Optimal Sales Strategies in Stochastic, Dynamic Environments

by

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Submitted to the Department of Electrical Engineering and Computer Science
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Abstract

In this thesis we study sales strategies that managers in service industries use when facing stochastic, dynamic environments. We analyze the application of three methods that are frequently used for matching supply and demand: downgrading, direct marketing and pricing strategies. Each of these methods is studied within the framework of a particular industry.

Chapter 1 studies optimal strategies for selling perishable products to various types of customers when all products perish at the same time (e.g., hotel rooms, restaurant tables, rental cars). Every time a customer requests a product, the manager has to decide whether or not to sell a product to that customer. When making this decision, the manager does not know how many additional customers will arrive, and whether among those who arrive there will be some who pay a higher price for the product currently being requested. We consider product downgrading, reservations before the planning horizon, and stochastic and dynamic arrival of customers with no particular order in the arrival of different types of customers.

Chapter 2 studies optimal mailing policies in the catalog sales industry when there is limited access to capital. An important factor that determines the total demand is the customers' response rate, which evolves endogenously. Initially it is low since firms are forced to rely on lists of names they rent. As time goes by, firms rely more and more on the names of customers that have bought its products, the house list, since the number of such customers is growing and their response rate is higher. It follows that companies in these industries typically lose money during their first years of operation and often go bankrupt. An appropriate cash flow management is therefore crucial to survive in this industry. We consider a stochastic environment given by the random responses of customers and a dynamic evolution of the customers in the house list.

Finally, Chapter 3 studies intertemporal pricing strategies when selling a perishable product. We consider a seller that faces a stochastic arrival of customers with heterogeneous valuations of the product. We analyze several pricing policies, as for example periodic pricing reviews and monotonically decreasing price path constraints.
We use the following general framework to study the application of the three sales strategies mentioned above. We first formulate the problems as stochastic dynamic models. The common uncertainty in the problems is given by the stochastic demand; managers must make their decisions knowing only the probability distributions for the number of requests. Secondly, when it is possible, we characterize the optimal policies. The qualitative properties of the optimal solutions allow us to develop efficient algorithms to find them and to gain useful insights for developing heuristics when exact solutions cannot be computed. Then, we solve the mathematical formulation; we either find the optimal policies or develop ad-hoc heuristics that lead to "good" approximations for the optimal solution. With the purpose of measuring the performance of the heuristics, we obtain upper bounds for the value of the objective functions. Finally, we use Monte Carlo simulations to study the heuristics' behavior in a wide variety of realistic scenarios.

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Introduction

One important factor that differentiates service industries from traditional manufacturing industries, is that in the service industry, the provision and delivery of the product occur simultaneously. Usually, services are produced and consumed at the same time; for example, hotel rooms have to be available when travelers arrive, receptionists have to receive telephone orders when clients want to buy, and airline seats must be empty when customers want to fly.

The impossibility of inventoring services leaves managers without the important buffer that is available for manufacturing managers to absorb fluctuations in demand. This inherent feature of services produces two main consequences. First, managers in service organizations are often faced with situations where their facilities are idle for long periods of time. For example, city hotels have a low occupancy rate during weekends, telephone operators receive few calls after midnight, and air conditioner technicians are rarely requested in winter. On the other hand, a queuing phenomenon is usually observed during peak times. For example, busy technical support lines at computer companies during working hours, waiting lines at restaurants on Saturday nights, and crowded retail stores during the Christmas season. Thus, managing supply and demand is a key task of the service managers. As stated in Sasser (1976), “when service managers plan rather than react, they can successfully fit their capacity to the demand for their products.”

There are two basic strategies for matching supply and demand: controlling supply and altering demand. The challenge for the service manager is to determine which is the combination of these strategies that best fits her company. Furthermore, she must choose among a variety of ways in which to implement these strategies. Some important alternatives are now described.

CONTROLLING SUPPLY

Managers have a direct influence in adjusting the supply of services. There are several
methods that can be used to adjust capacity to fluctuations in demand.

1. Floating Staff and Part-Time Employees: This alternative is useful to adjust for peak demand. The nature of peak demand varies with the type of business: it may correspond to certain hours of the day (telephone companies, subways), certain days during the week (restaurants), certain weeks during the month (banks) or certain months during the year (resorts, Christmas sales).

2. Cross-Training: This approach increases worker flexibility. Workers are trained so that they can work in bottleneck tasks during peak times and shift to their regular activities at off-peak times.

3. Modular Facility Design: This provides the ability to manage the capacity of the service facility. During off-peak times, parts of the plant can be closed, thereby reducing operational costs.

4. Sharing Capacity with Another Business: This is useful when there is expensive, underused equipment.

5. Extending Business Hours: This is used to meet growing demand without expanding the physical capacity. It is an attractive alternative when increases in demand are temporary.

**Altering Demand**

A proactive management of demand tends to reduce the degree of uncertainty in the number of requests. Thus, several techniques have been developed to make demand more predictable.

1. Reservation Systems: These systems presell the productive capacity of the service facility. Managers use them to shift excess demand by moving it to another period of time. This method is frequently used in restaurants, hotels,
rental cars companies, and hairdressers. The major problem present in reservation systems is that of "no-shows". Customers often make reservations that they do not use, and many times they do not incur any cost when failing to honor a reservation. Thus, service companies usually overbook their facilities, running the risk of having to turn down paying customers.

2. **PRICING STRATEGIES:** These are frequently used to shift demand from one period to another. Managers use a differential pricing scheme to encourage people to use the service facility at off-peak times. Examples include hotels, airlines, public facilities, matinee prices for movies and weekend and night rates for long-distance calls.

3. **INFORMATION SYSTEMS FOR CUSTOMERS:** For example, Fidelity Investments brokerage provides to its customers the distribution of calls received during the day so that they can adjust their calling times accordingly.

4. **DIRECT MARKETING:** For example, firms may use catalog sales to influence demand by controlling the timing and the quantity of their catalog mailings.

5. **PROMOTIONS:** These are used to boost demand, often during off-peak times. For example, hotels in cities offer mini-vacation packages on low demand periods such as weekends, and travel agencies offer special vacation packages during off-seasons.

6. **DOWNGRADING PRODUCTS:** Managers use this mechanism to meet fluctuations in demand that cannot be predicted. For example, the receptionist in a hotel can downgrade a suite to satisfy a request for a standard room, when the costs of turning down the customer is higher than the opportunity cost of the suite.

7. **PREVENTIVE MAINTENANCE:** This is used to reduce the demand during peak times. For example, companies that repair heat systems can do preventive maintenance in summer to reduce the number of break-offs during the winter time.
In recent years the quality of services has become an increasingly important factor in the service industry. More sophisticated customers and increased competitiveness have forced managers to focus on how they make short term operational decisions. These decisions now compare in importance with those made at a tactical and strategic level.

In this thesis we study optimal sales strategies in stochastic, dynamic environments. We analyze the application of three methods that are frequently used in the service industry to match supply and demand. The methods considered are downgrading, direct marketing and pricing strategies. Each of these methods is studied within the framework of a particular industry.

We use the following general methodology to study the application of the three methods mentioned above. We first model the problems using stochastic dynamic programming, where the stages correspond to the periods in the planning horizon at which decisions are made. The common uncertainty in the problems is given by the stochastic demand; managers must make their decisions knowing only the probability distributions for the number of requests. Secondly, when it is possible, we characterize the optimal policies. The qualitative properties of the optimal solutions allow us to develop efficient algorithms to find them and to gain useful insights for developing heuristics when exact solutions cannot be computed. The third step consists of solving the mathematical formulation; we either find the optimal policies or develop ad-hoc heuristics that lead to "good" approximations for the optimal solution. With the purpose of measuring the performance of the heuristics, we obtain upper bounds for the value of the objective functions. Finally, we use Monte Carlo simulations to study the heuristics' behavior in a wide variety of scenarios. Simulations are run with realistic data based on information given by managers in the industries under study or public domain data.

Chapter 1 studies the operational problem that arises in an industry where the producer facing a stochastic, dynamic demand sells a perishable good to customers paying different prices. For concreteness, we illustrate the problem considered in this
chapter by applying it to the hotel industry. Hotel managers must determine the optimal policy for renting rooms to various types of customers, taking into account that products perish at the same time; rooms that are not rented have a zero opportunity cost. Customers differ in the prices they pay for a given room (airline employees, tourists, etc.). Every time a customer requests a room, the manager has to decide whether or not to rent a room to that customer. When making this decision, the manager does not know how many additional customers will arrive that day, and whether among those who arrive there will be some who pay a higher price for the room currently being requested. The approach we take in this chapter focuses on the dynamic and stochastic nature of demand described above. Special emphasis is given to the manager’s ability to downgrade rooms, i.e., the manager may decide to give a customer a suite for the price of a standard room when the latter is not available.

The optimal policy is characterized by a collection of capacity threshold vectors that evolves over time. It is optimal to satisfy a customer’s request as long as the capacity vector at the moment the request is made has all components larger than or equal to at least one of the vectors in the current set of threshold capacities. Furthermore, the collection of capacity threshold vectors evolves over time in such a way that, for every class of customers and every capacity vector, there exists an instant in time beyond which it is optimal to satisfy the customer’s request.

Implementations of the optimal policy are also considered. The properties we derive for the optimal policy reduce significantly the computational effort needed to solve the problem, yet in the multiproduct case this is often not enough. Therefore, heuristics are developed to find good approximations for the optimal solution. Computational experiments show a satisfactory performance of the heuristics with respect to an upper bound for the optimal solution of the general optimization problem.

Chapter 2 studies mailing strategies in the catalog sales industry. The demand faced by a company in this industry is determined by the timing and the number of catalogs mailed, and by the response rate to these mailings. The response rate evolves endogenously. Initially it is low since firms are forced to rely on lists of names they rent; renting these names is also expensive. As time goes by, firms rely more and more
on the names of customers that have bought their products, since the number of such customers is growing and their response rate is higher. It follows that companies in this industry typically lose money during their first years of operation. With limited access to credit and inappropriate mailing strategies this often leads to bankruptcy. The problem considered in this chapter is how a manager with limited access to capital should plan her mailing strategy so as to maximize her present discounted profits. The fact that the stochastic and dynamic demand faced by the manager is determined by the timing and the number of catalogs mailed during the season plays a central role.

Given the size of real problems, it is not possible to compute the optimal mailing strategies. Therefore, we develop ad-hoc heuristics based on the optimal solutions of simplified versions of the problem studied. The performance of these heuristics is evaluated by comparing their outcome with upper bounds derived for the original problem. Computational experiments show that these heuristics behave satisfactorily.

Finally, Chapter 3 studies optimal intertemporal pricing strategies for a perishable product in a retail store. For example, fashionable clothing and food products for special holidays usually fall in this category. We consider a seller that faces a stochastic arrival of customers with heterogeneous valuations of the product, i.e., customers have different thresholds for the maximum price that they are willing to pay for the product. In this context, pricing strategies are used to boost demand when revenue slumps.

We characterize the optimal pricing policies as follows: for every outcome of the arrival process with the corresponding reservation prices, the optimal price is a decreasing function of time with jumps when goods are sold. We find necessary and sufficient conditions for the optimal pricing strategy, which are satisfied for a large group of distribution functions for the reservation price. For these distribution functions the optimal pricing strategy can be easily computed solving the first order condition backwards in time. This model is extended to include periodic pricing reviews and a monotonically decreasing price path constraint. Finally, we generalize the model to the case of a company that has two retail stores oriented to two differ-
ent market segments. We develop heuristics to solve real size problems when optimal solutions cannot be computed; these heuristics have a satisfactory performance with respect to an upper bound we derive for the optimization problem.
Chapter 1

Optimal Policies for Selling Perishable Goods Under Uncertain Demand

This chapter determines the optimal policy for selling perishable products to various types of customers when all products perish at the same time (e.g., hotel rooms, restaurant tables, rental cars). Product downgrading is considered, reservations (made before the planning horizon) that evolve stochastically are incorporated, and no particular order in the arrival of different types of customers is assumed.

The optimal policy is characterized by a collection of capacity threshold vectors that evolves over time. It is optimal to satisfy a customer's request as long as the capacity vector at the moment the request is made has all components larger than or equal to at least one of the vectors in the current set of threshold capacities. Furthermore, the collection of capacity threshold vectors evolves over time in such a way that, for every class of customers and every capacity vector, there exists an instant in time beyond which it is optimal to satisfy the customer's request.

Implementations of the optimal policy are also considered. The properties we derive for the optimal policy reduce significantly the computational effort needed to solve the problem, yet in the multiproduct case this is often not enough. Therefore, heuristics are developed to find "good" solutions for the general optimization prob-
lem. Finally, a family of upper bounds are derived which are useful when evaluating the performance of the heuristics. Computational experiments show a satisfactory performance of the heuristics under a large variety of realistic scenarios.

1.1 Introduction

There are several interesting problems where a producer facing a stochastic, dynamic demand sells a perishable good to customers paying different prices. Customers may pay different prices for the same product because of many reasons, such as (i) the product can be used for different purposes, (ii) customers have a special agreement with the producer, and (iii) producers may sell the same good at different outlets. An additional common characteristic of many of these problems is that the good being sold is perishable (in the sense that its residual value is eventually equal to zero), with all units perishing at the same time. Thus the producer must choose a sales-strategy in an environment where demand is stochastic and dynamic over the planning horizon, and goods have a finite common lifetime. Every time a customer arrives, the producer must decide whether to sell the good or not.

The following are examples of problems that can be formulated within the framework described above: (i) hotels rent rooms to customers that pay different prices, because they have special discounts as for example airline employees, government workers, and executives. (ii) restaurants prefer larger groups since they use their available space more efficiently, and (iii) rental car companies face different types of demand (e.g., credit card discounts and special agreements with travel agencies).

Frequently, specially in the service industry, more than one product is offered and “downgrading” is possible. This is the case when there exists a natural ordering of all products, so that every product is an acceptable substitute for those that are “worse” than it. For instance, in the previous examples, hotel suites are good substitutes for standard rooms, larger cars are acceptable substitutes for smaller cars, and larger restaurant tables are substitutes for smaller ones.

Finally, an additional characteristic is sometimes present in problems like those
described above; producers may accept reservations for the products being sold and these reservations are non-binding for the customer. The resulting number of reservations that turn into sales is a random variable. If the total number of reservations is known at the beginning of the planning horizon, the number of customers with reservations that will show up can be seen as a particular type of stochastic demand. This type of demand is particularly important in the hotel industry, where a significant number of rooms is reserved prior to the target date.

In this chapter, we first determine the optimal policy for selling perishable products when there are many types of customers that request the same product at different prices. Then, we extend the results for the case of multiple types of products with the possibility of downgrading. Finally, we allow a given number of reservations at the beginning of the planning horizon. The arrivals of different types of customers is stochastic. We make no assumptions concerning the particular order between the arrivals of different types of customers. We determine whether, at a given moment in time, the producer should sell a product to a particular type of customer, and, should the answer be positive, what product should be sold.

Our main theoretic contribution is the derivation of the optimal policy for this problem, which has a number of interesting properties. It is characterized by a collection of capacity threshold vectors that evolves over time (every vector in the set has as many components as the number of products considered). It is optimal to satisfy a customer's request as long as the actual capacity vector at the moment the request is made has all components larger than or equal to at least one of the vectors in the current set of threshold capacities. Furthermore, the collection of capacity threshold vectors evolves over time in such a way that, for every class of customers and every actual capacity vector, there exists an instant in time beyond which it is optimal to satisfy the customer's request.

The computational effort required to solve the problem can be significantly reduced using the properties described above, making problems with one product tractable. However, when real size problems with several types of products are considered, the computational burden becomes enormous even using this approach. Ad-hoc
heuristics are developed to handle those cases.

The remainder of this chapter is organized as follows. In Section 1.2 we present the literature review. In Section 1.3 we introduce a dynamic programming formulation for the general problem described above. We also characterize the optimal policy for selling products in this section. In Section 1.4 we find a family of upper bounds for the optimal solution. These bounds are useful when comparing various heuristics to solve the general problem. In Section 1.5 we describe the heuristics developed to solve the general optimization problem. Section 1.6 contains the computational experiments that show the performance of the different heuristics. Finally, in Section 1.7, we present the conclusions.

1.2 Literature Review

This chapter is motivated by the general problem of managing reservations and sales in the service industry. Decisions are made at two levels when solving this problem. The first level is tactical: the producer must decide the maximum number of reservations that she accepts at a given moment in time for a particular target date,¹ i.e., given that the producer has already accepted a number of reservations at a certain point in time and considering future stochastic requests and cancellations, she must decide whether or not to accept a specific reservation request. This problem is closely related with the second level at which decisions are made, namely the operational problem. The operational problem is the one described in the Introduction, i.e., during the target date (or planning horizon) the producer must decide whether or not to sell a product to a specific customer, considering the number of reservations made at the tactical level. Most of the literature that studies the tactical problem considers simplified dynamic relationships about what happens during the target date, for example, some assume that producers have perfect information on arrivals during the target date while others suppose that customers show up in a specific sequence (e.g., customers that pay less show up first).

¹The target date is the time period during which the product is sold.
Among the papers that consider the problem at a tactical level, Ladany (1976) proposes a dynamic programming formulation for managing reservations in the hotel industry; at each stage a random number of reservation requests and cancellations are received. He assumes that during the target day, all the customers arrive at the same time. Hence, the manager can do perfect price discrimination at the operational level. Alstrup et. al. (1986) studies the booking policy for a single flight leg with two types of customers. He also considers that during the target date, all customers arrive together. Bitran and Gilbert (1992) extend Ladany’s analysis to the case where customers do not arrive at the same time, assuming a specific order of arrival among various types of customers instead.

There has not been much work considering the problem at the operational level. Littlewood (1972) analyzes the problem of selling seats in the airline industry. He determines the maximum number of seats that can be sold to customers paying lower fares in order to maximize the expected profit; he assumes two classes of customers where discount customers book their seats first. Optimization is done over a single flight leg. Littlewood’s results have been extended in various directions: Belobaba (1989) considers an arbitrary number of fare classes, Brumelle et al. (1990) analyze separately the cases of dependent demand (the demand of discount fare customers gives information about the future demand of full fare customers), goodwill cost and spill rate (the proportion of full fare customers that is turned down), and upgrading (a discount fare customer decides to pay the full fare if there is no available seats).

All the models mentioned in the previous paragraph assume that different types of customers arrive in a preestablished order. Topkis (1968) considers the case where \( n \) demand classes of varying importance request the same single product and determines the optimal ordering policy and the optimal rationing policy between two reordering times. No specific order between the arrivals of customers of different types is assumed. His model is a dynamic programming formulation with continuous decision variables that correspond to the fraction of the demand in each class that is satisfied at each period of time and the amount of demand that is backlogged or lost. Our analysis goes beyond Topkis’ paper by considering many products with the possibility
of downgrading, and by incorporating reservations as a particular type of demand at the operational decision level. We also consider binary decision variables: a request is either accepted or rejected.

1.3 Mathematical Formulation

In this section we present the mathematical formulation of the problem of finding the optimal policy for selling perishable products to various classes of customers. We specify three different models, which vary in the degree of complexity in terms of the number of products and customers considered. The first model considers multiple types of customers that request a single product at different prices. The second model incorporates multiple products with the possibility of downgrading. Finally, we extend the latter model to the case where reservations are allowed prior to the planning horizon.

The general model is a dynamic and stochastic programming formulation, where the stages correspond to the periods in which the planning horizon is divided. The state of the system, at a given time, is determined by the remaining vector of capacities, the type of customer requesting the product, and the number of pending reservations if the producer accepts reservations prior to the planning horizon. We divide the planning horizon into time intervals small enough so that we can assume that the number of arrivals in each interval is either zero or one. For the case where reservations are not allowed, we assume that the demand in a time interval is independent of previous demands. An example of this type of demand is the non homogeneous Poisson arrival process. However, when we incorporate reservations, the demand associated to a customer type with reservations depends on all the previous requests in that segment of clients. For instance, if at a given period of time, all the customers with reservations have already arrived, the demand for this class of customers is zero from that period onwards. A natural model for the demand associated to a customer type with reservations is a Binomial distribution where the number of trials is equal to the total number of pending reservations.
Products are classified into ordered types \( s \in \{1, \ldots, m\} \), where product 1 is the "best product" (product 1 can substitute all other products) and \( m \) is the "worst." There are \( n \) classes of customers; in the most general case, a class is determined by the product type requested by the customer, the price that she pays, the rejection cost,\(^2\) and whether or not the customer has a reservation. Several classes of customers can request the same type of product. Therefore, we define \( A_s \) as the set of all types of customers that request product \( s \). Before presenting the mathematical models, we introduce the following notation:

1. \( \pi_i \): Price associated to a class \( i \) customer.

2. \( c_i \): Cost of rejecting a class \( i \) customer.

3. \( C \): Initial capacity or inventory. In the case of multiple products, \( C \) is an \( m \)-dimensional vector with \( C(s) \) equal to the number of available class \( s \) products.

4. \( T \): Planning horizon.

5. \( R \): Vector with pending reservations for each class of customer.

6. \( e_p \): Vector with a one in the \( p^{th} \) position and zeros elsewhere.

7. \( E_x[f(x, y)] \): Expected value of \( f(x, y) \) with respect to the random variable \( x \).

### 1.3.1 Model without reservations

In what follows, we analyze the case where reservations are not allowed. We first consider a single product, and later we extend the results to the multiple product case where downgrading is possible.

\(^2\)This is the cost of turning down the customer. In general, this cost is difficult to quantify because it is a combination of monetary and non-monetary costs. For example, if a customer with a reservation for an hotel room is turned down, the rejection cost may be the monetary cost of allocating that customer to another hotel combined with the non-monetary cost associated with the loss in trustworthiness for failing to honor a reservation.
Single product case

We define the function $F_t(c, i)$ as the maximum expected profit from period $t$ onwards if there are $c$ available products and a class $i$ customer requests a product at time $t$ ($i = 0$ corresponds to no arrivals). Without loss of generality we assume that the planning horizon is divided in time intervals of length equal to one. Therefore, the mathematical formulation corresponds to:

**Problem SP($C$):**

$$E_t[F_0(C, i)],$$

where:

1) If $c > 0$, $i \neq 0$:

$$F_t(c, i) = \max \begin{cases} \text{Sell a product:} & \pi_i + E_{j}[F_{t+1}(c - 1, j)], \\ \text{Reject the customer:} & -c_i + E_{j}[F_{t+1}(c, j)]. \end{cases}$$

2) If $c = 0$, $i \neq 0$:

$$F_t(c, i) = -c_i + E_{j}[F_{t+1}(c, j)].$$

3) If $i = 0$:

$$F_t(c, i) = E_{j}[F_{t+1}(c, j)].$$

Boundary conditions:

$$F_T(c, i) = \begin{cases} 0 & \text{if } i = 0, \\ \pi_i & \text{if } i \neq 0 \text{ and } c > 0, \\ -c_i & \text{if } i \neq 0 \text{ and } c = 0. \end{cases}$$

Hence, SP($C$) is equal to the maximum expected profit during the planning horizon, considering that the initial capacity is equal to $C$. Given an arrival and the current capacity, at every time period the model gives the optimal policy of whether or not to accept the request.

In what follows, we characterize the optimal policy given by SP($C$). The producer faces a non-trivial decision only when there is a positive capacity and someone requests...
a product. The objective function is this case is given by,

\[ F_i(c, i) = \max\{\pi_i + \mathcal{E}_j[F_{i+1}(c-1, j)], -c_i + \mathcal{E}_j[F_{i+1}(c, j)]\}. \]

Hence, the company sells the product if

\[ \pi_i + c_i \geq \mathcal{E}_j[F_{i+1}(c, j)] - \mathcal{E}_j[F_{i+1}(c-1, j)], \]

and rejects the customer if

\[ \pi_i + c_i < \mathcal{E}_j[F_{i+1}(c, j)] - \mathcal{E}_j[F_{i+1}(c-1, j)]. \]

Defining \( \alpha_t(c) = \mathcal{E}_j[F_{i+1}(c, j)] - \mathcal{E}_j[F_{i+1}(c-1, j)] \), we obtain that the company's policy for a class \( i \) request at time \( t \), given a capacity \( c_i \), is to sell the product if \( \pi_i + c_i \geq \alpha_t(c) \) and reject the customer if \( \pi_i + c_i < \alpha_t(c) \).

The following proposition shows that associated with every instant in time \( t \), there exists a threshold \( C^*_it \) such that a class \( i \) customer is accepted as long as the current capacity is larger than or equal to this threshold. The threshold \( C^*_it \) is determined by equating the expected profits from accepting and rejecting the customer's request, or equivalently: \( \alpha_t(C^*_it) = \pi_i + c_i \). The threshold \( C^*_it \) decreases when the opportunity cost of accepting the customer's request, \( \pi_i + c_i \), increases, i.e., if for a given capacity the optimal decision is to accept a specific type of customer then it is also optimal to accept any type of customer with a higher opportunity cost.

**Proposition 1.1** The function \( \alpha_t(c) \) is a non-increasing function of \( c \):

\[ \alpha_t(c-1) \geq \alpha_t(c). \]

**Proof:** See Appendix A.1. This is a particular case of Proposition 1.3. □

The following proposition shows that, for a given capacity \( c \) and type \( i \) customer, there exists a threshold time \( \tau_{ic} \) beyond which the customer's request is accepted. The
threshold \( \tau_c \) is obtained equating \( \alpha_t(c) \) and \( \pi_i + c_i \). As we show in proposition 1.2, \( \alpha_t(c) \) is a non-increasing function of \( t \). Hence, if we take two classes of customers \( i \) and \( j \) with \( \pi_i + c_i \geq \pi_j + c_j \), then \( \tau_c \leq \tau_{jc} \), i.e., the company sells the product to customers with a higher opportunity cost before selling it to customers that yield a lower opportunity cost.

**Proposition 1.2** The function \( \alpha_t(c) \) is decreasing in \( t \):

\[
\alpha_t(c) \geq \alpha_{t+1}(c)
\]

**Proof:** See Appendix A.1. This is a particular case of Proposition 1.4.

The simple structure of the optimal policy can be summarized as follows. For a given instant in time and customer class, there exists a capacity threshold such that a request within that class is accepted as long as the current capacity is larger than the threshold. Similarly, there exists a time threshold for every capacity and customer class, such that a request within that class that happens after this time threshold is always accepted.

The main characteristic of the optimal policy is its simplicity. This feature has an important consequence in terms of its practicality, because, in general, managers are more open to incorporate rules that are intuitive and easy to verify.

**The multiple product case**

The next model corresponds to the case of multiple products with the possibility of downgrading. Similarly to the previous case, reservations are not allowed. We define \( F_t(c, i) \) as the maximum expected profit from period \( t \) onwards when there are \( c \) available products and a class \( i \) customer arrives at time \( t \) (\( i = 0 \) corresponds to no arrivals). In this case \( c \) is a vector with as many components as product types.

The mathematical formulation for this model corresponds to:

**Problem MP\((C):**

\[
E_t[F_0(C, i)],
\]

where, assuming that a class \( i \) customer requests a type \( s \) product, i.e. \( i \in A_s \), we
have:

1) If $i \neq 0$ and there exists $c(k) > 0$ s.t. $k \leq s$:

$$F_t(c, i) = \max \left\{ \begin{array}{l}
\text{Sell a product: } \max_{1 \leq p \leq s} \pi_i + E_j[F_{t+1}(c - e_p, j)], \\
\text{Reject the customer: } -c_i + E_j[F_{t+1}(c, j)].
\end{array} \right. $$

2) If $i \neq 0$ and $c(k) = 0$ for all $k \leq s$:

$$F_t(c, i) = -c_i + E_j[F_{t+1}(c, j)].$$

3) If $i = 0$

$$F_t(c, i) = E_j[F_{t+1}(c, j)].$$

Boundary conditions:

$$F_T(c, i) = \left\{ \begin{array}{l}
\pi_i & \text{if } \exists k \in \{1, 2, \ldots, k\} \text{ s.t. } c(k) > 0 \text{ and } i \neq 0, \\
-c_i & \text{if } \forall k \leq s, c(k) = 0 \text{ and } i \neq 0, \\
0 & \text{if } i = 0.
\end{array} \right.$$ 

We observe that in the multiple product case the producer not only has to decide whether or not to accept the request but also what product to sell. However, using the property that the objective function is a non-decreasing function of the capacity (see Lemma A.1 in Appendix A.1) it is easy to prove that if the optimal decision is to sell a product to a class $i$ customer that requests a type $s$ product, then the company must sell the "lowest level" available product the customer is willing to buy (this product is given by $p = \max\{1, 2, \ldots, s\}$ s.t. $c(p) > 0$). That is, if the company decides to sell a product to a customer, it sells the product satisfying the customer's needs that is closest to the product requested by the customer.

In what follows we derive the properties that characterize the optimal policy. Similarly to the single product case, the manager faces a non trivial decision when a type $i$ customer requests product $s$ and there is an available product that satisfies this request. Then, the optimal decision is given by:
\[ F_t(c, i) = \max\{\pi_i + E_j[F_{t+1}(c - e_p, j)], -c_i + E_j[F_{t+1}(c, j)]\}, \]

where \( p = \max\{1, 2, \ldots, s\} \) subject to \( c(p) > 0 \). Hence, the company sells the product if:

\[ \pi_i + c_i \geq E_j[F_{t+1}(c, j)] - E_j[F_{t+1}(c - e_p, j)] \]

and rejects the customer otherwise.

Defining \( \alpha_t(c, p) = E_j[F_{t+1}(c, j)] - E_j[F_{t+1}(c - e_p, j)] \), the optimal policy consists of selling the product if \( \pi_i + c_i \geq \alpha_t(c, p) \), and rejecting the customer if \( \pi_i + c_i < \alpha_t(c, p) \).

The following two propositions characterize the optimal policy.

**Proposition 1.3:** The function \( \alpha_t(c, p) \) is non-increasing as a function of \( c \):

\[ \alpha_t(c - e_l, p) \geq \alpha_t(c, p) \quad \forall c, p, l, t. \]

**Proof:** See Appendix A.1. \( \blacksquare \)

**Proposition 1.4** The function \( \alpha_t(c, p) \) is non-increasing as a function of \( t \):

\[ \alpha_t(c, p) \geq \alpha_{t+1}(c, p) \quad \forall c, p, t. \]

**Proof:** See Appendix A.1. \( \blacksquare \)

We illustrate the properties described above with the following example. Consider a manager that sells two types of products, where product 1 can substitute product 2. For a given period of time, and a given class \( i \) of customers, figure 1-1 shows the collection of capacity threshold vectors. This corresponds to the curve that separates the acceptance and the rejection regions. If the current capacity is in the acceptance region, the manager accepts the class \( i \) request. Otherwise, she rejects the request. Assuming that class \( i \) customers request a type 2 product, downgrading only takes place when there is no type 2 products available in the acceptance region. As time goes by, the curve that corresponds to the set of capacity threshold vectors moves
down, and the rejection region becomes smaller.

![Graph showing acceptance and rejection regions for capacity threshold vectors](image)

**Figure 1-1: Example of a collection of capacity threshold vectors**

The two preceding propositions show how the results obtained for the single product case extend to the multiple product case with downgrading. The optimal policy for selling products in this case is characterized by a collection of capacity threshold vectors that evolves over time (every vector in the set has as many components as the number of product types considered). It is optimal to satisfy a customer's request as long as the capacity vector at the moment the request is made has all components larger than or equal to at least one of the vectors in the current set of threshold capacities. Furthermore, the collection of capacity threshold vectors evolves over time in such a way that, for every class of customers and every actual capacity vector, there exists an instant in time beyond which it is optimal to satisfy the customer's request.

Proposition 1.3 implies that if a customer's request is accepted when the capacity is equal to $c - e_l$ then it is also accepted when the capacity is equal to $c$, for all values of $c$ and $l$. Suppose that the request is satisfied with a type $p$ product when the capacity is $c - e_l$. When incrementing the capacity to $c$ this request will be satisfied with the same type $p$ product or with a type $l$ product if this product is closer to the
customer's need. This property is obtained by observing that:

$$
\alpha_t(c, p) \geq \alpha_t(c, I) \quad \forall p \leq I.
$$

The above inequality uses the fact that the function $F_t(c, i)$ is non-increasing in $c$ (see Lemma A.1 in Appendix A.1).

1.3.2 Model with reservations

In this section we consider the case where the company accepts a number of reservations before the planning horizon. At every instant in time, the demand associated to customers that have reservations depends on the number of pending reservations, i.e., the number of reservations that have not been canceled or requested previously. In general, the larger the number of pending reservations, the more likely it is that there will be a request from a customer with reservation in the next unit of time. It is also reasonable to assume that the probability that a customer with reservation arrives before the end of the planning horizon is a decreasing function of time. Therefore, in this case, the dynamic programming model has to include an additional state variable that corresponds to the number of pending reservations, $R$, which has as many components as the number of customer types.

We define $F_t(c, R, i)$ as the maximum expected profit from period $t$ onwards, if the company starts at period $t$ with $C$ available products, $R$ pending reservations and a class $i$ customer requests the product. We recall that the customer class determines the price, the rejection cost, the product that the customers requests, and whether she has a reservation or not. We define $\delta_i$ equal to 1 when class $i$ corresponds to customers with reservations and $\delta_i$ equal to 0 otherwise.

Considering the general multiple product case, the mathematical formulation is equal to:

**Problem MPR($C, R$):**

$$
E_t[F_0(C, R, i)],
$$

where, assuming that a class $i$ customer requests a type $s$ product, i.e. $i \in A_s$, we
have:

1) If \( i \neq 0 \) and there exists \( c(k) > 0 \) s.t. \( k \leq s \):

\[
F_t(c, R, i) = \max \begin{cases} 
\text{Sell a product: } \max_{1 \leq p \leq s}(\pi_i + E_j[F_{t+1}(c - e_p, R - \delta_i e_i, j)]), \\
\text{Reject the customer: } -c_i + E_j[F_{t+1}(c, R - \delta_i e_i, j)].
\end{cases}
\]

2) If \( i \neq 0 \) and \( c(k) = 0 \) for all \( k \leq s \):

\[
F_t(c, R, i) = -c_i + E_j[F_{t+1}(c, R - \delta_i e_i, j)].
\]

3) If \( i = 0 \)

\[
F_t(c, R, i) = E_j[F_{t+1}(c, R, j)].
\]

Boundary conditions:

\[
F_T(c, R, i) = \begin{cases} 
\pi_i & \text{if } \exists k \in \{1, 2, \ldots, k\} \text{ s.t. } c(k) > 0 \text{ and } i \neq 0, \\
-c_i & \text{if } \forall k \leq s, c(k) = 0 \text{ and } i \neq 0, \\
0 & \text{if } i = 0.
\end{cases}
\]

Similarly to the previous cases, we define:

\[
\alpha_t(c, p, R) = E_j[F_{t+1}(c, R, j) - F_{t+1}(c - e_p, R, j)].
\]

The company sells the product to a class \( i \) customer if \( \pi_i + c_i \geq \alpha_t(c, p, R - \delta_i e_i) \), and rejects the customer otherwise.

The following proposition shows that, similarly to the case without reservations, for a given customer class, instant in time, and number of pending reservations, there is a set of capacity threshold vectors such that the customer is accepted as long as the current capacity vector has all the components larger than or equal to at least one of the threshold vectors.

**Proposition 1.5** The function \( \alpha_t(c, p, R) \) is non-increasing as a function of the ca-
\[ \alpha_t(c, p, R) \leq \alpha_t(c - e_i, p, R) \quad \forall c, t, R, p, l. \]

**Proof:** See Appendix A.1. ■

We are not able to extend the result that the function \( \alpha_t(c, p, R) \) is decreasing as a function of time, using the same proof that we have used for the case without reservations, because the number of pending reservations varies as a function of the current arrival. For the particular case where the probability of an arrival in the next unit of time depends on the number of pending reservations but not on \( t \), we can obtain the result that \( \alpha_t(c, p, R) \) is decreasing as a function of time. This proof is done by induction using the hypothesis:

\[ F_t(c, R, i) - F_t(c - e_p, R, i) \geq F_{t+1}(c, R, i) - F_{t+1}(c - e_p, R, i). \]

Using the assumption that the probability of an arrival in the next unit of time depends only on the number of pending reservations, we take the expected value on both sides of the induction hypothesis, and obtain the desired result. We conjecture that the previous result still holds for the case where the arrival probabilities depends on \( t \). However, new approaches to prove this result are needed.

The optimal policy for the general problem (multiple products, several types of customers and reservations) can be found solving the dynamic programming formulation. However, this approach takes too long for most "real size problems." The computational effort can be significantly reduced using the properties characterizing the optimal policy derived in this section, making problems with one product tractable. However, when real size problems with several types of products are considered, the computational burden becomes intractable even using this approach. Therefore, it is necessary to develop ad-hoc heuristics that provide good approximations for the optimal policy. In the next section we present a family of upper bounds that will be useful to measure the quality of these heuristics.
1.4 Upper Bounds

In this section we describe a family of upper bounds for the optimization problem described Section 1.3, considering multiple types of products and reservations. These upper bounds are based on the general idea that expected optimal profits increase when the amount of information available about future demand becomes larger. For example, if at the beginning of the planning horizon the producer knows the total number of requests that will take place, she can make a better decision than the decision she would make without this information. Before presenting the model for the upper bound, we define the following additional notation:

1. \( R^k \): vector of pending reservations at the beginning of period \( k \).

2. \( s^k(R) \): vector with arrivals in period \( k \) if the initial vector of reservations is \( R \). The component \( s_i^k(R) \) is a random variable that represents the number of class \( i \) requests in period \( k \).

3. \( B_p \): set of customer classes that request a product type better than or equal to \( p \).

4. \( x^k \): vector of decision variables. The component \( x_i^k \) corresponds to the number of class \( i \) customers accepted in period \( k \).

The following problem provides an upper bound for the problem MPR(\( C, R \)) (multiple product problem with reservations).

**Problem UB_K(C^1, R^1)**: \( E_{s^k(R^1)}[G_1(C^1, R^1, s^1(R^1))] \),

where:

\[
G_k(C^k, R^k, s^k(R^k)) = \max_{x_1^k, \ldots, x_n^k} \sum_i \pi_i x_i^k - \sum_i c_i(s_i^k(R^k) - x_i^k) + \\
E_{s_{k+1}(R^k)}[G_{k+1}(C^{k+1}, R^{k+1}, s^{k+1}(R^{k+1}))]
\]

s.t. \( x_i^k \leq s_i^k(R^k), \quad \forall i = 1, \ldots, n \) (1.1)

\[
\sum_{i \in B_p} x_i^k \leq \sum_{i = 1}^p C^k(i), \quad \forall p = 1, \ldots, m, \quad (1.2)
\]
\[ R^{k+1} = R^k - f_1(s^k(R^k)), \]  
(1.3)

\[ C^{k+1} = C^k - f_2(x^k), \]  
(1.4)

\[ x_i^k \geq 0 \text{ and integer, } \forall i = 1, \ldots, n. \]  
(1.5)

The boundary condition is given by:

\[
G_K(C^K, R^K, s^K(R^K)) = \max_{s^K_1, \ldots, s^K_n} \sum_i x_i^k - \sum_i c_i(s^K_i(R^K) - x^K_i)
\]

s.t. \[ x^K_i \leq s^K_i(R^K), \quad \forall i = 1, \ldots, n, \]

\[ \sum_{i \in B_p} x^K_i \leq \sum_{i=1}^p C^K(i), \quad \forall p = 1, \ldots, m, \]

\[ x^K_i \geq 0 \text{ and integer, } \forall i = 1, \ldots, n. \]

The maximum number of customers that can be accepted in period \( k \) is bounded by the number of requests received in that period. This condition is given by constraint (1.1), for each class of customers. The set of constraints (1.2) corresponds to the capacity constraint for each class of products, considering that they can be downgraded. Constraints (1.3) are conservation constraints for the number of pending reservations, i.e., \( f_1(s^k(R^k)) \) is equal to zero for all components associated to customers without reservations and equal to the corresponding coordinate of \( s^k(R^k) \) for all components associated to customers with reservations. Finally, constraints (1.4) are conservation constraints for the capacity. Thus, the \( j \)th component of \( f_2(x^k) \) is equal to the number of type \( j \) products sold in period \( k \) considering that some of them were downgraded.

The index \( K \) of the family of upper bounds is given by the number of periods in which the planning horizon is divided. To prove that the problem \( UB_K(C, R) \) provides an upper bound for \( MPR(C, R) \) we have to apply several times the property that the maximum of the expected value is less than or equal to the expected value of the maximum. Instead of giving a formal proof, we explain the intuition underlying these upper bounds. The problem \( UB_1(C, R) \) corresponds to the extreme case where the producer has perfect information about the number of arrivals for each class of
customers in the planning period. Therefore, she can make a better decision than the decision she would make with less information. In problem $UB_K(C, R)$ we have divided the planning horizon in $K$ periods. At the beginning of every period the producer has perfect information about the customers that show up in that period. Hence, she can make a decision better than the decision she would make knowing the current arrival and the probability of potential arrivals in the future. In the limit, when $K$ is equal to the number of periods considered in the original problem (periods with at most one arrival), the upper bound is equal to the optimal solution of $MPR(C, R)$.

1.5 Heuristics

In this section we develop heuristics for finding "good" feasible solutions for the optimization problems formulated in Section 1.3. These heuristics are based on the characterization of the optimal policies studied in Section 1.3.

We introduce the following notation:

1. $\pi_i =$ price associated to a class $i$ customer.

2. $c_i =$ rejection cost of a class $i$ customer.

3. $D_i = \{j/\pi_j + c_j > \pi_i + c_i\}$.

4. $\text{pref}(i) =$ product type that a class $i$ customer requests.

5. $c =$ vector of current capacity.

6. $\lambda_i(t) =$ arrival rate of class $i$ customers at time $t$.

7. $T =$ planning horizon.

8. $t =$ current time.
1.5.1 Heuristic 1

This heuristic works as follows. If at time $t$ a customer requests a product, the manager calculates the expected number of arrivals that have an opportunity cost (price plus rejection cost) larger than the current request's opportunity cost, from $t$ to the end of the planning horizon. The current capacity is assigned optimally to this expected number of arrivals. If after the assignment there is a remaining product that satisfies the current request, then the product is sold to the customer. Otherwise, the manager rejects the request.

In order to assign optimally the products to the expected number of arrivals, the heuristic satisfies first the expected demand of customers with the largest opportunity cost, downgrading if it is necessary. Then it satisfies the expected demand associated to the second largest opportunity cost and so forth.

**Description of Heuristic 1 (HEUR1):** Suppose a class $i$ customer requests a type $s$ product at time $t$. The following algorithm is used to determine whether or not to accept the customer's request:

**Step 0:** Initialization

$$\text{Cap} = c.$$ 

**Step 1:** Determining the remaining capacity after assigning the products to the expected number of requests from $t$ to $T$ that have a larger opportunity cost than the current request's opportunity cost.

$$S_i = D_i$$

1.1 $j = \max_{k \in S_i} \{ \pi_k + c_k \}$

1.2 $\text{Exp}_s\text{arr} = \int_t^T \lambda_j(\tau) d\tau$

1.3 $p = \text{pref}(j)$

If $(\text{Cap}(p) > \text{Exp}_s\text{arr})$ then

$$\text{Cap}(p) = \text{Cap}(p) - \text{Exp}_s\text{arr}$$

$$\text{Exp}_s\text{arr} = 0$$
else
    \text{Exp}_{\text{arr}} = \text{Exp}_{\text{arr}} - \text{Cap}(p)
    
    \text{Cap}(p) = 0
endif

if \((p > 1)\) then \(p = p - 1\), goto 1.3

\(S_i = S_i - \{j\}\),

if \(S_i \neq \emptyset\) goto 1.1,

else goto Step 2.

\textbf{Step 2:} Checking if there is an available room for the class \(i\) request.

\(p = s\)

2.1 if \((\text{Cap}(p) > 0)\) then

    \textit{Optimal decision} \leftarrow \text{give product 1 to the current request.}

    go to step 3
endif

if \((p > 1)\) then \(p = p - 1\), goto 2.1

else, \textit{Optimal decision} \leftarrow \text{reject the current request.}

\textbf{Step 3:} STOP.

\textbf{1.5.2 Heuristic 2}

This heuristic is similar to the heuristic described above. The only difference is that instead of considering the expected number of arrivals from \(t\) to the end of the planning horizon, the heuristic considers the number of arrivals that occur with 90\% probability. \texttt{HEUR2} is used to denote heuristic 2.

\textbf{1.5.3 Heuristic 3}

This heuristic is developed for the single product case. It calculates the marginal expected profit of the product that satisfies the current request. If this marginal
value is larger than the certain opportunity cost associated to the current request, the manager sells the product. She rejects the request otherwise.

In order to compute the marginal expected value, we consider the customers that have a larger opportunity cost than the current request's net profit. For those customers, we calculate the probability that the number of requests is larger than or equal to the current capacity. The weighted opportunity cost for those customers times the above probability corresponds to the expected profit for this marginal product.

**Description of Heuristic 3 (HEUR3):** Suppose there is a class $i$ request at time $t$. The following algorithm is used to determine whether or not to accept the customer's request:

**Step 0:** Initialization

$$\text{Cap} = c,$$

**Step 1:** Computing the weighted opportunity cost

$$\text{Weighted}_i = \sum_{j \in D_i} (\pi_j + c_j) \times \lambda_j / \sum_{j \in D_i} \lambda_j$$

**Step 2:** Computing the probability that there are more or equal number of requests than the current capacity.

$$\text{Prob}_i = \Pr\{\sum_{j \in D_i} \text{ class } j \text{ requests in } (T-t) \geq \text{ Cap}\}$$

**Step 3:**

If $$(\text{Prob}_i \times \text{Weighted}_i > \pi_i + c_i)$$ then

Optimal decision ← reject the current request

else

Optimal decision ← accept the current request

Endif

**Step 4:** STOP.
1.5.4 Heuristic 4

In this heuristic the manager satisfies all the requests as long as there is an available product that satisfies the customer's need. The decisions are made without considering any additional information. This simple heuristic is only used for the purpose of comparison with the other heuristics. HEUR4 is used to denote heuristic 4.

1.6 Computational experiments

In this section we use Monte Carlo simulation to evaluate the performance of the heuristics described in Section 1.5. In order to have a concrete application, we use a hotel example where the rooms correspond to the products and the planning horizon is an operational day. We assume that customers arrive according to a Poisson process, and every time when a customer requests a room, the manager makes a decision following the rules given by the heuristics. We simulate the arrival process during an operational day and compute the total profit obtained under the application of the different heuristics. Averaging the profits given by repeated simulations, we obtained a statistical estimate of the expected profit.

The Poisson process is a natural model to represent the arrival process of customers. This model is commonly used in the literature; for example Alstrup et al. (1986) claim that airline ticket requests have this distribution. Rothstein (1974) and Bitran and Gilbert (1992) use a Poisson distribution to represent customer requests in the hotel industry.

We consider three groups of computational experiments. The first two sets of experiments correspond to relatively small applications that allow us to compute the optimal solutions. The parameters used in the third application are based on information given by the manager of a medium size hotel in Santiago, Chile. In this real application, we are not able to compute the optimal solution, therefore we compare the performance of the heuristics with the upper bounds for the general optimization problem.

A statistical estimate of the value of the upper bounds is also computed using
Monte Carlo simulation. Recall that the upper bounds correspond to the expected value of an optimization problem. The expectation is taken over all possible arrival combinations and therefore it is infeasible to compute its exact value.

The criterion to stop the simulations is given by a standard deviation of less than 0.1% of the expected value. In the experiments we compute two upper bounds. The first one, UB1, corresponds to the case where all the requests are known at the beginning of the planning horizon. In the second upper bound, UB2, the planning horizon is divided in two periods; the requests during each segment are known at their beginning.

In what follows, we define the booking index, OB, to measure the hotel’s capacity as a function of the expected number of requests.

Single product case: OB is equal to the total expected number of arrivals over the total capacity.

\[ OB = \frac{\sum_{i=1}^{n} \int_{t=1}^{T} \lambda_i(t)dt}{C} \]

Multiple product case: first we compute the booking index for each type of product, OBs, using the cumulative capacity and the cumulative number of arrivals. Then, we calculate OB as the weighted booking index, where the weights correspond to the expected number of arrivals that request each type of product.

\[ EA_s = \sum_{i \in A_s} \int_{t=1}^{T} \lambda_i(t)dt \]

\[ OB_s = \frac{\sum_{j=1}^{s} EA_j}{\sum_{j=1}^{s} C(j)} \]

and finally,

\[ OB = \sum_{s=1}^{m} OB_s \frac{EA_s}{\sum_{j=1}^{m} EA_j} \]
In the first group of computational experiments, we consider a single product, and 3 types of customers that arrive in a Poisson manner with arrival rates equal to 5, 3, and 2 customers per hour. The prices that they pay are $200, $120, and $85 respectively and there is no rejection costs. We can think on these customers as walk-ins that do not have a previous commitment with the hotel. The planning horizon is 12 hours. The performance of the heuristics is shown in table 1.1:

Table 1.1: Single Product Case

<table>
<thead>
<tr>
<th>Capacity # rooms</th>
<th>OB * 100</th>
<th>HEUR1 w/r opt. sol.</th>
<th>HEUR3 w/r opt. sol.</th>
<th>HEUR4 w/r opt. sol.</th>
<th>HEUR1 w/r UB2</th>
<th>HEUR3 w/r UB2</th>
<th>HEUR4 w/r UB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>240</td>
<td>99.9%</td>
<td>99.9%</td>
<td>76.7%</td>
<td>99.8%</td>
<td>99.8%</td>
<td>76.6%</td>
</tr>
<tr>
<td>60</td>
<td>200</td>
<td>98.9%</td>
<td>98.4%</td>
<td>78.5%</td>
<td>98.2%</td>
<td>97.7%</td>
<td>78.0%</td>
</tr>
<tr>
<td>70</td>
<td>171</td>
<td>98.5%</td>
<td>98.1%</td>
<td>82.2%</td>
<td>97.4%</td>
<td>97.0%</td>
<td>81.3%</td>
</tr>
<tr>
<td>80</td>
<td>150</td>
<td>99.1%</td>
<td>99.0%</td>
<td>86.0%</td>
<td>98.0%</td>
<td>97.8%</td>
<td>85.0%</td>
</tr>
<tr>
<td>90</td>
<td>133</td>
<td>99.6%</td>
<td>99.6%</td>
<td>89.5%</td>
<td>98.5%</td>
<td>98.5%</td>
<td>88.5%</td>
</tr>
<tr>
<td>100</td>
<td>120</td>
<td>99.6%</td>
<td>99.6%</td>
<td>93.3%</td>
<td>98.5%</td>
<td>98.6%</td>
<td>92.3%</td>
</tr>
<tr>
<td>110</td>
<td>109</td>
<td>99.9%</td>
<td>100.0%</td>
<td>96.9%</td>
<td>99.0%</td>
<td>99.1%</td>
<td>96.0%</td>
</tr>
</tbody>
</table>

The first column in table 1.1 is the hotel's capacity (number of available rooms). The next three columns have the performance of the heuristics compared with the optimal solution. Finally, in the last three columns appear the performance of the heuristics with respect to the upper bound UB2. The performance of HEUR4 (to give a room to all the requests as long as there is an available room) improves as long as the capacity increases. Naturally, if the capacity is very large, all the heuristics must lead to the optimal solution. However, for the cases where the capacity is an issue, HEUR4 performs much worse than the other heuristics. Therefore, in those cases, it is sensible to use additional information as for example the current capacity and the probability distribution of the future arrivals. We also observe that HEUR1 and HEUR3 perform almost the same; for very small or large capacities, the solution given by the heuristics is near to the optimal solution. When capacity is small, rooms
will be almost always rented to the higher opportunity cost customers. The exact behavior can be seen in Figure A-1 in Appendix A.2. The expected profits appear in Table A.1 in Appendix A.2.

In the second group of experiments we consider 2 types of rooms with the possibility of downgrading. We also consider 3 types of customers with arrival rates equal to 2, 3 and 5 customers per hour, prices equal to $200, $120, and $85, and room preferences equal to 1, 2 and 2 respectively. No rejection costs are considered and the planning horizon is 12 hours. The performance of the heuristics is shown in table 1.2:

Table 1.2: Multiple Product Case with Downgrading

<table>
<thead>
<tr>
<th>Capacity (# rooms)</th>
<th>OB = 100</th>
<th>HEUR1 w/r opt.sol.</th>
<th>HEUR2 w/r opt.sol.</th>
<th>HEUR4 w/r opt.sol.</th>
<th>HEUR1 w/r UB2</th>
<th>HEUR2 w/r UB2</th>
<th>HEUR4 w/r UB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,30)</td>
<td>850</td>
<td>99.8%</td>
<td>84.1%</td>
<td>85.6%</td>
<td>99.7%</td>
<td>84.0%</td>
<td>85.5%</td>
</tr>
<tr>
<td>(10,30)</td>
<td>528</td>
<td>99.8%</td>
<td>84.3%</td>
<td>84.3%</td>
<td>99.7%</td>
<td>84.2%</td>
<td>84.2%</td>
</tr>
<tr>
<td>(10,45)</td>
<td>463</td>
<td>98.5%</td>
<td>87.1%</td>
<td>90.1%</td>
<td>98.0%</td>
<td>86.6%</td>
<td>89.6%</td>
</tr>
<tr>
<td>(10,50)</td>
<td>448</td>
<td>98.7%</td>
<td>90.0%</td>
<td>91.9%</td>
<td>98.1%</td>
<td>89.5%</td>
<td>91.4%</td>
</tr>
<tr>
<td>(10,60)</td>
<td>425</td>
<td>99.0%</td>
<td>94.5%</td>
<td>94.6%</td>
<td>98.5%</td>
<td>94.1%</td>
<td>94.1%</td>
</tr>
<tr>
<td>(15,70)</td>
<td>305</td>
<td>99.4%</td>
<td>96.2%</td>
<td>96.4%</td>
<td>99.0%</td>
<td>95.8%</td>
<td>96.0%</td>
</tr>
<tr>
<td>(20,80)</td>
<td>240</td>
<td>99.6%</td>
<td>97.9%</td>
<td>97.8%</td>
<td>99.2%</td>
<td>97.5%</td>
<td>97.4%</td>
</tr>
<tr>
<td>(20,90)</td>
<td>231</td>
<td>99.9%</td>
<td>99.5%</td>
<td>99.3%</td>
<td>99.7%</td>
<td>99.2%</td>
<td>99.1%</td>
</tr>
<tr>
<td>(25,95)</td>
<td>195</td>
<td>100.0%</td>
<td>100.0%</td>
<td>100.0%</td>
<td>99.9%</td>
<td>100.0%</td>
<td>99.9%</td>
</tr>
</tbody>
</table>

The first column in Table 1.2 is the hotel’s capacity (number of rooms). The rooms are ordered according to their quality; thus the first number in the capacity vector corresponds to the suite rooms. The next three columns correspond to the performance of HEUR1, HEUR2, and HEUR4 compared with the optimal solution. Finally, the last three columns show the performance of these three heuristics with respect to the upper bound UB2. HEUR2 and HEUR4 have a similar behavior, HEUR4 being slightly better. However, these two heuristics have, in all the cases, a worse performance than heuristic 1. HEUR2 follows highly conservative rules, because a customer is rejected as long as the corresponding room can be rented to a higher net profit customer with 90% probability. Therefore, it is worthwhile to apply HEUR1.
because it gives better solutions to the optimization problem. We also observe from Table 1.2 that the performance of HEUR1 is almost optimal in both extremes; i.e. for very small and large capacities. When capacity is very small, is it always better not to downgrade rooms because better quality rooms will be rented to the highest opportunity cost customers. On the other hand, all the heuristics perform close to the optimal when the capacity increases to the limit that all the requests should be accepted. The expected profits for these experiments can be found in Appendix A.2, Table A.2 and the graphic behavior of heuristic 1 as a function of the capacity in Figure A-2, Appendix A.2.

For the last set of experiments, we use parameters based on data of a medium size hotel in Santiago, Chile. The hotel works with three main classes of customers: (i) guaranteed reservation customers that are guaranteed a room based on a previous arrangement, (ii) “6 p.m. hold” customers that are guaranteed a room based on a previous arrangement only if they arrive before 6 p.m., and (iii) “walk-ins” that request a room during the target date without a reservation. Within these categories, customers request standard and suite rooms. Therefore, the hotel faces the demand of 6 types of customers. The average percentage of requests that the hotel receives in a day is equal to 81% of guaranteed reservation customers, 14% of 6 p.m. hold customers, and 5% of walk-ins.

The parameters used in the following simulations can be found in Appendix A.2. In this case it is not feasible to compute the value of the optimal objective function, because it takes too long. We only compute the upper bounds UB1 and UB2 and the value of the objective function given by the heuristics 1, 2 and 4.

Table 1.3 shows the performance of HEUR1, HEUR2 and HEUR4 compared with the upper bounds given by UB1 and UB2. We observe that HEUR2 and HEUR4 have a similar behavior, HEUR2 being slightly better. Both heuristics improve their performances when the capacity increases, and in the limit when the capacity is large in comparison with the demand, they perform close to the optimal solution. However, HEUR1 has a better performance than the other heuristics in all the cases considered.
Table 1.3: A Real Case

<table>
<thead>
<tr>
<th>Capacity (# rooms)</th>
<th>OB * 100</th>
<th>HEUR1 w/r UB1</th>
<th>HEUR2 w/r UB1</th>
<th>HEUR4 w/r UB1</th>
<th>HEUR1 w/r UB2</th>
<th>HEUR2 w/r UB2</th>
<th>HEUR4 w/r UB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,130)</td>
<td>186</td>
<td>100.0%</td>
<td>83.9%</td>
<td>81.9%</td>
<td>100.0%</td>
<td>83.9%</td>
<td>82.0%</td>
</tr>
<tr>
<td>(20,150)</td>
<td>168</td>
<td>99.8%</td>
<td>87.1%</td>
<td>86.0%</td>
<td>99.9%</td>
<td>87.2%</td>
<td>86.1%</td>
</tr>
<tr>
<td>(20,190)</td>
<td>143</td>
<td>99.3%</td>
<td>93.3%</td>
<td>92.7%</td>
<td>99.8%</td>
<td>93.8%</td>
<td>93.2%</td>
</tr>
<tr>
<td>(30,200)</td>
<td>122</td>
<td>98.6%</td>
<td>95.1%</td>
<td>94.3%</td>
<td>99.2%</td>
<td>95.6%</td>
<td>94.8%</td>
</tr>
<tr>
<td>(40,210)</td>
<td>108</td>
<td>98.2%</td>
<td>97.0%</td>
<td>96.5%</td>
<td>98.7%</td>
<td>97.4%</td>
<td>96.9%</td>
</tr>
<tr>
<td>(40,220)</td>
<td>105</td>
<td>98.6%</td>
<td>98.1%</td>
<td>97.6%</td>
<td>99.0%</td>
<td>95.5%</td>
<td>98.0%</td>
</tr>
<tr>
<td>(60,220)</td>
<td>93</td>
<td>99.4%</td>
<td>99.3%</td>
<td>99.2%</td>
<td>99.6%</td>
<td>99.5%</td>
<td>99.4%</td>
</tr>
<tr>
<td>(80,220)</td>
<td>84</td>
<td>99.6%</td>
<td>99.7%</td>
<td>99.6%</td>
<td>99.8%</td>
<td>99.9%</td>
<td>99.8%</td>
</tr>
</tbody>
</table>

We also observe that the worst performance of HEUR1 is 98.7% compared with UB2, which is close to the optimal considering that the optimal solution is less than or equal to the upper bound.

1.7 Conclusions

This chapter has studied the optimal policies for selling perishable products to various types of customers. The general model has considered that different classes of customers are charged different prices for the products (airline employees, government workers, executives, etc.), reservations can be made before the planning horizon, product downgrading is possible, and there is no particular order in the arrival of different classes of customers. Even though exact solutions cannot be computed for the general case with multiple types of products, the optimal policies have a simple characterization: given a period of time, if a request is accepted for a certain capacity, then it is also accepted for any larger capacity. Furthermore, for every class of customers and every capacity vector, there exists an instant in time beyond which it is optimal to satisfy the customer's request. Ad-hoc heuristics were developed to find approximations for the optimal solution. Computational results showed that their performance is close to the optimal solution for a wide variety of realistic scenarios.
The main feature of these heuristics is their simplicity which is extremely important in terms of their practicality; managers are more open to use rules that are intuitive and easy to implement.
Chapter 2

Mailing Catalogs: An Optimization Approach

The catalog sales industry is one of the fastest growing business in the U.S. The most important asset a company in this industry has is its list of customers, called the house list. Building a house list is expensive, since the response rate of potential customers from rental lists is low. Therefore, limited access to capital plays a central role when these firms plan the strategies by which they will send catalogs.

This chapter studies the optimal mailing policies in the catalog industry taking into account cash flow constraints. We consider a stochastic environment given by the random responses of customers and a dynamic evolution of the house list. Given the size of real problems, it is not possible to compute the optimal solutions. Therefore, we develop ad-hoc heuristics based on the optimal solutions of simplified versions of the problem studied. The performance of these heuristics is evaluated by comparing their outcome with upper bounds derived for the original problem. Computational experiments show that these heuristics behave satisfactorily.

2.1 Introduction

One of the fastest growing segments within American business is the catalog sales industry. Companies that use catalogs as their sole marketing strategy sent a daily
average of 30 million catalogs during 1989, selling 50 billion US dollars during that
year (Holtz, 1990).

We define the catalog sales industry as those companies doing business by pursuing
prospective customers by means of a catalog. The catalog recipient can order a
product by mail, telephone, or any other alternative.

Initially, catalog companies were oriented toward rural areas, at a time where
access to stores was difficult and expensive. Nowadays, people continue buying via
catalogs for several reasons, as for example, the convenience of not having to leave
the home or office to buy a product, the popular notion that catalogs offer items more
cheaply than retail stores, and the exclusivity of the items offered. On the other hand,
customers often hesitate to buy by catalogs because they perceive disadvantages, as
for example, the delay experienced in the delivery of the product, the impossibility of
examining the product directly before buying it, and the inconvenience of returning
an item that, for some reason, is unsatisfactory.

Companies in the catalog sales industry divide the planning horizon into cam-
paigns, which usually coincide with the seasons of the year. During a campaign, all
the marketing effort is oriented to sell a specific set of products, which have common
characteristics that justifies promoting all of them in a same catalog. In what follows,
the terms campaign and season will be used interchangeably.

Companies in the catalog sales industry define a customer as a person that has
already bought products from the company; thus her name can be used – in the sense
of sending her new catalogs – as often as the company wants (unless the customer
explicitly demands the opposite). The list of customers is called the house list. The
typical behavior of a customer is that after buying a number of times from the com-
pany, she stops doing because her taste changes, she switches to the competition, etc.
Therefore a company selling by catalog must constantly add new customers to its
house list.

A rental list is a set of names with certain characteristics in common, such as, for
example, age, sex and income. These lists can be rented by the company, usually for
a one-time use, paying a price that depends on the number of names rented. These
names correspond to potential customers, their response rate — i.e., the percentage of people that respond with a sale — is usually much lower than that of customers on the house list. It follows that the company must use rental lists to obtain new customers, which it then may add to its house list. When a person from a rental list responds with a mail order, she can be incorporated into the house list and from that time onwards the name can be used without paying for doing so.

Usually, the house list is divided into segments (or states) according to a R-F-M classification, where Recency corresponds to the time since the last purchase, Frequency is equal to the number of purchases in a given period of time and Monetary amount is equal to the amount spent in the last purchase. However, recency is by far the most important factor to predict customers’ future behavior (Fleischmann, 1992). Thus, for example, the smaller the recency the larger the response rate, for a given frequency and order size. Therefore, the company faces a stochastic demand that depends directly on the number of catalogs mailed during the season and on the distribution of customers in the states of the house list. The company can also mail several catalogs to the same customers within a season, thereby increasing the response rate. This technique is called multiple mailing.

A large house list is one of the main assets a company in the catalog sales industry can have. This is expensive, specially at the beginning, since names of potential customers must be rented. Thus firms encounter serious cash flow restrictions during their first years of existence. Catalog companies usually lose money for two or three years until their house list reaches proportions which contribute more than the costs of acquiring new customers and overhead. When a company starts in the catalog business, it must invest an important fraction of its budget in acquiring customers. We illustrate this idea with the following example: suppose that the company rents a name from a rental list with an average response rate of 1%, and an average size of the order of $80. If we consider a cost of goods and fulfillment of $40 and a marketing cost (including the catalog, the mail, and the name’s rent) of $0.70 per customer, the company spends an average of $30 in adding a new customer into its house list. However, people from rental lists that respond with a sale become “good customers”
that are expected to generate profits over their lifetime in the house list.

One of the most important decisions that a manager faces in the catalog sales industry is to define the mailing policy, i.e., the fraction of the people in the rental lists and in the states of the house list that receive catalogs. The manager also has to decide the number of catalogs that a customer receives during the season if multiple mailings are allowed.

The purpose of this chapter is to determine the optimal mailing policy considering cash flow constraints. We study two tactical models which differ in the way that the supplier interacts with the catalog sales company. In the first model we consider a catalog company that manages the cash flow constraints incorporating the financial impact of carrying inventory. Hence, the model determines the optimal aggregate reordering policy together with the optimal mailing policy. Because the number of people that respond with a sale is a random variable, the manager faces a non-trivial decision when deciding how many products to have in store: having few products may lead to lost sales while having too many products implies spending money that can otherwise be utilized to send additional mailings. In the second model we consider a catalog company that is part of a major retail operation. Thus, all the inventory management decisions are made by the retail store. We also assume that the catalog sales are a small fraction of the total sales. Therefore, the catalog requests are always satisfied.

Most of the literature related to this topic gives a qualitative analysis describing the main characteristics of the industry, as for example Hill (1989). A good review of the catalog sales industry can be found in Holtz (1990). Bitran and Ramalho (1992) determine the optimal mailing policy in a deterministic environment. In this chapter we study optimal mailing and reordering policies in a stochastic environment, where uncertainty originates from customers’ random responses.

The remainder of the chapter is organized as follows. In Section 2.2 we introduce two dynamic programming formulations for the problems described above. We also present some properties of the optimal solutions. In Section 2.3 we develop a methodology to calculate the discounted net profit associated to a customer: Life-
time value of a customer. In Section 2.4 we describe various heuristics to solve the optimization problems. In Section 2.5 we find upper bounds for the dynamic and stochastic programming formulations, which are useful to measure the performance of the heuristics developed in the previous section. Section 2.6 contains the computational experiments that show the characteristics of the optimal policies and the performance of the heuristics. Finally, Section 2.7 presents extensions and conclusions.

2.2 Mathematical Models

In this section we describe two tactical models to maximize the total expected profit in the catalog sales industry, which differ in the relationship between the company and the supplier. The first model, the Catalog Mailing Problem with Aggregate Inventory Costs, corresponds to the case where companies must manage the cash flow incorporating the financial impact of carrying inventory. This model determines the optimal mailing policy together with the optimal reordering policy. We assume that the company can order only at the beginning of the season. Therefore, it must determine the optimal reordering amount taking into account the costs of having unsatisfied customers and carrying inventory from one season to another. The second model, the Catalog Mailing Problem, determines the optimal policy for sending catalogs, considering that the catalog company is part of a major retail store and that the catalog sales are a small fraction of the total sales. Therefore, as a good approximation of reality, we assume that all the mail orders received are satisfied. We also assume that the inventory management is carried out by the retail store.

Recency is the main factor considered by the managers in the catalog sales industry to determine the customers' future behavior. Hence, without loss of generality, we only consider the recency to describe the states in the house list. We also consider an average size of the order, common for all the states in the house list and the rental lists. In general, an order consists of several items. Hence, we define a unit equivalent as a unit that has the average ordered fraction of each item. We introduce
the following notation to show how to compute the average size of the order, the average cost of the goods and the proportion of each item in the unit equivalent. Using the company's relevant data for the past seasons we define:

\[ N = \text{total number of orders.} \]

\[ N_k = \text{total number of orders for item } k. \]

\[ OS_k = \text{price of item } k. \]

\[ g^k = \text{cost of item } k \text{ as a fraction of its price.} \]

\[ OS = \text{average size of the order.} \]

\[ g_1 = \text{average cost of goods as a fraction of the average sale per customer.} \]

Hence, a unit equivalent consists of \( N_k/N \) units of item \( k \), for all \( k \). We also define the average size of the order and the average costs of goods as follows:

\[ OS = \frac{\sum_k N_k OS_k}{N} \]

\[ g_1 = \frac{\sum_k g^k OS_k N_k}{OS N} \]

2.2.1 Catalog Mailing Problem with Aggregate Inventory Costs

The mathematical model corresponds to a stochastic and dynamic programming formulation where the objective function is to maximize the total expected profit during the planning horizon. The planning horizon is divided into seasons, with, generally, four seasons per year (some companies might consider five seasons including the Christmas sale).

One of the most important decisions that a manager faces in the catalog sales industry is to define the mailing policy, i.e., the fraction of the people in the rental lists and in the states of the house list that receive catalogs. The number of people in the house list that respond with an order is a random variable that depends on
the response rate of the corresponding state (the response rate increases with the number of mailings), and on the number of people that receive catalogs. Responses from rental lists have a similar behavior. However, in this case, only single mailing is possible because names are usually rented for one-time use. Therefore, the company faces a stochastic demand that depends directly on the marketing effort, i.e., the number of catalogs mailed in the season.

In practice companies can order a limited number of times from their suppliers. Frequently, they order one month before the beginning of the season and they reorder just once within the first three weeks of the season. Therefore, it is reasonable to assume that the company orders only once during every season, and that this order takes place at the beginning of the season. Additionally to the mailing policies we have to define the optimal reordering policies.

The model makes the following three assumptions:

- If a customer places an order her recency decreases to one independently of whether or not the request is satisfied.

- There is a monetary cost for not satisfying an order, which represents the company's loss of reputation.

- There is a holding cost for carrying inventory from one season to another.

In what follows we introduce the notation for the parameters, decision variables, and random variables.

PARAMETERS

\( p_{i,m} \) = probability that a customer in state \( i \) in the house list buys if she receives \( m \) catalogs during a season. This parameter is equivalent to the average response rate of state \( i \) in the house list, when customers receive \( m \) catalogs.

\( p_j \) = probability that a customer in rental list \( j \) buys if she receives a catalog. This is equivalent to the average response rate of rental list \( j \).

\( OS \) = average size of the order.
\( c_1 = \) variable marketing cost per customer in the house list (it includes printing and mailing).

\( c_2 = \) variable marketing cost per customer in the rental lists (it includes printing, mailing and renting the name).

\( g_1 = \) average cost of goods as a fraction of the average sale per customer.

\( g_2 = \) average variable cost associated to a request.

\( r = \) penalty for failing an order.

\( h = \) holding inventory cost per unit.

\( A_t = \) total amount of money that is available for additional investment at the beginning of season \( t \).

\( L_{j,t} = \) number of available names in rental list \( j \) during season \( t \).

\( \beta = \) discount rate per season.

\( I = \) total number of states in the house list.

\( M = \) maximum number of mailings within a season.

\( J = \) total number of available rental lists.

**Decision Variables**

\( H_{i,m,t} = \) total number of customers in state \( i \) in the house list that receive \( m \) catalogs during season \( t \).

\( M_{j,t} = \) number of people in rental list \( j \) that receive a catalog during season \( t \).

\( Z_t = \) Number of unit equivalents ordered at the beginning of season \( t \).

\( S_{i,t} = \) Number of sales associated to customers in state \( i \) at season \( t \).

\( S_{j,t} = \) Number of sales associated to rental list \( j \) at season \( t \).
RANDOM VARIABLES

\( X_{i,m,t} \) = total number of responses from people in state \( i \) in the house list that receive \( m \) catalogs during season \( t \). This variable has a binomial distribution with probability equal to \( p_{i,m} \) and total number of trials equal to \( H_{i,m,t} \).

\( X_{j,t} \) = number of responses from rental list \( j \) during season \( t \). This variable has a binomial distribution, with probability equal to \( p_j \) and total number of trials equal to \( M_{j,t} \).

\( I_t \) = inventory at the beginning of season \( t \).

\( Y_t \) = total amount of money available for investment at the beginning of season \( t \).

\( N_t \) = number of customers in the house list at season \( t \). \( N_t \) is a vector with as many elements as the number of states in the house list. Therefore, the element \( N_{i,t} \) of \( N_t \) corresponds to the number of customers in state \( i \) in the house list during season \( t \).

Finally, we define the function \( F_t(Y_t, I_t, N_t) \) as the maximum discounted expected profit from season \( t \) onwards if the company starts with \( Y_t \) dollars for investment, \( I_t \) products in inventory, and \( N_t \) customers in the house list at season \( t \).

The Model: the objective function at time \( t \) is given by the immediate profit during the current season plus the expected profit from the next season onwards. During season \( t \), the manager has to decide the optimal mailing and reordering policies satisfying the cash flow constraints. The set of constraints at season \( t \) is given by:

Cash flow constraint.

\[
OS g_t Z_t + \sum_{m=1}^{M} \sum_{i=1}^{I} m c_1 H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,t} \leq Y_t + A_t. \tag{2.1}
\]

Upper bound for the number of people in each state of the house list.

\[
\sum_{m=1}^{M} H_{i,m,t} \leq N_{i,t} \quad i = 1, \ldots, I. \tag{2.2}
\]
Upper bound for the number of available names in the rental lists.

\[ M_{j,t} \leq L_{j,t} \quad j = 1, \ldots, J. \tag{2.3} \]

Updating the number of customers in each house list segment.

- For state with recency equal to 1:

\[ N_{1,t+1} = \sum_{m=1}^{M} \sum_{i=1}^{I} X_{i,m,t} + \sum_{j}^{J} X_{j,t}. \tag{2.4} \]

- For states with recency larger than 1.

\[ N_{i,t+1} = N_{i-1,t} - \sum_{m=1}^{M} X_{i-1,m,t} \quad i = 2, \ldots, I. \tag{2.5} \]

Cash flow balance equation.

\[ Y_{t+1} = Y_{t} + A_{t} - OSg_{1}Z_{t} - \left( \sum_{m=1}^{M} \sum_{i=1}^{I} mc_{1}H_{i,m,t} + \sum_{j=1}^{J} c_{2}M_{j,t} \right) - \]
\[ g_{2} \left( \sum_{m=1}^{M} \sum_{i}^{I} X_{i,m,t} + \sum_{j}^{J} X_{j,t} \right) + OS \left( \sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} \right). \tag{2.6} \]

Upper bound for the number of sales.

\[ \sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} \leq I_{t} + Z_{t}, \tag{2.7} \]

where,

\[ S_{i,t} \leq \sum_{m=1}^{M} X_{i,m,t} \quad \forall m, i, \tag{2.8} \]

and,

\[ S_{j,t} \leq X_{j,t} \quad \forall j. \tag{2.9} \]
Inventory balance equation.

\[ I_{t+1} = I_t + Z_t - \sum_{i=1}^{I} S_{i,t} - \sum_{j=1}^{J} S_{j,t}. \] (2.10)

The optimization model at time \( t \) is given by the following stochastic and dynamic programming formulation:

\[
F_t(Y_t, I_t, N_t) = \max_{H_{i,m,t}, M_{j,t}, Z_t, V_{i,m,j}} \{ -OSg_1Z_t - \sum_{m=1}^{M} \sum_{i=1}^{I} mc_1H_{i,m,t} - \sum_{j=1}^{J} c_2M_{j,t} + \\
E_{X_{i,m,t}, X_{j,t}, Y_{j,i,m}} [G_t(Y_t, I_t, N_T, Z_t, M, H, X)] \}
\]

s.t. \ (2.1), \ (2.2), \ (2.3)

and,

\[
G_t(Y_t, I_t, N_t, Z_t, X, M, H) = \max_{S_{i,t}, S_{j,t}, V_{i,j}} OS(\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t}) - g_2(\sum_{m=1}^{M} \sum_{i=1}^{I} X_{i,m,t} + \sum_{j=1}^{J} X_{j,t}) - rl_t - hI_{t+1} + \beta F_{t+1}(Y_{t+1}, I_{t+1}, N_{t+1}).
\]

s.t. \ (2.4), \ (2.5), \ (2.6), \ (2.7), \ (2.8), \ (2.9), \ (2.10),

where \( l_t \) is equal to the amount of unsatisfied demand:

\[
l_t = \sum_{m=1}^{M} \sum_{i=1}^{I} X_{i,m,t} + \sum_{j=1}^{J} X_{j,t} - \sum_{i=1}^{I} S_{i,t} - \sum_{j=1}^{J} S_{j,t}.
\]

Boundary condition: at the end of the planning horizon the names in state \( i \) in the house list have a residual value equal to the lifetime value from \( T \) onwards, and the inventory has a residual value equal to \( v \). Therefore, the boundary condition is given by:

\[
F_T(Y_T, I_T, N_T) = \sum_{i=1}^{I} LF(i)N_{i,T} + vI_T
\]

The first constraint corresponds to the fact that the total marketing cost plus the
reordering cost must be less than or equal to the initial budget plus the exogenous investment. This constraint considers that when the optimal decision is to send $m$ catalogs to a group of customers, all of them receive $m$ mailings even though some customers respond after the first, second or $m-1^{th}$ mailing. Since the average number of responses is low, companies usually do not incur in the cost of purging from the second mailing list those customers that respond after the first mailing, but send a second catalog to the complete segment. The same situation happens with a larger number of mailings.

The second and the third constraints correspond to the upper bounds for the number of people in the house list and in the rental lists, respectively. In constraints (2.4) and (2.5), we update the number of people in the house list at the end of the season. This procedure has implicit a Markov chain representation for the behavior of the customers in the house list, where the states correspond to the recency of the customers. Thus, if a customer receives a catalog, she either decreases her recency to one (if she places an order) or increase her recency by one (if she does not respond with an order). The customer automatically increases her recency if she does not receive a catalog. In constraint (2.6), we update the budget at the end of the season. Constraint (2.7) corresponds to the upper bound for the number of sales; the total sale must be less than or equal to the total available inventory. Constraints (2.8) and (2.9) correspond to the upper bound in the number of sales in each state of the house list and in each rental list respectively, with respect to the number of requests. Finally, constraint (2.10) updates the inventory at the end of the season.

2.2.2 Catalog Mailing Problem

It is not unusual that catalog companies are part of major retail operations and that the inventory levels are managed by the retail stores. The next model assumes that the requests originated from catalogs are always satisfied because they represent a small fraction of the total sales. the model also assumes that all the costs associated to
managing inventory are considered in the global planning model for the retail stores.

**The Model**: the optimization model at time $t$ is given by the following stochastic and dynamic programming formulation (eliminating the presence of the decision variables $Z_t$ in the constraints):

$$F_t(Y_t, N_t) = \max_{H_{i,m,t}, M_{j,v}, V_{i,m,j}} \left\{ -\left( \sum_{m=1}^{M} \sum_{i=1}^{I} mc_i H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,v} \right) + \right.$$

$$\left. \text{Exp}[(OS - g_1 OS - g_2)(\sum_{m=1}^{M} \sum_{i=1}^{I} X_{i,m,t} + \sum_{j=1}^{J} X_{j,v}) + \beta F_{t+1}(Y_{t+1}, N_{t+1})] \right\}$$

s.t. (2.1), (2.2), (2.3), (2.4), (2.5), and

Cash flow balance equation.

$$Y_{t+1} = Y_t + A_t - \left( \sum_{m=1}^{M} \sum_{i=1}^{I} mc_i H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,v} \right) +$$

$$\left( OS - g_1 OS - g_2)(\sum_{m=1}^{M} \sum_{i=1}^{I} X_{i,m,t} + \sum_{j=1}^{J} X_{j,v}) \right). \quad (2.11)$$

Boundary condition:

$$F_T(Y_T, N_T) = \sum_{i=1}^{I} LF(i)N_{i,T}$$

In most applications, the dimensions of the dynamic programming model do not allow to solve optimally the mathematical formulation. Usually, the house list consists of several states with hundreds of customers each. Therefore, in real cases, the dimension of the state space in the mathematical formulation is large. Additionally, the size of the feasible region for the decision variables depends directly on the size of the state space; the larger the state space, the larger the feasible region. Finally, the set of possible outcomes associated with the decision variables also has a dimension that increases with the value of the optimal decision variable, i.e., the larger the number of people that receive catalogs, the larger the size of the set of possible outcomes for the random variable that represents the number of responses.
It is interesting to note that the solutions of the optimization problems described above do not always match the intuition. For instance, it is not always true that the company should spend currently available resources on sending catalogs as long as there remain profitable customers. We have constructed examples where it is better to save part of the current season’s budget to spend during the next season on customers whose profitability is larger than that of current customers. We have also found examples where, under the capital constraint, it is better to send a catalog to a customer with larger recency even though there is an “available customer” with a larger response rate. This case happens when the company is going to lose the customer if he is not activated by means of a catalog (this customer is in the last admissible state). Bitran and Ramalho (1992) state that the general practice of the catalog industry for the mailing strategy is focused in short term performance, i.e., it understates the the long-term impact of the current campaign. Hence, the two counterintuitive situations above show that a formal approach can lead to better decisions.

The following proposition shows that the expected number of customers in the house list converges to a constant when the available number of people in the rental lists is constant from a certain period in the planning horizon onwards. The proof of this proposition can be found in Appendix B.1.

**Proposition 2.1** *The number of customers in the house list converges to a constant (as time grows). The limit is given by:*

\[
N_1 = \frac{\sum_{i=1}^I L_i p_i}{1 - \sum_{j=1}^{i^*} \prod_{k=1}^{j-1} (1 - p_k)}
\]

\[
N_i = N_1 (1 - p_1)(1 - p_2) \ldots (1 - p_{i-1}) \quad \forall i = 2, \ldots, i^*.
\]

**Where:**

1. \( \Pi_{k=1}^{i^*} = 1 \) (by definition).
2. \( L_l \) = number of available names in rental list \( l \), which is constant for \( t > t_0 \), for some \( t_0 \).

3. \( p_l \) = probability that a person in rental list \( l \) responds with a sale.

4. \( p_i \) = probability that a customer in state \( i \) in the house list responds with a sale. For the multiple mailing case, it is the corresponding probability associated to the optimal number of mailings according to the lifetime value.

5. \( i^* \) = last profitable state in the house list (from state \( i^* + 1 \) onwards the lifetime value is zero).

**Proof:** See Appendix B.1.

### 2.3 Lifetime Value of a Customer

In what follows we describe a methodology to calculate the discounted net profit associated to a customer in the house list in an infinite planning horizon (lifetime value of a customer), with infinite budget. Calculating the lifetime value of a customer is equivalent to finding the optimal mailing policy for a customer when there are unlimited resources. This concept plays a central role in the heuristics that are described in the following section.

We consider a Markov chain to model the behavior of the customers in the house list, where the states are defined by their recency. In this representation there is a trapping state, a maximum admissible recency, that a customer reaches when she leaves the house list. In each state the customer can move to the next state increasing her recency or move to the first state with recency equal to one. The transition probabilities depend on the mailing policy. Therefore, the problem of calculating the lifetime value of a customer is equivalent to determining the optimal policy for sending catalogs to each state of the house list.

We define the additional notation:

1. \( LF(i) \) = lifetime value associated to a customer whose current state is \( i \).
2. \( \bar{r}_{i,m} \) = immediate expected profit if a customer in state \( i \) receives \( m \) catalogs during a season.

\[
\bar{r}_{i,m} = (OS - g_1 OS - g_2) p_{i,m} - mc_1
\]

Hence, the mathematical formulation to determine the lifetime value of a customer in state \( i \) is given by:

\[
LF(i) = \max \begin{cases} 
\text{Send } m \text{ catalogs:} & \bar{r}_{i,m} + \beta p_{i,m} LF(1) + \beta (1 - p_{i,m}) LF(i + 1) \forall m \\
\text{Do not send a catalog:} & \beta LF(i + 1).
\end{cases}
\]

(2.12)

**Proposition 2.2** If it is optimal not to send catalogs to a customer in state \( i \) then it is also optimal not to send catalogs to a customer in state \( i + 1 \), assuming that \( p_{i,m} \geq p_{i+1,m} \) for all \( m \).

**Proof:** See Appendix B.1.

The proposition above shows that the first \( i^* \) states are profitable, for some state \( i^* \). The remaining states from \( i^* + 1 \) onwards are not profitable and their lifetime values are equal to zero.

The optimality equation (2.12) has a unique solution, \( LF(i) \), \( \forall i \), that corresponds to the expected discount profit when the optimal stationary policy is implemented (see, e.g., Ross, 1983, Chapter II). In order to find the optimal policy and the value of the maximum expected profit it is necessary to solve the optimality equation (2.12). This can be done using one of several methods, as for example the Policy Improvement method and the Linear Programming method. In particular, in the computational experiments, we use the Policy Improvement method which is based on successive approximations to the optimal solution. The algorithm starts with a feasible policy and in iteration \( k \) computes the left hand side of equation (2.12) using the solution of iteration \( k - 1 \) in the right hand side (see, e.g., Ross (1983, p. 38) for a complete description of this algorithm). The algorithm converges to the optimal solution under
the following assumptions: (i) bounded rewards, (ii) discount factor less than one, and (iii) finite state space; all of them are satisfied in our formulation.

For the single mailing case (where at most one mailing takes place during every season), we develop a simpler algorithm where the maximum number of iterations is bounded by the number of states in the house list. The algorithm is described in Appendix B.1 and shows that an implicit expression for the lifetime value in terms of the last profitable state, $i^*$, is given by (to simplify the notation, the index that represents the number of mailings, $m=1$, is omitted):

$$LF(1) =$$

$$\frac{\bar{r}_1 + \beta(1-p_1)\bar{r}_2 + \beta^2(1-p_1)(1-p_2)\bar{r}_3 + \ldots + \beta^{i^*-1}(1-p_1)(1-p_2)\ldots(1-p_{i^*-1})\bar{r}_{i^*}}{1 - \beta p_1 - \beta^2(1-p_1)p_2 - \beta^3(1-p_1)(1-p_2)p_3 - \ldots - \beta^{i^*}(1-p_1)(1-p_2)\ldots(1-p_{i^*-1})p_{i^*}},$$

and,

$$LF(i) = \bar{r}_i + \beta p_i LF(1) + \beta(1-p_i)LF(i+1) \quad \forall i = 2, \ldots, i^*,$$

$$LF(i) = 0 \quad \forall i = i^* + 1, \ldots, I,$$

where $p_i$ is equal to the response rate if a customer in state $i$ receives one catalog.

For the particular case of $\beta = 1$, the expression above simplifies to:

$$LF(1) = \frac{\bar{r}_1}{(1-p_1)(1-p_2)\ldots(1-p_{i^*})} + \frac{\bar{r}_2}{(1-p_2)(1-p_3)\ldots(1-p_{i^*})} + \ldots + \frac{\bar{r}_{i^*}}{(1-p_{i^*})},$$

and,

$$LF(i) = \bar{r}_i + (1-p_i)\bar{r}_{i+1} + (1-p_i)(1-p_{i+1})\bar{r}_{i+2} + \ldots + (1-p_i)(1-p_{i+1})\ldots(1-p_{i^*-1})\bar{r}_{i^*} + [p_i + p_{i+1}(1-p_i) + p_{i+2}(1-p_i)(1-p_{i+1}) + \ldots + p_{i^*}(1-p_i)(1-p_{i+1})\ldots(1-p_{i^*-1})]LF(1).$$

The expressions above have the following interesting interpretation: $LF(i)$ is equal to the sum over all the profitable states in the house list of their immediate expected profits times the expected number of visits to those states starting from state $i$. In what follows we derive this result: suppose that a customer has a current state equal
to 1. Let $\hat{P}_1$ be the probability that this customer returns to state 1 before leaving the house list. This probability is equal to:

$$1 - \hat{P}_1 = (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_{i^*}).$$

Thus, the expected number of visits to state 1 before leaving the house list is given by:

$$\text{Exp[visits to state 1]} = \sum_{k=1}^{\infty} k\hat{P}_1^{k-1}(1 - \hat{P}_1) = \frac{1}{1 - \hat{P}_1}.$$ 

Hence, the expected profit associated to the visits to state 1 is equal to:

$$\frac{\bar{r}_1}{1 - \hat{P}_1} = \frac{\bar{r}_1}{(1 - p_1)(1 - p_2) \ldots (1 - p_{i^*})}.$$ 

Similarly, the number of visits to state $i$ before leaving the house list is given by:

$$\text{Exp[visits to state } i\text{]} = \sum_{k=1}^{\infty} k\hat{P}_i^{k-1}(1 - \hat{P}_i) = \frac{1}{1 - \hat{P}_i},$$ 

with $\hat{P}_i$ equal to the probability to return to state $i$ starting from state $i$ (from state 1 the customer visits state $i$ with probability 1). Thus,

$$1 - \hat{P}_i = (1 - p_i)(1 - p_{i+1}) \ldots (1 - p_{i^*}).$$

Therefore, the total expected profit associated to a customer in state 1 is equal to:

$$LF(1) = \sum_{i=1}^{i^*} \frac{\bar{r}_i}{1 - \hat{P}_i}.$$ 

Similarly, $LF(i)$ can be written as,

$$LF(i) = \hat{P}_i LF(1) + \bar{r}_i + (1 - p_i)\bar{r}_{i+1} + (1 - p_i)(1 - p_{i+1})\bar{r}_{i+2} + \ldots + (1 - p_i)(1 - p_{i+1}) \ldots (1 - p_{i^* - 1})\bar{r}_{i^*}.$$
where an equivalent expression for $\hat{P}_i$ is given by:

$$\hat{P}_i = p_i + p_{i+1}(1-p_i) + p_{i+2}(1-p_i)(1-p_{i+1}) + \ldots + p_{i+1} \ldots (1-p_{i+1} \ldots (1-p_{i+1}) \ldots (1-p_{i-1}))$$

Hence, the expected number of visits to any state from state $i$ is equal to the expected number of visits to that state from state 1 times the probability of returning to 1 from state $i\,(\hat{P}_i)$. In $LF(i)$ we also include the expected profits associated to states with larger recencies than state's $i$ recency that are visited before the first passage to state 1.

In the case of limited budget, it is possible to show that the lifetime value of a customer is an increasing and concave function of the initial budget. It is possible to show that there is a budget, $B^*$, such that the lifetime value of a customer starting with a budget greater than $B^*$ is equal to the lifetime value with unlimited budget.

### 2.4 Heuristics

In this section we describe the heuristics developed to find "good approximations" to the optimal solutions of the optimization problems described in Section 2.2. Initially, we describe two heuristics for the catalog mailing problem; later we introduce three heuristics for the general problem that includes the financial impact of inventory.

#### 2.4.1 Heuristics for the Catalog Mailing Problem

In what follows we describe two heuristics to determine mailing policies for the case where the catalog company is part of a major retail operations.

**HEURISTIC 1.1: HEUR 1.1**

This heuristic sorts the states in the house list and the rental lists in decreasing order of the lifetime values (the rental lists are considered as particular states where the customer leaves the house list immediately if she does not respond with a sale).
Catalogs are then sent according to this order until either the budget constraint is reached or there are no more available customers with positive lifetime value. The number of mailings for every customer that receives a catalog is also determined by the corresponding lifetime value (only single mailing is allowed for the rental lists). We introduce the following additional notation:

1. \( LF(j) = \) Life time value associated to rental list \( j \).

\[
LF(j) = \bar{r}_j + \beta_{p_j} LF(1) \quad \text{and} \quad \bar{r}_j = (OS - g_1 OS - g_2)p_j - c_2
\]

2. \( K = \) total number of rental lists and states in the house list \((K = I + J)\).

3. \( d(k) = \) number of people in state \( k \) that receive catalogs \((k \text{ can be a rental list})\).

4. \( dec(k) = \) optimal number of mailings to state \( k \) in the house list according to the lifetime value.

5. \( N(k) = \) number of people in state \( k \).

6. \( c(k) = \) marketing cost associated to state \( k \).

Description of heuristic 1.1 at the beginning of season \( t \).

Step 0: Sorting.

\( s(k) = \) state of the \( k^{th} \) largest life time value.

Step 1: Initialization.

\[
k = 1,
B = \text{Initial budget at season } t,
d(k) = 0 \quad \forall k.
\]

Step 2: Stop if the remaining budget is zero.

If \((B = 0)\) then GOTO Step 4.

Step 3: Determining the number of catalogs to send to state \( k \).
If \((LF(s(k)) > 0)\) then
\[d(s(k)) = \min \{N(s(k)); B/c(s(k))\}\]
\[B = B - c(s(k))d(s(k))\]
\[k = k + 1\]
if \((k > N)\) then
GOTO Step 4
else
GOTO Step 2.

Step 4: STOP.

The following proposition shows that heuristic 1.1 is optimal when there is no budget constraint.

**Proposition 2.3** Heuristic 1.1 is optimal when there is unlimited amount of money for investment in every season.

**Proof:** In the Catalog Mailing Problem there is only one constraint that involves more than one (in fact, all) clients. This restriction becomes redundant when unlimited amounts of money are available for investment. Therefore, the optimization problem separates by customers; these problems are equal to the optimization problems for calculating lifetime values. Hence, the optimal solution of the global problem is given by the optimal solution of each individual problem.  

**Heuristic 1.2: HEUR 1.2**

This heuristic is a modification of HEUR 1.1 to take into account the following effect. Suppose that the optimal decision, given by the lifetime value, is to send 2 catalogs to every customer in state \(i\). If at the current period, we only send one catalog to every customer in state \(i\), the lifetime value is smaller than it would be if the optimal policy is implemented, but we spend half of the money we would spend on marketing under the optimal policy. This extra amount of money can be spent on sending catalogs to additional (twice as many) customers. Therefore, there is a trade off between the decrement in the lifetime value when the optimal decision
is not implemented and the additional profit associated to more customers. This modification is only relevant when there is limited amount of money for investment. We present the following example to clarify the above tradeoff. Suppose there are two states in the house list; the second one is the trapping state. Let us consider a probability of 0.15 that a customer in state 1 responds with a sale if she receives one catalog, and a probability equal to 0.2 if she receives two catalogs, a marketing cost of $1.0, an average size of the order equal to $50, a variable cost per sale (including the cost of the good) equal to $25, and a discount rate of 1. In this case the optimal decision (when computing the lifetime value) is to send two catalogs to state 1 with a lifetime value equal to $3.75. In what follows assume that we have money to send only two catalogs and there are two customers in state 1. One alternative is to send two catalogs to only one customer with an expected profit equal to $3.75 or to send one catalog to each customer with a expected profit of $3.31 for each one (assuming that from the next period onwards we have unlimited budget for investment). Therefore, given the capital constraint, it is better in this case to send catalogs to both customers. The following heuristic captures this effect.

To simplify the notation, we assume that a maximum of two mailings can be sent in every season. We introduce the additional notation:

1. $LF(i, m) =$ lifetime value associated to state $i$ if we send $m$ catalogs in the current season (from the next season onwards the optimal decision is implemented).

2. $d(i, m) =$ number of people in state $i$ that receive $m$ catalogs.

We redefine the lifetime values for the states in the house list as follows:

If $(dec(i) = 0)LF(i, 1) = LF(i, 2) = -\infty$
If $(dec(i) = 1)LF(i, 1) = LF(i), and LF(i, 2) = -\infty$
If $(dec(i) = 2)$then
   
   $LF(i, 1) = -c_1 + (OS - g_1 OS - g_2)p_{i, 1} + \beta p_{i, 1} LF(1) + \beta(1 - p_{i, 1}) LF(i + 1)$
   
   $LF(i, 2) = LF(i).$
The value of the decision variables $d(i, m)$ is given by the solution of the following linear programming problem:

$$\max \sum_{i=1}^{I} \sum_{m=1}^{2} LF(i, m)d(i, m) + \sum_{j=1}^{J} LF(j)d(j)$$

s.t.

$$\sum_{i=1}^{I} \sum_{m=1}^{2} mc_1d(i, m) + \sum_{j=1}^{J} c_2 d(j) \leq Y_t$$

$$\sum_{i=1}^{I} d(i, m) \leq N_{i,t} \quad \forall i = 1, \ldots, I$$

$$d(j) \leq L_j \quad \forall j = 1, \ldots, J,$$

$$d(i, m) \geq 0, d(j) \geq 0 \quad \forall i, m, j.$$

### 2.4.2 Heuristics for the Catalog Mailing Problem with Aggregate Inventory Costs

In what follows we describe three heuristics for the problem with cash flow constraints and inventory level constraints. In these heuristics we have to decide simultaneously the mailing and reordering policies.

**Heuristic 2.1: HEUR 2.1**

This heuristic is based on the News Boy problem for a one period horizon. We first describe the probability distribution for the total demand. The number of requests associated to a state in the house list or to a rental list is a binomial random variable, whose parameters depend on the mailing policy and on the response rate. In general, more than a hundred people receive catalogs in each state of the house list and in each rental list. Therefore, a good approximation for the distribution of the total
demand is the normal distribution with mean, \( \mu \), and variance, \( \sigma^2 \), equal to:

\[
\mu = \sum_{m=1}^{M} \sum_{i=1}^{I} p_{i,m} H_{i,m,t} + \sum_{j=1}^{J} p_j M_{j,t},
\]

and,

\[
\sigma^2 = \sum_{m=1}^{M} \sum_{i=1}^{I} p_{i,m}(1 - p_{i,m}) H_{i,m,t} + \sum_{j=1}^{J} p_j (1 - p_j) M_{j,t}.
\]

In what follows we determine the optimal amount of goods to order in a one period problem, considering a normal distribution for the demand. We also consider no residual value for the unsold products. We define \( g(Z, I) \) as the total expected profit for a one period problem if we start with \( I \) units of products and order \( Z \) additional goods. Therefore, the function \( g(Z, I) \) is equal to:

\[
g(Z, I) = -g_1 OSZ + \int_0^{Z+I} (OSz - h(z+Z-I-x)) f(x) dx + \int_{Z+I}^{\infty} (OS(Z+I) - r(x-Z-I)) f(x) dx - g_2 \int_{0}^{\infty} x f(x) dx.
\]

where the demand has a probability density function equal to \( f(x) \). Hence, to obtain the "optimal reordering amount" we set the derivative of \( g(Z, I) \) with respect to \( Z \) equal to zero, which implies:

\[
F(Z + I) = \frac{OS + r - g_1 OS}{OS + r + h},
\]

where \( F(x) \) denotes the cumulative distribution function for the demand. Therefore, the optimal reordering amount, \( Z \), satisfies the following inequality:

\[
Pr\{x \leq Z + I\} = \frac{OS + r - g_1 OS}{OS + r + h}.
\]

Hence, using the normal distribution for the demand, the expression for the "optimal reordering amount" is given by:

\[
Z = \sigma a + \mu - I,
\]
and \( F_y(a) = (OS + r - g_1 OS)/(OS + r + h) \) where \( y \) has a standard normal distribution.

We notice that the reordering amount is a function of the number of catalogs to be sent in the current season, because the mean and the variance of the normal distribution are a function of the mailing policy. Therefore, the reordering amount in the cash flow constraint in the original problem is replaced by this "optimal reordering amount". The rest of the heuristic is similar to heuristic 1.1, i.e., we send catalogs according to the decreasing order of the lifetime values until either the cash flow constraint is binding or there are no more available customers.

**Description of heuristic 2.1 at the beginning of season \( t \).**

**Step 0: Sorting.**

\[ s(k) = \text{state of the } k^{th} \text{ largest life time value}. \]

**Step 1: Initialization.**

\[ k = 1, \]
\[ B = \text{Initial budget at season } t, \]
\[ d(k) = 0 \quad \forall k, \]
\[ I = \text{initial inventory at the beginning of season } t, \]
\[ Z_t = 0. \]

**Step 2: determining the number of catalogs to send to state \( k \).**

\[ d^* = \text{maximum value that satisfies the following 2 inequalities:} \]
\[ g_1 OS \left[ a \sqrt{\left( \sum_{l=1}^{k-1} p_{s(l)}(1 - p_{s(l)})d(s(l)) + p_{s(k)}(1 - p_{s(k)})d^*) + \sum_{l=1}^{k-1} p_{s(l)}d(s(l)) + p_{s(k)}d^* - I \right)} + \sum_{l=1}^{k-1} \text{dec}(s(l))c(s(l))d(s(l)) + \text{dec}(s(k))c(s(k))d^* \right] \leq B \] (i),
\[ \sum_{l=1}^{k-1} \text{dec}(s(l))c(s(l))d(s(l)) + \text{dec}(s(k))c(s(k))d^* \leq B \] (ii).
\[ d(s(k)) = \min\{d^*, N(s(k))\}. \]

**Step 3: Stopping criterion.**

\[ k = k + 1 \]
if \((k > N)\) then

\text{GOTO Step 4}

else

\text{GOTO Step 2.}

**Step 4:** Calculating the reordering amount.

\[
Z_t = \max\{0, a\sqrt{(\sum_{i=1}^{K} p_s(i)(1 - p_s(i))d(s(i))) + \sum_{i=1}^{K} p_s(i)d(s(i))} - I\}
\]

**Step 5:** STOP.

**Heuristic 2.2: HEUR 2.2**

This heuristic is similar to heuristic 2.1. The only difference is that, when solving the News Boy problem, it considers a positive residual value for the unsold products. This residual value is equal to the discounted cost of the goods that can be used to satisfy the demand in the next period. Therefore, the modified "optimal reordering amount" is given by the equation:

\[
F(Z + I) = \frac{OS + r - g_1OS}{OS + r - \beta g_1OS + h}.
\]

The heuristic 2.3 is equal to heuristic 2.2 replacing the reordering amount \(Z\) by this expression that includes the residual values for the unsold products.

**Heuristic 2.3: HEUR 2.3**

Finally, in this section we modify the heuristic 2.2 to include the same effect described in heuristic 1.2: the trade-off between the decrement in the lifetime value when the optimal mailing policy is not implemented and the additional profit associated to an extra customer that receives a catalog. In this case, before calculating the optimal mailing policy (as in heuristic 2.1, Step 2), we solve the same linear problem as in heuristic 1.2 to determine in which cases we send less catalogs than the optimal number of catalogs determined by the lifetime values.
2.5 Upper Bound

In this section we describe an upper bound for the optimization model, which is useful to determine the performance of the heuristics described in the previous section.

**Proposition 2.4** An upper bound for the optimization problems described in Section 2.2 is the solution of the deterministic versions of the stochastic models, where the random variables are replaced by their expected values.

**Proof:** The proof is straightforward and the details will be omitted. It is based on successive applications of Jensen's inequality and the concavity of the maximization of a linear programming problem as a function of the right hand side. 

Therefore, the upper bound for the catalog mailing problem with aggregate inventory costs is given by:

\[
UB = \max \{-OSg_1 \sum_{t=1}^{T} \beta^{t-1} Z_t - c_1 \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{i=1}^{I} \beta^{t-1} H_{i,m,t} - c_2 \sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} M_{j,t} +
\]

\[
OS \sum_{t=1}^{T} \sum_{i=1}^{I} \beta^{t-1} S_{i,t} + \sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} S_{j,t} - g_2 \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{i=1}^{I} \beta^{t-1} p_{i,m} H_{i,m,t} -
\]

\[
g_2 \sum_{i=1}^{T} \sum_{j=1}^{J} \beta^{t-1} p_j M_{j,t} - r \sum_{t=1}^{T} \beta^{t-1} l_t - k \sum_{t=1}^{T} \beta^{t-1} l_t + \sum_{i=1}^{I} \beta^T LF(i) N_{i,T+1}
\]

s.t.

\[
OSg_1 Z_t + \sum_{m=1}^{M} \sum_{i=1}^{I} c_1 H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,t} \leq Y_{t-1} + A_t \quad \forall t = 1, \ldots, T. \tag{2.13}
\]

\[
I_{t+1} - I_t - Z_t + \sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} = 0 \quad \forall t = 1, \ldots, T. \tag{2.14}
\]

\[
\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} - I_t - Z_t \leq 0 \quad \forall t = 1, \ldots, T. \tag{2.15}
\]
\[ S_{i,t} - \sum_{m=1}^{M} p_{i,m} H_{i,m,t} \leq 0 \quad \forall m, i, t. \]  
(2.16)

\[ S_{j,t} - p_{j} M_{j,t} \leq 0 \quad \forall j, t. \]  
(2.17)

\[ \sum_{m=1}^{M} H_{i,m,t} - N_{i,t} \leq \forall i, t. \]  
(2.18)

\[ M_{j,t} \leq L_{j,t} \quad \forall j, t. \]  
(2.19)

\[ N_{1,t+1} - \sum_{m=1}^{M} \sum_{i=1}^{I} p_{i,m} H_{i,m,t} - \sum_{j} p_{j} M_{j,t} = 0 \quad \forall t. \]  
(2.20)

\[ N_{i,t+1} - N_{i-1,t} + \sum_{m=1}^{M} p_{i-1,m} H_{i-1,m,t} \quad \forall i \neq 1, t. \]  
(2.21)

\[ Y_{t+1} - Y_{t} - A_{t} - OSg_{1} Z_{t} - (\sum_{m=1}^{M} \sum_{i=1}^{I} mc_{1} H_{i,m,t} + \sum_{j=1}^{J} c_{2} M_{j,t}) + \]
\[ - g_{2}(\sum_{m=1}^{M} \sum_{i} p_{i,m} H_{i,m,t} - \sum_{j} p_{j} M_{j,t}) - OS(\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} = 0 \quad \forall t. \]  
(2.22)

### 2.6 Computational Experiments

In this section we study the performance of the heuristics in several computational experiments. We use Monte Carlo simulations to estimate the company's expected profit during the planning horizon under the application of the different heuristics. For the catalog mailing problem the two heuristics described in Section 2.4 give the mailing policies. We use a binomial distribution to represent the number of people that place an order in each state of the house list and each rental list; the number of trials is equal to the total number of people that receive catalogs in each segment and the probability of success corresponds to the average response rate. For the catalog mailing model with aggregate inventory costs, the three heuristics described in Section 2.4 give additionally the reordering policies, i.e. the stock level to buy at the beginning of every season.

In the experiments, we implement the mailing and reordering policies given by the heuristics and simulate the number of responses according to each mailing policy.
Thus, we compute the profit for each particular outcome of the random variable during the planning horizon. Finally, averaging the profits given by repeated simulations we obtain an estimate of the expected profit. We stop the simulations when the coefficient of variation (standard deviation over expected profit) is less than 0.1%. The heuristics are compared with the corresponding upper bounds described in Section 2.5.

In the simulations, we use realistic data based on information given by the manager of a catalog sales company and public data obtained from 1990/1991 Statistical Fact Book, Direct Marketing Association.

The initial budget utilized as a reference point, \( Y_0 \), is equal to the minimum budget such that from that budget onwards the linear programming problem in the upper bound does not change its objective function.

We use the following set of data in the computational experiments:

Planning horizon: 18 seasons

Number of states in the house list: 12

One rental list with 100000 available names each season.

Marketing cost for the house list: $0.6

Marketing cost for the rental list: $0.7

Average size of the order: $70.

Cost of the goods: 35% of the average sale

Variable cost per request: $1

Penalty cost for rejecting an order: 0

Holding inventory cost: 25% of the good's cost per year

Initial inventory: 0

Exogenous Investment \( A_t = 0 \) \( t \in \{2, \ldots, T\} \)
Discount rate: 0.975

Response rate of the rental list: 1.5%

Response rates for the states in the house list in increasing order of the recency:

Single mailing: 4.4% 4.3% 4.0% 3.6% 3.3% 2.8% 2.5% 2.3% 2.0% 1.8% 1.0% 0.0%

Two mailings: 5.8% 5.6% 5.3% 4.9% 3.7% 3.0% 2.5% 2.3% 2.0% 1.8% 1.0% 0.0%

2.6.1 Catalog Mailing Problem

This case considers that all the demand is satisfied and there are no costs associated to managing inventory. In the data set, we use a maximum of two mailings per season for customers in the house list and a single mailing for customers in the rental list. The performance of the two heuristics is shown in Table 2.1.

Table 2.1: Catalog Mailing Model with Two Mailings

<table>
<thead>
<tr>
<th>((A_1/Y_0)\times 100)</th>
<th>(A_1=)Initial Budget (US$)</th>
<th>HEUR 1.1 (w/r) UB</th>
<th>HEUR 1.2 (w/r) UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1%</td>
<td>80</td>
<td>84.8%</td>
<td>87.7%</td>
</tr>
<tr>
<td>1%</td>
<td>800</td>
<td>94.8%</td>
<td>99.8%</td>
</tr>
<tr>
<td>10%</td>
<td>8000</td>
<td>96.8%</td>
<td>99.9%</td>
</tr>
<tr>
<td>25%</td>
<td>20000</td>
<td>96.7%</td>
<td>99.9%</td>
</tr>
<tr>
<td>50%</td>
<td>40000</td>
<td>97.7%</td>
<td>99.7%</td>
</tr>
<tr>
<td>100%</td>
<td>80000</td>
<td>99.9%</td>
<td>99.9%</td>
</tr>
</tbody>
</table>

The first column contains the initial budget with respect to the reference budget, \(Y_0\). The second column is the initial budget in dollars. Columns 3 and 4 contain the performance of heuristics 1.1 and 1.2 with respect to the upper bound. We observe that the behavior of the two heuristics improves as long as the initial budget increases, having heuristic 1.2 an excellent performance with an initial budget greater than or equal to 1% of \(Y_0\). We also observe that heuristic 1.2 is always better than heuristic 1.1, with a more significative difference when the initial budget is small. The expected profit associated to each experiment can be found in Table B.1 in Appendix B.2.
We also observe that the uncertainty in this formulation does not have an important impact when we solve “real problems”. In general, companies mail to a large number of customers. Therefore, the fraction of responses converges to a constant in probability. This effect can be specially appreciated when the initial budget is large, because the size of the mailings is also large. Moreover, it is possible to show that, with unlimited budget, the stochastic formulation for the catalog mailing problem is equal to the corresponding deterministic model. The proof is straightforward and is not presented in this chapter.

Finally, we remark the simplicity in the implementation of the heuristics. Both are based on a set of indices that are computed once at the beginning of the planning horizon. These indices are associated to each state of the house list and to each rental list. Therefore, to implement heuristic 1.1, the manager has to send catalogs to the customers according to the decreasing order of these indices until either he reaches the budget or there is no more available customers.

2.6.2 Catalog Mailing Problem with Aggregate Inventory Costs

For the catalog mailing problem with aggregate inventory costs, we present two sets of experiments. In the first set of experiments we consider the single mailing case. In the second group of experiments, we allow a maximum of two mailings for the customers in the house list.

For the single mailing case, the performance of heuristics 2.1, 2.2 is shown in Table 2.2 (heuristic 2.2 and 2.3 are the same for the single mailing case).

The first column is the initial budget with respect to the reference budget \( Y_0 \). The second column is the initial budget in dollars. Columns 3 and 4 show the behavior of the heuristics 2.1 and 2.2 with respect to the upper bound respectively. We observe that heuristic 2.2 has a better performance, with an achievement of more than 95% with respect to the upper bound when the budget is greater than 50% of \( Y_0 \). Heuristic 2.2 is better than heuristic 2.1 because it considers that the unsold products in a
Table 2.2: Catalog Mailing Model with Aggregate Inventory Costs: Single Mailing Case

<table>
<thead>
<tr>
<th>$(A_1/Y_0) \times 100$</th>
<th>$A_1=$Initial Budget (US$)</th>
<th>HEUR 2.1 w/r UB</th>
<th>HEUR 2.2 w/r UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>5900</td>
<td>65.0%</td>
<td>86.7%</td>
</tr>
<tr>
<td>10%</td>
<td>11800</td>
<td>75.6%</td>
<td>90.3%</td>
</tr>
<tr>
<td>25%</td>
<td>29500</td>
<td>84.0%</td>
<td>94.3%</td>
</tr>
<tr>
<td>50%</td>
<td>59000</td>
<td>88.6%</td>
<td>95.6%</td>
</tr>
<tr>
<td>75%</td>
<td>88500</td>
<td>92.2%</td>
<td>97.1%</td>
</tr>
<tr>
<td>100%</td>
<td>118000</td>
<td>94.0%</td>
<td>98.2%</td>
</tr>
</tbody>
</table>

season can be sold in the next period with the corresponding opportunity cost of the capital. However, the reordering policy in heuristic 2.1 considers that the unsold products in a season have a residual value equal to zero. The detailed information about the expected profits can be found in Appendix B.2, Table B.2.

In the second set of experiments, we allow a maximum of two mailings within a season for the house list. The performance of the four heuristics is shown in Table 2.3.

Table 2.3: Catalog Mailing Model with Aggregate Inventory Costs: Two Mailings.

<table>
<thead>
<tr>
<th>$(A_1/Y_0) \times 100$</th>
<th>$A_1=$Initial Budget (US$)</th>
<th>HEUR 2.1 w/r UB</th>
<th>HEUR 2.2 w/r UB</th>
<th>HEUR 2.3 w/r UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>6150</td>
<td>63.9%</td>
<td>83.4%</td>
<td>87.5%</td>
</tr>
<tr>
<td>10%</td>
<td>12300</td>
<td>72.9%</td>
<td>87.4%</td>
<td>90.8%</td>
</tr>
<tr>
<td>25%</td>
<td>30750</td>
<td>81.4%</td>
<td>89.2%</td>
<td>94.1%</td>
</tr>
<tr>
<td>50%</td>
<td>61500</td>
<td>86.0%</td>
<td>92.9%</td>
<td>94.1%</td>
</tr>
<tr>
<td>75%</td>
<td>92250</td>
<td>90.9%</td>
<td>95.7%</td>
<td>95.8%</td>
</tr>
<tr>
<td>100%</td>
<td>123000</td>
<td>94.1%</td>
<td>98.3%</td>
<td>98.4%</td>
</tr>
</tbody>
</table>

In this case we observe the same pattern of behavior for the three heuristics as in the previous set of experiments. In this case heuristic 2.3 gives better results than heuristics 2.2, because when the budget is small (specially during the first seasons in the planning horizon) it is better to activate more customers than to send the optimal
number of catalogs to few of them.

In the Catalog Mailing Model with Aggregate Inventory Costs, it is still true that with unlimited budget it is optimal to send catalogs to all the profitable customers. However, we cannot guarantee that the reordering policy given for the heuristics is the optimal reordering policy. Even, for the single period problem, the optimal mailing and reordering policy cannot be determined in a close form solution.

Finally, we remark that heuristics 2.2 and 2.3 have a very good performance, with solutions that are close to the solutions given by the upper bounds. The quality of the solutions improves as long as the initial budget increases. With a budget from 25% onwards with respect to the reference budget, heuristic 2.3 reaches more than 94% of the upper bound. It is reasonable to assume that in practice companies have access to loans when they have a profitable business. Therefore, they can finance at least a 25% of the total possible investment.

In what follows, we present a set of computational experiments to compare the effect of multiple mailings versus a single mailing. The following parameters are considered: 60 seasons, 12 states in the house list with zero initial customers, one rental list with 100000 names each season, an initial budget of US$60000, a marketing cost of $0.6 for the house list and $0.7 for the rental list, an average size of the order equal to $70, a variable cost per sale (including the cost of the good) equal to 35% of the average sale, a variable cost per sale equal to $1, a rejection cost of zero, and a discount rate of 0.975. The initial inventory is equal to zero.

The average response rate of the rental list is 1.5%. The response rates for the states in the house list in increasing order of the recency is equal to:

Single mailing : 3.0% 2.8% 2.6% 2.3% 2.0% 1.8% 1.5% 1.3% 1.0% 0.8% 0.1% 0.0%.
Two mailings : 5.8% 5.6% 5.3% 4.9% 4.7% 4.4% 3.5% 3.3% 2.5% 1.8% 1.0% 0.0%.

We compare the accumulated cash at the beginning of every season, the total number of people in the house list at the end of each season, and the number of catalogs mailed every season. The results are shown in figures 2-1, 2-2, 2-3 respectively. We observe that during the first five seasons the company loses money with the single mailing strategy; only after season thirteen it recovers the initial invest-
ment. However, with the two mailings strategy, the company recovers its investment in only six seasons, facing losses during the first three campaigns. The acceleration in the investment recovery period is due to the increment in the customer response rates produced by the multiple mailings. We remark that implementing the multiple mailing strategy, the company can always send a single mailing to some (or all) states during the planning horizon. Therefore, the multiple mailing strategy is at least as good as the single mailing strategy. We also observe that the number of people in the house list converges to a constant in the multiple mailing case after approximately season forty five. This effect is not observed in the single mailing case because we need a larger planning horizon to capture this asymptotic behavior. Finally, the number of catalogs mailed every season also converges to a constant, because the profitable customers do.

Informal evidence suggests that firms in the catalog sales industry often go bankrupt. The methodology developed in this chapter allows us to do risk analysis to determine what fraction of the times the company would run out of cash (at the beginning of some period in the planning horizon the company has no money left for new mailings). With this purpose, after each simulation, we could compute an index equal to one if the company runs out of cash for that specific outcome of the random demand or equal to zero otherwise. Averaging the indices given by repeated simulations we can obtain an estimate of the probability of going bankrupt.
Figure 2-1: Accumulated Cash at the End of Every Season

Figure 2-2: Total Number of People in the House List
2.7 Conclusions and Extensions

This chapter has presented two mathematical models, which vary in the way suppliers interact with the catalog companies, to find the mailing and reordering policies that maximize the expected profit of a company in the catalog sales industry. Stochastic demand and dynamic evolution of the customers within the house list were considered. Optimal solutions are hard to compute for real size problems. Therefore, ad-hoc heuristics were implemented based on the solutions of simplified versions of this problem. Computational experiments showed that these heuristics give satisfactory results. Without loss of generality, we only use recency to describe the states in the house list, which is by far the most important factor to predict customers' response rate.

The models proposed in this chapter allow us to study how the optimal solution changes with market conditions, as for example price, cost of goods, response rates, and mailing cost. The models can also be used to do risk analysis, i.e. to study what fraction of the times the company runs out of cash. Informal evidence indicates that firms in the catalog industry often go bankrupt. Running simulations, we can easily
compute if the company runs out of cash for every outcome of the stochastic demand during the planning horizon. With this information, we can estimate the probability of going bankrupt. This suggests that it would be interesting to incorporate the probability of bankruptcy into the firm's objective function; we leave this topic for future research.

The tactical model presented in this chapter establishes the optimal aggregated levels of reordering and the optimal number of catalogs to be sent to each segment in the house list and to each rental list at the beginning of every season. As a topic of future research, a hierarchical approach could be pursued to disaggregate the total reordering amount into individual items and to schedule the mailings during the corresponding season.

The current formulation considers a penalty cost for the unsatisfied demand. An alternative approach that could be studied is to replace the penalty cost by a service level constraint that assures the demand is satisfied with a given probability.
Chapter 3

Pricing a Perishable Product in a Retail Store

This chapter studies intertemporal pricing strategies when selling a perishable product. We consider a seller that faces a stochastic arrival of customers with heterogeneous valuations of the product. The optimal pricing strategies are characterized as functions of time and the inventory. We also find necessary and sufficient conditions for the optimal pricing strategy. This model is extended to include periodic pricing reviews and a monotonically decreasing price path constraint. Finally, we generalize the model to the case of a company that has two retail stores oriented to two different market segments. We develop heuristics to solve real size problems when optimal solutions cannot be computed; these heuristics have a satisfactory performance with respect to an upper bound we derive for the optimization problem.

3.1 Introduction

Pricing a product is one of the most important decisions a seller has to make. As quoted in Monroe (1979) “more and more, today’s pricing environment demands better, faster, and more frequent pricing decisions than ever before. It is also forcing companies to take a new look at pricing and its role in an increasingly complex marketing climate.” This statement is even more relevant nowadays.
In this chapter we study the optimal pricing strategies for a perishable product. We present a basic model that involves a retail store that sells a seasonal product. The seller faces a stochastic arrival of customers with heterogeneous valuation of the product, i.e., customers have different thresholds, reservation prices, for the maximum price that they are willing to pay for the product. Therefore, the seller has to decide the optimal price for the product at the beginning of every period during the planning horizon. We assume that prices are non-negotiable, i.e., haggling is not allowed. Riley and Zeckhauser (1980) show that if the seller can make a firm commitment in advance on his strategy (prices are credible), and buyers are risk-neutral, the take-it-or-leave-it price strategy is superior to any other possible selling strategy, whether there is learning or the distribution of buyer reservation prices is known, whether the buyer population is finite or infinite, whether there is one object for sale or many. We also assume that the customers' arrival process to the store is independent of the pricing policy; customers are shopping around in a mall, supermarket or commercial street. We characterize the optimal pricing strategies as functions of the inventory and time left in the planning horizon. We also formulate a necessary condition for the optimal solution which is sufficient for a large family of problems.

Later, we extend the basic model to incorporate two features that are often present in practical applications. Sellers usually prefer periodic pricing reviews instead of continuous pricing policies, because they reduce coordination problems and management costs. Secondly, price fluctuations could have a negative impact on consumers' perception of the product's value. Hence, many times, a non-increasing pricing policy is desirable. Optimal solutions are more difficult to compute in this case, specially when solving real size problems. We, therefore, develop heuristics that show a satisfactory performance with respect to an upper bound. Finally, we extend the models above to consider a company that has two retail stores oriented to two different market segments. Every store has its own inventory but a centralized management allows to move merchandise from one store to another at the end of each day taking into account the associated transportation costs. Hence, the goal is to find the pricing strategies for the two stores and the inventory management that maximize the company's total.
expected profit. We also develop an heuristic to find a "good approximation" of the optimal solution.

Traditional pricing strategies based on applying a mark-up to the cost of the article, see Gabor and Granger (1964) and Monroe (1979), are not well suited for our problem. The main factors that determine the pricing policy in our case are the finite planning horizon, the perishability of the products and the fact that after deciding the initial inventory, the cost of the goods are "sunk costs".

There are several papers that study intertemporal pricing models, where a monopolistic seller faces a market of consumers with heterogeneous valuations for the product. These models assume perfect information, i.e., potential consumers permanently know the price of the product. Thus, for example, Kalish (1983) assumes that for nondurable goods, customers buy as soon as the price falls below their reservation prices. Besanko and Winston (1990) extend the model to incorporate the assumption that consumers are intertemporal utility maximizers. These papers determine the optimal pricing policies assuming a known deterministic distribution for the reservation prices. They also consider the uncapacitated problem where the demand is always satisfied. Our problem differs from those described above in that it considers that customers do not have perfect information about the product's price before shopping at the store. Hence, sellers are not able to do a perfect price skimming. We also assume that the reservation price is stochastic, and sellers only know its probability distribution. In our model, a crucial factor that determines the pricing policy is the limited number of units in inventory.

Pasternack (1985) considers the pricing decision faced by a seller of perishable products. He assumes that a single price is posted during the planning horizon and focuses on the pricing and return policy which ensure channel coordination with the supplier. He considers a static pricing policy over the planning horizon.

The remainder of this chapter is organized as follows. In Section 3.2 we present the one store model and characterize the optimal pricing policy. We also extend the formulation to include periodic pricing reviews and non-increasing pricing policies. Section 3.3 formulates the two stores model and develops an heuristic to solve the
mathematical formulation. We also describe an upper bound for the problem that allows us to measure the performance of the heuristic. In Section 3.4 we present computational experiments to compare the pricing policies for the different models and the performance of the heuristics. Finally, in Section 3.5 we present the conclusions and extensions.

3.2 The One Store Model

In this section we study the case where a single retail store sells a seasonal product. For example, fashionable clothing and food products for special holidays usually fall in this category. We assume that the customers' arrival process to the store is independent of the pricing policy, i.e., customers are shopping around in a mall, supermarket, or commercial street. We also assume that customers have heterogeneous valuation of the product which define their reservation prices, i.e., the maximum price that they are willing to pay for the product. Hence, if the product's price is lower than their reservation prices, customers buy the product. We consider that the seller only knows the probability distribution of the customers' reservation price. Thus, she faces the trade-off of loosing a customer due to a high price and loosing the consumer surplus (the difference between the product's price and the customer's reservation price) due to a low price. Hence, at the beginning of every period the seller has to decide the product's price that maximizes the total expected profit during the planning horizon, given the current inventory and the probability distributions for the arrival process and reservation prices. We consider that the seller orders the product at the beginning of the planning horizon and reorders are not allowed.

We first present a model that determines the optimal pricing policy for perishable goods where all price paths are allowed. This model divides the planning horizon in periods of time small enough so that at most one arrival occurs in every time interval. In the limit, when the length of the time intervals goes to zero, we obtain a continuous time pricing model. The model also finds the optimal initial inventory. In
the second model, we incorporate two features that are usually present in practice: prices are decreasing in time and they can only be modified at specific instants during the planning horizon. We model the customers' arrival process as a Poisson process. Before presenting the models, we introduce the following notation:

\[ \lambda = \text{customers' arrival rate.} \]

\[ \Delta t = \text{time interval where at most one arrival occurs.} \]

\[ \beta = \text{discount rate.} \]

\[ C = \text{inventory at the beginning of the planning horizon.} \]

\[ T = \text{planning horizon.} \]

\[ p_{t,c} = \text{optimal price at period } t \text{ if the inventory at the beginning of period } t \text{ is equal to } c. \]

\[ r = \text{customers' reservation price.} \]

\[ f(r) = \text{probability density function for the reservation price.} \]

\[ F(r) = \text{cumulative distribution function for the reservation price.} \]

### 3.2.1 The Basic Model

We define the function \( V_t(c) \) as the maximum discounted expected revenue if the store starts with \( c \) units at the beginning of period \( t \). The objective function at period \( t \) is given by the immediate expected revenue of selling a product at period \( t \) plus the expected revenue from periods \( t - 1 \) onwards (we count the periods in the planning horizon backwards, i.e., 1 is the last period). The probability of selling a unit at price \( p \) in period \( t \) is equal to the probability of an arrival times the probability that the arrival's reservation price is higher than or equal to the current price \( p \). For simplicity,
we assume that the liquidation value of the products is zero. Hence, the mathematical model is given by the following stochastic and dynamic programming formulation:

\[ V_t(c) = \max_{p \geq 0} \{ \lambda \Delta t (1 - F(p))(p + \beta V_{t-1}(c - 1)) + \beta (1 - \lambda \Delta t (1 - F(p)))V_{t-1}(c) \}. \quad (3.1) \]

Boundary conditions:

\[ V_t(0) = 0 \quad \forall t, \]

and,

\[ V_0(c) = 0 \quad \forall c. \]

The following two propositions characterize the optimal pricing policy. The first proposition shows that for a given period in time, the larger the inventory, the smaller the optimal price. In our model, the only mechanism that the seller has to affect the demand is through the pricing policy. Hence, the seller has to reduce the price to increase the demand when the inventory increases. The second proposition shows that as long as the inventory remains constant, the optimal prices are decreasing in time; as time goes by the seller has less possibilities of selling the products. Hence, in general, the optimal pricing policies resulting from the model described above, are non monotonic during the planning horizon, i.e., for a particular outcome of the arrival process and the reservation prices, the optimal price is decreasing in time with jumps that correspond to the instances where the product is sold. Furthermore, the expected price, obtained by taking the expectation over all the possible outcomes for the arrival process and for the reservation prices, is not necessarily a decreasing function of time. We have constructed examples where the current optimal price is smaller than the expected optimal price for the next period (see Example 1 in Appendix C.1).

**Proposition 3.1** For a given period of time, the optimal price is a non-increasing function of the inventory.

\[ p_{t,c} \geq p_{t,c+1} \quad \forall t, c \]
**Proposition 3.2** For a given inventory, the optimal price is a non-increasing function of time.

\[ p_{t,c} \geq p_{t-1,c} \quad \forall t,c \]

**Proof:** See Appendix C.1.

Next we derive useful first order conditions for the problem described above. We first consider the single-period case, where the seller must price one good, there will be at most one buyer and the good perishes after one period. A necessary condition for \( p \) to be the optimal price is that the seller have no incentive to modify this price. The additional revenue obtained by increasing the price by a small amount \( dp \) comes from being able to sell the good at a higher price, the expected benefit this generates is equal to the probability that a customer's reservation price is above \( p \) times the price change: \( (1 - F(p))dp \). Yet this additional revenue comes at a cost since a fraction of customers who were willing to buy the good at the old price are no longer prepared to buy it. This fraction is equal to \( f(p)dp \), thus the seller expects to loose \( pf(p)dp \) because of them. For \( p \) to be the optimal price it must be the case that the gain and loss from a small change in price be the same. It follows that:

\[ pf(p) = 1 - F(p). \quad (3.2) \]

In the multi-period case the benefit associated to increasing the price (and therefore the right hand side of equation (3.2)) remains unchanged. Yet the loss associated by not selling the product is partly offset by the possibility of selling it in the future. Since the expected benefit of selling the product in the future is given by \( f(p)dp\beta(V_{t-1}(c) - V_{t-1}(c-1)) \) it follows that the first order condition in the general case is given by:

\[ (p - \beta(V_{t-1}(c) - V_{t-1}(c-1)))f(p) = 1 - F(p). \quad (3.3) \]

The optimal pricing policy can be found by solving the non linear equation (3.3)
backwards in time. Examples can be easily constructed to show that there may be values of \( p \) satisfying equation (3.3) that do not correspond to the optimal price. A sufficient condition for a price satisfying (3.3) to be optimal is given by requiring that the function \( (1 - F(p))^2 / f(p) \) be decreasing in \( p \). We summarize the results above in the following proposition:

**Proposition 3.3** Assuming that the cumulative distribution function for the reservation price is differentiable and that its density function is defined for all positive values of the price, a necessary condition for the price \( p \) to be optimal at time \( t \) given an inventory equal to \( c \) corresponds to:

\[
p = \frac{1 - F(p)}{f(p)} + \beta(V_{t-1}(c) - V_{t-1}(c - 1))
\]

If the function \( (1 - F(p))^2 / f(p) \) is decreasing in \( p \), then the first order condition has a unique solution and it corresponds to the optimal price.

**PROOF:** See Appendix C.1. 

There are several probability density functions for which the sufficient condition of Proposition 3.3 is satisfied. For example, the exponential and Weibull distributions \((k \geq 1)\).\(^1\) There are other density functions with bounded, convex support for which the first order condition leads to a feasible optimal solution. For example, the uniform distribution in \([0, b]\) has the property mentioned above. In general, when the density has a bounded support, the first order condition is given by the corresponding Kuhn Tucker system of equations.

A stronger sufficient condition for a price satisfying the necessary condition to be optimal is given by requiring that the hazard function associated with the reservation price distribution be increasing, where the hazard function \( H(p) \) is defined as \( H(p) = f(p) / (1 - F(p)) \). This function also has an interesting interpretation: given that the product's price is equal to \( p \) and that the reservation price associated to the current

---

\(^1\)The Weibull distribution with parameters \( k \) and \( \rho \) is equal to
\[
f(p) = k\rho(p)^{k-1}\exp(-\rho p^k) \quad \forall p > 0, \forall k > 0, \rho > 0.
\]
request is larger than or equal to \( p \), \( H(p)dp \) is approximately equal to the probability that the current customer’s reservation price is in the range \([p, p+dp]\). Hence the larger the hazard function, the more likely it is that the seller obtains all the consumer’s surplus,\(^2\) conditional on making the sale.

The optimal initial inventory, \( C^* \), can be found solving the problem:

\[
Z = \max_{C \in \mathbb{Z}^+} \{V_T(C) - gC\},
\]

where \( g \) is the unit cost of the goods. By lemma C.2 in Appendix C.1, we have that the function \( V_T(C) \) is concave as a function of the capacity. Hence, the optimal initial inventory is determined by the inventory such that the marginal increment in the function \( V_T(C) \) is equal to the unit good’s cost. The mathematical condition is given by:

\[
V_T(C^*) - V_T(C^* - 1) \geq g \geq V_T(C^* + 1) - V_T(C^*).
\]

Hence, when solving the dynamic programming formulation to find the optimal pricing policy, we can also compute the optimal initial inventory.

The model also has the property that the optimal pricing policy is constant over the planning horizon when the capacity is large enough. If the planning horizon is divided in \( T \) periods, then the total number of requests is bounded by \( T \). Therefore, for any initial inventory larger than or equal to \( T \), the optimal solution can be found solving the single period problem given by:

\[
\max_{p \geq 0} \{(1 - F(p))p\}.
\]

\(^2\)The consumer surplus is equal to the difference between the reservation price and the product’s price.
3.2.2 Extensions of the Basic Model

In this section we extend the basic model to incorporate two features that are usually present in practice. Often, retail stores sell many different products. Hence, it is difficult to implement a continuous pricing policy for every product. Therefore, sellers usually adopt a periodic pricing review for all products at the same time, where prices are modified once a day, week or month, reducing management costs and coordination problems. The second characteristic that many times is observed in practice is that the price is a non-increasing function of time. Despite the fact that a non-monotonic pricing policy could lead to a larger total profit, a monotonic pricing policy gives more consistent information to consumers about the product's value. In what follows we introduce a model that incorporates the two constraints described above.

We define the function $V_t(c, p_{t+1})$ as the maximum expected revenue from period $t$ onwards if the initial inventory is $c$ and the price in the previous period is $p_{t+1}$. The number of periods in this model is equal to the number of times the prices are allowed to change. Therefore, the mathematical formulation is given by:

$$V_t(c, p_{t+1}) = \max_{0 \leq p \leq p_{t+1}} \left\{ \sum_{j=0}^{\infty} \left[ \min(c, j)p + \beta V_{t-1}(c - \min(c, j), p) \right] \Pr\{j(p) = j\} \right\}. \quad (3.4)$$

Boundary conditions:

$V_t(0, p) = 0 \quad \forall t, p$

and,

$V_0(c, p) = 0 \quad \forall c, p.$

In equation (3.4), $j(p)$ denotes the random variable that represents the number of potential sales in period $t$ if the price is $p$. Its probability mass function is given by:

$$\Pr\{j(p) = j\} = \sum_{k=j}^{\infty} \frac{k!}{j!(k-j)!} (1 - F(p))^j (F(p))^{k-j} \exp^{-\lambda \Delta T} \frac{\lambda \Delta T^k}{k!},$$
where $\Delta T$ is the time interval when the prices remain constant.

Formulation (3.4) also has the property of a constant pricing policy when the capacity is large enough. To prove this property, we observe that the constant pricing policy obtained for formulation (3.1) is also feasible for formulation (3.4), therefore it is also optimal for (3.4).

The mathematical model described in (3.4) is difficult to solve, specially when working with real size problems. The degree of complexity increases for several reasons. First, at every stage of the dynamic programming formulation, we need to keep track of the price in the previous period. Thus, the state space includes an additional continuous variable given by the price. Second, the non-linear problem that is solved every period considers the monotonically decreasing price path constraint. Therefore, the first order condition for the optimal price is given by the Kuhn Tucker conditions instead of setting the first derivative equal to zero. Finally, the most serious difficulty is given by the dependence of $V_t(c, p)$ of $p$. The first order condition depends on the derivative of $V_t(c, p)$ with respect to $p$. It follows that the alternative of reducing the size of the state space by working with a discrete set of prices is not suitable because it is necessary to compute the derivatives mentioned above.

The difficulties described above motivate developing an heuristic to find "good approximations" for the optimal pricing policy. The heuristic is based on the optimal pricing policy for a one period problem, with the possibility of updating the prices after every period in the planning horizon.

**Description of the heuristic: HEUR1**

At period $t$, if the price is $p$, the expected number of buyers in the remainder of the planning horizon is $\lambda t (1 - F(p))$. If the current capacity is equal to $c$, then the total expected profit from period $t$ onwards is given by $p[\min(\lambda t (1 - F(p), c)]$. Therefore, the pricing policy at period $t$ is equal to the solution of the following non-linear problem:

$$\max_{0 \leq p \leq p_{t+1}} \{p \min[\lambda t (1 - F(p), c)]\},$$
where, \( p_{t+1} \) is the price in the previous period. We observe that in the limit when the capacity goes to infinity, the pricing policy given by the heuristic is constant and equal to the optimal pricing policy. We measure the performance of this heuristic in Section 3.4 using the optimal expected profit of the basic model as an upper bound for the optimal expected profit of the constrained problem.

### 3.3 The Two Stores Model

In this section we study the case of a company with two retail stores selling the product to two different market segments. Each store manages its own inventory, however, a global allocation allows to move merchandise from one store to another at the end of each day. The goal is to determine the pricing policies for the stores and the inventory management that maximize the company's total expected profit. We introduce the following additional notation:

\[
(c^1_t, c^2_t) = \text{inventory at the end of period } t \text{ in stores 1 and 2 respectively.}
\]

\[
I^i_t = \text{inventory at the beginning of period } t \text{ in store } i \text{ after adjusting the inventories,}\]

\[
i \in \{1, 2\}.
\]

\[
v = \text{moving cost per unit.}
\]

\[
z_t = \text{number of units moved from one store to another in period } t.
\]

\[
j_i(p) = \text{random variable that represents the number of buyers in store } i \text{ if the price is equal to } p \text{ during one period of time.}
\]

\[
F_i(p) = \text{cumulative density function for the reservation price in store } i, i \in \{1, 2\}.
\]

\[
V_t(c^1_{t+1}, c^2_{t+1}) = \text{maximum expected profit from period } t \text{ onwards if the company starts with an inventory equal to } (c^1_{t+1}, c^2_{t+1}).
\]

Hence, the objective function is given by the immediate expected profit in period \( t \) plus the expected profit from period \( t - 1 \) onwards. The immediate expected profit
is given by the price times the expected number of sales during one period of time. Because there are a limited number of units in inventory, the expected number of sales is the minimum between the inventory and the expected number of buyers. Thus, the model corresponds to the following mathematical formulation:

\[
V_t(c_{t+1}, c_{t+1}^2) = \max_{p_1, p_2, I_t^1, I_t^2} \left\{ p_1 \sum_{i=0}^{I_t^1} i \Pr\{j_1(p_1) = i\} + p_1 I_t^1 \sum_{i=I_t^1+1}^{\infty} \Pr\{j_1(p_1) = i\} + p_2 \sum_{i=0}^{I_t^2} i \Pr\{j_2(p_2) = i\} + p_2 I_t^2 \sum_{i=I_t^2+1}^{\infty} \Pr\{j_2(p_2) = i\} \right\} - v z_t + \beta \mathbb{E}_{j_1(p_1), j_2(p_2)}[V_{t-1}(c_t^1, c_t^2)] \\
\text{s.t.} \\
I_t^1 + I_t^2 = c_{t+1}^1 + c_{t+1}^2 \\
z_t \geq I_t^1 - c_{t+1}^1 \\
z_t \geq I_t^2 - c_{t+1}^2 \\
c_t^1 = I_t^1 - \min(I_t^1, j_1(p_1)) \\
c_t^2 = I_t^2 - \min(I_t^2, j_2(p_2))
\]

\[p_1 \geq 0, p_2 \geq 0, I_t^1 \in \mathbb{Z}^+ \text{ and } I_t^2 \in \mathbb{Z}^+,
\]

where the probability distribution for \(j_i(p)\) is given by:

\[
\Pr\{j_i(p) = j\} = \sum_{k=j}^{\infty} \frac{k!}{j!(k-j)!} (1 - F_i(p))^j (F_i(p))^{k-j} \frac{\exp(-\lambda_i) \lambda_i^k}{k!} \quad i = 1, 2.
\]

The first constraint corresponds to the balance equation for the inventory; the total initial inventory must be equal to the total inventory after moving merchandise from one store to another. Constraints (3.6) and (3.7) define the total number of units moved from one store to another at the beginning of period \(t\). Finally, constraints (3.8) and (3.9) update the inventory at the end of period \(t\). This model can also incorporate monotonically decreasing price path constraints.

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In what follows we present a model whose solution corresponds to an upper bound for the model presented above. In this formulation both stores share the inventory permanently during the planning horizon and prices can be updated after each $\Delta t$ units of time, where $\Delta t$ is small enough so that in total, considering both stores, at most one arrival occurs in every time period.

**Proposition 3.4** The solution of the following problem is an upper bound of the optimization problem described in this section.

\[
V_t(c_{t+1}) = \max_{p_1 \geq 0, p_2 \geq 0} \{ \lambda_1 \Delta t(1 - F_1(p_1))[p_1 + \beta V_{t-1}(c_{t+1} - 1)] + \lambda_2 \Delta t(1 - F_2(p_2))[p_2 + \beta V_{t-1}(c_{t+1} - 1)] + \beta[1 - \lambda_1 \Delta t(1 - F_1(p_1)) - \lambda_2 \Delta t(1 - F_2(p_2))]V_{t-1}(c_{t+1}) \}
\]

where $c_{t+1} = c_{t+1}^1 + c_{t+1}^2$.

**Proof:** The proof is straightforward if we note that any feasible outcome for the selling strategy in the optimization problem can be reproduced for the upper bound formulation, without the costs of moving merchandise from one store to another. Thus, for example, suppose that at time $t$ the optimal prices are $p_1$ and $p_2$ and the optimal initial inventories (after the adjustments) are $I^1$ and $I^2$ for stores 1 and 2, respectively. Then, the upper bound formulation can reproduce the same solution taking $p_1$ and $p_2$ as the optimal prices; if store $i$ sells $I^i$ units during period $t$ then $p_i$ is set equal to a large number ("infinity"). This avoids selling more units than those available at store $i$. Hence, any selling strategy for the optimization problem is also feasible for the upper bound formulation to a lower cost. ∎

The upper bound formulation is solved using the same approach used for the one store model: we solve the first order condition for the optimal pricing strategy backwards in time. This condition is given by:

\[
p^{i}_{c,t} = H_i^{-1}(p) + \beta(V_{t-1}(c) - V_{t-1}(c - 1)). \quad i = 1, 2
\]

We observe that the optimal price in store $i$ consists of the sum of two terms. The first term, $H_i^{-1}(p)$, depends only on the probability distribution for the reservation prices
of the i-th store’s customers. The second term, $\beta(V_{i-1}(c) - V_{i-1}(c - 1))$, takes into account the interaction between both stores. The optimal expected profit, $V_{i-1}(c)$, depends on the optimal pricing policies for both stores from period $t - 1$ onwards, for any initial capacity.

Next, we describe a heuristic developed to find a pricing policy for the two stores model. The heuristic evaluates all possible inventory adjustments at the beginning of each period and chooses the one that maximizes the total expected profit assuming that the prices must be held constant until the end of the planning horizon.

**Description of the heuristic: HEUR2**

The following heuristic determines the pricing policy at the beginning of period $t$ if the initial inventories are equal to $c_1$ and $c_2$ in stores 1 and 2 respectively.

**Step 0: Initialization**

$$\text{prof} = -\infty,$$

$$c = c_1 + c_2$$

$$\text{Inv}_1 = 0$$

$$\text{Inv}_2 = c$$

**Step 1:** Solve the non-linear programming problem below, to find the optimal prices at period $t$ if the inventories at stores 1 and 2 are $\text{Inv}_1$ and $\text{Inv}_2$ respectively.

$$D(\text{Inv}_1, \text{Inv}_2) = \max_{p^1, p^2} \{ p^1 \min [\lambda_1 t (1 - F_1(p^1)), \text{Inv}_1] + p^2 \min [\lambda_2 t (1 - F_2(p^2)), \text{Inv}_2] - v |\text{Inv}_1 - c_1| \}$$

s.t.

$$p^1 \geq 0, p^2 \geq 0.$$

**Step 2:** Check if the current solution leads to an improvement in the objective function.
If($D(Inv_1, Inv_2) > prof$) then

\[ I_1 = Inv_1 \]
\[ I_2 = c - Inv_1 \]
\[ p_t^1 = p^1 \]
\[ p_t^2 = p^2 \]

Endif

\[ Inv_1 = Inv_1 + 1 \]
\[ Inv_2 = Inv_2 - 1 \]

If($Inv_1 \leq c$) Goto Step1

**Step 3:** STOP: the prices at period $t$ in stores 1 and 2 are $p_t^1$ and $p_t^2$ respectively and the initial inventories are equal to $I_1$ and $I_2$.

### 3.4 Computational Experiments

In this section we present computational experiments that show the performance of the heuristics developed in the previous sections with respect to the corresponding upper bounds. We also compare the behavior of the pricing policies for the different models.

We use Monte Carlo simulations to estimate the expected profits given by the heuristics. At every period in time, we determine the pricing policy defined by the heuristics considering the available inventory and the remaining time in the planning horizon. Then, we simulate the arrival process during the next period in the planning horizon with the corresponding reservation prices. Finally, using this outcome, we update the inventory at the end of the period and apply the pricing policy again. Taking the average of the profits given by repeated simulations, we estimate the expected profit. The simulations stop when the standard deviation is less than or
equal to 0.1% of the expected profit. The upper bounds are computed solving the corresponding dynamic programming formulations.

We use a Poisson process to represent the customers' arrival process to the store and a Weibull distribution to represent the probability density function for the reservation prices. The latter unimodular distribution allows us to obtain a large variety of behavior for the reservation prices: symmetry with respect to the mean, a heavier right tail, a heavier left tail, etc. For example, a form of the Weibull distribution with shape parameter of 3.25 is almost identical with the unit normal distribution (Johnson and Kotz (1970)). When using a unimodular distribution for each store, implicitly we are assuming that the store faces a single market segment. This assumption is close to reality, specially for stores located in local malls. The Weibull density function with parameters $k, \rho$ and $\epsilon$ is given by:

$$f(p) = k \rho [\rho(p - \epsilon)]^{k-1} \exp(-[\rho(p - \epsilon)]^k) \quad \forall p > \epsilon, \forall k > 0, \rho > 0.$$  

Figure (3-1) shows the Weibull density functions for each of the two stores used in the computational experiments.

We define $C_o$ as the minimum capacity such that from that capacity onwards the value of the upper bound does not change. This quantity allows us to define an index of inventory availability equal to the initial inventory over $C_o$.

![Weibull distribution](https://via.placeholder.com/150)

**Figure 3-1: Weibull p.d.f.**
3.4.1 The One Store Model

In the first set of computational experiments we consider a single store that faces an average arrival rate of 15 customers per day for the product under consideration. The parameters for the Weibull distribution correspond to: \( k = 5, \rho = .010, \) and \( \epsilon = 0. \) For this set of experiments the value of \( C_o \) is equal to 400. For the purpose of comparison, additionally to heuristic HEUR1 described in Section 3.2, we implement a second heuristic called HEUR0. In this heuristic the seller posts a single price during the planning horizon which is determined by the single-period problem, where the seller must price one unit, there will be at most one buyer and the good perishes after one period. Mathematically, the price is determined by the following problem:

\[
\max_{p \geq 0} \{p(1 - F(p))\}
\]

Table 3.1 shows the computational results. The detailed information about expected profits can be found in Table C.1 in Appendix C.2.

<table>
<thead>
<tr>
<th>Initial Inventory C</th>
<th>((C/C_o) \times 100)</th>
<th>HEUR1 w/r BOUND1</th>
<th>HEUR0 w/r BOUND1</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>6.3%</td>
<td>99.4%</td>
<td>59.2%</td>
</tr>
<tr>
<td>50</td>
<td>12.5%</td>
<td>99.4%</td>
<td>62.4%</td>
</tr>
<tr>
<td>100</td>
<td>25.0%</td>
<td>99.4%</td>
<td>67.2%</td>
</tr>
<tr>
<td>200</td>
<td>50.0%</td>
<td>99.2%</td>
<td>76.1%</td>
</tr>
<tr>
<td>250</td>
<td>62.5%</td>
<td>99.0%</td>
<td>81.3%</td>
</tr>
<tr>
<td>300</td>
<td>75.0%</td>
<td>98.6%</td>
<td>87.7%</td>
</tr>
<tr>
<td>350</td>
<td>87.5%</td>
<td>98.1%</td>
<td>96.0%</td>
</tr>
<tr>
<td>370</td>
<td>92.5%</td>
<td>98.6%</td>
<td>98.6%</td>
</tr>
<tr>
<td>380</td>
<td>95.0%</td>
<td>99.3%</td>
<td>99.3%</td>
</tr>
<tr>
<td>400</td>
<td>100.0%</td>
<td>99.9%</td>
<td>99.9%</td>
</tr>
</tbody>
</table>

The first column contains the initial inventory in number of units. The second column corresponds to the index of inventory availability. The third and forth columns correspond to the performance of the heuristics. We observe that heuristic HEUR1
has an excellent performance with results that are close to the optimal solution for all the capacities considered. However, the single-period heuristic, HEUR0, behaves poorly when the capacity is small. Only when \((C/C_o) \times 100\) is larger than or equal to 87.5\%, HEUR0 has a similar performance to HEUR1's performance. As seen in Section 3.2, for a capacity large enough both heuristics lead to the optimal solution. Comparing the two heuristics we can observe that it is crucial to incorporate the current inventory and the remaining time in the planning horizon as factors in the pricing decision process. Only HEUR1 takes into account these factors when determining the pricing policy. We also observe that the performance of HUEER1 is good when the capacity is small. However, it deteriorates starting from \((C/C_o) \times 100\) equal to 50\%; only when the index of inventory availability is 95\% it reaches a performance of 99.3\% with respect to the upper bound again. We conjecture that this effect is due to a worsening of the upper bound. The upper bound is obtained by solving the problem where the prices are unconstrained during the planning horizon. As we showed in section 3.2, the optimal price in this case is decreasing in time with jumps when the products are sold. Therefore, the number of times that the optimal price becomes higher than the price in the previous period increases as a function of the capacity. Hence, the upper bound becomes looser when the capacity increases. For a large enough capacity this effect is eliminated because the optimal pricing policy becomes constant.

Figure 3-2 shows the expected price during the planning horizon for the one store problem. Curve UB1 corresponds to the basic unconstrained problem where the prices for a given outcome of the arrival process are decreasing in time with jumps at the instants when products are sold. In this example, we observe that the expected price, over all possible outcomes for the selling process, is non-monotonic as a function of time. The optimal expected price increases towards the end of the planning horizon because the optimal pricing strategy leads, in general, to a small inventory for the final periods. Hence, the seller is prepared to take the risk of selling the final units at a higher price. Curve HEUR1 corresponds to the expected price for the extended model where prices are decreasing in time for any outcome of the arrival process and
only periodic (daily) price reviews are allowed. We observe in this case that the initial expected price given by the heuristic is slightly higher than the expected price given by the upper bound. However, it decreases faster because this model does not allow price increases.

Figure 3-3 shows the price path for a particular outcome of the arrival process and the reservation prices. Curve UB1 shows the price for the basic unconstrained model. We observe that the price changes several times during the planning horizon including two or three changes during a single day. In practice, this solution is unrealistic because of the coordination and management costs associated to this type of strategy and the confusing information that customers receive about the product's value. HEUR1 leads to monotonically decreasing price paths which are easier to implement in practice. As we have shown in Table 3.1, the optimal expected profit decreases, in the worst case, by 2% with respect to the optimal solution of the unconstrained problem.

![Graph showing price evolution over time](image)

*Figure 3-2: Expected Price for the One Store Case (Initial Capacity: 50)*
3.4.2 Two Stores Model

In what follows we present the performance of the heuristic for two problems which only differ in the number of times the price can be modified. We consider a planning horizon of 30 days; in problem 1, P1, prices can be changed at the beginning of every day. In problem 2, P2, prices can be changed only four times during the planning horizon. The computational results are shown in table 3.2. The expected profits can be found in Appendix C.2, Table C.2. We consider arrival rates, for the product under consideration, of 15 and 8 customers per day to stores 1 and 2, respectively. The parameters of the Weibull distributions correspond to:

Store 1: \( k_1 = 5, \rho_1 = .010, \epsilon_1 = 0 \)

Store 1: \( k_2 = 4, \rho_2 = .007, \epsilon_2 = 0 \)

For this set of parameters, \( C_o \) is equal to 600.

The first column is the initial inventory. The second column contains the inventory availability index. Finally, the third and fourth columns contain the performance of the heuristics with respect to the upper bound in problems P1 and P2 respectively. We observe that the heuristic's performance improves as long as the capacity increases, leading to almost optimal solutions when the capacity is close to \( C_o \). This behavior
Table 3.2: The two stores problem

<table>
<thead>
<tr>
<th>Initial Inventory</th>
<th>((C/C_o) \times 100)</th>
<th>HUER2: P1 w/r BOUND2</th>
<th>HUER2: P2 w/r BOUND2</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1%</td>
<td>98.8%</td>
<td>94.5%</td>
</tr>
<tr>
<td>12</td>
<td>2%</td>
<td>99.0%</td>
<td>95.8%</td>
</tr>
<tr>
<td>30</td>
<td>5%</td>
<td>99.2%</td>
<td>97.1%</td>
</tr>
<tr>
<td>60</td>
<td>10%</td>
<td>99.3%</td>
<td>97.6%</td>
</tr>
<tr>
<td>150</td>
<td>25%</td>
<td>99.4%</td>
<td>98.2%</td>
</tr>
<tr>
<td>300</td>
<td>50%</td>
<td>99.6%</td>
<td>98.6%</td>
</tr>
<tr>
<td>360</td>
<td>60%</td>
<td>99.6%</td>
<td>98.8%</td>
</tr>
<tr>
<td>420</td>
<td>70%</td>
<td>99.6%</td>
<td>99.0%</td>
</tr>
<tr>
<td>480</td>
<td>80%</td>
<td>99.6%</td>
<td>99.0%</td>
</tr>
<tr>
<td>540</td>
<td>90%</td>
<td>99.6%</td>
<td>99.1%</td>
</tr>
<tr>
<td>600</td>
<td>100%</td>
<td>99.9%</td>
<td>99.7%</td>
</tr>
</tbody>
</table>

is consistent with the result shown in section 3.3, where we showed that the heuristic leads to the optimal pricing policy when the capacity goes to infinity (which is also the pricing policy given by the upper bound). We also observe that the behavior of the heuristic with respect to the upper bound is better in problem P1, where we modify the inventory and the prices every day during the planning horizon. However, this effect cannot be necessarily attributed to a worsening of the heuristic in P2. We have to take into account that the optimal pricing policy associated to problem P2 also leads to a lower total expected profit because we are solving a more constrained problem (prices and inventory can be modified only four times during the planning horizon). The overall performance of the heuristic is satisfactory with excellent results when the capacity is larger than or equal to 25% of \(C_o\).

3.5 Conclusions and extensions

This chapter has studied optimal pricing strategies for perishable products in retail stores. We have considered a continuous time problem where a seller faces a stochastic arrival of customers with heterogeneous valuation of the product. For this model we have characterized the optimal pricing policies as follows: for every outcome of the
arrival process with the corresponding reservation prices, the optimal price is a decreasing function of time with jumps when goods are sold. A necessary and sufficient condition for the optimal price is given, which is satisfied for a large group of distributions for the reservation price. For these distributions, the optimal pricing strategy can be easily computed solving the first order condition backwards in time. We have also extended this model to incorporate periodic pricing reviews and monotonic pricing policies which are features usually desired in practical applications. Finally, we have generalized the basic model to consider a company that has two retail stores in different markets. We have developed efficient heuristics to find approximations to the optimal pricing policies for these two extensions of the basic model, which are particularly useful when solving real size problems.

The models developed in this chapter can be extended to incorporate reservation prices that evolve over time. There are examples in practice where people are willing to pay less for the same item as time goes by. For example, winter clothing has less value for customers when the spring is coming. Another interesting extension is to consider bayesian updating of the parameters in the reservation price distribution functions. We leave the formal study of these two topics for future research.
Appendix A

Chapter 1

A.1 Chapter 1 proofs

Lemma A.1 The function $F_t(c, i)$ is monotonically increasing as a function of the capacity:

$$F_t(c, i) \geq F_t(c - e_p, i), \quad \forall c, p, t, i,$$

and

$$F_t(c - e_k, i) \geq F_t(c - e_p, i), \quad \forall k \geq p, \text{ and } \forall c, t, i.$$

The single product case corresponds to the particular case where $c$ is a scalar and $e_p = e_k = 1$.

**PROOF:** Suppose we have an optimal policy when we start with a capacity equal to $c - e_p$ at time $t$. This policy is also feasible when we start with a capacity equal to $c$ at time $t$ (and gives the same value for the objective function), but it is not necessarily optimal. Therefore, the maximum expected profit from time $t$ onwards starting with capacity $c - e_p$ is less than or equal to the maximum expected profit starting with capacity $c$. Hence,

$$F_t(c, i) \geq F_t(c - e_p, i), \quad \forall c, p, t, i.$$

The same argument can be used when we replace $c$ by $c - e_k$ with $k \geq p$, because product $p$ can be downgraded to be used as product $k$. —
Lemma A.2 The objective function satisfies the following inequality:

\[ F_t(c + e_t + e_p, i) - F_t(c + e_l, i) \leq F_t(c + e_p, i) - F_t(c, i), \quad \forall t, i, c, p, l. \tag{A.1} \]

**Proof:** In what follows we use induction in \( t \) to prove the lemma for the single product case. The objective function for the last period in the planning horizon is given by:

\[
F_T(c, i) = \begin{cases} 
\pi_i & \text{if } c > 0 \text{ and } i \neq 0, \\
-c_i & \text{if } c = 0 \text{ and } i \neq 0, \\
0 & \text{if } i = 0.
\end{cases}
\]

We observe that given \( i \), \( F_T(c, i) \) trivially satisfies the inequality (A.1), i.e.,

\[
F_T(c + 2, i) - F_T(c + 1, i) \leq F_T(c + 1, i) - F_T(c, i) \quad \forall c.
\]

In the induction hypothesis we assume that the inequality (A.1) is satisfied at period \( t + 1 \) and we prove that it is also satisfied at period \( t \). The objective functions at period \( t \) are given by (assuming that \( i \neq 0 \) and \( c > 0 \)):

\[
F_t(c, i) = \begin{cases} 
\pi_i + E_j[F_{t+1}(c - 1, j)] & (a), \\
-c_i + E_j[F_{t+1}(c, j)] & (b).
\end{cases}
\]

\[
F_t(c + 1, i) = \begin{cases} 
\pi_i + E_j[F_{t+1}(c, j)], & (c), \\
-c_i + E_j[F_{t+1}(c + 1, j)] & (d).
\end{cases}
\]

\[
F_t(c + 2, i) = \begin{cases} 
\pi_i + E_j[F_{t+1}(c + 1, j)] & (e), \\
-c_i + E_j[F_{t+1}(c + 2, j)] & (f).
\end{cases}
\]

In order to prove property (A.1) we have to consider all possible values for the functions above. Thus, we have to evaluate the 8 combinations.

1. Combination \((a) + (c) + (e)\): Replacing the objective functions for their corre-
sponding values in inequality (A.1), we get,

\[ \pi_i + E_j[F_{t+1}(c+1,j)] - \pi_i - E_j[F_{t+1}(c,j)] \leq \pi_i + E_j[F_{t+1}(c,j)] - \pi_i - E_j[F_{t+1}(c-1,j)], \]

which is true by the induction hypothesis. The same proof holds for combinations \((b) + (c) + (e), (b) + (d) + (e),\) and \((b) + (d) + (f).\)

2. Combination \((a) + (d) + (f):\) In this case, when the capacity is \(c\) we accept the request, i.e.,

\[ c_i + \pi_i \geq E_j[F_{t+1}(c,j)] + E_j[F_{t+1}(c-1,j)], \]

and when the capacity is \(c + 1,\) we reject the request, i.e.,

\[ c_i + \pi_i \leq E_j[F_{t+1}(c+1,j)] + E_j[F_{t+1}(c,j)]. \]

Both inequalities together lead to:

\[ E_j[F_{t+1}(c+1,j)] + E_j[F_{t+1}(c,j)] \geq E_j[F_{t+1}(c,j)] + E_j[F_{t+1}(c-1,j)], \]

which contradicts the induction hypothesis. Hence, the combination \((a) + (d) + (f)\) is infeasible. Combinations \((b) + (c) + (e), (b) + (d) + (e),\) and \((b) + (d) + (f)\) are also infeasibles. Therefore, the inequality (A.1) holds for period \(t.\) When \(c = 0\) the same proof holds. For the case when \(i = 0\) the inequality (A.1) holds trivially. \(\blacksquare\)

A similar proof holds for the two product case. We believe that the same proof can be generalized for the multiple product case with more than two products. This lemma says that the smaller the current inventory, the larger the profit associated to an extra unit of product. The intuition behind this lemma is that the opportunity cost of an extra unit decreases when the capacity increases because the stochastic process that determines the total demand does not change.

**Lemma A.3** The function \(F_t(c,R,i)\) is monotonically increasing as the function of the capacity and it satisfies the property given in Lemma A.2.
Proof: The proof is an extension of the multiple product case without reservations. ■

Proof of Proposition 1.3

In what follows we present the general proof for the multiple product case. By Lemma A.2, we know that the following inequality holds:

\[ F_{t+1}(c - e_t, j) - F_{t+1}(c - e_p - e_t, j) \geq F_{t+1}(c, j) - F_{t+1}(c - e_p, j) \quad \forall t, i, c, p, l. \]

The probability of a class \( i \) arrival in the next unit of time depends only on \( t \), hence we take the expected value in both sides and the inequality still holds:

\[ E_j[F_{t+1}(c - e_t, j) - F_{t+1}(c - e_t - e_p, j)] \geq E_j[F_{t+1}(c, j) - F_{t+1}(c - e_p, j)]. \]

Thus,

\[ \alpha_t(c - e_t, p) \geq \alpha_t(c, p). \quad ■ \]

Proof of Proposition 1.4

Assume that a type \( l \) product is sold to a class \( i \) customer when the capacity is equal to \( c \). Assuming that \( l \neq p \), the same product \( l \) is available for a class \( i \) customer when the initial capacity is \( c - e_p \), then:

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) = \max \{ \pi_i + E_j[F_{t+2}(c - e_t, j)], -c_i + E_j[F_{t+2}(c, j)] \} \\
- \max \{ \pi_i + E_j[F_{t+2}(c - e_p - e_t, j)], -c_i + E_j[F_{t+2}(c - e_p, j)] \} \]

We analyze the following two cases:

Case 1:

\[ \max \{ \pi_i + E_j[F_{t+2}(c - e_p - e_t, j)]; -c_i + E_j[F_{t+2}(c - e_p, j)] \} = -c_i + E_j[F_{t+2}(c - e_p, j)]. \]
In this case,

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq -c_i + E_j[F_{t+2}(c, j)] - (-c_i + E_j[F_{t+2}(c - e_p, j)]), \]

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq E_j[F_{t+2}(c, j)] - E_j[F_{t+2}(c - e_p, j)] = \alpha_{t+1}(c, p) \]

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \alpha_{t+1}(c, p). \quad (A.2) \]

**Case 2:**

\[
\max\{\pi_i + E_j[F_{t+2}(c - e_p - e_t, j)]; -c_i + E_j[F_{t+2}(c - e_p, j)]\} = \pi_i + E_j[F_{t+2}(c - e_p - e_t, j)].
\]

In this case,

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \pi_i + E_j[F_{t+2}(c - e_t, j)] - (\pi_i + E_j[F_{t+2}(c - e_p - e_t, j)]), \]

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq E_j[F_{t+2}(c - e_t, j)] - E_j[F_{t+2}(c - e_p - e_t, j)] = \alpha_{t+1}(c - e_t, p) \]

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \alpha_{t+1}(c - e_t, p). \quad (A.3) \]

Using Proposition 1.3, we obtain:

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \alpha_{t+1}(c - e_t, p) \geq \alpha_{t+1}(c, p). \]

Equations (A.2) and (A.3) yield:

\[ F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \alpha_{t+1}(c, p). \]

For \( i = 0 \), the previous result holds trivially. Therefore, using the fact that the probability of a new arrival depends only on \( t \), we take expected value on both sides of the inequality above, obtaining the desired result:

\[ E_t[F_{t+1}(c, i) - F_{t+1}(c - e_p, i)] \geq \alpha_{t+1}(c, p), \]
\[ \alpha_t(c, p) \geq \alpha_{t+1}(c, p). \]

For the special case when \( p = l \), we have two cases:

1. If the initial capacity \( c - e_p \) has an available type \( p \) product, then the proof above holds without any change.

2. If the initial capacity \( c - e_p \) does not have any type \( p \) product available, then, if the current request is accepted the customer receives a type \( k \) product. This product is better than a type \( p \) product, i.e., \( k < p \). In this case, the proof changes slightly, so we present the part of the proof that differs from the previous case. By definition we have:

\[
F_{t+1}(c, i) - F_{t+1}(c - e_p, i) = \max\{\pi_i + E_j[F_{t+2}(c - e_p, j)]; -c_i + E_j[F_{t+2}(c, j)]\}
\]

\[
- \max\{\pi_i + E_j[F_{t+2}(c - e_p - e_k, j)]; -c_i + E_j[F_{t+2}(c - e_p, j)]\}. \]

Similarly to the previous proof, we have to analyze two cases. The first one is analogous to the Case 1 above. The second case changes to the following:

\[
\max\{\pi_i + E_j[F_{t+2}(c - e_p - e_k, j)]; -c_i + E_j[F_{t+2}(c - e_p, j)]\} = \pi_i + E_j[F_{t+2}(c - e_p - e_k, j)].
\]

Hence,

\[
F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \pi_i + E_j[F_{t+2}(c - e_p, j)] - (\pi_i + E_j[F_{t+2}(c - e_p - e_k, j)]),
\]

\[
F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq E_j[F_{t+2}(c - e_p, j)] - E_j[F_{t+2}(c - e_p - e_k, j)] = \alpha_{t+1}(c - e_p, k).
\]

Using Proposition 1.3, we obtain:

\[
F_{t+1}(c, i) - F_{t+1}(c - e_p, i) \geq \alpha_{t+1}(c - e_p, k) \geq \alpha_{t+1}(c, k).
\]

Finally, using that the function \( F_{t+1}(c) \) is increasing as a function of the capacity,
we have:
\[ \alpha_{t+1}(c, k) \geq \alpha_{t+1}(c, p) \quad \text{if } k < p. \]

Therefore,
\[ F_{t+1}(c, i) - F_{t+1}(c - e, i) \geq \alpha_{t+1}(c, p). \]

The rest of the proof follows the same as in the previous case.

**Proof of Proposition 1.5**

Considering that the probability of a new arrival during the next time interval of length \( \Delta t \) depends only on \( t \) and on the number of pending reservations, we apply the expected value on both sides of the inequality given in Lemma A.3 to obtain the desired result.

**A.2 Chapter 1 tables and figures**

Table A.1 shows the results obtained for a single product case, 3 classes of customers that arrive in a Poisson manner with arrival rates equal to 5, 3, and 2 customers per hour. The prices that they pay are $200, $120, and $85 respectively and there is no rejection costs. The planning horizon is 12 hours.

<table>
<thead>
<tr>
<th>Capacity (# Prod.)</th>
<th>Optimal solution</th>
<th>UB1</th>
<th>UB2</th>
<th>HEUR1</th>
<th>HEUR3</th>
<th>HEUR4</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>9964</td>
<td>9975</td>
<td>9974</td>
<td>9956</td>
<td>9950</td>
<td>7644</td>
</tr>
<tr>
<td>60</td>
<td>11682</td>
<td>11764</td>
<td>11757</td>
<td>11550</td>
<td>11489</td>
<td>9171</td>
</tr>
<tr>
<td>70</td>
<td>13021</td>
<td>13182</td>
<td>13170</td>
<td>12821</td>
<td>12776</td>
<td>10703</td>
</tr>
<tr>
<td>80</td>
<td>14226</td>
<td>14407</td>
<td>14389</td>
<td>14103</td>
<td>14078</td>
<td>12228</td>
</tr>
<tr>
<td>90</td>
<td>15378</td>
<td>15563</td>
<td>15542</td>
<td>15309</td>
<td>15312</td>
<td>13760</td>
</tr>
<tr>
<td>100</td>
<td>16372</td>
<td>16593</td>
<td>16549</td>
<td>16299</td>
<td>16309</td>
<td>15274</td>
</tr>
<tr>
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<td>17221</td>
<td>17433</td>
<td>17381</td>
<td>17200</td>
<td>17219</td>
<td>16689</td>
</tr>
</tbody>
</table>

Table A.2 contains the expected profits for the case of 2 types of rooms with the
possibility of downgrading, 3 types of customers with arrival rates equal to 2, 3, and 5 customers per hour, prices equal to $200, $120, and $85, and room preferences equal to 1, 2, and 2 respectively. No rejection costs are considered and the planning horizon is 12 hours.

Table A.2: Expected Profits for the Multiple Product Case.

<table>
<thead>
<tr>
<th>Capacity (# rooms)</th>
<th>Optimal</th>
<th>UB1</th>
<th>UB2</th>
<th>HEUR1</th>
<th>HEUR2</th>
<th>HUER4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,30)</td>
<td>4578</td>
<td>4583</td>
<td>4583</td>
<td>4569</td>
<td>3850</td>
<td>3919</td>
</tr>
<tr>
<td>(10,30)</td>
<td>5578</td>
<td>5583</td>
<td>5585</td>
<td>5568</td>
<td>4593</td>
<td>4704</td>
</tr>
<tr>
<td>(10,45)</td>
<td>7036</td>
<td>7078</td>
<td>7075</td>
<td>6933</td>
<td>6127</td>
<td>6339</td>
</tr>
<tr>
<td>(10,50)</td>
<td>7463</td>
<td>7509</td>
<td>7506</td>
<td>7366</td>
<td>6717</td>
<td>6857</td>
</tr>
<tr>
<td>(10,60)</td>
<td>8314</td>
<td>8360</td>
<td>8357</td>
<td>8233</td>
<td>7860</td>
<td>7867</td>
</tr>
<tr>
<td>(15,70)</td>
<td>10158</td>
<td>10204</td>
<td>10202</td>
<td>10095</td>
<td>9775</td>
<td>9794</td>
</tr>
<tr>
<td>(20,80)</td>
<td>11934</td>
<td>11982</td>
<td>11977</td>
<td>11880</td>
<td>11679</td>
<td>11667</td>
</tr>
<tr>
<td>(20,90)</td>
<td>12672</td>
<td>12705</td>
<td>12697</td>
<td>12664</td>
<td>12603</td>
<td>12588</td>
</tr>
<tr>
<td>(25,95)</td>
<td>13615</td>
<td>13652</td>
<td>13625</td>
<td>13613</td>
<td>13623</td>
<td>13614</td>
</tr>
</tbody>
</table>

Table A.3 contains the expected profits for a real application. The planning horizon is a operational day that starts at 8 a.m. and finishes at 12 p.m. The parameters correspond to:

1. Number of customers : 6
2. Number of products : 2
3. Prices : (190, 200, 250, 70, 80, 110)
4. Rejection costs : (190, 100, 0 , 70, 40, 0)
5. Room preferences : (1, 1, 1, 2, 2, 2)
6. periods in the planning horizon : 8-10, 11-12, 12-16, 16-18, 18-20, 20-21, 21-24
7. Arrival rate (guarantee) : (3.1, 53.9, 3.1, 3.1, 3.1, 53.9, 3.1)
8. Arrival rate (6 p.m. ) : (.46, 16.7, .46, .46, 0, 0, 0)
9. Arrival rate (walk-ins): (.32, .32, .32, 3.6, 3.6, 3.6, 3.6)

Table A.3: Expected Profit for a Real Case.

<table>
<thead>
<tr>
<th>Capacity (#.Prod.)</th>
<th>UB1</th>
<th>UB2</th>
<th>HEUR1</th>
<th>HEUR2</th>
<th>HEUR4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,130)</td>
<td>5924</td>
<td>5923</td>
<td>5925</td>
<td>4970</td>
<td>4854</td>
</tr>
<tr>
<td>(20,150)</td>
<td>8740</td>
<td>8728</td>
<td>8722</td>
<td>7609</td>
<td>7512</td>
</tr>
<tr>
<td>(20,190)</td>
<td>13889</td>
<td>13813</td>
<td>13788</td>
<td>12953</td>
<td>12876</td>
</tr>
<tr>
<td>(30,200)</td>
<td>18725</td>
<td>18618</td>
<td>18465</td>
<td>17804</td>
<td>17656</td>
</tr>
<tr>
<td>(40,210)</td>
<td>22701</td>
<td>22601</td>
<td>22300</td>
<td>22014</td>
<td>21902</td>
</tr>
<tr>
<td>(40,220)</td>
<td>23548</td>
<td>23463</td>
<td>23228</td>
<td>23099</td>
<td>22990</td>
</tr>
<tr>
<td>(60,220)</td>
<td>25544</td>
<td>25491</td>
<td>25390</td>
<td>25374</td>
<td>25349</td>
</tr>
<tr>
<td>(80,220)</td>
<td>25849</td>
<td>25789</td>
<td>25747</td>
<td>25770</td>
<td>25744</td>
</tr>
</tbody>
</table>

Figure A-1: Performance of HEUR1 for the single product case
Figure A-2: Performance of HEUR1 for the multiple product case
Appendix B

Chapter 2

B.1 Chapter 2 proofs

Proof of Proposition 2.1

In the long run, the average number of customers in each state of the house list is given by (the optimal mailing policy is to send catalogs to all the profitable customers):

\[
N_{t,t} = \sum_{j=1}^{J} p_j M_j + \sum_{i=1}^{i^*} p_i N_{i,t-1},
\]

\[
N_{i,t} = (1 - p_i) N_{i-1,t-1} \quad \forall i = 2, \ldots, i^*.
\]

In matrix notation, the above system is equivalent to:

\[
N_t = AN_{t-1} + b,
\]

where,

\[
A = \begin{bmatrix}
 p_1 & p_2 & p_3 & \cdots & p_{i^*-1} & p_{i^*} \\
 1 - p_1 & 0 & 0 & \cdots & 0 & 0 \\
 0 & 1 - p_2 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & 1 - p_{i^*-1} & 0
\end{bmatrix}
\]
and,
\[ b = [\sum_{j=1}^{J} p_j M_j, 0, 0, \ldots, 0]. \]

Successive applications of equation (B.1) lead to:
\[ N_t = A^t N_0 + \sum_{i=0}^{t-1} A^i b. \]

Therefore, the limit of the customers in the house list is equal to:
\[ \lim_{t \to \infty} N_t = \lim_{t \to \infty} [A^t N_0 + \sum_{i=0}^{t-1} A^i b] = \lim_{t \to \infty} [A^t N_0 + (I - A)^{-1} (b - A^t b)]. \]

Therefore, the limit can be written as:
\[ \lim_{t \to \infty} N_t = \lim_{t \to \infty} A^t N_0 - \lim_{t \to \infty} (I - 1) A^t b + \lim_{t \to \infty} (I - A)^{-1} b. \]

Finally we prove that \( A \) is a linear contraction, i.e.:
\[ \lim_{t \to \infty} A^t x = 0 \quad \forall x \in \mathbb{R}^n. \]

**Lemma B.1** The matrix \( A \) is a linear contraction.

**Proof:** \( A \) is a linear contraction iff all their eigen values have absolute values less than 1 (See Hirsch and Smale(1974)). By definition an eigen value, \( \lambda \), satisfies the equation:
\[ AN = \lambda N \quad \text{for some } N \neq 0. \]

Replacing the value of \( A \) in the above equation, we obtain the following system of equation:
\[ p_1 N_1 + p_2 N_2 + \ldots + p_i N_i = \lambda N_1 \quad (a1) \]
\[ (1 - p_1) N_1 = \lambda N_2 \quad (a2) \]
\[ (1 - p_2) N_2 = \lambda N_3 \quad (a3) \]
\[ \vdots \]

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\[(1 - p_{i^*-1})N_{i^*-1} = \lambda N_{i^*}. \quad (ai^*).\]

Replacing \((a2), (a3), \ldots, (ai^*)\) in \((a1)\) we obtain:

\[LHS = \frac{p_1}{\lambda} + \frac{p_2(1 - p_1)}{\lambda^2} + \frac{p_3(1 - p_1)(1 - p_2)}{\lambda^3} + \ldots + \frac{p_{i^*}(1 - p_1)(1 - p_2) \ldots (1 - p_{i^*-1})}{\lambda^{i^*}} = 1.\]

For the purpose of contradiction we assume that there exists a real eigen value greater than or equal to 1. Hence, assuming that all the probabilities are strictly less than 1:

\[LHS \leq p_1 + p_2(1 - p_1) + p_3(1 - p_1)(1 - p_2) + p_{i^*}(1 - p_1)(1 - p_2) \ldots (1 - p_{i^*-1}) < 1,\]

which leads to a contradiction. Therefore, the largest real eigen values is strictly less than one. Using the fact that \(A\) is a positive matrix (all its elements are non negative with at least one strictly positive element) and the Frobenius theorem (Karlin and Taylor, page 547), we know that the absolute value of all its eigen value are less than or equal to the largest real eigen value. Therefore, all the eigen values of \(A\) have absolute values less than 1, or equivalently \(A\) is a linear contraction.

Therefore, using lemma 1, we conclude that the limit of the number of customers in the house list is given by:

\[\lim_{i \to \infty} N_i = (I - A)^{-1}b. \quad \blacksquare\]

**Proof of Proposition 2.2**

Consider that the optimal decision for state \(i\) is not to send catalogs. For the purpose of contradiction, suppose that it is optimal to send \(m\) mailings to state \(i + 1\). Therefore,

\[LF(i) = \beta LF(i + 1),\]

and

\[LF(i + 1) = \bar{r}_{i+1,m} + \beta p_{i+1,m} LF(1) + \beta(1 - p_{i+1,m}) LF(i + 2)\]
By definition of \( LF(i) \), we have the following inequality:

\[
LF(i) = \beta LF(i + 1) \geq \bar{r}_{i,m} + \beta p_{i,m} LF(1) + \beta (1 - p_{i,m}) LF(i + 1)
\]

Hence, we obtain:

\[
LF(i + 1) \geq \frac{\bar{r}_{i,m} + \beta p_{i,m} LF(1)}{\beta p_{i,m}}.
\]

(B.2)

By definition of \( LF(i + 1) \), we have:

\[
LF(i + 1) = \bar{r}_{i+1,m} + \beta p_{i+1,m} LF(1) + \beta (1 - p_{i+1,m}) LF(i + 2) \geq \beta LF(i + 2).
\]

Therefore, we obtain the following inequality:

\[
LF(i + 2) \leq \frac{\bar{r}_{i+1,m} + \beta p_{i+1,m} LF(1)}{\beta p_{i+1,m}}.
\]

(B.3)

Using (B.2) and the definition of \( LF(i + 1) \) we obtain:

\[
LF(i + 2) \geq \frac{\bar{r}_{i,m}/\beta p_{i,m} - \bar{r}_{i+1,m} + (1 - \beta p_{i+1,m}) LF(1)}{\beta (1 - p_{i+1,m})}.
\]

(B.4)

Finally, (B.3) and (B.4) together lead to the following inequality:

\[
\frac{\bar{r}_{i+1,m}}{p_{i+1,m}} \geq \frac{\bar{r}_{i,m}}{\beta p_{i,m}} + LF(1)(1 - \beta) \geq \frac{\bar{r}_{i,m}}{\beta p_{i,m}},
\]

which leads to a contradiction. \( \blacksquare \)

**Algorithm to Compute the Lifetime Value**

In what follows we describe the algorithm to compute the lifetime value of a customer for the single mailing case. From proposition 2, we know that there is a last profitable state that we denote by \( i^* \). Therefore, the lifetime value from \( i^* + 1 \) onwards is equal to zero.

\[
LF(i) = 0 \quad \forall i = i^* + 1, \ldots, I.
\]
The optimal decision in states 1 to \( i^* \) is to send one catalog. Hence,

\[
LF(1) = \bar{r}_1 + \beta p_1 LF(1) + \beta (1 - p_1) LF(2),
\]

or equivalently,

\[
LF(1)(1 - \beta p_1) = \bar{r}_1 + \beta (1 - p_1) LF(2).
\]

Replacing \( LF(2) \) by its optimal value, we get:

\[
LF(1)(1 - \beta p_1 - \beta (1 - p_1) p_2) = \bar{r}_1 + \beta (1 - p_1) \bar{r}_2 + \beta^2 (1 - p_1)(1 - p_2) LF(3)
\]

Replacing successively \( LF(3), LF(4), \ldots, LF(i^*) \) by their optimal values, we obtain the following expression for \( LF(1) \):

\[
LF(1) = \frac{\bar{r}_1 + \beta (1 - p_1) \bar{r}_2 + \beta^2 (1 - p_1)(1 - p_2) \bar{r}_3 + \ldots + \beta^{i^*-1} (1 - p_1)(1 - p_2) \ldots (1 - p_{i^*-1}) \bar{r}_{i^*}}{1 - \beta p_1 - \beta^2 (1 - p_1) p_2 - \beta^3 (1 - p_1)(1 - p_2) p_3 - \ldots - \beta^{i^*} (1 - p_1)(1 - p_2) \ldots (1 - p_{i^*-1}) p_{i^*}}.
\]

The last profitable state must be the last state satisfying the condition:

\[
LF(i^*) = \bar{r}_{i^*} + \beta p_{i^*} LF(1) > 0.
\]

Therefore, the algorithm to compute the optimal lifetime value of a customer works as follows:

**Step 1:** Initialization

\[
i^* = N - 1.
\]

**Step 2:** Computing the value of \( LF(1) \).

\[
LF(1) = \frac{\bar{r}_1 + \beta (1 - p_1) \bar{r}_2 + \ldots + \beta^{i^*-1} (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_{i^*-1}) \bar{r}_{i^*}}{1 - \beta p_1 - \beta^2 (1 - p_1) p_2 - \ldots - \beta^{i^*} (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_{i^*-1}) p_{i^*}}.
\]

**Step 3:** Checking if state \( i^* \) is the last profitable state.

If \((\bar{r}_{i^*} + \beta p_{i^*} LF(1) > 0)\) then
GOTO Step 4.

else

\[ i^* = i^* - 1 \]

GOTO Step 2.

**Step 4:** Computing the lifetime values.

\[ LF(i) = 0 \quad \forall i = i^* + 1, \ldots, I \]

\[ LF(i) = r_i + \beta p_i LF(1) + \beta (1 - p_i) LF(i + 1) \quad \forall i = 2, \ldots, i^*. \]

STOP.

**B.2 Chapter 2 tables**

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<tr>
<th>((A_1/Y_0) \times 100)</th>
<th>Initial Budget (A_1($))</th>
<th>HEUR 1.1</th>
<th>HEUR 1.2</th>
<th>BOUND</th>
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<td>1%</td>
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<td>179</td>
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Table B.2: Expected Profits for the Catalog Mailing Model with Aggregate Inventory Costs. (Single Mailing Case)

<table>
<thead>
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<th>((A_1/Y_0) \times 100)</th>
<th>Initial Budget (A_1($))</th>
<th>HEUR 2.1</th>
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<th>BOUND</th>
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<td>118000</td>
<td>127364</td>
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</table>
Table B.3: Expected Profits for the Catalog Mailing Model with Aggregate Inventory Costs. (Two Mailings Case)

<table>
<thead>
<tr>
<th>((A_1/Y_0) \times 100)</th>
<th>Initial Budget (A_1) ($)</th>
<th>HEUR 2.1</th>
<th>HEUR 2.2</th>
<th>HEUR 2.3</th>
<th>BOUND</th>
</tr>
</thead>
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<td>5%</td>
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<tr>
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<tr>
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<td>130306</td>
<td>136172</td>
<td>136287</td>
<td>138546</td>
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</tbody>
</table>
Appendix C

Chapter 3

C.1 Chapter 3 proofs

Lemma C.1 The function $V_t(c)$ is non-increasing as a function of time.

$$V_t(c) \geq V_{t-1}(c)$$

Proof: For $t = 1$ the inequality holds trivially. We assume that it holds for $t$, and prove it for $t + 1$.

$$V_{t+1}(c) = \max_{p \geq 0} \{\lambda \Delta t(1 - F(p))(p + \beta V_t(c - 1)) + \beta(1 - \lambda \Delta t(1 - F(p)))V_t(c)\},$$

using that the proposition is true for $t$, we obtain:

$$V_{t+1}(c) \geq \max_{p \geq 0} \{\lambda \Delta t(1 - F(p))(p + \beta V_{t-1}(c - 1)) + \beta(1 - \lambda \Delta t(1 - F(p)))V_{t-1}(c)\},$$

which is equal to:

$$V_{t+1}(c) \geq V_t(c). \quad \Box$$

Lemma C.2 The function $V_t(c)$ is a concave function of the capacity,

$$V_t(c + 1) - V_t(c) \geq V_t(c + 2) - V_t(c + 1) \quad \forall t, c$$
and the additional revenue given by an extra unit increases as long as the remaining time until the end of the planning horizon increases,

\[ V_{t+1}(c + 1) - V_{t+1}(c) \geq V_t(c + 1) - V_t(c) \quad \forall t, c \]

**Proof:** We define the following inequalities:

\[ I1(c, t) : V_{t+1}(c + 1) - V_{t+1}(c) \geq V_t(c + 1) - V_t(c) \quad \forall t, c \]

\[ I2(c, t) : V_{t+1}(c) - V_t(c) \geq V_{t+2}(c) - V_{t+1}(c) \quad \forall t, c \]

\[ I3(c, t) : V_t(c + 1) - V_t(c) \geq V_t(c + 2) - V_t(c + 1) \quad \forall t, c \]

The proof is done by induction in \( k = t + c \). The inequalities \( I1(c, t), I2(c, t), \) and \( I3(c, t) \) hold trivially for \( k = 0 \). We assume that the three inequalities are satisfied for all \( t + c < k \) and we prove that they hold for \( t + c = k \).

i) We prove that \( I1(c, t) \) holds when \( t + c = k \). For \( c = 0 \), using lemma 1 we obtain that \( I1(0, t) \) is true for all \( t \). Suppose \( c > 0 \), hence:

for some \( \bar{p} \), we have,

\[ V_{t+1}(c) = \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \beta \lambda \Delta t(1 - F(\bar{p}))V_t(c - 1) + \beta(1 - \lambda \Delta t(1 - F(\bar{p})))V_t(c) \]

subtracting \( V_t(c) \) in both sides of the equation above, we get,

\[ V_{t+1}(c) - V_t(c) = \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \lambda \Delta t(1 - F(\bar{p}))\beta(V_t(c - 1) - V_t(c)) \quad (C.1) \]

Because \( \bar{p} \) is feasible for \( V_{t+1}(c + 1) \) we have,

\[ V_{t+1}(c + 1) \geq \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \beta \lambda \Delta t(1 - F(\bar{p}))V_t(c) + \beta(1 - \lambda \Delta t(1 - F(\bar{p})))V_t(c + 1) \]

subtracting \( V_t(c + 1) \) in both sides of the previous equation, we get,

\[ V_{t+1}(c + 1) - V_t(c + 1) \geq \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \lambda \Delta t(1 - F(\bar{p}))\beta(V_t(c) - V_t(c + 1)) \quad (C.2) \]
By \( I3(c - 1, t) \) we know that the following inequality holds,

\[
V_t(c) - V_t(c + 1) \geq V_t(c - 1) - V_t(c)
\]

hence, replacing it in (C.2) we obtain:

\[
V_{t+1}(c + 1) - V_t(c + 1) \geq \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \lambda \Delta t(1 - F(\bar{p}))\beta(V_t(c - 1) - V_t(c)) \quad (C.3)
\]

Finally, (C.1) and (C.3) together lead to,

\[
I1(c, t) : V_{t+1}(c + 1) - V_t(c + 1) \geq V_{t+1}(c) - V_t(c).
\]

ii) We prove that \( I2(c, t) \) holds for \( t + c = k \).

For some \( \bar{p} \) we have,

\[
V_{t+2}(c) = \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \beta \lambda \Delta t(1 - F(\bar{p}))V_{t+1}(c - 1) + \beta(1 - \lambda \Delta t(1 - F(\bar{p})))V_{t+1}(c),
\]

subtracting \( V_{t+1}(c) \) in both sides of the equation above, we get,

\[
V_{t+2}(c) - V_{t+1}(c) = \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \lambda \Delta t(1 - F(\bar{p}))\beta(V_{t+1}(c - 1) - V_{t+1}(c)) \quad (C.4)
\]

Because \( \bar{p} \) is feasible for \( V_{t+1}(c) \) we have,

\[
V_{t+1}(c) \geq \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \beta \lambda \Delta t(1 - F(\bar{p}))V_t(c - 1) + \beta(1 - \lambda \Delta t(1 - F(\bar{p})))V_t(c),
\]

subtracting \( V_t(c) \) in both sides we get,

\[
V_{t+1}(c) - V_t(c) \geq \lambda \Delta t(1 - F(\bar{p}))\bar{p} + \lambda \Delta t(1 - F(\bar{p}))\beta(V_t(c - 1) - V_t(c)) \quad (C.5)
\]

By \( I1(c - 1, t) \) we know that,

\[
V_t(c - 1) - V_t(c) \geq V_{t+1}(c - 1) - V_{t+1}(c)
\]
Replacing this inequality in (C.5) we obtain,

\[ V_{t+1}(c) - V_t(c) \geq \lambda \Delta t(1 - F(\bar{p}))\beta + \lambda \Delta t(1 - F(\bar{p}))\beta(V_{t+1}(c - 1) - V_{t+1}(c)) \]  \hspace{1em} (C.6)

Finally, (C.4) and (C.6) lead to:

\[ I2(c, t) : V_{t+1}(c) - V_t(c) \geq V_{t+2}(c) - V_{t+1}(c). \]

iii) Finally, we prove that the inequality \( I3(c, t) \) holds for \( k = c + t \).

Similarly to the previous cases, for some \( \beta \), we get the following two inequalities:

\[ V_t(c+2) - V_{t-1}(c+1) = \lambda \Delta t(1 - F(\bar{p}))\beta + (1 - \lambda \Delta t(1 - F(\bar{p})))\beta(V_{t-1}(c+2) - V_{t-1}(c+1)) \]  \hspace{1em} (C.7)

and,

\[ V_{t+1}(c + 1) - V_t(c) \geq \lambda \Delta t(1 - F(\bar{p}))\beta + (1 - \lambda \Delta t(1 - F(\bar{p})))\beta(V_t(c + 1) - V_t(c)) \]  \hspace{1em} (C.8)

Using \( I1(c, t - 1) \) and \( I3(c, t - 1) \) we have the following inequalities,

\[ V_t(c + 1) - V_t(c) \geq V_{t-1}(c + 1) - V_{t-1}(c) \]

and,

\[ V_{t-1}(c + 1) - V_{t-1}(c) \geq V_{t-1}(c + 2) - V_{t-1}(c + 1) \]

hence,

\[ V_{t+1}(c + 1) - V_t(c) \geq \lambda \Delta t(1 - F(\bar{p}))\beta + (1 - \lambda \Delta t(1 - F(\bar{p})))\beta(V_{t-1}(c+2) - V_{t-1}(c+1)) \]  \hspace{1em} (C.9)

(C.7) and (C.9) lead to,

\[ V_{t+1}(c + 1) - V_t(c) \geq V_t(c + 2) - V_{t-1}(c + 1) \]  \hspace{1em} (C.10)
Additionally, by \( I2(c + 1, t - 1) \) we have,

\[
V_t(c + 1) - V_{t-1}(c + 1) \geq V_{t+1}(c + 1) - V_t(c + 1)
\]

or equivalently,

\[
2V_t(c + 1) \geq V_{t-1}(c + 1) + V_{t+1}(c + 1)
\]  \hspace{1cm} (C.11)

(C.10) and (C.11) together lead to the desire inequality,

\[
I3(c, t) : V_t(c + 1) - V_t(c) \geq V_t(c + 2) - V_t(c + 1). \quad \blacksquare
\]

**Proof of Proposition 3.1**

Defining the function \( h_t(p, c) \) equal to:

\[
h_t(p, c) = \lambda \Delta t(1 - F(p))p + \beta \lambda \Delta t(1 - F(p))V_{t-1}(c - 1) + \beta(1 - \lambda \Delta t(1 - F(p)))V_{t-1}(c),
\]

the maximization problem is equivalent to,

\[
V_t(c) = \max_{p \geq 0} \{ h_t(p, c) \}
\]

Let \( p_{t,c} \) be the optimal price at the beginning of period \( t \) when the initial inventory is \( c \). Hence, the following inequality holds for all \( p \):

\[
h_t(p_{c,t}, c) \geq h_t(p, c) \quad \forall p.
\]

Because \( p_{c,t} \) is feasible for the maximization problem starting with an inventory of \( c + 1 \) units, a sufficient condition for \( p_{t,c+1} \) to be smaller than or equal to \( p_{t,c} \) is:

\[
h_t(p_{c,t}, c + 1) \geq h_t(p, c + 1) \quad \forall p \geq p_{t,c}.
\]
Thus, a stronger sufficient condition is given by,

\[ h_t(p_{c,t}, c + 1) - h_t(p, c + 1) \geq h_t(p_{c,t}, c) - h_t(p, c) \quad \forall p \geq p_{t,c}. \]

Replacing the function \( h_t(p, c) \) by its corresponding value, we get that the sufficient condition is equivalent to,

\[ V_{t-1}(c) - V_{t-1}(c - 1) \geq V_{t-1}(c + 1) - V_{t-1}(c) \]

which is true by lemma C.2. Therefore, the optimal price is a non increasing function of the capacity.  

**Proof of Proposition 3.2**

Using the same notation as in the previous proof, we have:

\[ h_t(p_{c,t}, c) \geq h_t(p, c) \quad \forall p. \]

Because \( p_{c,t} \) is feasible for the maximization problem starting at period \( t - 1 \), a sufficient condition for \( p_{t-1,c} \) to be smaller than or equal to \( p_{t,c} \) is:

\[ h_{t-1}(p_{c,t}, c) \geq h_{t-1}(p, c) \quad \forall p \geq p_{t,c}. \]

Hence, a stronger sufficient condition is given by:

\[ h_{t-1}(p_{c,t}, c) - h_{t-1}(p, c) \geq h_{t}(p_{c,t}, c) - h_{t}(p, c) \quad \forall p \geq p_{t,c}. \]

Thus, replacing the value of \( h_t(p, c) \) we obtain the following sufficient condition:

\[ V_{t-1}(c - 1) - V_{t-2}(c - 1) \leq V_{t-1}(c) - V_{t-2}(c) \]

which is true by Lemma C.2. Hence, the optimal price is non increasing as a function of time.  

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Proof of Proposition 3.3

The first order condition for the optimal price given by:

\[ p = \frac{1 - F(p)}{f(p)} + \beta(V_{t-1}(c) - V_{t-1}(c - 1)), \]

is obtained setting the derivative of the objective function equal to zero. This equation has a unique solution if the function \( G(p) \) is increasing as a function of \( p \), where \( G(p) \) corresponds to:

\[ G(p) = p - \frac{1 - F(p)}{f(p)}. \]

After doing some basic algebra, an equivalent condition for \( G(p) \) to be an increasing function of \( p \) is given by:

\[ 2\theta[\log(1 - F(p))] \leq \theta[\log(f(p))] \quad \forall p, \]

or equivalently,

\[ \frac{(1 - F(p_2))^2}{f(p_2)} \leq \frac{(1 - F(p_1))^2}{f(p_1)} \quad \forall p_1 \leq p_2 \]

Hence, the function \( G(p) \) is increasing in \( p \) if and only if the function \( (1 - F(p))^2/f(p) \) is decreasing in \( p \). Hence, the first order condition has a unique solution if \( (1 - F(p))^2/f(p) \) is a decreasing function of \( p \). Finally, assuming that the probability density function for the reservation price is bounded, this unique solution must be the optimal solution. ■

Example 1

In what follows we present an example where the current optimal price is smaller than the expected optimal price in the next period of time. We consider that the reservation price has a uniform distribution in the interval \([0, b]\), the arrival process in each period of time is represented by a Bernoulli distribution where the probability
of one arrival is equal to \( p \), and a discount rate equal to one. For this data set, the optimal prices at periods 1, 2 and 3 correspond to:

\[
p_{1,c} = \frac{b}{2} \quad \forall c > 0.
\]

\[
p_{2,c} = \begin{cases} \frac{b}{2}(1 + \frac{c}{4}) & c = 1 \\ \frac{b}{4} & \forall c \geq 2. \end{cases}
\]

\[
p_{3,2} = \frac{b}{2}(1 + \frac{p^2}{8} - \frac{p^3}{64})
\]

Thus, assuming that the current capacity at period 3 is equal to 2 units, the expected price at period 2 corresponds to:

\[
p_2 = \frac{b}{2}(1 + \frac{p^2}{8} - \frac{p^3}{64} + \frac{p^4}{512})
\]

Therefore, for any \( p \in (0, 1] \), the expected price at period 2 is greater than the current optimal price at period 3.

### C.2 Chapter 3 tables

<table>
<thead>
<tr>
<th>Initial Inventory C</th>
<th>((C/C_0) \times 100)</th>
<th>BOUND1 ($)</th>
<th>HEUR1 ($)</th>
<th>HEUR0 ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>6.3%</td>
<td>3063</td>
<td>3044</td>
<td>1812</td>
</tr>
<tr>
<td>50</td>
<td>12.5%</td>
<td>5811</td>
<td>5776</td>
<td>3624</td>
</tr>
<tr>
<td>100</td>
<td>25.0%</td>
<td>10783</td>
<td>10717</td>
<td>7248</td>
</tr>
<tr>
<td>200</td>
<td>50.0%</td>
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<td>22071</td>
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<td>26195</td>
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<tr>
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<td>26461</td>
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<tr>
<td>400</td>
<td>100.0%</td>
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Table C.2: Expected profits for the two stores problem

<table>
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<tr>
<th>Initial Inventory C</th>
<th>((C/C_o) \times 100)</th>
<th>HEUR2 P1 ($)</th>
<th>HEUR2 P2 ($)</th>
<th>BOUND2 ($)</th>
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<td>6</td>
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<td>1158</td>
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<td>2122</td>
<td>2214</td>
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<td>5%</td>
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<td>5081</td>
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<td>10%</td>
<td>9180</td>
<td>9023</td>
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<td>25%</td>
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<td>19232</td>
<td>19595</td>
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<tr>
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<td>100%</td>
<td>45568</td>
<td>45436</td>
<td>45571</td>
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References


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15. HOLTZ, H. (1990), Starting & Building your Catalog Sales Business, John Wiley & Sons, Inc.


