MATHEMATICAL ANALYSIS OF STRESS AND DISPLACEMENT IN THIN SHALLOW SPHERICAL SHELLS

by

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SECTION I

INTRODUCTION

1. Purpose

The objectives of this study are:

a. to obtain a solution for the analysis of the stresses and displacements in thin-shallow spherical domes of constant thickness under symmetrical or unsymmetrical loading,

b. to demonstrate the application of this solution to the analysis of shells subjected to symmetrical or unsymmetrical vertical loading under various conditions of edge restraint, and

c. to reduce the labor involved in the design of thin-shallow shells subjected to symmetrical uniform vertical loading.

2. Historical Background

The problem of stress and displacement in spherical shells has been studied by many investigators.

In 1912, H. Reissner (1) starting with Love's equations of the general theory of shells of any shape gave the membrane solutions for the state of stress and displacement in a shell under symmetrical or unsymmetrical loading. For a shell with constant thickness under the action of a symmetrical system of edge forces, he showed that the equations of equilibrium reduced to two mutually symmetrical second order differential equations. He pointed out the use of the asymptotic integration procedure of O. Blumenthal (2) in the solution of these differential equations. He showed that it was possible to resolve the stress problem into a membrane stress and displacement analysis and a bending theory analysis for the determination of the edge forces re-

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(1) Refers to corresponding numbers in Bibliography

quired to satisfy the initial conditions of restraint of the shell.

In 1913, E. Meissner (3) demonstrated that H. Reissner's symmetrical differential equations could be solved exactly by a hypergeometric series. In 1914, Meissner (4) showed for the case of a symmetrically loaded shell of revolution with variable thickness, that two differential equations similar to those for the spherical shell with constant thickness were obtained and were solvable, provided the variation of shell thickness was taken in a certain way. Calculations using Meissner's solution for a constant thickness spherical shell were made by L. Bolle (5) in 1915.

E. Schwerin (6), in 1919, using the work of H. Reissner and E Meissner gave the solution for the symmetrically loaded shell by combining the membrane stress analysis with the edge effects solution and then applied his results in two numerical examples. He assumed a hemisphere in both cases. In the first example he considered the support to be hinged; in the second, to be fixed. The spherical shell was stressed by a uniform rise in temperature in both cases. He then developed the solution for a shell unsymmetrically loaded by wind pressure and then applied his results to two additional numerical problems. For these he assumed the hemispheres of the first two problems but now acted upon by an unsymmetrical wind pressure loading.

P. Pasternak (7), in 1926, applied the method of finite difference equations to the solution of H. Reissner's differential equations for the shell symmetrically loaded at the edge. He determined the formulas for the influence coefficients for the determination of the edge displacements due to edge forces or, conversely, the edge forces due to edge displacements.

J. W. Geckeler (8) in 1926, developed an approximate solution for the shell symmetrically loaded at the boundaries. Noting that the second derivatives were greater than the first derivatives and these in turn were greater than the functions themselves due to the exponential character of the functions at the boundaries, he neglected the functions and the first derivatives in the differential equations for the shell thus obtaining a simple approximate solution. These previous methods were unsuitable for the shallow dome inasmuch as the hypergeometric series and asymptotic integration procedure converge much too slowly as the dome becomes more shallow. Also, in the approximate methods, the neglection of the function or its derivatives leads to serious errors in the vicinity of the pole.

In 1930, Geckeler (9) developed a Bessel funtion solution applicable to the shallow spherical dome. He made a comparative study of his two previous approximate solutions and his new Bessel function solution. In discussing the degree of approximation of his two approximate solutions he pointed out their inapplicability to the case of the shallow dome. Geckeler obtained his Bessel function solution for shallow domes by making the assumption that in H. Reissner's basic differential equations for a symmetrically loaded shell the cotangent ϕ may be replaced by $\frac{1}{\phi}$ for small values of ϕ. This becomes more nearly correct as ϕ approaches zero. The solution of the differential equations for constant wall thickness and for symmetrical edge loading is then obtained in the form of Bessel functions. Geckeler applied this solutions to the determination of the state of stress and displacement due to a point load at the pole. In so doing he was forced to arbitrarily disregard part of the solution in order not to obtain infinite deflections. He demonstrated the transformation of his solution for the point load at the pole, restrained edges, into the solution for a circular flat plate under center point load with fixed edges as the shallow dome approaches the limiting case of the flat plate.

In 1935, A. Havers (10), developed an asymptotic method for the solution of the differential equations for a spherical shell with constant thickness subjected to an unsymmetrical boundary loading. He pointed out that this method is not valid for the case of the shallow dome.

A. Aas Jakobsen (11), in 1937, considered the case of the dome supported at various points along its edge using and simplifying the previous work of A. Havers.

In 1938, M. Hetényi (12) developed to completion an approximate solution obtained in (1) by neglecting only the functions themselves in the differential equations for a shell symmetrically loaded at the boundary. Hetényi made a comparative study of Geckeler's first approximate solution, this approximate solution and the exact hypergeometric series solution, by way of an example. He carried out the numerical calculations by the three methods for a shell with fixed edges under a uniform radial loading. Finally, the range of applicability of the approximate solutions was discussed and a scheme given to estimate the probable errors in these solutions. A comparison of the estimated versus the actual percentage errors in the numerical problem was made for the meridional moments and hoop forces. Hetenyi pointed out the inapplicability of the two approximate solutions for the case of the shallow dome.

In 1943, E. Reissner (13) utilized the basic assumption of shallowness of the shell and reduced the general equations of the shallow spherical shell to two differential equations of fourth order in terms of a stress function F and the vertical deflection W. He indicated the form F and W would take in the case of the unsymmetrically loaded shell and carried out the solution for the symmetrically loaded shell. He then applied it to the case of a concentrated load at the pole. The effects of the concentrated load as a point loading were studied as well as the effects of the concentrated load if distributed over a small area at the pole.

The present thesis is based upon this work of E. Reissner.

3. Scope of Thesis

The original work of this thesis properly commences with the solution for unsymmetrical loading of E. Reissner's basic differential equations for the shallow dome. However, for the sake of completeness, his development of the differential equations (13) is given nearly verbatim in Section II. His development has been modified therein to include the effect of temperature. Section III presents a solution of the differential equations suitable for unsymmetrical or symmetrical loading. Section IV gives the expressions for the stress resultants and reactions for unsymmetrical loading. Section V gives the expressions for the stress resultants and reactions for the symmetrical case. Section VI covers the general problem of the determination of the displacements with expressions for the unsymmetrical and symmetrical cases.

The applications of the results of Sections III, IV, V and VI are given in Sections VII through XII, inclusive. Before summarizing the contents of these sections the parameter, ξ_1, used throughout this thesis is now defined. See Figure 1.

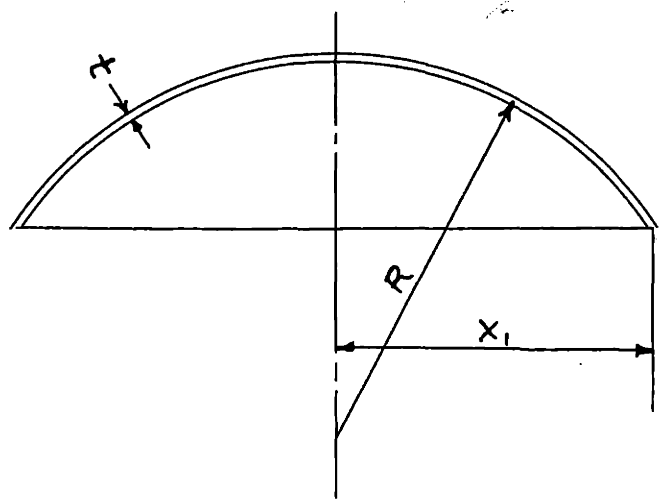


Figure 1. Notation for definition of ξ_1.

$$\xi_1 = \frac{x_1}{\ell} \qquad \text{(i)}$$

where

$$\ell = \frac{\sqrt{tR}}{\sqrt[4]{12(1-\nu^2)}} \qquad \text{(ii)}$$

and

x_1 = the half-span of the dome

t = the thickness of the shell

R = the radius of the shell

V = Poisson's ratio (taken equal to 0.2 for the numerical work of this thesis)

The results of Sections VII through XII, inclusive, are now summarized by section.

Section VII. In this case the edges are fully restrained and an uniform vertical load of intensity p is acting downward. The range of shells considered is given by $0 \leq \xi_1 \leq 25.0$. Direct, bending, combined (direct and bending) and maximum stress intensities have been computed, tabulated and plotted in dimensionless form. Shearing stress intensities were computed and tabulated also. For the same range of shells the vertical deflections at all points of the shell have been plotted and the variation of the vertical deflection at the center has been investigated with respect to ξ_1, the Bulge Ratio and R. A numerical example illustrating the use of the tables and graphs of this section is given.

Section VIII. No surface load is acting and the edges are considered to be loaded by an uniform edge moment M_E and an uniform horizontal edge force P_E. The range of shells considered is given by $0 \leq \xi_1 \leq 25.0$. Influence coefficients have been computed, tabulated and plotted for this range of shells. These coefficients when multiplied by the edge moments and horizontal forces acting will yield the values of the horizontal edge displacement and edge rotation due to that edge loading. Conversely, for a given edge horizontal displacement and rotation, the edge moments and thrusts may be determined. A numerical example illustrating the use of these influence coefficients is given.

Section IX. This case is analogous to that of Section VIII. The influence coefficients were computed, tabulated and plotted for the determination of the horizontal displacement and rotation of the edge of a lantern opening or center hole of a shell loaded by an uniform edge moment and edge horizontal force. The opening is measured by the parameter $\mu = \frac{x_o}{\ell}$ where x_o is the radius of opening or lantern circle and ℓ is as given for ξ_1. The range of openings is given by $0 \leq \mu \leq 5.0$. Due to the local effect of such an inner edge loading, the outer edge is considered to be an infinite distance away from the center of the shell.

Section X. This case illustrates an unsymmetrical hydrostatic loading condition, edges fully restrained, for a shell measured by $\xi_1 = 10.0$. This loading resolves itself into two parts; a uniform loading covered in Section VII and an unsymmetrical loading given by equation (208). This section gives the stresses and vertical deflections for the loading covered by equation (208). These results, given in tabular form, combined with the results for $\xi_1 = 10.0$ in Section VII give the desired values for the case of hydrostatic loading. The stresses and deflections are in dimensionless form.

Section XI. This is a numerical example to illustrate the techniques involved in the solution of a symmetrically loaded shell with a lantern opening, reinforced at the inner and outer edges by rings. A uniform vertical loading and a vertical lantern ring loading are individually considered. The results are given in tabular form.

Section XII. This case is analogous to those of Sections VIII and IX. In this section the procedure is outlined for the determination of influence coefficients for the calculation of the horizontal and circumferential displacements and rotation of the outer edge due to unsymmetrical horizontal and tangential (circumferential) forces and unsymmetrical moments acting on the outer edge. No values are computed.

The appendices include notes on Bessel functions and other miscellaneous topics.

4. Precision

Consideration of design and construction practice combined with the inaccuracies inherent in materials and actual loadings render a precision greater than that of a slide-rule impractical. Accordingly, four significant figures were maintained, except for small values of ξ, for Sections VII through X inclusive. The numerical values of Section XI are given to 1 psf and 0.001 inches. It is noted that for very flat shells, $\xi_1 \approx 0$, many significant figures are required in the values of the Bessel functions to maintain three or four significant figues in the results. This is due to the subtraction of one quantity from another of equal magnitude. This presents no formidable obstacle in practice due to the need to carry only a few terms in the series for the computation of the particular Bessel function when $\xi_1 \approx 0$.

5. Accuracy of Method.

The determination of the accuracy of this method of analysis is beyond the scope of this thesis. It suffices to point out that whereas all other methods, except that of Geckeler's (9) Bessel function solution for shallow domes, become more inaccurate as the shell approaches the limiting case of a flat plate, this method (and Geckeler's) become more accurate by virtue of the nature of the assumptions made. It is likewise felt that for the limiting case covered by this thesis; namely, a shell whose rise to span ratio is about one-eighth (1/8), the approximate method used herein gives results as good as any given by other methods.

6. Conclusions.

a. The rotationally symmetric case of the thin shallow spherical shell is capable of practical solution without serious mathematical difficulties. The shell may be analyzed directly under load with certain boundary conditions or may be considered as a combination of a shell in a membrane state upon which are superimposed the edge forces required to satisfy the initial conditions of restraint.

SECTION II

The Basic Equations of Shell Theory for a Shallow Spherical Dome.

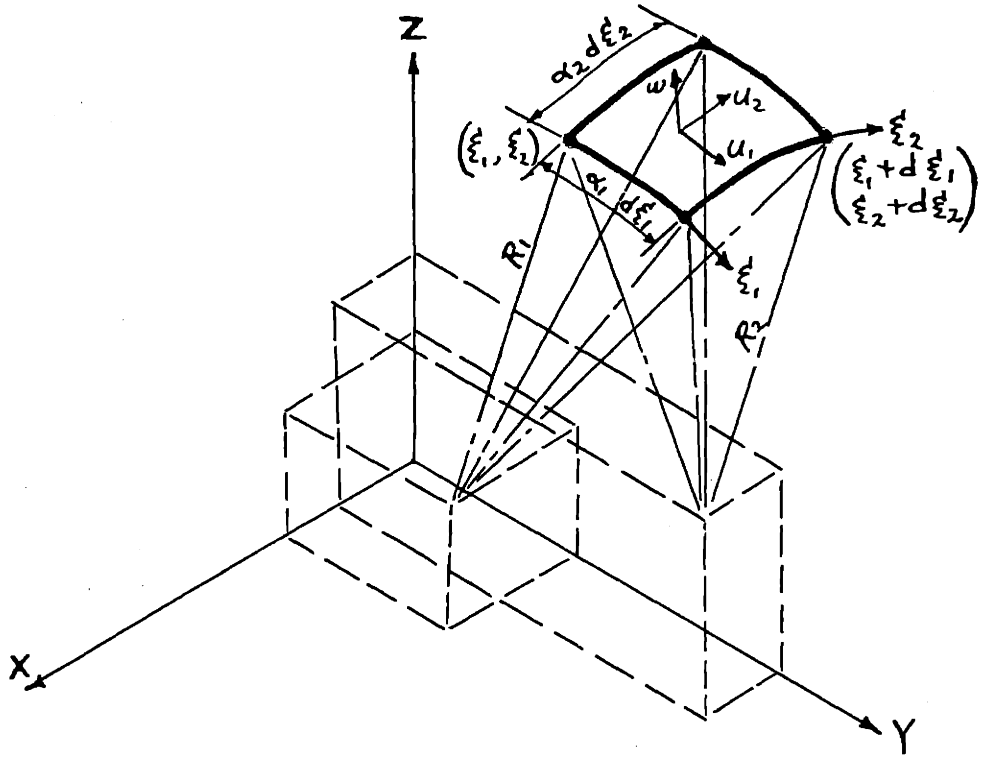
1. The Equations of General Shell Theory (a)

Love's equations of the general theory are the starting point for the derivation of the simplified shell equations of this report. E. Reissner (13) in a new derivations of these equations states them to be the following. The notation of Reissner's paper is employed. See Figure 2.

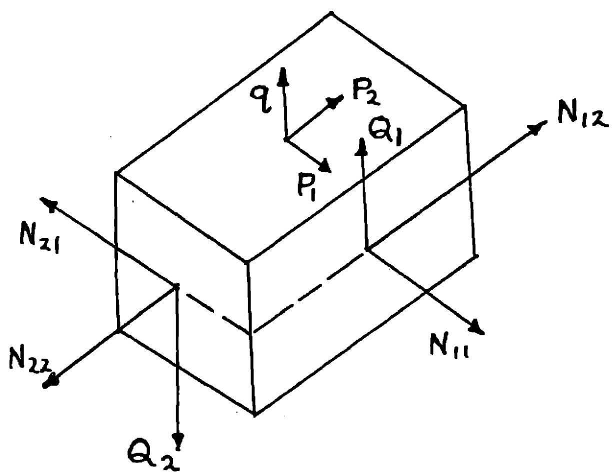
a. The Equations of Equilibrium for an Element of the Shell

$$\left.\begin{aligned}
&\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - N_{22}\frac{\partial \alpha_2}{\partial \xi_1} + Q_1 \frac{\alpha_1 \alpha_2}{R_1} + \alpha_1 \alpha_2 P_1 = 0 \\
&\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - N_{11}\frac{\partial \alpha_1}{\partial \xi_2} + Q_2 \frac{\alpha_1 \alpha_2}{R_2} + \alpha_1 \alpha_2 P_2 = 0 \\
&\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - N_{11}\frac{\alpha_1 \alpha_2}{R_1} - \frac{\alpha_1 \alpha_2}{R_2} N_{22} + \alpha_1 \alpha_2 q = 0 \\
&\frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} + N_{12} - N_{21} = 0
\end{aligned}\right\} \quad (A)$$

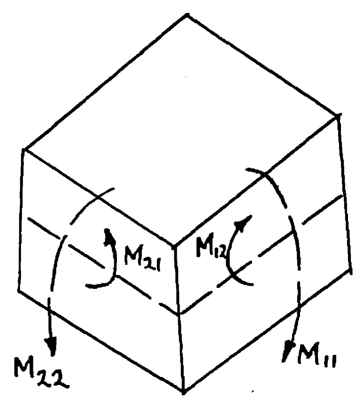
(a) The development of this section is based upon the work of E. Reissner (13) pp. 83-93. Reissner's demonstration has been modified to include the effects of temperature



(a) Geometry of an Element of the Shell



(b) Direct Stress Resultants and Surface Load Intensities



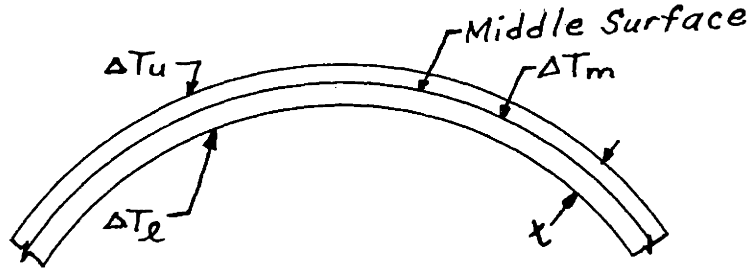
(c) Bending Stress Resultants

Figure 2. General Shell Theory Notation (from E. Reissner (13))

$$\left.\begin{aligned} &\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} - M_{11}\frac{\partial \alpha_1}{\partial \xi_2} + M_{21}\frac{\partial \alpha_2}{\partial \xi_1} \\ &\qquad\qquad - \alpha_1 \alpha_2 Q_2 = 0 \\ &\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} - M_{22}\frac{\partial \alpha_2}{\partial \xi_1} + M_{12}\frac{\partial \alpha_1}{\partial \xi_2} \\ &\qquad\qquad - \alpha_1 \alpha_2 Q_1 = 0 \end{aligned}\right\} \quad (A)$$

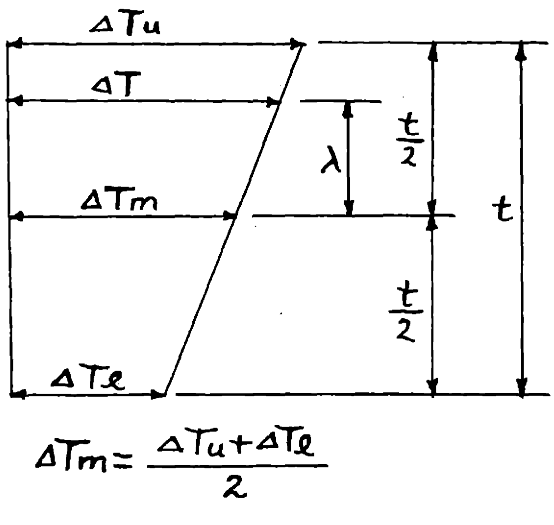
b. The Stress Resultant Strain Relations

The expressions given by E. Reissner (13) have been modified to include the effects of a temperature distribution assumed to vary linearly across the thickness of the shell. See Figure 3. An increase in temperature is positive. These temperature effects manifest themselves in the expressions for direct stress and bending stress only. The development of the temperature terms is to be found on page 204, Timoshenko's "Theory of Elasticity", McGraw-Hill, New York, 1934 and on page 54, Timoshenko's "Theory of Plates and Shells", McGraw-Hill, New York, 1940. β is the coefficient of thermal expansion.



ΔT_u , ΔT_ℓ are functions of position, (ξ^1, ξ^2) and are temperature changes. An increase in temperature is positive.

(a) Temperature Distribution at Shell Surfaces



(b) Temperature Distribution over Thickness t of Shell.

Figure 3. Notation for Temperature Effects.

The stress resultant strain relations are

$$\left.\begin{aligned}
N_{11} &= \frac{Et}{1-\nu^2}\left(\varepsilon_1^\circ + \nu\,\varepsilon_2^\circ\right) - \beta t \,\Delta T_m \\
N_{12} &= N_{21} = Gt\,\gamma_{12}^\circ \\
N_{22} &= \frac{Et}{1-\nu^2}\left(\varepsilon_2^\circ + \nu\,\varepsilon_1^\circ\right) - \beta t \,\Delta T_m \\
M_{11} &= D\left[\left(K_1 + \nu K_2\right) - \frac{\beta(1+\nu)}{t}\left(\Delta T_u - \Delta T_\ell\right)\right] \\
M_{12} &= M_{21} = \frac{(1-\nu)}{2} D\,\tau \\
M_{22} &= D\left[\left(K_2 + \nu K_1\right) - \frac{\beta(1+\nu)}{t}\left(\Delta T_u - \Delta T_\ell\right)\right]
\end{aligned}\right\} \quad (B)$$

where

$$D = \frac{Et^3}{12(1-\nu^2)}$$

and where the strains are given in terms of the displacements as follows

$$\left.\begin{aligned}
\varepsilon_1^{\circ} &= \frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1} \\
\varepsilon_2^{\circ} &= \frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2} \\
\gamma_{12}^{\circ} &= \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) \\
K_1 &= \frac{1}{\alpha_1}\frac{\partial u_1'}{\partial \xi_1} + \frac{u_2'}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} \\
K_2 &= \frac{1}{\alpha_2}\frac{\partial u_2'}{\partial \xi_2} + \frac{u_1'}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} \\
T &= \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2'}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1'}{\alpha_1}\right) \\
u_1' &= \frac{u_1}{R_1} - \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} \quad ; \quad u_2' = \frac{u_2}{R_2} - \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2}
\end{aligned}\right\} \qquad (C)$$

The basic assumptions made in the derivation of equations (A), (B) and (C) were

1. The thickness of the shell is small compared to the radii of curvature of its middle surface.
2. The stress components normal to the middle surface are small compared with the other stress components and may be neglected in the stress-strain relations.
3. The normals of the undeformed middle surface are deformed into the normals of the deformed middle surface.
4. The displacements are so small that the equilibrium conditions for deformed elements are the same as if the elements were not deformed.

2. Reduction of the Equations of the General Theory to the Equations for a Shallow Spherical Dome.

Considering a spherical shell, the following choice of coordinates is convenient. See Figure 4.

$\xi_1 = x$, the radial distance of the points of the shell from the axis of the shell.

$\xi_2 = \phi$, the angular coordinate of the points of the shell.

The equation of the shell surface may be written in the form

$$z = \sqrt{R^2 - x^2} - (R - h)$$

The radii of curvature of the shell are

$$R_1 = R_2 = R$$

The assumption of shallowness of the shell is expressed by the following order of magnitude relation

$$\frac{dz}{dx} = -\frac{x}{\sqrt{R^2 - x^2}} \approx -\frac{x}{R} \; ; \; -\frac{dz}{dx} \ll 1$$

Figure 4 shows, as may be verified by direct calculation, that except for negligibly small quantities the terms and occurring in equations (A) to (C) are given by

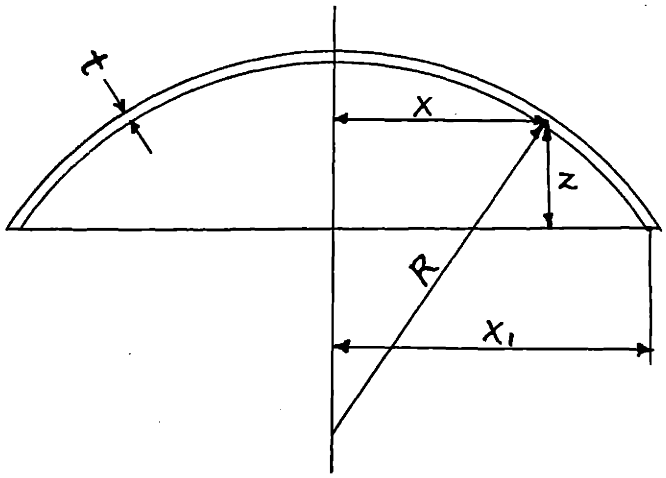
$$\alpha_1 = 1 \; ; \; \alpha_2 = x$$

with these values for α_1 and α_2 and with the specialized notation

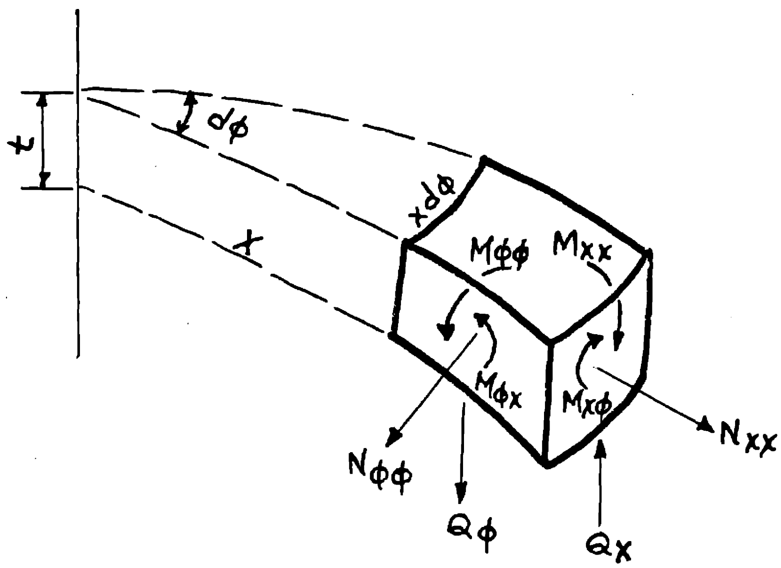
$$N_{11} = N_{xx} \; , \; N_{22} = N_{\phi\phi} \; , \; N_{12} = N_{21} = N_{x\phi}$$

$$M_{11} = M_{xx} \; , \; M_{22} = M_{\phi\phi} \; , \; M_{12} = M_{21} = M_{x\phi}$$

$$Q_1 = Q_x \; , \; Q_2 = Q_\phi \; , \; P_1 = P_x \; , \; P_2 = P_\phi \; , \; q = P$$



(a) Dimensions of Dome.



(b) Stress-Resultant Notations.

Figure 4. Key to Notation
(from E. Reissner (13))

the equilibrium equations (A) reduce to

$$\left.\begin{aligned} &\frac{\partial(xN_{xx})}{\partial x} + \frac{\partial N_{x\phi}}{\partial \phi} - N_{\phi\phi} + \frac{x}{R}Q_x + xP_x = 0 \\ &\frac{\partial(xN_{x\phi})}{\partial x} + \frac{\partial N_{\phi\phi}}{\partial \phi} + N_{x\phi} + \frac{x}{R}Q_\phi + xP_\phi = 0 \end{aligned}\right\} \qquad (1a,b)$$

$$\left.\begin{aligned} &\frac{\partial(xQ_x)}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} - (N_{xx} + N_{\phi\phi})\frac{x}{R} + xP = 0 \\ &\frac{\partial(xM_{xx})}{\partial x} + \frac{\partial M_{x\phi}}{\partial \phi} - M_{\phi\phi} - xQ_x = 0 \\ &\frac{\partial(xM_{x\phi})}{\partial x} + \frac{\partial M_{\phi\phi}}{\partial \phi} + M_{x\phi} - xQ_\phi = 0 \end{aligned}\right\} \qquad (2a,b,c)$$

It may be noted that when $R = \infty$ equations (1) become those of the theory of generalized plane stress while equations (2) become those of the theory of bending of flat plates.

The known treatments of the two separate limiting cases are used, in what follows, as guides towards a procedure applicable when the initial curvature of the shell couples the two systems of equations (1) and (2).

Next, it is necessary to reduce equations (C) for the components of strain to their form which is appropriate for the shallow spherical dome.

Writing for the meridional and circumferential displacement components

$$u_1 = u \quad , \quad u_2 = v$$

and changing slightly the notation for the components of direct strain, there follows for the components of direct strain

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{w}{R} \\ \varepsilon_\phi &= \frac{1}{x}\frac{\partial v}{\partial \phi} + \frac{u}{x} + \frac{w}{R} \\ \gamma_{x\phi} &= x\frac{\partial}{\partial x}\left(\frac{v}{x}\right) + \frac{1}{x}\frac{\partial u}{\partial \phi} \end{aligned}\right\} \qquad (3a,b,c)$$

For the quantities u_1' and u_2' occurring in the components of bending strain follows

$$\left.\begin{aligned} u_1' &= \frac{u}{R} - \frac{\partial w}{\partial x} \\ u_2' &= \frac{v}{R} - \frac{1}{x}\frac{\partial w}{\partial \phi} \end{aligned}\right\} \qquad (4a,b)$$

considering the fact that for shallow shells the deflections W, when it occurs at all, will presumably be large compared with U and V equations (4a, b) may be simplified to

$$u_1' = -\frac{\partial w}{\partial x} \quad , \quad u_2' = -\frac{1}{x}\frac{\partial w}{\partial \phi} \qquad (4c,d)$$

Introducing equations (4c, d) into the expressions for the bending strain components and making again use of the assumption leading from equations (4a, b) to equations (4c, d) there follows for the bending strains

$$\left.\begin{aligned} K_x &= -\frac{\partial^2 w}{\partial x^2} \\ K_\phi &= -\frac{1}{x}\frac{\partial w}{\partial x} - \frac{1}{x^2}\frac{\partial^2 w}{\partial \phi^2} \\ \tau &= -2\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial \phi}\right) \end{aligned}\right\} \qquad (5a,b,c)$$

It may be noted that these are the same expressions as those of the theory of flat plates. This conforms with previously known results for the buckling theory of circular cylindrical shells where during buckling the waves subdivide the circular shell into independent shallow panels.

In terms of the strains of equations (3) and (5) the stress resultant strain relations are now (see Figure 4)

$$\left.\begin{aligned} N_{xx} &= \frac{Et}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_\phi\right) - \beta t \Delta T_m \\ N_{\phi\phi} &= \frac{Et}{1-\nu^2}\left(\varepsilon_\phi + \nu\varepsilon_x\right) - \beta t \Delta T_m \\ N_{x\phi} &= G t \gamma_{x\phi} \end{aligned}\right\} \quad (6a,b,c)$$

$$\left.\begin{aligned} M_{xx} &= D\left[\left(K_x + \nu K_\phi\right) - \beta\frac{(1+\nu)}{t}\left(\Delta T_u - \Delta T_\ell\right)\right] \\ M_{\phi\phi} &= D\left[\left(K_\phi + \nu K_x\right) - \beta\frac{(1+\nu)}{t}\left(\Delta T_u - \Delta T_\ell\right)\right] \\ M_{x\phi} &= \frac{(1-\nu)}{2} D \tau \end{aligned}\right\} \quad (7a,b,c)$$

3. Reductions of Equations (1) to (7) to a System of Two Simultaneous Equations for a Stress Function F and the Deflection W.

To accomplish this reduction the observation is first made that, presumably, the vertical shear force terms in equations (1) are negligibly small. Proceeding on this assumption (which may be justified in any given case at the end by substituting in equations (1) the vertical shears obtained on the basis of neglecting them in equations (1) and by comparing the order of their magnitude with the order of magnitude of the other terms occurring in equations (1)) -equations (1) reduce to the equations of the theory of generalized plane stress.

$$\left.\begin{aligned} \frac{\partial(xN_{xx})}{\partial x} + \frac{\partial N_{x\phi}}{\partial \phi} - N_{\phi\phi} + x p_x = 0 \\ \frac{\partial(xN_{x\phi})}{\partial x} + \frac{\partial N_{\phi\phi}}{\partial \phi} + N_{x\phi} + x p_\phi = 0 \end{aligned}\right\} \qquad (8\,a,b)$$

As is known from the theory of plane stress, equations (8) may be satisfied by expressing the three stress resultants in terms of a stress function. This representation will be used here for the case only that the load terms p_x and p_ϕ are derivable from a load potential,

$$p_x = -\frac{\partial \Omega}{\partial x} \quad , \qquad p_\phi = -\frac{1}{x}\frac{\partial \Omega}{\partial \phi} \qquad (9)$$

Evidently, Ω vanishes when vertical loads p only are acting.

The representation in terms of the stress function F is

$$\left.\begin{aligned} N_{xx} &= \frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \phi^2} + \Omega \\ N_{\phi\phi} &= \frac{\partial^2 F}{\partial x^2} + \Omega \\ N_{x\phi} &= -\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial \phi}\right) \end{aligned}\right\} \qquad (10\,a,b,c)$$

It may be verified directly that equations (10) are in agreement with equations (8) and (9), for any choice of the function F.

<u>A differential equation for the F function is</u> obtained from the relevant equation of compatibility. To avoid reconverting to Cartesian coordinates, use may be made of the equations of compatibility in cylindrical coordinates. Of the equations of compatibility (16) the following one is used

$$\frac{\partial^2 \varepsilon_x^{(P)}}{\partial \phi^2} - x\frac{\partial \varepsilon_x^{(P)}}{\partial x} + \frac{\partial}{\partial x}\left(x^2 \frac{\partial \varepsilon_\phi^{(P)}}{\partial x}\right) = \frac{\partial^2 (x\gamma_{x\phi}^{(P)})}{\partial x \partial \phi} \tag{11}$$

$$\left.\begin{aligned} \varepsilon_x^{(P)} &= \frac{\partial u}{\partial x}, \quad \varepsilon_\phi^{(P)} = \frac{1}{x}\frac{\partial v}{\partial \phi} + \frac{u}{x} \\ \gamma_{x\phi}^{(P)} &= x\frac{\partial}{\partial x}\left(\frac{v}{x}\right) + \frac{1}{x}\frac{\partial u}{\partial \phi} \end{aligned}\right\} \tag{12}$$

Equation (11) is applied to equations (3) for the direct shell strains which, except for terms containing the deflection W, are identical with equations (12). Hence introducing equations (3) into equation (11) there follows

$$\frac{\partial^2 \varepsilon_x}{\partial \phi^2} - \frac{1}{R}\frac{\partial^2 w}{\partial \phi^2} - x\frac{\partial \varepsilon_x}{\partial x} + \frac{x}{R}\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(x^2\frac{\partial \varepsilon_\phi}{\partial x}\right)$$
$$- \frac{1}{R}\frac{\partial}{\partial x}\left(x^2\frac{\partial w}{\partial x}\right) = \frac{\partial^2 (x\gamma_{x\phi})}{\partial x \partial \phi} \tag{13}$$

The terms involving W may be combined to

$$-\frac{x^2}{R}\left(\frac{1}{x^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{1}{x}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) = -\frac{x^2}{R}\nabla^2 w \tag{14}$$

where ∇^2 is the symbol for the Laplace operator. Equation (13) may now be written

$$\frac{1}{x^2}\left[\frac{\partial^2 \varepsilon_x}{\partial \phi^2} - x\frac{\partial \varepsilon_x}{\partial x} + \frac{\partial}{\partial x}\left(x^2\frac{\partial \varepsilon_\phi}{\partial x}\right)\right.$$
$$\left. - \frac{\partial^2 (x\gamma_{x\phi})}{\partial x \partial \phi}\right] = \frac{1}{R}\nabla^2 w \tag{15}$$

In equation (15) introduce the strains in terms of the stress resultants, writing equations (6) in the form

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{tE}\left(N_{xx} - \nu N_{\phi\phi}\right) + \beta \Delta T_m \\ \varepsilon_\phi &= \frac{1}{tE}\left(N_{\phi\phi} - \nu N_{xx}\right) + \beta \Delta T_m \\ \gamma_{x\phi} &= \frac{1}{tG} N_{x\phi} \end{aligned}\right\} \qquad (16\,a,b,c)$$

substituting equations (16) in equation (15) and using the representation of the stress resultants by equations (10) there is obtained an equation connecting stress function F and deflection W. The calculations need not actually be carried out inasmuch as the result for $R = \infty$, that is, the value of the left hand side of the transformed equation (15) is known from the theory of plane stress (where it is derived by the use of Cartesian coordinates).

The resultant fundamental equation is

$$\nabla^2\nabla^2 F + (1-\nu)\nabla^2\Omega = \frac{tE}{R}\nabla^2 w - tE\beta\nabla^2 \Delta T_m \qquad (17)$$

To obtain a second differential equation involving W and F introduce, following the example of plate-bending theory, the moment equilibrium conditions (2b) and 2c) into the vertical force equilibrium condition (2a),

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial (x M_{xx})}{\partial x} + \frac{\partial M_{x\phi}}{\partial \phi} - M_{\phi\phi}\right) \\ &+ \frac{1}{x}\left(\frac{\partial (x M_{x\phi})}{\partial x} + \frac{\partial M_{\phi\phi}}{\partial \phi} + M_{x\phi}\right) \\ &- \frac{x}{R}\left(N_{xx} + N_{\phi\phi}\right) + xp = 0 \end{aligned} \qquad (18)$$

Introduce into equations (18) the values of M_{xx}, $M_{\phi\phi}$ and $M_{x\phi}$ from equations (7) and (5). The result is

$$-D\nabla^2\nabla^2 w - \frac{D\beta(1+\nu)}{t}\nabla^2(\Delta T_u - \Delta T_\ell)$$
$$-\frac{1}{R}(N_{xx} + N_{\phi\phi}) + p = 0 \qquad (19)$$

Into equation (19) introduce the values of N_{xx} and $N_{\phi\phi}$ from equations (10). With a slight change in notation we have as a final result

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F = p - \frac{2\Omega}{R} - \frac{\beta E t^2}{12(1-\nu)}\nabla^2(\Delta T_u - \Delta T_\ell) \qquad (20)$$

Equations (17) and (20) are the two fundamental differential equations for a thin shallow spherical dome.

To calculate the stresses and deflections of the dome it is necessary to obtain the solutions of equations (17) and (20) and to fit them to the prescribed boundary conditions. This task is materially helped by the fact that both equations involve only the Laplace operator ∇^2. Thus it is, for instance, possible to have solutions of the form

$$W_n = W_n(x)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix}$$
$$F_n = F_n(x)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix} \qquad (21\,a,b)$$

where the functions $W_n(x)$ and $F_n(x)$ are expressible in terms of Bessel functions. By combining terms with different values of n, solutions can be obtained for unsymmetrical loadings. Before proceeding with the solution of equations (17) and (20) the main results obtained so far may be summarized as follows

4. Resume of Basic Formulas for Shallow Spherical Domes.

a. Differential Equations

$$\nabla^2\nabla^2 F - \frac{tE}{R}\nabla^2 w = -(1-\nu)\nabla^2\Omega - tE\beta\nabla^2\Delta T_m \tag{17}$$

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F = p - \frac{2\Omega}{R} - \frac{\beta E t^2}{12(1-\nu)}\nabla^2(\Delta T_u - \Delta T_\ell) \tag{20}$$

b. Direct Stress Resultants

$$\left.\begin{aligned} N_{xx} &= \frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \phi^2} + \Omega \\ N_{\phi\phi} &= \frac{\partial^2 F}{\partial x^2} + \Omega \\ N_{x\phi} &= -\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial \phi}\right) \end{aligned}\right\} \quad (10\,a,b,c)$$

where $p_x = -\dfrac{\partial\Omega}{\partial x}$, $p_\phi = -\dfrac{1}{x}\dfrac{\partial\Omega}{\partial\phi}$

c. Bending Stress Resultants

$$\left.\begin{aligned} M_{xx} &= -D\Big[\frac{\partial^2 w}{\partial x^2} + \nu\left(\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w}{\partial \phi^2}\right) \\ &\qquad + \frac{\beta(1+\nu)}{t}(\Delta T_u - \Delta T_\ell)\Big] \\ M_{\phi\phi} &= -D\Big[\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w}{\partial \phi^2} + \nu\frac{\partial^2 w}{\partial x^2} \\ &\qquad + \frac{\beta(1+\nu)}{t}(\Delta T_u - \Delta T_\ell)\Big] \\ M_{x\phi} &= -(1-\nu)D\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial \phi}\right) \end{aligned}\right\} \quad (21\,c,d,e)$$

SECTION III

The Solution of the Shell Equations

1. Integration of the Differential Equations

From equation (17)

$$\nabla^2 F - \frac{tE}{R}w = \frac{tE}{R}\Phi - (1-\nu)\Omega - tE\beta\Delta T_m \tag{22}$$

where Φ is a solution of $\nabla^2\Phi = 0$ so that

$$\Phi = A_o + B_o \ln x + \sum_{n=1}^{n=\infty}\left(A_n X^n + B_n X^{-n}\right)\cos n\phi \tag{23}$$

It is to be noted that terms in sin $n\phi$ similar to the terms in cos $n\phi$ are possible solutions for Φ but are not relevant to the work of this thesis and accordingly are not included.

Substituting $\nabla^2 F$ from equation (22) in equation (20) we have

$$D\nabla^2\nabla^2 w + \frac{tE}{R^2}w = p - \frac{tE}{R^2}\Phi - \frac{(1+\nu)}{R}\Omega + \frac{tE\beta\Delta T_m}{R}$$

$$- \frac{t^2E\beta}{12(1-\nu)}\nabla^2(\Delta T_u - \Delta T_\ell) \tag{24}$$

Dividing through by D and introducing a new symbol

$$\frac{tE}{DR^2} = \frac{12(1-\nu^2)tE}{Et^3R^2} = \frac{12(1-\nu^2)}{t^2R^2} = \frac{1}{\ell^4}$$

$$\ell = \frac{\sqrt{tR}}{\sqrt[4]{12(1-\nu^2)}} \tag{25}$$

equation (24) may be written as

$$\nabla^2\nabla^2 w + \frac{1}{\ell^4} w = \frac{1}{D}\Big[P - \frac{(1+\nu)}{R}\Omega + \frac{tE\beta\Delta T_m}{R} - \frac{t^2 E\beta}{12(1-\nu)}\nabla^2(\Delta T_u - \Delta T_\ell)\Big] - \frac{\Phi}{\ell^4} \tag{26}$$

The solution of equation (26) may be considered as composed of three parts

$$w = w_h + w_p - \Phi \tag{27}$$

where W_h is the solution of the homogeneous equation

$$\nabla^2\nabla^2 w_h + \frac{1}{\ell^4} w_h = 0 \tag{28}$$

and where W_p is a particular integral of the equation

$$\nabla^2\nabla^2 w_p + \frac{1}{\ell^4} w_p = \frac{1}{D}\Big[P - \frac{(1+\nu)}{R}\Omega + \frac{tE\beta\Delta T_m}{R} - \frac{t^2 E\beta}{12(1-\nu)}\nabla^2(\Delta T_u - \Delta T_\ell)\Big] \tag{29}$$

The solution of equation (29) depends upon the loading of the shell and may often be found by inspection. The solutions of equation (28) are known but the process by which they are obtained will be outlined.

To solve equation (28) take

$$w_h = w_n(x)\cos n\phi \tag{30}$$

observing that

$$\nabla^2 w_h = \Big[\nabla^2 w_n(x) - \frac{n^2}{x^2} w_n(x)\Big]\cos n\phi \tag{31a}$$

$$\nabla^2\nabla^2 w_n = \Big[\nabla^2\nabla^2 w_n(x) - \nabla^2 \frac{n^2}{x^2} w_n(x) - \frac{n^2}{x^2}\nabla^2 w_n(x)$$

$$+ \frac{n^4}{x^4} w_n(x)\Big] \cos n\phi \qquad (31b)$$

Substituting equations (30) and (31b) into (28) we have

$$\Big[\nabla^2\nabla^2 w_n(x) - \nabla^2 \frac{n^2}{x^2} w_n(x) - \frac{n^2}{x^2}\nabla^2 w_n(x) + \frac{n^4}{x^4} w_n(x)$$

$$+ \frac{1}{\ell^4} w_n(x)\Big] \cos n\phi = 0 \qquad (32)$$

Dividing through by cos nϕ and factoring the expression in the brackets, we have

$$\Big[\nabla^2 + \Big(\frac{i}{\ell^2} - \frac{n^2}{x^2}\Big)\Big]\Big[\nabla^2 - \Big(\frac{i}{\ell^2} + \frac{n^2}{x^2}\Big)\Big] w_n(x) = 0 \qquad (33)$$

where $i = \sqrt{-1}$

Consider the solutions of

$$\Big[\nabla^2 - \Big(\frac{i}{\ell^2} + \frac{n^2}{x^2}\Big)\Big] w_n(x) = 0 \qquad (34)$$

From McLachlan "Bessel Functions for Engineers", Oxford Press, 1934, pp. 157, 168, 170 these are

$$\left.\begin{aligned} I_n\Big(\sqrt{i}\,\frac{x}{\ell}\Big) &= ber_n \frac{x}{\ell} + i\, bei_n \frac{x}{\ell} \\ K_n\Big(\sqrt{i}\,\frac{x}{\ell}\Big) &= ker_n \frac{x}{\ell} + i\, kei_n \frac{x}{\ell} \end{aligned}\right\} \qquad (35a,b)$$

Introducing equations (35) into (34) and combining the real and imaginary parts there follows

$$\left(\nabla^2 \mathrm{ber}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{ber}_n \frac{x}{\ell} + \frac{1}{\ell^2}\mathrm{bei}_n \frac{x}{\ell}\right)$$

$$+ i\left(\nabla^2 \mathrm{bei}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{bei}_n \frac{x}{\ell} - \frac{1}{\ell^2}\mathrm{ber}_n \frac{x}{\ell}\right)$$

$$+ \left(\nabla^2 \mathrm{ker}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{ker}_n \frac{x}{\ell} + \frac{1}{\ell^2}\mathrm{kei}_n \frac{x}{\ell}\right)$$

$$+ i\left(\nabla^2 \mathrm{kei}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{kei}_n \frac{x}{\ell} - \frac{1}{\ell^2}\mathrm{ker}_n \frac{x}{\ell}\right) = 0 \qquad (36)$$

whence

$$\left(\nabla^2 \mathrm{ber}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{ber}_n \frac{x}{\ell} + \frac{1}{\ell^2}\mathrm{bei}_n \frac{x}{\ell}\right) = 0 \qquad (37a)$$

$$\left(\nabla^2 \mathrm{bei}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{bei}_n \frac{x}{\ell} - \frac{1}{\ell^2}\mathrm{ber}_n \frac{x}{\ell}\right) = 0 \qquad (37b)$$

$$\left(\nabla^2 \mathrm{ker}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{ker}_n \frac{x}{\ell} + \frac{1}{\ell^2}\mathrm{kei}_n \frac{x}{\ell}\right) = 0 \qquad (37c)$$

$$\left(\nabla^2 \mathrm{kei}_n \frac{x}{\ell} - \frac{n^2}{x^2}\mathrm{kei}_n \frac{x}{\ell} - \frac{1}{\ell^2}\mathrm{ker}_n \frac{x}{\ell}\right) = 0 \qquad (37d)$$

To show that each one of the four functions $\mathrm{ber}_n \frac{x}{\ell}$, $\mathrm{bei}_n \frac{x}{\ell}$, $\mathrm{ker}_n \frac{x}{\ell}$ and $\mathrm{kei}_n \frac{x}{\ell}$ satisfies equation (28), the proof for $\mathrm{ber}_n \frac{x}{\ell}$, will be given.

Operate on equation (37a) with

$$\nabla^2 \nabla^2 \mathrm{ber}_n \frac{x}{\ell} - \nabla^2 \frac{n^2}{x^2}\mathrm{ber}_n \frac{x}{\ell} + \nabla^2 \frac{1}{\ell^2}\mathrm{bei}_n \frac{x}{\ell} = 0 \qquad (38)$$

Substituting equation (37a) to eliminate $\mathrm{bei}_n \frac{x}{\ell}$ we have

$$\nabla^2\nabla^2 \mathrm{ber}_n \frac{x}{\ell} - \nabla^2 \frac{n^2}{x^2} \mathrm{ber}_n \frac{x}{\ell} - \frac{n^2}{x^2} \nabla^2 \mathrm{ber}_n \frac{x}{\ell}$$

$$+ \frac{n^4}{x^4} \mathrm{ber}_n \frac{x}{\ell} + \frac{1}{\ell^4} \mathrm{ber}_n \frac{x}{\ell} = 0 \tag{39a}$$

or in more concise form

$$\nabla^2\nabla^2 \mathrm{ber}_n \frac{x}{\ell} + \frac{1}{\ell^4} \mathrm{ber}_n \frac{x}{\ell} = 0 \tag{39b}$$

This is equation (28) with $W_h = \mathrm{ber}_n \frac{x}{\ell}$. In similar fashion it can be shown that $\mathrm{bei}_n \frac{x}{\ell}$, $\mathrm{ker}_n \frac{x}{\ell}$ and $\mathrm{kei}_n \frac{x}{\ell}$ satisfy equation (28).

Thus the solution of the homogeneous equation (28) is

$$w_h = \left(C_{1n} \mathrm{ber}_n \frac{x}{\ell} + C_{2n} \mathrm{bei}_n \frac{x}{\ell} + C_{3n} \mathrm{ker}_n \frac{x}{\ell}\right.$$

$$\left. + C_{4n} \mathrm{kei}_n \frac{x}{\ell}\right) \cos n\phi \tag{40}$$

for all values of n.

To solve for F we substitute equations (27) and (40) into equation (22) obtaining

$$\nabla^2 F = \frac{tE}{R}\left[\left(C_{1n} \mathrm{ber}_n \frac{x}{\ell} + C_{2n} \mathrm{bei}_n \frac{x}{\ell} + C_{3n} \mathrm{ker}_n \frac{x}{\ell}\right.\right.$$

$$\left.\left. + C_{4n} \mathrm{kei}_n \frac{x}{\ell}\right) \cos n\phi + w_p\right] - (1-\nu)\Omega - tE\beta\Delta T_m \tag{41}$$

F may be written as

$$F = F_h + F_{p,1} + F_{p,2} \tag{42}$$

where F_h is the solution of

$$\nabla^2 F_h = 0 \tag{43}$$

and has the same form as Φ, given by equation (23).

$F_{p,2}$ is a particular solution of

$$\nabla^2 F_{p,2} = \frac{tE}{R} w_p - (1-\nu)\Omega - tE\beta\Delta T_m \tag{44}$$

and depends upon the loading of the shell.

$F_{p,1}$ is a particular solution of

$$\nabla^2 F_{p,1} = \frac{tE}{R}\Big(C_{1n}\,\mathrm{ber}_n\frac{x}{l} + C_{2n}\,\mathrm{bei}_n\frac{x}{l} + C_{3n}\,\mathrm{ker}_n\frac{x}{l} + C_{4n}\,\mathrm{kei}_n\frac{x}{l}\Big)\cos n\phi \tag{45}$$

Writing

$$F_{p,1} = \Big(R_n\,\mathrm{ber}_n\frac{x}{l} + S_n\,\mathrm{bei}_n\frac{x}{l} + V_n\,\mathrm{ker}_n\frac{x}{l} + Y_n\,\mathrm{kei}_n\frac{x}{l}\Big)\cos n\phi \tag{46}$$

we have

$$\nabla^2 F_{p,1} = \Big(\nabla^2 - \frac{n^2}{x^2}\Big)\Big[\Big(R_n\,\mathrm{ber}_n\frac{x}{l} + S_n\,\mathrm{bei}_n\frac{x}{l} + V_n\,\mathrm{ker}_n\frac{x}{l} + Y_n\,\mathrm{kei}_n\frac{x}{l}\Big)\cos n\phi\Big]$$

$$= \frac{tE}{R}\Big[\Big(C_{1n}\,\mathrm{ber}_n\frac{x}{l} + C_{2n}\,\mathrm{bei}_n\frac{x}{l} + C_{3n}\,\mathrm{ker}_n\frac{x}{l} + C_{4n}\,\mathrm{kei}_n\frac{x}{l}\Big)\cos n\phi\Big] \tag{47}$$

using relations (37) and equating equals to equals, we have

$$\frac{1}{\ell^2}\left(-R_n \operatorname{bei}_n \frac{x}{\ell} + S_n \operatorname{ber}_n \frac{x}{\ell} - V_n \operatorname{kei}_n \frac{x}{\ell} + Y_n \operatorname{ker}_n \frac{x}{\ell}\right)$$

$$= \frac{tE}{R}\left(C_{1n} \operatorname{ber}_n \frac{x}{\ell} + C_{2n} \operatorname{bei}_n \frac{x}{\ell} + C_{3n} \operatorname{ker}_n \frac{x}{\ell} + C_{4n} \operatorname{kei}_n \frac{x}{\ell}\right) \tag{48}$$

whence

$$\left.\begin{aligned} R_n &= -\frac{\ell^2 tE}{R} C_{2n} \\ S_n &= +\frac{\ell^2 tE}{R} C_{1n} \\ V_n &= -\frac{\ell^2 tE}{R} C_{4n} \\ Y_n &= +\frac{\ell^2 tE}{R} C_{3n} \end{aligned}\right\} \tag{49a,b,c,d}$$

Substituting equations (49) into equation (46)

$$F_{p,1} = -\frac{\ell^2 tE}{R}\left(C_{2n} \operatorname{ber}_n \frac{x}{\ell} - C_{1n} \operatorname{bei}_n \frac{x}{\ell} + C_{4n} \operatorname{ker}_n \frac{x}{\ell} - C_{3n} \operatorname{kei}_n \frac{x}{\ell}\right) \cos n\phi \tag{50}$$

2. Summary of Results.

With a slight modification in the form of the constants, from equations (27), (23) and (40)

$$w = \sum_{n=0}^{n=\infty}\left(C_{1n}\,\mathrm{ber}_n\frac{x}{\ell} + C_{2n}\,\mathrm{bei}_n\frac{x}{\ell} + C_{3n}\,\mathrm{ker}_n\frac{x}{\ell}\right.$$

$$\left. + C_{4n}\,\mathrm{kei}_n\frac{x}{\ell}\right)\cos n\phi - A_o - B_o\ln\frac{x}{\ell} + w_p$$

$$-\sum_{n=1}^{n=\infty}\left(C_{5n}\frac{x^n}{\ell^n} + C_{7n}\frac{\ell^n}{x^n}\right)\cos n\phi \qquad (51)$$

where

$$\nabla^2\nabla^2 w_p + \frac{1}{\ell^4}w_p = \frac{1}{D}\left[p - \frac{(1+\nu)}{R}\Omega + \frac{tE\beta\Delta T_m}{R}\right.$$

$$\left. - \frac{t^2E\beta}{12(1-\nu)}\nabla^2(\Delta T_u - \Delta T_\ell)\right] \qquad (29)$$

From equations (42), (43), and (50)

$$F = \sum_{n=0}^{n=\infty}\frac{\ell^2 tE}{R}\left(-C_{2n}\,\mathrm{ber}_n\frac{x}{\ell} + C_{1n}\,\mathrm{bei}_n\frac{x}{\ell} - C_{4n}\,\mathrm{ker}_n\frac{x}{\ell}\right.$$

$$\left. + C_{3n}\,\mathrm{kei}_n\frac{x}{\ell}\right)\cos n\phi + a_o + b_o\ln\frac{x}{\ell} + F_{p,2}$$

$$+\sum_{n=1}^{n=\infty}\left(C_{6n}\frac{x^n}{\ell^n} + C_{8n}\frac{\ell^n}{x^n}\right)\cos n\phi \qquad (52)$$

where

$$\nabla^2 F_{p,2} = \frac{tE}{R}w_p - (1-\nu)\Omega - tE\beta\Delta T_m \qquad (44)$$

3. Results for Rotational Symmetry without Radial Slit.

Of particular importance for the work that follows are the expressions for W and F for the above case.

First, it will be shown that in the expression for W, equation (51) B_o must be taken equal to zero for the case of rotational symmetry without radial slit in the shell. Considering equation (22) it is seen that the term $B_o \ln X$ in equation (23) becomes in plane stress a contribution to the stress function of the form $B_o X^2 \ln X$ and it is known that a term of this form leads to multi-valued displacements V and must therefore be discarded for shells without radial slit. The proof that follows is taken from E. Reissner (13). Temperature effects are not considered.

For rotationally symmetric displacements

$$\left.\begin{aligned} \varepsilon_x &= \frac{du}{dx} + \frac{\omega}{R} \\ \varepsilon_\phi &= \frac{u}{x} + \frac{\omega}{R} \end{aligned}\right\} \quad (53\,a,b)$$

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{tE}\left(N_{xx} - \nu N_{\phi\phi}\right) \\ \varepsilon_\phi &= \frac{1}{tE}\left(N_{\phi\phi} - \nu N_{xx}\right) \end{aligned}\right\} \quad (54\,a,b)$$

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{tE}\left[\nabla^2 F - (1+\nu)\frac{d^2F}{dx^2} + (1-\nu)\Omega\right] \\ \varepsilon_\phi &= \frac{1}{tE}\left[\nabla^2 F - (1+\nu)\frac{1}{x}\frac{dF}{dx} + (1-\nu)\Omega\right] \end{aligned}\right\} \quad (55\,a,b)$$

Substituting in equations (53) the value of $\frac{\omega}{R}$ from equation (22) and setting these equations equal to equations (55)

$$\frac{du}{dx} + \frac{1}{tE}\left[\nabla^2 F - \frac{tE}{R}\Phi + (1-\nu)\Omega\right]$$

$$= \frac{1}{tE}\left[\nabla^2 F - (1+\nu)\frac{d^2F}{dx^2} + (1-\nu)\Omega\right] \tag{56a}$$

$$\frac{u}{x} + \frac{1}{tE}\left[\nabla^2 F - \frac{tE}{R}\Phi + (1-\nu)\Omega\right]$$

$$= \frac{1}{tE}\left[\nabla^2 F - (1+\nu)\frac{1}{x}\frac{dF}{dx} + (1-\nu)\Omega\right] \tag{56b}$$

Cancelling the terms of both sides of equations (56)

$$\frac{du}{dx} - \frac{\Phi}{R} = -\frac{(1+\nu)}{tE}\frac{d^2F}{dx^2} \tag{57a}$$

$$\frac{u}{x} - \frac{\Phi}{R} = -\frac{(1+\nu)}{tE}\frac{1}{x}\frac{dF}{dx} \tag{57b}$$

Multiplying equation (57b) by X and differentiating with respect to X

$$\frac{du}{dx} - \frac{1}{R}\frac{d(x\Phi)}{dx} = -\frac{(1+\nu)}{tE}\frac{d^2F}{dx^2} \tag{58}$$

Comparing equation (58) with equation (57a) indicates

$$\Phi = \frac{d(x\Phi)}{dx} \tag{59}$$

Substituting in equation (59) the value of Φ given by equation (23) indicates the necessity of

$$B_o = 0 \tag{60}$$

We may now write the expressions for W and F for the case of rotational symmetry without radial slit. Temperature effects are not considered.

From equations (51) and (60) for n = o

$$w = C_{10}\,\mathrm{ber}\frac{x}{\ell} + C_{20}\,\mathrm{bei}\frac{x}{\ell} + C_{30}\,\mathrm{ker}\frac{x}{\ell} + C_{40}\,\mathrm{kei}\frac{x}{\ell} - A_o + w_p \tag{61}$$

where

$$\nabla^2\nabla^2 w_p + \frac{1}{\ell^4} w_p = \frac{1}{D}\left[p - \frac{(1+\nu)}{R}\Omega\right] \tag{62}$$

From equation (52) noting that only derivatives of F are relevant

$$F = \frac{\ell^2 tE}{R}\left(-C_{20}\,\mathrm{ber}\frac{x}{\ell} + C_{10}\,\mathrm{bei}\frac{x}{\ell} - C_{40}\,\mathrm{ker}\frac{x}{\ell} + C_{30}\,\mathrm{kei}\frac{x}{\ell}\right) + b_o \ln\frac{x}{\ell} + F_{p,2} \tag{63}$$

where

$$\nabla^2 F_{p,2} = \frac{tE}{R} w_p - (1-\nu)\Omega \tag{64}$$

SECTION IV

Expressions for Stress Resultants and Reactions

Primes indicate differentiation with respect to ξ . The abbreviation $\xi = \frac{x}{\ell}$ is used.

1. Bending Moments

From equations (21c, d)

$$\left.\begin{aligned} M_{xx} &= -D\left[\frac{\partial^2 \omega}{\partial x^2} + \nu\left(\frac{1}{x}\frac{\partial \omega}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \omega}{\partial \phi^2}\right)\right] \\ M_{\phi\phi} &= -D\left[\frac{1}{x}\frac{\partial \omega}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \omega}{\partial \phi^2} + \nu\frac{\partial^2 \omega}{\partial x^2}\right] \end{aligned}\right\} \qquad (65a,b)$$

These may be written as

$$\left.\begin{aligned} M_{xx} &= -D\left[\nabla^2 \omega - (1-\nu)\left(\frac{1}{x}\frac{\partial \omega}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \omega}{\partial \phi^2}\right)\right] \\ M_{\phi\phi} &= -D\left[\nu\nabla^2 \omega + (1-\nu)\left(\frac{1}{x}\frac{\partial \omega}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \omega}{\partial \phi^2}\right)\right] \end{aligned}\right\} \qquad (66a,b)$$

Using equations (37) we have

$$\nabla^2 \omega = \frac{1}{\ell^2}\sum_{n=0}^{n=\infty}\Big(C_{2n}\,\mathrm{ber}_n\,\xi - C_{1n}\,\mathrm{bei}_n\,\xi + C_{4n}\,\mathrm{ker}_n\,\xi$$

$$- C_{3n}\,\mathrm{kei}_n\,\xi\Big)\cos n\phi + \nabla^2 \omega_p \qquad (67)$$

Using equation (51) we have

$$\frac{1}{x}\frac{\partial \omega}{\partial x} = \frac{1}{\ell^2}\sum_{n=0}^{n=\infty}\Big(C_{1n}\frac{\mathrm{ber}_n'\xi}{\xi} + C_{2n}\frac{\mathrm{bei}_n'\xi}{\xi} + C_{3n}\frac{\mathrm{ker}_n'\xi}{\xi}$$

$$+ C_{4n}\frac{\mathrm{kei}_n'\xi}{\xi}\Big)\cos n\phi - \frac{B_0}{\ell^2\xi^2} + \frac{1}{\ell^2\xi}\,\omega_p'$$

$$-\frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n\left(C_{5n}\,\xi^{n-2} - C_{7n}\,\xi^{-(n+2)}\right)\cos n\phi \qquad (68)$$

$$\frac{1}{x^2}\frac{\partial^2 \omega}{\partial \phi^2} = -\frac{1}{\ell^2}\sum_{n=0}^{n=\infty}\frac{n^2}{\xi^2}\Big(C_{1n}\,\mathrm{ber}_n\,\xi + C_{2n}\,\mathrm{bei}_n\,\xi + C_{3n}\,\mathrm{ker}_n\,\xi$$
$$+ C_{4n}\,\mathrm{kei}_n\,\xi\Big)\cos n\phi + \frac{1}{\ell^2\xi^2}\frac{\partial^2 \omega_p}{\partial \phi^2}$$
$$+ \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n^2\left(C_{5n}\,\xi^{n-2} + C_{7n}\,\xi^{-(n+2)}\right)\cos n\phi \qquad (69)$$

2. Torsional Moment

$$M_{x\phi} = -(1-\nu)D\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \omega}{\partial \phi}\right) \qquad (21e)$$

or

$$M_{x\phi} = (1-\nu)D\left[\frac{1}{x^2}\frac{\partial \omega}{\partial \phi} - \frac{1}{x}\frac{\partial^2 \omega}{\partial \phi \partial x}\right] \qquad (70)$$

Using equation (51) we have

$$\frac{1}{x^2}\frac{\partial \omega}{\partial \phi} = -\frac{1}{\ell^2}\sum_{n=0}^{n=\infty}\frac{n}{\xi^2}\Big(C_{1n}\,\mathrm{ber}_n\,\xi + C_{2n}\,\mathrm{bei}_n\,\xi + C_{3n}\,\mathrm{ker}_n\,\xi$$
$$+ C_{4n}\,\mathrm{kei}_n\,\xi\Big)\sin n\phi + \frac{1}{\ell^2\xi^2}\frac{\partial \omega}{\partial \phi}$$
$$+ \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n\left(C_{5n}\,\xi^{n-2} + C_{7n}\,\xi^{-(n+2)}\right)\sin n\phi \qquad (71)$$

$$\frac{1}{x}\frac{\partial^2 \omega}{\partial \phi \partial x} = -\frac{1}{\ell^2}\sum_{n=0}^{n=\infty} n\Big(C_{1n}\frac{\mathrm{ber}_n'\xi}{\xi} + C_{2n}\frac{\mathrm{bei}_n'\xi}{\xi} + C_{3n}\frac{\mathrm{ker}_n'\xi}{\xi}$$
$$+ C_{4n}\frac{\mathrm{kei}_n'\xi}{\xi}\Big)\sin n\phi + \frac{1}{\ell^2\xi}\frac{\partial^2 \omega_p}{\partial \phi \partial \xi}$$
$$+ \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n^2\left(C_{5n}\,\xi^{n-2} - C_{7n}\,\xi^{-(n+2)}\right)\sin n\phi \qquad (72)$$

3. Direct Stress Resultants

$$\left.\begin{aligned} N_{xx} &= \frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \phi^2} + \Omega \\ N_{\phi\phi} &= \frac{\partial^2 F}{\partial x^2} + \Omega \end{aligned}\right\} \qquad (10\,a,b)$$

where we may write

$$N_{\phi\phi} = \nabla^2 F - \frac{1}{x}\frac{\partial F}{\partial x} - \frac{1}{x^2}\frac{\partial^2 F}{\partial \phi^2} + \Omega \qquad (73)$$

From equations (42), (43) and (45)

$$\nabla^2 F = \frac{tE}{R}\sum_{n=0}^{n=\infty}\Big(C_{1n}\,\mathrm{ber}_n\,\xi + C_{2n}\,\mathrm{bei}_n\,\xi + C_{3n}\,\mathrm{ker}_n\,\xi + C_{4n}\,\mathrm{kei}_n\,\xi\Big)\cos n\phi + \nabla^2 F_{p,2} \qquad (74)$$

From equation (52)

$$\frac{1}{x}\frac{\partial F}{\partial x} = \frac{tE}{R}\sum_{n=0}^{n=\infty}\Big(C_{1n}\frac{\mathrm{bei}_n'\,\xi}{\xi} - C_{2n}\frac{\mathrm{ber}_n'\,\xi}{\xi} + C_{3n}\frac{\mathrm{kei}_n'\,\xi}{\xi} - C_{4n}\frac{\mathrm{ker}_n'\,\xi}{\xi}\Big)\cos n\phi + \frac{b_o}{\ell^2\xi^2} + \frac{1}{\ell^2\xi}F_{p,2}' + \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n\Big(C_{6n}\,\xi^{n-2} - C_{8n}\,\xi^{-(n+2)}\Big)\cos n\phi \qquad (75)$$

$$\frac{1}{x^2}\frac{\partial^2 F}{\partial \phi^2} = \frac{tE}{R}\sum_{n=0}^{n=\infty}\frac{n^2}{\xi^2}\Big(C_{2n}\,\mathrm{ber}_n\,\xi - C_{1n}\,\mathrm{bei}_n\,\xi + C_{4n}\,\mathrm{ker}_n\,\xi - C_{3n}\,\mathrm{kei}_n\,\xi\Big)\cos n\phi + \frac{1}{\ell^2\xi^2}\frac{\partial^2 F_{p,2}}{\partial\phi^2} - \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n^2\Big(C_{6n}\,\xi^{n-2} + C_{8n}\,\xi^{-(n+2)}\Big)\cos n\phi \qquad (76)$$

4. Shear Resultant, $N_{x\phi}$

$$N_{x\phi} = -\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial \phi}\right) \qquad (10c)$$

or

$$N_{x\phi} = \frac{1}{x^2}\frac{\partial F}{\partial \phi} - \frac{1}{x}\frac{\partial^2 F}{\partial \phi \partial x} \qquad (77)$$

From equation (52)

$$\frac{1}{x^2}\frac{\partial F}{\partial \phi} = \frac{tE}{R}\sum_{n=0}^{n=\infty}\frac{n}{\xi^2}\Big(C_{2n}\,\mathrm{ber}_n\xi - C_{1n}\,\mathrm{bei}_n\xi + C_{4n}\,\mathrm{Ker}_n\xi - C_{3n}\,\mathrm{Kei}_n\xi\Big)\sin n\phi + \frac{1}{\ell^2\xi^2}\frac{\partial F_{P,2}}{\partial \phi} - \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n\Big(C_{6n}\,\xi^{n-2} + C_{8n}\,\xi^{-(n+2)}\Big)\sin n\phi \qquad (78)$$

$$\frac{1}{x}\frac{\partial^2 F}{\partial \phi \partial x} = \frac{tE}{R}\sum_{n=0}^{n=\infty} n\Big(C_{2n}\frac{\mathrm{ber}_n'\xi}{\xi} - C_{1n}\frac{\mathrm{bei}_n'\xi}{\xi} + C_{4n}\frac{\mathrm{Ker}_n'\xi}{\xi} - C_{3n}\frac{\mathrm{Kei}_n'\xi}{\xi}\Big)\sin n\phi + \frac{1}{\ell^2\xi}\frac{\partial^2 F_{P,2}}{\partial \phi \partial \xi} - \frac{1}{\ell^2}\sum_{n=1}^{n=\infty} n^2\Big(C_{6n}\,\xi^{n-2} - C_{8n}\,\xi^{-(n+2)}\Big)\sin n\phi \qquad (79)$$

5. Shear Resultant, Q_x

From equation (2b)

$$Q_x = \frac{1}{x}\left(\frac{\partial (x M_{xx})}{\partial x} + \frac{\partial M_{x\phi}}{\partial \phi} - M_{\phi\phi}\right) \qquad (80)$$

Substituting equations (5) and (7) into (80) we obtain

$$Q_x = -D\,\frac{\partial(\nabla^2 w)}{\partial x} \qquad (81)$$

This result is analogous to that obtained in plate theory.

From equation (67)

$$\frac{\partial(\nabla^2 \omega)}{\partial x} = \frac{1}{\ell^3}\sum_{n=0}^{n=\infty}\Big(C_{2n}\,ber_n'\xi - C_{1n}\,bei_n'\xi + C_{4n}\,ker_n'\xi$$

$$- C_{3n}\,kei_n'\xi\Big)\cos n\phi + \frac{1}{\ell}\left(\nabla^2\omega_p\right)' \qquad (82)$$

6. Shear Resultant, Q_ϕ

From equation (2c)

$$Q_\phi = \frac{1}{x}\left(\frac{\partial(x M_{x\phi})}{\partial x} + \frac{\partial M_{\phi\phi}}{\partial\phi} + M_{x\phi}\right) \qquad (83)$$

Substituting equations (5) and (7) into (83) we obtain

$$Q_\phi = -D\,\frac{\partial(\nabla^2\omega)}{x\,\partial\phi} \qquad (84)$$

This result is analogous to that obtained in plate theory.

From equation (67)

$$\frac{1}{x}\frac{\partial(\nabla^2\omega)}{\partial\phi} = \frac{1}{\ell^3}\sum_{n=0}^{n=\infty}\frac{n}{\xi}\Big(C_{1n}\,bei_n\xi - C_{2n}\,ber_n\xi + C_{3n}\,kei_n\xi$$

$$- C_{4n}\,ker_n\xi\Big)\sin n\phi + \frac{1}{\ell\xi}\,\frac{\partial(\nabla^2\omega_p)}{\partial\phi} \qquad (85)$$

7. Horizontal Reaction

$$P_h = N_{xx}\cos\phi + Q_x\sin\phi \approx N_{xx} + \frac{x}{R}Q_x \approx N_{xx} \qquad (86)$$

It should be noted that this simplification is implied by the approximations made to establish all preceding results.

8. Vertical Reaction

$$P_V = N_{xx}\sin\phi - Q_x\cos\phi \approx \frac{x}{R}N_{xx} - Q_x \qquad (87)$$

Using equations (10a), (75), (76), (81) and (82) we have

$$P_V = \frac{\ell t E}{R^2}\sum_{n=0}^{n=\infty}\frac{n^2}{\xi}\left(C_{2n}\,\mathrm{ber}_n\,\xi - C_{1n}\,\mathrm{bei}_n\,\xi + C_{4n}\,\mathrm{Ker}_n\,\xi - C_{3n}\,\mathrm{Kei}_n\,\xi\right)\cos n\phi + \frac{\ell^3 t E}{R^2}\frac{\partial(\nabla^2 w_p)}{\partial\xi} + \frac{b_o}{\ell R\xi}$$

$$-\frac{1}{\ell R}\sum_{n=1}^{n=\infty} n\left[(n-1)C_{6n}\,\xi^{n-1} + (n+1)C_{8n}\,\xi^{-(n+1)}\right]\cos n\phi$$

$$+\frac{1}{\ell R}\left(\frac{1}{\xi}\frac{\partial^2 F_{p,2}}{\partial\phi^2} + F_{p,2}'\right) + \frac{\ell\xi}{R}\Omega \qquad (88)$$

9. Tangential (Circumferential) Reaction

$$P_t = N_{x\phi} = -\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial\phi}\right) \qquad (10c)$$

$N_{x\phi}$ is given by equations (77), (78) and (79).

SECTION V

Expressions for Stress Resultants and Reactions for the Rotationally Symmetric Case.

Primes indicate differentiation with respect to ξ. The abbreviation $\xi = \frac{x}{\ell}$ is used. Temperature effects are not considered.

1. Bending Moments.

From equations (66a, b)

$$\left.\begin{aligned} M_{xx} &= -D\left[\nabla^2 w - (1-\nu)\frac{1}{x}\frac{dw}{dx}\right] \\ M_{\phi\phi} &= -D\left[\nu\nabla^2 w + (1-\nu)\frac{1}{x}\frac{dw}{dx}\right] \end{aligned}\right\} \qquad (89a,b)$$

From equation (67)

$$\nabla^2 w = \frac{1}{\ell^2}\left(C_{20}\,\mathrm{ber}\,\xi - C_{10}\,\mathrm{bei}\,\xi + C_{40}\,\mathrm{Ker}\,\xi - C_{30}\,\mathrm{kei}\,\xi\right) + \nabla^2 w_p \qquad (90)$$

From equation (68) and (60)

$$\frac{1}{x}\frac{dw}{dx} = \frac{1}{\ell^2}\left(C_{10}\frac{\mathrm{ber}'\xi}{\xi} + C_{20}\frac{\mathrm{bei}'\xi}{\xi} + C_{30}\frac{\mathrm{Ker}'\xi}{\xi} + C_{40}\frac{\mathrm{Kei}'\xi}{\xi}\right) + \frac{1}{\ell^2\xi}\left(w_p\right)' \qquad (91)$$

2. Torsional Moment, $M_{x\phi}$

From equation (70)

$$M_{x\phi} = 0 \qquad (92)$$

3. Direct Stress Resultants

From equations (10a, b) and (73)

$$\left.\begin{aligned} N_{xx} &= \frac{1}{x}\frac{dF}{dx} + \Omega \\ N_{\phi\phi} &= \nabla^2 F - \frac{1}{x}\frac{dF}{dx} + \Omega \end{aligned}\right\} \qquad (93a,b)$$

From equation (74)

$$\nabla^2 F = \frac{tE}{R}\left(C_{10}\,\mathrm{ber}\,\xi + C_{20}\,\mathrm{bei}\,\xi + C_{30}\,\mathrm{Ker}\,\xi + C_{40}\,\mathrm{Kei}\,\xi\right) + \nabla^2 F_{p,2} \qquad (94)$$

From equation (75)

$$\frac{1}{x}\frac{dF}{dx} = \frac{tE}{R}\left(C_{10}\frac{\mathrm{bei}'\xi}{\xi} - C_{20}\frac{\mathrm{ber}'\xi}{\xi} + C_{30}\frac{\mathrm{Kei}'\xi}{\xi} - C_{40}\frac{\mathrm{Ker}'\xi}{\xi}\right) + \frac{b_0}{\ell^2\xi^2} + \frac{1}{\ell^2\xi}\frac{dF_{p,2}}{d\xi} \qquad (95)$$

4. Shear Resultant, $N_{x\phi}$

From equation (77)

$$N_{x\phi} = 0 \qquad (96)$$

5. Shear Resultant, Q_x

From equation (81)

$$Q_x = -D\,\frac{d(\nabla^2 w)}{dx} \qquad (81)$$

From equation (82)

$$\frac{d(\nabla^2 \omega)}{dx} = \frac{1}{\ell^3}\Big(C_{20}\,\mathrm{ber}'\xi - C_{10}\,\mathrm{bei}'\xi + C_{40}\,\mathrm{Ker}'\xi - C_{30}\,\mathrm{kei}'\xi\Big) + \frac{1}{\ell}\left(\nabla^2 \omega_P\right)' \tag{97}$$

6. Shear Resultant, $Q\phi$

From equation (84)

$$Q_\phi = 0 \tag{98}$$

7. Horizontal Reaction

From equation (86)

$$P_h \approx N_{xx} \tag{86}$$

8. Vertical Reaction

From equation (87)

$$P_v \approx \frac{x}{R} N_{xx} - Q_x \tag{87}$$

From equation (88)

$$P_v = \frac{\ell^3 t E}{R^2}\frac{d(\nabla^2 \omega_P)}{d\xi} + \frac{1}{\ell R}\frac{dF_{P,2}}{d\xi} + \frac{b_o}{\ell R \xi} + \frac{\ell \xi}{R}\Omega \tag{99}$$

9. Tangential Reaction

From equation (10c)

$$P_t = 0 \tag{100}$$

SECTION VI

Expressions for the Determination of the Displacements.

1. Meridional Displacement, U_t

This quantity is determined from

$$\varepsilon_x = \frac{\partial u_t}{\partial x} + \frac{w}{R} = \frac{1}{tE}\left(N_{xx} - \nu N_{\phi\phi}\right) + \beta \Delta T_m \tag{101}$$

Using equations (10) we have

$$\frac{\partial U_t}{\partial x} = \frac{1}{tE}\left[\nabla^2 F - (1+\nu)\frac{\partial^2 F}{\partial x^2} + (1-\nu)\Omega\right] + \beta \Delta T_m - \frac{w}{R} \tag{102}$$

Using equation (22) we have

$$\frac{\partial U_t}{\partial x} = \frac{\Phi}{R} - \frac{(1+\nu)}{tE}\frac{\partial^2 F}{\partial x^2} \tag{103}$$

Integrating we obtain

$$U_t = \frac{1}{R}\int \Phi \, dx - \frac{(1+\nu)}{tE}\frac{\partial F}{\partial x} + f(\phi) \tag{104}$$

where $f(\phi)$ is a function of ϕ only

Noting that Φ is given as

$$\Phi = A_o + B_o \ln\frac{x}{\ell} + \sum_{n=1}^{n=\infty}\left(C_{5n}\frac{x^n}{\ell^n} + C_{7n}\frac{\ell^n}{x^n}\right)\cos n\phi \tag{105}$$

and using equation (52) we have in terms of ξ

$$U_t = A_o\frac{\ell\xi}{R} + B_o\frac{\ell}{R}\left(\xi \ln \xi - \xi\right) - \frac{(1+\nu)}{tE\ell}\frac{\partial F_{P,2}}{\partial \xi}$$
$$+ \frac{\ell}{R}\sum_{n=1}^{n=\infty}\left(C_{5n}\frac{\xi^{n+1}}{(n+1)} - C_{7n}\frac{\xi^{-(n-1)}}{(n-1)}\right)\cos n\phi$$

$$+\frac{(1+\nu)\ell}{R}\sum_{n=0}^{n=\infty}\Big(C_{2n}\,\mathrm{ber}_n'\,\xi - C_{1n}\,\mathrm{bei}_n'\,\xi + C_{4n}\,\mathrm{ker}_n'\,\xi$$

$$-C_{3n}\,\mathrm{kei}_n'\,\xi\Big)\cos n\phi - \frac{(1+\nu)\,b_0}{tE\ell\xi} + f(\phi)$$

$$-\frac{(1+\nu)}{tE\ell}\sum_{n=1}^{n=\infty} n\Big(C_{6n}\,\xi^{\,n-1} - C_{8n}\,\xi^{-(n+1)}\Big)\cos n\phi \qquad (106)$$

2. Circumferential Displacement, V

This quantity is determined from

$$\varepsilon_\phi = \frac{1}{X}\frac{\partial V}{\partial \phi} + \frac{U_t}{X} + \frac{\omega}{R}$$

$$= \frac{1}{tE}\Big(N_{\phi\phi} - \nu N_{xx}\Big) + \beta\Delta T_m \qquad (107)$$

Using equations (10) we have

$$\frac{\partial V}{\partial \phi} = \frac{X}{tE}\Big[\nabla^2 F - (1+\nu)\Big(\frac{1}{X}\frac{\partial F}{\partial X} + \frac{1}{X^2}\frac{\partial^2 F}{\partial \phi^2}\Big) + (1-\nu)\Omega\Big]$$

$$+\Big(\beta\Delta T_m - \frac{\omega}{R}\Big)X - U_t \qquad (108)$$

Using equations (22) and (104) we have

$$\frac{\partial V}{\partial \phi} = \frac{X}{R}\Phi - \frac{1}{R}\int \Phi\,dx - \frac{(1+\nu)}{tEX}\frac{\partial^2 F}{\partial \phi^2} - f(\phi) \qquad (109)$$

Integrating we obtain

$$V = \frac{1}{R}\int \Phi\,X\,d\phi - \frac{1}{R}\iint \Phi\,dx\,d\phi$$

$$-\frac{(1+\nu)}{tEX}\frac{\partial F}{\partial \phi} - \int f(\phi)\,d\phi + g(x) \qquad (110)$$

where g (x) is a function of x only.

Using equations (105) and (52) in equation (110) we have after carrying out the indicated operations, in terms of ξ,

$$V = \frac{(1+\nu)\ell}{R}\sum_{n=0}^{n=\infty} n\left(C_{1n}\frac{\mathrm{bei}_n \xi}{\xi} - C_{2n}\frac{\mathrm{ber}_n \xi}{\xi} + C_{3n}\frac{\mathrm{kei}_n \xi}{\xi} - C_{4n}\frac{\mathrm{ker}_n \xi}{\xi}\right)\sin n\phi - \frac{(1+\nu)}{tE\ell\xi}\frac{\partial F_{P,2}}{\partial \phi} + B_o\frac{\phi\ell\xi}{R}$$
$$+ \frac{\ell}{R}\sum_{n=1}^{n=\infty}\left(C_{5n}\frac{\xi^{n+1}}{(n+1)} + C_{7n}\frac{\xi^{-(n-1)}}{(n-1)}\right)\sin n\phi$$
$$+ \frac{(1+\nu)}{tE\ell}\sum_{n=1}^{n=\infty} n\left(C_{6n}\xi^{n-1} + C_{8n}\xi^{-(n+1)}\right)\sin n\phi$$
$$- \int f(\phi)\,d\phi + g(x) \qquad (111)$$

3. Horizontal Displacement, U_h

This quantity is determined from

$$\varepsilon_\phi = \frac{u_t}{x} + \frac{w}{R} + \frac{1}{x}\frac{\partial V}{\partial \phi} = \frac{u_h}{x} + \frac{1}{x}\frac{\partial V}{\partial \phi} \qquad (112)$$

whence

$$u_h = u_t + \frac{x}{R}w \qquad (113)$$

Using equations (106) and (51) we have in terms of

$$u_h = \frac{(1+\nu)\ell}{R}\sum_{n=0}^{n=\infty}\left(C_{2n}\mathrm{ber}_n'\xi - C_{1n}\mathrm{bei}_n'\xi + C_{4n}\mathrm{ker}_n'\xi - C_{3n}\mathrm{kei}_n'\xi\right)\cos n\phi - \frac{(1+\nu)}{tE\ell}\frac{\partial F_{P,2}}{\partial \xi}$$

$$+\frac{\ell}{R}\sum_{n=0}^{n=\infty}\xi\Big(C_{1n}\,\mathrm{ber}_n\xi + C_{2n}\,\mathrm{bei}_n\xi + C_{3n}\,\mathrm{ker}_n\xi$$

$$+ C_{4n}\,\mathrm{kei}_n\xi\Big)\cos n\phi + \frac{\ell\xi}{R}\left(w_p - B_o\right) - \frac{(1+\nu)\,b_o}{tE\ell\xi}$$

$$-\frac{\ell}{R}\sum_{n=1}^{n=\infty} n\left[\frac{C_{5n}\,\xi^{n+1}}{(n+1)} + \frac{C_{7n}\,\xi^{-(n-1)}}{(n-1)}\right]\cos n\phi + f(\phi)$$

$$-\frac{(1+\nu)}{tE\ell}\sum_{n=1}^{n=\infty} n\left(C_{6n}\,\xi^{n-1} - C_{8n}\,\xi^{-(n+1)}\right)\cos n\phi \qquad (114)$$

4. Determination of the Functions f (ϕ) and g (x)

These are determined from

$$\gamma_{x\phi} = \frac{N_{x\phi}}{Gt} \qquad (16c)$$

where

$$G = \frac{E}{2(1+\nu)}$$

From equation (3c)

$$\gamma_{x\phi} = x\frac{\partial}{\partial x}\left(\frac{v}{x}\right) + \frac{1}{x}\frac{\partial u_t}{\partial \phi} \qquad (3c)$$

From equation (77)

$$N_{x\phi} = \frac{1}{x^2}\frac{\partial F}{\partial \phi} - \frac{1}{x}\frac{\partial^2 F}{\partial \phi\,\partial x} \qquad (77)$$

Substituting equations (77) and (3c) into equation (16c), using equations (104) and (110) we have, after simplification,

$$0=\frac{\partial}{\partial X}\left(\int\frac{X\Phi}{R}\,d\phi\right)-\int\frac{\Phi}{R}\,d\phi-\frac{1}{X}\int\frac{X}{R}\,\Phi\,d\phi$$

$$+\frac{1}{X}\iint\frac{\Phi}{R}\,dx\,d\phi+\frac{1}{X}\frac{\partial}{\partial\phi}\left(\int\frac{\Phi}{R}\,dx\right)+\frac{\partial g(x)}{\partial X}$$

$$-\frac{g(x)}{X}+\frac{1}{X}\int f(\phi)\,d\phi+\frac{1}{X}\frac{\partial f(\phi)}{\partial\phi} \qquad (115)$$

Substituting equation (105) into equation (115) and carrying out the indicated operations we have

$$0=X\frac{\partial g(x)}{\partial X}-g(x)+\frac{\partial f(\phi)}{\partial\phi}+\int f(\phi)\,d\phi \qquad (116a)$$

or

$$X g(x)'-g(x)=-K \qquad (116b)$$

and

$$f(\phi)'+\int f(\phi)\,d\phi=+K \qquad (116c)$$

with the solutions

$$\left.\begin{aligned} g(x)&=K_1X+K\\ f(\phi)&=K_2\sin\phi+K_3\cos\phi\end{aligned}\right\} \qquad (117a,b)$$

in which K_1, K_2, K_3 and K are arbitrary constants, to be determined from the conditions of restraint.

5. Expressions for Displacements, Rotationally Symmetric Case.

a. Meridional Displacement, U_t

From equation (106), using equation (60), we have

$$U_t = \frac{(1+\nu)\ell}{R}\left(C_{20}\,\mathrm{ber}'\xi - C_{10}\,\mathrm{bei}'\xi + C_{40}\,\mathrm{Ker}'\xi - C_{30}\,\mathrm{Kei}'\xi\right) + \frac{A_0\ell\xi}{R} - \frac{(1+\nu)b_0}{tE\ell\xi} - \frac{(1+\nu)}{tE\ell}\frac{dF_{P,2}}{d\xi} \qquad (118)$$

b. Horizontal Displacement, U_h

From equation (114), using equation (60), we have

$$U_h = \frac{(1+\nu)\ell}{R}\left(C_{20}\,\mathrm{ber}'\xi - C_{10}\,\mathrm{bei}'\xi + C_{40}\,\mathrm{Ker}'\xi - C_{30}\,\mathrm{Kei}'\xi\right) + \frac{\ell\xi}{R}\left(C_{10}\,\mathrm{ber}\,\xi + C_{20}\,\mathrm{bei}\,\xi + C_{30}\,\mathrm{Ker}\,\xi + C_{40}\,\mathrm{Kei}\,\xi\right) - \frac{(1+\nu)}{tE\ell}\frac{dF_{P,2}}{d\xi} + \frac{\xi\ell}{R}\,w_P - \frac{(1+\nu)b_0}{tE\ell\xi} \qquad (119)$$

SECTION VII

Case I. Edges Fully Restrained. Uniform Vertical Loading of Intensity P.

The notation for this case is given in Figure 5.

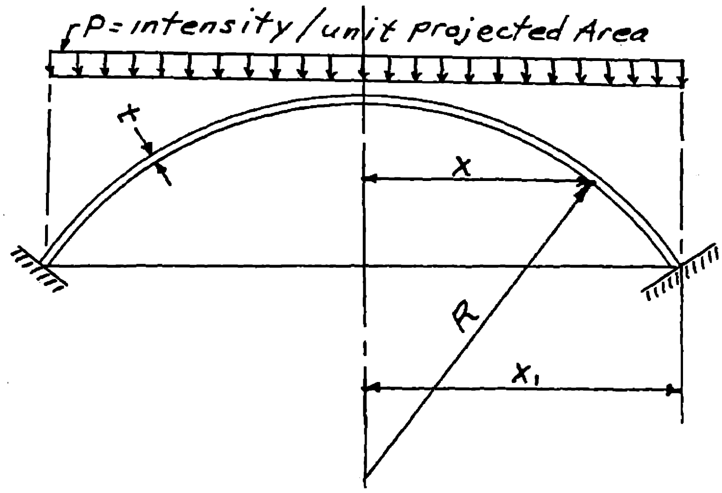


Figure 5. Notation and Loading, Case I.

1. Expressions for W and F.

The loading term $\frac{P}{D}$ given in equation (62) is directly applicable, theoretically, to a uniform radial loading of intensity P. However, it is consistent and permissible as a result of our basic assumption of the shallowness of the shell to consider the term P of equation (62) as equivalent to a uniform vertical loading of intensity P per unit of projected area. The approximation approaches the exact as the dome becomes shallower.

We have from before

$$w = C_{10}\,\mathrm{ber}\,\xi + C_{20}\,\mathrm{bei}\,\xi + C_{30}\,\mathrm{ker}\,\xi + C_{40}\,\mathrm{kei}\,\xi - A_0 + w_p \qquad (61)$$

where, from equation (62)

$$\nabla^2\nabla^2 w_P + \frac{1}{\ell^4} w_P = + \frac{P}{D} \qquad \text{(P is acting downward)} \tag{120}$$

also

$$F = \frac{\ell^2 t E}{R}\left(-C_{20}\,\mathrm{ber}\,\xi + C_{10}\,\mathrm{bei}\,\xi - C_{40}\,\mathrm{ker}\,\xi + C_{30}\,\mathrm{kei}\,\xi\right)$$

$$+ b_0 \ln \xi + F_{P,2} \tag{63}$$

where, from equation (64)

$$\nabla^2 F_{P,2} = \frac{tE}{R} w_P \tag{121}$$

2. Evaluation of w_P and $F_{P,2}$

From equation (120) we find by inspection that

$$w_P = + \frac{P\ell^4}{D} \tag{122}$$

is a particular integral satisfying equation (120).

From equation (121) we find by integration that

$$F_{P,2} = + \frac{tE\ell^4 P}{4DR} X^2 \tag{123a}$$

is the desired solution. Using equation (25) in equation (123a) we have

$$F_{P,2} = + \frac{RP}{4} X^2 = + \frac{RP\ell^2}{4} \xi^2 \tag{123b}$$

3. Boundary Conditions

See Figure 5.

at $x = 0$, $\xi = 0$:

$$\left\{\begin{array}{l} W \text{ is finite} \\ \text{stresses are finite} \\ \lim_{x\to 0}(2\pi x Q_x) \to 0 \end{array}\right. \qquad \begin{array}{r} (124) \\ (125) \\ (126) \end{array}$$

at $x = x_1$, $\xi = \xi_1$:

$$\left\{\begin{array}{l} W = 0 \\ \frac{dw}{dx} = 0 \\ U_h = 0 \end{array}\right. \qquad \begin{array}{r} (127) \\ (128) \\ (129) \end{array}$$

4. Equations for Determination of the Constants.

Application of condition (124) to equation (61), noting that ker ξ at $\xi = 0$ is equal to infinity results in

$$C_{30} = 0 \tag{130}$$

Application of condition (125) to equations (93), (94) and (95) results in

$$b_0 = 0 \tag{131}$$

Application of condition (126) using equations (81) and (97) noting that $\lim_{\xi\to 0}(C_{40}\,\xi\,\text{Ker}'\xi) \to -C_{40}$, results in

$$C_{40} = 0 \tag{132}$$

Letting

$$G_0 = A_0 - W_P \tag{133}$$

and applying condition (127) to equation (61) we have

$$0 = C_{10}\,\text{ber}\,\xi_1 + C_{20}\,\text{bei}\,\xi_1 - G_0 \tag{134}$$

Applying condition (128) to equation (61) we have

$$0 = C_{10}\,\mathrm{ber}'\xi_1 + C_{20}\,\mathrm{bei}'\xi_1 \tag{135}$$

Applying condition (129) to equation (119), using equations (123b) and (122) we have as the result

$$-\frac{(1-\nu)R^2p}{2Et} = C_{10}\left(\mathrm{bei}''\xi_1 - \frac{\nu\,\mathrm{bei}'\xi_1}{\xi_1}\right) - C_{20}\left(\mathrm{ber}''\xi_1 - \frac{\nu\,\mathrm{ber}'\xi_1}{\xi_1}\right) \tag{136}$$

Solution of equations (134), (135) and (136) suffice to determine the three constants C_{10}, C_{20}, and C_{s_0} . These constants have the dimension of a length. To transform the constants to a dimensionless form the following transformations are made

$$\left.\begin{aligned} K_1 &= C_{10}\left(\frac{2Et}{R^2p}\right) \\ K_2 &= C_{20}\left(\frac{2Et}{R^2p}\right) \\ K_3 &= C_{s_0}\left(\frac{2Et}{R^2p}\right) \end{aligned}\right\} \tag{137}$$

Using equations (137) and (37), equations (134), (135) and (136) become

$$K_3 = K_1\,\mathrm{ber}\,\xi_1 + K_2\,\mathrm{bei}\,\xi_1 \tag{138}$$

$$0 = K_1\,\mathrm{ber}'\xi_1 + K_2\,\mathrm{bei}'\xi_1 \tag{139}$$

$$-(1-\nu) = K_1\left[\text{ber}\,\xi_1 - (1+\nu)\frac{\text{bei}'\xi_1}{\xi_1}\right] + K_2\left[\text{bei}\,\xi_1 + (1+\nu)\frac{\text{ber}'\xi_1}{\xi_1}\right] \qquad (140)$$

Solution of equations (138), (139) and (140) yield K_1 , K_2 and K_3 in dimensionless form.

5. Reduced expressions for W and F

Utilizing the above results and equations (137) we have

$$\left(\frac{2Et}{R^2p}\right)w = K_1\,\text{ber}\,\xi + K_2\,\text{bei}\,\xi - K_3 \qquad (141)$$

Similarly, using equations (137) and (123b), equation (63) reduces to

$$F = \frac{\ell^2 Rp}{2}\left(-K_2\,\text{ber}\,\xi + K_1\,\text{bei}\,\xi + \frac{\xi^2}{2}\right) \qquad (142)$$

6. Expressions for Stresses, Displacements and Reactions

Using the results of Section V, we have as follows

(1) Bending Stresses

$$\sigma_{x,b} = \pm\frac{6M_{xx}}{t^2} = \mp\frac{6D}{t^2\ell^2}\left[\frac{d^2w}{d\xi^2} + \frac{\nu}{\xi}\frac{dw}{d\xi}\right] \qquad (143a)$$

$$\sigma_{\phi,b} = \pm\frac{6M_{\phi\phi}}{t^2} = \mp\frac{6D}{t^2\ell^2}\left[\frac{1}{\xi}\frac{dw}{d\xi} + \nu\frac{d^2w}{d\xi^2}\right] \qquad (143b)$$

where the ambiguous minus and plus signs refer to the upper and lower surfaces of the shell, respectively.

Now, noting that

$$D = \frac{Et^3}{12(1-\nu^2)} \qquad \ell^2 = \frac{tR}{\sqrt{12(1-\nu^2)}} \tag{144}$$

and using equations (144), (141) and (37) in equation (143) we obtain as the final result

$$\left(\frac{2t}{RP}\right)\sigma_{x,b} = \mp\sqrt{\frac{3}{(1-\nu^2)}}\left[-K_1\left(\operatorname{bei}\xi + (1-\nu)\frac{\operatorname{ber}'\xi}{\xi}\right) + K_2\left(\operatorname{ber}\xi - (1-\nu)\frac{\operatorname{bei}'\xi}{\xi}\right)\right] \tag{145a}$$

$$\left(\frac{2t}{RP}\right)\sigma_{\phi,b} = \mp\sqrt{\frac{3}{(1-\nu^2)}}\left[K_1\left((1-\nu)\frac{\operatorname{ber}'\xi}{\xi} - \nu\operatorname{bei}\xi\right) + K_2\left((1-\nu)\frac{\operatorname{bei}'\xi}{\xi} + \nu\operatorname{ber}\xi\right)\right] \tag{145b}$$

(2) Direct Stresses

$$\sigma_{x,d} = \frac{N_{xx}}{t} = \frac{1}{t\ell^2\xi}\frac{dF}{d\xi} \qquad \sigma_{\phi,d} = \frac{N_{\phi\phi}}{t} = \frac{1}{t\ell^2}\frac{d^2F}{d\xi^2} \tag{146a,b}$$

Using equations (142) and (37) in equation (146) we obtain as the final result

$$\left(\frac{2t}{RP}\right)\sigma_{x,d} = K_1\frac{\operatorname{bei}'\xi}{\xi} - K_2\frac{\operatorname{ber}'\xi}{\xi} + 1 \tag{147a}$$

$$\left(\frac{2t}{RP}\right)\sigma_{\phi,d} = K_1\left(ber\,\xi - \frac{bei'\xi}{\xi}\right) + K_2\left(bei\,\xi + \frac{ber'\xi}{\xi}\right) + 1 \qquad (147b)$$

(3) Shear Stress

$$\tau_x = \frac{Q_x}{t} = -\frac{D}{t\ell}\frac{d(\nabla^2 \omega)}{d\xi} = -\frac{D}{t\ell^3}\left(C_{20}\,ber'\xi - C_{10}\,bei'\xi\right) \qquad (148)$$

Using equations (144) and (137) in equation (148) we obtain as the final result

$$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x = \frac{1}{\sqrt[4]{12(1-\nu^2)}}\left(K_1\,bei'\xi - K_2\,ber'\xi\right) \qquad (149)$$

(4) Vertical Displacement

$$\left(\frac{2Et}{R^2P}\right)\omega = K_1\,ber\,\xi + K_2\,bei\,\xi - K_3 \qquad (141)$$

(5) Horizontal Reaction

$$P_h = \left(N_{xx}\right)_{\xi=\xi_1} = t\left(\sigma_{x,d}\right)_{\xi=\xi_1} \qquad (150)$$

where $\left(\sigma_{x,d}\right)_{\xi=\xi_1}$ is the value of $\sigma_{x,d}$ at $x = x_1, \xi = \xi_1$.

(6) Vertical Reaction

$$P_v = \frac{1}{\ell R}\frac{dF_{P,2}}{d\xi} \qquad (151)$$

Using equation (123b) in equation (151) we obtain

$$P_v = +\frac{\ell\xi}{2}P \qquad (152)$$

This same result may be obtained from the vertical statical equilibrium condition.

7. Calculation of K_1, K_2 and K_3

From equations (138), (139) and (140) by determinants we have

$$K_1 = \frac{+(1-\nu)\,bei'\xi_1}{(1+\nu)\left[\left(\frac{ber'\xi_1}{\xi_1}\right)^2+\left(\frac{bei'\xi_1}{\xi_1}\right)^2\right]+ber'\xi_1\,bei\,\xi_1-ber\,\xi_1\,bei'\xi_1} \qquad (153)$$

$$K_2 = \frac{-(1-\nu)\,ber'\xi_1}{(1+\nu)\left[\left(\frac{ber'\xi_1}{\xi_1}\right)^2+\left(\frac{bei'\xi_1}{\xi_1}\right)^2\right]+ber'\xi_1\,bei\,\xi_1-ber\,\xi_1\,bei'\xi_1} \qquad (154)$$

$$K_3 = \frac{-(1-\nu)(ber'\xi_1\,bei\,\xi_1-ber\,\xi_1\,bei'\xi_1)}{(1+\nu)\left[\left(\frac{ber'\xi_1}{\xi_1}\right)^2+\left(\frac{bei'\xi_1}{\xi_1}\right)^2\right]+ber'\xi_1\,bei\,\xi_1-ber\,\xi_1\,bei'\xi_1} \qquad (155)$$

These can be rendered more concise by use of the polar form of the Bessel function. Using relationships given in Appendix A, equations (153), (154) and (155) become

$$K_1 = \frac{+(1-\nu)\,bei'\xi_1}{M_1(\xi_1)\left[\frac{(1+\nu)}{\xi_1^2}M_1(\xi_1)-M_0(\xi_1)\sin\left(\theta_1-\theta_0-\frac{\pi}{4}\right)\right]} \qquad (156)$$

$$K_2 = \frac{-(1-\nu)\,ber'\xi_1}{M_1(\xi_1)\left[\frac{(1+\nu)}{\xi_1^2}M_1(\xi_1)-M_0(\xi_1)\sin\left(\theta_1-\theta_0-\frac{\pi}{4}\right)\right]} \qquad (157)$$

$$K_3 = \frac{+(1-\nu)\,M_0(\xi_1)\sin\left(\theta_1-\theta_0-\frac{\pi}{4}\right)}{\frac{(1+\nu)}{\xi_1^2}M_1(\xi_1)-M_0(\xi_1)\sin\left(\theta_1-\theta_0-\frac{\pi}{4}\right)} \qquad (158)$$

Substitution of the appropriate values from Table (A-1) of Appendix A into equations (153), (154) and (155) or from Table (A-5) of Appendix A into equations (156), (157) and (158) will yield the values for K_1, K_2 and K_3 required for the determination of the state of stress and displacement. These values have been computed and are tabulated in Table 1.

8. Determination of the Bending, Direct and Shear Stress Intensities.

These are obtained by substitution of the constant values for the pertinent value of ξ_1, tabulated in Table 1, into formulae (145), (147) and (149). These stresses have been computed for ξ_1 = 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 10.0, 15.0, 20.0 and 25.0. They are tabulated in the tables of this section and except for the shearing stress intensity have been plotted with respect to $\frac{x}{x_1}$ or $\frac{\xi}{\xi_1}$ on Graphs A, B, C and D. It is to be noted that σ_x, d, etc. are multiplied by $\left(\frac{2t}{PR}\right)$ and the shearing stress τ_x is multiplied by $\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}$.

The shearing stresses were not plotted inasmuch as they are of lower order of magnitude than the direct and bending stresses. This is consistent with the assumption made in the derivation of the theory and may be readily verified by a direct comparison of the numerical results found in the various tables of this section.

9. Determination of the Combined Bending and Direct Stress Intensities at the Upper and Lower Surfaces of the Dome.

These are obtained in the usual way by addition or substraction of the bending and direct stress intensities given in the tables of this section. These resultant combined values are tabulated herein. Likewise, as in Article 8 above they have been plotted with respect to $\frac{x}{x_1}$ or $\frac{\xi}{\xi_1}$ on Graphs E, F, G and H.

10. Maximum Tensile and Compressive Stress Intensities, Radial or Circumferential Direction.

From the theory of elasticity, the radial and tangential planes are planes of principal stress. A knowledge of the maximum stresses on these planes and their location are obviously of first importance to a design engineer in determining whether a proposed design will be stressed within the allowable stresses of the material used.

A study of the tables of combined stresses leads to the compilation of the maximum tensile and compressive stress intensities, radial or circumferential direction, which are tabulated in Table 34 and plotted with respect to ξ_1, on Graph I. The points at which the maximum stress values occur; namely, the value of ξ and the surface at which it acts, have been tabulated also in Table 34.

11. Determination of the Vertical Displacements.

These are obtained by substitution of the constant values found in Table 1 for the pertinent value of ξ_1, into formula (141). These results are tabulated in the tables of this section and are plotted with respect to $\frac{x}{x_1}$ or $\frac{\xi}{\xi_1}$ on Graph J.

12. Vertical Displacement at the Center of the Shell.

These values are obtained from the results of Article 11 above for the value $x = \xi = 0$ for the different boundary values and are plotted with respect to ξ_1 on Graph K. It is to be noted that W_c is multiplied by $\left(\frac{2Et}{R^2p}\right)$

It is desired to show the variation of the vertical deflection at the center, W_c, with the radius of the shell, R, for given values of t and x_1. To do this we proceed as follows: let

$$\left(\frac{2Et}{R^2p}\right) w_c = T \qquad (a)$$

where T represents the value of $\left(\frac{2Et}{R^2p}\right) w_c$ for a particular value of ξ_1 and is found on Graph K. Using the relationship

$$\xi_1 = \frac{\sqrt[4]{12(1-\nu^2)}\, X_1}{\sqrt{tR}} \tag{b}$$

in equation (a), we have

$$\left(\frac{E}{P}\right) w_c = \frac{R^2}{2t} T = \frac{6(1-\nu^2)\, X_1^{\,4}}{\xi_1^{\,4}\, t^3} T \tag{c}$$

Similarly from equation (b)

$$R = \frac{\sqrt{12(1-\nu^2)}\, X_1^{\,2}}{\xi_1^{\,2}\, t} \tag{d}$$

For the values $v = 0.2$, $t = 2"$, $x_1 = 20'$, equations(c) and (d) become

$$\left(\frac{E}{P}\right) w_c = \frac{1.991 \times 10^8}{\xi_1^{\,4}} T \text{ feet} \tag{e}$$

$$R = \frac{8.139 \times 10^3}{\xi_1^{\,2}} \text{ feet} \tag{f}$$

$\left(\frac{E}{P}\right) W_c$ and R have been tabulated in Table 35 for values of ξ_1 ranging from 0 to 25 using equations (e) and (f) and Graph K. $\left(\frac{E}{P}\right) W_c$ is plotted with respect to R on Graph K-a.

For ξ_1 equal to zero we have the case of the circular plate under uniform lateral load with clamped edges. Using the formula for the center deflection for this case given in Timoshenko "Theory of Plates and Shells", McGraw-Hill, 1940, page 60, we obtain $\left(\frac{E}{P}\right) W_c = 6.22 \times 10^5$. This value is approached by the flat shell as $\xi_1 \to 0$.

It is also desired to show the variation of the vertical deflection at the center, W_c, with the Bulge Ratio, $\frac{h}{t}$, for a given ratio $\frac{x_1}{t}$. To do this we proceed as follows; substituting relationship (b) into equation (a) we obtain

$$\left(\frac{E}{tP}\right)w_c = \frac{6(1-\nu^2)}{\xi_1^{\,4}}\left(\frac{x_1}{t}\right)^4 T \tag{g}$$

From equation (B-7), Appendix B, we have

$$B.R. = \frac{h}{t} = \frac{\xi_1^{\,2}}{4\sqrt{3(1-\nu^2)}} \tag{h}$$

For the values, $\nu = 0.2$, $\left(\frac{x_1}{t}\right) = 120$, equations (g) and (h) become

$$\left(\frac{E}{tP}\right)w_c = \frac{1.095\times10^9}{\xi_1^{\,4}}\,T \tag{i}$$

$$B.R. = \frac{h}{t} = 0.1473\,\xi_1^{\,2} \tag{j}$$

$\left(\frac{E}{tP}\right)W_c$ and $\frac{h}{t}$ have been computed according to equations (i) and (j) and are tabulated in Table 36. $\left(\frac{E}{tP}\right)W_c$ is plotted with respect to $\frac{h}{t}$ on Graph K-b. The values of T were taken from Graph K. For $\xi_1 = 0$; namely, the case of the flat plate, $\left(\frac{E}{tP}\right)W_c$ was determined by the formula given in Timoshenko "Theory of Plates and Shells", McGraw-Hill, 1940, page 60.

13. The Limiting Case of the Flat Plate, $\xi_1 = 0$, $R = \infty$.

For this case the fundamental differential equations (17) and (20) resolve themselves into the two well-known equations of plane stress and plate theory.

$$\nabla^2\nabla^2 F = 0 \tag{159a}$$

$$\nabla^2\nabla^2 w = \frac{P}{D} \tag{159b}$$

For an unloaded plate in plane stress equation (159a) leads to the result that all direct stresses are zero. Equation (159b) leads to the same results for bending stresses and vertical deflections given by equations (62), (63) and (64) and Figure 29 on pp. 60-61 of Timoshenko "Theory of Plates and Shells", McGraw-Hill, New York, 1940.

14. Comments on the Graphs.

It is easily shown that the membrane stress condition for a spherical shell of constant thickness loaded uniformly is given by

$$\left.\begin{aligned} \left(\frac{2t}{PR}\right)\sigma_{x,d} &= \left(\frac{2t}{PR}\right)\sigma_{\phi,d} = 1 \\ \left(\frac{2t}{PR}\right)\sigma_{x,b} &= \left(\frac{2t}{PR}\right)\sigma_{\phi,b} = 0 \end{aligned}\right\} \quad (160\,a,b)$$

Furthermore, as shown in Appendix B, the parameter ξ_1 measures the curvature of the shell for a given span, increasing with increasing curvature. For a flat plate $\xi_1 = 0$.

Graphs A through H, inclusive, verify the well-known shell action for as ξ_1 increases the area of the shell in a membrane stress condition increases. Furthermore, the effect of the edge restraint becomes more and more local as ξ_1 increases.

It may be shown by an examination of the equations for their determination that at the outer edge the following holds true:

$$\left.\begin{aligned} \sigma_{\phi,d} &= \nu\,\sigma_{x,d} \\ \sigma_{\phi,b} &= \nu\,\sigma_{x,b} \end{aligned}\right\} \quad (160\,c,d)$$

This is borne out by the tabulated and plotted results.

The apparent discontinuity in the curve for maximum circumferential tensile stress on Graph I is explained by Table 34 wherein we note the point of maximum stress changes abruptly from the lower surface center to the upper surface outer edge.

Graph J illustrates a phenomena common to Graphs A to H inclusive and Graph J. On Graph J, for example, we notice the $\xi_1 = 3.0$ curve intersecting the $\xi_1 = 10.0$ curve at $\frac{x}{X_1} = 0.20$ and as a result of such intersections we find the ordinates at some point such as the center increasing to some maximum value as ξ_1 increases and then diminishing as ξ_1 continues to increase. This is illustrated by Graph K. Physically, for a given X_1, t and v we would expect the ordinates for stresses and displacements (at a given $\frac{x}{X_1}$) to increase (or decrease on the other hand) with increasing ξ_1; that is, the ξ_1 curves would not intersect. The explanation is to be found in the dimensionless form of ordinate employed.

We may write for Graph J

$$\left(\frac{2Et}{R^2 p}\right) w = T \qquad (k)$$

where T is the value of the ordinate plotted on Graph J. Using relationship (b) in equation (k) we have

$$\left(\frac{E}{P}\right) w = \frac{6(1-\nu^2)X_1^4}{\xi_1^4 t^3} T \qquad (\ell)$$

or for given values of X_1, t and v

$$\left(\frac{E}{P}\right) w \sim \frac{T}{\xi_1^4} \qquad (m)$$

If for a given point $\frac{x}{X_1}$ (such as the center) we were to plot the variation of $\left(\frac{E}{P}\right)$ W with respect to ξ_1 or some equivalent parameter, we would find that $\left(\frac{E}{P}\right)$ W would increase (or decrease) continuously as the parameter increases. This is illustrated by Graphs K-a and K-b. In Graph K-a, $\left(\frac{E}{P}\right) W_c$ is plotted with respect to R for a given t, x_1 and v. In Graph K-b, $\left(\frac{E}{tP}\right) W_c$ is plotted with respect to $\frac{h}{t}$ for given $\left(\frac{x_1}{t}\right)$ and v. The above comments are pertinent to all the graphs of this section.

15. Numerical Problem

Given a shell with the following data

thickness t = $3\frac{1}{2}$ inches

half span, x_1, = 60.0 feet = 720 inches

rise, h = 15.0 feet = 180 inches

vertical loading = 150 lbs./sq. ft. (live + dead) downward

allowable compression = 1200 p.s.i. (concrete)

allowable tension = 0 p.s.i.

v = 0.20

E = 3,600,000 p.s.i.

To determine

(1) the maximum stresses and their location

(2) the regions within which tension acts (and which requires reinforcement)

(3) maximum deflection and its location

Solution

R is found by the use of the formula $R = \frac{x_1^2 + h^2}{2h}$ to be 127.5 feet. From equation (30) and the definition $\xi_1 = \frac{x_1}{\ell}$

$$\xi_1 = \frac{60\sqrt[4]{12(1-v^2)}}{\sqrt{\frac{3.5}{12} \times 127.5}} = 18.12 \text{ Say } 18.1$$

From Graph I for $\xi_1 = 18.1$ we find

$$+\left(\frac{2t}{PR}\right)\sigma_{x,\ \max} = 0.565$$

$$-\left(\frac{2t}{PR}\right)\sigma_{x,\ \max} = -2.42$$

$$+\left(\frac{2t}{PR}\right)\sigma_{\phi, max} = 0.115$$

$$-\left(\frac{2t}{PR}\right)\sigma_{\phi, max} = -1.06$$

whence using Table 34 noting that $\frac{PR}{2t} = 228$ (in these tables and graphs, downward p is assumed positive) we have

$+\sigma_{x, max} =$ 228 x 0.565 = 130 p.s.i. tension at outer edge of upper surface

$-\sigma_{x, max} =$ -228 x 2.42 = -550 p.s.i. compression at outer edge of lower surface

$+\sigma_{\phi, max} =$ 228 x 0.115 = 26 p.s.i. tension at outer edge of upper surface

$-\sigma_{\phi, max} =$ -228 x 1.06 = -240 p.s.i. compression on the upper surface at

$$x = \frac{18.12 - 4.00}{18.12} \times 60$$

$$= 46.8 \text{ ft.}$$

To determine the tensile regions use Graphs E, F, G and H.

From Graph E.

Upper Surface, Radial Direction

Tensions exists from the outer edge to a distance inward given by (1-0.983) x 60 = 1.0 foot.

From Graph F.

Lower Surface, Radial Direction

No tension region

From Graph G.

Upper Surface, Circumferential Direction

Tension exists from the outer edge to a distance inward given by (1-0.987) x 60 = 0.8 feet.

From Graph H.

<u>Lower Surface, Circumferential Direction</u>

No tension regions

To determine the maximum deflection and its location we use Graph J.

From Graph J for $\xi_1' = 18.12$ we estimate

$$\left(\frac{2Et}{R^2p}\right) W = -0.925 \text{ at } 0.775 \times 60 = x = 46.5' \text{ approx.}$$

That is

$$W = -\frac{127.5 \times 12}{3,600,000} \times 228 \times 0.925 \text{ at } x = 46.5' \text{ approx.}$$

or $W = -0.090$ inches downward at $x = 46.5'$ approx.

Table 1. Constant Values K_1, K_2 and K_3.

ξ_1	K_1	K_2	K_3
0.5	−1.9972	-6.2427×10^{-2}	−1.9991
1.0	−1.9539	-2.4531×10^{-1}	−1.9846
1.5	−1.7821	-5.1253×10^{-1}	−1.9272
2.0	−1.4153	-7.6101×10^{-1}	−1.8038
2.5	-9.1946×10^{-1}	-8.6908×10^{-1}	−1.6342
3.0	-4.5415×10^{-1}	-8.0972×10^{-1}	−1.4684
3.5	-1.2100×10^{-1}	-6.4935×10^{-1}	−1.3382
4.0	$+7.2415 \times 10^{-2}$	-4.6221×10^{-1}	−1.2453
4.5	$+1.5986 \times 10^{-1}$	-2.9234×10^{-1}	−1.1801
5.0	$+1.7890 \times 10^{-1}$	-1.5800×10^{-1}	−1.1330
6.0	$+1.2328 \times 10^{-1}$	-3.3311×10^{-3}	−1.0676
7.0	$+4.9865 \times 10^{-2}$	$+3.9681 \times 10^{-2}$	−1.0239
10.0	-8.0848×10^{-3}	$+3.0589 \times 10^{-3}$	-9.5006×10^{-1}
15.0	$+2.9565 \times 10^{-4}$	$+6.5856 \times 10^{-6}$	-8.9673×10^{-1}
20.0	-8.9309×10^{-6}	-3.8965×10^{-6}	-8.7136×10^{-1}
25.0	$+2.1784 \times 10^{-7}$	$+2.2544 \times 10^{-7}$	-8.5651×10^{-1}

$\xi_1 = 0.5$
$\nu = 0.20$

Table 2. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.0014	-0.0014	-0.0662	-0.0662	-0.0019	0
0.1	-0.0014	-0.0013	-0.0592	-0.0627	-0.0017	+0.0542
0.2	-0.0012	-0.0009	-0.0380	-0.0521	-0.0013	+0.1084
0.3	-0.0011	-0.0005	-0.0027	-0.0344	-0.0008	+0.1627
0.4	-0.0009	-0.0001	+0.0468	-0.0097	-0.0002	+0.2169
0.5	-0.0007	-0.0001	+0.1104	+0.0221	0	+0.2712

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 0.5$
$\nu = 0.20$

Table 3. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-0.0676	+0.0648	-0.0676	+0.0648
0.1	-0.0606	+0.0578	-0.0640	+0.0614
0.2	-0.0392	+0.0368	-0.0530	+0.0512
0.3	-0.0038	+0.0016	-0.0349	+0.0339
0.4	+0.0459	-0.0477	-0.0098	+0.0096
0.5	+0.1097	-0.1111	+0.0220	-0.0222

In using these values, downward p is positive.

$\xi_1 = 1.0$
$\nu = 0.20$

Table 4. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.0230	-0.0230	-0.2602	-0.2602	-0.0307	0
0.1	-0.0229	-0.0226	-0.2533	-0.2567	-0.0301	+0.0530
0.2	-0.0224	-0.0212	-0.2326	-0.2464	-0.0283	+0.1061
0.3	-0.0217	-0.0191	-0.1980	-0.2291	-0.0254	+0.1593
0.4	-0.0208	-0.0163	-0.1495	-0.2049	-0.0217	+0.2126
0.5	-0.0195	-0.0131	-0.0871	-0.1737	-0.0173	+0.2661
0.6	-0.0182	-0.0098	-0.0108	-0.1355	-0.0126	+0.3198
0.7	-0.0168	-0.0070	+0.0796	-0.0904	-0.0080	+0.3736
0.8	-0.0153	-0.0040	+0.1840	-0.0383	-0.0040	+0.4276
0.9	-0.0140	-0.0024	+0.3025	+0.0208	-0.0011	+0.4817
1.0	-0.0128	-0.0026	+0.4351	+0.0870	0	+0.5359

Note: Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 1.0$
$\nu = 0.20$

Table 5. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-0.2832	+0.2372	-0.2832	+0.2372
0.1	-0.2762	+0.2304	-0.2793	+0.2342
0.2	-0.2550	+0.2101	-0.2676	+0.2252
0.3	-0.2197	+0.1762	-0.2482	+0.2100
0.4	-0.1703	+0.1288	-0.2212	+0.1886
0.5	-0.1067	+0.0676	-0.1868	+0.1606
0.6	-0.0290	-0.0074	-0.1454	+0.1257
0.7	+0.0629	-0.0964	-0.0974	+0.0834
0.8	+0.1687	-0.1993	-0.0423	+0.0343
0.9	+0.2885	-0.3165	+0.0184	-0.0233
1.0	+0.4223	-0.4479	+0.0845	-0.0896

In using these values, downward p is positive.

$\xi_1 = 1.5$
$\nu = 0.20$

Table 6. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.1090	-0.1090	-0.5436	-0.5436	-0.1451	0
0.1	-0.1086	-0.1080	-0.5373	-0.5405	-0.1438	+0.0484
0.2	-0.1077	-0.1051	-0.5184	-0.5310	-0.1400	+0.0969
0.3	-0.1061	-0.1005	-0.4868	-0.5153	-0.1338	+0.1456
0.4	-0.1040	-0.0941	-0.4425	-0.4931	-0.1253	+0.1945
0.6	-0.0980	-0.0774	-0.3152	-0.4296	-0.1026	+0.2938
0.8	-0.0903	-0.0569	-0.1357	-0.3402	-0.0746	+0.3950
1.0	-0.0816	-0.0362	+0.0974	-0.2243	-0.0450	+0.4985
1.1	-0.0770	-0.0269	+0.2345	-0.1562	-0.0312	+0.5511
1.2	-0.0725	-0.0192	+0.3857	-0.0814	-0.0189	+0.6041
1.3	-0.0682	-0.0136	+0.5505	+0.0003	-0.0091	+0.6575
1.4	-0.0642	-0.0109	+0.7289	+0.0888	-0.0024	+0.7111
1.5	-0.0607	-0.0121	+0.9220	+0.1844	0	+0.7648

Note: Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 1.5$
$\nu = 0.20$

Table 7. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-0.6526	+0.4346	-0.6526	+0.4346
0.1	-0.6459	+0.4287	-0.6485	+0.4325
0.2	-0.6261	+0.4107	-0.6361	+0.4259
0.3	-0.5929	+0.3807	-0.6158	+0.4148
0.4	-0.5465	+0.3385	-0.5872	+0.3990
0.6	-0.4132	+0.2172	-0.5070	+0.3522
0.8	-0.2260	+0.0454	-0.3971	+0.2833
1.0	+0.0158	-0.1790	-0.2605	+0.1881
1.1	+0.1575	-0.3115	-0.1831	+0.1293
1.2	+0.3132	-0.4582	-0.1006	+0.0622
1.3	+0.4823	-0.6187	-0.0133	-0.0139
1.4	+0.6647	-0.7931	+0.0779	-0.0997
1.5	+0.8613	-0.9827	+0.1723	-0.1965

In using these values, downward p is positive.

$\xi_1 = 2.0$
$\nu = 0.20$

Table 8. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.2924	-0.2924	-0.8072	-0.8072	-0.3885	0
0.2	-0.2904	-0.2866	-0.7872	-0.7972	-0.3809	+0.0770
0.4	-0.2849	-0.2699	-0.7267	-0.7670	-0.3586	+0.1553
0.6	-0.2757	-0.2434	-0.6247	-0.7162	-0.3229	+0.2359
0.8	-0.2634	-0.2086	-0.4798	-0.6443	-0.2759	+0.3198
1.0	-0.2485	-0.1684	-0.2896	-0.5503	-0.2207	+0.4079
1.2	-0.2316	-0.1259	-0.0516	-0.4334	-0.1613	+0.5005
1.4	-0.2136	-0.0855	+0.2359	-0.2924	-0.1029	+0.5976
1.5	-0.2044	-0.0677	+0.3995	-0.2126	-0.0759	+0.6477
1.6	-0.1954	-0.0524	+0.5763	-0.1263	-0.0515	+0.6988
1.7	-0.1866	-0.0402	+0.7669	-0.0338	-0.0307	+0.7506
1.8	-0.1782	-0.0323	+0.9710	+0.0657	-0.0144	+0.8029
1.9	-0.1705	-0.0296	+1.1893	+0.1717	-0.0038	+0.8555
2.0	-0.1635	-0.0327	+1.421	+0.2843	0	+0.9081

Note: Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 2.0$
$\nu = 0.20$

Table 9. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-1.0996	+0.5148	-1.0996	+0.5148
0.2	-1.0776	+0.4968	-1.0838	+0.5106
0.4	-1.0116	+0.4418	-1.0369	+0.4971
0.6	-0.9004	+0.3490	-0.9596	+0.4728
0.8	-0.7432	+0.2164	-0.8529	+0.4357
1.0	-0.5381	+0.0411	-0.7187	+0.3819
1.2	-0.2832	-0.1800	-0.5593	+0.3075
1.4	+0.0223	-0.4495	-0.3779	+0.2069
1.5	+0.1951	-0.6039	-0.2803	+0.1449
1.6	+0.3809	-0.7717	-0.1787	+0.0739
1.7	+0.5803	-0.9535	-0.0740	-0.0064
1.8	+0.7928	-1.1492	+0.0334	-0.0980
1.9	+1.0188	-1.3598	+0.1421	-0.2013
2.0	+1.2575	-1.5845	+0.2516	-0.3170

In using these values, downward p is positive.

$\xi_1 = 2.5$
$\nu = 0.20$

Table 10. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.5403	-0.5403	-0.9218	-0.9218	-0.7147	0
0.2	-0.5381	-0.5338	-0.9088	-0.9153	-0.7061	+0.0501
0.4	-0.5317	-0.5145	-0.8693	-0.8956	-0.6803	+0.1017
0.6	-0.5210	-0.4832	-0.8020	-0.8622	-0.6384	+0.1560
0.8	-0.5065	-0.4409	-0.7054	-0.8146	-0.5817	+0.2143
1.0	-0.4884	-0.3896	-0.5766	-0.7515	-0.5122	+0.2777
1.2	-0.4672	-0.3314	-0.4123	-0.6718	-0.4328	+0.3471
1.4	-0.4433	-0.2692	-0.2098	-0.5741	-0.3468	+0.4230
1.6	-0.4177	-0.2069	+0.0357	-0.4569	-0.2587	+0.5057
1.8	-0.3909	-0.1486	+0.3271	-0.3186	-0.1737	+0.5951
2.0	-0.3642	-0.0996	+0.6685	-0.1578	-0.0980	+0.6903
2.1	-0.3511	-0.0806	+0.8585	-0.0685	-0.0659	+0.7397
2.2	-0.3385	-0.0663	+1.062	+0.0269	-0.0389	+0.7900
2.3	-0.3264	-0.0575	+1.278	+0.1285	-0.0182	+0.8409
2.4	-0.3151	-0.0553	+1.510	+0.2365	-0.0048	+0.8922
2.5	-0.3048	-0.0610	+1.754	+0.3509	0	+0.9433

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 2.5$
$\nu = 0.20$

Table 11. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-1.4621	+0.3815	-1.4621	+0.3815
0.2	-1.4469	+0.3707	-1.4491	+0.3815
0.4	-1.4010	+0.3376	-1.4101	+0.3811
0.6	-1.3230	+0.2810	-1.3454	+0.3790
0.8	-1.2119	+0.1989	-1.2555	+0.3737
1.0	-1.0650	+0.0882	-1.1411	+0.3619
1.2	-0.8795	-0.0549	-1.0032	+0.3404
1.4	-0.6531	-0.2335	-0.8433	+0.3049
1.6	-0.3820	-0.4534	-0.6638	+0.2500
1.8	-0.0638	-0.7180	-0.4672	+0.1700
2.0	+0.3043	-1.0327	-0.2574	+0.0582
2.1	+0.5074	-1.2096	-0.1491	-0.0121
2.2	+0.7235	-1.4005	-0.0394	-0.0932
2.3	+0.9516	-1.6044	+0.0710	-0.1860
2.4	+1.1949	-1.8251	+0.1812	-0.2918
2.5	+1.4492	-2.0588	+0.2899	-0.4119

In using these values, downward p is positive.

$\xi_1 = 3.0$
$\nu = 0.20$

Table 12; Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.7729	-0.7729	-0.8588	-0.8588	-1.0143	0
0.1	-0.7724	-0.7714	-0.8572	-0.8580	-1.0122	+0.0124
0.2	-0.7709	-0.7669	-0.8524	-0.8556	-1.0062	+0.0249
0.3	-0.7684	-0.7593	-0.8442	-0.8516	-0.9961	+0.0377
0.4	-0.7649	-0.7488	-0.8327	-0.8459	-0.9820	+0.0510
0.5	-0.7603	-0.7353	-0.8176	-0.8383	-0.9641	+0.0650
0.6	-0.7549	-0.7191	-0.7985	-0.8289	-0.9423	+0.0798
0.7	-0.7484	-0.7001	-0.7752	-0.8176	-0.9168	+0.0956
0.8	-0.7410	-0.6782	-0.7483	-0.8045	-0.8877	+0.1124
0.9	-0.7327	-0.6540	-0.7163	-0.7890	-0.8551	+0.1306
1.0	-0.7235	-0.6273	-0.6793	-0.7712	-0.8193	+0.1501
1.2	-0.7026	-0.5672	-0.5883	-0.7281	-0.7385	+0.1937
1.4	-0.6786	-0.5004	-0.4722	-0.6737	-0.6473	+0.2442
1.6	-0.6518	-0.4281	-0.3264	-0.6063	-0.5483	+0.3024
1.8	-0.6228	-0.3532	-0.1469	-0.5242	-0.4445	+0.3685
2.0	-0.5921	-0.2791	+0.0713	-0.4257	-0.3397	+0.4428
2.1	-0.5764	-0.2436	+0.1961	-0.3697	-0.2884	+0.4829
2.2	-0.5605	-0.2099	+0.3321	-0.3089	-0.2387	+0.5248
2.3	-0.5445	-0.1784	+0.4795	-0.2430	-0.1914	+0.5686
2.4	-0.5287	-0.1499	+0.6399	-0.1719	-0.1471	+0.6140
2.5	-0.5130	-0.1254	+0.8120	-0.0953	-0.1068	+0.6608
2.6	-0.4977	-0.1053	+0.9975	-0.0133	-0.0715	+0.7088
2.7	-0.4829	-0.0907	+1.196	+0.0747	-0.0420	+0.7578
2.8	-0.4687	-0.0823	+1.409	+0.1687	-0.0195	+0.8074
2.9	-0.4554	-0.0810	+1.634	+0.2686	-0.0051	+0.8573
3.0	-0.4430	-0.0887	+1.873	+0.3746	0	+0.9070

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 3.0$
$\nu = 0.20$

Table 13. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-1.6317	+0.0859	-1.6317	+0.0859
0.1	-1.6296	+0.0848	-1.6294	+0.0866
0.2	-1.6233	+0.0815	-1.6225	+0.0887
0.3	-1.6126	+0.0758	-1.6109	+0.0923
0.4	-1.5976	+0.0678	-1.5947	+0.0971
0.5	-1.5779	+0.0573	-1.5736	+0.1030
0.6	-1.5534	+0.0436	-1.5480	+0.1098
0.7	-1.5236	+0.0268	-1.5177	+0.1175
0.8	-1.4893	+0.0073	-1.4827	+0.1263
0.9	-1.4490	-0.0164	-1.4430	+0.1350
1.0	-1.4028	-0.0442	-1.3985	+0.1439
1.2	-1.2909	-0.1143	-1.2956	+0.1606
1.4	-1.1508	-0.2064	-1.1741	+0.1733
1.6	-0.9782	-0.3254	-1.0344	+0.1782
1.8	-0.7697	-0.4759	-0.8774	+0.1710
2.0	-0.5208	-0.6634	-0.7048	+0.1466
2.1	-0.3803	-0.7725	-0.6133	+0.1261
2.2	-0.2284	-0.8926	-0.5188	+0.0990
2.3	-0.0650	-1.0240	-0.4214	+0.0646
2.4	+0.1112	-1.1686	-0.3218	+0.0220
2.5	+0.2990	-1.3250	-0.2207	-0.0301
2.6	+0.4998	-1.4952	-0.1186	-0.0920
2.7	+0.7131	-1.6789	-0.0160	-0.1654
2.8	+0.9403	-1.8777	+0.0864	-0.2510
2.9	+1.1786	-2.0894	+0.1876	-0.3496
3.0	+1.4300	-2.3160	+0.2859	-0.4633

In using these values, downward p is positive.

$\xi_1 = 3.5$
$\nu = 0.20$

Table 14. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,d}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{pR}\right)\sigma_{x,b}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2p}\right)\omega$	$\frac{2}{p}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.9395	-0.9395	-0.6887	-0.6887	-1.2172	0
0.3	-0.9359	-0.9286	-0.6848	-0.6868	-1.2026	+0.0104
0.6	-0.9249	-0.8959	-0.6713	-0.6802	-1.1590	+0.0244
0.9	-0.9068	-0.8420	-0.6440	-0.6675	-1.0871	+0.0455
1.2	-0.8818	-0.7682	-0.5951	-0.6456	-0.9882	+0.0770
1.5	-0.8502	-0.6763	-0.5149	-0.6107	-0.8647	+0.1220
1.8	-0.8125	-0.5699	-0.3907	-0.5578	-0.7206	+0.1832
2.1	-0.7696	-0.4540	-0.2076	-0.4813	-0.5618	+0.2626
2.4	-0.7228	-0.3364	+0.0497	-0.3749	-0.3975	+0.3611
2.7	-0.6737	-0.2283	+0.3970	-0.2320	-0.2402	+0.4782
3.0	-0.6247	-0.1439	+0.8486	-0.0462	-0.1068	+0.6111
3.1	-0.6089	-0.1243	+1.0243	+0.0261	-0.0713	+0.6582
3.2	-0.5935	-0.1103	+1.2129	+0.1039	-0.0418	+0.7061
3.3	-0.5787	-0.1025	+1.415	+0.1875	-0.0194	+0.7546
3.4	-0.5647	-0.1022	+1.6301	+0.2767	-0.0051	+0.8034
3.5	-0.5515	-0.1103	+1.8585	+0.3717	0	+0.8520

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 3.5$
$\nu = 0.20$

Table 15. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-1.6282	-0.2508	-1.6282	-0.2508
0.3	-1.6207	-0.2511	-1.6154	-0.2418
0.6	-1.5962	-0.2536	-1.5761	-0.2157
0.9	-1.5508	-0.2628	-1.5095	-0.1745
1.2	-1.4769	-0.2867	-1.4138	-0.1226
1.5	-1.3651	-0.3353	-1.2870	-0.0656
1.8	-1.2032	-0.4218	-1.1277	-0.0121
2.1	-0.9772	-0.5620	-0.9353	+0.0273
2.4	-0.6731	-0.7725	-0.7113	+0.0385
2.7	-0.2767	-1.0707	-0.4603	+0.0037
3.0	+0.2239	-1.4733	-0.1901	-0.0977
3.1	+0.4154	-1.6332	-0.0982	-0.1504
3.2	+0.6194	-1.8064	-0.0064	-0.2142
3.3	+0.8363	-1.9937	+0.0850	-0.2900
3.4	+1.0654	-2.1948	+0.1745	-0.3789
3.5	+1.3070	-2.4100	+0.2614	-0.4820

In using these values, downward p is positive.

$\xi_1 = 4.0$
$\nu = 0.20$

Table 16. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0362	-1.0362	-0.4902	-0.4902	-1.3177	0
0.2	-1.0351	-1.0327	-0.4913	-0.4908	-1.3131	-0.0038
0.4	-1.0316	-1.0223	-0.4941	-0.4922	-1.2992	-0.0069
0.6	-1.0258	-1.0049	-0.4980	-0.4943	-1.2760	-0.0084
0.8	-1.0176	-0.9804	-0.5021	-0.4967	-1.2433	-0.0077
1.0	-1.0072	-0.9488	-0.5048	-0.4988	-1.2012	-0.0039
1.2	-0.9943	-0.9100	-0.5040	-0.4998	-1.1496	+0.0037
1.6	-0.9614	-0.8111	-0.4826	-0.4949	-1.0179	+0.0335
2.0	-0.9192	-0.6858	-0.4137	-0.4727	-0.8503	+0.0877
2.4	-0.8684	-0.5395	-0.2668	-0.4213	-0.6533	+0.1715
2.8	-0.8103	-0.3842	-0.0062	-0.3269	-0.4398	+0.2883
3.0	-0.7794	-0.3090	+0.1783	-0.2589	-0.3337	+0.3592
3.1	-0.7636	-0.2735	+0.2861	-0.2188	-0.2824	+0.3977
3.2	-0.7478	-0.2401	+0.4049	-0.1744	-0.2330	+0.4381
3.3	-0.7319	-0.2091	+0.5354	-0.1254	-0.1863	+0.4802
3.4	-0.7161	-0.1814	+0.6778	-0.0717	-0.1429	+0.5239
3.5	-0.7005	-0.1578	+0.8327	-0.0130	-0.1035	+0.5690
3.6	-0.6851	-0.1386	+1.000	+0.0508	-0.0691	+0.6153
3.7	-0.6702	-0.1251	+1.181	+0.1200	-0.0405	+0.6624
3.8	-0.6557	-0.1176	+1.374	+0.1944	-0.0188	+0.7101
3.9	-0.6419	-0.1175	+1.581	+0.2745	-0.0049	+0.7580
4.0	-0.6289	-0.1258	+1.802	+0.3603	0	+0.8058

Note;- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 4.0$
$\nu = 0.20$

Table 17. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-1.5264	-0.5460	-1.5264	-0.5460
0.2	-1.5264	-0.5438	-1.5235	-0.5419
0.4	-1.5257	-0.5375	-1.5145	-0.5301
0.6	-1.5238	-0.5278	-1.4992	-0.5106
0.8	-1.5197	-0.5155	-1.4771	-0.4837
1.0	-1.5120	-0.5024	-1.4476	-0.4500
1.2	-1.4983	-0.4903	-1.4098	-0.4102
1.6	-1.4440	-0.4788	-1.3060	-0.3162
2.0	-1.3329	-0.5055	-1.1585	-0.2131
2.4	-1.1352	-0.6016	-0.9608	-0.1182
2.8	-0.8165	-0.8041	-0.7111	-0.0573
3.0	-0.6011	-0.9577	-0.5679	-0.0501
3.1	-0.4775	-1.0497	-0.4923	-0.0547
3.2	-0.3429	-1.1527	-0.4145	-0.0657
3.3	-0.1965	-1.2673	-0.3345	-0.0837
3.4	-0.0383	-1.3939	-0.2531	-0.1097
3.5	+0.1322	-1.5332	-0.1708	-0.1448
3.6	+0.3149	-1.6851	-0.0878	-0.1894
3.7	+0.5108	-1.8512	-0.0051	-0.2451
3.8	+0.7183	-2.0297	+0.0768	-0.3120
3.9	+0.9391	-2.2229	+0.1570	-0.3920
4.0	+1.1731	-2.4309	+0.2345	-0.4861

In using these values, downward p is positive.

$\xi_1 = 4.5$
$\nu = 0.20$

Table 18. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0799	-1.0799	-0.3101	-0.3101	-1.3400	0
0.4	-1.0770	-1.0711	-0.3190	-0.3145	-1.3282	-0.0167
0.8	-1.0681	-1.0440	-0.3434	-0.3271	-1.2922	-0.0296
1.2	-1.0528	-0.9970	-0.3766	-0.3451	-1.2299	-0.0344
1.6	-1.0307	-0.9279	-0.4072	-0.3642	-1.1387	-0.0267
2.0	-1.0012	-0.8347	-0.4180	-0.3779	-1.0160	-0.0013
2.4	-0.9640	-0.7173	-0.3861	-0.3777	-0.8614	+0.0469
2.8	-0.9191	-0.5789	-0.2825	-0.3523	-0.6781	+0.1230
3.2	-0.8672	-0.4283	-0.0738	-0.2890	-0.4755	+0.2306
3.6	-0.8102	-0.2821	+0.2769	-0.1733	-0.2725	+0.3709
4.0	-0.7513	-0.1687	+0.8036	+0.0091	-0.1001	+0.5400
4.1	-0.7368	-0.1500	+0.9656	+0.0667	-0.0668	+0.5857
4.2	-0.7227	-0.1363	+1.141	+0.1292	-0.0392	+0.6322
4.3	-0.7089	-0.1290	+1.329	+0.1970	-0.0182	+0.6793
4.4	-0.6957	-0.1288	+1.531	+0.2704	-0.0047	+0.7266
4.5	-0.6832	-0.1366	+1.745	+0.3491	0	+0.7737

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 4.5$
$\nu = 0.20$

Table 19. Combined (Direct and Bending) Stress Intensities.

ξ_2	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-1.3900	-0.7698	-1.3900	-0.7698
0.4	-1.3960	-0.7580	-1.3856	-0.7566
0.8	-1.4115	-0.7247	-1.3711	-0.7169
1.2	-1.4294	-0.6762	-1.3421	-0.6519
1.6	-1.4379	-0.6235	-1.2921	-0.5637
2.0	-1.4192	-0.5832	-1.2126	-0.4568
2.4	-1.3501	-0.5779	-1.0950	-0.3396
2.8	-1.2016	-0.6366	-0.9312	-0.2266
3.2	-0.9410	-0.7934	-0.7173	-0.1393
3.6	-0.5333	-1.0871	-0.4554	-0.1088
4.0	+0.0523	-1.5549	-0.1596	-0.1778
4.1	+0.2288	-1.7024	-0.0833	-0.2167
4.2	+0.4183	-1.8637	-0.0071	-0.2655
4.3	+0.6201	-2.0379	+0.0680	-0.3260
4.4	+0.8353	-2.2267	+0.1416	-0.3992
4.5	+1.0618	-2.4282	+0.2125	-0.4857

In using these values, downward p is positive.

$\xi_1 = 5.0$
$\nu = 0.20$

Table 20. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0894	-1.0894	-0.1676	-0.1676	-1.312	0
0.1	-1.0894	-1.0892	-0.1679	-0.1682	-1.312	-0.0049
0.2	-1.0891	-1.0883	-0.1688	-0.1701	-1.310	-0.0097
0.3	-1.0886	-1.0868	-0.1704	-0.1732	-1.308	-0.0144
0.4	-1.0879	-1.0847	-0.1726	-0.1776	-1.306	-0.0191
0.5	-1.0870	-1.0819	-0.1754	-0.1832	-1.302	-0.0236
0.6	-1.0858	-1.0785	-0.1788	-0.1898	-1.297	-0.0280
0.7	-1.0845	-1.0744	-0.1827	-0.1976	-1.292	-0.0321
0.8	-1.0829	-1.0696	-0.1872	-0.2065	-1.285	-0.0360
0.9	-1.0811	-1.0640	-0.1922	-0.2163	-1.278	-0.0396
1.0	-1.0791	-1.0576	-0.1977	-0.2269	-1.270	-0.0429
1.2	-1.0743	-1.0421	-0.2100	-0.2505	-1.249	-0.0484
1.6	-1.0613	-0.9994	-0.2383	-0.3027	-1.194	-0.0532
2.0	-1.0431	-0.9378	-0.2683	-0.3527	-1.114	-0.0468
2.4	-1.0189	-0.8541	-0.2943	-0.3850	-1.006	-0.0246
2.8	-0.9880	-0.7467	-0.3085	-0.3788	-0.868	+0.0183
3.2	-0.9499	-0.6171	-0.3012	-0.3075	-0.700	+0.0870
3.6	-0.9049	-0.4717	-0.2604	-0.1390	-0.510	+0.1857
4.0	-0.8542	-0.3249	-0.1725	+0.1618	-0.312	+0.3165
4.1	-0.8409	-0.2907	-0.1416	+0.2618	-0.265	+0.3541
4.2	-0.8274	-0.2581	-0.1069	+0.3727	-0.219	+0.3935
4.3	-0.8138	-0.2281	-0.0677	+0.4953	-0.175	+0.4346
4.4	-0.8002	-0.2011	-0.0240	+0.6301	-0.134	+0.4772
4.5	-0.7866	-0.1779	+0.0243	+0.7765	-0.098	+0.5212
4.6	-0.7732	-0.1591	+0.0774	+0.9361	-0.065	+0.5664
4.7	-0.7599	-0.1454	+0.1354	+1.108	-0.038	+0.6124
4.8	-0.7470	-0.1377	+0.1988	+1.294	-0.018	+0.6591
4.9	-0.7346	-0.1371	+0.2669	+1.492	-0.005	+0.7059
5.0	-0.7227	-0.1444	+0.3407	+1.703	0	+0.7526

Note: Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 5.0$
$\nu = 0.20$

Table 21. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-1.2570	-0.9218	-1.2570	-0.9218
0.1	-1.2576	-0.9212	-1.2571	-0.9213
0.2	-1.2592	-0.9190	-1.2571	-0.9195
0.3	-1.2618	-0.9154	-1.2572	-0.9164
0.4	-1.2655	-0.9103	-1.2573	-0.9121
0.5	-1.2702	-0.9038	-1.2573	-0.9065
0.6	-1.2756	-0.8960	-1.2573	-0.8997
0.7	-1.2821	-0.8869	-1.2571	-0.8917
0.8	-1.2894	-0.8764	-1.2568	-0.8824
0.9	-1.2974	-0.8648	-1.2562	-0.8718
1.0	-1.3060	-0.8522	-1.2553	-0.8599
1.2	-1.3248	-0.8238	-1.2521	-0.8321
1.6	-1.3640	-0.7586	-1.2377	-0.7611
2.0	-1.3958	-0.6904	-1.2061	-0.6695
2.4	-1.4039	-0.6339	-1.1484	-0.5598
2.8	-1.3668	-0.6092	-1.0552	-0.4382
3.2	-1.2574	-0.6424	-0.9183	-0.3159
3.6	-1.0439	-0.7659	-0.7321	-0.2113
4.0	-0.6924	-1.0160	-0.4974	-0.1524
4.1	-0.5791	-1.1027	-0.4323	-0.1491
4.2	-0.4547	-1.2001	-0.3650	-0.1512
4.3	-0.3185	-1.3091	-0.2958	-0.1604
4.4	-0.1701	-1.4303	-0.2251	-0.1771
4.5	-0.0101	-1.5631	-0.1536	-0.2022
4.6	+0.1629	-1.7093	-0.0817	-0.2365
4.7	+0.3481	-1.8679	-0.0100	-0.2808
4.8	+0.5470	-2.0410	+0.0611	-0.3365
4.9	+0.7574	-2.2266	+0.1298	-0.4040
5.0	+0.9803	-2.4257	+0.1963	-0.4851

In using these values, downward p is positive.

$\xi_1 = 6.0$
$\nu = 0.20$

Table 22. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0616	-1.0616	-0.0035	-0.0035	-1.1909	0
0.3	-1.0616	-1.0616	-0.0075	-0.0054	-1.1908	-0.0100
0.6	-1.0615	-1.0612	-0.0192	-0.0114	-1.1903	-0.0200
0.9	-1.0613	-1.0601	-0.0387	-0.0212	-1.1889	-0.0299
1.2	-1.0607	-1.0574	-0.0659	-0.0348	-1.1857	-0.0395
1.5	-1.0596	-1.0521	-0.1002	-0.0521	-1.1793	-0.0485
1.8	-1.0576	-1.0429	-0.1410	-0.0729	-1.1681	-0.0563
2.1	-1.0545	-1.0281	-0.1869	-0.0966	-1.1502	-0.0622
2.4	-1.0499	-1.0059	-0.2359	-0.1227	-1.1234	-0.0650
2.7	-1.0434	-0.9744	-0.2844	-0.1499	-1.0853	-0.0636
3.0	-1.0344	-0.9318	-0.3283	-0.1768	-1.0339	-0.0561
3.3	-1.0227	-0.8766	-0.3614	-0.2013	-0.9669	-0.0407
3.6	-1.0077	-0.8076	-0.3758	-0.2208	-0.8829	-0.0151
3.9	-0.9892	-0.7247	-0.3616	-0.2318	-0.7815	+0.0228
4.2	-0.9670	-0.6289	-0.3071	-0.2304	-0.6636	+0.0752
4.5	-0.9410	-0.5234	-0.1989	-0.2118	-0.5320	+0.1441
4.8	-0.9115	-0.4133	-0.0221	-0.1705	-0.3924	+0.2307
5.1	-0.8790	-0.3072	+0.2384	-0.1011	-0.2538	+0.3350
5.2	-0.8677	-0.2746	+0.3465	-0.0707	-0.2100	+0.3735
5.3	-0.8562	-0.2444	+0.4658	-0.0363	-0.1682	+0.4137
5.4	-0.8446	-0.2171	+0.5970	+0.0022	-0.1293	+0.4555
5.5	-0.8330	-0.1933	+0.7406	+0.0451	-0.0939	+0.4986
5.6	-0.8214	-0.1738	+0.8965	+0.0762	-0.0628	+0.5430
5.7	-0.8099	-0.1595	+1.065	+0.1261	-0.0369	+0.5882
5.8	-0.7986	-0.1509	+1.247	+0.2015	-0.0171	+0.6341
5.9	-0.7876	-0.1493	+1.443	+0.2632	-0.0044	+0.6802
6.0	-0.7770	-0.1553	+1.651	+0.3302	0	+0.7263

Note:-- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 6.0$
$\nu = 0.20$

Table 23. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-1.0651	-1.0581	-1.0651	-1.0581
0.3	-1.0691	-1.0541	-1.0670	-1.0562
0.6	-1.0807	-1.0423	-1.0726	-1.0498
0.9	-1.1000	-1.0226	-1.0813	-1.0389
1.2	-1.1266	-0.9948	-1.0922	-1.0226
1.5	-1.1598	-0.9594	-1.1042	-1.0000
1.8	-1.1986	-0.9166	-1.1158	-0.9700
2.1	-1.2414	-0.8676	-1.1247	-0.9315
2.4	-1.2858	-0.8140	-1.1286	-0.8832
2.7	-1.3278	-0.7590	-1.1243	-0.8245
3.0	-1.3627	-0.7061	-1.1086	-0.7550
3.3	-1.3841	-0.6613	-1.0779	-0.6753
3.6	-1.3835	-0.6319	-1.0284	-0.5868
3.9	-1.3508	-0.6276	-0.9565	-0.4929
4.2	-1.2741	-0.6599	-0.8593	-0.3985
4.5	-1.1399	-0.7421	-0.7352	-0.3116
4.8	-0.9336	-0.8894	-0.5838	-0.2428
5.1	-0.6406	-1.1174	-0.4083	-0.2061
5.2	-0.5212	-1.2142	-0.3453	-0.2039
5.3	-0.3904	-1.3220	-0.2807	-0.2081
5.4	-0.2476	-1.4416	-0.2149	-0.2193
5.5	-0.0924	-1.5736	-0.1482	-0.2384
5.6	+0.0751	-1.7179	-0.0976	-0.2500
5.7	+0.2551	-1.8749	-0.0334	-0.2856
5.8	+0.4484	-2.0456	+0.0506	-0.3524
5.9	+0.6554	-2.2306	+0.1139	-0.4125
6.0	+0.8740	-2.4280	+0.1749	-0.4855

In using these values, downward p is positive.

$\xi_1 = 7.0$
$\nu = 0.20$

Table 24. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)w$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0249	-1.0249	+0.0421	+0.0421	-1.0738	0
0.1	-1.0250	-1.0250	+0.0419	+0.0421	-1.0739	-0.0014
0.2	-1.0250	-1.0252	+0.0414	+0.0417	-1.0742	-0.0027
0.3	-1.0252	-1.0256	+0.0405	+0.0413	-1.0747	-0.0041
0.4	-1.0253	-1.0261	+0.0392	+0.0407	-1.0753	-0.0055
0.5	-1.0255	-1.0268	+0.0376	+0.0399	-1.0762	-0.0069
0.6	-1.0258	-1.0275	+0.0356	+0.0389	-1.0772	-0.0084
0.7	-1.0261	-1.0284	+0.0332	+0.0377	-1.0784	-0.0099
0.8	-1.0265	-1.0294	+0.0304	+0.0363	-1.0798	-0.0115
0.9	-1.0269	-1.0305	+0.0272	+0.0347	-1.0813	-0.0131
1.0	-1.0273	-1.0317	+0.0236	+0.0329	-1.0829	-0.0148
1.2	-1.0282	-1.0343	+0.0148	+0.0287	-1.0864	-0.0184
1.6	-1.0304	-1.0395	-0.0087	+0.0173	-1.0938	-0.0264
2.0	-1.0326	-1.0434	-0.0413	+0.0017	-1.1000	-0.0354
2.4	-1.0345	-1.0437	-0.0840	-0.0185	-1.1022	-0.0450
2.8	-1.0355	-1.0373	-0.1365	-0.0435	-1.0967	-0.0539
3.2	-1.0348	-1.0204	-0.1970	-0.0729	-1.0792	-0.0604
3.6	-1.0316	-0.9889	-0.2609	-0.1054	-1.0444	-0.0618
4.0	-1.0250	-0.9382	-0.3198	-0.1385	-0.9871	-0.0542
4.5	-1.0104	-0.8422	-0.3658	-0.1745	-0.8764	-0.0253
5.0	-0.9871	-0.7068	-0.3442	-0.1926	-0.7178	+0.0350
5.5	-0.9541	-0.5376	-0.2010	-0.1753	-0.5156	+0.1369
6.0	-0.9118	-0.3554	+0.1301	-0.0999	-0.2911	+0.2873
6.1	-0.9024	-0.3209	+0.2251	-0.0756	-0.2472	+0.3232
6.2	-0.8927	-0.2879	+0.3309	-0.0479	-0.2046	+0.3610
6.3	-0.8829	-0.2573	+0.4481	-0.0163	-0.1641	+0.4005
6.4	-0.8729	-0.2294	+0.5772	+0.0192	-0.1262	+0.4415
6.5	-0.8628	-0.2050	+0.7184	+0.0587	-0.0917	+0.4840
6.6	-0.8527	-0.1848	+0.8723	+0.1026	-0.0614	+0.5278
6.7	-0.8426	-0.1696	+1.039	+0.1508	-0.0361	+0.5725
6.8	-0.8326	-0.1602	+1.219	+0.2038	-0.0167	+0.6178
6.9	-0.8228	-0.1576	+1.412	+0.2612	-0.0043	+0.6635
7.0	-0.8134	-0.1627	+1.617	+0.3235	0	+0.7091

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 7.0$
$\nu = 0.20$

Table 25. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-0.9828	-1.0670	-0.9828	-1.0670
0.1	-0.9831	-1.0669	-0.9829	-1.0671
0.2	-0.9836	-1.0664	-0.9835	-1.0669
0.3	-0.9847	-1.0657	-0.9843	-1.0669
0.4	-0.9861	-1.0645	-0.9854	-1.0668
0.5	-0.9879	-1.0631	-0.9869	-1.0667
0.6	-0.9902	-1.0614	-0.9886	-1.0664
0.7	-0.9929	-1.0593	-0.9907	-1.0661
0.8	-0.9961	-1.0569	-0.9931	-1.0657
0.9	-0.9997	-1.0541	-0.9958	-1.0652
1.0	-1.0037	-1.0509	-0.9988	-1.0646
1.2	-1.0134	-1.0430	-1.0056	-1.0630
1.6	-1.0391	-1.0217	-1.0222	-1.0568
2.0	-1.0739	-0.9913	-1.0417	-1.0451
2.4	-1.1185	-0.9505	-1.0622	-1.0252
2.8	-1.1720	-0.8990	-1.0808	-0.9938
3.2	-1.2318	-0.8378	-1.0933	-0.9475
3.6	-1.2925	-0.7707	-1.0943	-0.8835
4.0	-1.3448	-0.7052	-1.0767	-0.7997
4.5	-1.3762	-0.6446	-1.0167	-0.6677
5.0	-1.3313	-0.6429	-0.8994	-0.5142
5.5	-1.1551	-0.7531	-0.7129	-0.3623
6.0	-0.7817	-1.0419	-0.4553	-0.2555
6.1	-0.6773	-1.1275	-0.3965	-0.2453
6.2	-0.5618	-1.2236	-0.3358	-0.2400
6.3	-0.4348	-1.3310	-0.2736	-0.2410
6.4	-0.2959	-1.4501	-0.2102	-0.2486
6.5	-0.1444	-1.5812	-0.1463	-0.2637
6.6	+0.0196	-1.7250	-0.0822	-0.2874
6.7	+0.1964	-1.8816	-0.0188	-0.3204
6.8	+0.3864	-2.0516	+0.0436	-0.3640
6.9	+0.5892	-2.2348	+0.1036	-0.4188
7.0	+0.8036	-2.4304	+0.1608	-0.4862

In using these values, downward p is positive.

$\xi_1 = 10.0$
$\nu = 0.20$

Table 26. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-0.9960	-0.9960	+0.0032	+0.0032	-0.9420	0
0.1	-0.9960	-0.9960	+0.0033	+0.0033	-0.9420	+0.0002
0.2	-0.9960	-0.9960	+0.0034	+0.0033	-0.9420	+0.0004
0.3	-0.9960	-0.9960	+0.0035	+0.0034	-0.9420	+0.0007
0.4	-0.9960	-0.9961	+0.0037	+0.0035	-0.9421	+0.0009
0.5	-0.9960	-0.9961	+0.0040	+0.0036	-0.9422	+0.0011
0.6	-0.9960	-0.9962	+0.0043	+0.0038	-0.9423	+0.0013
0.7	-0.9961	-0.9963	+0.0046	+0.0039	-0.9424	+0.0015
0.8	-0.9961	-0.9964	+0.0050	+0.0041	-0.9425	+0.0017
0.9	-0.9961	-0.9965	+0.0055	+0.0044	-0.9427	+0.0019
1.0	-0.9962	-0.9966	+0.0060	+0.0046	-0.9429	+0.0021
1.2	-0.9963	-0.9970	+0.0072	+0.0052	-0.9433	+0.0024
1.6	-0.9966	-0.9981	+0.0100	+0.0067	-0.9447	+0.0030
2.0	-0.9970	-0.9998	+0.0132	+0.0084	-0.9470	+0.0032
2.4	-0.9977	-1.0025	+0.0163	+0.0102	-0.9503	+0.0030
2.8	-0.9987	-1.0062	+0.0186	+0.0119	-0.9549	+0.0020
3.2	-0.9999	-1.0111	+0.0193	+0.0131	-0.9611	+0.0002
3.6	-1.0015	-1.0172	+0.0171	+0.0134	-0.9688	-0.0029
4.0	-1.0034	-1.0244	+0.0105	+0.0122	-0.9778	-0.0074
4.5	-1.0062	-1.0337	-0.0067	+0.0077	-0.9900	-0.0152
5.0	-1.0094	-1.0413	-0.0370	-0.0014	-1.0008	-0.0255
5.5	-1.0125	-1.0435	-0.0832	-0.0164	-1.0060	-0.0372
6.0	-1.0148	-1.0344	-0.1455	-0.0378	-0.9992	-0.0481
6.5	-1.0154	-1.0066	-0.2193	-0.0651	-0.9720	-0.0542
7.0	-1.0129	-0.9515	-0.2924	-0.0954	-0.9145	-0.0492
7.5	-1.0060	-0.8608	-0.3413	-0.1226	-0.8169	-0.0246
8.0	-0.9931	-0.7302	-0.3282	-0.1363	-0.6734	+0.0300
8.5	-0.9729	-0.5639	-0.1992	-0.1210	-0.4869	+0.1251
9.0	-0.9451	-0.3815	+0.1125	-0.0566	-0.2767	+0.2682
9.1	-0.9387	-0.3465	+0.2032	-0.0358	-0.2352	+0.3028
9.2	-0.9321	-0.3128	+0.3046	-0.0118	-0.1950	+0.3391
9.3	-0.9253	-0.2812	+0.4174	+0.0154	-0.1566	+0.3773
9.4	-0.9182	-0.2522	+0.5419	+0.0460	-0.1205	+0.4171
9.5	-0.9111	-0.2266	+0.6786	+0.0803	-0.0877	+0.4584
9.6	-0.9039	-0.2048	+0.8280	+0.1184	-0.0588	+0.5010
9.7	-0.8966	-0.1879	+0.9904	+0.1604	-0.0346	+0.5446
9.8	-0.8893	-0.1768	+1.1658	+0.2065	-0.0161	+0.5890
9.9	-0.8820	-0.1721	+1.3544	+0.2568	-0.0042	+0.6339
10.0	-0.8749	-0.1750	+1.5564	+0.3113	0	+0.6788

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 10.0$
$\nu = 0.20$

Table 27. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-0.9928	-0.9992	-0.9928	-0.9992
0.1	-0.9927	-0.9993	-0.9927	-0.9993
0.2	-0.9926	-0.9994	-0.9927	-0.9993
0.3	-0.9925	-0.9995	-0.9926	-0.9994
0.4	-0.9923	-0.9997	-0.9926	-0.9996
0.5	-0.9920	-1.0000	-0.9925	-0.9997
0.6	-0.9917	-1.0003	-0.9924	-1.0000
0.7	-0.9915	-1.0007	-0.9924	-1.0002
0.8	-0.9911	-1.0011	-0.9923	-1.0005
0.9	-0.9906	-1.0016	-0.9921	-1.0009
1.0	-0.9902	-1.0022	-0.9920	-1.0012
1.2	-0.9891	-1.0035	-0.9918	-1.0022
1.6	-0.9866	-1.0066	-0.9914	-1.0048
2.0	-0.9838	-1.0102	-0.9914	-1.0082
2.4	-0.9814	-1.0140	-0.9923	-1.0127
2.8	-0.9801	-1.0173	-0.9943	-1.0181
3.2	-0.9806	-1.0192	-0.9980	-1.0242
3.6	-0.9844	-1.0186	-1.0038	-1.0306
4.0	-0.9929	-1.0139	-1.0122	-1.0366
4.5	-1.0129	-0.9995	-1.0260	-1.0414
5.0	-1.0464	-0.9724	-1.0427	-1.0399
5.5	-1.0957	-0.9293	-1.0599	-1.0271
6.0	-1.1603	-0.8693	-1.0722	-0.9966
6.5	-1.2347	-0.7961	-1.0717	-0.9415
7.0	-1.3053	-0.7205	-1.0469	-0.8561
7.5	-1.3473	-0.6647	-0.9834	-0.7382
8.0	-1.3213	-0.6649	-0.8665	-0.5939
8.5	-1.1721	-0.7737	-0.6849	-0.4429
9.0	-0.8326	-1.0576	-0.4381	-0.3249
9.1	-0.7345	-1.1419	-0.3823	-0.3107
9.2	-0.6275	-1.2367	-0.3246	-0.3010
9.3	-0.5079	-1.3427	-0.2658	-0.2966
9.4	-0.3763	-1.4601	-0.2062	-0.2982
9.5	-0.2325	-1.5897	-0.1463	-0.3069
9.6	-0.0759	-1.7319	-0.0864	-0.3232
9.7	+0.0938	-1.8870	-0.0275	-0.3483
9.8	+0.2765	-2.0551	+0.0297	-0.3833
9.9	+0.4724	-2.2364	+0.0847	-0.4289
10.0	+0.6815	-2.4313	+0.1363	-0.4863

In using these values, downward p is positive.

$\xi_1 = 15.0$
$\nu = 0.20$

Table 28. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0001	-1.0001	0	0	-0.8970	0
0.2	-1.0001	-1.0001	0	0	-0.8970	0
0.4	-1.0001	-1.0001	0	0	-0.8970	0
0.6	-1.0001	-1.0001	0	0	-0.8970	0
0.8	-1.0001	-1.0001	-0.0001	0	-0.8970	-0.0001
1.0	-1.0001	-1.0001	-0.0001	0	-0.8970	-0.0001
1.2	-1.0001	-1.0001	-0.0001	-0.0001	-0.8970	-0.0001
1.6	-1.0001	-1.0001	-0.0003	-0.0001	-0.8970	-0.0001
2.0	-1.0001	-1.0001	-0.0004	-0.0002	-0.8970	-0.0002
2.4	-1.0001	-1.0000	-0.0006	-0.0003	-0.8969	-0.0002
2.8	-1.0001	-0.9999	-0.0007	-0.0004	-0.8968	-0.0002
3.2	-1.0001	-0.9998	-0.0009	-0.0005	-0.8966	-0.0001
3.6	-1.0000	-0.9996	-0.0009	-0.0005	-0.8963	-0.0001
4.0	-1.0000	-0.9993	-0.0009	-0.0006	-0.8960	+0.0001
4.5	-0.9999	-0.9989	-0.0006	-0.0005	-0.8955	+0.0003
5.0	-0.9997	-0.9984	+0.0002	-0.0004	-0.8949	+0.0007
5.5	-0.9996	-0.9980	+0.0016	0	-0.8944	+0.0012
6.0	-0.9995	-0.9979	+0.0038	+0.0007	-0.8941	+0.0017
6.5	-0.9994	-0.9982	+0.0067	+0.0017	-0.8943	+0.0023
7.0	-0.9993	-0.9995	+0.0103	+0.0030	-0.8955	+0.0026
7.5	-0.9994	-1.0020	+0.0140	+0.0044	-0.8982	+0.0024
8.0	-0.9997	-1.0063	+0.0166	+0.0057	-0.9027	+0.0014
8.5	-1.0002	-1.0125	+0.0163	+0.0064	-0.9095	-0.0011
9.0	-1.0011	-1.0206	+0.0107	+0.0058	-0.9184	-0.0056
9.5	-1.0024	-1.0295	-0.0036	+0.0030	-0.9287	-0.0124
10.0	-1.0040	-1.0375	-0.0301	-0.0033	-0.9381	-0.0215
10.5	-1.0057	-1.0407	-0.0718	-0.0140	-0.9431	-0.0323
11.0	-1.0072	-1.0338	-0.1295	-0.0299	-0.9377	-0.0427
11.5	-1.0079	-1.0094	-0.1997	-0.0506	-0.9140	-0.0491
12.0	-1.0070	-0.9586	-0.2711	-0.0737	-0.8623	-0.0454
12.5	-1.0035	-0.8730	-0.3216	-0.0943	-0.7731	-0.0235
13.0	-0.9962	-0.7472	-0.3144	-0.1035	-0.6400	+0.0272
13.1	-0.9930	-0.7172	-0.3017	-0.1028	-0.6081	+0.0417
13.2	-0.9919	-0.6858	-0.2841	-0.1010	-0.5745	+0.0579
13.3	-0.9895	-0.6531	-0.2610	-0.0980	-0.5393	+0.0758
13.4	-0.9869	-0.6192	-0.2320	-0.0936	-0.5028	+0.0956
13.5	-0.9840	-0.5842	-0.1965	-0.0876	-0.4650	+0.1172
13.6	-0.9809	-0.5484	-0.1539	-0.0798	-0.4261	+0.1407
13.7	-0.9776	-0.5120	-0.1036	-0.0703	-0.3864	+0.1662
13.8	-0.9741	-0.4753	-0.0449	-0.0586	-0.3462	+0.1937
13.9	-0.9704	-0.4386	+0.0226	-0.0448	-0.3058	+0.2232
14.0	-0.9665	-0.4023	+0.0997	-0.0285	-0.2656	+0.2546
14.1	-0.9624	-0.3669	+0.1868	-0.0097	-0.2260	+0.2880
14.2	-0.9581	-0.3327	+0.2847	+0.0118	-0.1875	+0.3234
14.3	-0.9536	-0.3004	+0.3938	+0.0363	-0.1507	+0.3605

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 15.0$
$\nu = 0.20$

Table 28. (cont.) Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
14.4	-0.9489	-0.2705	+0.5146	+0.0639	-0.1162	+0.3992
14.5	-0.9441	-0.2438	+0.6477	+0.0946	-0.0846	+0.4396
14.6	-0.9393	-0.2208	+0.7935	+0.1288	-0.0568	+0.4813
14.7	-0.9343	-0.2025	+0.9522	+0.1665	-0.0335	+0.5241
14.8	-0.9293	-0.1895	+1.1243	+0.2078	-0.0156	+0.5678
14.9	-0.9243	-0.1830	+1.3097	+0.2523	-0.0041	+0.6120
15.0	-0.9194	-0.1839	+1.5087	+0.3017	0	+0.6563

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 15.0$
$\nu = 0.20$

Table 29. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-1.0001	-1.0001	-1.0001	-1.0001
0.2	-1.0001	-1.0001	-1.0001	-1.0001
0.4	-1.0001	-1.0001	-1.0001	-1.0001
0.6	-1.0001	-1.0001	-1.0001	-1.0001
0.8	-1.0002	-1.0000	-1.0001	-1.0001
1.0	-1.0002	-1.0000	-1.0001	-1.0001
1.2	-1.0002	-1.0000	-1.0002	-1.0000
1.6	-1.0004	-0.9998	-1.0002	-1.0000
2.0	-1.0005	-0.9997	-1.0003	-0.9999
2.4	-1.0007	-0.9995	-1.0003	-0.9997
2.8	-1.0008	-0.9994	-1.0003	-0.9995
3.2	-1.0010	-0.9992	-1.0003	-0.9993
3.6	-1.0009	-0.9991	-1.0001	-0.9991
4.0	-1.0009	-0.9991	-0.9999	-0.9987
4.5	-1.0005	-0.9993	-0.9994	-0.9984
5.0	-0.9995	-0.9999	-0.9988	-0.9980
5.5	-0.9980	-1.0012	-0.9980	-0.9980
6.0	-0.9957	-1.0033	-0.9972	-0.9986
6.5	-0.9927	-1.0061	-0.9965	-0.9999
7.0	-0.9890	-1.0096	-0.9965	-1.0025
7.5	-0.9854	-1.0134	-0.9976	-1.0064
8.0	-0.9831	-1.0163	-1.0006	-1.0120
8.5	-0.9839	-1.0165	-1.0061	-1.0189
9.0	-0.9904	-1.0118	-1.0148	-1.0264
9.5	-1.0060	-0.9988	-1.0265	-1.0325
10.0	-1.0341	-0.9739	-1.0408	-1.0342
10.5	-1.0775	-0.9339	-1.0547	-1.0267
11.0	-1.1367	-0.8777	-1.0637	-1.0039
11.5	-1.2076	-0.8082	-1.0600	-0.9588
12.0	-1.2781	-0.7359	-1.0323	-0.8849
12.5	-1.3251	-0.6819	-0.9673	-0.7787
13.0	-1.3106	-0.6818	-0.8507	-0.6437
13.1	-1.2947	-0.6913	-0.8200	-0.6144
13.2	-1.2760	-0.7078	-0.7868	-0.5848
13.3	-1.2505	-0.7285	-0.7511	-0.5551
13.4	-1.2189	-0.7549	-0.7128	-0.5256
13.5	-1.1805	-0.7875	-0.6718	-0.4966
13.6	-1.1348	-0.8270	-0.6282	-0.4686
13.7	-1.0812	-0.8740	-0.5823	-0.4417
13.8	-1.0190	-0.9292	-0.5339	-0.4167
13.9	-0.9478	-0.9930	-0.4836	-0.3938
14.0	-0.8668	-1.0662	-0.4308	-0.3738
14.1	-0.7756	-1.1492	-0.3766	-0.3572
14.2	-0.6734	-1.2428	-0.3209	-0.3445

In using these values, downward p is positive.

$\xi_1 = 15.0$
$\nu = 0.20$

Table 29. (Cont.) Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
14.3	-0.5598	-1.3474	-0.2641	-0.3367
14.4	-0.4343	-1.4635	-0.2066	-0.3344
14.5	-0.2964	-1.5918	-0.1492	-0.3384
14.6	-0.1458	-1.7328	-0.0920	-0.3496
14.7	+0.0179	-1.8865	-0.0360	-0.3690
14.8	+0.1950	-2.0536	+0.0183	-0.3973
14.9	+0.3854	-2.2340	+0.0693	-0.4353
15.0	+0.5893	-2.4281	+0.1178	-0.4856

In using these values, downward p is positive.

$\xi_1 = 20.0$
$\nu = 0.20$

Table 30. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0000	-1.0000	0	0	-0.8714	0
0.2	↑	↑	↑	↑	-0.8714	↑
0.4					-0.8714	
0.6					-0.8714	
0.8					-0.8714	
1.0					-0.8714	
1.2					-0.8713	
1.6					-0.8713	
2.0					-0.8713	
2.4					-0.8714	
2.8					-0.8714	
3.2					-0.8714	
3.6					-0.8714	
4.0		↓			-0.8714	
4.5		-1.0000			-0.8714	
5.0		-1.0001	↓		-0.8714	
5.5		-1.0001	0		-0.8714	↓
6.0		-1.0001	-0.0001	↓	-0.8715	0
6.5		-1.0001	-0.0002	0	-0.8715	-0.0001
7.0		-1.0001	-0.0003	-0.0001	-0.8715	-0.0001
7.5		-1.0000	-0.0005	-0.0001	-0.8714	-0.0001
8.0		-0.9999	-0.0006	-0.0002	-0.8713	-0.0001
8.5		-0.9997	-0.0008	-0.0003	-0.8711	-0.0001
9.0	↓	-0.9994	-0.0008	-0.0003	-0.8708	0
9.5	-1.0000	-0.9991	-0.0006	-0.0003	-0.8704	+0.0002
10.0	-0.9999	-0.9986	+0.0001	-0.0002	-0.8699	+0.0006
10.5	-0.9998	-0.9983	+0.0013	+0.0001	-0.8694	+0.0010
11.0	-0.9998	-0.9981	+0.0032	+0.0006	-0.8692	+0.0015
11.5	-0.9997	-0.9983	+0.0058	+0.0013	-0.8694	+0.0020
12.0	-0.9996	-0.9994	+0.0091	+0.0023	-0.8704	+0.0023
12.5	-0.9997	-1.0016	+0.0126	+0.0034	-0.8727	+0.0022
13.0	-0.9998	-1.0055	+0.0152	+0.0044	-0.8767	+0.0013
13.5	-1.0001	-1.0114	+0.0152	+0.0049	-0.8828	-0.0010
14.0	-1.0007	-1.0190	+0.0103	+0.0043	-0.8910	-0.0050
14.5	-1.0014	-1.0277	-0.0028	+0.0017	-0.9005	-0.0114
15.0	-1.0025	-1.0355	-0.0276	-0.0038	-0.9094	-0.0200
15.5	-1.0036	-1.0392	-0.0672	-0.0132	-0.9141	-0.0303
16.0	-1.0046	-1.0332	-0.1226	-0.0272	-0.9092	-0.0404
16.5	-1.0052	-1.0102	-0.1907	-0.0452	-0.8868	-0.0468
17.0	-1.0047	-0.9616	-0.2610	-0.0653	-0.8376	-0.0436
17.5	-1.0024	-0.8784	-0.3119	-0.0830	-0.7522	-0.0228
18.0	-0.9973	-0.7552	-0.3074	-0.0898	-0.6239	+0.0260
18.1	-0.9959	-0.7257	-0.2955	-0.0888	-0.5930	+0.0401
18.2	-0.9943	-0.6948	-0.2788	-0.0868	-0.5605	+0.0558
18.3	-0.9926	-0.6624	-0.2568	-0.0836	-0.5264	+0.0733

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 20.0$
$\nu = 0.20$

Table 30. (cont.) Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
18.4	-0.9907	-0.6288	-0.2290	-0.0791	-0.4909	+0.0925
18.5	-0.9887	-0.5941	-0.1947	-0.0732	-0.4542	+0.1136
18.6	-0.9865	-0.5586	-0.1535	-0.0656	-0.4164	+0.1366
18.7	-0.9841	-0.5224	-0.1046	-0.0563	-0.3778	+0.1615
18.8	-0.9815	-0.4857	-0.0476	-0.0450	-0.3386	+0.1884
18.9	-0.9788	-0.4491	+0.0183	-0.0317	-0.2993	+0.2173
19.0	-0.9759	-0.4128	+0.0937	-0.0161	-0.2600	+0.2482
19.1	-0.9729	-0.3771	+0.1790	+0.0019	-0.2214	+0.2811
19.2	-0.9697	-0.3427	+0.2751	+0.0224	-0.1838	+0.3158
19.3	-0.9664	-0.3101	+0.3823	+0.0457	-0.1478	+0.3524
19.4	-0.9629	-0.2797	+0.5013	+0.0719	-0.1140	+0.3907
19.5	-0.9593	-0.2524	+0.6325	+0.1011	-0.0831	+0.4305
19.6	-0.9557	-0.2288	+0.7764	+0.1334	-0.0558	+0.4718
19.7	-0.9519	-0.2096	+0.9334	+0.1691	-0.0329	+0.5142
19.8	-0.9481	-0.1958	+1.1037	+0.2082	-0.0153	+0.5575
19.9	-0.9443	-0.1883	+1.2876	+0.2508	-0.0040	+0.6014
20.0	-0.9405	-0.1881	+1.4851	+0.2970	0	+0.6455

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 20.0$
$\nu = 0.20$

Table 31. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
0	-1.0000	-1.0000	-1.0000	-1.0000
0.2	↑	↑	↑	↑
0.4				
0.6				
0.8				
1.0				
1.2				
1.6				
2.0				
2.4				
2.8				
3.2				
3.6				
4.0			↓	↓
4.5			-1.0000	-1.0000
5.0	↓	↓	-1.0001	-1.0001
5.5	-1.0000	-1.0000	-1.0001	-1.0001
6.0	-1.0001	-0.9999	-1.0001	-1.0001
6.5	-1.0002	-0.9998	-1.0001	-1.0001
7.0	-1.0003	-0.9997	-1.0002	-1.0000
7.5	-1.0005	-0.9995	-1.0001	-0.9999
8.0	-1.0006	-0.9994	-1.0001	-0.9997
8.5	-1.0008	-0.9992	-1.0000	-0.9994
9.0	-1.0008	-0.9992	-0.9997	-0.9991
9.5	-1.0006	-0.9994	-0.9994	-0.9988
10.0	-0.9998	-1.0000	-0.9988	-0.9984
10.5	-0.9985	-1.0011	-0.9982	-0.9984
11.0	-0.9966	-1.0030	-0.9975	-0.9987
11.5	-0.9939	-1.0055	-0.9970	-0.9996
12.0	-0.9905	-1.0087	-0.9971	-1.0017
12.5	-0.9871	-1.0123	-0.9982	-1.0050
13.0	-0.9846	-1.0150	-1.0011	-1.0099
13.5	-0.9849	-1.0151	-1.0065	-1.0163
14.0	-0.9904	-1.0110	-1.0147	-1.0233
14.5	-1.0042	-0.9986	-1.0260	-1.0294
15.0	-1.0301	-0.9749	-1.0393	-1.0317
15.5	-1.0708	-0.9364	-1.0524	-1.0260
16.0	-1.1272	-0.8820	-1.0604	-1.0060
16.5	-1.1959	-0.8145	-1.0554	-0.9650
17.0	-1.2657	-0.7437	-1.0269	-0.8963
17.5	-1.3143	-0.6905	-0.9614	-0.7954
18.0	-1.3047	-0.6899	-0.8450	-0.6654

In using these values, downward p is positive.

$\xi_1 = 20.0$
$\nu = 0.20$

Table 31. (Cont.) Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,u}$	$\left(\frac{2t}{PR}\right)\sigma_{x,l}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
18.1	-1.2914	-0.7004	-0.8145	-0.6369
18.2	-1.2731	-0.7155	-0.7816	-0.6080
18.3	-1.2494	-0.7358	-0.7460	-0.5788
18.4	-1.2197	-0.7617	-0.7079	-0.5497
18.5	-1.1834	-0.7940	-0.6673	-0.5209
18.6	-1.1400	-0.8330	-0.6242	-0.4930
18.7	-1.0887	-0.8795	-0.5787	-0.4661
18.8	-1.0291	-0.9339	-0.5307	-0.4407
18.9	-0.9605	-0.9971	-0.4808	-0.4174
19.0	-0.8822	-1.0696	-0.4289	-0.3967
19.1	-0.7939	-1.1519	-0.3752	-0.3790
19.2	-0.6946	-1.2448	-0.3203	-0.3651
19.3	-0.5841	-1.3487	-0.2644	-0.3558
19.4	-0.4616	-1.4642	-0.2078	-0.3516
19.5	-0.3268	-1.5918	-0.1513	-0.3535
19.6	-0.1793	-1.7321	-0.0954	-0.3622
19.7	-0.0185	-1.8853	-0.0405	-0.3787
19.8	+0.1556	-2.0518	+0.0124	-0.4040
19.9	+0.3433	-2.2319	+0.0625	-0.4391
20.0	+0.5446	-2.4256	+0.1089	-0.4851

In using these values, downward p is positive.

$\xi_1 = 25.0$
$\nu = 0.20$

Table 32. Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
0	-1.0000	-1.0000	0	0	-0.8565	0
0.4						
0.8						
1.2						
1.6						
2.0						
2.4						
2.8						
3.2						
3.6						
4.0						
4.5						
5.0						
5.5						
6.0						
6.5						
7.0						
7.5						
8.0						
8.5						
9.0						
9.5					-0.8565	
10.0		-1.0000			-0.8566	
10.5		-1.0001	0		-0.8566	
11.0		-1.0001	-0.0001		-0.8566	0
11.5		-1.0001	-0.0001	0	-0.8566	-0.0001
12.0		-1.0001	-0.0003	-0.0001	-0.8566	-0.0001
12.5		-1.0000	-0.0004	-0.0001	-0.8566	-0.0001
13.0		-0.9999	-0.0006	-0.0002	-0.8565	-0.0001
13.5		-0.9998	-0.0007	-0.0002	-0.8563	-0.0001
14.0		-0.9995	-0.0007	-0.0002	-0.8560	+0.0000
14.5	-1.0000	-0.9992	-0.0005	-0.0002	-0.8556	+0.0002
15.0	-0.9999	-0.9987	+0.0000	-0.0001	-0.8552	+0.0005
15.5	-0.9999	-0.9984	+0.0011	+0.0002	-0.8548	+0.0009
16.0	-0.9998	-0.9982	+0.0029	+0.0006	-0.8545	+0.0014
16.5	-0.9998	-0.9984	+0.0054	+0.0012	-0.8547	+0.0018
17.0	-0.9998	-0.9994	+0.0086	+0.0020	-0.8556	+0.0022
17.5	-0.9998	-1.0015	+0.0119	+0.0029	-0.8578	+0.0021
18.0	-0.9999	-1.0052	+0.0145	+0.0038	-0.8616	+0.0012
18.5	-1.0001	-1.0108	+0.0146	+0.0042	-0.8674	-0.0009
19.0	-1.0005	-1.0182	+0.0100	+0.0036	-0.8751	-0.0048
19.5	-1.0010	-1.0267	-0.0024	+0.0011	-0.8842	-0.0109
20.0	-1.0018	-1.0344	-0.0263	-0.0040	-0.8927	-0.0192
20.5	-1.0026	-1.0382	-0.0647	-0.0128	-0.8973	-0.0292

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 25.0$
$\nu = 0.20$

Table 32. (cont.) Direct, Bending and Shear Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{PR}\right)\sigma_{x,b}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$
21.0	-1.0034	-1.0327	-0.1188	-0.0256	-0.8926	-0.0391
21.5	-1.0039	-1.0106	-0.1856	-0.0424	-0.8710	-0.0454
22.0	-1.0036	-0.9632	-0.2551	-0.0609	-0.8232	-0.0426
22.5	-1.0018	-0.8816	-0.3062	-0.0769	-0.7399	-0.0223
23.0	-0.9980	-0.7599	-0.3031	-0.0823	-0.6144	+0.0254
23.1	-0.9969	-0.7307	-0.2917	-0.0812	-0.5841	+0.0392
23.2	-0.9957	-0.7001	-0.2756	-0.0790	-0.5522	+0.0546
23.3	-0.9943	-0.6680	-0.2542	-0.0757	-0.5187	+0.0718
23.4	-0.9929	-0.6346	-0.2270	-0.0711	-0.4839	+0.0907
23.5	-0.9913	-0.6001	-0.1936	-0.0652	-0.4478	+0.1115
23.6	-0.9895	-0.5647	-0.1532	-0.0577	-0.4107	+0.1342
23.7	-0.9877	-0.5286	-0.1052	-0.0485	-0.3727	+0.1588
23.8	-0.9857	-0.4921	-0.0490	-0.0374	-0.3342	+0.1854
23.9	-0.9835	-0.4555	+0.0159	-0.0244	-0.2954	+0.2139
24.0	-0.9812	-0.4191	+0.0902	-0.0092	-0.2568	+0.2445
24.1	-0.9788	-0.3834	+0.1745	+0.0084	-0.2186	+0.2770
24.2	-0.9763	-0.3489	+0.2694	+0.0284	-0.1816	+0.3114
24.3	-0.9736	-0.3160	+0.3755	+0.0510	-0.1461	+0.3476
24.4	-0.9709	-0.2854	+0.4933	+0.0764	-0.1127	+0.3856
24.5	-0.9680	-0.2577	+0.6234	+0.1047	-0.0822	+0.4252
24.6	-0.9651	-0.2337	+0.7662	+0.1360	-0.0552	+0.4662
24.7	-0.9623	-0.2141	+0.9220	+0.1706	-0.0326	+0.5083
24.8	-0.9593	-0.1998	+1.0913	+0.2084	-0.0152	+0.5514
24.9	-0.9560	-0.1916	+1.2741	+0.2495	-0.0040	+0.5951
25.0	-0.9529	-0.1907	+1.4707	+0.2941	0	+0.6390

Note:- Bending stresses are given for the upper surface. For the lower surface reverse the sign only.

In using these values, downward p is positive.

$\xi_1 = 25.0$
$\nu = 0.20$

Table 33. Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
0	-1.0000	-1.0000	-1.0000	-1.0000
0.4				
0.8				
1.2				
1.6				
2.0				
2.4				
2.8				
3.2				
3.6				
4.0				
4.5				
5.0				
5.5				
6.0				
6.5				
7.0				
7.5				
8.0				
8.5				
9.0				
9.5				
10.0			-1.0000	-1.0000
10.5	-1.0000	-1.0000	-1.0001	-1.0001
11.0	-1.0001	-0.9999	-1.0001	-1.0001
11.5	-1.0001	-0.9999	-1.0001	-1.0001
12.0	-1.0003	-0.9997	-1.0002	-1.0000
12.5	-1.0004	-0.9996	-1.0001	-0.9999
13.0	-1.0006	-0.9994	-1.0001	-0.9997
13.5	-1.0007	-0.9993	-1.0000	-0.9996
14.0	-1.0007	-0.9993	-0.9997	-0.9993
14.5	-1.0005	-0.9995	-0.9994	-0.9990
15.0	-0.9999	-0.9999	-0.9988	-0.9986
15.5	-0.9988	-1.0010	-0.9982	-0.9986
16.0	-0.9969	-1.0027	-0.9976	-0.9988
16.5	-0.9944	-1.0052	-0.9972	-0.9996
17.0	-0.9912	-1.0084	-0.9974	-1.0014
17.5	-0.9879	-1.0117	-0.9986	-1.0044
18.0	-0.9854	-1.0144	-1.0014	-1.0090
18.5	-0.9855	-1.0147	-1.0066	-1.0150
19.0	-0.9905	-1.0105	-1.0146	-1.0218
19.5	-1.0034	-0.9986	-1.0256	-1.0278
20.0	-1.0281	-0.9755	-1.0384	-1.0304
20.5	-1.0673	-0.9379	-1.0510	-1.0254

In using these values, downward p is positive.

$\xi_1 = 25.0$
$\nu = 0.20$

Table 33. (Cont.) Combined (Direct and Bending) Stress Intensities.

ξ	$\left(\frac{2t}{pR}\right)\sigma_{x,u}$	$\left(\frac{2t}{pR}\right)\sigma_{x,l}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,u}$	$\left(\frac{2t}{pR}\right)\sigma_{\phi,l}$
21.0	-1.1222	-0.8846	-1.0583	-1.0071
21.5	-1.1895	-0.8183	-1.0530	-0.9682
22.0	-1.2587	-0.7485	-1.0241	-0.9023
22.5	-1.3080	-0.6956	-0.9585	-0.8047
23.0	-1.3011	-0.6949	-0.8422	-0.6776
23.1	-1.2886	-0.7052	-0.8119	-0.6495
23.2	-1.2713	-0.7201	-0.7791	-0.6211
23.3	-1.2485	-0.7401	-0.7437	-0.5923
23.4	-1.2199	-0.7659	-0.7057	-0.5635
23.5	-1.1849	-0.7977	-0.6653	-0.5349
23.6	-1.1427	-0.8363	-0.6224	-0.5070
23.7	-1.0929	-0.8825	-0.5771	-0.4801
23.8	-1.0347	-0.9367	-0.5295	-0.4547
23.9	-0.9676	-0.9994	-0.4799	-0.4311
24.0	-0.8910	-1.0714	-0.4283	-0.4099
24.1	-0.8043	-1.1533	-0.3750	-0.3918
24.2	-0.7069	-1.2457	-0.3205	-0.3773
24.3	-0.5981	-1.3491	-0.2650	-0.3670
24.4	-0.4776	-1.4642	-0.2090	-0.3618
24.5	-0.3446	-1.5914	-0.1530	-0.3624
24.6	-0.1989	-1.7313	-0.0977	-0.3697
24.7	-0.0403	-1.8843	-0.0435	-0.3847
24.8	+0.1320	-2.0506	+0.0086	-0.4082
24.9	+0.3181	-2.2301	+0.0579	-0.4411
25.0	+0.5178	-2.4236	+0.1034	-0.4848

In using these values, downward p is positive.

Table 34. Values and Location of Maximum Stresses.

ξ_1	Radial-Tensile			Circumferential-Tensile		
	$\left(\frac{2t}{PR}\right)\sigma_{x,max}$	Surface	at ξ	$\left(\frac{2t}{PR}\right)\sigma_{\phi,max}$	Surface	at ξ
0.5	0.1097	Upper	Outer Edge	0.0648	Lower	0.0
1.0	0.4223	"	" "	0.2372	"	0.0
1.5	0.8613	"	" "	0.4346	"	0.0
2.0	1.2575	"	" "	0.5148	"	0.0
2.5	1.4492	"	" "	0.3815	"	0.0
3.0	1.4300	"	" "	0.2859	Upper	Outer Edge
3.5	1.3070	"	" "	0.2614	"	" "
4.0	1.1731	"	" "	0.2345	"	" "
4.5	1.0618	"	" "	0.2125	"	" "
5.0	0.9803	"	" "	0.1963	"	" "
6.0	0.8740	"	" "	0.1749	"	" "
7.0	0.8036	"	" "	0.1608	"	" "
10.0	0.6815	"	" "	0.1363	"	" "
15.0	0.5893	"	" "	0.1178	"	" "
20.0	0.5446	"	" "	0.1089	"	" "
25.0	0.5178	"	" "	0.1034	"	" "

In using these values, downward p is positive.

Table 34. (Cont.) Values and Location of Maximum Stresses.

ξ_1	Radial - Compressive			Circumferential Compressive		
	$\left(\frac{2t}{PR}\right)\sigma_{x,max}$	Surface	at ξ	$\left(\frac{2t}{PR}\right)\sigma_{\phi,max}$	Surface	at ξ
0.5	-0.1111	Lower	Outer Edge	-0.0676	Upper	0.0
1.0	-0.4479	"	" "	-0.2832	"	0.0
1.5	-0.9827	"	" "	-0.6526	"	0.0
2.0	-1.5845	"	" "	-1.0996	"	0.0
2.5	-2.0588	"	" "	-1.4621	"	0.0
3.0	-2.3160	"	" "	-1.6317	"	0.0
3.5	-2.4100	"	" "	-1.6282	"	0.0
4.0	-2.4309	"	" "	-1.5264	"	0.0
4.5	-2.4282	"	" "	-1.3900	"	0.0
5.0	-2.4257	"	" "	-1.2573	"	0.5
6.0	-2.4280	"	" "	-1.1286	"	2.4
7.0	-2.4304	"	" "	-1.0943	"	3.6
10.0	-2.4313	"	" "	-1.0722	"	6.0
15.0	-2.4281	"	" "	-1.0637	"	11.0
20.0	-2.4256	"	" "	-1.0604	"	16.0
25.0	-2.4236	"	" "	-1.0583	"	21.0

In using these values, downward p is positive.

Table 35. Values of $\left(\frac{E}{P}\right) W_c$ and R for t = 2 inches, x_1 = 20 feet.

ξ_1	T	$\left(\frac{E}{P}\right) W_c$ in Feet	R in Feet
0	0	-62.2×10^5	∞
0.5	-0.0019	-61.5×10^5	32560
1.0	-0.0307	-61.1×10^5	8139
1.5	-0.1451	-57.1×10^5	3617
2.0	-0.3885	-48.4×10^5	2035
2.5	-0.7147	-36.4×10^5	1302
3.0	-1.0143	-25.0×10^5	904
3.5	-1.2172	-16.2×10^5	665
4.0	-1.3177	-10.3×10^5	509
4.5	-1.3400	-6.50×10^5	402
5.0	-1.3120	-4.19×10^5	325
6.0	-1.1909	-1.83×10^5	226
7.0	-1.0738	-0.894×10^5	166
10.0	-0.9420	-0.188×10^5	81
15.0	-0.8970	-0.0353×10^5	36
20.0	-0.8714	-0.0108×10^5	20
25.0	-0.8565	-0.00437×10^5	13

In using these values, downward p is positive.

Table 36. Values of $\left(\frac{E}{tp}\right) W_c$ and $\frac{h}{t}$ for $\frac{x_1}{t} = 120$.

ξ_1	T	$\left(\frac{E}{tp}\right) w_c$	B.R. = $\frac{h}{t}$
0	0	-373×10^5	0
0.5	-0.0019	-369×10^5	0.0368
1.0	-0.0307	-366×10^5	0.147
1.5	-0.1451	-342×10^5	0.332
2.0	-0.3885	-290×10^5	0.590
2.5	-0.7147	-218×10^5	0.921
3.0	-1.0143	-150×10^5	1.33
3.5	-1.2172	-97.2×10^5	1.81
4.0	-1.3177	-61.8×10^5	2.36
4.5	-1.3400	-39.0×10^5	2.98
5.0	-1.3120	-25.2×10^5	3.68
6.0	-1.1909	-11.0×10^5	5.31
7.0	-1.0738	-5.37×10^5	7.22
10.0	-0.9420	-1.13×10^5	14.7
15.0	-0.8970	-0.212×10^5	33.2
20.0	-0.8714	-0.0648×10^5	59.0
25.0	-0.8565	-0.0262×10^5	92.0

In using these values, downward p is positive.

SECTION VIII

Case II. Outer Edge Effects. Uniform radial edge moment M_E and / or an uniform horizontal force P_E acting all along the outer edge. No surface loading ($P = 0$).

(1) Discussion

It is possible to determine influence coefficients such that for a known radial edge moment and / or a known horizontal edge force we can determine the change of slope $\frac{dw}{dx}$ and / or the horizontal displacement at the outer edge due to these known forces.

Likewise, it is possible to determine, by these same influence coefficients, the radial moment and the horizontal force acting at the outer edge due to a known change of slope $\frac{dw}{dx}$ and a known horizontal displacement of the outer edge.

This section deals with the determination of the above mentioned influence coefficients.

We write

$$\frac{dw}{dx} = a_1 M_E + a_2 P_E \tag{161}$$

$$u_h = b_1 M_E + b_2 P_E \tag{162}$$

where a_1, a_2, b_1, and b_2 are influence coefficients and M_E and P_E are a uniform radial edge moment and a uniform horizontal edge force, respectively.

2. Boundary Conditions.

See Figure 6.

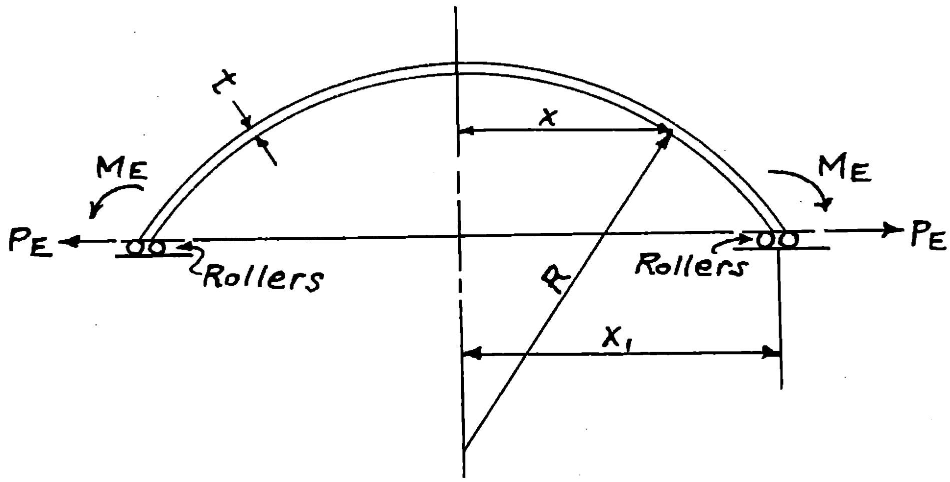


Figure 6. Notation for Shell of Section VIII.

at $x = 0$, $\xi = 0$:

$$W = \text{finite} \tag{163}$$

$$\text{stresses are finite} \tag{164}$$

$$\lim_{x\to 0}(2\pi x Q_x) \to 0 \tag{165}$$

at $x = x_1$, $\xi = \xi_1$:

$$W = 0 \tag{166}$$

$$M_{xx} = M_E \tag{167}$$

$$P_h = P_E \tag{168}$$

3. Equations for the Determination of the Constants

We use the results of Case I in applying conditions (163), (164), (165), noting that W_p and F_p do not exist, and determine as before that

$$C_{30} = C_{40} = C_0 = 0 \tag{169}$$

Noting that $C_0^s = A_0$ and applying condition (166) we have

$$0 = C_{10}\, ber\, \xi_1^d + C_{20}\, bei\, \xi_1 - C_0^s \tag{170}$$

From equations (89a), (90) and (91), applying condition (167) we have

$$-\frac{\ell^2}{D} M_E = C_{10}\left(ber''\xi_1^d + \frac{\nu\, ber'\xi_1^d}{\xi_1^d}\right) + C_{20}\left(bei''\xi_1^d + \frac{\nu\, bei'\xi_1^d}{\xi_1^d}\right) \tag{171}$$

From equations (86), (93a) and (95), applying condition (168) we have

$$\frac{R}{tE} P_E = C_{10}\frac{bei'\xi_1^d}{\xi_1^d} - C_{20}\frac{ber'\xi_1^d}{\xi_1^d} \tag{172}$$

4. Solutions for C_{10} and C_{20}

Solution of equations (171) and (172) by determinants, using equations (37) gives as a result

$$C_{10} = \frac{-\frac{\ell^2}{D} M_E\left(\frac{ber'\xi_1}{\xi_1}\right) + \frac{R}{tE} P_E\left[ber\,\xi_1^d - (1-\nu)\frac{bei'\xi_1}{\xi_1}\right]}{\frac{1}{\xi_1}\left(ber\,\xi_1\, bei'\xi_1 - ber'\xi_1\, bei\,\xi_1\right) - \frac{(1-\nu)}{\xi_1^2}\left[\left(ber'\xi_1\right)^2 + \left(bei'\xi_1\right)^2\right]} \tag{173}$$

$$C_{20} = \frac{-\frac{\ell^2}{D} M_E\left(\frac{bei'\xi_1}{\xi_1}\right) + \frac{R}{tE} P_E\left[bei\,\xi_1 + (1-\nu)\frac{ber'\xi_1}{\xi_1}\right]}{\frac{1}{\xi_1}\left(ber\,\xi_1\, bei'\xi_1 - ber'\xi_1\, bei\,\xi_1\right) - \frac{(1-\nu)}{\xi_1^2}\left[\left(ber'\xi_1\right)^2 + \left(bei'\xi_1\right)^2\right]} \tag{174}$$

The denominators of the above equations may be simplified with the aid of the polar forms of Bessel functions.

From Bessel function theory we may write in polar form (see Appendix A)

$$\left(ber'\xi_1\right)^2 + \left(bei'\xi_1\right)^2 = M_1^2(\xi_1) \tag{175}$$

$$\mathrm{ber}\,\xi_1\,\mathrm{bei}'\xi_1 - \mathrm{ber}'\xi_1\,\mathrm{bei}\,\xi_1 = M_0(\xi_1)\,M_1(\xi_1)\,\mathrm{Sin}\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \qquad (176)$$

where $M_0(\xi_1), M_1(\xi_1), \theta_0(\xi_1)$ and $\theta_1(\xi_1)$ are tabulated functions to be found in various references. (see Appendix A).

Using equations (175) and (176) equations (173) and (174) become

$$C_{10} = \frac{-\frac{\ell^2}{D} M_E\left(\frac{\mathrm{ber}'\xi_1}{\xi_1}\right) + \frac{R}{tE} P_E\left[\mathrm{ber}\,\xi_1 - (1-\nu)\frac{\mathrm{bei}'\xi_1}{\xi_1}\right]}{\frac{M_1(\xi_1)}{\xi_1}\left[(\nu-1)\frac{M_1(\xi_1)}{\xi_1} + M_0(\xi_1)\,\mathrm{Sin}\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right)\right]} \qquad (177)$$

$$C_{20} = \frac{-\frac{\ell^2}{D} M_E\left(\frac{\mathrm{bei}'\xi_1}{\xi_1}\right) + \frac{R}{tE} P_E\left[\mathrm{bei}\,\xi_1 + (1-\nu)\frac{\mathrm{ber}'\xi_1}{\xi_1}\right]}{\frac{M_1(\xi_1)}{\xi_1}\left[(\nu-1)\frac{M_1(\xi_1)}{\xi_1} + M_0(\xi_1)\,\mathrm{Sin}\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right)\right]} \qquad (178)$$

5. Determination of the influence Coefficients a_1 and a_2 for change of Slope $\frac{dw}{dx}$.

From equations (61) and (169) and differentiating with respect to x, we have (at $\xi = \xi_1$)

$$\frac{dw}{dx} = \frac{1}{\ell}\left(C_{10}\,\mathrm{ber}'\xi_1 + C_{20}\,\mathrm{bei}'\xi_1\right) \qquad (179)$$

Substituting in equation (179) the values of C_{10} and C_{20} given by equations (177) and (178) and noting further from Bessel function theory in polar form (see Appendix A) that

$$\mathrm{ber}\,\xi_1\,\mathrm{ber}'\xi_1 + \mathrm{bei}\,\xi_1\,\mathrm{bei}'\xi_1 = M_0(\xi_1)\,M_1(\xi_1)\,\mathrm{Cos}\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \qquad (180)$$

we get as the result

$$\frac{dw}{dx} = \frac{-\frac{\ell}{D} M_E \left(\frac{M_1^2(\xi_1)}{\xi_1}\right) + \frac{R}{tE\ell} P_E \left[M_0(\xi_1) M_1(\xi_1) \cos(\theta_1 - \theta_0 - \frac{\pi}{4}) \right]}{\frac{M_1(\xi_1)}{\xi_1} \left[(\nu - 1) \frac{M_1(\xi_1)}{\xi_1} + M_0(\xi_1) \sin(\theta_1 - \theta_0 - \frac{\pi}{4}) \right]} \qquad (181)$$

Now comparing equation (161) with equation (181) we have

$$a_1 = \frac{-\frac{\ell}{D} M_1(\xi_1)}{M_0(\xi_1) \sin(\theta_1 - \theta_0 - \frac{\pi}{4}) - (1-\nu) \frac{M_1(\xi_1)}{\xi_1}} \qquad (182)$$

$$a_2 = \frac{\frac{R}{tE\ell} \xi_1 M_0(\xi_1) \cos(\theta_1 - \theta_0 - \frac{\pi}{4})}{M_0(\xi_1) \sin(\theta_1 - \theta_0 - \frac{\pi}{4}) - (1-\nu) \frac{M_1(\xi_1)}{\xi_1}} \qquad (183)$$

6. Determination of the Influence Coefficients b_1 and b_2 for Horizontal Displacement U_h

From equation (119) we have

$$U_h = \frac{\ell}{R} \left[C_{10} (\xi_1 \operatorname{bei}'' \xi_1 - \nu \operatorname{bei}' \xi_1) \right] - \frac{\ell}{R} \left[C_{20} (\xi_1 \operatorname{ber}'' \xi_1 - \nu \operatorname{ber}' \xi_1) \right] \qquad (184)$$

Using equations (37) equation (184) becomes

$$U_h = \frac{\ell \xi_1}{R} \left[C_{10} \left\{ \operatorname{ber} \xi_1 - (1+\nu) \frac{\operatorname{bei}' \xi_1}{\xi_1} \right\} + C_{20} \left\{ \operatorname{bei} \xi_1 + (1+\nu) \frac{\operatorname{ber}' \xi_1}{\xi_1} \right\} \right] \qquad (185)$$

Substituting in equation (185) the values of C_{10} and C_{20} given by equations (177) and (178) and noting further from Bessel function theory in polar form (see Appendix A) that

$$(\text{ber}\,\xi_1)^2 + (\text{bei}\,\xi_1)^2 = M_0^2(\xi_1) \qquad (186)$$

we get as the result

$$U_h = \frac{-\frac{\ell^3}{RD} M_E\, \xi_1 M_0(\xi_1) M_1(\xi_1) \cos\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) + \frac{\ell}{tE} P_E\, \xi_1^2 M_0^2(\xi_1)}{M_1(\xi_1)\left[M_0(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) - (1-\nu)\frac{M_1(\xi_1)}{\xi_1}\right]}$$

$$+ \frac{\frac{\ell}{tE} P_E\left[(1-\nu^2)M_1^2(\xi_1) - 2\xi_1 M_0(\xi_1) M_1(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right)\right]}{M_1(\xi_1)\left[M_0(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) - (1-\nu)\frac{M_1(\xi_1)}{\xi_1}\right]} \qquad (187)$$

Now comparing equation (162) with equation (187) we have

$$b_1 = \frac{-\frac{\ell^3}{RD}\, \xi_1 M_0(\xi_1) \cos\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right)}{M_0(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) - (1-\nu)\frac{M_1(\xi_1)}{\xi_1}} \qquad (188)$$

$$b_2 = \frac{\frac{\ell}{tE}\left[(1-\nu^2)M_1(\xi_1) - 2\xi_1 M_0(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right)\right]}{M_0(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) - (1-\nu)\frac{M_1(\xi_1)}{\xi_1}}$$

$$+ \frac{\frac{\ell}{tE}\, \xi_1^2 M_0^2(\xi_1)}{M_1(\xi_1)\left[M_0(\xi_1)\sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) - (1-\nu)\frac{M_1(\xi_1)}{\xi_1}\right]} \qquad (189)$$

7. Application of the Influence Coefficients.

The influence coefficients given by equations (186), (187), (188) and (189) have been computed with the use of the data given in Tables A-5 of Appendix A. These results have been tabulated in Table 37 and plotted with respect to ξ_1 on Graph L for a range of ξ_1 given by $0 \leq \xi_1 \leq 25.0$. It is to be observed that for small values of ξ_1 the precision offered by the graph is insufficient and one is advised to use the tabulated values to plot these coefficients to a more suitable scale for use.

The fundamental types of problems solved by these influence coefficients are two in number; namely,

(1) Given the edge moment M_E and the edge horizontal force P_E (or N_{xx}), to determine the horizontal displacement and rotation of the edge induced by these forces.

(2) Given the horizontal displacement and rotation of the edge, to determine the edge moment M_E and edge horizontal force P_E induced by these displacements.

The second type of problem is the one most likely to be encountered. It permits one to analyze a shell as a membrane, determine the displacements at the boundary due to membrane action and then by the influence coefficients to determine the edge moment and horizontal edge force required to nullify the membrane edge displacements to the extent required and thus satisfy the original conditions of restraint. Inasmuch as in a shell subjected to edge restraint, the maximum tensile and compressive stresses most likely occur in a radial sense at the outer edge, application of these influence coefficients permits one to determine the radial stresses at the outer edge and hence for a preliminary design to decide upon the feasibility of a particular shell.

8. Numerical Example

To illustrate the technique employed in the use of the influence coefficients we will determine the edge moments and horizontal reaction in a shell under uniform radial loading with fixed edges by superposition upon the membrane stress condition of the edge moment and horizontal force calculated by influence coefficients as necessary to satisfy the restraint conditions. These results will be compared with those obtained directly from the results of Section VII.

Data. See Figure a.

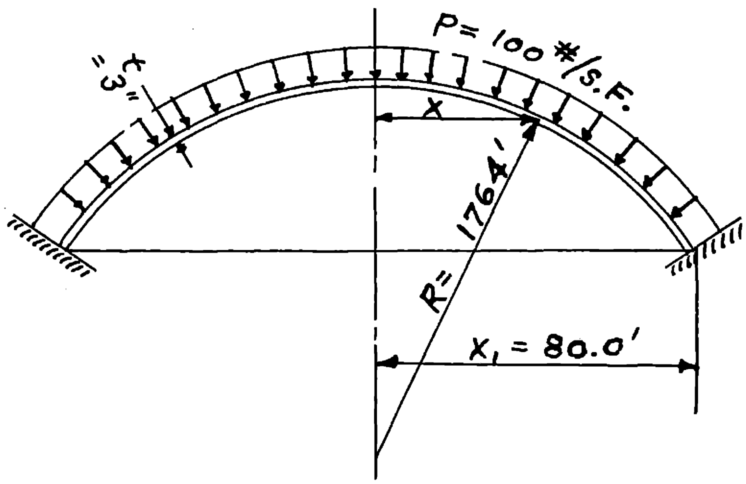


Figure a. Shell Data.

$$\xi_1 = \frac{\sqrt[4]{12(1-\nu^2)}\; X_1}{\sqrt{tR}} = 7.0$$

Membrane Stresses and Displacements

$$N_{xx} = N_{\phi\phi} = \frac{RP}{2} \qquad (a)$$

Displacement inward is

$$\Delta = \varepsilon_x X_1 = \frac{X_1}{tE}\left(N_{xx} - \nu N_{\phi\phi}\right) \qquad (b)$$

Using equations (a) and (b) and substituting numerical data we have

$$\Delta = -0.0436 \text{ feet (inward)} \qquad (c)$$

For membrane stress condition

$$\left(\frac{d\omega}{dx}\right)_{x=x_1} = 0 \qquad (d)$$

From Table 37 for $\xi_1 = 7.0$

$$\left.\begin{aligned} Et^2\left(\frac{t}{R}\right)^{\frac{1}{2}} a_1 &= -9.3401 \\ Et\left(\frac{t}{R}\right)^{\frac{1}{2}} a_2 &= 13.493 \\ E(t)\left(\frac{t}{R}\right)^{\frac{1}{2}} b_1 &= -13.493 \\ E\left(\frac{t}{R}\right)^{\frac{1}{2}} b_2 &= 36.039 \end{aligned}\right\} \qquad (e)$$

Substituting numerical data in equations (e) we have

$$\left.\begin{aligned} a_1 &= -2.42 \times 10^{-5} \\ a_2 &= 8.74 \times 10^{-6} \\ b_1 &= -8.74 \times 10^{-6} \\ b_2 &= 5.83 \times 10^{-6} \end{aligned}\right\} \qquad (f)$$

Substituting equations (f) into equations (161) and (162) using equations (c) and (d) we have

$$\left.\begin{aligned} 0 &= -2.42 \times 10^{-5} M_E + 8.74 \times 10^{-6} P_E \\ +0.0436 &= -8.74 \times 10^{-6} M_E + 5.83 \times 10^{-6} P_E \end{aligned}\right\} \qquad (g)$$

with solution

$$\left.\begin{aligned} P_E &= +16{,}300 \text{ lbs./L. F.} \\ M_E &= +\ 5{,}900 \text{ ft.-lbs./L. F.} \end{aligned}\right\} \qquad (h)$$

For the positive directions of P_E and M_E see Figure 6.

We may now compute the direct and bending stresses at the upper surface (or lower surface) of the shell at the outer edge.

<u>at upper surface</u>

$$\sigma_{x,b} = +\frac{6M_E}{t^2} = +3930 \text{ p.s.i.} \qquad (ia)$$

$$\sigma_{x,d} = +\frac{PR}{2t} + \frac{P_E}{t}$$

$$= -2450 + 450 = -2000 \text{ p.s.i.} \qquad (i_b)$$

Direct Method (Section VII)

From Table (24), for $\xi_1 = 7.0$.

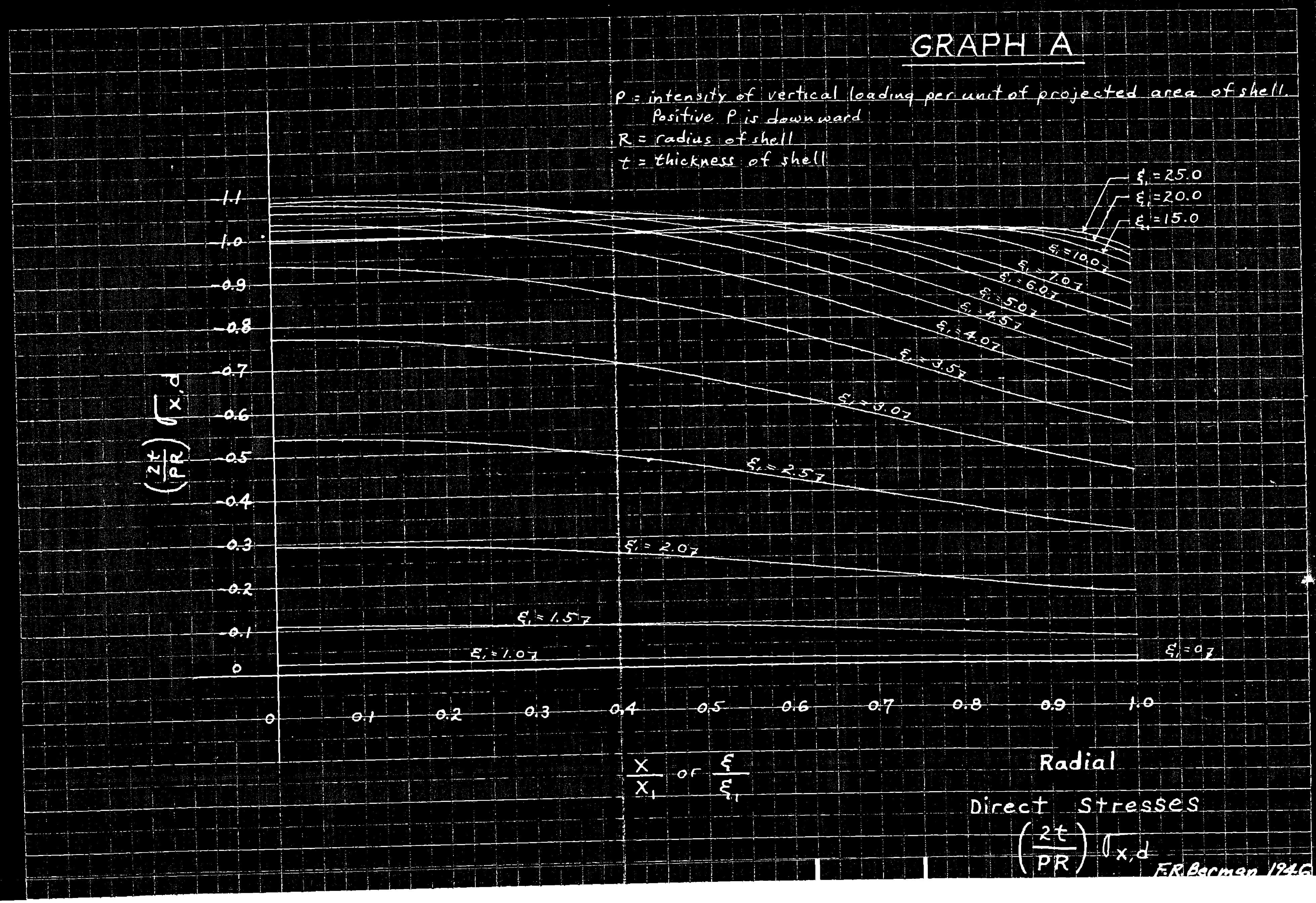
at upper surface

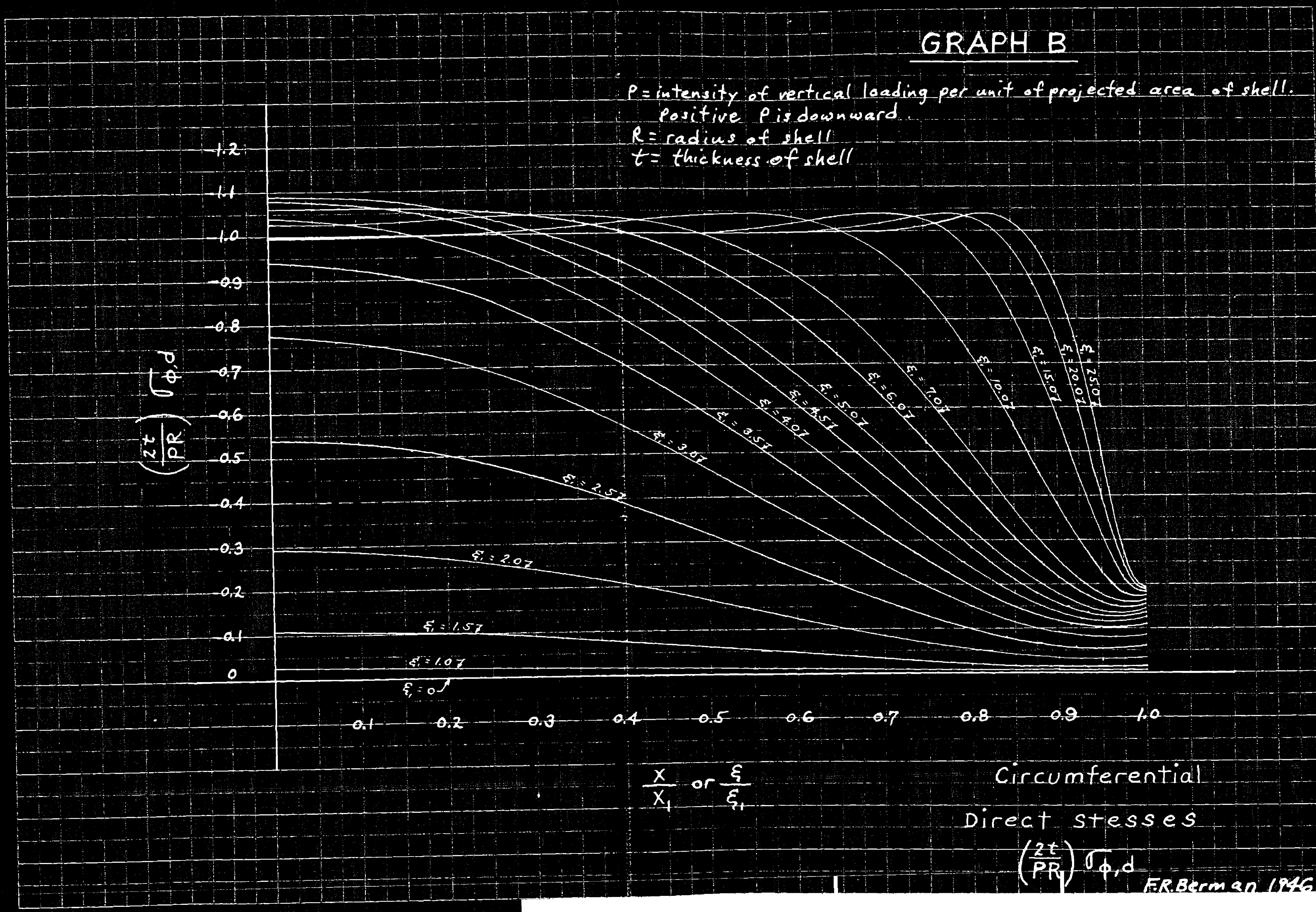
$$\left.\begin{aligned} \sigma_{x,b} &= +\frac{PR}{2t} \times 1.617 = +3950 \text{ p.s.i.} \\ \sigma_{x,d} &= -\frac{PR}{2t} \times 0.8134 = -2000 \text{ p.s.i.} \end{aligned}\right\} \qquad (j)$$

The agreement of results (i) with results (j) are within slide-rule precision.

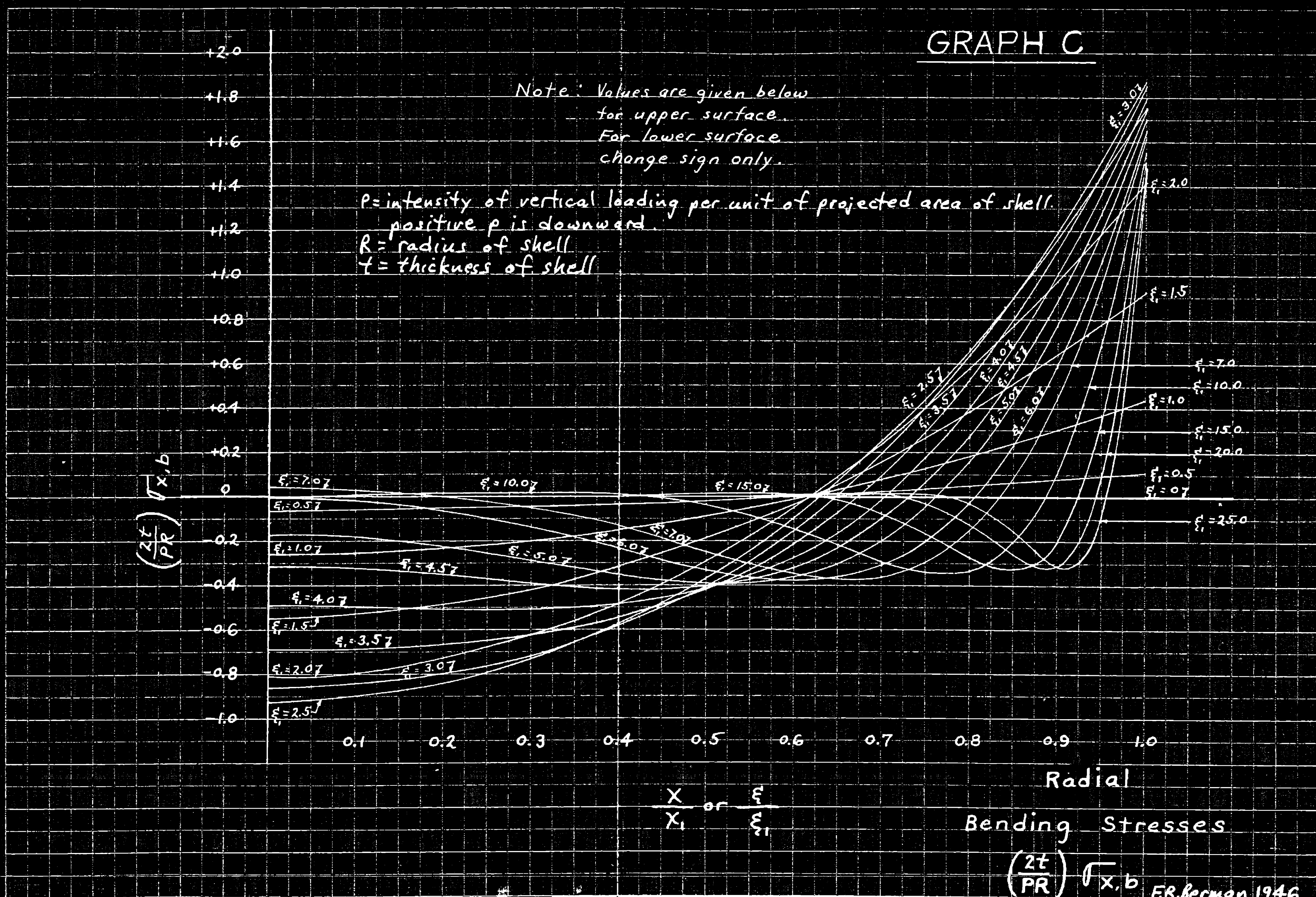
Table 37. Influence Coefficients (Symmetrical Outer Edge Effects).

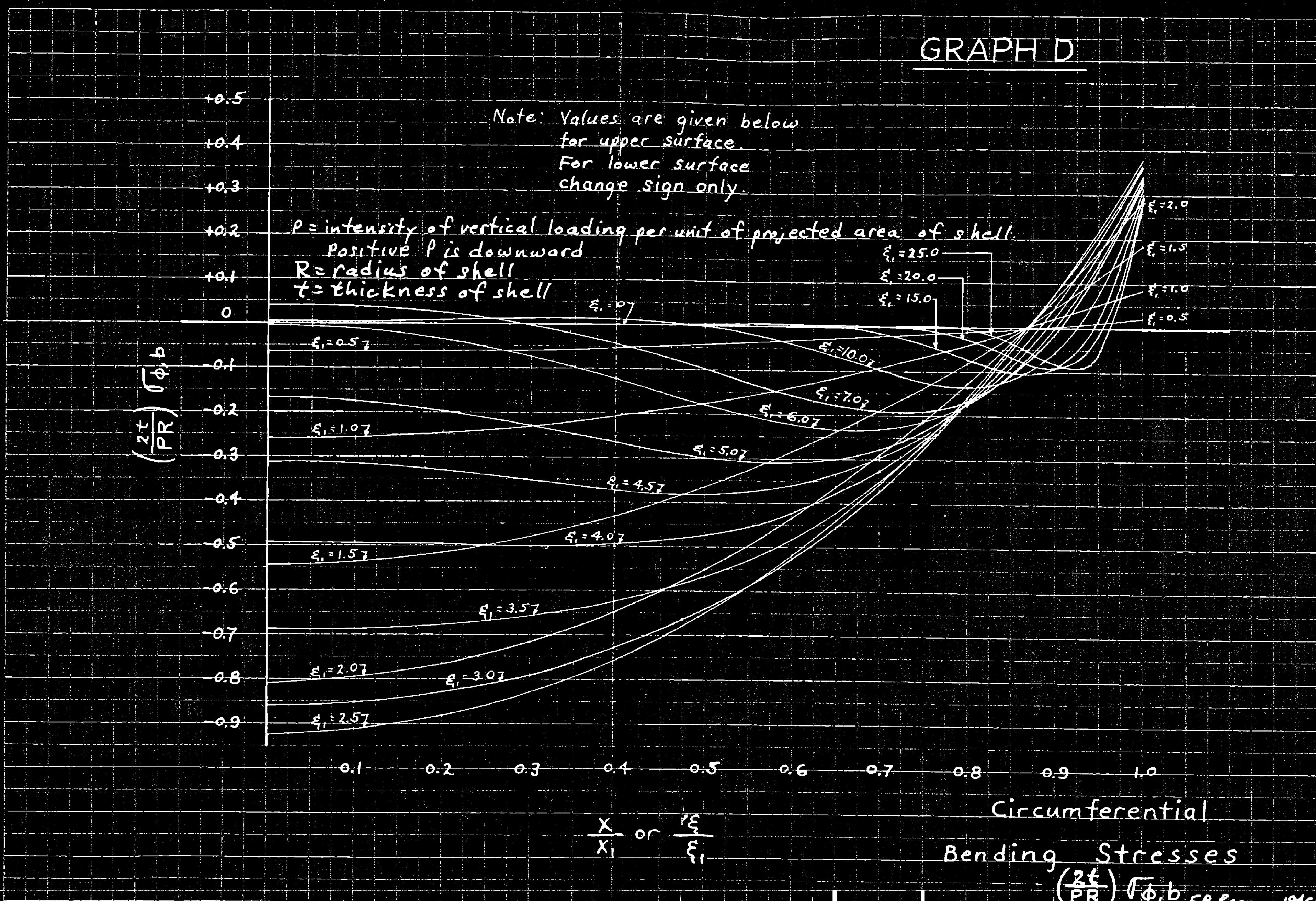
ξ	$Et^2\left(\frac{t}{R}\right)^{\frac{1}{2}}a_1$	$Et\left(\frac{t}{R}\right)^{\frac{1}{2}}a_2$	$Et\left(\frac{t}{R}\right)^{\frac{1}{2}}b_1$	$E\left(\frac{t}{R}\right)^{\frac{1}{2}}b_2$
0	0	0	0	0
0.5	-2.604	0.04794	-0.04794	0.2180
1.0	-5.166	0.3795	-0.3795	0.4678
2.0	-8.0526	2.2802	-2.2802	1.553
3.0	-10.217	5.7241	-5.7241	5.547
4.0	-9.7872	7.9164	-7.9164	11.084
5.0	-9.5158	9.7425	-9.7425	17.806
6.0	-9.4095	11.607	-11.607	26.001
7.0	-9.3401	13.493	-13.493	36.039
8.0	-9.2826	15.363	-15.363	47.088
9.0	-9.2363	17.225	-17.225	59.936
10.0	-9.1988	19.083	-19.083	74.303
11.0	-9.1681	20.938	-20.938	90.213
12.0	-9.1425	22.789	-22.789	107.66
13.0	-9.1206	24.641	-24.641	126.63
14.0	-9.1012	26.489	-26.489	147.15
15.0	-9.0850	28.338	-28.338	169.19
16.0	-9.0706	30.186	-30.186	192.78
17.0	-9.0575	32.034	-32.034	217.88
18.0	-9.0462	33.878	-33.878	244.55
19.0	-9.0356	35.726	-35.726	272.73
20.0	-9.0262	37.570	-37.570	302.48
21.0	-9.0181	39.414	-39.414	333.73
22.0	-9.0099	41.260	-41.260	366.50
23.0	-9.0024	43.106	-43.106	400.83
24.0	-8.9968	44.952	-44.952	436.71
25.0	-8.9899	46.793	-46.793	474.08


GRAPH A
P = intensity of vertical loading per unit of projected area of shell.
Positive P is downward
R = radius of shell
t = thickness of shell
ξ₁ = 25.0
ξ₁ = 20.0
ξ₁ = 15.0
ξ₁ = 10.0
ξ₁ = 7.0
ξ₁ = 6.0
ξ₁ = 5.0
ξ₁ = 4.5
ξ₁ = 4.0
ξ₁ = 3.5
ξ₁ = 3.0
ξ₁ = 2.5
ξ₁ = 2.0
ξ₁ = 1.5
ξ₁ = 1.0
ξ₁ = 0
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
(2t/PR) σx,d
X/X₁ or ξ/ξ₁
Radial
Direct Stresses
(2t/PR) σx,d
F.R. Berman 1946

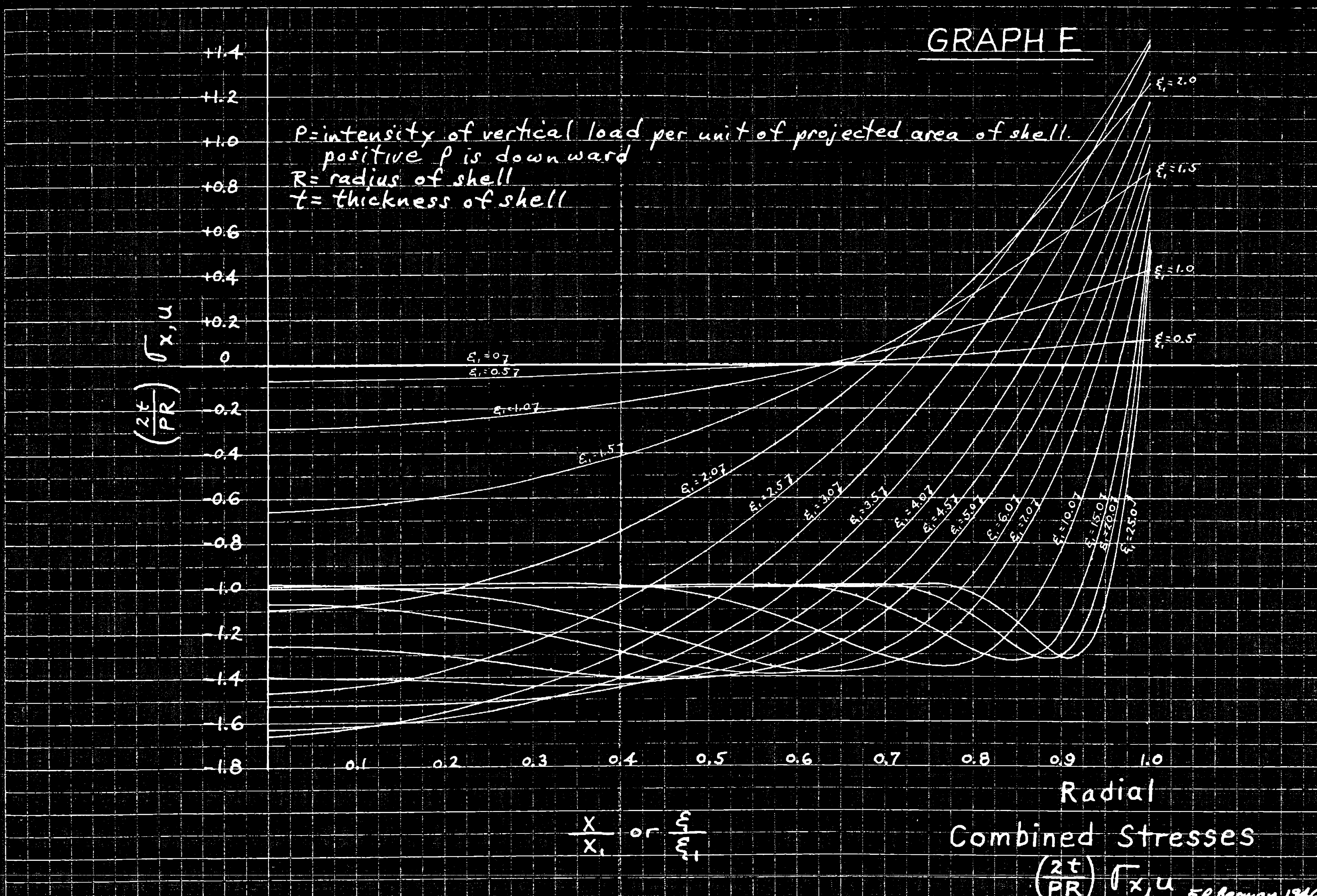


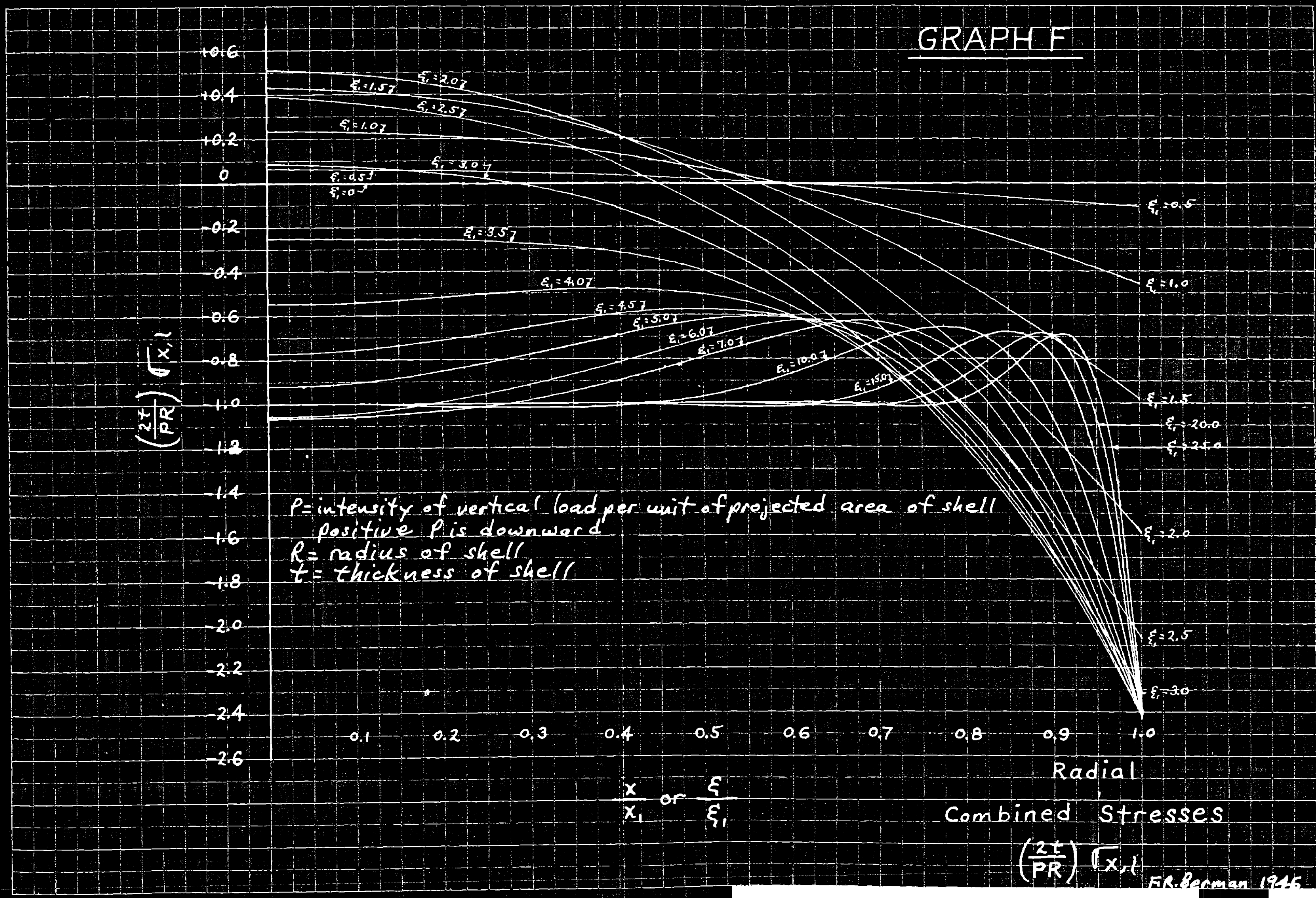
GRAPH B
P = intensity of vertical loading per unit of projected area of shell.
Positive P is downward
R = radius of shell
t = thickness of shell
$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
$\xi_1 = 25.0\gamma$
$\xi_1 = 20.0\gamma$
$\xi_1 = 15.0\gamma$
$\xi_1 = 10.0\gamma$
$\xi_1 = 7.0\gamma$
$\xi_1 = 6.0\gamma$
$\xi_1 = 5.0\gamma$
$\xi_1 = 4.5\gamma$
$\xi_1 = 4.0\gamma$
$\xi_1 = 3.5\gamma$
$\xi_1 = 3.0\gamma$
$\xi_1 = 2.5\gamma$
$\xi_1 = 2.0\gamma$
$\xi_1 = 1.5\gamma$
$\xi_1 = 1.0\gamma$
$\xi_1 = 0$
Circumferential
Direct stesses
$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$
F.R. Berman 1946


GRAPH C
Note: Values are given below for upper surface. For lower surface change sign only.
P = intensity of vertical loading per unit of projected area of shell
positive P is downward.
R = radius of shell
t = thickness of shell
+2.0
+1.8
+1.6
+1.4
+1.2
+1.0
+0.8
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\left(\frac{2t}{PR}\right)\sigma_{x,b}$
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
$\xi_1 = 7.0$
$\xi_1 = 0.5$
$\xi_1 = 1.0$
$\xi_1 = 4.5$
$\xi_1 = 4.0$
$\xi_1 = 1.5$
$\xi_1 = 3.5$
$\xi_1 = 2.0$
$\xi_1 = 3.0$
$\xi_1 = 2.5$
$\xi_1 = 10.0$
$\xi_1 = 5.0$
$\xi_1 = 6.0$
$\xi_1 = 15.0$
$\xi_1 = 2.5$
$\xi_1 = 3.5$
$\xi_1 = 4.0$
$\xi_1 = 4.5$
$\xi_1 = 5.0$
$\xi_1 = 6.0$
$\xi_1 = 3.0$
$\xi_1 = 2.0$
$\xi_1 = 1.5$
$\xi_1 = 7.0$
$\xi_1 = 10.0$
$\xi_1 = 1.0$
$\xi_1 = 15.0$
$\xi_1 = 20.0$
$\xi_1 = 0.5$
$\xi_1 = 0$
$\xi_1 = 25.0$
Radial
Bending Stresses
$\left(\frac{2t}{PR}\right)\sigma_{x,b}$
F.R. Berman 1946

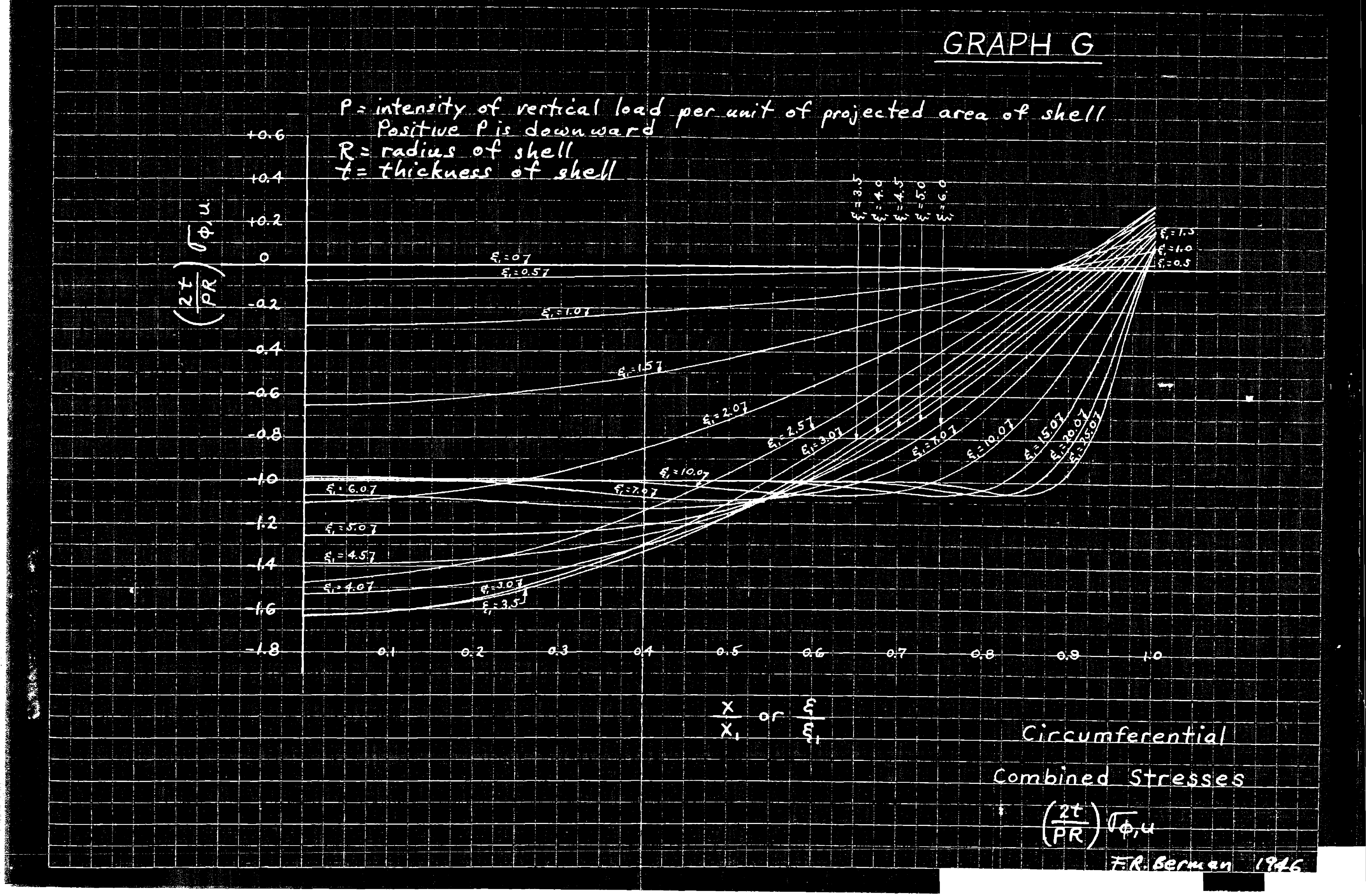


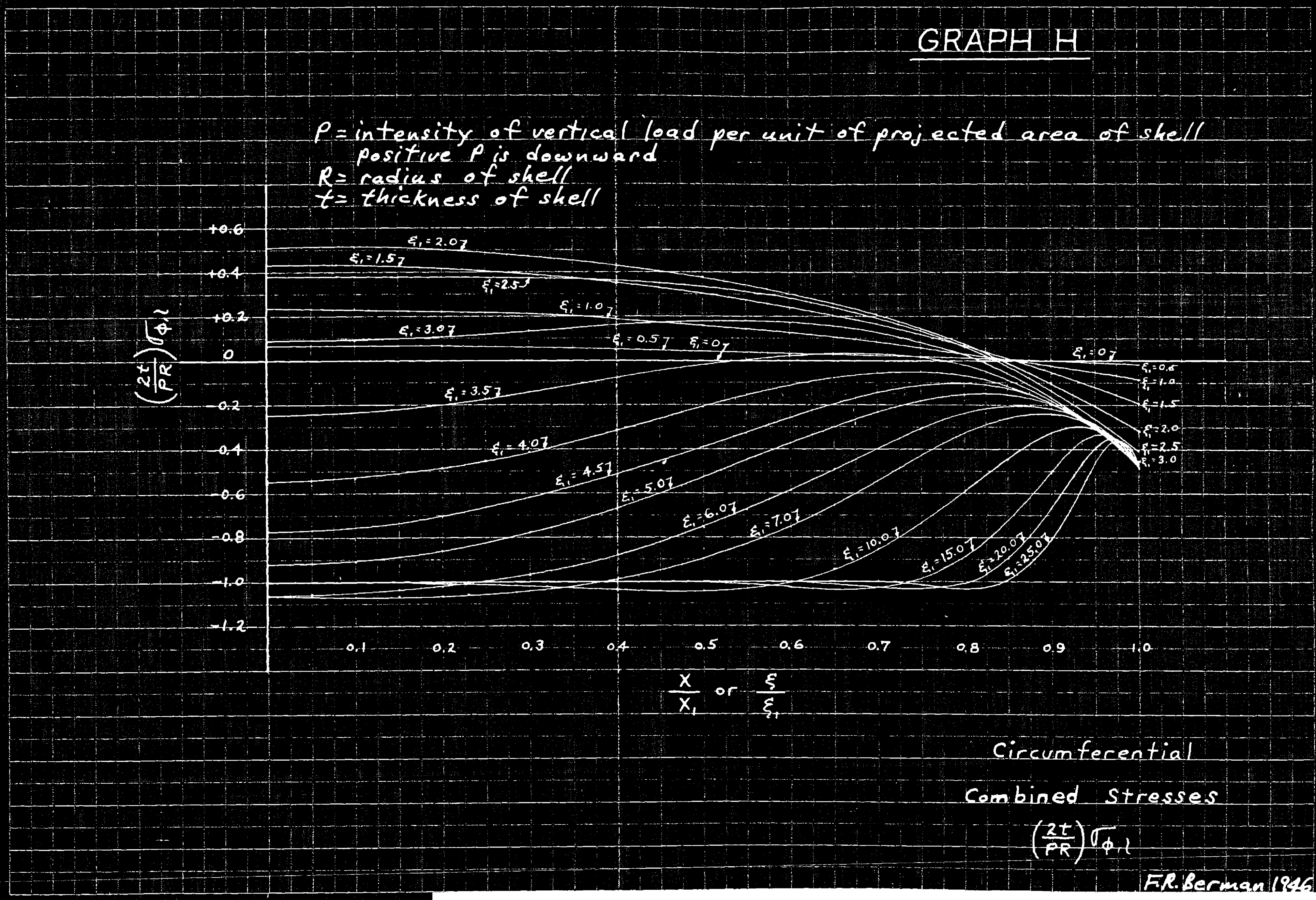
GRAPH D
Note: Values are given below for upper surface. For lower surface change sign only.
p = intensity of vertical loading per unit of projected area of shell. Positive p is downward
R = radius of shell
t = thickness of shell
$\xi_1 = 25.0$
$\xi_1 = 20.0$
$\xi_1 = 15.0$
$\xi_1 = 2.0$
$\xi_1 = 1.5$
$\xi_1 = 1.0$
$\xi_1 = 0.5$
$\xi_1 = 0.5$
$\xi_1 = 1.0$
$\xi_1 = 1.5$
$\xi_1 = 2.0$
$\xi_1 = 2.5$
$\xi_1 = 3.0$
$\xi_1 = 3.5$
$\xi_1 = 4.0$
$\xi_1 = 4.5$
$\xi_1 = 5.0$
$\xi_1 = 6.0$
$\xi_1 = 7.0$
$\xi_1 = 10.0$
$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$
+0.5
+0.4
+0.3
+0.2
+0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
Circumferential Bending Stresses
$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$
F.R. Berman 1946

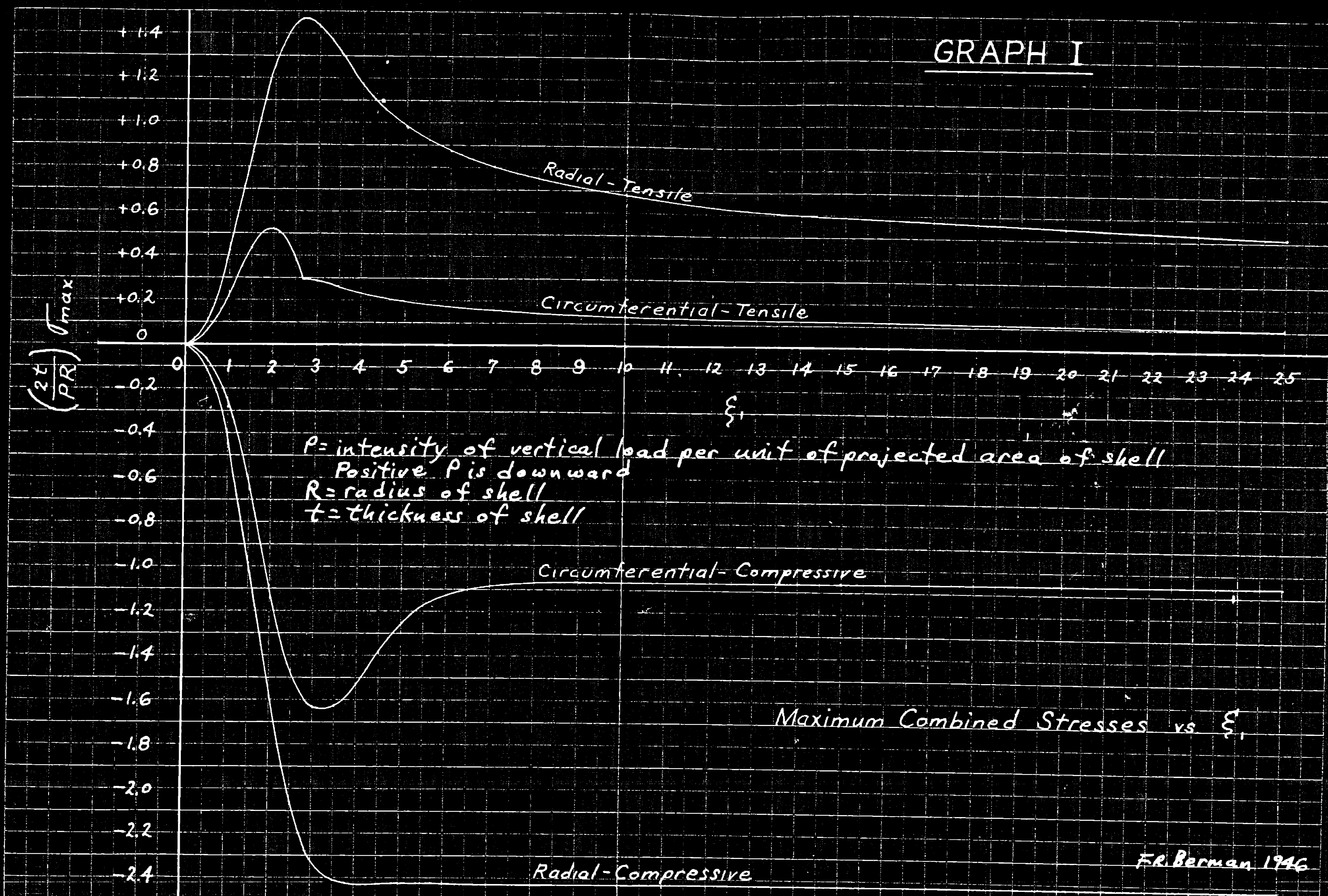

GRAPH E
P=intensity of vertical load per unit of projected area of shell.
positive P is downward
R= radius of shell
t= thickness of shell
(2t/PR) σx,u
+1.4
+1.2
+1.0
+0.8
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X/X1 or ξ/ξ1
ξ1=2.0
ξ1=1.5
ξ1=1.0
ξ1=0.5
ξ1=0
ξ1=0.5
ξ1=1.0
ξ1=1.5
ξ1=2.0
ξ1=2.5
ξ1=3.0
ξ1=3.5
ξ1=4.0
ξ1=4.5
ξ1=5.0
ξ1=6.0
ξ1=7.0
ξ1=10.0
ξ1=15.0
ξ1=20.0
ξ1=25.0
Radial
Combined Stresses
(2t/PR) σx,u
F.R. Berman 1946



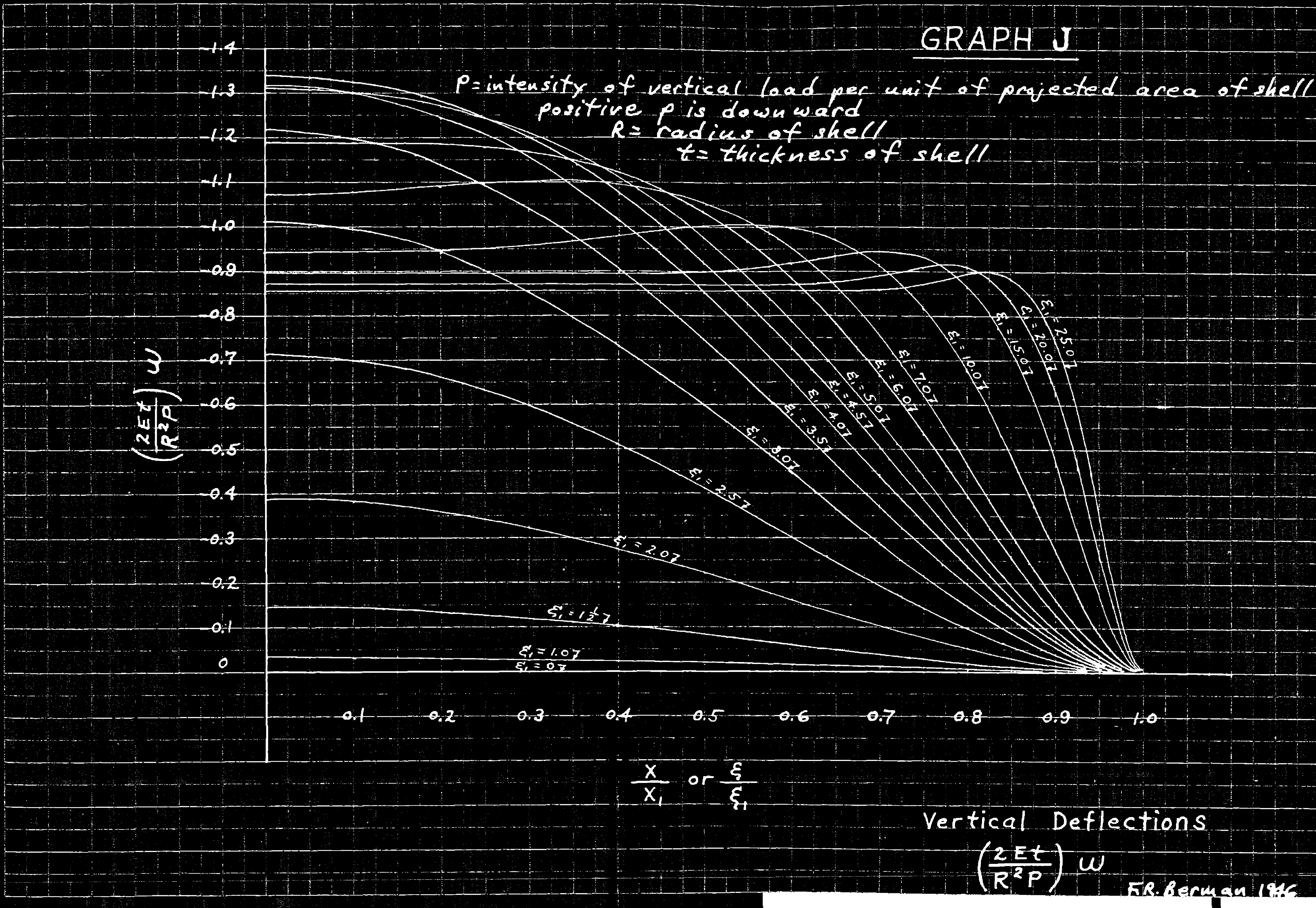
GRAPH F
$\left(\frac{2t}{PR}\right)\sigma_{x,1}$
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-2.4
-2.6
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{x}{x_1}$ or $\frac{\xi}{\xi_1}$
$\xi_1 = 2.0$
$\xi_1 = 1.5$
$\xi_1 = 2.5$
$\xi_1 = 1.0$
$\xi_1 = 3.0$
$\xi_1 = 0.5$
$\xi_1 = 0$
$\xi_1 = 3.5$
$\xi_1 = 4.0$
$\xi_1 = 4.5$
$\xi_1 = 5.0$
$\xi_1 = 6.0$
$\xi_1 = 7.0$
$\xi_1 = 10.0$
$\xi_1 = 15.0$
$\xi_1 = 0.5$
$\xi_1 = 1.0$
$\xi_1 = 1.5$
$\xi_1 = 20.0$
$\xi_1 = 25.0$
$\xi_1 = 2.0$
$\xi_1 = 2.5$
$\xi_1 = 3.0$
P= intensity of vertical load per unit of projected area of shell
positive P is downward
R= radius of shell
t= thickness of shell
Radial
Combined Stresses
$\left(\frac{2t}{PR}\right)\sigma_{x,1}$
F.R. Berman 1945

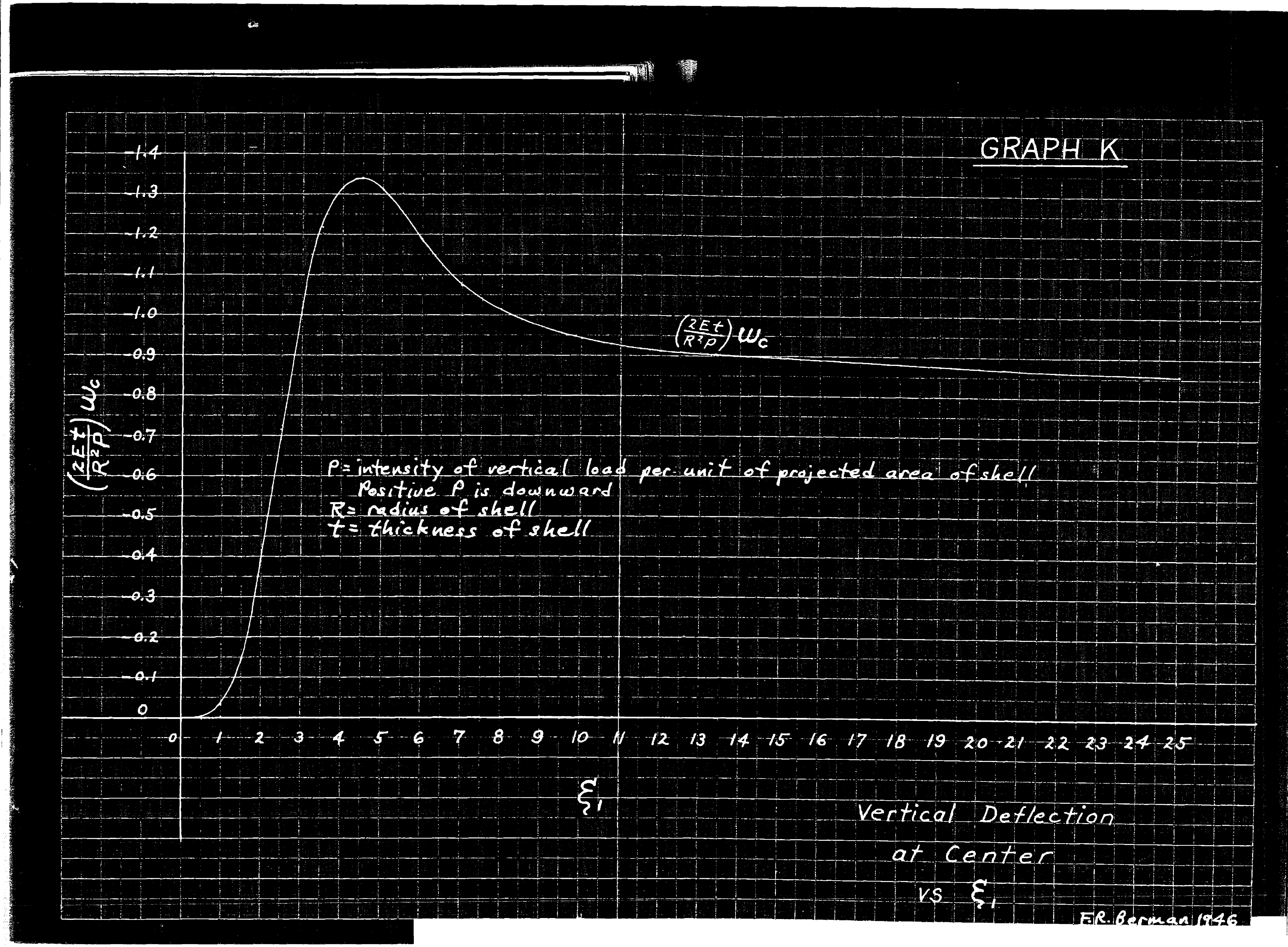

GRAPH G
P = intensity of vertical load per unit of projected area of shell
Positive P is downward
R = radius of shell
t = thickness of shell
$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
$\xi_1 = 0$
$\xi_1 = 0.5$
$\xi_1 = 1.0$
$\xi_1 = 1.5$
$\xi_1 = 2.0$
$\xi_1 = 2.5$
$\xi_1 = 3.0$
$\xi_1 = 3.5$
$\xi_1 = 4.0$
$\xi_1 = 4.5$
$\xi_1 = 5.0$
$\xi_1 = 6.0$
$\xi_1 = 7.0$
$\xi_1 = 10.0$
$\xi_1 = 15.0$
$\xi_1 = 20.0$
$\xi_1 = 25.0$
Circumferential
Combined Stresses
$\left(\frac{2t}{PR}\right)\sigma_{\phi,u}$
F.R. Berman 1946


GRAPH H
P = intensity of vertical load per unit of projected area of shell
positive P is downward
R = radius of shell
t = thickness of shell
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
ξ₁ = 2.0
ξ₁ = 1.5
ξ₁ = 2.5
ξ₁ = 1.0
ξ₁ = 3.0
ξ₁ = 0.5
ξ₁ = 0
ξ₁ = 3.5
ξ₁ = 4.0
ξ₁ = 4.5
ξ₁ = 5.0
ξ₁ = 6.0
ξ₁ = 7.0
ξ₁ = 10.0
ξ₁ = 15.0
ξ₁ = 20.0
ξ₁ = 25.0
ξ₁ = 0.5
ξ₁ = 1.0
ξ₁ = 1.5
ξ₁ = 2.0
ξ₁ = 2.5
ξ₁ = 3.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
Circumferential
Combined Stresses
$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$
F.R. Berman 1946

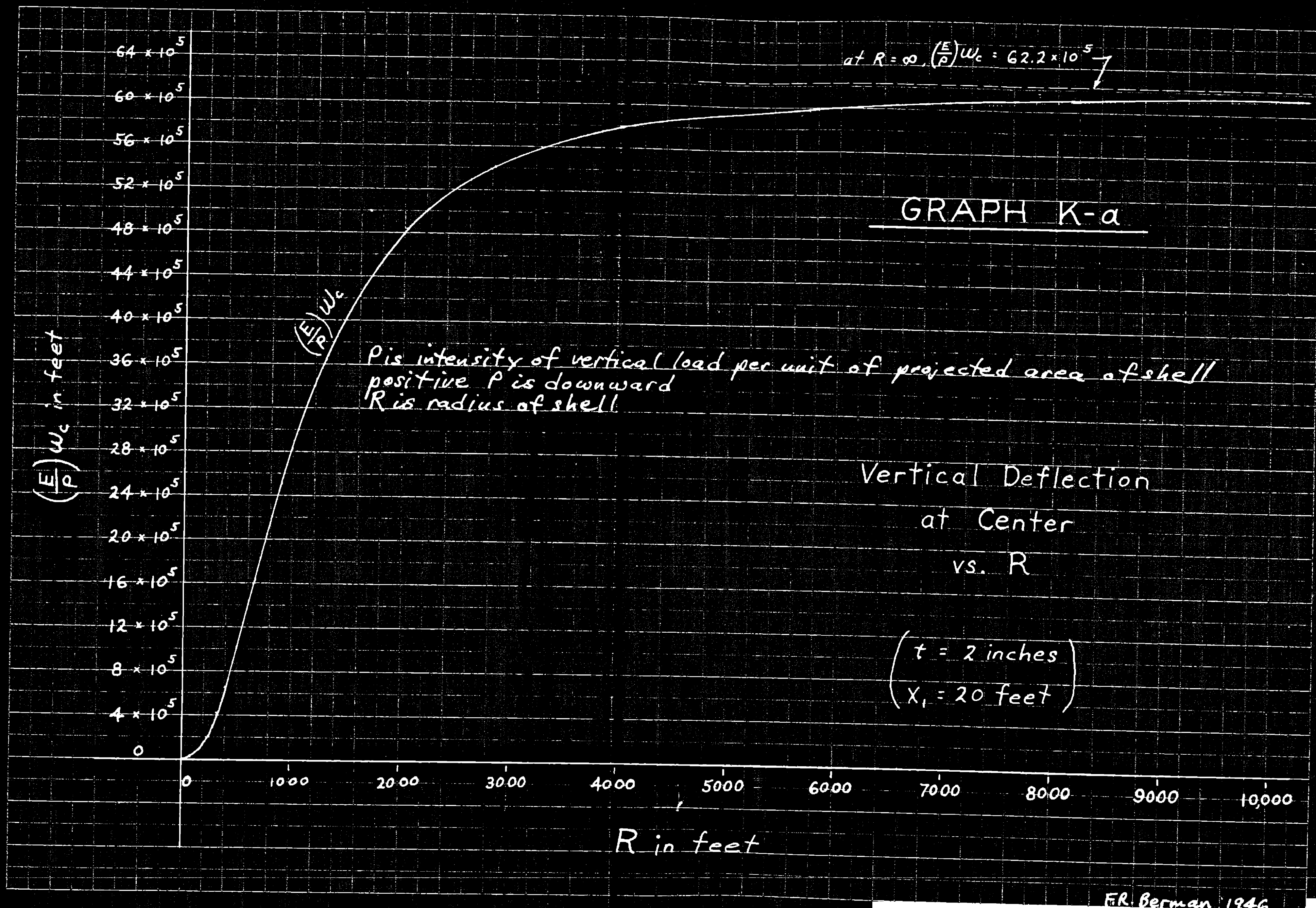


GRAPH I
+1.4
+1.2
+1.0
+0.8
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-2.4
$\left(\frac{2t}{PR}\right)\sigma_{max}$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
ξ_1
Radial - Tensile
Circumferential - Tensile
Circumferential - Compressive
Radial - Compressive
P = intensity of vertical load per unit of projected area of shell
Positive P is downward
R = radius of shell
t = thickness of shell
Maximum Combined Stresses vs ξ_1
F.R. Berman 1946

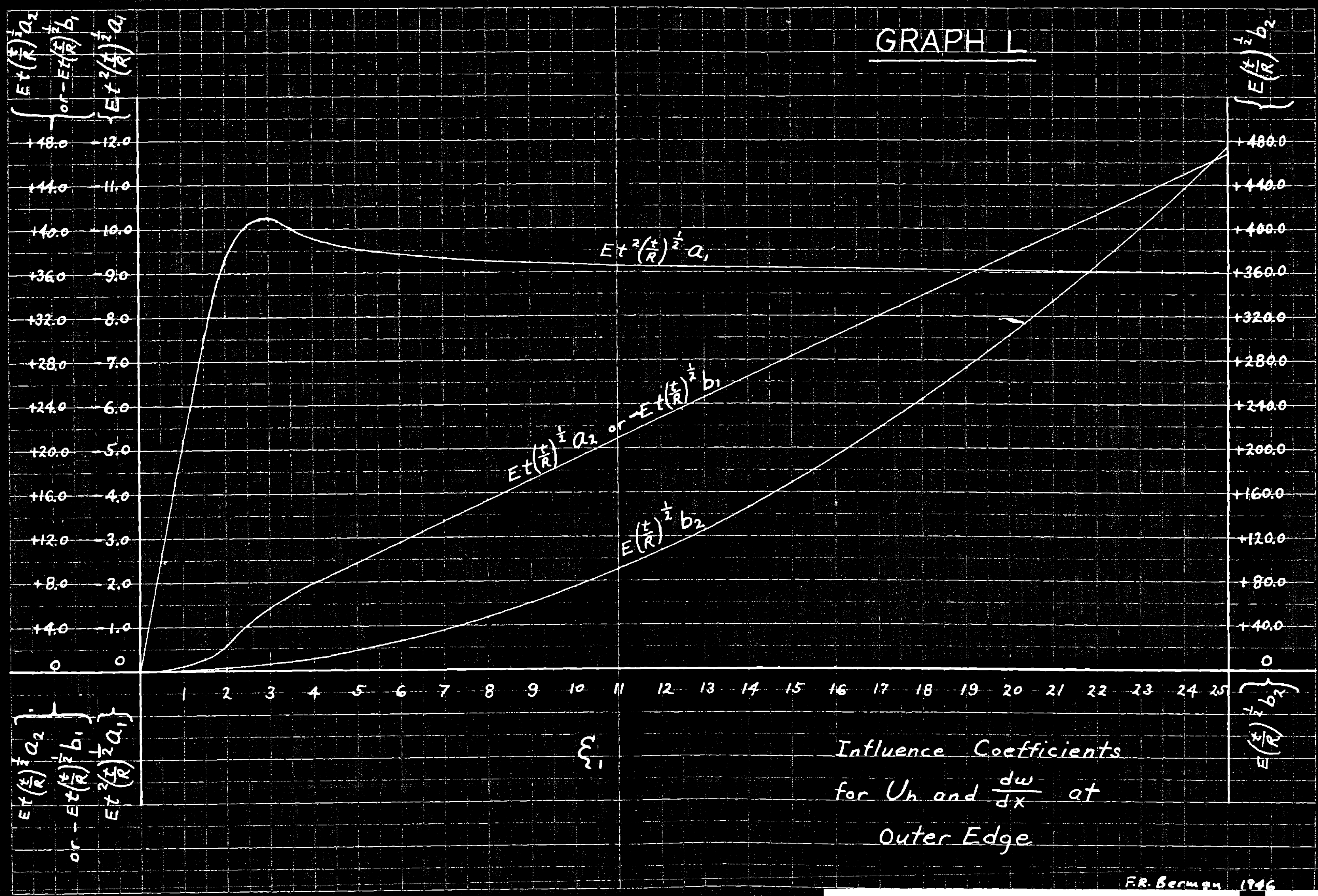

GRAPH J
P = intensity of vertical load per unit of projected area of shell
positive P is downward
R = radius of shell
t = thickness of shell
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$\left(\frac{2Et}{R^2P}\right)\omega$
$\xi_1 = 25.0\gamma$
$\xi_1 = 20.0\gamma$
$\xi_1 = 15.0\gamma$
$\xi_1 = 10.0\gamma$
$\xi_1 = 7.0\gamma$
$\xi_1 = 6.0\gamma$
$\xi_1 = 5.0\gamma$
$\xi_1 = 4.5\gamma$
$\xi_1 = 4.0\gamma$
$\xi_1 = 3.5\gamma$
$\xi_1 = 3.0\gamma$
$\xi_1 = 2.5\gamma$
$\xi_1 = 2.0\gamma$
$\xi_1 = 1\frac{1}{2}\gamma$
$\xi_1 = 1.0\gamma$
$\xi_1 = 0\gamma$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
Vertical Deflections
$\left(\frac{2Et}{R^2P}\right)\omega$
F.R. Berman 1946



GRAPH K
$\left(\frac{2Et}{R^2p}\right) w_c$
-1.4
-1.3
-1.2
-1.1
-1.0
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
ξ_1
$\left(\frac{2Et}{R^2p}\right) w_c$
p = intensity of vertical load per unit of projected area of shell
Positive p is downward
R = radius of shell
t = thickness of shell
Vertical Deflection
at Center
vs ξ_1
F.R. Berman 1946



GRAPH K-a
Vertical Deflection
at Center
vs. R
(t = 2 inches
X1 = 20 feet)
at R = ∞, (E/P) Wc = 62.2 × 10^5
P is intensity of vertical load per unit of projected area of shell
positive P is downward
R is radius of shell
(E/P) Wc
(E/P) Wc in feet
64 × 10^5
60 × 10^5
56 × 10^5
52 × 10^5
48 × 10^5
44 × 10^5
40 × 10^5
36 × 10^5
32 × 10^5
28 × 10^5
24 × 10^5
20 × 10^5
16 × 10^5
12 × 10^5
8 × 10^5
4 × 10^5
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10,000
R in feet
F.R. Berman 1946


GRAPH L
$Et\left(\frac{t}{R}\right)^{\frac{1}{2}}a_2$ or $-Et\left(\frac{t}{R}\right)^{\frac{1}{2}}b_1$
$Et^2\left(\frac{t}{R}\right)^{\frac{1}{2}}a_1$
$E\left(\frac{t}{R}\right)^{\frac{1}{2}}b_2$
+48.0 +44.0 +40.0 +36.0 +32.0 +28.0 +24.0 +20.0 +16.0 +12.0 +8.0 +4.0 0
12.0 11.0 10.0 9.0 8.0 7.0 6.0 5.0 4.0 3.0 2.0 1.0 0
+480.0 +440.0 +400.0 +360.0 +320.0 +280.0 +240.0 +200.0 +160.0 +120.0 +80.0 +40.0 0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
ξ_1
Influence Coefficients
for U_h and $\frac{dw}{dx}$ at
Outer Edge
F.R. Berman 1946

SECTION IX

Case III. Inner Edge Effects. Uniform radial edge moment M_F and / or an uniform horizontal force P_F acting all along the edge of a central hole. No Surface Loading ($P = 0$).

(1) Discussion

This case is analogous to Case II. The stresses and deflections in the shell due to a loading along the edge of a central hole decay so rapidly as we proceed radially outward that we may treat the outer edge as being at infinity.

We write

$$\frac{d\omega}{dx} = a_3 M_F + a_4 P_F \qquad (190)$$

$$u_h = b_3 M_F + b_4 P_F \qquad (191)$$

where a_3, a_4, b_3 and b_4 are influence coefficients and M_F and P_F are an uniform radial edge moment and an uniform horizontal edge force, respectively.

2. Boundary Conditions

See Figure 7.

at $x = x_o$, $\xi = \mu$:

$$Q = 0 \qquad (192)$$

$$P_h = P_F \qquad (193)$$

$$M_{xx} = M_F \qquad (194)$$

at $x = \infty$, $\xi = \infty$:

Stresses vanish

Deflections vanish (195)

$$\lim_{x \to \infty} (2\pi x P_V) \to 0 \qquad (196)$$

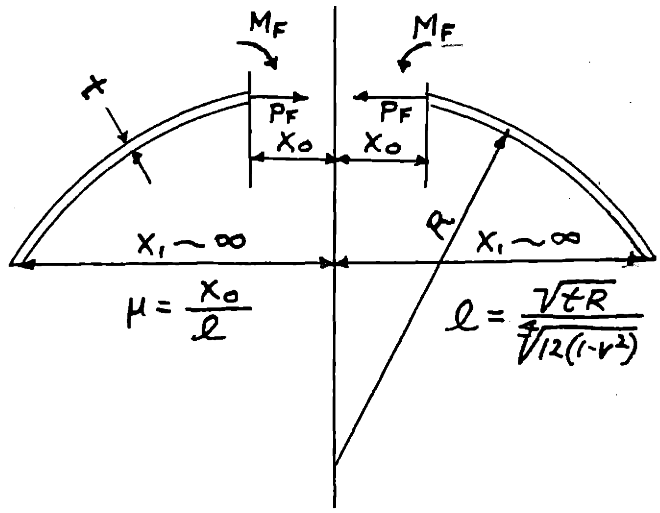


Figure 7. Notation for Shell of Section IX.

It is noted that condition (196) is not required. However, it is simpler to use (196) in lieu of (192).

3. Expressions for W and F

Noting that W_p and $F_{p,2}$ are zero and using the notation of equation (133), we have from equations (61) and (63)

$$w = C_{10}\,\mathrm{ber}\,\xi + C_{20}\,\mathrm{bei}\,\xi + C_{30}\,\mathrm{ker}\,\xi + C_{40}\,\mathrm{kei}\,\xi - C_{50} \qquad (197)$$

$$F = \frac{\ell^2 tE}{R}\Big(-C_{20}\,\mathrm{ber}\,\xi + C_{10}\,\mathrm{bei}\,\xi - C_{40}\,\mathrm{ker}\,\xi$$

$$+ C_{30}\,\mathrm{kei}\,\xi\Big) + b_0 \ln \xi \qquad (198)$$

4. Equations for the Determination of the Constants.

Applying condition (195); namely, $W(\infty) = 0$, to equation (197) we obtain the requirement

$$C_{10} = C_{20} = C_{50} = 0 \tag{199}$$

Applying condition (196) to equation (99) noting that

$$\lim_{x\to\infty}\left(\frac{2\pi x b_0}{xR}\right) \rightarrow \frac{2\pi}{R} b_0 \tag{200a}$$

we obtain the requirement

$$b_0 = 0 \tag{200b}$$

From equations (89a), (90) and (91), applying condition (194) we have

$$-\frac{\ell^2}{D} M_F = C_{30}\left(\mathrm{ker}''\mu + \frac{\nu\,\mathrm{ker}'\mu}{\mu}\right) - C_{40}\left(\mathrm{kei}''\mu + \frac{\nu\,\mathrm{kei}'\mu}{\mu}\right) \tag{201}$$

From equations (86), (93a) and (95), applying condition (193) we have

$$\frac{R}{tE} P_F = C_{30}\frac{\mathrm{kei}'\mu}{\mu} - C_{40}\frac{\mathrm{ker}'\mu}{\mu} \tag{202}$$

Equations (201) and (202) are seen to be identical with equations (171) and (172) if we replace $\mathrm{ker}''\mu$ by $\mathrm{ber}''\xi_1$, etc. Likewise, expressions for $\frac{dw}{dx}$ and U_h would be found to be similarly identical.

Accordingly, the solutions for a_3, a_4, b_3 and b_4 may be obtained directly from equations (182), (183), (188) and (189), respectively by replacing (see Appendix A)

$$\left.\begin{array}{l} \xi_1 \text{ with } \mu \\ M_0(\xi_1) \text{ with } N_0(\mu) \\ M_1(\xi_1) \text{ with } N_1(\mu), \text{ etc.} \end{array}\right\} \quad (203)$$

The results follow.

5. Influence Coefficients a_3 and a_4 for change of slope $\frac{dw}{dx}$

$$a_3 = \frac{-\frac{\ell}{D} N_1(\mu)}{N_0(\mu)\,\mathrm{Sin}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)-(1-\nu)\,\frac{N_1(\mu)}{\mu}} \quad (204)$$

$$a_4 = \frac{\frac{R}{tE\ell}\,\mu\, N_0(\mu)\,\mathrm{Cos}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)}{N_0(\mu)\,\mathrm{Sin}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)-(1-\nu)\,\frac{N_1(\mu)}{\mu}} \quad (205)$$

6. Influence Coefficients b_3 and b_4 for Horizontal Displacement U_h

$$b_3 = \frac{-\frac{\ell^3}{RD}\,\mu\, N_0(\mu)\,\mathrm{Cos}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)}{N_0(\mu)\,\mathrm{Sin}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)-(1-\nu)\,\frac{N_1(\mu)}{\mu}} \quad (206)$$

$$b_4 = \frac{\frac{\ell}{tE}\left[(1-\nu^2)N_1^2(\mu)-2\mu N_0(\mu)N_1(\mu)\,\mathrm{Sin}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)+\mu^2 N_0^2(\mu)\right]}{N_1(\mu)\left[N_0(\mu)\,\mathrm{Sin}\left(\phi_1-\phi_0-\frac{\pi}{4}\right)-(1-\nu)\,\frac{N_1(\mu)}{\mu}\right]} \quad (207)$$

7. Application of the Influence Coefficients.

The influence coefficients given by equations (204), (205), (206) and (207) have been computed with the use of the data given in Table A-6 of Appendix A. These results have been tabulated in Table 38 and plotted with respect to μ on Graph M for a range of μ given by $0 \leq \mu \leq 5.0$.

The comments and applications covered by Section VIII -7 are directly applicable to the use of the influence coefficients of this section.

Table 38. Influence Coefficients (Symmetrical Inner Edge Effects).

μ	$Et^2\left(\frac{t}{R}\right)^{\frac{1}{2}}a_3$	$Et\left(\frac{t}{R}\right)^{\frac{1}{2}}a_4$	$Et\left(\frac{t}{R}\right)^{\frac{1}{2}}b_3$	$E\left(\frac{t}{R}\right)^{\frac{1}{2}}b_4$
0	0	0	0	0
0.1	0.77443	0.0055693	-0.0055693	-0.065581
0.2	1.5112	0.031905	-0.031905	-0.13394
0.3	2.1881	0.083468	-0.083468	-0.20793
0.4	2.7969	0.15915	-0.15915	-0.29008
0.5	3.3372	0.25586	-0.25586	-0.40357
0.6	3.8135	0.37018	-0.37018	-0.48662
0.7	4.2321	0.49881	-0.49881	-0.60375
0.8	4.6001	0.63901	-0.63901	-0.73462
0.9	4.9239	0.78843	-0.78843	-0.87998
1.0	5.2099	0.94523	-0.94523	-1.0401
1.2	5.6875	1.2752	-1.2752	-1.4058
1.4	6.0675	1.6204	-1.6204	-1.8331
1.6	6.3743	1.9753	-1.9753	-2.3227
1.8	6.6251	2.3366	-2.3366	-2.8747
2.0	6.8339	2.7023	-2.7023	-3.4895
2.2	7.0096	3.0706	-3.0706	-4.1666
2.4	7.1584	3.4405	-3.4405	-4.9058
2.6	7.2860	3.8117	-3.8117	-5.7075
2.8	7.3973	4.1841	-4.1841	-6.5716
3.0	7.4942	4.5562	-4.5562	-7.4976
3.5	7.6893	5.4884	-5.4884	-10.085
4.0	7.8369	6.4205	-6.4205	-13.059
4.5	7.9513	7.3516	-7.3516	-16.417
5.0	8.0445	8.2828	-8.2828	-20.163

SECTION X

Case IV. Edges fully restrained. Hydrostatic loading over entire surface varying from zero to a maximum intensity P.

1. Discussion

This case may be analyzed as a combination of an uniform loading of intensity p as given under Case I and an unsymmetrical loading with an intensity of loading given by

$$P = p\frac{x}{x_1}\cos\phi \tag{208}$$

This treatment is shown by Figure 8.

This case is analyzed in this way for a shell with $\xi_1 = 10.0$. The development that follows is for the determination of the stresses and displacements due to that part of the loading given by equation (208).

2. Boundary Conditions.

See Figure 8.

$$\text{at} \quad \begin{matrix} x = 0 \\ \xi = 0 \end{matrix} \left\{ \begin{matrix} \text{Stresses} \\ \text{and} \\ \text{Deflections} \\ \text{are finite} \end{matrix} \right. \tag{209}$$

$$\text{at} \quad \begin{matrix} x = x_1 \\ \xi = \xi_1 \end{matrix} \left\{ \begin{array}{lr} w = 0 & (210) \\ \dfrac{\partial w}{\partial x} = 0 & (211) \\ U_h = 0 & (212) \\ V = 0 & (213) \end{array} \right.$$

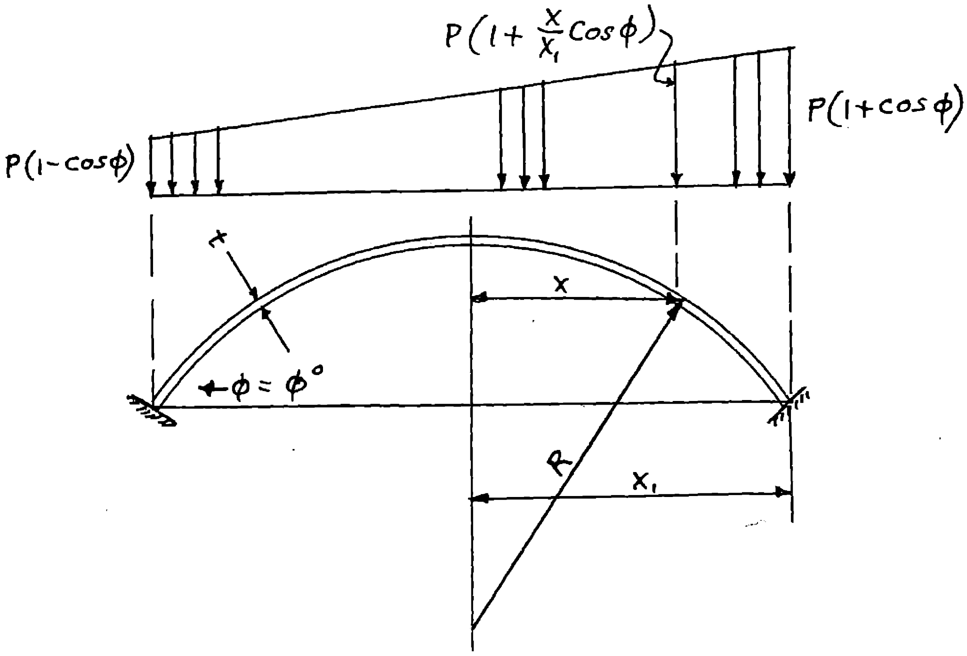


Figure 8a. Notation and Loading for Shell of Section X. (For Meridian Plane $\emptyset = \emptyset^\circ$).

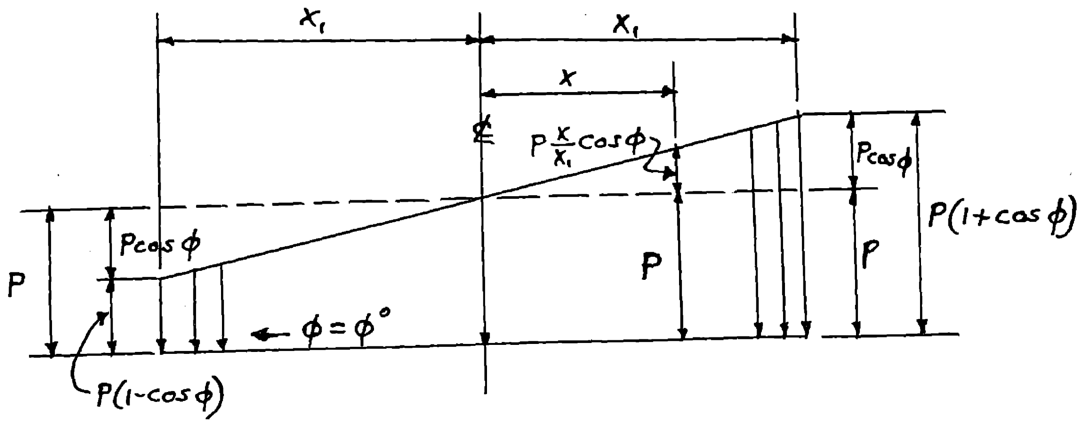


Figure 8b. Resolution of Hydrostatic Loading into an Uniform Load and a Linearly Varying Load. (Meridian Plane $\emptyset = \emptyset^\circ$).

3. Expressions for W and F

From equation (51), n = 1

$$w = \left(C_{11}\,\mathrm{ber}_1\,\xi + C_{21}\,\mathrm{bei}_1\,\xi + C_{31}\,\mathrm{ker}_1\,\xi + C_{41}\,\mathrm{kei}_1\,\xi - C_{51}\,\xi - C_{71}\,\xi^{-1}\right)\cos\phi + w_p \qquad (214)$$

where w_p is given by

$$\nabla^2\nabla^2 w_p + \frac{1}{\ell^4} w_p = \frac{p}{D} \qquad (215)$$

From equation (52), n = 1

$$F = \frac{\ell^2 tE}{R}\left(-C_{21}\,\mathrm{ber}_1\,\xi + C_{11}\,\mathrm{bei}_1\,\xi - C_{41}\,\mathrm{ker}_1\,\xi + C_{31}\,\mathrm{kei}_1\,\xi\right)\cos\phi + \left(C_{61}\,\xi + C_{81}\,\xi^{-1}\right)\cos\phi + F_{p,2} \qquad (216)$$

where $F_{p,2}$ is given by

$$\nabla^2 F_{p,2} = \frac{tE}{R} w_p \qquad (217)$$

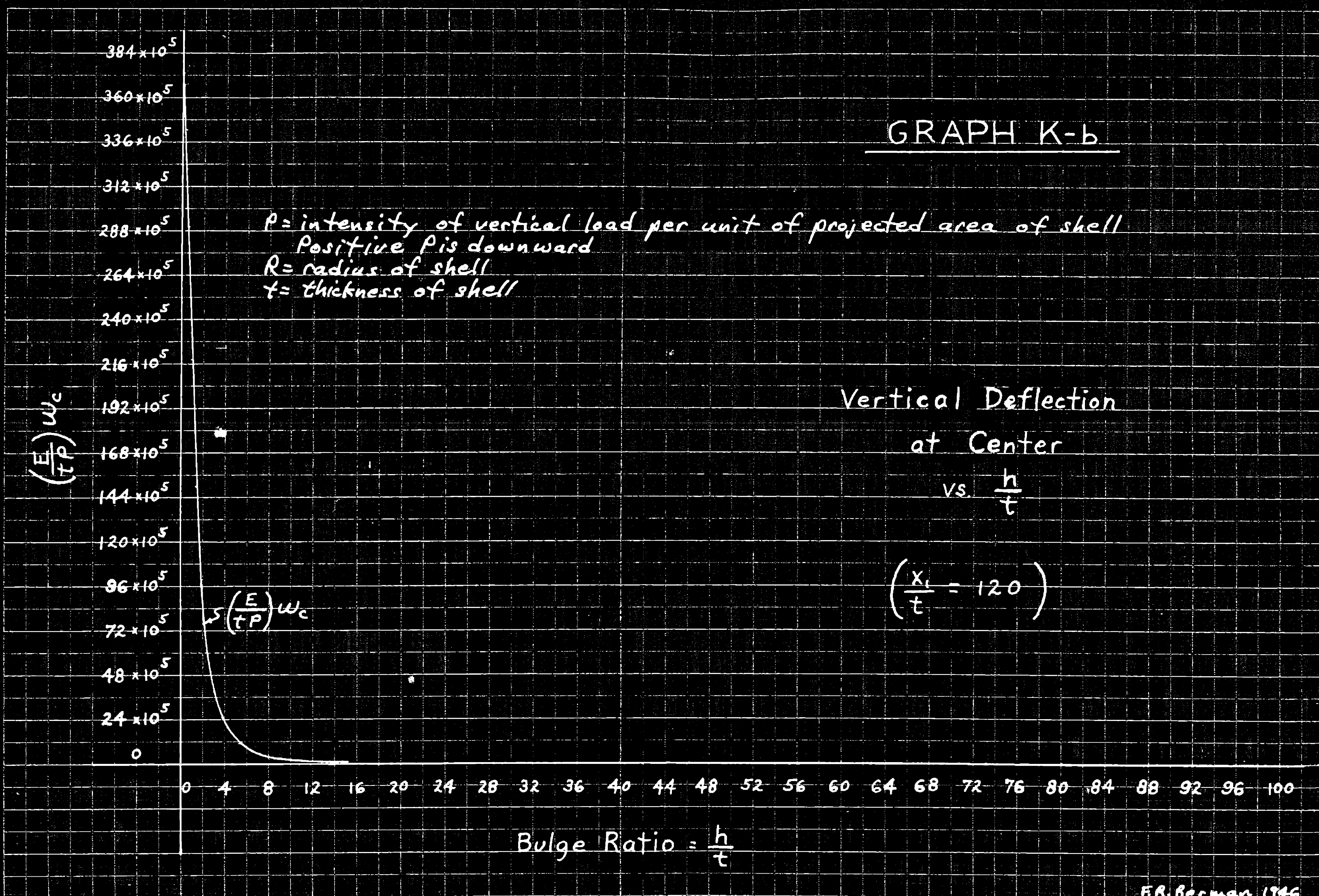
4. Solutions for W_p and $F_{p,2}$

By inspection of equation (215), considering (208), the particular integral is given by

$$w_p = \frac{p\ell^4}{D} = \frac{p\ell^4}{D}\frac{x}{x_1}\cos\phi$$

or

$$w_p = \frac{p\ell^4}{D}\frac{\xi}{\xi_1}\cos\phi \qquad (218)$$



GRAPH K-b
P= intensity of vertical load per unit of projected area of shell
Positive P is downward
R= radius of shell
t= thickness of shell
Vertical Deflection
at Center
vs. h/t
(x1/t = 120)
(E/tP) wc
384×10^5
360×10^5
336×10^5
312×10^5
288×10^5
264×10^5
240×10^5
216×10^5
192×10^5
168×10^5
144×10^5
120×10^5
96×10^5
72×10^5
48×10^5
24×10^5
0
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 84 88 92 96 100
Bulge Ratio = h/t
F.R. Berman 1946

It is easily verified that the solution of equation (217) using equation (218) is

$$F_{p,z} = \frac{tE\ell^4}{8RD} p \frac{X^3}{X_1} \cos\phi \qquad (219a)$$

or using equation (25)

$$F_{p,z} = \frac{RP}{8} \frac{X^3}{X_1} \cos\phi = \frac{R\ell^2 p}{8} \frac{\xi^3}{\xi_1} \cos\phi \qquad (219b)$$

5. Determination of the Equations for the Constants.

Applying conditions (209) to equations (214) and (216) we obtain

$$C_{31} = C_{41} = C_{71} = C_{81} = 0 \qquad (220)$$

Applying condition (210) to equation (214) using (220) we have after division by cos ϕ

$$0 = C_{11}\, ber_1 \xi_1 + C_{21}\, bei_1 \xi_1 - C_{51}\, \xi_1 + \frac{p\ell^4}{D} \qquad (221)$$

Applying condition (211) to equation (214) using (220) we have after multiplication by $\frac{\xi_1 \ell}{\cos\phi}$

$$0 = C_{11}\, \xi_1 ber_1' \xi_1 + C_{21}\, \xi_1 bei_1' \xi_1 - C_{51}\, \xi_1 + \frac{p\ell^4}{D} \qquad (222)$$

To evaluate the constants K_1, K_2, K_3 and K appearing in the displacement equations (111), (114) and (117) we use condition (212) and symmetry to obtain the additional conditions

$$at \begin{Bmatrix} x = x_1 \\ \phi = \frac{\pi}{2} \end{Bmatrix} \; ; \; u_h = 0 \tag{223}$$

$$at \;\; \phi = 0° \;\; ; \;\; v = 0 \tag{224a}$$

$$at \;\; \phi = 0° \;\; ; \;\; \frac{\partial v}{\partial x} = 0 \tag{224b}$$

Substituting equation (117b) into equation (114) using equation (220) and applying condition (223) we have

$$K_2 = 0 \tag{225}$$

Substituting equations (117) into equation (111) using equations (220) and (225) and applying conditions (224) we have

$$K = K_1 = 0 \tag{226}$$

The terms involving K_3 and C_{61} in equations (111) and (114) are functions of cos ϕ only. These constants appear in the displacement equations only and can be combined into one constant by the transformation

$$+\frac{(1+\nu)}{tE\ell} C_{61} - K_3 = \frac{\ell}{R} C_{91} \tag{227}$$

Applying condition (212) to equation (114) using equations (117b), (218), (219b), (220), (225), (226), (227) and (25) we have after multiplication by $\frac{R}{\ell \cos\phi}$

$$0 = C_{11}\left[\xi_1 \,\mathrm{ber}_1 \xi_1 - (1+\nu)\,\mathrm{bei}_1' \xi_1\right] + C_{21}\left[\xi_1 \,\mathrm{bei}_1 \xi_1 + (1+\nu)\,\mathrm{ber}_1' \xi_1\right]$$

$$- C_{51}\frac{\xi_1^2}{2} - C_{91} + \frac{R^2 p}{8tE}(5-3\nu)\,\xi_1 \qquad (228)$$

Applying condition (213) to equation (111) using equations (117), (219b), (220), (225), (226) and (227) we have after multiplication by $\frac{R}{\ell \sin\phi}$

$$0 = (1+\nu)\,C_{11}\frac{\mathrm{bei}_1 \xi_1}{\xi_1} - (1+\nu)\,C_{21}\frac{\mathrm{ber}_1 \xi_1}{\xi_1} + C_{51}\frac{\xi_1^2}{2} + C_{91}$$

$$+ \frac{R^2 p}{8tE}(1+\nu)\,\xi_1 \qquad (229)$$

Equations (221), (222), (228) and (229) provide four equations for the determination of the constants C_{11}, C_{21}, C_{51} and C_{91}. The evaluation of these constants will permit the determination of the state of stress and displacement in the shell under investigation.

6. Equations for the Constants in Dimensionless Form.

To transform the constants to the dimensionless form we use notation as before; namely,

$$\frac{D}{p\ell^4} = \frac{tE}{R^2 p} \qquad (25)$$

$$\left(\frac{D}{p\ell^4}\right) C_{11} = K_{11} \qquad (230a)$$

$$\left(\frac{D}{p\ell^4}\right) C_{21} = K_{21} \qquad (230b)$$

$$\left(\frac{D}{P\ell^4}\right) C_{51} = K_{51} \tag{230c}$$

$$\left(\frac{D}{P\ell^4}\right) C_{91} = K_{91} \tag{230d}$$

Using equations (230) in equations (221), (222), (228) and (229) we have

$$0 = K_{11}\, ber_1 \xi_1 + K_{21}\, bei_1 \xi_1 - K_{51}\, \xi_1 + 1 \tag{231}$$

$$0 = K_{11}\, \xi_1\, ber_1' \xi_1 + K_{21}\, \xi_1\, bei_1' \xi_1 - K_{51}\, \xi_1 + 1 \tag{232}$$

$$0 = K_{11}\left[\xi_1\, ber_1 \xi_1 - (1+\nu)\, bei_1' \xi_1\right] + K_{21}\left[\xi_1\, bei_1 \xi_1 + (1+\nu)\, ber_1' \xi_1\right]$$
$$- K_{51}\frac{\xi_1^2}{2} - K_{91} + \frac{(5-3\nu)}{8}\xi_1 \tag{233}$$

$$0 = K_{11}(1+\nu)\frac{bei_1 \xi_1}{\xi_1} - K_{21}(1+\nu)\frac{ber_1 \xi_1}{\xi_1} + K_{51}\frac{\xi_1^2}{2}$$
$$+ K_{91} + \frac{(1+\nu)}{8}\xi_1 \tag{234}$$

Equations (231), (232), (233) and (234) constitute the solution for the constants K_{11}, K_{21}, K_{51} and K_{91} in dimensionless form.

7. Expressions for Stresses, Displacements and Reactions

Using the results of Sections IV and VI we have

a. Bending Stresses

$$\left(\frac{2t}{PR}\right)\sigma_{x,b} = \frac{-}{+}\frac{2\sqrt{3}}{\sqrt{1-\nu^2}}\left\{K_{11}\left[(1-\nu)\left(\frac{ber_1 \xi}{\xi^2} - \frac{ber_1' \xi}{\xi}\right) - bei_1 \xi\right]\right.$$

$$+K_{21}\left[(1-\nu)\left(\frac{bei_1\xi}{\xi^2}-\frac{bei_1'\xi}{\xi}\right)+ber_1\xi\right]\Bigg\}\cos\phi \qquad (235a)$$

$$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}=\mp\frac{2\sqrt{3}}{\sqrt{1-\nu^2}}\Bigg\{K_{11}\left[(1-\nu)\left(\frac{ber_1'\xi}{\xi}-\frac{ber_1\xi}{\xi^2}\right)-\nu bei_1\xi\right]$$

$$+K_{21}\left[(1-\nu)\left(\frac{bei_1'\xi}{\xi}-\frac{bei_1\xi}{\xi^2}\right)+\nu ber_1\xi\right]\Bigg\}\cos\phi \qquad (235b)$$

b. Torsional Stresses

$$\left(\frac{2t}{PR}\right)\tau_{x\phi,t}=\pm 2\sqrt{2}\left[K_{11}\left(\frac{ber_1'\xi}{\xi}-\frac{ber_1\xi_1}{\xi^2}\right)\right.$$

$$\left.+K_{21}\left(\frac{bei_1'\xi}{\xi}-\frac{bei_1\xi}{\xi^2}\right)\right]\sin\phi \qquad (236)$$

c. Direct Stresses

$$\left(\frac{2t}{PR}\right)\sigma_{x,d}=2\left[K_{11}\left(\frac{bei_1'\xi}{\xi}-\frac{bei_1\xi}{\xi^2}\right)+K_{21}\left(\frac{ber_1\xi}{\xi^2}-\frac{ber_1'\xi}{\xi}\right)\right.$$

$$\left.+\frac{1}{4}\frac{\xi}{\xi_1}\right]\cos\phi \qquad (237a)$$

$$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}=2\left[K_{11}\left(\frac{bei_1\xi}{\xi^2}-\frac{bei_1'\xi}{\xi}+ber_1\xi\right)\right.$$

$$\left.+K_{21}\left(\frac{ber_1'\xi}{\xi}-\frac{ber_1\xi}{\xi^2}+bei_1\xi\right)+\frac{3}{4}\frac{\xi}{\xi_1}\right]\cos\phi \qquad (237b)$$

d. Shearing Stress due to $N_{x\phi}$.

$$\left(\frac{2t}{PR}\right)\tau_{x\phi}=+2\left[K_{11}\left(\frac{bei_1'\xi}{\xi}-\frac{bei_1\xi}{\xi^2}\right)+K_{21}\left(\frac{ber_1\xi}{\xi^2}-\frac{ber_1'\xi}{\xi}\right)\right.$$

$$\left.+\frac{1}{4}\frac{\xi}{\xi_1}\right]\sin\phi \qquad (238)$$

e. Shearing stress due to Qx.

$$\frac{2}{p}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x = +\frac{\sqrt{2}}{\sqrt[4]{3(1-\nu^2)}}\left(K_{11}\,bei_1'\xi - K_{21}\,ber_1'\xi\right)\cos\phi \qquad (239)$$

f. Shearing stress due to Qϕ.

$$\frac{2}{p}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_\phi = +\frac{\sqrt{2}}{\sqrt[4]{3(1-\nu^2)}}\left(-K_{11}\frac{bei_1\xi}{\xi} + K_{21}\frac{ber_1\xi}{\xi}\right)\sin\phi \qquad (240)$$

g. Vertical Deflection.

$$\left(\frac{2Et}{R^2p}\right)w = 2\left(K_{11}\,ber_1\xi + K_{21}\,bei_1\xi - K_{51}\xi + \frac{\xi}{\xi_1}\right)\cos\phi \qquad (241)$$

h. Horizontal Reaction.

$$p_h = N_{xx} = t\,\sigma_{x,d} \qquad (242)$$

i. Tangential (circumferential) Reaction.

$$p_t = N_{x\phi} = t\,\tau_{x\phi} \qquad (243)$$

j. Vertical Reaction.

$$p_v = \frac{\sqrt{tR}}{\sqrt[4]{12(1-\nu^2)}}\,p\left(K_{21}\frac{ber_1\xi_1}{\xi_1} - K_{11}\frac{bei_1\xi_1}{\xi_1} + \frac{\xi_1}{4}\right)\cos\phi \qquad (244)$$

8. Evaluation of the Constants K_{11}, K_{21}, K_{51} and K_{91}.

Subtracting equation (232) from (231) we have

$$0 = K_{11}\left(ber_1\xi_1 - \xi_1\,ber_1'\xi_1\right) + K_{21}\left(bei_1\xi_1 - \xi_1\,bei_1'\xi_1\right) \qquad (245)$$

Adding equation (233) to (234) we have

$$0 = K_{11}\left[\xi_1 \, ber_1 \xi_1 - (1+\nu) bei_1' \xi_1 + (1+\nu)\frac{bei_1 \xi_1}{\xi_1}\right]$$
$$+ K_{21}\left[\xi_1 \, bei_1 \xi_1 + (1+\nu) ber_1' \xi_1 - (1+\nu)\frac{ber_1 \xi_1}{\xi_1}\right]$$
$$+ \frac{(3-\nu)}{4}\xi_1 \qquad (246)$$

Substituting the values of the Bessel functions for $\xi_1 = 10.0$ from the tables in Appendix A, equations (245) and (246) become

$$\left.\begin{aligned} 0 &= +12.6140\, K_{11} - 3.19394\, K_{21} \\ -0.7 &= -63.3103\, K_{11} + 116.742\, K_{21} \end{aligned}\right\} \qquad (247)$$

with solution

$$\left.\begin{aligned} K_{11} &= -0.00175993 \\ K_{21} &= -0.00695058 \end{aligned}\right\} \qquad (248)$$

Substituting equations (248) into (232) and using the result in equation (234) we obtain

$$\left.\begin{aligned} K_{51} &= +0.0188039 \\ K_{91} &= -2.36273 \end{aligned}\right\} \qquad (249)$$

9. Evaluation of the Stresses and Displacements.

Substitution of the constant values given by equations (248) and (249) into the expressions for stresses and displacements given in Article 7 of this section will yield the desired values for the loading given by equation (208). This has been done and the values tabulated in tables 39 and 40. These results are for $\phi = 0^\circ$. To obtain them for any other ϕ, these values are multiplied by $\cos\phi$ or $\sin\phi$ as required by the pertinent formula.

To obtain the total stresses and displacements due to hydrostatic loading we need only combine the results given in tables 39 and 40 for the loading given by equation (208) with the results for a symmetrical loading given under Case I for $\xi_1 = 10.0$ in table 26. It must be kept in mind that the factors cos ϕ or sin ϕ must be taken into account in the use of tables 39 and 40.

$\phi = 0°$

Table 39. Direct and Bending Stress Intensities. Vertical Deflections.

ξ	$\left(\frac{2t}{PR}\right)\sigma_{x,d}$	$\left(\frac{2t}{PR}\right)\sigma_{\phi,d}$	$\left(\frac{2t}{RP}\right)\sigma_{x,b}$	$\left(\frac{2t}{RP}\right)\sigma_{\phi,b}$	$\left(\frac{2Et}{R^2P}\right)\omega$
0	0	0	0	0	0
0.5	0.0246	0.0737	-0.0045	-0.0022	0.0795
1.0	0.0492	0.1479	-0.0094	-0.0046	0.1595
1.5	0.0741	0.2232	-0.0150	-0.0073	0.2408
2.0	0.0992	0.3001	-0.0213	-0.0103	0.3241
2.5	0.1249	0.3794	-0.0274	-0.0133	0.4103
3.0	0.1511	0.4617	-0.0316	-0.0159	0.5001
3.5	0.1781	0.5475	-0.0302	-0.0170	0.5940
4.0	0.2058	0.6364	-0.0208	-0.0152	0.6918
4.5	0.2343	0.7271	0.0039	-0.0087	0.7921
5.0	0.2632	0.8162	0.0493	0.0046	0.8913
5.5	0.2920	0.8980	0.1206	0.0268	0.9831
6.0	0.3198	0.9634	0.2199	0.0590	1.0576
6.5	0.3454	1.0002	0.3420	0.1007	1.1012
7.0	0.3668	0.9937	0.4698	0.1482	1.0973
7.5	0.3818	0.9291	0.5683	0.1925	1.0288
8.0	0.3881	0.7962	0.5789	0.2179	0.8834
8.5	0.3838	0.5970	0.4151	0.2005	0.6611
9.0	0.3681	0.3573	-0.0348	0.1077	0.3870
9.1	0.3637	0.3092	-0.1702	0.0768	0.3307
9.2	0.3589	0.2626	-0.3356	0.0411	0.2755
9.3	0.3538	0.2183	-0.4941	0.0002	0.2224
9.4	0.3484	0.1772	-0.6854	-0.0463	0.1721
9.5	0.3428	0.1404	-0.8966	-0.0986	0.1258
9.6	0.3369	0.1088	-1.1290	-0.1570	0.0847
9.7	0.3310	0.0839	-1.3832	-0.2217	0.0501
9.8	0.3250	0.0669	-1.6599	-0.2930	0.0234
9.9	0.3192	0.0593	-1.9595	-0.3712	0.0062
10.0	0.3134	0.0627	-2.2822	-0.4564	0

Note:- The above values are for $\phi = 0°$. For any other $\phi = \phi°$, the above values are to be multiplied by cos ϕ. Values of bending stress are for the upper surface. For the lower surface reverse the sign only.

$\phi = 0^{\circ}$

Table 40. Shearing Stress Intensities.

ξ	due to $N_{x\phi}$ $\left(\frac{2t}{PR}\right)\tau_{x\phi}$	due to $M_{x\phi}$ $\left(\frac{2t}{PR}\right)\tau_{x\phi,t}$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_x$	$\frac{2}{P}\left(\frac{t}{R}\right)^{\frac{1}{2}}\tau_\phi$
0	0	0	-0.0033	-0.0033
0.5	0.0246	0.0011	-0.0035	0.0034
1.0	0.0492	0.0023	-0.0040	0.0036
1.5	0.0741	0.0036	-0.0046	0.0038
2.0	0.0992	0.0050	-0.0049	0.0041
2.5	0.1249	0.0066	-0.0043	0.0042
3.0	0.1511	0.0079	-0.0022	0.0041
3.5	0.1781	0.0090	-0.0024	0.0035
4.0	0.2058	0.0091	0.0103	0.0023
4.5	0.2343	0.0079	0.0223	0.0003
5.0	0.2632	0.0043	0.0384	-0.0027
5.5	0.2920	-0.0022	0.0576	-0.0069
6.0	0.3198	-0.0125	0.0766	-0.0119
6.5	0.3454	-0.0269	0.0894	-0.0174
7.0	0.3668	-0.0452	0.0863	-0.0226
7.5	0.3818	-0.0657	0.0536	-0.0260
8.0	0.3881	-0.0851	-0.0262	-0.0255
8.5	0.3838	-0.0978	-0.1716	-0.0185
9.0	0.3681	-0.0956	-0.3980	-0.0021
9.1	0.3637	-0.0924	-0.4536	0.0026
9.2	0.3589	-0.0881	-0.5126	0.0078
9.3	0.3538	-0.0826	-0.5749	0.0136
9.4	0.3484	-0.0756	-0.6403	0.0199
9.5	0.3428	-0.0673	-0.7086	0.0268
9.6	0.3369	-0.0574	-0.7796	0.0342
9.7	0.3310	-0.0458	-0.8530	0.0423
9.8	0.3250	-0.0325	-0.9283	0.0510
9.9	0.3192	-0.0172	-1.0051	0.0603
10.0	0.3134	0	-1.0829	0.0701

Note:- The above values are for $\phi = 0^{\circ}$. For any other $\phi = \phi^{\circ}$, the above values excluding that for τ_x are to be multiplied by $\sin\phi$. Values for τ_x are to be multiplied by $\cos\phi$. Values of $\left(\frac{2t}{PR}\right)\tau_{x\phi,t}$ due to $M_{x\phi}$ are for the top surface. For the lower surface reverse the sign only.

SECTION XI

Case V. Lantern Opening. Reinforced by Rings at Inner and Outer Edges. Symmetrical Vertical Loading. $\xi_1 = 10.0$.

Two loadings will be investigated. Case A is for a concentrated load all along the lantern ring. Case B is for a uniform vertical loading.

1. Discussion

It is required that the circumferential strain at the inner and outer edges of the shell be equal to the circumferential strain in the ring at that edge. This is equivalent to the requirement that the horizontal displacement of the shell at an edge be equal to the radial displacement of the ring at that edge. It is assumed that only the steel in a tension ring is effective in carrying tensile stresses and that the compression ring is wholly concrete with the same modulus of elasticity as that of the concrete in the shell.

2. Determination of the Radial Displacement of a Ring due to an Uniform Thrust Nxx (per unit length) all along the Circumference.

The notation is given in Figure 9 below.

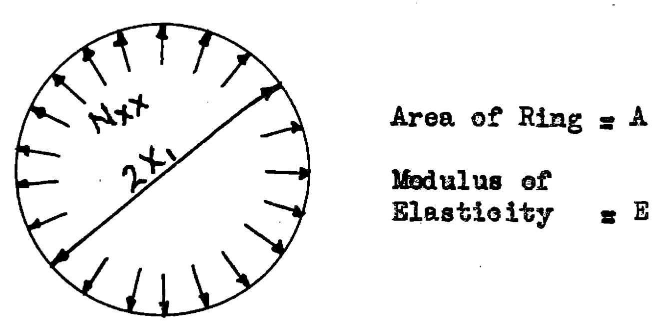


Figure 9. Notation for Ring under Uniform Pressure.

Tension in Ring $= T = \frac{N_{xx} X_1}{A}$ (250)

Circumferential Strain $= \varepsilon_\phi = \frac{T}{E} = \frac{N_{xx} X_1}{AE}$ (251)

Radial Displacement $= \varepsilon_\phi X_1 = \frac{N_{xx} X_1^2}{AE}$ (252)

Since our sign conventions give a positive displacement outward we require, under the assumption that the circumferential strains of the shell and ring at an edge being equal, that

at the outer edge

$$U_h = - \frac{N_{xx} X_1^2}{A_1 E_s} \qquad (253a)$$

at the inner edge

$$U_h = + \frac{N_{xx} X_o^2}{A_o E_c} \qquad (253b)$$

For the shallow dome, we use Nxx as the horizontal forces acting upon the ring inasmuch as the contrbution of the vertical shear forces is negligible and was so considered in the determination of our horizontal reactions in our previous work.

Case A. Concentrated Load all along the Lantern Ring.

3. Data. See Figure 10.

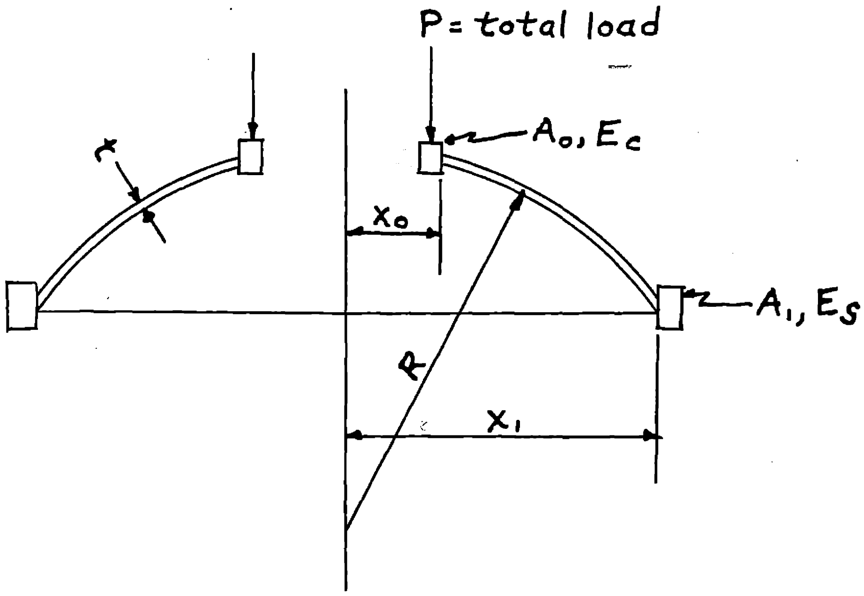


Figure 10. Notation for Case A.

Assumed Data

$P = 1000$ lbs. Total	$A_o = 36$ sq. ins.
$\xi_1 = 10.0$	$E_c = 3.6 \times 10^6$ p.s.i.
$x_1 = 30' = 360''$	$A_1 = 7.2$ sq. ins.
$x_o = 1.5' = 18''$	$E_S = 30 \times 10^6$ p.s.i.
$t = 2''$	$V = 0.2$

Computed Data

$$\ell = \frac{x_1}{\xi_1} = 36'' \qquad \mu = \frac{x_o}{\ell} = 0.5$$

$$R = \frac{x_1^2\sqrt{12(1-V^2)}}{\xi_1^2 t} = 2199.386''$$

$$D = \frac{E_c t^3}{12(1-V^2)} = 2.5 \times 10^6 \text{ in.-lbs.}$$

$$n = \frac{E_s}{E_c} = 8\tfrac{1}{3}$$

4. Expressions for W and F

From equation (61) noting that $W_p = 0$

$$w = C_{10}\,\mathrm{ber}\,\xi + C_{20}\,\mathrm{bei}\,\xi + C_{30}\,\mathrm{Ker}\,\xi + C_{40}\,\mathrm{kei}\,\xi - A_0 \qquad (254)$$

From equation (63)

$$F = \frac{\ell^2 tE}{R}\left(-C_{20}\,\mathrm{ber}\,\xi + C_{10}\,\mathrm{bei} - C_{40}\,\mathrm{ker}\,\xi + C_{30}\,\mathrm{kei}\,\xi\right)$$

$$+ b_0 \ln \xi \qquad (255)$$

5. Boundary Conditions

Inasmuch as the effects of a concentrated load decay very rapidly and are negligible a short distance away from the point of loading, the problem is simplified considerably by considering the outer boundary in this case as being at ∞.

Hence, our boundary conditions are

$$\text{at} \quad \begin{matrix} x = x_0 \\ \xi = \mu \end{matrix} \quad \begin{cases} M_x = 0 & (256) \\ 2\pi x_0 Q_x = P & (257) \\ U_h = + \dfrac{x_0^2 N_{xx}}{A_0 E} & (258) \end{cases}$$

$$\text{at} \quad \begin{matrix} x = \infty \\ \xi = \infty \end{matrix} \quad \begin{cases} \left.\begin{matrix}\text{Stress and Displacements} \\ \text{vanish}\end{matrix}\right\} & (259) \\ \lim\limits_{x \to \infty} (2\pi x P_v) \to P & (260) \end{cases}$$

Condition (257) is valid in that the shell is so flat at this point that the contribution of Nxx to the vertical equilibrium requirement is negligible.

The lantern ring is assumed to possess no torsional resistance as given by condition (256). The horizontal reaction is measured by N_{xx} due to the extreme flatness of the shell at this point. This is expressed in condition (258).

6. Equations for Determination of the Constants.

Application of condition (259) to equation (254), noting that ber ξ and bei $\xi \rightarrow \infty$ at $\xi \rightarrow \infty$, requires that

$$C_{10} = C_{20} = A_0 = 0 \tag{261}$$

Application of condition (256), using equations (89a), (90), (91), (254) and (261), results in

$$0 = C_{30}\left[\frac{(1-\nu)}{\mu}\,\mathrm{Ker}'\mu + \mathrm{Kei}\,\mu\right] + C_{40}\left[\frac{(1-\nu)}{\mu}\,\mathrm{Kei}'\mu - \mathrm{Ker}\,\mu\right] \tag{262}$$

Application of condition (257), using equations (81), (97), (254) and (261), results in

$$\frac{P\ell^2}{2\pi\mu D} = C_{30}\,\mathrm{Kei}'\mu - C_{40}\,\mathrm{Ker}'\mu \tag{263}$$

Application of condition (258), using equations (93a), (95), (255) and (261), results in

$$\begin{aligned} 0 = {} & C_{30}\left[\left(\frac{\mu\ell t}{A_0} + 1 + \nu\right)\frac{\mathrm{Kei}'\mu}{\mu} - \mathrm{Ker}\,\mu\right] \\ & - C_{40}\left[\left(\frac{\mu\ell t}{A_0} + 1 + \nu\right)\frac{\mathrm{Ker}'\mu}{\mu} + \mathrm{Kei}\,\mu\right] \\ & + \frac{b_0 R}{E_c \ell \mu}\left[\frac{1}{A_0} + \frac{(1+\nu)}{t\ell\mu}\right] \end{aligned} \tag{264}$$

Equations (261), (262), (263) and (264) provide the solution for the six constants C_{10}, C_{20}, C_{30}, C_{40}, A_o and b_o.

7. Expressions for Stresses and Displacements

From Section V we have

(1) Bending Stresses

$$\sigma_{x,b} = \pm \frac{6M_{xx}}{t^2}$$

$$= \mp \frac{\sqrt{3}}{\sqrt{1-\nu^2}} \frac{E}{R} \left\{ -C_{30}\left[(1-\nu)\frac{\ker'\xi}{\xi} + \operatorname{kei}\xi\right] - C_{40}\left[(1-\nu)\frac{\operatorname{kei}'\xi}{\xi} - \ker\xi\right]\right\} \tag{265}$$

$$\sigma_{\phi,b} = \pm \frac{6M_{\phi\phi}}{t^2}$$

$$= \mp \frac{\sqrt{3}}{\sqrt{1-\nu^2}} \frac{E}{R} \left\{ C_{30}\left[(1-\nu)\frac{\ker'\xi}{\xi} - \nu \operatorname{kei}\xi\right] + C_{40}\left[(1-\nu)\frac{\operatorname{kei}'\xi}{\xi} + \nu\ker\xi\right] \right. \tag{266}$$

(2) Direct Stresses

$$\sigma_{x,d} = \frac{N_{xx}}{t}$$

$$= \frac{E}{R}\left[C_{30}\frac{\operatorname{kei}'\xi}{\xi} - C_{40}\frac{\ker'\xi}{\xi}\right] + \frac{b_o}{t\ell^2\xi^2} \tag{267}$$

$$\sigma_{x,d} = \frac{N\phi\phi}{t} = \frac{E}{R}\left[C_{30}\left(\ker\xi - \frac{\mathrm{kei}'\xi}{\xi}\right) + C_{40}\left(\mathrm{kei}\,\xi + \frac{\ker'\xi}{\xi}\right)\right] - \frac{b_0}{t\ell^2\xi^2} \qquad (268)$$

(3) Shearing Stress

$$\tau_x = \frac{Q_x}{t}$$

$$= \frac{E}{\sqrt[4]{12(1-\nu^2)}\,R}\left(\frac{t}{R}\right)^{\frac{1}{2}}\left[C_{30}\,\mathrm{kei}'\xi - C_{40}\ker'\xi\right] \qquad (269)$$

(4) Vertical Deflection

$$\omega = C_{30}\ker\xi + C_{40}\,\mathrm{kei}\,\xi \qquad (270)$$

8. Evaluation of the Constants C_{30}, C_{40}, b_0

Substitution of the numerical values from page 143 and Table A-2 into equations (262), (263) and (264) lead to the following three equations in C_{30}, C_{40} and b_0.

$$\left.\begin{aligned} +3.58326\,C_{30} + 0.322780\,C_{40} &= 0 \\ +0.333204\,C_{30} + 1.81980\,C_{40} &= 0.165012 \\ +0.610192\,C_{30} + 8.67870\,C_{40} - 2.07418 \times 10^{-6}\,b_0 &= 0 \end{aligned}\right\} \qquad (271\,a,b,c)$$

Solution of these equations gives

$$\left.\begin{aligned} C_{30} &= 0.092197 \\ C_{40} &= -0.0083051 \\ b_0 &= -3.8332 \times 10^{5} \end{aligned}\right\} \qquad (272)$$

9. Determination of the Stresses and Displacements

Substitution of equations (272) into the expressions (265) through (270) inclusive will give the desired stresses and vertical displacements. This has been done and the results are tabulated in Table 41.

Case B. Uniform Vertical Loading.

10. Data. See Figure 11. This differs from Figure 10 only in loading.

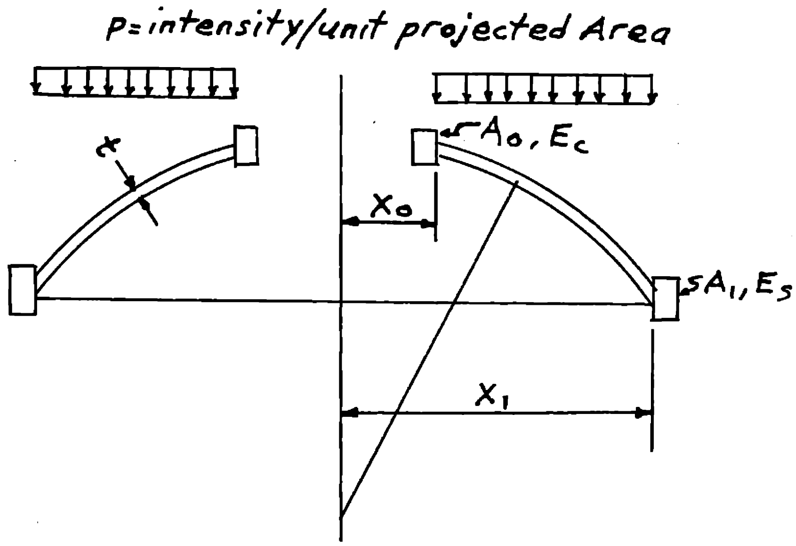


Figure 11. Notation for Case B.

Assumed Data

P = -30 p.s.f. = -0.208333 p.s.i.

ξ_1 = 10.0

x_1 = 30' = 360"

x_o = 1.5 = 18"

t = 2"

A_o = 36 sq. ins.

E_c = 3.6×10^6 p.s.i.

A_1 = 7.2 sq. ins.

E_s = 30×10^6 p.s.i.

V = 0.2

Computed Data

$$\ell = \frac{x_1}{\xi_1} = 36''$$

$$\mu = \frac{x_0}{\ell} = 0.5$$

$$R = \frac{x_1^2\sqrt{12(1-\nu^2)}}{\xi_1^2\, t} = 2199.386''$$

$$D = \frac{E_c t^3}{12(1-\nu^2)} = 2.5 \times 10^6 \text{ in.-lbs.}$$

$$n = \frac{E_s}{E_c} = 8\frac{1}{3}$$

11. Expressions for W and F

Noting that W_p and $F_{p,2}$ are given by equations (122) and (123b), using the notation of equation (133) we have from equation (61)

$$w = C_{10}\,\text{ber}\,\xi + C_{20}\,\text{bei}\,\xi + C_{30}\,\text{ker}\,\xi + C_{40}\,\text{kei}\,\xi - C_{50} \qquad (273)$$

From equation (63)

$$F = \frac{\ell^2 t E}{R}\left(-C_{20}\,\text{ber}\,\xi + C_{10}\,\text{bei}\,\xi - C_{40}\,\text{ker}\,\xi + C_{30}\,\text{kei}\,\xi\right)$$

$$+\, b_0 \ln \xi + F_{p,2} \qquad (274)$$

12. Boundary Conditions

$$\text{at} \quad \begin{matrix} x = x_0 \\ \xi = \mu \end{matrix} \begin{cases} M_x = 0 & (275) \\ 2\pi x_0 Q_x = 0 & (276) \\ U_h = + \dfrac{x_0^2 N_{xx}}{A_0 E_c} & (277) \end{cases}$$

$$\text{at} \quad \begin{matrix} x = x_1 \\ \xi = \xi_1 \end{matrix} \left\{ \begin{aligned} W &= 0 && (278) \\ \frac{dw}{dx} &= 0 && (279) \\ U_h &= -\frac{X_1^2 N_{xx}}{A_1 E_s} && (280) \end{aligned} \right.$$

The lantern ring is assumed to possess no torsional resistance as expresses by condition (275). The horizontal reaction at the edges is given by Nxx due to the flatness of the dome. This is expressed by conditions (277) and (280). At the outer edge, the support offered the edge is assumed to prevent rotation of the ring. This is given in condition (279). Condition (276) is similar to condition (257) of Case A.

13. Equations for Determination of the Constants.

At $x = x_0$, using equations (89a), (90), (91) and (273) application of condition (275) gives as a result

$$0 = C_{10}\left[\frac{(1-\nu)}{\mu} ber'\mu + bei\,\mu\right] + C_{20}\left[\frac{(1-\nu)}{\mu} bei'\mu - ber\,\mu\right]$$
$$+ C_{30}\left[\frac{(1-\nu)}{\mu} ker'\mu + kei\,\mu\right] + C_{40}\left[\frac{(1-\nu)}{\mu} kei'\mu - ker\,\mu\right] \qquad (281)$$

Application of condition (276), using equations (81), (97) and (273) gives as a result

$$0 = C_{10}\, bei'\mu - C_{20}\, ber'\mu + C_{30}\, kei'\mu - C_{40}\, ker'\mu \qquad (282)$$

Application of condition (277), using equations (93a), (123b), (95) and (274) gives as a result

$$-\frac{R^2 p}{2E_c}\left[\frac{\mu l}{A_o}-\frac{(1-\nu)}{t}\right]=\frac{b_o R}{E_c l\mu}\left[\frac{1}{A_o}+\frac{(1+\nu)}{t l\mu}\right]$$
$$+C_{10}\left[\left(\frac{\mu l t}{A_o}+1+\nu\right)\frac{bei'\mu}{\mu}-ber\mu\right]$$
$$-C_{20}\left[\left(\frac{\mu l t}{A_o}+1+\nu\right)\frac{ber'\mu}{\mu}+bei\mu\right]$$
$$+C_{30}\left[\left(\frac{\mu l t}{A_o}+1+\nu\right)\frac{kei'\mu}{\mu}-ker\mu\right]$$
$$-C_{40}\left[\left(\frac{\mu l t}{A_o}+1+\nu\right)\frac{ker'\mu}{\mu}+kei\mu\right] \qquad (283)$$

At $x = x_1$ using equations (273), condition (278) gives

$$0 = C_{10}\,ber\,\xi_1 + C_{20}\,bei\,\xi_1 + C_{30}\,ker\,\xi_1 + C_{40}\,kei\,\xi_1 - C_0 \qquad (284)$$

Application of condition (279) to equation (273) gives

$$0 = C_{10}\,ber'\xi_1 + C_{20}\,bei'\xi_1 + C_{30}\,ker'\xi_1 + C_{40}\,kei'\xi_1 \qquad (285)$$

Application of condition (280), using equations (93a), (95), (123b) and (274) gives

$$+\frac{R^2 p}{2E_c}\left[\frac{(1-\nu)}{t}+\frac{\xi_1 l}{A_1 n}\right]=\frac{b_o R}{E_c l\xi_1}\left[\frac{(1+\nu)}{t l\xi_1}-\frac{1}{A_1 n}\right]$$
$$+C_{10}\left[\left(-\frac{\xi_1 l t}{A_1 n}+1+\nu\right)\frac{bei'\xi_1}{\xi_1}-ber\,\xi_1\right]$$
$$-C_{20}\left[\left(-\frac{\xi_1 l t}{A_1 n}+1+\nu\right)\frac{ber'\xi_1}{\xi_1}+bei\,\xi_1\right]$$

$$+C_{30}\left[\left(-\frac{\xi_1 \ell t}{A_1 n}+1+\nu\right)\frac{\mathrm{kei}'\xi_1}{\xi_1}-\mathrm{ker}\,\xi_1\right]$$

$$-C_{40}\left[\left(-\frac{\xi_1 \ell t}{A_1 n}+1+\nu\right)\frac{\mathrm{ker}'\xi_1}{\xi_1}+\mathrm{kei}\,\xi_1\right] \qquad (286)$$

Solution of the six equations (281) through (286), inclusive, provides the values of C_{10}, C_{20}, C_{30}, C_{40}, b_0 and G_0 use of which will determine the state of stress and displacement in the shell under investigation.

14. Expressions for Stresses and Displacements

From Section V we have

(1) Bending Stresses

$$\sigma_{x,b} = \pm\frac{6M_{xx}}{t^2} = \mp\frac{\sqrt{3}}{\sqrt{1-\nu^2}}\frac{E}{R}\Big\{-C_{10}\left[(1-\nu)\frac{\mathrm{ber}'\xi}{\xi}+\mathrm{bei}\,\xi\right]$$

$$-C_{20}\left[(1-\nu)\frac{\mathrm{bei}'\xi}{\xi}-\mathrm{ber}\,\xi\right]-C_{30}\left[(1-\nu)\frac{\mathrm{ker}'\xi}{\xi}+\mathrm{kei}\,\xi\right]$$

$$-C_{40}\left[(1-\nu)\frac{\mathrm{kei}'\xi}{\xi}-\mathrm{ker}\,\xi\right]\Big\} \qquad (287)$$

$$\sigma_{\phi,b} = \pm\frac{6M_{\phi\phi}}{t^2} = \mp\frac{\sqrt{3}}{\sqrt{1-\nu^2}}\frac{E}{R}\Big\{C_{10}\left[(1-\nu)\frac{\mathrm{ber}'\xi}{\xi}-\nu\,\mathrm{bei}\,\xi\right]$$

$$+C_{20}\left[(1-\nu)\frac{\mathrm{bei}'\xi}{\xi}+\nu\,\mathrm{ber}\,\xi\right]+C_{30}\left[(1-\nu)\frac{\mathrm{ker}'\xi}{\xi}-\nu\,\mathrm{kei}\,\xi\right]$$

$$+C_{40}\left[(1-\nu)\frac{\mathrm{kei}'\xi}{\xi}+\nu\,\mathrm{ker}\,\xi\right] \qquad (288)$$

(2) Direct Stresses

$$\sigma_{x,d} = \frac{N_{xx}}{t}$$

$$= \frac{E}{R}\left[C_{10}\frac{\mathrm{bei}'\xi}{\xi} - C_{20}\frac{\mathrm{ber}'\xi}{\xi} + C_{30}\frac{\mathrm{kei}'\xi}{\xi} - C_{40}\frac{\mathrm{ker}'\xi}{\xi}\right] + \frac{b_0}{t\ell^2\xi^2} + \frac{RP}{2t} \tag{289}$$

$$\sigma_{\phi,d} = \frac{N_{\phi\phi}}{t}$$

$$= \frac{E}{R}\left[C_{10}\left(\mathrm{ber}\,\xi - \frac{\mathrm{bei}'\xi}{\xi}\right) + C_{20}\left(\mathrm{bei}\,\xi + \frac{\mathrm{ber}'\xi}{\xi}\right) + C_{30}\left(\mathrm{ker}\,\xi - \frac{\mathrm{kei}'\xi}{\xi}\right) + C_{40}\left(\mathrm{kei}\,\xi + \frac{\mathrm{ker}'\xi}{\xi}\right)\right] - \frac{b_0}{t\ell^2\xi^2} + \frac{RP}{2t} \tag{290}$$

(3) Shearing Stress

$$\tau_x = \frac{Q_x}{t}$$

$$= \frac{E}{\sqrt[4]{12(1-\nu^2)}\,R}\left(\frac{t}{R}\right)^{\frac{1}{2}}\left[C_{10}\mathrm{bei}'\xi - C_{20}\mathrm{ber}'\xi + C_{30}\mathrm{kei}'\xi - C_{40}\mathrm{ker}'\xi\right] \tag{291}$$

(4) Vertical Deflection

$$w = C_{10}\,\mathrm{ber}\,\xi + C_{20}\,\mathrm{bei}\,\xi + C_{30}\,\mathrm{ker}\,\xi + C_{40}\,\mathrm{kei}\,\xi - C_{50} \tag{292}$$

15. Evaluation of the Constants C_{10}, C_{20}, C_{30}, C_{40}, b_0, and Cs_0.

Substitution of the numerical values from pages 148 and 149 and Tables A-1 and A-2 into equations (281) through (286) inclusive lead to the following six equations in C_{10}, C_{20}, C_{30}, C_{40}, b_0 and Cs_0.

C_{10}	C_{20}	C_{30}	C_{40}	b_0	Cs_0	
+0.049994	-0.599154	-3.58326	-0.322780			0
+0.249919	+0.0078121	+0.333204	+1.81980			0
+0.100618	-0.0281201	+0.610192	+8.67870	$+2.07418 \times 10^{-6}$		+0.0139968
+51.1953	+135.309	-3.15597×10^{-4}	$+1.40914 \times 10^{-4}$			0
+138.840	+56.3705	$+1.29466 \times 10^{-4}$	-3.07524×10^{-4}		-1.00000	0
-284.975	-1.07958	-2.81653×10^{-4}	-3.33208×10^{-5}	-2.54559×10^{-8}		-0.895796

(293)

Solution of these equations gives

$$
\left.
\begin{aligned}
C_{10} &= +0.00314715 \\
C_{20} &= -0.00119075 \\
C_{30} &= +0.000286207 \\
C_{40} &= -0.000479500 \\
b_0 &= +8500. \\
Cs_0 &= +0.3698
\end{aligned}
\right\} \qquad (294)
$$

It is to be noted that the above six constants could have been computed rather closely by utilization of two sets of three equations rather than one set of six equations. This is based upon the realization that the effect of the lantern opening is local as is the effect of the outer boundary. Thus by considering the outer boundary at infinity we will obtain the values of three of the six constants. Then considering the shell closed at the crown and the outer boundary to be at x_1, we obtain the values of the other three constants. This is tantamount to the superposition of two shells which together are equivalent to the given shell.

16. Determination of the Stresses and Displacements.

Substitution of equations (294) into the expressions for stresses and vertical displacement is now made. The results are tabulated in Table 42.

17. Discussion of the Results.

The membrane displacement of the outer edge for this case would be inward. However, the results of Table 42 are consistent with a displacement of the outer ring outward. Furthermore, at the outer edge $\sigma_{\phi_1 d} = +411$ p.s.i. whereas $\sigma_{x_1 d} = -35$ p.s.i. which seems strange. It seems reasonable, however, that as the shell displaces outward the $\sigma_{x_1 d}$ stresses should approach zero as the condition of a free boundary ($\sigma_{x_1 d} = 0$); i.e. no ring, is attained. It appears, then, for $\sigma_{x_1 d} = 0$, that $\sigma_{\phi_1 d}$ should be a tensile stress greater than +411 p.s.i. A calculation based upon a closed shell with the outer boundary conditions: $W = 0$, $\frac{d\omega}{dx} = 0$ and $\sigma_{x_1 d} = 0$ under the same vertical loading gave $\sigma_{\phi_1 d} = +641.2$ p.s.i. at the outer edge. The results appear rational and it is plausible that the outward displacement is due to condition (279); namely, that $\frac{d\omega}{dx} = 0$ at $x = x_1$.

Table 41. Direct, Bending and Shear Stresses. Vertical Deflections. Concentrated Lantern Ring Loading. Case A.

ξ	$\sigma_{x,d}$ (P.S.I.)	$\sigma_{\phi,d}$ (P.S.I.)	$\sigma_{x,b}$ (P.S.I.)	$\sigma_{\phi,b}$ (P.S.I.)	ω (in.)	τ_x (P.S.I.)
0.5	-51	-62	0	-255	-0.069	+4
1.0	-48	-31	+24	-101	-0.048	+2
1.5	-39	-12	+40	- 47	-0.031	+1
2.0	-30	0	+40	- 21	-0.018	0
2.5	-23	+ 8	+34	- 9	-0.010	0
3.0	-18	+11	+25	- 3	-0.004	0
3.5	-14	+12	+17	0	-0.001	0
4.0	-10	+11	+11	+ 1	+0.001	0
4.5	- 8	+10	+ 6	+ 1	+0.001	0
5.0	- 6	+ 8	+ 3	+ 1	+0.001	0
5.5	- 5	+ 7	+ 1	0	+0.001	0
6.0	- 4	+ 5	0	0	+0.001	0
6.5	- 4	+ 4	- 1	0	0	0
7.0	- 3	+ 3	- 1	0	0	0
7.5	- 3	+ 3	- 1	0	0	0
8.0	- 2	+ 2	0	0	0	0
8.5	- 2	+ 2	0	0	0	0
9.0	- 2	+ 2	0	0	0	0
9.5	- 2	+ 2	0	0	0	0
10.0	- 1	+ 1	0	0	0	0

Bending stress values are for the upper surface only. For the lower surface, reverse the sign.

Table 42. Direct, Bending and Shear Stresses. Vertical Deflections. Uniform Loading. Case B.

ξ	$\sigma_{x,d}$ (p.s.i.)	$\sigma_{\phi,d}$ (p.s.i.)	$\sigma_{x,b}$ (p.s.i.)	$\sigma_{\phi,b}$ (p.s.i.)	w (in.)	τ_x (p.s.i.)
0.5	-101	-122	0	+ 6	-0.366	0
1.0	-109	-115	+ 3	+ 4	-0.367	0
1.5	-111	-114	+ 5	+ 4	-0.367	0
2.0	-112	-115	+ 8	+ 5	-0.369	0
2.5	-113	-117	+ 11	+ 7	-0.369	0
3.0	-114	-120	+ 12	+ 8	-0.373	0
3.5	-115	-125	+ 11	+ 9	-0.376	0
4.0	-117	-130	+ 7	+ 8	-0.381	0
4.5	-118	-136	- 4	+ 5	-0.385	0
5.0	-120	-141	- 24	- 1	-0.390	0
5.5	-122	-142	- 53	- 10	-0.392	- 1
6.0	-124	-137	- 93	- 24	-0.389	- 1
6.5	-124	-119	-140	- 41	-0.378	- 1
7.0	-123	- 84	-186	- 61	-0.356	- 1
7.5	-118	- 26	-217	- 78	-0.318	0
8.0	-110	+ 57	-209	- 87	-0.262	+ 1
8.5	- 97	+163	-127	- 77	-0.189	+ 2
9.0	- 80	+280	+ 72	- 36	-0.108	+ 5
9.1	- 75	+302	+129	- 23	-0.092	+ 6
9.2	- 71	+323	+194	- 8	-0.076	+ 7
9.3	- 67	+343	+266	+ 10	-0.061	+ 7
9.4	- 62	+362	+345	+ 29	-0.047	+ 8
9.5	- 58	+378	+432	+ 51	-0.034	+ 9
9.6	- 53	+392	+528	+ 75	-0.023	+10
9.7	- 49	+403	+631	+102	-0.013	+10
9.8	- 44	+410	+743	+132	-0.006	+11
9.9	- 39	+413	+863	+164	-0.002	+12
10.0	- 35	+411	+992	+198	0	+13

Bending stress values are for the upper surface only. For the lower surface, reverse the sign.

SECTION XII

Case VI. Outer Edge Effects. Unsymmetrical Edge Loading.

1. Discussion

In a manner analogous to that given in Case II for symmetrical edge loading, it is possible to determine influence coefficients whereby any desired displacement of the edge (i.e. ratation, horizontal displacement, circumferential displacement or vertical deflection) may be evaluated for any given unsymmetrical edge loading.

Having found these influence coefficients the converse of the above problem may be solved; namely, for any given edge displacement the edge forces required may be determined, rigid-body displacements being discounted.

In this section the procedure will be outlined and the fundamental equations developed. It is not considered feasible in a work of this scope to compute numerical values of the influence coefficients as was done for the symmetrical case, due to the extreme amount of labor involved and the lack of tabulated values of the $n^{\underline{th}}$ order Bessel functions.

For simplicity, the vertical edge deflection is assumed equal to zero. Furthermore, rigid-body displacements were not considered in this treatment inasmuch as they do not effect the values of the influence coefficients.

Given the general edge loadings

$$\left.\begin{aligned} M_E &= M_n \cos n\phi \\ P_{ER} &= P_{nR} \cos n\phi \\ P_{ET} &= P_{nT} \sin n\phi \end{aligned}\right\} \qquad (295)$$

where M_n is the amplitude of the edge moment loading

P_{nR} is the amplitude of the edge radial horizontal loading

P_{nT} is the amplitude of the edge tangential loading we may write

$$\frac{\partial \omega}{\partial x} = a_{1n} M_E + a_{2n} P_{ER} + a_{3n} P_{ET} \tag{296a}$$

$$u_h = b_{1n} M_E + b_{2n} P_{ER} + b_{3n} P_{ET} \tag{296b}$$

$$V = C_{1n} M_E + C_{2n} P_{ER} + C_{3n} P_{ET} \tag{296c}$$

This section concerns itself with the procedures involved in the determination of the influence coefficients a_{1n}, b_{1n}, c_{1n}, etc.

2. Boundary Conditions

$$\text{at} \quad \begin{matrix} x = 0 \\ \xi = 0 \end{matrix} \quad \left\{ \begin{array}{l} \text{Stresses} \\ \text{and} \\ \text{Deflections} \\ \text{are finite} \end{array} \right. \tag{297}$$

$$\text{at} \quad \begin{matrix} x = x_1 \\ \xi = \xi_1 \end{matrix} \quad \left\{ \begin{array}{l} w = 0 \\ M_{xx} = M_E = M_n \cos n\phi \\ P_h = P_{ER} = P_{nR} \cos n\phi \\ P_t = P_{ET} = P_{nT} \sin n\phi \end{array} \right\} \tag{298 a,b,c,d}$$

3. Equations for the Determination of the Constants. n = 2

Application of conditions (297), using equations (66a), (68), (69), (67), (10a), (75) and (76), noting that W_p and $F_{p,2}$ do not exist, lead to the requirement that,

$$C_{3n} = C_{4n} = C_{7n} = C_{8n} = 0 \tag{299}$$

Application of condition (298a) to equation (51) using equation (299) leads to the result, after division by cos $n\phi$,

$$0 = C_{1n}\,\mathrm{ber}_n\,\xi_1' + C_{2n}\,\mathrm{bei}_n\,\xi_1' - C_{5n}\,\xi_1'^{\,n} \tag{300}$$

Application of condition (298b) to equation (66a) using equations (299), (68), (69) and (67) leads to the result, after division by cos $n\phi$,

$$M_n = -\frac{(1-\nu)D}{\ell^2}\Big[C_{1n}\Big(-\frac{\mathrm{ber}_n'\,\xi_1'}{\xi_1'} + \frac{n^2\,\mathrm{ber}_n\,\xi_1'}{\xi_1'^{\,2}} - \mathrm{bei}_n\,\xi_1' - \nu\,\mathrm{bei}_n\,\xi_1'\Big)$$
$$+ C_{2n}\Big(-\frac{\mathrm{bei}_n'\,\xi_1'}{\xi_1'} + \frac{n^2\,\mathrm{bei}_n\,\xi_1'}{\xi_1'^{\,2}} + \mathrm{ber}_n\,\xi_1' + \nu\,\mathrm{ber}_n\,\xi_1'\Big)$$
$$- n(n-1)\,C_{5n}\,\xi_1'^{\,n}\Big] \tag{301}$$

Application of condition (298c) to equation (86) using equations (299), (10a), (75) and (76) leads to the result, after division by cos $n\phi$,

$$P_{nR} = \frac{tE}{R}\Big[C_{1n}\Big(\frac{\mathrm{bei}_n'\,\xi_1'}{\xi_1'} - \frac{n^2\,\mathrm{bei}_n\,\xi_1'}{\xi_1'^{\,2}}\Big)$$
$$+ C_{2n}\Big(\frac{n^2\,\mathrm{ber}_n\,\xi_1'}{\xi_1'^{\,2}} - \frac{\mathrm{ber}_n'\,\xi_1'}{\xi_1'}\Big)\Big] - \frac{n(n-1)}{\ell^2}\,C_{6n}\,\xi_1'^{\,n-2} \tag{302}$$

Application of condition (298d) to equation (10c) using equations (78), (79) and (299) leads to the result, after division by sin $n\phi$,

$$P_{nT} = \frac{tEn}{R}\Big[C_{1n}\Big(\frac{\mathrm{bei}_n'\,\xi_1'}{\xi_1'} - \frac{\mathrm{bei}_n\,\xi_1'}{\xi_1'^{\,2}}\Big)$$
$$+ C_{2n}\Big(\frac{\mathrm{ber}_n\,\xi_1'}{\xi_1'^{\,2}} - \frac{\mathrm{ber}_n'\,\xi_1'}{\xi_1'}\Big)\Big] + \frac{n(n-1)}{\ell^2}\,C_{6n}\,\xi_1'^{\,n-2} \tag{303}$$

Solution of the four equations (300), (301), (302) and (303) will yield the solution for C_{1n}, C_{2n}, C_{5n} and C_{6n} in terms of Mn, P_{nR} and P_{nT}. Substitution of these values into the equations for $\frac{\partial \omega}{\partial x}$ (obtained by differentiating equation (61)), U_h (equation (114)) and V (equation (111)) will result in three equations of the form given by equations (296). We need only equate the coefficients of Mn, P_{nR} and P_{nT} to the respective symbols a_{1n}, a_{2n}, a_{3n}, b_{1n}, etc. to obtain expressions for the influence coefficients. This is analogous to the procedure followed for symmetrical edge loading given in Case II.

It is to be pointed out that the above four equations for the constants exist for $n \geq 2$. The case for $n = 1$ will now be discussed.

4. Equations for the Determination of the Constants, n = 1

An examination of the equations (61), (111) and (114) will disclose that the four constants C_{11}, C_{21}, C_{51}, C_{61} are required for their determination. However, substitution of $n = 1$ into equations (300), (301), (302) and (303) will result in the disappearance of terms involving C_{61} leaving four equations in three unknowns C_{11}, C_{21} and C_{51} implying the existence of dependence among two of the four equations and the necessity of a fourth condition to determine this dependency. Examination of the problem will show that conditions (298c) and (298d) are bound together by the condition that the summation of all the horizontal edge forces along any horizontal axis must vanish. That is, the condition of horizontal statical equilibrium is violated for $n = 1$ unless conditions (298c) and (298d) are so selected that the summation of their components along the axis $\phi = 0$ is equal to zero. This will be shown satisfied if $P_{nR} = P_{nT}$ for $n = 1$. For $n = 2$ or more, conditions (298c) and (298d) each in themselves satisfy the requirement of statical equilibrium.

Thus for $n = 1$

$$P_{1R} = P_{1T} \tag{304}$$

For a demonstration of this relationship see Appendix C.

Appendix A

Notes on Properties of the Bessel Functions

References to McLachlan "Bessel Functions for Engineers" are given in square brackets thus [-------] .

n = 0

The following series representations hold [p. 168]

$$\mathrm{ber}\,\xi = 1 - \frac{\xi^4}{2^2 \cdot 4^2} + \frac{\xi^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \cdots \qquad (A-1)$$

$$\mathrm{bei}\,\xi = \frac{\xi^2}{2^2} - \frac{\xi^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{\xi^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \cdots \qquad (A-2)$$

$$\mathrm{Ker}\,\xi = -(\ln \xi - 0.1159)\,\mathrm{ber}\,\xi + \frac{\pi}{4}\,\mathrm{bei}\,\xi - \frac{\xi^4}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right) + \frac{\xi^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \cdots \qquad (A-3)$$

$$\mathrm{kei}\,\xi = -(\ln \xi - 0.1159)\,\mathrm{bei}\,\xi - \frac{\pi}{4}\,\mathrm{ber}\,\xi + \frac{\xi^2}{2^2} - \frac{\xi^6}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots \qquad (A-4)$$

Accordingly for small values of ξ

$$\mathrm{ber}\,\xi \approx 1 - \frac{\xi^4}{64} \qquad (A-5)$$

$$\mathrm{bei}\,\xi \approx \frac{\xi^2}{4} - \frac{\xi^6}{64 \cdot 36} \qquad (A-6)$$

$$\mathrm{Ker}\,\xi \approx -\ln \xi + 0.0059 + \frac{\pi}{16}\,\xi^2 \qquad (A-7)$$

$$\mathrm{kei}\,\xi \approx -\frac{\xi^2}{4}\ln\xi - \frac{\pi}{4} + 1.1159\,\frac{\xi^2}{4} \qquad (A-8)$$

For large values of ξ the following relations hold according to Gray-Mathews "Bessel Functions", London 1931, pp. 58 and 59

$$\mathrm{ber}\,\xi \approx \frac{e^{\xi/\sqrt{2}}}{\sqrt{2\pi\xi}}\cos\left(\frac{\xi}{\sqrt{2}} - \frac{\pi}{8}\right) \qquad (A-9)$$

$$\mathrm{bei}\,\xi \approx \frac{e^{\xi/\sqrt{2}}}{\sqrt{2\pi\xi}}\sin\left(\frac{\xi}{\sqrt{2}} - \frac{\pi}{8}\right) \qquad (A-10)$$

$$\mathrm{ker}\,\xi \approx \sqrt{\frac{\pi}{2\xi}}\,e^{-\xi/\sqrt{2}}\cos\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (A-11)$$

$$\mathrm{kei}\,\xi \approx \sqrt{\frac{\pi}{2\xi}}\,e^{-\xi/\sqrt{2}}\sin\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (A-12)$$

For large values of ξ the following more accurate relations hold according to H. G. Savidge, Phil. Mag. V 6 (19) - 1910, P. 49 et seq.

$$\mathrm{ber}\,\xi = \frac{e^{\alpha}}{\sqrt{2\pi\xi}}\cos\beta \qquad (A-13)$$

$$\mathrm{bei}\,\xi = \frac{e^{\alpha}}{\sqrt{2\pi\xi}}\sin\beta \qquad (A-14)$$

$$\mathrm{ber}'\xi = \frac{e^{\eta}}{\sqrt{2\pi\xi}}\cos\omega \qquad (A-15)$$

$$\mathrm{bei}'\xi = \frac{e^{\eta}}{\sqrt{2\pi\xi}}\sin\omega \qquad (A-16)$$

where

$$\alpha = \frac{\xi}{\sqrt{2}} + \frac{1}{8\sqrt{2}\,\xi} - \frac{25}{384\sqrt{2}\,\xi^3} - \frac{13}{128\,\xi^4} \;----\qquad (A-17)$$

$$\beta = \frac{\xi}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8\sqrt{2}\,\xi} - \frac{1}{16\,\xi^2} - \frac{25}{384\sqrt{2}\,\xi^3} \;----\qquad (A-18)$$

$$\eta = \frac{\xi}{\sqrt{2}} - \frac{3}{8\sqrt{2}\,\xi} + \frac{21}{128\sqrt{2}\,\xi^3} + \frac{27}{128\,\xi^4} \;----\qquad (A-19)$$

$$\omega = \frac{\xi}{\sqrt{2}} + \frac{\pi}{8} + \frac{3}{8\sqrt{2}\,\xi} + \frac{3}{16\,\xi^2} + \frac{21}{128\sqrt{2}\,\xi^3} \;----\qquad (A-20)$$

Values of ber ξ, bei ξ, ber$'\xi$ and bei$'\xi$ for $0 \leq \xi \leq 20.0$ were taken directly from Dwight's "Mathematical Tables" p. 214 et seq. and are tabulated in Table A-1. Values of these functions for $20.0 < \xi \leq 25.0$ were computed with the aid of equations (A-13) to (A-16). The results are tabulated in Table A-1. Values of α, β, η and W used in these calculations are tabulated in Table A-4.

Values of ker ξ, kei ξ, ker$'\xi$ and kei$'\xi$ for $0 \leq \xi \leq 10.0$ were taken directly from Dwight's "Tables of Integrals and Other Mathematical Data" p. 213 et seq. and are tabulated in Table A-2.

To calculate the second derivatives of these functions we use equations (42), n = 0, and thus we have

$$\text{ber}''\xi = -\frac{1}{\xi}\,\text{ber}'\xi - \text{bei}\,\xi \qquad (A-21)$$

$$\text{bei}''\xi = -\frac{1}{\xi}\,\text{bei}'\xi + \text{ber}\,\xi \qquad (A-22)$$

$$\ker''\xi = -\frac{1}{\xi}\ker'\xi - \mathrm{kei}\,\xi \qquad (A-23)$$

$$\mathrm{kei}''\xi = -\frac{1}{\xi}\mathrm{kei}'\xi + \ker\xi \qquad (A-24)$$

In general, in the main text of this work these relations were utilized to express the formulae in terms of the functions and their first derivatives only.

$n \neq 0$

The formulae for $\mathrm{ber}_n\,\xi$, $\mathrm{bei}_n\,\xi$, $\ker_n\,\xi$, $\mathrm{kei}_n\,\xi$ and their derivatives are not given here inasmuch as they are not used in that form in this text. They are extremely long and cumbersome. See equations (52), (53) p. 132., equations (185), (186), (187), (188) p. 169., equations (226) through (233) pp. 171-172 for their formulae. Short tables of these functions may be found on p. 181 for n = 1 to 5.

To compute $\mathrm{ber}_1\,\xi$, $\mathrm{bei}_1\,\xi$, $\mathrm{ber}_1'\,\xi$ and $\mathrm{bei}_1'\,\xi$ for use in Case IV, Hydrostatic Loading we have from equations (46), (46) p. 131

$$\mathrm{ber}_1\,\xi = \frac{1}{\sqrt{2}}\left(\mathrm{ber}'\xi - \mathrm{bei}'\xi\right) \qquad (A-25)$$

$$\mathrm{bei}_1\,\xi = \frac{1}{\sqrt{2}}\left(\mathrm{ber}'\xi + \mathrm{bei}'\xi\right) \qquad (A-26)$$

from equation 51 p. 132 , n = 1

$$\mathrm{ber}_1'\xi = -\frac{1}{\xi}\mathrm{ber}_1\,\xi - \frac{\sqrt{2}}{2}\left(\mathrm{ber}\,\xi + \mathrm{bei}\,\xi\right) \qquad (A-27)$$

$$\mathrm{bei}_1'\xi = -\frac{1}{\xi}\mathrm{bei}_1\,\xi + \frac{\sqrt{2}}{2}\left(\mathrm{ber}\,\xi - \mathrm{bei}\,\xi\right) \qquad (A-28)$$

Values of $ber_1 \xi$, $bei_1 \xi$, $ber_1' \xi$ and $bei_1' \xi$ computed by equations (A-25) through (A-28) are tabulated in Table A-3.

Polar Form of the Bessel Functions

Since $J_n\left(\xi i^{3/2}\right) = i^n I_n\left(\xi i^{3/2}\right) = ber_n \xi + i\, bei_n \xi$ [p. 122 et seq.] it is frequently advantageous to express the function in terms of its modulus and phase.

Thus we have, see Figure A-1,

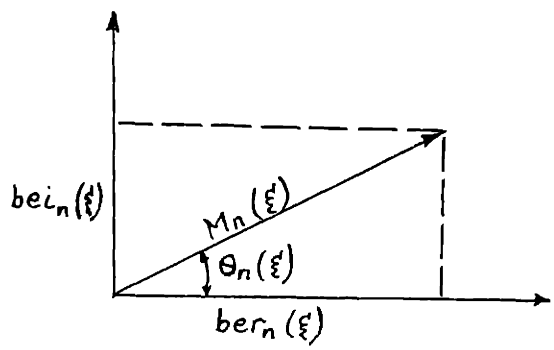


Figure A-1. Complex Representation of $J_n\left(\xi i^{\frac{3}{2}}\right) = ber_n \xi + i\, bei_n \xi$

$$M_n(\xi)\, e^{i\theta_n(\xi)} = J_n\left(\xi i^{3/2}\right) \tag{A-29}$$

$$M_n(\xi)\, e^{i\theta_n(\xi)} = \sqrt{(ber_n \xi)^2 + (bei_n \xi)^2}\left\{\cos\theta_n(\xi) + i \sin\theta_n(\xi)\right\} \tag{A-30}$$

where

$$M_n(\xi) = \sqrt{(ber_n \xi)^2 + (bei_n \xi)^2} \tag{A-31}$$

$$ber_n(\xi) = M_n(\xi) \cos\theta_n(\xi) \tag{A-32}$$

$$bei_n(\xi) = M_n(\xi)\,\sin\theta_n(\xi) \tag{A-33}$$

$$\theta_n(\xi) = \tan^{-1}\frac{bei_n(\xi)}{ber_n(\xi)} \tag{A-34}$$

In particular

$$ber\,\xi = M_0(\xi)\,\cos\theta_0(\xi) \tag{A-35}$$

$$bei\,\xi = M_0(\xi)\,\sin\theta_0(\xi) \tag{A-36}$$

$$\theta_0(\xi) = \tan^{-1}\frac{bei\,\xi}{ber\,\xi} \tag{A-37}$$

Similarly we write, see Figure A-2,

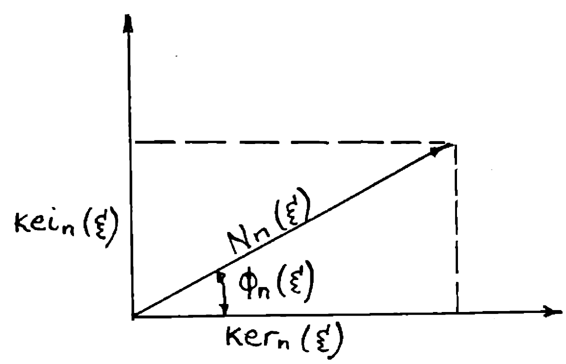


Figure A-2. Complex Representation of $i^{-n}K_n(\xi i^{\frac{1}{2}}) = ker_n\,\xi + i\,kei_n\,\xi$

$$i^{-n} K_n\left(\xi i^{\frac{1}{2}}\right) = \ker_n \xi + i \operatorname{kei}_n \xi \tag{A-38}$$

$$= \sqrt{(\ker_n \xi)^2 + (\operatorname{kei}_n \xi)^2}\left\{\cos\phi_n(\xi) + i \sin\phi_n(\xi)\right\} \tag{A-39}$$

$$= N_n(\xi)\, e^{i\phi_n(\xi)} \tag{A-40}$$

where

$$N_n(\xi) = \sqrt{(\ker_n \xi)^2 + (\operatorname{kei}_n \xi)^2} \tag{A-41}$$

$$\ker_n(\xi) = N_n(\xi) \cos\phi_n(\xi) \tag{A-42}$$

$$\operatorname{kei}_n(\xi) = N_n(\xi) \sin\phi_n(\xi) \tag{A-43}$$

$$\phi_n(\xi) = \tan^{-1}\frac{\operatorname{kei}_n(\xi)}{\ker_n(\xi)} \tag{A-44}$$

In particular

$$\ker \xi = N_0(\xi) \cos\phi_0(\xi) \tag{A-45}$$

$$\operatorname{kei} \xi = N_0(\xi) \sin\phi_0(\xi) \tag{A-46}$$

$$\phi_0(\xi) = \tan^{-1}\frac{\operatorname{kei} \xi}{\ker \xi} \tag{A-47}$$

The following relations can be obtained [pp. 170, 172]

$$\mathrm{ber}_n^2 \xi + \mathrm{bei}_n^2 \xi = M_n^2(\xi) \qquad \text{(A-48)}$$

$$\mathrm{ber}^2 \xi + \mathrm{bei}^2 \xi = M_0^2(\xi) \qquad \text{(A-49)}$$

$$(\mathrm{ber}' \xi)^2 + (\mathrm{bei}' \xi)^2 = M_1^2(\xi) \qquad \text{(A-50)}$$

$$\mathrm{ber}\, \xi\, \mathrm{bei}' \xi - \mathrm{bei}\, \xi\, \mathrm{ber}' \xi = M_0(\xi) M_1(\xi) \sin\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \qquad \text{(A-51)}$$

$$\mathrm{ber}\, \xi\, \mathrm{ber}' \xi + \mathrm{bei}\, \xi\, \mathrm{bei}' \xi = M_0(\xi) M_1(\xi) \cos\left(\theta_1 - \theta_0 - \frac{\pi}{4}\right) \qquad \text{(A-52)}$$

$$\mathrm{ker}_n^2 \xi + \mathrm{kei}_n^2 \xi = N_n^2(\xi) \qquad \text{(A-53)}$$

$$(\mathrm{ker}' \xi)^2 + (\mathrm{kei}' \xi)^2 = N_1^2(\xi) \qquad \text{(A-54)}$$

$$\mathrm{ker}\, \xi\, \mathrm{kei}' \xi - \mathrm{kei}\, \xi\, \mathrm{ker}' \xi = N_0(\xi) N_1(\xi) \sin\left(\phi_1 - \phi_0 - \frac{\pi}{4}\right) \qquad \text{(A-55)}$$

$$\mathrm{ker}\, \xi\, \mathrm{ker}' \xi + \mathrm{kei}\, \xi\, \mathrm{kei}' \xi = N_0(\xi) N_1(\xi) \cos\left(\phi_1 - \phi_0 - \frac{\pi}{4}\right) \qquad \text{(A-56)}$$

Values of $M_0(\xi)$, $M_1(\xi)$, $\theta_0(\xi)$, $\theta_1(\xi)$ are tabulated in Table A-5. Values of $N_0(\xi)$, $N_1(\xi)$, $\phi_0(\xi)$, $\phi_1(\xi)$ are tabulated in Table A-6.

In solving problems by aid of ber, bei, ker, and kei functions, the notation to be used depends upon the type of problem. Sometimes it is convenient to work in ber, bei and then express the results in $Me^{i\theta}$ notation. Also there are cases where the use of $Me^{i\theta}$ notation permits one to manipulate the equations more easily.

Table A-1. Values of the Bessel Functions.

ξ	ber ξ	bei ξ	ber' ξ
0	1.00000	0	0
0.1	1.00000	0.002500	-0.0000625
0.2	0.99998	0.010000	-0.0005000
0.3	0.99987	0.022500	-0.0016875
0.4	0.99960	0.039998	-0.004000
0.5	0.99902	0.062493	-0.007812
0.6	0.99798	0.089980	-0.013498
0.7	0.99625	0.12245	-0.021433
0.8	0.99360	0.15989	-0.031989
0.9	0.98975	0.20227	-0.045537
1.0	0.98438	0.24957	-0.062446
1.2	0.96763	0.35870	-0.10781
1.4	0.94008	0.48673	-0.17093
1.5	0.92107	0.55756	-0.21001
1.6	0.89789	0.63273	-0.25454
1.8	0.83672	0.79526	-0.36118
2.0	0.75173	0.97229	-0.49307
2.1	0.69869	1.0654	-0.56906
2.2	0.63769	1.1610	-0.65200
2.3	0.56805	1.2585	-0.74202
2.4	0.48905	1.3575	-0.83920
2.5	0.39997	1.4572	-0.94358
2.6	0.30009	1.5569	-1.0551
2.7	0.18871	1.6557	-1.1738
2.8	0.065112	1.7529	-1.2993
2.9	-0.071368	1.8472	-1.4314
3.0	-0.22138	1.9376	-1.5698
3.2	-0.56438	2.1016	-1.8636
3.5	-1.1936	2.2832	-2.3361
3.6	-1.4353	2.3199	-2.4983
4.0	-2.5634	2.2927	-3.1347
4.1	-2.8843	2.2309	-3.2818
4.2	-3.2195	2.1422	-3.4200
4.3	-3.5679	2.0236	-3.5465
4.4	-3.9283	1.8726	-3.6588
4.5	-4.2991	1.6860	-3.7537
4.6	-4.6784	1.4610	-3.8280
4.7	-5.0639	1.1946	-3.8782
4.8	-5.4531	0.88366	-3.9006
4.9	-5.8429	0.52515	-3.8911
5.0	-6.2301	0.11603	-3.8453
5.5	-7.9736	-2.7890	-2.9070
6.0	-8.8583	-7.3347	-0.29308
6.1	-8.8491	-8.4545	0.49429
6.2	-8.7561	-9.6437	1.3835

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	$bei'\xi$	$\frac{ber'\xi}{\xi}$	$\frac{bei'\xi}{\xi}$
0	0	0	0.50000
0.1	0.050000	-0.000625	0.50000
0.2	0.099999	-0.0025000	0.50000
0.3	0.14999	-0.0056250	0.49997
0.4	0.19997	-0.010000	0.49992
0.5	0.24992	-0.015624	0.49984
0.6	0.29980	-0.022497	0.49967
0.7	0.34956	-0.030619	0.49937
0.8	0.39915	-0.039986	0.49894
0.9	0.44846	-0.050597	0.49829
1.0	0.49740	-0.062446	0.49740
1.2	0.59352	-0.089842	0.49460
1.4	0.68601	-0.12209	0.49001
1.5	0.73025	-0.14001	0.48683
1.6	0.77274	-0.15909	0.48296
1.8	0.85093	-0.20066	0.47274
2.0	0.91701	-0.24654	0.45850
2.1	0.94418	-0.27098	0.44961
2.2	0.96661	-0.29636	0.43937
2.3	0.98361	-0.32262	0.42766
2.4	0.99443	-0.34967	0.41435
2.5	0.99827	-0.37743	0.39931
2.6	0.99426	-0.40581	0.38241
2.7	0.98149	-0.43474	0.36351
2.8	0.95897	-0.46404	0.34249
2.9	0.92566	-0.49359	0.31919
3.0	0.88048	-0.52327	0.29349
3.2	0.74992	-0.58238	0.23435
3.5	0.43530	-0.66746	0.12437
3.6	0.29366	-0.69397	0.081572
4.0	-0.49114	-0.78368	-0.12278
4.1	-0.74817	-0.80044	-0.18248
4.2	-1.0319	-0.81429	-0.24569
4.3	-1.3433	-0.82477	-0.31240
4.4	-1.6833	-0.83155	-0.38257
4.5	-2.0526	-0.83416	-0.45613
4.6	-2.4520	-0.83217	-0.53304
4.7	-2.8818	-0.82515	-0.61315
4.8	-3.3422	-0.81262	-0.69629
4.9	-3.8331	-0.79410	-0.78227
5.0	-4.3541	-0.76906	-0.87082
5.5	-7.3729	-0.52855	-1.3405
6.0	-10.846	-0.048847	-1.8077
6.1	-11.547	0.081031	-1.8930
6.2	-12.235	0.22315	-1.9734

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	ber ξ	bei ξ	ber′ ξ
6.3	-8.5688	-10.901	2.3802
6.4	-8.2762	-12.223	3.4899
6.5	-7.8669	-13.607	4.7174
6.6	-7.3287	-15.047	6.0675
6.7	-6.6492	-16.538	7.5442
6.8	-5.8155	-18.074	9.1510
6.9	-4.8146	-19.644	10.891
7.0	-3.6329	-21.239	12.765
7.5	5.4550	-29.116	24.130
8.0	20.974	-35.017	38.311
8.5	43.936	-35.298	53.442
9.0	73.936	-24.713	65.601
9.1	80.576	-20.724	67.145
9.2	87.350	-15.976	68.246
9.3	94.208	-10.412	68.831
9.4	101.10	-3.9693	68.821
9.5	107.95	3.4106	68.132
9.6	114.70	11.787	66.674
9.7	121.26	21.218	64.353
9.8	127.54	31.758	61.070
9.9	133.43	43.459	56.720
10.0	138.84	56.370	51.195
10.5	153.77	140.32	1.9344
11.0	132.95	257.21	-94.212
11.5	49.517	400.08	-250.37
12.0	-128.51	546.95	-472.57
12.5	-432.56	653.56	-750.87
13.0	-882.65	646.64	-1047.3
13.1	-990.17	622.87	-1102.7
13.2	-1103.1	589.42	-1154.8
13.3	-1221.0	545.22	-1202.7
13.4	-1343.4	489.19	-1245.3
13.5	-1469.8	420.18	-1281.5
13.6	-1599.5	337.04	-1309.9
13.7	-1731.5	238.57	-1329.2
13.8	-1865.0	123.55	-1337.7
13.9	-1998.7	-9.210	-1333.9
14.0	-2131.3	-160.94	-1316.1
14.1	-2261.3	-332.82	-1282.3
14.2	-2387.1	-526.02	-1230.7
14.3	-2506.8	-741.65	-1159.1
14.4	-2618.2	-980.75	-1065.4
14.5	-2719.1	-1244.3	-947.37
14.6	-2806.8	-1533.1	-802.69
14.7	-2878.6	-1847.9	-628.96

Table A-1 (Cont.) Values of the Bessel Functions.

ξ	$bei'\,\xi$	$\frac{ber'\,\xi}{\xi}$	$\frac{bei'\,\xi}{\xi}$
6.3	-12.901	0.37781	-2.0478
6.4	-13.536	0.54530	-2.1150
6.5	-14.129	0.72575	-2.1737
6.6	-14.670	0.91932	-2.2227
6.7	-15.146	1.1260	-2.2606
6.8	-15.543	1.3457	-2.2857
6.9	-15.847	1.5784	-2.2967
7.0	-16.041	1.8236	-2.2916
7.5	-14.736	3.2173	-1.9648
8.0	-7.6603	4.7889	-0.95754
8.5	8.2895	6.2873	0.97524
9.0	36.299	7.2890	4.0332
9.1	43.583	7.3786	4.7893
9.2	51.460	7.4180	5.5935
9.3	59.936	7.4012	6.4447
9.4	69.012	7.3214	7.3417
9.5	78.684	7.1718	8.2825
9.6	88.940	6.9452	9.2646
9.7	99.763	6.6343	10.285
9.8	111.12	6.2316	11.339
9.9	122.99	5.7293	12.423
10.0	135.31	5.1195	13.531
10.5	201.30	0.18423	19.171
11.0	264.12	-8.5647	24.011
11.5	300.29	-21.771	26.112
12.0	272.67	-39.381	22.722
12.5	129.49	-60.070	10.359
13.0	-192.61	-80.562	-14.816
13.1	-284.38	-84.176	-21.708
13.2	-386.45	-87.485	-29.277
13.3	-499.28	-90.429	-37.540
13.4	-623.27	-92.933	-46.513
13.5	-758.77	-94.926	-56.205
13.6	-906.08	-96.315	-66.624
13.7	-1065.4	-97.022	-77.766
13.8	-1236.9	-96.935	-89.630
13.9	-1420.5	-95.964	-102.19
14.0	-1616.1	-94.007	-115.44
14.1	-1823.5	-90.943	-129.33
14.2	-2042.3	-86.669	-143.82
14.3	-2272.0	-81.056	-158.88
14.4	-2511.6	-73.986	-174.42
14.5	-2760.4	-65.336	-190.37
14.6	-3016.9	-54.979	-206.64
14.7	-3279.9	-42.786	-223.12

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	ber ξ	bei ξ	ber' ξ
14.8	-2931.6	-2189.2	-423.74
14.9	-2962.3	-2557.4	-184.56
15.0	-2967.3	-2952.7	91.056
15.5	-2466.1	-5319.6	2100.3
16.0	-659.50	-8190.7	5349.3
16.5	3110.8	-11094	9952.9
17.0	9484.5	-13087	15683
17.5	18849	-12619	21711
18.0	30962	-7454.3	26298
18.1	33619	-5616.0	26807
18.2	36317	-3452.4	27115
18.3	39034	-938.55	27189
18.4	41746	1950.9	26993
18.5	44423	5241.1	26492
18.6	47033	8956.4	25645
18.7	49539	13121	24411
18.8	51901	17757	22745
18.9	54072	22886	20602
19.0	56003	28527	17934
19.1	57640	34697	14691
19.2	58921	41409	10823
19.3	59782	48674	6279.0
19.4	60152	56497	1006.0
19.5	59957	64879	-5048.2
19.6	59115	73816	-11935
19.7	57540	83297	-19706
19.8	55143	93303	-28411
19.9	51826	103810	-38095
20.0	47489	114780	-48803
20.5	6749.4	174570	-118870
21.0	-76155	233700	-217320
21.5	-214450	271420	-338580
22.0	-415520	253880	-463870
22.5	-672690	132730	-554480
23.0	-953540	-152740	-545340
23.1	-1007100	-235330	-523700
23.2	-1058000	-327610	-493430
23.3	-1105400	-430050	-453600
23.4	-1148400	-543070	-403190
23.5	-1185700	-667080	-341180
23.6	-1216200	-802370	-266510
23.7	-1238500	-949230	-178080
23.8	-1251300	-1107800	-74794
23.9	-1253000	-1278200	44480
24.0	-1241800	-1460400	180850

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	$bei'\xi$	$\frac{ber'\xi}{\xi}$	$\frac{bei'\xi}{\xi}$
14.8	-3547.4	-28.631	-239.69
14.9	-3817.5	-12.387	-256.21
15.0	-4087.8	6.0704	-272.52
15.5	-5332.7	135.50	-344.05
16.0	-5999.5	334.33	-374.97
16.5	-5303.9	603.21	-321.45
17.0	-2155.5	922.53	-126.79
17.5	4774.9	1240.6	272.85
18.0	16841	1461.0	935.61
18.1	19968	1481.0	1103.2
18.2	23345	1489.8	1282.7
18.3	26975	1485.7	1474.0
18.4	30857	1467.0	1677.0
18.5	34987	1432.0	1891.2
18.6	39360	1378.8	2116.1
18.7	43967	1305.4	2351.2
18.8	48793	1209.8	2595.4
18.9	53821	1090.1	2847.7
19.0	59029	943.89	3106.8
19.1	64390	769.16	3371.2
19.2	69871	563.70	3639.1
19.3	75433	325.34	3908.4
19.4	81030	51.856	4176.8
19.5	86609	-258.88	4441.5
19.6	92111	-608.93	4699.5
19.7	97468	-1000.3	4947.6
19.8	102600	-1434.9	5181.8
19.9	107430	-1914.3	5398.5
20.0	111860	-2440.2	5593.0
20.5	123920	-5798.7	6044.9
21.0	105770	-10349	5036.8
21.5	33875	-15748	1575.6
22.0	-120190	-21085	-5463.4
22.5	-384900	-24643	-17107
23.0	-779080	-23711	-33873
23.1	-873550	-22671	-37816
23.2	-972840	-21269	-41933
23.3	-1076600	-19468	-46208
23.4	-1184500	-17230	-50621
23.5	-1296000	-14518	-55149
23.6	-1410400	-11293	-59763
23.7	-1527000	-7514.1	-64431
23.8	-1644900	-3142.6	-69114
23.9	-1763100	1861.1	-73769
24.0	-1880300	7535.4	-78347

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	ber ξ	bei ξ	ber' ξ
24.1	-1216200	-1654200	335420
24.2	-1174100	-1859300	509260
24.3	-1113600	-2075300	703410
24.4	-1032700	-2301500	918840
24.5	-929140	-2537100	1156500
24.6	-800660	-2780900	1417100
24.7	-644940	-3031700	1701300
24.8	-459590	-3288000	2009800
24.9	-242160	-3547700	2342800
25.0	9797.6	-3808800	2700500

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	$bei'\xi$	$\frac{ber'\xi}{\xi}$	$\frac{bei'\xi}{\xi}$
24.1	-1995300	13918	-82792
24.2	-2106500	21044	-87044
24.3	-2212100	28947	-91033
24.4	-2310300	37658	-94685
24.5	-2399000	47203	-97917
24.6	-2475700	57604	-100640
24.7	-2538100	68880	-102760
24.8	-2583200	81040	-104160
24.9	-2608100	94088	-104740
25.0	-2609600	108020	-104380

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	ber ξ	bei ξ	ber' ξ
1.1	0.97714	0.30173	-0.083082
1.3	0.95543	0.42041	-0.13697
1.7	0.86997	0.71204	-0.30484
1.9	0.79752	0.88212	-0.42384
3.1	-0.38553	2.0228	-1.7141
3.3	-0.75841	2.1723	-2.0177
3.4	-0.96804	2.2334	-2.1755
3.7	-1.6933	2.3413	-2.6608
3.8	-1.9674	2.3454	-2.8222
3.9	-2.2576	2.3300	-2.9807
5.1	-6.6107	-0.34666	-3.7589
5.2	-6.9803	-0.86584	-3.6270
5.3	-7.3344	-1.4443	-3.4445
5.4	-7.6674	-2.0845	-3.2064
5.6	-8.2466	-3.5597	-2.5410
5.7	-8.4794	-4.3986	-2.1024
5.8	-8.6644	-5.3068	-1.5855
5.9	-8.7937	-6.2854	-0.98438

Table A-1. (Cont.) Values of the Bessel Functions.

ξ	$bei'\xi$	$\frac{ber'\xi}{\xi}$	$\frac{bei'\xi}{\xi}$
1.1	0.54581	-0.075529	0.49619
1.3	0.64034	-0.10536	0.49257
1.7	0.81310	-0.17932	0.47829
1.9	0.88574	-0.22307	0.46618
3.1	0.82230	-0.55294	0.26526
3.3	0.66214	-0.61142	0.20065
3.4	0.55769	-0.63985	0.16403
3.7	0.13149	-0.71914	0.035538
3.8	-0.052527	-0.74268	-0.013823
3.9	-0.25965	-0.76428	-0.066577
5.1	-4.9046	-0.73704	-0.96169
5.2	-5.4835	-0.69750	-1.0545
5.3	-6.0892	-0.64991	-1.1489
5.4	-6.7199	-0.59378	-1.2444
5.6	-8.0454	-0.45375	-1.4367
5.7	-8.7336	-0.36884	-1.5322
5.8	-9.4333	-0.27336	-1.6264
5.9	-10.139	-0.16684	-1.7185

Table A-2. Values of the Bessel Functions.

ξ	ker ξ	kei ξ	ker' ξ	kei' ξ
0	$+\infty$	-0.785398	$-\infty$	0
0.1	2.42047	-0.776851	-9.96096	0.145975
0.2	1.73314	-0.758125	-4.92295	0.222927
0.3	1.33722	-0.733102	-3.21987	0.274292
0.4	1.06262	-0.703800	-2.35207	0.309514
0.5	0.855906	-0.671582	-1.81980	0.333204
0.6	0.693121	-0.637450	-1.45654	0.348164
0.7	0.561378	-0.602176	-1.19094	0.356310
0.8	0.452882	-0.566368	-0.987335	0.359042
0.9	0.362515	-0.530511	-0.825869	0.357443
1.0	0.286706	-0.494995	-0.694604	0.352370
1.2	0.168946	-0.426164	-0.494643	0.334474
1.4	0.0851260	-0.361665	-0.351055	0.309642
1.5	0.0529349	-0.331396	-0.294182	0.295608
1.6	0.0260299	-0.302566	-0.245115	0.280904
1.8	-0.0146961	-0.249417	-0.165942	0.250438
2.0	-0.0416645	-0.202400	-0.106601	0.219808
2.2	-0.0583388	-0.161431	-0.062337	0.190114
2.4	-0.0673735	-0.126242	-0.029712	0.162107
2.5	-0.0696880	-0.110696	-0.0169298	0.148895
2.6	-0.0708257	-0.0964429	-0.0061358	0.136269
2.8	-0.0702963	-0.0715707	0.0103990	0.112875
3.0	-0.0670292	-0.0511219	0.0214762	0.0920431
3.2	-0.0619848	-0.0345823	0.0283603	0.0737752
3.4	-0.0558966	-0.0214463	0.0320662	0.0579881
3.5	-0.0526393	-0.0160026	0.0329886	0.0509821
3.6	-0.0493156	-0.0112311	0.0334087	0.0445394
3.8	-0.0426469	-0.0034867	0.0330400	0.0332480
4.0	-0.0361788	0.00219840	0.0314785	0.0239106
4.2	-0.0301076	0.00619361	0.0291324	0.0163137
4.4	-0.0245569	0.00882562	0.0263187	0.0102433
4.5	-0.0219999	0.00972092	0.0248145	0.00771543
4.6	-0.0195950	0.0103789	0.0232791	0.00549226
4.8	-0.0152482	0.0110974	0.0201939	0.00186478
5.0	-0.0115117	0.0111876	0.0171934	-0.00081998
5.5	-0.00463216	0.00971631	0.0105763	-0.00444016
6.0	-0.00065304	0.00721649	0.00563171	-0.00522392
6.5	0.00127808	0.00472399	0.00234900	-0.00460003
7.0	0.00192202	0.00270036	0.000420510	-0.00345951
7.5	0.00185989	0.00126687	-0.000538787	-0.00229590
8.0	0.00148583	0.000369584	-0.000879724	-0.00133631
8.5	0.00103735	-0.000114902	-0.000874656	-0.000646733
9.0	0.000637164	-0.000319153	-0.000711231	-0.000208079
9.5	0.000333029	-0.000355743	-0.000504544	0.000034158
10.0	0.000129466	-0.000307524	-0.000315597	0.000140914

Table A-3. Values of the Bessel Functions.

ξ	$ber_1 \xi$	$bei_1 \xi$	$ber_1' \xi$	$bei_1' \xi$
0	0	0	- 0.353553	+ 0.353553
0.5	- 0.182243	+ 0.171195	- 0.386120	+ 0.319837
1.0	- 0.395868	+ 0.307557	- 0.476665	+ 0.212036
1.5	- 0.664865	+ 0.367865	- 0.602308	+ 0.011799
2.0	- 0.997078	+ 0.299775	- 0.720531	- 0.305846
2.5	- 1.37310	+ 0.038668	- 0.763963	- 0.763031
3.0	- 1.73264	- 0.487454	- 0.635995	- 1.36414
3.5	- 1.95964	- 1.34404	- 0.210603	- 2.07449
4.0	- 1.86925	- 2.56382	+ 0.658745	- 2.79283
4.5	- 1.20282	- 4.10568	+ 2.11501	- 3.31973
5.0	+ 0.359777	- 5.79791	+ 4.25133	- 3.32780
5.5	+ 3.15785	- 7.26902	+ 7.03614	- 2.34444
6.0	+ 7.46220	- 7.87668	+ 10.2065	+ 0.23545
6.5	+13.3267	- 6.65532	+ 13.1337	+ 5.08242
7.0	+20.3689	- 2.31717	+ 14.6776	+ 12.7807
7.5	+27.4822	+ 6.64293	+ 13.0664	+ 23.5594
8.0	+32.5069	+ 21.6735	+ 5.86638	+ 36.8822
8.5	+31.9274	+ 43.6505	- 9.86427	+ 50.8912
9.0	+20.7192	+ 72.0543	- 37.1080	+ 61.7490
9.1	+16.6608	+ 78.2964	- 44.1532	+ 63.0259
9.2	+11.8699	+ 84.6448	- 51.7589	+ 63.8623
9.3	+ 6.29017	+ 91.0518	- 59.9296	+ 64.1871
9.4	- 0.134871	+ 97.4626	- 68.6649	+ 63.9243
9.5	- 7.46143	+103.814	- 77.9584	+ 62.9927
9.6	-15.7448	+110.0360	- 87.7977	+ 61.3063
9.7	-25.0385	+116.047	- 98.1627	+ 58.7744
9.8	-35.3939	+121.759	-109.026	+ 55.3010
9.9	-46.8590	+127.073	-120.350	+ 50.7865
10.0	-59.4776	+131.879	-132.087	+ 45.1272

Table A-3. (Cont.) Values of the Bessel Functions.

ξ	$\frac{ber_1 \xi}{\xi^2}$	$\frac{bei_1 \xi}{\xi^2}$	$\frac{ber_1' \xi}{\xi}$	$\frac{bei_1' \xi}{\xi}$
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$
0.5	-0.728972	+0.684780	- 0.772239	+0.639673
1.0	-0.395868	+0.307557	- 0.476665	+0.212037
1.5	-0.295496	+0.163496	- 0.401538	+0.007866
2.0	-0.249270	+0.074944	- 0.360266	-0.152923
2.5	-0.219696	+0.006187	- 0.305586	-0.305212
3.0	-0.192516	-0.054162	- 0.211998	-0.454712
3.5	-0.159971	-0.109718	- 0.060172	-0.592712
4.0	-0.116828	-0.160239	+ 0.164686	-0.698208
4.5	-0.059398	-0.202750	+ 0.470003	-0.737718
5.0	+0.014391	-0.231916	+ 0.850266	-0.665560
5.5	+0.104392	-0.240298	+ 1.27930	-0.426261
6.0	+0.207283	-0.218797	+ 1.70109	+0.039242
6.5	+0.315425	-0.157522	+ 2.02057	+0.781911
7.0	+0.415692	-0.047289	+ 2.09679	+1.82581
7.5	+0.488572	+0.118097	+ 1.74218	+3.14126
8.0	+0.507920	+0.338648	+ 0.733298	+4.61028
8.5	+0.441902	+0.604159	- 1.16050	+5.98720
9.0	+0.255793	+0.889559	- 4.12311	+6.86000
9.1	+0.201193	+0.945495	- 4.85200	+6.92592
9.2	+0.140240	+1.00006	- 5.62597	+6.94155
9.3	+0.072727	+1.05274	- 6.44405	+6.90184
9.4	-0.001526	+1.10302	- 7.30478	+6.80046
9.5	-0.082675	+1.15029	- 8.20614	+6.63082
9.6	-0.170842	+1.19397	- 9.14559	+6.38608
9.7	-0.266112	+1.23336	-10.1199	+6.05921
9.8	-0.368533	+1.26779	-11.1251	+5.64296
9.9	-0.478104	+1.29653	-12.1565	+5.12995
10.0	-0.594776	+1.31879	-13.2087	+4.51273

Table A-4. Values for the Computation of ber ξ, bei ξ, ber' ξ and bei' ξ for $20.5 \leq \xi \leq 25.0$.

ξ	α	β	η	ω
20.5	14.50000	14.09852	14.48277	14.90178
21.0	14.85344	14.45219	14.83663	15.25501
21.5	15.20690	14.80585	15.19048	15.60825
22.0	15.56036	15.15950	15.54431	15.96150
22.5	15.91383	15.51315	15.89813	16.31477
23.0	16.26730	15.86679	16.25194	16.66805
23.1	16.33799	15.93752	16.32270	16.73871
23.2	16.40868	16.00825	16.39346	16.80936
23.3	16.47938	16.07898	16.46422	16.88002
23.4	16.55007	16.14970	16.53498	16.95068
23.5	16.62078	16.22043	16.60574	17.02134
23.6	16.69146	16.29116	16.67649	17.09200
23.7	16.76216	16.36189	16.74725	17.16266
23.8	16.83285	16.43262	16.81801	17.23332
23.9	16.90355	16.50334	16.88877	17.30398
24.0	16.97424	16.57407	16.95952	17.37464
24.1	17.04494	16.64480	17.03028	17.44531
24.2	17.11563	16.71552	17.10104	17.51597
24.3	17.18633	16.78625	17.17179	17.58663
24.4	17.25702	16.85698	17.24255	17.65729
24.5	17.32772	16.92770	17.31330	17.72796
24.6	17.39842	16.99843	17.38406	17.79862
24.7	17.46911	17.06916	17.45481	17.86929
24.8	17.53981	17.13988	17.52556	17.93995
24.9	17.61051	17.21061	17.59632	18.01062
25.0	17.68120	17.28133	17.66707	18.08128

Table A-5. Values of the Bessel Functions in Polar Form.

ξ	$M_0(\xi)$	$M_1(\xi)$	$\theta_0(\xi)$ degrees	$\theta_1 - \frac{\pi}{4}$ degrees	$\sin(\theta_1 - \theta_0 - \frac{\pi}{4})$	$\cos(\theta_1 - \theta_0 - \frac{\pi}{4})$
0	1.0000	0	0	90.000	1.00000	0
0.1	1.0000	0.05000	0.143	90.072	1.00000	0.00124
0.2	1.0000	0.10000	0.573	90.286	0.99999	0.00501
0.3	1.0001	0.15000	1.289	90.465	0.99990	0.01438
0.4	1.0004	0.20001	2.291	91.146	0.99980	0.01998
0.5	1.0010	0.25004	3.579	91.790	0.99952	0.03122
0.6	1.0020	0.30010	5.152	92.578	0.99899	0.04491
0.7	1.0037	0.35022	7.007	93.509	0.99813	0.06101
0.8	1.0064	0.40043	9.141	94.582	0.99683	0.07948
0.9	1.0102	0.45077	11.550	95.798	0.99497	0.10022
1.0	1.0155	0.50130	14.226	97.156	0.99239	0.12308
1.1	1.0227	0.55209	17.160	98.655	0.98900	0.14790
1.2	1.0320	0.60323	20.340	100.295	0.98468	0.17442
1.3	1.0438	0.65482	23.750	102.074	0.97931	0.20238
1.4	1.0586	0.70698	27.373	103.991	0.97285	0.23144
1.5	1.0767	0.75985	31.188	106.045	0.96528	0.26123
1.6	1.0984	0.81358	35.172	108.232	0.95661	0.29137
1.7	1.1242	0.86837	39.299	110.551	0.94694	0.32141
1.8	1.1544	0.92441	43.545	112.999	0.93639	0.35096
1.9	1.1892	0.98192	47.883	115.572	0.92513	0.37964
2.0	1.2290	1.0412	52.290	118.266	0.91338	0.40712
2.1	1.2741	1.1024	56.743	121.077	0.90133	0.43313
2.2	1.3246	1.1659	61.221	124.001	0.88926	0.45741
2.3	1.3808	1.2321	65.708	127.030	0.87733	0.47989
2.4	1.4429	1.3012	70.188	130.161	0.86579	0.50040
2.5	1.5111	1.3736	74.651	133.387	0.85478	0.51898
2.6	1.5855	1.4498	79.090	136.701	0.84443	0.53566
2.7	1.6665	1.5300	83.498	140.098	0.83485	0.55048
2.8	1.7541	1.6148	87.873	143.570	0.82607	0.56357
2.9	1.8486	1.7046	92.213	147.110	0.81812	0.57505
3.0	1.9502	1.7999	96.518	150.713	0.81101	0.58503
4.0	3.4391	3.1729	138.191	188.905	0.77399	0.63320
5.0	6.2312	5.8091	178.933	228.551	0.76175	0.64788
6.0	11.501	10.850	219.625	268.452	0.75273	0.65833
7.0	21.548	20.500	260.294	308.510	0.74566	0.66633
8.0	40.818	39.070	300.920	348.693	0.74049	0.67207
9.0	77.957	74.974	341.518	28.957	0.73656	0.67638
10.0	149.85	144.67	22.098	69.275	0.73345	0.67973
11.0	289.54	280.42	62.665	109.631	0.73095	0.68243
12.0	561.84	545.59	103.222	150.015	0.72889	0.68464
13.0	1094.2	1064.9	143.773	190.420	0.72713	0.68649
14.0	2137.3	2084.2	184.318	230.842	0.72566	0.68805
15.0	4186.1	4088.8	224.859	271.276	0.72437	0.68941
16.0	8217.2	8038.0	265.397	311.721	0.72326	0.69058

Table A-5. (Cont.) Values of the Bessel Functions in Polar Form.

ξ	$M_0(\xi)$	$M_1(\xi)$	$\theta_0(\xi)$ degrees	$\theta_1 - \frac{\pi}{4}$ degrees	$\sin(\theta_1-\theta_0-\frac{\pi}{4})$	$\cos(\theta_1-\theta_0-\frac{\pi}{4})$
17.0	16163.	15831.	305.931	352.174	0.72228	0.69160
18.0	31847	31229	346.463	32.635	0.72142	0.69249
19.0	62851	61693	26.994	73.101	0.72063	0.69331
20.0	124210	122040	67.522	113.572	0.71995	0.69403
21.0	245790	241690	108.049	154.048	0.71933	0.69467
22.0	486940	479190	148.575	194.527	0.71875	0.69526
23.0	965710	950990	189.100	235.009	0.71824	0.69580
24.0	1917000	1889000	229.624	275.494	0.71776	0.69629
25.0	3808800	3755400	270.147	315.981	0.71733	0.69674

Table A-6. Values of the Bessel Functions in Polar Form.

ξ	$N_0(\xi)$	$N_1(\xi)$	$\phi_0(\xi)$ degrees	$\phi_1(\xi)$ degrees	$\sin(\phi_1-\phi_0-\frac{\pi}{4})$	$\cos(\phi_1-\phi_0-\frac{\pi}{4})$
0	∞	∞	- 0.000	225.000	0	-1.0000
0.1	2.5421	9.9620	- 17.794	224.160	-0.29161	-0.95654
0.2	1.8917	4.9280	- 23.626	222.407	-0.35890	-0.93337
0.3	1.5250	3.2315	- 28.733	220.131	-0.40457	-0.91451
0.4	1.2746	2.3723	- 33.517	217.503	-0.43868	-0.89864
0.5	1.08794	1.8501	- 38.119	214.624	-0.46550	-0.88504
0.6	0.94168	1.4976	- 42.604	211.557	-0.48726	-0.87326
0.7	0.82326	1.24310	- 47.008	208.344	-0.50531	-0.86293
0.8	0.72517	1.05059	- 51.353	205.016	-0.52054	-0.85383
0.9	0.64254	0.89990	- 55.654	201.597	-0.53362	-0.84572
1.0	0.57203	0.77887	- 59.920	198.101	-0.54494	-0.83847
1.2	0.45843	0.59711	- 68.375	190.934	-0.56366	-0.82601
1.4	0.37155	0.46810	- 76.755	183.587	-0.57845	-0.81571
1.6	0.30368	0.37281	- 85.083	176.108	-0.59048	-0.80705
1.8	0.24985	0.30043	- 93.372	168.529	-0.60043	-0.79967
2.0	0.20664	0.24429	-101.632	160.872	-0.60882	-0.79331
2.2	0.17165	0.20007	-109.869	153.154	-0.61598	-0.78776
2.4	0.14309	0.16481	-118.088	145.386	-0.62216	-0.78289
2.6	0.11966	0.13641	-126.293	137.578	-0.62757	-0.77856
2.8	0.100319	0.113353	-134.485	129.736	-0.63231	-0.77471
3.0	0.084299	0.094515	-142.668	121.866	-0.63654	-0.77124
3.5	0.055018	0.060724	-163.090	102.095	-0.64526	-0.76396
4.0	0.036246	0.039530	-183.477	82.220	-0.65206	-0.75816
4.5	0.024052	0.025986	-203.839	62.272	-0.65752	-0.75343
5.0	0.016052	0.017213	-224.182	42.270	-0.66200	-0.74951

Appendix B

Relationship between ξ_1 and the Bulge Ratio $\frac{h}{t}$.

For a given span, measured by x_1 or ξ_1, we may measure the amount of curvature by the use of the Bulge Ratio. See Figure B-1.

$$B.R. = \frac{h}{t} \qquad (B-1)$$

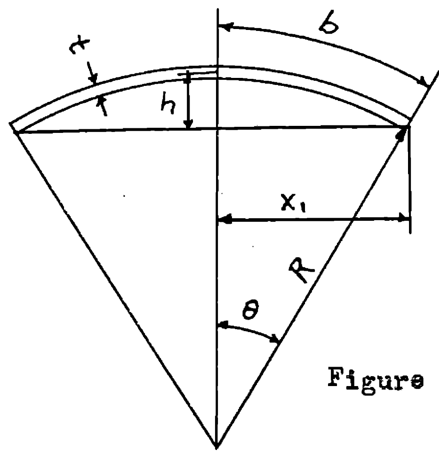


Figure B-1. Notation.

For shallow domes

$$h = R(1-\cos\theta) \approx \frac{R\theta^2}{2} \qquad (B-2)$$

$$\theta = \frac{b}{R} \approx \frac{x_1}{R} \qquad (B-3)$$

Using these relationships in equation (B-1) we have

$$B.R. = \frac{x_1^2}{2Rt} \qquad (B-4)$$

Now

$$\xi_1 = \frac{x_1}{\ell} = \frac{\sqrt[4]{12(1-\nu^2)}\; x_1}{\sqrt{Rt}} \qquad (B-5)$$

or

$$\xi_1^{\,2} = \sqrt{12(1-\nu^2)}\ \frac{x_1^{\,2}}{Rt} \qquad \text{(B-6)}$$

From equations (B-4) and (B-6)

$$\xi_1^{\,2} = 2\sqrt{12(1-\nu^2)}\ (B.R.) \qquad \text{(B-7)}$$

or

$$\xi_1 = k\ (B.R.)^{\frac{1}{2}} \qquad \text{(B-8)}$$

where k is constant and is equal to $2\sqrt[4]{3(1-\nu^2)}$.

Thus for a given span x_1, ξ_1 is a measure of the bulge ratio as given by (B-8). An increase in ξ_1 indicates an increase in the bulge ratio or a decrease in the radius of curvature. A value of $\xi_1 = 0$ may indicate a flat plate (bulge ratio equal to zero) or that $x_1 = 0$.

Appendix C

Demonstration that $P_{1R} = P_{1T}$, Equation (304), is Valid.

In the discussion on edge effects, unsymmetrical edge loading it was stated that for n = 1, the tangential edge loading $P_{1T} \sin \phi$ and the horizontal edge loading $P_{1R} \cos \phi$ must satisfy the requirement of statical equilibrium and that this requires that

$$P_{1R} = P_{1T} \tag{304}$$

The validity of this assertion will be shown in this section.

Let

$$P_{ER} = P_{nR} \cos n\phi \tag{C-1}$$

$$P_{ET} = P_{nT} \sin n\phi \tag{C-2}$$

be the prescribed edge loadings where P_{nR} is the amplitude of the edge horizontal radial loading and P_{nT} is the amplitude of the edge tangential loading positive P_{nR} acts radially outward and positive P_{nT} acts tangentially in the direction of increasing ϕ.

Then the requirement for statical horizontal equilibrium is expressed by summing the components parallel to the axis $\phi = 0°$. Thus, see Figure C-1

$$0 = \int_0^{2\pi} P_{nR} \cos n\phi \cos\phi \, d\phi - \int_0^{2\pi} P_{nT} \sin n\phi \sin\phi \, d\phi \tag{C-3}$$

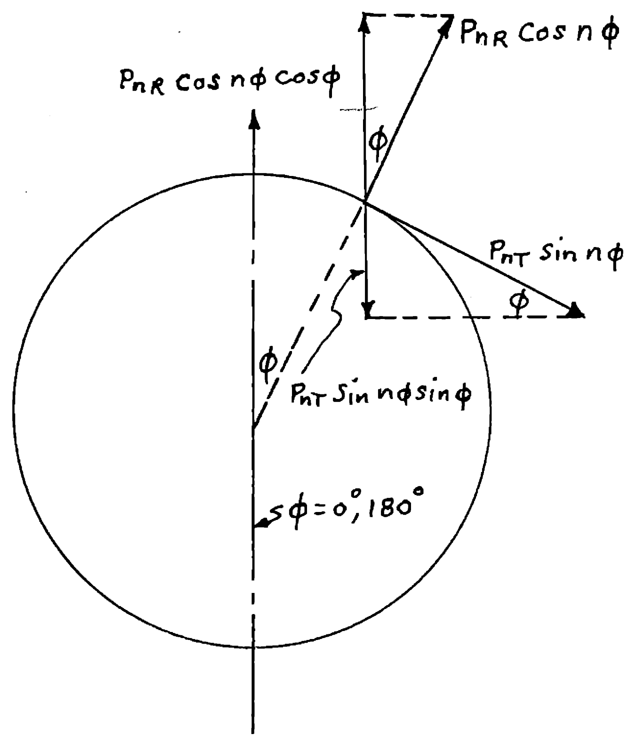


Figure C-1. Prescribed Edge Loading Components in Direction $\phi = 0°$.

Now

$$\int_0^{2\pi} \cos n\phi \cos\phi \, d\phi = \int_0^{2\pi} \sin n\phi \sin\phi \, d\phi \qquad (C-4)$$

and

$$\int_0^{2\pi} \cos n\phi \cos\phi \, d\phi = \begin{cases} 0 & n = 2, 3, 4, 5, \cdots \infty \\ \pi & n = 1 \end{cases} \qquad (C-5)$$

Thus we see that for n = 2, 3, 4,··· ∞ each of the edge loadings as given by each term in equation (C-3) satisfies the equilibrium condition. For n = 1, equations (C-3) and (C-5) require that

$$P_{1R} = P_{1T} \qquad (304)$$

Our assertion has been validated.

To demonstrate the corollary that the edge stresses satisfy the conditions of statical equilibrium we note from equations (86), (10a), (10c) and (77)

$$P_h = N_{xx} = \frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \phi^2} \qquad (C-6)$$

$$P_t = N_{x\phi} = \frac{1}{x^2}\frac{\partial F}{\partial \phi} - \frac{1}{x}\frac{\partial^2 F}{\partial \phi \partial x} \qquad (C-7)$$

Representing F by equation (21b)

$$F_n = F_n(x) \cos n\phi \qquad (21b)$$

equations (C-6) and (C-7) become

$$P_h = \frac{1}{x}\frac{\partial F_n(x)}{\partial x}\cos n\phi - \frac{n^2}{x^2} F_n(x)\cos n\phi \qquad (C-8)$$

$$P_t = -\frac{n}{x^2} F_n(x) \sin n\phi + \frac{n}{x}\frac{\partial F_n(x)}{\partial x}\sin n\phi \qquad (C-9)$$

The equation of statical horizontal equilibrium is similar to that given by equation (C-3) thus

$$0 = \int \frac{1}{x}\frac{\partial F_n(x)}{\partial x}\cos n\phi \cos\phi \, d\phi - \int \frac{n^2}{x^2} F_n(x) \cos n\phi \cos\phi \, d\phi$$
$$- \int -\frac{n}{x^2} F_n(x) \sin n\phi \sin\phi \, d\phi - \int \frac{n}{x}\frac{\partial F_n(x)}{\partial x} \sin n\phi \sin\phi \, d\phi \qquad (C-10)$$

Using equation (C-5) we have as before that for $n = 2, 3, 4, \cdots \infty$ each type of stress satisfies the equilibrium condition by itself.

For $n = 1$ using equation (C-5) we see equation (C-10) satisfied by any function $F_n(x)$.

In the above discussion it is assumed that no resultant horizontal loading of the shell area exists. Obviously, if a resultant horizontal loading exists, the statical equilibrium condition as given by equation (304) must be modified to take this into account. We need only satisfy the condition that the sum of the horizontal forces must equal zero for equilibrium.

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BIOGRAPHY

EDUCATION

1. Massachusetts Institute of Technology, 1932-1936. S. B. in Civil Engineering.

2. Massachusetts Institute of Technology, 1937-1938. S. M. in Civil Engineering.

3. Graduate Student in Civil Engineering, Harvard Graduate School of Engineering, November, 1943 to June, 1944.

4. Civil Engineering, Massachusetts Institute of Technology, July 1944 to June 1946.

PROFESSIONAL EXPERIENCE

1. General practice, 1936-1937.

2. Fay, Spofford and Thorndike, Boston, Structural Designer, 1938.

3. U. S. Navy Department, Bureau of Construction and Repair, Hull Section, Boston, Jr. Naval Architect, 1938-1940.

4. Fay, Spofford and Thorndike, Boston, Structural Engineer, 1940-1941.

5. U. S. Army, Captain, Corps of Engineers, Construction Division and later with 306th Engr. Bn. (Combat), 81st Inf. Div., 1941-1943.

6. Consultant, U. S. War Department, Washington, D. C., 1943.

7. Instructor (part-time), ASTP, Civil Engineering, Harvard Graduate School of Engineering, Sept. 1943 to June 1944.

8. Instructor (part-time), Department of Mathematics, Massachusetts Institute of Technology, July 1944 to November 1944.

9. Cram and Ferguson, Boston, Structural Engineer, January 1946 to date.

ABSTRACT

Mathematical Analysis of Stress and Displacement in Thin Shallow Spherical Shells

I. Purpose.

The objectives of this study are

a. to obtain a solution for the analysis of the stresses and displacements in thin shallow spherical shells of constant thickness under symmetrical or unsymmetrical loading,

b. to demonstrate the application of this solution to the analysis of shells subjected to symmetrical or unsymmetrical vertical loading under various conditions of edge restraint, and

c. to reduce the labor involved in design of thin shallow shells subjected to symmetrical uniform vertical loading.

II. Historical Backround.

Numerous investigators have studied the problem of the spherical thin shell. E. Reissner (1), in 1912, obtained the basic results for the rotational symmetric case which constituted the starting point for later studies. The non-rotationally symmetrical case has been studied by a number of investigators, principally by E. Schwerin (2), A. Havers (3) and A. A. Jakobsen (4).

E. Reissner (5,6), in 1943, utilizing the basis assumption of the shallowness of the shell, reduced the general equations of the shallow spherical shell to two differential equations of fourth order in terms of a stress function F and the vertical deflection w and carried out the solution for the rota-

tional symmetrical case.

This thesis is based upon the fundamental research of E. Reissner.

III. Results.

The original work of this thesis properly commences with the solution for unsymmetrical loading of E. Reissner's basic differential equations for the shallow dome. However, for the sake of completeness, his development of the differential equations (5,6) is given nearly verbatim. His development has been modified therein to include the effect of temperature. A solution of the differential equations suitable for symmetrical or unsymmetrical loading is presented. The functions F and w constituting the solution except for the particular integral are combinations of elementary functions and of Bessel functions with argument proportional to $\sqrt{i}$. The expressions for the stress resultants, reactions and displacements for the symmetrical and unsymmetrical are then given.

The applications of these theoretical results are given in six cases. Before summarizing the contents of these cases, the parameter, ξ_1, used throughout this thesis is defined as

$$\xi_1 = \frac{\sqrt[4]{12(1-v^2)}\, x_1}{\sqrt{tR}}$$

where

x_1 = the half-span of the shell

t = the thickness of the shell

R = the radius of the shell

v = Poisson's ratio (taken equal to 0.2 for the numerical work of this thesis)

<u>Case I.</u> In this case the edges are fully restrained and an uniform vertical load of intensity p is acting downward. The range of shells considered is given by $0 \leq \xi_1 \leq 25.0$. Direct, bending, combined (direct and bending) and maximum stress intensities have been computed, tabulated and plotted in dimensionless form. Shearing stress intensities were computed and tabulated also. For the same range of shells the vertical deflections at all points of the shell have been plotted and the variation of the vertical deflection as the center has been investigated with respect to ξ_1, the Bulge Ratio and R. A numerical example illustrating the use of the tables and graphs for this case is given.

<u>Case II.</u> No surface load is acting and the edges are considered to be loaded by an uniform edge moment M_E and an uniform horizontal edge force P_E. The range of shells considered is given by $0 \leq \xi_1 \leq 25.0$. Influence coefficients have been computed, tabulated and plotted for this range of shells. These coefficients when multiplied by the edge moments and horizontal forces acting will yield the values of the horizontal edge displacement and edge rotation due to that edge loading. Conversely, for a given edge horizontal displacement and rotation, the edge moments and thrusts may be determined. A numerical example illustrating the use of these influence coefficients is given.

<u>Case III.</u> This case is analogous to that of Case II. The influence coefficients were computed, tabulated and plotted for the determination of the horizontal displacement and rotation of the edge of a lantern opening or center hole of a shell loaded by an uniform edge moment and edge horizontal force. The opening is measured by the parameter μ defined as

$$\mu = \frac{\sqrt{12(1-\nu^2)}\; x_o}{\sqrt{tR}}$$

where x_o is the radius of opening or lantern circle and t, R and v are as given above for the definition of ξ_1. The range of openings is given by $0 \leq \mu \leq 5.0$. Due to the local effect of such an inner edge loading, the outer edge is considered to be an infinite distance away from the center of the shell.

Case IV. This case illustrates an unsymmetrical hydrostatic loading condition, edges fully restrained, for a shell measured by $\xi_1 = 10.0$ The stresses and the deflections are given in tables in dimensionless form.

Case V. This is a numerical example to illustrate the techniques involved in the solution of a symmetrically loaded shell with a lantern opening, reinforced at the inner and outer edges by rings. An uniform vertical loading and a vertical lantern ring loading are individually considered. The results are given in tabular form.

Case VI. This is analogous to Cases II and III. In this case the procedure is outlined for the determination of influence coefficients for the calculation of the horizontal and circumferential displacements and rotation of the outer edge due to unsymmetrical horizontal and tangential (circumferential) forces and unsymmetrical moments acting on the outer edge. No values are computed.

IV. Conclusions.

1. The rotationally symmetric case of the thin shallow spherical shell is capable of practical solution without serious mathematical difficulties. The shell may be analyzed directly under load with certain boundary conditions or may be considered as a combination of a shell in a membrane state

upon which are superimposed the edge forces required to satisfy the initial conditions of restraint.

2. The non-rotationally symmetric case of the thin shallow spherical shell is capable of solution with the following qualifications:

a. The determination of the particular integral, which depends upon the loading, requires considerable study. The evaluation of the particular integral will, in general, present difficulty and recourse to numerical integration procedures may be required. It is felt, however, that with the advance in mechanical integration methods this difficulty can be overcome.

b. n^{th} order values of the ber, bei, ker and kei functions are required for the non-rotationally symmetrical case. The tabulated values for these functions are inadequate and hence further numerical studies of this case will require more extensive tables of these functions.

3. Though a study of the precision and accuracy of the methods of this thesis was beyond the scope of this work, it is felt that the theoretical results contained herein are applicable to shells whose ratio of height to base diameter is less than, say, 1/8. Also, the results will be applicable to the pole region of much steeper shells. It is obvious from the nature of the assumption of shallowness made that as the shell approaches the flat plate, the equations yield more accurate results. A study of the precision and accuracy of the methods used herein is obviously necessary in future research.

Frank R. [illegible]

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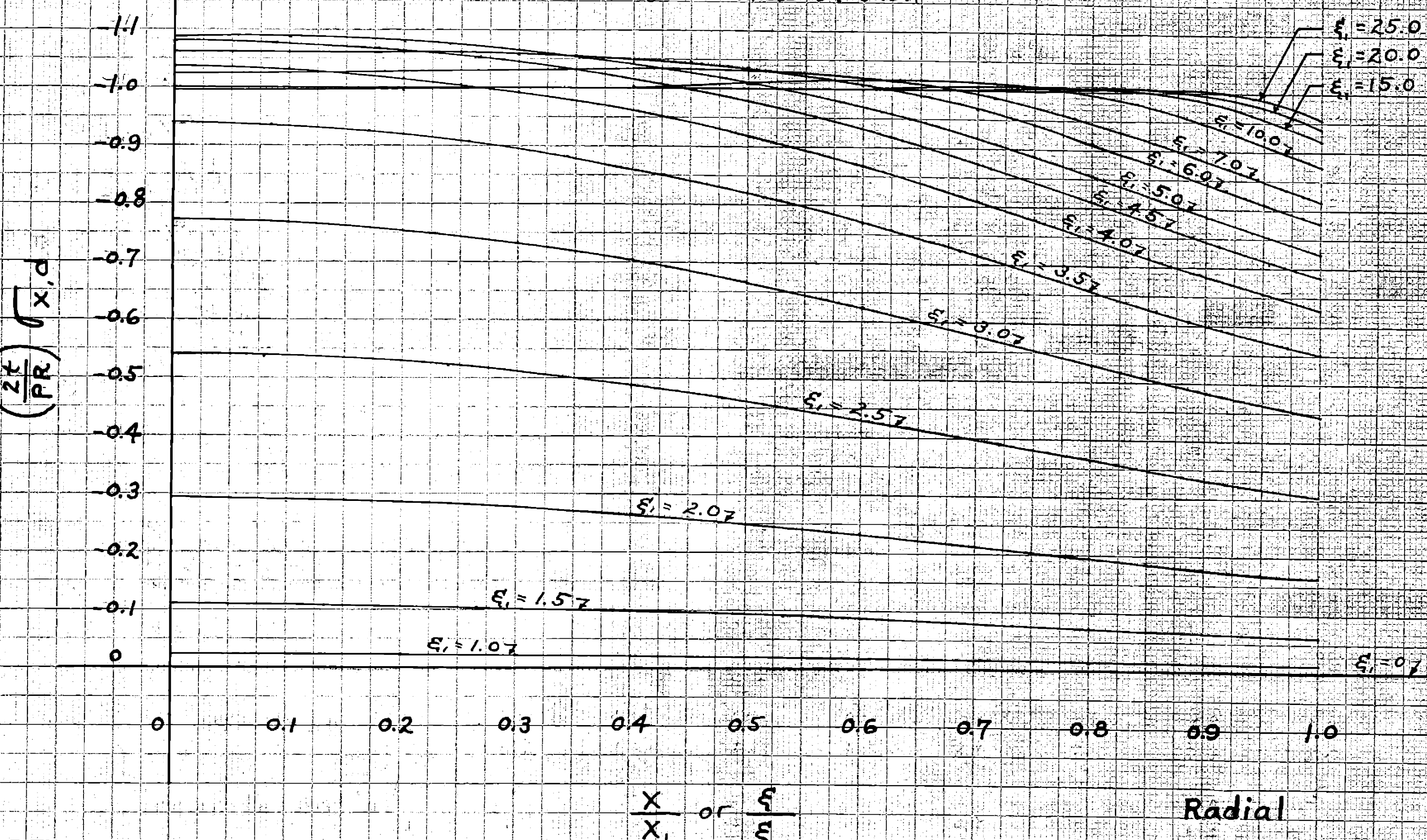
GRAPH A

P = intensity of vertical loading per unit of projected area of shell.
Positive P is downward
R = radius of shell
t = thickness of shell

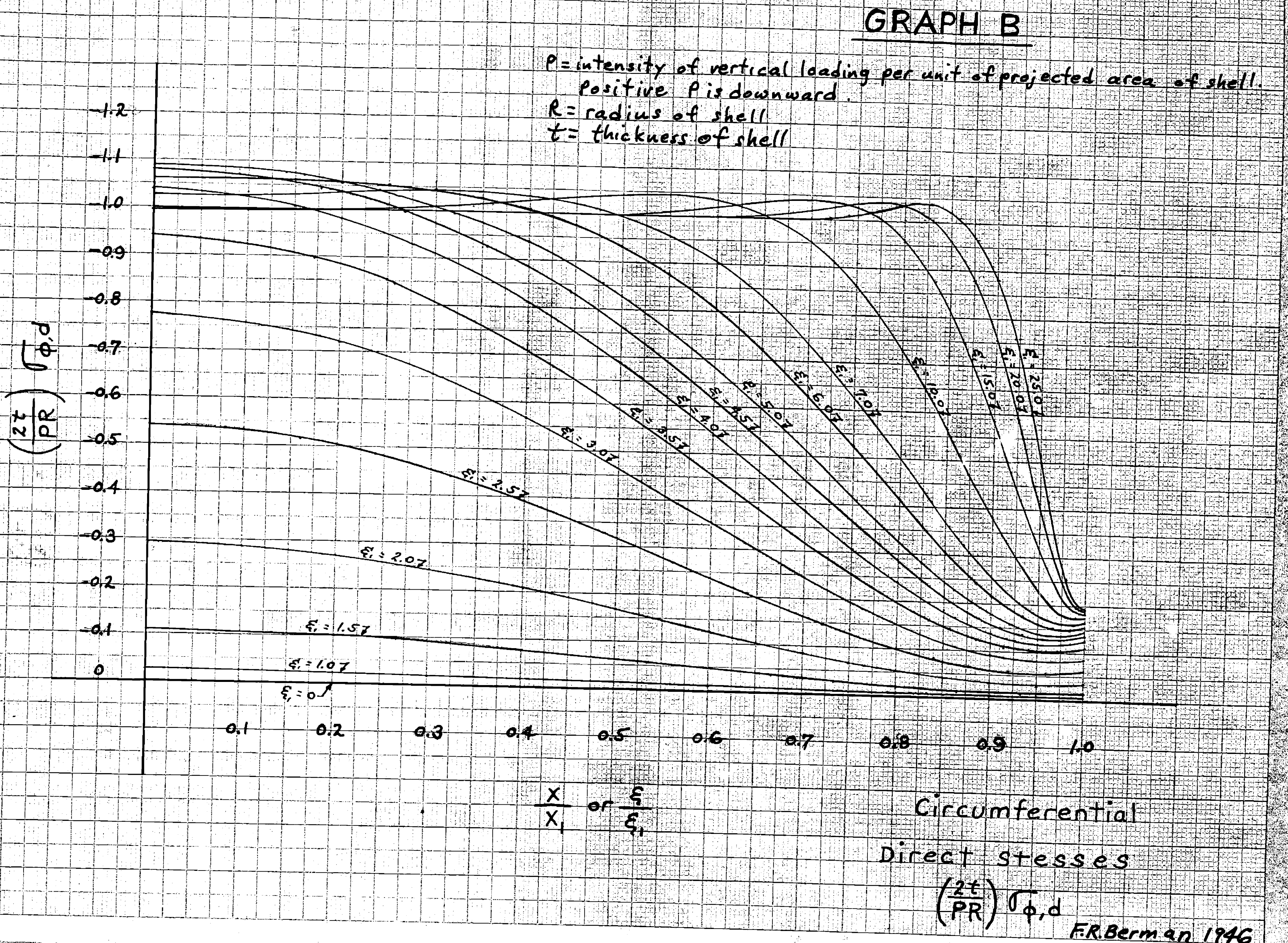


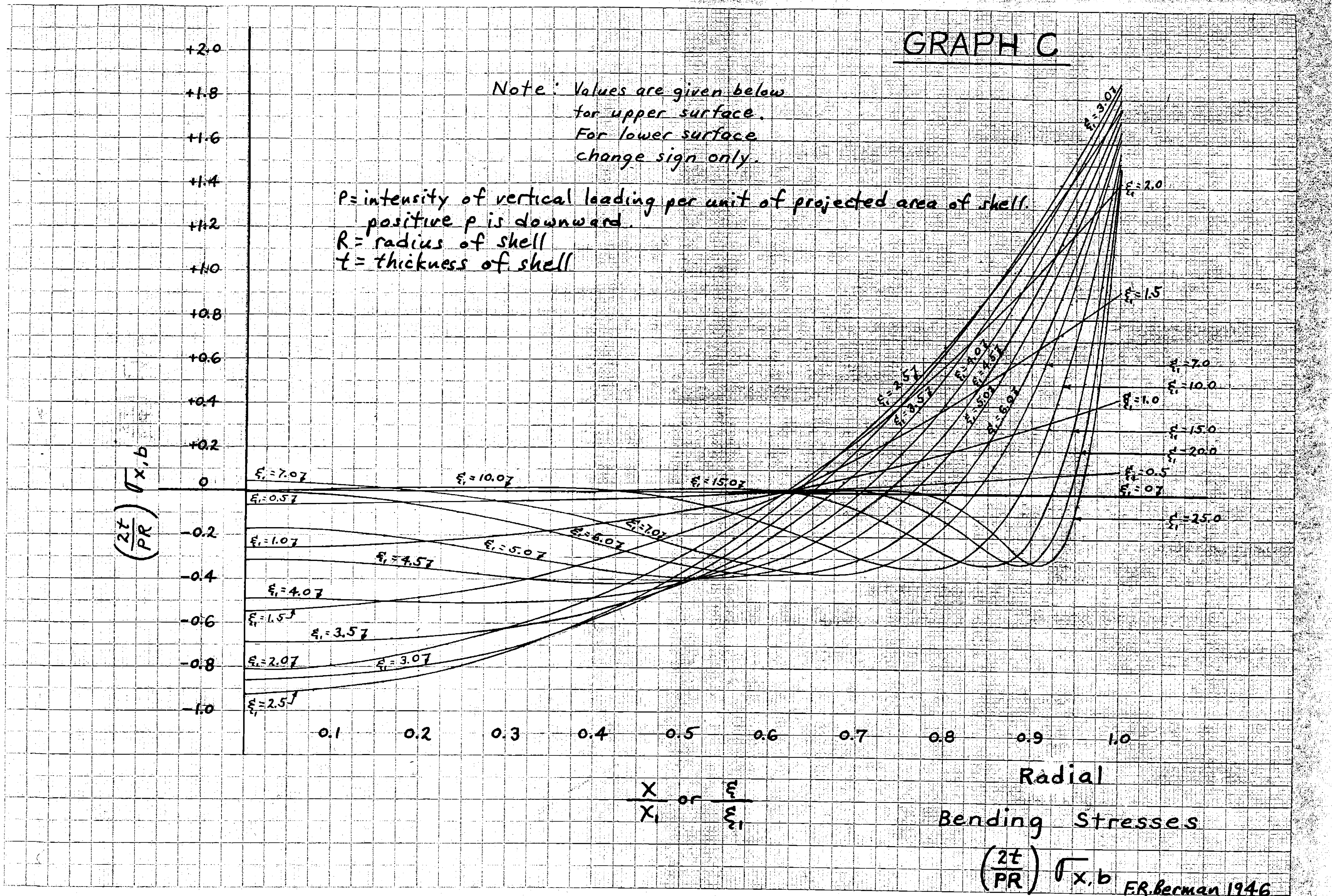
Radial

Direct Stresses

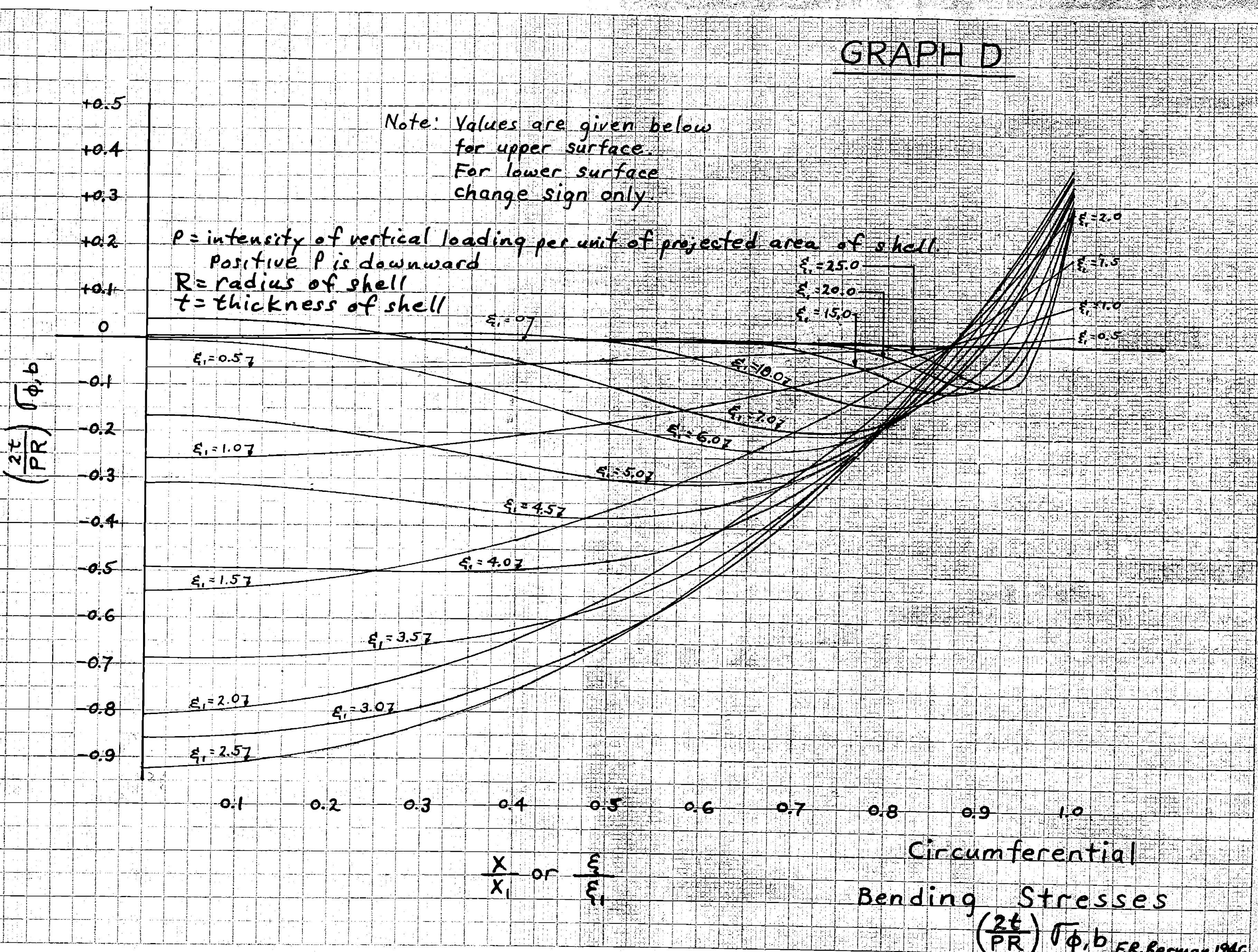
$\left(\frac{2t}{PR}\right)\sigma_{x,d}$

F.R. Berman 1946

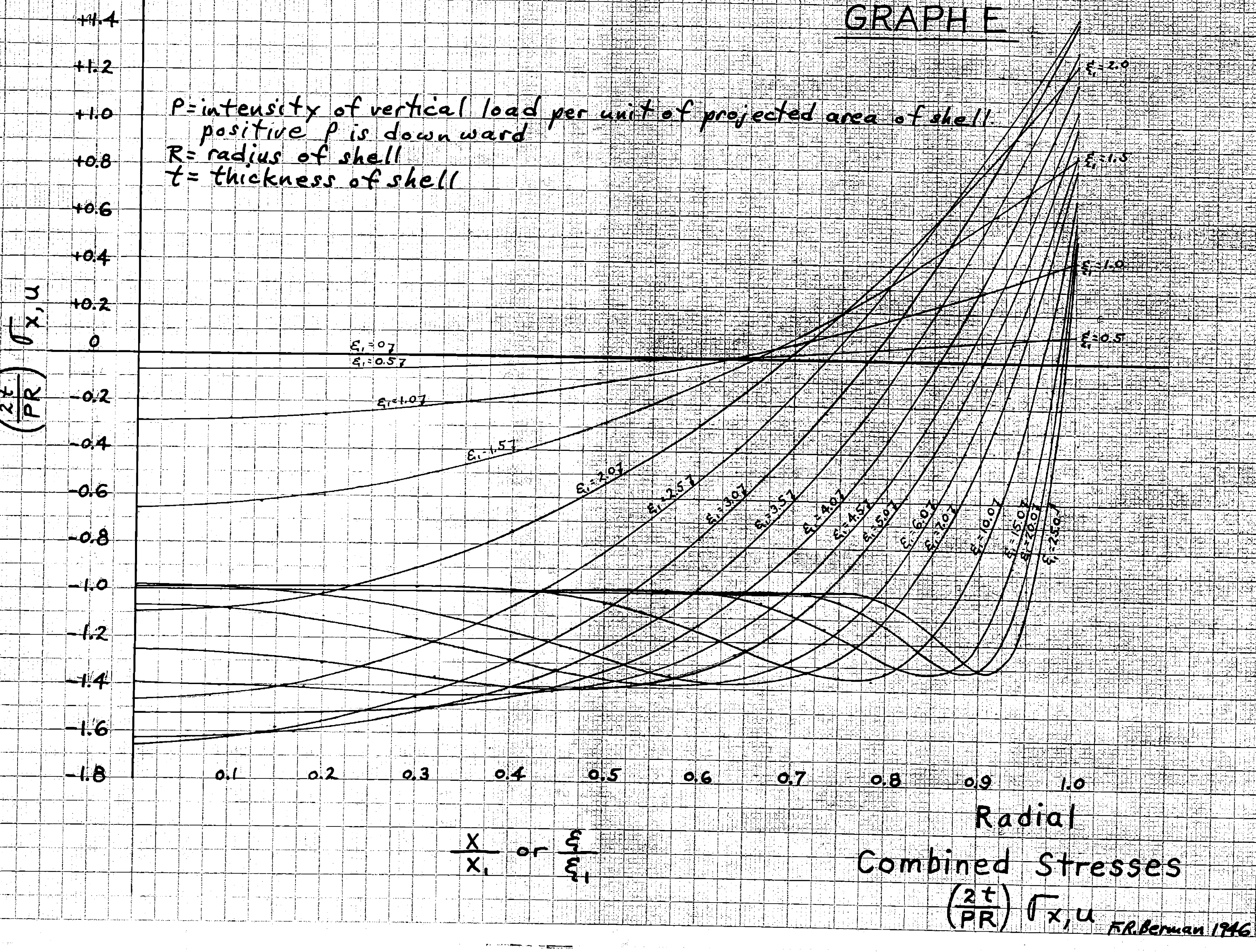

GRAPH B
P = intensity of vertical loading per unit of projected area of shell.
Positive P is downward.
R = radius of shell
t = thickness of shell
-1.2
-1.1
-1.0
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
(2t/PR) σφ,d
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X/X1 or ξ/ξ1
ξ1 = 25.0
ξ1 = 20.0
ξ1 = 15.0
ξ1 = 10.0
ξ1 = 7.0
ξ1 = 6.0
ξ1 = 5.0
ξ1 = 4.5
ξ1 = 4.0
ξ1 = 3.5
ξ1 = 3.0
ξ1 = 2.5
ξ1 = 2.0
ξ1 = 1.5
ξ1 = 1.0
ξ1 = 0
Circumferential
Direct stesses
(2t/PR) σφ,d
F.R.Berman 1946

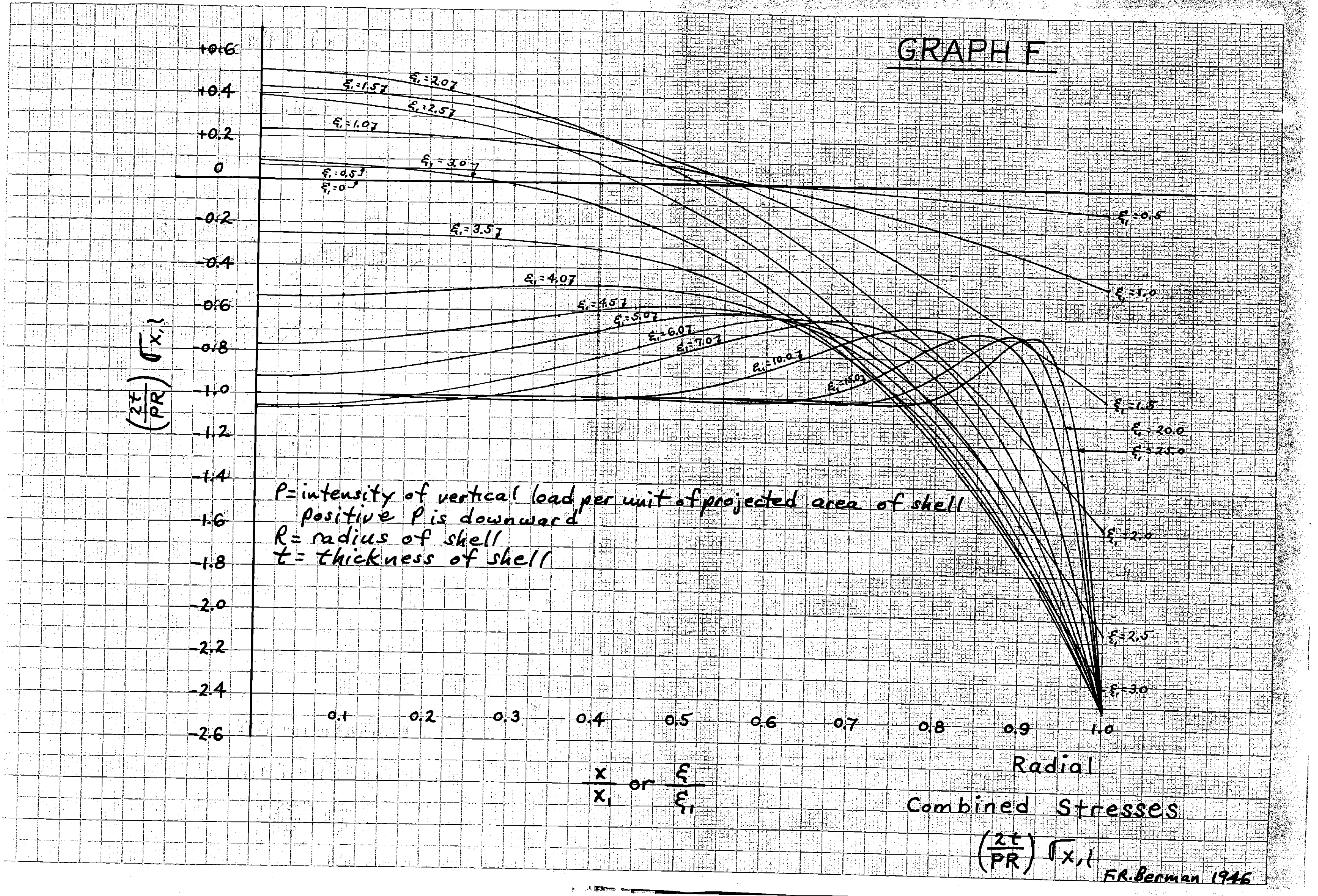


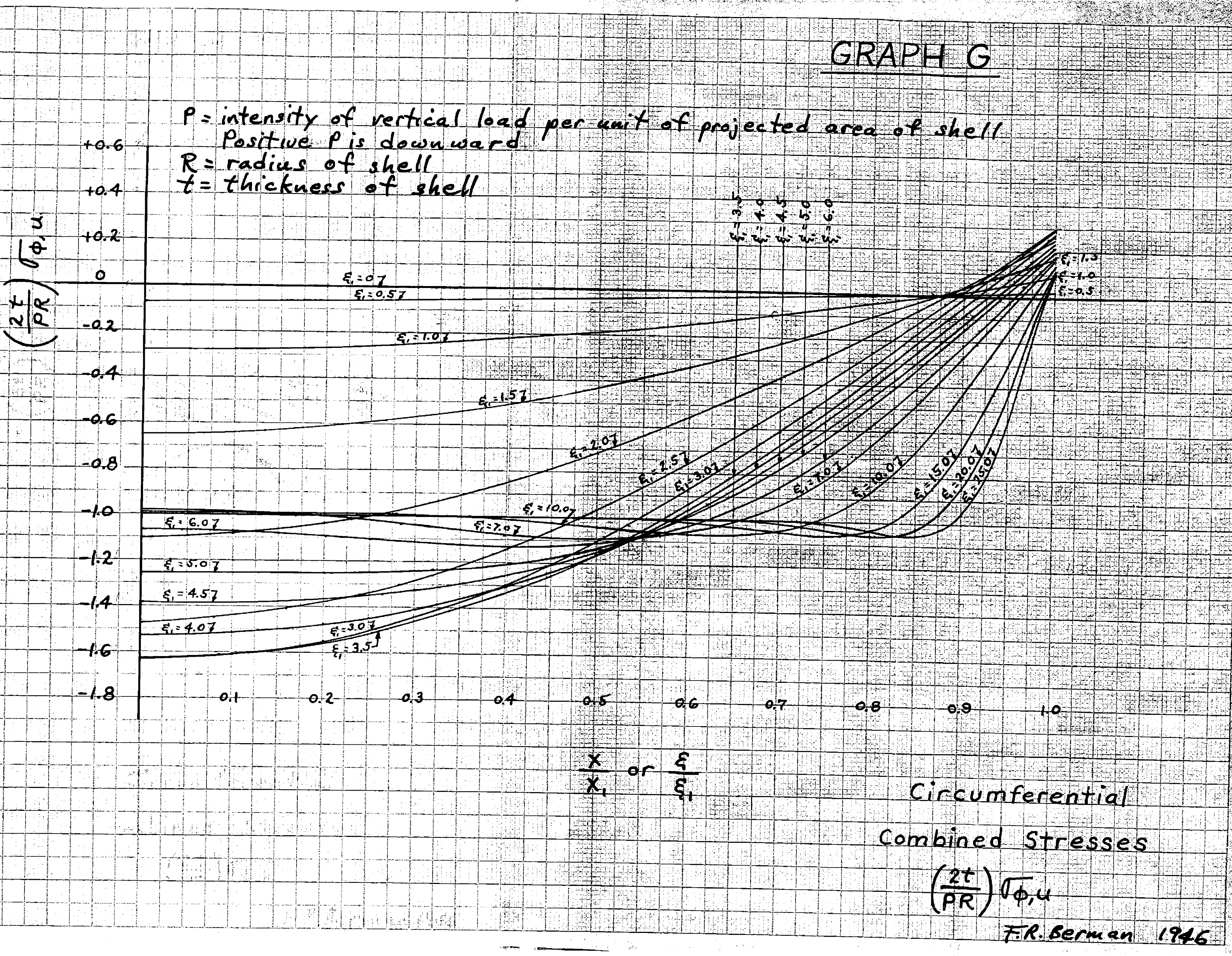
GRAPH C
Note: Values are given below for upper surface. For lower surface change sign only.
p = intensity of vertical loading per unit of projected area of shell. positive p is downward.
R = radius of shell
t = thickness of shell
+2.0
+1.8
+1.6
+1.4
+1.2
+1.0
+0.8
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X/X1 or ξ/ξ1
Radial Bending Stresses
(2t/PR) σx,b
F.R. Berman 1946



GRAPH D
Note: Values are given below for upper surface. For lower surface change sign only.
p = intensity of vertical loading per unit of projected area of shell. Positive p is downward
R = radius of shell
t = thickness of shell
+0.5
+0.4
+0.3
+0.2
+0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
$\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{X}{X_1}$ or $\frac{\xi}{\xi_1}$
$\xi_1 = 0.5$
$\xi_1 = 1.0$
$\xi_1 = 1.5$
$\xi_1 = 2.0$
$\xi_1 = 2.5$
$\xi_1 = 3.0$
$\xi_1 = 3.5$
$\xi_1 = 4.0$
$\xi_1 = 4.5$
$\xi_1 = 5.0$
$\xi_1 = 6.0$
$\xi_1 = 7.0$
$\xi_1 = 10.0$
$\xi_1 = 15.0$
$\xi_1 = 20.0$
$\xi_1 = 25.0$
Circumferential Bending Stresses $\left(\frac{2t}{PR}\right)\sigma_{\phi,b}$
F.R. Berman 1946


GRAPH E
P= intensity of vertical load per unit of projected area of shell.
positive P is downward
R= radius of shell
t= thickness of shell
+1.4
+1.2
+1.0
+0.8
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
(2t/PR) σx,u
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X/X1 or ξ/ξ1
ξ1 = 2.0
ξ1 = 1.5
ξ1 = 1.0
ξ1 = 0.5
ξ1 = 0.7
ξ1 = 0.57
ξ1 = 1.07
ξ1 = 1.57
ξ1 = 2.07
ξ1 = 2.57
ξ1 = 3.07
ξ1 = 3.57
ξ1 = 4.07
ξ1 = 4.57
ξ1 = 5.07
ξ1 = 6.07
ξ1 = 7.07
ξ1 = 10.07
ξ1 = 15.07
ξ1 = 20.07
ξ1 = 25.07
Radial
Combined Stresses
(2t/PR) σx,u
F.R. Berman 1946

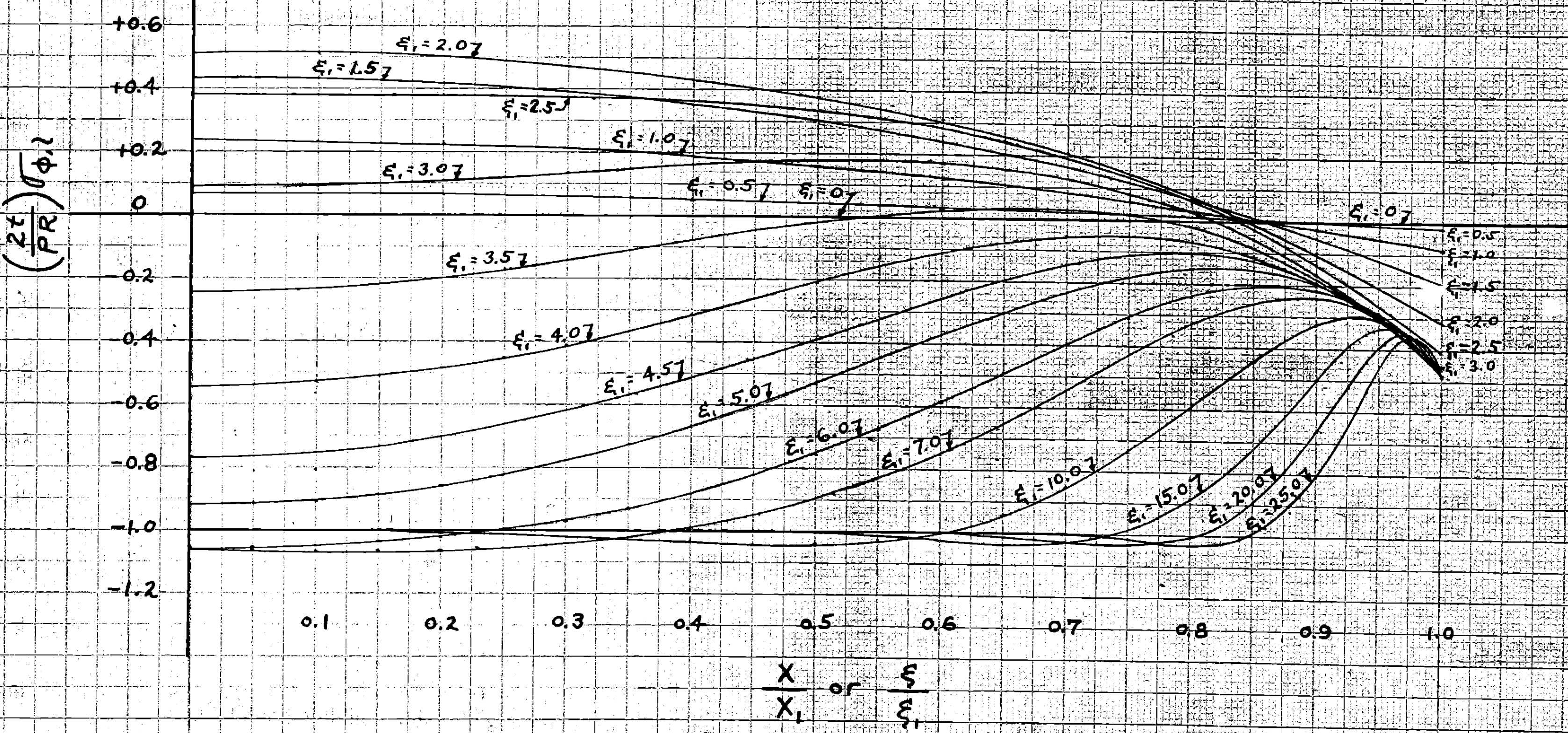

GRAPH F
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-2.4
-2.6
$\left(\frac{2t}{PR}\right)\sigma_{x,i}$
ξ₁=1.5
ξ₁=2.0
ξ₁=2.5
ξ₁=1.0
ξ₁=3.0
ξ₁=0.5
ξ₁=0
ξ₁=3.5
ξ₁=4.0
ξ₁=4.5
ξ₁=5.0
ξ₁=6.0
ξ₁=7.0
ξ₁=10.0
ξ₁=15.0
ξ₁=0.5
ξ₁=1.0
ξ₁=1.5
ξ₁=20.0
ξ₁=25.0
ξ₁=2.0
ξ₁=2.5
ξ₁=3.0
P= intensity of vertical load per unit of projected area of shell
positive P is downward
R= radius of shell
t= thickness of shell
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{x}{x_1}$ or $\frac{\xi}{\xi_1}$
Radial
Combined Stresses
$\left(\frac{2t}{PR}\right)\sigma_{x,i}$
F.R. Berman 1946



GRAPH G
P = intensity of vertical load per unit of projected area of shell
Positive P is downward
R = radius of shell
t = thickness of shell
(2t/PR) σφ,u
+0.6
+0.4
+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
ξ₁ = 0.7
ξ₁ = 0.57
ξ₁ = 1.07
ξ₁ = 1.57
ξ₁ = 2.07
ξ₁ = 2.57
ξ₁ = 3.07
ξ₁ = 3.5
ξ₁ = 4.0
ξ₁ = 4.5
ξ₁ = 5.0
ξ₁ = 6.0
ξ₁ = 7.07
ξ₁ = 10.07
ξ₁ = 15.07
ξ₁ = 20.07
ξ₁ = 25.07
ξ₁ = 1.5
ξ₁ = 1.0
ξ₁ = 0.5
ξ₁ = 6.07
ξ₁ = 5.07
ξ₁ = 4.57
ξ₁ = 4.07
x/x₁ or ξ/ξ₁
Circumferential
Combined Stresses
(2t/PR) σφ,u
F.R. Berman 1946

GRAPH H

P = intensity of vertical load per unit of projected area of shell
positive P is downward
R = radius of shell
t = thickness of shell



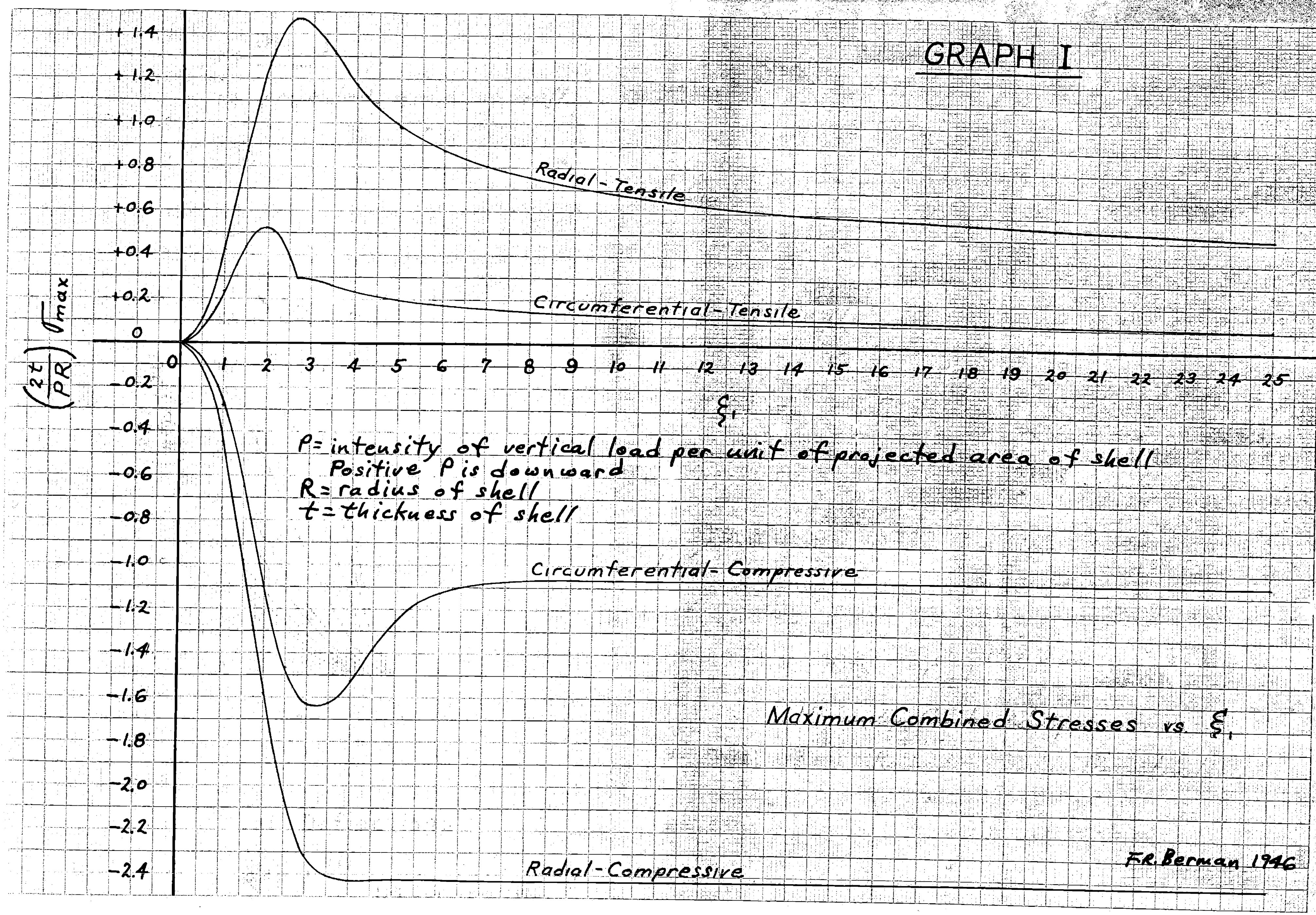
Circumferential

Combined Stresses

$\left(\frac{2t}{PR}\right)\sigma_{\phi,l}$

F.R. Berman 1946

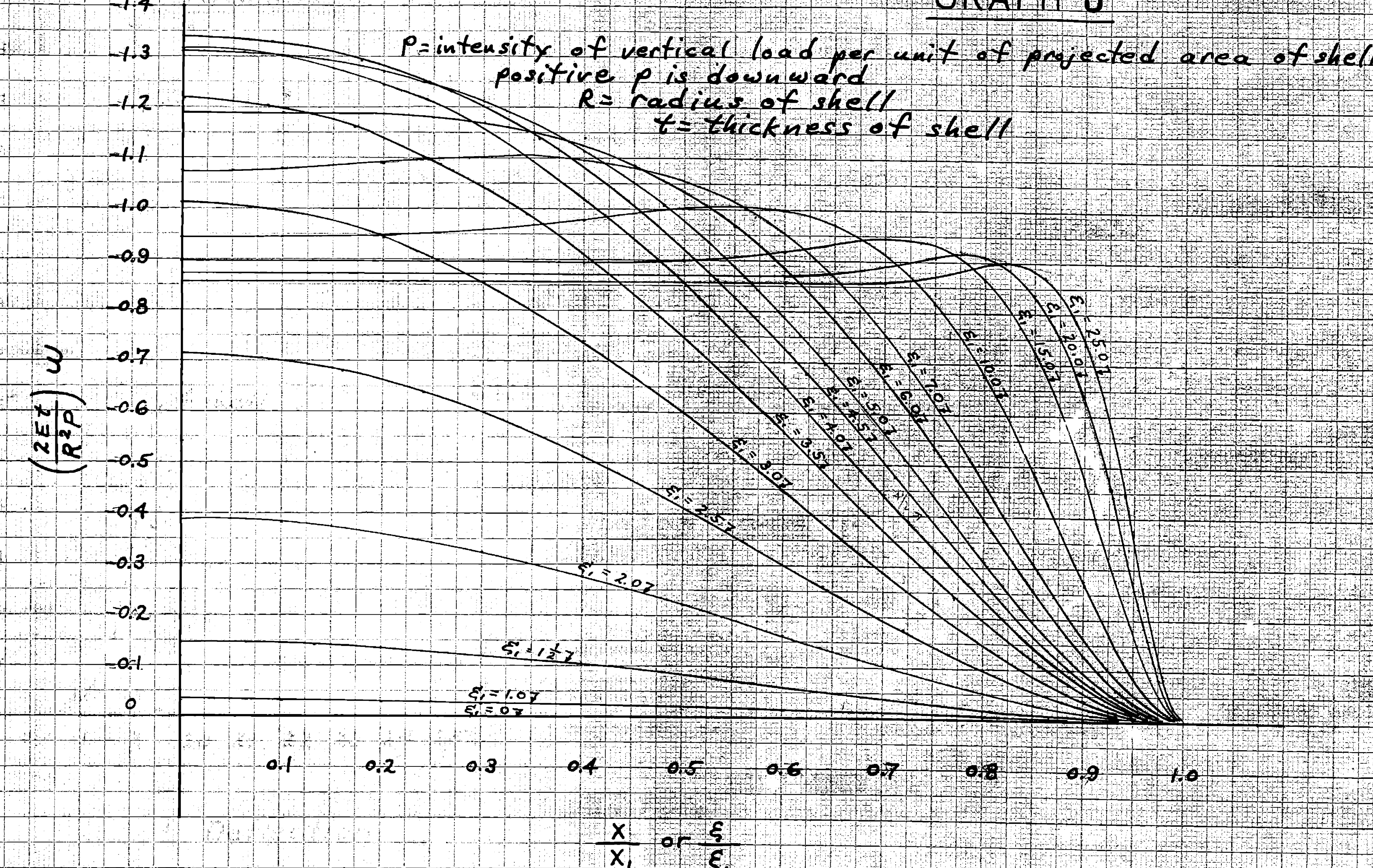
GRAPH I



Maximum Combined Stresses vs. ξ_1

F.R. Berman 1946

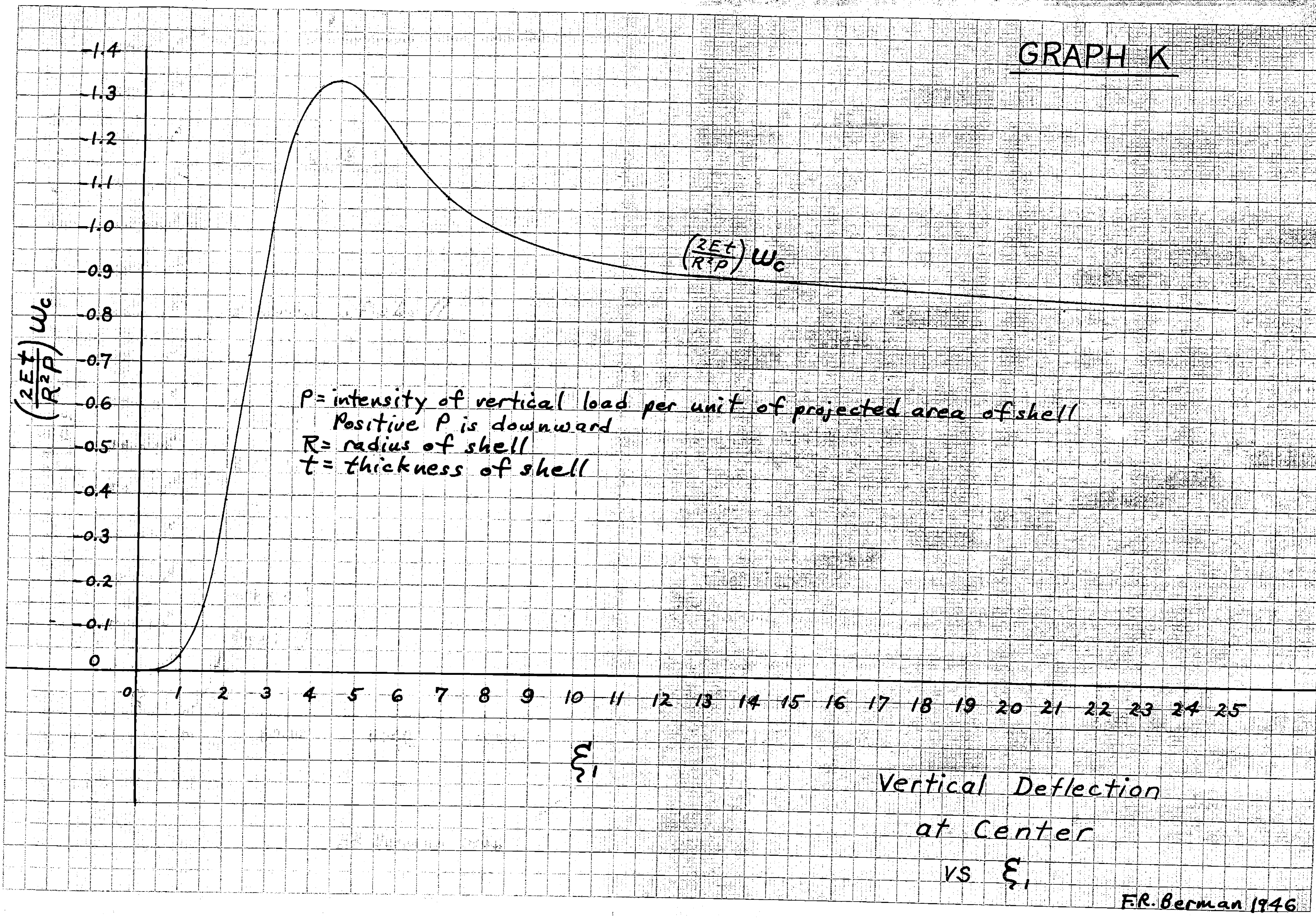
GRAPH J



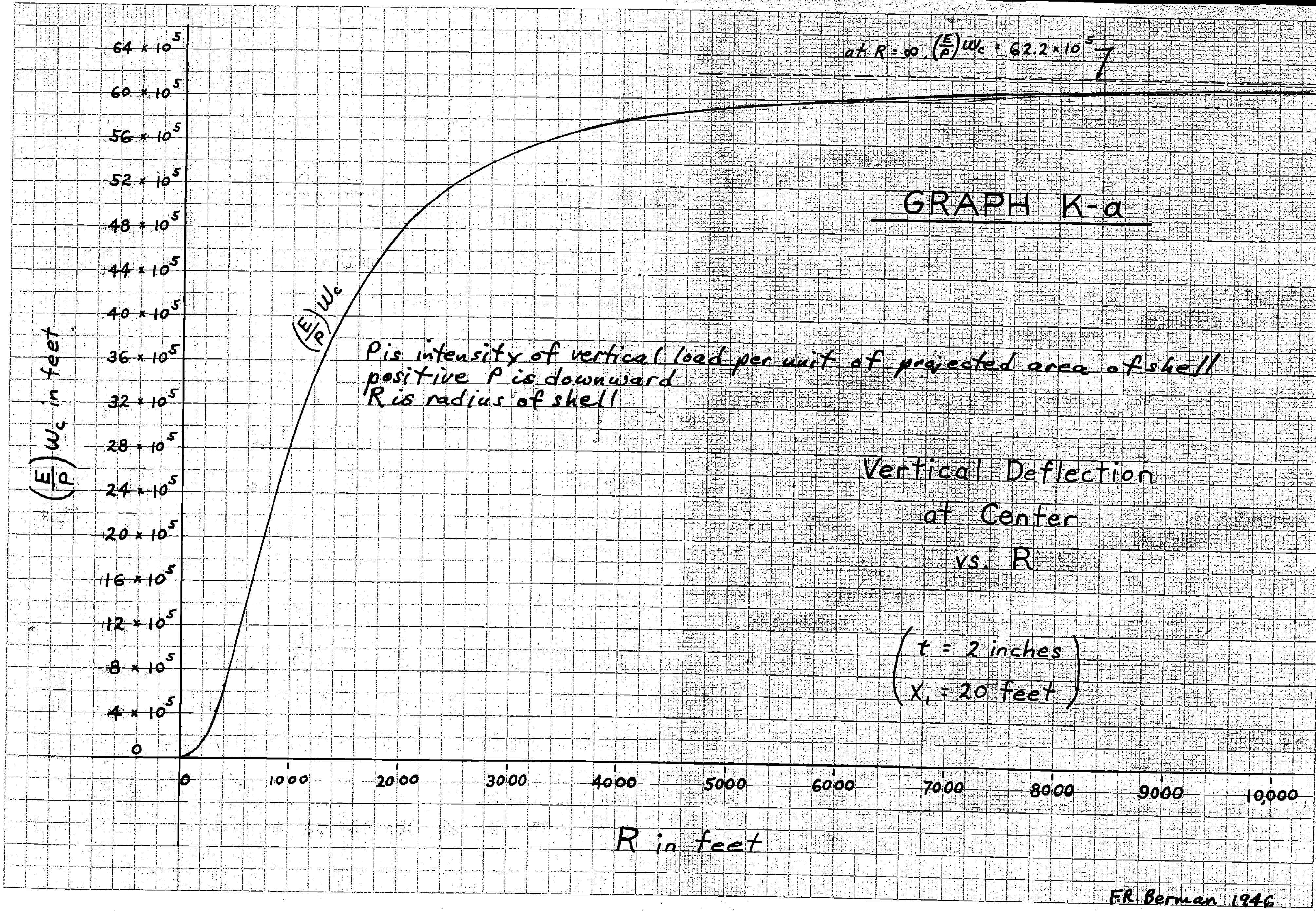
Vertical Deflections

$\left(\frac{2Et}{R^2P}\right)\omega$

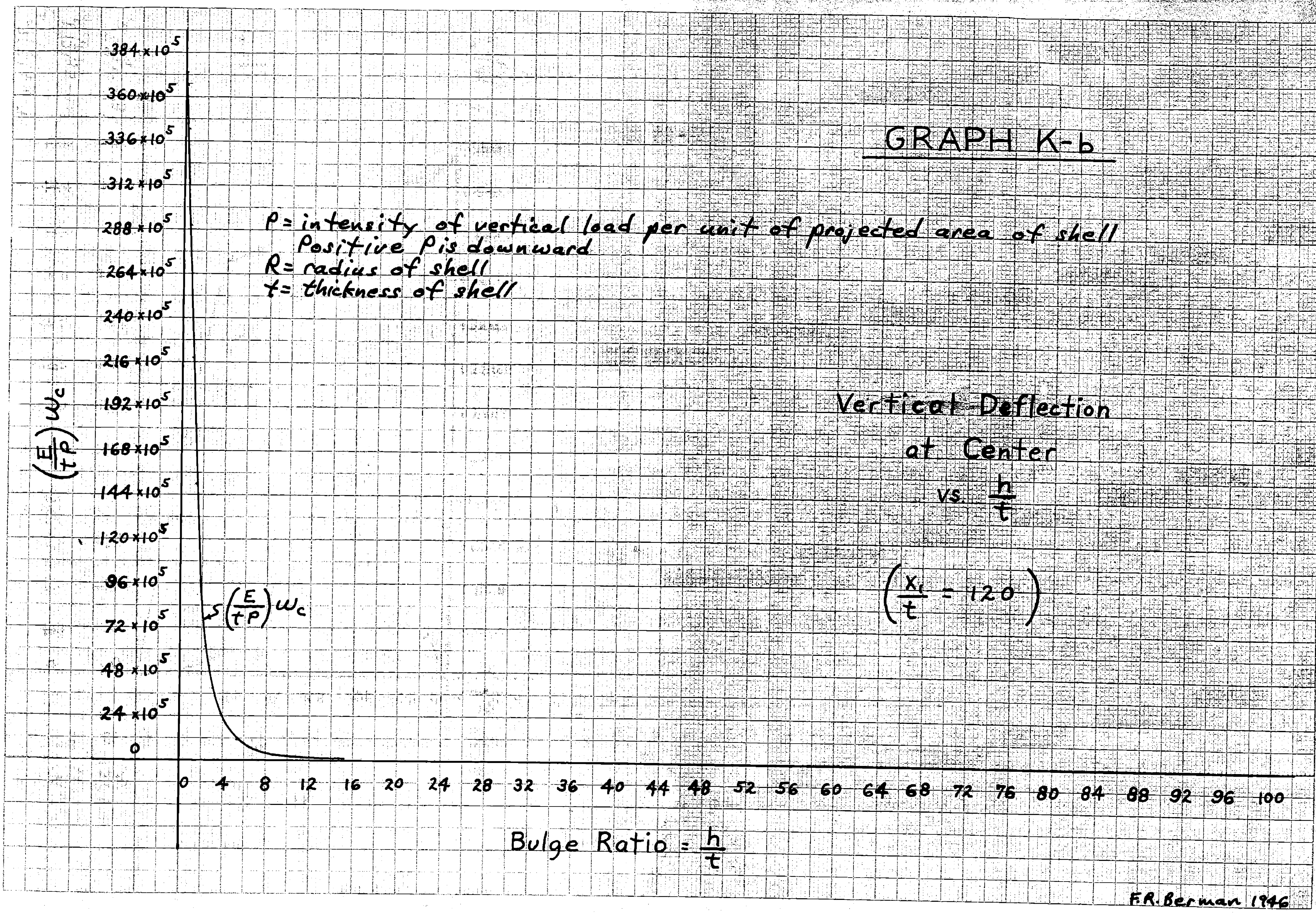
F.R. Berman 1946



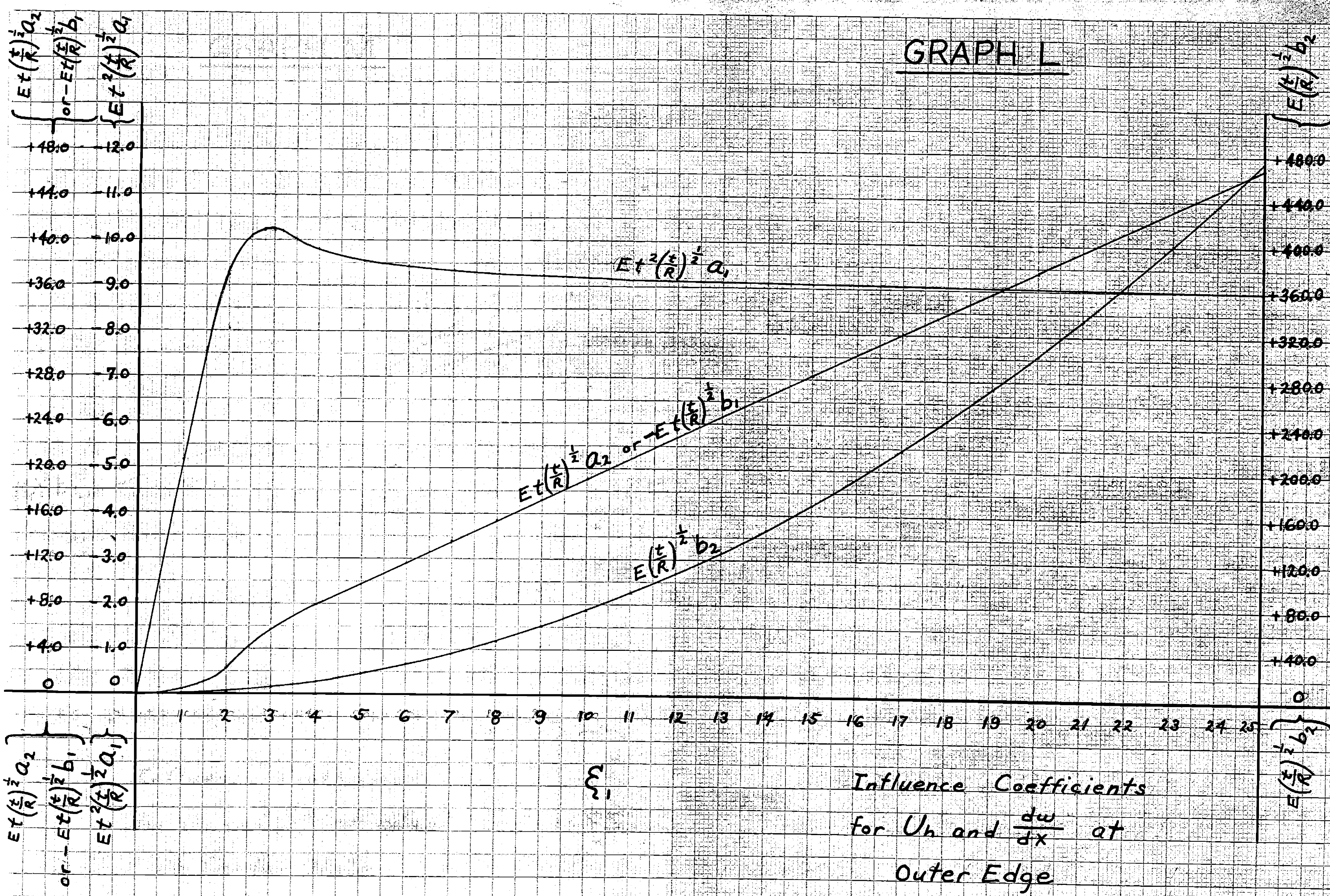
GRAPH K
(2Et/R²P) wc
P = intensity of vertical load per unit of projected area of shell
Positive P is downward
R = radius of shell
t = thickness of shell
-1.4
-1.3
-1.2
-1.1
-1.0
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
ξ1
Vertical Deflection
at Center
vs ξ1
F.R. Berman 1946



GRAPH K-a
at R = ∞, (E/P) w_c = 62.2 × 10^5
(E/P) w_c in feet
64 × 10^5
60 × 10^5
56 × 10^5
52 × 10^5
48 × 10^5
44 × 10^5
40 × 10^5
36 × 10^5
32 × 10^5
28 × 10^5
24 × 10^5
20 × 10^5
16 × 10^5
12 × 10^5
8 × 10^5
4 × 10^5
0
(E/P) w_c
P is intensity of vertical load per unit of projected area of shell
positive P is downward
R is radius of shell
Vertical Deflection
at Center
vs. R
t = 2 inches
X_1 = 20 feet
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10,000
R in feet
F.R. Berman 1946
KEUFFEL & ESSER CO. N. Y.


GRAPH K-b
P = intensity of vertical load per unit of projected area of shell
Positive P is downward
R = radius of shell
t = thickness of shell
Vertical Deflection
at Center
vs. h/t
(X₁/t = 120)
S (E/tP) ω_c
(E/tP) ω_c
384×10⁵
360×10⁵
336×10⁵
312×10⁵
288×10⁵
264×10⁵
240×10⁵
216×10⁵
192×10⁵
168×10⁵
144×10⁵
120×10⁵
96×10⁵
72×10⁵
48×10⁵
24×10⁵
0
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 84 88 92 96 100
Bulge Ratio = h/t
F.R. Berman 1946

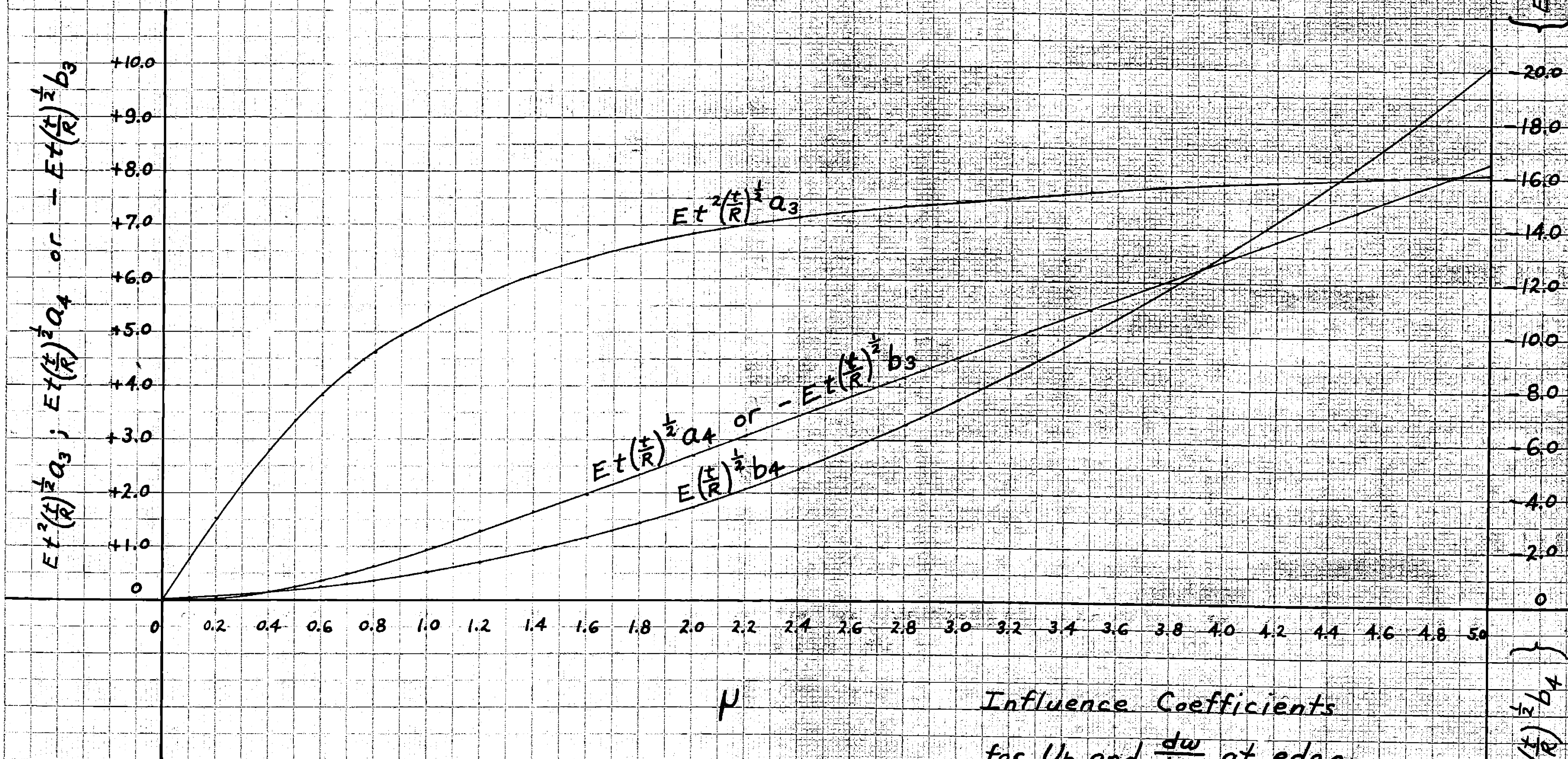
GRAPH L



Influence Coefficients for U_h and $\frac{dw}{dx}$ at Outer Edge

F.R. Berman 1946

GRAPH M



Influence Coefficients for U_h and $\frac{dw}{dx}$ at edge of Center Hole

F.R. Berman 1946