ON THE STABILITY OF CERTAIN TWO-

DIMENSIONAL UNSYMMETRIC PARALLEL FLOWS

by

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§1. INTRODUCTION

This study is directed at solving some of the problems which arise in the attempt to extend existing results on the stability of two-dimensional parallel flows to cases of unsymmetric velocity distribution, especially jet flow from a narrow slit. The value of a minimum critical Reynolds number at which instability or turbulence begins is of great interest in such problems, and its calculation for unsymmetric velocity profiles requires the use of certain asymptotic expansions having complicated behavior in the neighborhood of two points in the complex plane. The principal theorem of this study is on the determination of a path around these two points along which the asymptotic expansions of certain solutions of the stability equation do not change; i.e., we can avoid the so-called Stokes phenomenon. Applying this conclusion to jet flow (symmetric or unsymmetric), it will be shown that the effect of viscosity cannot be brought in through the "viscous solutions", as was done in stability problems previously investigated.

There exists an extensive literature* on the stability of essentially parallel flows, i.e., on the eigenvalue problem associated with the linearized stability equation:

$$(w-c)\,(\varphi''-\alpha^2\varphi)-w''\varphi=-\frac{i}{\alpha R}\,(\varphi^{iv}-2\alpha^2\varphi''+\alpha^4\varphi), \quad (1.1)$$

which is an equation for $\varphi(y)$ with suitable boundary conditions. The derivation of (1.1) can be found in [1], and here we merely define the necessary symbols. In the derivation the stream function $\psi(x,y,t)$ has been represented as the sum of a steady main flow $\Psi(x,y)$ and a disturbance function $\psi'(x,y,t)$. The main flow is taken as $w(y)$, an analytic function assumed to be given, and the small two-dimensional periodic disturbance superposed onto the main flow is represented by

$$\psi'(x,y,t)=\varphi(y)\,e^{i\alpha(x-ct)}. \quad (1.2)$$

The disturbance velocity components are

$$u=\frac{\partial\psi'}{\partial y},\ v=-\frac{\partial\psi'}{\partial x}. \quad (1.3)$$

We can take α as always real and positive, while in general $c=c_r+ic_i$. Finally $R=Ul/\nu$ is the Reynolds number in

* See [1]. Numbers in brackets refer to References at the end of this paper .

terms of a characteristic velocity U and characteristic length l, and ν is the kinematical viscosity. All velocities and lengths are customarily referred to these, making (1.1) non-dimensional.

The eigenvalue problem has been substantially solved [1] for certain cases of flow by the determination of the neutral curve

$$c_i(\alpha, R) = 0$$

in the α-R plane. Such a curve separates the region of stability from the region of instability. Thus, existence of a real c implies existence of a neutral disturbance. We note [1] that only two-dimensional disturbances are considered since Squire and Hollingdale proved that such disturbances are less stable than three-dimensional disturbances.

We shall see that one pair of solutions in a fundamental system for (1.1) has exponential asymptotic character. Therefore in the jet profile we must eliminate both of these solutions whether or not we have very large Reynolds number in the flow since one solution will diverge for positive y and the other for negative y when y becomes infinite. This is an important distinction of the jet problem from the channel flow problem and means that the only way to bring in the effect of viscosity is to use higher order terms in the expansions of the other pair of solutions of (1.1).

Further we shall see that when the Reynolds number is infinite the other pair of solutions for (1.1) gives a fundamental system for the inviscid equation:

$$(w-c)(\varphi''-\alpha^2\varphi)-w''\varphi = 0\,, \qquad (1.4)$$

which could be obtained from (1.1) by formally passing to the limit for infinitely large Reynolds number. Because of this, the equation (1.4) plays an important part in discovering certain properties of the flows under consideration. The theorems of §2 and §3 will refer directly to (1.4) and its solutions, but because of both this limit relationship and the main theorem proved in §4, we also refer indirectly to solutions of (1.1) which can be used in the associated eigenvalue problem.

§2. THE INVISCID EQUATION

This section concerns both symmetric and unsymmetric jet flow; these types of real $w(y)$ are illustrated in Fig. 1, Fig. 2 respectively for real y. In these flows the x-axis is in

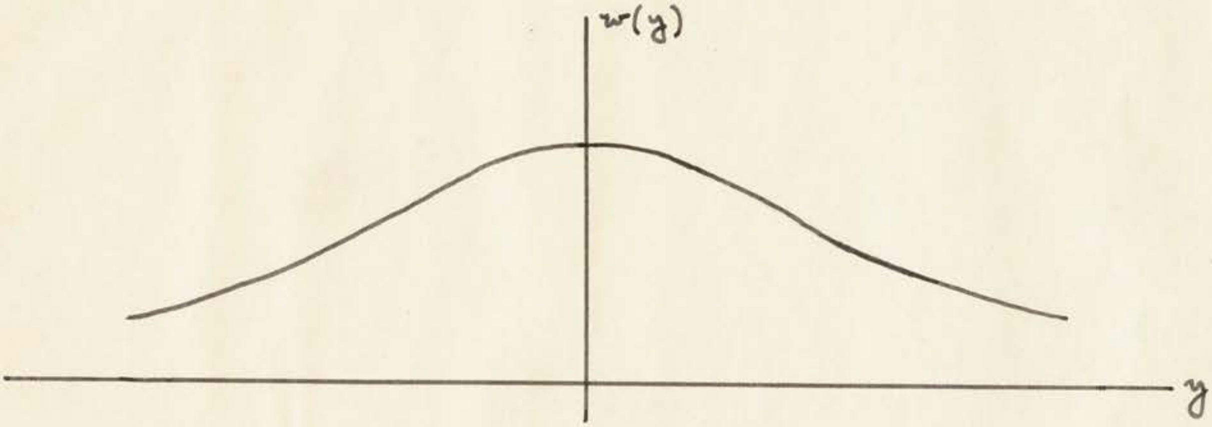

Fig. 1.

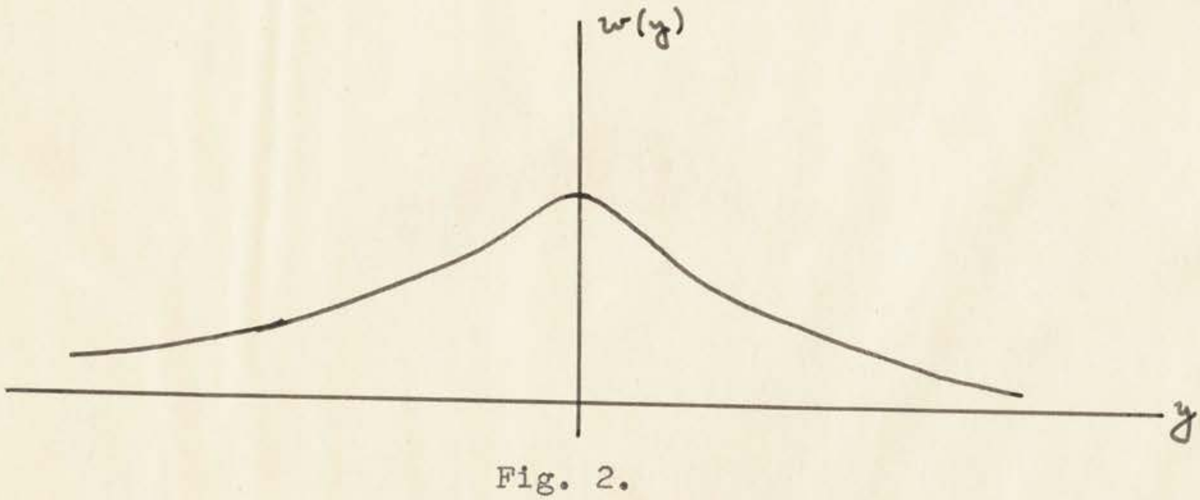

Fig. 2.

the direction of the main flow. We note both cases have exactly two points of inflection, and each has zero slope at $y = \pm\infty$ and $y = 0$. In each case $w - c$ is assumed to have exactly two simple zeros in the complex plane. For symmetric flows only one side of the profile, and one zero, need be considered.

Suppose $w(y_c) - c = 0$ and write (1.4) in the normal form

$$(y - y_c)^2 \varphi'' - (y - y_c)^2 (\alpha^2 + \frac{w''}{w - c}) \varphi = 0 . \qquad (2.1)$$

But on using (2.3), (2.4) we have

$$(y - y_c)^2 \frac{w''}{w - c} = \frac{w_c''(y - y_c) + w_c'''(y - y_c)^2 + \cdots}{w_c' + \frac{w_c''}{2}(y - y_c) + \cdots} = \frac{w_c''}{w_c'}(y - y_c) + \cdots .$$

Hence the coefficients in (2.1) are analytic so that y_c is a regular singularity of (1.4). The general indicial equation is

$$\alpha^2 + \alpha(p_0 - 1) + q_0 = 0 \quad ,$$

where $p_0 = q_0 = 0$ are the first terms of the coefficients of φ', φ respectively. Hence

$$\alpha^2 - \alpha = 0 , \qquad (2.2)$$

and the roots of (2.2) differ by an integer, so that y_c is a logarithmic branch point of solutions of (1.4).

The explicit form of the general solution will now be derived [3]. The series solution is of the form

$$\varphi_1 = a_0 + a_1 (y - y_c) + \cdots .$$

The coefficients of (1.4) may be expanded:

$$w - c = w_c' (y - y_c) + \frac{w_c''}{2} (y - y_c)^2 + \cdots \qquad (2.3)$$

$$w'' = w_c'' + w_c''' (y - y_c) + \cdots . \qquad (2.4)$$

Substitution into (1.4) gives

$$w_c'' a_0 + \left[2w_c' a_2 - w_c'' a_1 - w_c''' a_0 \right] (y - y_c) + \cdots = 0 ,$$

so that we may take $a_0 = 0$, $a_1 = 1$, since the latter is arbitrary. Hence

$$\varphi_1(y) = (y - y_c) + \cdots . \qquad (2.5)$$

Another solution has the form

$$\varphi_2 = b_0 + b_1 (y - y_c) + \cdots + C \varphi_1 \ln(y - y_c) .$$

Substitution into (1.4) gives

$$C w_c' - w_c'' b_0 + \cdots = 0 ,$$

so that taking $b_0 = 1$ gives $C = w_c'' / w_c'$. Hence

$$\varphi_2(y) = 1 + \cdots + \frac{w_c''}{w_c'} \varphi_1 \ln(y - y_c) . \qquad (2.6)$$

Therefore the general solution of (1.4) is

$$\varphi(y) = A\varphi_1 + B\varphi_2 = A\varphi_1 + B\varphi_3 + B\frac{w_c''}{w_c'} \varphi_1 \ln(y - y_c) , \qquad (2.7)$$

where φ_3 is the power series part of φ_2. In general A and B are complex.

In the following Theorem 1. we suppose the velocity profile is symmetric and that c is real, so that y_c is real. The boundary conditions [1] are:

$$1) \lim_{y\to-\infty}\left[\frac{\varphi'(y)}{\varphi(y)}\right]=\alpha \ , \ \lim_{y\to-\infty} w(y)=0 \ ,$$

$$\lim_{y\to-\infty} u=0 \ , \ \lim_{y\to-\infty} v = 0 \qquad (2.8)$$

2a) $\varphi'(0)=0$, if $\varphi(y)$ is even.

2b) $\varphi(0)=0$, if $\varphi(y)$ is odd.

By (1.2) and (1.3) the physical significance of an even $\varphi(y)$ is that the disturbance is antisymmetric, while an odd $\varphi(y)$ implies a symmetric disturbance. In Fig. 3 we have assumed that the flex, at y_s, and the

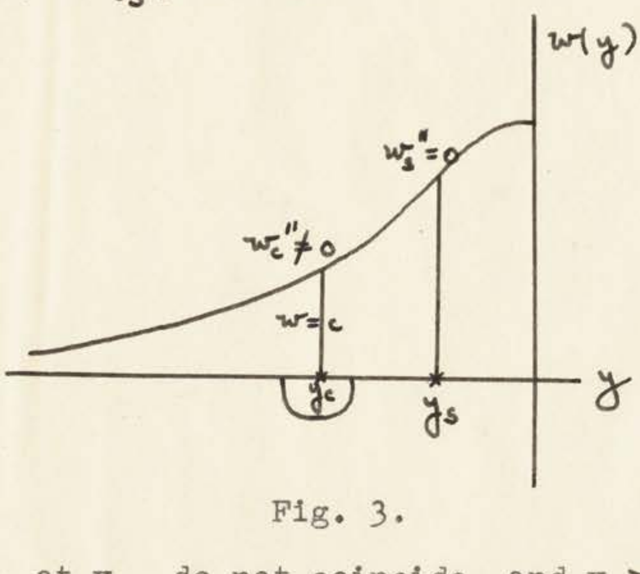

Fig. 3.

branch point, at y_c, do not coincide, and $y_s > y_c$. But the following argument does not depend on their relative positions.

THEOREM 1. If there exists a neutral disturbance with wave velocity c for the symmetric jet profile, then either

1) The point y_c must coincide with the point y_s , or

2) The inviscid equation has the trivial solution $\varphi = w - c$ with $\alpha = 0$.

PROOF: Multiply (1.4) by $-\bar{\varphi}$ and integrate:

$$\int_{-\infty}^{0} \left[-\bar{\varphi}\varphi'' + \alpha^2 |\varphi|^2 + \frac{w''}{w-c} |\varphi|^2 \right] dy = 0 . \quad (2.9)$$

The first term can be integrated by parts, and using (2.8) we have

$$-\int_{-\infty}^{0} \bar{\varphi}\varphi'' dy = -\bar{\varphi}\varphi' \Big]_{-\infty}^{0} + \int_{-\infty}^{0} |\varphi'|^2 dy = \int_{-\infty}^{0} |\varphi'|^2 dy .$$

Then (2.9) becomes

$$\int_{-\infty}^{0} \left[|\varphi'|^2 + \alpha^2 |\varphi|^2 + \frac{w''}{w-c} |\varphi|^2 \right] dy = 0 . \quad (2.10)$$

In (2.10) we may use a path in the complex plane given by the three lines:

$$\left.\begin{array}{ll} \text{a)} & -\infty < y \leq y_c - \epsilon \\ \text{b)} & y - y_c = \epsilon e^{i\theta} , \qquad -\pi \leq \theta \leq 0 \\ \text{c)} & y_c + \epsilon \leq y \leq 0 \quad . \end{array}\right\} \quad (2.11)$$

On each of these three lines the integrand of (2.10) is analytic, so the integral exists, is analytic, and is finite for each $\epsilon > 0$. In particular the imaginary part of (2.10) has these properties. But (2.10) is real on each of (2.11a), (2.11c), so that on path (2.11b) we have

$$\mathrm{Im} \int_{\circlearrowright} \left[|\varphi'|^2 + \alpha^2 |\varphi|^2 + \frac{w''}{w-c} |\varphi|^2 \right] dy = 0 \, . \tag{2.12}$$

Each of these three terms is evaluated individually. Now $\varphi(y)$, $\varphi_1(y)$, and $\varphi_3(y)$ have known forms in (2.5), (2.6), (2.7). Using (2.11b) in these with small ϵ we obtain

$$\begin{aligned} \varphi_1 &= \epsilon e^{i\theta} + O(\epsilon^2) \, , & \varphi_3 &= 1 + b_1 \epsilon e^{i\theta} + O(\epsilon^2) \\ \varphi_1' &= 1 + O(\epsilon) \, , & \varphi_3' &= b_1 + O(\epsilon) \end{aligned} \tag{2.13}$$

And then

$$\begin{aligned} \varphi'(y) &= A + Bb_1 + \cdots + B\frac{w_c''}{w_c'}\Big[1 + 2a_2 \epsilon e^{i\theta} + \cdots + \\ &\quad + \ln(\epsilon e^{i\theta}) + a_2 \epsilon e^{i\theta} \ln(\epsilon e^{i\theta}) + \cdots \Big] \\ &= c_0 + c_1 \ln\epsilon + c_2 \theta + \big[\epsilon (c_3 + c_4 \ln\epsilon + c_5 \theta) + \cdots \big] e^{i\theta} \, . \end{aligned} \tag{2.14}$$

In the last equation the omitted terms are of higher order in ϵ.

From (2.12), for fixed ϵ:

$$\mathrm{Im} \int_{\circlearrowright} |\varphi'(y)|^2 dy \le \left| \int_{\circlearrowright} |\varphi'(y)|^2 dy \right| \le \left\{ \mathrm{Max}\, |\varphi'(y)|^2 \right\} \pi\epsilon \, . \tag{2.15}$$

Hence to show that this term vanishes with ϵ we need only a form of the Max $|\varphi'(y)|^2$ on the designated path that will behave well when multiplied by the ϵ of (2.15). It is clear from (2.14) that no term, for example $\frac{1}{\epsilon}\ln\epsilon$, can appear in Max $|\varphi'(y)|^2$ which might destroy the convergence desired in (2.15).

Therefore we conclude that

$$\operatorname{Im}\int_{\circlearrowleft} |\varphi'(y)|^2 dy = 0 .$$

It is evident that $|\varphi(y)|^2$ can be treated in the same way, and we obtain the form:

$$\varphi(y) = B + (c_0\epsilon + c_1\epsilon\ln\epsilon + c_2\epsilon\theta + \cdots)e^{i\theta} , \qquad (2.16)$$

where $\varphi(y_c) = B$. Again it is clear that no term of $|\varphi(y)|^2$ can diverge as ϵ approaches zero; here the aid of the ϵ in (2.15) is not needed.

Therefore

$$\operatorname{Im}\int_{\circlearrowleft} \alpha^2|\varphi(y)|^2 dy = 0 . \qquad (2.17)$$

Finally, on combining (2.3) and (2.4) with the third term of (2.10):

$$\operatorname{Im}\int_{\circlearrowleft} \frac{w''}{w-c}|\varphi(y)|^2 dy = \operatorname{Im}\int_{\circlearrowleft}\left\{\frac{w_c''}{w_c'(y-y_c)} + \frac{2w_c'w_c''' - (w_c'')^2}{2(w_c')^2} + \right.$$
$$\left. + O(y-y_c)\right\}|\varphi(y)|^2 dy = 0 .$$

The second term of this expression yields zero in the same way as discussed above for (2.17). The third term is also zero since the function $(y-y_c)|\varphi(y)|^2$ is of higher order in ϵ than the terms for (2.17). Hence there remains:

$$\operatorname{Im}\int_{\circlearrowleft} \frac{w_c''}{w_c'}|\varphi(y)|^2 \frac{dy}{y-y_c} = 0 .$$

Then (2.11b) yields

$$\frac{dy}{y-y_c} = i d\theta ,$$

and from (2.16):

$$|\varphi(y)|^2 = |\varphi(y_c)|^2 + O(\epsilon) = |B|^2 + O(\epsilon) \ .$$

Therefore, as ϵ approaches zero:

$$\mathrm{Im}\oint \frac{w_c''}{w_c'}|B|^2 i\,d\theta = \int_{-\pi}^{0} \frac{w_c''}{w_c'}|B|^2 d\theta = \frac{w_c''}{w_c'}|B|^2\pi = 0 \ .$$

Hence we have either $w_c'' = 0$, which yields $y_s = y_c$, or we have $B = 0$. In the latter case (2.7) gives $\varphi = A\varphi_1$ so that (2.5) gives $\varphi(y_c) = 0$. Thus we may write

$$\varphi = A\varphi_1 = A(w-c)\ g(y) \ .$$

Using this in (1.4) we can determine $g(y)$ [2]. The equation (1.4) becomes

$$\left[(w-c)^2 g'\right]' - \alpha^2 (w-c)^2 g = 0 \ .$$

On multiplying this by $-\bar{g}$ and integrating this equation we can reduce the first term as in the step preceding (2.10) and obtain

$$\int_{-\infty}^{0} (w-c)^2 \left[|g'|^2 + \alpha^2 |g|^2\right] dy = 0 \ .$$

Since c is taken here as real, all terms in this integrand are positive so that if $\alpha \neq 0$ then $g = g' = 0$, whence $\varphi(y) \equiv 0$. That is, there is no solution with $\alpha \neq 0$. If $\alpha = 0$, then $g' = 0$ or $g = \text{const.} = k$, whence $\varphi(y) = Ak(w-c)$, and $\varphi = w - c$ is a solution. This completes the proof of the theorem. The Case 2 of the theorem is a rather trivial case of a disturbance since $\alpha = 0$ prevents a periodic disturbance in either x or t.

The situation for unsymmetric jet flow is quite different and we proceed with several lemmas to that result. In the following we assume the situation of Fig. 4, i.e., an unsymmetric profile.

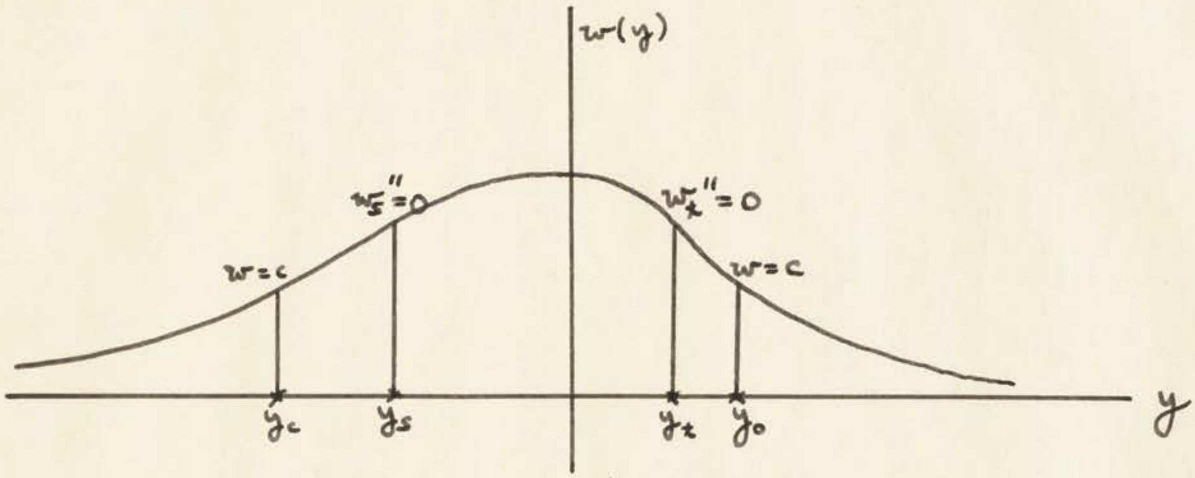

Fig. 4.

The x-average of the product of the disturbance velocity components in a flow is proportional to the Reynolds shear stress and is always an important physical quantity in the study of turbulence. In the present case these components are the real parts of the complex velocities (1.3). If we put for the general solution

$$\varphi(y) = \varphi_r(y) + i\varphi_i(y)$$

then we can prove a lemma known to Tollmien [4] for channel flow, but extended to the case $c = c_r + ic_i$ and to jet flow.

<u>LEMMA 1</u>. For jet flow the x-average of $(u'v')$, denoted by $\overline{u'v'}$, is proportional to the Wronskian of φ_i, φ_r when $c = c_r + ic_i$.

<u>PROOF</u>: We have

$$u' = \mathrm{Re}(\psi_y') = \mathrm{Re}\left[\varphi'(y)e^{i\alpha(x-ct)}\right]$$
$$= \mathrm{Re}\left\{(\varphi_r' + i\varphi_i')\left[\cos\alpha(x-c_rt) + i\,\sin\alpha(x-c_rt)\right]e^{c_it}\right\},$$

or $$u' = \left[\varphi_r' \cos\alpha(x - c_r t) - \varphi_i' \sin\alpha(x - c_r t)\right] e^{c_i t} .$$

Similarly

$$v' = \mathrm{Re}(-\psi_x') = \left[\alpha\varphi_i \cos\alpha(x - c_r t) + \alpha\varphi_r \sin\alpha(x - c_r t)\right] e^{c_i t}.$$

Hence

$$\begin{aligned}\overline{u'v'} &= \frac{1}{x}\int_0^x u'v'\,dx \\ &= \frac{\alpha e^{2c_i t}}{x}\int_0^x \left[\varphi_r' \cos\alpha(x - c_r t) - \varphi_i' \sin\alpha(x - c_r t)\right] \times \\ &\qquad \times \left[\varphi_i \cos\alpha(x - c_r t) + \varphi_r \sin\alpha(x - c_r t)\right] dx .\end{aligned}$$

On multiplying and integrating we obtain

$$\begin{aligned}\overline{u'v'} = \frac{e^{2c_i t}\alpha}{x} \Bigg\{ &\varphi_r'\varphi_i \tfrac{1}{2}\Big[\sin\alpha(x - c_r t)\ \cos\alpha(x - c_r t) + \alpha(x - c_r t) + \\ &\qquad + \sin\alpha c_r t\ \cos\alpha c_r t + \alpha c_r t\Big] + \\ &+ (\varphi_r'\varphi_r - \varphi_i'\varphi_i)\ \tfrac{1}{2}\left[\sin^2\alpha(x - c_r t) - \sin^2\alpha c_r t\right] + \\ &- \varphi_i'\varphi_r \tfrac{1}{2}\Big[-\sin\alpha(x - c_r t)\cos\alpha(x - c_r t) + \\ &\qquad + \alpha(x - c_r t) - \sin\alpha c_r t.\ \cos\alpha c_r t + \alpha c_r t\Big] \Bigg\} .\end{aligned}$$

On combining terms by various trigonometric formulae we have

$$\begin{aligned}\overline{u'v'} = \frac{e^{2c_i t}\alpha}{2x} \Bigg\{ &\varphi_r'\varphi_i \left[\sin\alpha x\ \cos\alpha(x - 2c_r t) + \alpha x\right] + \\ &+ (\varphi_r'\varphi_r - \varphi_i'\varphi_i)\left[\sin\alpha(x - 2c_r t)\ \sin\alpha x\right] + \\ &- \varphi_i'\varphi_r\left[-\sin\alpha x\ \cos\alpha(x - 2c_r t) + \alpha x\right] \Bigg\} .\end{aligned}$$

$$= \frac{\alpha e^{2c_i t}}{2x}\Big\{(\varphi_r'\varphi_i - \varphi_i'\varphi_r)\alpha x +$$

$$+ \Big[(\varphi_r'\varphi_i + \varphi_i'\varphi_r)\cos\alpha(x - 2c_r t) + (\varphi_r'\varphi_r - \varphi_i'\varphi_i)\sin\alpha(x - 2c_r t)\Big]\sin\alpha x\Big\}$$

Hence for the limit of infinitely large x we have

$$\overline{u'v'} = \frac{\alpha^2 e^{2c_i t}}{2}(\varphi_r'\varphi_i - \varphi_i'\varphi_r) = \frac{\alpha^2 e^{2c_i t}}{2} W(\varphi_i, \varphi_r) \;. \quad (2.18)$$

This lemma gives a form of the average which can be calculated since $\varphi(y)$ is known in the form (2.7). The next lemma also was known to Tollmien [4] for the case of channel flow.

LEMMA 2. If there exists a neutral disturbance the jump along the real axis in the value of $W(\varphi_i, \varphi_r)$ across the singular point y_c is

$$\Big[\Big[\varphi_r'\varphi_i - \varphi_r\varphi_i'\Big]\Big] = |\varphi_c|^2 \pi \frac{w_c''}{w_c'} \;.$$

PROOF: Since y_c and the path along which the jump is taken are real, so are all the symbols on the right side of (2.7) with the possible exception of the constants A, B . Hence

$$\varphi = \varphi_r + i\varphi_i = (A_r + iA_i)\varphi_1 + (B_r + iB_i)\varphi_3 + \\ + \varphi_1 \frac{w_c''}{w_c'}(B_r + iB_i)\ln(y - y_c) \;. \quad (2.19)$$

For $y > y_c$, (2.19) holds unchanged and the Wronskian is

$$
\begin{aligned}
\varphi_r' \varphi_i - \varphi_r \varphi_i' = \Big\{ \Big[& A_r \varphi_1' + B_r \varphi_3' + B_r \frac{w_c''}{w_c'} \varphi_1' \ln(y - y_c) + \\
& + B_r \frac{w_c''}{w_c'} \varphi_1 \frac{1}{y - y_c} \Big] \times \\
\times \Big[& A_i \varphi_1 + B_i \varphi_3 + B_i \frac{w_c''}{w_c'} \varphi_1 \ln(y - y_c) \Big] \Big\} + \\
- \Big\{ \Big[& A_r \varphi_1 + B_r \varphi_3 + B_r \frac{w_c''}{w_c'} \varphi_1 \ln(y - y_c) \Big] \times \\
\times \Big[& A_i \varphi_1' + B_i \varphi_3' + B_i \frac{w_c''}{w_c'} \varphi_1' \ln(y - y_c) + \\
& + B_i \frac{w_c''}{w_c'} \varphi_1 \frac{1}{y - y_c} \Big] \Big\} .
\end{aligned}
\tag{2.20}
$$

For $y < y_c$, to evaluate $\ln(y - y_c)$ we must go <u>below</u> y_c in the complex plane for reasons stated in §4. Thus

$$
\ln(y - y_c) = \ln(y_c - y) - i\pi
$$

is to be used in (2.19) and the Wronskian in this case is

$$
\begin{aligned}
\varphi_r' \varphi_i - \varphi_r \varphi_i' = \Big\{ \Big[& A_r \varphi_1' + B_r \varphi_3' + B_r \frac{w_c''}{w_c'} \varphi_1' \ln(y_c - y) + \\
& - B_r \frac{w_c''}{w_c'} \varphi_1 \frac{1}{y_c - y} + B_i \pi \frac{w_c''}{w_c'} \varphi_1' \Big] \times \\
\times \Big[& A_i \varphi_1 + B_i \varphi_3 + B_i \frac{w_c''}{w_c'} \varphi_1 \ln(y_c - y) - B_r \frac{w_c''}{w_c'} \pi \varphi_1 \Big] \Big\} + \\
- \Big\{ \Big[& A_r \varphi_1 + B_r \varphi_3 + B_r \frac{w_c''}{w_c'} \varphi_1 \ln(y_c - y) + B_i \pi \frac{w_c''}{w_c'} \varphi_1 \Big] \times \\
\times \Big[& A_i \varphi_1' + B_i \varphi_3' + B_i \frac{w_c''}{w_c'} \varphi_1' \ln(y_c - y) + \\
& - B_i \frac{w_c''}{w_c'} \varphi_1 \frac{1}{y_c - y} - B_r \frac{w_c''}{w_c'} \pi \varphi_1' \Big] \Big\} .
\end{aligned}
\tag{2.21}
$$

Now subtract (2.20) from (2.21) and let y approach y_c as a limit. In this process we suppose that the y on the left and the y on the right to be always at the same distance from y_c. This gives for the jump

$$\left[\left[\varphi_r'\varphi_i - \varphi_r\varphi_i'\right]\right] \equiv$$

$$\lim_{y\to y_c}\Big\{ -2B_r\frac{w_c''}{w_c'}\varphi_1\frac{1}{|y-y_c|}\Big[A_i\varphi_1 + B_i\varphi_3 + B_i\frac{w_c''}{w_c'}\varphi_1\ln|y-y_c|\Big] +$$

$$-B_r\frac{w_c''}{w_c'}\pi\varphi_1\Big[A_r\varphi_1' + B_r\varphi_3' + B_r\frac{w_c''}{w_c'}\varphi_1'\ln(y_c-y)\Big] +$$

$$+ B_r^2\pi\left(\frac{w_c''}{w_c'}\right)^2\varphi_1^2\frac{1}{(y_c-y)} +$$

$$+ B_i\frac{w_c''}{w_c'}\pi\varphi_1'\Big[A_i\varphi_1 + B_i\varphi_3 + B_i\frac{w_c''}{w_c'}\varphi_1\ln(y_c-y) - B_r\frac{w_c''}{w_c'}\pi\varphi_1\Big] +$$

$$+ 2B_i\frac{w_c''}{w_c'}\varphi_1\frac{1}{|y-y_c|}\Big[A_r\varphi_1 + B_r\varphi_3 + B_r\frac{w_c''}{w_c'}\varphi_1\ln|y-y_c|\Big] +$$

$$+ B_r\frac{w_c''}{w_c'}\pi\varphi_1'\Big[A_r\varphi_1 + B_r\varphi_3 + B_r\frac{w_c''}{w_c'}\varphi_1\ln(y_c-y)\Big] +$$

$$- B_i\frac{w_c''}{w_c'}\pi\varphi_1\Big[A_i\varphi_1' + B_i\varphi_3' + B_i\frac{w_c''}{w_c'}\varphi_1'\ln(y_c-y) +$$

$$- B_i\frac{w_c''}{w_c'}\varphi_1\frac{1}{(y_c-y)} - B_r\frac{w_c''}{w_c'}\pi\varphi_1'\Big]\Big\} .$$

On expanding this it is found that all terms involving $y-y_c$ or y_c-y will cancel. The remaining terms are

$$\left[\left[\varphi_r'\varphi_i - \varphi_r\varphi_i'\right]\right] = \lim_{y\to y_c}\Big\{(B_r^2+B_i^2)\pi\frac{w_c''}{w_c'}(\varphi_1'\varphi_3 - \varphi_1\varphi_3') + \pi\frac{w_c''}{w_c'}(B_r^2+B_i^2)\varphi_1^2\Big\} . \qquad (2.22)$$

But inspection of (2.5), (2.6) and their derivatives shows

$$\lim_{y\to y_c} \varphi_1(y) = 0 \; ; \; \lim_{y\to y_c} \varphi_1'(y) = 1 \; ;$$

$$\lim_{y\to y_c} \varphi_3(y) = 1 \; ; \; \lim_{y\to y_c} \varphi_3'(y) = \text{const.} = b_1 \; ,$$

so that

$$\lim_{y\to y_c} \varphi_1'\varphi_3 = 1; \; \lim_{y\to y_c} \varphi_1\varphi_3' = 0 \; ; \; \lim_{y\to y_c} \varphi_1^2 = 0 \; ,$$

and from (2.7)

$$\varphi(y_c) \equiv \varphi_c = \lim_{y\to y_c} \varphi(y) = B \; ,$$

whence

$$B_r^2 + B_i^2 = |\varphi_c|^2 \; .$$

Therefore (2.22) gives the desired result:

$$\left[\left[\varphi_r'\varphi_i - \varphi_r\varphi_i'\right]\right] = |\varphi_c|^2 \pi \frac{w_c''}{w_c'}$$

<u>THEOREM 2</u>. If there exists a neutral disturbance, a jet flow with symmetric velocity profile has $\overline{u'v'} = 0$ throughout the flow.

<u>PROOF</u>: Since the wave velocity c is real, Lemmas 1 and 2 yield

$$\left[\left[\overline{u'v'}\right]\right] = \frac{\alpha^2}{2} |\varphi_c|^2 \pi \frac{w_c''}{w_c'} \; .$$

Now Theorem 1. allows $\varphi_c = \varphi(y_c) = 0$; clearly in this case no change in the value of the average can occur at any finite y, since at $y = \pm\infty$ we have $u' = v' = 0$, then $\overline{u'v'} \equiv 0$. The other case of Theorem 1. is $y_s = y_c$, so that $w_c'' = 0$, and again $\overline{u'v'} \equiv 0$, proving the theorem.

For the unsymmetric profile we refer to Fig. 4 of page 14 for definiteness. Because of the asymmetry, if $w(y_s) = C$, then $w(y_t) \neq C$, and vice versa, so that Theorem 1 does not hold for this case. We still have $\overline{u'v'} = 0$ for $y = \pm\infty$, and this with Lemmas 1, 2 show that there must be a jump in the value of $\overline{u'v'}$ at each of y_s , y_t , and that these must be equal in magnitude. There is no loss in generality in assuming $w_s > w_t$. For definiteness we take $w_o = w_c < w_t$. At y_c we already have for the jump

$$\left[\left[\overline{u'v'}\right]\right] = \frac{\alpha^2}{2}\left|\varphi^{(1)}(y_c)\right|^2 \pi \frac{w_c''}{w_c'} ,$$

where $\varphi^{(1)}(y)$ is the $\varphi(y)$ of (2.7).

But since all the preceding work holds for every singular point, at y_o the jump is

$$\left[\left[\overline{u'v'}\right]\right] = \frac{\alpha^2}{2}\left|\varphi^{(2)}(y_o)\right|^2 \pi \frac{w_o''}{w_o'} ,$$

where

$$\varphi^{(2)}(y) = A'\varphi_1 + B'\varphi_3 + B'\frac{w_o''}{w_o'}\varphi_1 \ln(y - y_o) .$$

Thus

$$\left|\varphi_c^{(1)}\right|^2 \frac{w_c''}{w_c'} = \left|\varphi_o^{(2)}\right|^2 \frac{w_o''}{w_o'} . \qquad (2.23)$$

From Fig. 4 the derivatives in (2.23) have the behavior:

$$w'(y)\begin{cases} > 0, & -\infty < y < 0 \\ < 0, & 0 < y < \infty \end{cases}; \quad w''(y)\begin{cases} > 0, & -\infty < y < y_s \\ < 0, & y_s < y < y_t \\ > 0, & y_t < y < \infty \end{cases}.$$

Thus (2.23) shows the assumed positions of y_c, y_o are impossible since $\frac{w_c''}{w_c'}$, $\frac{w_o''}{w_o'}$ are of opposite sign.

Similarly if $w_o = w_c > w_s$, these ratios are again of opposite sign. Therefore we conclude with

THEOREM 3. If there exists a neutral disturbance, a jet flow with unsymmetric velocity profile can only have $w_s > w_c > w_t$, or $w_s < w_c < w_t$.

§3. THE BOUNDARY VALUE PROBLEM

We shall see in §4 that there exists a fundamental system of (1.1) containing two solutions φ_3, φ_4 with exponential asymptotic behavior:

$$\begin{aligned} \varphi_3 &= e^{\lambda Q(y)} \left[\eta(y) + O(\tfrac{1}{\lambda}) \right] \\ \varphi_4 &= e^{-\lambda Q(y)} \left[\eta(y) + O(\tfrac{1}{\lambda}) \right], \end{aligned} \tag{3.1}$$

where $\lambda^2 = \alpha R$ is a large and positive parameter, and when y is real, $yQ(y) > 0$; and $Q'(y) = \sqrt{i(w-c)}$. From (3.1) it is clear that even when λ is not infinite we must have $c_3 = c_4 = 0$ in the general solution

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 \tag{3.2}$$

if we take the velocity profile as a whole to get boundary conditions. This holds for both the symmetric and the unsymmetric profile. However, the boundary conditions (2.8) are for the symmetric profile and use only the part of the profile where φ_3 does not diverge, and it appears we need not take $c_3 = 0$. If this were true we could conveniently bring in the effect of viscosity through φ_3 , rather than through the higher order terms in the expansions of the other pair of solutions:

$$\varphi_1 = u(y) + O(\tfrac{1}{\lambda^2}) \ , \ \varphi_2 = v(y) + O(\tfrac{1}{\lambda^2}) \ , \qquad (3.3)$$

where u, v form a fundamental system of (1.4). We proceed to show that even when the boundary conditions are taken as in (2.8) the solution φ_3 does not contribute to the viscosity effect.

The boundary condition at $y_1 = -\infty$ can be written

$$\begin{aligned} \varphi'_{11} - \alpha\varphi_{11} &= 0 \\ \varphi'_{21} - \alpha\varphi_{21} &= 0 \end{aligned} \qquad (3.4)$$

where $\varphi'_{11} = \varphi'_1(y_1)$, $\varphi_{11} = \varphi_1(y_1)$, etc. For case 2a) of (2.8) we have also $\varphi'''(o) = 0$, and using this case with

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$$

at $y_2 = 0$ we obtain

$$\begin{aligned} c_1\varphi'_{12} + c_2\varphi'_{22} + c_3\varphi'_{32} &= 0 \\ c_1\varphi'''_{12} + c_2\varphi'''_{22} + c_3\varphi'''_{32} &= 0 \ . \end{aligned}$$

Treating (3.4) similarly to get a third condition on the coefficients, the only way these three relations for c_i, $i = 1,2,3$, can hold is that

$$\begin{vmatrix} \varphi'_{11} - \alpha\varphi_{11} & \varphi'_{21} - \alpha\varphi_{21} & 0 \\ \varphi'_{12} & \varphi'_{22} & \varphi'_{32} \\ \varphi'''_{12} & \varphi'''_{22} & \varphi'''_{32} \end{vmatrix} = 0 \ . \qquad (3.5)$$

It is understood that the fundamental system has dependence on all the parameters α, c, αR appearing in (1.1), so that (3.5) is a relation

$$F_1(\alpha, c, \alpha R) = 0$$

for the eigenvalue problem.

In the same manner for Case 2b) of (2.8) we obtain

$$F_2(\alpha, c, \alpha R) = \begin{vmatrix} \varphi_{11}' - \alpha\varphi_{11} & \varphi_{21}' - \alpha\varphi_{21} & 0 \\ \varphi_{12} & \varphi_{22} & \varphi_{32} \\ \varphi_{12}'' & \varphi_{22}'' & \varphi_{32}'' \end{vmatrix} = 0 \ . \quad (3.6)$$

Suppose $\phi = k_1\varphi_1 + k_2\varphi_2$ is a solution of (1.4) satisfying the boundary condition at $y_1 = -\infty$. In (3.6) if we take this linear combination of the first two columns and replace the first column with the result we obtain

$$\begin{vmatrix} 0 & \varphi_{21}' - \alpha\varphi_{21} & 0 \\ \phi_2 & \varphi_{22} & \varphi_{32} \\ \phi_2'' & \varphi_{22}'' & \varphi_{32}'' \end{vmatrix} = 0 \ ,$$

where $\phi_2 = \phi(y_2)$. From this we have the eigenvalue condition as

$$\phi_2 = \phi_2'' \frac{\varphi_{32}}{\varphi_{32}''} \ . \quad (3.7)$$

Using (3.1), (3.3), (1.4) we have respectively

$$\frac{\varphi_{32}}{\varphi_{32}''} = \frac{1}{i(w-c)_2\alpha R}\left[1 + O\left(\frac{1}{\sqrt{\alpha R}}\right)\right] \quad (3.8)$$

$$\phi_2 = \phi_2^{(0)} + \frac{\phi_2^{(1)}}{\alpha R} + O\left(\frac{1}{\alpha R}\right)^2 \quad (3.9)$$

$$\phi_2^{(0)\,''} = \left(\alpha^2 + \frac{w''}{w-c}\right)_2 \phi_2^{(0)} \ . \quad (3.10)$$

Substitution into (3.7) gives

$$\phi_2^{(0)} + \frac{\phi_2^{(1)}}{\alpha R} + \cdots =$$

$$\left[\phi_2^{(0)\,''} + \frac{\phi_2^{(1)\,''}}{\alpha R} + \cdots\right] \frac{1}{i(w-c)_2\alpha R}\left[1 + O\left(\frac{1}{\sqrt{\alpha R}}\right)\right] \ . \quad (3.11)$$

From (3.11) we see that if terms of order $O(\frac{1}{\alpha R})$ are neglected we approximate (3.7) with $\phi_2^{(0)} = 0$, i.e., $\phi_2^{(0)}$ is of order $O(\frac{1}{\alpha R})$. Then (3.10) shows that $\phi_2^{(0)''}$ is of order $O(\frac{1}{\alpha R})$. Thus, if in (3.11) we include $\phi_2^{(1)}$ we still have no effect from φ_3 since the lowest order term on the right is $O(\frac{1}{\alpha R})^2$. This puts (3.7), the case of symmetric disturbance, in the form

$$\phi_2^{(0)} + \frac{\phi_2^{(1)}}{\alpha R} = 0 \,. \tag{3.12}$$

If we wish to carry the process further we can say (3.12) is of order $O(\frac{1}{\alpha R})^2$ and substitute (3.12) into the recursion formula [1] for higher approximations to (3.3) and find that $\phi_2^{(0)''} + \frac{\phi_2^{(1)''}}{\alpha R}$ is also of order $O(\frac{1}{\alpha R})^2$, so that taking $\phi_2^{(2)}$ on the left of (3.11) does not allow φ_3 to introduce viscosity at this order of terms, since the right side now has order $O(\frac{1}{\alpha R})^3$.

Finally we note that we can perform the same steps with (3.5), the case of antisymmetric disturbance, since the condition corresponding to (3.7) is

$$\phi_2' = \phi_2''' \, \frac{\varphi_{32}'}{\varphi_{32}'''} \,,$$

and we find

$$\phi_2^{(0)'''} = \left(\frac{w''}{w-c}\right)_2' \phi_2^{(0)} + \left(\alpha^2 + \frac{w''}{w-c}\right)_2 \phi_2^{(0)'} \,.$$

But w'' and $w-c$ are even functions of y, so that their ratio is even and its first derivative is odd and hence zero at $y_2 = 0$. Thus

$$\phi_2^{(0)\,'''} = \left(\alpha^2 + \frac{w''}{w-c}\right)_2 \phi_2^{(0)\,'} ,$$

and the calculations go through in exactly the same way. Therefore we have:

THEOREM 4. The solutions of the stability equation having exponential asymptotic behavior cannot be used in the eigenvalue problem for either symmetric or unsymmetric jet flow.

4. AN EXISTENCE THEOREM

If the velocity distribution w(y) is an analytic function of y, the stability equation (1.1) has a fundamental system of four solutions analytic in y and in the parameters α, c, $\alpha R \equiv \lambda^2$. These solutions can be found as power series of some suitable small parameter, but for most numerical purposes their asymptotic expansions in inverse powers of λ or of λ^2 are found more useful. A set of formal asymptotic expansions have been calculated [1] and the sector of their validity found near a zero of $w(y) - c$. It was mentioned in §1 that a fundamental system for (1.4) forms a first approximation to the asymptotic expansion of two solutions of (1.1). We noted in §2 that a zero of first order is a logarithmic branch point of solutions of the inviscid equation (1.4).

Recently [5] the validity of the formal expansions was established and further insight was given into the behavior of the asymptotic representations in certain sectors of the y-plane with boundaries starting from the zero of $w - c$. However, all the results heretofore mentioned are found with only one singular point of (1.4) considered. This is satisfactory for those hydrodynamics problems in which the velocity profile is either symmetric or monotonic, since in the first case the boundary values may be taken at one end and at the center of the profile, while the second case can have only one point where $w-c=0$ for each value of c. But an unsymmetric profile leads to two singular points of (1.4), each of which must be considered. For this case the central problem is the determination of a method for going around the two points by means of the asymptotic expansions proper to the neighborhood of each point. The main theorem of this section is on this question and is proved with an equation more general than (1.1) and for two zeros of $w-c$ in the complex plane. We retain the notation y_c, y_o for the zeros. In the interest of generality we allow λ to take on complex values, although αR is a real and positive number in the hydrodynamics problem, and write $\lambda = \rho\lambda_0$ where λ_0 is a complex constant different from zero, and $\rho > 1$.

Following the notation of Wasow [5], for this theorem we consider the somewhat more general equation

$$N(\varphi) + \lambda^2 M(\varphi) = 0 \tag{4.1}$$

where $\varphi = \varphi(y)$, and $N(\varphi)$, $M(\varphi)$ are linear differential expressions of order four and two respectively. The leading coefficient of $M(\varphi)$ may be taken as the function $-i(w-c)$ without loss of generality. (Wasow uses $b_0(x)$). The leading coefficient of $N(\varphi)$ does not vanish at the zeros of $w - c$. We define

$$Q_c(y) = \int_{y_c}^{y} \sqrt{i(w-c)}\, dy \tag{4.2}$$

$$Q_0(y) = \int_{y_0}^{y} \sqrt{i(w-c)}\, dy \quad , \tag{4.3}$$

and now introduce results we need from [5]. Due to the zero of first order at y_c there are three curves C_j, $j = 1, 2, 3$, meeting at y_c, along which $\mathrm{Re}\left[\lambda Q_c(y)\right] = 0$. These curves in general are not straight lines. Each curve, near y_c, makes an angle of $\frac{2\pi}{3}$ with the other two curves. They divide the doubly connected domain S_c defined by

$$0 < \left| Q_c(y) \right| < K$$

into three curvilinear sectors S_j, $j = 1, 2, 3$. The subscripts are so chosen that S_j is bounded by the two arcs C_i, $i \neq j$, and these arcs are considered to be part of S_j. Similarly at y_0 there are three curves $\tilde{C}_j$, $j = 1, 2, 3$, dividing the domain S_0 defined by

$$0 < \left| Q_0(y) \right| < K$$

into three sectors $\tilde{S}_j$, j 1, 2, 3. The constant K is so chosen that neither domain contains a zero of $w - c$, so that we can take

$$K = \left| Q_c(y_0) \right| = \left| \int_{y_c}^{y_0} \sqrt{i(w-c)}\, dy \right| \quad . \tag{4.4}$$

If we put $E(T)$ as a symbol denoting any function of y and λ which, along with all its y-derivatives, is bounded, uniformly in λ, in every closed subdomain of T, then from [5] at y_c we have four theorems:

A. There exist solutions $A_j(y,\lambda)$, $j = 1, 2, 3$, of (4.1) with asymptotic representations

$$A_j(y,\lambda) = e^{\lambda Q_c(y)}\left[\eta(y) + \frac{E(S_c - C_j)}{\lambda}\right] , \qquad (4.5)$$

where $\lambda Q_c(y)$ is taken with the determination that gives a negative real part in S_j.

B. There exist solutions $U_j(y,\lambda)$, $j = 1, 2, 3$, of (4.1) such that

$$U_j(y,\lambda) = u(y) + \frac{E(S_c - S_j)}{\lambda^2} , \qquad (4.6)$$

where $u(y)$ is a solution of $M(\varphi) = 0$.

C. If $u(y)$ is multivalued near y_c then the corresponding $U_j(y,\lambda)$ tending to $u(y)$ in $S_c - S_j$ will diverge at every interior point of S_j.

D. If $v(y)$ is a solution of $M(\varphi) = 0$ regular at y_c and if not all solutions of $M(\varphi) = 0$ are singlevalued at y_c, then there exists a solution $V(y,\lambda)$ of (4.1) such that

$$V(y,\lambda) = v(y) + \frac{E(S_c)}{\lambda^2} . \qquad (4.7)$$

The function $\eta(y)$ has an explicit form in terms of the coefficients of (4.1) and is regular in $S_c - C_j$. The solutions $A_j(y,\lambda)$ are termed dominant or subdominant according as their corresponding exponent $\lambda Q_c(y)$ has a positive or negative real part respectively. From Theorem A., clearly A_j is subdominant in S_j and dominant in the other two sectors since the curves C_j are where $\mathrm{Re}\left[\lambda\, Q_c(y)\right] = 0$. Nothing is stated about the asymptotic character of A_j on C_j, and crossing C_j will cause the asymptotic form of A_j to change abruptly since we have to change from one branch of $\lambda Q_c(y)$ to the other. It is well to state explicitly that the U_j, by Theorems B. and C., are of known asymptotic form in just two sectors each and that such a relation as $U_2 \sim u$ is meant to hold only in S_1 and S_3.

We have similar theorems and conditions at y_0. These solutions are $\tilde{A}_j(y,\lambda)$, $\tilde{U}_j(y, \ )$, $j = 1, 2, 3$, and $\tilde{V}(y,\lambda)$.

Using (4.2) and (4.3):

$$Q_c(y) = \int_{y_c}^{y_0} \sqrt{i(w-c)}\; dy + \int_{y_0}^{y} \sqrt{i(w-c)}\; dy = Q_c(y_0) + Q_0(y). \quad (4.8)$$

When $\mathrm{Re}\left[\lambda\, Q_c(y_0)\right] = 0$, then for each y $\mathrm{Re}\left[\lambda\, Q_c(y)\right] = \mathrm{Re}\left[\lambda\, Q_0(y)\right]$. In particular, when y travels a C_j it also is traveling a $\tilde{C}_j$ so that the configuration is topologically like that of Fig. 5. We have the same configuration if only a single intersection point is known, and $\mathrm{Re}\left[\lambda\, Q_c(y_0)\right] = 0$ is necessary as well as sufficient for the situation of Fig. 5; for when y travels a C_j we have $\operatorname{Re}\left[\lambda Q_c(y)\right] = 0$, and

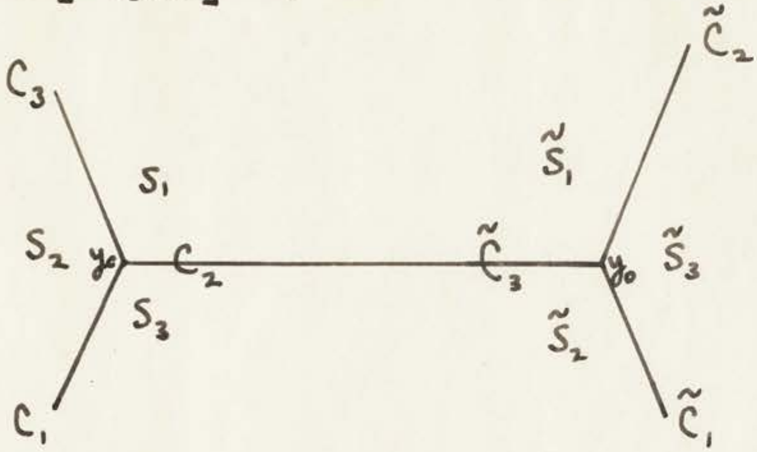

Fig. 5.

(4.8) shows $\operatorname{Re}\left[\lambda Q_0(y)\right]$ is a constant, which must be zero since at the point of intersection $y = y_1$ we have $\operatorname{Re}\left[\lambda Q_c(y_1)\right] = 0 = \operatorname{Re}\left[\lambda Q_0(y_1)\right]$. Hence C_j is also some $\tilde{C}_j$.

On the other hand, the necessary and sufficient condition that no C_j intersects any $\tilde{C}_j$ is $\operatorname{Re}\left[\lambda Q_c(y_0)\right] \neq 0$. For if the inequality holds then (4.8) shows that $\operatorname{Re}\left[\lambda Q_c(y)\right]$ and $\operatorname{Re}\left[\lambda Q_0(y)\right]$ cannot be zero simultaneously, and conversely. This is the situation of Fig. 6.

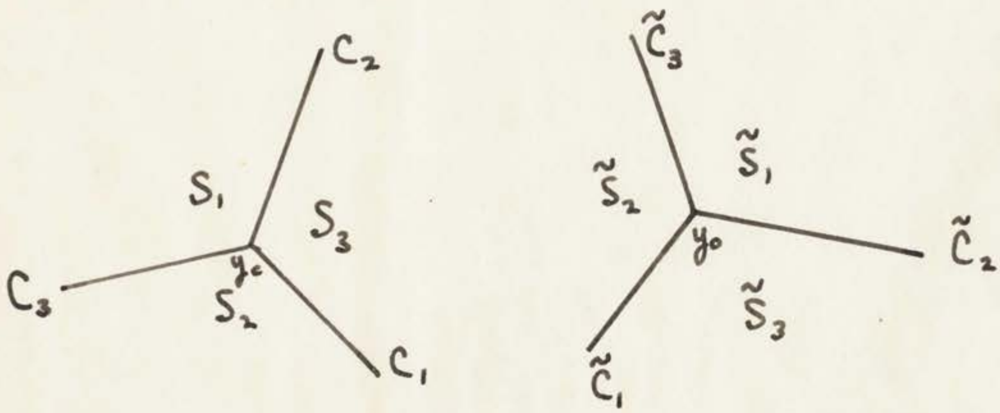

Fig. 6.

This is the general case and holds in the hydrodynamics problem when y_c and y_o are real, or when they have small imaginary parts. This will be taken up in more detail following proof of the main theorem.

In the general case we note from (4.8) that the curve $\mathrm{Re}\left[\lambda Q_c(y)\right] =$ constant includes as a special case the curve $\mathrm{Re}\left[\lambda Q_c(y)\right] = \mathrm{Re}\left[\lambda Q_c(y_o)\right]$, which is just one or more of the $\tilde{C}_j$; also the C_j belong to the family $\mathrm{Re}\left[\lambda Q_o(y)\right] =$ constant. Hence the general configuration for these curves is as is given in Fig. 7.

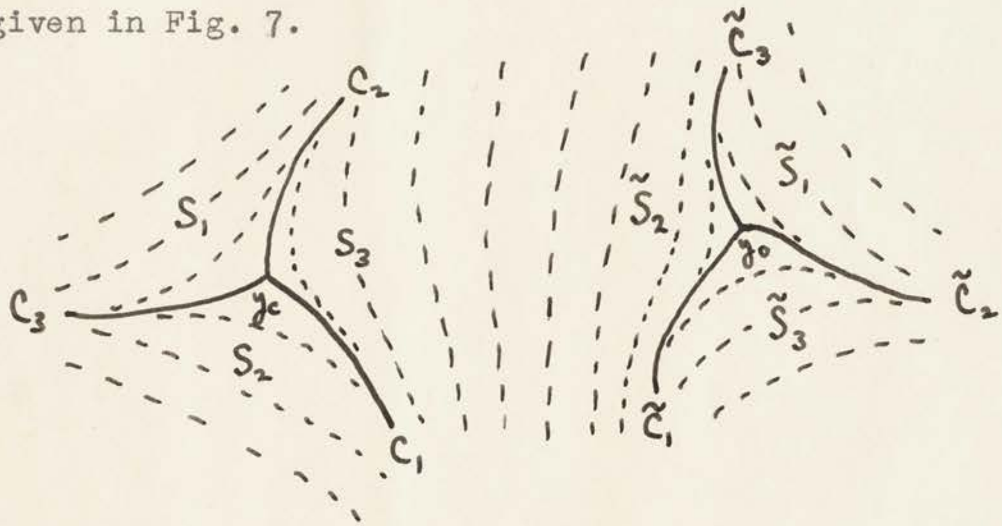

Fig. 7.

We can also make use of (4.8) in determining the nature of the $\tilde{A}_j$, $j = 1, 2, 3$, if we take a path crossing the C_j, $j = 1, 2, 3$. For example, consider $\tilde{A}_2$ on a path in $\tilde{S}_2$ approaching C_2. In $\tilde{S}_2$ we know $\mathrm{Re}\left[\lambda Q_o(y)\right] < 0$ since $\tilde{A}_2$ is subdominant there, and this real part will decrease monotonically, in crossing the contours of Fig. 7, from zero at

$\tilde{C}_3$ to $-\mathrm{Re}\left[\lambda Q_c(y_0)\right] < 0$ at C_2, because on C_2 $\mathrm{Re}\left[\lambda Q_c(y)\right] = 0$. Thus $\mathrm{Re}\left[\lambda Q_c(y_0)\right] > 0$ and (4.8) shows $\mathrm{Re}\left[\lambda Q_c(y)\right] > 0$ for y in $\tilde{S}_2$ or in S_3 . Whether $\mathrm{Re}\left[\lambda Q_0(y)\right]$ continues to decrease on crossing C_2 depends on the way $\mathrm{Re}\left[\lambda Q_c(y)\right]$ behaves across C_2 , since the variable real parts differ by the constant $\mathrm{Re}\left[\lambda Q_c(y_0)\right]$. For $y = y_1$ on C_2 we shall see in (4.23) that for y_1 near y_c we shall have

$$\mathrm{ang}\left[\lambda Q_c(y_1)\right] = \beta_1 + \tfrac{3}{2}\theta_1 \ ,$$

where β_1 is a constant and $\theta_1 = \mathrm{ang}(y_1 - y_c)$. Thus we have

$$\mathrm{Re}\left[\lambda Q_c(y_1)\right] = \cos(\beta_1 + \tfrac{3}{2}\theta_1) = 0 \ .$$

Now if

$$\delta = \mathrm{ang}(y - y_c) - \mathrm{ang}(y_1 - y_c)$$

takes on both positive and negative values then for y crossing C_2 the sign of $\mathrm{Re}\left[\lambda Q_c(y)\right]$ is governed by

$$\cos(\beta_1 + \tfrac{3}{2}\theta_1 + \tfrac{3}{2}\delta) = -\sin(\beta_1 + \tfrac{3}{2}\theta_1)\sin\tfrac{3}{2}\delta,$$

which changes sign with δ so that in S_1 we have $\mathrm{Re}\left[\lambda Q_c(y)\right] < 0$, which proves that $\mathrm{Re}\left[\lambda Q_0(y)\right]$ continues to decrease on crossing C_2. Exactly the same procedure proves $\mathrm{Re}\left[\lambda Q_0(y)\right]$ continues to decrease on crossing C_1 into S_2 . In the latter case $\mathrm{Re}\left[\lambda Q_c(y)\right]$ changes from positive to negative on crossing C_1 , so that (4.8) gives

$$A_2 = e^{\lambda Q_c(y_0)}\tilde{A}_2 \ , \tag{4.9}$$

where $\mathrm{Re}\left[\lambda Q_c(y_0)\right] > 0$.

For the general case we now determine the directions near y_0 which the boundary of S_c must have. First some definitions:

$$w_c' = |w_c'|\, e^{i\theta_c} \quad , \quad w_0' = |w_0'|\, e^{i\theta_0} \tag{4.10}$$

$$\left.\begin{aligned} &Q_c(y_0) = Ke^{i\varphi_c} = -Q_0(y_c) = Ke^{i(\varphi_0+\pi)} \\ &\text{whence} \quad \varphi_c = \pi + \varphi_0 \end{aligned}\right\} \tag{4.11}$$

Expanding $w-c$ near y_0 we have

$$w - c = w_0'(y - y_0) + \cdots .$$

Near y_0 , on taking

$$y - y_0 = \rho e^{i\theta}, \tag{4.12}$$

and using (4.10) and (4.12):

$$(w-c)^{1/2} \cong |w_0'|^{1/2}\, e^{i\frac{\theta_0}{2}} (y - y_0)^{1/2}. \tag{4.13}$$

We can put

$$\Delta Q_c(y) \equiv Q_c(y) - Q_c(y_0) = \int_{y_0}^{y} \sqrt{i(w-c)}\, dy = Q_0(y) \; . \tag{4.14}$$

Then using (4.12) and (4.13) with (4.14) gives:

$$\begin{aligned} \Delta Q_c(y) &\cong e^{i\frac{\pi}{4}} \int_{y_0}^{y} |w_0'|^{1/2} e^{i\frac{\theta_0}{2}} (y - y_0)^{1/2}\, dy \\ &= e^{i\left(\frac{\pi}{4} + \frac{\theta_0}{2} + \frac{3}{2}\theta\right)} |w_0'|^{1/2}\, \frac{2}{3}\, \rho^{3/2} \; . \end{aligned} \tag{4.15}$$

Hence

$$\arg \Delta Q_c(y) \cong \frac{\pi}{4} + \frac{1}{2}\theta_0 + \frac{3}{2}\theta \cong \arg Q_0(y) \; . \tag{4.16}$$

The problem is to determine the angle θ at which y can leave y_0 so that, referring to Fig. 8, ΔQ_c does not leave

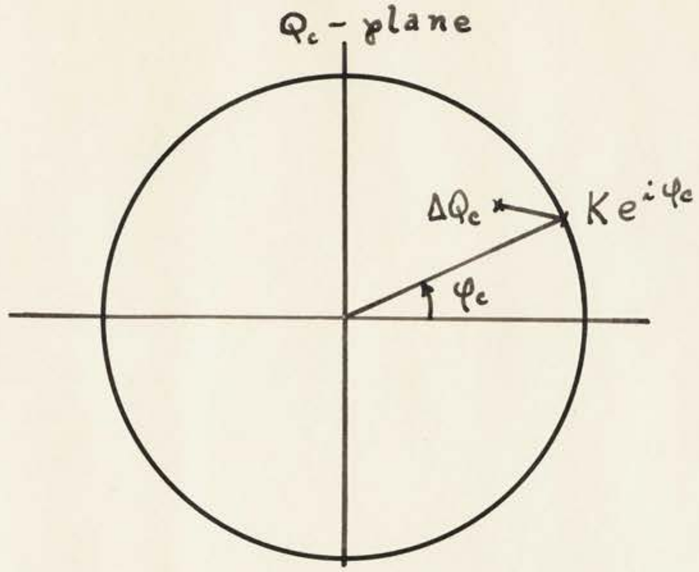

Fig. 8.

the circle, for this will mean y is not outside the boundary of S_c . It is clear from Fig. 8 that we should have

$$\varphi_c - \frac{\pi}{2} > \text{ang}\,\Delta Q_c > \varphi_c - \frac{3\pi}{2} \quad . \tag{4.17}$$

On inserting (4.16) into (4.17) the result may be written

$$\left(\frac{2\varphi_c - \theta_0}{3} - \frac{5\pi}{6}\right) + \frac{\pi}{3} > \theta > \left(\frac{2\varphi_c - \theta_0}{3} - \frac{5\pi}{6}\right) - \frac{\pi}{3} \tag{4.18}$$

From (4.18) we conclude that S_c near y_0 is like a sector of angle $\frac{2\pi}{3}$ and that the two lines tangent to the boundary at y_0 are symmetric with respect to the line having angle $\frac{2\varphi_c - \theta_0}{3} - \frac{5\pi}{6}$. In general this sector will not coincide at y_0 with a pair of the $\tilde{C}_j$, and this general case is illustrated in Fig. 9 with $\tilde{C}_4$ and $\tilde{C}_5$ as the two new lines including a part of $\tilde{C}_1$. It is easy to see that (4.18) gives the shape of S_0 near y_c if we interchange subscripts.

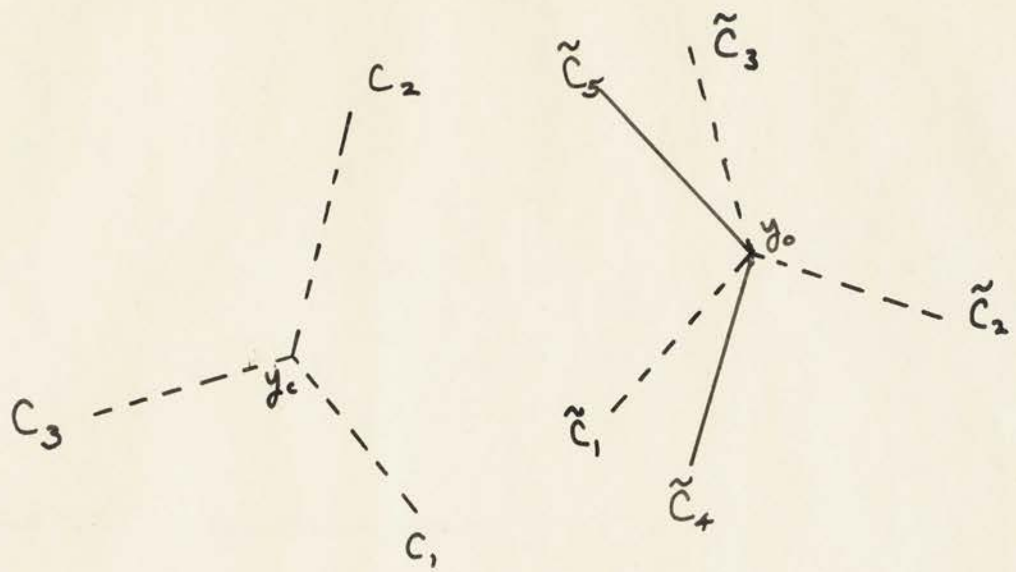

Fig. 9.

We dispose of the special case where, for example, $\tilde{C}_3$ and $\tilde{C}_5$ coincide, and $\tilde{C}_1$, $\tilde{C}_4$ coincide by determining a necessary and sufficient condition for this to happen. This, however, is done under the assumption that the two singularities of $M(\varphi) = 0$ are of the logarithmic type. It has been established [1] that a pair of the $\tilde{C}_j$, say $\tilde{C}_2$ and $\tilde{C}_3$, have angles near y_0 given by

$$\begin{aligned} \theta_{\tilde{C}_2} &= \frac{\pi}{6} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda = \left(-\frac{\pi}{2} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda\right) + \frac{2\pi}{3} \\ \theta_{\tilde{C}_3} &= -\frac{7\pi}{6} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda = \left(-\frac{\pi}{2} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda\right) - \frac{2\pi}{3} \end{aligned} \tag{4.19}$$

From this we have

$$\theta_{\tilde{C}_1} = -\frac{\pi}{2} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda \ .$$

Hence for sector $\tilde{S}_2$ near y_0 :

$$-\frac{\pi}{2} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda < \operatorname{ang}(y - y_0) < -\frac{7\pi}{6} - \frac{\theta_0}{3} - \frac{2}{3}\operatorname{ang}\lambda \tag{4.20}$$

If this is to coincide with the sector of (4.18), we must have

$$\varphi_c = -\operatorname{ang}\lambda \quad . \tag{4.21}$$

The condition (4.21) is also sufficient to make the sectors coincide, for if it is put into (4.20), the result is (4.18). In a similar manner we find that the sector (4.18) coincides with $\tilde{S}_3$ near y_0 when

$$\varphi_c = -\operatorname{ang}\lambda + \pi ,$$

and that it coincides with $\tilde{S}_1$ when

$$\varphi_c = -\operatorname{ang}\lambda + 2\pi .$$

The last equation becomes

$$\varphi_c = -\operatorname{ang}\lambda - \pi$$

if we go around y_0 in the other direction, since φ_c changes by 3π when y changes by 2π . (See (4.22) below). Henceforth we ignore these special cases.

The special case of Fig. 5 now can be treated in more detail to find the orientation of the boundaries of S_c near y_0 . The result will be that these boundaries are symmetric with respect to the common curve C between y_c and y_0 . Referring to Fig. 10, suppose C enters y_c at angle θ_1

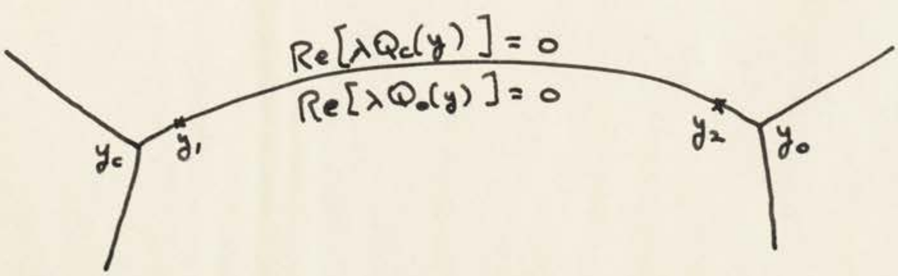

Fig. 10

and y_0 at angle θ_2. For y_1 near y_c and on C, corresponding to (4.12) and (4.13) we have

$$(w-c)^{1/2} = |w_c'|^{1/2} e^{i\frac{\theta_c}{2}} (y_1 - y_c)^{1/2}$$
$$y_1 - y_c = \rho_1 e^{i\theta_1} .$$

Then

$$\left.\begin{aligned} Q_c(y_1) &= \int_{y_c}^{y_1} \sqrt{i(w-c)}\, dy = \frac{2}{3} |w_c'|^{1/2} \rho_1^{3/2} e^{i\left(\frac{\pi}{4} + \frac{\theta_c}{2} + \frac{3}{2}\theta_1\right)} \\ Q_0(y_2) &= \frac{2}{3} |w_0'|^{1/2} \rho_2^{3/2} e^{i\left(\frac{\pi}{4} + \frac{\theta_0}{2} + \frac{3}{2}\theta_2\right)} \quad . \end{aligned}\right\} \qquad (4.22)$$

But for all y on C: $\text{Re}\left[\lambda Q_c(y)\right] = \text{Re}\left[\lambda Q_0(y)\right] = 0$, so that $\text{ang}Q_c(y)$ and $\text{ang}Q_0(y)$ are constant along C. Hence, using (4.11) and (4.22) we have

$$\left.\begin{aligned} \varphi_c &= \frac{\pi}{4} + \frac{\theta_c}{2} + \frac{3}{2}\theta_1 \\ \varphi_0 &= \frac{\pi}{4} + \frac{\theta_0}{2} + \frac{3}{2}\theta_2 \end{aligned}\right\} \qquad (4.23)$$

Subtracting equations (4.23) and using (4.11):

$$\theta_c - \theta_0 = 2\pi - 3(\theta_1 - \theta_2) \ ,$$

so that

$$2\varphi_c - \theta_0 = \frac{\pi}{2} + (\theta_c - \theta_0) + 3\theta_1 = \frac{5\pi}{2} + 3\theta_2 \quad ,$$

and using this in (4.18) yields

$$\theta_2 + \frac{\pi}{3} > \theta > \theta_2 - \frac{\pi}{3} \ ,$$

which shows that the sector S_c near y_0 is symmetric with respect to C. Similar computation shows S_0 near y_c is symmetric to C.

We now proceed with the more general configuration, a combination of Fig. 7 and Fig. 9. This corresponds to the flows of §2. The lines $\tilde{C}_4$ and $\tilde{C}_5$ appear to limit the region of validity of the asymptotic expansions of the solutions A_j, U_j, $j = 1, 2, 3$, V. However, we can prove this is not actually the case. All the solutions of (4.1) are analytic functions of y and of λ so that any one solution is a linear combination of a fundamental system. We recall the assumption that S_c near y_o includes part of $\tilde{C}_1$ in its interior and for convenience select the fundamental system

$$\tilde{A}_2 \;, \tilde{A}_3 \;, \tilde{U}_1 \;, \tilde{V} \;, \tag{4.24}$$

where we choose $\tilde{U}_1$, $\tilde{V}$ so that the following asymptotic relationships hold for each suitable y:

$$\tilde{U}_1 \sim \tilde{u} \;, \quad \tilde{V} \sim \tilde{v} \;, \tag{4.25}$$

where $\tilde{u}(y)$ is the solution of the reduced equation

$$M(\varphi) = 0 \tag{4.26}$$

multivalued at y_o, and $\tilde{v}(y)$ is the solution of (4.26) regular at y_o. Each solution from the fundamental system,

$$A_2 \;, A_3 \;, U_1 \;, V \;, \tag{4.27}$$

can be written as a linear combination of the solutions (4.24); for example,

$$U_1 = k_1\tilde{U}_1 + k_2\tilde{A}_2 + k_3\tilde{A}_3 + k_4\tilde{V} \;. \tag{4.28}$$

This is an analytic function and (4.28) holds anywhere. The coefficients k_i, $i = 1, 2, 3, 4$, are functions of λ, and something of their asymptotic behavior can be inferred from the known behavior of the solutions in the region common to S_c and S_o. We may choose U_1 so that $U_1 \sim \tilde{u}$ by (4.6) since the solutions of (4.26) are analytic in any simply connected region not containing y_c or y_o. Using this and (4.25) in the region common to S_c and S_o gives

$$k_1 \sim 1 \ , \ k_4 \sim 0 \ , \ (k_2 \tilde{A}_2) \sim 0 \ , \ (k_3 \tilde{A}_3) \sim 0 \ . \qquad (4.29)$$

Now if we take a point y_1 in the sector $\tilde{C}_1$, $\tilde{C}_4$ it is clear that (4.29) and the expansions of the functions in (4.28) will continue to be valid on the curve $\mathrm{Re}\left[\lambda Q_o(y)\right] = \mathrm{Re}\left[\lambda Q_o(y_1)\right]$, wherever y is in S_o and on the same segment of the curve as y_1. The curve $\mathrm{Re}\left[\lambda Q_o(y)\right] = \mathrm{Re}\left[\lambda Q_o(y_1)\right]$ is like the curves of Fig. 7. This extends the asymptotic expansion of U_1 into a certain subset of $\tilde{S}_3$ with $\tilde{C}_2$ as one boundary. The solution V extends into a subset of $\tilde{S}_3$ in exactly the same way. Using equation (4.9) we can identify A_2 with $\tilde{A}_2$ in $\tilde{S}_3$ and clearly a similar equation will identify A_3 with $\tilde{A}_2$ in $\tilde{S}_3$. The exact shape and extent of the subset of $\tilde{S}_3$ into which the expansions of the solutions (4.27) may be extended is not important and we shall identify it with the symbol $\tilde{S}_3'$.

Similarly the expansions of the solutions (4.27) can be extended across $\tilde{C}_5$ to points arbitrarily near $\tilde{C}_3$, and we shall use the symbol $\tilde{S}_2'$ to identify the subset of $\tilde{S}_2$ in which the expansions are valid. This gives

LEMMA 3. When the domain S_c includes a part of $\tilde{C}_1$, the asymptotic expansions of the fundamental system (4.27) for equation (4.1) are valid in the four sectors S_2, S_3, $\tilde{S}_2'$, $\tilde{S}_3'$, where the last two sectors are certain subsets of $\tilde{S}_2$ and $\tilde{S}_3$ partly bounded by $\tilde{C}_3$ and $\tilde{C}_2$ respectively.

It is clear that an analogy of this lemma also holds for the special case of Fig. 5 since S_c near y_0 is symmetric with respect to the common curve.

Independently of the particular notation selected here we can assert that there is a fundamental system of solutions for (4.1) with asymptotic expansions valid on a path through two sectors at y_c and through two sectors at y_0, when $\mathrm{Re}\left[\lambda Q_c(y_0)\right] \neq 0$. The particular pair of sectors at y_0 will have as a common boundary the $\tilde{C}_j$ which is partly included in S_c. This lemma indicates that the boundaries of S_c and of S_0 are not necessarily bars to the asymptotic expansions and reference to the proofs in [5] indicate that the main requirement in establishing a domain of validity of the expansions is to have monotonicity of $\mathrm{Re}\left[\lambda Q(y)\right]$.

Finally we shall prove the

THEOREM 5. There exists a solution $Q_3(y,\lambda)$ of (4.1) having an unchanging asymptotic expression valid in sectors S_2, S_3, $\tilde{S}_2'$, $\check{S}_1$ when the domain S_c includes a part of $\tilde{C}_1$.

The effect of this is to state that we can cross $\tilde{C}_3$. We shall find $Q_3 \sim \tilde{u}$; it is then clear from the theorems of page 30 that Q_3 must diverge in $\tilde{S}_3$.

Choose a new fundamental system $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{U}_3$, $\tilde{V}$, where $\tilde{U}_3 \sim \tilde{u}$ and $\tilde{V} \sim \tilde{v}$. For the proof we shall use a fixed arbitrary path C passing through the sectors of the Theorem and crossing C_1 and $\tilde{C}_3$. Define a solution $Q_1(y,\lambda)$ of (4.1) in terms of (4.27):

$$\begin{aligned} &Q_1 = n_1 U_1 + n_2 A_2 + n_3 V \\ &Q_1 \sim \tilde{u} \text{ in } S_2 \text{ and } S_3 \, . \end{aligned} \tag{4.30}$$

Then by Lemma 3, $Q_1 \sim \tilde{u}$ holds also in $\tilde{S}_2'$ and $\tilde{S}_3'$, and this carries the expansion of Q_1 up to $\tilde{C}_3$ along path C. Now $Q_1 - \tilde{U}_1$ is a solution so that

$$\begin{aligned} &Q_1 - \tilde{U}_1 = c_1 \tilde{A}_1 + c_2 \tilde{A}_2 + c_3 \tilde{U}_3 + c_4 \tilde{V} \\ &(Q_1 - \tilde{U}_1) \sim 0 \text{ in } \tilde{S}_2' \text{ and } \tilde{S}_3' \, . \end{aligned} \tag{4.31}$$

Thus as we traverse C the asymptotic relation in (4.31) begins to hold when C enters $\tilde{S}_2'$ and then extends to $\tilde{C}_3$. In $\tilde{S}_2$ the expansion of $\tilde{A}_1$ diverges so that (4.4), (4.5) and (4.31) give $c_1 \sim e^{-K\lambda}$ and $c_3 \sim 0$, $c_4 \sim 0$. The relations for the c_i , $i = 1,3,4$, then hold throughout S_0 . For $\tilde{A}_2$ we note the three properties: 1) $c_2 \tilde{A}_2$ diverges in $\tilde{S}_1$ to reflect the divergence of $\tilde{U}_1$ there, by (4.31); 2) $\tilde{A}_2$ is subdominant in $\tilde{S}_2$; 3) Both $\tilde{U}_3$ and $\tilde{A}_2$ diverge in $\tilde{S}_3$ and in (4.31) cancel each other there. These properties indicate no definite asymptotic form for c_2, but it shall be sufficient to note that c_2 does not dominate $\tilde{A}_2$ where the latter is subdominant.

Again, $\tilde{U}_1 - \tilde{U}_3$ is a solution of (4.1) so that

$$\left.\begin{aligned} &\tilde{U}_1 - \tilde{U}_3 = m_1\tilde{A}_1 + m_2\tilde{A}_2 + m_3\tilde{U}_3 + m_4\tilde{V} \\ &(\tilde{U}_1 - \tilde{U}_3) \sim 0 \text{ in } \tilde{S}_2' . \end{aligned}\right\} \qquad (4.32)$$

In particular, the asymptotic relation in (4.32) holds on path C after C enters $\tilde{S}_2'$, and holds up to $\tilde{C}_3$. Then $m_1 \sim e^{-K\lambda}$ in $\tilde{S}_2'$ since otherwise $m_1\tilde{A}_1$ might diverge there. Again, m_2 does not dominate $\tilde{A}_2$ in $\tilde{S}_2$, and we have $m_3 \sim o$, $m_4 \sim o$. The relations concerning m_i, $i = 1, 3, 4$, hold throughout S_0. The right side of (4.32) again reflects the divergence of $\tilde{U}_1$ or of $\tilde{U}_3$ in the sectors $\tilde{S}_1$ and $\tilde{S}_3'$ respectively.

Now we add (4.31) and (4.32) to obtain

$$Q_1 - \tilde{U}_3 = (c_1 + m_1)\tilde{A}_1 + (c_2 + m_2)\tilde{A}_2 + (c_3 + m_3)\tilde{U}_3 + (c_4 + m_4)\tilde{V}. \quad (4.33)$$

In this expression all the coefficients have known asymptotic behavior in $\tilde{S}_2'$ except $(c_2 + m_2)$, and this cannot dominate $\tilde{A}_2$ for large λ in $\tilde{S}_2'$, i.e., in S_3, since the discussion preceding (4.9) has established this for $\tilde{A}_2$. The functions $\tilde{A}_1$, $\tilde{U}_3$, $\tilde{V}$ are well behaved in both $\tilde{S}_2'$ and $\tilde{S}_1$ and they do not change asymptotic representation on crossing $\tilde{C}_3$.
Hence we define a solution of (4.1):

$$Q_3(y,\lambda) \equiv Q_1(y,\lambda) - (c_2 + m_2)\tilde{A}_2(y,\lambda) \qquad (4.34)$$

and we observe that in $\tilde{S}_2$ (and S_3) $Q_3 \sim \tilde{u}$, since $Q_1 \sim \tilde{u}$ there and $\left[(c_2 + m_2)\tilde{A}_2\right] \sim o$. From (4.33) we have $Q_3 \sim \tilde{u}$ in $\tilde{S}_1$.
Finally, in S_2 we have $Q_3 \sim \tilde{u}$ since $Q_1 \sim \tilde{u}$, and it was noted in (4.9) that $\tilde{A}_2$ retains its subdominant behavior in S_2.
This makes $Q_3 \sim \tilde{u}$ at all points of such a path as C and proves the theorem.

Using (4.9) we can write Q_3 as a linear combination of the fundamental system (4.27), although we may not be able to determine explicitly all the coefficients.

We can be somewhat more definite about the orientation of the sectors S_j, $\tilde{S}_j$, $j = 1, 2, 3$, if we return to the hydrodynamics situation of §2, i.e., assume an unsymmetric profile with real y_c and y_o. However, if y_c and y_o have small imaginary parts the sectors are rotated only slightly, since $w(y)$ in general will not be changing rapidly near the real axis, and the configuration remains basically the same. By [1] at y_c we can find solutions corresponding to (4.27) valid in

$$-\frac{7\pi}{6} < \arg(a\eta) < \frac{\pi}{6},$$

where $a = (w_c')^{1/3}$ and $\eta = (\alpha R)^{1/3}(y - y_c)$. In this, $w_c' > 0$ so that we may take ang $w_c' = 0$. This is also the form of the sector at y_o, but we must determine $\arg(w_o')^{1/3}$ properly, where $w_o' < 0$. By examining three power series expansions of $w'(y)$ near 1) a negative y_1, 2) the origin, 3) a positive y_2, we can find $\arg(w_o')^{1/3} = -\frac{\pi}{3}$, i.e., we go into the lower half of the complex plane on changing from positive to negative slope of the velocity profile. This gives

$$-\frac{3\pi}{2} < \arg(y - y_o) < -\frac{\pi}{6}$$

as a sector for validity of the asymptotic expansions of solutions in a fundamental system at y_o.

By examining power series for $w - c$ we find:

$$\arg(w-c) = \begin{cases} 0, & y_c < y < y_o \;. \\ \theta = \arg(y - y_c), & \text{near } y_c \;. \\ -\pi + \theta \;, & \text{near } y_o, \text{ where } \theta = \arg(y - y_o). \end{cases}$$

These relations make it possible to find φ_c and φ_o so that we can obtain the directions of S_c and S_o near y_o and y_c respectively. These appear as dashed lines in Fig. 11. The

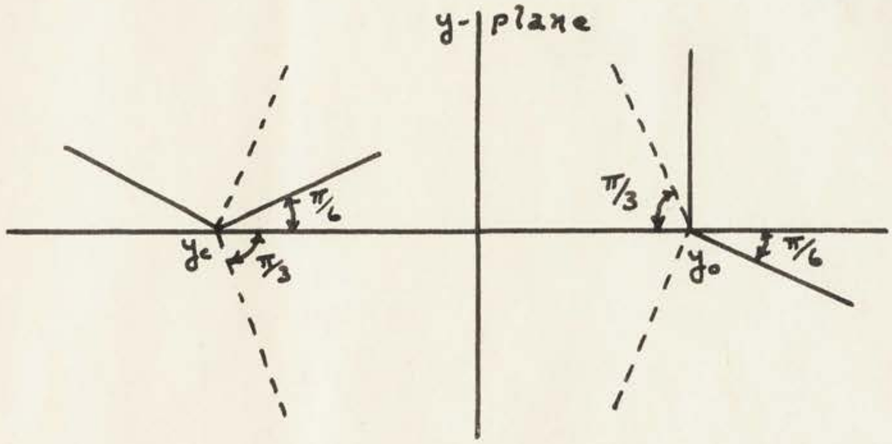

Fig. 11.

solid lines are for the above sectors. The diagram explains the necessity of taking a path <u>below</u> y_c , as we did in the proof of Lemma 2.

In conclusion we mention some of the problems yet to be solved in connection with the jet problem. First, the existence of a neutral disturbance might be demonstrated, and for this one might extend existing results concerning channel flow. Then the calculations for the eigenvalue problem of §3 should be carried out to the extent that the curve $c_i(\alpha, R) = 0$ is found; in this connection one might use the methods already used for the monotonic profile extending to infinity in both directions. In such a calculation the theorem of §4 would be assumed to hold along the real axis to infinity. There is indication that a theorem might be found justifying the use of the asymptotic expansions along such a path, since it appears that the essential requirement is the monotonicity of $\mathrm{Re}\left[\lambda Q(y)\right]$. Finally one might correlate the present hydrodynamics theory with existing theory to help complete the classification of unsymmetric velocity profiles for channel flow.

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ABSTRACT

The results contained in this paper form the mathematical basis for the investigation of the stability of two-dimensional parallel (symmetric or unsymmetric) jet flow from a narrow slit and related problems. An investigation of the asymptotic properties of the solutions of the stability equation for large Reynolds numbers leads to a complete formulation of the eigenvalue problem in a form suitable for detailed calculations. As usual, only periodic disturbances are studied. It is shown that the effect of viscosity enters the problem in a manner completely different from the stability problems of boundary layer flow and channel flow. Certain general conclusions are reached for the "inviscid case."

The velocity profile representing a jet flow extends to infinity in both directions and has two points of inflection. These properties make the problem differ greatly from stability problems previously investigated. The fourth order stability equation has two solutions with exponential asymptotic behavior which cannot be used in the boundary value problem whether or not we have very large Reynolds number in the flow. The other pair of solutions tend in the

limit of infinitely large Reynolds number to a fundamental system of the second order equation of inviscid flow. To bring in the effect of viscosity we can only take higher order terms in the asymptotic expansions of this pair. Further, the stability problem tends for large Reynolds number to that of the inviscid case. For these reasons a study is made of the inviscid equation. As is well known, the complete stability equation has no singularity, while the inviscid equation has a leading coefficient with one or more simple zeros, which become logarithmic branch points of the solutions of the inviscid equation.

In the case of the symmetric velocity profile it is proved that for a neutral disturbance (if one exists) the wave velocity must be equal to the flow velocity at the points of inflection of the velocity profile. The only exceptional case is the trivial case of a steady deviation. For more general velocity distributions, a form of the x-average of the disturbance velocity product $\overline{u'v'}$ is found in terms of the Wronskian of a certain pair of functions related to the general solution of the inviscid equation. In the case of a neutral disturbance these functions are also solutions of the inviscid equation and the discontinuity in the value of the Wronskian across a singular point of the inviscid equation is calculated, with the result that a symmetric profile has $\overline{u'v'} = 0$ throughout the flow, and that for an unsymmetric profile the value of the wave velocity lies between the values of the two ordinates to the profile at the two points of inflection.

For the purpose of investigating the analytical properties of the solutions, the stability equation is replaced by a somewhat more general equation, $N[\varphi]+\lambda^2 M[\varphi]=0$. Wasow has proved four theorems concerning the asymptotic expansions of a fundamental system for this equation. The domain of validity of these expansions is a sector in a certain annulus around and excluding a first order zero of the leading coefficient of $M[\varphi]=0$, which corresponds to the inviscid equation. Several choices of a fundamental system exist, each differing from another only in location of sector of validity around the annulus. The expansions can change form abruptly in crossing three curves which divide the annulus into three sectors. These are the curves corresponding to the usual Stokes phenomenon. Using two such simple zeros, a criterion is found which allows determination of the general relative orientation of the three curves originating at each zero. The special case wherein one curve from one zero coincides with a curve from the second zero is noted and the directions of the outer boundary of the annulus around the first zero are determined near the second zero. It is shown that this boundary near the second zero is not actually a bar to the asymptotic expansions. For the more general orientation of the six curves, a criterion is found for the coincidence of the outer boundary around the first zero with a pair of curves originating at the second zero. The general case is taken as the case where this coincidence does not occur and where there is no intersection of a curve from one zero with a curve from the other zero. Again it is found that for the general case the outer boundary near the second zero is not a bar to the asymptotic expansions. Further it is proved that there exists a solution of the complete equation having an unchanging asymptotic expression valid in two sectors at each zero; i.e., the Stokes phenomenon can be avoided, and this with a solution of the reduced equation $M[\varphi]=0$.

The general case is illustrated by the hydrodynamics problem of the unsymmetric jet profile with a real value of the wave velocity assumed.

BIOGRAPHICAL NOTE

Joe Reeder Foote was born in Amarillo, Texas, on August 17, 1919, and is the youngest of a family of seven. His father was a Methodist minister so that he attended school in many towns of West Texas, finally graduating from high school at Slaton. He played a trombone in high school and college bands for recreation, but his main interest at college was in mathematics and physics, He attended Texas Technological College, graduating first in his Class of 1940. This honor was easily lived down, however, the next year at the University of Texas Graduate School. He was accepted as Aviation Cadet in September 1941 and inducted in January 1942 and trained as a single engine fighter pilot. He participated in action conducted in the Aleutians, and on returning to the U. S. he instructed in fighter pilot training and attended several Air Force schools. Upon discharge in September 1945 he became for one year an instructor of mathematics at the University of Oklahoma, in view of his experience in teaching at the University of Texas. He was an instructor of mathematics at M.I.T. the next three years, and has accepted an assistant professorship in mathematics at Iowa State College, Ames, Iowa. He is still single.