In fact, we show that although there is only one kind of good and one trading period, the innovation of an additional security can still make all investors worse off, if the investors have differential information. This is a counterpart of Hart's (1975) example in the case of differential information. Finally, we show that with differential information among the investors, the security-innovation that Pareto-improves the investors welfare can be missed as long as the securities are created by the private innovators that seek their own private benefits. This observation can justify the need of intervention by the authority in the security-innovation when there is asymmetry of the information among the investors.

The rest of the paper is organized as follows. In section 3.2, we formulate a model of security-innovation possibly with differential information among the investors. In section 3.3, we discuss our choice of the necessary condition for any reasonable innovation-criterion. In section 3.4, we characterize this necessary condition in terms of optimality of the investors' initial endowments, and discuss its relation with no-trade theorems. In section 3.5, we analyze the number of securities that can be, by any chance, created by private security-innovators. In section 3.6, we investigate the relation between the security-innovation and the investors' ex-ante welfare. Finally, in section 3.7, we offer some concluding remarks.
3.2 A Model

We consider an economy with three dates \{0, 1, 2\} facing uncertainty, which is described by a complete probability space \((\Omega, \mathcal{F}, Pr)\). A single good is available in each event at the terminal date. Hence, we take the commodity space \(X \equiv L^2(\Omega, \mathcal{F}, Pr)\).

There are two types of agents, innovators and investors. The innovators create securities and the investors trade them. The timing is as follows: At date 0, ex-ante, every agent has the same information about the characteristics of the economy, and the innovators create securities. At date 1, interim, the investors receive both their private and public information, and trade the created securities. At date 2, ex-post, uncertainty is resolved, the traded securities are liquidated, and consumption occurs.

More precisely, the information structure of the agents is described as follows: At date 0, every agent has the same information of a trivial \(\sigma\)-field \(\mathcal{F}_0\) (i.e., \(\mathcal{F}_0\) is the \(\sigma\)-field generated by \(\{\phi, \Omega, \text{all } Pr\text{-null sets}\}\)). At date 1, each consumer \(k\) \((k = 1, \ldots, K)\) receives private information \(\mathcal{F}_k \subset \mathcal{F}\) and public information \(\mathcal{F}^u \subset \mathcal{F}\). The public information \(\mathcal{F}^u\) can depend on the innovation of the securities, and hence can be interpreted as the information inferred from the prices and the transactions of the created securities. Finally at date 2, uncertainty is completely resolved and each agent has the complete information \(\mathcal{F}\).

There are \(K\) investors. Each \(k\) \((= 1, \ldots, K)\) has the initial endowment \(e_k \in X\) of date 2 risky income. His preference depends on the available information \(\mathcal{G} \subset \mathcal{F}\), and is described by a VNM-utility \(U_k : X \times \mathcal{S} \to L^2(Pr)\), \(U_k(x \mid \mathcal{G}) = E[u_k(x) \mid \mathcal{G}]\) \((x \in X, \mathcal{G} \subset \mathcal{S})\) where \(\mathcal{S}\) is the set of sub-\(\sigma\)-fields of \(\mathcal{F}\) and \(u_k : R \to R\) is continuous, strictly increasing, and strictly concave. Since investor \(k\) receives private information \(\mathcal{F}_k \subset \mathcal{F}\) at date 1, each \(k\) is described by the triple \((U_k, e_k, \mathcal{F}_k)\). We call the collection \((U_k, e_k, \mathcal{F}_k)_{k=1,\ldots,K}\) an economy.
Innovators, knowing the economy \((U_k, \epsilon_k, \mathcal{F}_k)_{k=1,...,K}\), create securities at date 0.\(^1\) We denote by \(f_i \in X\) \((i = 0, 1,..., I)\) the payoff of security \(i\), and assume that \(f_0 = 1_\Omega\) a.e. is always available. We call an array \(F = (f_0, f_1, ..., f_I)^T \in \prod_{i=0}^{I} X\) a security structure, and denote by \(F\) the set of the feasible security-structure.

We call a collection \(x = (x_1, ..., x_K)^T \in \Pi_{k=1}^{K} X\) an allocation if \(\sum_{k=1}^{K} x_k = \sum_{k=1}^{K} \epsilon_k\) a.e. Define \(G_k = \mathcal{F}_k \vee \mathcal{F}^u\) for all \(k = 1,..., K\). \(G_k\) is the coarsest \(\sigma\)-field containing both \(\mathcal{F}_k\) and \(\mathcal{F}^u\), and represents the information available to investor \(k\) at date 1. We call a collection \(G = (G_1, ..., G_K)\) information distribution. Finally, we call a collection \(\{(U_k, \epsilon_k, \mathcal{F}_k)_{k=1,...,K}, (F, \mathcal{F}^u)\}\) an economy with security structure \(F\) and public information \(\mathcal{F}^u\). Because the investors' evaluation of the allocation depends on the available information, for each economy with given \((F, \mathcal{F}^u)\), we distinguish three concepts of optimality of the allocation constrained to the transactions of the given security-structure \(F\). This definition follows Holmstrom-Myerson (1983), and Laffont (1985, 1989).

**Definition 3.2.1.**

1. An allocation \(x = (x_1, ..., x_K)^T\) is **ex-ante constrained optimal** w.r.t. a security-structure \(F\), if there exists no \(y \equiv (y_1, ..., y_K)\) s.t.

\[
\begin{align*}
&\forall k \quad y_k \in R^I, \quad y_k^T F \in X \text{ and } \sum_{k=1}^{K} y_k = 0 \\
&(\forall k) \quad U_k(x_k + y_k^T F) \geq U_k(x_k), \text{ and} \\
&(\exists k') \quad U_{k'}(x_{k'} + y_{k'}^T F) > U_{k'}(x_{k'})
\end{align*}
\]

2. An allocation \(x = (x_1, ..., x_K)^T\) is **interim constrained optimal** w.r.t. a security structure \(F\) and information distribution \(G = (G_1, ..., G_K)\), if there exists no \(y \equiv (y_1, ..., y_K)\) s.t. \((\forall k)\ y_k \in G_k y_k^T F \in X\) and \(\sum_{k=1}^{K} y_k = 0\) a.e.

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\(^1\)In this paper, we will not define the explicit objective of the innovators. Instead, we consider a necessary condition that any reasonable objective of the investors must satisfy, and ask what kind of security structure can be, in general, consistent with this condition.
\[(\forall k) \ U_k(x_k + y_k^T F \mid G_k) \geq U_k(x_k \mid G_k) \text{ a.e., and} \]
\[(\exists k') \ U_{k'}(1_A x_{k'} + y_{k'}^T F \mid G_{k'}) > U_{k'}(1_A x_{k'} \mid G_{k'}) \text{ for some } A \in G_k^2 \]

(3) An allocation \( x = (x_1, \ldots, x_K)^T \) is ex-post constrained optimal w.r.t. a security structure \( F \), if there exists no \( y \equiv (y_1, \ldots, y_K) \) s.t.
\[(\forall k) \ y_k \in \mathcal{F} \ y_k^T F \in X \text{ and } \sum_{k=1}^K y_k = 0 \text{ a.e.} \]
\[(\forall k) \ U_k(x_k + y_k^T F \mid \mathcal{F}) \geq U_k(x_k \mid \mathcal{F}) \text{ a.e., and} \]
\[(\exists k') \ U_{k'}(1_A x_{k'} + y_{k'}^T F \mid \mathcal{F}) > U_{k'}(1_A x_{k'} \mid \mathcal{F}) \text{ for some } A \in \mathcal{F} \]

In this paper, we assume that the markets for the securities \( F \) open at date 1. That is, the investors receive the private and public information before they trade, and they cannot commit not to utilize such information for trading. Each security \( f_i \in X \) is traded at the price \( p_i \in X \) \((i = 1, \ldots, I)\). Obviously, without arbitrage, \( p_0 = 1 \) a.e. Define \( P = (p_0, p_1, \ldots, p_I) \), and \( \mathcal{F}_P \equiv \sigma(P) \) i.e., the \( \sigma \)-field generated by \( P \).

For each \( k = (1, \ldots, K) \), we call \( y_k \) an optimal trading plan of investor \( k \) w.r.t. a security-structure \( F \) and a price system \( P \), if \( y_k \in G_k, y_k^T P = 0 \text{ a.e., and } y_k^T F \in X \) s.t. \( U_k(e_k + y_k^T F \mid G_k) \geq U_k(e_k + y_k'^T F \mid G_k) \) for all \( y_k' \in G_k, y_k'^T P = 0 \text{ a.e., and } y_k'^T F \in X \).

Define \( y = (y_1, \ldots, y_K)^T \) and \( \mathcal{F}_y \equiv \sigma(y) \) i.e., the \( \sigma \)-field generated by \( y \). A market equilibrium w.r.t. a security structure \( F \) is a pair \((y, P)\) of optimal trading plans and prices such that (1) the markets clear i.e., \( \sum_{k=1}^K y_k = 0 \text{ a.e., and} \) (2) the information transmitted by the prices and the transactions is public to the investors i.e., \( \mathcal{F}_P, \mathcal{F}_y \subset \mathcal{F}_u \subset G_k \) \((\forall k)\).

Now, let us consider the innovation of the securities \( F \). Recall that these securities are created by some private security-innovators that seek their own benefits.

\footnote{U_k(1_A x) \equiv E[u_k(1_A x) \mid G_k]}
Since any reasonable objective of such security-innovation requires the transactions of the created securities in an equilibrium, the following condition is necessary for the security-innovation in general.

**Definition 3.2.2.**

A set of securities $F \in \Pi_{i=0}^T X$ is called **possibly created**, if and only if there exists a market equilibrium $(y, P)$ w.r.t. $F$ such that $Pr(\{\sum_{k=1}^{K} | y_k | > 0\}) > 0$

Clearly, all objectives of the innovators used in the papers above on futures innovation, such as the maximization of the expected trading volume, or that of the general transaction fee revenue, satisfy this condition. Note also that any security-innovation with lump-sum participation fee charge satisfies this condition as necessary, as long as the investors' participation in the markets is verified by their transactions of the securities. Other innovation-criteria that necessarily require the transactions of the created securities, such as the maximization of the sales value of the assets (Allen-Gale (1988)), and the maximization of the benefit from the trade (Demange-Laroque (1992)), satisfy this condition, too.

Note that this condition simply requires that at least some of the securities are traded in an equilibrium. In this weak sense, this is necessary for any reasonable objective of the security-innovation\(^5\). Obviously, it is unlikely for us to observe the securities that are not possibly created, simply because none of such securities are traded. Our purpose is to characterize such possibly created securities, and investigate its implication on the security-innovation.

\(^3\) For a vector $y$, $y > 0$ implies that $(\forall i)y_i \geq 0$ and $(\exists i)y_i > 0$, where $y_i$ is the $i$-th element of $y$.

\(^4\) Equivalently, a set of securities is called possibly created if and only if at least some of the securities have positive expected trading volume in an equilibrium.

\(^5\) One may strengthen this definition by excluding all the securities that neither are traded nor have any new informational content i.e., all the securities that are redundant for the hedging and informational roles. However, this modification does not add any substantially new result for the analysis here.
Now, one may wonder whether this condition is too weak, and may argue that we should use a stronger, and more intuitive, condition such as the following:

**Definition 3.2.3.**

A set of securities \( F \in \prod_{i=0}^{l} X \) is called **possibly created in the strong sense**, if and only if there exists a market equilibrium \((y, P)\) w.r.t. \( F \) such that \( Pr(\{\text{Min}_{i=1,\ldots,l} \sum_{k=1}^{K} |y_{ik}| > 0\}) > 0 \)

That is, the securities are possibly created in the strong sense, if and only if each of the securities is traded with positive probability. This seems more natural. However, this condition turns out too strong to be a necessary condition for the general innovation-criterion, if the investors have differential information. In the following, we first discuss why we need the weaker condition of Definition 3.2.2. We then characterize this necessary condition of the security-innovation in general, and apply the result to give some restrictions on the security-structure that we are likely to observe.

### 3.3 A Necessary Condition for Security-Innovation

When the investors have differential information, a security plays two roles: First, it provides the risk-sharing opportunities. Second, it transmits the information among the investors. Since the price transmits the information, though a security is not traded in an equilibrium, the security can still play the informational role. The following example illustrates this point. That is, a security that is never traded in the equilibrium, but that transmits the information through its price, can facilitate the transactions of another security. In this situation, if a security-innovator can create both of these securities to maximize, say, the expected trading volume, it
decides to create this untraded security. However, this implies that even the maximization of the expected trading volume need not satisfy the stronger condition, $Pr(\{Min_{i=1,...,I} \sum_{k=1}^{K} y_{ik} | > 0\}) > 0$, of the possibly created security-structure. That is, the stronger condition, Definition 3.2.3, need not be necessary for some innovation-criteria, if the investors have differential information.

**Example 3.3.1: A Necessary Condition of Security-Innovation**

Consider an economy with two random shocks $\tilde{z}_1$ and $\tilde{z}_2$. These shocks are correlated as follows: $\tilde{z}_1 = \tilde{z}_2 + \tilde{c}$ where $\tilde{c}$ is another random variable. There are two investors with the identical CARA utility function $U(\tilde{x}) \equiv E[-exp(-(1/r)\tilde{x})]$. Investor 1 has no initial endowment i.e., $\tilde{c}_1 = 0$. Investor 2 has a random endowment $\tilde{c}$ of the random shock $\tilde{z}_1$ so that his initial endowment of date 2 risky income is $\tilde{c}_2 = \tilde{c}\tilde{z}_1$. We assume that $(\tilde{z}_2, \tilde{c}, \tilde{c}) \sim N(0, I_3)$ where $I_3$ is a $3 \times 3$-identity matrix. Finally, we assume that investor 1 is uninformed, but that investor 2 knows the realization of $\tilde{z}_2$ at date 1 before trading.

Suppose now that an innovator can create two types of futures contracts with the payoffs, $\tilde{f}_1 = \tilde{z}_1$, and $\tilde{f}_2 = \tilde{z}_2$. Clearly, even if it is created, the second contract $\tilde{f}_2$ will not be traded, simply because investor 1 knows that investor 2 knows the realization of its payoff. However, the presence of the second futures can facilitate the trading of the first futures in the following way: Though it is not traded, the price of the second futures transmits the information about the part of the payoff of the first contract. Hence, if investor 1 can observe the price of the second futures and learn the realization of $\tilde{z}_2$, the risk that investor 1 faces in trading $\tilde{f}_1$ decreases, and he trades $\tilde{f}_1$ more. Since there is no change in the private information of investor 2, with the second futures $\tilde{f}_2$, the expected trading volume of the first futures $\tilde{f}_1$ eventually goes up. Consequently, the innovator, who maximize the expected sum of the trading volume, is better off by creating $\tilde{f}_2$ which is not traded in an equilibrium.
Indeed, the calculation shows that, without the second contract, the equilibrium position of the first contract $\tilde{f}_1$ is given by $-\tilde{y}_1 = \tilde{y}_2 = r(\tilde{z}_2 - \tilde{p}) - \tilde{c}$ where $\tilde{p} = \alpha(\tilde{z}_2 + k\tilde{e})$, $\alpha = \frac{2(1+k^2)}{2+3k^2}$, $k = -1/r$, and hence the expected trading volume is equal to $\sqrt{\frac{2}{\pi} \frac{2\sqrt{1+r^2}}{3+2r^2}}$. On the other hand, with the second contract $\tilde{f}_2$, the equilibrium position of the first contract $\tilde{f}_1$ is given by $-\tilde{y}_1 = \tilde{y}_2 = r(\tilde{z}_2 - \tilde{p})$ where $\tilde{p} = \tilde{z}_2 - \tilde{c}/(2r)$, and hence the expected trading volume is equal to $\sqrt{\frac{2}{\pi}}$. Obviously, the latter is larger than the former. □

Hence, under differential information among the investors, the security-innovation that maximizes the expected trading volume need not satisfy the stronger condition in Definition 3.2.3. Thus, as a necessary condition of the general security-innovation, we have to use the weaker condition of Definition 3.2.2.

Two things are worth noting. First, if the investors have symmetric information, since the securities play only the role to provide the risk-sharing opportunities, and risk-sharing necessarily causes the transactions of the securities, the stronger definition of the security-innovation may make more sense. Second, this example suggests that the informational role played by the securities can be beneficial to the society, but this benefit may not be internalized by the private security-innovators. That is, although opening a market for a security does the investors good by transmitting more information, the innovators may have no incentive to do so. This discrepancy of interest may cause inefficient innovation of the securities. We will come back to these points in section 3.6.
3.4 Characterization of General Security-Innovation

3.4.1 Possibly Created Securities

We now characterize the necessary condition of the general security-innovation. Recall that if the securities are possibly created by the security-innovators, some of the securities are necessarily traded in an equilibrium with positive probability. Because the transactions are voluntary, this implies that, in some events and for some investors, there are gains from trading the created securities. Furthermore, since these transactions occur in the interim stage i.e., after the investors receive the information, these gains are subject to the information available in the equilibrium. Thus, for a necessary condition of the security-innovation in general, we obtain the following characterization:

Proposition 3.4.1.

If a set of securities $F$ is possibly created by some security-innovators, then the allocation of the investors' initial endowments $e = (e_1, ..., e_K)$ is not constrained optimal in the interim sense w.r.t. $F$ and $G$.

Therefore, in general, we are likely to observe the securities for which the investors' initial endowments are not constrained optimal in the interim sense. The converse claim, however, is difficult to make especially when the investors have differential information so that the prices and the transactions of the securities transmit non-trivial public information.

This proposition shows that we can characterize the general security-innovation from the viewpoint of Holmstrom-Myerson (1983) about common knowledge and no-trade theorem at the interim stage. There is, however, subtle difference between these two issues: First, in no-trade theorem literature, the main concern is the existence and the characterization of an optimal decision rule i.e., a mechanism to attain the
optimal allocation, given the distribution of information among the investors. On the other hand, in the security-innovation, the decision rule is restricted to the market mechanism, and is determined by the choice of the created securities. Second, unlike the no-trade theorem literature, the information available to the investors generally depends on the innovation of the securities, or the choice of the decision rule, through the release of information by the prices and the transactions of the created securities. This fact makes the characterization of security-innovation a unique problem, and differentiates this issue from no-trade theorem.

Now, the following examples show some of the security structures that attain interim constrained optimality, and hence are not possibly created. Thus, as long as the securities are created by private innovators, we are unlikely to observe such securities. The first example considers the case of symmetric information among the investors (i.e., $(\forall k)\mathcal{F}_k = \mathcal{F}^u = \mathcal{F}_0$), and the second example considers that of differential information.

**Example 3.4.1: Not Possibly Created under Symmetric Information**

Consider an economy where uncertainty is described by a vector of two Gaussian random variables $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2)^T$, $\tilde{Z} \sim N(0, I_2)$, where $I_2$ is the $2 \times 2$ identity matrix, which we call the risk factors. There are two investors $k = 1, 2$ with the identical CARA utility function $E[-exp(-(1/r)x)] (x \in X)$. Investor 1 has an initial endowment of date 2 risky income, which is equal to one unit of the risk factor 1 i.e., $e_1 = \tilde{z}_1$. Investor 2 has no initial endowment i.e., $e_2 = 0$ a.e.. There is no private information. The distribution of the risk factors, the initial endowments, and the utility functions are the public information.
Now, suppose that, besides the riskless asset, a single security can be created and that its payoff $f$ must be a linear combination of the risk factors i.e., $f = a\tilde{z}_1 + b\tilde{z}_2$ ($a, b \in R$).

![Diagram of securities creation](image)

**Fig. 3.4.1: Not Possibly Created Securities**

Since both investors have the same (i.e., zero) initial endowments of risk factor 2, the initial endowments are interim constrained optimal w.r.t. the securities with the payoffs $f = b\tilde{z}_2$ ($b \in R$). There is no gains for the investors from trading the risk in the direction of the risk factor 2, and the securities that only allows such trade are not traded. In this example, the securities that we are likely to observe should satisfy $a \neq 0$ □

When the investors have differential information, the prices and the transactions of the securities transmit the information. This release of public information have non-trivial effect on whether a set of securities is possibly created or not. The next example illustrates this point.
Example 3.4.2: Not Possibly Created under Differential Information

Let us consider an economy similar to that in example 4.1, but assume that investor 1 (resp. 2) knows the realization of the risk factor 1 (resp. 2). Let us also assume that investor 1 (resp. 2) has a random initial endowment of risk factor 1 (resp. 2) $\tilde{e}_1 \sim N(0,1)$ (resp. $\tilde{e}_2 \sim N(0,1)$) so that his initial endowment of date 2 risky income is $\tilde{e}_1\tilde{z}_1$ (resp. $\tilde{e}_2\tilde{z}_2$).

Now, suppose again that, besides the riskless asset, a single security can be created, and that its payoff $f$ should be a linear combination of the risk factors i.e., $f = a\tilde{z}_1 + b\tilde{z}_2$ ($a, b \in \mathbb{R}$).

![Diagram](attachment:image.png)

Fig.3.4.2: Not Possibly Created Securities

Now, the securities with the payoffs collinear either to the risk factor 1, i.e., $f = a\tilde{z}_1$ ($a \in \mathbb{R}$) or to the risk factor 2, i.e., $f = b\tilde{z}_2$ ($b \in \mathbb{R}$), is not possibly created by any security-innovator. This is because, given the knowledge that investor 1 learns the realization of $\tilde{z}_1$, investor 2 knows that he always loses by trading a security $f = a\tilde{z}_1$ with investor 1, and vice versa. Hence, the initial endowments are interim constrained optimal w.r.t. the securities with payoffs $f = a\tilde{z}_1$ and $f = b\tilde{z}_2$ ($a, b \in \mathbb{R}$) and the corresponding information distribution. In this example, the security that we are likely to observe should have a payoff $f = a\tilde{z}_1 + b\tilde{z}_2$ ($a \neq 0, b \neq 0$). □
3.4.2 Possibly Created Securities and Ex-ante Constrained Optimality

In order to tell whether or not a set of securities is possibly created, it is important to know how the information is transmitted through the transactions of the securities. To understand this point, we investigate the relation between the possibly created security-structure and constrained optimality of the allocation w.r.t. the ex-ante transactions of the securities i.e., the trades that utilize neither private nor public information. (See Definition 3.2.1. (1).)

Note that this characterization of the possibly created securities is valuable in the sense that it allows the security-innovators to give some restrictions on the possible innovation without knowing the distribution of information among the investors which is induced by the security-innovation. Unfortunately, it turns out that we can obtain such characterization only for a limited case.

In the following, we say that a security-structure $F$ completes the markets, if for any consumption $x \in X$, there exists some date 0 portfolio $y$ of $F$ such that the payoff of the portfolio is equal to $x$ i.e., $(\forall x \in X)(\exists y \in \mathcal{F}_0) \ y^T F = x$. When a security structure $F$ completes the markets, we obtain a simple relationship between the possibly created security-structure and the ex-ante (constrained) optimality w.r.t. these securities.

Proposition 3.4.2.

Suppose that a security structure $F$ completes the markets. Then, if $F$ is possibly created by some security-innovators, the initial endowments $e = (e_1, ..., e_K)$ is not ex-ante (constrained) optimal w.r.t. $F$.

\[ \text{If } \Omega \text{ is finite, this definition of market completeness is clearly meaningful. If } \Omega \text{ is infinite so that infinite number of securities are needed, we appropriately change the way to index the securities and the corresponding transactions, and define completeness of the markets.} \]
This proposition connects our characterization of the security-innovation with no-trade theorem of Milgrom-Stoky (1982) in terms of ex-ante optimality. When the markets are complete, ex-ante optimality implies interim optimality. Hence, this claim follows. However, when a security structure $F$ does not complete the markets, we lose this simple relationship, and ex-ante constrained optimality is not useful to predict whether the securities are possibly created or not. Information, which differentiates interim from ex-ante, comes to play an important role to determine the possibly created security-structure. The following example illustrates this point:

**Example 3.4.3: Ex-ante Constrained Optimal, but Possibly Created**

Consider an economy with four states of the world i.e., $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. They are equally likely. There are two investors who receive, at date 1, the information about which of the events $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$ occurs. Both investors have an identical risk-averse utility, $E[u(\bar{x})]$. Investor 1 has the initial endowment $e_1 = (1,-1,-1,1)$ (i.e., 1 unit of the good in $\omega_1$, -1 unit in $\omega_2$, and so on). On the other hand, investor 2 has the initial endowment $e_2 = (-1,1,1,-1)$.

Now, consider a futures contract with the payoff $f = (1,-1,1,-1)$. Clearly, $(e_1,e_2)$ is ex-ante constrained optimal w.r.t. $f$. However, at date 1, after information is received, the investors are willing to trade this futures contract to attain the full risk-sharing i.e., $f$ is possibly created. $\square$

Furthermore, the converse of this proposition need not be true.
Example 3.4.4: Not Ex-ante Optimal, but Not Possibly Created

Consider an economy with two states of the world i.e., \( \Omega = \{ \omega_1, \omega_2 \} \). There are two investors. Investor 1 has the initial endowment \( e_1 = (1, 0) \) (i.e. 1 unit of the good in \( \omega_1 \), and 0 in \( \omega_2 \)), and knows, at date 1, whether state \( \omega_1 \) or \( \omega_2 \) occurs. Investor 2 has the initial endowment \( e_2 = (0, 1) \), and has no private information about which state occurs. Now, suppose that, at date 0, Arrow-Debreu securities, \( F = (f_1, f_2)^T \) where \( f_1 = (1, 0) \) and \( f_2 = (0, 1) \), are available. Obviously, \( e \) is not ex-ante (constrained) optimal w.r.t. \( F \). However, at date 1, investor 1 knows the true state for sure, and the prices of the securities \( F \) fully reveals the true state. Thus, as long as the investors have monotonic utility functions, there is no trade of \( F \). \( \square \)

The result in Example 3.4.4. does not depend on the completeness of the markets.

Example 3.4.5: Not Ex-ante Optimal, but Not Possibly Created

Consider an economy with a risk factor \( \bar{\varepsilon} \sim N(0,1) \). There are two investors. Investor 1 has the initial endowment of one unit of the risk factor and learns the realization of the risk factor before trading, while investor 2 has neither initial endowment nor private information about the risk factor.

Now, consider a futures contract with the payoff equal to the risk factor i.e., \( \bar{f} = \bar{\varepsilon} \). Ex-ante, before investor 1 receives the private information, the initial endowments are not ex-ante constrained optimal w.r.t. \( \bar{f} \). Thus, if the investors can relocate their date 2 risky income in the direction of the futures’ payoff, then the investors’ welfare Pareto-improves in ex-ante sense. However, at date 1, since investor 1 know the realization of the futures’ payoff completely, there is no trade of this contract between the investors. Hence, \( \bar{f} \) is not possibly created. \( \square \)
From these examples, we can see that although relocation of consumption in a certain direction can Pareto-improve the welfare in the ex-ante sense, the presence of private information may prevent the innovator from creating the securities that makes such relocation possible. These results suggest the importance of the role of information to determine the possibly created security-structure. That is, we have to understand how information is transmitted and how its distribution is determined in order to predict the types of the securities that we are likely to observe. The following section concerns this point, in particular, how many securities are likely to be created.

3.5 Security-Innovation and the Degree of Market Incompleteness

In this section, we apply the characterization above, and obtain some restrictions on the structure of possibly created securities. Since we have already seen how the design of securities matters, here, we are especially interested in the restriction on the number of possibly created securities. This question is natural because the information transmitted through the security-markets depends on both the design and the number of the available securities. The following proposition shows one case where the number of the possibly created securities, or the degree of the market-incompleteness, can be predicted from the distribution of information.

Proposition 3.5.1.

Suppose that uncertainty is described by $N$ risk factors $\tilde{Z} = (\tilde{z}_1, ..., \tilde{z}_N)^T$ i.e., $\mathcal{F} = \sigma(\tilde{Z})$, and that, except the riskless asset, each security must have the payoff linear in the risk factors i.e., $(\forall i \neq 0) \ f_i = \theta_i^T \tilde{Z} \ (\theta_i \in R^N)$.

Define $\Theta \equiv [\theta_1, ..., \theta_I] \in R^{N \times I}$.

If, for any risk factor $\tilde{z}_n$, there is some investor $k_n$ who knows the realization of $\tilde{z}_n$, then $\text{Rank}[\Theta] < N$ whenever $F \equiv (f_0, f_1, ..., f_I)^T$ is possibly created by some security-innovators.
Hence, if the aggregate information completely reveals the realization of uncertainty, the securities that complete the markets cannot be possibly created by any private security-innovator. This is for the following reason: If the markets are complete, all information is aggregated and reflected in the security-prices. Hence, if the aggregate information reveals the realization of the uncertainty, all payoffs of the securities are known deterministically to the investors. Since there is only one good, and the investors' utilities are monotonically increasing in this good, the securities with the deterministic payoffs are not traded.

This result is an analogy to Grossman (1977, 1980) in the context of the security-innovation by an innovator. Grossman argued that the complete markets cannot exist under differential (asymmetric) information if the acquisition of information is costly. This is because, then, the security prices are so informative that all the advantages and the incentives to be informed with cost are eliminated. However, this is contradictory to the assumption that the information is acquired with cost. On the other hand, in this paper, the information is costlessly endowed, but securities are created by some innovators. The complete markets still may not be possible, if the innovator creates securities for their own benefits so that at least some of them are traded. This is because, under differential information, creating an additional security transmits an additional piece of information, but this release of new public information can reduce the trade of the securities, in the extreme case, to zero. The following example illustrates this point:

Example 3.5.1: A Bound on the Number of the Securities

Consider an economy similar to the previous examples in which all the random variables are Gaussian. Uncertainty is described by two risk factors $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2)^T$ where $\tilde{z}_n \sim N(0,1)$ ($n = 1, 2$). There are two investors, 1, 2. Investor 1 (resp. 2) knows the realization of $\tilde{z}_1$ (resp. $\tilde{z}_2$) at date 1, but is uninformed about that of $\tilde{z}_2$ (resp. $\tilde{z}_1$). Investor 1 (resp. 2) has the random endowment $\tilde{e}_{1n}$ (resp. $\tilde{e}_{2n}$)
\( \sim N(0, \eta^2) \) of the risk factor \( n (n = 1, 2) \). We assume that all random variables \( (\hat{z}_1, \hat{z}_2, \hat{e}_{11}, \hat{e}_{12}, \hat{e}_{21}, \hat{e}_{22}) \) are independent. Finally, investor \( k (= 1, 2) \) has a CARA utility \( E[-\exp(-(1/r)x)] \) over the date 2 consumption \( x \).

Suppose that securities can be created. Besides the riskless asset, we assume that the payoff of each security must be normalized to have the form: \( a\hat{z}_1 + b\hat{z}_2 \) (\( a^2 + b^2 = 1 \)). Furthermore, we assume only the following set of design is possible: \( (a, b) = (1, 0), (0, 1), (1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}) \). We ask which design of the securities and how many of them are possibly created by some security-innovators.

Consider first the innovation of a single security. Since investor 1 knows the realization of the risk factor 1, as is in Example 3.4.2, any initial endowment is interim constrained optimal w.r.t. the security design \((1, 0)\) and its corresponding information distribution. Hence, the security with the design \((1, 0)\) is not traded and hence cannot be possibly created. Similarly, the security with the design \((0, 1)\) cannot be possibly created, either.

On the other hand, the security with the design \((1/\sqrt{2}, 1/\sqrt{2})\) can be possibly created. Assuming the usual price function of the futures \( \bar{p} = \alpha[\hat{z}_1 + \hat{z}_2 + l(\hat{e}_{12} + \hat{e}_{21})] \), one can show that, if \( r < \eta \), then given the security design \((1/\sqrt{2}, 1/\sqrt{2})\), there exist the noisy rational expectation equilibria with trading in which \( l = \frac{-1}{r} \pm \sqrt{\frac{1}{r^2} - \frac{1}{\eta^2}} \) and \( \alpha = \frac{1}{4\sqrt{2}}(4 + rl) \). Similarly, the security design \((-1/\sqrt{2}, 1/\sqrt{2})\) also can be possibly created. With one of these securities, the security price does not transmit too much information, and hence the initial endowment is not interim constrained optimal. Consequently, transactions can occur. Thus, under differential information, the feasibility of a security can depend on its design.

Now, suppose that the security with the design \((1/\sqrt{2}, 1/\sqrt{2})\) has already been created. Consider the introduction of an additional security \((-1/\sqrt{2}, 1/\sqrt{2})\). Although each of them is possibly created when created alone, this additional creation makes
the security structure not possibly created. This is because, with two securities, the
security prices transmits all the information that the investors have so that there
is no uncertainty left. Consequently, with this information restriction, all initial
endowments are interim constrained optimal. Thus, there is no trade and these
securities are not possibly created.

Hence, we saw that, in this example, the security with the design \((1/\sqrt{2}, 1/\sqrt{2})\)
or \((-1/\sqrt{2}, 1/\sqrt{2})\) is possibly created only when one of them is created. That is, the
upper bound on the number of possibly created securities is one. \(\square\)

3.6 Security-Innovation and Investors’ Welfare

3.6.1 Welfare Effects of Additional Securities

In this section, we investigate the effect of security-innovation on the investors’ ex-ante
utility levels. Notice first that being possibly created is just a necessary condition,
and hence is too weak to describe any meaningful efficiency result on the security-
innovation by itself. However, since the transactions of securities are voluntary so
that the investors can always choose not to trade, the following claim should be true
for all possibly created securities:

Lemma 3.6.1.

Any equilibrium allocation resulting from the trade of the possibly created
securities is not Pareto-dominated by the investors’ initial endowment
allocation.

That is, the security-innovation by the innovators, at least, does not let the in-
vestors' welfare down below their status quo i.e., the utility levels attained by the
investors' initial endowment allocation.
Now, let us consider how the innovation of an additional security affects the investors' welfare. When the investors have symmetric information, we know from the famous Hart's (1975) example that if there are more than two kinds of goods or more than two trading periods, the introduction of new security can make all investors worse off. If there are only one kind of good and one trading period, the innovation of a new security leads to a little better result.

**Lemma 3.6.2.**

Suppose that there are one kind of good and one trading period of the securities, and that the investors have symmetric information. Then, the investors' welfare level in the equilibrium after the innovation of additional securities will not be Pareto-dominated by that before the innovation of these additional securities.

When the investors have differential information, however, even with only one kind of good and one trading period, the innovation of the additional securities can still make all investors worse off. The following example shows this claim:

**Example 3.6.1: Welfare Reduction Due to Security-Innovation — A Counterpart of Hart's Example —**

Consider an economy which is similar to that in Example 3.5.1., but whose investors receive noisy information about the state of the world. That is, uncertainty is described by two risk factors \( \bar{Z} = (\bar{z}_1, \bar{z}_2)^T \sim N(0, I_2) \) where \( I_2 \) is the \( 2 \times 2 \)-identity matrix. Investor 1 (resp. 2) receives a signal \( \bar{s}_1 = \bar{z}_1 + \bar{\varepsilon}_1 \) (resp. \( \bar{s}_2 = \bar{z}_2 + \bar{\varepsilon}_2 \)) at date 1, but is uninformed about the risk factor \( \bar{z}_2 \) (resp. \( \bar{z}_1 \)). We assume that \( \bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2)^T \sim N(0, \sigma^2 I_2) \). Investor 1 (resp. 2) has the random endowment \( \bar{\varepsilon}_{1n} \) (resp. \( \bar{\varepsilon}_{2n} \)) \( \sim N(0, \eta^2) \) of the risk factor \( n \) (\( n = 1, 2 \)) with the distribution
\( \tilde{e}_1 = (\tilde{e}_{11}, \tilde{e}_{12})^T \sim N(0, \eta^2 I_2) \) and \( \tilde{e}_2 = (\tilde{e}_{21}, \tilde{e}_{22})^T \sim N(0, \eta^2 I_2) \). We assume that all random variables \((\tilde{Z}, \tilde{e}_1, \tilde{e}_2)\) are independent. Finally, investor \( k = 1, 2 \) has a CARA utility \( E[-\exp(-1/r x)] \) over the date 2 consumption \( x \).

Suppose that, originally, only a futures contract with payoff \( \tilde{f}_1 = (1/\sqrt{2})\tilde{z}_1 + (1/\sqrt{2})\tilde{z}_2 \) is available, and consider the innovation of an additional contract with payoff \( \tilde{f}_2 = a\tilde{z}_1 + b\tilde{z}_2 \) \((a^2 + b^2 = 1, (a, b) \neq \pm(1/\sqrt{2}, 1/\sqrt{2}))\). We can show that the introduction of such an additional contract may make all investors worse off for the following reason: When there is differential information, a security transmits information among the investors. Hence, with a new contract \( \tilde{f}_2 \), the public information that the futures prices transmit becomes more accurate than that with only one contract \( \tilde{f}_1 \). However, this release of more accurate public information may not be desirable to the investors, because it may deprive the investors of the risk-sharing opportunities, which otherwise they could have enjoyed. (That is, Hirschleifer effect.) If this negative effect of public information is large enough, welfare effects of the innovation of the additional futures contract is also negative to the investors.

Indeed, by taking \( r = 5, \eta = 4.5 \), and \( \sigma = 0.4 \), we can show that investor \( k \)'s ex-ante utility level with only \( \tilde{f}_1 \), \( U^{(1)} = -2.981 \), and with both \( \tilde{f}_1 \) and \( \tilde{f}_2 \), \( U^{(2)} = -4.269 \). That is, \( U^{(1)} > U^{(2)} \), and the innovation of an additional security makes all investors worse off in the ex-ante sense. \( \square \)

This result is a counterpart of Hart's (1975) example. The mechanism of the welfare reduction is similar in both cases: The introduction of a new security changes the equilibrium prices, which reduce the investors' risk-sharing opportunities. However, we have to emphasize the difference in the causes of these welfare reductions. In Hart's (1975) case, this reduction is due to the change of the spanning of the securities resulting from the change of the relative prices of the goods. On the other hand, in this example, the loss of the risk-sharing opportunities is due to informational reason i.e., the release of new public information by the new prices with the additional secu-
rities deprives the investors of the risk-sharing opportunities which, otherwise, they could have enjoyed.

The reduction of the welfare due to the release of public information and the loss of the risk-sharing opportunities has been considered by many including Hirschleifer (1971), and Milne–Shefrin (1987) with the exogenously given securities. However, to my knowledge, this is the first example to show that the change of the security-structure causes the change of available public information, and hence can make the investors worse off, exactly as Hart (1975) shows in the case of symmetric information.

3.6.2 Inefficiency of Security-Innovation

When the investors have symmetric information, if the securities $F$ complete the markets, and there exists an equilibrium given $F$, the equilibrium allocation is Pareto-efficient. As long as each security has the same price as this equilibrium price with the complete markets, we can eliminate the untraded securities, and still can implement this Pareto-optimal allocation in an equilibrium with only the traded securities. Hence, under symmetric information, creating all traded securities is sufficient for efficiency. This result supports, in some sense, a naive conjecture that as long as all traded securities are created, the investors can attain the efficient allocation.

When the investors have differential information, this naive argument need not hold. Since the security-price can transmit information among the investors, even the innovation of an untraded security can improve the investors’ welfare. However, such a security tends not to be created, because it is generally difficult for the private security-innovators to benefit from creating untraded securities. Consequently, the security-innovation that Pareto-improves the investors’ welfare can be missed, and the innovation can be inefficient if the security-innovators seek their own private benefits. This type of inefficiency can easily occur especially in an economy with production or several goods. (Although these cases deviate from the assumptions in this paper,
it should be noted that our characterization of possibly created securities can be easily extended to accommodate these cases with production and several goods.) The following examples illustrate this point:

**Example 3.6.2: Inefficient Security-Innovation with Production**

Consider an economy in which there are two types of producers 1 and 2. Both producers has the same initial endowments, the identical risk-averse preferences, the same constant to scale production technology of the good with return \( \tilde{z}_1 + \tilde{z}_2 \) per unit of input. However, they are different in information. That is, investor 1 (resp. 2) receives private signal about the realization of \( \tilde{z}_1 \) (resp. \( \tilde{z}_2 \)) before input, but is uninformed about \( \tilde{z}_2 \) (resp. \( \tilde{z}_1 \)).

Now, suppose that securities are created. Then, any security whose payoff is a function of \((\tilde{z}_1, \tilde{z}_2)\) transmits and symmetrizes the information that the producers have. Since the producers are different only in information, it follows that no security is traded between these producers. However, creating such securities may improve the welfare of the producers by transmitting the more accurate information and facilitating the production. □

**Example 3.6.3: Inefficient Security-Innovation with Two Goods**

Consider an economy in which there are two kinds of physically distinct goods. There are two dates 0 and 1. Consumption occurs only at date 1. There are two consumers 1 and 2. Each consumer \( k \) (\( k = 1, 2 \)) has the initial endowment of the goods \( e_k \in R^2 \), and \( k \)'s preference is represented by a utility function \( u_k : R^2 \to R \), which is strictly concave. There is no intrinsic risk i.e., \( e_k \) and \( u_k \) are deterministic for each \( k \). We assume that there are spot markets for goods 1 and 2 at date 1, and that only financial securities i.e., the securities that pay units of account at date 1, can

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\(^7\)I thank Atsushi Kajii and Jiang Wang for their comments about these possibilities.
be created and traded at date 0. We assume that given \((u_k, c_k)_{k=1,2}\), there are three distinct equilibria for the spot markets of good 1 and 2, whose prices of good 2 relative to good 1 and allocation of the goods are given by \((p_1, (x_{11}, x_{12}))\), \((p_2, (x_{21}, x_{22}))\), and \((p_3, (x_{31}, x_{32}))\) where \(p_i, x_{ik} \in \mathbb{R}^2\).

Now, consider the extrinsic risk or sunspots i.e., a publicly observed random event that is irrelevant to the payoffs i.e., the initial endowments and the utilities, and assume that this sunspot signal takes the realization \((h, t)\) or \((\text{head, tail})\) with probability \(q\), and \(1 - q\). Suppose that the equilibria realizing at date 1 are affected by the sunspots i.e., \((p_1, (x_{11}, x_{12}))\) emerges when \(h\) occurs and \((p_2, (x_{21}, x_{22}))\) emerges when \(t\) occurs. Note that from the date 0 viewpoint, this allocation is Pareto-inefficient. (We can confirm this claim by calculating the expected utility w.r.t. the sunspot events \(q\) and \(1 - q\), applying Jensen's inequality to the strictly concave utility functions, and using the fact that the convex combination of feasible allocation is feasible.) Thus, for the equilibrium to be Pareto-efficient in the ex-ante sense, the equilibrium should be independent of the sunspots.

Now, consider the innovation of financial securities that span the sunspots. (That is, the securities that enable, at date 0, the consumers to transfer the wealth between the event \(h\) and \(t\). One can imagine the innovation of two contingent commodities that pay 1 unit of goods 1 depending on which of \(h\) or \(t\) occurs. Kajii (1991) considers the innovation of option on the relative price of good 2 w.r.t. good 1.) With these securities, the markets are dynamically complete in Arrow's sense, and by the first fundamental theorem, the equilibrium allocation should be Pareto-efficient at date 0. Hence, at date 1, independently of the sunspots, only one of the three equilibria should emerge.

However, such securities will never be traded in an equilibrium, since there is no need for the securities that span the sunspot events to attain the equilibrium which is independent of the sunspots. Hence, these securities can never be created by any
security-innovator that seeks his own private benefit, although such innovation is
desirable for the consumers to avoid inefficient allocation.

The idea of Example 3.6.3. is from Kajii (1991), which analyzes the welfare effects
of the innovation of options in the more general set-up. This example, however,
points out that, despite the desirable role that these securities play for efficiency,
such securities may be unlikely to be created as long as the securities are created by
the private innovators for their own private benefits.

These results suggests the need of appropriate regulations for the security-innovation
to be efficient. Since inefficient innovation can occur under any private innovation
mechanism, intervention by the authority in the financial innovation may be justified
in some circumstances under differential information.

3.7 Concluding Remarks

(1) In the real world, securities may not be created though they can be traded. To
analyze this moderate, but more interesting, case, we have to consider more
explicit mechanism of securities innovation. For the case of futures innovation,
it is, for example, the maximization of the trading volume or some functions of
the transactions. One may prefer to interpret such criteria as an approximation
of some transaction fee maximizing behavior. It seems that only a quadratic fee
schedule provides an analytically tractable model that is consistent with this
interpretation, if we define the innovators' behavior as the limit of decreasing
positive fee charge. (See Ōhashi (1992)). Under the assumption that the innova-
tor (futures exchange) creates securities (futures contracts) to maximize the
expected sum of the square trading volume, Ōhashi (1993) conducts a system-
atic analysis of the endogenous determination of the market incompleteness.

(2) We point out that the presence of private information may prevent the security-
innovator from creating the securities that would allow the investors to attain ex-ante optimal allocation if the ex-ante transaction were possible. In the more moderate situation where an informed agent receives a noisy signal about the realization of the asset’s return, Demange-Laroque (1992) studies a similar question. They show that the informed agent, who creates the security to hedge the risk which he is informed of, designs a security so that its payoff contains noise. This is because such design of the security mitigates the adverse selection against the uninformed investor, and facilitates the transaction of the security. Consequently, the created security, which we are likely to observe, is different from the one that would allow the investors to attain the ex-ante optimal allocation if it were not for the private information.

(3) We analyze an exchange economy with only one good and one trading date. As we have seen in the end of section 3.6, the deviation from this restrictive setting enriches the predistion of the model. With some exception (Rahi (1992) considers the security-innovation in a production economy), few work has been done. Further research in this direction is needed.
3.8 Appendix 3

Proof of Proposition 3.4.1.

By assumption, \((\forall k)\) \(U_k(e_k + y_k^\top F \mid G_k) \geq U_k(e_k \mid G_k)\). Define \(\hat{y}_k = (1/2)y_k\) for all \(k\).

Then, by concavity of the utility, \((\forall k)\)

\[
U_k(e_k + \hat{y}_k^\top F \mid G_k) = E[u_k(e_k + \hat{y}_k^\top F) \mid G_k]
\]

\[
\geq (1/2) E[u_k(e_k + y_k^\top F) \mid G_k] + (1/2) E[u_k(e_k) \mid G_k]
\]

\[
\geq E[u_k(e_k) \mid G_k] = U_k(e_k \mid G_k)
\]

But, since \(Pr(\{\sum_{k=1}^K |y_k| > 0\}) > 0\), \((\exists k)\) \(A' \equiv \{ |y_{k'}| > 0 \} \subseteq G_k\), \(Pr(A') > 0\), and

\[
U_{k'}(1_{A'}[e_{k'} + \hat{y}_{k'}^\top F] \mid G_{k'})
\]

\[
> (1/2) U_{k'}(1_{A'}[e_{k'} + y_{k'}^\top F] \mid G_{k'}) + (1/2) U_{k'}(1_{A'}e_{k'} \mid G_{k'})
\]

\[
\geq U_1(1_{A'}e_{k'} \mid G_{k'})
\]

Hence, \(e\) is not interim constrained optimal w.r.t. \(F\) and \(G\). \(\square\)

Proof of Proposition 3.4.2.

We only have to show that, given the assumption of market completeness, ex-ante constrained optimality w.r.t. \(F\) implies interim constrained optimality w.r.t. \(F\) and \(G\) for any information distribution \(G\).

Now, given \(F\) and \(G\), suppose that \(e\) is not interim constrained optimal w.r.t. \(F\) and \(G\). Then, by definition of interim constrained optimality, and by taking the expectations, we have

\((\forall k)\) \(U_k(x_k + y_k^\top F) \geq U_k(x_k)\), and

\((\exists k')\) \(U_{k'}(x_{k'} + y_{k'}^\top F) > U_{k'}(x_{k'})\)

Since \((\forall k)y_k^\top F \in X\), we can find, for \(k = 2, \ldots, K\), \(\bar{y}_k \in F_o\), \(\bar{y}_k^\top F = y_k^\top F\). Define \(\bar{y}_1 \equiv -\sum_{k=2}^K \bar{y}_k\). Then, \(\bar{y}_1^\top F = -\sum_{k=2}^K \bar{y}_k^\top F = -\sum_{k=2}^K y_k^\top F = y_1^\top F\) a.e. Substituting \(\bar{y}_k^\top F = y_k^\top F\) for all \(k\) in the inequalities above, we prove \(e\) is not ex-ante constrained optimal w.r.t. \(F\). \(\square\)
Proof of Proposition 3.5.1.

Assume that \( \text{Rank}[\Theta] \geq N \). Then, for each \( n \in N \), we can always create a portfolio with payoff equal to \( \tilde{z}_n \). Let \( q_n \) be the value of the portfolio with the payoff \( \tilde{z}_n \). Since consumer \( k_n \) knows the realization \( z_n \) of \( \tilde{z}_n \), without arbitrage, \( q_n = z_n \) for all \( n \). But, this implies that \( \sigma(P) = \mathcal{F} \) i.e., the security prices transmits the all the information and there is no uncertainty left at interim stage. Hence, given a security structure with \( \text{Rank}[\Theta] \geq N \), the initial endowments \( e \) is interim constrained optimal w.r.t. \( F \) and \( G \), and there is no trade. Thus, such securities are not innovatable. \( \Box \)

Proof of Lemma 3.6.1.

Suppose that a set of securities \( F \) is possibly created. Then, by Definition 3.2.2, there exists a market equilibrium \((y, P)\) w.r.t. \( F \) s.t. \( U_k(e_k + y_k^T F \mid G_k) \geq U_k(e_k \mid G_k) \) for all \( k \). Taking expectations, we get \( U_k(e_k + y_k^T F) \geq U_k(e_k) \) for all \( k \). \( \Box \)

Proof of Lemma 3.6.2.

Let \( F \) be the security-structure before the innovation of additional securities, and \( F' \) be the security-structure after the innovation of the additional securities. If \( F^{\text{add}} \) is the set of these additional securities, \( F' = [F, F^{\text{add}}] \). Define \( F_{-0} \) such that \( F = [f_0, F_{-0}] \), and \( F' = [f_0, F'_{-0}] \). Let \( P \) be the equilibrium prices corresponding to \( F \), and \( P' \) corresponding to \( F' \). Also, let \( y_k \) be investor \( k \)'s equilibrium security net trade corresponding to \( F \), and \( y'_k \) corresponding to \( F' \). We define that \( P_{-0} \) such that \( P = [p_0, P_{-0}], P'_{-0} \) such that \( P' = [p_0, P_{-0}], y_{-0k} \) such that \( y_k = [y_{0k}, y_{-0k}] \), and \( y'_{-0k} \) such that \( y'_k = [y'_{0k}, y'_{-0k}] \). Finally define \( P'^{-\text{add}} \) and \( P^{\text{add}} \) such that \( P' = [P'^{-\text{add}}, P^{\text{add}}] \) where \( P^{\text{add}} \) is the price of \( F^{\text{add}} \).

Now, suppose that the equilibrium allocation with \( F \) Pareto-dominates that with \( F' \) i.e.,

\[
(\forall k) \ U_k(e_k + y_k^T F) \geq U_k(e_k + y'_k^T F')
\]

\[
(\exists k') \ U_{k'}(e_{k'} + y_{k'}^T F) > U_{k'}(e_{k'} + y'_{k'}^T F')
\]

Note that under symmetric information, \( G_k = \mathcal{F}_0 \) for all \( k \). Since \( P'_{-0} y'_k = 0 \) i.e., \( y_0 = -P_{-0} y_{-0k} \), we have
\[ U_k(e_k + y_k^T F') \geq U_k(e_k + y_{-0k}^T (F_{-0}' - P_{-0}')) \geq U_k(e_k + y_{-0k}^T (F_{-0}' - P_{-0}')) \]
\[ = U_k(e_k + y_{-0k}^T (F_{-0}' - P_{-0}) + y_k^T (P_{-0}' - P_{-0})) \]
\[
\text{Since } y_k^T (P_{-0}' - P_{-0}) \in R, \text{ and } U_k(\cdot) \text{ is monotonically increasing, for the above inequality to be true, for all } k \neq k' \ y_k^T (P_{-0}' - P_{-0}) \leq 0. \text{ But, by the market clearing condition } \sum_k y_k = 0, \text{ this implies that } y_k^T (P_{-0}' - P_{-0}) = -\sum_{k \neq k'} y_k^T (P_{-0}' - P_{-0}) \geq 0. \text{ By monotonicity of the utility function, this contradicts the assumption that } (\exists k') U_{k'}(e_{k'} + y_{k'}^T F') > U_{k'}(e_{k'} + y_{k'}^T F') \text{. Hence, the welfare associated with the equilibrium allocation under } F' \text{ will not Pareto-dominated by that under } F. \quad \square

\textbf{Proof of the Claim in Example 3.6.1.}

When only a futures contract with payoff \( \tilde{f}_1 = (1/\sqrt{2})\tilde{z}_1 + (1/\sqrt{2})\tilde{z}_2 \) is created, there exists a noisy rational expectation equilibrium (NREE) with a price \( \tilde{p} = \alpha[\tilde{s}_1 + \tilde{s}_2 + l_1(\tilde{\varepsilon}_{11} + \tilde{\varepsilon}_{22}) + l_2(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21})] \) where \( \alpha = \frac{2(2+\sigma^2)+rl_i}{4\sqrt{2}(1+\sigma^2)}, \quad l_i = (-2\sigma^2)/r, \) and \( l_2 \) is the solution of \( r\eta^2l^3 + 2(1+\sigma^2)\eta^2l^2 + r(1+\sigma^2+l_i^2\eta^2)l^2 + 2(1+\sigma^2)(\sigma^2 + l_i^2\eta^2) = 0. \)

Assuming that \( r > \eta, \) the ex-ante utility level of both investors are given by

\[
U_k^{(1)} = -\sqrt{\frac{1}{2V_1} \left( \frac{\sigma^2}{1+\sigma^2} + \frac{\sigma^2 + l_i^2\eta^2 + l_2^2\eta^2}{1+\sigma^2 + l_i^2\eta^2 + l_2^2\eta^2} \right)} \times \frac{1}{\eta^2 \sqrt{A_1 C_1 - B_1^2}}
\]

where \( A_1 = (1/\eta^2) - (1/r^2) + (1/V_1)\left\{l_1\alpha + (1/r)(1/\sqrt{2} - \alpha)\right\}^2, \)

\[ C_1 = (1/\eta^2) - (1/r^2) + (1/V_1)\left\{l_2\alpha + (1/r)(1/\sqrt{2} - \alpha)\right\}^2, \]

\[ B_1 = (1/V_1)\left\{l_2\alpha + (1/r)(1/\sqrt{2} - \alpha)\right\} \left\{l_2\alpha + (1/r)(1/\sqrt{2} - \alpha)\right\}, \]

\[ V_1 = (\sigma^2 - rl_2/2)^2/\left\{4(1+\sigma^2)^2 + \alpha^2(2\sigma^2 + (l_i^2 + l_2^2)\eta^2) \right\}^2 \]

When, additionally, a futures contract with payoff \( \tilde{f}_2 = a\tilde{z}_1 + b\tilde{z}_2 \) (\( a^2 + b^2 = 1 \)) and \( a \neq 1/\sqrt{2} \) is created, the NREE-prices of \( \tilde{p}_1, \tilde{p}_2 \) of the contracts \( \tilde{f}_1, \tilde{f}_2 \) are given by \( \tilde{p}_1 = (1/\sqrt{2})\tilde{q}_1 + (1/\sqrt{2})\tilde{q}_2, \tilde{p}_2 = a\tilde{q}_1 + b\tilde{q}_2 \) where

\[
\tilde{q}_1 = h_1(\tilde{s}_1 + l_1\tilde{\varepsilon}_{11}) + h_2\tilde{\varepsilon}_{21}, \quad \tilde{q}_2 = h_1(\tilde{s}_2 + l_1\tilde{\varepsilon}_{22}) + h_2\tilde{\varepsilon}_{12},
\]

\[
h_1 = \frac{2\sigma^2 + l_i^2\eta^2}{\sigma^2 (1+\sigma^2 + l_i^2\eta^2) + (1+\sigma^2)(\sigma^2 + l_i^2\eta^2)}, \quad \text{and}
\]

\[
h_2 = \frac{-2\sigma^2 (\sigma^2 + l_i^2\eta^2)}{r(\sigma^2 (1+\sigma^2 + l_i^2\eta^2) + (1+\sigma^2)(\sigma^2 + l_i^2\eta^2)).}
\]
Under the assumption that $r > \eta$, the ex-ante utility level of both investors are given by

$$U_k^{(2)} = -\sqrt{\frac{\sigma^2 (\sigma^2 + l_1^2 \eta^2)}{V_1 (1 + \sigma^2) (1 + \sigma^2 + l_1^2 \eta^2)}} \times \frac{1}{\eta^2 \sqrt{A_1 C_1}}$$

where $A_2 = (1/\eta^2) - (1/r^2) + \frac{(h_1 l_1 + (1/r)(1-h_1))^2}{(1-h_1)^2 + h_1^2 \sigma^2 + h_2^2 \eta^2}$,

$C_2 = (1/\eta^2) - (1/r^2) + \frac{(h_2 + (1/r)(1-h_1))^2}{(1-h_1)^2 + h_1^2 \sigma^2 + h_2^2 l_1^2 \eta^2}$, and

$V_2 = \{(1-h_1)^2 + h_1^2 \sigma^2 + h_2^2 \eta^2\} \{(1-h_1)^2 + h_1^2 \sigma^2 + h_2^2 l_1^2 \eta^2\}$.

The claim follows from direct computation. \(\square\)
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