Spatial Diversity in Cellular Networks

by

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Submitted to the department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

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Chapter 1

Introduction

1.1 Cellular Networks and Spatial Diversity

Current cellular networks were designed more than 10 years ago. They allow a mobile to make or receive phone calls via airwaves. Nowadays, these networks are analog and are based on Frequency Division Multiple Access (FDMA), see [8].

Over the past decade the demand for cellular phones grew dramatically leading to insufficient capacity in cities like New York, Chicago, and Los Angeles in the United States and London in Europe. Moreover the industry of cellular phones forecasts a steady growth of demand until year 2000 and beyond. All these reasons urge the need for new systems that increase network capacity.

The new generation of cellular phones increases the capacity by switching from analog to digital technology. Two systems are competing in the United States to become the North American standard. They are: Time Division Multiple Access (TDMA) and Code Division Multiple Access (CDMA, also known as Spread Spectrum, SSMA). In Europe the standard has already been defined as TDMA mixed with Slow Frequency Hopping (Groupe Special Mobile, GSM).

One often refers to the available resources in a communication system as having 2 dimensions: Time and Frequency (see [Fig. 1.1]). One can separate 2 users transmitting on the same channel if:

- they transmit on two distinct frequencies (FDMA);
- they transmit in two distinct time slots (TDMA);
- they transmit on the same bandwidth at the same time and use different codes (CDMA), see [10].

But in reality, there are 3 available resources in a communication system: Time, Frequency and Space (see [Fig. 1.2]).

Space provides an additional degree of freedom to help separate different users. If the receiver
Figure 1-1: Multiple Access Schemes in Time and Frequency

wants to listen to mobile X and knows (or can estimate) where mobile X and mobile Y are, then it can listen predominantly in the direction of X. By doing so, the receiver increases its SNR and, in a sense, separates X from Y. This is referred to as Space Diversity Multiple Access (SDMA).

Because SDMA uses Space Diversity and not Spectrum or Time, it can be used either on top of TDMA or CDMA systems to further increase the current capacity of existing cellular networks.

Therefore, SDMA is a very attractive and elegant solution in a world eager for erlang per Hertz per square feet!

1.2 Organization of The Work

Chapter 2 describes arrays of sensors as means to explore Spatial Diversity and explains their implementation in a cellular network.

Chapter 3 introduces some notions of information theory, which provide the framework for the later chapters. TDMA, CDMA and TDMA/SDMA Multiple Access schemes are also presented.

Chapter 4 summarizes the results for Multiple Access Channels (MAC). It also gives the basis for
Channel with 2 users, X and Y. If X and Y are at 2 different positions, the receiver must be able to separate them.

Figure 1-2: Multiple Access Schemes with Spatial Diversity

a fair comparison among Multiple Access schemes by carefully comparing the capacity for TDMA and CDMA.

Chapter 5 investigates MAC with Spatial Diversity. We derive the capacity region and show that it expands and varies with the relative position of the mobiles. We also compare the different Multiple Access schemes with each other.

Finally, in chapter 6, we conclude and propose some possible further research.

In this present work, we limit our investigation to a 1-cell network without multipath. This model may seem restrictive, especially given that interference from neighboring cells as well as multipath are critical parameters in the performance of a cellular network. However our motivation was based on the fact that simple models reveal fundamental properties and help to get a good understanding of phenomena. We hope that this goal has been reached.
Chapter 2

Adaptive Arrays

2.1 The Idea

In current cellular networks, each mobile has an omnidirectional antenna on board, whereas the cell site has either an omnidirectional antenna or a sectored antenna. We propose replacing the antenna at the cell site by an array of sensors while leaving unchanged the antenna on board the mobiles. The motivation is that the extra processing needed by the array is not compatible with the requirements for mobiles’ handsets (lightness, compactness, cost effectiveness), whereas there are no such constraints at the cell site.

Omnidirectional and Sectored Antennae

An omnidirectional antenna covers the whole plane ([0, 2π] in azimuth), whereas a sectored antenna contains p antennae, each of them covering an angle ([2π(k−1)/p, 2πk/p], k ∈ [1, p]). Usually p = 3 and the capacity per cell is increased by a factor less than 3. Although there is a gain in capacity by using a sectored antenna rather than an omnidirectional antenna, neither of these systems takes real advantage of spatial diversity among users. With sectored antennae, one subdivides a cell in subcells whose boundaries are fixed. Therefore it does not adapt to arbitrary mobile positions (see [Fig. 2-1]).

Arrays of Sensors

An array is made of several sensors (see [Fig. 2-2]). At the output, the receiver combines the signals from each sensor. With an appropriate adaptive processing of the output, the receiver can separate the mobiles from each other: this is known as spatial filtering or adaptive array signal processing. Arrays have already been widely used to solve detection problems (radar, sonar,...). See [9] for an exhaustive presentation.
1 omnidirectional  3 sector antenna  ADAPTIVE ARRAY

Note that the sectored antenna does not adapt to the positions of the mobiles: 1 sector is heavily loaded with 3 mobiles whereas the other 2 are not.

Figure 2-1: A cell with different antennae

2.2 Uniform Linear Array

Uniform Linear Arrays (ULA) are a special kind of arrays. Their straightforward geometry makes the processing algorithms simple and fast. Throughout this work, without loss of essential generality, we will use an ULA in our models.

The Geometry

A \((K, d)\) uniform linear array is made of \(K\) identical isotropic sensors uniformly placed every \(d\) cm (see [Fig. 2-2]).

Received Signals

Signals for cellular phones are of the form \(s(t)e^{-j2\pi f_c t}\), where:

- \(s(t)\) is the baseband signal;
- \(f_c\) is the carrier frequency.

The sources are supposed to be far from the array (far-field sources) so that the wavefronts are planar.

TDMA and CDMA signals are narrowband, e.g. the base band signal does not vary over the length of the ULA. For example, in CDMA, the carrier is 900 MHz and the base band signal is 1.5 MHz. If \(d = .15\) meter (half of the carrier wave length) and \(K = 20\) (number of sensors), the array is 3 meters across but the base band wave length is 180 meters.

Since signals are narrowband, they are received with phase shifts at each sensor of the ULA. The phase shifts are functions of:

- \(d\), the space between two consecutive sensors;
- \(f_c (\lambda_c)\), the carrier frequency (wavelength) and
\[ O = \sum_{i=1}^{M} s_i d_i + N \]
\[ ... = DS + N, \]
\[ S = (s_1, ..., s_M, ..., s_M)^T, \ (M \ vector), \]
\[ D = (d_1, ..., d_M, ..., d_M), \ (K \times M \ DOA \ matrix) \]
\[ R_S = \text{matrix } (E[s_i s_j]/\sigma_N^2) = \text{matrix } (\alpha_{ij}), \]
\[ R_O = \sigma_N^2(DRSD^1 + I), \ E[O] = 0. \]

\( N \) is assumed white Gaussian noise (in space and time domain), \( N = N(0, \sigma_N^2 I) \). This assumption is reasonable if most of the noise is generated inside the sensors, since they are taken identical to each other.

For a \((K,d)\) ULA, we have (see [Fig. 2-2]):

\[ d_i = \left( 1, \cdots, e^{-jK\Phi_i}, \cdots, e^{-j(K-1)\Phi_i} \right)^T \]
\[ \Phi_i = \frac{2\pi d \sin \theta_i}{\lambda_e} \]
Direction of Arrival

In a cellular network, the DOAs, $\theta_i$, are constant over many bits. Indeed, the bandwidth of the baseband signal is approximatively $200 \text{ kHz}$ for TDMA and $1.5 \text{ MHz}$ for CDMA system. Let $R$ be the distance to the cell site. We assume that a mobile cannot be closer than 10 meters to the cell site: $10 \leq R$. We consider a mobile which drives around the cell site (circle of radius $R$) at 3 different speeds (50 km/h, 100 km/h, 150 km/h). We plotted the number of bits sent over a period where the DOA varies over less than one degree (bits/deg). The worst case, where $R = 10$ meters, gives:

<table>
<thead>
<tr>
<th>speed (km/h)</th>
<th>50</th>
<th>100</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>bit/deg TDMA</td>
<td>2500</td>
<td>1200</td>
<td>840</td>
</tr>
<tr>
<td>CDMA</td>
<td>18850</td>
<td>9420</td>
<td>6280</td>
</tr>
</tbody>
</table>

The number of bits is large enough to assume that the DOAs, $\{\theta_i\}$, and the vectors, $\{d_i\}$, are known and constant over many bits.

2.3 Adaptive Array Processing

Esprit and Music are 2 algorithms developed for array signal processing. We will briefly present the idea behind these algorithms and the reader can refer to [9] and [11] for more details. Hereafter, the number of users, $M$, is strictly less than the number of sensors, $K$.

Case Without Multipath

Finding the Signal Subspace: With no multipath, the $RX$ matrix is diagonal. Given the structure of the vectors, $\{d_i\}$, and under the assumption of distinct DOAs, $(\theta_i \neq \theta_j, i \neq j)$, the set $\{d_i, 1 \leq i \leq M\}$ spans a M-dimensional space called the signal subspace. The rank of $R_S$ is $M$.

The orthogonal complement subspace is called the noise subspace.

signal subspace $\oplus$ noise subspace = the entire $K$ space

$R_0$ is a $K \times K$ hermitian matrix. It can be diagonalized in an orthonormal basis. The eigenvectors from the noise subspace have eigenvalues, $\{\lambda_p, M + 1 \leq p \leq K\}$, corresponding to the noise power:

$$\lambda_p = \sigma_n^2$$
The eigenvectors from the *signal subspace* have eigenvalues, \( \{ \lambda_q, 1 \leq q \leq M \} \), corresponding to the signal power (plus some noise power) and therefore

\[
\lambda_q > \sigma_N^2, \quad 1 \leq q \leq M
\]

Using this property, one can find the *signal subspace*.

**Finding the DOAs**: Searching for the DOAs is the same as searching for the \( \theta_s \) or the vectors, \( d(\theta) \), where

\[
\begin{align*}
d(\theta) &= (1, \ldots, e^{-jK\theta}, \ldots, e^{-j(K-1)\theta})^T \\
\Phi &= \frac{2\pi d \sin \theta}{\lambda_e}
\end{align*}
\]

Using the orthogonal complement property, one makes \( \theta \) vary from 0 to \( \pi (2\pi) \). The DOAs are the values of \( \theta \) for which \( d(\theta) \) is orthogonal to the *noise subspace*.

See [14] for a different algorithm.

**The Signal Copy**: One projects \( \mathbf{O} \) on a direction, \( \mathbf{W}_i \), in order to recover the signal \( s_i \). The appropriate \( \mathbf{W}_i \) is derived by maximizing the SNR.

**Case With Multipath**

An array of sensors can resolve multipath at a cost in computation and separation ability.

Indeed the algorithms are much more complex. Moreover each path is treated like it were a source, therefore diminishing the overall ability of separating a given number of users.

### 2.4 Remarks

**limitation**: \( M < K \)

As seen in the description above, in order to find the DOAs, the number of users must be less than the number of sensors, \( (M < K) \). That is at most \( (K - 1) \) users can transmit simultaneously.

Actually, in our work, we will assume that:

- for TDMA signals, the limitation holds, i.e., per time slot at most \((K - 1)\) users can transmit;
- for CDMA signals, on the contrary, more than \((K - 1)\) users can transmit simultaneously.

The motivation is that we can envision a CDMA receiver that processes the received signals both in time (with a processing gain from spreading) and in space (with a processing gain
from array processing) whereas TDMA signals have just a processing gain from the array. The study of such a receiver is beyond this work, but nevertheless this assumption seems legitimate.

Gain for TDMA and CDMA

Besides the gain in separation ability, the use of an array can:

- help TDMA systems in resolving multipath.
  Indeed, TDMA systems encounter correlated multipath which creates severe fading (30 dB).
  To combat fading, one must increase the transmitting power. This increases the interference from other cells and makes the reuse factor smaller (1/7 in current systems) which severely limits the capacity.
  Since multiple paths come from different DOAs, they could be resolved by using an array;

- make CDMA systems more robust to power control.
  The processing gain from the array signal processing helps to protect the whole cell from a user received with too much power. With an array, only users 'close' to the high power user are penalized.

In this chapter, we have shown how arrays of sensors can be implemented in a cellular network to explore Spatial Diversity among the mobiles.

In the next chapter, we will present a corresponding model for an information theoretic analysis.
Chapter 3

Multiple Access Channels

A Multiple Access Channel, referred to as a MAC, is a channel shared by geometrically distributed users (sources). Hereafter we will assume no multipath.

Cellular Networks have a star geometry. The mobiles (users) transmit on the uplink (also called the reversed link) to the cell site (the hub). This can be modeled as a Multiple Access Channel, MAC. On the downlink (forward link), the cell site broadcasts the signals to all the mobiles. This is known as a Broadcast Channel.

We will focus our work on the Uplink, i.e., the MAC. Indeed the uplink is more severe than the downlink (interference of other users, etc ...). For example, in TDMA systems, interference occurs when one user is not well synchronized and its time slot overlaps with another slot. It is easier to synchronize the transmissions for the downlink (originated from the hub) than for the uplink (originated from distributed users). In CDMA systems, one controls the interference by an appropriate control of the received power. On the downlink, only the users with small received signal are penalized. Conversely, on the uplink, if the signal received from one user is too high, all the users are penalized.

Cellular Networks have several cells. However, in an attempt to simplify the modeling, we will restrict our model to a 1-cell network. By doing so we neglect the interference created by neighboring cells. If we wish to extend the results to a multiple cell network, we could include the interference as part of the surrounding noise. One can assume that the interference is a Gaussian process in time and space. This is legitimate if there are many cells with a large number of users in each of them. Then if the noise is assumed AWGN, the 'new’ noise (noise + interference) is still AWGN. Hereafter we will use the word interference to talk about interference among users of the same cell (i.e., our 1-cell network).

Finally, in order to simplify the analysis, we assume that the channel is time synchronized, i.e.,
code words (see below) originating from different sources arrive at the receiver at the same time. In practice, distributed sources have different [user-to-cell-site] travel time. Moreover if they are mobile, like in cellular networks, the travel times vary. However the assumption we make is not critical, especially since, in the later chapters, we will be only interested in independent decoding. The reader can refer to [7] for a study on this issue.

In this chapter, we will present a model for the MAC: the Discrete Time Memoryless MAC. We will show that real world channels (Continuous Time MACs) can be represented by parallel Discrete Time Memoryless MACs and, finally, we will model the Multiple Access Schemes (TDMA, CDMA and TDMA/SDMA).

The reader can see [4],[3] and [2] for a more complete presentation.

3.1 Discrete Time Memoryless MAC

3.1.1 Sources with Constrained Energy

![](image)

Figure 3-1: Source $i$ with its encoder

Codes:

Let each source $s$, $s \in [1, M]$, have a set, $\{m^s_i, i \in [1, M^s]\}$, of messages.

The upper index is the source's index and the lower index gives the message number. Hereafter without important lost of generality, we look only at source 1 ($s = 1$) and the source's index will be omitted when it doesn't impair comprehension.

We will assume that each source has equally likely messages.

Definition 1 Code of length $n$:

A one to one mapping is generated between the message set, $\{m^s_i, i \in [1, M^s]\}$, and the code book, $\{X^s_i, i \in [1, M^s]\}$. The one-to-one mapping is called the coding function, $E^1$, explained below. The code book is the resulting set of code words of length $n$: $X^s_i = (x^s_{1i}, ..., x^s_{ni})$. 

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The code word components, $x_{ij}$, $j \in [1, n]$, are called *letters*. Letters are real or complex numbers. The coding function, $E^1$, is a one to one mapping between the set of messages and the code book:

<table>
<thead>
<tr>
<th>$m^1_i$</th>
<th>$z_{i1} \ldots z_{in}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$m^1_i$</td>
<td>$z_{i1} \ldots z_{in}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$m^1_M$</td>
<td>$z_{M1} \ldots z_{Mn}$</td>
</tr>
</tbody>
</table>

**Definition 2: Code Rate**:

The rate of a code book is $R = \frac{\log_M M}{n}$

---

Hereafter, we introduce the idea of an ensemble of randomly generated codes. The motivation for analysing an ensemble of codes is simple: it is much easier to compute the average probability of error (for decoding) over an ensemble of randomly generated codes than for a given code. Moreover, there is at least one code in the ensemble with a decoding probability of error better than or equal to the average probability of error.

We define an ensemble of codes as the following:

**Definition 3: Ensemble of Codes $(n, 2^{nR}, Q^1_X(\vec{x}))$**:

Let $Q^1_X(\vec{x})$ be an arbitrary probability function on the set of channel input sequences of length $n$. The ensemble of codes, $(n, 2^{nR}, Q^1_X(\vec{x}))$, is the set of all codes whose code words are independent and distributed according to $Q^1_X(\vec{x})$.

---

Let $Q^1_X(\vec{x})$ be the probability assignment for the letters and let letters be independent from each other. Then the probability function for the code words, $Q^1_X(\vec{x})$, is given by:

$$Q^1_X(\vec{x}) = P(\vec{X} = \vec{x}) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} Q^1_X(x_i)$$

When $Q^1_X(\vec{x})$ is a Gaussian distribution, the code words are called Gaussian code words.

In this paragraph, we give some insight into the notion of independent decoding that we will use intensively in the following sections.

The receiver performs independent decoding when it knows the code book of only one user. The performance of this decoder is evaluated over the ensemble of other users' code books (plus the noise). The letters received from other users are interference. They are randomly distributed according to the distribution $Q^1_X(\vec{x})$. If $Q^1_X(\vec{x})$ is Gaussian, the interference is Gaussian.
Constrained Energy:

To represent the fact that transmissions are always limited in power, one constrains the energy of a code word to be $\mathcal{E}$:

$$E[||\vec{x}||^2] = \sum_{i=1}^{n} E[X_i^2] = nE[X^2] \leq \mathcal{E}$$

In what follows we assume that the $M$ sources have the same code word length, $n$.

3.1.2 The Channel

![Figure 3-2: The Discrete Time Memoryless Multiple Access Channel](image)

**Definition 4** Discrete Time Multiple Access Channel: $[(\vec{x}_1, ..., \vec{x}_M), \vec{o}]$

A channel with $M$ different independent inputs (sources), $(\vec{x}_1, ..., \vec{x}_M)$, is called a Multiple Access Channel (MAC or $M$-MAC if there are $M$ sources). Its output is given by:

$$\vec{o} = \sum_{i=1}^{M} \vec{x}_i + \vec{n}$$

It is statistically described by the conditional probability:

$$P_{\vec{o}|(\vec{x}_1, ..., \vec{x}_M)}(\vec{o}|\vec{z}_1, ..., \vec{z}_M) = P_{\vec{n}}(\vec{o} - \sum_{i=1}^{M} \vec{x}_i)$$

where $\vec{n}$ is the noise.

Let $S$ be a subset of $(1, ..., M)$ and $S^c$ be the complement of $S$ (i.e., $S \cap S^c = \emptyset$, $S \cup S^c = (1, ..., M)$). It is often convenient to work with the probability of the output conditional on a subset $S$ of inputs where the complementary subset of inputs, $S^c$, are randomly chosen over the ensembles of codes. This has the form:

$$P_{\vec{o}|(\vec{x}_1, ..., \vec{x}_M), s \in S} = P_{\vec{n}_S}(\vec{o} - \sum_{i \in S} \vec{x}_i)$$
where $\tilde{N} = \tilde{N} + \sum_{i \in S^t} \tilde{X}^i$.

For example, let us consider 2 sources ($M = 2$), $X$ and $Y$. The various choices of $P_{O|\tilde{S}}$ are:

- $P_{\tilde{O}|\tilde{X}}, \tilde{S} = \tilde{X}, \tilde{S}^c = \tilde{Y}$
- $P_{\tilde{O}|\tilde{Y}}, \tilde{S} = \tilde{Y}, \tilde{S}^c = \tilde{X}$
- $P_{\tilde{O}|\tilde{X},\tilde{Y}}, \tilde{S} = (\tilde{X} \tilde{Y}), \tilde{S}^c = \emptyset$

Moreover, if we know $(Q_{\tilde{X}}, Q_{\tilde{Y}})$ and $P_{\tilde{O}|\tilde{X},\tilde{Y}}$, the channel is completely described.

$$P_{\tilde{O}|\tilde{X}} = \int Q_{\tilde{Y}} P_{\tilde{O}|\tilde{X},\tilde{Y}} ... \text{ etc}$$

When independent decoding is done at the receiver, decoder 1 knows only the code book of source 1. Other sources look like interference: $\tilde{S} = (X^1)$ and $\tilde{S}^c = (X^2, \ldots, X^M)$.

**Definition 5** Memoryless MAC

A channel is said to be memoryless when:

$$P_{\tilde{O}|\tilde{S}O} = \prod_{i=1}^{n} P_{O_i|S_i}(o|s) = \prod_{i=1}^{n} P_{N_i+S_i^t}(n+s^t)$$

where $O = (O_1, \ldots, O_n)$, $S_i$ and $S_i^t$ are the same as $S$ and $S^t$ but for letters instead of vectors, and $(N_1, \ldots, N_i, \ldots, N_M)$ are the noise components in the letters of the code words.

A memoryless channel can be represented by a channel with letters for inputs. Then its output is given by:

$$O_i = \sum_{i=1}^{M} X_i^t + N_i$$

For memoryless channels, the output letter $O_i$ depends only upon the corresponding input letters and not upon what letters have been sent before. Moreover this dependence is the same for each output letter; it is independent of time.

In cellular networks, the probabilistic characteristics of the channel vary in time and this variation is different from one user to the other. Our model can be viewed as picturing the 'average' of the probabilistic characteristics or even the worst case depending on what we adopt for conditional probabilities.

**Definition 6** Gaussian MAC:

When the noise, $N$, is Gaussian distributed the channel is called a Gaussian Multiple Access Channel.
3.1.3 The Receiver

The receiver guesses from the output of the channel, \( \tilde{O} \), the messages that have been sent by the sources. Its guess is done according to a decoding rule and the knowledge of the code books. When no guess is possible, the receiver gives an erasure message. It implements the mapping function:

\[
\tilde{O} \rightarrow ([m_1^1, ..., m_1^M], \text{ erasure})
\]

Decoding Rules:

The decoding rule can be:

- the Maximum A Posteriori (MAP) rule, (optimal). It minimizes the error probability but requires the knowledge of the input distribution.

- the Maximum Likelihood rule, (suboptimal). It is the MAP when all inputs are equiprobable (our case).

Joint Decoding:

The receiver includes only one decoder for all the sources.

![Joint Decoding Diagram](image)

**Figure 3-3: Joint decoding: The Receiver and its Decoder**

**Definition 7 Joint Decoding:**

When the receiver knows the \( M \) code books, it can use maximum likelihood decoding jointly over the \( M \) sources.

Obviously this is the optimum decoding scheme since no information is 'thrown away', but its implementation is hard. Indeed, let us assume \( M \) users with convolutional codes of constraint length \( n \). The 'optimum' Viterbi decoder has a trellis with \( 2^M n \) states at each step. We see that even for small values of \( M \) the computation time is too large (exponential).

Independent Decoding:

The receiver contains as many receivers as there are sources.
**Definition 8 Independent Decoding:**

When a decoder knows only user 1's code book, it decodes only source 1 regarding the other signals as noise.

This is obviously suboptimal since the decoders do not cooperate. Other sources are interference. With gaussian code ensembles (i.e., Gaussian code words), the interference is Gaussian noise to the receiver. In the case of independent decoding, the Multiple Access Channel is an Interference Channel.

### 3.1.4 Capacity Region and Achievable Rates

In this section, we introduce the notions of capacity for discrete time channels, capacity region for Multiple Access Channels and achievable rate points, \((R_1, \ldots, R_M)\).

**Definition 9 Entropy:**

The entropy of a random vector, \(X\), with probability assignment \(P(X)\) is defined to be:

\[
H(X) = \sum_X P(X) \log_2 \frac{1}{P(X)}
\]

We used \(\log_2\) in the definition and therefore the entropy is in bits. Entropy can be viewed as the average number of bits of information needed to specify \(X\).

The definitions of conditional entropy and joint entropy follow immediately from above:

\[
H(X|Y) = \sum_{X,Y} P(XY) \log_2 \frac{1}{P(X|Y)}
\]

\[
H(X,Y) = \sum_{X,Y} P(XY) \log_2 \frac{1}{P(XY)}
\]
The next 2 propositions will be intensively used throughout the next sections.

**Proposition 1 Entropy of a Normal Vector \( \mathbf{W} \):**

Let \( \mathbf{W} \) be a normal distributed vector of dimension \( n \) : \( \mathcal{N}(E[\mathbf{W}], E[(\mathbf{W} - E[\mathbf{W}])(\mathbf{W} - E[\mathbf{W}])^\dagger]) \), then \( H(\mathbf{W}) = \frac{1}{2} \log(2\pi e)^n \det(E[(\mathbf{W} - E[\mathbf{W}])(\mathbf{W} - E[\mathbf{W}])^\dagger]) \).

**Proposition 2 Maximum Entropy of a Random Vector, \( \mathbf{W} \), with Given First 2 Moments**

Let \( \mathbf{W} \) be a random vector with zero mean, \( E[\mathbf{W}] = 0 \), and given covariance matrix, \( E[(\mathbf{W} - E[\mathbf{W}])(\mathbf{W} - E[\mathbf{W}])^\dagger] \), then \( H(\mathbf{W}) \) is maximized iff \( \mathbf{W} \) is normal distributed.

**Definition 10 Mutual Information**:

The Mutual Information between two random vectors, \( \mathbf{X} \) and \( \mathbf{Y} \), is defined to be:

\[
I(\mathbf{X}; \mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y} | \mathbf{X}) = H(\mathbf{X}) - H(\mathbf{X} | \mathbf{Y})
\]

In the next sections, we look at the mutual information between the inputs and the output of a MAC:

\[
I(X^1, ..., X^M; O)
\]

Mutual Information can be viewed as the average number of bits of information that we obtain about the inputs to the channel by knowing the output.

From the above we see that mutual information depends upon:

- the channel characteristics, \( P(O | X^1, ..., X^M) \), and

- the input probability assignments, \( P(X^1, ..., X^M) = P(X^1)...P(X^M) \) (inputs from different sources are independent).

**Definition 11 Capacity Region**:

The Capacity Region is defined as the convex closure of the class of rate vectors \( (R_1, ..., R_M) \) such that

\[
\sum_{r \in S} R_r \leq \max I(O; X^r, s \in S | X^u, u \in S^c)
\]

for all subsets \( S \) of \( \{1, ..., M\} \) and for some choice of \( (Q_{X^s}, s \in [1, M]) \).

**Definition 12 Average probability of error**:

Let \( \hat{M} = (\hat{m}_1, ..., \hat{m}_M) \) or \( \hat{M} = \) (erasure) be the estimate (guess) at the output of the decoder.

An erasure message occurs when the decoder can not guess what messages were sent. This is not an
error since we assume that the receiver can always ask the sources to send the messages again.

An error occurs when the decoder makes a wrong guess, \( (\hat{m}^1, ..., \hat{m}^M) \neq (\tilde{m}^1, ..., \tilde{m}^M) \).

We denote by \( P_e \) the average probability of error.

**Definition 13** Achievable rate point \( (R^1, ..., R^M) \):

A rate point \( (R^1, ..., R^M) \) is achievable iff, for any \( \epsilon > 0 \) and for \( n \) sufficiently large, there are codes \( ((n, 2^{nR^1}, E^1), ..., (n, 2^{nR^M}, E^M)) \) such that \( P_e < \epsilon \).

The Coding Theorem states that any rate point in the interior of the capacity region is an achievable rate point.

In the later sections, we will study different scenarii of transmission and, in the special case of a Gaussian ensemble of codes, we will refer to:

- symmetric channel capacity as \( I(O; X^1, ..., X^M) \) where there is power control and fairness among users, i.e., \( R_1 = ... = R_M \);

- user capacity for TDMA as \( I(O; X^s|X^u, u \in [1, M], u \neq s) \) where \( s \in [1, M] \);

- user capacity for CDMA as \( I(O; X^s) \) where \( s \in [1, M] \).

### 3.2 Continuous Time MAC

#### 3.2.1 Sources with Constrained Power

![Diagram](image)

Figure 3-5: Source \( i \) : the Coder and the Modulator

In real systems, source 1 generates a message, \( m^1 \), which is coded into a code word, \( \tilde{X}^1 \), by the coder. Then in the modulator, \( \tilde{X}^1 \) is mapped into a waveform, \( x^1(t) \), and is sent into the channel. Thus the inputs to the channel are not discrete inputs but rather continuous time waveforms.

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The waveform Decomposition:

Let the channel be bandlimited to $[-W, +W]$ and $T$ be the finite duration of a code word. The waveform associated with a code word should be bandlimited and time limited at the same time. We know that this is impossible.

Moreover, waveforms have randomness characteristics that must be described one way or another. The model we adopt is based on the decomposition of signals over a complete set, $\Theta$, of real orthonormal functions $\theta_i(t)$. For $\Theta$, we take the set of sampling functions:

$$\Theta = \{ \theta_i(t) \mid \theta_i(t) = \sqrt{\frac{2}{2W}} \sin \frac{2\pi W(t - \frac{i}{2W})}{2\pi W(t - \frac{i}{2W})}, i \in [-\infty, +\infty] \}$$

The $\theta_i$ are bandlimited to $[-W, +W]$. A waveform, $x^1(t)$, is given by:

$$x^1(t) = \sum_{0}^{n-1} x^1_i \theta_i(t), \ n = 2WT$$

$$x^1_i = \sqrt{\frac{1}{2W}} x^1(\frac{i}{2W})$$

$x^1(t)$ is bandlimited to $[-W, +W]$ because of the way the $\theta$s are constructed. $x^1(t)$ is also 'quasi' time limited to $[0, T]$ because the sum is finite with $n = 2WT$.

$n$ is the number of degrees of freedom and the length of the code words. Note that to increase the degrees of freedom, one can either increase the bandwidth $W$, i.e., spread in frequency, or increase the code word duration $T$, i.e., spread in time.

The coefficients are chosen to be IID random variables according to a distribution, $Q_X^1(z)$. Thus the waveforms themselves are random waveforms.

The channel modulator is defined as:

$$X^1 = (x^1_1, ..., x^1_n) \longrightarrow x^1(t) = \sum_{0}^{n-1} x^1_i \theta_i(t)$$

When $Q_X^1(z)$ is a Gaussian distribution or equivalently when the ensemble of codes is Gaussian, the sequence of waveforms from successive code words is a stationary Gaussian process with a flat spectral density over $[-W, +W]$.

This model is not completely realistic because of the absence of time limits, but it provides us with some insight to justify our approach of continuous time Multiple Access Channels. Refer to [4] for a more rigorous model.
Constrained Power:

The inputs to the channel are power limited to $S$. The power $S$ is $\frac{E}{T}$, where $E$ and $T$ are respectively the energy and the duration of a code word.

$$\sum_{i=1}^{n} ||\mathbf{x}_i||^2 \leq \int_0^T ||\mathbf{z}^1(t)||^2 dt \leq ST = E$$

3.2.2 The Channel

![Channel diagram]

The channel is bandlimited to $[-W, +W]$. The noise $N(t)$ is additive, Gaussian and independent of the channel inputs. The unit gain filter is placed to remove the noise outside $[-W, +W]$.

Figure 3-6: The continuous time channel

The channel is bandlimited to $[-W, +W]$. The noise, $N(t)$, is assumed additive AWNG and independent of the channel inputs. It is given by:

$$N(t) = \sum_{i=0}^{+\infty} n_i \theta_i(t)$$

where $n_i$ are IID Gaussian random variables.

3.2.3 The Receiver

![Receiver diagram]

Figure 3-7: The Receiver for independent decoding.

The receiver includes a demodulator and a decoder. The demodulator projects the continuous time input, $O(t)$, over each of the $\theta_i(.)$ to obtain the received code word, $O$. Then the decoder guesses the corresponding messages.
For joint decoding the receiver is:

\[ O(t) \rightarrow Q = (O_1, \ldots, O_n) \rightarrow (\tilde{m}^1, \ldots, \tilde{m}^M, \text{erasure}) \]

For independent decoding the receiver is:

\[ O(t) \rightarrow Q = (O_1, \ldots, O_n) \rightarrow \{(m^s, \text{error}), \ s \in [1, M]\} \]

The projections use the orthonormal property of the \( \theta_i(.) \)s and give:

\[ O_i = \int_0^T O(t) \theta_i(t) dt, \ i \in [0, 2WT] \]

### 3.2.4 The Discrete Time Model

Between the output of the coders and the input of the decoder, the channel can be represented as parallel discrete time channels where the \( i \)th channel is:

\[ O_i = \sum_{s=1}^{s=M} x_i^s + N_i \]

\( x_i^s \) for a given \( s \) are IID by construction.

\( N_i \) are IID because the noise is assumed Gaussian (Identical random variables) and white (Independent random variables).

\( N_i \) and \( (x_i^s, \ s \in [1, M]) \) are independent by assumption.

### 3.3 Multiple Access Schemes

#### 3.3.1 Sources

We will consider a fixed number, \( M \), of sources. For the derivation of the capacity region, we will assume:

- Code ensembles, \( \{[n, 2^nR, Q_X], \ s \in [1, M]\} \),

Then to compare the different multiple access schemes with each other, we will make the following additional assumptions:

- Gaussian code ensembles, \( \{[n, 2^nR, \mathcal{N}(0, \alpha^s)], \ s \in [1, M]\} \),

- perfect power control unless otherwise. This means that users are received with the same power, \( S_1 = \ldots = S_M \),
• optimal coding. In each code ensemble, \( ([n, 2^R, N(0, \alpha_s)], s \in [1, M]) \), we use only 'good' code books, \( ([n, 2^n R, E_s], s \in [1, M]) \), such that the capacity is approached.

• fairness (each user has the same rate, \( \{R_u = R_s, s \in [1, M], u \in [1, M]\} \)). This is a requirement in current cellular networks.

Sometimes we consider different rates among users. This will be important in multiple service networks of the future (voice, data, ...).

### 3.3.2 Time Division Multiple Access, TDMA

In pure TDMA, only one user transmits at a time. During the time slot allocated to user \( i \), user \( i \) transmits at full capacity while other users remain silent.

**Our Model**:

The system has the following characteristics:

• Gaussian code ensembles : \( ([n, 2^n R, N(0, \alpha_s)], s \in [1, M]) \);

• Bandwidth : \( [-W, +W] \);

• Time sharing is optimal : we assume the system is perfectly time synchronized so that no guard time slots are needed between two consecutive slots.

Within time slot \( s \), the network approaches capacity:

• \( C_s = I(O; X_s|X^u, u \in [1, M], u \neq s) \).

Let \( T_s \) be the normalized time duration of slot \( s \), \( \sum_{s=1}^{M} T_s = 1 \); Then the operating rate point is given by:

• \( \{T_1 C_1, ..., T_M C_M\} \), (see point \( T \) in [Fig.4-1]).

Obviously, real systems have suboptimal codes and they need guard time slots between consecutive time slots.

### 3.3.3 Code Division Multiple Access, CDMA

In CDMA, all the \( M \) users transmit at the same time.

**Our Model**:

The system has the following characteristics:

• Gaussian code ensembles : \( ([n, 2^n R, N(0, \alpha_s)], s \in [1, M]) \);
• Bandwidth: \([-W, +W]\);

• Power control: first, we will assume perfect power control, and derive the capacity, then we will look at the degradation when some users are received at a higher power than others.

The operating rate points are:

• for joint decoding (assuming power control and fairness)

\[
\left\{ \left( \sum_{i=1}^{M} R_i = (I(O; X^1, ..., X^M)) \right) \right\} \cap \left\{ R_1 = ... = R_M \right\} = \left( \frac{I(O; X^1, ..., X^M)}{M}, ..., \frac{I(O; X^1, ..., X^M)}{M} \right)
\]

(see point J in [Fig.4-1]),

• for independent decoding

\[
(R_1, ..., R_M) = (I(O; X^1), ..., I(O; X^M))
\]

(see point C in [Fig.4-1]).

In real systems, the codes are suboptimal (they do not reach full capacity), the source encoder has a higher rate (or equivalently, the spreading is done after the encoding of the source) and thus the code (of each user) is less efficient. Finally power control is usually far from optimal.

### 3.3.4 TDMA and Space Division Multiple Access, TDMA/SDMA

In TDMA/SDMA, only a subset of the \(M\) users transmit at a time. This can be viewed as a pure TDMA network whose omnidirectional antenna is replaced by an array of sensors. In order to minimize the cost, the Multiple Access characteristics (such as time slot duration, modulation ...) remain unchanged. The capacity is then increased by allowing more than one user to transmit in the same slot.

Users are partitioned in subsets. Users from the same subset transmit in the same time slot and take advantage of the spatial diversity.

Our Model:

The system is a mix of TDMA and SDMA. Different subsets are separated in time since they use different time slots. Users in the same subset are separated by Space Division.

The model has the following characteristics:

• Code ensembles: \([n, 2^n R_s, \mathcal{N}(0, \alpha^s)], s \in [1, M] \)

• Bandwidth: \([-W, +W]\).
• We assume that the number of subsets, \( N_{\text{subset}} \), is fixed and that the \( i^{th} \) subset is denoted by \( S_i \) (its complement is denoted by \( S_i^c \)). This implies that \( M = N_{\text{subset}} \cdot |S_i| \), where \( M \) is the total number of users in the cell.

• We assume optimal time sharing and finally, we assume that the time slots have the same duration.

The operating points for TDMA/SDMA are given by:

• for joint decoding (assuming power control and fairness)

\[
(R_1, ..., R_M)
\]

where, for \( s \in S_i \)

\[
R_s = \frac{1}{|S_i|} I(O; X^v, v \in S_i | X^u, u \in S_i^c)
\]

• for independent decoding

\[
(R_1, ..., R_M)
\]

where, for \( s \in S_i \)

\[
R_s = I(O; X^v | X^u, u \in S_i^c)
\]

(see point T in [Fig. 5-7]).

Optimum TDMA/SDMA:

We do optimum TDMA/SDMA, when the partition into subsets is realized as follow:

• at any time, among all possible partitions, the system will choose the partition that maximizes the minimum rate over all users, ie:

\[
\text{arg}(\max_{\text{all partitions}} \min_{\text{all users}} \text{rate})
\]

We will compare optimum TDMA/SDMA with non optimum TDMA/SDMA where the partition is fixed.

From now on, we will consider only independent decoding.

In this chapter, we have set the basis to build a model for Multiple Access Channels. Then we defined the different multiple access schemes that we will be studying in the next chapters.


Chapter 4

Multiple Access Channel Without Spatial Diversity

In this chapter, we consider either an omnidirectional antenna or a sectored antenna at the cell site (see [sec 2.1]).

We will derive the capacity region for a Multiple Access Channel without Spatial Diversity. Then we will look at TDMA and CDMA, and we will develop the basis for a fair comparison between multiple access schemes.

The reader can refer to [6], [5], [3], [12], and [1] for more details.

4.1 The 2-Multiple Access Channel, [(X, Y), O]

The output, O, of the channel is given by:

\[ O = X + Y + N \]

In this section, we will use the same character for the source and for the letters generated by the source.

X and Y are sources independent of each other. They generate IID letters, X and Y. Letters have zero mean and variance \( \alpha_X \) and \( \alpha_Y \).

N is AWGN with distribution \( \mathcal{N}(0, \sigma_N^2) \). N is independent of inputs X and Y.

We can normalize the noise variance to 1 without modifying the problem. Indeed, as we will see in the next paragraphs, the capacity depends only upon the ratio (signal energy) to (noise energy) or equivalently on the SNR.

Let \( \alpha_X = \frac{\sigma_X^2}{\sigma_N^2} \) and \( \alpha_Y = \frac{\sigma_Y^2}{\sigma_N^2} \). \( \alpha_X \) and \( \alpha_Y \) are normalized energies.
Finally we have the equivalent channel:

- $N = \mathcal{N}(0,1)$,
- $X$ and $Y$ are identically distributed and zero mean with variance ($\alpha_X$ and $\alpha_Y$).

### 4.1.1 Capacity Region

In all the following formulas, we will use $[\max I(O; . | .)]$ instead of $[\max_{Q_X, Q_Y} I(O; . | .)]$ when the maximization is implicitly over all possible input distributions $(Q_X, Q_Y)$.

Any achievable rate pair, $(R_X, R_Y)$, must satisfy:

\begin{align*}
R_X &\leq \max I(O; X|Y) \leq \frac{1}{2} \log(1 + \alpha_X) \\
R_Y &\leq \max I(O; X|X) \leq \frac{1}{2} \log(1 + \alpha_Y) \\
R_X + R_Y &\leq \max I(O; X,Y) \leq \frac{1}{2} \log(1 + \alpha_X + \alpha_Y)
\end{align*}

(4.1)  
(4.2)  
(4.3)

The region delimited by (4.1, 4.2, 4.3) is the capacity region of the 2 Multiple Access Channel.

They are several ways of proving this result. A proof can be found in [5] and [3].

The following remarks provide some insight about these 3 inequalities:

- $R_X \leq \max I(X; O|Y)$ is the maximum rate that $X$ can reach when its decoder knows what is sent by $Y$. That is, either $Y$ doesn’t transmit (rate is zero) or the decoder first decodes $Y$ and then decodes $X$ knowing the value of $Y$.
- $R_Y \leq \max I(Y; O|X)$ is the same as above but for $Y$.
- $R_X + R_Y \leq \max I(X, Y; O)$ is the maximum rate that can be reached when $X$ and $Y$ transmit jointly, i.e., $X$ and $Y$ are considered as one transmitter with total energy ($\alpha_X + \alpha_Y$).

The second part of the inequalities comes from the fact that, among all zero mean distributions with given second moment, the entropy is maximized by a Gaussian distribution (see [sec. 3.1.4, prop. 2]).

### 4.1.2 Operating Points

Hereafter points on the graph are designated by their coordinates $(R_X, R_Y)$. In the expression of the capacity, $\max_{Q_X, Q_Y} I(O; . | .)$, we will omit the maximization sign for, implicitly, we operate at capacity.

**Pure TDMA:**

TDMA systems do time sharing between:
Figure 4-1: Operating Points for a Given Pair \((\alpha_X, \alpha_Y)\)

- point \(A = (0, I(0; Y|X)) = (0, \frac{1}{2} \log(1 + \alpha_Y))\) where only \(Y\) is transmitting and
- point \(F = (I(0; X|Y), O) = (\frac{1}{2} \log(1 + \alpha_X), 0)\) where only \(X\) is transmitting.

Under the assumptions of equal energy, \(\alpha, (\alpha_X = \alpha_Y = \alpha)\) and fairness \((R_X = R_Y)\), the operating point, \(T\), is reached by equally sharing time between \(X\) and \(Y\):

- point \(T = (\frac{1}{4} \log(1 + \alpha), \frac{1}{4} \log(1 + \alpha)).\)

Note that by appropriate time sharing any point between \(A\) and \(F\) can be reached.

Decoding in Sequence:

At the receiver, the decoder of \(X\) (respectively \(Y\)) first decodes \(Y\) (\(X\)), then subtracts \(Y\) (\(X\)) from the output \(O\) and finally decodes \(X\) (\(Y\)). Meanwhile, the decoder for \(Y\) (\(X\)) decodes \(Y\) (\(X\)) considering \(X\) (\(Y\)) as interference. The corresponding operating point is \(D\) (\(B\)):

- point \(D = (I(O; X|Y), I(0; Y)) = (\frac{1}{2} \log(1 + \alpha_X), \frac{1}{2} \log(1 + \frac{\alpha_X}{1 + \alpha_Y}))\) and
- point \(B = (I(O; X), I(0; Y|X)) = (\frac{1}{2} \log(1 + \frac{\alpha_Y}{1 + \alpha_X}), \frac{1}{2} \log(1 + \alpha_Y)).\)

For example, in a channel with high interference \((\alpha_X \ll \alpha_Y)\), the decoder first decodes \(Y\), then subtract \(Y\) from \(O\) and then decodes \(X\). Point \(B\) is the operating point. In practice implementation is somewhat difficult because signals are continuous time waveforms and the decoder must reconstruct \(Y\), in phase and amplitude, before any subtraction can be done.

Note that by an appropriate time sharing point \(J\) can be reached.
Joint Decoding:

When optimal joint decoding is done, the receiver decodes $X$ and $Y$ at the same time. Under power control, the operating point, $J$, is given by (4.3) and ($R_X = R_Y$):

- $J = (\frac{1}{2} I(O; X, Y), \frac{1}{2} I(O; X, Y)) = (\frac{1}{2} \log(1 + \alpha_X + \alpha_Y), \frac{1}{2} \log(1 + \alpha_X + \alpha_Y))$

Point $J$ corresponds to CDMA systems under the assumption of joint decoding.

Independent Decoding:

Independent decoding means that each decoder decodes its own signal and considers the other users' signals as noise:

- $C = (I(O; X), I(O; Y)) = (\frac{1}{2} \log(1 + \frac{\alpha_X}{1+\alpha_Y}), \frac{1}{2} \log(1 + \frac{\alpha_Y}{1+\alpha_X}))$.

With power control we get:

- $C = (\frac{1}{2} \log(1 + \frac{\alpha}{1+\alpha}), \frac{1}{2} \log(1 + \frac{\alpha}{1+\alpha}))$.

Independent decoding is suboptimal since point $C$ lies inside the capacity region.
Point $C$ corresponds to ideal CDMA systems under the assumption of independent decoding.

Suboptimal Coding:

Point $H$ lies inside the capacity region and is suboptimal: indeed, for example, we can increase $R_X$ without changing $R_Y$ by using a better code for $X$. By doing so we reach point $E$. $E$ is also suboptimal since we can increase $R_Y$ with $R_X$ unchanged up to point $D$ which is optimal.

4.2 The M-Multiple Access Channel, $[(X^1, \ldots, X^M), O]$

We use $S$ to designate a subset of $[1, \ldots, M]$ and $S^c$ for its complement ($S \cap S^c = \emptyset$, $S \cup S^c = [1, \ldots, M]$).

An achievable rate M-tuple, $(R_1, \ldots, R_M)$, must satisfy:

$$
\sum_{s \in S} R_s \leq \max \{ I(O; X^s, s \in S), u \in S^c \} \leq \frac{1}{2} \log(1 + \sum_{s \in S} \alpha_s) \tag{4.4}
$$

for all subsets $S$.

The convex hull of the region delimited by equations (4.4) is the capacity region of the M-Multiple Access Channel.

A proof can be found in [5] and [3].
4.3  A Fair Comparison Between TDMA and CDMA

![Diagram showing comparison between TDMA and CDMA](image-url)

Depending upon the SNR CDMA is better than TDMA.

\[ SNR = \sigma = \frac{\sigma^2}{N_0 T} \]

C=CDMA and T=TDMA.

Figure 4-2: Discrete Time Model : (a) low SNR, (b) high SNR.

In cellular networks, TDMA and CDMA systems have completely different characteristics. CDMA systems are spread over a much larger bandwidth (1 MHz vs 200 kHz) and have a much smaller SNR than TDMA systems.

Discrete time models do not contain any information about the bandwidth and thus must be compared carefully. Also, depending upon SNR (\( \sigma \), CDMA or TDMA look more efficient (see [Fig. 4-2]).

We shall develop a Bandlimited Continuous Time model within which we can interpret and compare the discret time models carefully.

4.3.1  Continuous Time Bandlimited MAC

Relevant Parameters :

In this paragraph we give the relevant parameters for a Bandlimited Continuous Time model. The parameters are :

- the energy of a code word, \( E_c \);
- the length of a code word, also called the degrees of freedom, \( n \);
- the duration of a code word, \( T \);
• the power, $S$, of the source. Note that this is the power at the input to the decoder, and thus it is the received power;

• the bandwidth, $[-W, +W]$ and

• the noise power spectral density, $N_0 = E[N^2]$.

We have the following relationships:

$$\mathcal{E} = nE[X^2]$$  \hspace{1cm} (4.5)

$$S = \frac{\mathcal{E}}{T}$$  \hspace{1cm} (4.6)

$$n = 2WT$$  \hspace{1cm} (4.7)

**The Capacity as a Function of $W$ and $S$**

In order to simplify the presentation, we derive the capacity for the case of only 2 users $X$ and $Y$. The generalisation to $M$ users is immediate. We assume that the 2 users share the same bandwidth $[-W, +W]$.

In the discrete time model, TDMA is time sharing between the 2 points:

$$\left(\frac{1}{2} \log(1 + \alpha_X), 0\right)$$

and

$$\left(0, \frac{1}{2} \log(1 + \alpha_Y)\right)$$

In the discrete time model, CDMA (with independent decoding) is given by:

$$\left(\frac{1}{2} \log(1 + \frac{\alpha_X}{1 + \alpha_Y}), \frac{1}{2} \log(1 + \frac{\alpha_Y}{1 + \alpha_X})\right)$$

In these expressions, capacities are given in bits per channel use, i.e., how many bits of information can be sent each time the sources use the channel to send a letter.

We can express $\alpha_X$ and $\alpha_Y$ as:

$$\alpha_X = \frac{E[X^2]}{E[N^2]} = \frac{\mathcal{E}_X}{N_0^2 n_X} = \frac{S_X}{N_0 W}$$

$$\alpha_Y = \frac{E[Y^2]}{E[N^2]} = \frac{\mathcal{E}_Y}{N_0^2 n_Y} = \frac{S_Y}{N_0 W}$$

If both sources are received with the same power, $S = S_X = S_Y$, then

$$\alpha_X = \alpha_Y = \frac{S}{N_0 W}$$
Now, from the expressions for the discrete time model, we want to derive the capacity in bits per second.

For user $X$, the number of bits of information that $X$ can send per letter (or channel use) is:

- in the case of TDMA, $\frac{1}{2} \log(1 + \alpha_X)$ and
- in the case of CDMA, $\frac{1}{2} \log(1 + \frac{S_X}{N_0 W})$.

$X$ sends $\frac{W}{2}$ letters per second. From [eq. 4.7], $\frac{W}{2} = 2W$. Therefore the capacity for $X$ is:

- TDMA: $\frac{W}{2} \log(1 + \alpha_X) = W \log(1 + \frac{S_X}{N_0 W})$ while $X$ is transmitting;
- CDMA: $\frac{W}{2} \log(1 + \frac{S_X}{N_0 W}) = W \log(1 + \frac{S_X}{N_0 W + S_Y})$.

User $Y$ sends $\frac{W}{2}$ letters per second and $\frac{W}{2} = 2W$. The results are immediate.

We obtain the capacity in bits per second.

TDMA is time sharing between the 2 points:

$$(W \log(1 + \frac{S_X}{N_0 W}), 0)$$

and

$$(0, W \log(1 + \frac{S_Y}{N_0 W}))$$

CDMA is given by:

$$(W \log(1 + \frac{S_X}{N_0 W + S_Y}), W \log(1 + \frac{S_Y}{N_0 W + S_X}))$$

**Capacity for 2 Users with Power Control**:

If there is power control, $S_X = S_Y = S$, and if TDMA shares time equally between $X$ and $Y$, we get for TDMA:

$$(\frac{W}{2} \log(1 + \frac{S}{N_0 W}), \frac{W}{2} \log(1 + \frac{S}{N_0 W}))$$

and for CDMA:

$$(W \log(1 + \frac{S}{N_0 W + S}), W \log(1 + \frac{S}{N_0 W + S}))$$

**Capacity for $M$ Users with Power Control**:

The result for the case with $M$ users can be derived from above. For TDMA, we get:

$$(\frac{W}{M} \log(1 + \frac{S}{N_0 W}), \ldots, \frac{W}{M} \log(1 + \frac{S}{N_0 W}))$$

and for CDMA:

$$(W \log(1 + \frac{S}{N_0 W + (M-1)S}), \ldots, W \log(1 + \frac{S}{N_0 W + (M-1)S}))$$

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Spectral Efficiency:

We define the Spectral Efficiency as the capacity in bits per second per Hertz. That is, the Spectral Efficiency $= \frac{C}{W}$.

For TDMA:

$$\left( \frac{1}{M} \log(1 + \frac{S}{N_0W^2}), \ldots, \frac{1}{M} \log(1 + \frac{S}{N_0W^2}) \right) = \left( \frac{1}{M} \log(1 + \alpha), \ldots, \frac{1}{M} \log(1 + \alpha) \right)$$

For CDMA:

$$\left( \log(1 + \frac{S}{N_0W + (M-1)S}), \ldots, \log(1 + \frac{S}{N_0W + (M-1)S}) \right) = \left( \log(1 + \frac{\alpha}{1 + (M-1)\alpha}), \ldots, \log(1 + \frac{\alpha}{1 + (M-1)\alpha}) \right)$$

4.3.2 TDMA vs CDMA

In this paragraph, we show that depending upon the power, $S$, and the bandwidth, $W$, CDMA or TDMA will perform better.

We also show that the performance depends upon the number of users and we briefly explain why perfect power control is a critical assumption.

Power Limited Systems

The power is fixed. We see in [Fig. 4-3] that CDMA is more efficient than TDMA for large bandwidth $W$:

- Large $W$, referred to as wide band, corresponds to many degrees of freedom per unit of time. In this case, code words from different users are almost orthogonal to each other and CDMA works efficiently.
  
  On the contrary, TDMA forces each user to be orthogonal in time with each other. By doing so, it wastes the power of each user when it is not transmitting. For large $W$, forcing separation among users is not optimal since users are already almost orthogonal.

- Small $W$, known as narrow band, corresponds to few degrees of freedom per unit of time. Code words are far from orthogonal. Thus CDMA is very poor.
  
  On the contrary, TDMA is efficient since it forces orthogonality among users.

Band Limited Systems

We see in [Fig. 4-4] that TDMA is more efficient than CDMA for high power:

- High power: For CDMA, the interference is large relative to the noise. Therefore the $\text{SNR}$ is
We arbitrarily set $\frac{\xi}{N_0} = 1$. We can view the capacity as been in Hz per unit of $\frac{\xi}{N_0}$.

Figure 4-3: Spectral Efficiency as a function of the bandwidth for a 2 user channel.

Low and CDMA performs poorly.

Of course, this does not happen with TDMA thanks to the separation in time.

- Low power: Here, the interference from other users is much lower than the background noise, thus the $SNR$ for CDMA is very close to the $SNR$ for TDMA and CDMA performs much better than TDMA.

**Influence of the number of users:**

With fixed power and fixed bandwidth, we would like to know how both systems behave when the number of users increases. That is, with more and more users, are we better off doing TDMA or CDMA?

The intuition is that two phenomena are competing:
We arbitrary set $N_0 W = 1$. For large $S$:
- CDMA approaches $1$ bits/sec/Hz
- TDMA approaches $\frac{1}{2} \log_2 (\frac{S}{N_0 W})$ bits/sec/Hz

Figure 4-4: Spectral Efficiency as a function of the power for a 2 user channel.

- since degrees of freedom are fixed (fixed bandwidth), the more users there are in the network, the less codes words are approximately orthogonal;

- since power is fixed, as the number of users increases, TDMA is more and more penalized by wasting the power of each user when it is not transmitting.

To show this, we will study the behavior of point $E$ (see Fig. 4-4). Point $E$ is defined by the $SNR$, $\alpha_E$, where TDMA and CDMA have the same spectral efficiency.

$\alpha_E$ is given by:

$$\frac{1}{M} \log_2 (1 + \alpha_E) = \log_2 (1 + \frac{\alpha_E}{1 + (M - 1) \alpha_E})$$
we have, as $M \to \infty$:

$$M \log_2(1 + \frac{\alpha_E}{1 + (M - 1)\alpha_E}) \to \frac{1}{\ln 2}$$

This gives, as $M \to \infty$:

$$\alpha_E \to e - 1 = 1.718$$

For example: for $M = 2$, $\alpha_E = 1.618$ and for $M = 6$, $\alpha_E = 1.682$ (we will need this result in [sec. 5.5.4]).

Our intuition was correct; $\alpha_E$ depends, but only weakly, upon the number of users because the two phenomena that we presented above cancel each other.

**Imperfect power control:**

Let us suppose that power control is imperfect and that user 1 is received with a power $10^3$ times larger than other users.

For TDMA, we get:

$$\left(\frac{W}{M} \log(1 + \frac{10^3 S}{N_0 W}), ..., \frac{W}{M} \log(1 + \frac{S}{N_0 W})\right)$$

and for CDMA:

$$\left(W \log(1 + \frac{10^3 S}{N_0 W + (M - 1)S}), ..., W \log(1 + \frac{S}{N_0 W + ((M - 2) + 10^3)S})\right)$$

We see that with TDMA other users are not penalized because they are separated (in time) from user 1.

But with CDMA, the SNR of other users drops drastically ($\frac{S}{N_0 W + ((M - 2) + 10^3)S}$). Other users are severely penalized.

**CDMA is not as robust as TDMA to imperfect power control.**

**Conclusion:**

We have seen that:

- We can not compare TDMA with CDMA without taking into account the available bandwidth as well as the available power.

- The limit where CDMA becomes more efficient than TDMA, depends, but only very weakly, upon the number of users in the cell.

- Perfect power control is a dramatic assumption since CDMA degrades as power control becomes imperfect.
Recall, however, that this analysis ignores adjoining cells, which is one of the major advantages of CDMA (CDMA has a reuse factor of 1 whereas TDMA has a reuse factor of $\frac{1}{f}$).

In the next chapter, we will look at CDMA and TDMA/SDMA. We want to answer the following questions:

- How does TDMA/SDMA perform compared to pure TDMA?

- Is TDMA/SDMA better than CDMA? To answer this question, we will look at the behavior of both systems for a $SNR$ equal to $\alpha_E$. (where $\alpha_E$ is the $SNR$ with which TDMA and CDMA are equivalent (point E in [Fig. 4-4])).

- With spatial diversity, does CDMA gain in robustness against imperfect power control?

We briefly outline the results we will get:

- Both pure TDMA and TDMA/SDMA encounter dramatic $SNR$ enhancement thanks to the multiple receivers but only TDMA/SDMA takes advantage of the separation ability.

- TDMA/SDMA is better than CDMA for minimum capacity but CDMA is better for average capacity.

- CDMA gains in robustness against imperfect power control.
Chapter 5

MAC with Spatial Diversity

To take advantage of the Spatial Diversity among users, we put an Array of Sensors (see [chap. 2]) at the front end of the receiver. We will consider an array with K identical and isotropic sensors and without loss of essential generality we will use a \((K, d)\) Uniform Linear Array.

We assume that the Directions of Arrival, DOAs, are known (see [chap. 2]).

The reader can refer to [13] for a study on a Multiple User channel with several receivers.

5.1 The Model, \(\{(X^1, \ldots, X^M), Q\}\):

5.1.1 K Parallel Channels:

We model the Multiple Access Channel as K parallel discrete time MACs:

\[\{(X^1, \ldots, X^M), O_k, k \in [1, K]\}\]

The K channels are identical:

- They have the same noise characteristics: \(N_k\) is AWGN, \(\mathcal{N}(0, 1)\). We implicitly assume that the noise is white in time and also in space. Among other things this means that the noise generated by the sensors are identical (and independent, see below).

- Each source transmits the same letter on each channel with the same energy per channel. The only difference is that each letter is weighted by a different complex factor: for example, source \(i\) has its letter multiplied by \(d_{i,k} = \exp(-jk\Phi_i)\), depending upon its position and the channel number \(k\).

The K channels are independent:

- The noise components, \((N_k, k \in [1, K])\), are independent of each other.
Figure 5-1: K parallel Multiple Access channels and the simplified vector model.

- The channels are parallel, they do not interfere with one another.

5.1.2 Simplified Model:

The model can be simplified to a unique Multiple Access channel with $M$ inputs and 1 vector output (see [Fig. 5-1]):

$$ [(X_1, ..., X_M), O] $$

The output, $O$, of the channel is given by:

$$ O = DX + N $$

$X$ is the vector of input sources, it is of dimension $M$. We assume that the sources are independent and zero mean.

$$ X = (X_1, ..., X_M)^T $$

$$ E[X] = 0 $$

$$ R_X = E[(X_i, X_j)] = \text{diag}\{\alpha_1, ..., \alpha_M\} = \Delta $$

$D$ is the ($K$ by $M$) DOA matrix. Its columns are the DOA vectors $d_m$, $m \in \{1, \ldots, M\}$.

$$ D = (d_1, ..., d_M) $$
\( \mathbf{N} \) is the noise vector, it is of dimension \( K \). We assume that the noise is white in space.

\[
\mathbf{N} = (N_1, ..., N_K)^T \\
E[\mathbf{N}] = 0 \\
R_{\mathbf{N}} = E[(N_iN_j)] = I
\]

\( \mathbf{O} \) is the output vector of the channel. We assume that the noise and the sources are uncorrelated.

\[
E[\mathbf{O}] = 0 \\
R_{\mathbf{O}} = I + D R_X D^T
\]

Let \( S \) denote a subset of \( (1, ..., M) \) and \( S^c \) its complement (i.e., \( S \cap S^c = \emptyset, S \cup S^c = (1, ..., M) \)). We define:

- \( D_S \), respectively \( D_{S^c} \), to be the matrix whose columns are the DOA vectors of the sources in \( S \) (\( S^c \)).

- \( \Delta_S \), respectively \( \Delta_{S^c} \), to be the diagonal matrix whose elements are the energies of the sources in \( S \) (\( S^c \)).

**Independent decoding:**

Note that in the case of independent decoding the model can be simplified (see [Fig. 5-2]). For each source \( i \), there are \( K \) channels with 1 input and 1 vector output:

\[
\{X_i, O_k\}, \ k \in [1, K].
\]

The noise components are independent but the interference components are correlated. We have \( K \) parallel channels with colored interference.

### 5.2 The 2-Multiple Access Channel: \([X, Y), O]\)

For \( M = 2 \), there are 2 sources \( X = X^1 \) and \( Y = X^2 \). The output, \( O \), of the array is given by:

\[
O = D(X, Y)^T + N = X \cdot d_X + Y \cdot d_Y + N
\]
The noise is white in space and time : \( E[N] = 0 \), \( E[NN^\dagger] = \sigma_N I_d \).
The interference is white in time but colored in space : its components are correlated : \( E[I] = 0 \), \( E[II^\dagger] = \text{non diagonal} \).

Figure 5.2: Independent Decoding : K Interference Channels W/O Multipath and the model simplified

5.2.1 Capacity Region

Here we explicitly show the content of each subset \( S \), i.e., \( (X, Y) \), or \( XY \).

Any achievable rate pair, \((R_X, R_Y)\), must satisfy :

\[
R_X \leq \max I(X; Y | O) \leq \frac{1}{2} \log (\det[I + \alpha_X D_X D_X^\dagger]) \quad (5.1)
\]

\[
R_Y \leq \max I(Y; X | O) \leq \frac{1}{2} \log (\det[I + \alpha_Y D_Y D_Y^\dagger]) \quad (5.2)
\]

\[
R_X + R_Y \leq \max I(X, Y; O) \leq \frac{1}{2} \log (\det[I + D_{XY} \Delta_{XY} D_{XY}^\dagger]) \quad (5.3)
\]

The region delimited by \((5.1, 5.2, 5.3)\) is the capacity region of the 2-Multiple Access Channel.

\textbf{proof:}

- see the general proof for M users in [sec. 5.3.1].

For any subset \( S \) of \( (1, 2) \), the matrices \( I + D_S \Delta_S D_S^\dagger \) are full rank hermitian matrices. Their eigenvectors are real. Their determinants can be derived and are given by :

\[
\det[I + \alpha_X D_X D_X^\dagger] = 1 + K \alpha_X
\]

\[
\det[I + \alpha_Y D_Y D_Y^\dagger] = 1 + K \alpha_Y
\]

\[
\det[I + D_{XY} \Delta_{XY} D_{XY}^\dagger] = 1 + K(\alpha_X + \alpha_Y) + K^2 \alpha_X \alpha_Y (1 - \cos^2(d_X, d_Y))
\]
where

\[
\cos^2(d_x, d_y) = \frac{\sqrt{d_x \cdot d_y} \sqrt{d_x \cdot d_y^*}}{||d_x||^2 ||d_y||^2} \\
||d_x||^2 = ||d_y||^2 = K
\]

This finally gives for the capacity region (see [Fig. 5-3]):

\[
R_X \leq \frac{1}{2} \log(1 + K\alpha) \tag{5.4}
\]

\[
R_Y \leq \frac{1}{2} \log(1 + K\beta) \tag{5.5}
\]

\[
R_X + R_Y \leq \frac{1}{2} \log(1 + K(\alpha + \beta) + K^2\alpha\beta(1 - \cos^2(d_1, d_2))) \tag{5.6}
\]

We recognize in (5.4, 5.5, 5.6) the parametric form: Capacity \( \leq \frac{1}{2} \log(1 + SNR) \).

![Diagram](image)

**Figure 5-3:** Capacity region for 2 users and a K sensor array.

**Independent Decoding:**

We assume that the inputs are gaussian distributed. The pair \((R_X, R_Y)\) corresponding to independent decoding is given by:

\[
R_X = \frac{1}{2} \log\left(\frac{\det[I + D_{XY} \Delta_{x,y} D_{X,Y}^*]}{\det[I + \alpha_Y D_Y D_Y^*]}\right) \tag{5.7}
\]
\[ R_Y = \frac{1}{2} \log \left( \frac{\det[I + D_{XY}^1 \Delta_{\alpha_x, \alpha_y} D_{XY}^1]}{\det[I + \alpha_x D_X D_X^1]} \right) \] (5.8)

**proof:**

- see the general proof for M users in [sec 5.3.1].

This gives:

\[ R_X = \frac{1}{2} \log \left( 1 + \frac{K\alpha_x + K^2\alpha_x \alpha_y (1 - \cos^2(d_X, d_Y))}{1 + K\alpha_y} \right) \] (5.9)

\[ R_Y = \frac{1}{2} \log \left( 1 + \frac{K\alpha_y + K^2\alpha_x \alpha_y (1 - \cos^2(d_X, d_Y))}{1 + K\alpha_x} \right) \] (5.10)

### 5.2.2 Boundaries of the Region

We observe that, SDMA actually changes the channel used by multiple access. Thus, whereas TDMA and CDMA refer to different ways of using a given channel, SDMA provides a new channel and thus a new capacity region.

In this section, we assume a Gaussian ensemble of codes.

**Orthogonal Position : the outer boundary**

When \( \theta_X \) and \( \theta_Y \) are such that \( (1 + \ldots + e^{-jK(\theta_Y - \theta_X)} + \ldots + e^{-j(K-1)(\theta_Y - \theta_X)} = 0) \),
\[ d_X^{-1}d_Y = 0, \cos^2(d_X, d_Y) = 0 \] and \( X \) is orthogonal to \( Y \).

Then the capacity region is maximal (see [Fig. 5.3]), and is given by:

\[ I(X; O|Y) = \frac{1}{2} \log(1 + K\alpha_x) \]

\[ I(Y; O|X) = \frac{1}{2} \log(1 + K\alpha_y) \]

\[ I(X, Y; O) = \frac{1}{2} \log(1 + K\alpha_x) + \frac{1}{2} \log(1 + K\alpha_y) \]

For independent decoding:

\[ I(X; O) = \frac{1}{2} \log(1 + K\alpha_x) \]

\[ I(Y; O) = \frac{1}{2} \log(1 + K\alpha_y) \]

The capacity region has the well known rectangular shape because user \( X \) and user \( Y \) are orthogonal to each other (in space) and therefore they do not interfere. One can separate them exactly by using SDMA.
Aligned Position: the inner boundary

When $\theta_x = \pm \theta_y$ then $d_x = d_y$, $\cos^2(d_x, d_y) = 1$ and X is aligned with Y.

The capacity region is minimal (see [Fig. 5.3]), and is given by:

\[
\begin{align*}
\max I(X; Q|Y) &= \frac{1}{2} \log(1 + K\alpha_x) \\
\max I(Y; Q|X) &= \frac{1}{2} \log(1 + K\alpha_y) \\
\max I(X, Y; Q) &= \frac{1}{2} \log(1 + K(\alpha_x + \alpha_y))
\end{align*}
\]

For independent decoding:

\[
\begin{align*}
\max I(X; Q) &= \frac{1}{2} \log(1 + \frac{K\alpha_x}{1 + K\alpha_y}) \\
\max I(Y; Q) &= \frac{1}{2} \log(1 + \frac{K\alpha_y}{1 + K\alpha_x})
\end{align*}
\]

This aligned position has the smallest capacity region over all possible positions $(d_x, d_y)$. By using SDMA one cannot separate user X from user Y. The interference is maximum.

However, one gains in Signal to Noise Ratio compared with the case without SDMA (point D compared with point D' in [Fig. 5.3]). The enhancement of a factor K comes from the $(K,d)$ array of sensors. The receiver gets K replicas of the signal and adds them up coherently whereas the noise adds up incoherently.

Separation Ability:

We can make the following remarks:

- as the 2 mobiles move, the capacity region varies between the inner boundary and the outer boundary;

- as $K \to \infty$ the separation ability increases. Unless X and Y are aligned, we have:

\[
\cos^2(d_x, d_y) = \frac{\sqrt{d_x} \cdot d_y}{\sqrt{d_y} \cdot d_x^t} \to 0
\]

\[
\begin{align*}
I(X; Q) &\to I(X; Q|Y) \\
I(Y; Q) &\to I(Y; Q|X)
\end{align*}
\]

This means that even if X and Y are close to each other, if K is large, the capacity region will look rectangular: the separation ability is excellent.

This means also that the performance of independent decoding is close to the performance of
joint decoding. We don’t gain too much by doing joint decoding.

However, we must emphasize that $K$ can not be too large or the assumptions of narrowband signals and far-field sources (see [sec 2.2]) break down. $K = 50$ is a reasonable upper limit and for 2 users, it gives excellent separation ability.

5.3 The M-Multiple Access Channel, $[(X^1, \ldots, X^M), Q]$

Let $M$ be the number of sources, $[1, \ldots, M]$. The output of the array, $Q$, is given by:

$$Q = DX + N = \sum_{i=1}^{M} X_i d_i + N$$

5.3.1 Capacity Region

Any achievable rate $M$-tuple, $(R_1, \ldots, R_M)$, must satisfy:

$$\sum_{s \in S} R_s \leq \max_{s \in S} I(Q_s; X^s, s \in S | X^u, u \in S^c) \leq \frac{1}{2} \log(\det[I + D_s \Delta_s D_s^T])$$  \hspace{1cm} (5.11)

for all subsets $S$ of $[1, \ldots, M]$.

The convex hull of the region delimited by equation (5.11) is the capacity region.

**proof:**

The first part of the inequality comes from [chap. 3, def. 11].

For the second part, we have:

- $I(Q_s; X^s, s \in S | X^u, u \in S^c) = H(Q_s | X^u, u \in S^c) - H(Q_s | X^s, s \in [1, M]),$ from [chap. 3, def. 19];

- $H(Q_s | X^s, s \in [1, M]) = H(N) = \frac{1}{2} \log(2\pi e)^K,$ from [chap. 3, prop. 1] and the assumption of Gaussian noise with $R_N = I$;

- $H(Q_s | X^u, u \in S^c) \leq \frac{1}{2} \log((2\pi e)^K \det[I + D_s \Delta_s D_s^T]),$ from [chap. 3, prop. 1 and 2].

**QED**

**Independent Decoding:**

We assume that the inputs are Gaussian distributed. For any subset $S$ with a single user ($S = (i), i \in [1, M]$), the rate $R_s$ corresponding to independent decoding is given by:

$$R_s = I(Q_s; X^s) = \frac{1}{2} \log(\frac{\det[I + D_{(1, \ldots, M)} \Delta_{(1, \ldots, M)} D_{(1, \ldots, M)}^T]}{\det[I + D_s \Delta_s D_s^T]}).$$  \hspace{1cm} (5.12)

**proof:**
5.3.2 Inner Boundary, Outer Boundary

In this paragraph, we assume perfect power control, \( \Delta_{(1...M)} = \alpha I \), and fairness among users, \( R_1 = ... = R_M \).

We looked at the symmetric channel capacity \( I(Q; X^s, s \in [1, M]) \) (see [sec. 3.1.4]).

When \( M \leq K \), the number of users \leq the number of sensors [see [Fig. 5-4]):

\( M \leq K \) is the TDMA/SDMA case (see [chap. 2]). It is also the case of lightly loaded CDMA networks.

**Proposition 3** The symmetric channel capacity is minimal when all the \( M \) users are aligned with one another (aligned DOA vectors). It is maximal when the \( M \) users are orthogonal to each other (the DOA vectors are orthogonal to each other) and then the region is the cube defined as \( R_s \leq \frac{1}{2} \log(1 + \alpha K), s \in [1, M] \).

**Proof:** Let us use the lower index \( r \) in expressions where the \( M \) users are orthogonal to each other.

We first show that, for all \( i \) in \([1, M]\),

\[
I(Q; X^i)_{r} = I(Q; X^i|X^u, u \in [1, M], u \neq i)_r = \frac{1}{2} \log(1 + \alpha K) \quad (5.13)
\]

To show this, we observe that

- \( \det[I + \alpha D_{(1...M)}D_{(1...M)}^\dagger]_{r} = \prod_{s=1}^{M} (1 + \alpha K) \)
- \( \det[I + D_{S^c} \Delta_{S^c} D_{S^c}^\dagger]_{r} = \prod_{s=1}^{M-1} (1 + \alpha K), \) where \( S^c \) is defined in [eq. 5.12]

Using [eq. 5.12], we obtain

\[
I(Q; X^i)_{r} = \frac{1}{2} \log(1 + \alpha K) = I(Q; X^i|X^u, u \in [1, M], u \neq i)_r
\]

Now, we want to prove:

- **proposition (a):**
  \[
  \arg(\max I(Q; X^s, s \in [1, M])) = \text{positions where all the users are orthogonal to each other};
  \]
• proposition (b):
  \[ \arg\min I(O; X^s, s \in [1, M]) = \text{positions where all the users are aligned with each other.} \]

We will prove proposition (a) by induction. Proposition (b) is the same idea and therefore will not be presented.

Note that when \( M \leq K \), there always exist positions where the users are orthogonal to each other.

- We have seen in [sec 5.2.2] that proposition (a) is true for \( M = 2 \).
- Let us suppose it is true for \( M - 1 \).
- Let us suppose that there is a position \((1, \ldots, M)\), where the users are not orthogonal to each other, such that:
  \[ I_M(O; X^u, u \in [1, M]) > I_M(O; X^s, s \in [1, M])_{T} \]

Here, we index by \( M \) the mutual information when there are \( M \) users in the network.

This gives, for any \( i \in [1, M] \)

\[
I_M(O; X^i) + I_M(O; X^u, u \in [1, M], u \neq i|X^i) > \\
I_M(O; X^i)_{T} + I_M(O; X^s, s \in [1, M], s \neq i|X^i)_{T} \tag{5.14}
\]

From [eq. 5.13]

\[
I_M(O; X^i) \leq I_M(O; X^i|X^s, s \in [1, M], s \neq i)_{T} = I_M(O; X^i)_{T} \tag{5.15}
\]

Since the multiple access channel is additive, we have
\[
I_M(O; X^u, u \in [1, M], u \neq i|X^i) = I_{M-1}(O; X^u, u \in [1, M], u \neq i) \text{ and} \\
I_M(O; X^s, s \in [1, M], s \neq i|X^i)_{T} = I_{M-1}(O; X^s, s \in [1, M], s \neq i)_{T}
\]

Since proposition (a) is true for \( M - 1 \), we get

\[
I_M(O; X^u, u \in [1, M], u \neq i|X^i) \leq I_{M-1}(O; X^t, s \in [1, M], s \neq i)_{T} \\
= \max_{t \in [1, M], s \neq i} I_{M-1}(O; X^t, s \in [1, M])
\]

Combining this with [eq. 5.15], we get a contradiction to [eq. 5.14].

This complete the proof for proposition (a).

Finally from [eq. 5.13], the capacity region is the cube, \( R_s \leq \frac{1}{2} \log(1 + \alpha K) \), \( s \in [1, M] \).

**QED**
When $M > K$, the number of users > the number of sensors (see [Fig. 5.5]):

$M > K$ is the case of medium to heavy loaded CDMA networks.

**Proposition 4** The symmetric channel capacity is minimal when all the $M$ users are aligned with one another (aligned DOA vectors). It is maximal when the $M$ users are such that the rows of the DOA matrix are orthogonal to each other.

**proof:**

We have for any $K$ by $M$ matrix $A$, for any $M$ by $K$ matrix $B$ and for any real $\alpha$

$$
\det(I_K + \alpha AB) = \det(I_M + \alpha BA)
$$

We can prove this as follows:

$$
\begin{align*}
\det \left( \begin{array}{cc}
I_K & A \\
-\alpha B & I_M
\end{array} \right) &= \det \left( \begin{array}{cc}
I_K & 0 \\
\alpha B & I_M
\end{array} \right) \left( \begin{array}{cc}
I_K & A \\
-\alpha B & I_M
\end{array} \right) \\
&= \det \left( \begin{array}{cc}
I_K & A \\
0 & I_M + \alpha BA
\end{array} \right) \\
&= \det(I_M + \alpha BA)
\end{align*}
$$

and

$$
\begin{align*}
\det \left( \begin{array}{cc}
I_K & A \\
-\alpha B & I_M
\end{array} \right) &= \det \left( \begin{array}{cc}
I_K & -A \\
0 & I_M
\end{array} \right) \left( \begin{array}{cc}
I_K & A \\
-\alpha B & I_M
\end{array} \right) \\
&= \det \left( \begin{array}{cc}
I_K + \alpha AB & 0 \\
-\alpha B & I_M
\end{array} \right) \\
&= \det(I_K + \alpha AB)
\end{align*}
$$

Therefore

$$
\det(I_K + \alpha DD^\dagger) = \det(I_M + \alpha D^\dagger D)
$$

Note that each column of $D^\dagger$ has a fixed norm $M$.

Applying the previous proof to $D^\dagger$, we get the result.

**QED**

**5.3.3 The 3-Multiple Access Channel** : $[(X, Y, Z), Q]$

The 3 user channel is convenient because:
• the capacity region, with its inner and outer bounds, can easily be plotted. This provides us with some insight about how SDMA works;

• we can see graphically what CDMA and TDMA/SDMA correspond to.

Capacity Region:

The cube arises when X, Y, and Z are orthogonal to each other. The polyhedron inside is when they are aligned.
When the mobiles move the capacity varies within this 2 boundaries.

Figure 5-4: Outer Bound and Inner Bound of the Capacity Region for 3 users and (3 ≤ K) sensors

We see that when M ≤ K, there are positions such that all users are orthogonal to each other. Then the capacity region is maximal and in the case (M = 3 ≤ K) it is the cube in [Fig. 5-4].

On the contrary, when M > K, there are not such positions where all users are orthogonal to each other. Then the maximal capacity region is a polyhedron (see [Fig. 5-5]).

In [Fig. 5-6], we show, for 3 users, the capacity region and some special points.

Multiple Access Schemes:
The outer polyhedron is the outer boundary. The inner polyhedron inside is when they are aligned.
When the mobiles move the capacity varies within these 2 boundaries.

Figure 5-5: Outer Bound and Inner Bound of the Capacity Region for 3 users and \( K = 2 \) sensors

In [Fig. 5.7], we show the operating point for TDMA/SDMA. We assume that we have 2 time slots and initially 2 users, X and Y. Then, we consider an array and add one more user, Z. We have 3 possible partitions:

- \( Z|XY|Z|XY|... \), see (point T [Fig. 5-7]);
- \( Y|ZX|Y|ZX|... \);
- \( X|YZ|X|YZ|... \).

For example, in the first partition (point T), we do equal time sharing between point B, where Z transmits, and point A, where X and Y transmit with independent decoding at the reception.
With non optimal TDMA/SDMA, the partition is fixed whereas with optimal TDMA/SDMA the system chooses the partition that ensures that the minimum rate is maximized.
Finally, [Fig. 5.8] shows the operating point for CDMA with independent decoding.
5.4 Gain in Capacity

5.4.1 The Gain with a K Sensor Array

By replacing the omnidirectional antenna by an array with \( K \) sensors, one gains:

- a factor of \( K \) in \( SNR \):

  When other users are silent, we look at the capacity of the \( i^{th} \) user \( (S = (i)) \).

  For an omnidirectional antenna, we have:

  \[
  R_i \leq \frac{1}{2} \log(1 + \alpha_i)
  \]

  For a \( K \) sensor array, we have:

  \[
  R_i \leq \frac{1}{2} \log(\det[I + \alpha_i D_i D_i^T]) = \frac{1}{2} \log(1 + K \alpha_i)
  \]
\( B \) = only \( Z \) transmits; 
\( A \) = \( X \) and \( Y \) transmit and the receiver does independent decoding. 
\( T \) = time sharing between \( B \) and \( A \).

**Figure 5-7: Operating Point for TDMA/SDMA.**

- in separation ability:

  As shown above, the capacity region varies depending upon the relative position (in space) of the mobiles. We say that SDMA is efficient when the capacity is close to its maximum (close to the outer boundary).

## 5.5 Different Multiple Access Schemes

### 5.5.1 Simulation

To get some insight on how the different Multiple Access schemes behave, we first simulate a cell with 3 users and 1 omnidirectional antenna. Then we simulate an array of 3 sensors and increase the number of users from 3 to 6.

The corresponding Multiple Access schemes are:

- pure TDMA : 1 | 2 | 3 | 1 | 2 | 3 | ... , without the array;
Figure 5-8: Operating Point for CDMA with independent decoding.

- TDMA/SDMA : $1'22'33'11'22'33'11'$..., with the array;
- pure CDMA : .....$1 + 2 + 3$.....$1 + 2 + 3$....., without the array;
- CDMA : .....$1 + 2 + 3 + 1' + 2' + 3'$....., with the array.

We will look at the capacity in bits per second per Hertz of each user, also called the user capacity. With spatial diversity, the capacity depends upon the relative position of the mobiles. We will compute the distribution of the capacity over all relative positions of the mobiles in the cell. We will plot the minimum rate, the mean rate and the maximum rate over all 6 users.

For each simulation, we will take the DOAs, $(\theta_i)$, uniformly distributed over $[0, 2\pi]$ and run 8000 trials.

The program was written in Matlab.
5.5.2 TDMA/SDMA vs Optimal TDMA/SDMA

With non optimal TDMA/SDMA, the partition into subsets is fixed throughout the simulation. With optimal TDMA/SDMA (see [sec. 3.3.4]) at any time, among all possible partitions, the system chooses the partition that maximizes the minimum rate over all users, i.e.,

$$\arg\left( \max_{\text{all partitions}} \min_{\text{all users}} \text{rate} \right)$$

In [Fig. 5-9], we see that the improvement for the minimum rate is spectacular. It is interesting to note that optimal TDMA/SDMA also performs better in average and for the maximum rate. We note that optimum TDMA/SDMA forces orthogonality in space by allocating, when it is possible, different time slots to users that are close to one another. This is a feature that regular TDMA/SDMA does not have. As we will see in [sec 5.5.4], CDMA does not have this characteristic either.

From now on, we will always refer to optimum TDMA/SDMA as TDMA/SDMA.

5.5.3 TDMA vs TDMA/SDMA, CDMA with and without Spatial Diversity

In [Fig. 5-10], we see that:

- with TDMA/SDMA we can double the number of users compared to TDMA with a single antenna and still have a better quality (capacity) for most of the time (or for most of the positions in the cell); the average rate is 50% higher for half of the time. However in this case, there is a probability of 19% that the minimum rate will drop below the initial rate. This means that 19% of the time the service will be degraded for at least 1 user. This happens when more than 3 users are close to each other and is known as the near-by problem. Then the separation ability breaks down and because there are only 3 time slots, the system can not force orthogonality in space by placing each one of the near-by users in a different time slot.

  We can see by simulation that this probability decreases as the number of sensors, K, increases. It shows that the separation ability increases with K.

- the comparison with TDMA with 6 users and a single antenna is similar. The quality (capacity) increases most of the time but there is a possible degradation in the service quality.

In [Fig. 5-11], we compare TDMA, with 3 antennae, with TDMA/SDMA. We have the same characteristics as above, i.e., most of the time, the capacity with TDMA/SDMA is much better, but there is a non zero probability (near-by situation) that TDMA/SDMA is worst than pure TDMA.
(10% of the time).

For CDMA (see [Fig. 5-12]):

- we observe the same phenomena as above when we compare CDMA (6 users, 3 sensor array) with CDMA (3 users, 1 antenna);

- however CDMA (6 users, 3 sensor array) always performs better than CDMA (6 users, 1 antenna). Indeed in the near-by case, where all the users are aligned, CDMA (6 users, 3 sensor array) cannot take advantage of any spatial diversity, but it can benefit from the improvement in SNR. In a sense, CDMA is more robust against the near-by problem than TDMA/SDMA.

5.5.4 TDMA/SDMA vs CDMA

In [Fig. 5-13], we compare TDMA/SDMA vs CDMA for the case where TDMA and CDMA, without spatial diversity, are equivalent (point E in ([sec 4.3.2], Fig. 4-4)). With 6 users, TDMA and CDMA have the same efficiency for $SNR = 1.6825$.

We have plotted TDMA/SDMA and CDMA for this parameter. We observe that:

- TDMA/SDMA has a better minimum rate. Indeed Optimum TDMA/SDMA forces separation in space by allocating different time slots to users that are close to each other. CDMA cannot do this;

- however, given its structure, CDMA is better than TDMA/SDMA for average capacity.

5.5.5 CDMA and TDMA/SDMA with imperfect Power Control

We have simulated the case where the cell site receives 1 user at a power 10 times higher than the 5 other users. We see in [Fig. 5-14] that:

- without spatial diversity:
  - The 5 users are penalized, and their capacity drops from 0.11 to 0.05 (bit/sec/Hertz).
  - One user has its capacity increased to 0.71.

  This is the well known drawback of CDMA: it is not robust to unequal powers (see [sec. 4.3.2]).

- with spatial diversity:
  - the minimum rate is much better than before and it is even sometimes better than if power control were perfect. This means that the user that is the most penalized, is less penalized than before or even not penalized at all.
the average rate is much higher than the rate with perfect power control. This indicates that not all the 5 users are penalized.

We can conclude that CDMA gains in robustness against unequal powers.

On the contrary, we must emphasize that TDMA/SDMA requires more careful power control than pure TDMA. Indeed, with pure TDMA, only one user transmits at a time so that power control is not a critical issue (except for neighboring cells). Conversely, with TDMA/SDMA, there is more than 1 user transmitting at a time and power control becomes very important.
We consider:

- non optimal TDMA/SDMA (dashdot curves) with 6 users and an array of 3 sensors.
- optimal TDMA/SDMA (solid curves) with 6 users and an array of 3 sensors.

For each scheme, we plot the minimal, the mean and the max user capacity (in bits per second per Hertz).

Figure 5-9: Regular TDMA/SDMA vs Optimal TDMA/SDMA
We consider:

- (optimal) TDMA/SDMA with 6 users and an array of 3 sensors (solid curves).
- TDMA with 6 users and 1 antenna (dashdot curve).
- TDMA with 3 users and 1 antenna (dotted curve).

We plot the user capacity (in bits per second per Hertz).

**Figure 5-10: TDMA without Spatial Diversity vs TDMA/SDMA**
We consider:
- optimal TDMA/SDMA with 6 users and an array of 3 sensors (solid curves).
- TDMA with 6 users and 3 antennae (dotted curve).

We plot the user capacity (in bits per second per Hertz).

Figure 5-11: TDMA with 3 antennae vs TDMA/SDMA
We consider:

- CDMA with 6 users and an array of 3 sensors (dashdot curves),
- CDMA with 3 users and 1 antenna (solid curve),
- CDMA with 6 users and 1 antenna (dotted curve).

We plot the user capacity (in bits per second per hertz).

Figure 5-12: CDMA without an array vs CDMA with an array
We consider:

- (optimal) TDMA/SDMA with 6 users and an array of 3 sensors (solid curves).
- CDMA with 6 users and an array of 3 sensors (dashdot curves).
- TDMA and CDMA with 6 users and 1 antenna (dotted curve). With $SNR = 1.682$, they have the same capacity (see sec. 4.3.2).

We plot the user capacity (in bits per second per Hertz).

Figure 5-13: TDMA/SDMA vs CDMA
We consider:

- CDMA with 6 users, 1 antenna and Power Control (solid curve).
- CDMA with 6 users, 1 antenna and no Power Control (dotted curves). 1 user has a capacity of 0.71 whereas the other users are penalized with a capacity of only 0.05.
- CDMA with 6 users, an array of 3 sensors and no Power Control (dashdot curves).

We plot the user capacity (in bits per second per Hertz).

Figure 5.14: CDMA with imperfect power control
Chapter 6

Conclusion

6.1 Summary

In chapter 2, we explained what an array of sensors is [sec 2.2, 2.3] and we proposed replacing the omnidirectional antenna at the cell site by an array in order to take advantage of spatial diversity among mobiles.

Then we presented an information theoretic model for the corresponding Multiple Access Channel [sec 5.1]. We derived its capacity region [sec 5.3], based on the case without spatial diversity [sec 4.2].

We showed that there is a gain in:

- $SNR$, [sec 5.4.1], and
- Separation Ability. The capacity region expands and varies depending upon the relative positions of the mobiles. It is bounded by the inner boundary and the outer boundary, [sec 5.3.2 and 5.4.1].

We presented the case with 2 users, [sec 5.2], and 3 users, [sec 5.3.3], which provide insight on how the capacity region expands.

We also introduced a new Multiple Access scheme, TDMA/SDMA, [sec 3.3.4], in addition to the existing schemes, TDMA and CDMA [sec 3.3].

In [sec 4.3], we explained how to make a fair comparison between the different Multiple Access Schemes:

- we must use a Continuous Time Bandlimited model and
- compare the spectral efficiencies, or capacities in bits per second per Hertz.

We also explained [sec 4.3.2] why:
• TDMA is more efficient in narrowband systems or high power systems;

• whereas CDMA is more efficient in broadband systems or low power systems.

Using a Continuous Time Bandlimited model, we ran simulations [sec 5.5.1] to get some insight on how each scheme behaves. We found that:

• Optimum TDMA/SDMA, where the partition into subsets is optimum, performs much better than regular TDMA/SDMA, [sec 5.5.2];

• TDMA/SDMA with an array, respectively CDMA with an array, has a higher quality and capacity (in number of users) than TDMA with a single antenna and TDMA with an array (CDMA with a single antenna). The pitfall is that there is a non zero probability that the performance drops below the initial capacity. This degradation happens when many mobiles are close to each other (near-by problem) and the separation in space breaks down. The probability of this degradation, however, decreases as the number of sensors increases: this shows again that the separation ability increases with $K$. [sec 5.5.3].

Moreover, CDMA, given its structure, appears more robust to the near-by problem than TDMA/SDMA;

• TDMA/SDMA has better minimal user capacity than CDMA because it forces separation in space. Yet CDMA, given its structure, has a better average user capacity. [sec 5.5.4];

• CDMA with an array is much more robust to power control than without. TDMA/SDMA must have a better power control than TDMA. [sec 5.5.5].

### 6.2 Further Work

We suggest some potential areas for further research:

• Since we have seen that Spatial Diversity makes CDMA more robust against power control, it would be interesting to know how much TDMA/SDMA gains in robustness against multipath, [sec 2.4]. Channels with multipath correspond to channels with time varying and frequency varying fading. This effectively makes the noise colored. The case of colored noise for point to point communication is treated in [4] and [3]. Applying this method to the case of arrays, we get a (time, space) matrix which makes the analysis appear to be complicated.

• A study of algorithms to approach the optimal TDMA/SDMA scheme would be of the greatest importance;
Finally as mentioned in [sec 2.4], to our knowledge, nobody has yet explored how to build a receiver for CDMA systems where the number of users is larger than the number of sensors.
Bibliography


